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Aspects of Entanglement Entropy in $3d$ $\mathcal{N} = 2$ Field Theories

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Abstract

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Entanglement entropy is an extremely important quantity in quantum field theory which specifies the amount of entanglement of quantum mechanical degrees of freedom across a spacelike entangling surface. On the other hand, supersymmetric localization is a powerful technique which reduces path integrals in supersymmetric gauge theories to finite dimensional integrals. In this thesis, we explore the possibilities of computing entanglement entropy (EE) in $3d\ \mathcal{N} = 2$ quantum field theories using localization. Firstly, examples of theories where the conformal symmetry is broken are studied, and the corrections to the EE along the induced RG flows are determined. Then, the focus will be mainly on understanding contributions to the EE due to deformations of the entangling region and to changes of its topology. This leads to evidence that the round disk maximizes the EE among entangling regions with non-simply connected topologies. More importantly, it is found that EE is independent of smooth deformations of the boundary of the entangling region due to powerful restrictions imposed by supersymmetry. Motivated by this and other considerations, we are led to a novel computation of EE of $3d\ \mathcal{N} = 2$ superconformal gauge theories at arbitrary Yang-Mills coupling which reveals a particular UV divergence in its structure. Its presence results from sectors in the algebra of local operators of the theory which are charged under a gauge or global symmetry.

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Contents

1 Introduction

Computing observables in quantum field theory (QFT) which give us important information such as its dynamics, behaviour along renormalization group (RG) flows or dependence on the background spacetime geometry, etc., in a fully non-perturbative way is often an overly demanding task. In particular, finding which observables constitute a suitable measure of these properties is a challenge in itself. In this sense, entanglement entropy is an especially useful theoretical quantity. It is a non-local object containing information about either long or short-distance quantum correlations between degrees of freedom contained in different sectors of the Hilbert space. It characterises a quantum theory by quantifying how much a given quantum state differs from a classical one, given that purely quantum correlations arise in QFT or quantum mechanics due to the allowed linear superspositions of states. In the case of QFT, the correlations we are interested in measuring through entanglement entropy are between subspaces of the Hilbert space corresponding to degrees of freedom of fields localized in different spacelike regions of spacetime.

To name some of the most striking applications of entanglement entropy in QFT, we first mention that early in the century it was realised as an order parameter for quantum phase transitions and topological order in condensed matter systems, being particularly useful to detect and characterise critical phases in addition to describing their non-equilibrium dynamics [1, 2, 3, 4, 5, 6]. Furthermore, entanglement entropy has led to significant progress in understanding how degrees of freedom of a QFT evolve along RG flows. Important irreversibility theorems were proved by constructing monotonic functions along RG flows through entropic quantities [7, 8]. Particularly, in three spacetime dimensions it was realised that such a quantity is related to the free energy on the three-sphere [9], making it possible to use the free energy as a measure of coarse-graining of degrees of freedom. Also from the field theory point of view, entanglement has been used as a tool to describe confinement/ deconfinement phase transitions in gauge theories [10, 11], which is inaccessible by conventional methods.

Arguably the most prolific application of entanglement in high-energy physics has been its realisation in the context of the AdS/CFT correspondence/ gauge-gravity duality [12] through the Ryu-Takayanagi formula [13]. Among many aspects, this connection has highlighted how spacetime in a gravitational theory emerges from entanglement in a dual conformal field theory, both being related through the above correspondence [14]. Importantly, it has also provided numerous non-trivial checks of the correspondence which build our understanding of it. See [15] for a beautiful review of this subject.

On the other hand, supersymmetric gauge theories are relevant in their own right. They exhibit a plethora of interesting opportunities which are worth studying in order to better understand the properties of theories we can test at our accessible energy scales: they serve as candidates for beyond the Standard Model theories [16], in part due to their desirable non-renormalization properties [17]. They have given us access to a lot of interesting phenomena regarding dynamics of gauge theories which are difficult to observe in QFT's without supersymmetry and are important to fully understand the phase diagram of QCD or $SU(N)$ Yang-Mills in general (see [18] for a comprehensive review). They have also been the prototypical theories (although not the only ones) providing a testing ground for gauge-gravity dualities and improving our understanding of the connection between degrees of freedom in gravitational theories and in field theories, owing, for instance, to the full conformal symmetry of the maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions, or alternatively the superconformal $U(N)_k \times U(N)_{-k}$ ABJM theory [19] in three dimensions.

A wide variety of exact results in supersymmetric field theories have been obtained in the last 15 years. These are due to the method of supersymmetric localization, first pioneered in [20] where the exact partition function of superconformal gauge theories on S^4 and the respective Wilson loops were computed. This technique relies on the fact that the fermionic symmetries present in the theory allow us to express infinite dimensional path integrals of the theory in terms of finite dimensional integrals in matrix models. Similar results were subsequently obtained in three dimensions [21]. These methods have been applied in many contexts, most notably in establishing infrared dualities between supersymmetric gauge theories [22, 23] and in computing the free energy of $\mathcal{N} = 2$ $U(N)_k \times U(N)_{-k}$ ABJM theory [19] on S^3 at arbitrary coupling [24]; this was then matched with the free energy of its conjectured supergravity dual (through the AdS/CFT correspondence) on $\mathrm{AdS}_4 \times \mathrm{CP}^3$ at strong field theory coupling.

In the midst of such developments, Nishioka and Yaakov obtained the exact partition function of superconformal gauge theories on a branched three-sphere with certain conical singularities, which allowed for the computation of Rényi entropy in these theories [25]. This was made possible by Cardy and Calabrese's ingenious replica trick [1], as well as Casini, Huerta and Myers' formulation of Rényi entropy of spheres in flat space in terms of a thermal partition on hyperbolic cylinders [9]. The latter was useful to establish a duality between spacetimes with hyperbolic black holes and supersymmetric gauge theories [26, 27], by making use of the large N limit of Nishioka and Yaakov's result. Other applications include the exact result for the entanglement entropy of a quark in SYM [28], and extensions of their methods to higher dimensions have allowed further checks of gauge-gravity dualities in, for instance, $4d$ $\mathcal{N} = 4$ SYM [29].

Motivation and outline of this thesis

The main distinct feature of Nishioka and Yaakov's supersymmetric definition of Rényi entropy [25] is that it involves the introduction of a background R-symmetry gauge field in order to preserve supersymmetry on the branched sphere. This renders the definition of Rényi entropy in $3d$ $\mathcal{N} = 2$ superconformal theories inherently dependent on this deformation of their field content. In addition, it is also highly dependent on conformality of the theory and on a high degree of symmetry of the entangling region in order to map the computation of flat space Rényi entropy to a partition function on the branched three-sphere. One may therefore ask the following questions:

- Does the formulation of Rényi entropy just described lead to physical consequences on the entanglement structure of our superconformal theories which are not observed in the absence of supersymmetry?
- Is it possible to compute entanglement entropy in situations where conformal symmetry is broken? If so, how do the corrections along RG flows behave?

- Is entanglement entropy computable under deformations of the entangling region? What about changes of its topology? If so, what can we learn from this?

In light of the questions posed above, this thesis aims to extend our understanding of the entanglement structure of $3d$ $\mathcal{N} = 2$ supersymmetric theories which characterises their vacuum state. Our general guiding principle is to extract physical information from the entanglement entropy which is typically observed in field theories without supersymmetry. In this sense, we attempt to fill some gaps in the literature, which does not provide extensive comments on the physical behaviour of these theories reflected in the entanglement entropy, and has mainly focused on studying theories with conformal symmetry and their Rényi entropies across circles in flat space. We propose to study essentially two distinct aspects: how entanglement entropy behaves in supersymmetric theories when conformal invariance is broken and its dependence on shape deformation of the entangling region, namely continuous deformations of its boundary and changes of its topology.

The outline of this thesis is as follows: In Chapter 2 we give an overview of the basic concepts of entanglement entropy in QFT, including the relation between Rényi entropy of circles and thermal partition functions, as well as some explicit computations of examples which are relevant in the following sections. In Chapter 3 we review some basic concepts in supersymmetric field theories, in addition to the constructions necessary to perform supersymmetric localization on the branched cover of S^3. Along the way we will reproduce Nishioka and Yaakov's result for supersymmetric Rényi entropy with a fair amount of computational details which are skipped in [25], thus facilitating the understanding for newcomers to the subject. In Chapter 4 we study entanglement entropy along RG flows induced by two different kinds of relevant deformations, FI and mass terms; we apply the former to deform the $U(1)_k \times U(1)_{-k}$ ABJM theory. In Chapter 5 we compute some corrections to the free energy and entanglement entropy of a free chiral multiplet when considering entangling regions which are not simply connected. The defect operator formulation of supersymmetric Rényi entropy is employed here, and this is reviewed in the beginning of the chapter. In Chapter 6 we review extremely powerful results which constrain the dependence of supersymmetric partition functions on parameters of the background geometry. These are then used to study the dependence of supersymmetric Rényi entropy on smooth deformations of the entangling surface and determine the leading perturbative correction for a class of non-trivial deformations. Motivated by some intriguing results found in the previous chapters, we go on to compute entanglement entropy in superconformal gauge theories using a combination of localization and the heat kernel expansion in Chapter 7. Chapter 8 contains a general discussion of our results and a brief outlook.

2 Entanglement in Quantum Field Theory

2.1 Measures of entanglement in QFT and the replica trick

Consider the vacuum state $|0\rangle$ of a quantum field theory[1] with Hilbert space $\mathcal{H}$. Here and throughout this thesis we will be interested in the vacuum entanglement entropy of a spacelike region $\mathcal{A}$ given by a disk of radius R on the Cauchy surface of 2 + 1-dimensional flat space determined by $t = 0$. By considering a spatial splitting of the Cauchy surface between $\mathcal{A}$ and its complement $\mathcal{B} = \overline{\mathcal{A}}$, we will consider measures of entanglement between degrees of freedom in $\mathcal{A}$ and $\mathcal{B}$.

Path integral representation of density matrices

Consider now the density matrix of the vacuum state, generically written as $\rho = |0\rangle\langle 0|$. Its matrix elements are $\langle \phi \;\; |0\rangle\langle 0|\phi^{+}\rangle$, where $|\phi \;\;\rangle, |\phi^{+}\rangle$ are eigenvectors of the field operator of the theory at time $t = 0$, $\hat{\phi}(\vec{x}, 0)|\phi \;\;\rangle = \phi \;\; (\vec{x})|\phi \;\;\rangle$. We can view $\langle 0|\phi_0\rangle \equiv$ $[\phi_0(\vec{x})]$ as a vacuum wavefunctional (a functional of the field configuration $\phi_0(\vec{x})$ at $t = 0$), whose path integral representation is

$$[\phi_0(\vec{x})] = \int_{t= \quad 1}^{(x;t=0)= \;\; _0(x)} \mathcal{D}\phi(\vec{x}, t)e^{\;\; S[\;\;]}. \tag{2.1}$$

Analogously, its conjugate wavefunctional $\langle \phi_0|0\rangle \equiv \quad [\phi_0(\vec{x})]$ is represented by an Euclidean time evolution of the field configuration $\phi_0(\vec{x})$ at $t = 0$ until $t = +\infty$. With this construction, the vacuum density matrix is obtained by a path integral over all field configurations at times $t \in \mathrm{R} - \{0\}$, and it is a functional of the field configurations at $t = 0 \quad$ and $t = 0^{+}$:

$$\langle \phi \;\; (x)|\rho|\phi^{+}(x)\rangle = \frac{1}{Z}\int_{(t \;\; ;x)= \quad ;\; (t^{+};x)= \;\; ^{+}} \mathcal{D}\phi(t, x)e^{\;\; S[\;\;]}, \tag{2.2}$$

where Z is the partition function. It is then clear that ρ is normalized such that its trace is unity:

$$\mathrm{T}r[\rho] = \frac{1}{Z}\int \mathcal{D}\phi_0(x)\int_{(t=0;x)= \;\; _0(x)} \mathcal{D}\phi(t, x)e^{\;\; S[\;\;]} = \frac{1}{Z}\int \mathcal{D}\phi(t, x)e^{\;\; S[\;\;]} = 1. \tag{2.3}$$

The object which quantifies the entanglement of degrees of freedom of $\mathcal{A}$ with those of $\mathcal{B}$ is the von Neumann entropy,

$$S_A = -\mathrm{Tr}_A[\rho_A \log \rho_A], \tag{2.4}$$

[1] We will always consider field theories in Euclidean signature, which is convenient for the path integral representation of density matrices using Euclidean time evolution.

where $\rho_A = \mathrm{Tr}_B[\rho]$ is the reduced density matrix of $\mathcal{A}$, obtained by tracing the density matrix of the vacuum state, $\rho = |0\rangle\langle 0|$, on the subspace of $\mathcal{H}$ corresponding to the degrees of freedom of $\mathcal{B}$. More concretely, its matrix elements with states whose support in position space is contained in $\mathcal{A}$ are given by

$$\langle \phi^-(x)|\rho_A|\phi^+(x)\rangle = \frac{1}{Z}\int_{\phi(t^-,x)=\phi^-;\ \phi(t^+,x)=\phi^+;\ x\in A} \mathcal{D}\phi(t,x)e^{-S[\phi]}. \tag{2.5}$$

Pure and entangled quantum mechanical states

There are many ways to see that the von Neumann entropy is an adequate measure of entanglement. It vanishes for the limiting case of product states of the form $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ ($|\psi\rangle_{A,B} \in \mathcal{H}_{A,B}$) for which it is clear that the reduced density matrix is $\rho_A = |\psi_A\rangle\langle\psi_A|$; in particular, it is zero on the subspace $\mathcal{H}_B$. This signals that there is no entanglement between degress of freedom of the state $|\psi\rangle$ contained in $\mathcal{A}$ and $\mathcal{B}$, and the corresponding von Neumann entropy vanishes. In fact, it can be shown that $S_A = 0$ if and only if the pure state ρ is separable.[2] Therefore, entanglement entropy can be thought of as a quantity that measures how much a state differs from a product state with no correlations between subregions $\mathcal{A}$ and $\mathcal{B}$. It can also be shown that for a pure state $S_A = S_{\overline{A}}$.

Let us look at two examples of entangled states, for which it is simpler to consider finite dimensional Hilbert spaces. An entangled state is a state which is not separable, and it is expressed in the form

$$|\psi\rangle = \sum_{i,j} c_{ij}|\phi_i\rangle_A \otimes |\phi_j\rangle_B, \tag{2.6}$$

where $|\phi_i\rangle_{A,B}$ is an orthonormal basis of states of $\mathcal{H}_{A,B}$, and c_{ij} are complex coefficients. It can be shown that (see [30]), upon performing a Schmidt decomposition, the reduced density matrix takes the form

$$\rho_A = \sum_{k=1}^{\min(d_A,d_B)} p_k|\psi_k\rangle_A\langle\psi_k|_A \Rightarrow S_A = -\sum_{k=1}^{\min(d_A,d_B)} p_k \log p_k, \tag{2.7}$$

where $\sum_k p_k = 1$ and $d_{A,B} = \dim(\mathcal{H}_{A,B})$.

Now let $\mathcal{H}$ be the Hilbert space of a quantum mechanical system of two spin-$\frac{1}{2}$ particles labelled A and B. The bipartition of the Hilbert space we consider is between both subspaces of each spin given by $\{|\uparrow\rangle, |\downarrow\rangle\}$. For a ground state of some spin Hamiltonian given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B\right), \tag{2.8}$$

we have the following reduced density matrix and von Neumann entropy of subsystem A:

$$\rho_A = \frac{1}{2}\left(|\downarrow\rangle_A\langle\downarrow|_A + |\uparrow\rangle_A\langle\uparrow|_A\right) \equiv \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix} \Rightarrow S_A = \log 2. \tag{2.9}$$

[2] A state is called pure if its density matrix takes the form $\rho = |\psi\rangle\langle\psi|$ for some vector $|\psi\rangle$ in the Hilbert space; if a state is not pure it is said to be mixed. If $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ is separable then the reduced density matrix ρ_A has precisely one eigenvalue equal to one and the remaining are zero; since $\lim_{x\to 0} x\log x = 0$ we obtain $S_A = 0$.

This is in fact the maximal Shannon entropy 2.7 for a bipartition of the Hilbert space in two subspaces of dimension 2. It signals a two-fold uncertainty in measuring the z-projection of the spin of any of the two particles (since it is certain that they must have opposite projections).

Another prototypical example of a mixed state with non-zero entanglement is a thermal state. A quantum system in thermodynamic equilibrium in the canonical ensamble at inverse temperature β can be described by the density matrix

$$\rho = \frac{1}{Z}\sum_n e^{-\beta E_n}|n\rangle\langle n| \equiv \frac{1}{Z}e^{-\beta H}, \qquad Z = \mathrm{Tr}\left(e^{-\beta H}\right), \tag{2.10}$$

where $\{|n\rangle\}$ is the set of energy eigenstates and H is the Hamiltonian operator. The entanglement entropy is

$$S = -\mathrm{Tr}\,[\rho(-\beta H - \log Z)] = \beta(\langle H\rangle - F). \tag{2.11}$$

This is nothing but the thermodynamic entropy of the ensemble, with the thermal free energy $F = -T\log Z$ and the mean energy $E = \langle H\rangle$. This shows that the entanglement entropy captures the thermodynamic order of the system if it is immersed in a heat bath at finite temperature.

We can alternatively think of a microcanonical ensemble, where all states (of total number N) at energy E have equal probability of being measured, so that

$$\rho = \frac{1}{N}\mathbb{1} \Rightarrow S = \log N. \tag{2.12}$$

This gives us the Boltzmann entropy for the microcanonical ensemble; similarly to the above case for a spin-$\frac{1}{2}$ system, it reflects an N-fold uncertainty in measuring a state at energy E.[3]

No product states in QFT: the Reeh-Schlieder theorem

While in systems with a finite-dimensional Hilbert space we can easily construct product states resulting from a bipartition of the Hilbert space of the form $\mathcal{H}_A \otimes \mathcal{H}_B$ (take, for instance, the state $|\uparrow\rangle_A \otimes |\downarrow\rangle_B$ in the spin-$\frac{1}{2}$ system described above), this turns out to be forbidden in the infinite dimensional Hilbert space $\mathcal{H}$ of a quantum field theory, even in non-interacting theories. This is due to the powerful Reeh-Schlieder theorem [32], which states the following: given any region V with finite spacetime volume, acting on the vacuum state with field operators supported in V creates a set of states which is dense in the entire Hilbert space of the QFT. This means that, given some norm on $\mathcal{H}$, the set of states of the form

$$\hat{\phi}(x_1)...\hat{\phi}(x_n)|0\rangle, \quad x_i \in V \tag{2.13}$$

is arbitrarily close to *any* state in $\mathcal{H}$, including states created by acting on $|0\rangle$ with operators supported outside V.[4]

This result forbids the existence of spacetime regions having no entanglement with its complement. The reason is that it shows that any product of operators supported in

[3] There are numerous important properties associated to entanglement entropy as an information measuring quantity; see [31] for a thorough review.

[4] This is proven by arguing that, for any state $|\psi\rangle \in \mathcal{H}$, if the correlation function $\langle\psi|\phi(x_1)...\phi(x_n)|0\rangle$ vanishes for $x_i \in V$ then it must vanish for x_i located on the entire spacetime (due to analyticity of the correlator), and thus $|\psi\rangle = 0$. Thus, there are no states orthogonal to states of the form 2.13, so the set of these is dense in $\mathcal{H}$.

either region will have non-zero vacuum expectation value, implying that the density matrix $|0\rangle\langle 0|$ has no non-zero eigenvalues. Equivalently, the Hilbert space of a QFT cannot factorise into a tensor product, that is, $\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}}$ for any region $\mathcal{A}$. Knowing that the vacuum state of a QFT necessarily possesses entanglement across a $d-2$-dimensional spacelike region makes it interesting to study the dependence of this quantity on interactions of the theory or on the shape of this region.

Rényi entropy and the replica trick

Another useful measure of entanglement is an integer parameter generalization of the von Neumann entropy, the Rényi entropy:

$$S_A^{(n)} = \frac{1}{1-n} \log \mathrm{Tr}_A[\rho_A^n]. \tag{2.14}$$

This quantity has the useful property that it reduces to the entanglement entropy upon taking the limit $n \to 1$:

$$\begin{aligned} \lim_{n \to 1} S_A^{(n)} &= \lim_{n \to 1} \frac{\log \mathrm{Tr}_A[\rho_A^n] - \log \mathrm{Tr}_A[\rho_A]}{1-n} \\ &= -\partial_n \log \mathrm{Tr}_A[\rho_A^n]\Big|_{n=1} \\ &= -\mathrm{Tr}_A[\rho_A \log \rho_A]. \end{aligned} \tag{2.15}$$

A path integral representation of the Rényi entropy can be obtained by inserting a resolution of the identity between each pair of copies of ρ_A:

$$\begin{aligned} \mathrm{Tr}_A[\rho_A^n] &= \int \left(\prod_{k=0}^{n-1} \mathcal{D}\psi_k\right) \langle\psi_0|0\rangle\langle 0|\psi_1\rangle\langle\psi_1|0\rangle \dots \langle 0|\psi_{n-1}\rangle\langle\psi_{n-1}|0\rangle\langle 0|\psi_0\rangle \\ &= \frac{1}{Z^n} \int \left(\prod_{k=0}^{n-1} \mathcal{D}\psi_k\right) \prod_{i=0}^{n-1} \mathcal{D}\phi_k e^{-S[\phi_k]} \delta\left(\phi_k(\vec{x}, 0^-) = \psi_k(\vec{x})\right) \delta\left(\phi_k(\vec{x}, 0^+) = \psi_{k+1}(\vec{x})\right), \end{aligned} \tag{2.16}$$

where $\psi_n(\vec{x}) \equiv \psi_0(\vec{x})$. Now, a key insight lies in realising that the path integral in the expression above is actually computing a partition function on our original spacetime $\mathcal{M}$ replicated n times, with the field configurations infinitesimally above and below $\mathcal{A}$ (in the timelike direction) being identified across the copies of $\mathcal{M}$ [1]. This is equivalent to having a single copy of the field content of the theory placed on a copy of the spacetime with a conical singularity of defect angle $2\pi(1-n)$ along $\partial\mathcal{A}$. This is because, starting from a point on $\mathcal{M}$ close to $\partial\mathcal{A}$, a rotation of 2π along the timelike direction centered at $\partial\mathcal{A}$ does not bring us back to the same point, but to the equivalent point on the next copy; it follows that only a rotation of $2\pi n$ brings us to the same point. Therefore, the required field identifications across the copies can be expressed in terms of inserting such a conical singularity on the co-dimension two surface $\partial\mathcal{A}$. Denoting our spacetime with this conical singularity by $\mathcal{M}_n$, we conclude that

$$S_A^{(n)} = \frac{1}{1-n} \log \mathrm{Tr}_A[\rho_A^n] = \frac{1}{1-n} \log\left(\frac{Z_{\mathcal{M}_n}}{Z_{\mathcal{M}}^n}\right). \tag{2.17}$$

This reduces the task of computing the von Neumann entropy of a state in a QFT, which consists of a path integral with non-trivial boundary conditions, to computing partition functions on spacetimes with singular points, which are easier objects to handle.

Example: Entanglement entropy of interval in CFT_2 at finite temperature

As an example of the applicability of the replica trick, we look at a class of theories which are highly tractable analytically, two-dimensional conformal field theories. We consider an entangling region given by the interval $\mathcal{A} \equiv [v_1, v_2] \in \mathbb{R}$ at $t = 0$, immersed in a heat bath of inverse temperature β. The result at finite temperature is obtained from the result of the same quantity obtained in a vacuum zero-temperature CFT. Due to the analytical tools at our disposal in $2d$ CFT's, the replica trick allows us to compute the partition function on the q-covering space, Z_{M_q}, in terms of the two-point function of local operators (called twist operators) inserted at the ends of the interval. Essentially, this comes about because it is possible to implement the required boundary conditions at the boundary of the entangling surface in terms of local, primary operators, precisely because in this case $\partial\mathcal{A}$ is zero-dimensional. This correlator is computed on the complex plane $\mathbb{C}$, with complex coordinate $w = v + it$. We have [1]

$$\mathrm{Tr}\rho_A^q = \frac{Z_{M_q}}{Z_M^q} = \langle \mathcal{T}(v_1)\mathcal{T}^{\ 1}(v_2)\rangle_{\mathbb{C}}. \tag{2.18}$$

Then, the conformal transformation $w \to z = \left(\frac{w\ \ v_1}{w\ \ v_2}\right)^{\frac{1}{q}}$ maps the covering space $\mathcal{M}_q$ to a complex plane where the branch cut singularities are no longer at located at $\{v_1, v_2\}$ but at $\{0, \infty\}$. Thus, on this new space we have $\langle T(z)\rangle = 0$, with $T(z)$ the holomorphic component of the energy-momentum tensor of the CFT_2. This is used to determine $\langle T(w)\rangle_{M_q} = \langle T(w)\mathcal{T}(v_1, 0)\mathcal{T}^{\ 1}(v_2, 0)\rangle_{\mathbb{C}}$. By use of a conformal Ward identity, the conformal dimensions of the holomorphic and anti-holomorphic twist operators are determined to be $\ _q = \ _q = \frac{c}{24}\left(q - \frac{1}{q}\right)$. Using the standard scaling behaviour of two-point functions in CFT, the vacuum Rényi entropy is

$$S_A^{(q)} = \frac{1}{1-q}\log\left[c_q \frac{\delta_{\ _q;\ _q}\delta_{\ _q;\ _q}}{l^{2\ _q} l^{2\ _q}}\epsilon^{2\ _q+2\ _q}\right] = \frac{c}{6}\left(q + \frac{1}{q}\right)\log\frac{l}{\epsilon} + \frac{\log c_q}{1-q}, \tag{2.19}$$

where a UV cutoff ϵ is included to ensure correct dimensionality, and c_q is a constant.

Suppose now that we wish to find the behaviour of the above two point function on a space corresponding to a system in a heat bath of finite inverse temperature β. This amounts to Wick rotating to Euclidean time, $t \to t_E = -it$, so that the metric becomes

$$ds_E^2 = dt_E^2 + dv^2, \qquad t \sim t + \beta \tag{2.20}$$

The mapping from $\mathcal{M}_q$ to this space (isomorphic to an infinitely long cylinder) is

$$w \to w^0 = \frac{\beta}{2\pi}\log w, \tag{2.21}$$

This is because, writing $w^0 = v + it_E$, we have $w = e^{\frac{2\pi}{\beta}(v+it_E)} = e^{\frac{2\pi}{\beta}v + i\frac{2\pi}{\beta}(t_E+\ )}$, and therefore t_E is periodically identified as $t \sim t + \beta$, while the spatial dimension parametrized by v in the w-plane now has the range $e^{\frac{2\pi}{\beta}} \in (0, +\infty)$. We should keep in mind that our region $\mathcal{A}$ is now an interval of length $l = w_2^0 - w_1^0$ at $t_E = 0$ on the cylinder parallel to its axis.

Now, because $\mathrm{Tr}\rho_A^q$ scales as a 2-point function of primary operators under a holomorphic change of coordinates, we have

$$\mathrm{Tr}\rho_A^q = \langle \mathcal{T}(w_1')\mathcal{T}^{-1}(w_2')\rangle = \left(\frac{\partial w}{\partial w'}\right)^{-2\Delta_q}\Bigg|_{w_1'}\left(\frac{\partial w}{\partial w'}\right)^{-2\Delta_q}\Bigg|_{w_2'}\langle \mathcal{T}(w_1)\mathcal{T}^{-1}(w_2)\rangle_{\mathbb{C}}. \tag{2.22}$$

Replacing $\frac{\partial w}{\partial w'}(w') = \frac{2\pi}{\beta}e^{\frac{2\pi}{\beta}w'}$, we get

$$\begin{aligned}\mathrm{Tr}\rho_A^q &= \left(\frac{2\pi}{\beta}\right)^{4\Delta_q} e^{2\Delta_q\frac{2\pi}{\beta}(w_1'+w_2')}\frac{c_q\epsilon^{2\Delta_q+2\Delta_q}}{(w_1-w_2)^{2\Delta_q}(w_1-w_2)^{2\Delta_q}}\\ &= c_q\left(\frac{\epsilon\pi}{\beta}\right)^{4\Delta_q}\left[\frac{e^{\frac{\pi}{\beta}(w_2'-w_1')}-e^{\frac{\pi}{\beta}(w_1'-w_2')}}{2}\right]^{-4\Delta_q}\\ &= c_q\left[\frac{\beta}{\epsilon\pi}\sinh\left(\frac{\pi l}{\beta}\right)\right]^{-4\Delta_q}.\end{aligned} \tag{2.23}$$

The entanglement entropy of $\mathcal{A}$ on the cylinder of perimeter β is then

$$\begin{aligned}S_A &= \lim_{q\to 1}\frac{1}{1-q}\log c_q\left[\frac{\beta}{\epsilon\pi}\sinh\left(\frac{\pi l}{\beta}\right)\right]^{-4\Delta_q}\\ &= \lim_{q\to 1}\frac{1}{q-1}\frac{c}{6}\left(q-\frac{1}{q}\right)\log\left[\frac{\beta}{\epsilon\pi}\sinh\left(\frac{\pi l}{\beta}\right)\right]+\lim_{q\to 1}\frac{\log c_q}{1-q}\\ &= \frac{c}{3}\log\left[\frac{\beta}{\epsilon\pi}\sinh\left(\frac{\pi l}{\beta}\right)\right]+c_1'.\end{aligned} \tag{2.24}$$

Note that, for very low temperatures, taking the limit of large β as $l/\beta << 1$ gives

$$\sinh\left(\frac{\pi l}{\beta}\right)\approx\frac{\pi l}{\beta}\Rightarrow S_A\approx\frac{c}{3}\log\frac{l}{\epsilon}+c_1', \tag{2.25}$$

from which we recover the result for $S(\mathcal{A})$ of a vacuum CFT$_2$ [1]! This result has the interesting feature that it only depends on the properties of the theory through the central charge c.

2.2 From spherical entangling surface to heat bath on hyperbolic space

In the beautiful paper [9], an important formulation of vacuum Rényi entropy of a spherical entangling surface on flat space is given in terms of a thermal partition function on a hyperbolic space, which is then made to correspond to a partition function on (a branched cover of) the three-sphere. This will be extremely useful for our purposes; it is thanks to this formulation that the first proposal for computing entanglement entropy of supersymmetric theories was put forward, which we review below and in the next chapter. Moreover, this formulation also sprawled a vast number of opportunities for studying the holographic behaviour of entanglement entropy [9, 33], due to the relation between thermal entropy of a CFT$_d$ on $\mathrm{H}^d\times S^1$ and the horizon entropy of black holes in asymptotically AdS$_{d+1}$ bulk spacetimes, in the context of the AdS/CFT correspondence.

We start with a conformal field theory defined on d-dimensional flat spacetime; assuming time translation invariance of the theory, we perform a Wick rotation to

Euclidean time. The metric in spherical coordinates is

$$ds^2 = dt^2 + d\rho^2 + \rho^2 d\Omega_{d-2}^2. \tag{2.26}$$

We also consider a $(d-2)$-dimensional spherical entangling region on a Cauchy time slice, defined by

$$\mathcal{A} = \{x \in \mathbb{R}^d \,:\, t = 0,\ \rho \leq R\}. \tag{2.27}$$

The goal is to write a conformal transformation of coordinates which maps this spacetime to a hyperbolic cylinder.

Introducing the complex coordinates, $w = \rho + it$, $\bar{w} = \rho - it$, the line element reads

$$ds^2 = dw d\bar{w} + \left(\frac{w+\bar{w}}{2}\right)^2 d\Omega_{d-2}^2. \tag{2.28}$$

Now define the new complex coordinate $\sigma = u + i\tau$ according to

$$e^{-\sigma} = \frac{R-w}{R+w} \iff w = R\frac{1-e^{-\sigma}}{1+e^{-\sigma}} = R\tanh\frac{\sigma}{2}. \tag{2.29}$$

We immediately note that this transformation maps the boundary entangling region at $w = R$ to $u = +\infty$ for any τ. Furthermore, it is evident that the coordinate τ is periodically identified as $\tau \sim \tau + 2\pi$.

We are working in dimension $d \geq 3$, where the conformal group is not infinite dimensional ($i.e.$ not given by all holomorphic coordinate transformations $w \to w'(w)$), so we do not know at this point whether 2.29 is a conformal transformation. However, this can be readily confirmed by computing the line element in the new coordinates:

$$\begin{aligned}
dw &= \frac{R}{2}\frac{1}{\cosh^2\frac{\sigma}{2}}d\sigma \\
\Rightarrow ds^2 &= \frac{R^2}{4}\left(\cosh^{-2}\frac{\sigma}{2}d\sigma\right)\otimes\left(\cosh^{-2}\frac{\bar{\sigma}}{2}d\bar{\sigma}\right) + \frac{R^2}{4}\left(\tanh\frac{\sigma}{2} + \tanh\frac{\bar{\sigma}}{2}\right)^2 d\Omega_{d-2}^2 \\
&= \frac{R^2}{4}\frac{1}{(\cosh\frac{\sigma}{2}\cosh\frac{\bar{\sigma}}{2})^2}\left[d\sigma d\bar{\sigma} + \sinh\left(\frac{\sigma+\bar{\sigma}}{2}\right) d\Omega_{d-2}^2\right] \\
&= \Omega^2(\tau,u)\left[d\tau^2 + du^2 + \sinh^2 u\, d\Omega_{d-2}^2\right],
\end{aligned} \tag{2.30}$$

where $\Omega(\tau,u) = \frac{R}{\cosh u + \cos\tau}$. We then see that the region $\rho \leq R$ of $\mathbb{R}^d$ is conformally equivalent to $S^1 \times \mathbb{H}^{d-1}$, with the Euclidean time compactified along S^1 with length 2π. We therefore have a heat bath of temperature $T = \frac{1}{2\pi}$ on this latter space.

As explained in the previous section, in order to obtain Rényi entropies we are interested in computing the partition function on the q-covering of $\mathbb{R}^d$ branched along a sphere S^{d-2} at $t = 0$, to be denoted by $\mathbb{R}^d_q$. This space can be mapped to a hyperbolic space by assigning $\sigma = u + i\tau/q$, which changes the period of τ to $2\pi q$.[5] The conical singularity at $w = R$ is mapped to $u = +\infty$. Note that, before removing the conformal factor by a Weyl transformation, the metric in hyperbolic coordinates becomes degenerate (singular) at $u = +\infty$ for any τ, which is precisely the location of the entangling region.

Because partition functions of conformal theories are invariant under conformal transformations of spacetime, the desired partition function should be equal to a

[5] Or alternatively maintaining the period 2π but modifying the line element above as $d\tau^2 \to q^2 d\tau^2$; either way, the presence of a conical singularity is explicit.

thermal partition function on $S^1 \times \mathbb{H}^{d-1}$ with temperature $T_q = \frac{1}{2\pi q}$. In [9] the relation $Z_{S^1 \times \mathbb{H}^{d-1}}(\frac{1}{2\pi q}) = Z_{\mathbb{R}^d_q}$ was concretely established using modular flows of density matrices. Specifically, it was found that

$$\rho_D = \frac{1}{Z} U^{-1} e^{-\frac{1}{T_q} H_\tau} U, \tag{2.31}$$

where $\mathcal{D}$ is the causal development of $\mathcal{A}$[6], U is the symmetry operator implementing a conformal transformation on the fields and H_τ is the generator of τ translations on the hyperbolic space. This means that the vacuum density matrix on $\mathcal{D}$ is related to the thermal density matrix on hyperbolic space by a unitary transformation, leading to the equivalence of von Neumann entropies computed with both density matrices.

There is a further conformal transformation which maps $S^1 \times \mathbb{H}^{d-1}$ to the compact three-sphere. Starting from the metric on $S^1 \times \mathbb{H}^{d-1}$, we now consider the coordinate transformation

$$\sinh u = \cot\theta \Rightarrow \cosh u\, du = -\frac{1}{\sin^2\theta} d\theta. \tag{2.32}$$

This transformation compactifies $\mathbb{H}^{d-1}$ into a region of finite volume. Since $\theta(u = 0) = \frac{\pi}{2}$ and $\theta(u = +\infty) = 0$, we have $0 \leq \theta \leq \frac{\pi}{2}$. This yields a line element

$$\begin{aligned} ds^2 &= d\tau^2 + \frac{1}{(1+\cot^2\theta)\sin^4\theta} d\theta^2 + \cot^2\theta\, d\Omega^2_{d-2} \\ &= \frac{1}{\sin^2\theta}\left[d\theta^2 + \sin^2\theta\, d\tau^2 + \cos^2\theta\, d\Omega^2_{d-2}\right]. \\ &\sim d\theta^2 + \sin^2\theta\, d\tau^2 + \cos^2\theta\, d\Omega^2_{d-2}, \end{aligned} \tag{2.33}$$

where $\sim$ denotes equivalence by a Weyl transformation. This space is nothing but S^d if the Euclidean time coordinate τ has period 2π.[7] For integer $q \geq 2$ we obtain a q-fold covering of S^d, which is the desired space on which we will compute partition functions. As we have argued above, this amounts to computing the thermal partition function on S^d as

$$Z_{S^d}(2\pi q) = \mathrm{Tr}\left(e^{-2\pi q H}\right), \qquad H = \int_{S^d} d^{d-1}x \sqrt{g}\, T_{\tau\tau}(x). \tag{2.34}$$

In section 4 of [9], H is computed in terms of a compact expression for the trace anomaly of any CFT in even dimensions. See section VI of [30] for the role of the conformal anomaly in computing entanglement entropies.

Supersymmetric Rényi Entropy

Even though we have simplified the problem of computing von Neumann entropies in QFT to computing partition functions on branched spacetimes, it is still a challenge to determine these objects exactly, unless the theory is free (see [34] for exact Rényi entropies of free scalar and fermion fields). In certain classes of superconformal gauge theories, *supersymmetric localization* allows one to compute partition functions

[6] The causal development of a Cauchy hypersurface A is the set of points $p \in M$ in the entire spacetime M such that any causal curve passing through p must intersect A.

[7] To see this: for the case of d odd, we have $S^d = \{(z_1, \ldots, z_{\frac{d+1}{2}}) \in \mathbb{C}^{\frac{d+1}{2}} \,|\, |z_1|^2 + \ldots + |z_{\frac{d+1}{2}}|^2 = 1\}$ parametrized as, for instance, $z_1 = \cos\theta\, e^{i\tau}$, $z_i = \sqrt{\frac{2}{d-1}} \sin\theta\, e^{i\varphi_i}$, $i = 2, \ldots, d-2$; the observation for even d is similar.

exactly, provided that the theory is placed on a manifold which preserves some supersymmetry. We will learn about this technique in the next chapter, in particular by paying attention to how supersymmetry can be preserved on the branched sphere with conical singularities.

It follows that the Rényi entropy of $3d$ $\mathcal{N} = 2$ gauge theories across a circular entangling surface on flat space can be computed exactly by applying supersymmetric localization to compute the corresponding partition functions on branched covers of S^3. This led the authors of [25] to define *supersymmetric Rényi entropy* as

$$S_n^{\text{susy}} = \frac{1}{1-n} \log \left| \frac{Z_{S^3_n}}{Z^n_{S^3}} \right| . \tag{2.35}$$

As remarked in the Introduction, this quantity is only defined when introducing a background gauge field for the R symmetry on the branched sphere in order to preserve supersymmetry in the presence of the conical singularities. In this way, S_n^{susy} does not exactly correspond to the usual definition of Rényi entropy through the replica trick, and we will use this fact to justify some results at several points in this thesis.

2.3 Structure of UV divergences

One of the main features of entanglement entropy in QFT is that it is a purely divergent quantity, and it is ill-defined in the continuum limit. In order to define it one must set a spatial UV cutoff, ϵ. It should be noted, however, that this divergence is physical (unlike the ones occurring, for instance, in the renormalization of coupling constants in interacting QFT's), originating in the entanglement between an infinite amount of degrees of freedom across $\partial\mathcal{A}$ which emerge as we take $\epsilon \to 0$.

The divergent terms should be expressed as a power series in ϵ^{-1}. The general form for the entanglement entropy may be written as

$$S_A = \sum_i C_i(\partial\mathcal{A})\epsilon^{-i} + C_0 \log(\epsilon R) + S_{A;0}. \tag{2.36}$$

The coefficient C_0 is only non-zero is even spacetime dimensions, R being some macroscopic length scale of the geometry or of the theory; we comment on it below. In odd dimensions, $S_{A;0}$ is a finite, cutoff-independent term and therefore contains non-local information about the entanglement of the given state. The coefficients $C_i(\partial\mathcal{A})$ are non-universal as they are not invariant under a redefinition of the cutoff ϵ (for example, a multiplicative rescaling). We will see that they must be expressed as integrals along the boundary of $\mathcal{A}$. Moreover, they depend on physics at small distances in the UV limit of the theory, and because at very short distances (much smaller than any correlation length) any state can be approximated by the vacuum state, the $C_i(\partial\mathcal{A})$ are state independent.

To see that the C_i's only depend on the geometry of the boundary of $\mathcal{A}$, consider the following argument: if the effective action of the theory at scale ϵ^{-1} is regarded as a functional of the spacetime metric $g_{\mu\nu}$, then we are able to classify the UV divergent terms by diffeomorphism invariant integrals of local curvatures. In the case of a CFT$_d$, we can write [30, 31]

$$\log Z[g_{\mu\nu}] = \sum_{i=0}^{d-1} c_{d-i} \int_M d^d x \sqrt{g} \frac{\mathcal{R}^i}{\epsilon^{d-i}} + (-1)^{\frac{d-1}{2}} F[g_{\mu\nu}], \tag{2.37}$$

where the last term represents the renormalized free energy, and is only present for odd d; $\mathcal{R}^i$ denote diffeomorphism invariant curvatures of scale dimension i. If we consider the same effective action computed on an n-branched cover of $\mathcal{M}$ with appropriate conical singularities along $\partial\mathcal{A}$, we can compute the Rényi entropy. The coefficient of ϵ^{2i-d} will only depend on the difference $\int_{M_n} \mathcal{R}^i_n - n \int_M \mathcal{R}^i$. Because $\mathcal{M}_n$ is locally diffeomorphic to $\mathcal{M}^n$ away from the singularity localized at $\partial\mathcal{A}$, the contribution to this difference should be expressed as an integral over only $\partial\mathcal{A}$. In particular, it vanishes for $i = 0$, since $\int_{M_n} d^dx\sqrt{g} = n\int_M d^dx\sqrt{g}$.

We can further constrain the C_i's by noting that, if the state is pure, then we must have $S_A = S_{\overline{A}}$. Therefore, curvatures of odd dimension which change sign under a change in the orientation of the manifold should be dropped, implying $C_i(\partial\mathcal{A}) = 0$ for i odd.

As noted in 2.36, it is also possible to have logarithmic divergences in S_A of the form $C_0(\partial\mathcal{A})\log(\epsilon R)$. These are only present in even dimensions (as seen in the example in 2.1 for a CFT$_2$): here C_0 must be an integral over the $(d-2)$-dimensional surface $\partial\mathcal{A}$ of a $(d-2)$-dimensional curvature, and such a coefficient is only allowed to be dimensionless for even d. Note however, that this coefficient is universal, since a redefinition of ϵ cannot be absorbed in C_0. It can, however, be absorbed by a finite term proportional to $\log R$, and this means that finite terms in S_A in even dimensions are not universal. Conversely, the absence of logarithmic terms in odd dimensions guarantees that $S_{A;0}$ cannot be changed by a redefinition of ϵ, as alluded to above.

We conclude that

$$S_A = \begin{cases} \frac{C_{d-2}}{\epsilon^{d-2}} + \frac{C_{d-4}}{\epsilon^{d-4}} + \ldots + C_0\log(\epsilon R) + \ldots & d \text{ even} \\ \frac{C_{d-2}}{\epsilon^{d-2}} + \frac{C_{d-4}}{\epsilon^{d-4}} + \ldots + (-1)^{\frac{d-1}{2}}F & d \text{ odd} \end{cases} \tag{2.38}$$

where the ellipses in the first line denote non-universal constant terms and F the universal, renormalized free energy. Note that C_{d-2} is always given by the integral along the boundary of the entangling surface of only the volume form, so it is proportional to Area($\partial\mathcal{A}$); this is the well known "area law" of entanglement entropy.

The fact that C_0 and F are universal means that they hold relevant information intrinsic to the theory. In fact, these entropic quantities have been shown to obey monotonicity conditions under renormalization group flow, much like the one established in the celebrated proof of the c-theorem for $2d$ CFT's [35]. Importantly for our context, a c-like function can be constructed in three spacetime dimensions, which obeys monotonicity and equals the regularized free energy on S^3 at the conformal fixed points. Some evidence was gathered for this in theories with and without supersymmetry [36, 37, 38], before a fascinating complete non-perturbative proof was given in [7]. In four dimensions, a similar result was obtained for the logarithmic coefficient [8], which relates to the stress tensor anomaly.

2.4 Relevant examples

We conclude this chapter by presenting detailed computations of entanglement entropy of particular classes of theories which will make appearences throughout this thesis: topological field theories (in particular, $U(1)$ Chern-Simons theory at level k) and field theories with a mass gap.

2.4.1 Topological entanglement entropy

In general, if our entangling region is a disk of length $L >> \xi$, with ξ some correlation length of the theory, the entanglement entropy $S(\rho) = -\mathrm{tr}(\rho \log \rho)$ in 2 + 1 dimensions takes the general form

$$S(\rho) = \alpha L - \gamma. \tag{2.39}$$

α is a UV divergent quantity, which appears due highly entangled degrees of freedom at short length scales across the boundary of the entangling region; as explained in 2.3, it is non-universal. γ is identified as the *topological entanglement entropy*, a universal quantity which represents the entanglement of degrees of freedom that survives at long distances (much larger than ξ) and is therefore characteristic of topological field theories; in particular, it is cutoff independent. Its properties were first studied in [3, 39], where its general expression in terms of quantities of an effective topological field theory was determined as

$$\gamma = \log \mathcal{D} = \log \sqrt{\sum_a d_a^2}, \tag{2.40}$$

where $\mathcal{D}$ is the total quantum dimension of the system and d_a is the dimension of the Hilbert subspace of each topological sector.

This formula is firstly derived by topological considerations. The Cauchy timeslice of the system is divided into four disjoint regions much larger than any ξ and which meet at double and triple intersections (see [3]). The combination

$$S_{\text{top}} = S_A + S_B + S_C + S_{ABC} - S_{AB} - S_{AC} - S_{BC} \tag{2.41}$$

will depend only on the universal coefficient in the entropy since the terms proportional to the length of A, B, C and D all cancel out. Moreover it is topologically invariant, in the sense that small deformations of the boundaries between any of the regions does not affect $S(\rho)$. For instance, deforming the boundary between C and D does not change the entropy since we have

$$S_{\text{top}} = (\quad S_{ABC} - \quad S_{BC}) - (\quad S_{AC} - \quad S_C) = 0, \tag{2.42}$$

since the quantity $\quad S$ computed for some region should not change when appending to it another region along a common boundary which is not being deformed.

Derivation from thermal $2d$ CFT

We can argue for 2.40 in the following way: we assume that, at length scales $>> L$ where scale dependent phenomena are not relevant, the density matrix can be expressed as $\rho = e^{\quad H}$, where H is taken to be the Hamiltonian of a (1+1)-dimensional CFT. Such an ansatz should reproduce the behaviour represented by the non-local universal term, but not by the area term dependent on local physics. To compute the Euclidean partition function for a (1 + 1)d system with an anyon of charge a,[8] we represent it as a path integral on a torus of length β in the Euclidean time direction and length L in spatial direction, in the presence of a Wilson loop of charge a. We

[8] We refer as anyons to excitations of some topological sector of the theory which is charged under a symmetry group. The reader may keep the $U(1)_k$ Chern-Simons theory in mind during this section, since it will be the relevant example for us.

perform a modular transformation,

$$Z_a = \sum_b \mathcal{S}_a^b Z_b, \tag{2.43}$$

where $\mathcal{S}$ is the modular S-matrix of the CFT [40], and in the limit $L \to \infty$ the sum is dominated by the trivial block Z_1. This is because, by modular invariance, the limit $L \to \infty$ is equivalent to having a partition function whose leading contribution in the trace of the density matrix comes from the action of the Virasoro generators on the ground state.[9] We will review this argument below, but for now recall that the modular S-matrix of a CFT is defined as the matrix satisfying

$$\chi(-1/\tau) = \mathcal{S}\chi(\tau), \tag{2.44}$$

where χ are the characters of the CFT and $\tau \mapsto -1/\tau$ is a modular S transformation acting on the modular parameter τ of the torus. We then have

$$\log Z_a \approx \log(\mathcal{S}_a^0 Z_0) \approx \log \mathcal{S}_a^0 + \frac{\pi}{12}(c + c)L/\beta. \tag{2.45}$$

The entanglement entropy coincides with the thermodynamic entropy (due to 2.11), and reads:

$$S(\rho) = \frac{\partial}{\partial T}(T \log Z) = \alpha L + \underbrace{\log \mathcal{S}_a^0}_{\log(d_a = D)}. \tag{2.46}$$

In the vacuum state, a refers to the identity sector, and thus $d_a = 1$, yielding 2.40.

CFT partition function Z_0 in the $L/\beta \to \infty$ limit

We now give the argument for 2.45. In the thermodynamic limit, we take the $\text{Im}(\tau) = \beta \to 0$ limit (with $L = 1$ the period of the torus in the spatial direction). Assume that the variable α conjugate to the momentum P in the partition function is negligible, so that τ is small in absolute value. Then, $|\tau| \approx \text{Im}\tau << 1$. $\text{Im}\tau$ small implies $|\text{Im}(-1/\tau)|$ large, so that $Z(-1/\tau, -1/\tau)$ effectively corresponds to a partition function for a system at very low temperature, whose main contribution comes from the ground state. To see this, use $\frac{1}{\tau} = \frac{\text{Re} \quad i\text{Im}}{|\tau|^2}$ to write

$$
\begin{aligned}
Z(\tau,\tau) &= Z(-1/\tau,-1/\tau) \\
&= \sum_{h^0;h^0} e^{2 \quad i\frac{1}{\tau}(h^0 \quad \frac{c}{24})} e^{2 \quad i\frac{1}{\tau}(h^0 \quad \frac{c}{24})} \\
&= \sum_{h^0;h^0} e^{2 \quad i\frac{i\text{Im}\tau}{|\tau|^2}h^0 + 2 \quad i\frac{c}{24\tau}} e^{2 \quad i\frac{i\text{Im}\tau}{|\tau|^2}h^0 \quad 2 \quad i\frac{c}{24\tau}} e^{2 \quad i\frac{\text{Re}\tau}{|\tau|^2}h^0 + 2 \quad i\frac{c}{24\tau}} e^{2 \quad i\frac{\text{Re}\tau}{|\tau|^2}h^0 \quad 2 \quad i\frac{c}{24\tau}} \\
[|\tau| \approx \text{Im}\tau] \quad &\approx \sum_{h^0;h^0} e^{2 \quad \frac{1}{\text{Im}\tau}h^0 + 2 \quad i\frac{c}{24\tau}} \underbrace{e^{2 \quad \frac{1}{\text{Im}\tau}h^0 \quad 2 \quad i\frac{c}{24\tau}}}_{= e^{2\pi \frac{1}{\text{Im}\tau}h^0 \quad 2\pi i\frac{c}{24\tau}}}.
\end{aligned}
\tag{2.47}
$$

Terms in the above sum are exponentially suppressed for greater values of h^0 and h^0. Thus, the leading contribution to the partition function comes from the ground state

[9] The motivation for performing this modular transformation is precisely because we are allowed to do it by modular invariance of the partition function on the torus, and this leads to the stated simplification in the thermodynamic limit.

with $h^0 = \bar{h}^0 = 0$ (since $L_0|0\rangle = \bar{L}_0|0\rangle = 0$). We are left with

$$Z(\tau,\bar{\tau}) = Z(-1/\tau,-1/\bar{\tau}) \approx e^{2\ i\frac{c}{24\tau}} e^{\ 2\ i\frac{\bar{c}}{24\bar{\tau}}} \Rightarrow$$
$$\Rightarrow \log \mathcal{Z}_0 = \frac{\pi}{\beta}\frac{c+\bar{c}}{12}. \tag{2.48}$$

Description in terms of characters of the edge CFT

Another way to understand long range topological order in terms of field theoretic degrees of freedom is through the fact that that quasiparticle excitations in the $(2+1)d$ topological field theory are in $1-1$ correspondence with primary operators in a rational CFT defined on a $(1+1)d$ boundary, as explained in [41].

