Universiteit Utrecht

Graduate School of Natural Sciences

On h-fulness in Number Fields

Master Thesis

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July 13, 2024

Abstract

Consider a number field K with ring of integers $\mathcal{O}_K$. Call an ideal of $\mathcal{O}_K$ h-ful if all exponents in its prime factorization are at least h. In this thesis, we will find an asymptotic expression with one main term for the number of h-ful ideals of norm at most $x \geq 0$. To do this, we will use the two papers [ES34] and [BG58] which have explored this problem in $K = \mathbb{Q}$, and we will adapt their methods to arbitrary number fields.

Furthermore, we will consider elements $\alpha \in \mathcal{O}_K$ where the principal ideal (α) is h-ful. We will explore but not fully solve the problem of finding the number of such α of height $\leq x$ for some $x \geq 0$, with height function given by

$$H(\alpha) = \prod_{v \in \Omega_\infty} \max(1, ||\alpha||_v),$$

where Ω_∞ denotes the set of infinite places.

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1 Introduction

Mathematicians have often been interested in numbers that have various properties, and their distribution within the integers or reals. This thesis starts with just such a question. Namely, given some positive integer h, consider the set of numbers $k \in \mathbb{Z}$ such that if a prime p divides k, then p^h also divides k. Alternatively, this can be phrased as the set of numbers for which all (nonzero) exponents in their prime factorization are at least h. Numbers with this property are often called h-ful or h-full, and we will use the former term in this thesis. The question that starts our thesis is the following: how many of these numbers are less than some bound x?

In 1933, Erdős and Szekeres decided to investigate this question (along with counting the number of abelian groups with a given cardinality), and they published a paper [ES34] concerning these two problems, where they use manipulation of nested sums. Their findings are as follows.

Denote

$$N_h(x) = \#\{k \in \mathbb{Z}_{>0} : k \leq x, p \mid k \implies p^h \mid k\}.$$

Then Erdős and Szekeres find the following result.

Theorem 1.1 (Erdős, Szekeres)

$$N_h(x) = C_1(h)x^{\frac{1}{h}} + O\left(x^{\frac{1}{h+1}}\right),$$

where

$$C_1(h) = \prod_p \left(1 + \sum_{j=h+1}^{2h-1} p^{-j/h}\right).$$

Erdős and Szekeres actually represented $C_1(h)$ somewhat differently, but the expression used here is easier to state and is equal in value.

In 1957, Bateman and Grosswald decided to expand upon this result in their paper [BG58]. Rephrasing the manipulation of nested sums into sums of coefficients of Dirichlet series, they proved a more detailed asymptotic expression.

Theorem 1.2 (Bateman, Grosswald)

$$N_h(x) = C_1(h)x^{\frac{1}{h}} + C_2(h)x^{\frac{1}{h+1}} + O\left(x^{\frac{1}{h+2}}\right),$$

where

$$C_1(h) = \prod_p \left(1 + \sum_{j=h+1}^{2h-1} p^{-j/h}\right),$$

and where

$$C_2(h) = \zeta\left(\frac{h}{h+1}\right) \prod_p \left(1 + \sum_{j=h+2}^{2h-1} p^{-\frac{j}{h}} - \sum_{j=2h+2}^{3h} p^{-\frac{j}{h+1}}\right).$$

Just like how mathematicians are interested in numbers with certain properties, they are also interested in generalization. To what extent does this problem rely on the specific structure of the integers? Can we adapt it to other settings?

In this thesis, we find that we can, in fact, adapt the problem to other settings. Namely, to rings of integers of various number fields. After all, these have a lot of properties in common with $\mathbb{Z}$, but they also have certain new difficulties. For example, in general, rings of integers of number fields do not have a finite number of units, they do not have unique factorization of elements, and they do not have a total ordering that is compatible with the ring structure.

We will see that, despite these complications, we can have a sensible analogue to the h-ful numbers that are less than or equal to some $x \geq 0$, by looking at ideals. Namely, we can have h-ful ideals (which will be defined more precisely in Section 3), and we can use the ideal norm to only consider ideals of norm at most $x \geq 0$.

