Utrecht University

Graduate School of Natural Sciences

Theoretical Physics master's thesis

Carrollian Physics and its Thermodynamics description

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Abstract

Carroll symmetry emerges as a consequence of the limit where the speed of light tends to zero, starting from Poincaré symmetry. Further, it is expected that the Carrollian thermodynamics description in the strict Carroll limit in different frameworks gives a cosmological equation of state $\mathcal{E} + \mathcal{P} = 0$. To establish a rigorous thermodynamic setup, both for the individual particles and massless scalar quantum field theories we employ an imaginary chemical potential conjugate to momentum. Then we focus on the diagonality condition of the stress energy tensor in the Carroll regime and the leading term in $\left(\frac{c}{v}\right)$- expansion. The conjecture linking Carroll conformal field theories with flat space holography regarding the BMS_3 group, potentially extending to de Sitter spacetime is gaining traction. This study holds seeds for the relevance of Carroll physics to dark energy and inflation. To make this master thesis self-consistent there is a brief review including Carroll particles and Carroll quantum field theories that contains some new material. The electric and magnetic sectors are presented and in thermodynamics, they appear to exhibit uniform behavior in the massless case.

Acknowledgments

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Contents

Chapter 1

Introduction

In inflationary cosmology, the pressure and the energy density are related by the equation of state. This relation determines the time evolution of an expanding universe via the Friedmann-Lemaître-Robertson-Walker(FLRW) metric and the evolution through Einstein field equations. Moreover, in a universe solely driven by the cosmological constant, often denoted by Λ, the energy-momentum tensor acquires the form of a perfect fluid and the equation of state reads

$$\mathcal{E} + \mathcal{P} = 0 \quad . \tag{1.1}$$

Depending on the constituents of the universe, the relation between energy density and pressure varies. To parametrize this variance, the equation of state parameter is introduced that measures the relation between the two quantities

$$\mathcal{P} = w\mathcal{E} \quad . \tag{1.2}$$

The case with $w = -1$ produces an exponentially expanding universe with a scale factor $a(t) \sim e^{H_0 t}$ with H_0 denoting the Hubble constant and the scale factor defined within the line element for FLRW spacetime

$$ds^2 = -c^2 dt^2 + a(t) dx_i dx^i, \qquad i = 1, ..., d \quad . \tag{1.3}$$

An exponentially expanding universe introduces a horizon beyond which information cannot reach a static observer. It is formulated by the introduction of the Hubble radius

$$R_H = \frac{c}{H_0} \quad . \tag{1.4}$$

Every particle situated beyond the Hubble radius cannot be observed. Effectively, any particle could be viewed as a tachyon when lying outside of the Hubble radius, moving with recessional speeds greater than the speed of light, $v \gg c$ or equivalently $\frac{c}{v} \ll 1$.

From this perspective, the super-Hubble scales allow for a Carrollian physics description. In Carrollian physics the strict limit $c \to 0$, keeping H_0 fixed, becomes manifest and an intuitive way to think about it is to state all the particle speeds in the spacetime to be much greater than the speed of light, hence tachyonic (Figure 1.1).

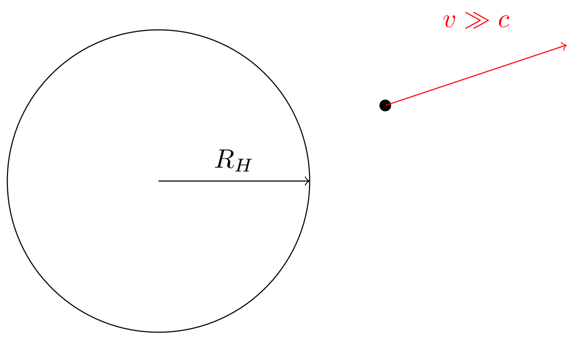


Figure 1.1: Tachyonic speeds in super-Hubble scales.

Ultimately, when the strict Carroll limit is considered, the whole Hubble radius shrinks down to a single point and all of the spacetime becomes super-Hubble. Therefore, it can be described by a Carroll framework.

An additional motivation to make such an attempt is the fact that if instead of a cosmological constant Λ, a scalar field is introduced to parametrize the exponential expansion, for instance, the inflaton, then the equation of state parameter becomes

$$w = \frac{\frac{1}{2}c^2\pi_\phi^2 - V(\phi)}{\frac{1}{2}c^2\pi_\phi^2 + V(\phi)} \overset{c\to 0}{\to} -1 \quad . \tag{1.5}$$

Here the π_ϕ denotes the conjugate momentum to the scalar field, defined by $\pi_\phi = \frac{1}{c^2}\dot{\phi}^2$. And in the strict Carroll limit, $c \to 0$, it acquires the very same form as the equation of state parameter for an exponentially expanding universe [1], provided that in the limiting procedure, it is kept fixed. The same result is extracted for slow-roll inflation in which the scalar field is slowly varying, typically when the condition $\dot{\phi} \approx 0$ is satisfied.

When Carroll conditions are applied to an energy-momentum tensor it obtains a diagonal form that is needed in a spacetime dominated by dark energy. Different frameworks for which Carroll conditions are satisfied and Carroll symmetry is present should lead to a model for inflationary cosmology.

As was initially presented in [1] the Carroll limit is also relevant to de Sitter spacetime and therefore it provides seeds to study de Sitter features through a Carrollian model. Therefore, besides the BMS (Bondi-Metzner-Sachs) algebra matching the Carroll algebra, indicating a potential duality to asymptotically flat space [2, 3], de Sitter space and cosmology and their relation to Carrollian physics gain traction.

The intention of this master thesis is to provide a well-defined thermodynamics description for two settings: the single particle and the conformal field theory (CFT), under the Carroll limit. Following the establishment of thermodynamics and after writing down a partition function for each case, the Carroll limit is taken in order to conclude an equation of state and check the validity of $\mathcal{E} + \mathcal{P} = 0$. The limits we take have to be on dimensionless quantities to make sense, thus some extra care is required than naively writing $c \to 0$. This point will be analyzed separately for the single particle and the conformal field theory (CFT).

Some quantities diverge in the strict Carroll limit $c \to 0$. Some results will be presented in the leading order of the $\left(\frac{c}{v}\right)$-expansion which we will refer to as the Carroll regime. The Carroll regime appears as the opposite procedure to the classical or Galilei limit. The Galilei limit describes the transition from relativistic mechanics to classical or Newtonian mechanics. The strict Galilei limit leads to divergences in physical quantities. Thus, special care is required when taking the limit.

To begin with, this thesis project starts with a brief review of Carroll particles and Carroll quantum field theories, adding reason to the results in the main body and making it overall self-consistent.

Chapter 2

Carroll symmetry and particles

2.1 Carroll boosts

The Carroll limit modifies the Lorentz transformations and changes the Poincare symmetry to the so-called Carroll symmetry [2, 4]. The lightcone under the Carroll limit closes up. However, there is still some room for formulating a sensible concept of particles and the respective laws they are subjected to. The Carroll symmetry can be viewed as the opposite limit to the limit $c \to \infty$ that

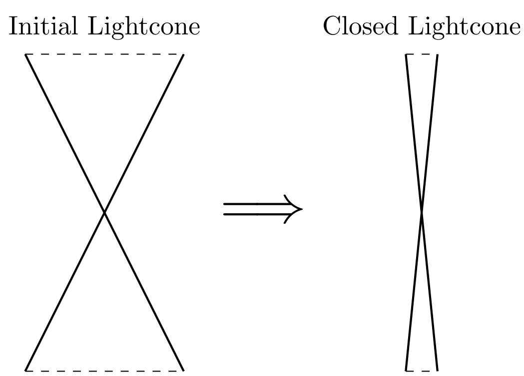


Figure 2.1: Light Cone as $c \to 0$.

gives the Galilei group. As argued in [5], starting off from the Poincare group, a different symmetry group arises, the Carroll group. For example in 1+1 dimensions, the Lorentz transformations are

$$x' = \gamma\big(x - t\frac{c^2}{v}\big) \ , \tag{2.1}$$

$$t' = \gamma\big(t - \frac{x}{v}\big) \ , \tag{2.2}$$

where v is the rescaled velocity $v \to \frac{c^2}{v}$ compared to the usual Lorentz boost parameter and the Lorentz factor $\gamma = \frac{1}{\sqrt{1-\frac{c^2}{v^2}}}$. The rescaled velocity has the interpretation of the rate of motion of an event measured at a different frame at a fixed instant of time, $\Delta t' = 0$. The initial value was valid in the range $0 \leq |v| < c$, while the rescaled lies in the range $|v| > c$. The Carroll limit can be obtained by considering the limit to the dimensionless ratio $\frac{|v|}{c} \to \infty$, yielding

$$x' = x \ , \tag{2.3}$$

$$t' = t - \frac{x}{v} \ . \tag{2.4}$$

In higher spatial dimensions, a reciprocal vector $\vec{b}$ to the velocity $\vec{v}$ that we call the *Carroll boost parameter* is suitable, such that the Lorentz transformations become

$$\vec{x'} = \vec{x} \ , \tag{2.5}$$

$$t' = t - \vec{b} \cdot \vec{x} \ . \tag{2.6}$$

As opposed to a Galilei universe where time is absolute, in a Carroll universe space is absolute. In this way, the time ordering of events does not necessarily work for different observers.

Together with spatial rotations, the Lorentz boosts form a group. Taking the Carroll limit in d spatial dimensions, the general transformations that form the Carroll group read

$$\begin{cases} \vec{x'} = R\vec{x} \\ t' = t - \vec{b} \cdot \vec{x} \end{cases} \tag{2.7}$$

where R is a three-dimensional orthogonal matrix. The transformation matrix to the basis $(ct, \vec{x})$ is

$$U(\vec{a}, R) = \begin{pmatrix} 1 & \vec{a}^T \\ 0 & R \end{pmatrix} \tag{2.8}$$

where $\vec{a} := \frac{\vec{b}}{c}$. The set of matrices $U(\vec{a}, R)$ form a group with the properties:

- unit element: $U(0, I)$
- inverse element: $U(\vec{a}, R)^{-1} = U(-R\vec{a}, R^T)$
- group composition: $U(\vec{a'}, R') \cdot U(\vec{a}, R) = U(R^T\vec{a'} + \vec{a}, R'R)$

The elements $U(\vec{a}, I)$ form a subgroup that is isomorphic to the Abelian group of d-dimensional vector space. The total group is then realized as being an extension of the Abelian group of d-dimensional vector space by the operator group of d-dimensional orthogonal matrices.

2.2 Velocity transformation

The velocity of a Carroll particle can be readily derived by writing

$$\vec{u'} = \frac{d\vec{x'}}{dt'} = \frac{\vec{u}}{1 - \vec{b} \cdot \vec{u}} \quad . \tag{2.9}$$

The transformation in the velocity of a particle exhibits different behavior depending on whether the particle velocity is zero or non-zero. If it is zero, it will remain zero regardless of Carroll boosts (as analyzed in [6]). For non-zero velocities, the Carroll boost may result in arbitrarily large velocities. It can even flip the sign and change the direction of the vector if the denominator in the transformation becomes negative, namely for $\vec{b} \cdot \vec{u} > 1$.

2.3 Energy, momentum & tachyons

Following the analysis as presented in [1] and departing from the Lorentz transformations on time and position, the transformations on energy and spatial momentum read

$$\vec{p_{\parallel}}' = \gamma\big(\vec{p}_{\parallel} - \vec{b}E\big), \qquad \vec{p_{\perp}}' = \vec{p}_{\perp} \quad , \tag{2.10}$$

$$\frac{E'}{c} = \gamma\big(\frac{E}{c} - c\vec{b} \cdot \vec{p}\big) \quad . \tag{2.11}$$

Then in the Carroll limit these transformations acquire the form

$$\vec{p'} = \vec{p} - \vec{b}E \quad , \tag{2.12}$$

$$E' = E \quad . \tag{2.13}$$

These are the transformations dictated by the Carroll group as viewed from the representation $U(\vec{a}, I)$ defined in equation (2.8). Hence, without taking into account the extension through the d-dimensional orthogonal matrices.

One thing noted here is that energy is a Carroll invariant. Again there are two distinct cases similar to the velocity transformation rule to be taken into account: $E = 0$ and $E \neq 0$.

1. For $E = 0$, the momentum becomes Carroll invariant and the states can be described exclusively by the momentum and the helicity around the axis parallel to the momentum direction.

2. For $E \neq 0$, we can Carroll boost to vanishing momentum and the states are fully described by the spin and the spin along a single axis.

In both cases, the energy and spatial momentum can be deduced from special relativity through the well-known relations

$$\begin{cases} \vec{p} = \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \\ E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} \end{cases} \tag{2.14}$$

where $u = \left|\frac{d\vec{x}}{dt}\right| = \sqrt{\vec{u} \cdot \vec{u}}$ and denotes the velocity of a particle in the usual way that is presented for relativistic particles. The reason we review these relations at this point is to highlight the appropriate way to make the transition from relativistic particles to Carrollian particles.

The denominator shared among the energy and spatial momentum expressions does not have a well-defined behavior under the strict Carroll limit. The quantity in the denominator should not be confused with the Lorentz factor, since it contains the velocity of a particle instead of a boost parameter. The leading term in the Carroll expansion in powers of c gives the value of the denominator as

$$\sqrt{1 - \frac{u^2}{c^2}} \to \pm i \frac{u}{c} \tag{2.15}$$

The energy and momentum therefore vanish in the strict Carroll limit $c \to 0$. In the leading contribution, they generically acquire complex values. There are two interesting cases to discuss at this point:

- For $\vec{u} \to 0$ in the strict Carroll limit the particles are at rest and if we decide to keep the rest energy fixed in the strict Carroll limit then

$$E = mc^2, \qquad \vec{p} = 0 \tag{2.16}$$

- For $\vec{u} \neq 0$ and in the leading term expansion that was discussed we get

$$\vec{p} = \pm imc\frac{\vec{u}}{u}, \qquad E = 0 \tag{2.17}$$

In the latter, the energy vanishes due to the relativistic dispersion relation:

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 \implies E^2 = -m^2 c^4 + m^2 c^4 = 0 \tag{2.18}$$

In this case, the combination mc has to be kept fixed in the Carroll limit and the velocity has to be made imaginary for the momentum to make sense as a physical quantity. Namely, the mass of the particle becomes imaginary since $m^2 < 0$ is the only resolution. This requirement is also needed for (2.10) expansion to be valid with $u^2 > c^2$. Then, the module of the velocity $\vec{u}$ is interpreted as independent from the momentum and arbitrarily valued, but non-zero. It only assigns to the momentum a direction through the unit vector $\frac{\vec{u}}{u}$. These particles keep on moving and they continue moving after any Carroll boost. A similar result works also for massless particles for which the relativistic dispersion relation reads

$$E = pc \tag{2.19}$$

These particles will become important in a later discussion on this thesis.

Chapter 3

Carroll quantum field theories

In this Chapter, we present the main observables of massless Carroll quantum field theories to review the well-known classification of the ”electric” and ”magnetic” sectors that emerge. A combination of magnetic and electric Lagrangian terms builds up the sector of combinations and they will not be reviewed in this thesis. The intention is to build up a setting that allows for a thermal quantum field theory assigned to the two sectors to be deduced later on. This classification in ”electric” and ”magnetic” theories arises from considering the Carroll limit to the Maxwell theory of electrodynamics. Two Carrollian limits result in Maxwell field equations to reduce to a vanishing electric field and a vanishing magnetic field respectively [1, 4, 7].