We will then be looking at $2d$ CFT's whose Hilbert space states can be understood in terms of characters. The matrix elements of the corresponding modular $\mathcal{S}$ matrix will give us information about each of the topological sectors of the theory, in particular about the dimension of the corresponding Hilbert subspace, which is related to its quantum dimension. The quantum dimension d_a for a quasiparticle sector a of the TQFT is defined such that

$$H_a(\mathcal{N}) \propto d_a^N, \tag{2.49}$$

for $\mathcal{N}$ very large, where $H_a(\mathcal{N})$ is the dimension of the Hilbert subspace for $\mathcal{N}$-quasiparticle states of type a. A more concrete expression in terms of matrix elements of $\mathcal{S}$ is available, by considering its relation to the fusion matrix through the Verlinde formula. We know from the Verlinde formula that the modular $\mathcal{S}$ matrix is precisely the matrix which diagonalizes the fusion matrices $(\bar{N}_i)_{jk} = N_{ij}^k$, which are defined as $\phi_i \cdot \phi_j = \sum_k N_{ij}^k \phi_k$ for ϕ_i primary operators of the theory. This means that

$$\bar{N}_i = \mathcal{S} D_i \mathcal{S}^{\ 1}, \quad D_i = \text{diag}\left(\frac{\mathcal{S}_{i0}}{\mathcal{S}_{00}}, ..., \frac{\mathcal{S}_{im}}{\mathcal{S}_{0m}}\right). \tag{2.50}$$

This can be seen by explicit use of the Verlinde formula when acting with the fusion matrices on the vector $\frac{\mathcal{S}_{ka}}{\mathcal{S}_{0a}}$:

$$\sum_k (\bar{N}_i)_{jk}\frac{\mathcal{S}_{ka}}{\mathcal{S}_{0a}} = \sum_{k;m}\frac{\mathcal{S}_{im}\mathcal{S}_{jm}\mathcal{S}_{mk}}{\mathcal{S}_{0m}}\frac{\mathcal{S}_{ka}}{\mathcal{S}_{a0}} = \sum_m \frac{\mathcal{S}_{im}\mathcal{S}_{jm}\delta_{ma}}{\mathcal{S}_{0m}\mathcal{S}_{a0}} = \frac{\mathcal{S}_{ia}}{\mathcal{S}_{0a}}\frac{\mathcal{S}_{ja}}{\mathcal{S}_{0a}}. \tag{2.51}$$

The quantum dimension d_a is now defined as the largest eigenvalue of the matrix $\bar{N}_i$. Heuristically this is because when we consider a fusion of M ϕ_a fields:

$$\phi_a \cdot ... \cdot \phi_a = N_{aa}^{c_1} N_{ac_1}^{c_2} ... N_{ac_{M\ 2}}^{c_{M\ 1}} \phi_{c_{M\ 1}}, \tag{2.52}$$

the largest eigenvalue of $\bar{N}_a$ will dominate this product in the $M \to \infty$ limit, and thus gives a measure of the dimension of the Hilbert subspace spanned by the primary operator ϕ_a according to the definition given above. This $\frac{\mathcal{S}_{ak}}{\mathcal{S}_{0k}}$ is largest for $k = 0$, so we arrive at

$$d_a = \text{max}_a\left\{\frac{\mathcal{S}_{ak}}{\mathcal{S}_{0k}}\right\} = \frac{\mathcal{S}_{a0}}{\mathcal{S}_{00}}. \tag{2.53}$$

Since the $\mathcal{S}$ matrix is symmetric and unitary we have

$$\mathcal{D} = \sqrt{\sum_j |d_j|^2} = \frac{1}{\mathcal{S}_0^0}\sqrt{\sum_j \mathcal{S}_{0j}\mathcal{S}_{j0}^y} = \frac{1}{\mathcal{S}_{00}}. \tag{2.54}$$

The sum is over all topological sectors of the theory (in the case of the FQH state at filling fraction $\nu = 1/m$ described below, there are m such sectors since this is the number of primary fields in the edge theory). Thus, the topological entanglement entropy is

$$S_{\text{top}} = -\gamma = -\log \mathcal{D} = \log \mathcal{S}_{00}. \tag{2.55}$$

Physical example: fractional quantum Hall eect and bosonic CFT on a torus

The fractional quantum Hall eect can be described by a Laughlin state at filling fraction $\nu = 1/m$. Such a state allows for the creation of quasiparticles of charge e/m. It turns out that the excitations of the edge modes when the system has finite size can be described in terms of a bosonic CFT defined compactified on a torus (with compactification radius $R = \sqrt{m}$), or equivalently a $U(1)_m$ Kac-Moody CFT. The corresponding bulk theory is a $U(1)_m$ Chern-Simons theory. See [42] for more details.

To obtain the matrix elements of $\mathcal{S}$ we must consider the partition function on the torus. We start with the complex plane, where the mode expansion for a bosonic field reads [40]

$$X(z, z) = x_0 - i\,(j_0 \log z + j_0 \log z) + i \sum_{n \neq 0} \frac{1}{n} \left(j_n z^{\;\;n} + j_n z^{\;\;n} \right), \tag{2.56}$$

where the currents are related to X by

$$j(z) = i\partial X(z, z) = \sum_{n 2 Z} j_n z^{\;\;n \;\; 1}. \tag{2.57}$$

Next, we require that under rotations in the complex plane, $z \mapsto e^{2\;\;i} z$, the field $X(z, z)$ is invariant, but we impose the stronger condition that this invariance is up to translations of $2\pi R n \in 2\pi R \mathbb{Z}$ is field space:

$$X(e^{2\;\;i} z, e^{\;\;2\;\;i} z) = X(z, z) + 2\pi R n \Rightarrow j_0 - \bar{j}_0 \in R\mathbb{Z}. \tag{2.58}$$

This implies that the ground state of the CFT (with conformal weight $h = 0$) is non-trivially charged under j_0, so $|h = 0\rangle$ will be labelled by this charge as well as the winding $n \in \mathbb{Z}$ of X around the circle. We denote

$$j_0 |h = 0, \;\; , n\rangle = \;\; |h = 0, \;\; , n\rangle, \quad \bar{j}_0 |h = 0, \;\; , n\rangle = (\;\; - Rn)\, |h = 0, \;\; , n\rangle. \tag{2.59}$$

To obtain the partition function on the torus, we compute

$$\begin{aligned} \operatorname{Tr}\left(q^{L_0 \;\; \frac{c}{24}} \bar{q}^{\bar{L}_0 \;\; \frac{c}{24}} \right) &= \mathcal{Z}_{\text{bos}}(\tau, \bar{\tau}) \sum_{;n} \langle \;\; , n | q^{\frac{1}{2} j_0^2} \bar{q}^{\frac{1}{2} \bar{j}_0^2} | \;\; , n \rangle \\ &= \frac{1}{|\eta(\tau)|^2} \sum_{;n} q^{\frac{1}{2} \;\; 2} \bar{q}^{\frac{1}{2}(\;\; Rn)^2}, \end{aligned} \tag{2.60}$$

where $\mathcal{Z}_{\text{bos}}(\tau, \bar{\tau})$ is the partition function for a non-compact boson and $q = e^{2\;\;i}$. Under T-transformations on the torus,

$$\mathcal{Z}_{\text{circ}}(\tau + 1, \bar{\tau} + 1) = \frac{1}{|\eta(\tau)|^2} \sum_{;n} q^{\frac{1}{2} \;\; 2} \bar{q}^{\frac{1}{2}(\;\; Rn)^2} e^{2\;\; i\left(\;\; R \;\; \frac{R^2 n}{2} \right)}. \tag{2.61}$$

Therefore, T-invariance implies

$$e^{2\pi i\left(\alpha R - \frac{R^2 n}{2}\right)} = 1 \Rightarrow \alpha = \frac{m}{R} + \frac{Rn}{2}, \quad m \in \mathbb{Z}. \tag{2.62}$$

With this restriction we can write the (chiral) characters of this CFT as

$$\chi_n(\tau) = \sum_{l \in \mathbb{Z}} \frac{q^{\frac{1}{2}\left(\frac{n}{R} + \frac{lR}{2}\right)^2}}{\eta(\tau)} = \sum_{l \in \mathbb{Z}} \frac{q^{\frac{m}{8}\left(\frac{2n}{m} + l\right)^2}}{\eta(\tau)}. \tag{2.63}$$

Note that they are indexed by n since this is the label of the winding of the vacuum state, while l simply labels a degeneracy in this state $|\alpha, n\rangle$. We obtain the matrix elements of $\mathcal{S}$ through Poisson resummation:

$$\begin{aligned}
\chi_n(-1/\tau) &= \frac{1}{\eta(-1/\tau)} \sum_{l \in \mathbb{Z}} \exp\left(-\frac{\pi i m}{4\tau}\left(\frac{2n}{m} + l\right)^2\right) \\
&= \frac{1}{\sqrt{-i\tau}\,\eta(\tau)} \sqrt{\frac{-i\tau}{m}} \sum_{n' \in \mathbb{Z}} \exp\left(\frac{i\pi\tau m}{4}\left(\frac{n'}{m}\right)^2 + 4i\pi\frac{n}{m}n'\right) \\
&= \frac{1}{\eta(\tau)}\frac{1}{\sqrt{m}} \sum_{n'=-k+1}^{k} \sum_{l' \in \mathbb{Z}} \exp\left(\frac{i\pi\tau m}{4}\left(l' + \frac{2n'}{m}\right)^2 - 4i\pi n\frac{ml' + 2n'}{m}\right) \\
&= \frac{1}{\eta(\tau)}\frac{1}{\sqrt{m}} \sum_{n'=-k+1}^{k} \sum_{l' \in \mathbb{Z}} \exp\left(\frac{i\pi\tau m}{4}\left(l' + \frac{2n'}{m}\right)^2 - 8i\pi\frac{n'n}{m}\right) \\
&= \sum_{n'} \mathcal{S}^{(m)}_{nn'} \chi_{n'}(\tau),
\end{aligned} \tag{2.64}$$

where in the third line we have substituted $n' = -(ml' + 2n')$, and

$$\mathcal{S}^{(m)}_{nn'} = \frac{1}{\sqrt{m}} \sum_{l' \in \mathbb{Z}} \exp\left(-8i\pi\frac{n'n}{m}\right). \tag{2.65}$$

From this we can extract the total quantum dimension:

$$\mathcal{D}^{(m)} = \frac{1}{\mathcal{S}^{(m)}_{00}} = \sqrt{m}, \tag{2.66}$$

The topological entanglement entropy for the Laughlin state at filling fraction $\nu = 1/m$ is

$$S^{(m)}_{\text{top}} = -\gamma = -\frac{1}{2}\log m. \tag{2.67}$$

It is interesting to note the correspondence between edge modes and bulk modes in the case of the FQH effect. Because there are m types of quasiparticle excitations in the theory, each one of charge ae/m, $0 \leq a \leq m-1$, there will be m primary operators in the $U(1)_m$ CFT on the edge. Moreover, particle excitations with charge equal to an integer multiple of e correspond to acting on the state with electron creation/annihilation operators, which are not part of the topological sector of the theory, and such excitations correspond to the identity sector of the theory.

This description also gives a heuristic derivation of the total quantum dimension: m primary operators in the edge CFT correspond to m types of anyonic excitations.

The quantum dimension of each quasiparticle sector in an abelian theory is $d_a = 1$[10]. This immediately gives

$$\mathcal{D} = \sqrt{\sum_{a=0}^{m\ 1} |d_a|^2} = \sqrt{m}. \tag{2.68}$$

Thermodynamic entropy in the edge $2d$ CFT = Topological entanglement entropy in the $3d$ TQFT

In [41] it is argued that it is no coincidence that the topological entanglement entropy of the $(2+1)d$ system in from [3] is equivalent to that derived from a $2d$ CFT living on the edge, with a definition of the quantum dimension in terms of objects related to the Hilbert space of this CFT. We quote:

"The latter is defined for a system with a real edge, while the former is defined on the boundary between two regions A and B, not a physical edge. Nevertheless, one expects the two to be the same. Edge modes arise formally in a topological theory to cancel the chiral anomaly [43]. The topological entanglement entropy arises formally by integrating out the system beyond the boundary, *i.e.* subsystem B. When integrating out the degrees of freedom of B, one of course integrates out an anomaly-free theory, so this must include the edge modes of B as well. The remaining A theory then, in some sense, must include edge modes as well which are required to cancel those on B. This required cancellation is one way of seeing that systems A and B are indeed entangled. Thus, the value of the topological entanglement entropy should be identical to our computation of the edge entropy [through the thermodynamic limit of the partition function on the torus]".

2.4.2 Massive quantum field theories

As a second example, we want to know how the entanglement entropy behaves in a general massive field theory as a function of the mass gap. This can be estimated in a $(1+1)$-dimensional QFT for an entangling region given by $\mathcal{A} = \{t = 0, x \geq 0\}$, whose causal domain is the Rindler wedge [44]. In this case, the vacuum density matrix is a thermal density matrix with respect to the boost generator in Rindler space, so we can calculate $S(\mathcal{A})$ as thermodynamic entropy. The thermoynamic entropy is calculated from the physical temperature measure by the observer in Rindler space at constant acceleration a [45], which is $T_{\text{phys}} = \frac{1}{2\ r} = \frac{a}{2}$.

For a field of mass $m = \frac{1}{\ }$, the entropy density $s(T)$ is negligible for temperatures below the scale set by m, since in this regime the fields are frozen out and their degrees of freedom do not propagate. This is analogous to stating that there is no significant entanglement between excitations of the fields across distances much larger than the characteristic length of these excitations, ξ. This means we shoud consider integrating the entropy density for temperatures larger than $\frac{1}{2\ }$ in order to obtain the total entropy:

$$S(\mathcal{A}) \approx \int s(T_{\text{phys}}(r)) \propto \int dr\frac{1}{r} = \log\frac{\xi}{\epsilon} = -\log(m\epsilon). \tag{2.69}$$

[10] If we consider quasiparticle excitations of charge $ae{=}m$ to be represented by line/ surface operator insertions, then for an N quasiparticle-state in an abelian theory $hW_{\gamma_1}(a_1):::W_{\gamma_N}(a_N)i$ is independent of the ordering $fa_1:::a_N g$, and thus corresponds to the same N quasiparticle-state for any ordering.

This tells us that the entropy gets a positive contribution, which is larger as ξ becomes bigger compared to the cutoff scale ϵ; equivalently, as the fields get lighter they become entangled with degrees of freedom localized at increasingly larger distances.

Let's now look at a computation for a massive scalar in $2d$, following [1]. In the singular replicated geometry, we can fix the cut arbitrarily to lie on the positive real axis. We use

$$\frac{\partial}{\partial m^2}\log Z_n = -\frac{1}{2}\int G_n(x,x)d^2x, \tag{2.70}$$

where $G_n(x,x^0)$ is the two-point Green's function on the n-sheeted geometry, which has an expansion in terms of Bessel functions. Upon integrating over the angular coordinate, we have

$$\begin{aligned}\frac{\partial}{\partial m^2}\log Z_n &= -\frac{1}{2}\int_0^{1} dr\sum_{k=0}^{1} d_k I_{k=n}(mr)K_{k=n}(mr)\\ &= \frac{1}{4nm^2}\underbrace{\sum_k d_k k}_{2\ (\ 1)=\ 1=6} = -\frac{1}{24nm^2},\end{aligned} \tag{2.71}$$

leading to

$$\frac{\partial}{\partial m^2}\log Z^n = -n\frac{1}{24m^2}. \tag{2.72}$$

To integrate this equation in m^2, we impose an inferior limit of integration $m^2 = \frac{1}{2}$ determined by a spatial cutoff ϵ, so that

$$\log(\mathrm{tr}\rho^n) = \log\left(\frac{Z_n}{Z^n}\right) = \frac{\log m^2a^2}{24}\left(n-\frac{1}{n}\right). \tag{2.73}$$

The entanglement entropy is

$$\begin{aligned}S &= \lim_{n\,!\,1}\frac{1}{1-n}\log\left(\frac{Z_n}{Z^n}\right)\\ &= -\partial_n\left(\frac{\log m^2a^2}{24}\left(n-\frac{1}{n}\right)\right)\Big|_{n=1}\\ &= -\frac{1}{6}\log ma = \frac{1}{6}\log\frac{\xi}{a},\end{aligned} \tag{2.74}$$

which agrees with the result derived above in Rindler space.

3 Supersymmetry and Localization

In this work we propose to study entanglement entropy in quantum field theories with a symmetry which enlarges the usual spacetime Poincaré symmetry group, supersymmetry (SUSY). This chapter introduces the concept of supersymmetry and the relevant mathematical objects which appear in field theories enjoying this symmetry. We go on to introduce supersymmetric localization and review how it is used in the exact computation of supersymmetric partition functions in three dimensions, as well as the respective Rényi entropies. In particular, we carefully review Yaakov's computation [25] of their proposal for a supersymmetric definition of Rényi entropy, including some technical details which were not made explicit in this work.

3.1 Basic concepts in supersymmetric field theories

Supersymmetry presents itself as the symmetry of spacetime which extends the Poincaré group of symmetries, present in any relativistic field theory. It evades the remarkable Coleman-Mandula theorem [46], which states that the allowed symmetry group of the S-matrix of a QFT, and therefore of 1-particle states in its Hilbert space, must factorize into a product of the Poincaré group with an internal (semisimple) symmetry group, whose generators commute with P and M [1]. Such evasion is possible because the generators of supersymmetry are fermionic, that is, they carry half-integer spin. This means that, when acting on states in the Hilbert space, they change its statistics. The algebra of a theory with such generators will be specified by anti-commutators (since fermionic objects are anti-commuting) as well as the commutators defining the Poincaré group, and this is something the Coleman-Mandula theorem does not account for. Specifically, supersymmetry generators form a structure of a graded Lie algebra, to be specified below, which allows for a weakening of the constraints of the theorem. It was then showed [48] that supersymmetry is the only allowed symmetry of the S-matrix when considering a theory with fermionic symmetry generators.

3.1.1 The supersymmetry algebra

Consider for now theories in four-dimensional Minkowski spacetime dimensions. Let us start by recalling the structure of the Poincaré algebra, consisting of the generators of the symmetries of this spacetime. These are P and M, where P is the 4-momentum operator and M contains the boost and angular momentum operators,[2] the generators of the Lorentz group $SO(1,3)$. Accordingly, the latter satisfy

$$[M\ , M\ ] = -i\eta\ \ M\ \ - i\eta\ \ M\ \ + i\eta\ \ M\ \ + i\eta\ \ M\ \ , \tag{3.1}$$

[1] This is proved under reasonable physical assumptions - locality, causality, positivity of energy and finiteness of number of particles. A short and sweet proof can be given by realising that any symmetric and traceless symmetry generator $Q_{\mu\nu} \neq M_{\mu\nu}$ with $[Q_{\mu\nu}; P_\rho] \neq 0$ must give rise to trivial scattering among two one-particle states [47].

[2] $M_{0i} = K_i$, M_{ij} = "$_{ijk}J_k$; the two copies of $SU(2)$ are generated by $J_i = \frac{1}{2}(J_i \quad iK_i)$.

which can be recast in terms of the commutation relations of two sets of generators of $SU(2)$,

$$[J_i \ , J_j \ ] = i\varepsilon_{ijk}J_k \ , \quad [J_i \ , J_j \ ] = 0. \tag{3.2}$$

The above reflects the fact that, in Lorentzian signature, $SO(1,3) \cong SU(2) \times SU(2)$, which can be shown to be equivalent to the fact that $SL(2, \mathrm{C})$ is a double cover of $SO(1,3)$.

This leads to the existence of two dimensional representations of the Lorentz group, left- and right- handed Weyl spinors, which are representations of $SL(2, \mathrm{C})$ conjugate to each other. They transform under Lorentz transformations according to

$$\psi \ \to S[\ ] \ \ \psi \ , \quad \bar{\psi}_{-} \to S \ [\ ]_{-}^{\ -}\bar{\psi}_{-}, \tag{3.3}$$

and $S[\ ]$ are the generators of the spinor representation of Lorentz transformations, satisfying the above commutator algebra of $SO(1,3)$.

We now introduce the supersymmetry generator Q , the *supercharge*, a left-handed Weyl spinor in the $\left(\frac{1}{2}, 0\right)$ representation of the Lorentz group. There exists another supercharge $\bar{Q}_{-}$ in the complex conjugate representation, $\left(0, \frac{1}{2}\right)$. As fittingly phrased by Tong [18], at the heart of the supersymmetry algebra is the anti-commutation relation

$$\{Q \ , \bar{Q}_{-}\} = 2\sigma \ _{-} P \tag{3.4}$$

Moreover, because supercharges are Weyl spinors they transform non-trivially under rotations or boosts. This is reflected in the following commutators:

$$[M \ \ , Q \ ] = i(\sigma \ \ ) \ \ Q \ , \quad [M \ \ , \bar{Q}^{-}] = i(\sigma \ \ )^{-}{}_{-}\bar{Q}^{-} \tag{3.5}$$

In fact, this is what guarantees that supersymmetry provides a non-trivial extension of the usual spacetime symmetries in the sense of the Coleman-Mandula theorem; it would be classified as an internal symmetry if the above commutators vanished.

The algebra shown above has plenty of important consequences. For instance, it implies that acting with supercharges on particle states changes the spin, or the quantum number of the z-projection of the angular momentum operator, by $\frac{1}{2}$:

$$[M_{12}, Q_{1;2}] = [J_3, Q \ ] = \pm\frac{1}{2}Q \ , \quad [M_{12}, \bar{Q}_{1;2}] = [J_3, \bar{Q}_{1;2}] = \mp\frac{1}{2}\bar{Q}_{1;2}. \tag{3.6}$$

It also leads to the non-negativity of energy of any one-particle state by evaluating the anticommutator of the complex conjugate supercharges in the rest frame:

$$0 \leq |Q \ \ |\psi\rangle|^2 + |\bar{Q}_{-}|\psi\rangle|^2 = \langle\psi|\{Q \ , \bar{Q}_{-}\}|\psi\rangle = \sigma^0 \ _{-}\langle\psi|H|\psi\rangle, \tag{3.7}$$

and $\langle\psi|H|\psi\rangle \geq 0$ follows from taking the trace over the indices of the $SL(2, \mathrm{C})$ representation, using $\sigma^0 = 1$. This fact is widely used when building supersymmetry breaking models, where potentials with non-zero vacuum expectation value may lift the ground state energy to non-zero values.

Another important ingredient in the algebra of supersymmetric theories is the generator of R-*symmetry* transformations. R-symmetry is a global symmetry acting on the supercharges, which is special in the sense that it is internal (it commutes with all Poincaré generators) but commutes non-trivially with the supercharges. This is because it acts on the supercharges according to

$$Q \ \to e^{\ \ i} \ Q \ , \quad \bar{Q}_{-} \to e^{i} \ \bar{Q}_{-}, \tag{3.8}$$

giving, at the level of infinitesimal generators,

$$[R, Q\ ] = -Q\ ,\quad [R, \bar{Q}\ ] = \bar{Q}\ . \tag{3.9}$$

This implies that the charge under $U(1)_R$ of one-particle states varies by ∓ 1 when acted on by Q or $\bar{Q}$, respectively.

3.1.2 Representations of one-particle states and $\mathcal{N} = 2$ supersymmetry

When talking about one-particle states in the above, what is actually being referred to is irreducible representations of the Poincaré algebra, a collection of which forms an irreducible representation of the supersymmetry algebra. Such a collection will consist of an equal number of bosonic and fermionic states, related by the action of Q and $\bar{Q}$. Since this is a crucial aspect, let us work it out. It follows from computing the trace (over the Hilbert space) of the supercharge anti-commutator, dressed with the fermion number operator which satisfies $\{(-1)^F, Q\ \} = 0$:

$$\mathrm{tr}(-1)^F\{Q\ , \bar{Q}_{\dot{\alpha}}\} = \mathrm{tr}[\underbrace{(-1)^F Q\ }_{Q_\alpha(\ \ 1)^F} \bar{Q}_{\dot{\alpha}} + (-1)^F \bar{Q}\ Q\ ] = 0, \tag{3.10}$$

where the cyclicity of the trace was used. This implies

$$\sigma\ \ p\ \ \mathrm{tr}(-1)^F = 0 \Rightarrow n_B - n_F = 0. \tag{3.11}$$

It is a fundamental fact that irreducible representations of the Poincaré group are labelled by their mass and spin, which are eigenvalues of the two Casimir elements of the corresponding algebra, $P\ P$ and $W\ W$.[3] The latter is no longer a Casimir of the supersymmetry algebra (due to $[M_{12}, Q\ ] \neq 0$), justifying that irreducible representations of the SUSY algebra contain particles with distinct values of spin. Note also that, because $[P\ , Q\ ] = [P\ , \bar{Q}\ ] = 0$, all particle states contained in the same irreducible representation of the SUSY algebra have the same mass. We will call these representations *supermultiplets*, or multiplets, in short.

Given that $P\ P$ remains a Casimir element, one can then consider massive or massless supermultiplets. They can be obtained by constructing a set of creation and annihilation operators which build a Fock space when acting on the vacuum state $|0\rangle$. Also, depending on the maximum helicity we allow for particles in the multiplet, labelled by h, we get distinct types of multiplets. Let us illustrate this for massless representations, for which $p\ = (E, 0, 0, E)$ in the rest frame. We have

$$Q\ , \bar{Q}\ = 2\sigma\ \ P\ = 2E(1 + \sigma^3) = 4E\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.12}$$

Computing the expectation value of the above equation on the state $|p\ , h\rangle$ and using 3.7, we see that Q_2 and $\bar{Q}_2$ are trivial operators, while

$$a = \frac{Q_1}{\sqrt{4E}},\quad a^{y} = \frac{\bar{Q}_1}{\sqrt{4E}} \tag{3.13}$$

obey the anti-commutation relations of fermionic Fock space creation and annihilation operators. This space consists of $\{|p\ , h\rangle, a^{y}|p\ , h\rangle\}$, where $a|p\ , h\rangle = 0$; because

[3] For a careful treatment of this subject based on the Casimirs of the Poincaré algebra, see [18, 49].

$[M_{12}, Q_1] = -\frac{1}{2}Q_1$, the helicity of the latter state is $h - \frac{1}{2}$. The supermultiplet consists of these two states, together with their CPT conjugates.[4]

We are now ready to construct representations of the supersymmetry algebra. Two of the most important multiplets which appear in SUSY theories are the following: the *chiral multiplet* is obtained by choosing $h = \frac{1}{2}$, and the representations of the Poincaré group contained in it are two scalars and two fermionic states which combine into a Weyl fermion:

$$\psi \quad \equiv |p \ , \frac{1}{2}\rangle, \ \psi\text{-} \equiv |p \ , -\frac{1}{2}\rangle, \ \phi \equiv \frac{Q_1}{\sqrt{4E}}|p \ , \frac{1}{2}\rangle, \ F \equiv \frac{Q_1}{\sqrt{4E}}|p \ , -\frac{1}{2}\rangle. \tag{3.14}$$

This is the multiplet which accomodates matter fields. We can also have a multiplet containing a gauge field of spin 1, added to additional fermionic superpartners. This is the *vector multiplet*, obtained by setting $h = 1$:

$$A \quad \equiv \{|p \ , 1\rangle; |p \ , -1\rangle\}, \ \lambda \quad \equiv \frac{Q_1}{\sqrt{4E}}|p \ , 1\rangle, \ \lambda\text{-} \equiv \frac{Q_1}{\sqrt{4E}}|p \ , -1\rangle, \tag{3.15}$$

where the notation $(|p \ , 1\rangle; |p \ , -1\rangle)$ refers to the two transverse degrees of freedom of a massless photon in 4 spacetime dimensions.

These are the simplest representations of supersymmetry, and they correspond to theories with one set of preserved supercharges $Q \ , Q_{\_}$. We can also build representations of *extended supersymmetry* when we have N sets of such supercharges, $Q^I \ , Q^I_{\_}$, $I = 1, ..., N$. The same commutation relations with the Poincaré generators hold, and in particular

$$\{Q^I \ , Q^J_{\_}\} = 2\sigma_{\_} \ P \ \delta^{IJ}. \tag{3.16}$$

Having laid out some basic ingredients of supersymmetric theories, we will now skip the construction of supersymmetric Lagrangians using superfields and the superspace formalism. Excellent expositions can be found in [49, 18].

3.2 $3d$ $\mathcal{N} = 2$ supersymmetric gauge theories with $U(1)_R$ symmetry

3.2.1 Rigid supersymmetry in curved space

We will be interested in computing observables of a supersymmetric theory on a compact manifold in order to use localization. Therefore, we need to understand how to place a supersymmetric field theory on a curved background in a way consistent with supersymmetry.

Preserving supersymmetry on curved manifolds

Placing any supersymmetric theory on a curved Riemannian manifold (with a metric tensor g) involves coupling conserved spacetime vectors/ tensors to the metric tensor. This is a general procedure in quantum field theory: in order to probe the behaviour of our original theory when a background gauge field is turned on (in this case, the metric tensor), we couple this field to conserved quantities, leading to invariance of the theory under gauge transformations of the background field.

[4] It is usually required that representations of the supersymmetry algebra are CPT invariant. Because these are labelled by *h*, which transforms as *h* ! *h* under CPT, we must include particles with opposite helicty but equal 4-momentum.

When considering supersymmetric theories the issue is more subtle, because if we simply couple the metric minimally to the theory then supersymmetry is lost. Minimal coupling would involve adding to the Lagrangian density a term

$$\mathcal{L} \propto \delta g \quad (T \quad + \mathcal{O}[\partial \ \partial \quad - \delta \quad \partial^2]), \tag{3.17}$$

where $\delta g \quad = g \quad - \eta$ and $\mathcal{O}[\partial \ \partial \quad - \delta \quad \partial^2]$ denotes any combination of operators depending on the gauge-invariant and symmetric $(2, 0)$ spacetime tensor $\partial \ \partial \ - \delta \quad \partial^2$. This would immediately break supersymmetry because, as we will see below, T has non-trivial transformation properties under supersymmetry transformations, that is, [50],

$$\delta_Q T \quad = [\zeta \ \ \mathcal{Q} \ , T \quad] \neq 0. \tag{3.18}$$

This means that linearizing the action around the flat metric already leads to corrections with non-trivial supersymmetry transformations. Also, the above variation does not give rise to total spacetime derivatives, but rather transforms T into other non-trivial conserved currents of the theory, as we will learn below.

Another way to understand this is the following: supersymmetry variations of the Lagrangian can be expressed in terms of the divergence of the supersymmetry current S , which always exists and is conserved on-shell,

$$\delta\mathcal{L} = \zeta \ \ \partial \ S \ . \tag{3.19}$$

We assume all fields and conserved currents vanish at the boundary of flat spacetime, so that the variation of the action vanishes if and only if $\partial \ \zeta \quad = 0$. This is the condition that there exists one strictly constant spinorial transformation parameter for each flat space supercharge which implements a symmetry of the theory, and it is precisely the condition for *global* supersymmetry.

If we carry out the same analysis starting from a SUSY theory on a curved manifold $\mathcal{M}$ without boundary, we must consider a supersymmetry current which is covariantly conserved on-shell and such that, in the same manner as above,

$$\delta S = 0 \Rightarrow \int_M d^d x \zeta \ \ \nabla \ \ S \quad = 0 \Rightarrow \nabla \ \ \zeta \quad = 0. \tag{3.20}$$

The above would result from minimal coupling of the theory (that is, replacing $\delta \quad \to g$ and $\partial \ \to \nabla$ in conservation laws), in which case each conserved supercharge on $\mathcal{M}$ would require a *covariantly constant* spinorial SUSY transformation parameter. This condition heavily restricts the allowed compact manifolds we can consider (in particular, $\mathcal{M}$ must be flat in $3d$, see footnote 4 of [51]). It is for this reason that we will study SUSY theories that are non-minimally coupled to the metric tensor.

This can be achieved by coupling to theory to a rigid limit of off-shell supergravity [52]. The idea is to embed the metric tensor in an off-shell supergravity multiplet and the energy-momentum tensor in a conserved current multiplet (a sub-multiplet of the $\mathcal{S}$ multiplet [53]). Then, the limit $M_P \to \infty$ is taken, with M_P the Planck mass. This decouples all fluctuations of the brackground fields from the supergravity multiplet (we do not want to define a theory on a background with a dynamical metric), and crucially these need not be in on-shell configurations. What Festuccia and Seiberg have beautifully shown in [52] is that, taking this limit in a generic supergravity theory coupled to matter, the resulting Lagrangian can be expressed in terms of a coupling between the supergravity multiplet and the $\mathcal{S}$ multiplet, where the gravitino is set to zero and the bosonic fields of the former satisfy conditions imposed by

$$\delta \quad = 0, \quad \tilde{\delta} \; = 0. \tag{3.21}$$

The general procedure is then to consider all the fields in the supermultiplet of T , denoted schematically by $\mathcal{J}^i_B$ (bosonic) and $\mathcal{J}^i_F$ (fermionic), as well as background fields $\mathcal{B}^i_B, \mathcal{B}^i_F$, and deform the Lagrangian according to

$$\mathcal{L} \propto \delta g \quad T \quad + \sum_i \mathcal{B}^i_B \mathcal{J}^i_B, \tag{3.22}$$

up to gauge transformations of T , $\mathcal{B}^i_B$, where the fields $\mathcal{B}^i_B, \mathcal{B}^i_F$ are contained in the background supergravity multiplet. Note that terms with $\mathcal{B}^i_F$ were immediately excluded from $\mathcal{L}$, since, as mentioned above, we impose that the background fields are in explicitly supersymmetric configurations, and in general $\delta_Q \mathcal{B}^i_B = 0 \iff \mathcal{B}^i_F = 0$. On the other hand, maintaining $\mathcal{B}^i_F = 0$ after a SUSY transformation is equivalent to the condition $\delta_Q \mathcal{B}^i_B = 0$. The number of degrees of freedom (parametrized by spacetime dependent spinors ζ , $\tilde{\zeta}$) available to perform SUSY transformations satisfying the previous conditions determine the number of preserved supercharges when the theory is placed on the curved spacetime. These will be solutions to the *Killing spinor equation*, which we will now derive.

3.2.2 3d $\mathcal{N}$ = 2 theories with $U(1)_R$ symmetry: the $\mathcal{R}$ multiplet and coupling to supergravity

We will now work out the precise form of the coupling between the rigid supergravity theory and conserved currents in the case of three-dimensional $\mathcal{N}$ = 2 theories with a (non-anomalous) $U(1)_R$ symmetry.

Recall first that, by definition of the supersymmetry current and energy-momentum tensor,

$$Q \quad = \int d^{d-1}x S \;^0(x), \qquad P \quad = \int d^{d-1}x T \;^0(x). \tag{3.23}$$

We can then write a local version of the anti-commutator $\{Q \; , \bar{Q} \} = 2\sigma \; P$, which expresses the energy-momentum tensor as the SUSY variation of the supersymmetry current:

$$\{\bar{Q} , S \quad \} = 2\sigma \quad T \quad . \tag{3.24}$$

Therefore, T and S should belong to the same supersymmetry multiplet, the $\mathcal{S}$ multiplet, which exists in every supersymmetric field theory [5].

If the theory contains a continuous R-symmetry, then there exists a conserved R-symmetry current $j^{(R)}$, which becomes the bottom component of the supercurrent multiplet. In this case, improvement transformations can be performed on $\mathcal{S}$, and we obtain the $\mathcal{R}$*-multiplet*. It satisfies (in three dimensions)

$$\bar{D} \; \mathcal{R} \quad = -4i\bar{D} \; \mathcal{J}^{(Z)}, \quad D \; \mathcal{R} \quad = 4iD \; \mathcal{J}^{(Z)}, \quad D^2\mathcal{J}^{(Z)} = \bar{D}^2\mathcal{J}^{(Z)} = 0. \tag{3.25}$$

Note that the $\mathcal{R}$ multiplet is independent of a Lagrangian description of the theory. In superspace components, it is given by

$$\mathcal{R} \quad = j^{(R)} - i\theta S \quad - i\bar{\theta}\bar{S} \quad - (\theta\gamma \; \bar{\theta}(2T \quad + i\varepsilon \quad \partial \; J^{(Z)}) - i\theta\bar{\theta}(2j^{(Z)} + i\varepsilon \quad \partial \; j^{(R)} \;) + \ldots, \tag{3.26}$$

[5] This was constructed in [53], imposing the requirements $T_{\mu\nu}$ and $S_{\mu\alpha}$ are the only operators in S with spin larger than one and that S is indecomposable.

with

$$\mathcal{J}^{(Z)} = J^{(Z)} - \frac{1}{2}\theta\gamma\ S\ + \frac{1}{2}\bar{\theta}\gamma\ \bar{S}\ + i\theta\bar{\theta}T\ - (\theta\gamma\ \bar{\theta})j^{(Z)} + \ldots, \tag{3.27}$$

where the ellipses denote terms that can be absorbed by improvement transformations. Note that the operators $J^{(Z)}$ and $j^{(Z)}$ belong to the same irreducible multiplet as the trace of the energy-momentum tensor, T . This means that in a superconformal theory these become redundant operators (they are related by supersymmetry transformations to an operator that identically vanishes, so they must vanish themselves).

J is a scalar operator and $j^{(Z)}$ is the conserved central charge current associated to the conservation of the central charge quantum number, z, and it is unique to multiplets of theories with $\mathcal{N} \geq 2$ because only these admit a central charge in their anti-commutator algebra,

$$\{Q^I, Q^J\} = \epsilon\ \ Z^{IJ}, \quad \{\overline{Q}^I, \overline{Q}^J\} = \epsilon\ (Z^y)^{IJ}. \tag{3.28}$$

The bosonic fields $j^{(R)}, T\ , j^{(Z)}$ and J contain, respectively, (3 − 1) + (6 − 3) + (3 − 1) + 1 = 8 independent components (the currents obey a conservation equation, T is symmetric and conserved and J is a real scalar). The supersymmetry current contains 3 × 4 − 1 × 4 = 8 fermionic components, which matches the number of bosonic degrees of freedom.

The background supergravity multiplet contains the fields that gauge the symmetries corresponding to each field in $\mathcal{R}$. In Wess-Zumino gauge, it is given by [54]

$$\mathcal{H}\ = \frac{1}{2}(\theta\gamma\ \bar{\theta})(h\ + B\ ) - \frac{i}{2}\theta\bar{\theta}C\ - \frac{i}{2}\theta^2\bar{\theta}\bar{\psi}\ + \frac{i}{2}\bar{\theta}^2\theta\psi\ + \frac{1}{2}\theta^2\bar{\theta}^2(A\ - V\ ). \tag{3.29}$$

Here, h is the linearized metric perturbation, B is a two-form gauge field, C and A are abelian gauge fields and ψ , $\bar{\psi}$ are the gravitini.

We are now ready to write the coupling of the conserved currents to the corresponding supergravity gauge fields. At the linearized level, it is expressed in superspace as

$$\delta\mathcal{L} = 2\int d^4\theta\mathcal{R}\ \mathcal{H}\ . \tag{3.30}$$

This gives

$$\begin{aligned}\delta\mathcal{L} = 2\int d^4\theta\Big[&j^{(R)} - i\theta S\ - i\bar{\theta}\bar{S} - (\theta\gamma\ \bar{\theta})(2T\ + i\varepsilon\ \ \partial\ J^{(Z)}) - i\theta\bar{\theta}(2j^{(Z)} + i\varepsilon\ \ \partial\ j^{(R)}\ )\Big]\times\\ &\times\left[\frac{1}{2}(\theta\gamma\ \bar{\theta})(h\ + B\ ) - \frac{i}{2}\theta\bar{\theta}C\ - \frac{i}{2}\theta^2\bar{\theta}\bar{\psi}\ + \frac{i}{2}\bar{\theta}^2\theta\psi\ + \frac{1}{2}\theta^2\bar{\theta}^2(A\ - V\ )\right]\\ =\ & j^{(R)}(A\ - V\ ) - \frac{1}{2}S\ \psi\ + \frac{1}{2}\bar{S}\ \bar{\psi}\ + j^{(Z)}C\ + i\varepsilon\ \ \partial\ j^{(R)}\ C\ +\\ &-\frac{1}{2}(2T\ + i\varepsilon\ \ \partial\ J^{(Z)})(h\ + B\ ).\end{aligned} \tag{3.31}$$

Now we rewrite

$$\begin{aligned} i\varepsilon\ \ \partial\ j^{(R)}\ C\ &= -(-i\varepsilon\ \ \partial\ C\ j^{(R)}\ ) + (\text{tot. derivative}) \sim -V\ j^{(R)}\ ,\\ -\frac{i}{2}\varepsilon\ \ \partial\ J^{(Z)}B\ &= -\frac{i}{2}\varepsilon\ \ \partial\ B\ \ J^{(Z)} + (\text{tot. derivative}) \sim J^{(Z)}H, \end{aligned} \tag{3.32}$$

where we neglect total derivative terms in the Lagrangian since we are considering $\mathcal{M}$ without boundary. We have also defined

$$V \quad = -i\varepsilon \quad \partial \, C \;, \quad H = \frac{i}{2}\varepsilon \quad \partial \, B \quad . \tag{3.33}$$

Using the integration rule for Grassman variables $\int d^2\theta \equiv \frac{1}{4}\varepsilon \quad \frac{@}{@ \;\alpha} \frac{@}{@ \;\beta}$, we have performed manipulations such as

$$\begin{aligned}
\int d^4\theta(\theta S \;)(\bar\theta^2\theta\psi \;) &= \int d^2\theta d^2\bar\theta\theta \; S \quad \bar\theta\bar\theta \; \theta \; \psi \\
&= \int d^2\theta(-1)^2 S \quad (\int d^2\bar\theta\bar\theta\bar\theta \;)\theta \;\; \theta \;\; \psi \\
&= S \quad \frac{1}{4}\varepsilon \quad \partial \; \partial \; \theta \;\; \theta \;\; \psi \\
&= S \quad \frac{1}{4}\partial \; (\varepsilon \quad \theta \;\; - \varepsilon \quad \theta \;)\psi \\
&= S \quad \frac{1}{4}(\varepsilon \quad - \varepsilon \quad)\psi \\
&= -\frac{1}{2}S \quad \varepsilon \quad \psi \quad = -\frac{1}{2}S \; \psi \;\; ,
\end{aligned} \tag{3.34}$$

and to deal with the γ matrices,

$$\begin{aligned}
\int d^4\theta(\theta\gamma \;\; \bar\theta)(\theta\gamma \;\; \bar\theta) &= \frac{1}{4}\varepsilon \; \bar\partial \; \bar\partial \; \frac{1}{4}\varepsilon \quad \partial \; \partial \; (\theta \;\; \gamma \;\; \bar\theta)(\theta \;\; \gamma \;\; \bar\theta) \\
&= \frac{1}{16}\varepsilon \; \varepsilon \quad \{-\gamma \;\; \gamma \;\; + \gamma \;\; \gamma \;\; + \gamma \;\; \gamma \;\; - \gamma \;\; \gamma \;\; \} \\
&= \frac{1}{8}\varepsilon \; \varepsilon \quad \{\gamma \;\; \gamma \;\; - \gamma \;\; \gamma \;\; \} \\
&= -\frac{1}{4}(\delta \quad \delta \quad - \delta \quad \delta \quad)\gamma \;\; \gamma \\
&= -\frac{1}{4}((\gamma \;) \quad (\gamma \;) \quad - (\gamma \; \gamma \;) \quad) \\
&= \frac{1}{4}(\delta \quad + i\varepsilon \quad \gamma \;) \quad = \frac{1}{2}\delta \quad .
\end{aligned} \tag{3.35}$$

In a similar way,

$$\int d^4\theta(\theta\gamma \;\; \bar\theta)(\theta\bar\theta) = -\frac{1}{4}\underbrace{\varepsilon \; \varepsilon \quad \delta}_{(\varepsilon^2)\varrho\alpha = \;\; \varrho\alpha} \;\; \gamma \;\; = -\frac{1}{4}(\gamma \;) \quad = 0, \tag{3.36}$$

since γ are traceless. The latter ensures there is no coupling between C and T or $J^{(Z)}$. We arrive at

$$\delta\mathcal{L} = -T \quad h \quad -\frac{1}{2}S \;\; \psi \;\; + \frac{1}{2}\bar S \;\; \bar\psi \;\; + j^{(R)}(A \;\; - \frac{3}{2}V \;\;) + j^{(Z)}C \;\; + J^{(Z)}H. \tag{3.37}$$

This is explicitly invariant (up to total derivative terms) under gauge transformations of fields in $\mathcal{H}$ of the form

$$\begin{aligned} \delta h \quad &= \partial \quad ^{(h)} - \partial \quad ^{(h)}, \quad \delta B \quad = \partial \quad ^{(B)} - \partial \quad ^{(B)}, \\ \delta C \quad &= \partial \quad ^{(C)}, \quad \delta A \quad = \partial \quad ^{(A)}, \\ \delta \psi \quad &= \partial \; \varepsilon \; , \quad \delta \tilde{\psi} \quad = \partial \; \tilde{\varepsilon} \; . \end{aligned} \tag{3.38}$$

Therefore, h indeed corresponds to a metric field which, as expected, must couple to the energy-momentum tensor in the linearized Lagrangian variation; $^{(h)}$ parametrize spacetime di eomorphisms. We can also see that A gauges local R-symmetry transformations (it couples to $j^{(R)}$), C gauges the central charge current $j^{(Z)}$ and B gauges the current $i\varepsilon \quad \partial \; J^{(Z)}$.

This is a gauge fixed form of the Lagrangian that is invariant under global supersymmetry transformations followed by a gauge transformation to go back to Wess-Zumino gauge. The variation of the gravitini under such global transformations is [54]

$$\delta \psi \quad = -i\varepsilon \quad \partial \; h \quad \gamma \; \zeta - 2iA \; \zeta + H\gamma \; \zeta + \varepsilon \quad V \; \gamma \; \zeta. \tag{3.39}$$

The above results from a variation of the components of the metric superfield under a global supersymmetry transformation. When local supersymmetry is considered, an extra term $\partial \; \zeta$ appears in the variation of the gravitino. In fact, under a local supersymmetry transformation the Lagrangian has a variation

$$\delta \mathcal{L} = S \; \partial \; \zeta - \tilde{S} \; \partial \; \tilde{\zeta}, \tag{3.40}$$

and the term contracted with the supercurrents must be absorbed in the variations of the gravitini. We then have

$$\delta \psi \quad = 2(\partial \; \zeta - \frac{i}{2}\varepsilon \quad \partial \; h \quad \gamma \; \zeta) - 2iA \; \zeta + H\gamma \; \zeta + \varepsilon \quad V \; \gamma \; \zeta. \tag{3.41}$$

Now, this represents the gravitino variation under a linearized metric perturbation. Any additional terms that we must add in the non-linear case are constrained either by covariance of the expression under di eomorphisms or by dimensional analysis. The former allows us to replace the term in parenthesis by $\nabla \; \zeta$, since it indeed corresponds to the linearized covariant derivative. On the other hand, by analysing the mass dimensions of every field and knowing that $\delta \psi$ must be proportional to ζ, we can conclude that all the allowed terms are already contained in the linearized gravitino variation.[6] Setting this variation to zero gives the condition for residual local supersymmetry to hold, the Killing spinor equation:

$$\begin{aligned} \delta \psi \quad &= 0 \Rightarrow (\nabla \quad - iA \quad)\zeta = -\frac{1}{2}H\gamma \quad - iV \; \zeta - \frac{1}{2}\varepsilon \quad V \; \gamma \; \zeta, \\ \delta \tilde{\psi} \quad &= 0 \Rightarrow (\nabla \quad + iA \quad)\tilde{\zeta} = -\frac{1}{2}H\gamma \quad + iV \; \zeta + \frac{1}{2}\varepsilon \quad V \; \gamma \; \zeta. \end{aligned} \tag{3.42}$$

The equation for the complex conjugate Weyl spinor $\tilde{\zeta}$ was included.

The conclusion is the following: given a spacetime manifold $\mathcal{M}$ with metric g and some background configuration of the fields A , V and H (which need not be on-shell), the number of globally defined solutions of 3.42 determines the number of

[6]Since [$_\mu$] = $\frac{1}{2}$, [ ] = $\frac{1}{2}$, $[A_\mu] = [V_\mu] = [H] = 0$, the only allowed term not already present in the linearized variation is a term proportional to ~ ; however = ~ = 0 in the rigid supergravity limit.

supercharges that generate supersymmetries of the $3d$ $\mathcal{N} = 2$ theory when it is placed on $\mathcal{M}$.

3.3 Supersymmetry on the branched three-sphere

3.3.1 Killing spinors on S^3

A fairly simple application of the Killing spinor equation arises in the case $\mathcal{M} = S^3$. Solutions of R-charges ± 1 of both equations 3.42 can be obtained by simply turning on $H = \mp i$ and setting the remaining background fields to zero; this leads to a KSE

$$\nabla \ \varepsilon = \pm \frac{i}{2} \sigma \ \varepsilon \tag{3.43}$$

The simplest solutions are obtained for a choice of vierbein given by the left- or right-invariant vector fields e_i on S^3,[7] see [21, 25] This is because the spin connection greatly simplifies in this basis:

$$\omega_{ij} = \pm \varepsilon_{ijk} e^k \tag{3.44}$$

The first KSE in 3.42 can be solved by any constant spinor in the left-invariant basis by turning on $H = -i$:

$$\nabla \ \varepsilon = \left(\partial \ + \frac{1}{8} e^k \varepsilon_{ijk} [\sigma^i, \sigma^j] \right) \varepsilon = \left(\partial \ + \frac{i}{2} \sigma \ \right) \varepsilon \tag{3.45}$$

This gives two linearly independent constant Killing spinors. Analgously, there are two other Killing spinors which solve the second KSE in 3.42 by considering now the right-invariant vierbein basis with $H = i$, for which

$$\nabla \ \varepsilon = \left(\partial \ - \frac{i}{2} \sigma \ \right) \varepsilon. \tag{3.46}$$

This shows that the three-sphere preserves all four (real) supercharges of the $\mathcal{N} = 2$ algebra.

3.3.2 Killing spinors on S^3_q

As explained in 2.2, we are interested in computing supersymmetric observables on a branched cover of the three-sphere. This requires a careful analysis of the preserved supersymmetries on this background. In this section, we will explicitly analyse how the introduction of specific background configurations of the fields from the supergravity multiplet can lead to globally defined solutions of the Killing spinor equation on the branched sphere.

The metric we consider is

$$ds^2 = d\theta^2 + q^2 \sin^2 \theta d\tau^2 + p^2 \cos^2 \theta d\phi^2, \quad \theta \in \left[0, \frac{\pi}{2}\right], \tau, \phi \in [0, 2\pi). \tag{3.47}$$

This is a two-parameter generalization of the line element for S^3 in Hopf coordinates, which is recovered for $q = p = 1$. For $q \neq 1$ or $p \neq 1$ it possesses a conical singularity

[7] Given that S^3, as a manifold, is isomorphic to $SU(2)$, the left-invariant vector fields on S^3 may be defined as the vector fields whose action (as differential operators) on elements of $SU(2)$ reproduces the action of $SU(2)$ on itself; these are $L_i \in \mathfrak{X}(S^3)$ such that $_i\ (x) = L_i^\mu \partial_\mu\ (x)$, where $: S^3 \to SU(2)$ is an isomorphism that is usually taken to be the one parametrizing matrices of $SU(2)$ through Euler angles on S^3.

at $\theta = 0$ or $\theta = \frac{}{2}$, respectively. An immediate way to see this is to write the metric for $\theta \approx 0$ or $\theta \approx \frac{}{2}$. For instance,

$$ds^2\Big|_{\quad 0} = d\theta^2 + q^2\theta^2 d\tau^2, \tag{3.48}$$

and analogously for $\theta \approx \frac{}{2}$. This local form of the metric represents flat space in polar coordinates, where the angular coordinate has an excess angle of $2\pi(q-1)$. This leads to a singularity at $\theta = 0$ if $q \neq 1$. See Apendix C of [25] for a careful derivation of the singularity of the Ricci scalar, where it is shown that $R(\theta \approx 0) \simeq 2\frac{q\ \ 1}{q}\frac{(\ )}{}$.

The background gauge fields we can turn on are A , V and H. We first note that, setting them all to zero leads to a covariantly constant Killing spinor (along the fibers of the spin manifold):

$$\begin{aligned}\nabla\ \zeta = 0 &\Rightarrow \left(\partial\ + \frac{1}{4}\omega\ {}^{ab}\gamma_{ab}\right)\zeta = 0\\ &\Rightarrow \left(\partial\ + \frac{1}{4}\left(\varepsilon^{abc}e\ {}_c - \varepsilon^{ab3}((q-1)\delta\ {}_{;2} + (p-1)\delta\ {}_{;3})\right) i\varepsilon_{ab}{}^d\sigma_d\right)\zeta = 0\\ &\Rightarrow \left(\partial\ + \frac{1}{4}\left(2i\delta^{cd}e\ {}_c\sigma_d - 2i\delta^{d;3}\sigma_d((q-1)\delta\ {}_{;2} + (p-1)\delta\ {}_{;3})\right)\right)\zeta = 0\\ &\Rightarrow \left(\partial\ + \frac{i}{2}\gamma\ - \frac{i}{2}((q-1)\delta\ {}_{;}\ + (p-1)\delta\ {}_{;}\ )\sigma_3\right)\zeta = 0,\end{aligned} \tag{3.49}$$

where the conventions of (A.1), (A.2), (B.4) and (B.5) in [25] were used. This condition is overly restrictive on the solutions ζ; in fact it has no solutions for most arbitrary values that we choose for p and q. This shows that we need to consider non-zero values for the background fields in order to preserve supersymmetry.