In this thesis, we will work towards the following result (which is Theorem 5.6):

Theorem 1.3 (Main Theorem) *Let K be a number field of degree d_K with ring of integers $\mathcal{O}_K$. Let Z_K be the residue of the Dedekind zeta function $\zeta_K(s)$ at $s = 1$. Let $N_{K,h}(x)$ denote the number of h-ful ideals I of $\mathcal{O}_K$ of norm at most $x \geq 0$. Then we have*

$$N_{K,h}(x) = Z_K \gamma x^{\frac{1}{h}} + O\left(x^{\alpha}(\log(x+1))^{\beta}\right),$$

where

$$\gamma = \prod_{\mathfrak{p}} \left(1 + \sum_{m=h+1}^{2h-1} N(\mathfrak{p})^{-\frac{m}{h}}\right),$$

where

$$\alpha = \begin{cases} \frac{1}{h+1} & \text{if } 2h+1 \geq d_K \\ \frac{d_K - 1}{h(d_K+1)} & \text{if } 2h+1 < d_K \end{cases},$$

and where

$$\beta = \begin{cases} 0 & \text{if } 2h+1 > d_K \\ d_K & \text{if } 2h+1 = d_K \\ d_K - 1 & \text{if } 2h+1 < d_K \end{cases}.$$

After this, we will consider elements α of the ring of integers that generate h-ful ideals, rather than looking at the ideals themselves. Instead of using a norm function like we did for ideals, we will use a height function, defined as

$$H(\alpha) = \prod_{v \in \Omega_\infty} \max(1, ||\alpha||_v),$$

where Ω_∞ denotes the set of infinite places of K.

In this setting, we cannot easily use similar methods to those of [ES34] and [BG58], so we will try various new methods. We will explore in what ways the number of h-ful elements α of height at most x differs from the number of h-ful ideals of norm at most x, though we will not find a general asymptotic formula for this quantity.

The division into chapters is as follows.

Chapter 2 will introduce the notation used throughout the thesis.

Chapter 3 will introduce the necessary definitions to talk about h-ful ideals of bounded norm in number fields, so that we can properly define the quantity we want to find an asymptotic formula for.

Chapter 4 will adapt the methods from Erdős and Szekeres, though not yet reach a desirable asymptotic formula, and then will rephrase the problem in the language of Bateman and Grosswald using Dirichlet series, along with proving the tools necessary to get a nice asymptotic function.

Chapter 5 will finally state and prove the asymptotic formula we are after, by showing that the conditions for using the tools from Chapter 4 are fulfilled, and then applying them.

Chapter 6 will introduce the aforementioned height function, then explore in which ways it is similar or different to the ideal norm, and explore various avenues that bridge one or more differences between the number of h-ful ideals of bounded norm and the number of h-ful elements of bounded height, unfortunately without providing a full solution.

2 Notation

In this section, we define some notation and conventions that will be used throughout the thesis.

If $a, b \in \mathbb{Z}$, then we use $a \mid b$ to indicate that a divides b, and we use $a \nmid b$ to indicate a does not divide b. This notation will extend to ideals once we define division for them.

Every product $\prod_p$ ranges over all positive prime numbers in $\mathbb{Z}$.

Given two sets A, B, we will use $A \setminus B$ to indicate the set difference $\{a \in A : a \notin B\}$.

A ring is assumed to have a multiplicative identity 1.

In a number field K, its ring of integers is always denoted by $\mathcal{O}_K$, its Dedekind zeta function is denoted by $\zeta_K(s)$, and its degree is denoted by d_K. The set of (integral) ideals of $\mathcal{O}_K$ is denoted by $\mathcal{I}_K$. The number of real embeddings is denoted by r_1 and the number of pairs of complex embeddings is denoted by r_2, so that $d_K = r_1 + 2r_2$. The residue of the Dedekind zeta function at $s = 1$ is denoted by Z_K. The class number of K is denoted by h_K.

The class of an ideal I in the ideal class group is denoted by $[I]$.