In a more general setting, these two distinct limiting procedures correspond to the timelike or spacelike structure of the relativistic theory right before taking the Carroll limit. As a result, ”electric” theories are ultralocal in space, meaning that different space points are independent of each other in terms of their dynamics, while ”magnetic” theories exhibit a non-trivial space dependence.

In this Chapter and this thesis, only massless scalar fields will be discussed. Hence, the subset of QFTs where conformal symmetry is manifest; namely conformal Carroll field theories (CFTs). Aspects of Carroll CFTs have been studied in [8–13]. Higher spin representations of the Carroll algebra have also been studied. For instance, Carroll fermions have been studied in [14–17], Carroll Yang-Mills in [18] and super symmetry (SUSY) results were presented in [19].

3.1 Electric massless scalar field

Starting from the Klein-Gordon field ϕ with action in Hamiltonian formalism

$$S = \int dt d^d x \Big(\pi_\phi \dot{\phi} - H \Big), \quad \text{with} \quad H = \int d^d x \frac{1}{2} \big(c^2 \pi_\phi^2 + \partial_i \phi \partial^i \phi \big) \tag{3.1}$$

and $i = 1, 2, ..., d$ where the indices are contracted with the flat metric. Rescaling the fields as

$$\phi \to c\phi, \quad \pi_\phi \to \frac{1}{c} \pi_\phi, \tag{3.2}$$

and taking the Carroll limit $c \to 0$, the electric limit is concluded

$$S_e = \int dt d^d x \Big(\pi_\phi \dot{\phi} - \frac{1}{2} \pi_\phi^2 \Big) \ . \tag{3.3}$$

On a flat Carroll background the action of a massless scalar field that corresponds to the electric limit and an ultra-local field theory can be expressed by reducing the canonical momentum field π_ϕ as

$$S_e = \int dt d^d x \frac{1}{2} \dot{\phi}^2 \ . \tag{3.4}$$

The equations of motion, derived as a result of the variation of the action give

$$\ddot{\phi} = 0 \tag{3.5}$$

and therefore we can compute the propagator by the corresponding Green's function to assign it the most general solution.

$$\partial_t^2 G(\vec{x} - \vec{x}'; t - t') = \delta^d (\vec{x} - \vec{x}'; t - t') \ . \tag{3.6}$$

It is easy to solve this equation in momentum space and then transfer back to position.

$$\begin{aligned} G(\vec{x} - \vec{x}'; t - t') &= \int d\omega \int d^d x \Big(- \frac{1}{\omega^2 + \epsilon^2} \Big) e^{i\omega(t-t')} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \\ &= \frac{i}{2} \Big(\frac{1}{\epsilon} - |t - t'| \Big) \delta^d (\vec{x} - \vec{x}') \ . \end{aligned} \tag{3.7}$$

This is divergent, primarily because the theory is ultra-local since the Lagrangian density does not contain any spatial derivatives on the scalar field. A way to regulate this result is to omit the divergent term of the propagator

$$G(\vec{x}-\vec{x}';t-t') = -\frac{i}{2}|t-t'|\delta^d(\vec{x}-\vec{x}') \ . \tag{3.8}$$

This analysis provides the physical part of the correlator of the electric Carroll sector for a massless scalar field. A similar approach was followed in [20] to argue about the correlators of asymptotically flat space, with a vision of a holographic description of the electric theories on the null boundary of a (d+1) flat space to correlators in the bulk.

Remarks:

- The ultralocality of the electric Carroll theories makes the integral in (3.4) diverge. To make a physical connection with the result, the divergent term gets omitted by the propagator expression.
- The time dependence of the correlator is a unique feature of the massless Carrollian CFT but appears useful for flat space holography.
- The three-point function vanishes as one can show, while the higher order correlators can be inferred by considering symmetry arguments, following from a discussion in [21].

3.2 Magnetic massless scalar field

Departing again from the Klein-Gordon field (3.1) and without performing field rescalings this time, the Carroll limit $c \to 0$ results in

$$S_m = \int dtd^dx\Big(\pi_\phi\dot{\phi} - \frac{1}{2}\partial_i\phi\partial^i\phi\Big) \ . \tag{3.9}$$

By introducing an auxiliary field χ to make contact with the reference [22]

$$\pi_\phi = \chi + \pi_{\mathcal{L}}(\phi) \ , \tag{3.10}$$

the general form of a magnetic scalar field theory is described by the action

$$S = \int dtd^dx(\chi\dot{\phi} + \mathcal{L}(\phi)) \ , \tag{3.11}$$

where $\mathcal{L}$ can be any term in the Lagrangian density that depends only on the scalar field. For simplicity we can consider the simplest example, omitting the arbitrary Lagrangian part

$$S = \int dt d^d x \chi \dot{\phi} \ \ . \tag{3.12}$$

It is a Carroll boost invariant since it transforms as a total derivative. The Green's function between the fields is again through a similar approach to the electric case

$$G_{\chi\phi}(\vec{x}-\vec{x}'; t-t') = \frac{i}{2} sgn(t-t')\delta^d(\vec{x}-\vec{x}') \ \ . \tag{3.13}$$

Remarks:

- The $\chi\phi$ correlator exhibits time discontinuity. This is a general feature of magnetic theories as can be seen in [8].
- The correlator containing only ϕ fields vanishes in a general magnetic field theory, whereas the one containing only χ fields generically exhibits a polynomial form in time as shown in [22].

The correlators were also derived in [23] where it is also shown that the two sectors; ”electric” and ”magnetic” originate from one higher dimensional Bargmann invariant action.

3.3 Unified Lagrangian parametrization of electric and magnetic sectors

To illustrate the thermodynamics of the electric and magnetic massless scalars we can write the Lagrangian density as

$$\mathcal{L} = a^2\dot{\phi}^2 - b^2(\nabla\phi)^2 \ \ . \tag{3.14}$$

To understand how it leads to the magnetic case, it is convenient to rewrite the Lagrangian density with the auxiliary field $\mathcal{L}_\chi = \chi\dot{\phi} - \frac{1}{4a^2}\chi^2$, whose equations of motion can be used to eliminate it. Then to reach the two sectors, the following limits to the parameters a and b have to be considered:

- For the electric case: $b \to 0$
- For the magnetic case: $a \to \infty$

- For the relativistic case: $a \to \frac{1}{c}$ and $b \to 1$

This is a parametrization initially used in [22] that provides a straightforward formulation for the limits in the thermodynamics of the two sectors.

Comment: Both electric and magnetic sectors obtained through the limiting procedure that was discussed are Carroll invariant theories. There are two ways to achieve Carroll invariance; either the spatial derivatives have to be dropped and only the time derivative remains, or in other words the canonical momentum field (electric limit). Or the canonical momentum field has to be eliminated while keeping the spatial derivatives (magnetic limit).

The kinetic term $\pi_\phi \dot{\phi}$ is present in both sectors. The difference lies in the energy densities $\mathcal{E}$ that remain. Both of them satisfy the Poisson bracket relation $[\mathcal{E}(x), \mathcal{E}(x')] = 0$ which is key to Carroll invariance [7]. Moreover, both the kinetic energy density and potential energy density are independently invariant under translations and rotations. Therefore, the electric and magnetic sectors are overall Carroll invariant.

Chapter 4

Partition function for massless Carroll particles

4.1 A massless relativistic particle

Starting with a simple example, we consider the free massless[1] relativistic particle. Initially, a relativistic particle has a Hamiltonian $H_1(p) = \sqrt{p^2c^2 + m^2c^4}$.
Thus, the constituents of a Boltzmann gas for free relativistic particles have the following expression for their 4-momentum and the 4-velocity:

$$
\begin{aligned}
U^\mu &= \gamma(1, v^i) \ , \\
P_\mu &= (-E, p_i) \ ,
\end{aligned}
\tag{4.1}
$$

with the square of the 4-momentum satisfying $P^2 = -m^2c^2$. Then the partition function for a single particle is defined with the introduction of a generalized chemical potential v^i conjugate to the conserved spatial momentum p_i

$$
Z_1 = \frac{1}{h^d}\int d^dx \int d^dp e^{\tilde{\beta}U^\mu P_\mu} = \frac{V}{h^d}\int d^3p e^{-\beta H_1(p) + \beta v^i p_i} \ . \tag{4.2}
$$

At this point, we can define a modified temperature $\beta = \frac{1}{k_BT} = \gamma\tilde{\beta} = \frac{\gamma}{k_B\tilde{T}}$ which makes the partition function Lorentz invariant (more details in Appendix B). By setting the mass equal to

[1] for simplification and easier to make the transition to the conformal case.

zero we have $H_1(p) = \sqrt{p^2c^2} = c|\vec{p}|$ for the Hamiltonian which results in the partition function in three spatial dimensions acquiring the following form:

$$
\begin{aligned}
Z_1 &= \frac{V}{h^3}\int d^3p e^{-\beta c|\vec{p}|+\beta v^i p_i} \\
&= \frac{2\pi V}{h^3}\int_0^\infty d\vec{p}\int_{-1}^{+1} d(cos\theta)|\vec{p}|^2 e^{-\beta c|\vec{p}|+\beta|\vec{v}||\vec{p}|cos\theta} \\
&= \frac{2\pi V}{h^3}\int_0^\infty dpp^2\frac{1}{\beta vp}[e^{\beta(v-c)p} - e^{-\beta(v+c)p}] \\
&= \frac{4\pi V}{h^3\beta v}\Big[-\frac{e^{\beta(v-c)p}}{\beta^2(v-c)^2} + \frac{e^{-\beta(v+c)p}}{\beta^2(v+c)^2}\Big]_0^\infty
\end{aligned} \tag{4.3}
$$

- for chemical potential $v < c$: the partition function takes eventually the final form

$$
\begin{aligned}
Z_1 &= \frac{4\pi V}{h^3\beta^3 v}\frac{1}{2}\frac{v^2+c^2+2vc-v^2-c^2+2vc}{(v^2-c^2)^2} \\
&= \frac{8\pi V}{h^3\beta^3 v}\frac{vc}{(v^2-c^2)^2} \\
&= \frac{8\pi V\gamma^4}{h^3\beta^3 c^3}, \qquad \gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}
\end{aligned} \tag{4.4}
$$

- for chemical potential $v > c$: the integral does not converge. One way to make it work is to choose β and v to lie in the complex plane. Then, the problematic exponential $e^{\beta(v-c)x}$ as $x \to \infty$ may still converge, if the condition $Re[\beta(v-c)] < 0$ is fulfilled.

In other words, if the following relation is satisfied [22]:

$$
\begin{aligned}
&Re(\beta v) - Re(\beta c) < 0 \Rightarrow \\
&Re(\beta)Re(v - Im(\beta)Im(v) - cRe(\beta) < 0 \Rightarrow \\
&Re(\beta)Re(v-c) - Im(\beta)Im(v) < 0
\end{aligned} \tag{4.5}
$$

One way to satisfy the above condition is to keep real values for the temperature (real β) and at the same time switch to purely imaginary chemical potential v while keeping c non-zero[2] as suggested in [22]. It is no longer appropriate to think about v as a particle velocity as in the 4-velocity (4.1), but rather as an imaginary chemical potential.

[2] Later we would have to consider the Carroll regime (leading term in $c \to 0$ expansion) and not the strict Carroll limit for such a consideration.

In that case, to extend to a converging and sensible expression for the partition function of massless Carroll particles, a separate treatment is needed for $v > c$ for which the partition function is

$$Z_1 = \frac{V}{h^d} \int d^3 p e^{-\beta c|\vec{p}| + i\beta v^i p_i} \quad . \tag{4.6}$$

Here each component $v^i \in \mathbb{R}$ and the imaginary unit i is explicitly written in the front of the term in the exponential. As will be shown in section 4.4, in $d = 3$ spatial dimensions, the partition function with an imaginary chemical potential is

$$Z_1 = \frac{8\pi V \gamma^4}{h^3 \beta^3 c^3}, \quad \gamma = \frac{1}{\sqrt{1 + \left(\frac{v}{c}\right)^2}} \quad . \tag{4.7}$$

Hence, the difference lies in the change to the sign of the γ factor, or in other words it can be reached by an analytic continuation to the chemical potential.

4.2 Many particles- Boltzmann gas

Using the canonical ensemble, we could generalize the Carroll particles partition function to N particles (in such non-discredited system of relativistic particles). Namely, one way to express it is by writing

$$Z(N, T, V, v^i) = \frac{1}{N!} (Z_1)^N \quad . \tag{4.8}$$

Then, adding a chemical potential μ to the description of the gas dynamics, or in other words manifesting the grand canonical potential, the suitable partition function becomes:

$$Z_{gr}(\mu, T, V, v_i) = \sum_{N=0}^{\infty} \frac{1}{N!} e^{\beta\mu N} Z(N, T, V, v_i) = \sum_{N=0}^{\infty} \frac{1}{N!} (e^{\beta\mu} Z_1)^N \quad . \tag{4.9}$$

Therefore, the partition function describing the gas would be

$$log Z_{gr}(\mu, T, V, v_i) = e^{\beta\mu} Z_1(T, V, v_i) \tag{4.10}$$

and the corresponding grand potential obeys the following rules

$$\Omega_{gr} = -PV \quad , \tag{4.11}$$

$$\Omega_{gr} = -k_B T log Z_{gr} \quad . \tag{4.12}$$

Combining them together, we arrive to

$$PV = -k_B T log Z_{gr}(\mu, T, V, v_i) \tag{4.13}$$

The thermodynamic quantities of the gas macroscopically can be obtained by the 1st thermodynamic law, expressed in the following way following the definition of the grand potential

$$d\Omega_{gr} = -SdT - PdV - p_i dv^i - Nd\mu \tag{4.14}$$

For instance, the momentum of the center of mass of the Boltzmann gas is

$$-p_i = \frac{\partial \Omega_{gr}}{\partial v^i}\Big|_{T,V,\mu} = -k_B T e^{\beta\mu} \frac{8\pi V}{h^3 \beta^3 c^3} 4\gamma^3 \frac{\partial \gamma}{\partial v^i} = -k_B T e^{\beta\mu} \frac{32\pi V}{h^3 \beta^3 c^3} \gamma^6 \frac{v_i}{c^2}$$

For the two velocity cases that were discussed the 1st law of thermodynamics holds, given the form of (4.10) for both $v < c$ and $v > c$. In the end, all the thermodynamic quantities become real-valued. The difference that was manifest in the two cases of the analysis is the fact that the Lorentz factor is defined as $\gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$ for $v < c$, while for $v > c$, the minus sign gets converted to a plus in practice, namely $\gamma = \frac{1}{\sqrt{1+(\frac{v}{c})^2}}$.