The first possibility to consider is the supergravity background which solves the KSE on S^3. Setting

$$H = -i, \quad A\ = V\ = 0 \tag{3.50}$$

is the simplest choice of background fields for which the KSE is immediately satisfied. This configuration already allows us to easily generate solutions, provided that they are chosen so that a term $\frac{i}{2}((q-1)\delta\ {}_{;}\ + (p-1)\delta\ {}_{;}\ )\sigma_3$ is obtained when the partial derivative acts on the spinor. We are led to

$$\zeta(\tau,\phi) = \tilde{\zeta}(\tau,\phi) = \begin{pmatrix} c_+ e^{\frac{i}{2}((q\ \ 1)\ \ +(p\ \ 1)\ \ )} \\ c\ \ e^{\ \ \frac{i}{2}((q\ \ 1)\ \ +(p\ \ 1)\ \ )}\end{pmatrix}, \tag{3.51}$$

where it was noted that for $A\ = V\ = 0$, ζ and $\tilde{\zeta}$ obey the same equation.

Although this solution is θ-independent, its validity breaks down at the regions where the curvature becomes singular. This can be seen from an integrability condition which is implied by the KSE in the presence of two independent solutions of each of the equations 3.42, which is the case here. Given a solution ζ and using that $\varepsilon\ \ [\nabla\ ,\nabla\ ]\zeta = \frac{i}{2}(2R\ \ - Rg\ \ )\gamma\ \zeta$, we have [54]

$$\left[\frac{i}{2}(2R\ \ - Rg\ \ )\gamma\ + iH^2\gamma\ \right]\zeta = 0. \tag{3.52}$$

Due to the presence of $\delta(\theta)$ and $\delta(\theta - \frac{}{2})$ at $\theta = 0$ and $\theta = \frac{}{2}$, respectively, the solutions 3.51 are not regular at these regions of the branched sphere, and therefore are not globally defined.

We should then search for a background field configuration which cancels these singularities in the integrability condition. H is a scalar field, so the only way to tune it for this purpose would be to add δ-functions to it, and we do not want a singular background field configuration. There is a way to achieve this by endowing ζ and $\tilde{\zeta}$ with opposite and non-zero R-charges, in which case we turn on a non-zero value for A , which gauges the R-symmetry. The latter can be done while retaining a vanishing central charge of the supersymmetry algebra, meaning that V = 0. The integrability condition in this case becomes

$$\left[\frac{i}{2}(2R \quad - Rg \quad)\gamma \quad - 2i\varepsilon \quad \nabla A \quad + iH^2\gamma \right]\zeta = 0. \tag{3.53}$$

The reason we can find a solution without being left with an explicitly singular background field configuration is that A appears in 3.53 only through its field strength. Indeed,

$$\begin{aligned}
\varepsilon \quad \nabla A &= \varepsilon \quad \nabla (e\ _a A^a) \\
&= \varepsilon \quad \left((\partial\ e\ _a - \omega\ ^b\ _a e\ _b)A^a + e\ _a(\partial\ A^a + \omega\ ^a\ _b A^b)\right) \\
&= \varepsilon \quad \left(\partial\ \underbrace{(e\ _a A^a)}_{=A^\rho} \underbrace{-\omega\ _{ba} e\ ^b A^a + \omega\ _{ab} e\ ^a A^b}_{=0}\right) \\
&= \varepsilon \quad \partial\ A \quad = \frac{1}{2}\varepsilon \quad F \quad ,
\end{aligned} \tag{3.54}$$

where we have assumed A an abelian $U(1)$ gauge field, so that $A \wedge A = 0 \Rightarrow F = \mathrm{d}A$. Now the point is that we know how to construct regular one-form gauge fields whose field strength is singular. This is achieved by

$$\begin{aligned}
A &= \frac{q-1}{2}\mathrm{d}\tau + \frac{p-1}{2}\mathrm{d}\phi \Rightarrow \\
\Rightarrow F &= \frac{q-1}{2}\delta(\theta)\mathrm{d}\theta \wedge \mathrm{d}\tau + \frac{p-1}{2}\delta\left(\theta - \frac{\pi}{2}\right)\mathrm{d}\theta \wedge \mathrm{d}\phi.
\end{aligned} \tag{3.55}$$

The divergences in the Ricci scalar at the regions $\theta = 0$, $\theta = \frac{\pi}{2}$ can be understood as the coordinate θ being ill-defined at these points. We must keep this in mind when determining the field strength, since the following computation is only valid outside the regions with $\theta = 0$ or $\theta = \frac{\pi}{2}$:

$$\begin{aligned}
\mathrm{d}A\Big|_{\theta\neq 0;\frac{\pi}{2}} &= \partial\ A\ \mathrm{d}x\ \wedge \mathrm{d}x \\
&= \left(\partial\ \frac{q-1}{2}\mathrm{d}\theta + \partial\ \frac{q-1}{2}\mathrm{d}\phi\right) \wedge \mathrm{d}\tau + \left(\partial\ \frac{p-1}{2}\mathrm{d}\theta + \partial\ \frac{p-1}{2}\mathrm{d}\phi\right) \wedge \mathrm{d}\tau = 0.
\end{aligned} \tag{3.56}$$

Since the coordinate ϕ is well-defined everywhere, $\partial\ \frac{q-1}{2}\mathrm{d}\phi = \partial\ \frac{p-1}{2}\mathrm{d}\theta = 0$ for all points on the branched sphere. The same cannot be said for the derivatives with respect to θ, so the two-form F will have coe□cients only in $\mathrm{d}\theta \wedge \mathrm{d}\tau$ and $\mathrm{d}\theta \wedge \mathrm{d}\phi$, denoted by $f\ (\theta)$ and $f\ (\theta)$. To determine them, we use Stokes' theorem by integrating F along the codimension-1 surfaces spanned by $\theta \in [0, \frac{\pi}{2}]$, $\tau \in [0, 2\pi q]$ and $\theta \in [0, \frac{\pi}{2}]$, $\phi \in [0, 2\pi]$, whose boundaries are the codimension two regions $\gamma \ \equiv \{\theta = \frac{\pi}{2}, \tau \in$

$[0, 2\pi q]\}$ and $\gamma \equiv \{\theta = 0, \phi \in [0, 2\pi]\}$, respectively:

$$\int_{\tau} A = \int_0^{2\ q} \frac{q-1}{2} \mathrm{d}\tau \Rightarrow \int_0^{2\ q} \int_0^{\frac{\pi}{2}} F \quad \mathrm{d}\theta \wedge \mathrm{d}\tau = \int_0^{2\ q} \frac{q-1}{2} \mathrm{d}\tau$$
$$\Rightarrow \int_0^{2\ q} \left(\int_0^{\frac{\pi}{2}} f\ (\theta) d\theta \right) \wedge \mathrm{d}\tau = 2\pi q \frac{q-1}{2}. \tag{3.57}$$

This is solved by $f\ (\theta) = \frac{q\ 1}{2}\delta(\theta)$. By an analogous argument, we have $f\ (\theta) = \frac{p\ 1}{2}\delta\left(\theta - \frac{}{2}\right)$.

Let us now check that the singularities are cancelled from the integrability condition (using (C.6) and (C.7) from [25]), keeping $H = -i$ and $V = 0$. We look at the ϕ component of 3.53:

$$\left[\frac{i}{2}(2R \quad - Rg \quad)\gamma \quad - i\varepsilon \quad F \quad - i\gamma \quad \right]\zeta =$$
$$= \left[- 2i\varepsilon \quad F \quad - i\gamma \quad + i\left(-2p^2\cos^2\theta + p^2\cot\theta\left(\frac{1}{p} - 1\right)\delta(\theta - \frac{\pi}{2})\right) g \quad \gamma \quad + \right.$$
$$\left. + i\left(3 + \frac{1}{\sin\theta\cos\theta}\left(\frac{1}{q} - 1\right)\delta(\theta) - \frac{1}{\sin\theta\cos\theta}\left(\frac{1}{p} - 1\right)\delta\left(\theta - \frac{\pi}{2}\right)\right) g \quad \gamma \quad \right]\zeta. \tag{3.58}$$

Near $\theta = 0$, this becomes

$$\left[- 2i\varepsilon \quad F \quad + \frac{i}{\sin\theta\cos\theta}\left(\frac{1}{q} - 1\right)\delta(\theta)\gamma \quad \right]\zeta. \tag{3.59}$$

It can easily be checked that

$$\varepsilon \quad = e^{\ a} e^{\ b} e^{\ c}\varepsilon_{abc} = \underbrace{\varepsilon_{123}}_{=1} pq\cos\theta\sin\theta \Rightarrow \varepsilon \quad F \quad = \frac{p}{q}\cot\theta F \quad , \tag{3.60}$$

as well as

$$\gamma \quad = e^{\ a}\sigma_a = \begin{pmatrix} p\cos^2\theta & -p\frac{\sin 2}{2}e^{\ i(\ +\ )} \\ -p\frac{\sin 2}{2}e^{i(\ +\ )} & -p\cos^2\theta \end{pmatrix}. \tag{3.61}$$

The integrability condition is then

$$0 = \left[- 2i\frac{p}{q}\cot\theta F \quad + i\left(\frac{1}{q} - 1\right)\begin{pmatrix} p\cot\theta & -pe^{\ i(\ +\ )} \\ -pe^{i(\ +\ )} & -p\cot\theta \end{pmatrix}\delta(\theta)\right]\zeta$$
$$= \left[- F \quad + \frac{1-q}{2}\begin{pmatrix} 1 & \frac{1}{\cot}e^{\ i(\ +\ )} \\ \frac{1}{\cot}e^{i(\ +\ )} & -1 \end{pmatrix}\delta(\theta)\right]\zeta \tag{3.62}$$
$$\simeq \left[\frac{1-q}{2}\delta(\theta)\sigma_3 - F \quad \right]\zeta,$$

where the last equality holds up to o-diagonal terms multiplying $\cot^{\ 1}\theta$ which vanish in the $\theta \to 0$ limit. The remaining non-trivial component of 3.53 is the τ component, and in a completely analogous manner as above it can be shown that near $\theta = \frac{}{2}$ we have

$$\left[\frac{1-p}{2}\delta\left(\theta - \frac{\pi}{2}\right)\sigma_3 + F \quad \right]\zeta = 0. \tag{3.63}$$

We now write the KSE on this background to find the spinor solutions. Denoting the Killing spinors of R-charges ± 1 by $\zeta \equiv \zeta_+$, $\tilde\zeta \equiv \zeta$, we have

$$\begin{aligned}&(\nabla \ \mp iA \ )\zeta \ = -\frac{1}{2}H\gamma \ \zeta \ \Rightarrow \\ \Rightarrow &\left(\partial \ + \frac{i}{2}\gamma \ - \frac{i}{2}((q-1)\delta \ ; \ + (p-1)\delta \ ; \ )\sigma_3 \mp \frac{i}{2}((q-1)\delta \ ; \ + (p-1)\delta \ ; \ )\right)\zeta \ = \frac{i}{2}\gamma \ \zeta \ .\end{aligned} \tag{3.64}$$

We can find a 2-parameter family of solutions given by constant spinors which satisfy $\sigma_3\zeta \ = \mp\zeta$, and are thus parametrized as

$$\zeta_+ = \begin{pmatrix} 0 \\ c_+ \end{pmatrix}, \quad \zeta \ = \begin{pmatrix} c \\ 0 \end{pmatrix}. \tag{3.65}$$

This is because the previous action of σ_3 on the Killing spinors leads to the cancelation of the term in the spin connection dependent on p and q with the components of A , resulting in

$$\begin{aligned}&\left(\partial \ + \frac{i}{2}\gamma \ \pm \frac{i}{2}((q-1)\delta \ ; \ + (p-1)\delta \ ; \ )\mp \frac{i}{2}((q-1)\delta \ ; \ + (p-1)\delta \ ; \ )\right)\zeta \ = \\ &= \partial \ \zeta \ + \frac{i}{2}\gamma \ \zeta \ = \frac{i}{2}\gamma \ \zeta \ .\end{aligned} \tag{3.66}$$

On the other hand, we can also find solutions which are not constant, taking the form

$$\zeta_+ = \begin{pmatrix} c_+^1 e^{i((q \ \ 1) \ \ +(p \ \ 1) \ )} \\ c_+^2 \end{pmatrix}, \quad \zeta \ = \begin{pmatrix} c^1 \\ c^2 \ e \ \ ^{i((q \ \ 1) \ \ +(p \ \ 1) \ )} \end{pmatrix}. \tag{3.67}$$

This is because

$$\begin{aligned}&\left(\partial \ + \frac{i}{2}\gamma \ - \frac{i}{2}((q-1)\delta \ ; \ + (p-1)\delta \ ; \ )(\sigma_3+1)\right)\zeta_+ = \\ &= \begin{pmatrix}(i((q-1)\delta \ ; \ + (p-1)\delta \ ; \ ) - i((q-1)\delta \ ; \ + (p-1)\delta \ ; \ ))c_+^1 e^{i((q \ \ 1) \ \ +(p \ \ 1) \ )} \\ 0\end{pmatrix} + \frac{i}{2}\gamma \ \zeta_+ \\ &= \frac{i}{2}\gamma \ \zeta_+,\end{aligned} \tag{3.68}$$

and a similar cancellation occurs for ζ . Having found four independent Killing spinors, we arrive at a contradiction with the results of [54]. Namely, a generic background preserving four independent supercharges must satisfy the following restriction implied by the integrability condition:

$$\partial \ (A \ - V \ ) - \partial \ (A \ - V \ ) = 0. \tag{3.69}$$

Since our R-symmetry gauge field di□ers from $V \ = 0$ not by a flat connection but by a singular one, the above is not satisfied. This means that the branched sphere 3.47 is not an appropriate space to preserve supersymmetry in the presence of the chosen supergravity background.

This motivates us to consider a regularized version of this space,

$$ds^2 = \frac{1}{f(\theta)}d\theta^2 + q^2\sin^2\theta d\theta^2 + p^2\cos^2\theta d\phi^2, \quad f(\theta) = \begin{cases} \frac{1}{q^2}, & \theta \to 0 \\ \frac{1}{p^2}, & \theta \to \frac{\ }{2} \\ 1, & \epsilon < \theta < \frac{\ }{2} - \epsilon, \end{cases} \tag{3.70}$$

as well as a regularized supergravity background,

$$H = -i\sqrt{f(\theta)}, \quad A = \frac{q\sqrt{f(\theta)} - 1}{2}\mathrm{d}\tau + \frac{p\sqrt{f(\theta)} - 1}{2}\mathrm{d}\phi, \quad V = 0. \tag{3.71}$$

The new spin connection di￿ers from the previous one by replacing $p \to \sqrt{f}\,p$, $q \to \sqrt{f}\,q$. The KSE becomes

$$\begin{aligned} &(\nabla \ \mp iA\ )\zeta \ = -\frac{1}{2}H\gamma\ \zeta \ \Rightarrow \\ &\Rightarrow \Big(\partial \ + \frac{i}{2}\gamma \ - \frac{i}{2}\Big((q\sqrt{f} - 1)\delta\ ; \ + (p\sqrt{f} - 1)\delta\ ;\ \Big)\sigma_3 \mp \\ &\quad \mp \frac{i}{2}\Big((q\sqrt{f} - 1)\delta\ ; \ + (p\sqrt{f} - 1)\delta\ ;\ \Big)\Big)\zeta = \frac{i}{2}\sqrt{f}\gamma\ \zeta\ . \end{aligned} \tag{3.72}$$

Two constant spinors of the form 3.65 are still preserved. Although we can still have two independent solutions of the form 3.68 by doing the replacements above the previous equation, no contradiction with [54] arises this time, since the field strength for $A\ \ - V$ is no longer singular, but rather everywhere vanishing.

To summarise: in order to rest the restrictive condition $\nabla\ \zeta = 0$ while preserving supersymmetry, we must deform the theory by adding couplings between conserved currents of the field theory and certain background gauge fields, but these must belong to a supermultiplet which contains g . This is why we consider such background fields as embedded in a supergravity multiplet. This means that we can tune either A ,V or H in order to cancel the singularities from the Ricci scalar in the integrability condition, leading to globally defined Killing spinors on the branched sphere.

3.4 Supersymmetric Localization

We have established that there exist preserved supercharges on the branched sphere, which is an essencial requirement if we wish to localize supersymmetric gauge theories on this space. Let us now review the basics about supersymmetric localization.

Suppose we want to compute the partition function of a theory on a spacetime in Euclidean signature with action S,

$$Z = \int \mathcal{D}\varphi e^{\ S[']}, \tag{3.73}$$

where φ denotes the collection of fields of the theory. Suppose now that the theory enjoys a fermionic symmetry generated by an operator $\mathcal{Q}$, which may be viewed as a di￿erential operator in superspace coordinates. In particular, either $\mathcal{Q}^2 = 0$ or $\mathcal{Q}$ squares to some bosonic symmetry of the theory, δ_B. Consider now the one-parameter deformation of the partition function

$$Z(t) = \int \mathcal{D}\varphi e^{\ S[']\ \ tQV[']}, \tag{3.74}$$

where $V[\varphi]$ is some functional of the fields. We assume that $V[\varphi]$ is invariant under the bosonic symmetry transformations of S, $\delta_B V[\varphi] = 0$. Also, the fact that $\mathcal{Q}$ generates a symmetry of the theory means that $\mathcal{Q}S[\varphi] = 0$. Under small deformations of the parameter t, the variation of the partition function is given by

$$\begin{aligned}\frac{\partial Z}{\partial t} &= -\int \mathcal{D}\varphi \mathcal{Q}V[\varphi] e^{-S[\varphi]-tQV[\varphi]} \\ &= -\int \mathcal{D}\varphi \mathcal{Q}\left(V[\varphi] e^{-S[\varphi]-tQV[\varphi]}\right).\end{aligned} \tag{3.75}$$

By definition, a symmetry of the action is a field redefinition $\varphi \to \varphi' = \mathcal{Q}\varphi$ such that $S[\varphi] = S[\varphi']$; we also require an anomaly free measure, meaning that $\mathcal{D}\varphi = \mathcal{D}\varphi'$. The action of $\mathcal{Q}$ is implemented as $S[\varphi'] = S[\varphi] + \mathcal{Q}S[\varphi]$ and $V[\varphi'] = V[\varphi] + \mathcal{Q}V[\varphi]$. We then have

$$\begin{aligned}\int \underbrace{\mathcal{D}\varphi'}_{=\mathcal{D}\varphi} V[\varphi'] e^{-S[\varphi']-tQV[\varphi']} &= \int \mathcal{D}\varphi (V[\varphi] + \mathcal{Q}V[\varphi]) e^{-S[\varphi]-tQ(V[\varphi]+QV[\varphi])} \\ &= \int \mathcal{D}\varphi (V[\varphi] + \mathcal{Q}V[\varphi]) e^{-S[\varphi]-tQV[\varphi]}.\end{aligned} \tag{3.76}$$

The first term in parenthesis is equal to the first term on the left-hand side, giving

$$\begin{aligned}&\int \mathcal{D}\varphi \mathcal{Q}V[\varphi] e^{-S[\varphi]-tQV[\varphi]} = 0 \Rightarrow \\ \Rightarrow &\int \mathcal{D}\varphi \mathcal{Q}\left(V[\varphi] e^{-S[\varphi]-tQV[\varphi]}\right) = 0 \Rightarrow \frac{\partial Z}{\partial t} = 0.\end{aligned} \tag{3.77}$$

This establishes that $Z(t)$ is independent of the parameter t. Exactly the same argument goes through when computing correlation functions (in the t deformed theory) of invariant operators under the fermionic symmetry,

$$\mathcal{Q}\mathcal{O} = 0 \Rightarrow \frac{\partial}{\partial t}\int \mathcal{D}\varphi \mathcal{O} e^{-S[\varphi]-tQV[\varphi]} = 0. \tag{3.78}$$

This means that the partition function or correlation functions such as the above can be computed in the limit $t \to +\infty$, in which case contributions to the path integral coming from field configurations φ for which $\mathcal{Q}V[\varphi] \neq 0$ are exponentially suppressed. However, for this argument to work it is crucial that the bosonic part of $\mathcal{Q}V[\varphi]$ is a positive definite functional. A weaker condition which does the job is that $\mathcal{Q}V[\varphi] \geq 0$ along the contour of the path integral in field space. We will see later that we can construct such positive definite terms in $3d$ $\mathcal{N} = 2$ theories which correspond to physically relevant actions. Conversely, such a term does not exist for every field theory in arbitrary dimensions or with arbitrary amount of supercharges.

Call $\{\varphi_0\}$ the set of field configurations for which $\mathcal{Q}V[\varphi_0] = 0$. In our cases of interest, there will be finitely many configurations. The contributions to the path integral which are not infinitely suppressed will then come from the action $S[\varphi_0]$ evaluated at φ_0, as well as from an appropriate expansion of $t\mathcal{Q}V[\varphi]$ in which powers of t are eliminated. More precisely, there is a field contour for which the contribution from field fluctuations around φ_0 can be evaluated exactly by a saddle point approximation. This is

$$\varphi = \varphi_0 + \frac{\tilde{\varphi}}{\sqrt{t}}, \tag{3.79}$$

and in the $t \to +\infty$ limit this leads to

$$S[\varphi] + \mathcal{Q}V[\varphi] = S[\varphi_0] + (\mathcal{Q}V)^{(2)}[\tilde{\varphi}] + \mathcal{O}(t^{-\frac{1}{2}}), \tag{3.80}$$

where $(\mathcal{Q}V)^{(2)}[\varphi] = \frac{\delta^2}{\delta\varphi^2}\mathcal{Q}V[\varphi]\Big|_{\varphi_0}\varphi^2$ is the quadratic expansion of the functional $\mathcal{Q}V$ around φ_0. The partition function in this limit is then

$$\begin{aligned} Z(t \to +\infty) &= \int d\varphi_0 \int \mathcal{D}\varphi e^{-S[\varphi_0] - (\mathcal{Q}V)^{(2)}[\varphi]} \\ &= \int d\varphi_0 e^{-S[\varphi_0]} \frac{1}{\mathrm{SDet}'(\mathcal{Q}V)^{(2)}_{\varphi_0}}, \end{aligned} \tag{3.81}$$

where SDet$'$ denotes the superdeterminant of $(\mathcal{Q}V)^{(2)}$, *i.e.* the ratio of determinants of fermionic and bosonic parts of the quadratic operator, with the zero modes removed. Note that the superdeterminant is a functional of the zero modes $\{\varphi_0\}$. This is because the path integral over the fluctuations of the fields around $\{\varphi_0\}$ is gaussian, thanks to all terms in the expansion of $\mathcal{Q}V$ other than the quadratic one vanishing in this limit. In other words, the saddle point approximation of the path integral is *exact*.

We note that if φ is some BPS field configuration, meaning that $\delta_Q\varphi = 0$ under supersymmetry transformations, then we have $\mathcal{Q}V[\varphi_0] = 0$.[8] In the examples we will study here, the set $\{\varphi_0\}$ will always correspond to BPS configurations.

We now apply this to our theories of interest. Supersymmetric theories possess fermionic generators such as the $\mathcal{Q}$ above. These are generated by supercharges parametrized by the Killing spinors of the background geometry $\mathcal{M}$. It is important to stress that this is the reason why we need to analyse the supersymmetries that are preserved on $\mathcal{M}$ before proceding with localization. On the other hand, choosing a single Killing spinor suffices even if there is more than one preserved supercharge on $\mathcal{M}$. The supercharge we will use to localize on the branched sphere is

$$\delta_Q = \delta_\zeta + \delta_{\tilde{\zeta}}, \tag{3.82}$$

with $\zeta, \tilde{\zeta}$ as in 3.65.

3.4.1 $3d$ $\mathcal{N} = 2$ Supersymmetric Multiplets and Lagrangians

We will now construct the relevant superfields and corresponding Lagrangians. We follow [54]. In flat space, the components of a supermultiplet transform under the action of supersymmetry according to the flat space supersymmetry algebra described earlier in the chapter. Placing the theory on a curved manifold $\mathcal{M}$ leads to deformations of this algebra by terms which vanish in the flat space limit. Moreover, because the supersymmetric background on $\mathcal{M}$ is obtained from the rigid limit of a supergravity multiplet, then the SUSY algebra on $\mathcal{M}$ should also correspond to the rigid limit of an appropriate algebra of a supergravity theory. Given solutions $\zeta, \eta, \tilde{\zeta}, \tilde{\eta}$ of 3.42, and considering components $\varphi_{(r;z)}$ of a superfield of R-charge r and central charge z, the $\mathcal{N} = 2$ algebra on $\mathcal{M}_3$ was established to be

$$\begin{aligned} \{\delta_\zeta, \delta_{\tilde{\zeta}}\}\varphi_{(r;z)} &= -2i\left(\mathcal{L}'_K\varphi_{(r;z)} + \zeta\tilde{\zeta}(z - rH)\varphi_{(r;z)}\right), \\ \{\delta_\zeta, \delta_\zeta\}\varphi_{(r;z)} &= 0, \quad \{\delta_{\tilde{\zeta}}, \delta_{\tilde{\zeta}}\}\varphi_{(r;z)} = 0, \end{aligned} \tag{3.83}$$

[8] It turns out that the converse is also true, see footnote 14 of [51].

where $K = \zeta\gamma\ \zeta$ is a Killing vector for $\mathcal{M}$ (this can be seen by using the fact that $\zeta, \tilde{\zeta}$ satisfy the KSE) and $\mathcal{L}^0_K$ is a modified Lie derivative along K,

$$\mathcal{L}^0_K \varphi_{(r;z)} = \mathcal{L}_K \varphi_{(r;z)} - irK \left(A - \frac{1}{2} V \right) \varphi_{(r;z)} - izK\ C\ \varphi_{(r;z)}. \tag{3.84}$$

Indeed, when setting $H = A = C = V = 0$ we obtain the flat space superalgebra, with $\delta, \delta_{\tilde{}}$ given by a translation along K plus the eigenvalue under the central charge.

The most general irreducible representations of $\mathcal{N} = 2$ supersymmetry obeying the above algebra take the form of supermultiplets with 16 + 16 bosonic and fermionic (independent) components; we write them as

$$\mathcal{S} = (C, \chi, \tilde{\chi}, M, \tilde{M}, a, \sigma, \lambda, \tilde{\lambda}, D). \tag{3.85}$$

Because $[R, Q^I] = Q^I$, $[R, \tilde{Q}^I] = \tilde{Q}^I$ and $[Q^I, Z] = [\tilde{Q}^I, Z] = 0$, all fields in the multiplet have the same central charge z but varying R-charge relative to r (with $R(C) = 0$). $\mathcal{S}$ can be expanded in terms of $\mathcal{N} = 1$ superspace coordinates as[9]

$$\begin{aligned}\mathcal{S}(\theta, \tilde{\theta}, x) =& C + i\theta\chi + i\tilde{\theta}\tilde{\chi} + \frac{i}{2}\theta^2 M + \frac{i}{2}\tilde{\theta}^2 \tilde{M} + (\theta\gamma\ \tilde{\theta})a - i\theta\tilde{\theta}\sigma \\ &+ i\tilde{\theta}^2\theta\left(\lambda - \frac{i}{2}\gamma\ \partial\ \tilde{\chi}\right) - i\theta^2\tilde{\theta}\left(\tilde{\lambda} + \frac{i}{2}\gamma\ \partial\ \chi\right) - \frac{1}{2}\theta^2\tilde{\theta}^2\left(D + \frac{1}{2}\partial^2 C\right)\end{aligned} \tag{3.86}$$

This general $\mathcal{N} = 2$ multiplet must accomodate both matter fields and gauge vector fields (the highest helicity of the components of $\mathcal{S}$ is 1). These will be generalized chiral and vetor multiplets, whose components are deformed by supergravity background fields relative to their flat space counterparts. We can construct them by imposing distinct constraints on 3.85. Demanding consistency with 3.83 then fixes the form of the remaining components. We will be interested in the following multiplets:

Chiral/ Anti-chiral Multiplet: Given $\mathcal{S}$ with R-charge r and central charge z, imposing $\tilde{\chi} = 0$ or $\chi = 0$ leads, respectively, to a chiral multiplet

$$\begin{aligned} = \Big(&\phi, -\sqrt{2}i\psi, 0, -2iF, 0, -iD\ \phi, (z - rH)\phi, 0, 0 \\ &\frac{r}{4}(R - 2V\ V - 6H^2)\phi - (z - rH)H\phi\Big)\end{aligned} \tag{3.87}$$

and an anti-chiral multiplet

$$\begin{aligned}\tilde{} = \Big(&\tilde{\phi}, 0, \sqrt{2}i\tilde{\psi}, 0, -2i\tilde{F}, iD\ \tilde{\phi}, (z - rH)\tilde{\phi}, 0, 0 \\ &\frac{r}{4}(R - 2V\ V - 6H^2)\tilde{\phi} - (z - rH)H\tilde{\phi}\Big).\end{aligned} \tag{3.88}$$

Vector multiplet: A vector/ gauge multiplet $\mathcal{V}$ is a multiplet $\mathcal{S}$ of vanishing r and z charges, subject to the gauge freedom

$$\delta\mathcal{V} = \ + \tilde{}, \tag{3.89}$$

with $\ , \tilde{}$ chiral and anti-chiral superfields, respectively, with $r = z = 0$. The components $C, \chi, \tilde{\chi}, M, \tilde{M}$ of $\mathcal{V}$ can be set to zero by an appropriate choice of the

[9]See [54] for the explicit transformation rules of the components satisfying the algebra 3.83.

corresponding components of and $\tilde{}$. A gauge fixed form of $\mathcal{V}$ is then

$$\mathcal{V} = \Big(0,0,0,0,0,a \ ,\sigma,\lambda \ ,\tilde{\lambda} \ ,D\Big). \tag{3.90}$$

By looking at the bottom and vector components of $\ ,\tilde{}$, we see that if we choose $C \ = \frac{i}{2} \ ^{(a)}$ and $C_{\tilde{}} = -\frac{i}{2} \ ^{(a)}$, there exists a residual gauge freedom preserving the form of $\mathcal{V}$ and which transforms $\delta a \ = \partial \ ^{(a)}$. We have then constructed a gauge multiplet containing an abelian gauge field a .

Field strength multiplet: As in flat space, a gauge-invariant multiplet containing the field strenght of an abelian gauge field is a real linear multiplet. This is obtained by taking $\mathcal{S}$ of vanishing r and z charges by imposing $M = \tilde{M}$ and that its vector component is covariantly conserved. Imposing as well that the bottom component is the σ field from $\mathcal{V}$, the remaining components are fixed by the supersymmetry transformations of the components of $\mathcal{V}$ to be

$$= \Big(\sigma, i\tilde{\lambda} \ ,-i\lambda \ ,0,0,\frac{i}{2}\varepsilon \quad f \quad - V \ \sigma, -D-\sigma H, \frac{1}{2}H\lambda \ + i\gamma \ (D \ - \frac{1}{2}V \ \lambda \ , \frac{1}{2}H\tilde{\lambda} \ + i\gamma \ (D \ - \frac{1}{2}V \ \tilde{\lambda} \ , \frac{i}{2}V^{mu}\varepsilon \quad f \quad - H(D+\sigma H) - \nabla^2\sigma - V \ V \ \sigma\Big). \tag{3.91}$$

The fact that the field strength $f \ = \partial \ a \ - \partial \ a$ appears in this multiplet is a consequence of the following property satisfied by the fermions in the vector multiplet:

$$\delta \ _{+\tilde{}}\lambda = i\zeta(D+\sigma H) - \frac{i}{2}\varepsilon \quad \gamma \ \zeta f \quad + -\gamma \ \zeta(i\partial \ \sigma - V \ \sigma). \tag{3.92}$$

Background vector multiplets: We can also consider non-dynamical gauge fields contained in background vector multiplets coupled to a global flavour symmetry. The respective components need to explicitly saitisfy the BPS condition, that is, $\lambda = \tilde{\lambda} = 0$ and $\delta\lambda = \delta\tilde{\lambda} = 0$. From 3.92 we see that this implies

$$a \ = -\sigma C \ + a^{\text{flat}}, \ \ \partial \ \sigma = 0, \ \ D = -\sigma H. \tag{3.93}$$

As advocated earlier, we will see that these conditions are precisely those obeyed by the fields from $\mathcal{V}$ on the localization locus.

We can now construct supersymmetric Lagrangians constructed out of the multiplets presented above. These will be D-terms of superfields of vanishing r and z charge. In curved space, the superspace integral of such a D-term is not invariant under supersymmetry, contrary to the case in flat space, since

$$\delta \ _{+\tilde{}}D = -iV \ (\zeta\gamma \ \tilde{\lambda} + \zeta\gamma \ \lambda) - H(\zeta\tilde{\lambda} - \zeta\lambda) + \nabla \ (\zeta\gamma \ \tilde{\lambda} - \zeta\gamma \ \lambda). \tag{3.94}$$

Instead, we have the following curved space generalization of the D-term:

$$\mathcal{L}_D = -\frac{1}{2}(D - a \ V \ - \sigma H). \tag{3.95}$$

This is indeed supersymmetric:

$$\begin{aligned}\delta\mathcal{L}_D &= -\frac{1}{2}\Big\{D\ (\xi\gamma\ \tilde{\lambda} - \tilde{\xi}\gamma\ \lambda) - iV\ (\xi\gamma\ \tilde{\lambda} + \tilde{\xi}\gamma\ \lambda) - H(\xi\tilde{\lambda} - \tilde{\xi}\lambda) - \\ &- \Big(-i(\xi\gamma\ \tilde{\lambda} + \tilde{\xi}\gamma\ \lambda) + D\ (\xi\chi - \tilde{\xi}\tilde{\chi})\Big)V\ \ - \Big(-\xi\tilde{\lambda} + \tilde{\xi}\lambda\Big)H\Big\} \\ &= -\frac{1}{2}\Big\{\nabla\ (\xi\gamma\ \tilde{\lambda} - \tilde{\xi}\gamma\ \lambda) - \nabla\ (\xi\chi - \tilde{\xi}\tilde{\chi})V\ \ \Big\}.\end{aligned} \tag{3.96}$$

The remaining terms should be set to zero by requiring that ξ and $\tilde{\xi}$ satisfy the KSE. The relevant Lagrangians now follow from the multiplets discussed above:

FI terms: obtained by applying 3.95 to the multiplet $-2\xi\mathcal{V}$, giving

$$\mathcal{L}_{FI} = \xi(D - a\ \ V\ \ \ - \sigma H). \tag{3.97}$$

Note that $\mathcal{V}$ has $r = z = 0$, implying that $\mathcal{L}_{FI}$ is gauge-invariant up to a total spacetime derivative.

Gauge-gauge Chern-Simons terms: obtained by applying 3.95 to $\frac{k}{2}\mathcal{V}$, giving

$$\mathcal{L}_{CS} = \frac{k}{4\pi}(i\varepsilon\ \ \ \ a\ \partial\ a\ \ - 2D\sigma + 2i\tilde{\lambda}\lambda). \tag{3.98}$$

$\mathcal{L}_{CS}$ is explicitly gauge-invariant under abelian transformations as above. Note that this is equal to the usual flat space supersymmetrization of the Chern-Simons Lagrangian.

Yang-Mills Lagrangian: this is the quadratic action for the real linear multiplet , obtained by applying $\mathcal{L}_D$ to $-\frac{1}{g_{YM}}\ \ ^2$:

$$\begin{aligned}\mathcal{L}_{YM} = \frac{1}{2g_{YM}^2}\Big\{&\frac{1}{2}f\ \ \ f\ \ \ + D\ \sigma D\ \sigma - 2i\tilde{\lambda}\gamma\ (D\ \ + \frac{i}{2}V\ )\lambda + \\ &+ i\sigma\varepsilon\ \ \ V\ f\ \ \ - V\ \ V\ \sigma^2 - (D + \sigma H)^2 + iH\tilde{\lambda}\lambda\Big\}.\end{aligned} \tag{3.99}$$

Matter Lagrangian: this is the Kähler potential for the chiral superfield, $\mathcal{K} = \tilde{\ }\ $, where $R(\ ) = -R(\tilde{\ }) = r$, $Z(\ ) = -Z(\tilde{\ }) = z$, and is charged under the abelian gauge symmetry, with charge q. Appying $\mathcal{L}_D$ to $\mathcal{K}$ gives

$$\begin{aligned}\mathcal{L}_K = \mathcal{D}\ \tilde{\phi}\mathcal{D}\ \phi - i\tilde{\psi}\gamma\ \mathcal{D}\ \psi - \tilde{F}F + q(D + \sigma H)\tilde{\phi}\phi - 2(r-1)H(z - q\sigma)\tilde{\phi}\phi + \\ + \left((z - q\sigma)^2 - \frac{r}{4}R + \frac{1}{2}(r - \frac{1}{2})V\ V\ \ + r(r - \frac{1}{2})H^2\right)\tilde{\phi}\phi + \\ + \left(z - q\sigma - (r - \frac{1}{2})H\right)i\tilde{\psi}\psi + \sqrt{2}iq(\tilde{\phi}\lambda\psi + \phi\tilde{\lambda}\tilde{\psi}).\end{aligned} \tag{3.100}$$

When $z = 0$ and $r = \frac{1}{2}$, the fields are conformally coupled to the metric [55]. Note also the covariance of this Lagrangian with respect to all gauge symmetries. We can also note that real constant values of σ result in real mass terms for ϕ and ψ. This can be implemented by coupling to a background gauge multiplet, that is, by charging it under the corresponding global flavour symmetry. As discussed above, such gauge multiplets have constant σ-components.

Importantly for our purposes, we can show that a D-term Lagrangian of the form 3.95 is always Q-exact provided that $V = 0$. Consider the variation

$$\begin{aligned}\delta\, \delta_{\tilde{}} \left(-\frac{i}{2}\sigma - iHC\right)\Big|_{r=z=0} &= \delta \left(-\frac{i}{2}\zeta\lambda - iHi\zeta\chi\right) = -\frac{i}{2}\zeta\delta\,\lambda - iHi\zeta\delta\,\chi \\ &= \zeta\zeta\frac{1}{2}(D + \sigma H) + \frac{1}{2}\varepsilon \quad \zeta\gamma\,\zeta\nabla\ a \ + \frac{1}{2}\zeta\gamma\,\zeta\nabla\ \sigma + H\left(-\zeta\zeta\sigma - \zeta\gamma\,\zeta(\nabla\ C - ia\ )\right) \\ &= \zeta\zeta\frac{1}{2}(D - \sigma H) - iHK\ a\ - \frac{1}{2}\varepsilon \quad (\nabla\ K\ )a\ + \frac{1}{2}\nabla\ (\varepsilon \quad K\ a\ + K\ \sigma - HK\ C),\end{aligned} \tag{3.101}$$

in the fourth line we have used $[\zeta\gamma\,\zeta = \zeta\gamma\,\zeta]$. The total covariant derivative term vanishes when integrated over a manifold without boundary. Multiplying the KSE 3.42 on the left by $\zeta\gamma$ or $\zeta\gamma$ we can obtain an expression for $\nabla\ K$:

$$\begin{aligned}&\begin{cases}(\nabla\ - iA\ )\zeta = -\frac{1}{2}H\gamma\,\zeta - iV\,\zeta - \frac{1}{2}\varepsilon \quad V\,\gamma\,\zeta \\ (\nabla\ + iA\ )\zeta = -\frac{1}{2}H\gamma\,\zeta + iV\,\zeta + \frac{1}{2}\varepsilon \quad V\,\gamma\,\zeta\end{cases} \Rightarrow \\ &\Rightarrow \zeta\gamma\,(\nabla\ \zeta) + \zeta\gamma\,(\nabla\ \zeta) = -\frac{1}{2}H(\zeta\gamma\,\gamma\,\zeta + \zeta\gamma\,\gamma\,\zeta) - \varepsilon \quad V\,(\zeta\gamma\,\gamma\,\zeta - \zeta\gamma\,\gamma\,\zeta) \\ &\Rightarrow \nabla\ K\ - \zeta(\nabla\ \gamma\ )\zeta = -\frac{1}{2}H\zeta\underbrace{[\gamma\ ,\gamma\ ]}_{2i''_{\mu\rho\nu}\ \nu}\zeta + \varepsilon \quad V\,\zeta\underbrace{\{\gamma\ ,\gamma\ \}}_{\sigma_{\rho}}\zeta \\ &\Rightarrow \nabla\ K\ - \zeta(\nabla\ \gamma\ )\zeta = iH\varepsilon \quad K\ - \varepsilon \quad V\,\zeta\zeta \\ &\Rightarrow \nabla\ K\ = iH\varepsilon \quad K\ - \varepsilon \quad V\,\zeta\zeta,\end{aligned} \tag{3.102}$$

where we eliminated $\nabla\ \gamma\ = - \quad \gamma\ = - _{(\ \ )}\gamma$ by antisymmetrizing the LHS, since the RHS is proportional to ε which is antisymmetric in $(\mu\rho)$. Taking $V = 0$, we obtain

$$\begin{aligned}\delta\,\delta_{\tilde{}}\left(-\frac{i}{2}\sigma - iHC\right)\Big|_{r=z=0} &\sim \zeta\zeta\frac{1}{2}(D - \sigma H) - iHK\ a\ - \frac{1}{2}\varepsilon \quad (\nabla\ K\ )a \\ &= \zeta\zeta\frac{1}{2}(D - \sigma H) - iHK\ a\ - \frac{i}{2}H\underbrace{\varepsilon \quad \varepsilon}_{2\ \nu_{\kappa}}K\ a \\ &= \zeta\zeta\frac{1}{2}(D - \sigma H) = \zeta\zeta\mathcal{L}_D\Big|_{V_\mu=0}.\end{aligned} \tag{3.103}$$

By the localization argument this guarantees that, choosing $V = 0$, supersymmetric observables are independent of couplings of D-terms of superfields with $r = z = 0$, and they can be localized on the corresponding zero modes.

3.4.2 Localizing 3d $\mathcal{N} = 2$ theories on the branched sphere

We choose $\mathcal{L}_K$ and $\mathcal{L}_{YM}$ as the localizing terms (striping g_{YM}^2 out of $\mathcal{L}_{YM}$):

$$\mathcal{L}_{\text{loc}} = \frac{1}{g_{YM}^2}\mathcal{L}_{YM} + \frac{1}{g_K^2}\mathcal{L}_K. \tag{3.104}$$

For $V = 0$, the bosonic part of $\mathcal{L}_{\text{loc}}$ is positive definite. We will then compute the partition function in the limit $g_{YM}, g_K \to 0$. This corresponds to embedding the 3d $\mathcal{N} = 2$ gauge theory in the IR limit of an RG flow from a UV free theory, the latter corresponding precisely to the limit $g_{YM}, g_K \to 0$ of the IR theory.[10] Depending on the central charge and R-charge of the chiral multiplets, the UV theory can be massless and conformally coupled to the curvature of the background $\mathcal{M}$.

The localization locus is obtained by setting the bosonic part of $\mathcal{L}_{\text{loc}}$ to zero. Since this is a sum of positive definite terms, each must vanish separately, giving

$$\begin{aligned} \{\varphi_0\}_V &= \{a \ \ , \sigma, D \,|\, f \ \ \ = 0, \ \ D \ \ \sigma = 0, \ \ D = -H\sigma\}, \\ \{\varphi_0\} \ \ &= \{\phi = F = 0\}. \end{aligned} \tag{3.105}$$

On a smooth and simply connected manifold, a flat gauge connection has zero holonomy around any loop, so we can set $a \ \ = 0$. This implies for the adjoint scalar σ that

$$D \ \ \sigma = 0 \iff \partial \ \ \sigma - i[a \ \ , \sigma] = 0 \iff \partial \ \ \sigma = 0 \iff \sigma = \sigma_0, \tag{3.106}$$

where σ_0 is constant Lie algebra-valued matrix. It follows that D is constant as well, with $D = -H\sigma_0$. This means that the integration over BPS configurations is given by an integration over Lie algebra-valued constant modes:

$$\int d\varphi_0 \equiv \int_{\text{Lie}(G)} [d\sigma_0] = \int \prod_{i=1}^{\text{rank}G} \frac{(d\sigma_0)_i}{\text{Vol}(G)}. \tag{3.107}$$

Following [25], we will now compute the one-loop determinant for the quadratic fluctuations of the matter fields, $Z^{\text{1-loop}}_{\text{matter}} = (\text{SDet}^0(\mathcal{L}_K)^{(2)}) \ \ ^1$ on the regularized branched sphere. The result on S^3 will follow from the special case $q = p = 1$. The gauge one-loop determinant is computed in a similar way but is more involved, and we will not make use of its explicit form in this thesis.

We first perform an integration by parts to write $\mathcal{L}_K = \bar{\phi} \ \ \ \phi + \bar{\psi} \ \ \ \psi - \bar{F}F$, with the quadratic operators for the dynamical scalar and spinor given by

$$\begin{aligned} &= -\nabla^2 + r^2 A \ \ A \ \ + ((r-1)H + \sigma_0)^2 - H^2 - \frac{r}{4}(R - 6H^2), \\ &= -i\gamma \ \ (\nabla \ \ - i(r-1)A \ \ ) - i\left((r - \frac{1}{2})H + \sigma_0\right). \end{aligned} \tag{3.108}$$

Because we are working on a compact space, we will find a spectrum for ∇ and ∇ given by discrete eigenmodes. Given that the non-trivial coordinate dependence of the Laplacian on S^3 (or S^3_q) is in θ, we make use of the following ansatz for the scalar eigenmodes:

$$\phi(\theta, \tau, \phi) = e^{i(m \ \ +n \ \ )}\phi(\theta), \quad m, n \in \mathbb{Z}. \tag{3.109}$$

The eigenvalue problem for the scalar field, $\phi = \lambda_S \phi$, then takes the form

$$\phi^{00}(\theta) + \left(2\cot(2\theta) + \frac{f^0(\theta)}{2f(\theta)}\right)\phi^0(\theta) + \left(\ \ _1 - \frac{\ \ _2^2}{\sin^2\theta} - \frac{\ \ _3^2}{\cos^2\theta}\right)\phi(\theta) = 0, \tag{3.110}$$

with

[10] Recall that Yang-Mills theories are asymptotically free, hence the identification of the IR and UV limits of the theory with the strong and weak coupling limits, respectively.

$$
\begin{aligned}
{}_1 &\equiv \frac{\lambda^2}{f(\theta)} - 1 - \frac{(\sigma_0 - i(r-1)\sqrt{f(\theta)})^2}{f(\theta)} + \frac{f^0(\theta)}{2f(\theta)} r\cot(2\theta),\\
{}_2 &\equiv \frac{2m - r(q\sqrt{f(\theta)}-1)}{2q}, \qquad {}_3 \equiv \frac{2n - r(p\sqrt{f(\theta)}-1)}{2p}.
\end{aligned}
\tag{3.111}
$$

For the spinor we use the following ansatz:

$$
\psi(\theta,\tau,\phi) = e^{i(m\ \ +n\ \ )}\begin{pmatrix}\psi_1(\theta)\\ e^{i(\ \ +\ \ )}\psi_2(\theta)\end{pmatrix}, \quad m, n \in Z, \tag{3.112}
$$

and the corresponding eigenvalue problem, $\psi = \lambda_f \psi$, is

$$
\begin{aligned}
&\psi_1^0(\theta) + (r\cot(2\theta) + c_1\tan\theta - c_2\cot\theta)\psi_1(\theta) + \left(\frac{\lambda_f + i\sigma_0}{\sqrt{f(\theta)}} + c_1 + c_2\right)\psi_2(\theta) = 0,\\
&\psi_2^0(\theta) + ((2-r)\cot(2\theta) - c_1\tan\theta + c_2\cot\theta)\psi_2(\theta) + \left(-\frac{\lambda_f + i\sigma_0}{\sqrt{f(\theta)}} + 2 - 2r + c_1 + c_2\right)\psi_1(\theta) = 0,
\end{aligned}
\tag{3.113}
$$

with

$$
c_1 \equiv \frac{2n + r}{2p\sqrt{f(\theta)}}, \quad c_2 \equiv \frac{2m + r}{2q\sqrt{f(\theta)}}. \tag{3.114}
$$

When both $\psi_1 \neq 0$ and $\psi_2 \neq 0$, we can solve for ψ_2 in the first equation. Replacing ψ_2 in the second one gives us a second order differential equation for ψ_1:

$$
\psi_1^{00}(\theta) + \left(2\cot(2\theta) + \frac{f^0(\theta)}{2f(\theta)}\right)\psi_1^0(\theta) + \left(\tilde{}_1 - \frac{{}_2^2}{sin^2\theta} - \frac{{}_3^2}{\cos^2\theta}\right)\psi_1(\theta) = 0. \tag{3.115}
$$

We see that $\psi_2(\theta)$ satisfies the same differential equation as $\phi(\theta)$, with ${}_1 \leftrightarrow \tilde{}_1$ and

$$
\tilde{}_1 \equiv \frac{(\lambda_f + (r-1)\sqrt{f(\theta)} + i\sigma_0)^2}{f(\theta)} - 1 + \frac{f^0(\theta)}{2f(\theta)} r\cot(2\theta). \tag{3.116}
$$

We then impose that, due to supersymmetry, the eigenvalues of these non-trivial momentum eigenfunctions match those of the scalar eigenfunctions; to 0-th order in ϵ, we have the relation between scalar and fermion eigenvalues

$$
{}_1 = \tilde{}_1 \iff \lambda_s = \lambda_f(\lambda_f + 2(r-1) + 2i\sigma_0). \tag{3.117}
$$

This means that, given a scalar eigenvalue λ_s, there exist two distinct eigenvalues λ_f of spinor eigenfunctions with $\psi_1, \psi_2 \neq 0$, satisfying $-\lambda_f^+\lambda_f = \lambda_s$, where $\lambda_f = -\lambda_f^+ - 2(r-1) - 2i\sigma_0$. This can also be seen by noticing that, from 3.116, if λ_f is a solution of the spinor eigenvalue problem, which exists if and only if there exists λ_s solving the analogous scalar eigenvalue problem, then $-\lambda_f - 2(r-1) - 2i\sigma_0$ is also a solution to the same equation. In particular, this is the case if and only if $\psi_1, \psi_2 \neq 0$. From this we conclude that in the superdeterminant of $\mathcal{L}_K$ there is a cancellation of scalar eigenvalues with all spinor eigenvalues whose corresponding eigenfunctions satisfy $\psi_1, \psi_2 \neq 0$.

The remaining eigenvalues contributing to this superdeterminant will come from spinor eigenmodes with either $\psi_1 \neq 0$, $\psi_2 = 0$ or $\psi_1 = 0$, $\psi_2 \neq 0$. In order for 3.113

to be satisfied we must have, respectively (again, to 0-th order in ϵ),

$$\begin{aligned}
\lambda_f^{(1)} &= \frac{n}{p} + \frac{m}{q} - i\sigma_0 + \frac{r}{2}\left(\frac{1}{p} + \frac{1}{q}\right) - 2(r-1), \\
\lambda_f^{(2)} &= -\frac{n}{p} - \frac{m}{q} - i\sigma_0 - \frac{r}{2}\left(\frac{1}{p} + \frac{1}{q}\right).
\end{aligned} \tag{3.118}$$

These are obtained by setting either ψ_1 or ψ_2 to zero in the conditions 3.113; one equation gives the above conditions on the eigenvalues, and the other gives a first order di erential equation for the non-zero modes whose solution yields regularity conditions at $\theta = 0$ and $\theta = \frac{}{2}$, resulting in $m > -1, n > -1$ for $\lambda_f^{(1)}$ and $m < 0, n < 0$ for $\lambda_f^{(2)}$.

There is still some pairing left to consider between scalar and spinor eigenvalues. This is because 3.115 is still valid when $\psi_1 \neq 0$, $\psi_2 = 0$, and the matching with the solution to the corresponding scalar di erential equation gives

$$\lambda_S = \lambda_f^{(1)}(\lambda_f^{(1)} + 2(r-1) + 2i\sigma_0). \tag{3.119}$$

On the other hand, $\lambda_f^{(2)}$ has no pairing with scalar eigenvalues. The chiral 1-loop determinant then reads

$$\begin{aligned}
Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, q, p) &= \frac{\det}{\det} = \prod_{\substack{(1) \\ f}\,;\,\substack{(2) \\ f}} \frac{\lambda_f^{(1)}\lambda_f^{(2)}}{\lambda_f^{(1)}(\lambda_f^{(1)} + 2(r-1) + 2i\sigma_0)} \\
&= \prod_{m;n\ \ 0} \frac{\frac{n}{p} + \frac{m}{q} - i\sigma_0 - \frac{r\ 2}{2}\left(\frac{1}{p} + \frac{1}{q}\right)}{\frac{n}{p} + \frac{m}{q} + i\sigma_0 + \frac{r}{2}\left(\frac{1}{p} + \frac{1}{q}\right)} \\
&= {}_h\left(-\sigma_0 + \frac{ir}{2}\left(\frac{1}{p} + \frac{1}{q}\right); \frac{i}{p}, \frac{i}{q}\right)
\end{aligned} \tag{3.120}$$

If we set $p = q = 1$, we obtain the 1-loop determinant for a chiral multiplet on S^3 (see for instance $(A.2.12)$ of [56]).