The notation $\sum_{N(I) \leq x}$ is shorthand for $\sum_{I \in \mathcal{I}_K, N(I) \leq x}$. Every product $\prod_{\mathfrak{p}}$ ranges over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$. If $a \in \mathcal{O}_K$, then let (a) denote the ideal generated by a. The notation $I \neq 0$ is shorthand for $I \neq (0)$ and the notation $\sum_{I \neq 0}$ is shorthand for $\sum_{I \in \mathcal{I}_K, I \neq 0}$. Similarly, $\mathcal{I}_K \setminus \{0\}$ is shorthand for $\mathcal{I}_K \setminus \{(0)\}$.

Let Ω_K denote the set of infinite (Archimedean) places of K, and for all $\alpha \in K$, denote $||\alpha||_v := v(\alpha)$. These places are normalized in such a way that if $v \in \Omega_K$ corresponds to a real embedding, then $||k||_v = |k|$ for all $k \in \mathbb{Z}$, and if v corresponds to a pair of complex embeddings, then $||k||_v = |k|^2$ for all $k \in \mathbb{Z}$.

3 Defining h-fulness and norm in number fields

In the introduction, we considered the problem of finding an asymptotic formula for the number of h-ful numbers that are less than or equal to some $x \geq 0$. If we want to translate this problem to number fields, the first thing we have to do is to define what h-fulness means in a number field.

Let K be a number field with ring of integers $\mathcal{O}_K$. Then we know that primes are, in general, no longer numbers, but ideals in the ring of integers. Since $\mathbb{Z}$ is a principal ideal domain, the prime ideals correspond to the positive prime numbers, but such a correspondence is not possible for all rings of integers.

We see that prime ideals in rings of integers have some desirable properties with these classical lemmas.

Lemma 3.1 *Let K be a number field with ring of integers $\mathcal{O}_K$. Then all nonzero prime ideals are maximal.*

Proof. For a proof, see any book on Algebraic Number Theory or Commutative Algebra, for example [ST02, §5.2, Theorem 5.3]. □

Definition 3.2 Let R be a ring and let $I, J \subseteq R$ be two ideals. We call I and J **comaximal** if $I + J = R$. If R is the ring of integers of a number field, we may use the term coprime as well, since all nonzero prime ideals are maximal, and all maximal ideals are nonzero primes.

Remark 3.3 Since $1 \in R$, two ideals I and J being comaximal implies there are $r \in I, s \in J$ such that $r + s = 1$. If $I = (a)$ and $J = (b)$ are principal, then this implies there exist $m, n \in R$ such that $ma + nb = 1$, which matches the definition of coprimality in $\mathbb{Z}$.

Lemma 3.4 *Let R be a ring and let $I, J \subseteq R$ be two comaximal ideals. Then for all $m, n \in \mathbb{Z}_{>0}$, we have that I^m and J^n are comaximal.*

Proof. Consider $(I + J)^{m+n} = R$, due to comaximality of I and J. We also see

$$(I + J)^{m+n} = I^{m+n} + \cdots + I^m J^n + \cdots + J^{m+n},$$

where all terms contain a factor of I^m or J^n. Hence, each term is contained within either of those two, and it follows that all terms are contained in $I^m + J^n$. Therefore, $(I + J)^{m+n} \subseteq I^m + J^n$ and thus $I^m + J^n = R$, proving comaximality. □

Corollary 3.5 *Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Let $\mathfrak{p}, \mathfrak{q} \in \mathcal{I}_K$ be two prime ideals. Then for all $m, n \in \mathbb{Z}_{>0}$, we have that $\mathfrak{p}^m$ and $\mathfrak{q}^n$ are comaximal.*

To define h-fulness on a number field, we will also have to define divisibility of ideals, since the primes are now ideals rather than numbers.

Definition 3.6 Let R be a ring, with $I, J \subseteq R$ two ideals. Then we say that **I divides J** or **I | J** if $J \subseteq I$.

Remark 3.7 If I, J are principal, such that $I = (r)$ and $J = (s)$, then $J \subseteq I$ implies that there exists $k \in R$ such that $s = kr$, in other words, $r \mid s$. So this is equivalent to the usual definition in principal ideal domains.