4.3 Partition function in $d = 2$

We investigate the tachyonic form of the partition function for single particles in two spatial dimensions. We consider the Hamiltonian

$$H_1(p) = c|\vec{p}| \quad . \tag{4.15}$$

Then the partition function takes the form

$$Z_1(V,T,v) = \frac{V}{h^2}\int d^2p e^{-\beta c|\vec{p}|+i\beta\vec{p}\cdot\vec{v}}$$
$$= \frac{V}{h^2}\int_0^\infty pdp\int_0^{2\pi} d\theta e^{\beta cp+i\beta cpv_1\cos\theta+i\beta pv_2\sin\theta} \quad . \tag{4.16}$$

To integrate over the angle θ, we have to make use of the integral representation of the Bessel function of the first kind

$$J_0(z) = \frac{1}{\pi}\int_0^\pi e^{iz\cos\theta}d\theta \quad . \tag{4.17}$$

This observation leads to the expression

$$Z_1(V,T,v) = \frac{V}{h^2}\int_0^\infty pdpe^{-\beta cp}2\pi J_0(\beta p|\vec{v}|) \quad . \tag{4.18}$$

Finally, to treat the integration on the radial p, we make use of the Laplace transform of the Bessel function of the first kind. This can be obtained by applying the Laplace transform to the Bessel differential equation; as pointed out in the textbook of reference [24]. The result then is

$$Z_1(V,T,v) = \frac{2\pi V}{h^2}\frac{1}{c^2(1+\frac{v^2}{c^2})^{\frac{3}{2}}\beta^2} = \frac{2\pi V}{c^2h^2\beta^2}\gamma^3 \quad . \tag{4.19}$$

This result is compatible with the one produced by the authors of [22] for an arbitrary number of dimensions.

An alternative approach would be to view the initial integral as a multivariable Fourier transform, by defining the Fourier transform to be $\mathcal{F}[f(\vec{x})](\vec{v}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\vec{v}\cdot\vec{x}}$. That concludes as a result that

$$Z_1(V,T,v) = \frac{V}{h^2}\frac{2\pi}{\beta^2}\mathcal{F}[e^{-\beta c|\vec{x}|}](\vec{v})$$
$$= \frac{2\pi V}{c^2h^2\beta^2}\gamma^3 \quad , \tag{4.20}$$

which straightforwardly agrees with the result obtained by using the first integration method.

Again here, what changes from the case with $v < c$ to the one with chemical potential values $v > c$ is the definition of the gamma factor γ. For the latter, the usual expression $\gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$ turns into $\gamma = \frac{1}{\sqrt{1+(\frac{v}{c})^2}}$.

Comment: The partition function is no longer Lorentz invariant. However, we can still preserve

Lorentz invariance if instead of β we introduce an effective inverse temperature $\tilde{\beta} = \gamma\beta$ which adopts the transformation law under a Lorentz boost of a usual Lorentz factor γ. More details are explained in Appendix B.

4.4 Generalizing to arbitrary dimensions

By working in 2 spatial dimensions we realized that the integral is a multivariable Fourier transform. Namely, it can be written as

$$Z_1 = \frac{V}{h^d} \int d^d p e^{-\beta c|\vec{p}| + i\beta \vec{v}\cdot\vec{p}} \ . \tag{4.21}$$

To be more specific, it is the Fourier transform of a radial function (one that only depends on the modulus of the integration variable). Using spherical coordinates in d dimensions we may write:

$$Z_1 = \frac{V}{h^d} \int_0^\infty dp p^{d-1} e^{\beta cp} \Big(S_{d-2} \int_0^\pi d\theta sin^{d-2}\theta e^{i\beta upcos\theta} \Big) \ . \tag{4.22}$$

Here S_{d-2} is the area of the $\mathbb{S}^{d-2}$ sphere. The inner integral in the angle θ can be deduced using the representation of Bessel functions

$$\int_0^\pi d\theta sin^{d-2}\theta e^{i\beta upcos\theta} = \frac{2^{\frac{d-2}{2}}\sqrt{\pi}}{(pu\beta)^{\frac{d}{2}}} J_{\frac{d-2}{2}}(pu\beta)\Gamma\Big(\frac{d-1}{2}\Big) \ . \tag{4.23}$$

The partition function of the single relativistic particle induces now a Hankel transform. A Hankel transform of a radial function $f(r)$ is defined by the integral

$$\mathcal{H}_\nu[f(r)](k) = \int_0^\infty dr r f(r) J_\nu(kr) \ . \tag{4.24}$$

In our case the function to be Hankel transformed is $f(p) = p^{\frac{d-2}{2}} e^{-\beta cp}$. The way to explicitly perform the integral is challenging and it involves the series expansion of the Bessel function. This step induces a Laplace transform for each term of the series. More details are discussed in Appendix A. Using also the fact that the area of a $\mathbb{S}^{d-2}$ sphere is $S_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}$ the final result for the partition function of a single massless relativistic particle becomes

$$Z_1(V,T,v) = \frac{V 2^d \pi^{\frac{d-1}{2}}}{(\beta hc)^d} \gamma^{d+1} \Gamma\Big(\frac{d+1}{2}\Big) \ , \tag{4.25}$$

which agrees with the results we produced for the $d = 2$ and $d = 3$ spatial dimensions cases and with the result presented in reference [22].

The strict Carroll limit $v \to 0$ makes the partition function vanish for non-zero chemical potential v. To be more concrete

$$Z_1 \propto \frac{\gamma^{d+1}}{c^d} = \frac{c}{\left(v^2 + c^2\right)^{\frac{d+1}{2}}} \stackrel{c \to 0}{\to} 0 \ . \tag{4.26}$$

However, for vanishing chemical potential v, the partition function diverges. This fact stresses the difference between $v = 0$ and $v \neq 0$ particles if v is interpreted as a velocity.

In the Carroll regime, or in other words in the leading term of $\left(\frac{c}{v}\right)$ expansion, the partition function is

$$Z_1(V, T, v) = \frac{V 2^d \pi^{\frac{d-1}{2}}}{(\beta h v)^d} \frac{c}{v} \Gamma\left(\frac{d+1}{2}\right) \ . \tag{4.27}$$

4.5 Equation of state

In the canonical ensemble, the energy of the gas can be expressed in the following way

$$E = V\mathcal{E} = U - N v^i < p_i > \ , \tag{4.28}$$

where we denote by U the internal energy of the system, v^i the chemical potential (associated with the mean velocity of the particles of the gas), N the number of particles (which is fixed in a thermal bath and not allowed to fluctuate) and $< p_i >$ is the mean momentum of each of the constituents of the gas.

The internal energy is $U = N \frac{\partial}{\partial \beta} \log Z_1$, while the mean momentum can be computed by the relation $< p_i >= \frac{1}{\beta} \frac{\partial}{\partial v^i} \log Z_1$.

In the Carroll regime (where $v \gg c$ and $v \neq 0$), the linear approximation of the Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1 + (v/c)^2}} \stackrel{v \gg c}{\approx} \left|\frac{c}{v}\right| \tag{4.29}$$

and then the energy of the system reads

$$V\mathcal{E} = E = -d\frac{N}{\beta} + (d+1)\frac{N}{\beta}\frac{v^i v_i}{|v|^2} = \frac{N}{\beta} \ . \tag{4.30}$$

Using the fact that $\mathcal{P} = \frac{N}{\beta}\frac{\partial \log Z_1}{\partial V} = \frac{N}{V\beta}$, with $\mathcal{P}$ denoting the pressure, we finally get the equation state that is established for the Carroll regime:

$$\mathcal{E} + \mathcal{P} = 0 \quad . \tag{4.31}$$

4.6 Stress-energy tensor

The stress-energy tensor for a perfect fluid can be written with translation and rotation symmetry. For what follows there is no further assumption for boost invariance of any kind. The components of the stress-energy tensor are written as studied in [25]

$$T^0_0 = -\mathcal{E} \quad , \tag{4.32}$$

$$T^0_j = P_j \quad , \tag{4.33}$$

$$T^i_0 = -(\mathcal{E} + \mathcal{P})v^i \quad , \tag{4.34}$$

$$T^i_j = \mathcal{P}\delta^i_j + v^i P_j \quad . \tag{4.35}$$

In inflationary cosmology the stress-energy tensor describing the matter distribution ought to be diagonal. Therefore, it is expected that the off-diagonal components vanish in the strict Carroll limit.
As we have seen, the equation of state in the strict Carroll limit is of the following form

$$\mathcal{E} + \mathcal{P} = 0 \quad . \tag{4.36}$$

Thus, straightforwardly, the energy flux vanishes. One way to acquire a diagonal structure is by a vanishing momentum P_j. This condition is fulfilled for the particles description that we have constructed

$$P_j = \pm\frac{N}{\beta}(d+1)v_j\gamma^2 \quad . \tag{4.37}$$

Analyzing the gamma factor in the strict Carroll limit for the case $v > c$ shows that it straightforwardly vanishes

$$\gamma^2 = \Big(\frac{c}{v}\Big)\Big(\big(\frac{c}{v}\big)^2 + 1\Big)^{-1} \overset{\frac{c}{v}\to 0}{\to} 0 \quad . \tag{4.38}$$

That is to be expected because from Lorentz symmetry $T_{\mu\nu} = T_{\nu\mu}$ and additionally the Carroll Ward identity that corresponds to the Carroll boost generator is

$$\begin{aligned}&\partial_\mu(T^\mu_\nu C^\nu_i) = \partial_\mu(T^\mu_0 C^0_i) = \\ &\partial_\mu(T^\mu_0 x^i) = (\partial_\mu T^\mu_0)x^i + T^\mu_0 \partial_\mu x^i \ \ .\end{aligned} \tag{4.39}$$

The first term vanishes because of the conservation law of the stress energy tensor and only the second one survives and has to be equated to zero.

$$T^\mu_0 \partial_\mu x^i = T^\mu_0 \delta^i_\mu = T^i_0 = 0 \ \ . \tag{4.40}$$

As a conclusion, the stress-energy tensor becomes diagonal in the strict Carroll limit

$$T^\mu_\nu = \begin{pmatrix} -\mathcal{E} & 0 & 0 & 0 \\ 0 & \mathcal{P} & 0 & 0 \\ 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & 0 & \mathcal{P} \end{pmatrix} \ \ . \tag{4.41}$$

4.7 First order correction to the equation of state

As we have seen, starting with a Hamiltonian $H_1(p) = c|\vec{p}|$ and employing an imaginary chemical potential, the partition function ends up taking the following form:

$$Z_1(V, T, v_i) = \frac{V 2^d \pi^{\frac{d-1}{2}}}{(\beta h c)^d} \gamma^{d+1} \Gamma\Big(\frac{d+1}{2}\Big) \ \ , \tag{4.42}$$

where the gamma factor gets a plus sign compared to the real chemical potential case, namely

$$\gamma = \frac{1}{\sqrt{1 + \left(\frac{v}{c}\right)^2}} \ \ . \tag{4.43}$$

Without considering any Carroll limit yet, we aim to derive an equation of state for the general d dimensional case. First, we define the generalized energy

$$\tilde{E} = E - iP_i v^i \ \ , \tag{4.44}$$

where $P_i = N < p_i >_1$ the generalized momentum. Giving it a closer look, $\tilde{E} = N\sum_s P_s E_s$ counts all the microscopic configurations indexed by s and weighs them with their probability P_s. This analysis reveals that the generalized energy of the system is obtained by the derivative of the logarithm of Z_1 with respect to β

$$\tilde{E} = N\frac{1}{Z}\sum_s E_s e^{-\beta E_s} = -\frac{N}{Z}\frac{\partial}{\partial\beta}Z_1(V,T,v_i) = -N\frac{\partial \log Z}{\partial\beta} \quad . \tag{4.45}$$

In the particular case of partition function (4.36)

$$\tilde{E} = -N\frac{\partial}{\partial\beta}\log Z_1 = \frac{d}{\beta}N \quad . \tag{4.46}$$

The microscopic description is of the form $E_s = E_s^{(0)} + \lambda A_s$ for the energies of each particle and with a similar analysis $< A >= \sum_s A_s P_s = -\frac{1}{\beta}\frac{\partial}{\partial\lambda}\log Z_1$. Within our context $\lambda = -iv^i$ and $A_s = p_i$,

$$< p_i >_1 = \frac{1}{\beta}\frac{1}{i}\frac{\partial}{\partial v^i}\log Z_1 \quad . \tag{4.47}$$

Therefore, equation (4.47) adjusts the single particle mean momentum by including the factor i. Finally, using the ideal gas law, equation (4.46) takes the form:

$$\tilde{E} - d\mathcal{P} = 0 \quad . \tag{4.48}$$

Using equation (4.47)

$$\begin{aligned} < p_i >_1 &= \frac{1}{i\beta}\frac{\partial}{\partial v^i}\log\gamma^{d+1} \\ &= \frac{1}{i\beta}\frac{d+1}{\gamma^{d+1}}\gamma^d\Big(-\frac{1}{2}\Big)\big(1+(\frac{v}{c})^2\big)^{-3/2}\frac{2v_i}{c^2} \\ &= i\frac{v_i}{\beta c^2}\gamma^2(d+1) \quad . \end{aligned} \tag{4.49}$$

Combining equations (4.44),(4.46), and (4.48) we may conclude the equation of state in the following lines

$$
\begin{aligned}
d\mathcal{P}V &= E + \frac{N}{\beta c^2} v^i v_i \gamma^2 (d+1) \\
&= E + \frac{\mathcal{P}V}{c^2} v_i v^i \gamma^2 (d+1) \\
&= E + \frac{\mathcal{P}V}{c^2} v_i v^i \frac{c^2}{v_i v^i} \Big(1 - \frac{1}{2}\big(\frac{c}{v}\big)^2\Big)^2 (d+1) \\
&= E + \mathcal{P}V(d+1)\Big[1 - \big(\frac{c}{v}\big)^2 + \mathcal{O}\Big(\big(\frac{c}{v}\big)^4\Big)\Big] \\
&= E + \mathcal{P}V(d+1) - \big(\frac{c}{v}\big)^2 \mathcal{P}V(d+1) \ \ ,
\end{aligned} \tag{4.50}
$$

where we have used the fact that the pressure is derived from the relation:

$$
\mathcal{P} = N \frac{1}{\beta} \frac{\partial log Z_1}{\partial V} = \frac{N}{V} \frac{1}{\beta} \tag{4.51}
$$

that resembles the ideal gas law. Finally, dividing by the volume we end up with the equation of state:

$$
\mathcal{E} + \mathcal{P} = \frac{c^2}{v^2} \mathcal{P}(d+1) + \mathcal{O}\Big(\big(\frac{c}{v}\big)^4\Big) \ \ . \tag{4.52}
$$

In this way, we realize the correction to the Carroll regime that relaxes the strict Carroll limit.

An important remark can be made at this point; the pressure is positive and realised by equation (4.51). Therefore, for certain values of the chemical potential, the energy density is positive and after exceeding a critical value it becomes negative. This critical velocity is $v_c = \sqrt{d}c$ and it depends only on the dimensions of the theory.

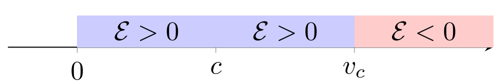


Figure 4.1: Energy density sign vs particle velocity.

Comment: In Figure 4.1 the transition for the sign of the energy density is highlighted with blue color for the relativistic regime and the region where $c < v < v_c$ and with red for the part of the Carroll regime where $v > v_c$.

4.8 Equation of state related to the chemical potential

As it has already been obtained for d spatial dimensions the pressure and the energy density in a single particle state is of the following form:

$$\mathcal{E} = \frac{d}{\beta}\frac{N}{V} \pm \frac{Nv^2}{V\beta c^2}\gamma^2(d+1) \ ,$$
$$\mathcal{P} = \frac{N}{V}\frac{1}{\beta} \ . \tag{4.53}$$

The plus sign in the energy density corresponds to velocities less than the speed of light, while the minus sign to tachyonic velocities. The plus/minus sign originates from the momentum derivative.

$$< p_1 >_1 \propto \frac{\partial}{\partial v^i} log\gamma \ , \tag{4.54}$$

where the gamma factor is the normal Lorentz factor for values of the chemical potential less than the speed of light and for values greater than the speed of light it is the one that we can obtain through analytic continuation of the first

$$\gamma = \begin{cases} \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} & v < c \\ \frac{1}{\sqrt{1+\left(\frac{v}{c}\right)^2}} & v > c \end{cases} \ . \tag{4.55}$$

We are interested in the equation of state parameter which is given by the ratio of pressure over energy density.