In the next chapter we will make use of this expression to compute the partition function of $U(1)$ chiral multiplets coupled to Chern-Simons interactions. Here we just note the result for a free chiral multiplet. Setting $p = 1$, the partition function on the branched sphere is

$$\begin{aligned}
Z^{\text{1-loop}}_{\text{matter}}(q) &= {}_h\left(\frac{ir}{2}\left(1 + \frac{1}{q}\right); i, \frac{i}{q}\right) \\
&= \exp\left(\int_0^{+1} \frac{dx}{x}\left(\frac{(1-r)\omega}{x} - \frac{\sinh(2x\omega(1-r))}{2\sin(\sqrt{q}x)\sin(\sqrt{q^{\ 1}}x)}\right)\right),
\end{aligned} \tag{3.121}$$

where $\omega \equiv \frac{\sqrt{q}+\sqrt{q^{\ 1}}}{2}$. The lowest order term in an expansion around $q = 1$ is of order $\mathcal{O}((q-1)^2)$. Therefore, the entanglement entropy is simply determined by (minus) the free energy,

$$S^{\text{free chiral}} = \log Z^{\text{1-loop}}_{\text{matter}}(q = 1) = -\frac{1}{2}\log 2, \tag{3.122}$$

This is in agreement with the well known fact that, in a conformal field theory, the universal term of the entanglement entropy across a spherical $S^{d\ 2}$ region is equivalent to the Euclidean free energy on the S^d for odd d, as noted in 2.38. This means that there are no non-trivial contributions from the branching when performing the replica trick on a CFT. Furthermore, it is interesting to note that this result is reproduced by the sum of the regularized free energies of one free scalar and one free fermion on the three-sphere. In [34], these are determined to be

$$\begin{aligned} S^{\text{fermion}} &= -\frac{1}{4}\log 2 - \frac{3\zeta(3)}{8\pi^2} \\ S^{\text{scalar}} &= -\frac{1}{4}\log 2 + \frac{3\zeta(3)}{8\pi^2}. \end{aligned} \tag{3.123}$$

4 Entanglement Along Renormalization Group Flows

In this chapter we analyse several examples of relevant deformations of the superconformal theories considered in the previous chapter which break conformal invariance. Such deformations will introduce a mass scale in the theory and induce an RG flow into the IR. We will be interested in answering the following:

1. In which cases is it possible to use localization and the proposal of [25] to compute Rényi entropy of $3d$ $\mathcal{N} = 2$ supersymmetric theories which do not preserve conformal invariance?

2. What is the behaviour of the entanglement entropy along the RG flow? Is it consistent with the ones observed in non-supersymmetric theories?

4.1 Superconformal transformations of relevant perturbations

4.1.1 Setup

We want to consider deformations of the Lagrangian for $\mathcal{N} = 2$ super Yang-Mills in flat Minkowski space which break conformal invariance. Since we rely on mapping the theory to S^3 by a conformal transformation, we wish to determine what form these deformations will take after performing the conformal transformation.

The deformations will be formulated in terms of superfields, so in order to do this precisely we need to determine the form of our transformation not only in spacetime coordinates but in *all of superspace coordinates*. This will allow us to determine the conformal factor (depending on $(x\ ,\theta\ )$) which implements the transformation properties of quasi-primary superfields. From this we can write the action of the deformed theory on S^3. We use on the formalism developed in [57].

Assuming time translation symmetry, we first Wick rotate the metric of three-dimensional Minkowski space to Euclidean time. The conformal transformation of interest will be the Casini-Huerta-Myers (CHM) map [9] introduced in 2.2, which maps the causal development of the two-dimensional ball of radius R to S^3. Starting from the flat space metric

$$ds^2 = dt^2 + d\rho^2 + \rho^2 d\phi^2, \tag{4.1}$$

we apply the sequence of coordinate transformations presented in 2.2, which we state here once more for convenience:

$$w = \rho + it \longrightarrow e^{\ (u+i\ =q)} = \frac{R-w}{R+w} \longrightarrow \sinh u = \cot\theta. \tag{4.2}$$

This results in the following transformations of the line element:

$$
\begin{aligned}
ds^2 &= dt^2 + d\rho^2 + \rho^2 d\phi^2 \to \\
&\to \left(\frac{R}{\cosh u + \cos\tau}\right)^2 \left[d\tau^2 + du^2 + \sinh^2 u d\phi^2\right] \\
&\to \left(\frac{R}{\cosh u(\theta) + \cos\tau}\right)^2 \left(\frac{1}{\sin^2\theta}\right)\left[d\theta^2 + q^2\sin^2\theta d\tau^2 + \cos^2\theta d\phi^2\right],
\end{aligned} \tag{4.3}
$$

We will look at two kinds of supersymmetric actions which break conformal invariance. The simplest such deformation is a Fayet-Iliopulos (FI) term. Consider the expansion of the $\mathcal{N} = 2$ vector multiplet 3.89 in Wess-Zumino gauge in terms of $\mathcal{N} = 1$ superspace coordinates,

$$V_{WZ}(x,\theta,\bar\theta) = \theta\sigma\ \bar\theta a\ (x) - i\theta\bar\theta\sigma - i\theta^2\bar\theta\bar\lambda(x) + i\bar\theta^2\theta\lambda(x) - \frac{1}{2}\theta^2\bar\theta^2 D(x). \tag{4.4}$$

One can construct a gauge invariant action out of the D-term of the vector superfield[1]: On flat space,

$$\int_{\mathbb{R}^3} d^3x \mathcal{L}_{FI}(x) = 2\xi \int_{\mathbb{R}^{3|2}} d^3x d^2\theta d^2\bar\theta V_{WZ}(x,\theta,\bar\theta) = \xi\int_{\mathbb{R}^3} d^3x D(x). \tag{4.5}$$

The second relevant deformation to be considered is a mass term for the matter fields. However, this will require a di erent treatment which we postpone to 4.3.

4.1.2 Superconformal transformations of quasi-primary superfields

We will now learn how superfields in a SCFT transform under superconformal transformations. In the superconformal theory, we assume the existence of quasi-primary superfields which, under $z = (x,\theta,\bar\theta) \to z' = (x',\theta',\bar\theta') \equiv g(z)$, transform according to

$$(z) \to {}'(z') = {}^J(z) D_J{}^I(z), \tag{4.6}$$

with

$$D_J{}^I(z) = D\ \ (\ \ ; g)\ \ (z)\ \ , \tag{4.7}$$

where $D\ \ (\ \ ; g)$ implements the action of the Lorentz group on the field according to its Lorentz representation, and is the conformal weight of (see A for details on representations of the superconformal algebra). We consider scalar superfields. One important aspect to note is that supersymmetric actions remain supersymmetric under these superconformal transformations, by virtue of expressing them in superspace and not only in spacetime. For example, an infinitesimal supersymmetry transformation on the FI action written after performing a superconformal transformation takes the

[1] This is only possible when the gauge group of the theory contains $U(1)$ factors. V_{WZ} is the vector multiplet whose components are in the adjoint representation of these abelian factors. Gauge invariance follows from the fact that a supersymmetric gauge transformation is implemented as

$$V^I_{WZ}\ !\ V^I_{WZ}\ \ i\ \ + i\tilde{\ },$$

(I labels the abelian factors; , ~ are chiral and anti-chiral, respectively) under which the D-term transforms as a total spacetime derivative, $D^I\ !\ D^I + @_\mu @^\mu(:::)$.

form

$$\delta \int d^3x^0 d^4\theta^0 \underbrace{\quad (g \;\; ^1 z^0) V^0_{WZ}(z^0)}_{=V_{WZ}(g \;\; ^1 z)} = i\zeta Q \int d^3x d^4\theta \quad (g \;\; ^1 z^0) V^0_{WZ}(z^0)$$

$$= i\zeta \left(-i\partial \;\; - \gamma \quad \theta \bar{} \partial \; \right) \int d^3x d^4\theta \;\; (g \;\; ^1 z) \;\; V_{WZ}(z), \tag{4.8}$$

which is zero up to spacetime boundary terms.

Infinitesimally, the transformation of the field evaluated at z is obtained by expanding to first order both sides of 4.6:

$$\delta \quad (z) = -(\mathcal{L} + \quad \hat{\lambda}(z)) \quad (z), \quad \mathcal{L} = h \;\; \partial \;\; - \delta\theta \;\; D \tag{4.9}$$

where $h \;\; = \delta x \;\; - i\bar{\theta}\gamma \;\; \delta\theta$, and $\mathcal{L}$ is the generator of infinitesimal superconformal transformations (if we ignore θ dependence it corresponds to the usual generator of infinitesimal transformations on spacetimes fields, $\delta x \;\; \partial$). $\hat{\lambda}(z) = \lambda + 2x \cdot b - 2\bar{\theta}\rho$ contains the possible scaling coe□cients coming from dilations and special conformal transformations, parametrized by the scalar λ, the vector b and the spinor ρ .

As an example, let us work the transformation of the FI action under an infinitesimal superspace dilation. The superspace coordinates transform as

$$z \to z + \quad z \iff \delta x \;\; = \;\; x \;\; , \; \delta\theta \;\; = \frac{1}{2} \;\; \theta \;\; . \tag{4.10}$$

Keeping only the resulting D-term, we compute

$$\begin{aligned}
[\mathcal{L} V_{WZ}(x,\theta)]\Big|_{\,2\,\bar{}2} &= \left\{ \left(\delta x \;\; - i\bar{\theta}\gamma \;\; \delta\theta \right) \partial \;\; - \delta\theta \;\; \left(-\partial \;\; + i(\bar{\theta}\gamma \;\;) \;\; \partial \;\; \right) \right\} \times \\
&\times \left(\theta\sigma \;\; \bar{\theta} v \;\; (x) + i\theta^2 \bar{\theta}\bar{\lambda}(x) - i\bar{\theta}^2 \theta\lambda(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x) \right) \Big|_{\,2\,\bar{}2} \\
&= \frac{1}{2} \;\; x \;\; \partial \;\; D(x) - \frac{1}{4} \;\; \theta \;\; \partial \;\; (\theta^2\bar{\theta}^2 D(x)) - \frac{i}{2} \left(\bar{\theta}\gamma \;\; \theta + \theta \;\; (\bar{\theta}\gamma \;\;) \;\; \right) \partial \;\; \theta\gamma \;\; \bar{\theta} v \;\; (x) \\
&= \frac{1}{2} \;\; x \;\; \partial \;\; D(x) - \frac{1}{2} \;\; D(x),
\end{aligned} \tag{4.11}$$

where we have used the identities

$$\bar{\theta}\gamma \;\; \theta = \bar{\theta}\bar{}(\gamma \;\;) \quad \theta \;\; = -\theta \;\; \bar{\theta}\bar{}(\gamma \;\;) \quad = -\theta \;\; (\bar{\theta}\gamma \;\;) \tag{4.12}$$

and

$$\frac{\partial}{\partial\theta}\theta^2 = \varepsilon \quad \frac{\partial}{\partial\theta}\theta \;\; \theta \;\; = \varepsilon \;\; (\delta \;\; \theta \;\; - \theta \;\; \delta \;\;) = \varepsilon \;\; \theta \;\; - \varepsilon \;\; \theta \;\; = 2\theta \;\; . \tag{4.13}$$

The transformation of the vector superfield is then

$$\begin{aligned}
V^0_{WZ}(z^0) &= \left[V_{WZ}(g \;\; ^1 z^0) + \delta V_{WZ}(g \;\; ^1(z^0)) \right] \Big|_{\,2\;2} = \\
&= (1 + \quad - \quad) D((1 - \quad)x^0) + \;\; x^0 \;\; \partial \;\; D((1 - \quad)x^0) + \mathcal{O}(\;\; ^2).
\end{aligned} \tag{4.14}$$

Up to boundary terms, the FI term is mapped according to (the superspace integral measure is invariant)

$$\begin{aligned}\xi \int d^3x d^2\theta d^2\bar\theta V_{WZ}(z) &= \xi \int d^3x' d^2\theta' d^2\bar\theta' V'_{WZ}(z') \\ &= \xi \int d^3x \, \{(1 + \quad - \quad) D((1 - \quad)x) + \quad x \; \partial \; D((1 - \quad)x)\} \\ &\sim \xi \int d^3x \, (1 + \quad - \quad) D(x),\end{aligned} \tag{4.15}$$

where the last relation holds up to boundary terms. We see that already in this simple example of an infinitesimal dilation there is a non-trivial additional contribution coming from the action of the conformal transformation on the Grassman coordinates of superspace, as opposed to the usual spacetime dilation where the transformation of a scalar is of the form $D \to (1 + \quad)D$.

4.1.3 Finite superconformal transformation: the CHM map

We now want to work out the conformal factor (z), associated to our conformal transformation of interest, the CHM map. For this we decompose it into the fundamental finite conformal transformations, and write the conformal factor associated to each one. For this we consider the transformation under superconformal transformations of the invariant line element in superspace, $e \;(z) = dx \; - i\bar\theta\gamma \; d\theta$ [57]:

$$e \;(z') = e \;(z) R \;\;(z) = e \;(z) \left(\frac{\partial x'}{\partial x} - i\bar\theta'\gamma \; \frac{\partial \theta'}{\partial x} \right), \tag{4.16}$$

and analyse the the supersymmetric line element $e^2(z)$ changes under $z \to z'$ using

$$e^2(z') = \;\; {}^2(z) e^2(z) \tag{4.17}$$

The relevant finite transformations will be the following:

Finite supertranslations

The operator implementing the finite superconformal transformations is $e^{i(a_\mu P^\mu + Q_\alpha \;\; ^\alpha + \bar Q^\alpha \tilde{\;}_\alpha)}$. According to (2.15) of [57], the coordinate transformation that is consistent with the supersymmetry algebra is

$$\begin{cases} x' \;(x, \theta) = x \;\; + a \;\; + i\theta\gamma \;\; \lambda \\ \theta' \;(x, \theta) = \theta \;\; + \lambda \;, \end{cases} \tag{4.18}$$

which implies

$$R \;\;(z) = \delta \quad \Rightarrow e \;(z') = e \;(z). \tag{4.19}$$

This renders a trivial conformal factor.

Finite dilations

Because an infinitesimal dilation is implemented as

$$\begin{cases} \delta x \;\; = \lambda x \\ \delta\theta = \frac{1}{2}\lambda\theta, \end{cases} \tag{4.20}$$

we have $(x' \;, \theta' \;) = (e \;\; x \;, e^{\frac{\lambda}{2}}\theta \;\;)$ and thus $\;\;(z) = e \;\;$.

Finite superinversions

A superinversion is defined as $z \to z'$ such that

$$\mathrm{x}' \;\; = -\mathrm{x} \;\; ^{1}, \quad \theta' = i\mathrm{x} \;\; ^{1}\theta, \tag{4.21}$$

where $\mathrm{x} \;\; = x \;\; \gamma \;\; \pm \frac{i}{2}\bar{\theta}\theta 1$. The authors in [57] work out that the associated conformal factor is

$$\;\;(z) = \frac{1}{x^2 + \frac{1}{4}(\bar{\theta}\theta)^2}. \tag{4.22}$$

Determining $\;\;(z)$

We start from the flat space metric 4.1 in coordinates $w = \rho + it$, $\overline{w} = \rho - it$,

$$ds^2 = dwd\overline{w} + \left(\frac{w + \overline{w}}{2}\right)^2 d\phi^2. \tag{4.23}$$

Consider the transformation $\eta = \frac{R \;\; w}{R + w}$, where $\eta = e \;\;\;$. It is convenient to write $w = R + \delta w$, so that

$$\eta = -\frac{\delta w}{2R + \delta w} = -\frac{1}{1 + \frac{2R}{w}}. \tag{4.24}$$

The complete sequence of transformations involved in the mapping $\delta w \to \eta$ is:

$$\delta w \xrightarrow{\text{dilation}} \frac{1}{2R}\delta w \xrightarrow{\text{inversion}} \frac{2R}{\delta w} \xrightarrow{\text{translation}} 1 + \frac{2R}{\delta w} \xrightarrow{\text{inversion}} \frac{1}{1 + \frac{2R}{w}} \xrightarrow{\text{rotation}} -\frac{1}{1 + \frac{2R}{w}}. \tag{4.25}$$

According to the above, the resulting conformal factor is

$$\frac{1}{\;\;(w,\theta)} = \left[2R\left(\left(\frac{w - R}{2R}\right)^2 + \frac{1}{4}(\bar{\theta}\theta)^2\right)\left(1 + \left(\frac{2R}{w - R}\right)^2 + \frac{1}{4}(\bar{\theta}\theta)^2\right)\right]^{\;\;1} \Rightarrow$$

$$\Rightarrow \quad \;\;(w,\theta) = 2R\left[\left(\frac{w - R}{2R}\right)^2 + \frac{1}{4}(\bar{\theta}\theta)^2\right]\left[\left(1 + \frac{2R}{w - R}\right)^2 + \frac{1}{4}(\bar{\theta}\theta)^2\right]$$

$$= 2R\left(\frac{w - R}{2R}\right)^2\left(1 + \frac{2R}{w - R}\right)^2 + \frac{R}{2}(\bar{\theta}\theta)^2\left[\left(\frac{w - R}{2R}\right)^2 + \left(1 + \frac{2R}{w - R}\right)^2\right]. \tag{4.26}$$

We can already see that there will be no contribution to V'_{WZ} from the remaining coordinate redefinitions $\eta = e^{\;\;u \;\; i}$ and $\cot\theta = \sinh u$ besides the conformal factor coming from the spacetime transformations. This is because V_{WZ} has no bottom component which is independent of θ or $\bar{\theta}$ and the only θ-dependence coming from conformal factors is due to superspace inversions, which always equals $(\bar{\theta}\theta)^2$; this factor therefore vanishes when multiplying V_{WZ}.

We already know the form of spacetime dependent conformal factor, which is determined from

$$\underbrace{\left(\frac{R^2}{\sin\theta(1 + \sin\theta\cos\tau)}\right)}_{= \;\;\; ^2} g' \;\; (\tau, \theta, \phi) = g \;\;\; (t, \rho, \phi). \tag{4.27}$$

This leads to

$$V'_{WZ}(\tau,\theta,\phi,\theta,\bar{\theta}) = \quad (t,\rho,\phi,\theta,\bar{\theta}) \quad V_{WZ}(t,\rho,\phi,\theta \ ,\bar{\theta}\text{-}). \tag{4.28}$$

We can finally write down the transformation of the FI action, $S_{FI} = \int d^3xd^2\theta d^2\bar{\theta}\mathcal{L}_{FI}$. It reads

$$\begin{aligned} &\xi \int_{\mathbb{R}^3} d^3xd^2\theta d^2\bar{\theta}\sqrt{g_{\mathbb{R}^3}}V_{WZ}(z) = \\ &= \xi \int_{S^3} d^3x'd^2\theta' d^2\bar{\theta}'\sqrt{g_{S^3}} \quad {}^3(z)V_{WZ;S^3}(g \ {}^1z') \\ &= \xi \int_{S^3} d^3x\theta d^2\bar{\theta}\sqrt{g_{S^3}}d^2 \quad {}^3(z)V'_{WZ;S^3}(z) \\ &= \xi \int_{S^3} d^3x\sqrt{g_{S^3}}\left(\frac{R^2}{\sin\theta(1+\sin\theta\cos\tau)}\right)^{-\frac{}{2}+\frac{3}{2}} (D'(x) - H\sigma'(x)). \end{aligned} \tag{4.29}$$

There are two things to note here: on the right-hand side we are evaluating the action of a vector superfield *defined on* S^3. We must therefore replace D by the curved space generalization of the D-term of a vector superfield when mapping V_{WZ} to S^3, as in 3.97.[2] We have also considered that, although the integral measure $\int d^3x\sqrt{g}$ is invariant under coordinate transformations, we must perform an additional Weyl rescaling to obtain the metric on S^3, which is

$$\left(\frac{R^2}{\sin\theta(1+\sin\theta\cos\tau)}\right) g' \quad (\tau,\theta,\phi) \to g' \quad (\tau,\theta,\phi). \tag{4.30}$$

4.2 Localizing the FI deformation

4.2.1 The partition function

We generically denote the action of the superconformal theory which we perturb by S_{SYM}, which may contain the Lagrangians $\mathcal{L}_K$, $\mathcal{L}_{YM}$ or $\mathcal{L}_{CS}$. The full theory on flat space we will work with is then[3]

$$S[\varphi] = S_{SYM}[\varphi] + \frac{\xi}{2\pi}\int_{\mathbb{R}^3} d^3x \int d^2\theta d^2\bar{\theta}^2 V_{WZ}(x,\theta) \tag{4.31}$$

Following the previous section, mapping the partition function from $\mathbb{R}^3$ to S^3 proceeds as

$$\begin{aligned} Z &= \int \mathcal{D}\varphi \exp\left(-S_{SYM}[\varphi] - \frac{\xi}{2\pi}\int_{\mathbb{R}^3} d^3x \int d^2\theta d^2\bar{\theta}^2 V_{WZ}(x,\theta)\right) \\ &= \int \mathcal{D}\varphi \exp\left(-S_{SYM;S^3}[\varphi] - \frac{\xi}{2\pi}\int_{S^3} d^3x\sqrt{g_{S^3}}\left(\frac{R^2}{\sin\theta(1+\sin\theta\cos\tau)}\right)^{-\frac{}{2}+\frac{3}{2}} (D'(x) - H\sigma'(x))\right) \end{aligned} \tag{4.32}$$

[2] One way to think about this is to consider that the term H is always present in the D-term but flat space supersymmetry is preserved with $H = 0$, while to preserve supersymmetry on S^3 we must turn on H.

[3] We ommit the trace of V_{WZ} over the abelian factors of the gauge group, which is needed to assuring gauge invariance if the group is not simply $U(1)$.

There is a very important aspect to note here. In order to write the superconformal transformation of the vector superfield, we needed to assume that it is a superconformal primary superfield. This is true, but only in the superconformal theory (there is no notion of primary operators if the theory is not conformal). This implies that the above manipulation is only valid if it is expressed as a perturbative expansion of the partition function around the superconformal theory S_{SYM}, which can immediately be resummed. Of course, this is only valid if $\xi << 1$, meaning that our treatment is necessarily perturbative, even though it retains information about all perturbation orders in this regime.[4]

If we use $\mathcal{L}_{YM}$ as the localizing term in S_{SYM}, then the computation of the one-loop determinants coming from quadratic fluctuations of the gauge fields and matter fields goes through exactly as in [25]. Although $\mathcal{L}_{FI}$ is Q-exact, it does not contribute with additional orbits to the set of localizing field configurations. It appears in the partition function as the classical action, denoted by $S[\varphi_0]$ in 3.81. Thus, we simply evaluate D and σ on the localization orbits,

$$D = -H\sigma, \quad \sigma = \sigma_0 = \text{constant}, \tag{4.33}$$

where the dimensions in the background field H will be recovered as $H = -\frac{i}{l}$. The result is (substituting $\sqrt{g_{S^3}} = \frac{1}{2}\sin(2\theta)$)

$$Z(q=1,\xi) = \int [d\sigma_0] Z^{\text{1-loop}}_{\text{matter}}(\sigma_0) Z^{\text{1-loop}}_{\text{gauge}}(\sigma_0) e^{i\ k\ {}^2_0} \times$$
$$\times \exp\left\{ -i\frac{\xi}{4\pi l}\sigma_0 \left[l^3 \int_{S^3} d^3x \sin(2\theta) \left(\frac{l^2}{R^2}\sin\theta(1+\sin\theta\cos\tau) \right)^{-\frac{D}{2}\ \frac{3}{2}} \right] \right\}. \tag{4.34}$$

We now extend R^3 to a q-branched cover of Euclidean space and introduce conical singularities with conical defect $2\pi(q-1)$ at $\rho = R$ on each copy of R^3. By the CHM map, this is conformally mapped to S^3_q, the q-branched cover of S_3, with conical singularities at $\theta = 0$ on each copy of S_3. This has the eect of replacing $\sqrt{g_{S^3}} \to q\sqrt{g_{S^3}}$, and the one-loop determinants are now evaluated on the replicated sphere with the conical singularity. The partition function on the branched sphere becomes

$$Z(q,\xi) = \int [d\sigma_0] Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, q) Z^{\text{1-loop}}_{\text{gauge}}(\sigma_0, q) \times$$
$$\times \exp\left\{ -\frac{i\xi}{4\pi l}\sigma_0 \left[q l^3 \int_{S^3} d^3x \sin(2\theta) \left(\frac{l^2}{R^2}\sin\theta(1+\sin\theta\cos\tau) \right)^{-\frac{D}{2}\ \frac{3}{2}} \right] \right\}. \tag{4.35}$$

We can immediately note the dierence with (3.62) of [25]: there, an FI term is added after the calculation is mapped to S^3_q, so it is treated as a deformation of the theory on S^3_q. Because of this they obtain the simpler form for the FI term, namely

$$-iq\frac{\xi}{4\pi}\sigma_0 \int_0^{2} d\phi \int_0^{\frac{\pi}{2}} d\theta \int_0^{2} d\tau \sin(2\theta) = -2\pi i \xi q \sigma_0. \tag{4.36}$$

We then conclude the following: in a perturbative regime (but to all orders in perturbation theory), mapping the partition function of super Yang-Mills on $\mathcal{M}_q$ with an FI deformation to S^3_q is equivalent to deforming the theory directly on S^3_q but

[4] I thank Guim Planella for clarifying this specific point.

with the FI coupling being renormalized according to

$$8\pi^2\xi \longrightarrow \frac{R}{2l}\xi \int_0^{2} d\phi \int_0^{2} d\tau \int_0^{\frac{\pi}{2}} d\theta \cos\theta \sqrt{\frac{\sin\theta}{1+\sin\theta\cos\tau}} \approx 4.879\pi\frac{R}{l}\xi, \tag{4.37}$$

where we have inserted $\ _D = 2$. In other words, the operations of conformal mapping and D-term deformation commute up to a rescaling of the coupling ξ (again, at a perturbative level). Denoting

$$\xi_q^0 \equiv \frac{4.879}{4}\frac{qR}{l}\xi, \tag{4.38}$$

the Rényi entropy is now determined by the following expectation value in the σ_0 matrix model:

$$\begin{aligned} Z_q &= \int [d\sigma_0] Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, q) Z^{\text{1-loop}}_{\text{gauge}}(\sigma_0, q) e^{\ il^2 \ {}^0_q \ {}_0} = \left\langle e^{\ il^2 \ {}^0_q \ {}_0} \right\rangle_q \Rightarrow \\ \Rightarrow S_q^{\text{susy;FI}}(\xi) &= \frac{1}{1-q}\text{Re}\left[\log\left\langle e^{\ il^2 \ {}^0_q \ {}_0}\right\rangle_q - q\log\left\langle e^{\ il^2 \ {}^0_1 \ {}_0}\right\rangle_1\right] \\ &= \frac{1}{1-q}\text{Re}\left[\log\left\langle e^{\ il^2 \ {}^0_q \ {}_0}\right\rangle_q - \log\left\langle e^{\ il^2 \ {}^0_1 \ {}_0}\right\rangle_1 + (1-q)\log\left\langle e^{\ il^2 \ {}^0_1 \ {}_0}\right\rangle_1\right]. \end{aligned} \tag{4.39}$$

The entanglement entropy is now obtained by the general expression

$$\begin{aligned} S_A^{\text{susy,FI}}(\xi) = \lim_{q\,!\,1} S_q^{\text{susy,FI}}(\xi) &= \text{Re}\left[\log\left\langle e^{\ il^2 \ {}^0_1 \ {}_0}\right\rangle_1 - \lim_{q\,!\,1}\frac{1}{q-1}\left(\log\left\langle e^{\ il^2 \ {}^0_q \ {}_0}\right\rangle_q - \log\left\langle e^{\ il^2 \ {}^0_1 \ {}_0}\right\rangle_1\right)\right] \\ &= \text{Re}\left[(1-\partial_q|_{q=1})\left\langle e^{\ il^2 \ {}^0_q \ {}_0}\right\rangle_q\right]. \end{aligned} \tag{4.40}$$

4.2.2 Examples

Pure $U(1)_k$ Chern-Simons theory

For a $U(1)$ gauge group, the gauge 1-loop determinant is trivial and is given by[25]

$$Z^{\text{1-loop}}_{\text{gauge};U(1)}(q) = 2\pi\sqrt{q}. \tag{4.41}$$

The partition function amounts to integrating the Chern-Simons and FI actions along the gauge moduli. We use the previous section to write the partition function on S_q^3 after conformally mapping the theory from flat space to the sphere. The Chern-Simons action 3.98 evaluated at the classical localization locus is

$$S_{CS} = \frac{k}{4\pi}\int_{S^3} 2H\sigma_0^2 = iq\pi k\sigma_0^2, \tag{4.42}$$

and therefore we have[5]

[5] We set $l = 1$ for simplicity. The integral measure on the gauge moduli is normalized by $\text{Vol}(U(1))^{\ 1} = (2\ )^{\ 1}$, which cancels against the factor of 2 in $Z^{\text{1-loop}}_{\text{gauge},U(1)}$.

$$
\begin{aligned}
Z_{U(1)_k}(q,\xi) &= \sqrt{q}\int_{-\infty}^{+\infty} d\sigma_0 e^{i\pi q k\sigma_0^2 - i\xi'_q\sigma_0} \\
&= \sqrt{q}\int_{-\infty}^{+\infty} d\sigma_0 e^{i\pi q k\left(\sigma_0 - \frac{\xi'_1}{2\pi k}\right)^2 - \frac{iq}{4\pi k}(\xi'_1)^2}.
\end{aligned}
\tag{4.43}
$$

To solve this integral, we first shift the integration variable $\sigma_0 \to \sigma_0 - \frac{\xi'_1}{2\pi k}$. We then slightly rotate the real axis clockwise, so that σ_0 picks up an imaginary phase which makes the integral convergent (this is the case if $k > 0$, otherwise we rotate anticlockwise). This results in

$$
Z_{U(1)_k}(q,\xi) = \frac{1}{\sqrt{k}} e^{-\frac{iq}{4\pi k}(\xi'_1)^2} \tag{4.44}
$$

In the $q = 1$ limit we recover the topological entanglement entropy of $U(1)_k$ Chern-Simons theory derived in 2.4, $-\frac{1}{2}\log k$:

$$
S^{\text{susy}}_{U(1)_k} = -\frac{1}{2}\log k. \tag{4.45}
$$

The ξ dependence contributes with an imaginary term to $\log Z_{U(1)_k}(q,\xi')$, and therefore the entanglement entropy is the same for any value of ξ'_q. This means that the entanglement structure of the $U(1)_k$ theory is blind to the RG flow to the IR induced by the relevant perturbation. The conformality-breaking FI term therefore does not introduce any relevant entanglement of degrees of freedom in the vacuum state of the topological $U(1)_k$ theory, which should be attributed to the absence of matter fields.

$U(1)_k \times U(1)_{-k}$ ABJM model

We take two pairs of bi-fundamental chiral multiplets in the $1\times\overline{1}$ and $\overline{1}\times 1$ representations of $U(1)_k \times U(1)_{-k}$. The weights of these representations are, respectively, $\sigma - \tilde{\sigma}$ and $\tilde{\sigma} - \sigma$, where $\sigma, \tilde{\sigma}$ parametrize the gauge moduli of both $U(1)$ factors. Again, the gauge 1-loop determinant is trivial. According to the rescaling of σ by $q^{-\frac{1}{2}}$ performed in section 3.5 of [25],[6] the partition function is

$$
\begin{aligned}
Z_{\text{ABJM}}(q,\xi) &= (2\pi)^2\int_{-\infty}^{+\infty}\frac{d\sigma d\tilde{\sigma}}{2\text{Vol}(U(1))} e^{i\pi k(\sigma^2-\tilde{\sigma}^2)} e^{-i\xi\sqrt{q}(\sigma+\tilde{\sigma})}\,\Gamma_h\left[\sigma-\tilde{\sigma}+\frac{i\omega}{2}\right]^2\Gamma_h\left[\tilde{\sigma}-\sigma+\frac{i\omega}{2}\right]^2 \\
&= \pi\int_{-\infty}^{+\infty} d\sigma d\tilde{\sigma}\exp\left\{i\pi k\left(\left(\sigma-\frac{\xi\sqrt{q}}{2\pi k}\right)^2-\left(\tilde{\sigma}+\frac{\xi\sqrt{q}}{2\pi k}\right)^2\right)\right\}\times \\
&\quad\times\Gamma_h\left[\sigma-\tilde{\sigma}+\frac{i\omega}{2}\right]^2\Gamma_h\left[\tilde{\sigma}-\sigma+\frac{i\omega}{2}\right]^2.
\end{aligned}
\tag{4.46}
$$

Shifting the integration variables as

$$
\sigma \to \sigma' = \sigma - \frac{\xi\sqrt{q}}{2\pi k}, \qquad \tilde{\sigma}\to\tilde{\sigma}' = \tilde{\sigma} + \frac{\xi\sqrt{q}}{2\pi k} \tag{4.47}
$$

this becomes

[6] The hyperbolic Gamma function is invariant under a rescaling of its arguments, $\Gamma_h(z;\omega_1;\omega_2) = \Gamma_h(\lambda z;\lambda\omega_1;\lambda\omega_2)$, $\lambda\in\mathbb{C}$. We use this to rescale the arguments of Γ_h in 3.121 by $\sqrt{q}$, and subsequently rescale the integration variable as mentioned.

$$Z_{\text{ABJM}}(q,\xi) = \pi \int_{-\infty}^{+\infty} d\sigma d\tilde{\sigma} e^{i\pi k(\sigma^2 - \tilde{\sigma}^2)} \; h \left[\sigma - \tilde{\sigma} + \frac{\xi \phi_{\bar{q}}}{\pi k} + \frac{i\omega}{2}\right]^2 \; h \left[\tilde{\sigma} - \sigma - \frac{\xi \phi_{\bar{q}}}{\pi k} + \frac{i\omega}{2}\right]^2 \tag{4.48}$$

Performing now the coordinate transformation (with unit Jacobian)

$$z \equiv \sigma^0 + \tilde{\sigma}^0, \quad \tilde{z} \equiv \sigma^0 - \tilde{\sigma}^0, \tag{4.49}$$

this becomes

$$\begin{aligned}
&\pi \int_{-\infty}^{+\infty} dz d\tilde{z} e^{i\pi k z\tilde{z}} \; h \left[\tilde{z} + \frac{\xi \phi_{\bar{q}}}{\pi k} + \frac{i\omega}{2}\right]^2 \; h \left[-\tilde{z} - \frac{\xi \phi_{\bar{q}}}{\pi k} + \frac{i\omega}{2}\right]^2 \\
=&\pi \int_{-\infty}^{+\infty} dz \underbrace{\delta(\pi k \tilde{z})}_{\frac{1}{\pi |k|}\delta(\tilde{z})} \exp\left\{-2\int_0^{\infty} \frac{dx}{x}\left(\frac{\omega}{x} - \frac{\sinh[x(\omega + 2i(\tilde{z} + \frac{\phi_{\bar{q}}}{k}))] + \sinh[x(\omega - 2i(\tilde{z} + \frac{\phi_{\bar{q}}}{k}))]}{2\sinh(bx)\sinh(x/b)}\right)\right\} \\
=& \frac{1}{k}\exp\left\{-2\int_0^{\infty} \frac{dx}{x}\left(\frac{\omega}{x} - \frac{\sinh(x\omega)\cos\left(x\frac{2\phi_{\bar{q}}}{k}\right)}{\sinh(bx)\sinh(x/b)}\right)\right\}.
\end{aligned} \tag{4.50}$$

The resulting Rényi entropy reads

$$\begin{aligned}
S_q^{\text{ABJM};N=1}(\xi) =& \frac{1}{1-q}\text{R}\left[(1-q)\log\left(\frac{1}{k}\right)\right. \\
&\left. -2\int_0^{\infty}\frac{dx}{x}\left(\frac{\omega - q}{x} - \frac{\sinh(x\omega)\cos\left(x\frac{2\phi_{\bar{q}}}{k}\right)}{\sinh(bx)\sinh(x/b)} + \frac{q\cos\left(x\frac{2\phi_1}{k}\right)}{\sinh(x)}\right)\right] \\
=& \text{R}\left[-\log k - \frac{2}{1-q}\int_0^{\infty}\frac{dx}{x}\left(\frac{\omega - q}{x} - \frac{\sinh(x\omega)\cos\left(x\frac{2\phi_{\bar{q}}}{k}\right)}{\sinh(bx)\sinh(x/b)} + \frac{q\cos\left(x\frac{2\phi_1}{k}\right)}{\sinh x}\right)\right].
\end{aligned} \tag{4.51}$$

Since we are interested in the entanglement entropy, we should simplify the integral in the limit $q \to 1$. Let us first look at the case $\xi = 0$, which reproduces the example from section 5.3 of [25]. To do this we take $q = 1 + \epsilon$, where $\epsilon << 1$. In order to properly take the $\epsilon \to 0$ limit we must retain all the terms in the integral of order $\mathcal{O}(\epsilon)$, since we have $\frac{1}{\epsilon}$ multiplying it; higher powers of ϵ will vanish when the limit is taken. We have

$$\begin{aligned}
&b = \sqrt{q} \simeq 1 + \frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2), \quad \omega = \frac{\sqrt{q} + \frac{1}{\sqrt{q}}}{2} \simeq \frac{1}{2}(1 + \frac{\epsilon}{2} + 1 - \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2)) = 1 + \mathcal{O}(\epsilon^2), \\
&\sinh(x\omega) \simeq \sinh x + x(\omega - 1)\cosh x + \mathcal{O}(\epsilon^2) = \sinh x + \mathcal{O}(\epsilon^2) \\
&\sinh(bx) \simeq \sinh x + x(b-1)\cosh x + \mathcal{O}(\epsilon^2) = \sinh x + \frac{x\epsilon}{2}\cosh x + \mathcal{O}(\epsilon^2) \\
&\sinh(x/b) \simeq \sinh x + x(\frac{1}{b} - 1)\cosh x + \mathcal{O}(\epsilon^2) = \sinh x - \frac{x\epsilon}{2}\cosh x + \mathcal{O}(\epsilon^2),
\end{aligned} \tag{4.52}$$

and the integral in 4.51 becomes

$$
\begin{aligned}
&-\frac{2}{1-q}\int_0^1 \frac{dx}{x}\left(\frac{1-q}{x} - \frac{\sinh x + \mathcal{O}(\epsilon^2)}{(\sinh^2 x - \frac{1}{4}x^2\epsilon^2\cosh^2 x)} + \frac{q}{\sinh x}\right) \\
&= -\frac{2}{1-q}\int_0^1 \frac{dx}{x}\left(\frac{1-q}{x} + (q-1)\frac{1}{\sinh x} + \mathcal{O}(\epsilon^2)\right) \\
&= -2\int_0^1 \frac{dx}{x}\left(\frac{1}{x} - \frac{1}{\sinh x}\right) = -2\log 2.
\end{aligned}
\tag{4.53}
$$

The integral performed here is the same one which reproduces the result from section 5.1 of [25] for one single chiral multiplet. Indeed, in this computation the Chern-Simons sector decouples from the matter sector, and the contribution from the matter fields is equivalent to that of 4 free chiral multiplets, each contributing $-\frac{1}{2}\log 2$ to the entanglement entropy. Note also that in this case where the theory is conformal there are no $\mathcal{O}(\epsilon)$ corrections to the Rényi entropy. This means that the entanglement entropy is determined the free energy (as remarked at the end of 3).

For $\xi \neq 0$, we must also consider the following expansion to order $\mathcal{O}(1-q)$ around $q = 1$:

$$
\cos\left(x\frac{2\xi^{\flat}_{\bar q}}{\pi k}\right) = \cos\left(x\frac{2\xi_1^0}{\pi k}\right) - \epsilon\frac{x\xi_1^0}{\pi k}\sin\left(x\frac{2\xi_1^0}{\pi k}\right) + \mathcal{O}(\epsilon^2). \tag{4.54}
$$

In the $\epsilon \to 0$ limit, the integral which computes the contribution from the matter sector is:

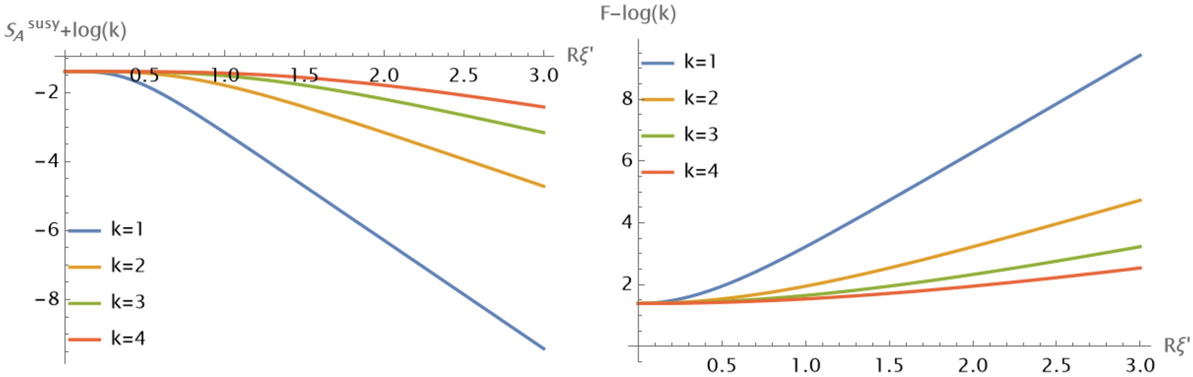
$$
\begin{aligned}
&\int_0^1 \frac{dx}{x}\left(\frac{1}{x} - \frac{\cos\left(x\frac{2\ {}^0_1}{k}\right)}{\sinh x} - \frac{1}{-\epsilon}\frac{-\epsilon\frac{x\ {}^0_1}{k}\sin\left(x\frac{2\ {}^0_1}{k}\right)}{\sinh x}\right) = \\
&= \int_0^1 dx\left(\frac{1}{x^2} - \frac{1}{x}\frac{\cos\left(x\frac{2\ {}^0_1}{k}\right)}{\sinh x} - \frac{\xi_1^0}{\pi k}\frac{\sin\left(x\frac{2\ {}^0_1}{k}\right)}{\sinh x}\right).
\end{aligned}
\tag{4.55}
$$

The final expression for the entanglement entropy reads

$$
\boxed{S^{\text{susy}}_{\text{ABJM};N=1}(\xi,k) = -\log k - 2\int_0^1 dx\left(\frac{1}{x^2} - \frac{1}{x}\frac{\cos\left(x\frac{2\ {}^0_1}{k}\right)}{\sinh x} - \frac{\xi_1^0}{\pi k}\frac{\sin\left(x\frac{2\ {}^0_1}{k}\right)}{\sinh x}\right)}. \tag{4.56}
$$

Recalling that $\xi_1^0 \propto \frac{R}{l}\xi$, we see that there are several physical parameters of the theory which determine the correction to the entropy, and they are all manifest in a single dimensionless quantity: the strength of the FI perturbation ξ_1^0 (renormalized due to the conformal mapping), the ratio between the two relevant length scales (radii of $\partial\mathcal{A}$ R, and the 3-sphere l) and the Chern-Simons level k.

The variation of the entanglement entropy with $R\xi$ is shown in 4.1, as well as the free energy (simply determined by $F = -\log Z_{\text{ABJM}}(q = 1)$, setting $q = 1$ in 4.50). The behaviour we obtain is quite unexpected: we see a decrease in the universal piece of the entanglement entropy as $R\xi$ increases, and it appears that $S^{\text{susy}}_{\text{ABJM};N=1}$ diverges to $-\infty$ for large $R\xi$. We do not have a physical explanation for this behaviour; one should expect that, as the theory is driven towards the IR and the correlation lengths of the fields get much smaller than the length scale of the entangling region, the non-local entanglement of degrees of freedom across the boundary measured by the finite piece of the entanglement entropy would be suppressed. If this result is indeed correct, we are led to conclude that the deformation of the field background by $A^{(R)}$



(a) Entanglement entropy

(b) Free energy

Figure 4.1: Entanglement entropy and free energy of FI-perturbed $U(1)_k \times U(1)_{-k}$ model, as a function of a dimensionless parameter proportional to $R\xi$.

which defines supersymmetric Rényi entropy leads very significant consequences for the entanglement entropy which are not expected in the absence of supersymmetry (see for instance the example of a free massive $2d$ QFT in 2.4).

On the other hand, the condition proven in [7] for the entanglement entropy of circles in 2 + 1 dimensions,

$$\frac{d^2 S(R)}{dR^2} \leq 0, \tag{4.57}$$

is satisfied. In contrast, the usual statement of the F-theorem, from which 4.57 is derived, is not obeyed: we observe a monotonic increase of the free energy along the RG flow from the UV instead of a monotonic decrease.

In Appendix B we present another solution which results from a di□erent limiting procedure for the partition function on the branched sphere as $q \to 1$ and reproduces the result we had expected. We hope this turns out to be correct, although at the moment the analytically correct procedure seems to us to be the one shown in the present section.

4.3 Massive deformation

The second type of deformations we consider is a mass term for the chiral multiplet. This is a relevant deformation setting a finite correlation length in the theory, thus inducing an RG flow into the IR. We will see below that mapping the partition function from flat space to the three-sphere does not allow for localization of this deformation as for the FI term. We will first present the behaviour of the entanglement entropy when the mass term is introduced after performing the localization on S^3. Of course, this does not possess a direct physical meaning since we are ultimately interested in the behaviour of the theory on flat space. This will nevertheless provide us with an interpolating function for the entropy between the UV and IR.

4.3.1 Deformation of the matrix model on S^3

Recalling the form of the matter Lagrangian on S^3, 3.100, real mass terms can be implemented by setting σ to a (real) constant value. More precisely, real mass terms can be generated by coupling the matter fields to a background vector multiplet associated to a flavour global symmetry. As discussed around 3.93 such a vector multiplet is set to an explicitly supersymmetric configuration:[7]

$$\mathcal{V}_F = \left(\sigma^{(F)}, A^{(F)}, D^{(F)}, \bar{\lambda}^{(F)}, \lambda^{(F)}\right), \quad \sigma^{(F)} = m,\ A^{(F)} = 0,\ D^{(F)} = -\sigma^{(F)} H. \tag{4.58}$$

Note that on flat space, or in the limit where the radius of the sphere is infinite, we have $D^{(F)} = 0$. The deformation of the flat space Lagrangian resulting from the coupling to this global symmetry is then

$$\mathcal{L}_m = m^2 \bar{\phi}\phi - im\bar{\psi}\psi \subset \mathcal{L}_K. \tag{4.59}$$

We will analyse the case of a free chiral multiplet, since it is both analytically and numerically tractable, while displaying the expected behaviour for the entanglement entropy of a theory with a mass gap (already observed in 2.4). On the branched sphere, the partition function of a free chiral multiplet 3.121 coupled to 4.58 is (setting $r = \frac{1}{2}$)

$$\begin{aligned} Z^{\text{1-loop}}_{\text{matter}}(q,m) &= \ h\left(m + \frac{i}{4}\left(1 + \frac{1}{q}\right); i, \frac{i}{q}\right) = \ h\left(m\sqrt{q} + \frac{i}{4}\left(\sqrt{q} + \frac{1}{\sqrt{q}}\right); i\sqrt{q}, \frac{i}{\sqrt{q}}\right) \\ &= \exp\left(i\int_0^{+1} \frac{dx}{x}\left(\frac{m\sqrt{q} - \frac{i}{2}\omega}{-x} - \frac{\sin(2x(m\sqrt{q} - \frac{i}{2}\omega))}{2\sin(i\sqrt{q}x)\sin(i\sqrt{q}\ ^{1}x)}\right)\right), \end{aligned} \tag{4.60}$$

As seen in the previous section, for $m = 0$ lowest order term of the expansion of the integrand above around $q = 1$ is of order $\mathcal{O}((q-1)^2)$. Then, we need only focus on the q-dependent terms multiplying m in the analytic continuation of $Z^{\text{1-loop}}_{\text{matter}}(q,m)$ at $q \to 1$. This is

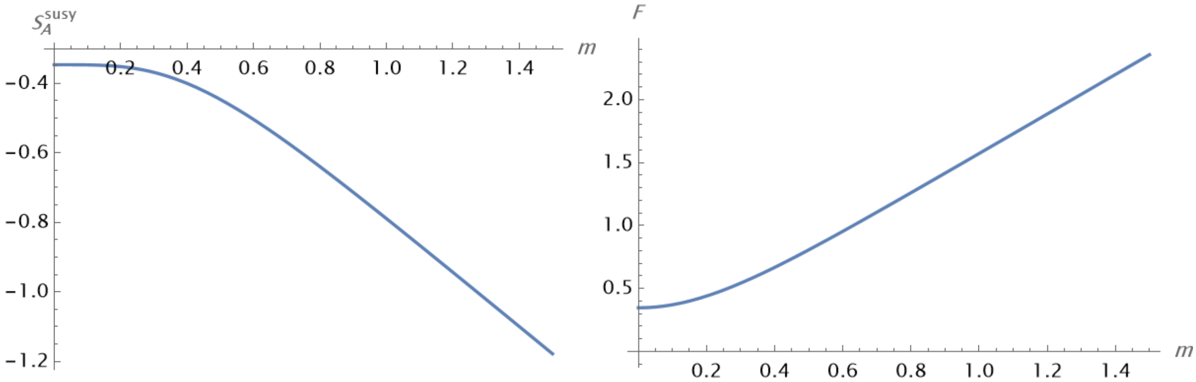
$$\begin{aligned} \log Z^{\text{1-loop}}_{\text{matter}}(q,m) &= i\int_0^{+1} \frac{dx}{x}\left(\frac{m - \frac{i}{2}\omega}{-x} - \frac{\sin(2x(m - \frac{i}{2}\omega))}{2\sin^2(ix)}\right) + \\ &\quad + \frac{1}{2}(q-1)i\int_0^{+1}\frac{dx}{x}\left(\frac{m}{-x} - 2xm\frac{\cos(2x(m - \frac{i}{2}\omega))}{2\sin^2(ix)}\right) + \mathcal{O}((q-1)^2). \end{aligned} \tag{4.61}$$

The entanglement entropy is then

$$\boxed{S^{\text{susy}}_{\text{free chiral}}(m) = i\int_0^{+1} dx\left(\frac{-m + i\omega}{2x^2} - \frac{\sin(2x(m - \frac{i}{2}\omega)) - xm\cos(2x(m - \frac{i}{2}\omega))}{2x\sin^2(ix)}\right).} \tag{4.62}$$

This interpolating function is plotted in 4.2. Its behaviour is very similar to the one found for the ABJM theory in 4.1, which is no surprise given that in both situations we have perturbed the SCFT with a relevant term in the Lagrangian. However, we would expect a very different behaviour for the reasons stated in the previous section under 4.1. Not only that, but since we are looking at a massive free chiral multiplet on S^3, we could also expect the behaviour to match the one found in [34] for the regularized entanglement entropy of free scalars and fermions on S^3, up to some corrections imposed by supersymmetry. We observe that such corrections are rather drastic, since in the cited example the entropy is suppressed to zero for large mass, while we are observing a divergence to $-\infty$, as in the case of the FI perturbation. We are then led to conclude that the deformation of the action on S^3 by $A^{(R)}$ drastically

[7] It should be stressed that when V^F is not restricted to a fixed background configuration, its components transform under the superconformal algebra and the action for the gauge-coupled matter fields is conformally invariant.



(a) Entanglement entropy

(b) Free energy

Figure 4.2: Interpolating function for the entanglement entropy of a massive free chiral multiplet on S^3.

alters the entanglement entropy relative to the theory without supersymmetry.[8] As for the FI perturbation, we have found a different limiting procedure as $q \to 1$ which reproduces the expected behaviour but which appears to be incorrect; we leave it to B.

4.3.2 Deformation in the flat space theory: perturbative treatment

We now explicitly deform the flat space super Yang-Mills theory. We will not procede as in the last section by mapping the partition function to S^3. The reason is that this would lead to an additional spacetime dependent term in the quadratic operators for the scalar and fermion fields due to their transformation under a conformal rescaling of the metric. To see why, consider, for instance, the transformation of the scalar term in 4.59[9] (see A for some properties of conformal primaries in SCTF's)

$$m^2 \int_{\mathbb{R}^3} d^3x \tilde{\phi}(x)\phi(x) = m^2 \int_{S^3} dx' \sqrt{g_{S^3}} \quad {}^{2\ \ \phi\ \ 3}(x)\tilde{\phi}'(x')\phi'(x'). \tag{4.63}$$

Contrary to what occurs for the FI perturbation, the integrand does not become constant on the localization orbits; it is instead quadratic in the fluctuations of the scalar field around the localization orbits (when multiplied by the overall coupling constant of the matter Lagrangian). This would in turn lead to the modification of the scalar quadratic scalar operator in 3.108 according to (setting $\quad = \frac{1}{2}$)

$$\longrightarrow \quad + m^2 \frac{\cos\theta}{1+\cos\tau\sin\theta}, \tag{4.64}$$

[8] As remarked in [25], such a deformation can be understood as a change of boundary conditions for the fields on the branched sphere.