We will use the following definition for h-fulness in number fields:

Definition 3.8 Let K be a number field with ring of integers $\mathcal{O}_K$. Let $I \in \mathcal{I}_K \setminus \{0\}$ be an ideal. Then we call I ***h*-ful** if for all prime ideals $\mathfrak{p} \subseteq \mathcal{O}_K$, we have $\mathfrak{p} \mid I \implies \mathfrak{p}^h \mid I$.

Ideals are a logical choice in this definition, because ideals have unique factorization into prime ideals, which mirrors the unique factorization that is useful for integers and h-fulness.

Lemma 3.9 *Let K be a number ring, with ring of integers $\mathcal{O}_K$. Let $I \in \mathcal{I}_K \setminus \{0\}$ be an ideal. Then there exist unique $e_\mathfrak{p} \in \mathbb{Z}_{\geq 0}$, with only finitely many nonzero $e_\mathfrak{p}$, such that*

$$I = \prod_{\mathfrak{p}} \mathfrak{p}^{e_\mathfrak{p}}.$$

Proof. See for example [ST02, §5.2, Theorem 5.6]. □

Next, we need to find an analogue to the bound we had in $\mathbb{Z}$, where we would estimate the number of h-ful numbers up to x. But number fields do not necessarily come with such an ordering, especially as they can contain complex numbers, which have no natural ordering. The simplest thing to do would be to find a map from K to $\mathbb{R}$ and then do comparisons there.

Definition 3.10 Let K be a number field, with ring of integers $\mathcal{O}_K$, and let $I \in \mathcal{I}_K \setminus \{0\}$ be an ideal. Then the **norm** $N(I)$ of the ideal I is given by $N(I) := [\mathcal{O}_K : I] = \#(\mathcal{O}_K/I)$.

Remark 3.11 The norm is finite for all nonzero ideals in a ring of integers (see for example the proof of part (d) of [ST02, §5.2, Theorem 5.3]). In $\mathbb{Z}$, we have that if $I = (k)$, then $N(I) = |k|$, so it reduces to the norm we already know.

The ideal norm is also multiplicative.

Lemma 3.12 *Let K be a number field, with ring of integers $\mathcal{O}_K$, and let I, J be two nonzero ideals. Then $N(IJ) = N(I)N(J)$.*

Proof. See for example [ST02, §5.3, Theorem 5.12]. □

With the ideal norm and h-fulness condition we now have for number fields, we can define the following quantities.

Definition 3.13 Let

$$S_{K,h} = \{I \in \mathcal{I}_K \setminus \{0\} : \mathfrak{p} \mid I \implies \mathfrak{p}^h \mid I\}$$

and

$$N_{K,h}(x) = \#\{I \in \mathcal{I}_K \setminus \{0\} : N(I) \leq x, \mathfrak{p} \mid I \implies \mathfrak{p}^h \mid I\}.$$

We will try to find an asymptotic formula for $N_{K,h}(x)$ in the next chapter.

4 Tools for approximating $N_{K,h}(x)$

In this section, we will first consider the methods of [ES34] and how they transfer to number fields. Then, we will see how this is in fact a specific case of the methods in [BG58] and explore those, which will also give us a nicer expression for $N_{K,h}(x)$.

4.1 $N_{K,h}(x)$ as sum

To start, we recall Lemma 3.9: every ideal can be written uniquely as a product of nonnegative powers of prime ideals. For h-ful ideals, we can separate all the nonzero exponents into a nonnegative multiple of h and remainder that is strictly between h and $2h$. In other words, if $I \in \mathcal{I}_K \setminus \{0\}$ is h-ful, then

$$I = \prod_{\substack{\mathfrak{p} \\ e_\mathfrak{p} \neq 0}} \mathfrak{p}^{e_\mathfrak{p}} = \prod_{\substack{\mathfrak{p} \\ e_\mathfrak{p} \neq 0}} \mathfrak{p}^{k_\mathfrak{p} h} \prod_{\substack{\mathfrak{p} \\ e_\mathfrak{p} \neq 0}} \mathfrak{p}^{e'_\mathfrak{p}},$$

with $k_\mathfrak{p} \in \mathbb{Z}_{\geq 0}$, $e'_\mathfrak{p} \in \mathbb{Z}$, $h+1 \leq e'_\mathfrak{p} \leq 2h-1$, and $k_\mathfrak{p} h + e'_\mathfrak{p} = e_\mathfrak{p}$ for all $\mathfrak{p}$.