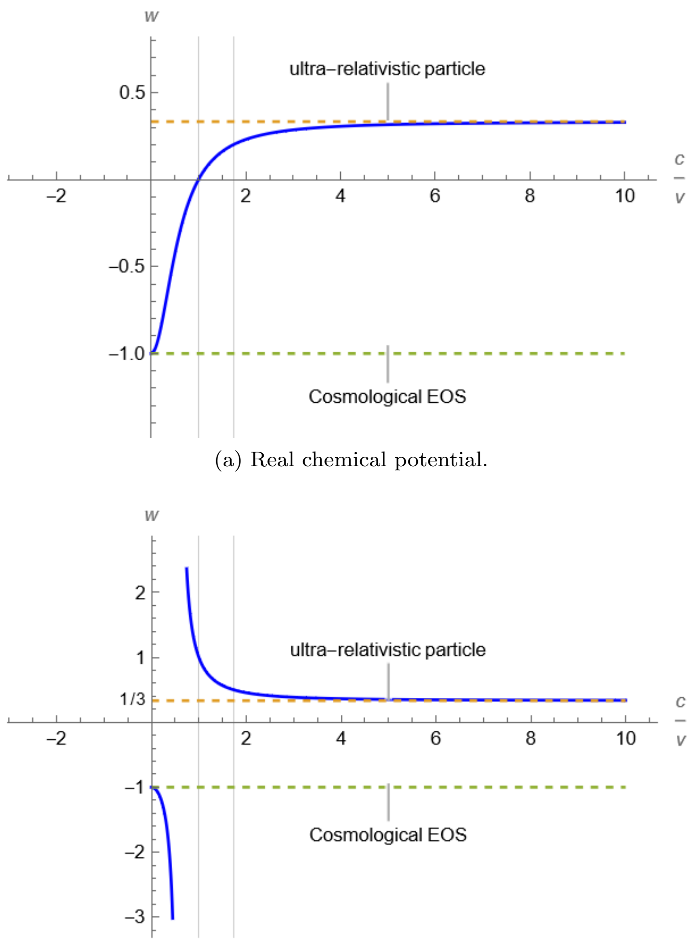
$$w = \frac{\mathcal{P}}{\mathcal{E}} \ . \tag{4.56}$$

Hence, we may readily substitute into equation (4.56) the results for pressure and energy density that are already known, to derive the equation of state parameter with respect to the velocity of the single particle

$$w\left(\frac{c}{v}\right) = \frac{1}{d \pm \frac{d+1}{(\frac{c}{v})^2 \mp 1}} \ . \tag{4.57}$$

Specializing for spatial dimensions $d = 3$ the equation of state parameter can be plotted vs the chemical potential.

As can be seen in Figure 4.2 (a) and (b), for vanishing velocity the equation of state parameter is $w = \frac{1}{3}$ as expected, since for that value of v it can be derived from the partition function for an



(a) Real chemical potential.

(b) Imaginary chemical potential.

Figure 4.2: Equation of state parameter for the two cases of real/imaginary potential.

ultra-relativistic particles gas as the imaginary chemical potential is approaching zero

$$Z = Tr\big(e^{-\beta c|\vec{p}|}\big) \quad . \tag{4.58}$$

On the other side, we can notice the transition where the chemical potential is becoming equal to the speed of light and analytical continuation is utilized.

For values of the chemical potential greater than the critical velocity $v_{cr} = \sqrt{d}c \overset{d=3}{=} \sqrt{3}c$ (equivalently for $\frac{c}{v} < \frac{1}{\sqrt{3}}$ it is smoothly approaching the equation of state parameter that corresponds to the de Sitter space, which describes the idealized version of the universe that undergoes an accelerating expansion with $w = -1$.

Chapter 5

Carroll conformal field theory

5.1 Second quantization of the scalar field and mode expansions

We start with the general Lagrangian density description for the real scalar

$$\mathcal{L} = a^2\dot{\phi}^2 - b^2(\partial_x\phi)^2 \quad . \tag{5.1}$$

The equations of motion are acquired by varying the action with respect to the field

$$a^2\ddot{\phi} - b^2\partial_x^2\phi = 0 \tag{5.2}$$

and Fourier transforming the equation of motion of the field gives the relation in momentum space

$$a^2(-\omega^2) - b^2(-k^2) = 0 \Rightarrow \omega^2(k) = \frac{b^2k^2}{a^2} \tag{5.3}$$

We recognize that (5.2) represents a harmonic oscillator, for which we know the solution. In fact, the field assigns a harmonic oscillator to each space point.
If we further impose periodic conditions or in other words make the identification $\phi(x) = \phi(x+2\pi R)$, we discretize the momenta. For each harmonic oscillator

$$e^{ikx-i\omega t} = e^{ik(x+2\pi R)-i\omega t} \Rightarrow e^{ik2\pi R} = 1 \Rightarrow k2\pi R = 2\pi n \Rightarrow k = \frac{n}{R}, n \in \mathbb{Z} \quad . \tag{5.4}$$

The most general solution for the quantized field is the following

$$\hat{\phi}(x) = \sum_{n\in\mathbb{Z}} \hat{a}^{\dagger}_{k_n} e^{i(k_n x+\omega(k_n)t)} + h.c. \quad . \tag{5.5}$$

Furthermore, this Lagrangian density is translation invariant. The spatial translation transformation $x \to x+\alpha, \alpha \in \mathbb{R}$ is a symmetry. The field for an infinitesimal transformation becomes $\phi(x) \to \phi'(x') = \phi'(x) + \alpha\partial_x\phi \Rightarrow \delta\phi = \alpha\partial_x\phi$ and the Lagrangian density

$$\begin{aligned}
\delta\mathcal{L} &= a^2\delta\dot{\phi}^2 - b^2\delta(\partial_x\phi)^2 \\
&= 2a^2\dot{\phi}\delta\dot{\phi} - 2b^2\partial_x\phi\delta(\partial_x\phi) \\
&= 2\alpha a^2\dot{\phi}(\partial_x\phi) - 2\alpha b^2(\partial_x^2\phi)(\partial_x\phi) \\
&= \partial_x\big(\alpha a^2\dot{\phi}^2 - \alpha b^2(\partial_x\phi)^2\big) \quad ,
\end{aligned} \tag{5.6}$$

which indeed shows that the Lagrangian density transforms as a total derivative and therefore it is translation invariant.

The Noether current from this symmetry is

$$j^0 = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\Delta\phi - J^0 \quad , \tag{5.7}$$

with $\Delta\phi$ and J^μ defined in the relations

$$\delta\mathcal{L} = \alpha\partial^\mu J_\mu \quad , \tag{5.8}$$

$$\Delta\phi = \alpha\delta\phi \quad . \tag{5.9}$$

So $J^0 = 0$ and the conserved current acquires the form

$$j^0 = 2a^2\dot{\phi}(\partial_x\phi) \quad , \tag{5.10}$$

which we identify as the spatial momentum operator. To proceed, we have to figure out the mode expansions of the Hamiltonian and the spatial momentum. The goal is to find a basis that enables us to simultaneously diagonalize P and H.

We begin by writing the mode expansion of the derivatives of the field

$$\hat{\dot{\phi}}(x) = \sum_{n\in\mathbb{Z}} (i\omega)\left[\hat{a}_{k_n} e^{i(k_n x+\omega t)} - h.c.\right] \ , \tag{5.11}$$

$$\partial_x \hat{\phi}(x) = \sum_{n\in\mathbb{Z}} (ik)\left[\hat{a}_{k_n} e^{i(k_n x+\omega t)} - h.c.\right] \ . \tag{5.12}$$

Then our goal is to express the $\hat{P}$ using the mode expansions.

$$\begin{aligned}
\hat{P} = Q^0 = \int dx j^0 \\
= -2a^2 \int dx \sum_{n\in\mathbb{Z}} \sum_{n'\in\mathbb{Z}} \omega k' \left[\hat{a}_k e^{i(kx+\omega t)} - \hat{a}_k^\dagger e^{-i(kx+\omega t)}\right]\left[\hat{a}_{k'} e^{i(k'x+\omega' t)} - \hat{a}_{k'}^\dagger e^{-i(k'x+\omega' t)}\right] \\
= -2a^2 \sum_{n,n'\in\mathbb{Z}} \int dx \omega k' \Big[\hat{a}_k \hat{a}_{k'} e^{i(k+k')x} e^{i(\omega+\omega' t)} + \hat{a}_k^\dagger \hat{a}_{k'}^\dagger e^{-i(k+k')x} e^{-i(\omega+\omega')} - \\
\hat{a}_k \hat{a}_{k'}^\dagger e^{i(k-k')x} e^{i(\omega-\omega' t)} - \hat{a}_k^\dagger \hat{a}_{k'} e^{-i(k-k')x} e^{i(-\omega-\omega' t)}\Big] \ .
\end{aligned} \tag{5.13}$$

In the previous, we used the short-hand notation $k = k_n$, $k' = k_{n'}$, $\omega = \omega(k_n)$, and $\omega' = \omega(k_{n'})$. To proceed, we perform the x integration, which results in Kronecker deltas.

$$\begin{aligned}
\hat{P} &= -2a^2(2\pi) \sum_{n\in\mathbb{Z}} \omega k \Big[- \hat{a}_k \hat{a}_{-k} e^{-2i\omega t} - \hat{a}_k^\dagger \hat{a}_{-k}^\dagger e^{2i\omega t} - \hat{a}_k \hat{a}_k^\dagger - \hat{a}_k^\dagger \hat{a}_k \Big] \\
&= 2a^2(2\pi) \sum_{n\in\mathbb{Z}} (2\hat{a}_k^\dagger \hat{a}_k + [\hat{a}_k, \hat{a}_k^\dagger]) k\omega(k) \\
&= 8\pi a^2 \sum_{n\in\mathbb{Z}} k\omega(k) \hat{a}_k^\dagger \hat{a}_k \ .
\end{aligned} \tag{5.14}$$

In the first line, the first two terms in the bracket are odd and so they vanish. The commutator of the creation with the annihilation operator in the second line does not contribute to the zero point energy of each harmonic oscillator, because it is an odd function together with the coefficients. Similarly, we can compute the Hamiltonian in terms of the modes that lead to the result that

$$\hat{H} = 8\pi a^2 \sum_{n\in\mathbb{Z}} \left(\omega^2(k) \hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\omega^2(k)\right) \ . \tag{5.15}$$

5.2 Quantum thermal partition function

The ultimate intention is to manage to express the partition function

$$Z = Tr[e^{-\beta H + i\beta v P}] \tag{5.16}$$

using a basis for the trace. By the mode expansions that we computed, we notice that there is a basis that could diagonalize both H and P simultaneously. To be more precise:

$$\hat{\hat{a}}_k^\dagger \hat{\hat{a}}_k \left|\tilde{n}\right\rangle = \tilde{n} \left|\tilde{n}\right\rangle \tag{5.17}$$

the number density operator eigenstates will be used. In expression (5.17) the creation and annihilation operators are rescaled compared to the mode coefficients as $\hat{\hat{a}}_k = 2\alpha\sqrt{2\pi\omega}\hat{a}$. This is determined in such a way because the Hamiltonian acting on the ground state obeys $H\hat{\hat{a}}_k^\dagger \left|0\right\rangle = \omega\hat{\hat{a}}_k^\dagger \left|0\right\rangle$ and $H\left|\tilde{n}\right\rangle = H(\hat{\hat{a}}_k^\dagger)^{\tilde{n}} \left|0\right\rangle = \tilde{n}\omega(\hat{\hat{a}}_k^\dagger)^{\tilde{n}} \left|0\right\rangle$. The integers $\tilde{n}$ are the eigenvalues of the number density operator.

The trace can be rewritten as the sum in the diagonal of the matrix elements in these eigenstates $\left|\tilde{n}\right\rangle$ in the following way.

$$Z = \sum_{\tilde{n}} \langle\tilde{n}|e^{-\beta H + i\beta v P}|\tilde{n}\rangle \quad . \tag{5.18}$$

Using the expressions (5.16) and (5.17) we deduce that

$$\begin{aligned} Z &= Tr\left[e^{-\beta H + i\beta v P}\right] \\ &= Tr\left[e^{-\beta \sum_{n\in\mathbb{Z}} \omega(k_n)\hat{\hat{a}}_k^\dagger \hat{\hat{a}}_k + i\beta v \sum_{n\in\mathbb{Z}} k_n \hat{\hat{a}}_k^\dagger \hat{\hat{a}}_k}\right] \\ &= Tr\left[\prod_{n\in\mathbb{Z}} e^{-\beta\omega(k_n) + i\beta v k_n}\right] \quad . \end{aligned} \tag{5.19}$$

Here we remark that the modes are independent and the trace can be performed separately for the microstates of each harmonic oscillator. The trace then has to sum over the modes of the Hilbert space of the k-th mode.

$$Z = \prod_{n\in\mathbb{Z}} Tr_{k_n}\left(e^{-\beta\omega(k_n)\omega(k_n)\hat{\hat{a}}_k^\dagger \hat{\hat{a}}_k + i\beta k_n \omega(k_n)\hat{\hat{a}}_k^\dagger \hat{\hat{a}}_k}\right) \quad . \tag{5.20}$$

Finally, this individual tracing procedure can be performed by summing over the trace representation realized by the number density operator

$$
\begin{aligned}
&Tr_{k_n}\left(e^{-\beta\omega(k_n)\omega(k_n)\hat{a_k}^{\dagger}\hat{a_k}+i\beta k_n\omega(k_n)\hat{a_k}^{\dagger}\hat{a_k}}\right) = \\
&\sum_{\tilde{n}}\langle\tilde{n}|(e^{-\beta\omega(k_n)\omega(k_n)\hat{a_k}^{\dagger}\hat{a_k}+i\beta k_n\omega(k_n)\hat{a_k}^{\dagger}\hat{a_k}}|\tilde{n}\rangle = \\
&\sum_{\tilde{n}}e^{-\beta\omega(k_n)\tilde{n}+i\beta k_n\tilde{n}} = \frac{1}{1-e^{-\beta\omega(k_n)+i\beta k_n}} \quad .
\end{aligned} \tag{5.21}
$$

In the last step, the geometric series with a complex argument was used. An important point to keep in mind is that the quantum Hamiltonian carries information about the vacuum energy. The essence of it gets captured within the notion of Casimir energy E_C, which is expressed by the sum in the zero point energies of all the harmonic oscillators $E_c = \frac{1}{2}\sum_n \omega(k_n)$ and it requires a regularization to make sense. The quantum partition function takes the final form:

$$Z = e^{-\beta E_C}\prod_{n\in\mathbb{Z}}\frac{1}{1-e^{-\beta\omega(k_n)+i\beta v k_n}} \quad . \tag{5.22}$$

5.3 Necessity of an imaginary chemical potential

One may wonder why we introduced an imaginary chemical potential in the partition function. The main reason is that ultimately we would like to explore the convergence domain for values of $v > c$ or equivalently $|v| > b/a$ (because $b/a = c$ in the relativistic case).