[9] There are two ways to implement a massive deformation in the theory: we can couple the matter fields to a global flavour symmetry as explained above, which adds this deformation to the Lagrangian as part of the Kähler potential $K = \ \ \tilde{}$; alternatively, we can consider adding a quadratic superpotential of the form

$$\int d^2 \ W(\ ) = \int d^2 \ \frac{i}{2}(\ \ 4iF \ \ + 2i \ \ ^2) = 2F \qquad ^2,$$

together with its anti-chiral counterpart. However, because the matter fields are zero on the localization orbits, the superpotential itself vanishes when performing localization, since when expanding the fields around the orbits as $\ ! \ \ _0 + \frac{1}{\sqrt{t}}\ '$, we would have $W(\ )$ of order $O(t^{\ 1})$.

which severely complicates the determination of the eigenvalues used to compute the one-loop determinant.[10]

We therefore take a simpler approach. We compute the first non-vanishing contribution to the entanglement entropy in perturbation theory in the limit of small mass, by using the usual form of two point functions of primary operators in flat space CFT:

$$\langle\bar{\phi}(x)\phi(y)\rangle_0 = C \frac{1}{|x-y|^{2\ \phi}}, \tag{4.65}$$

where the subscript $_0$ indicates an expectation value in the unperturbed SCFT. In fact, this allows us to directly perform the calculation on R^3 without having to rely on a conformal transformation to S^3 (this would be redundant since we will not be using localization).

The perturbative expansion we consider takes the form

$$\begin{aligned} Z(m) &= \int \mathcal{D}\bar{\phi}\mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{F}\mathcal{D}F e^{\ \int d^3x(L_{K,m=0}+L_m)} \\ &= \int \mathcal{D}\bar{\phi}\mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{F}\mathcal{D}F \left(\sum_{n=0}^{1}\frac{1}{n!}(-\int d^3x m^2\bar{\phi}(x)\phi(x) - mi\bar{\psi}(x)\psi(x))^n\right) e^{\ \int d^3x L_{K,m=0}} \end{aligned} \tag{4.66}$$

In a conformal field theory, single-point correlation functions vanish by conformal invariance. Also, for a free chiral multiplet we are able to factorise n-point functions by Wick contractions. As a result, the lowest order contribution in m in the perturbative expansion is

$$= Z(0)\left(1 + \int d^3x d^3y \left(m^4\langle\bar{\phi}(x)\phi(y)\rangle_0^2 - m^2\langle\bar{\psi}(x)\psi(y)\rangle_0^2\right) + \dots\right) \tag{4.67}$$

Introducing an IR cuto scale $\ = \frac{1}{m}$, the integrated scalar two-point function can be written as

$$\begin{aligned} \int_{\mathrm{R}^3} d^3x d^3y\langle\bar{\phi}(x)\phi(y)\rangle_0^2 &= C^2\int_{\mathrm{R}^3} d^3x d^3y\left(\frac{1}{|x-y|^{2\frac{1}{2}}}\right)^2 \\ &= C^2\mathrm{Vol}(\mathrm{R}^3)\int_{\mathrm{R}^3} d^3x\frac{1}{|x|^2} \\ &= C^2\mathrm{Vol}(\mathrm{R}^3)4\pi\int_0 dr r^2\frac{1}{r^2} \\ &= C^2\mathrm{Vol}(\mathrm{R}^3)4\pi\frac{1}{m}. \end{aligned} \tag{4.68}$$

The spinor two point function is also fixed by conformal invariance [57, 58]:

$$\langle\bar{\psi}\_(x)\psi\ (y)\rangle = C\ \frac{\sigma\_\ (x-y)}{(|x-y|^2)^{\ \psi+\frac{1}{2}}}, \tag{4.69}$$

[10] This in fact seems to be impossible without attempting to solve for the full spectrum of the quadratic operator, and the complication lies in the introduction of -dependence. This is because a matching of fermion and scalar eigenvalue problems as in 3.117 leads to a and dependence in the relation between $_s$ and $_f$.

where, because ψ belongs to the same multiplet as ϕ, the coe□cient C is determined by supersymmetry to be $C = 4iC$ [58]. Writing $\langle\bar{\Psi}(x)\psi(y)\rangle_0^2 = (\varepsilon\,\bar{\Psi}(x)\psi(y))^2$,

$$\int_{\mathrm{R}^3} d^3x d^3y \langle\bar{\Psi}(x)\psi(y)\rangle_0^2 = \mathrm{Vol}(\mathrm{R}^3)\int_{\mathrm{R}^3} d^3x C^2 \left(\frac{\varepsilon\ \sigma\ x}{(|x|^2)^{\psi+\frac{1}{2}}}\right)^2. \tag{4.70}$$

Writing the integrand in spherical coordinates with $= 1$,

$$r^2\left(\varepsilon\ \frac{\sigma^1 \sin\theta\cos\varphi + \sigma^2 \sin\theta\sin\varphi + \sigma^3 \cos\theta}{r^3}\right)^2, \tag{4.71}$$

the integrated two-point function reads

$$\mathrm{Vol}(\mathrm{R}^3)C^2\int_0^{2} d\varphi\int_0 d\theta\sin\theta\int_0^{+1} dr\frac{r^4}{r^6}\left(\varepsilon\ \left(\sigma^1 \sin\theta\cos\varphi + \sigma^2 \sin\theta\sin\varphi + \sigma^3 \cos\theta\right)\right)^2. \tag{4.72}$$

The crossed terms in the square contain vanishing integrals either over θ or φ. For the remaining terms, we must evaluate the expression $\varepsilon\ (\sigma\ )$, which vanishes for $\mu = 1, 3$ and equals $-2i$ for $\mu = 2$. We conclude

$$\begin{aligned}\int_{\mathrm{R}^3} d^3x d^3y \langle\bar{\Psi}(x)\psi(y)\rangle_0^2 &= \mathrm{Vol}(\mathrm{R}^3)C^2\int dr\frac{1}{r^2}(-4)\int_0 d\theta\sin^3\theta\int_0^{2}\sin^2\varphi \\ &= -\frac{64}{3}C^2\mathrm{Vol}(\mathrm{R}^3)(m-\frac{1}{\epsilon}).\end{aligned} \tag{4.73}$$

Even when introducing an IR cuto□ , both correlators computed above are still divergent by a factor proportional to the entire volume of (Euclidean) spacetime. This is a physical divergence, occuring due to the translational symmetry of the theory across the entire spacetime. In order to deal with finite quantities, we look at the correlation density that we obtain from this perturbative correction, by simply dividing our results by $\mathrm{Vol}(\mathrm{R}^3)$. Additionally, the fermion correlator has a UV divergence; because this is an additive divergence we can simply introduce a counterterm in the Lagrangian which, at energy scale m, cancels this divergence.

On the n-branched cover of R^3 we assume that the correlation functions do not receive any non-trivial contribution from the conical singularities. This is justified because conformal invariance of the theory on R^3 extends to conformal invariance of the theory on its branched cover, so the correlators 4.65 and 4.69 take the same form, being multiplied by n when integrated over R^3_n. The regularized partition function is then

$$Z_n(m) = Z_n(0)\left(1 + nm^3C^2\left(4\pi + \frac{64}{3}\right) + \mathcal{O}(m^6)\right), \tag{4.74}$$

and we define lowest order correction to its logarithm as

$$\delta_m \log Z_n(m) \equiv \log\left(1 + nm^3C^2\left(4\pi + \frac{64}{3}\right) + \mathcal{O}(m^6)\right). \tag{4.75}$$

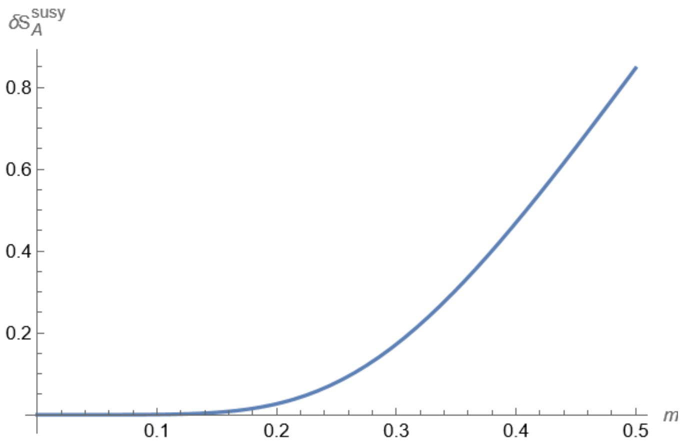


Figure 4.3: Lowest order correction to S_A^{susy} for a massive perturbation around the SCFT.

To obtain the correction to the entanglement entropy, we consider

$$\begin{aligned}\delta_m S_A &= \delta_m \lim_{n \to 1} \frac{1}{1-n} \log\left(\frac{Z_n(m)}{Z(0)^n}\right) \\ &= \lim_{n \to 1} \frac{1}{1-n}\Big[\delta_m \log Z_n(m)\big|_{n=1} + (n-1)\partial_n|_{n=1}\delta_m \log Z_n(m)\big|_{n=1} + \\ &\quad + \mathcal{O}((1-n)^2) - n\delta_m \log Z_1(m)\Big] \\ &= \delta_m \log Z_1(m) - \frac{m^3 C^2\left(4\pi + \frac{64}{3}\right)}{1 + m^3 C^2\left(4\pi + \frac{64}{3}\right)}.\end{aligned} \tag{4.76}$$

In 4.3 we see that this correction reproduces the convex and monotonically growing behaviour of the free energy instead of the monotonically decreasing behaviour of the entanglement entropy. We take this to mean that either it is not enough to consider a lowest order approximation in flat space to reproduce the behaviour of the interpolating function on S^3, or some non-trivial contributions from the branching singularities need to be taken into account in the perturbative approximation of the partition function on $\mathbb{R}^3_n$, 4.74.

5 Defect Operators and Non-trivial $\pi_1(\mathcal{A})$

Having studied examples of how the entanglement entropy behaves under a breaking of conformal symmetry of the $\mathcal{N}$ = 2 SCFT's described in 3, we now consider breaking another requirement for the equivalence between flat space entanglement entropy and partition functions on S^3_q. Namely, we will consider entangling regions $\mathcal{A}$ which are no longer simply connected. This is a relevant aspect to take into account if we wish to understand the entanglement structure of any quantum theory, since in general we expect that the von Neumann entropy receives corrections in a non-trivial way under changes of $\pi_1(\mathcal{A})$. In particular, by removing a subregion $\mathcal{B} \subset \mathcal{A}$ from the entangling region, the quantity $S_{A=B}$ will no longer measure entanglement of degrees of freedom of quantum fields contained in $\mathcal{B}$ with those in $\overline{\mathcal{A}}$, and the difference is not simply $S_A - S_B$.

The treatment here will still rely on the conformal mapping of the entangling region to a compact space through the CHM map, thus we will restric to superconformal theories in this chapter. However, the resulting manifold will now be a 3-sphere with boundaries. We will make use of the description of supersymmetric Rényi entropy in terms of expectation values of certain defect operators inserted on the boundary of the entangling surface at each copy of the branched background. We review this formalism, introduced in [59], in the first section. This is followed by a review of the localization procedure on S^3 with boundaries, after which our computation is carried out.

5.1 Defect operators and localization

We now describe a formulation of Rényi entropy which is equivalent to placing the theory on an n-branched cover of spacetime. This consists in defining the theory on a non-branched space and replicating the field content n times. Additionally, there is a codimension-one defect acting between the different field copies of the theory, implementing the necessary boundary conditions infinitesimally above and below the entangling surface (in the timelike direction). We will refer to the latter description as the *n-copy theory*.

Now, the supersymmetric Rényi entropy 2.35 may be computed by coupling the n-copy theory to a Z_n gauge theory. We know that the n-copy theory is invariant under the replica Z_n symmetry, which exchanges the copies of the theory cyclically from $_i$ to $_{i+1}$, with $n+1 \equiv 1$ (thus preserving the identification of boundary conditions between consecutive copies). Gauging this symmetry by coupling the theory to a Z_n gauge field allows us to implement the necessary boundary conditions on the fields at each copy of the theory (this will be made precise below). Computing supersymmetric Rényi entropy with this procedure requires that the boundary conditions corresponding to the action of the co-dimension one defect preserve some supersymmetry. This will reproduce the e□ect of coupling the single copy theory to a background R-symmetry gauge field on the branched S^3.

5.1.1 Z_n gauge theory, defect operators and supersymmetry

Consider a non-supersymmetric theory with an abelian gauge field A whose field strength is $F = dA$, and couple it another abelian $U(1)$ gauge field B, which in 3 dimensions we may take to be a 1-form. The coupling is topological and takes the form of a BF action [60]:

$$S_{BF} = \frac{in}{2\pi}\int F \wedge B. \tag{5.1}$$

The way we require B to become a Z_n gauge field is by restricting its allowed gauge transformations, or equivalently its holonomy, to lie in a set isomorphic to Z_n. This places a restriction on the coupling constant of S_{BF}, since under a gauge transformation of B,[1]

$$S_{BF} \to S_{BF} + \frac{in}{2\pi}\int_M dA \wedge d\alpha = 2\pi\frac{in}{2\pi}\int_{@M} dA \in in2\pi Z, \tag{5.2}$$

For $\exp(-S_{BF})$ to be gauge-invariant (and consequently the Euclidean partition function), we impose that $n \in Z$.

The codimension one defect is realized by prescribing a singular configuration for A (more precisely, for its field strength) along a one-cycle ∂ corresponding to the entangling surface. This is achieved through the insertion of the operator[2]

$$\exp\left(ik\oint_{@} B\right) = \exp\left(ik\int B \wedge \delta_{@}\right). \tag{5.3}$$

We can now treat B as an auxiliary field, which after being integrated out gives the following background configuration for F:[3] (this will be sometimes denoted below by *vortex charge*)

$$\frac{\delta}{\delta B}\left(\frac{in}{2\pi}\int F \wedge B + ik\int B \wedge \delta_{@}\right) = 0 \Rightarrow F = \frac{2\pi k}{n}\delta_{@}\ . \tag{5.4}$$

In our supersymmetric setting, the action of the theory takes the form

$$\int d^3x \int d^2\theta d^2\bar{\theta}\left(-\frac{1}{e^2}\ ^2 + \tilde{}\ \ + \frac{k}{2\pi}\mathcal{V}\right). \tag{5.5}$$

In order to incorporate a coupling with a BF-type theory, the gauge field A in S_{BF} must sit in an additional vector multiplet

$$\mathcal{V}^A = \{A\ , \sigma^A, \lambda^A, \bar{\lambda}^A, D^A\}. \tag{5.6}$$

[1] Here we use that the periods of B are constrained to be multiples of 2 and that the integral along a compact manifold of the first Chern class of the principal $U(1)$ bundle, whose connection is locally represented by A, is an integer multiple of 2 .

[2] For an abelian connection there is no need to path order the exponential.

[3] Note that in the pure BF theory the equations of motion for A and B imply that their field strengths both vanish: $dA = 0$, $dB = 0$. This eliminates all dynamical degrees of freedom and emphasises the topological nature of the theory.

The auxiliary gauge field B is similarly included in a background abelian vector multiplet $\mathcal{V}^B = \{B\ , \sigma^B, \lambda^B, \bar{\lambda}^B, D^B\}$, whose field strength is contained in the corresponding linear multiplet $\ ^B$.[4] The topological coupling is then

$$\begin{aligned} S^{\text{susy}}_{BF} &= \frac{in}{2\pi}\int_{S^3}\int d^2\theta d^2\bar{\theta}\left[\ ^A\mathcal{V}^B\right] \\ &= \frac{in}{2\pi}\int_{S^3}\left(\mathrm{d}A\wedge B + \sigma^A D^B + \sigma^B D^A + \text{fermions}\right), \end{aligned} \tag{5.7}$$

where we again have the restriction $n \in \mathbb{Z}$ since gauge transformations for the auxiliary vector multiplet, $\mathcal{V}^B \to \mathcal{V}^B + \ + \tilde{\ }$, are such that the bottom component ω of chiral multiplet is a $\mathbb{Z}_n$-valued function. This is a mixed Chern-Simons term, and in the first line above it is written in an explicitly supersymmetric form.

The defect operator which implements the codimenison one defect is realized by the insertion of the supersymmetric Wilson line along a path $x(\tau)$,

$$W^{\text{susy}}(k) \equiv \exp\left[ik\oint(B - i[\gamma]\sigma^B)\right] = \exp\left[ik\oint d\tau(B\ \underline{x}\ - i\sigma^B|\underline{x}|)\right], \tag{5.8}$$

where γ is a maximal $S^1 \subset S^3$, with $[\gamma]$ its volume form. Choosing a maximal S^1 at $\theta = 0$ along $d\phi \equiv dx^3$ is a simple way to preserve supersymmetry of $W^{\text{susy}}(k)$, since in this case $\underline{x} = Re_3$ and[5]

$$\begin{aligned} \delta_{"}(B\ \underline{x}\ - i\sigma^B|\underline{x}|) &= -\frac{i}{2}(\bar{\varepsilon}\gamma\ \lambda^B + \bar{\lambda}^B\gamma\ \varepsilon)\underline{x}\ - \frac{i}{2}(\bar{\varepsilon}\lambda^B - \bar{\lambda}^B\varepsilon)|\underline{x}| \\ &= (\gamma\ \underline{x}\ + |\underline{x}|)\bar{\varepsilon} + (\gamma\ \underline{x}\ - |\underline{x}|)\varepsilon. \end{aligned} \tag{5.9}$$

For this variation to vanish, both terms in ε and $\bar{\varepsilon}$ must vanish separately, giving

$$(\gamma_3 - 1)\varepsilon = (\gamma_3 + 1)\bar{\varepsilon} = 0, \tag{5.10}$$

which is the half-BPS condition, meaning that it breaks the supersymmetries generated by half of the Killing spinors on the round S^3.

Because B is abelian and the exponential in $W^{\text{susy}}(k)$ need not be path ordered, we include its integrand in the BF action and integrate out the auxiliary vector field once more. In superspace this process schematically takes the form

$$\frac{\delta}{\delta\mathcal{V}^B}\left(\frac{in}{2\pi}\int\ ^A\mathcal{V}^B + ik\int\mathcal{V}^B\delta\ \right) = 0 \Rightarrow\ ^A = \frac{2\pi k}{n}\delta\ . \tag{5.11}$$

We can work out this expression explicitly in terms of components, since in the supersymmetric theory the integration over the fields of the vector multiplet reduces to an integration over constant Lie algebra-valued modes σ_0. On the localization orbits, the only non-zero auxiliary field is σ^B. Integrating it out results in

$$\frac{\delta}{\delta\sigma^B}\left(-\frac{in}{\pi}\int_{S^3}\sigma^A\sigma^B + k\int_{=0}\sigma^B\right) = 0 \Rightarrow \sigma^A = -i\frac{k}{n}. \tag{5.12}$$

[4]Recall that a real linear multiplet containing the field strength of a vector superfield is defined such that its bottom component is , and with the SUSY variations defined by V the completion of becomes $= f\ ;i\tilde{\ };\ \ i\ ;\ \ \frac{i}{2}"_{\mu\nu\rho}F^{\nu\rho};Dg$.

[5]Here we are implicitly choosing a diagonal vierbein; for supersymmetry variations of components of vector and chiral multiplets on S^3 see, for instance, [61].

5.1.2 From $\mathbb{Z}_n$ gauge theory to S_q^{susy}

We now need to understand the following:

1. how to couple this background configuration for $\mathcal{V}^A$ to the dynamical fields in the Lagrangian of the theory in a gauge-invariant manner;
2. which is the correct background configuration for $\mathcal{V}^A$ that computes supersymmetric Rényi entropy.

Because localization reduces the calculation of the 1-loop determinants to a free field computation, we should expect that the Rényi entropy is obtained in the same way as for a usual free QFT, where one considers an n-copy of the field content $\vec{\ } = (\ _1, \ldots, \ _n)$ which acquires a monodromy $\beta_k = e^{i\frac{2\pi k}{n}}$ as one crosses the entangling surface from $t = 0^+$ to $t = 0$. This is equivalent to having co-dimension two defects localized along ∂ , thus reproducing the effects of coupling the free fields to a background gauge field with holonomy $i\frac{2\ k}{n}$ around ∂ for each field copy.

Such a monodromy can be implemented by an imaginary supersymmetric mass term, interpreted either as a Higgs-type VEV for the fields in the matter multiplet or as a mass term associated to a background $U(1)$ flavour symmetry for chiral multiplets. The latter is of the form

$$\mathcal{L}_m = \tilde{\phi}\sigma^2\phi - i\tilde{\psi}\sigma\psi. \tag{5.13}$$

Such a mass term can effectively be included by gauging the $\mathbb{Z}_n$ symmetry in the BF action after coupling the chiral multiplet to $\mathcal{V}^A$.

Let us consider the partition function on S^3, whose contribution from the matter one-loop determinant is

$$Z^{\text{1-loop}}_{\text{chiral}}(\sigma_0) = \ _h(i\rho(\sigma_0) + r|1,1). \tag{5.14}$$

In order to obtain the correct background vortex charge that computes the supersymmetric Rényi entropy, we take the n-copy theory and couple each copy to a background vector multiplet $\mathcal{V}^A_{(k)}$, $k \in \{0, \ldots, n-1\}$, carrying vortex charges

$$q^{(k)}_{\text{vortex}} = \frac{r}{2}\left(\frac{1}{n} - 1\right) + \frac{k}{n}. \tag{5.15}$$

The first r-dependent factor added to the $\frac{k}{n}$ term derived initially can be interpreted as the R-symmetry twist of the theory that is necessary to maintain supersymmetry on the n-copy theory. Explicitly, the vortex configuration is imposed by adding to the k-th copy of the theory the action

$$S^{\text{susy}}_{BF;(k)} = \frac{in}{2\pi}\int_{S^3}\left(\mathrm{d}A_{(k)} \wedge B_{(k)} + \sigma^A_{(k)}D^B_{(k)} + \sigma^B_{(k)}D^A_{(k)} + \text{fermions}_{(k)}\right), \tag{5.16}$$

together with the insertion of the Wilson loop along the entangling surface $\theta = 0$

$$W^{\text{susy}}_{(k)}\left(\frac{}{2}(1-n) + k\right) \equiv \exp\left[i\left(\frac{}{2}(1-n) + k\right)\oint(B_{(k)} - i[\gamma]\sigma^B_{(k)})\right]. \tag{5.17}$$

The procedure for endowing $\mathcal{V}^A_{(k)}$ with the desired singular vortex configuration is by integrating out the $\mathbb{Z}_n$ gauge fields $B_{(k)}$ when computing the following expectation value

$$\left\langle \prod_{k=0}^{n\ 1} e^{\ S^{\text{susy}}_{BF,(k)}} W^{\text{susy}}_{(k)}\left(\frac{}{2}(1-n) + k\right)\right\rangle_{(n)}, \tag{5.18}$$

where the subscript (n) stands for an expectation value in the n-copy theory.

Because $Z^{\text{1-loop}}_{\text{matter}}$ depends on the constant adjoint valued field σ_0 from the vector multiplet, its value on each copy of the theory will depend on distinct constant modes $\sigma_{0(k)}$. However, due to the identifications of all fields at the lower and upper (in the timelike direction) boundaries of the entangling region implied by the original codimension one defect (which was translated into the codimension two defect implemented by the Z_n gauge coupling), we see that, precisely because the modes are constant, we have $\sigma_{0(i)} = \sigma_{0(j)}$ for all $i, j \in \{0, ..., n-1\}$. This can be rephrased in terms of the insertion of a determinant in the integration over the Lie algebra at each copy, which represents invariance of the theory under a moduli space of transformations acting on the σ_0 fields on every copy; this determinant is

$$\delta_{\text{moduli}}(\sigma_{0(k)}) = \delta(\sigma_{0(k)} - \sigma_{0(k+1)}). \tag{5.19}$$

We are now ready to write the matter one-loop chiral determinant of the n-copy theory, where the integration over the moduli space of σ_0 configurations is left implicit:

$$\begin{aligned} Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, n) &= \prod_{k=0}^{n \quad 1} {}_h \left(i\rho(\sigma_0) + \frac{}{2}\left(\frac{1}{n} - 1\right) + \frac{k}{n} + \quad \middle| 1, 1 \right) \\ &= {}_h \left(i\rho(\sigma_0) + \frac{}{2}\left(\frac{1}{n} + 1\right) \middle| 1, \frac{1}{n} \right). \end{aligned} \tag{5.20}$$

The second step is due to properties of the hyperbolic Gamma function summarized in Appendix D of [25]. This corresponds to the result for the $3d$ $\mathcal{N} = 2$ supersymmetric Rényi entropy. We conclude that this quantity is computed by considering an n-copy theory with an insertion of an abelian vortex of charge $q^{(k)}_{\text{vortex}}$ in each copy. For future purposes let us rewrite the above superdeterminant in terms of products over eigenmodes:

$$\begin{aligned} Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, n)_{S^3} &= \prod_{k=0}^{n \quad 1} \prod_{m;l \quad 0} \frac{i\rho(\sigma_0) + \frac{}{2}\left(\frac{1}{n} - 1\right) + \frac{k}{n} - \quad + 2 + m + l}{-i\rho(\sigma_0) - \frac{}{2}\left(\frac{1}{n} - 1\right) - \frac{k}{n} + \quad + m + l}, \\ &= \prod_{m;l \quad 0} \frac{i\rho(\sigma_0) - \frac{2}{2}\left(1 + \frac{1}{n}\right) + \frac{m}{n} + l}{-i\rho(\sigma_0) + \frac{}{2}\left(1 + \frac{1}{n}\right) + \frac{m}{n} + l}. \end{aligned} \tag{5.21}$$

5.2 Localization on S^3 with boundaries

We will now review the calculation from [61] for the partition function of $3d$ $\mathcal{N} = 2$ super Yang-Mills on S^3 with a boundary along $\theta = \theta_0$. This result will allow us to compute Rényi entropy for non-simply connected regions $\mathcal{A}$ in the next section.

5.2.1 Preserved supersymmetries on S^3 with boundary

Let us first discuss preserved supersymmetries on S^3 with boundaries. We will essentially perform the same analysis as in 3.3.1 with a distinct vierbein, which will be more convenient to implement boundary conditions. We choose a diagonal vierbein:

$$e^1 = \cos\theta \mathrm{d}\varphi, \quad e^2 = \sin\theta \mathrm{d}\tau, \quad e^3 = \mathrm{d}\tau. \tag{5.22}$$

On the round S^3, one considers the following Killing spinor equation (setting $l = 1$):

$$D \varepsilon = \frac{i}{2l}\gamma \Rightarrow (\partial + \frac{i}{4}\omega^{ab}\varepsilon_{abc}\gamma^{c})\varepsilon = \frac{i}{2}\gamma\ \varepsilon. \tag{5.23}$$

After computing the Christo□el symbols

$$\quad = \quad = -\tan\theta, \quad = \frac{1}{2}\sin(2\theta), \quad = -\frac{1}{2}\sin(2\theta), \quad = \quad = \cot\theta, \tag{5.24}$$

we find a spin connection given by

$$\begin{aligned} \omega^{12} &= e^{1} g \nabla e^{2} \mathrm{d}x = 0, \\ \omega^{13} &= e^{1} g \nabla e^{3} \mathrm{d}x = -\sin\theta \mathrm{d}\varphi, \\ \omega^{23} &= e^{2} g \nabla e^{3} \mathrm{d}x = \cos\theta \mathrm{d}\tau. \end{aligned} \tag{5.25}$$

It is easy to check that we find the independent solutions:

$$\varepsilon_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{\frac{i}{2}(\quad)} \\ e^{\frac{i}{2}(\quad + \quad)} \end{pmatrix}, \quad \varepsilon_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{i}{2}(\quad + \quad)} \\ e^{\frac{i}{2}(\quad)} \end{pmatrix}. \tag{5.26}$$

For instance, the φ component of the Killing spinor equation for ε_1 is

$$\begin{aligned} &\left(\partial + 2\frac{i}{4}(\gamma^{3}\omega^{12} - \gamma^{2}\omega^{13} + \gamma^{1}\omega^{23}) - \frac{i}{2}\cos\theta\gamma^{1}\right)\varepsilon_1 = \\ &= \left(-\frac{i}{2} + \frac{i}{2}\sin\theta\gamma^{2} - \frac{i}{2}\cos\theta\gamma^{1}\right)\varepsilon_1 \\ &= \left(-\frac{i}{2} + \frac{i}{2}\begin{pmatrix} 0 & e^{i} \\ e^{i} & 0 \end{pmatrix}\right)\varepsilon_1 = 0, \end{aligned} \tag{5.27}$$

with similar cancellations occuring for the remaining components and for the ε_2 equation.

On a manifold without boundaries, the actions constructed out of $\mathcal{L}_{\text{matter}}$, $\mathcal{L}_{YM}$ and $\mathcal{L}_{CS}$ are immediately invariant under supersymmetry, since the variation of these Lagrangians is zero up to total spacetime derivatives. Obviously, this does not ensure supersymmetry of these actions when the theory is placed on a manifold with boundaries. We can find appropriate boundary conditions for the fields of the theory by requiring that the boundary terms in the SUSY variations δS_{matter}, δS_{YM} and δS_{CS} vanish, and that the Killing spinors obey the same boundary conditions as the fermionic fields. For the vector multiplet, for instance, these are:

$$A_{\tilde{\mu}}\big|_{0} = a_{\tilde{\mu}}, \quad \sigma\big|_{0} = \sigma_0, \quad -e^{i(\quad)}\gamma\ \lambda\big|_{0} = \bar{\lambda}, \tag{5.28}$$

where $\tilde{\mu}$ denotes coordinates orthogonal to the boundary; $a_{\tilde{\mu}}$ and σ_0 are constant modes valued in the Cartan, and therefore $[a_{\tilde{\mu}}, \sigma_0] = F_{\tilde{\mu}\tilde{\nu}} = 0$. It is easily checked that the solutions 5.26 satisfy the gaugino boundary conditions:

$$-e^{i(\quad)}\gamma\ \varepsilon_1 = \varepsilon_2. \tag{5.29}$$

As an example, we can verify supersymmetry of S_{YM}. Inserting $D\ \sigma\big|_0 = F_{\sim\sim}\big|_0 = 0$ in eq. (B.1) of [61], we immediately obtain

$$\int d^3x\sqrt{g}\delta\mathcal{L}_{YM} \propto (\bar{\lambda}\gamma\ \varepsilon_1 - \underbrace{\lambda\gamma\ \bar{\varepsilon}_1}_{=\ ''_2\ \theta}) = e^{i(\ '\quad)}(\bar{\lambda}\varepsilon_2 + \varepsilon_2\bar{\lambda}) = 0. \tag{5.30}$$

5.2.2 Localization and matter one-loop determinant

Let us now review the argument for the cancellation bewteen bosonic and fermionic eigenvalues in the matter one-loop determinant, which is slightly distinct from the one on the round S^3. Suppose there is a fermionic eigenmode for ψ satisfying $\psi = \nu\psi$. Then, defining

$$\omega \equiv i\sigma_0 - \left(r - \frac{1}{2}\right), \tag{5.31}$$

$\phi_1 \equiv \bar{\varepsilon}\psi$ is a scalar eigenmode with eigenvalue equation $\phi_1 = \nu(\nu-2\omega)\phi_1$ satisfying the boundary condition. Conversely, given a scalar eigenmode $\phi \equiv M^2\phi$, we find that $\psi = \left(\nu - \omega + \frac{1}{2}\right)\varepsilon\phi - i\gamma\ \varepsilon D^{(a)}\phi$ is a fermionic eigenmode with eigenvalue $\nu = \omega \pm \sqrt{M^2 + \omega^2}$.

The conditions for the unpaired fermionic modes are such that $\phi_1 \equiv \bar{\varepsilon}\psi = 0$. For these modes, we may take $\psi = \bar{\varepsilon}$ as a $\bar{\delta}$-exact field, for some scalar . Now, ϕ_1 is vanishing but ψ is still a non-zero fermionic eigenmode. This gives

$$\psi = \nu\psi \Rightarrow \left(-i\not{D}^{(a)} + \omega - \frac{1}{2}\right)\bar{\varepsilon} = \nu\bar{\varepsilon}\ . \tag{5.32}$$

We obtain two independent conditions on by contracting this equation with $\bar{\varepsilon}$ and ε separately; one of them is

$$\bar{\varepsilon}\not{D}^{(a)}\bar{\varepsilon} = 0 \Rightarrow v\ D^{(a)} = 0, \tag{5.33}$$

with $v^a \equiv \bar{\varepsilon}\gamma^a\bar{\varepsilon} = (i\sin\theta, i\cos\theta, -1)e^{i(\ '\quad)}$. This gives

$$\begin{aligned} &v\ D\ \ + v^{'} D_{'}\ \ + v\ D\ \ = 0 \Rightarrow \\ &\Rightarrow -\partial\ \ + i\tan\theta(\partial_{'} - ia_{'})\ \ + i\cot\theta\partial\ \ = 0. \end{aligned} \tag{5.34}$$

The previous equation is satisfied by the ansatz $(\theta,\tau,\varphi) \propto \cos^{m\ \ a_\varphi}\theta\sin^n\theta e^{im'\ \ in}$, and this solution is unique since 5.34 becomes a sum of linearly independent powers of trigonometric functions. Indeed, the left-hand side reads

$$\begin{aligned} &e^{im'\ \ in}\ + ((m - a_{'})\sin^{n+1}\theta\cos^{m\ \ a_\varphi\ \ 1}\theta - n\sin^{n\ \ 1}\theta\cos^{m\ \ a_\varphi+1}\theta) + \\ &+ e^{im'\ \ in}\ \cos^{m\ \ a_\varphi}\theta\sin^n\theta(-(m - a_{'})\tan\theta + n\cot\theta) = 0. \end{aligned} \tag{5.35}$$

We must set $m \in \mathbb{Z}, n \geq 0$ to have globally regular solutions.

The second condition resulting from 5.33 determines the eigenvalues of the unpaired modes:

$$
\begin{aligned}
\varepsilon\left(-i\not{D}^{(a)}+\omega+1\right)\tilde{\varepsilon} &= \nu\underbrace{\varepsilon\tilde{\varepsilon}}_{=1} && \Rightarrow -iv\ \ D^{(a)} = (\nu-\omega-1)\\
&&& \Rightarrow -i\frac{-\cos\theta}{\cos\theta}(\partial\cdot - ia\cdot)\ \ -i\frac{\sin\theta}{\sin\theta}\partial \ \ = \left(\nu-\omega+\frac{1}{2}\right)\\
&&& \Rightarrow (m-a\cdot)\ \ +n \ \ = (\nu-\omega-1)\\
&&& \Rightarrow \nu = i\sigma_0 - r + 2 + m + n - a\cdot,
\end{aligned}
\tag{5.36}
$$

where $v^a \equiv \varepsilon\gamma^a\tilde{\varepsilon} = (-\sin\theta, \cos\theta, 0)$.

5.3 Inserting vortices on S^3 with a boundary

We now extend the calculation of the previous section to obtain the partition function on the n-branched cover of S^3 with a boundary along $\theta = \theta_0$, in the presence of a conical singularity along $\theta = 0$. We do this by using the defect operator prescription which implements the correct singular configurations for a background vector multiplet which computes supersymmetric Rényi entropy. In the cases where the theory is superconformal, this computation will correspond to the Rényi entropy in flat space for an entangling region given by the ring

$$
\mathcal{A} = \{(t=0, r, \phi) : R\ \ \le r \le R, \phi \in [0, 2\pi)\}, \tag{5.37}
$$

with $0 < R\ \ < R$. By the CHM map, this region is conformally mapped to S^3 with a boundary at $\theta = \theta_0$, given by the metric

$$
ds^2 = d\theta^2 + \sin^2\theta d\tau^2 + \cos^2\theta d\phi^2, \tag{5.38}
$$

with $\theta \in [0,\theta_0]$, where $\theta_0 < \ \frac{}{2}$ is the image of $r = R$ under this map.

5.3.1 Boundary at $\theta_0 = \frac{}{2}$

For $\theta \in [0, \frac{}{2})$ our entangling region is $0 < \rho < R$, a circle from which the point in the centre has been removed. Given the result for the partition function on S^3 with a boundary along $\theta_0 = \frac{}{2}$ [61],

$$
Z^{\text{1-loop}}_{\text{matter}}(\sigma_0) = \prod_{l\ \ 0} \frac{\prod_{m\ \ 0}(i\rho(\sigma_0) - r + 2 + m + l)}{\prod_{m\ \ 1}(-i\rho(\sigma_0) + r + m + l)}, \tag{5.39}
$$

we can attempt to obtain the corresponding supersymmetric Rényi entropy by taking n copies of this theory and inserting a background vortex configuration of charge 5.15 which reproduces the correct holonomy around ∂ while maintaining supersymmetry. This is achieved by[6]

[6] The apparent di□erence in signs in front of ($_0$) relative to the result from [25], but this is not relevant since $_0$ is being integrated over the Cartan; we made use of this fact to change the signs of ($_0$) according to [25], and then we perform the shift $^{(k)}_0$! $^{(k)}_0$ $iq^{(k)}_{\text{vortex}}$. Note that the additional term we have here seems to break the n ! 1=n duality that is observed in the Rényi entropy in 3d and 4d.

$$
\begin{aligned}
Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, n)\Big|_{\theta_0=\frac{\pi}{2}} &= \prod_{k=0}^{n-1}\prod_{l\geq 0}\frac{\prod_{m\geq 0}\left(-i\rho(\sigma_0) - \frac{r}{2}\left(\frac{1}{n}-1\right) - \frac{k}{n} - r + 2 + m + l\right)}{\prod_{m\geq 1}\left(i\rho(\sigma_0) + \frac{}{2}\left(\frac{1}{n}-1\right) + \frac{k}{n} + r + m + l\right)} \\
&= \left[\prod_{k=0}^{n-1}\prod_{m;l\geq 0}\frac{-i\rho(\sigma_0) - \frac{r}{2}\left(\frac{1}{n}-1\right) + \frac{k}{n} - r + 2 + m + l}{i\rho(\sigma_0) + \frac{r}{2}\left(\frac{1}{n}-1\right) + \frac{k}{n} + r + m + l}\right] \times \\
&\quad \times \prod_{k=0}^{n-1}\prod_{l\geq 0}\left(-i\rho(\sigma_0) + \frac{r}{2}\left(\frac{1}{n}-1\right) + \frac{k}{n} + r + l\right) \\
&= Z^{\text{1-loop}}_{\text{chiral}}(\sigma_0, n)_{S^3}\prod_{k=0}^{n-1}\prod_{l\geq 0}\left(i\rho(\sigma_0) + \frac{}{2}\left(\frac{1}{n}-1\right) + \frac{k}{n} + r + l\right) \\
&\equiv Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, n)_{S^3} Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(\sigma_0, n).
\end{aligned}
\tag{5.40}
$$

This manipulation shows that the result for the boundary considered here nicely factorises into a product of the partition function on S^3 with a contribution from the presence of the boundary.

$\theta_0 = \frac{\pi}{2}$ boundary entropy = Chiral multiplet at the boundary ?

It seems that the additional term $Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(\sigma_0, n)$ relative to the partition function on S^3 term can be realized by including an extra chiral multiplet, at each copy of the theory, living on the great circle $\theta = \frac{\pi}{2}$, with BPS configurations such that $\phi = 0$ and ψ as in [61], so that the 1-loop determinant for this new chiral multiplet essentially reduces to a fermionic determinant. The expansion into one dimensional Fourier modes reproduces the desired correction. In addition, this extra chiral multiplet should be still coupled to a background abelian vector multiplet (independent from the $\mathcal{V}^A$ above) with a vortex configuration on this circle, whose bottom component has a background configuration determined by the vortex charge. However, the correct vortex charge that should be coupled to this new chiral multiplet at each copy is now

$$
q_{\text{vortex}} = \frac{}{2}\left(\frac{1}{n}-1\right) + \frac{k}{n} - 2, \tag{5.41}
$$

where the -2 cancels the $+2$ coming from the fermionic 1-loop determinant, and allows us to reproduce the desired determinant.

The author is unsure about the validity of this procedure with regards to supersymmetry preservation of an action whose fields are in configurations as described above. However, if this is correct we can attribute the degrees of freedom measured by the $Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(\sigma_0, n)$, and consequently the boundary contribution to the entanglement entropy derived below, to fermions propagating on the boundary $\theta_0 = \frac{\pi}{2}$.

Boundary entropy of free $\mathcal{N} = 2$ chiral multiplet

From 5.40 we can compute the correction to the entanglement entropy relative to the one for a disk-shaped entangling region. We consider a free chiral multiplet for simplicity, allowing us to set $\sigma_0 = 0$. Such a correction takes the general form

$$\begin{aligned} S^{\theta_0=\frac{\pi}{2}}_{\text{boundary}} &= \lim_{n\to 1}\frac{1}{1-n}\Big\{\log Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(1) + (n-1)\frac{d}{dn}\log Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(n)\Big|_{n=1} + \\ &\quad + \mathcal{O}((n-1)^2) - n\log Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(1)\Big\} \\ &= \log Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(1) - \frac{d}{dn}\log Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(n)\Big|_{n=1}. \end{aligned} \tag{5.42}$$

- $Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(n)$ at $n=1$:

The boundary contribution to the logarithm of the partition function for the single copy theory is

$$\log\left[r\prod_{l\geq 1}(r+l)\right] = \log\left[r\frac{\sqrt{2\pi}}{\Gamma(1+r)}\right], \tag{5.43}$$

where we have used the ζ-regularized infinite product $\prod_{n\in\mathbb{N}}(an+b) = a^{\frac{1}{2}-\frac{b}{a}}\frac{\sqrt{2\pi}}{\Gamma(\frac{b}{a}+1)}$. Replacing the value for the superconformal R-charge in 3 dimensions, $r=\frac{1}{2}$ [55], the above gives a contribution to minus the free energy of (inserting $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$)

$$\log Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(\sigma_0,1) = \log\left(\frac{1}{2}\frac{\sqrt{2\pi}}{\frac{\sqrt{\pi}}{2}}\right) = \frac{1}{2}\log 2. \tag{5.44}$$

This is an intriguing result - for a free chiral multiplet, the contribution to the partition function on S^3 resulting from including a boundary at $\theta_0 = \frac{\pi}{2}$ is exactly the same as the one for the partition function on the round S^3 but with the opposite sign. Note that this result is dependent on the choice of R-charge at the UV which here corresponds to matter fields conformally coupled to the metric.

- $Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(n)$ at $\mathcal{O}(1-n)$:

The boundary contribution to the logarithm of the partition function on the n-copy theory is

$$\log\left[\prod_{k=0}^{n-1}\prod_{l\geq 0}\left(\frac{r}{2}\left(\frac{1}{n}-1\right)+\frac{k}{n}+r+l\right)\right]. \tag{5.45}$$

Since the product over l does not formally converge, we should first perform this product and explicitly regularize it before performing the product over k. We again employ the infinite sum identity used in 5.43 to write

$$\begin{aligned} \prod_{l\geq 0}\left(\frac{r}{2}\left(\frac{1}{n}-1\right)+\frac{k}{n}+r+l\right) &= \left(\frac{r}{2}\left(\frac{1}{n}-1\right)+\frac{k}{n}+r\right)\prod_{l\in\mathbb{N}}\left(\frac{r}{2}\left(\frac{1}{n}-1\right)+\frac{k}{n}+r+l\right) \\ &= \left(\frac{r}{2}\left(\frac{1}{n}-1\right)+\frac{k}{n}+r\right)\frac{\sqrt{2\pi}}{\Gamma\left(\frac{r}{2}\left(\frac{1}{n}-1\right)+\frac{k}{n}+r+1\right)}. \end{aligned} \tag{5.46}$$

For the product over copies, we use the multiplication theorem for the Gamma function,

$$\prod_{k=0}^{n-1}\Gamma\left(\frac{r}{2}\left(\frac{1}{n}-1\right)+\frac{k}{n}+r+1\right) = (2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-n\left(\frac{r}{2}\left(\frac{1}{n}-1\right)+r+1\right)}\Gamma\left[n\frac{r}{2}\left(\frac{1}{n}-1\right)+nr+n\right]. \tag{5.47}$$

Another identity for the Pochhammer symbol, $\prod_{k=0}^{n-1}(a+k) = \frac{\Gamma(a+n)}{\Gamma(a)}$, gives

$$\begin{aligned}\prod_{k=0}^{n-1}\left(\frac{r}{2}\left(\frac{1}{n}-1\right)+\frac{k}{n}+r\right) &= n^{-n}\prod_{k=0}^{n-1}\left(n\frac{r}{2}\left(\frac{1}{n}-1\right)+nr+k\right)\\ &= n^{-n}\frac{\Gamma\left[n+n\frac{r}{2}\left(\frac{1}{n}-1\right)+nr\right]}{\Gamma\left[n\frac{r}{2}\left(\frac{1}{n}-1\right)+nr\right]}.\end{aligned} \tag{5.48}$$

Putting everything together, we find

$$\log\left[\prod_{k=0}^{n-1}\prod_{l\geq 0}\left(\frac{r}{2}\left(\frac{1}{n}-1\right)+\frac{k}{n}+r+l\right)\right] = n^{-n}\frac{(2\pi)^{\frac{n}{2}}(2\pi)^{-\frac{n-1}{2}}n^{-\frac{1}{2}+n\left(\frac{r}{2}\left(\frac{1}{n}-1\right)+r+1\right)}}{\Gamma\left[n\frac{r}{2}\left(\frac{1}{n}-1\right)+nr\right]}. \tag{5.49}$$

We now compute the $\mathcal{O}(1-n)$ contribution to the Rényi entropy. As noted in 5.42, this is given by the derivative of 5.49 at $n=1$. Using $\frac{d}{dz}\log\Gamma(z) = \psi^{(0)}(z)$, this yields

$$\begin{aligned}&\frac{d}{dn}\Big|_{n=1}\left\{-n\log n+\left(-\frac{1}{2}+n\left(\frac{r}{2}\left(\frac{1}{n}-1\right)+r+1\right)\right)\log n-\right.\\ &\qquad\left.-\log\Gamma\left[n\frac{r}{2}\left(\frac{1}{n}-1\right)+nr\right]\right\} =\\ &= -1-\frac{1}{2}+r+1-\psi^{(0)}(r)\left(-\frac{r}{2}+r\right)\\ &= -\frac{1}{2}+r-\frac{r}{2}\psi^{(0)}(r).\\ &= \frac{1}{2}\log 2+\frac{\gamma}{4},\end{aligned} \tag{5.50}$$

where in the last step we have inserted $r=\frac{1}{2}$, together with $\psi^{(0)}\left(\frac{1}{2}\right) = -2\log 2-\gamma$.

We conclude that the boundary contribution to the entanglement entropy is

$$S^{\theta_0=\frac{\pi}{2}}_{\text{boundary}} = \log Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(1) - \frac{d}{dn}\log Z^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}(n)\Big|_{n=1} = -\frac{\gamma}{4}, \tag{5.51}$$

and the total result for the entanglement entropy of a free $3d$ $\mathcal{N}=2$ chiral multiplet on a punctured disk is

$$\boxed{\begin{aligned}S^{\text{susy}}_{\theta_0=\frac{\pi}{2}} &= S^{\text{susy}}_{S^3} + S^{\theta_0=\frac{\pi}{2}}_{\text{boundary}}\\ &= -\frac{1}{2}\log 2-\frac{1}{4}\gamma \approx -0{,}49.\end{aligned}} \tag{5.52}$$

This represents quite a large difference in vacuum entanglement when only one point was removed from the entangling region. We can interpret such a variation in the value of the entropy to be associated to the change in the topology of the entangling region (it is no longer simply connected).

$U(1)_k \times U(1)_{-k}$ ABJM theory

The above calculation can easily be applied to the ABJM model we have already studied in 4. In this case we cannot immediately separate the boundary contribution since the whole expression is integrated in σ. However, the contribution to the free energy from each chiral multiplet remains the same as in 5.44: ($z \equiv \sigma + \tilde{\sigma}, z \equiv \sigma - \tilde{\sigma}$)

$$\begin{aligned}
Z^{\text{1-loop}}_{\text{chiral}}(\sigma_0, 1)\Big|_{0=\frac{\pi}{2}} &= \int_{1}^{1} d\sigma d\tilde{\sigma} e^{i\ k(\ ^2\ \tilde{\ }^2)}\ _h[\sigma - \tilde{\sigma} + i]^2\ _h[\tilde{\sigma} - \sigma + i]^2 \times \\
&\times \left[\prod_{l\ 0}(-i(\sigma - \tilde{\sigma}) + \quad + l)\right]^2 \left[\prod_{l\ 0}(-i(\tilde{\sigma} - \sigma) + \quad + l)\right]^2 \\
&= \frac{1}{|\pi k|}\int_{1}^{+1} dz\delta(z)\ _h[z+i]^2\ _h[-z+i]^2 \left[\prod_{l\ 0}(-iz + \quad + l)\right]^2 \left[\prod_{l\ 0}(iz + \quad + l)\right]^2 \\
&= Z_{\text{ABJM},S^3}(k)\left[\prod_{l\ 0}(\quad + l)\right]^4 .
\end{aligned} \tag{5.53}$$

The correction to minus the free energy is identical to the previous case, with each chiral multiplet contributing with $\frac{1}{2}\log 2$.

5.3.2 Boundary at $\theta_0 < \frac{}{2}$

The partition function for the free chiral multiplet (meaning we consider $\rho(a\,\cdot) = \rho(\sigma_0) = 0$) now reads [61]

$$Z^{\text{1-loop}}_{\text{chiral}} = \prod_{m}\prod_{n\ 0}(-i\rho(\sigma_0) - \quad + 2 + m + n)\,, \tag{5.54}$$

where we have retained the σ_0 dependence as a reference to know which sign to give to q_{vortex} below (and also employing the same sign convention as in [25] for consistency with the above). The partition function of a fre chiral multiplet on the n-copy theory is then

$$Z_{\ 0<\frac{\pi}{2}}(n) = \prod_{k=0}^{n\ 1}\prod_{m2Z}\prod_{l\ 0}\left(-\frac{}{2}\left(\frac{1}{n} - 1\right) - \frac{k}{n} - \quad + 2 + m + l\right). \tag{5.55}$$

In this case we will not be able to factorise the partition function into a contribution from the boundary and the result on S^3, so we compute the entire products explicitly.

- $Z_{\ 0<\frac{\pi}{2}}(n)$ at $n = 1$:

The free energy is computed by

$$\begin{aligned}
Z_{\ 0<\frac{\pi}{2}}(1) &= \prod_{m2Z}\prod_{l\ 0}(- \quad + 2 + m + l) \\
[\quad = \tfrac{1}{2}] &= \prod_{m2Z}\left(\frac{3}{2} + m\right)\frac{\sqrt{2\pi}}{[\frac{3}{2} + m + 1]} . \\
[m \to m - 2] &= \prod_{m2Z}\left(-\frac{1}{2} + m\right)\frac{\sqrt{2\pi}}{[\frac{1}{2} + m]} .
\end{aligned} \tag{5.56}$$

The product of the first term in parenthesis is

$$\prod_{m2Z}\left(-\frac{1}{2}+m\right) = -\frac{1}{2}\prod_{m>0}(-1)\left(-\frac{1}{2}+m\right)\left(\frac{1}{2}+m\right) = \frac{1}{2}\frac{2\pi}{\left(\frac{1}{2}\right)\ \left(\frac{3}{2}\right)}\prod_{m\ \ 0}(-1) = 2\prod_{m\ \ 0}(-1), \tag{5.57}$$

For the numerator, we use the ζ-regularised product $\prod_{n2N} a = a^{\ \frac{1}{2}}$ and the formulas for the Gamma function evaluated at half-integer arguments,

$$\left(\frac{1}{2}+n\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \qquad \left(\frac{1}{2}-n\right) = \frac{(-1)^n 2^n}{(2n-1)!!}\sqrt{\pi}, \tag{5.58}$$

giving

$$\prod_{m2Z}\frac{\sqrt{2\pi}}{\left[\frac{1}{2}+m\right]} = \sqrt{2}\left[\prod_{m>0}\frac{2\pi}{\left[\frac{1}{2}+m\right]\ \left[\frac{1}{2}-m\right]}\right] = \sqrt{2}\left(\prod_{m2N}\frac{2\pi}{(-1)^m\pi}\right) = \prod_{m>0}(-1)^m. \tag{5.59}$$

The contribution to the Rényi entropy at $\mathcal{O}((1-n)^0)$ is then

$$\log Z^{\ 0<\frac{\pi}{2}}(n=1) = \log 2 + \dots, \tag{5.60}$$

where the ellipses denote an imaginary contribution proportional to $\log(-1)$, which we may discard.

Unfortunately, we find that the ζ function regularization does not give unambiguous result for the quantity just computed. This is because we could consider performing first the product over the $m \in$ Z eigenmodes intead ov the product over l, since both products are formally divergent and there is no predetermined order in which they should be performed. This allows us to shift $m \to m + l + 2$ for every $l \geq 0$. We then have

$$\begin{aligned} Z^{\ 0<\frac{\pi}{2}}(1) &= \prod_{l\ \ 0}\prod_{m2Z}(-\quad + m + l) \\ [\quad = \frac{1}{2}] &= \prod_{l\ \ 0}\prod_{m2Z}\left(-\frac{1}{2}+m\right) \\ &= 2\prod_{l2N} 2 + \dots = \sqrt{2} \Rightarrow \\ \Rightarrow \log Z^{\ 0<\frac{\pi}{2}}(n=1) &= \frac{1}{2}\log 2, \end{aligned} \tag{5.61}$$

which, again, holds up to imaginary terms.

We have not been able to solve for the $\mathcal{O}(1-n)$ contribution to the entanglement entropy. Its computation relies on performing an infinite product of Gamma functions, for which we have not found a closed form.

5.3.3 Comments (disk topology maximizes S_A^{susy}?)

With either choice of regularization 5.60 or 5.61, from the previous two subsections we can draw the conclusion that the free energy of a conformal free chiral multiplet decreases by multiples of $\log 2$ when the entangling region is a non-simply connected disk. This confirms that the universal entanglement of this theory is characterised by this quantity, $\log 2$, and it would therefore be interesting to understand its meaning.[7] Moreover, we should emphasize that the results collected in these limited cases show that the round disk maximizes the free energy among non-simply connected spacetimes (and the free energy decreases when the dimension of the boundary is increased). Although a computation of the full entanglement entropy was only achieved in the $\theta_0 = \frac{\pi}{2}$ case, it seems natural to conjecture that the round disk maximizes S_A^{susy} in 2 + 1 dimensions among entangling regions given by non-simply connected disks. One reason for this suggestion is that we observe it to be true for the $\theta_0 = \frac{\pi}{2}$ case, where both the free energy and the entanglement entropy are decreased relative to the round disk, while for the $\theta_0 < \frac{\pi}{2}$ case we again observe a decrease in free energy relative to the $\theta_0 = \frac{\pi}{2}$ case. We warn that the analogous statement for entangling regions with conical topology will be contradicted in 6.3. However, one would not necessarily expect the behaviour of the entanglement entropy to be similar when introducing conical (curvature) singularities or smooth boundaries inside the entangling region, given that these are drastically distinct geometric operations.

Furthermore, the claim of the previous paragraph is in agreement with the results found in [62], which studied the universal term of the entanglement entropy in generic $3d$ CFT's for different topologies of the entangling region. Nevertheless, note that such an agreement would not necessarily be expected because, as we have noted below 2.35, the supersymmetric definition of Rényi entropy we are working with differs from von Neumann entropy in generic CFT's due to the background deformation of the theory on the branched sphere. In fact, in the next chapter we will find a disagreement between the behaviour of S_A^{susy} and the one previously demonstrated for von Neumann entropies in CFT's when considering smooth deformations of $\partial\mathcal{A}$.

[7] It is tempting to attribute the gradual increase of the free energy in the sequence of cases disk $\to$ boundary at $\theta_0 = \frac{\pi}{2}$ $\to$ boundary at $\theta_0 < \frac{\pi}{2}$ to degrees of freedom of additional modes propagating on the boundaries we introduce on the disk, and whose contribution increases in the latter transition due to the increase in the dimension of the boundary; this was also suggested around 5.41. Such an interpretation would hint towards topological behaviour of S_A^{susy}, and we will have a lot more to say about this in the next chapter.

6 Parameter Dependence of Supersymmetric Entanglement Entropy

We will now move on to study the dependence of entanglement entropy in $3d$ $\mathcal{N}$ = 2 superconformal theories on continuous deformations of the boundary of the entangling region. As in the previous chapter, we will rely on conformally mapping this region to a compact space using the CHM map. By making use of know results in the literature about the dependence of superymmetric partition functions on parameters of the background geometry of compact three-manifolds, we will draw conclusions on the shape dependence of entanglement entropy. We will first review general results regarding supersymmetry on three-manifolds.

6.1 Geometry of three-manifolds preserving two supercharges

Transversely holomorphic foliations

Consider a compact, oriented Riemannian three-manifold $\mathcal{M}_3$ which preserves some supersymmetry. We will focus on the case where there exist at least two Killing spinors of opposite R-charge, ζ and ζ. It can then be shown, using the Killing spinor equations 3.42, that the following vector on $\mathcal{M}_3$ is Killing:

$$K \ = \zeta\gamma\ \zeta \Rightarrow \nabla\ K\ + \nabla\ K\ = 0. \tag{6.1}$$

We can additionally define the following tensors on $\mathcal{M}_3$:

$$\eta\ = \frac{1}{\ }K\ , \qquad = \varepsilon\ \ \eta\ , \tag{6.2}$$

where $= K\ K$ is a normalization factor such that $\eta\ \eta\ = 1$. It is immediate to see that satisfies

$$= -\delta\ + \eta\ \eta\ . \tag{6.3}$$

The condition 6.3 defines η as an *almost contact metric structure* on $\mathcal{M}_3$; this is a three-dimensional analogue of a complex structure on four-dimensional manifolds, in the sense that it acts on the sub-bundle of $T\ \mathcal{M}_3$ whose fibers are orthogonal to η as $^2 = -$Id. This, together with the fact that the following integrability condition holds,

$$(\mathcal{L}_K\ ) \ = 0, \tag{6.4}$$

implies that the triple ($K\ , \eta\ ,$) defines a *tranversely holomorphic foliation* (THF) on $\mathcal{M}_3$. The fact that $\mathcal{M}_3$ is endowed with this structure implies that is can be covered by coordinate charts (ψ, z, z) [54]. Here, $\psi \in$ R and $z \in$ C, and there exists a change of coordinates map $(\psi, z, z) \to (\psi', z', z')$ such that

$$\psi' = \psi + t(z, z), \quad z' = f(z), \tag{6.5}$$

where $t(z, z)$ is real and $f(z)$ is holomorphic. More importantly, due to the presence of the Killing vector K which we choose to be real (so that it generates one single isometry) we can always choose coordinates such that $K = \partial$. The metric in the adapted coordinates (ψ, z, z) takes the form

$$ds_{M_3} = \big(d\psi + h(z, z)dz + h(z, z)dz\big)^2 + c(z, z)^2 dz dz. \tag{6.6}$$

In particular, the one-form η satisfies

$$\eta \; dx = d\psi + h(z, z)dz + h(z, z)dz. \tag{6.7}$$

This means that the nowhere vanishing Killing vector $K = g \; \eta$ determines a fibration over the complex two-dimensional manifold parametrized by (z, z).

One type of manifolds admitting a THF is Seifert manifolds, which are three-manifolds endowed with a locally free $U(1)$ action on the fibers of a circle bundle over a two-dimensional Riemann surface :

$$S^1 \longrightarrow \mathcal{M}_3 \longrightarrow \tag{6.8}$$

Deformations of the THF on $\mathcal{M}_3$

We now look at supersymmetry-preserving deformations of the THF. These are infinitesimal deformations K , η and which satisfy the almost contact structure condition 6.3 and the integrability condition 6.4. In terms of the adapted coordinates (ψ, z, z), it is found that these two conditions imply [63]

$$\begin{aligned} (\quad - ih \; K^z + ih \; K^z)\partial \; h &= 0, \\ \partial \; (\quad {}^z{}_z - ih \; K^z) + 2i\partial_z \; K^z &= 0. \end{aligned} \tag{6.9}$$

The first condition can always be satisfied by tuning , while the second condition gives us more non-trivial information. To see this, consider the following anti-holomorphic one-form[1]

$$^z = -2i \; K^z(d\psi + hdz) + (\quad {}^z{}_z - ih \; \psi^z)dz. \tag{6.10}$$

In [63] the authors construct a de-Rham-like differential operator

$$\bar\partial^{p,q} \to \quad ^{p,q+1}, \quad \bar\partial\omega^{p,q} = d\omega^{p,q}|_{\;p,q+1}, \tag{6.11}$$

[1] On a THF there is a natural notion of (anti-)holomorphic one-forms, given that it is a three-dimensional analogue of a complex structure. These are defined by their eigenvalue under the projector

$$^\mu{}_\nu = \frac{1}{2}(\; ^\mu{}_\nu \quad i \; ^\mu{}_\nu \quad K^\mu \; _\nu), \qquad ^\mu{}_\nu \; ^\nu{}_\rho = \; ^\mu{}_\rho.$$

A holomorphic one-form in the holomorphic cotangent bundle, $\omega^{1,0} \in {}^{1,0}$, satisfies $\omega^{1,0}_\mu \; ^\mu{}_\nu = \omega^{1,0}_\nu$, and thus it can be written in adapted coordinates as $\omega^{1,0} = \omega^{1,0}_z dz$. The fact that it is holomorphic is garenteed by its transformation under changes of coordinate patches, $(\omega'^{1,0})_{z'} = \frac{1}{f'(z)}\omega^{1,0}_z$. Conversely, anti-holomorphic one-forms $\omega^{0,1} \in {}^{0,1}$ satisfy $\omega^{0,1}_\mu \; ^\mu{}_\nu = 0$, and are given by $\omega^{0,1} = \omega^{0,1}_\psi(d\psi + hdz) + \omega^{0,1}_z dz$.

acting on (0,1)-forms in $\;^{0,1}$ according to

$$\bar{\partial}\left(\omega^{0,1}(d\psi + hdz) + \omega_{\bar{z}}^{0,1}d\bar{z}\right) = \left(\partial \;\; \omega_{\bar{z}}\omega_{\bar{z}}^{0,1} - \partial_{\bar{z}}\omega^{0,1}\right)(d\psi + hdz) \wedge d\bar{z}. \tag{6.12}$$

Then, the second integrability condition for the deformations of the THF in 6.9 can be expressed in terms of the condition that $\;^{z}$ is $\bar{\partial}$-closed:

$$\bar{\partial} \;\;^{z} = 0. \tag{6.13}$$

This is quite remarkable, since non-trivial deformations of the THF can now be parametrized by the $\bar{\partial}$-cohomology classes

$$[\;\;^{z}] \in H^{0,1}(\mathcal{M}_3, T^{1,0}\mathcal{M}_3). \tag{6.14}$$

Below we will work out the linearized variation of the Lagrangian under a deformation of the THF. We will be able to know whether such deformations a ect the partition function by looking at whether cohomology classes determined by deformation parameters are trivial or not.

Lagrangian deformations

By considering the flat space supersymmetry algebra of the components of the $\mathcal{R}$-multiplet 3.26, we can construct all the bosonic Q-exact operators of the form $\{Q, \mathcal{S} \;\;\}$ (see section 6.1 of [63], where these are spelled out). As an example, one of them is

$$\mathcal{T}_{zz} = T_{zz} - \frac{i}{2}\partial_z j_z^{(R)} - \frac{1}{4}\partial \;\; J^{(Z)}. \tag{6.15}$$

This corresponds to contracting ζ with the $\mu = z$ component of the anticommutator

$$\{Q, \mathcal{S} \;\;\} = \zeta \;\left(2j^{(Z)} + i\varepsilon \;\;\; \partial \;\, j^{(R)}\right) + (\gamma \;\,\zeta) \;\left(2iT \;\;\; + \partial \;\, j^{(R)} - \varepsilon \;\;\; \partial \;\, J^{(Z)}\right). \tag{6.16}$$

Using that the Killing spinors are anticommuting ($\zeta \;\,\zeta \;\; = \zeta \;\,\varepsilon \;\;\; \zeta \;\; = 0$), together with the fact that the one-form $P \;\; = \zeta\gamma \;\,\zeta$ is anti-holomorphic (see for instance section 3 of [54]):

$$\zeta \;\, \{Q, \mathcal{S} \;\,_z\} = P^z\left(2iT_{zz} + \partial_z j_z^{(R)} - \partial \;\; J^{(Z)}\right). \tag{6.17}$$

Now recall the linearized variation of the Lagrangian under metric deformations around flat space by $\;\; g \;\;$, 3.37. The fields from the background supergravity multiplet in this expression can be expressed in terms of the infinitesimal deformation parameters of the THF. Substituting the resulting form for the background fields into $\;\mathcal{L}$, the following expression is obtained [63]:

$$\begin{aligned}\mathcal{L} = &-4 \;\; g_{zz}\mathcal{T}_{zz} - 2 \;\; \eta_z(\mathcal{T} \;\,_z - i\mathcal{J}_z) - 2 \;\; \eta_z\mathcal{T} \;\,_z +\\ &+ \;\; K \;\; \mathcal{T} \;\;\; + \;\; K^z\mathcal{T} \;\,_z - i \;\;\; {}^{z}{}_{z}\mathcal{T}_{zz} + \mathcal{C} \;\, j^{(Z)} + i\kappa J^{(Z)} +\\ &+ \;\; K^z\left(T \;\,_z + \frac{1}{2}\partial_z J^{(Z)}\right) + i \;\;\; {}^{z}{}_{z}\left(T_{zz} + \frac{i}{2}\partial_z j_z^{(R)}\right).\end{aligned} \tag{6.18}$$

Terms of the form $\mathcal{T}$ are Q-exact, just like 6.15. By the localization argument given in 3.4, Z_{M_3} is independent of their coe cients. In particular, it is independent of deformations of the g_{zz} component of the metric, and therefore of $c(z, \bar{z})$ in 6.6. Moreover, it can be shown that the g_{zz} component of the adapted metric is deformed as

$$g_{zz} = \frac{ic^2}{2} \quad {}^{z}{}_{z} - \frac{hc^2}{2} \quad K^z + 2h \quad \eta_z. \tag{6.19}$$

Independence of Z_{M_3} on η_z, K^z and ${}^{z}{}_{z}$ implies independence on g_{zz}, and therefore on $h(z, z)$ in 6.6.

The above analysis applies to deformations of the THF around flat space. However, it can in fact be shown that it applies to deformations around an arbitrary THF [64]. This is achieved by using a twisted description of the theory, which consists in redefining the fields in the $\mathcal{S}$ multiplet using the Killing spinors ζ and the one-form

$$p \equiv P_z = \zeta\gamma_z\zeta \in \quad (L^2 \otimes \overline{\mathcal{K}}), \tag{6.20}$$

where L is the R-symmetry line bundle over $\mathcal{M}_3$ and $\overline{\mathcal{K}}$ is the line bundle of anti-holomorphic one-forms. Such a redefinition has the effect that the new field variables have vanishing R-charge, and all fields in the same multiplet can be grouped into pairs $(\mathcal{X}, \mathcal{Y})$ such that

1. $\mathcal{X}$ and $\mathcal{Y}$ transform as sections of exactly the same bundle[2] $\mathcal{K}^{\frac{r}{2}} \otimes {}^{n;m}$ and
2. Supersymmetry transformations of this pair take the (extremely simple) form

$$\delta \ \mathcal{X} = \mathcal{Y}, \qquad \delta \ \mathcal{Y} = 0. \tag{6.21}$$

The above two facts guarantee that the supercharge defined by δ is a *scalar* under transformations of adapted coordinates. The paper [64] goes on to show that the same conclusions regarding the parameter dependence of Z_{M_3} at the linearized level that were studied above also hold in this formalism. Since δ is a scalar, δ-exact terms in $\mathcal{L}_{\mathbb{R}^3}$ remain δ-exact terms in the Lagrangian variation around $\mathcal{L}_{M_3}$ for any $\mathcal{M}_3$.

A consequence of this is that Z_{M_3} is also independent of deformations of the THF around $(K + \ K, \eta + \ \eta, \ + \ )$, since this is a well-defined THF by virtue of K, η, satisfying the integrability conditions 6.9. This means that Z_{M_3} remains invariant after applying an arbitrary number of infinitesimal deformations to the THF.

6.2 Trivial deformations of S^3 and entangling surfaces

Having studied which deformations of the adapted metric on $\mathcal{M}_3$ preserve supersymmetry while having no effect on the partition function Z_{M_3}, we will conformally map them to flat space to understand what kind of deformations of the entangling surface are trivial. In particular, the disk-shaped entangling region $\mathcal{A}$ is conformally related to the Seifert manifold S^3. Therefore, by studying the supersymmetry-preserving deformations of the THF around S^3, we will be able to determined the allowed deformations of $\mathcal{A}$, and whether they produce a change in the entanglement entropy.

To be clear, denoting by $\tilde{\mathcal{A}}$ any entangling region which is smoothly deformed with respect to the disk, we want to find regions $\tilde{\mathcal{A}}$ such that

$$Z_{\mathrm{CHM}[D_{\tilde{A}}]} = Z_{S^3}, \tag{6.22}$$

[2] This relies on the fact that, because and consequently p are globally defined, $L^2 \quad \overline{K}$ has a nowhere vanishing section and is therefore trivial. This allows the identification $L \quad (\overline{K}^{\frac{1}{2}})$. This captures the idea that we can "convert" non-trivial transformation properties of the fields under R-symmetry transformations into non-trivial transformations under adapted changes of coordinates, leading to condition 1.

where CHM[$\mathcal{D}_{\bar{A}}$] denotes the image of the causal development of $\bar{A}$ under the CHM map.

An important aspect regarding the validity of this procedure is that the KMS property of the CHM map, as shown to hold in [9] for circular entangling surfaces, is still obeyed if we apply it to a deformed entangling surface. This is because it relies on the modular flow induced from the modular Hamiltonian H, defined as $\rho = e^{\quad H}$, which always exists in a relativistic field theory and need not be local, allowing us to depart from circular entangling regions.

The strategy will be to consider deformations of the adapted metric on S^3 under small deformations of $c(z, z)$ and $h(z, z)$, which we already know leaves Z_{M_3} invariant. By applying the CHM map to the coordinates (θ, τ, φ) and performing a Weyl rescaling, we will obtain deformations of the flat metric on R^3. Schematically,

$$ds^2_{S^3} + \quad {}_c ds^2_{S^3} + \quad {}_h ds^2_{S^3} \xrightarrow{\text{CHM}} \quad {}^2\left[ds^2_{R^3} + \quad {}_c ds^2_{R^3} + \quad {}_h ds^2_{R^3}\right] \tag{6.23}$$

6.2.1 Deformations of $c(z, z)$

Consider the metric is deformed by the line element

$$c dz dz = \quad c\left(f'^2 d\theta^2 + f^2(d\varphi^2 + d\tau^2 - 2d\varphi d\tau)\right). \tag{6.24}$$

We perform the sequence of transformations of the CHM map presented in 2.2, keeping track of the required conformal factors which allow for the transformation between $ds^2_{S^3}$ and the flat space metric, as represented in 6.23. We start with the transformation that maps S^3 to H$^2 \times S^1$:

$$\begin{aligned} {}_c ds^2_{S^3} &= \quad c \sin^2\theta\Big[f'^2 \underbrace{\frac{d\theta^2}{\sin^2\theta}}_{=du^2} + \cosh^2 u f^2(d\varphi^2 + d\tau^2 - 2d\varphi d\tau)\Big] \\ &= \quad c\sin^2\theta\cosh^2\frac{\sigma}{2}\cosh^2\frac{\sigma}{2}\left[\frac{f'^2}{4}\frac{(d\sigma + d\sigma)^2}{\cosh^2\frac{}{2}\cosh^2\frac{}{2}} + \right. \\ &\left. + \frac{\cosh^2\left(\frac{+}{2}\right)}{\cosh^2\frac{}{2}\cosh^2\frac{}{2}} f^2\left(d\varphi^2 - \frac{1}{4}(d\sigma - d\sigma)^2 + i d\varphi(d\sigma - d\sigma)\right)\right]. \end{aligned} \tag{6.25}$$

We rewrite

$$\frac{\cosh^2\left(\frac{+}{2}\right)}{\cosh^2\frac{}{2}\cosh^2\frac{}{2}} = \left(1 + \tanh\frac{\sigma}{2}\tanh\frac{\sigma}{2}\right)^2 = \left(1 + \frac{|w|^2}{R^2}\right)^2 = \left(1 + \frac{\rho^2 + t^2}{R^2}\right)^2 \tag{6.26}$$

and

$$\frac{(d\sigma + d\sigma)^2}{\cosh^2\frac{}{2}\cosh^2\frac{}{2}} = \frac{4}{R^2}\frac{(\cosh^2\frac{}{2}dw + \cosh^2\frac{}{2}dw)^2}{\cosh^2\frac{}{2}\cosh^2\frac{}{2}} = \frac{4}{R^2}\left(\frac{\cosh\frac{}{2}}{\cosh\frac{}{2}}dw + \frac{\cosh\frac{}{2}}{\cosh\frac{}{2}}dw\right)^2. \tag{6.27}$$

At this point we could rewrite all the hyperbolic functions in terms of w using

$$e^{\frac{\sigma}{2}} + e^{\quad\frac{\sigma}{2}} = \sqrt{\frac{R + w}{R - w}} + \sqrt{\frac{R - w}{R + w}}. \tag{6.28}$$

However, the manipulations become significantly simplified if we consider that we will only be concerned with the entangling surface at the timeslice $t = 0$, and we may therefore restrict the line element to $t = 0$.[3] In this case, we may set $w = w$, implying $\sigma = \sigma$. In the end, we obtain the following deformation of the $t = 0$ spacelike surface in flat space:

$$ {}_c ds^2 = \quad c^2 \left[f'^2 d\rho^2 + f^2 \frac{R^2}{4} \left(1 + \frac{\rho^2}{R^2}\right)^2 d\varphi^2 \right]. \tag{6.29}$$

Now recall that the only restriction placed on $c(z, z)$ is reality. Therefore, c^2 is arbitrary, as long as it is a small deformation and it is real. In particular, we can give c arbitrary φ dependence, which will result in a non-symmetric profile for the entangling region around the origin of the plane. However, we must ensure that the θ dependence of c^2 in (τ, θ, φ) coordinates contains a factor $\sin^2\theta$, so that the above line element does not diverge as we restrict it to the entangling surface $\theta = 0$ (because $f \propto \tan\theta$). In particular, at $t = 0$ the function $f(\theta)$ is written in terms of ρ as (see Appendix C for details)

$$ f^2(\theta)\Big|_{t=0} = 4\left(\frac{R^2 - \rho^2}{R^2 + \rho^2}\right)^2, \quad f'^2(\theta)\Big|_{t=0} = \frac{16R^2\rho^2}{(R^2 + \rho^2)^2}. \tag{6.30}$$

This gives

$$ {}_c ds^2 = \quad c^2 \left[\frac{16R^2\rho^2}{(R^2+\rho^2)^2} d\rho^2 + R^2 \left(\frac{R^2 - \rho^2}{R^2 + \rho^2}\right)^2 \left(1 + \frac{\rho^2}{R^2}\right)^2 d\varphi^2 \right]. \tag{6.31}$$

The singularity of the metric noted above at $\theta = 0$ is manifested in the singularity at $\rho = 0$ in (t, ρ, φ) coordinates.

6.2.2 Trivial di□eomorphisms and integrability conditions

In order to conclude about the allowed deformations of the entangling surface which do not change the entanglement entropy, we wish to study deformations of the background metric only up to those induced by di□eomorphisms $x \to x + \xi (x)$. Nontrivial deformations of the line element which are not generated by di□eomorphisms correspond to explicit deformations of the background spacetime of our quantum field theory, which is not what we are after. This means that we want to find the spacetime vectors ξ such that

$$ {}_c g \quad = 2\nabla_{(} \ \xi \ {}_{)}, \tag{6.32}$$

where ∇ is the covariant derivative, which reduces to the partial derivative on flat space. From 6.31 we see that 6.32 will only possess diagonal components, resulting in the following integrability conditions on the di□eomorphism parameter ξ :

$$ \begin{cases} \partial \ \xi \ (\rho, \varphi) = \quad c^2 \frac{16R^2 \ {}^2}{(R^2 + \ {}^2)^2} \\ \partial_{'} \xi_{'} (\rho, \varphi) = \quad c^2 R^2 \left(\frac{R^2 \ \ {}^2}{R^2 + \ {}^2}\right)^2 \left(1 + \frac{{}^2}{R^2}\right)^2 \\ \xi_t(\rho, \varphi) = 0, \end{cases} \tag{6.33}$$

[3]One might worry that this may no longer correspond to a spacelike surface because we have now acquired a crossed term $d'dt$ in the flat space metric, altering the norm of the normal vector $_\mu = (1; 0; 0)$ in the timelike direction. However, because the metric deformation is small, by continuity c there will be no change in the sign of this norm, and the $t = 0$ hypersurface remains spacelike.

where we have imposed that timelike translations of the entangling surface are not allowed, and time dependence of ξ is neglected since we are working on a fixed time slice $t = 0$.

Finally, we want to analyse how these diffeomorphisms act on the boundary of the entangling surface. For this, we solve the integrability conditions 6.33 on the region $\rho = R$ of the $t = 0$ surface. The second condition can be trivially solved as long as c^2 is an integrable function of φ:

$$\begin{aligned} g_{\cdot\cdot}\xi^{\cdot}(\rho,\varphi) &= \left(\int d\varphi \quad c^2\right) R^2 \left(\frac{R^2-\rho^2}{R^2+\rho^2}\right)^2 \left(1+\frac{\rho^2}{R^2}\right)^2 \Rightarrow \\ \Rightarrow \xi^{\cdot}(\rho = R,\varphi) &= k(\rho = R), \end{aligned} \tag{6.34}$$

where $k(\rho)$ is some arbitrary function of ρ; the undeformed metric on S^3 is used to map forms to vectors. Because of the symmetry of the entangling surface along φ, the action of the diffeomorphism along the φ direction is trivial and leaves the entangling surface invariant.

We may also restrict to the case where c^2 has no ρ dependence (such a dependence would only contribute with some constant rescaling of ξ , given that we will evaluate it at fixed ρ). This way, the first condition can be integrated to give

$$\begin{aligned} \underbrace{\xi}_{= \ \rho}(t,\rho,\varphi) &= 16R^2 \quad c^2(t,\varphi) \int d\rho \frac{\rho^2}{(R^2+\rho^2)^2} \\ \Rightarrow \xi \ (\rho = R,\varphi) &= 2(\pi-2)R \quad c^2(\varphi) \propto \quad c^2(\varphi), \end{aligned} \tag{6.35}$$

where any residual t or ρ dependence from integration constants have been ignored. Note in particular that ξ is directly proportional to R.

We have obtained a diffeomorphism acting on the boundary of the entangling region whose radial component has an arbitrary angular dependence. This implements arbitrary smooth deformations of $\partial\mathcal{A}$. Because this results from deformations of the THF on S^3 and S^3_b which do not affect the partition functions Z_{S^3} and $Z_{S^3_b}$,[4] we conclude that

Vacuum entanglement entropy of $3d$ $\mathcal{N} = 2$ theories is independent of arbitrary smooth deformations of the boundary of the circular entangling region on flat space.

It is worth emphasizing that, due to the comment at the end of the previous section (namely that Z_{M_3} is independent of $h(z,z)$ and $c(z,z)$ and not just of infinitesimal variations of these parameters), this holds not only for infinitesimal deformations but for arbitrary ones.

Deformations of $h(z,z)$ can be analysed in a similar manner. The resulting diffeomorphisms lead to similar conclusions as above. We leave this to D.

[4] The analysis presented in the previous section regarding the independence of Z_{M_3} on $h(z;z)$ and $c(z;z)$ also holds for deformations around the THF S^3_b, provided that we regularize its metric as in 3.70. This is because, as was seen in 3, only on a regularized background do we have a well-defined integrability condition for the KSE, and therefore we need a regularized S^3_b in order for conditions such as 6.9 to make sense.

6.3 Squashed spheres and finite corrections to $S_{\mathcal{A}}^{\text{susy}}$

Having found that a large class of diffeomorphisms preserve the disk entanglement entropy, one can now ask which types of deformations of S^3 affect the partition function, and consequently lead to deformations of the entanglement across deformed circles. The first part of this question was answered by [63]; such deformations turn out to be realized by *squashed spheres*, whose metric reads

$$ds^2 = d\theta^2 + b^{-2}\cos^2\theta d\varphi^2 + b^2\sin^2\theta d\tau^2, \tag{6.36}$$

for some $b \in \mathbb{R}$. From 3.3 we already know that this metric possesses a conical singularity with deficit angle $2\pi(1-b^2)$ around $\theta = 0$ and $(1-b^{-2})2\pi$ around $\rho = \frac{\pi}{2}$. These are mapped by CHM to conical singularities at $\rho = R$ and $\rho = 0$, respectively. The latter is produced by a twisting of the $t = 0$ timeslice of flat space along the angular direction φ.

6.3.1 Non-trivial deformations of the THF on S^3

As reviewed in 6.1, the non-trivial deformations of the THF are parametrized by the cohomology classes of the (0, 1)-form 6.10 with coefficients in the holomorphic tangent bundle $T^{1,0}\mathcal{M}_3$. It is shown in [63] that there are only two distinct cohomology classes, given by[5]

$$\Theta^{(\gamma)} = \gamma X \otimes \eta, \quad X = \begin{cases} z(\partial_z - h\partial_\psi) \\ (\partial_z - h\partial_\psi) \text{ or } z^2(\partial_z - h\partial_\psi). \end{cases} \tag{6.37}$$

All the known examples of squashed spheres preserving two supercharges are realized by deformations of the first type above, and these are the ones we will focus on.

In the second type, the moduli γ can in fact be absorbed by a rescaling of the holomorphic coordinate on $\mathbb{CP}^1$, since under $z \to \gamma z$ or $z \to \gamma^{-1}z$, the vector fields $(\partial_z - h\partial_\psi)$ and $z^2(\partial_z - h\partial_\psi)$, respectively, are rescaled by a factor of γ^{-1}. For this reason, the second type does not seem to correspond to a physically relevant deformation (at least in a conformal theory). This is because we expect that such deformations can be recast in terms of partial derivatives of the free energy of the theory with respect to the moduli, which is equivalent to the insertion of some integrated operators;[6] we will see how this works below for first type.

Knowing the relevant class of deformations of S^3 which Z_{S^3} depends on, we can determine the corresponding deformations around flat space. For this we choose to work with with the metric and the one form $\Theta^{(\gamma)}$ in coordinates adapted to the THF.

For the deformation of interest, the holomorphic component of $\Theta^{(\gamma)}$ is

$$\Theta^z = \gamma z(d\psi + hdz + \bar{h}d\bar{z}). \tag{6.38}$$

[5] This is determined by first writing the adapted metric of S^3 as a circle fibration over a $\mathbb{CP}^1$ base parametrized by $(z, \bar{z})$ and realising that trivial cohomology classes in $H^{(0,1)}(S^3)$ are parametrized by (0, 1)-forms $\omega^{(0,1)} = \omega_\psi^{0,1}(d\psi + hdz) + \omega_{\bar{z}}^{0,1}$ which satisfy $\partial_\psi\omega_{\bar{z}}^{0,1} = \partial_{\bar{z}}\omega_\psi^{0,1}$, up to $\bar{\partial}$-exact elements of the form $\omega_\psi^{0,1} = \partial_\psi\epsilon$, $\omega_{\bar{z}}^{0,1} = \partial_{\bar{z}}\epsilon$ for some $\epsilon(\psi, z, \bar{z}) \in C^\infty(S^3)$. It can then be argued that both sides of the previous condition must vanish separately, from which it follows that $\omega_\psi^{0,1}$ must be a holomorphic function on $\mathbb{CP}^1$, implying it is constant, $\omega_\psi^{0,1} = \gamma \in \mathbb{C}$. On the other hand, $\omega_{\bar{z}}^{0,1}$ can be shown to equal $h\omega_\psi^{0,1}$ up to an exact $\partial_{\bar{z}}\epsilon$. The conclusion is that $\omega^{0,1} = \gamma\eta$, and the corresponding cohomology class in $H^{0,1}(S^3, T^{1,0}S^3)$ is obtained by tensoring the form $\gamma\eta$ with a holomorphic vector field $X = X^z(\partial_z - h\partial_\psi)$ on $\mathbb{CP}^1$, of which there are only three.

[6] I thank Itamar Yaakov for clarifying this point in particular.

Comparing this with the generic expression 6.10 in terms of the deformation parameters of the THF, we identify

$$\begin{cases} -2i \quad K^z = \gamma z \\ \quad {}^z{}_z - ih \quad K^z = \gamma z h \end{cases} \Rightarrow \begin{cases} \quad K^z = \frac{i}{2}\gamma z \\ \quad {}^z{}_z = -\frac{1}{2}\gamma z h. \end{cases} \tag{6.39}$$

Note that, by construction, these deformations satisfy the integrability conditions 6.9. These parameters correspond to an explicit deformation of the Killing vector K which determines the orbits of the Seifert fibration:

$$K = \partial \quad \longrightarrow K + \quad K = \partial \quad + \frac{i}{2}\gamma z \partial_z. \tag{6.40}$$

In turn, this induces a deformation of the dual one-form η and, consequently, of the adapted metric. To first order in the deformations, we have

$$\begin{aligned} \eta \quad &= (g \quad + \quad g \quad)(K \quad + \quad K \quad) \Rightarrow \\ \Rightarrow \quad \eta &= g \quad K \; dx \quad + \quad g \quad K \; dx \\ &= \frac{i}{2}\gamma z \left((c + 2|h|^2)dz + h^2 dz + 2h d\psi\right) + \quad g \quad dx \quad . \end{aligned} \tag{6.41}$$

Because Z_{M_3} is independent of $\quad K^z$ (or depends holomorphically on $\quad K$), we can set $\quad K^z = 0$. The same applies to ${}^z{}_z$ and its complex conjugate ${}^z{}_z$. Such a choice does not affect the reality of the partition function: $\quad g \quad$ written below will be real, because so is the combination $ihz = \frac{i}{2}(ig_2(\theta)e^{\quad i} \quad)f(\theta)e^{i}$.

The requirement that $g \quad + \quad g$ remains a transversely Hermitean metric leads to (see section 5.4 of [63])

$$\begin{cases} g \quad = -2\eta \quad K \quad = -i\gamma h z \\ g \;{}_z = \quad \eta_z - h\eta \quad K \quad - \frac{c^2}{2} \quad K^z = \frac{i}{2}\gamma z h^2 + \quad g \;{}_z - \frac{i}{2}\gamma h^2 z = \quad g \;{}_z \\ g_{zz} = \frac{ic^2}{2} \quad {}^z{}_z - \frac{hc^2}{2} \quad K^z + 2h \quad \eta_z = i\gamma z h^3 + \quad g \;{}_z. \end{cases} \tag{6.42}$$

The second condition is satisfied trivially for any $\quad g \;{}_z$. This is a reflection of the fact that the parameter $\quad \eta_z$ is a moduli of the geometry, since Z_{M_3} does not depend on it. This is equivalent to stating independence of Z_{M_3} on small deformations of $h(z, z)$. Therefore $\quad g \;{}_z$ can be set to zero, or it can also be tuned to set $\quad g_{zz} = 0$.

We conclude that the moduli deformation $\;{}^{(\;)}$ leads to a variation of the adapted metric which always includes non-zero component $\quad g \quad = -i\gamma h z$. Applying the CHM map, the resulting line element at $t = 0$ is

$$\begin{aligned} -i\gamma h z d\psi^2\Big|_{t=0} &= \gamma \frac{\cos 2\theta}{4}(d\varphi^2 + d\tau^2 + 2d\varphi d\tau)\Big|_{t=0} \\ &= \gamma \left(1 - 2\left(\frac{R^2 - \rho^2}{R^2 + \rho^2}\right)^2\right) \frac{R^2}{16}\left(1 + \frac{\rho^2}{R^2}\right)^2 d\varphi^2 . \end{aligned} \tag{6.43}$$

At $\rho = R$, this gives precisely the line element of a circle with a defect angle proportional to γ. The corresponding diffeomorphisms satisfy

$$\partial \cdot \xi \cdot (t, \rho, \varphi) = \gamma \left(1 - 2\left(\frac{R^2 - \rho^2}{R^2 + \rho^2}\right)^2\right) \frac{R^2}{16}\left(1 + \frac{\rho^2}{R^2}\right)^2 \Rightarrow \xi\,'(\rho = R, \varphi) = \frac{\gamma}{4}\varphi. \tag{6.44}$$

This (local) diffeomorphism rotates the boundary of the entangling surface on itself in the angular direction, so it does not alter its shape. However, because φ is an angular coordinate identified as $\varphi \sim \varphi + 2\pi$, ξ' is not continuous at $\varphi = 0$ and therefore does not represent a smooth deformation of $\partial\mathcal{A}$. This discontinuity signals the presence of a conical singularity at $\rho = 0$, which we had already predicted due to the conical singularity of the standard squashed sphere metric 6.36 at $\theta = \frac{}{2}$.

This is further support to our claim at the end of the previous section. Indeed, we find that the only instance in which small deformations of the circular entangling surface affect the vacuum entanglement is when these correspond to inserting a conical singularity at the centre of the disk. In this case, the topology of $\mathcal{A}$ is no longer that of a disk and the entanglement entropy remains invariant under infinitesimal smooth deformations of its boundary.

6.3.2 Corrections to S_A^{susy} under a conical deformation

We will now compute the correction to the entanglement entropy under the circular non-trivial deformation of the entangling surface, the conical defect at $\rho = 0$. The knowledge of this correction will completely determine the entanglement strucutre of our SCFT's for infinitesimal deformations of $\mathcal{A}$.

Instead of working with the Lagrangian variation $\mathcal{L}$ written in terms of adapted coordinates on the Seifert manifold, we can take a more direct approach and consider a particular squashing of S^3 in Hopf coordinates considered in [54], where most of the relevant computations have been done; we review them below. The metric is[7]

$$ds^2 = \frac{r^2}{4}\left(h^2(d\psi + 2\sin^2\frac{\theta}{2}d\phi)^2 + (d\theta^2 + \sin^2\theta d\phi^2)\right), \tag{6.45}$$

with $h \equiv \frac{1}{2}(b + b^{\ 1})$.

It is possible to determine exactly the supergravity background, since this squashed sphere falls into the category of three-manifolds for which the nowhere vanishing Killing vector V determines a fibration over a surface of constant curvature,[8] either S^2, T^2 or H^2 [54]. It is given by

$$V\ \partial\ = \frac{2V\ V}{hr}\partial\ ,\quad H = \frac{ih}{r},\quad V\ V\ = \frac{4(h^2-1)}{r^2}. \tag{6.46}$$

We will express the squashed sphere metric 6.45 in coordinates which reduce to stereographic coordinates on S^3 when $b = 1$:

$$g\ \ = \ ^2\delta\ \ + \left(\frac{b-b^{\ 1}}{b+b^{\ 1}}\right)^2 v\ v\ ,\qquad = \frac{2}{1+x^2}. \tag{6.47}$$

Here, v relates to the nowhere vanishing Killing vector V which determines the fibration of 6.45 over an S^2 (see section 8 of [54] for the explicit expression for v).

Before proceeding, we should make sure that this corresponds to a deformation of the THF of the first type in 6.42, since, as emphasised in [63], there exist squashings which do not change the THF but only the adapted metric, and therefore they leave the partition function invariant. In Appendix E of [63], a two-parameter squashed sphere is constructed, and the corresponding THF is seen to be realised by a deformation

[7] For consistency with the notation in the references, here we are using 2 [0;], ; 2 [0; 2).

[8] This arises from one of the conditions for two pairs of Killing spinors of opposite R-charge to be preserved on M_3:

$$R_{\mu\nu} = V_\mu V_\nu + g_{\mu\nu}(V^2 + 2H^2).$$

of the first type around the three-sphere. There, the case $\gamma_i = 0$ corresponds to the squashed sphere with $SU(2) \times U(1)$ isometry presented in 6.45.[9]

Another way to see this is the following: consider the linearized (to first order in δb, where $\delta b = b - 1$) variation of the Lagrangian and recall that, under a linearized variation of the metric, it takes the form 3.37. Because the first order term in δb of the expansion of g is zero, at $\mathcal{O}(\delta b)$ we have

$$\delta\mathcal{L}^{(\ b)} = \left(A \quad - \frac{3}{2}V \quad \right) j^{(R)} + j^{(Z)} C \quad = \delta b v \quad (i j^{(Z)} - j^{(R)}). \tag{6.48}$$

This cannot correspond to a Q-exact Lagrangian variation, because neither of the Q-exact bosonic operators which we can form with our superalgebra ((6.3) of [63]) contain components of $j^{(R)}$, only of its derivatives.

We emphasize that the energy-momentum tensor is not included in $\delta\mathcal{L}$ because the leading order term in δb of the metric is of order $(\delta b)^2$. In the case of a superconformal theory which we are considering, the operator $j^{(Z)}$ belongs to the multiplet $\mathcal{J}$ 3.27 containing T , and is therefore redundant. Morevover, one-point functions of operators in a conformal field theory vanish in general.[10] The lowest order correction to the partition function is then

$$\begin{aligned} Z^{(\ b)} &= \int \mathcal{D}\varphi e^{\ S \ \int d^3x \sqrt{g}\ L^{(\delta b)}} \\ &= Z^{(0)} \left(1 + \delta b^2 \int d^3x\sqrt{g} \int d^3y\sqrt{g} v \quad (x) v \quad (y) \langle j^{(R)} \quad (x) j^{(R)} \quad (y)\rangle_{S^3}\right) \\ &= Z^{(0)}(1 + \delta b^2 \mathcal{I}_{S^3}). \end{aligned} \tag{6.49}$$

Following section 8 of [54], the result of the integral above upon regularizing divergent contributions from contact terms equals

$$\mathcal{I}_{S^3} = - \left.\frac{\partial^2 F}{\partial b^2}\right|_{b=1} = -\frac{\pi^2}{2}\tau_{rr} + \text{(imaginary terms)}, \tag{6.50}$$

where $\tau_{rr} > 0$ determines the two point function of the R-symmetry current at separated points, which in flat space takes the form

$$\langle j^{(R)}(x) j^{(R)}(0)\rangle = \frac{\tau_{rr}}{16\pi^2}(\delta \quad \partial^2 - \partial \quad \partial \quad)\frac{1}{x^2}. \tag{6.51}$$

The imaginary terms originate from background Chern-Simons terms which must be included to regularize divergent contributions from contact terms while giving an imaginary contribution to the free energy, see [36, 65].

On the branched cover of the three sphere, we assume that there are no non-trivial contributions at $\mathcal{O}(1 - n)$ from the conical singularities to the correlation function

[9] This can be seen by directly substituting $\ _i = 0$, from which one obtains the line element in the form $ds^2 = \ ^2 + 2g_{zz}dzdz$, with

$$= \frac{2}{4 + \frac{2}{r}}\left(\cos^2 \frac{}{2} d \quad + \sin^2 \frac{}{2} d \quad \right), \quad 2g_{zz} = \sin \quad (d \quad + d \quad)$$

[10] We can consider the case of spacetime translations, x^μ ! $x^\mu + a^\mu$, under which the one-point function should remain invariant (such transformations have trivial Jacobian); then, $hO(x)i = hO(x + a)i$, implying that $hO(x)i$ should be constant.

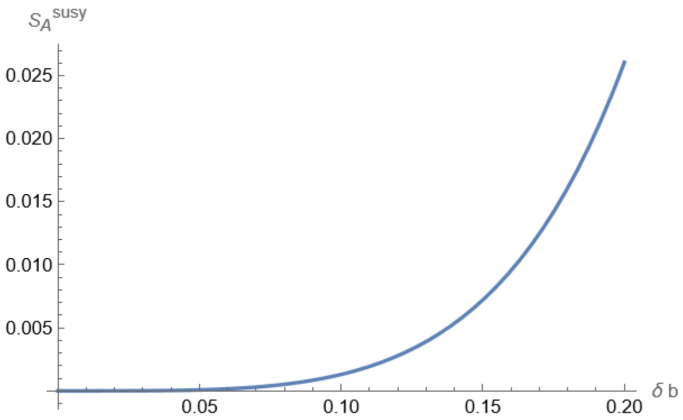


Figure 6.1: Correction to supersymmetric entanglement entropy as a function of the excess angle of the conical defect introduced at the center of $\mathcal{A}$.

$\langle j^{(R)}\ (x) j^{(R)}\ (y)\rangle_{S^3}$. This means that[11]

$$\mathcal{I}_{S^3_n} = n\mathcal{I}_{S^3} \tag{6.52}$$

To obtain the correction to the entanglement entropy, we consider

$$\begin{aligned}
\delta_b S_A &= \lim_{n \to 1} \frac{1}{1-n} \log \left(\frac{Z^{(\delta b)}_{S^3_n}}{(Z^{(\delta b)}_{S^3})^n} \right) \\
&= \lim_{n \to 1} \frac{1}{1-n} \Big[\log(1 + (\delta b)^2 \mathcal{I}_{S^3_n})|_{n=1} + (n-1)\partial_n|_{n=1} \log(1 + (\delta b)^2 \mathcal{I}_{S^3_n}) + \\
&\quad + \mathcal{O}((1-n)^2) - n \log(1 + (\delta b)^2 \mathcal{I}_{S^3}) \Big] \\
&= \log(1 + (\delta b)^2 \mathcal{I}_{S^3}) - \frac{(\delta b)^2 \mathcal{I}_{S^3}}{1 + (\delta b)^2 \mathcal{I}_{S^3}}.
\end{aligned} \tag{6.53}$$

As shown in 6.1, the behaviour of this lowest order correction to S_A^{susy} is simple: it is a convex function of the excess angle introduced by the conical singularity (or equivalently, of the squashing parameter of the squashed sphere conformally related to $\mathcal{A}$).[12] This behaviour is opposite to the one we have gathered evidence for in the previous chapter, although there is no apparent contradiction. As remarked in 5.3.3, one should not expect the behaviour of the entanglement entropy when introducing a curvature singularity in the entangling region as in the present case to mimic the one we have observed when introducing smooth internal boundaries of the disk. In a way, there is some consistency with the previous results, given that in 5 we observed a decrease in both the free energy and the entanglement entropy, while here we observe a simultaneous increase of both in an expansion around the disk entropy.

It is interesting to note that this behaviour is in agreement with the result from [66], namely that the spherical entangling surfaces locally minimizes the universal term in the entanglement entropy of a CFT. There, it was shown that for infinitesimal

[11] In fact, it was shown in [26] that the partition function on the branched cover of a squashed sphere equals that of a branched sphere; because a branched sphere can be expressed in terms of squashing, we do not expect that there should be any non-trivial dependence on the branching when performing the replica trick for the partition function on the squashed sphere.

[12] We have set $\;_{rr} = 1$, but its particular value is not relevant here since it is positive definite.

smooth deformations of $\partial\mathcal{A}$ around a spherical entangling surface (parametrized by ϵ), the universal piece of the CFT entanglement entropy in d dimensions takes the form

$$S(\partial\mathcal{A}) = S_d^{\text{sphere}} + \epsilon^2 C_T F[\partial\mathcal{A}] + \mathcal{O}(\epsilon^3), \tag{6.54}$$

where C_T, determined by the stress tensor two-point function, and $F[\partial\mathcal{A}]$, a functional of the shape deformation, are positive semi-definite coe□cients. This leads to the conclusion that the sphere locally minimizes the universal term of the entanglement entropy among all shapes with sphere topology. Of course, the case we are considering here does not correspond to a deformation of $\mathcal{A}$ with sphere topology. However, this result still serves to demonstrate that, in three-dimensional superconformal theories, spherical entangling regions locally minimize the universal term of entanglement entropy among regions with conical topology which reduce to spheres in the limit of vanishing deformation. In this sense, it extends the conclusions of [66] to another type of topology of $\mathcal{A}$.[13] In particular, the conclusions of the previous section which apply to deformations of $\mathcal{A}$ with sphere topology are also in agreement with 6.54, given that they establish that $F[\partial\mathcal{A}]$ vanishes for 3d $\mathcal{N}$ = 2 superconformal theories. Although the results of the previous section were determined for linearized Lagrangian deformations, it is still true that the quadratic term in the expansion 6.54 should vanish. Here is a short argument:

Consider a perturbative expansion of the partition function of the theory defined by a deformed Lagrangian up to $\mathcal{O}(\epsilon^2)$:

$$\mathcal{L} = \mathcal{L}^{(0)} + \epsilon \quad \mathcal{L}^{(1)} + \epsilon^2 \quad \mathcal{L}^{(2)} + \mathcal{O}(\epsilon^3). \tag{6.55}$$

Contributions from $\mathcal{L}^{(1)}$ to Z_{M_3} have been seen to vanish identically (non-perturbatively), and in particular they vanish at order $\mathcal{O}(\epsilon^2)$. The only remaining contributions to Z_{M_3} at this order in ϵ must come from $\mathcal{L}^{(2)}$. However, we do not need the explicit form of $\mathcal{L}^{(2)}$, since in perturbation theory its $\mathcal{O}(\epsilon^2)$ contribution would come from the one-point function of local operators, namely $\int d^3x\langle \quad \mathcal{L}^{(2)}\rangle_{SCFT}(x)$, which must vanish by conformal invariance, independently of the operator content of $\mathcal{L}^{(2)}$. This proves $F[\partial\mathcal{A}] = 0$.

6.3.3 Comments (S_A^{susy} is topological?)

The above statements may seem to be in contradiction with the result from [66] just cited, where the functional $F[\partial\mathcal{A}]$ was determined to be strictly positive for a large class of deformations whose angular dependence is given by superpositions of spherical harmonics. Although the computations done there were purely holographic, [67] proved the result using CFT techniques, by expressing corrections to the density matrix, and therefore to the von Neumann entropy, in terms of stress tensor correlators. Crucially, they have not made any assumptions on the details of the field theory. We now argue why this should not contradict the results we have obtained in this section.

We have concluded about deformation independence of supersymmetric entanglement entropy while relying on its formulation in terms of partition functions on the branched cover of the three-sphere, 2.35. As we have seen in detail in 3, this crucially relied on deforming the theory with a background configuration for the R-symmetry gauge field $A^{(R)}$ in order to preserve supersymmetry on a manifold with conical singularities. It is in this sense that S_n^{susy} departs from the usual definition of von Neumann entropy in terms of the partition functions on replicated spaces via the

[13] Our expression 6.53 does not take the form 6.54, due to non-vanishing contributions to S_A when the limit n ! 1 is taken in the Rényi entropy.

replica trick, where the conical singularities arising in the replicated manifolds can be dealt with without changing the background of the theory when supersymmetry is absent. Such a departure should be responsible for the deformation invariance observed in this chapter, which is certainly uncharacteristic of entanglement entropy in other CFT's, as shown in [67]. We therefore make the suggestion that the background field deformation of $3d$ $\mathcal{N} = 2$ superconformal theories required to define supersymmetric Rényi entropy renders it a topological quantity. Of course, this statement applies only when talking about the dependence of S_A^{susy} on the geometry of $\mathcal{A}$, since it should also depend on the field content of the theory and on the gauge group, as we will see explicitly in Chapter 7.

If S_A^{susy} indeed represents the true (finite piece of the) entanglement entropy, we are led to conclude that supersymmetry places restrictions on the theory in such a way that the correlation of degrees of freedom across the boundary is highly insensitive to the geometry of $\mathcal{A}$. Given a fixed field content, gauge group and correlation length of the theory, the correlations measured by S_A^{susy} are dependent only on the topology of $\mathcal{A}$ and therefore highly non-local. Another way to think about this is to consider the *mutual information*, defined as [31]

$$I(\mathcal{A},\mathcal{B}) = S_A + S_B - S_{A\cup B} - S_{A\cap B}. \tag{6.56}$$

This quantity is cutoff-independent, and is usually taken as a measure of universal correlations captured by the entanglement entropy.[14] If we consider $\mathcal{A}$ and $\mathcal{B}$ to be adjoint regions smoothly deformed with respect to one another (and with respect to the disk), then all four terms above equal the disk supersymmetric entanglement entropy (in absolute value), implying that $I(\mathcal{A},\mathcal{B}) = 0$. The fact that the mutual information not only is invariant but actually vanishes for a large family of pairs of entangling regions signals that there is indeed a high suppression of quantum correlations across the boundary due to supersymmetry.

The suggestion presented above is interesting from the point of view of our earlier investigations in Chapters 4 and 5. In the latter, we have seen that S_A^{susy} must definitely depend on the topology of $\mathcal{A}$, while the negativity of its finite piece observed in the former may be the smoking gun of long range topological order charcateristic of topological entanglement entropy 2.40. In the next chapter we will attempt to better understand the nature of such finite terms through another method of computing S_A^{susy}.

[14] A plethora of exact results on mutual information in quantum field theory have been obtained; see for instance [68, 69, 70, 71]

7 Supersymmetric Entanglement Entropy From Heat Kernel

Following the discussion at the end of the previous chapter, we would like to gain more insight into the entanglement structure of our $3d$ $\mathcal{N} = 2$ superconformal theories. Especially, it would be interesting to make contact between the results obtained from Nishioka and Yaakov's prescription and the known structure of entanglement entropy in $3d$ QFT's, 2.38. Namely, an answer to the following questions is called for:

- Is it possible to determine the terms dependent on the UV cutoff ϵ of the theory which are expected in three dimensions?
- Is it possible to match the finite terms resulting from Nishioka and Yaakov's prescription with results computed by some other field theory method in order to better understand their nature?

Answers to the above questions may help us gain insight into the features we have found regarding S_A^{susy}, namely its negativity and topological invariance.