The analog of the thermal quantum partition function with a real chemical potential excluding the zero mode is

$$
\begin{aligned}
Z = Tre^{-\beta H+\beta vp} &= \sum_{n>0}\langle n|e^{-\beta\omega(k_n)+\beta v k_n}|n\rangle \\
&= \prod_{k\in\mathbb{N}^*}\frac{1}{1-e^{-\beta/R(\frac{b}{a}-v)k}}\prod_{k\in\mathbb{N}^*}\frac{1}{1-e^{-\beta/R(\frac{b}{a}+v)k}} \quad ,
\end{aligned} \tag{5.23}
$$

where $\mathbb{N}^*$ denotes the set of positive integers. The convergence of the products above is determined by the convergence domain of the Euler function

$$\phi(q) := \prod_{k\in\mathbb{N}^*}(1-q^k) \quad , \tag{5.24}$$

which is given by the condition $|q| < 1$. For the two products in (5.23) to converge the following two conditions have to be satisfied simultaneously:

$$\begin{cases} \left|e^{-\beta/R(\frac{b}{a}-v)}\right| < 1 \\ \left|e^{-\beta/R(\frac{b}{a}+v)}\right| < 1 \end{cases} , \tag{5.25}$$

which subsequently implies the condition

$$-\frac{b}{a} < v < \frac{b}{a} \implies -c < v < c \ . \tag{5.26}$$

We infer the requirement of an imaginary chemical potential introduction to investigate the behavior of thermodynamic quantities and ultimately the equation of state for the region $|v| > c$.

The thermodynamic large R limit corresponds to the asymptotic expansion of the Euler function (in terms of q-series)[26].

$$\phi(q) = (q;q)_\infty = \prod_{k=1}^{\infty} \left(1 - q^k\right) \ , \tag{5.27}$$

with the following asymptotic result:

$$\phi(q) = \sqrt{\frac{2\pi}{t}} exp\Big(- \frac{\pi^2}{6t} + \frac{t}{24}\Big) + \mathcal{O}(1) \ , \tag{5.28}$$

where $q = e^{-t}$ and we denoted exp the exponential function. In our case, we apply this result to the arguments $t = \frac{\beta}{R}\left(\frac{b}{a} - v\right)$ and $\tilde{t} = \frac{\beta}{R}\left(\frac{b}{a} + v\right)$.

Then, the partition function in the large R limit obtains the form following the expression for the asymptotic result:

$$\begin{aligned} Z = \frac{1}{(q;q)_\infty(\tilde{q};\tilde{q})_\infty} &\overset{R\to\infty}{\to} \sqrt{\frac{\frac{\beta}{R}\left(\frac{b}{a} - v\right)}{2\pi}} \sqrt{\frac{\frac{\beta}{R}\left(\frac{b}{a} + v\right)}{2\pi}} \times \\ &exp\left(\frac{\pi^2}{\frac{\beta}{R}\left(\frac{b}{a} - v\right)} - \frac{\frac{\beta}{R}\left(\frac{b}{a} - v\right)}{24} + \frac{\pi^2}{\frac{\beta}{R}\left(\frac{b}{a} + v\right)} - \frac{\frac{\beta}{R}\left(\frac{b}{a} + v\right)}{24} \right) \\ &= \frac{\beta}{2\pi R} \sqrt{\frac{b^2}{a^2} - v^2} \times exp\left(\frac{\pi^2}{3} \frac{\frac{b}{a}}{\frac{\beta}{R}\left(\frac{b^2}{a^2} - v^2\right)} - \beta \frac{b}{12aR} \right) \ . \end{aligned} \tag{5.29}$$

At this point, we introduce the dimensionless quantities on which we are allowed to consider limits later on.

$$x = \frac{\beta b}{aR} \quad , \tag{5.30}$$

$$z = \frac{\beta v}{R} \quad . \tag{5.31}$$

It follows straightforwardly that the logarithm of the partition function in the large R limit is

$$\log Z = \log \sqrt{x^2 - z^2} + \frac{\pi^2}{3} \frac{x}{x^2 - z^2} - \frac{x}{12} \quad . \tag{5.32}$$

The first term is subleading in the large R limit. The second term is the extension of the Cardy result after the addition of the chemical potential v. And the third term is the inverse temperature times the Casimir energy.[1] The more appropriate answer then is

$$\log Z = \log \sqrt{\frac{x^2 - z^2}{x}} + \frac{\pi^2}{3} \frac{x}{x^2 - z^2} - \frac{x}{12} \quad . \tag{5.33}$$

This will be discussed more in detail in section 5.9 where the modular invariance will be introduced for the imaginary chemical potential case. The zero mode contribution is the same in the real and imaginary chemical potential cases.

5.4 Electric and magnetic cases

For well-defined thermodynamics, we would like the logarithm of the partition function to be extensive. For now, we may ignore the Casimir energy and focus on the rest of the expression. We are going to investigate what happens when we consider the limits that turn the theory into the "electric" case ($b \to 0$) and the "magnetic" case ($a \to \infty$).

The limit $b \to 0$ corresponds to an electric quantum field theory, or in terms of dimensionless parameters $x = \frac{b\beta}{aR} \to 0$. Then the partition function takes the following form

$$Z = \prod_{n \in \mathbb{Z}} \frac{1}{1 - e^{\frac{i\beta v n}{R}}} = \prod_{n \in \mathbb{Z}} \frac{1}{1 - e^{izn}} \quad , \tag{5.34}$$

where the second dimensionless parameter $z = \frac{\beta v}{R}$ is involved.

In the magnetic case, by considering the limit $b \to \infty$, the dimensionless parameter $x \to 0$,

[1] If the finite part of the zero mode is included there is a contribution to the first term which still keeps it subleading in the large R expansion

and the same holds in the electric case where the partition function looks like the expression (5.34) again.

We conclude that thermodynamics in these two limits still stands a chance to be well-defined since there is still a dependence on the dimensionless parameter z which contains the inverse temperature and volume. However, as will become clear later in section 5.10, the partition function can be expressed in terms of Dedekind eta functions that are defined for strictly $x > 0$, and the partition function becomes zero. As a consequence, we can only look at the Carroll regime which translates to the leading contribution of $x \to 0$. This means that in the strict Carroll limit, we do not achieve sensible thermodynamics.

Comment: Although the internal energy seems to generate imaginary values by $U = -\frac{\partial \log Z}{\partial \beta}$, a manipulation in the summation shows that it is real by separating the positive and negative modes

$$U = \frac{iv}{R} \sum_{n \in \mathbb{Z}} \frac{n}{1 - e^{i\frac{\beta v}{R} n}} = -\frac{v}{R} \sum_{n \in \mathbb{N}^*} \frac{n \sin(\frac{\beta v}{R} n)}{1 - \cos(\frac{\beta v}{R} n)} \quad . \tag{5.35}$$

5.5 Turning on a mass term

If we were to add a mass term to the Lagrangian density discussed in the previous analysis, then the only thing that changes is the dispersion relation. Namely, instead of $\omega^2(k) = \frac{b^2 k^2}{a^2}$, it generalizes to $\omega^2(k) = \frac{b^2 k^2 + m^2}{a^2}$.

In the electric case, $b \to 0$ there is an extra dependence on the dimensionless parameter $y = \frac{m\beta}{a}$ added. While for the magnetic case, $a \to \infty$ the partition function can depend on the ratio x/y that remains finite and the dimensionless parameter z.

5.6 Expanding the logarithm

We may omit the Casimir energy term as subleading in the large volume expansion and ignore it from the beginning. Later on, this will be formally justified in Section 5.8 through a CFT analysis. This approach considers as a starting point the partition function expression after the resummation

$$\log Z = -\sum_{k \in \mathbb{Z}^*} \log \Big(1 - exp\big(-\frac{\beta b |k|}{aR} + i\frac{\beta v k}{R}\big)\Big) \quad . \tag{5.36}$$

The Taylor expansion of the logarithm yields

$$\log Z = \sum_{k\in\mathbb{Z}^*}\sum_{n>0}\frac{1}{n}e^{-\frac{\beta b|k|}{aR}n+i\frac{\beta vk}{R}n} \quad . \tag{5.37}$$

This expansion is valid for large values of the volume R, in such a way that the module of the exponential's argument is less than 1 and the Taylor expansion holds within the convergence region.

Next, we approximate the sum on $k \in \mathbb{Z}^*$ by an integral:

$$logZ \approx \sum_{n>0}\frac{1}{n}\int dkexp\Big\{-\frac{\beta b|k|}{aR}n + i\frac{\beta vk}{R}n\Big\} \quad . \tag{5.38}$$

At this point, we remind the dimensionless quantities on which we are allowed to consider limits.

$$x = \frac{\beta b}{aR} \quad , \tag{5.39}$$

$$z = \frac{\beta v}{R} \quad . \tag{5.40}$$

In terms of these dimensionless parameters, the partition function becomes

$$\begin{aligned}\log Z &\approx \sum_{n>0}\frac{1}{n}\int_{-\infty}^{\infty}dkexp(-x|k|n + izkn)\\ &\approx \sum_{n>0}\Big[\frac{1}{n}\int_{0}^{\infty}dkexp(-xkn + izkn) + \int_{0}^{\infty}dkexp(-xkn - izkn)\Big]\\ &\approx \sum_{n>0}\frac{2}{n}\int_{0}^{\infty}dke^{-xkn}cos(zkn) \quad .\end{aligned} \tag{5.41}$$

The integral converges for any value of the integer n because the argument of the exponential is always negative and yields the following

$$\begin{aligned}\log Z &\approx 2\sum_{n>0}\frac{x}{z^2+x^2}\frac{1}{n^2}\\ &\approx \frac{\pi^2}{3}\frac{x}{x^2+z^2} \quad .\end{aligned} \tag{5.42}$$

This answer is essentially the generalization of the standard Cardy result for a $c = 1$ CFT that does not contain a chemical potential v. Comparing to the real chemical potential case of chapter 5.3 we notice that it is possible to achieve the result of an imaginary chemical potential by analytically continuing the chemical potential $v \to iv \implies z \to iz$.

Recovering the initial parameters

$$\log Z \approx \frac{\pi^2}{3}\frac{\frac{\beta b}{aR}}{\left(\frac{\beta v}{R}\right)^2+\left(\frac{\beta b}{aR}\right)^2} = \frac{R\pi^2}{3\beta}\frac{\frac{b}{a}}{v^2+\frac{b^2}{a^2}} \quad . \tag{5.43}$$

At this stage, by keeping the leading term in the Carroll regime, we have achieved extensive thermodynamics because the logarithm of the partition function is linear in R in the large volume limit.

5.7 Equation of State

From classical thermodynamics, the Euler equation is known to relate different thermodynamic quantities. In the case in which we have introduced an imaginary chemical potential as one of the parameters, it follows that

$$\mathcal{E}+\mathcal{P} = \frac{1}{\beta R}S + \frac{iv}{R}P \quad . \tag{5.44}$$

It provides a useful way to derive the equation of state in the large R limit starting with the terms on the right-hand side: the entropy S and the momentum P (conjugate to the chemical potential v).

First, we evaluate the entropy:

$$S = (1-\beta\partial_\beta)\log Z = \frac{\pi^2 R}{3\beta}\frac{\frac{b}{a}}{v^2+\frac{b^2}{a^2}} + \frac{\pi^2 R}{3\beta}\frac{\frac{b}{a}}{v^2+\frac{b^2}{a^2}} = \frac{2\pi^2 R}{3\beta}\frac{\frac{b}{a}}{v^2+\frac{b^2}{a^2}} \quad , \tag{5.45}$$

while the momentum is evaluated by

$$P = \frac{1}{i\beta}\frac{\partial log Z}{\partial v} = -\frac{1}{i\beta}\frac{2\pi^2 R}{3\beta}v\frac{\frac{b}{a}}{\left(v^2+\frac{b^2}{a^2}\right)^2} \quad . \tag{5.46}$$

Plugging the last two answers into the right-hand side of the Euler equation (5.44), we obtain

$$\mathcal{E}+\mathcal{P} = \frac{2\pi^2}{3\beta^2}\frac{\left(\frac{b}{a}\right)^3}{\left(v^2+\frac{b^2}{a^2}\right)^2} \quad . \tag{5.47}$$

Recovering the dimensionless parameter x which is relevant to the Carroll limit, the equation of state can be written as

$$\mathcal{E}+\mathcal{P} = \frac{2\pi^2}{3\beta^2}\frac{\left(\frac{xR}{\beta}\right)^3}{\left(v^2+\left(\frac{xR}{\beta}\right)^2\right)^2} \xrightarrow{x\to 0} 0 \quad . \tag{5.48}$$

This is an expected result for any Carroll theory and at this point, we remind that the dimensionless quantity x when approaching zero represents both the electric and magnetic case for the scalar quantum field theory.

Another important feature of this result is that in the case of vanishing chemical potential, it is not continuously connected to the non-vanishing case. This is also a general characteristic of Carrollian theories. We recovered the same conclusion to the vanishing/non-vanishing chemical potential as for the Carroll massless particles. The cases of vanishing and non-vanishing chemical potential should always be studied separately.

5.8 Hamiltonian and momentum revisited

The quantum field theory that is studied here has a special feature. Namely, it does not intrinsically contain a scale. It is hence a conformal field theory (CFT). In a CFT the partition function should only depend on a modular parameter $\tau \in \mathbb{C}$ and that should relate to a combination of the physical quantities of our model. The purpose of this section is to provide these identifications.

Firstly, we consider the generators of the Witt algebra or to be more specific the zero modes of the Laurent expansion for the energy-momentum field

$$
\begin{aligned}
l_0 &= -z\partial_z \ , \\
\bar{l_0} &= -\bar{z}\partial_{\bar{z}} \ .
\end{aligned}
\tag{5.49}
$$

We then perform a change of variables

$$
z = re^{i\phi} \ , \tag{5.50}
$$

after which we gain a bit more of the geometric intuition for those generators

$$
\begin{aligned}
l_0 &= -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\phi \ , \\
\bar{l_0} &= -\frac{1}{2}r\partial_r - \frac{i}{2}\partial_\phi \ .
\end{aligned}
\tag{5.51}
$$

These lead to the linear combinations of the operators

$$
\begin{aligned}
l_0 + \bar{l_0} &= -r\partial_r \ , \\
i(l_0 - \bar{l_0}) &= -\partial_\phi
\end{aligned}
\tag{5.52}
$$

and we recognize that they generate 2-dimensional dilations and rotations respectively.