Motivated by these questions, we will use the heat kernel approach to compute vacuum entanglement entropy of superconformal gauge theories, by computing the necessary heat kernel coefficients on a branched cover of the three-sphere. Such a space has conical singularities along the branching region, which we choose to be the entangling surface $\partial\mathcal{A}$ along the great circle $\theta = 0$. The possibility of using the heat kernel method relies on the fact that the localization computation allows us to obtain the partition function of a strongly coupled IR gauge theory from a computation in a UV free theory to which the IR $\mathcal{N} = 2$ theory flows. We can therefore compute the 1-loop determinants arising from the localization procedure through the heat kernel expansion. This approach will allow us to explicitly keep track of the UV cutoff dependence of the entanglement entropy.

We are, of course, ultimately interested in the entanglement entropy in the flat space theory. Thanks to the equivalence between flat space Rényi entropy and a partition function on a branched 3-sphere described in 2.2, the former is exactly what we will compute with the prescription described above. This follows from the fact that 2.35, where $Z_{S^3_n}$ is a partition function on S^3 with a conical singularity along $\theta = 0$ of excess angle $2\pi(n-1)$, is exactly the Rényi entropy of the theory on S^3, where the entangling region is the one bounded by the great circle along $\theta = 0$ (this was also pointed out in [26]). We then have the following sequence of equalities:

$$
\begin{aligned}
\frac{1}{1-n}\log\left|\frac{Z_{S^3_n}}{Z^n_{S^3}}\right| &= \text{Rényi entropy on } S^3 \text{ across } S^1 \text{ at } \theta = 0 \\
&= \text{Rényi entropy on } \mathrm{R}^3 \text{ across } S^1 \text{ at } \rho = R.
\end{aligned}
\tag{7.1}
$$

A similar approach has been taken by Fursaev in $4d$ $\mathcal{N} = 4$ superconformal theories [72]. There, the computation is reduced to one-loop determinants because it is performed at zero coupling between matter and gauge fields. Here we will go further and perform the computation at arbitrary coupling, which is a consequence of embedding the theory in the IR limit of an RG flow to a free UV theory through localization.

7.1 Entanglement Entropy and the Heat Kernel

Consider a non-interacting quantum field theory defined on $\mathcal{M}_n$, an n-branched cover of a smooth Riemannian manifold $\mathcal{M}$ with a conical singularity along a given codimension-two surface Σ. If the quadratic operator for a given field is of the form $\mathcal{D} + m^2$, with $\mathcal{D}$ some differential operator, the one-loop effective action on $\mathcal{M}_n$ is [30]

$$\log Z_n = -\frac{1}{2}\log\det(\mathcal{D} + m^2) = \frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{ds}{s}\mathrm{tr}K_{\mathcal{M}_n}(s)e^{-m^2 s}, \tag{7.2}$$

and the heat kernel operator has the following expansion:

$$\mathrm{tr}K_{\mathcal{M}_n}(s) = \mathrm{tr}e^{-sD} = \frac{1}{(4\pi s)^{d/2}}\sum_{i=0}^{\infty} a_i(\mathcal{M}_n)s^i. \tag{7.3}$$

The heat kernel coefficients $a_i(\mathcal{M}_n)$ allow for an expansion of the one-loop effective action in the $n \to 1$ limit, which computes the entanglement entropy. This originates in an expansion of the coefficients of the form

$$a_i = a_i^{\text{bulk}} + (1-n)a_i^{\Sigma} + \mathcal{O}((1-n)^2), \tag{7.4}$$

where a_i^{bulk} do not include any contribution from the singularity and are given by an integral over the smooth part of $\mathcal{M}_n$; therefore $a_i^{\text{bulk}}(\mathcal{M}_n) = n a_i^{\text{bulk}}(\mathcal{M})$. The surface parts a_i^{Σ} represents contributions from the conical singularity at the spacelike hypersurface Σ; they contain the non-trivial information which computes the entanglement entropy. We will additionally need to consider a branched manifold with regularized conical singularities, $\tilde{\mathcal{M}}_n$. By computing $a_i^{\text{bulk}}(\tilde{\mathcal{M}}_n)$, the surface parts a_i^{Σ} can be extracted by reading off its coefficient in $(1-n)$. This is because $a_i^{\text{bulk}}(\tilde{\mathcal{M}}_n)$ can be integrated over the entire manifold, and so it will include terms in $(1-n)$ which diverge when removing the regularization and contain the non-trivial contributions from the conical singularity.

Upon performing the s integral in 7.2, the final form for the entanglement entropy, explicitly dependent on the UV cutoff, is

$$\begin{aligned} S_A &= \lim_{n\to 1}\frac{1}{1-n}\left(\log Z_n - n\log Z_1\right) \\ &= \frac{1}{2(4\pi)^{d/2}}\sum_{i=0}^{\infty}\frac{a_i^{\Sigma}}{m^{2i-d}}\,\Gamma\left(i-\frac{d}{2}, m^2\epsilon^2\right), \end{aligned} \tag{7.5}$$

where $\Gamma(a,x)$ is the incomplete Gamma function. Let us determine which coefficients a_i we must retain in three dimensions. From an expansion of the incomplete Gamma function around $\epsilon = 0$, we have

$$
\begin{aligned}
\left(-\frac{3}{2}, m^2\epsilon^2\right) &= (m\epsilon)^{\ \ 3}\left(\frac{2}{3} - 2m^2\epsilon^2 + \mathcal{O}(\epsilon^4)\right) = \frac{2}{3m^3\epsilon^3} + \frac{2}{m\epsilon} + \mathcal{O}(\epsilon) \\
\left(-\frac{1}{2}, m^2\epsilon^2\right) &= (m\epsilon)^{\ \ 1}\left(2 + \mathcal{O}(\epsilon^2)\right) = \frac{2}{m\epsilon} + \mathcal{O}(\epsilon).
\end{aligned} \tag{7.6}
$$

The combination of coe□cients we want is then[1]

$$
S_A = \frac{1}{2(4\pi)^{3=2}}\left[a_0\left(\frac{2}{3\epsilon^3} + \frac{2}{m\epsilon}\right) + a_1\frac{2}{\epsilon}\right] + \mathcal{O}(\epsilon) = \frac{1}{(4\pi)^{3=2}}\frac{a_1}{\epsilon}. \tag{7.7}
$$

This tells us that in $3d$ we are only interested in a_1 (a_0 will be seen to vanish).

7.2 Heat coe□cients of our SCFT's and the zero coupling limit

We will collect the bulk and surface parts of the heat coe□cients for the quadratic operators which compute the one-loop determinants after localization on S^3_n, where is the entangling surface at $\theta = 0$. We do this by writing the Lagrangians for the UV free theory to which our IR $\mathcal{N} = 2$ SCFT flows by virtue of localization, and extract from them the relevant heat coe□cients.

Scalar heat coe□cients

We use the regularization of the branched three-sphere from [25],

$$
ds^2 = \frac{1}{f(\theta)}d\theta^2 + q^2\sin^2\theta d\theta^2 + p^2\cos^2\theta d\phi^2, \quad f(\theta) = \begin{cases} \frac{1}{q^2}, & \theta \to 0 \\ \frac{1}{p^2}, & \theta \to \frac{\ }{2} \\ 1, & \delta < \theta < \frac{\ }{2} - \delta. \end{cases} \tag{7.8}
$$

Upon localizing our SCFT we have a free theory with one scalar and one fermionic quadratic action (the remaining scalar field from the chiral multiplet is non-dynamical and can be trivially integrated out). We use the regularized supergravity background from 3.71 which preserves 2 supercharges of opposite R-charge. The scalar quadratic operator in the matter Lagrangian is[2]

$$
\begin{aligned}
(\mathcal{D} + m^2) &\equiv -\nabla^2 + \frac{1}{4}A\ \ A\ \ + \left(\frac{1}{2}H + \sigma_0\right)^2 - H^2 - \frac{1}{8}(R - 6H^2) \\
&= -\nabla^2 + \frac{1}{l^2}\left(\frac{1}{n^2\sin^2\theta}\frac{(n\sqrt{f(\theta)} - 1)^2}{16} - il\sigma_0 + l^2\sigma_0^2 - \frac{1}{8}l^2R\right),
\end{aligned} \tag{7.9}
$$

We can use the results from [73] for the surface heat co□cients for a scalar quadratic operator of the form $\mathcal{D} = -\nabla^2 + V$, where we identify V with the potential term in

[1] Note that this quantity is dimensionless, since a_1 has dimensions of length.

[2] Choosing the superconformal R-charge $r = \frac{1}{2}$ and keeping an explicit dependence on l, the radius of the S^3.

brackets in 7.9. These are

$$a_0 = 0, \quad a_1 = \frac{2\pi}{3}(1-6\xi)\int 1, \quad a_2 = \frac{2\pi}{3}\int \left(\frac{1}{6}R - V + \frac{1}{30}(2R_{ijij} - R_{ii} - 2\text{Tr}k^2 + \frac{1}{2}k^2)\right), \tag{7.10}$$

where ξ is the coupling of ϕ to the Ricci scalar. We will not need a_2, but it is important to keep in mind that V will not contribute to the surface part of the coe□cients only because we are in $d < 4$, where a_2 does not appear in the heat kernel expansion.

Note that the above expression for a_1 is can be recovered from the one for the bulk coe□cients presented in [74]:[3]

$$a_1^{\text{bulk}}(\mathcal{M}_n) = n\int_M \sqrt{g}d^dx\left(\frac{1}{6}R - V\right) \tag{7.11}$$

Because $(\mathcal{D} + m^2)$ contains $-\frac{1}{8}R$ in the potential, the singularities in the Ricci scalar give the following overall contribution to the surface coe□cients:

$$a = \left(1 + \frac{6}{8}\right)\frac{2\pi}{3}\int 1 = \frac{7\pi}{6}\int 1. \tag{7.12}$$

Now, it is crucial to note that the potentials we will consider here depend

1. on the regularization of the manifold through $f(\theta)$;
2. explicitly on n, due to the presence of $A^{(R)}$.

Therefore, terms $\propto (1-n)$ could arise when integrating V over $\tilde{S}^3_n$ in 7.11, which should be included in the surface coe□cients.

To see how this works, let us recall how the surface term coming exclusively from the Ricci scalar (which is precisely a_1) is obtained, as outlined in Appendix C of [25]. The integral $\int_{S^3_n}\sqrt{g}R$ is computed with the regularized expressions for $\sqrt{g}$ and R depending explicitly on f. The appearence of the term $\propto (1-n)$ relies on an integration by parts due to the presence of a term f' in R, and this results in $f\,|\,_{=0}^{=} =$ $(1-1/n)$ as a contribution from the conical singularity. Because our potential does not contain any derivatives of f, terms due to 1) do not appear, since we cannot perform an integration by parts as in the case just mentioned; this in fact allows us to simply take the limit $\delta \to 0$. The result of the integration over $\tilde{S}^3_n$ is therefore equal to the result on S^3_n, and no surface terms occur. However, there is still a dependence on $(1-n)$ to second order due to 2). One might think this would give contributions not to S_A, but rather to $S_A^{(n)}$. However, this contribution actually vanishes. It is given by (including the regularized configuration for $A^{(R)}$)

$$\begin{aligned}\int_{S^3_n}\sqrt{g}\frac{1}{n^2\sin^2\theta}\left(n\sqrt{f}-1\right)^2 &= \int_{S^3_n}\frac{\cot\theta}{n^2\sqrt{f}}\left(n\sqrt{f}-1\right)^2 \\ [\cot\theta \simeq \frac{1}{\theta}] &= \int_0 d\theta\frac{1}{n^2\theta\sqrt{f}}\left(n\sqrt{f}-1\right)^2 + \int^{\frac{\pi}{2}} d\theta\cot\theta(n^2-1) \\ &= \log\left(\sin\left(\frac{\pi}{2}\right)\right)(n^2-1) - \log\epsilon + \log\epsilon = 0,\end{aligned} \tag{7.13}$$

[3] The factor $\frac{2\pi}{3}\int$ 1 from a_1 in 7.10 is exactly the $(1 \quad n)$ contribution coming from the term $\frac{1}{6}R$ in this equation, when it is integrated over the regularized S^3_n. This can be checked explicitly in our case, making use of the Ricci scalar of the regularized S^3_n presented in Appendix C of [25]. Notice in particular that the factor $\frac{1}{6}$ appears in this expression for any dimension, so only for a conformally coupled scalar in 4*d* does the expression actually vanish.

where we used that $\sqrt{f\,(\theta=\delta)} = 1$ and for $\delta > \theta \to 0$ the integrand behaves as

$$\frac{1}{\theta}(n\sqrt{f}-1)^2 = \frac{1}{\theta}(n\frac{1}{n} + \mathcal{O}(\theta) - 1)^2 = \mathcal{O}(\theta^2) \to 0. \tag{7.14}$$

Since contributions to a_1 would come from extracting terms $\propto (1-n)$ in 7.13, we conclude that there are no surface contributions from the potential in the scalar Lagrangian and only the Ricci-like surface terms 7.12 remain. The contribution of the scalar field to the entanglement entropy is then[4]

$$\frac{1}{(4\pi)^{3=2}}\frac{a_{1;}}{\epsilon} = \frac{1}{(4\pi)^{3=2}}\frac{1}{\epsilon}\frac{7\pi}{6}\int_0^{2} d\phi\sqrt{l^2} = \frac{7\sqrt{\pi}}{24}\frac{l}{\epsilon}. \tag{7.15}$$

Spinor heat coe cients

The generalization of 7.11 to fields of higher spin is [74]

$$a^{\text{bulk}}_{1;(j)} = \int_M \left(\frac{N^{(j)}}{6} R\,\text{Tr}1 - \text{Tr}_i X^{(j)}\right), \tag{7.16}$$

where $N^{(j)}$ is the dimension of the representation of the Lorentz group of the field of spin j, Tr_i is the trace over the indices of the representation, and $X^{(j)}$ is a potential. With the superconformal R-charge $r = \frac{1}{2}$, the quadratic operator for the spinor fields in $\mathcal{L}_K$ is

$$\begin{aligned}(\mathcal{D}+m^2) &\equiv -i\nabla\!\!\!/ - \frac{1}{2}\gamma\ A\ - i\sigma_0 \\ &= -i\nabla\!\!\!/ - \frac{1}{2}\frac{n}{n^2}\begin{pmatrix}1 & \cot\theta e^{-i(\tau+\phi)} \\ \cot\theta e^{i(\tau+\phi)} & -1\end{pmatrix}\frac{n\sqrt{f\,(\theta)}-1}{2} - i\sigma_0.\end{aligned} \tag{7.17}$$

A comment on the spinor operator is in order here. In [74] the heat coe cients for the free fermion Laplacian are determined by taking its square,

$$(-i\nabla\!\!\!/)^2 = -\nabla\ \nabla\ + \frac{1}{4}R, \tag{7.18}$$

and dividing the heat coe cients of the resulting operator by 2 (since the logarithm of the eigenvalues of $-i\nabla\!\!\!/$ is half of the logarithm of eigenvalues of $(-i\nabla\!\!\!/)^2$). Any other terms that we add to the spinor operator can be treated as explained above, by including their contribution to the coe cients in the term $-\int_M \text{Tr}_i(X^{(j)})$.

The trace over spinor indices that arises when computing $\text{Tr}_i X^{(j)}$ is

$$(\gamma\ )\ \ = \varepsilon\ \ (\gamma\ )\ \ = 2in\cot\theta\sin(\tau+\phi), \tag{7.19}$$

and this gives

$$\int_{S^3_n} \text{Tr}_i(X^{(j)}) = -i\int_0^{\frac{\pi}{2}} d\theta \int_0^{2} d\phi \int_0^{2} d\tau \frac{\sin\theta\cos\theta}{n\sqrt{f}}\cot\theta\sin(\tau+\phi)\frac{n\sqrt{f}-1}{2} = 0. \tag{7.20}$$

[4] It is curious that this quantity is model independent, depending only on the background geometry. Even including a mass term in the scalar quadratic operator does not a ect 7.15, as was seen above. This means that all three-dimensional free scalar field theories with arbitrary mass gap possess the same UV divergent coe cient in the entanglement entropy.

Again, no surface terms arise due to the potential. The surface coe□cient for spin $\frac{1}{2}$ fields is found in a similar way to $a_{1;}$ by (inserting $\xi = \frac{1}{4}$)

$$a_{1;} = -\frac{2^{b3=2c}}{2}\left(1-\frac{6}{4}\right)\frac{2\pi}{3}\int 1 = \frac{2\pi^2 l}{3}, \tag{7.21}$$

so that the contribution from the fermions to the entanglement entropy is

$$\frac{1}{(4\pi)^{3=2}}\frac{a_{1;}}{\epsilon} = \frac{\sqrt{\pi}}{12}\frac{l}{\epsilon}. \tag{7.22}$$

Vector heat coe□cients

After gauge fixing the path integral and integrating out ghost fields and the quadratic fluctuations in σ, the quadratic operator for the fields in the vector multiplet is [25]

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= \text{Tr}\left[B \quad {}_V B \quad -[B \ ,\sigma_0]^2 - i\bar{\lambda}\gamma \ (\nabla \ + iA \ )\lambda - i\bar{\lambda}[\sigma_0,\lambda] + \frac{1}{2}\bar{\lambda}\lambda\right] = \\ &= \sum_{i=1}^{2}\text{Tr}\left[B^i \quad {}_V B_i \ - i\bar{\lambda}_i\gamma \ (\nabla \ + iA \ - \frac{i}{2})\lambda_i\right] + \\ &\quad + \sum \text{Tr}\left[B \quad (\ {}_V + \alpha(\sigma_0)^2)B \ - i\bar{\lambda} \quad \gamma \ (\nabla \ + iA \ )\lambda \ + \bar{\lambda} \quad (-i\alpha(\sigma_0) + \frac{1}{2})\lambda \ \right].\end{aligned} \tag{7.23}$$

Here ${}_V = \star d \star d + d \star d \star$ and B is the divergenceless part of the photon field a , that is, $d \star B = 0$. Moreover, the fields from the gauge multiplet have been decomposed in the root spaces of the adjoint representation, as well as in the Cartan basis.

The gaugino quadratic determinant gives the same heat coe□cients as the fermions in the chiral multiplet, and the surface parts vanish for the same reasons as above. It remains to include the heat coe□cients for the vector field

$$a_{1;B} = N^{(1)}a_{1;} = \frac{7\pi}{3}\int 1, \tag{7.24}$$

where $N^{(1)} = 2$ since the divergenceless vector B has two o□-shell degrees of freedom.[5] In the end we still need to multiply the coe□cients of the photon and gauginos by the dimension of the Cartan subalgebra plus the number of root subspaces α of g, which we denote by $N_{\text{Cartan}} + N$.

Putting everything together, we have the following UV divergent contributions in the entanglement entropy of the matter and gauge Lagrangians at zero coupling:

$$\boxed{\begin{aligned} S^{(\ )}_{A;} &= \frac{1}{(4\pi)^{3=2}}\frac{1}{\epsilon}(a_{1;} \ + a_{1;} \ ) = \frac{3\sqrt{\pi}}{8}\frac{l}{\epsilon}, \\ S^{(\ )}_{A;V} &= \frac{1}{(4\pi)^{3=2}}\frac{1}{\epsilon}(N_{\text{Cartan}} + N \ )(a_{1;} \ + a_{1;B}) = \frac{35\sqrt{\pi}}{48}(N_{\text{Cartan}} + N \ )\frac{l}{\epsilon}. \end{aligned}} \tag{7.25}$$

[5] In [74] there is an additional term $4 \int 1$, but this comes from the presence of $R_{\mu\nu}$ in the vector Laplacian, which is not present in our case; its contribution is precisely the $(1 \quad n)$ term in $\int_{S^3_n} \text{Tr}R_{\mu\nu} = \int_{S^3_n} g^{\mu\nu}R_{\mu\nu} = 4 \int$, so we remove this.

Finite terms

In [75] a finite, mass-dependent contribution to the EE is computed for general smooth geometries, of the form

$$\gamma(m) = (1-6\xi)\gamma_{d=3}(m^2)^{\frac{1}{2}}\int 1, \qquad (7.26)$$

where, again, the -6ξ factor is included to account for the contribution from the Ricci scalar coupling. Finite terms are extracted by taking derivatives of the heat kernel sum with respect to m^2. This isolates finite contributions for which the ϵ dependence is cancelled, which are then integrated back in m^2. Explicitly, for each field of spin i in 3d,

$$\begin{aligned} \frac{\partial S^{(i)}}{\partial(m^2)} &= \frac{\partial}{\partial m^2}\frac{1}{2}\int_{2}^{1}\frac{ds}{s}\frac{1}{(4\pi s)^{3=2}}\left(a_{0;i}+a_{1;i}s\right)e^{\;m^2 s} \\ &= -\frac{a_{1;i}}{2(4\pi)^{3=2}}\int_{2}^{1} dse^{\;m^2 s}s^{\;\frac{1}{2}} \\ [\epsilon\to 0] &= -\frac{a_{1;i}}{2(4\pi)^{3=2}}\sqrt{\frac{\pi}{m^2}} \Rightarrow \\ \Rightarrow S^{\text{finite}}_{m^2} &= \int d(m^2)\frac{\partial S}{\partial(m^2)} = -\frac{a_i}{(4\pi)^{3=2}}\sqrt{\pi}m. \end{aligned} \qquad (7.27)$$

Summing the surface heat coe□cients determined above for all spins i gives the total result. How to write the full result including divergent and this finite contributions?

This means that we can immediately write down a correction to the entanglement entropy of a free chiral multiplet coupled to a real mass term. Recall that such terms can be implemented by simply introducing a background gauge multiplet whose configuration can be fixed by the insertion of a vortex operator along the entangling surface such that the σ field takes an imaginary value. This gives

$$S^{\text{finite}}_{+m^2} = -\frac{a_{1;i}}{(4\pi)^{3=2}}\sqrt{\pi}m. \qquad (7.28)$$

Of course, such terms only appear in non-conformal theories with a mass gap. In the absence of mass terms, this method does not allow us to extract any finite piece in the entropy. We will see below that this not the only finite piece that the heat kernel method allows us to extract.

It is curious to note that this linear behaviour in m with a negative slope is similar to the one we have found for the mass deformation of a free conformal chiral multiplet on S^3 in 4.2. However, because the function in that case is not linear for small m, we will not make a correspondence between that case and the formula above.

7.3 Finite gauge coupling: handling the integration over σ_0

All of the above is valid only at zero coupling, when the partition function is a product of one-loop determinants for the matter and gauge Lagrangians.[6] We will now consider the case when the matter content of the theory is coupled to gauge fields. The main di□erence this brings is that the partition function no longer factorizes into a product

[6] This is only in the case of an abelian gauge theory, where $_0$ does not appear in L_{YM}.

of one-loop determinants, but there is the overall integration over the localization orbits,

$$Z_n = \int [d\sigma_0] e^{-S_{\text{cl}}(\sigma_0;n)} Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, n) Z^{\text{1-loop}}_{\text{gauge}}(\sigma_0, n). \tag{7.29}$$

In this situation we cannot compute $\log Z_n$ by simply taking the logarithm of the full expression and retaining the surface heat coefficients, due to the overall integration over g. However, we can still make use of the heat kernel expansion for each individual 1-loop determinant by rewriting the above as

$$Z_n = \int [d\sigma_0] e^{-S_{\text{cl}}(\sigma_0;n)} \exp\left(\log Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, n)\right) \exp\left(\log Z^{\text{1-loop}}_{\text{gauge}}(\sigma_0, n)\right). \tag{7.30}$$

The price to pay is that we must consider the full expansion 7.4 incuding the bulk coefficients, since their contribution doesn not necessarily vanish when computing the Rényi entropy. However, it remains sufficient to consider its expansion up to first order in $(1 - n)$ if we are only interested in the entanglement entropy.

7.3.1 Examples: Abelian gauge groups and topological sectors

Single $U(1)$ chiral multiplet

Let us first consider a $U(1)$ gauge group, where the σ_0 dependence drops out of $\mathcal{L}_{\text{gauge}}$ 7.23 since the commutators of fields in the adjoint representation vanish. We have

$$\begin{aligned} a^{\text{bulk}}_{1;}(S^3_n) &= n a^{\text{bulk}}_{1;}(S^3) \\ &= n \int_{S^3} \sqrt{g} d^d x \left(\frac{1}{6} R - \left(\frac{1}{2} H + \sigma_0 \right)^2 + H^2 + \frac{1}{8}(R - 6H^2) \right) \\ [R(S^3) = -6/l^2] &= n \text{Vol}(S^3) \left(\frac{1}{l^2} \left(-1 - \frac{6}{8} \right) + \frac{i}{l}\sigma_0 - \sigma_0^2 \right) \\ &= n \text{Vol}(S^3) \left(-\frac{7}{4l^2} + \frac{i}{l}\sigma_0 - \sigma_0^2 \right). \end{aligned} \tag{7.31}$$

The quadratic operator of the fermion field in the chiral multiplet on S^3 is $\mathcal{D} \equiv -i\nabla\!\!\!/ - i\sigma_0$. In the previous section we have related the coefficients of the Dirac operator with no mass terms, $-i\nabla\!\!\!/$, to those of $(-i\nabla\!\!\!/)^2 = -\nabla^2 + \frac{1}{4}R$. In the presence of a mass term this procedure is not so simple, since squaring $-i\nabla\!\!\!/ - i\sigma_0$ would give us both second and first order terms in $\nabla\!\!\!/$. We could try to solve for the coefficients of

$$(-i\nabla\!\!\!/)^2 + \sigma_0^2 = (-i\nabla\!\!\!/ - i\sigma_0)(-i\nabla\!\!\!/ + i\sigma_0), \tag{7.32}$$

but then we would not be able to relate them to the heat coefficients of $\mathcal{D}$ alone. Instead, we make the change of integration variables $\sigma_0 \to i\sigma_0$, and the operator becomes $\mathcal{D} \equiv -i\nabla\!\!\!/ - \sigma_0 1$. This is no longer an operator with purely imaginary eigenvalues; we can write

$$\mathcal{D}^\dagger \mathcal{D} = (-i\nabla\!\!\!/ - \sigma_0)(i\nabla\!\!\!/ - \sigma_0) = -(i\nabla\!\!\!/)^2 + \sigma_0^2 = \nabla^2 - \frac{1}{4}R + \sigma_0^2, \tag{7.33}$$

The eigenvalues of this operator are the square of the absolute value of the eigenvalues of $\mathcal{D}$. We then take the heat coefficients of $\mathcal{D}$ as half of the heat coefficients of $\mathcal{D}^\dagger \mathcal{D}$. Explicitly, the identification we are making in order to compute the heat coe□cients is

$$\lambda_{D_\psi}\lambda_{D_\psi} \longleftrightarrow (\lambda_{D_\psi})^2. \tag{7.34}$$

These two quantitites di□er by a complex phase, $\lambda\ \lambda = e^{\ 2i\arg(\ )}\lambda^2$, meaning that the di□erence between applying $\log\det$ to $\mathcal{D}^y\mathcal{D}$ and to $\mathcal{D}$ is, besides the factor of 2, a purely imaginary additive quantity which we are not concerned with and may safely disregard. The spinor heat coe□cients then become

$$\begin{aligned} a_{1;}^{\text{bulk}}(S_n^3) &= -\frac{2^{[3=2]}}{2} n \int_{S^3} \sqrt{g} d^d x \left(-\frac{1}{6}R - \left(-\frac{1}{4}R + \sigma_0^2\right)\right) \\ &= -n\text{Vol}(S^3)\left(\frac{6}{l^2}\left(\frac{1}{6}-\frac{1}{4}\right) - \sigma_0^2\right). \end{aligned} \tag{7.35}$$

We are now ready to write the full heat kernel expansion, keeping in mind that the σ_0 terms present in the scalar bulk coe□cient must be complexified. We have

$$\begin{aligned} \exp\left(\log Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, n)\right) &= \exp\left(n\frac{\text{Vol}(S^3)}{(4\pi)^{3=2}}\left(-\frac{7}{4l^2} - \frac{1}{l}\sigma_0 + \sigma_0^2\right)\frac{1}{\epsilon} + (1-n)\frac{7\sqrt{\pi}}{24}\frac{l}{\epsilon} + \right. \\ &\quad \left. + n\frac{\text{Vol}(S^3)}{(4\pi)^{3=2}}\left(\frac{1}{2l^2} + \sigma_0^2\right)\frac{1}{\epsilon} + (1-n)\frac{7\sqrt{\pi}}{48}\frac{l}{\epsilon}\right) \\ &= \exp\left(n\frac{\text{Vol}(S^3)}{(4\pi)^{3=2}}\left(-\frac{5}{4l^2} + 2\sigma_0^2 - \frac{1}{l}\sigma_0\right)\frac{1}{\epsilon} + (1-n)\frac{21\sqrt{\pi}}{48}\frac{l}{\epsilon}\right) \\ &= \exp\left(n\frac{\sqrt{\pi}l^2}{4}\left(-\frac{5}{4l^2} + 2(\sigma_0 - \frac{1}{4l})^2 - \frac{1}{8l^2}\right)\frac{l}{\epsilon} + (1-n)S^{(\ )}_{A;}\right). \end{aligned} \tag{7.36}$$

Since this is all the σ_0 dependence that is present in the UV partition function, we can perform the integral over the Lie algebra. For the integral to converge, we go back to the original real contour through $\sigma_0 \to -i\sigma_0$. This results in the following σ_0-dependent integral[7]

$$\int_{1}^{+1} d\sigma_0 e^{\ n^{p}-\frac{l^3}{2\epsilon}(\ _0\ \ \frac{i}{4l})^2} = \frac{1}{l}\sqrt{\frac{2\sqrt{\pi}\epsilon}{nl}}, \tag{7.37}$$

where we have performed a constant imaginary shift of the integration variable on the complex plane.

Now, our quantity of interest is the Rényi entropy in the $n \to 1$ limit.[8] This is

$$\begin{aligned} &\frac{1}{1-n}\left[\log\int_{1}^{+1}[d\sigma_0]\exp\left(\log Z^{\text{1-loop}}_{\text{matter}}(\sigma_0,n)\right) - n\log\int_{1}^{+1}[d\sigma_0]\exp\left(\log Z^{\text{1-loop}}_{\text{matter}}(\sigma_0,1)\right) + (\text{gauge})\right] \\ &= \frac{1}{1-n}\left[-\frac{1}{2}\log\frac{nl}{2\sqrt{\pi}\epsilon} + n\frac{1}{2}\log\frac{l}{2\sqrt{\pi}\epsilon} + (1-n)(S^{(\ )}_{A;} + S^{(\ )}_{A;V} - \log\text{Vol}(U(1))) + \mathcal{O}((1-n)^2)\right] \\ &= \frac{1}{1-n}\left[(1-n)\left(\frac{3\sqrt{\pi}}{8}\frac{l}{\epsilon} - \frac{1}{2}\log\frac{l}{2\sqrt{\pi}\epsilon} + \frac{1}{2} + (S^{(\ )}_{A;} + S^{(\ )}_{A;V} - \log 2\pi)\right) + \mathcal{O}((1-n)^2)\right], \end{aligned} \tag{7.38}$$

[7] The overall integral is multiplied by a factor of $l^{\ 1}$, due to the fact that has dimensions of inverse length. In what follows we will not make this l dependence explicit (in fact, it can simply be set to 1).

[8] If we wish to study higher order Rényi entropies, then this method is insu□cient. The reason is that the expansion of the heat coe□cients $a_i(M_n)$ only tells us about contributions of the conical singularity to first order in $(1\ \ n)$.

where we have recognised that the contribution from the gauge multiplet does not depend on σ_0, and therefore it amounts to only $S^{(\ )}_{A;V}$. The terms inside the exponential proportional to n not contributing to the σ_0 integral will vanish in the expression for the entropy. This results in (with $S^{(\ )}_{A;V}$ computed with $N\ \ = 0$, $N_{\text{Cartan}} = 1$)

$$
\begin{aligned}
S_A^{U(1)} &= S^{(\ )}_{A;\ } + S^{(\ )}_{A;V} - \frac{1}{2}\log\frac{l}{2\sqrt{\pi}\epsilon} - \log 2\pi + \frac{1}{2} \\
&= \frac{7\sqrt{\pi}}{6}\frac{l}{\epsilon} - \frac{1}{2}\log\frac{l}{\epsilon} - \frac{1}{2}\log 2 - \frac{3}{4}\log\pi + \frac{1}{2}.
\end{aligned}
\tag{7.39}
$$

It is interesting to note that this computation is unchanged if we take the limit of infinite radius, $l \to +\infty$. In particular,

$$
\exp\left(\log Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, n)\right)_{l\ !\ +1} = \exp\left(n\frac{\text{Vol}(S^3)}{(4\pi)^{3=2}}\frac{1}{\epsilon}2\sigma_0^2 + (1-n)(S^{(\ )}_{A;\ } + S^{(\ )}_{A;V})\right). \tag{7.40}
$$

This leads to the same σ_0-integral and to the same final expression for $S_A^{U(1)}$. Therefore, the only e□ects produced by taking $l \to +\infty$ are the IR divergences in the logarithmic term and in the linear term in $\epsilon\ ^1$ (including the discarded term resulting from the σ_0 integral, see footnote 7). This can be seen as a consequence of the conformal symmetry of the theory, in the sense that changing the macroscopic scale of the geometry l has no direct physical consequences and can be re-expressed in terms of a change of the microscopic UV cuto□.[9]

Before attempting to make sense of this result, let us work out several more examples with distinct gauge groups in order to study the general structure of the terms resulting from this computation.

$\underbrace{U(1) \times ... \times U(1)}_{N}$ chiral multiplet/ N $U(1)$ chiral multiplets

In this case the partition function simply factorizes into a product of N integrals over $\sigma_0^{(1)}, ..., \sigma_0^{(N)}$, each contributing

$$
\left(\int_{\ 1}^{+\ 1}\frac{d\sigma_0}{\text{Vol}(U(1))}e^{\ n^{p}-\frac{l^3}{2\epsilon}(\ _0\ \ \frac{i}{4l})^2}\right)^N = \left(\frac{1}{2\pi l}\sqrt{\frac{2\sqrt{\pi}\epsilon}{nl}}\right)^N, \tag{7.41}
$$

and the entropy is simply multiplied by N relative to the previous case:

$$
S_A^{U(1)^N} = N\left[\frac{7\sqrt{\pi}}{6}\frac{l}{\epsilon} - \frac{1}{2}\log\frac{l}{\epsilon} - \frac{1}{2}\log 2 - \frac{3}{4}\log\pi + \frac{1}{2}\right]. \tag{7.42}
$$

Two chiral multiplets in the 1 and $\overline{1}$ representations of $U(1)$ (opposite electric charges)

Before the complexification of the σ_0 variables, the relevant coe□cients are

$$
a_{1;\ }^{\text{bulk}}(S_n^3) = n\text{Vol}(S^3)\left(-\frac{7}{4l^2} \pm \frac{i}{l}\sigma_0 - \sigma_0^2\right), \quad a_{1;\ }^{\text{bulk}}(S_n^3) = a_{1;\ }^{\text{bulk}}(S_n^3), \tag{7.43}
$$

[9] Recall that l is related to the radius of the circle R in flat space, but can always be absorbed by a Weyl transformation of the metric in the conformal theory.

where ϕ denotes the field in the 1 or $\overline{1}$ representation. Going through the same steps as in the single $U(1)$ chiral multiplet, this leads to

$$\exp\left(\log Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, n)\right) = \exp\left(\frac{n\sqrt{\pi}}{4}\left(-\frac{5}{2} + 4l^2\sigma_0^2\right)\frac{l}{\epsilon} + 2(1-n)\frac{3\sqrt{\pi}}{8}\frac{l}{\epsilon}\right). \quad (7.44)$$

The integral over σ_0 is now

$$\int_{1}^{+1} \frac{d\sigma_0}{\mathrm{Vol}(U(1))} e^{\ n\sqrt{\pi}\frac{l^3}{\epsilon}(\ _0\ \frac{i}{4l})^2} = \frac{1}{2\pi l}\sqrt{\frac{\sqrt{\pi}\epsilon}{nl}}, \quad (7.45)$$

which results in[10]

$$S_A^{N_c=1;N_f=1} = 2S^{(\ )}_{A;\ } + S^{(\ )}_{A;V} - \frac{1}{2}\log\frac{l}{\epsilon} - \log 2 - \frac{3}{4}\log\pi + \frac{1}{2}. \quad (7.46)$$

This represents a diﬀerence of $-\frac{1}{2}\log 2$ relative to the finite part of the previous case. Note that the coeﬃcient of $S_{A;V}$ has not doubled, since we still have only one abelian gauge multiplet with $N_{\text{Cartan}} = 1$ and $N\ = 0$. More interestingly, the coeﬃcient of the logarithmic divergence has remained the same as for the case of a single chiral multiplet in a representation of $U(1)$. This seems to suggest that this coeﬃcient only depends on the rank of the gauge group. This would be a natural conclusion, since the logarithmic divergence only occurs when gauge fields are present at non-zero coupling with the matter fields.

Single $U(1)$ chiral multiplet + $U(1)_k$ Chern-Simons

The integral over u(1) is modified if we include a Chern-Simons action, which introduces a term dependent on the classical action in the integrand, $e^{\ S_{\text{cl}}(\ _0;n)} = e^{in\ k\ ^2_0}$. Setting $a \equiv \frac{\mathrm{Vol}(S^3)}{l(4\ )^{3/2}} = \frac{\sqrt{\ } l^2}{4}$, it reads

$$(-i)\int_{1}^{+1} \frac{d\sigma_0}{\mathrm{Vol}(U(1))}\exp\left(n\left(2a\frac{l}{\epsilon} - i\pi k\right)\sigma_0^2 - na\frac{\sigma_0}{l}\frac{l}{\epsilon}\right) =$$

$$= (-i)\int_{1}^{+1} \frac{d\sigma_0}{\mathrm{Vol}(U(1))}\exp\left(n\left(2a\frac{l}{\epsilon} - i\pi k\right)\left(\sigma_0 - \frac{1}{2}\frac{a}{2a\frac{l}{\ } - i\pi k}\right)^2 - \frac{1}{4}\frac{na^2}{2a\frac{l}{\ } - i\pi k}\right)$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{2na\frac{l}{\ } - in\pi k}}\, e^{\ \frac{1}{4}\frac{na^2}{2a\frac{l}{\epsilon}\ i\pi k}}. \quad (7.47)$$

The contribution from the exponential term will cancel in the expression for the entropy since its argument is linear in n. The diﬀerence relative to the above case lies in the imaginary term in the denominator of the square root, which adds to the entropy

[10] Below we denote by N_f the number of flavours of the theory, which we consider as the number of pairs of chiral multiplets in the fundamental and anti-fundamental representations of the guage group. N_c denotes the number of colors of the theory, *i.e.* the fields are charged under $U(N_c)$ and transform under representations of rank N_c.

$$\begin{aligned}
&\lim_{n\to 1}\frac{1}{1-n}\left[-\frac{1}{2}\log\left(\frac{n}{2\sqrt{\pi}}\frac{l}{\epsilon}-ink\right)+\frac{1}{2}n\log\left(\frac{1}{2\sqrt{\pi}}\frac{l}{\epsilon}-ik\right)-(1-n)\log 2\pi\right]=\\
&=-\frac{1}{2}\log\left(\sqrt{\left(\frac{1}{2\sqrt{\pi}}\frac{l}{\epsilon}\right)^2+k^2}\exp\left(i\arctan\frac{-2\sqrt{\pi}k\epsilon}{l}\right)\right)-\log 2\pi+\frac{1}{2}\\
&=-\frac{1}{4}\log\left(\left(\frac{1}{2\sqrt{\pi}}\frac{l}{\epsilon}\right)^2+k^2\right)-\log 2\pi+\frac{1}{2}+\ldots,
\end{aligned} \tag{7.48}$$

where the ellipses denote imaginary terms which we may discard. This means that the Chern-Simons sector has no significant effect on the entropy in the presence of a $U(1)$ chiral multiplet (SQED) in the UV limit $\epsilon\to 0$.

Chiral multiplet in the bifundamental $1\times\overline{1}$ representation of $U(1)_k\times U(1)_{-k}$

As in the computation performed in 4.2.2, the weight of the $1\times\overline{1}$ representation of $U(1)_k\times U(1)_{-k}$ which enters in the Lagrangian is $\sigma-\tilde{\sigma}$, with $\sigma,\tilde{\sigma}\in\mathbb{R}$. The classical partition function is now $e^{-S_{\text{cl}}(\sigma_0;n)}=e^{in\pi k(\sigma^2-\tilde{\sigma}^2)}$. We have

$$\begin{aligned}
&(-i)\int_{-\infty}^{+\infty}\frac{d\sigma d\tilde{\sigma}}{2\mathrm{Vol}(U(1))}\exp\left(-in\pi k(\sigma^2-\tilde{\sigma}^2)+2na(\sigma-\tilde{\sigma})^2\frac{l}{\epsilon}-na\frac{1}{l}(\sigma-\tilde{\sigma})\frac{l}{\epsilon}\right)=\\
&=\int_{-\infty}^{+\infty}\frac{d\sigma_+ d\sigma_-}{2\mathrm{Vol}(U(1))}\exp\left(in\pi k\sigma_+\sigma_- -2na\sigma_-^2\frac{l}{\epsilon}+na\frac{1}{l}\sigma_-\frac{l}{\epsilon}\right)\\
&=\frac{1}{4\pi}\int_{-\infty}^{+\infty}d\sigma_+\delta(n\pi k\sigma_-)\exp\left(-2na\sigma_-^2\frac{l}{\epsilon}+na\frac{1}{l}\sigma_-\frac{l}{\epsilon}\right)\\
&=\frac{1}{4n\pi^2 k}.
\end{aligned} \tag{7.49}$$

With this gauge group we obtain a finite contribution to the topological entanglement entropy coming from the Chern-Simons sector, which is

$$\lim_{n\to 1}\frac{1}{1-n}\left[-\log n\pi k+n\log\pi k\right]=-\log k-2\log 2-2\log\pi+1. \tag{7.50}$$

The first term precisely matches the Chern-Simons contribution for this gauge group encountered in 4.2.2.

Comments

On the one hand, this method does not seem to produce very accurate results when considering topological (Chern-Simons) sectors. This is because for the $U(1)_k$ chiral multiplet we do not obtain the expected additive contribution of $-\frac{1}{2}\log k$, while for the $U(1)_k\times U(1)_{-k}$ bifundamental multiplet we indeed obtain such a contribution but the divergent piece is not retained in the final result.

Another setback which the reader might be worrying about is that if a logarithmic divergence is present in our final result, then any finite terms resulting from the computation can always be absorbed or shifted by terms proportional to $\log l$ resulting from arbitrary redefinitions of the UV cutoff ϵ.[11] This is indeed true and prevents us from attributing a concrete interpretation to the value of such finite terms. However,

[11] In contrast, the coefficient of the logarithmic divergence is universal.

a quantity which is cutoff-independent is the *difference*, at fixed ϵ, of the finite terms observed between theories with different numbers of matter and gauge multiplets and/ or different gauge groups. This effectively allows us to deduce the universal contribution to the entanglement when varying these parameters of the theory.

We see that, in these simple setups of abelian groups, this is achieved with some accuracy: the contribution of each chiral multiplet (charged under $U(1)$) is $-\frac{1}{2}\log 2$, in agreement with the examples studied in 3 and 4 computed with Nishioka and Yaakov's formula for the finite piece of S_A^{susy}! Moreover, we are able to extract the amount of universal entanglement due to each $U(1)$ gauge multiplet, namely $-\frac{3}{4}\log\pi + \frac{1}{2} \approx -0{,}36$.

The two previous conclusions may be drawn from looking at the first three examples presented above: for each chiral multiplet transforming under an independent representation of $U(1)$ (meaning that for each such multiplet there is a corresponding $U(1)$ gauge multiplet) there is an increment of $S_A^{U(1)}$, wheras if a chiral multiplet is added which does not transform under an independent representation of $U(1)$ we only get an increment of $-\frac{1}{2}\log 2$ in the entropy. We should also note that with Nishioka and Yaakov's formula it is not straightforward to obtain a closed expression for the entanglement entropy of a single $U(1)$ chiral multiplet.[12]

Last but not least, from the first three examples of this section we see that the logarithmic divergence resulting from our calculations depends exclusively on the number of charge generators, or the rank of the gauge group, and not on the number of matter multiplets. Indeed, when computing the entropy of two chiral multiplets with opposite electric charge, this term remains the same as in the case of a single $U(1)$ chiral multiplet due to the fact that the number of charge generators is unchanged (and only the coefficient of $S_{A;}$ doubles).

7.3.2 Examples: Non-abelian unitary groups

Warm-up: $U(2)$ chiral multiplet

We take the Cartan subalgebra as the set of diagonal matrices $\sigma_0 = \text{diag}(\sigma_1, \sigma_2)$, $\sigma_1, \sigma_2 \in \mathbb{R}$. The roots and weights of $U(2)$ are

$$\begin{aligned} \rho_i(\sigma_0) &= \sigma_i, \quad i \in \{1, 2\}, \\ \alpha_{12}(\sigma_0) &= \sigma_1 - \sigma_2 = -\alpha_{21}(\sigma_0), \end{aligned} \tag{7.51}$$

so that the measure on the space of localization zero modes is[13]

$$[d\sigma] = \frac{d\sigma_1 d\sigma_2}{\text{Vol}(U(2))} \prod_{>0} \alpha(\sigma_0)^2 = \frac{d\sigma_1 d\sigma_2}{\text{Vol}(U(2))}(\sigma_1 - \sigma_2)^2. \tag{7.52}$$

We now write the one-loop determinants expanded in heat coefficients. The determinant from the chiral sector factorises into a product of determinants depending

[12] Even *Mathematica* fails to perform the numerical integration of the chiral one-loop determinant 3.121 over σ_0. Of course, if we have a pair of chiral multiplets in conjugate representations then the computation can be done analytically as in 4.2.

[13] The volume of $U(N)$ is defined as $\text{Vol}(U(N)) = \frac{(2\pi)^{\frac{1}{2}N(N+1)}}{G_2(N+1)}$, where $G_2(z)$ is the Barnes function.

on each of the weights of $U(2)$,

$$\exp\left(\log Z^{\text{1-loop}}_{\text{matter}}(\sigma_0, n)\right) = \prod_{i=1}^{2} \exp\left(n\frac{\text{Vol}(S^3)}{(4\pi)^{3/2}}\left(-\frac{5}{4l^2} + 2(\sigma_i - \frac{1}{4l})^2 - \frac{1}{8l^2}\right)\frac{1}{\epsilon} + (1-n)\frac{3\sqrt{\pi}}{8}\frac{l}{\epsilon}\right). \tag{7.53}$$

Additionally, there is now a potential in $\mathcal{L}_{\text{gauge}}$ which must be accounted for in the corresponding heat coe□cients. This is due to the non-trivial root spaces of $U(2)$ along which the fields in the gauge multiplet are expanded (see [21]). In this case we have N = 2 and N_{Cartan} = 2, so that

$$\begin{aligned} a^{\text{bulk}}_{1;B}(S^3_n) &= n a^{\text{bulk}}_{1;B}(S^3) \\ &= nN^{(1)}(N_{\text{Cartan}} + N)\int_{S^3}\sqrt{g}\frac{1}{6}R - nN^{(1)}\sum\int_{S^3}\sqrt{g}\alpha(\sigma_0)^2 \\ &= -6n\text{Vol}(S^3) - 4n\text{Vol}(S^3)(\sigma_1-\sigma_2)^2. \end{aligned} \tag{7.54}$$

For the gaugino sector we square the Dirac operator analogously to what was done in the $U(1)$ case. For the components λ of the gaugino along each root space labelled by α, this operator is

$$\mathcal{D}_{\alpha} = -i\nabla\!\!\!/ + 1\left(-i\alpha(\sigma_0) + \frac{1}{2}\right). \tag{7.55}$$

A similar complexification of the Cartan subalgebra will be necessary in order not to obtain both second and first order diferential operators in the square of $\mathcal{D}$. We consider the heat coe□cients of the following operator:

$$\mathcal{D}^{y}_{\alpha}\mathcal{D}_{\alpha} \equiv \left(-i\nabla\!\!\!/ + \alpha(\sigma_0) + \frac{1}{2}\right)\left(i\nabla\!\!\!/ + \alpha(\sigma_0) + \frac{1}{2}\right) = \nabla^2 - \frac{1}{4}R + \left(\alpha(\sigma_0) + \frac{1}{2}\right)^2, \tag{7.56}$$

giving

$$\begin{aligned} a^{\text{bulk}}_{1;}(S^3_n) &= -n(N_{\text{Cartan}} + N)\frac{2^{b3/2c}}{2}\int_{S^3}\sqrt{g}d^3x\left(-\frac{1}{6}R - \left(-\frac{1}{4}R\right)\right) + \\ &\quad + n\sum_{2\ 12;\ 21}\frac{2^{b3/2c}}{2}\int_{S^3}\sqrt{g}d^3x\left(\alpha(\sigma_0) + \frac{1}{2}\right)^2 \\ &= -n\text{Vol}(S^3)\frac{6}{l^2}\left(\frac{1}{6} - \frac{1}{4}\right) + n\text{Vol}(S^3)\left(2(\sigma_1-\sigma_2)^2 + \frac{1}{2}\right). \end{aligned} \tag{7.57}$$

Then, the terms from the bulk coe□cients contributing to the σ integral amount to (recall that we have defined $a \equiv \frac{\text{Vol}(S^3)}{l(4\ )^{3/2}} = \frac{p_{-l^2}}{4}$)

$$\begin{aligned} &(-i)^2\int_{1}^{+1} d\sigma_1 d\sigma_2(\sigma_1-\sigma_2)^2\exp\left[na\left(4(\sigma_1-\sigma_2)^2 + 2\left(\sigma_1 - \frac{1}{4}\right)^2 + 2\left(\sigma_2 - \frac{1}{4}\right)^2\right)\frac{l}{\epsilon}\right] = \\ &= \int_{1}^{+1} d\sigma_1 d\sigma_2(\sigma_1-\sigma_2)^2\exp\left[na\left(-4(\sigma_1-\sigma_2)^2 - 2\sigma_1^2 - 2\sigma_2^2\right)\frac{l}{\epsilon}\right]. \end{aligned} \tag{7.58}$$

To solve this integral, we can introduce the (unitary) change of variables $\sigma_+ \equiv \sigma_1+\sigma_2$, $\sigma \equiv \sigma_1 - \sigma_2$, which gives

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d\sigma_+ d\sigma_- \; \sigma_-^2 \exp\left[na\left(-4\sigma_-^2 - 2\left(\frac{\sigma_+ + \sigma_-}{2}\right)^2 - 2\left(\frac{\sigma_+ - \sigma_-}{2}\right)^2\right)\frac{l}{\epsilon}\right] \\
&= \int_{-\infty}^{+\infty} d\sigma_+ d\sigma_- \; \sigma_-^2 \exp\left[na\left(-5\sigma_-^2 - \sigma_+^2\right)\frac{l}{\epsilon}\right] \\
&= \sqrt{\frac{\pi}{na}\frac{\epsilon}{l}} \int_{-\infty}^{+\infty} d\sigma_- \; \sigma_-^2 \exp\left[-5na\sigma_-^2 \frac{l}{\epsilon}\right] \\
&= \frac{\pi}{2\cdot 5^{3/2}}\left(\frac{\epsilon}{nal}\right)^2 = \frac{8}{5^{3/2}n^2 l^4}\left(\frac{\epsilon}{l}\right)^2 .
\end{aligned} \tag{7.59}$$

Upon taking logarithms to obtain the entanglement entropy, we arrive at the following contribution from the integration over u(2) (again neglecting a factor proportional to $\log l$):

$$\begin{aligned}
S_A^{U(2)} &= \lim_{n \to 1} \frac{1}{1-n}\left[\log\left(\frac{8}{5^{3/2}n^2 l^4}\left(\frac{\epsilon}{l}\right)^2\right) - n\log\left(\frac{8}{5^{3/2} l^4}\left(\frac{\epsilon}{l}\right)^2\right) + (1-n)(S_{A;\,}^{(\,)} + 4S_{A;V}^{(\,)})\right] \\
&= S_{A;\,}^{(\,)} + 4S_{A;V}^{(\,)} - 2\log\frac{l}{\epsilon} + 3\log 2 - \frac{3}{2}\log 5 + 2,
\end{aligned} \tag{7.60}$$

Note that the coefficient of the logarithmic divergence is $-2 = -\frac{4}{2}$. So far we are verifying that this coefficient behaves as $-\frac{\dim g}{2}\log\frac{l}{\epsilon}$.

Analogously to what was observed in the abelian case, considering a chiral multiplet charged under the gauge group $U(2)^N$ (or equivalently, N chiral multiplets charged under independent representations of $U(2)$) results in $S_A^{U(2)^N} = N S_A^{U(2)}$.

Two chiral multiplets in the 2 and $\overline{2}$ representations of $U(2)$

Gathering the bulk coefficients for each of chiral multiplets in conjugate representations of $U(2)$ as in the abelian example, the integral we must solve is

$$\begin{aligned}
&\int_{-\infty}^{+\infty} d\sigma_+ d\sigma_- \; \sigma_-^2 \exp\left[na\left(-4\sigma_-^2 - 4\left(\frac{\sigma_+ + \sigma_-}{2}\right)^2 - 4\left(\frac{\sigma_+ - \sigma_-}{2}\right)^2\right)\frac{l}{\epsilon}\right] \\
&= \int_{-\infty}^{+\infty} d\sigma_+ d\sigma_- \; \sigma_-^2 \exp\left[na\left(-6\sigma_-^2 - 2\sigma_+^2\right)\frac{l}{\epsilon}\right] \\
&= \frac{2}{3^{3/2}n^2 l^4}\left(\frac{\epsilon}{l}\right)^2 ,
\end{aligned} \tag{7.61}$$

and thus

$$S_A^{N_c=2;N_f=1} = 2S_{A;\,}^{(\,)} + 4S_{A;V}^{(\,)} - 2\log\frac{l}{\epsilon} + \log 2 - \frac{3}{2}\log 3 + 2 \tag{7.62}$$

It is now certainly possible to read off the amount of universal entanglement coming from each $U(2)$ chiral multiplet as in the abelian case, as well as the one for each vector multiplet by considering gauge group $U(2)^N$. For instance, the former equals $-2\log 2 - \frac{3}{2}\log\frac{3}{5} \approx -0.62 < -\frac{1}{2}$, signalling that a $U(2)$ chiral multiplet introduces higher universal entanglement in the vacuum state compared to a $U(1)$ chiral multiplet; this should likely be attributed to its non-abelian statistics. However, due to our current inability to precisely interpret these finite values we obtain, we will refrain from performing this analysis for $U(N)$.

Generalization: $U(N)$ gauge group with N_f flavours

The integral 7.59 can easily be computed when the gauge group is $U(N)$, for arbitrary N. We simply need to notice that we can introduce the variables

$$\sigma_{ij}^{\pm} \equiv \sigma_i \pm \sigma_j, \quad 1 \leq i \neq j \leq N, \tag{7.63}$$

and that each term of the form σ_i^2 can be written as

$$\frac{N-1}{N-1}\sigma_i^2 = \frac{1}{N-1}\sum_{j \neq i}\left(\frac{\sigma_{ij}^{+} + \sigma_{ij}^{-}}{2}\right)^2. \tag{7.64}$$

Then, the resulting integration over u(N) is equivalent to having an integral of the type 7.59 for each pair i, j. In particular, the number of such integrals we get is

$$\binom{N}{2} = \sum_{k=1}^{N\ 1}(N-k) = N(N-1) - \frac{N(N-1)}{2} = \frac{N(N-1)}{2}, \tag{7.65}$$

taking the form

$$\begin{aligned}
&\int_{\mathrm{u}(N)}[d\sigma]\exp\left\{na\left(-4(\sigma_{ij})^2 - \frac{2}{N-1}\left(\frac{\sigma_{ij}^{+} + \sigma_{ij}}{2}\right)^2 - \frac{2}{N-1}\left(\frac{\sigma_{ij}^{+} - \sigma_{ij}}{2}\right)^2\right)\frac{1}{\epsilon}\right\}\\
&= \int_{\mathrm{u}(N)}[d\sigma]\exp\left\{na\left(-\left(4 + \frac{1}{N-1}\right)(\sigma_{ij})^2 - \frac{1}{N-1}(\sigma_{ij}^{+})^2\right)\frac{1}{\epsilon}\right\}\\
&= \frac{1}{\mathrm{Vol}(U(N))}\left\{\frac{\pi}{2}\left(4 + \frac{1}{N-1}\right)^{\ \frac{3}{2}}(N-1)^{\frac{1}{2}}\left(\frac{(4\pi)^{3=2}}{n\mathrm{Vol}(S^3)}\epsilon\right)^2\right\}^{\frac{N(N\ 1)}{2}}\\
&= \frac{1}{\mathrm{Vol}(U(N))}\left\{\frac{8}{n^2 l^4}(4N-3)^{\ \frac{3}{2}}(N-1)^2\left(\frac{\epsilon}{l}\right)^2\right\}^{\frac{N(N\ 1)}{2}},
\end{aligned} \tag{7.66}$$

where

$$\int_{\mathrm{u}(N)}[d\sigma] \equiv \frac{1}{\mathrm{Vol}(U(N))}\int_{\ 1}^{+\ 1}\prod_{1\ i<j\ N} d\sigma_{ij}^{+}d\sigma_{ij}(\sigma_{ij})^2. \tag{7.67}$$

This expression leads to the following entanglement entropy:

$$\begin{aligned}
S_A^{U(N)} =& \frac{N(N-1)}{2}\left[-2\log\frac{l}{\epsilon} + 3\log 2 + \log\frac{(N-1)^2}{(4N-3)^{\frac{3}{2}}} + 2\right] +\\
&+ S_{A;}^{(\ )} + N(N-1)S_{A;V}^{(\ )} - \log(\mathrm{Vol}(U(N))).
\end{aligned} \tag{7.68}$$

To observe the growing complexity of this expression for higher ranks, let us write down the following explicit results:

$$\begin{aligned}
S_A^{U(3)} &= S_{A;}^{(\ )} + 6S_{A;V}^{(\ )} - 6\log\frac{l}{\epsilon} - 9\log 3 - 6\log\pi + 6;\\
S_A^{U(4)} &= S_{A;}^{(\ )} + 12S_{A;V}^{(\ )} - 12\log\frac{l}{\epsilon} - 5\log 2 - 9\log 13 + 13\log 3 - 10\log\pi + 12.
\end{aligned} \tag{7.69}$$

We also note that our expression has the following behaviour in the large N limit:

$$N^2\left[-2\log\frac{l}{\epsilon}+\frac{1}{4}+\frac{5}{2}\log N-\log\pi\right], \tag{7.70}$$

where the asymptotic expansion of the Barnes function was used and only terms of $\mathcal{O}(N^2)$ or $\mathcal{O}(N^2\log N)$ were kept.

We finally note the expression for the entanglement entropy of N_f flavours of chiral multiplets charged under $U(N_c)$, which follows from applying 7.61 for a flavour of $U(2)$ in the general integral 7.66. Since all the steps are analogous to the manipulations above, we simply state the result:

$$\begin{aligned} S_A^{U(N)} =& \frac{N_c(N_c-1)}{2}\left[(3-N_f)\log 2-2\log\frac{l}{\epsilon}+2+\log\frac{(N_c-1)^2}{(4(N_c-1)+N_f)^{\frac{3}{2}}}\right] \\ &+ S_{A;}^{(\ )}+N_c(N_c-1)S_{A;V}^{(\ )}-\log(\mathrm{Vol}(U(N_c))). \end{aligned} \tag{7.71}$$

7.4 Superselection sectors and the origin of the logarithmic divergence

The most striking feature of the results of the previous section is the presence of a logarithmic divergence in the entanglement entropy when the matter content of our $\mathcal{N}=2$ SCFT is coupled to gauge fields. As alluded to above, such a divergence is not explained by the general structure of entanglement entropy in three dimensions which is predicted by the geometric considerations presented in 2.3.

7.4.1 Orbifold entropy from superselection sectors

Intertwiners and the loss of duality

Following [76], we will describe the consequences at the level of entanglement entropy of von Neumann algebras of having a theory with superselection sectors (SS), *i.e.* when the full algebra of operators of the theory is labelled by sectors corresponding to their charge under some global symmetry of the theory. When this is the case, the physical theory which is tested in a laboratory is the algebra of operators of null charge, since we do not want to measure operators which change a conserved quantum number.

In general, it is relevant to study the consequences of superselection sectors because these can affect:

- Relations between algebras and regions; for topologically non-trivial regions there is a conflict between preserving additivity (the property that the algebra $\mathcal{O}$ of a region A is the union of algebras of smaller balls contained inside A) and Haag duality (the property that $\mathcal{O}$ consists of all the operators which commute with the operators in A^0)[14]; this arises due to the possibility of choosing more than one macroscopic algebra of operators in the continuum limit.

- Charged sectors allow for the possibility of vacuum fluctuations through the creation of charge-anti-charge virtual pairs; this has an effect on the entanglement entropy which can be measured through the moments of some probability distribution of charge fluctuations along the boundary of the region.

[14] In this section, A^0 denotes the spatial complement of a spacelike region A.

Let us first define the concept of SS. For some theory, given a total algebra $\mathcal{F}$ which contains all the possible operators (including the ones which are not invariant under the action of G, that is, which can map a charged sector to a different one), we define the *orbifold* theory $\mathcal{O} = \mathcal{F}/G$ as the one consisting of all operators invariant under G. The precise meaning of this is that any $O \in \mathcal{O}$ transforms in the trivial (or identity) representation of G:

$$O \to U^{\dagger}(g) O U(g) = O, \quad \forall g \in G, \ \forall O \in \mathcal{O}. \tag{7.72}$$

Conversely, when a theory is allowed to have charged states under some global symmetry group G, the Hilbert space of the total algebra $\mathcal{F}$ factorises into sets of states of a given charge, the *superselection sectors*:

$$\mathcal{H}_F = \bigoplus_{r,i} \mathcal{H}_{r,i}. \tag{7.73}$$

These can be represented by irreducibe representations of (the Lie algebra of) G acting on the vacuum state; each of which is closed under the action of local, uncharged operators of the algebra. Specifically, these states are of the form $(\psi^{r,i})^{\dagger}|0\rangle$, with $\psi^{r,i}$ transforming in unitary representations of G: given any $g \in G$,

$$U(g)^{\dagger} \psi^{r,i} U(g) = D_r(g)^{ij} \psi^{r,j}. \tag{7.74}$$

Given these SS and irreducible representations $\{r\}$ of G, there exist two classes of operators which give rise to the tension between additivity and duality of operator algebras.

1. Firstly, we have the *intertwiners*:

$$\mathcal{I}_r = \sum_i \psi_1^{i,r} (\psi_2^{i,r})^{\dagger} \in \mathcal{O}, \tag{7.75}$$

where i runs over the indices of the representation r and $\psi_1^r \in \mathcal{F}_1$, $\psi_2^r \in \mathcal{F}_2$ are charge creating operators supported inside disjoint balls R_1, R_2. Since $[\mathcal{F}(R), \mathcal{F}(R')]$ for any ball R, $\mathcal{I}_r$ commutes with operators additively generated in $(R_1 \cup R_2)'$. However, it cannot be generated additively by operators in $\mathcal{O}_1$ and $\mathcal{O}_2$, since charged operators do not belong to the algebra of the orbifold theory. Therefore, additivity fails for two disjoint balls when there are non-trivial SS.

2. Dually, we have the *twist operators*. These are unitary operators which implement group transformations on elements of $\mathcal{F}$, in the sense that they measure the total charge of some state in A'. They take the form

$$\tau = e^{i \int d^{d-1}x \, \alpha(x) J^0(x)}, \tag{7.76}$$

where $J(x)$ is the charge density operator taking the form $J^0(x) =: \psi^{\dagger}(x)\psi(x):$ and $\alpha(x)$ is a smearing function which vanishes on R_2. The reason for defining α in this way is that it will enable us to detect the failure of duality for τ precisely due to the existence of intertwiners in the union of balls $R_1 \cup R_2$. Let us see how this works: τ commutes with operators of $\mathcal{O}_2$ but anticommutes with operators in $\mathcal{O}_1$. It is in this sense that τ implements group transformations on $\mathcal{O}_{A'}$, given that $\tau \psi^{\dagger}(x) \tau^{-1} = -\psi^{\dagger}(x)$ for $x \in R_1$.[15] This means that τ,

[15] Since we are also assuming a smearing function normalized as $\int d^{d-1}x \, \alpha(x)$, the actions of $\psi^{\dagger}(x)$ and $-\psi^{\dagger}(x)$ on any state differ by a phase $e^{i\pi} = -1$.

although τ commutes with the neutral algebra $\mathcal{O}_1$, it will not commute with the intertwiners $\mathcal{I}_r$, $[\tau, \mathcal{I}_r] \neq 0$, since they are built out of charged operators in R_1. Therefore, twist operators acting on $R_1 \cup R_2$ non-trivially will commute with $\mathcal{O}_1 \cup \mathcal{O}_2$, but not all operators in $(\mathcal{O}_1 \cup \mathcal{O}_2)'$ (the commutant of $\mathcal{O}_1 \cup \mathcal{O}_2$) commute with $\mathcal{O}_1 \cup \mathcal{O}_2$, since the former includes the twists and the latter includes the intertwiners. Therefore, the existence of twist operators leads to the failure of duality for disconnected regions.

In summary, if we wish to retain an additive algebra of G-invariant operators $\mathcal{O}_{\text{add}}$ (which necessarity excludes the intertwiners $\mathcal{I}_r$) either in $R_1 \cup R_2$ or in $(R_1 \cup R_2)'$, we have[16]

$$\begin{aligned} (\mathcal{O}_{\text{add}}(R_1 \cup R_2))' &= \mathcal{O}_{\text{add}}((R_1 \cup R_2)') \cup \{\tau_c\} \\ \mathcal{O}_{\text{add}}((R_1 \cup R_2)') &= \mathcal{O}_{\text{add}}(R_1 \cup R_2) \cup \{\mathcal{I}_r\}, \end{aligned} \tag{7.77}$$

The first and second lines above show that *we cannot retain simultaneously additivity and duality for the regions* $(R_1 \cup R_2)'$ *and* $R_1 \cup R_2$, respectively. This serves as a motivation for the construction developed below, and will have important consequences for the entanglement entropy measured in the orbifold theory $\mathcal{O}$.

Mutual information of the orbifold theory

A useful order parameter which is sensitive to the presence of SS is the diﬀerence of mutual informations of the *additive* parts of the algebras $\mathcal{F}$ and $\mathcal{O}$:

$$I \equiv I_F(R_1, R_2) - I_O(R_1, R_2). \tag{7.78}$$

This is the case because $\mathcal{O}_{\text{add}}(R_1 \cup R_2)$ does not contain the intertwiners supported on $R_1 \cup R_2$ but $\mathcal{F}_{\text{add}}(R_1 \cup R_2)$ does. We will review the evaluation of this quantity presented in [76], which relies on determining lower and upper bounds. We start with the case in which the global symmetry group is finite.

For an uncharged vacuum state, one should be able to express mutual information of the algebra $\mathcal{F}$ in terms of quantities measured by $\mathcal{O}$. In other words, non-zero vacuum expectation values of operators in $\mathcal{F}$ correspond to those of $\mathcal{O}$. Following this idea, I_F can be expressed in terms of relative entropies of $\mathcal{O}$. This is a valuable insight, since it allows us to use monotinicity of relative entropy to obtain a lower bound for I.

Call ω_{12} the vacuum state in $\mathcal{F}_1 \otimes \mathcal{F}_2$ and $E_{12} : \mathcal{F}_1 \otimes \mathcal{F}_2 \to \mathcal{O}_1 \otimes \mathcal{O}_2$ a surjective map which projects out the charged operators.[17] It can then be shown that[18][76]

$$\begin{aligned} I_F(R_1, R_2) - I_O(R_1, R_2) &\geq S(\omega_{12} | \omega_{12} \circ E_{12})\Big|_{C_{12}} \\ &= S(\omega_{12} \circ E_{12})\Big|_{C_{12}} - S(\omega_{12})\Big|_{C_{12}}, \end{aligned} \tag{7.79}$$

for some subalgebra $\mathcal{C}_{12} \subset \mathcal{F}_1 \otimes \mathcal{F}_2$ satisfying $E_{12}(\mathcal{C}_{12}) \subset \mathcal{C}_{12}$. This diﬀerence is positive because charged operators in R_1 and R_2 can have entanglement in vacuum, which is reflected in expectation values of intertwiners; this makes the first term larger (there is more uncertainty in the state that is measured when considering only operators producing states with zero global charge).

[16] $_c$ refers to the twist operators invariant under conjugation, with c labelling the conjugacy classes of the group, defined by $gcg\ ^1 = c$.

[17] This must have the properties of s a conditional expectation map; see [76] for details.

[18] This follows from showing that I itself can be expressed as a relative entropy in the full algebra F, $I = S_F(!_{12} j !_{12} \ E_{12})$.

A universal lower bound for a finite group G can be obtained by taking R_1 and R_2 to be complementary regions to each other, which maximizes the expectation values of intertwiners. This is because intertwiners can be thought of as measuring charge pair fluctuations in some state, with each charge created or in R_1 and R_2. Then, the closer the boundaries of R_1 and R_2 are located, the higher the expectation values of the intertwiners will be. By computing the density matrix of the state $\phi \equiv \omega_{12} \circ E_{12}$ in terms of a unitary transformation of the density matrix of ω_{12}, it is determined that

$$\rho = \bigoplus_r \underbrace{\frac{n_r d_r}{N}}_{q_r} \left[\frac{1}{n_r} (1)_{n_r \quad n_r} \oplus (0)_{n_r^2 \quad n_r \quad n_r^2 \quad n_r} \right] \otimes \left[\frac{1}{d_r^2} \mathrm{Id}_{d_r^2 \quad d_r^2} \right]. \tag{7.80}$$

The corresponding von Neumann entropy is

$$S(\phi) = -\sum_r q_r \log q_r + \sum_r q_r \log d_r^2, \tag{7.81}$$

which is maximized for $q_r = \frac{d_r^2}{|G|}$, giving $S(\phi) = \log |G|$. From this we may arrive at

$$I_F(R_1, R_2) - I_O(R_1, R_2) \geq \log |G|. \tag{7.82}$$

An upper bound can be given by making use of the following property of relative entropy for states φ and σ_i in an algebra:

$$\sum_i \lambda_i S(\sigma_i|\varphi) - S\left(\sum_i \lambda_i \sigma_i | \varphi\right) \leq -\sum_i \lambda_i \log \lambda_i, \tag{7.83}$$

where $\sum_i \lambda_i = 1$. Applying this to to the states $\sigma_i \equiv \omega \circ g_i$ and

$$\varphi \equiv \omega_{12} \circ E_{12} = \frac{1}{|G|^2} \sum_{g_1, g_2 \in G} \omega \circ (g_1 g_2) = \frac{1}{|G|} \sum_{g_1 \in G} \omega \circ g_1, \tag{7.84}$$

where the last equality follows from invariance of the vacuum state ω under group transformations, and g_i refers to a group operation acting on the region R_i. Using $\lambda_i = \frac{1}{|G|}$, we get $I \leq \log |G|$.

As R_2 approaches the complement of R_1 we saturate both bounds, and we conclude that

$$I_F(R_1, R_2) - I_O(R_1, R_2) = \log |G|. \tag{7.85}$$

It should be noted that such a lower bound can be obtained thanks to the presence of intertwiners in the algebra $\mathcal{F}(R_1, R_2)$, while the upper bound is obtained due to the presence of twist operators which implement group transformations on ω; this is further emphasized in detail in [76].

For Lie groups we expect that I diverges as R_2 approaches R_1' due to the large amount of intertwiners. We will be able to extrapolate to the case of non-abelian groups from the $U(1)$ case, for which the twist operators are of the form

$$\tau = e^{ikQ_1} = e^{ik \int d^{d-1}x \ (x) J^0(x)}, \quad k \in (-\pi, \pi). \tag{7.86}$$

Contributions to expectation values of τ will come from non-zero charge fluctuations localized inside the ball R_1. Therefore, the leading contribution would arise due to charge-anticharge pairs created along ∂R_1 where one charge lies inside R_1 and the other outside. Such fluctuations can be expressed as random variables, whose sum will give the total charge contributing to Q_1. In other words, the large number of independent intertwiners along the boundary act as random variables. Furthermore, their independence comes from the fact that short distance fluctuations should not affect each other at macroscopic length scales either inside the ball or across its boundary.

Due to the central limit theorem, such a sum of random variables will give rise to a Gaussian distribution of charges of the form

$$p_q = \frac{1}{\sqrt{2\pi\langle Q_1^2\rangle}} e^{-\frac{q^2}{2\langle Q_1^2\rangle}}, \tag{7.87}$$

where q refers to the charge measured by the operator Q_1 inside the ball R_1.

It is shown in [76] that when considering probability distributions of charges measured by twist operators, the entropy of this distribution gives an upper bound for I, in this case given by

$$-\sum_q p_q \log p_q = \frac{1}{2}\log\langle Q_1^2\rangle + \text{const.} \tag{7.88}$$

We see that an upper bound on I can be obtained if we consider the lowest value of $\langle Q_1^2\rangle$. In order to estimate this, we may look at the intertwiners 7.86, whose expectation value is

$$\langle \tau_k\rangle = \sum_q p_q e^{ikq} \simeq e^{-\frac{1}{2}k^2\langle Q_1^2\rangle}, \tag{7.89}$$

where the approximation is for $\langle Q_1^2\rangle >> 1$. Then, the lowest value of $\langle Q_1^2\rangle$ corresponds to the most spread-out smearing function $\alpha(x)$ across the ball R_1 in the definition of the twist operator 7.86. Fluctuations inside the ball have zero total charge and will not contribute significantly to $\langle Q_1^2\rangle$ in this case. Conversely, the probability of charge fluctuations with one charge inside R_1 and the other outside R_1 will be evenly distributed along ∂R_1, and thus the leading contribution will be proportional to the area of the boundary, that is, $\langle Q_1^2\rangle \sim A/\epsilon^{d-2}$, leading to

$$I \leq \frac{1}{2}\log\frac{A}{\epsilon^{d-2}} + \text{const.} \tag{7.90}$$

On the other hand, we have a lower bound due to the contribution of the intertwiners. For each abelian subalgebra of the group labelled by integer charge Q, the effect of the intertwiners is to create charge-anticharge pairs of charge Q across the boundary between R_1 and R_2. This is measured by computing transition probabilities from some state of charge q close to the boundary to a state with charge $q+Q$:

$$\begin{aligned}\langle \mathcal{I}_Q\rangle &\sim \sum_q \left(\sqrt{p_q}\langle q|_1 \otimes \langle -q|_2\right)\mathcal{I}_Q\left(\sqrt{p_q}|q\rangle_1 \otimes |-q\rangle_2\right)\\ &= \sum_q \left(\sqrt{p_q}\langle q|_1 \otimes \langle -q|_2\right)\left(\sqrt{p_q}|q+Q\rangle_1 \otimes |-q-Q\rangle_2\right)\\ &= \sum_q \sqrt{p_q p_{q+Q}} = e^{-\frac{Q^2}{8\langle Q_1^2\rangle}}\end{aligned} \tag{7.91}$$

An algebraic argument can be used to show that the tightest lower bound to the mutual information has a leading contribution given by

$$I \simeq \frac{1}{2} \log \frac{A}{\epsilon^{d-2}}. \tag{7.92}$$

For a non-abelian global symmetry group G, [76] also gives two arguments for why, in the limit of large charge fluctuations across the boundary, $\langle Q_1^2 \rangle >> 1$, the charge fluctuations generated by $\mathcal{G}_i$, $i = 1, ..., \text{dim(g)}$ (the generators of the Lie algebra of G) should behave as those generated by dim(g) abelian generators; in other words, non-commutative effects become negligeble when $\epsilon \to 0$ and charge fluctuations across the boundary are very relevant. The above determines that, up to a subleading constant, the difference between the mutual information in the full algebra and the orbifold algebra is

$$I_F(R_1, R_2) - I_O(R_1, R_2) \simeq \frac{d-2}{2} \text{dim(g)} \log \frac{R}{\epsilon}, \tag{7.93}$$

with R the radius of R_1.

I is topological

Importantly for our context, the quantity 7.93 is positive and should be attributed to a *negative* term in $I_O(R_1, R_2)$ instead of a positive term in $I_F(R_1, R_2)$. This directly identifies a very specific divergence in the mutual information when this is computed in the theory whose algebra of local operators is the orbifold $\mathcal{O}$. In particular, this logarithmic divergence is present in any dimension $d > 2$, and therefore it is not dependent on integrals if local curvatures on the boundary (these determine the structure of divergences derived in 2.3). It is also not dependent on the specific dynamics or interactions of the theory at short or large distances or mass scales, having been derived using only properties of charge generators of the global symmetry group of the orbifold algebra of operators.

The same observations hold for the derivation of 7.85 in the case of finite G. In fact, a direct correspondence with topological entanglement entropy [3] can be made by writing

$$|G| = \sum_r d_r^2 \Rightarrow \quad I = 2 \log \mathcal{D}, \tag{7.94}$$

where $\mathcal{D} = \sqrt{\sum_r d_r^2}$ is the total quantum dimension introduced in 2.40. This leads to a finite term in the difference in entanglement entropies $S = \frac{I}{2} = \log \mathcal{D}$, resulting from a negative contribution $-\log \mathcal{D}$ to S_O.

7.4.2 Where is the global symmetry?

The construction described above was derived for theories with global symmetry groups. Nevertheless, it predicts the logarithmic divergence that was encountered in the entanglement entropy of our $\mathcal{N} = 2$ gauge theories computed in the previous section, including the negative sign and the dependence on the number of group generators in the coefficient of this divergence. In principle, the Lie group symmetry of these theories should be local (gauged) instead of global. We now argue that the above construction should still apply since the symmetry of our theory is indeed global in the regime where the computation of the previous section is done.

The computation of entanglement entropy of the previous section is done when the theory is embedded in the deep IR of an RG flow from a UV free theory through localization. Indeed, we have computed the heat coefficients of a free theory to which our IR SCFT of interest flows, and the partition functions of both theories coincide by virtue of localization. This phenomenon effectively reduces the full gauge symmetry in the IR to a global symmetry in the UV due to the condition a = 0 on the localization orbits; it is also crucial to note that gauge of a has been fixed and the ghosts integrated out, so that the condition $a = 0$ is not only up to exact one-forms. This means that, in the UV limit of the theory, the field content still transforms in the adjoint representation of the gauge group G. However, the symmetry implemented by G is no longer local because of a = 0, implying that the symmetry transformation parameters must be constant functions of spacetime. In this sense, the effective symmetry group G in the UV limit of the theory is global, and the analysis of [76] applies.

However, this is only a partial answer as to why our logarithmic divergences are explained by [76]. For this to be the case, we must be computing entanglement entropy measured by an algebra of local operators of the orbifold theory. We can indeed convince ourselves that this is the case, given that we are computing entanglement of the vacuum state of a gauge theory. Whether we are viewing the symmetry implemented by G as local or global, according to the explanation of the previous paragraph, the vacuum state $|0\rangle$ is always uncharged under this symmetry. This implies that only uncharged operators contribute to the vacuum entanglement, since vacuum expectation values of charged operators (operators which do not transform trivially under G, contrary to 7.72) vanish.

A perhaps sharper argument for this can be stated as follows: the vacuum reduced density matrix of $\mathcal{A}$ should have zero eigenvalues on the Hilbert subspace corresponding to charged operators smeared inside $\mathcal{A}$. Therefore, $S(\rho)$ should only pick up contributions either from uncharged operators smeared in $\mathcal{A}$ acting on $|0\rangle$ or from intertwiners whose global charge is zero (therefore they do not annihilate $|0\rangle$) but are the product of charged operators localized in $\mathcal{A}$ and in its complement. The latter leads to the logarithmic contribution, as shown in this section.

We can then assume that, although no explicit quotient of the operators under G has been made in our treatment, the entanglement entropy computed here indeed corresponds to that of an orbifold theory.[19]

We thus conclude that the mechanism of supersymmetric localization which embeds strongly coupled SCFT's in the IR regime of an RG flow from a UV free theory heavily constrains its entanglement structure according to the number of generators of the gauge group which rotate the matter fields in the adjoint representation.

[19] I am thankful to Horacio Casini for providing useful comments regarding this particular aspect. And, of course, without his, Huerta, Magán and Pontello's gorgeous work in [76], my interpretation given here of the results in this chapter would not have been possible.

8 Discussion

In this thesis we have studied several aspects of entanglement entropy in $3d\ \mathcal{N} = 2$ supersymmetric field theories, attempting to extend our knowledge of the vacuum entanglement structure of these theories.

First of all, in Chapter 4 we have seen from two examples that, when the theory is perturbed by a relevant deformation which breaks conformality, the finite piece of the entanglement entropy appears to diverge to $-\infty$ as the theory flows to the IR, which goes against physical expectations from observations in, for instance, two-dimensional massive free QFT's (see 2.4).[1] We take this to mean that the deformation of the theory on the branched sphere by $A^{(R)}$ introduces drastic modifications in universal behaviour of SCFT's as they flow from the UV to the IR.

We went on to study corrections to the entanglement entropy when deforming the disk-shaped entangling region; in Chapter 5 we have computed these corrections for two types of non-simply connected regions, which suggest that disks in 2 + 1 dimensions maximize the entanglement entropy among disk-like entangling regions with internal boundaries.[2] In Chapter 6 we have studied smooth deformations of the boundary of the entangling region around a circle; it was found that S_A^{susy} is independent of smooth deformations of the boundary. These two chapters gather evidence that S_A^{susy} is a topological quantity, which is attributed to the deformation of the background field content of the theory which is necessary in order to map flat space Rényi entropy to a partition function on the branched sphere while preserving supersymmetry. This marks a contrast with the usual von Neumann entropy in field theories without supersymmetry, for which the conclusions of Chapter 6 in particular do not apply.

In an attempt to elucidate the entanglement structure of our SCFT's in light of the previous conclusions, we have computed entanglement entropy using a combination of localization and the heat kernel expansion at arbitrary coupling. This has revealed a logarithmic UV divergence whose presence can be understood from the gauge symmetry of the theory at the IR, which is manifest as a global symmetry at the UV. In particular, this divergence carries a universal coe□cient depending exclusively on the number of charge generators of the theory. Through a purely field theoretic computation, we have thus accessed universal information contained in the entanglement entropy which tells us about internal symmetries of our SCFT's, and in particular about charge fluctuations in their highly entangled vacuum state. This computation has also allowed us to extract the contribution to the universal finite piece of the entanglement entropy added by each chiral and vector multiplet, thus rederiving the $-\frac{1}{2}\log 2$ contribution from each $U(1)$ chiral multiplet and producing predictions for these contributions for gauge group $U(N)$.

There are several aspects which have been left unclarified throughout this thesis. These include:

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[1]We recall here that the expected behaviour can be obtain by a slight modification of the calculation presented in B, which however does not appear to be correct.

[2]Recall, however, that the opposite was found when studying singular regions with a conical topology in 6.3.

1. Negativity and topological character. As reviewed in 2.4, a negative finite part of the entanglement entropy of 2 + 1 dimensional theories is usually associated to long range topological order. Also, one of the puzzling features found in the examples studied in this thesis is the negativity of the finite piece of the entanglement entropy. It would be interesting to understand whether there is a concrete relation between these two facts, further reinforcing the topological character of S_A^{susy} that we have found evidence for.

   While such a connection is tempting, we should note that the conclusions of Chapter 6 were only valid in conformal theories. Moreover, we have seen that S_A^{susy} is sensitive to the appearance of length scales in the theory, even at the perturbative level, which is not characteristic of topological EE in the sense of [3]. The negativity of entanglement entropy in a free scalar + free fermion theory was already observed in [34] in the absence of supersymmetry, so it does not seem reasonable to immediately make the connection between negativity of S_A^{susy} and its topological character.

2. Logarithmic divergences and finite terms. Although we have found a coe□cient of the logarithmic divergence depending only on the rank of the gauge group, it disagrees with the value $-\frac{\dim(\mathfrak{g})}{2}$ predicted in [76] (the fact that the coe□cients agree for $\mathfrak{g} = \mathfrak{u}(1), \mathfrak{u}(2)$ may simply be a coincidence). It was also not clear how to interpret the universal contribution to the entanglement from each $U(N)$ chiral and vector multiplet for $N \geq 2$. In fact, a precise explanation of the origin of the $-\frac{1}{2}\log 2$ contribution from a $U(1)$ or even a free chiral multiplet is lacking. It seems that this result corresponds to some Shannon entropy of finitely many degrees of freedom resulting from the cancellation of infinitely many propagating modes in the one-loop determinants due to supersymmetry. An interpretation of this sort would surely clarify the values which are obtained for higher rank gauge groups, as well as the results obtained in Chapter 5 for the free energy of free chiral multiplets when considering non-simply connected spacetimes.

Several directions were left unexplored in this thesis, which might help elucidate some of the above aspects, namely:

1. Mutual information and higher genus surfaces. It would be interesting to be able to discuss mutual information in this framework, as well as entanglement entropy across higher genus surfaces. The challenge in computing these quantities using localization is essentially the same, and lies in finding a conformal transformation mapping the causal development of the union of two disconnected disks or of a higher genus $2d$ surface to a compact $3d$ manifold. In the event that such a manifold corresponds to a Seifert manifold given by a circle fibration over a higher genus Riemann surface, then we would be able to calculate the partition function on this space using the techiques of [77].

2. What about holography? At this point there is a pretty large elephant in the room which the reader might be wondering about, namely the possibility to test some of the results in this thesis through holographic computations. Indeed, there exists a holographic dual to $3d$ $\mathcal{N} = 2$ $U(N)_k \times U(N)_{-k}$ ABJM theory given by a supergravity theory on $AdS_4 \times CP^3$ [24]. It would be excellent to verify the deformation independence of Chapter 6 directly by computing deformations to Ryu-Takayanagi surfaces, and this has not been considered.

One could also consider computing the holographic entanglement entropy in this supergravity theory and matching it to the predictions for the UV divergent terms of the entanglement entropy computed in 7.2 at large N for the ABJM theory, similarly to what was done in [72] for $4d$ $\mathcal{N} = 4$ SYM (the universal terms were already matched in [24]). However, it is not clear how such a matching would be achieved, given that we have obtained a scaling of N^2 of these terms at large N, as opposed to the $N^{3/2}$ scaling on the gravity side (which is simply due to the fact that $G_4^{-1} \propto N^{3/2}$). We would certainly like to understand how to reconcile these two facts.

3. Further tests of our heat kernel computation. More confirmations of our predictions for the universal contribution to the entanglement of each chiral/ vector multiplet are called for. One way to obtain this would be to exploit the countour integration techniques involving JK residues presented also in [77] in order to compute the exact one-loop determinants arising from localization.

We have by now seen that, by trying to naïvely explore as many ways as possible to break the CHM map which the definition of supersymmetric Rényi entropy from [25] relies on, we have uncovered intriguing facts about entanglement entropy in $3d$ field theories with $\mathcal{N} = 2$ supersymmetry, having perhaps raised more questions than they have answered. Nevertheless, we hope that this thesis has helped to further elucidate our understanding of the vacuum entanglement structure of these theories.

A Representations of the superconformal algebra

For a thorough discussion on the superconformal algebra see, for instance, [78]. Here we will simply discuss how to determine the scaling dimension of an arbitrary field in a supermultiplet, or equivalently how it transforms under the action of the superconformal algebra. The generators of this algebra can be either bosonic or fermionic and organized in the following schematic operator acting on the Z_2-graded Hilbert space,

$$\begin{pmatrix} D, P\ , K\ , M & Q\ , \bar{S} \\ \bar{Q}, S & T^A, R \end{pmatrix}, \tag{A.1}$$

where Q , $\bar{Q}$ generate the usual Poincaré supersymmetry transformations, and we now include the conformal supercharges S and $\bar{S}$ which generate superconformal transformations. These must be present in the superconformal algebra because K does not commute with the Poincaré supercharges Q , $\bar{Q}$.[1] Their commutators must be proportional to additional (fermionic) symmetry generators, the conformal supercharges S , $\bar{S}$. The spacetime conformal symmetry group, $SO(2,d) \simeq SU(2,d-2)$, together with the full R-symmetry group, $SU(\mathcal{N})$, forms the superconformal global symmetry group $SU(2,d-2|\mathcal{N})$.

Fields in a given supermultiplet are obtained from one another by acting with the fermionic operators in the above algebra. In order to classify the (unrenormalized) scaling dimensions of operators in a given multiplet, we need to know how these commute with the operator that generates spacetime dilations. This is relevant because the scale dimensions of operators in a CFT completely determines their correlators. These are

$$[D, Q\ ] = \frac{1}{2}Q\ , \quad [D, S\ ] = -\frac{1}{2}S\ , \quad [D, P\ ] = P\ , \quad [D, K\ ] = -K\ , \tag{A.2}$$

while

$$[D, T^A] = [D, M\ \ ] = 0. \tag{A.3}$$

Next, we classify the operators in a supermutliplet into primary and descendant operators. Recall that primary operators in a non-supersymmetric CFT may be defined as the operators whose corresponding state (by the state-operator map) is annihilated by K , which must exist for the spectrum of conformal weights to be bounded from below. This is due to the last commutator in A.2, which implies that the operator K carries negative scaling dimension. In the case of a superconformal theory, we have an additional operator which carries negative scaling dimension, S . We then define

[1] Given a primary operator , the state $|0\rangle$ is annihilated by K_μ and therefore $Q_\alpha K_\mu(\ |0\rangle) = 0$. On the other hand, the supersymmetry variation of results in another operator in the same supermultiplet as which need not be primary. In such cases, $K_\mu Q_\alpha(\ |0\rangle) \neq 0$, so that we must have $[Q_\alpha; K_\mu] \neq 0$.

a superconformal primary operator to be the non-vanishing operator $\mathcal{O}$ obeying

$$[S, \mathcal{O}]_{\pm} = 0, \tag{A.4}$$

where $\pm$ denotes commutator/ anti-commutator. Equivalently, $\mathcal{O}$ is the operator of lowest scaling dimension in a supermultiplet. Descendant operators of $\mathcal{O}$ are defined to be the ones that are generated from $\mathcal{O}$ by action of the Poincaré supercharge,

$$\mathcal{O}' = [Q, \mathcal{O}] \quad \Rightarrow \quad \Delta_{O'} = \Delta_O + \frac{1}{2}. \tag{A.5}$$

How to apply this to our $3d$ $\mathcal{N} = 2$ supermultiplets of interest? For the chiral multiplet $\Phi = (\phi, \psi, F)$, the scaling dimensions of scalar operator ϕ can be read off from the Lagrangian in the usual fashion, giving $\Delta_\phi = \frac{1}{2}$ in three dimensions. Successive action of Q on ϕ gives the remaining fields ψ and F, so

$$\Delta_\psi = \Delta_\phi + \frac{1}{2} = 1, \quad \Delta_F = \Delta_\psi + \frac{1}{2} = \frac{3}{2}. \tag{A.6}$$

Similarly for the field strenght vector multiplet $\Sigma = (\sigma, i\bar{\lambda}, -i\lambda, -\frac{i}{2}\varepsilon f, D + \sigma H)$, we can read off from $\mathcal{L}_K$ that $\Delta_D = 2$, implying that $\Delta_\sigma = \frac{\Delta_D}{2} = 1$, and therefore

$$\Delta_\lambda = \frac{3}{2}, \quad \Delta_{f^{\mu\nu}} = 2 = \Delta_D, \tag{A.7}$$

where we wrote $\Delta_{f^{\mu\nu}} = \Delta_D$ because both f and D apear on the RHS of $\delta\lambda = \{\zeta Q, \lambda\}$, so they do not differ by an action of S.

B Expected behaviour for $S^{\mathbf{susy}}_{ABJM,N=1}(\xi)$ and $S^{\mathbf{susy}}_{\mathbf{free\ chiral}}(m)$

The analytical origin of the behaviour observed in 4.1 seems to be that the ξ couples to $\sqrt{q}$ insted of q in the matrix model on S^3_q. This introduces a factor of $\frac{1}{2}$ in the $\mathcal{O}(1-q)$ contribution to the entanglement entropy. Had we expanded the $Z_{\text{ABJM}}(q,\xi)$ around $q = 1$ to order first order in $(\sqrt{q}-1)$ instead, we would find

$$S^{\text{susy}}_{\text{ABJM};N=1}(\xi,k) = -\log k - 2\int_0^1 dx\left(\frac{1}{x^2} - \frac{1}{x}\frac{\cos\left(x\frac{2\,{}^{0}_{1}}{k}\right)}{\sinh x} - \frac{2\xi^0_1}{\pi k}\frac{\sin\left(x\frac{2\,{}^{0}_{1}}{k}\right)}{\sinh x}\right), \tag{B.1}$$

In this case the universal piece of the entanglement entropy would tend to zero for all values of the Chern-Simons level, as the coupling $R\xi^0_1$ is increased. We claim that this is the physical behaviour we should typically observe: because the FI perturbation sets an energy scale in the theory, increasing values of $R\xi$ correspond to either an increase in the energy scale set by ξ such that its characteristic length is much smaller compared to R, or equivalently a much larger size of the entangling surface compared to the one set by ξ. In either case, the entanglement of degrees of freedom across the boundary will be suppressed. This is analogous to the behaviour of a massive $2d$ CFT observed in 2.4.

An interesting eect of the Chern-Simons sector on the entanglement can also be noted. A slower variation of the entropy towards zero is observed when k is increased (additionally to the constant shift $-\log \pi k$). This is telling us that, in some sense, the two physical couplings of the system have opposing eects when it comes to driving the system along the RG flow. This can be interpreted by stating that, the larger k is, the more topological entanglement of the ground state we are measuring. Recall the

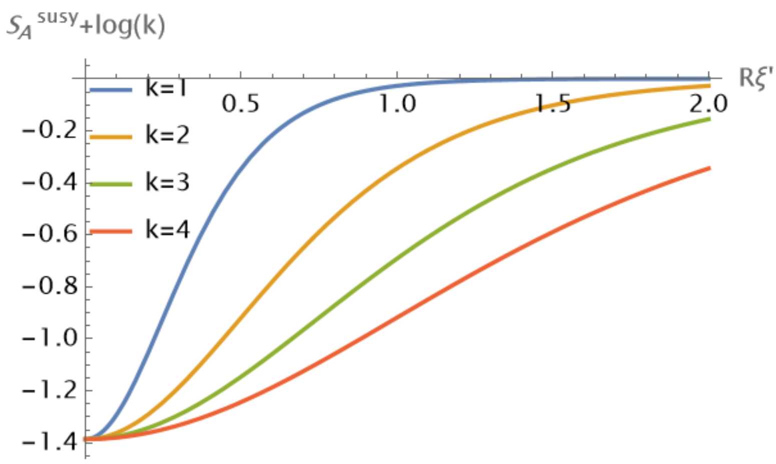

Figure B.1: Expected behaviour for $S^{\text{susy}}_{\text{ABJM},N=1}(\xi,k)$ obtained via a dierent limiting procedure.

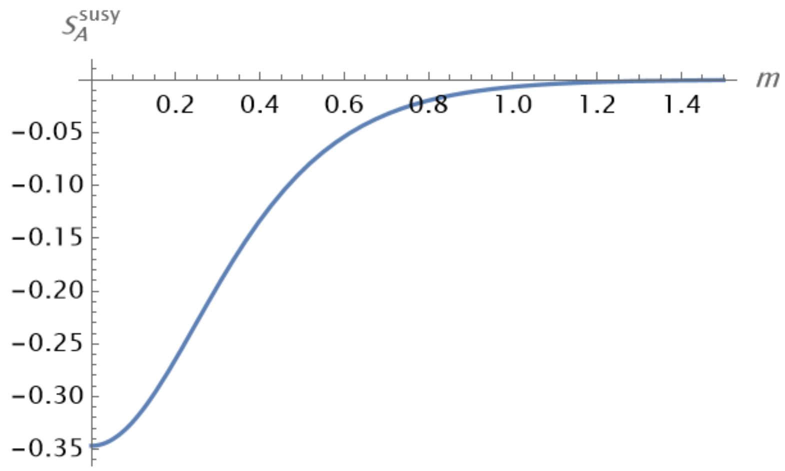

Figure B.2: Expected behaviour for $S^{susy}_{\text{free chiral}}(m)$ obtained via a different limiting procedure.

discussion of topological entanglement entropy in 2.4, this is a topological order of the system which survives at length scales much larger than any correlation length (hence the constant shift $-\log \pi k$) and thus the more the ground state wants to depart from an unentangled state for a given $R\xi$.

On the other hand, we would not satisfy the condition $\frac{d^2S(R)}{dR^2} \leq 0$ in this case (at least for small $R\xi$, where the computation is to be trusted). As noted before, there may be ambiguities introduced by zeta function-regularization in computing the matter one-loop determinants which would lead to this as well as to the opposite monotonic behaviour of the free energy we have already noted.

A very similar observation holds for the interpolating function of the free massive chiral multiplet on S^3. If we drop a factor of $\frac{1}{2}$ in the $\mathcal{O}(q-1)$ contribution to the entanglement entropy by expanding the partition function on the branched sphere to first order in $(\sqrt{q}-1)$, we would obtain

$$S^{susy}_{\text{free chiral}}(m) = i\int_0^{+\infty} dx \left(\frac{-m+i\omega}{2x^2} - \frac{\sin(2x(m-\frac{i}{2}\omega)) - 2xm\cos(2x(m-\frac{i}{2}\omega))}{2x\sin^2(ix)} \right). \tag{B.2}$$

Here we see that the more massive the matter fields become, the entanglement of degrees of freedom across the boundary gets suppressed, by virtue of a larger mass gap being equivalent to a shorter correlation length. As the correlation length gets shorter compared to the radius of the entangling circle and the fields more spatially localized, the entropy decreases.

C THF on the three-sphere

We first want to express the metric on S^3_b in terms of the metric of a Seifert manifold,

$$ds^2_{M_3} = \eta^2 + c^2(z,z)dzdz, \quad \eta = \eta\ dx\ = d\psi + h(z,z)dz + h(z,z)dz. \tag{C.1}$$

Using Hopf coordinates on S^3, we start from[1]

$$ds^2_{S^3} = d\theta^2 + b^2\sin^2\theta d\tau^2 + b\ ^2\cos^2\theta d\varphi^2, \tag{C.2}$$

The adapted coordinates on the THF (ψ, z, z) can be expressed in terms of (θ, τ, φ) through [79]

$$z = f(\theta)e^{i}\ , \quad \begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}b\ ^{1} & -b\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\varphi\\ \tau\end{pmatrix}, \tag{C.3}$$

We now want to find the functions $f(\theta)$, $h(z,z)$ and $c(z,z)$ such that C.1 and C.2 agree. The one form η corresponding to the fibers of the Seifert manifold reads

$$\eta = \frac{b}{2}d\varphi + \frac{b\ ^{1}}{2}d\tau + 2\mathrm{Re}[h(f^0 e^{i}\ d\theta + ife^{i}\ d\phi)]. \tag{C.4}$$

Take $h(\theta,\phi) = \frac{1}{2}e\ ^{i}\ (g_1(\theta) + ig_2(\theta))$. Then,

$$2\mathrm{Re}[h(f^0 e^{i}\ d\theta + ife^{i}\ d\phi)] = g_1 f^0 d\theta - g_2 f d\phi. \tag{C.5}$$

This must equal

$$\mathcal{C} = -(b - b\ ^{1} - (b + b\ ^{1})\cos(2\theta))d\phi, \tag{C.6}$$

which in the case of S^3 (with $b = 1$) corresponds to $g_1(\theta) = 0$, $g_2(\theta)f(\theta) = 2\cos\theta$. Moreover, the part of the metric parametrizing the base Riemann surface is

$$c^2 dzdz = c^2(f^{02}d\theta^2 + f^2 d\phi^2), \tag{C.7}$$

which in the case of S^3 must equal

$$2g_{zz}dzdz = d\theta^2 + \sin^2(2\theta)d\phi^2. \tag{C.8}$$

We can find c and f by solving

$$\begin{aligned}
&\begin{cases} cf^0 = 1 \\ cf = \sin(2\theta) \end{cases} \quad &&\Rightarrow cf^0 + c^0 f = 2\cos(2\theta)\\
& &&\Rightarrow 1 + \frac{c^0}{c}\sin(2\theta) = 2\cos(2\theta)\\
& &&\Rightarrow c = \cos\theta \Rightarrow f = 2\sin\theta.
\end{aligned} \tag{C.9}$$

This determines the THF and the adapted metric of the Seifert fibration describing

[1] We could allow for a dependent coefficient in $d\ ^2$, but it can be checked that this can always be absorbed in a redefinition of u when mapping S^3 to H S^1.

the manifold S^3. Note that because $c = 0$ at $\theta = \frac{}{2}$, the step performed in the second line above is not valid at this value of θ. However, this does not concern us for the purposes of 6, since we will be looking at the induced metric at the entangling surface, or equivalently, at $\theta \to 0$.

For the purposes of the calculations in 6.2, we express $f(\theta)$ in terms of flat space coordinates at $t = 0$ related, to θ through the CHM map. Because

$$\sinh u\Big|_{t=0} = \frac{1}{2}\left(\frac{R+\rho}{R-\rho} - \frac{R-\rho}{R+\rho}\right) = \frac{2R\rho}{R^2-\rho^2}, \tag{C.10}$$

we have

$$\sin^2\theta\Big|_{t=0} = \frac{1}{\cosh^2 u}\Big|_{t=0} = \left(\frac{R^2-\rho^2}{R^2+\rho^2}\right)^2 \Rightarrow \cos^2\theta\Big|_{t=0} = \frac{4R^2\rho^2}{(R^2+\rho^2)^2}. \tag{C.11}$$

D Deformations of $h(z,\bar{z})$

We analyse deformations of the one form η induced by $h(z,z)$ and its complex conjugate $h(z,z)$. Restricting g to first order in h[1],

$$\begin{aligned}\eta &= \frac{1}{2}(d\varphi + d\tau) + \mathcal{C} + h(g_1 f^0 d\theta - g_2 f d\phi) \Rightarrow \\ \Rightarrow \quad g &= 2(g_1 f^0 d\theta - g_2 f d\varphi + g_2 f d\tau)(\cos^2\theta d\varphi + \sin^2\theta d\tau).\end{aligned} \tag{D.1}$$

A deformation in $g_2(\theta)$ results in

$$\begin{aligned}{}_{g_2} ds^2_{S^3} &= -2 \quad g_2 f \sin^2\theta \left[\sinh^2 u d\varphi^2 - d\tau^2 - (\sinh^2 u - 1)d\varphi d\tau\right] \\ &= -2 \quad g_2 f \sin^2\theta \cosh^2\frac{\sigma}{2}\cosh^2\frac{\sigma}{2}\left[\frac{\sinh^2\left(\frac{+}{2}\right)}{\cosh^2\frac{}{2}\cosh^2\frac{}{2}}d\varphi^2 + \right. \\ &\left. + \frac{1}{4\cosh^2\frac{}{2}\cosh^2\frac{}{2}}(d\sigma - d\sigma)^2 - \frac{1}{\cosh^2\frac{}{2}\cosh^2\frac{}{2}}(\sinh^2\left(\frac{\sigma+\sigma}{2}\right) - 1)d\varphi d\tau\right].\end{aligned} \tag{D.2}$$

As explained below 6.28, we may restrict the line element to $t = 0$; factors of $d\tau$ will vanish since $\sigma = \sigma$. This gives (up to the conformal factor)

$${}_{g_2} ds^2_{S^3} = -2 \quad g_2 f R^2 \tanh^2\frac{\sigma}{2}d\varphi^2 = (-2 \quad g_2 f)\rho^2 d\varphi^2 = - \quad g_2\frac{4R\rho}{R^2+\rho^2}\rho^2 d\varphi^2. \tag{D.3}$$

This again leads to a constant $\xi^{'}$ component (at $\rho = R$), with no restriction on ξ .

Similarly to what we have seen for the c deformation, an arbitrary φ dependence of g_2 gives an arbitrary angular profile for the entangling region (small deformations). A new term that can appear here which did not appear in the c deformation is $\sim d\theta d\varphi$, if we turn on $g_1 \neq 0$. In this case, (neglecting right away the $d\tau$ term for the chosen spacelike region)

$$\begin{aligned} g &= 2 \quad g_1 f^0 \cos^2\theta d\theta d\varphi \\ &= 2 \quad g_1 f^0 \sin^2\theta\left[-\tanh^2 u \cosh u du d\varphi\right] \\ &= 2 \quad g_1 f^0 \sin^2\theta \cosh^4\frac{\sigma}{2}\left[-2\frac{\sinh^2\sigma}{\cosh^4\frac{}{2}\cosh\sigma}d\sigma d\varphi\right] \\ &\sim -2 \quad g_1 f^0 2\frac{R^2\tanh^2\frac{}{2}}{e \ + e}\frac{2}{R}\cosh^2\frac{\sigma}{2}dw d\varphi \\ &= - \quad g_1\frac{32}{R}\frac{R^2\rho^2(R^2-\rho^2)}{(R^2+\rho^2)^3}d\rho d\varphi.\end{aligned} \tag{D.4}$$

[1] In fact, second order contributions would only give extra terms d'^2 and d^2 relative to what is shown below; these do not lead to non-trivial di□eomorphisms acting on @A.

The corresponding diffeomorphisms are found by the integrability condition

$$\frac{1}{2}(\partial \, \xi_{'} + \partial_{'} \xi \;) = - \;\; g_1 \frac{32}{R} \frac{R^2 \rho^2 (R^2 - \rho^2)}{(R^2 + \rho^2)^3}. \tag{D.5}$$

This seems to have many possible solutions. A reasonable restriction to place on ξ is $\partial \, \xi^{'} = 0$, and this leads to a constant ξ, which simply implements a trivial dilation of the circular entangling surface.

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