The procedure that we have followed in the previous chapters to derive the thermal quantum partition function involves periodic conditions that essentially map the complex plane to the cylinder with radius R. To realize the correspondence between the two pictures we have to tailor the above transformation to the mapping

$$z = e^{\frac{x^0}{R} + i\frac{x^1}{R}} \quad . \tag{5.53}$$

Here it is easy to check that this transformation ultimately maps the complex plane to the cylinder of radius R through the identification $x^1 \sim x^1 + 2\pi R n$ with $n \in \mathbf{Z}$ for the spatial coordinate. Furthermore, we have defined that

$$x^0 = \frac{b}{a} t \tag{5.54}$$

for two reasons:

- To restore the relativistic case with a factor of c (the speed of light in vacuum),
- To match the dimensionality of the x^1 spatial coordinate

Then, going back to the generator combinations we conclude the following

$$\begin{aligned} l_0 + \bar{l}_0 &= -e^{\frac{x^0}{R}} R e^{-\frac{x^0}{R}} \partial_{x^0} \\ &= -R \frac{a}{b} \partial_t \end{aligned} \tag{5.55}$$

and also in a similar manner

$$l_0 - \bar{l}_0 = R\partial_{x^1} \quad . \tag{5.56}$$

Expressions (5.55) and (5.56) lead to the definition of the Hamiltonian and the spatial momentum since they are the generators of time and space translations respectively. We arrive at the conclusion that

$$\begin{aligned} H &= \frac{b}{Ra}(l_0 + \bar{l}_0) + \text{central charge} \quad , \\ P &= \frac{1}{R}(l_0 - \bar{l}_0) \quad . \end{aligned} \tag{5.57}$$

The extension to the Witt algebra that captures the quantum nature of the theory is the Virasoro algebra Vir_c characterized by the central charge c. It contributes to the Hamiltonian that gives the zero point energy or Casimir energy and can be evaluated via

$$E_C = \frac{b}{Ra}\langle (T_{cyl})_{00}\rangle = \frac{b}{Ra}\Big[\langle T_{cyl}\rangle + \langle \bar{T_{cyl}}\rangle\Big] = -\frac{b}{Ra}2\frac{c}{24} = -\frac{c}{12}\frac{b}{Ra} \quad , \tag{5.58}$$

where for the first equality the following properties were used

$$\begin{aligned} T_{cyl} &= \frac{1}{2}(T_{00} - iT_{10}) \quad , \\ \bar{T_{cyl}} &= \frac{1}{2}(T_{00} + iT_{10}) \quad , \end{aligned} \tag{5.59}$$

while for the last equation we used that

$$T_{cyl}(w) = \Big(\frac{\partial f(w)}{\partial w}\Big)^2 T(f(w)) + \frac{c}{12}S(f(w), w) = z^2T(z) - \frac{c}{24} \quad , \tag{5.60}$$

with $z = f(w) = e^w$ and the Schwarzian derivative $S(z, w) = -\frac{1}{2}$ for that case.

5.9 Modular parameter

Going back to the partition function we have ended up with for the conformal bosonic case in terms of the infinite product

$$Z' = e^{-\beta E_C} \prod_{n\in\mathbb{Z}^*} \frac{1}{1 - exp(-\beta\omega(k_n) + i\beta v k_n)} \quad , \tag{5.61}$$

with $\omega(k_n) = \left|\frac{bk_n}{a}\right|$ and $k_n = \frac{n}{R}$ and excluding the zero mode.

On the other hand, the partition function from the conformal field theory analysis (excluding the zero mode) is in terms of the modular parameter τ

$$Z'(\tau, \bar{\tau}) = \frac{1}{|\eta(q)|^2} \quad , \tag{5.62}$$

with η denoting the Dedekind eta function and $q = e^{2\pi i\tau}$. That leads to the identification $\tau_1 = \frac{\beta v}{2\pi R}$ and $\tau_2 = \frac{\beta b}{2\pi a R}$ that forms the modular parameter $\tau = \tau_1 + i\tau_2$ with $\tau_1, \tau_2 \in \mathbb{R}$.

Given those identifications, the partition function corresponds to the one that can be defined through the modular parameter in the following way

$$Z(\tau) = Tr_{\mathcal{H}}(e^{-2\pi\frac{Ra}{b}\tau_2 H + i2\pi\tau_1 RP}) \quad . \tag{5.63}$$

Recovering the Virasoro algebra generators, the partition function is taking the usual form for a bosonic CFT.

$$Z(\tau) = Tr_{\mathcal{H}}\left(q^{L_0-\frac{c}{24}}\bar{q}^{-\bar{L_0}-\frac{c}{24}}\right) \ . \tag{5.64}$$

Attempting to include the finite part of the zero mode back to the partition function, we have to think that the wavefunction of the zero mode needs to have a free waveform e^{ipx_0}. This then leads to determining the contribution of it by $L_0 = \bar{L}_0 = p^2/2$. A way to think about it is to remember the operator-state correspondence that maps the state e^{ipx_0} to the normal ordered operator $:e^{ipX}:$ (whose weight coincides with the Virasoro generators L_0 and $\bar{L_0}$). The other modes get mapped one-to-one to the local operators $\partial^n X$, $n \in \mathbb{N}^*$. Given the equation (5.64) we realise that the zero mode behaves like a massless free particle

$$Z_0 = \int dp q^{p^2/2}\bar{q}^{p^2/2} = \int dp e^{-2\pi\tau_2 p^2} = \frac{A}{\sqrt{\tau_2}} \ , \tag{5.65}$$

where A is a factor that depends on the normalization and is not relevant to our analysis since it is a constant. The full partition function that now incorporates the holomorphic part, anti-holomorphic part and the zero mode reads

$$Z = \frac{1}{\sqrt{\tau_2}}\frac{1}{|\eta(q)|^2} \ . \tag{5.66}$$

Finally, provided that for a conformal bosonic theory, the central charge is $c = 1$, the Casimir energy is taking the form

$$E_C = -\frac{b}{12Ra} = -\frac{\pi}{6\beta}\tau_2 \ . \tag{5.67}$$

The partition function now has become modular invariant. It ought to satisfy this condition, since $SL(2,\mathbb{Z})/\mathbb{Z}_2$ transformations do not change the cylinder. Therefore, the partition function $Z(\tau,\bar{\tau})$ in the CFT should be modular invariant.

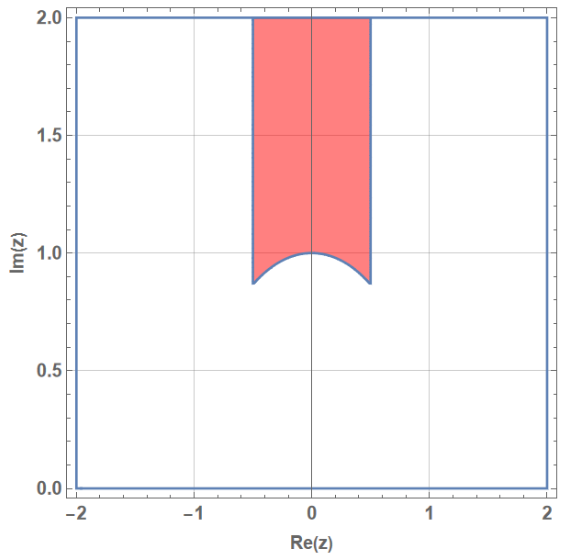


Figure 5.1: Fundamental Domain of $\mathrm{SL}(2,\mathbb{Z})$ in the Complex Plane.

To determine the partition function we only need to study it within the red domain that is highlighted in Figure 5.1. For any other value of the modular parameter in the upper half plane, we can find a value of it within the fundamental domain that gives the partition function for any other value and can be obtained by an element of $SL(2,\mathbb{Z})$.

5.10 Dedekind eta function expansion

Another way to extract the behavior of the quantum partition function in the large volume limit is to consider the expansion of the Dedekind eta function for $R \to \infty$ or equivalently $\tau \to 0$.

To achieve this, we write down the generating function of partitions $p(n)$ with $q = e^{2\pi i\tau}$.

$$P(\tau) = \sum_{n=1}^{\infty} p(n) q^n = \frac{e^{\pi i \tau/12}}{\eta(\tau)} \quad . \tag{5.68}$$

The above is defined in the domain of convergence of the upper half plane $\mathbb{H} := \{\tau \in \mathbb{C}/\mathcal{I}m(\tau) > 0\}$. The Dedekind eta function is a modular form of weight 1/2, which means that it satisfies the following two properties:

$$\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau) \ , \tag{5.69}$$

$$\eta\big(-\frac{1}{\tau}\big) = \sqrt{-i\tau}\eta(\tau) \ . \tag{5.70}$$

Therefore, it is possible to rewrite the generating function of partitions using the second modular property

$$P(\tau) = e^{\frac{\pi i\tau}{12}} P\big(-\frac{1}{\tau}\big)\sqrt{-i\tau}e^{\frac{\pi i}{12\tau}} \ . \tag{5.71}$$

From the above an asymptotic expansion can be obtained for small values of the modulus of the modular parameter, namely:

$$P(\tau) \to \sqrt{-i\tau}e^{-\frac{\pi i}{12\tau}} \approx \frac{e^{\pi i\tau/12}}{\eta(\tau)} \ . \tag{5.72}$$

The total partition function contains both the positive modes part (”chiral”) and the negative modes part (”anti-chiral”).

$$Z = \frac{1}{\sqrt{\tau_2}}\frac{1}{|\eta(\tau)|^2} \overset{\tau\to 0}{\to} \frac{|\tau|}{\sqrt{\tau_2}}e^{\frac{\pi i}{12}(\frac{1}{\tau}-\frac{1}{\bar\tau})}e^{-\pi i(\tau-\bar\tau)/12} = \frac{|\tau|}{\sqrt{\tau_2}}e^{\frac{\pi Im(\tau)}{6|\tau|^2}}e^{-\pi Im\tau/6} \ . \tag{5.73}$$

The logarithm of the partition function in the large R limit is

$$\log Z \overset{\tau\to 0}{\to} \log\frac{|\tau|}{\sqrt{\tau_2}} + \frac{\pi\tau_2}{6|\tau|^2} - \frac{\pi\tau_2}{6} = \log\sqrt{\frac{x^2+z^2}{x}} + \frac{\pi^2}{3}\frac{x}{x^2+z^2} - \frac{x}{12} \ , \tag{5.74}$$

a result that coincides with the one obtained for the expansion of the logarithm (5.42) in section 5.6 for the leading term and the analytic continuation of the result for a real chemical potential (5.33) in section 5.3. Again, the first term is subleading in the large R limit and the last one is the Casimir term.

5.11 Equation of state parameter

The model describing a general massless scalar quantum field theory in 1+1 dimensions is described by the Lagrangian

$$\mathcal{L} = a^2(\partial_t\phi)^2 - b^2(\partial_x\phi)^2 \ . \tag{5.75}$$

The equation of state after the substitutions for a relativistic setting with parameters

$$a \to \frac{1}{c} \ , \tag{5.76}$$

$$b \to 1 \tag{5.77}$$

is the following

$$\mathcal{E} + P = \frac{2\pi^2}{3\beta^2} \frac{c^3}{\left(v^2 + c^2\right)^2} \ . \tag{5.78}$$

Individually the pressure is extracted from the already known partition function in the large R limit

$$P = -\frac{1}{\beta} \frac{\partial log Z}{\partial R} = \frac{\pi^2}{3\beta^2} \frac{c}{v^2 + c^2} \ . \tag{5.79}$$

Therefore, the equation of state parameter is concluded as a function of the velocity v

$$w = \frac{v^2 + c^2}{v^2 - c^2} = \frac{1 + \left(\frac{v}{c}\right)^2}{1 - \left(\frac{v}{c}\right)^2} \ . \tag{5.80}$$

As we commented in the Carroll particles analysis; for small values of the velocity v compared to the speed of light, the equation of state parameter is $w = 1$ as expected ($w = \frac{1}{d}$) and in the Carroll limit $w = -1$ as can be noticed in Figure 5.2.

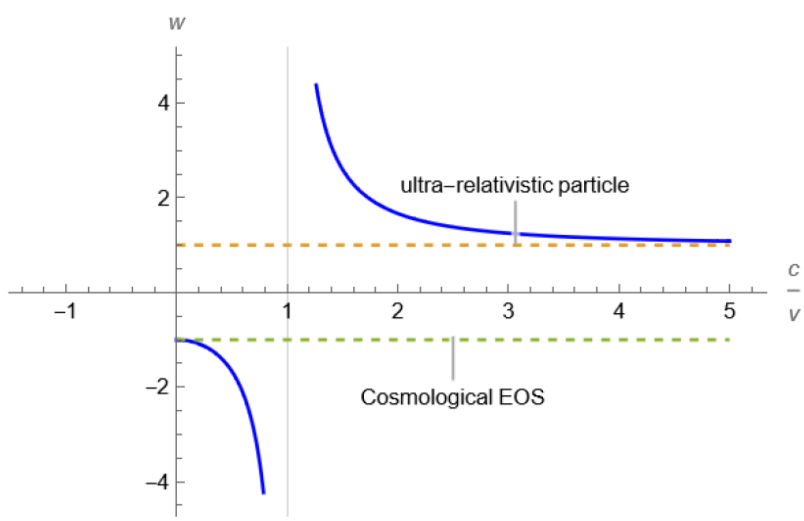


Figure 5.2: Equation of state parameter vs velocity.

5.12 Generalizing the dimensions

This section intends to provide the generalization to an arbitrary number of spatial dimensions for the analogous to a CFT discussed in the previous sections. One way to achieve it is to study the theory on $\mathbb{R}^{d-1} \times S^1$. In other words, we place the theory in an infinite volume with two parallel hyperplanes which we identify with each other. Then essentially one of the dimensions is considered small compared to the other directions. The approach that we follow is equivalent to the one used in [27] and [28].

Similarly to the prescription for one spatial dimension, the Hamiltonian and spatial momentum (conjugate to the chemical potential) are given in terms of the modes:

$$H = \sum_{n_i} \omega(k_{n_i}) \hat{a_{k_i}}^\dagger \hat{a}_{k_i} + E_c^d \quad , \tag{5.81}$$

$$P = \sum_{n_i} k_d \hat{a_{k_i}}^\dagger \hat{a}_{k_i} \quad . \tag{5.82}$$

At this point, the sum notation is a generalization to include the quantized k's and the continuous ones, for which the sum is an integral.

Without loss of generality, we assume that the small dimension is the one carrying the label d. The dispersion relation is given by

$$\omega(k_{n_i}) = \frac{b}{a}\sqrt{k^2 + k_d^2} \quad , \tag{5.83}$$

where k^2 contains all the k's associated with the large dimensions.

Similarly, we can use the number operators (one for each dimension) to perform the trace in the partition function

$$Z = e^{-\beta E_c^d} \prod_{n_i \in \mathbb{Z}^d/\{0\}} \sum_{N_i \in N} e^{-\beta(\omega(k_{n_i}) - ivk_d)N_i} = e^{-\beta E_c^d} \prod_{n_i \in \mathbb{Z}^d/\{0\}} \frac{1}{1 - e^{-\beta(\omega(k_{n_i}) - ivk_d)}} \quad . \tag{5.84}$$

In the large dimensions, k's can be integrated over, because they are continuous and the logarithm of the partition gains the form

$$\begin{aligned}\log Z &= -\beta E_c^d - \frac{V_{d-1}}{(2\pi)^{d-1}}\int_{\infty}^{\infty} d^{d-1}k \sum_{n_d\in\mathbb{Z}} \log(1-e^{-\beta(\omega(k_{n_i})-ivk_d)} \\ &= -\beta E_c^d - \frac{V_{d-1}}{(2\pi)^{d-1}}\frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\int_0^{\infty} dk k^{d-2} \sum_{n^d\in\mathbb{Z}} \log(1-e^{-\beta(\frac{b}{a}\sqrt{k^2+k_d^2}-ivk_d)}) \ ,\end{aligned} \tag{5.85}$$

where in the last line we performed the angular integrals that give the volume of a sphere in $(d-1)$-dimensions. Splitting the zero mode:

$$\begin{aligned}\log Z =& -\beta E_c^d + \frac{V_{d-1}\Gamma(d-1)\zeta(d)}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})\beta^{d-1}(\frac{b}{a})^{d-1}} \\ &- \frac{V_{d-1}}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})}\sum_{n_d\in\mathbb{Z}/\{0\}}\int_0^{\infty} dk k^{d-2}\log(1-e^{-\beta(\frac{b}{a}\sqrt{k^2+k_d^2}-ivk_d)}) \ ,\end{aligned} \tag{5.86}$$

where we used the integral

$$\int_0^{\infty} dk k^{d-2}\log(1-e^{-\lambda k}) = -\frac{1}{(d-1)\lambda^{d-1}}\int_0^{\infty} dx\frac{x^{d-1}}{e^x-1} = -\frac{1}{\lambda^{d-1}}\Gamma(d-1)\zeta(d) \tag{5.87}$$

for the parameter value $\lambda = \beta\frac{b}{a}$. The remaining integral cannot be computed in a closed form and will keep it as a series after expanding the logarithm. The result is involving modified Bessel function K_ν after performing integration

$$\begin{aligned}\log Z =& -\beta E_c^d + \frac{V_{d-1}\Gamma(d-1)\zeta(d)}{2^{d-2}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})\beta^{d-1}(\frac{b}{a})^{d-1}} \\ &+ 2\frac{V_{d-1}}{R^{d/2}\beta^{\frac{d-2}{2}}\left(\frac{b}{a}\right)^{\frac{d-2}{2}}}\sum_{n_d\in\mathbb{Z}/0}\sum_{m\in\mathbb{N}/0}\left(\frac{|n_d|}{m}\right)^{d/2}K_{\frac{d}{2}}\left(2\pi|m|\frac{\beta}{R}\right)e^{2\pi i m|n_d|\beta\frac{v}{R}} \ .\end{aligned} \tag{5.88}$$

The transformations of the modular group imply for the parameters of the model the following:

$$\tau' = \tau+1 \iff \begin{cases}\left(\frac{\beta\frac{b}{a}}{R}\right)' = \frac{\beta\frac{b}{a}}{R} \\ \left(\frac{\beta v}{R}\right)' = \frac{\beta v}{R}+1\end{cases} \tag{5.89}$$

and

$$\tau' = -\frac{1}{\tau} \iff \begin{cases}\left(\frac{\beta\frac{b}{a}}{R}\right)' = \frac{R\frac{a}{b}}{\beta((\frac{b}{a})^2+v^2)} \\ \left(\frac{\beta v}{R}\right)' = -\frac{R}{\beta((\frac{b}{a})^2+v^2)}\end{cases} \ . \tag{5.90}$$

Under the second transformation (5.86) the logarithm of the partition function changes in the following way:

$$\log Z' = \frac{\beta^{d-1}}{R^{d-1}} \Big((\frac{b}{a})^2 + v^2 \Big)^{\frac{d-1}{2}} \log Z \quad . \tag{5.91}$$

Using the explicit form for the left-hand side of (5.87) and solving for $\log Z$ we obtain the expression

$$\begin{aligned} \log Z = & \frac{V_{d-1} R}{\beta^d \big(\frac{b}{a}\big)^{-1}} \frac{\Gamma\big(\frac{d+1}{2}\big)\zeta(d+1)}{\pi^{\frac{d+1}{2}}} \Big((\frac{b}{a})^2 + v^2 \Big)^{-\frac{d+1}{2}} \\ & + \frac{V_{d-1}}{\big(\frac{b}{a}\big)^{d-1} R^{d-1}} \frac{\Gamma\big(\frac{d}{2}\big)\zeta(d)}{\pi^{d/2}} \Big((\frac{b}{a})^2 + v^2 \Big)^{\frac{d-1}{2}} + 2 \Big(\frac{\beta}{R}\Big)^{-\frac{d}{2}} \Big((\frac{b}{a})^2 + v^2 \Big)^{\frac{d-2}{2}} \times \\ & \sum_{m \in \mathbb{N}/\{0\}} \sum_{n_d \in \mathbb{Z}/\{0\}} \Big(\frac{|n_d|}{m} \Big)^{d/2} K_{d/2} \Big(2\pi |m| \frac{R}{\beta} \frac{1}{\big(\frac{b}{a}\big)^2 + v^2} \Big) e^{-2\pi i m |n_d| \frac{R}{\beta} \frac{1}{(\frac{b}{a})^2 + v^2}} \quad . \end{aligned} \tag{5.92}$$

In the first term, the Legendre duplication formula for the gamma function was used. The second term is the part deriving from the evaluation of Casimir energy in d spatial dimensions, while the double summation is exponentially suppressed in the limit of $R \to \infty$. The Casimir energy can be computed by evaluating the zero point energy based on the Hamiltonian which is defined in (5.81).

$$\begin{aligned} E_c^d = \langle 0 | H | 0 \rangle = & \frac{1}{2} \frac{b}{a} \sum_{n_i \in \mathbb{Z}^d/\{0\}} \omega_{k_{n_i}} = \frac{1}{2} \frac{b}{a} \sum_{n_i \in \mathbb{Z}^d/\{0\}} \sqrt{k^2 + k_d^2} \\ = & \frac{1}{2} \frac{b}{a} Vol(S_{d-2}) \sum_{n_d \in \mathbb{Z}/\{0\}} \int_0^\infty \frac{dk}{(2\pi)^{d-1}} \sqrt{k^2 + k_d^2} \\ = & -\frac{b}{a} \frac{V_{d-1}}{R^d} \frac{\Gamma\big(-\frac{d}{2}\big)\zeta(-d)}{\pi^{-d/2}} = -\frac{b}{a} \frac{V_{d-1}}{R^d} \frac{\Gamma\big(\frac{d+1}{2}\big)\zeta(d+1)}{\pi^{(d+1)/2}} \quad . \end{aligned} \tag{5.93}$$

This deduction involves zeta function regularization and the Legendre duplication formula for the gamma function once again.

We desire to deduce the equation of state formula for arbitrary dimensions in the large volume limit, hence sending $R \to \infty$. By using the proper form of the Euler equation that relates thermodynamic quantities:

$$\mathcal{E} + \mathcal{P} = \frac{1}{\beta V} S + \frac{iv}{V} P \tag{5.94}$$

and the leading term in the expansion for large R in the logarithm of the partition function

$$S = (1-\beta\partial_\beta)logZ = \frac{V_{d-1}R}{\left(\frac{b}{a}\right)\beta^d}(d+1)\frac{\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{(d+1)/2}}\left(\left(\frac{b}{a}\right)^2+v^2\right)^{-\frac{d+1}{2}} \ , \tag{5.95}$$

$$P = \frac{1}{i\beta}\partial_v logZ = \frac{1}{i\beta}\frac{V_{d-1}R}{\left(\frac{b}{a}\right)\beta^d}(-d-1)\frac{\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{(d+1)/2}}v\left(\left(\frac{b}{a}\right)^2+v^2\right)^{-\frac{d+3}{2}} \ . \tag{5.96}$$

Inserting these results in the Euler equation and trading $\frac{b}{a} \to c$, which represents a relativistic setting, we acquire the equation of state below

$$\mathcal{E}+\mathcal{P} = \frac{d+1}{\beta^{d+1}}\frac{\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{\frac{d+1}{2}}}\frac{c^3}{\left(v^2+c^2\right)^{\frac{d+3}{2}}} \propto \frac{1}{\beta^{d+1}}\frac{c^3}{\left(v^2+c^2\right)^{\frac{d+3}{2}}} \ . \tag{5.97}$$

Taking the strict Carroll limit (electric or magnetic), meaning $x \to 0 \implies c \to 0$, in this stage forces the equation of state to take the form of the one present in inflationary cosmology

$$\mathcal{E}+\mathcal{P} \overset{c\to 0}{\to} 0 \ . \tag{5.98}$$

Furthermore, if we use the leading order term in the large R expansion to compute the pressure $\mathcal{P}$ separately we can infer the energy density of the framework and compare it to the Carroll particles' case

$$\mathcal{P} = \frac{1}{V_{d-1}}\frac{\partial(\beta^{-1}logZ)}{\partial R} = \frac{1}{\beta^{d+1}}\frac{\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{\frac{d+1}{2}}}\frac{c}{\left(v^2+c^2\right)^{\frac{d+1}{2}}} \tag{5.99}$$

and combining with the equation (5.97)

$$\mathcal{E} = \frac{1}{\beta^{d+1}}\frac{\Gamma\left(\frac{d+1}{2}\right)\zeta(d+1)}{\pi^{\frac{d+1}{2}}}\frac{c}{\left(v^2+c^2\right)^{\frac{d+3}{2}}}(dc^2-v^2) \ , \tag{5.100}$$

which yields the same result for the critical velocity as extracted for the Carroll particles case. Namely, that the critical value for the chemical potential is $v_c = \sqrt{d}c$.

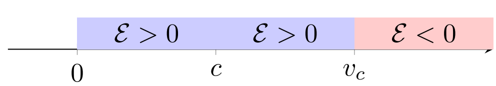


Figure 5.3: Energy density sign vs chemical potential.

Comments:

- The critical value for the chemical potential coincides with the result that we extracted for the Carroll particles description $v_c = \sqrt{d}c$, which only depends on the dimensions.

- The equation of state is also the same under the strict Carroll limit followed by the large volume expansion. Intuitively, this is to be expected due to the decoherence of the quantum effects in the large volume limit.

Chapter 6

Conclusion and outlook

In this thesis several aspects and consequences of Carroll physics were presented. It is shown how Carroll symmetry algebra arises from Poincare symmetry algebra by taking the limit $c \to 0$. This limit however results in the appearance of tachyons. This fact did not come as a surprise because the Carroll limit has to be ultimately taken into a dimensionless parameter. It thus translates to comparing with velocities $v \gg c$. Effectively, what becomes manifest in this thesis is the comparison to superluminal recessional velocities in an expanding universe (de Sitter spacetime). The concept of thermodynamics within Carrollian physics was investigated. Following the logic of Carrollian physics principles, we dived into two different cases:

1. Carroll Particles (Chapter 3)

2. Carroll conformal field theory (Chapter 4)

In the Carroll particles case, we computed the partition function by adding an imaginary chemical potential that represents the particle's velocity, as the conjugate quantity to the momentum. The result is that thermodynamics can be described for finite value of the speed of light, thermodynamic quantities can be evaluated and in the end, the Carrollian limit can be reached from the relativistic regime. The equation of state parameter is evaluated and exhibits the same form as in cosmology, $\mathcal{E} + \mathcal{P} = 0$ and therefore an equation of state parameter $w = -1$. Further, there is a special value for the particle velocity; the critical velocity $v_c = \sqrt{d}c$ which depends on the spatial dimensions d and determines the sign of the energy density.

In the conformal field theory, we began by canonical quantization for a massless scalar field that was initially equipped with periodic conditions for in space. Then we carried on to compute the thermal partition function in $1+1$ dimensions and continued by generalizing the number of spatial dimensions with one spatial dimension being periodic for the field. Eventually, the large volume limit has to be considered to avoid boundary effects and on this limit thermodynamics is extensive, equivalently the logarithm of the partition function is proportional to the volume and generalizes the Cardy answer with a correction originating from the introduction of an imaginary chemical potential. Again, the equation of state in the strict Carroll limit is of the form $\mathcal{E}+\mathcal{P}=0$ and the critical value for the imaginary chemical potential exhibits the same behavior as in the particles' case. Namely, the critical value is $v_c=\sqrt{d}c$ when we approach the electric/magnetic sectors from the relativistic regime.

An important highlight is that the electric and magnetic sectors behave the same under the thermodynamic formulation that we presented, but it is important to keep in mind that the two cases are not equivalent in general. As was shown in Chapter 3, the two theories exhibit fundamental differences even from the point of 2-point correlators. It is however expected to end up with the very same result for the equation of state in the strict Carroll limit as explained in Chapter 1.

These results verify the relevance of Carrollian frameworks; both semi-classical and quantum to de Sitter spacetime. There are however several more aspects to be investigated to formally establish dS/CFT (de Sitter/ Conformal field theory correspondence); linking these two descriptions. Some parts also require some further motivation to acquire a proper physical interpretation. For instance, the Carroll particles description is set to define a tachyon gas which lacks physical context. Another challenge is to assign a meaningful interpretation to the formulation of Carroll field theories in a similar way to the standard quantum field theories where special relativity clashes with the principles of quantum mechanics. A better understanding in this direction would enable an improved explanation to our results and the relevance between Carrollian frameworks and de Sitter spacetime.

Certainly, more research is yet to come to give light to the principles of Carroll physics and the interplay with an expanding universe which looks significantly like our own. The beginning has been done and this research direction seems promising.

Appendix A

Hankel Transformation

To be more concrete on the evaluation of the integral encountered in section 4.4 involving Bessel functions, we provide this descriptive section. The Hankel transform that we had to compute has the form:

$$\begin{aligned} I &= \int_0^\infty x^n e^{-ax} J_n(sx) dx = \frac{1}{s^{n+1}} \int_0^\infty t^n J_n(t) e^{-a/st} dt \\ &= \frac{1}{s^{n+1}} \mathcal{L}[t^n J_n(t)](p = a/s) \quad , \end{aligned} \tag{A.1}$$

where by $\mathcal{L}$ we denote the Laplace transform. To proceed, we expand the function which has to be Laplace transformed in a series.

$$t^n J_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+2n}}{k!\Gamma(k+n+1)2^{2k+n}} \quad . \tag{A.2}$$

The Laplace transform then yields

$$\begin{aligned} \mathcal{L}[t^n J_n](p) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+n+1)2^{2k+n}} \mathcal{L}[t^{2k+2n}](p) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(2k+2n+1)}{k!\Gamma(k+n+1)2^{2k+n} p^{2k+2n+1}} \\ &= \frac{2^n}{\sqrt{\pi} p^{2n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k+n+1/2)}{k!} \Big(\frac{1}{p^2}\Big)^k \quad . \end{aligned} \tag{A.3}$$

In the last step we used the Legendre duplication formula for the gamma function. To write the sum in an analytic form we have to first notice that the coefficients resemble the binomial coefficients

$$\binom{-l}{m} = \frac{(-1)^m l(l+1)\dots(l+m-1)}{m!} = \frac{(-1)^m \Gamma(m+l)}{m!\Gamma(l)} \tag{A.4}$$

Putting everything back together, the result derived in (4.21) for the partition function of Carroll particles is concluded. Techniques presented in [24] were used in this Appendix.

Appendix B

Lorentz invariance

B.1 Total phase space measure

In this section, we would like to demonstrate the transformation law of the partition function of free massless relativistic particles with a real chemical potential $\vec{v}$

$$Z(\beta, v, V) = Tre^{-\beta(H-\vec{v}\cdot\vec{p})} = \frac{1}{h^d}\int d^dxd^dpe^{-\beta(H-\vec{v}\cdot\vec{p})} \quad . \tag{B.1}$$

To begin with, we may study the behavior of the measure of the total phase space. First of all, we have to keep in mind that d^dx and d^dp are invariant under rotations and we can decide on the initial conditions before boosting. Boosting along the first spatial dimension x leads to:

$$p'_x = \gamma_u(p_x - \frac{u_x}{c^2}E) \quad , \tag{B.2}$$

$$p'_i = 0 \qquad\qquad i \neq x \quad , \tag{B.3}$$

The determinant of the Jacobian corresponding to the transformation in spatial momenta is therefore determined by

$$\det J = \begin{vmatrix} \gamma_u\big(1-\frac{u}{c^2}\frac{\partial E}{\partial p_x}\big) & -\gamma_u\frac{u}{c^2}\frac{\partial E}{\partial p_y} & -\gamma_u\frac{u}{c^2}\frac{\partial E}{\partial p_z} & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix} \qquad = \gamma_u\big(1-\frac{u}{c^2}\frac{\partial E}{\partial p_x}\big)$$

With energy determined by the usual form for a free massless particle

$$E = c|\vec{p}| = c\sum_{i=1}^{d}\sqrt{p_i^2} \ \ . \tag{B.4}$$

Therefore, the transformation law is obtained to be

$$d^dp' = |detJ|d^dp = \gamma_u\big(1 - c\frac{p_x}{E/c}\frac{u}{c^2}\big) = \frac{E'}{E}d^dp \ \ , \tag{B.5}$$

from which we deduce that the combination

$$\frac{d^dp}{E} \tag{B.6}$$

is a Lorentz invariant. Now we have to determine the transformation law for the position of the measure d^dx. Using the Hamilton equations for the Hamiltonian that describes the free massless particle, we obtain

$$\dot{x} = \frac{\partial H}{\partial p_x} \implies \dot{x} =: v = \frac{c^2 p_x}{E} \tag{B.7}$$

and

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}} = 0 \implies \vec{p}(t) = \text{constant}. \tag{B.8}$$

Their combination results in the trajectory of the particles, which is

$$x(t) = vt + x_0 \ \ . \tag{B.9}$$

Now, assuming two worldlines given by

$$x_1(t_1) = vt_1 \quad ,$$
$$x_2(t_2) = vt_2 + \Delta x \quad . \tag{B.10}$$

In a primed frame of reference that moves along the x-direction, the usual Lorentz transformations apply for the x position and time. Working explicitly for the first worldline:

$$x'_1 = \gamma_u(x_1 - ut_1) \quad ,$$
$$t'_1 = \gamma_u(t_1 - \frac{ux_1}{c^2}) \quad .$$

Inserting the equations for $x_1(t_1)$ that describe a uniformly moving particle, we get

$$x'_1 = \gamma_u(v - u)t_1 \quad ,$$
$$t'_1 = \gamma_u(1 - \frac{uv}{c^2})t_1 \quad , \tag{B.11}$$

and eliminating t we get the following formula that amounts to the transformation law of the velocity

$$x'_1 = \frac{v - u}{1 - \frac{uv}{c^2}} t'_1 \quad . \tag{B.12}$$

Working in an analogous manner for the second worldline we achieve the following:

$$x'_2 = \frac{v - u}{1 - \frac{uv}{c^2}} t'_2 + \frac{1}{\gamma_u(1 - \frac{uv}{c^2})} \Delta x \quad , \tag{B.13}$$

It essentially means that the separation of the worldlines as viewed from the primed frame relative to the unprimed along the x-direction is given by:

$$\Delta x' = \frac{\Delta x}{\gamma_u(1 - \frac{uv}{c^2})} = \frac{E}{E'} \Delta x \quad . \tag{B.14}$$

Hence, the separation between the worldlines is

$$\Delta x' = \frac{E}{E'} \Delta x \quad , \tag{B.15}$$
$$\Delta x'_i = \Delta x_i \quad \text{for} \quad i \neq x \quad . \tag{B.16}$$

The conclusion that follows for the position measure is

$$d^d x' = \frac{E}{E'} d^d x \quad . \tag{B.17}$$

Combining the two results, the total measure of the phase space is invariant

$$d^d x d^d p = \textit{Lorentz invariant} \quad , \tag{B.18}$$

B.2 General Hamiltonian, velocity, and momentum transformations

To induce the transformation of the argument of the exponential we will have to consider the general transformations of the Hamiltonian, the velocity and the spatial momentum.

The Hamiltonian together with the spatial momentum form a four-vector and their general transformations follow the analogous transformation rule to four-position transformation. Namely,

$$H' = \gamma_u (H - \vec{u} \cdot \vec{p}) \quad , \tag{B.19}$$

$$\vec{p'} = \vec{p} + \frac{\gamma_u^2}{\gamma_u + 1} (\vec{p} \cdot \vec{u}) \frac{\vec{u}}{c^2} - \gamma_u \frac{H}{c^2} \vec{u} \quad . \tag{B.20}$$

Someone could easily verify that

$$H'^2/c^2 - \vec{p'}^2 = H^2/c^2 - \vec{p}^2 \tag{B.21}$$

is satisfied given the aforementioned transformation rule. By analyzing the velocity to its parallel and perpendicular components with respect to the boost parameter we further obtain the general transformation rule for the velocity

$$\vec{v'} = \frac{1}{1 - \frac{\vec{v} \cdot \vec{u}}{c^2}} \left[\gamma_u^{-1} \vec{v} - \vec{u} + \frac{1}{c^2} \frac{\gamma_u}{\gamma_u + 1} (\vec{v} \cdot \vec{u}) \vec{u} \right] \quad . \tag{B.22}$$

Next, we would like to determine the transformation law for the combination of the two terms involved in the exponential of the partition function

$$
\begin{aligned}
&H' - \vec{p}' \cdot \vec{v}' = \\
&\gamma_u (H - \vec{u}\cdot\vec{p}) - \frac{1}{1-\frac{\vec{v}\cdot\vec{u}}{c^2}} \left[\gamma_u^{-1}\vec{v} - \vec{u} + \frac{1}{c^2}\frac{\gamma_u}{\gamma_u+1}(\vec{v}\cdot\vec{u})\vec{u} \right] \cdot \\
&\cdot \left[\vec{p} + \frac{\gamma_u^2}{\gamma_u+1}(\vec{p}\cdot\vec{u})\frac{\vec{u}}{c^2} - \gamma_u \frac{H}{c^2}\vec{u} \right]
\end{aligned}
\tag{B.23}
$$

To simplify the expression we are going to treat the terms containing the Hamiltonian separately to the ones containing the momentum. Starting by the terms that multiply the Hamiltonian:

$$
\begin{aligned}
&\frac{1}{1-\frac{\vec{v}\cdot\vec{u}}{c^2}} \left[\frac{\vec{u}\cdot\vec{v}}{c^2} - \frac{u^2\gamma_u}{c^2} + \frac{1}{c^4}\frac{\gamma_u^2}{\gamma_u+1}(\vec{v}\cdot\vec{u})u^2 \right] + \gamma_u = \\
&\frac{1}{1-\frac{\vec{v}\cdot\vec{u}}{c^2}} \left[\gamma_u \Big(\frac{\vec{u}\cdot\vec{v}}{c^2} - \frac{u^2}{c^2} \Big) + \gamma_u - \gamma_u \frac{\vec{u}\cdot\vec{v}}{c^2} \right] = \\
&\frac{1}{1-\frac{\vec{v}\cdot\vec{u}}{c^2}} \gamma_u \big(1 - \frac{\vec{u}\cdot\vec{v}}{c^2}\big) = \\
&\gamma_u^{-1}\big(1 - \frac{\vec{u}\cdot\vec{v}}{c^2}\big)^{-1} \quad .
\end{aligned}
\tag{B.24}
$$

Then, we rearrange the terms containing momentum:

$$
\begin{aligned}
&-\gamma_u \vec{u}\cdot\vec{p} + \frac{1}{1-\frac{\vec{v}\cdot\vec{u}}{c^2}} \left[\vec{p} - \frac{\gamma_u^2}{\gamma_u+1}(\vec{p}\cdot\vec{u})\frac{\vec{u}}{c^2} \right] \cdot \left[\gamma_u^{-1}\vec{v} - \vec{u} + \frac{1}{c^2}\frac{\gamma_u}{\gamma_u+1}(\vec{v}\cdot\vec{u})\vec{u} \right] = \\
&-\gamma_u \vec{u}\cdot\vec{p} + \frac{1}{1-\frac{\vec{v}\cdot\vec{u}}{c^2}} \Big[-\gamma_u^{-1}\vec{p}\cdot\vec{v} + \vec{p}\cdot\vec{u} - \frac{2}{c^2}\frac{\gamma_u}{\gamma_u+1}(\vec{v}\cdot\vec{u})(\vec{p}\cdot\vec{u}) + \\
&\qquad \frac{\gamma_u^2}{\gamma_u+1}(\vec{p}\cdot\vec{u})\frac{u^2}{c^2} - \frac{\gamma_u^3}{(\gamma_u+1)^2}\frac{1}{c^2}(\vec{p}\cdot\vec{u})(\vec{u}\cdot\vec{v})\frac{u^2}{c^2} \Big] = \\
&\frac{1}{1-\frac{\vec{v}\cdot\vec{u}}{c^2}} \Big[-\gamma_u^{-1}\vec{p}\cdot\vec{v} + \frac{1}{c^2}\Big(-\frac{\gamma_u}{\gamma_u+1} - \frac{\gamma_u^2}{\gamma_u+1} + \gamma_u \Big)(\vec{p}\cdot\vec{u})(\vec{u}\cdot\vec{v}) \Big] = \\
&-\gamma_u^{-1}\big(1 - \frac{\vec{u}\cdot\vec{v}}{c^2}\big)^{-1}(\vec{v}\cdot\vec{p}) \quad .
\end{aligned}
\tag{B.25}
$$

Putting everything back together we infer the transformation law for the terms in the exponential, namely

$$
H' - \vec{v}'\cdot\vec{p}' = \gamma_u^{-1}\big(1 - \frac{\vec{u}\cdot\vec{v}}{c^2}\big)^{-1}(H - \vec{v}\cdot\vec{p}) \quad . \tag{B.26}
$$

B.3 The general transformation of the Lorentz factor

We are already familiar with the way that the parallel and perpendicular components of the velocity with respect to the boost direction transform. To be specific

$$\vec{v'}_{\parallel} = \frac{\vec{v}_{\parallel} - \vec{u}}{1 - \frac{\vec{u}\cdot\vec{v}}{c^2}} \quad , \tag{B.27}$$

$$\vec{v'_{\perp}} = \gamma_u^{-1} \frac{\vec{v}_{\perp}}{1 - \frac{\vec{u}\cdot\vec{v}}{c^2}} \quad . \tag{B.28}$$

Analyzing the velocity within the Lorentz factor in the parallel and perpendicular components and taking into account that their dot product vanishes,

$$\begin{aligned}
\gamma_{v'} &= \left[1 - \left(\frac{\vec{v_{\parallel}}/c - \vec{u}/c}{1 - \frac{\vec{u}\cdot\vec{v}}{c^2}}\right)^2 - \left(\frac{\sqrt{1 - \left(u/c\right)^2}\,\vec{v_{\perp}}/c}{1 - \frac{\vec{u}\cdot\vec{v}}{c^2}}\right)^2\right]^{-1/2} = \\
&= \frac{1 - \frac{\vec{u}\cdot\vec{v}}{c^2}}{\sqrt{\left(1 - \left(\frac{v_{\parallel}}{c}\right)^2 - \left(\frac{v_{\perp}}{c}\right)^2\right)\left(1 - \left(\frac{u}{c}\right)^2\right)}} \\
&= \gamma_v \gamma_u \left(1 - \frac{\vec{u}\cdot\vec{v}}{c^2}\right) \quad .
\end{aligned} \tag{B.29}$$

Here the parallel component of the velocity is

$$\vec{v}_{\parallel} = \frac{\vec{u}\cdot\vec{v}}{v^2}\vec{v} \quad , \tag{B.30}$$

while the perpendicular is given by the difference between the total vector minus the parallel part:

$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel} \quad . \tag{B.31}$$

Multiplying the Lorentz factor with the exponential in the partition function could cancel the transformation law of the remaining and make the whole partition function Lorentz invariant. One way to make use of this fact would be to introduce an effective inverse temperature

$$\tilde{\beta} = \gamma\beta \quad , \tag{B.32}$$

that acquires the transformation of the Lorentz gamma factor. At this point, there is no need to give a physical interpretation to the parameter $\tilde{\beta}$. We can still take derivatives with respect to it and obtain the expression for the generalized energy.

This approach would mean that every β parameter should get substituted by a $\tilde{\beta}$ parameter in our calculations if we wish to preserve Lorentz invariance. Hence, the partition function for massless free relativistic particles would result in

$$Z = 2^d V \frac{\pi^{\frac{d-1}{2}}}{(hc\tilde{\beta})^d} \gamma_v^{d+1} \Gamma\Big(\frac{d+1}{2}\Big) \quad . \tag{B.33}$$

Double-checking the final formula, the result is Lorentz invariant as expected; there are d Lorentz factors that cancel the transformation laws of d factors of $\tilde{\beta}$ and in the end, all that remains is the boost transformation contributions from the volume and a single gamma factor. The transformation in equation (B.29) implies that the remaining factors cancel as well.

B.4 Analytical continuation

The partition function for a massless free relativistic particle is given by equation (B.33) for arbitrary spatial dimensions and it is invariant under the proper orthochronous Lorentz group $SO^+(1,d)$. This invariance is convenient to assign a Lorentz invariant interpretation to temperature later on. It would also be convenient to be aware of whether there is a symmetry for chemical potential values $v > c$. In other words, to know whether the partition function with imaginary chemical potential iv is invariant under the action of another symmetry group.

For the following, we will simplify and specialize the problem in $1+1$ spacetime dimensions. The question then becomes: what is the analogue to the group $SO(1,1)$?

We take a close look at the real matrix representation of the group $SO(1,1)$ in terms of the boost parameter

$$\Lambda = \begin{pmatrix} \gamma & -\frac{u}{c}\gamma \\ -\frac{u}{c}\gamma & \gamma \end{pmatrix} \in SO(1,1) \quad ,$$

which transforms the spacetime coordinates

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix} \quad .$$

These matrices satisfy the following two relations that are crucial to form the group of transformations

$$\Lambda^{\top}\eta\Lambda = \eta \ , \tag{B.34}$$

$$det\Lambda = 1 \ , \tag{B.35}$$

where η is the Minkowski metric in $1+1$ dimensions. To proceed, we perform an analytical continuation to the boost parameter $u \to iu$ and then the Λ matrix representation becomes

$$\tilde{\Lambda} = \begin{pmatrix} \tilde{\gamma} & -\frac{iu}{c}\tilde{\gamma} \\ -\frac{iu}{c}\tilde{\gamma} & \tilde{\gamma} \end{pmatrix} \ ,$$

in which the Lorentz factor denoted by $\tilde{\gamma} = \frac{1}{\sqrt{1+\left(\frac{u}{c}\right)^2}}$ acquires the usual plus sign in the square root that we have encountered before. These matrices now belong to another group that admits complex matrix representations. This group is $U(1)$ and to get there we have to also perform a Wick rotation. Hence, we need to change the metric

$$ds^2 = -dt^2 + dx^2 \to ds^2 = dt^2 + dx^2 \ . \tag{B.36}$$

These requirements are enough to satisfy the two necessary conditions for an element to belong in $U(1)$. Namely,

$$\tilde{\Lambda}^{\dagger}\tilde{\Lambda} = \mathbb{1} \ , \tag{B.37}$$

$$det\tilde{\Lambda} = 1 \ . \tag{B.38}$$

The conclusion that follows is that the partition function for imaginary chemical potential is invariant under $U(1) \simeq SO(2)$ transformations. The transition from $SO(1,1)$ requires both an analytical continuation to the boost parameter and a Wick rotation.

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