Define

$$D_{K,h} = \{I \in \mathcal{I}_K \setminus \{0\} : I = \prod_{\mathfrak{p}} \mathfrak{p}^{e_\mathfrak{p}}, e_\mathfrak{p} = 0 \text{ or } h+1 \leq e_\mathfrak{p} \leq 2h-1 \text{ for all } \mathfrak{p}\}.$$

We can now write the following:

$$N_{K,h}(x) = \sum_{\substack{N(I) \leq x \\ I \in S_{K,h}}} 1 = \sum_{\substack{N(I) \leq x \\ I \in D_{K,h}}} \sum_{\substack{N(J) \leq x/N(I) \\ \exists \tilde{J}:\ J = \tilde{J}^h}} 1$$

$$= \sum_{\substack{N(I) \leq x \\ I \in D_{K,h}}} \sum_{N(J) \leq (x/N(I))^{1/h}} 1 = \sum_{\substack{N(I) \leq x \\ I \in D_{K,h}}} \# \left\{ J \in \mathcal{I}_K \setminus \{0\} : N(J) \leq \left(\frac{x}{N(I)} \right)^{\frac{1}{h}} \right\}.$$

So if we have an expression for $\#\{J \in \mathcal{I}_K \setminus \{0\} : N(J) \leq y\}$ for arbitrary y, then we can get a formula for $N_{K,h}(x)$. Luckily, this is a fairly classic problem, and there are many papers that include the result we desire. For example, we can use Theorem 3 from [LDTT22]:

Theorem 4.1 (Lowry-Duda, Taniguchi, Thorne) *Let K be a number field of degree d_K. Let Δ_K be the discriminant of K and recall from Section 2 that we set $Z_K := \text{res}_{s=1}\zeta_K(s)$. Then, for $y \geq 2$,*

$$\#\{J \in \mathcal{I}_K \setminus \{0\} : N(J) < y\} = Z_K y + O\left(|\Delta_K|^{\frac{1}{d_K+1}} y^{\frac{d_K-1}{d_K+1}} (\log y)^{d_K-1} \right),$$

where the implied constant depends only on d_K.

We can immediately remark that the theorem also applies to $\#\{J \in \mathcal{I}_K \setminus \{0\} : N(J) \leq y\}$. Namely, if $y \notin \mathbb{Z}$, then this is equal to $\#\{J \in \mathcal{I}_K \setminus \{0\} : N(J) < y\}$, and if y is an integer, we can say $\#\{J \in \mathcal{I}_K \setminus \{0\} : N(J) \leq y\} = \#\{J \in \mathcal{I}_K \setminus \{0\} : N(J) < y + \varepsilon\}$ for some $0 < \varepsilon < 1$ and since we assume $y \geq 2$, we have

$$O\left(|\Delta_K|^{\frac{1}{d_K+1}} (y+\varepsilon)^{\frac{d_K-1}{d_K+1}} (\log(y+\varepsilon))^{d_K-1} \right) = O\left(|\Delta_K|^{\frac{1}{d_K+1}} y^{\frac{d_K-1}{d_K+1}} (\log y)^{d_K-1} \right).$$

Since $\#\{J \in \mathcal{I}_K \setminus \{0\} : N(J) \leq 1\} = 1$ and $\#\{J \in \mathcal{I}_K \setminus \{0\} : N(J) \leq 0\} = 0$, we may extend the result to $y \in \mathbb{R}_{\geq 0}$ by using $\log(x+1)$ rather than $\log(x)$ in the error term. For our purposes, the independence of the implied constant of Δ_K is also not important, so we may as well use the following corollary:

Corollary 4.2 (Lowry-Duda, Taniguchi, Thorne) *Let K be a number field of degree d_K and recall from Section 2 that we set $Z_K := \text{res}_{s=1}\zeta_K(s)$. Then, for $y \in \mathbb{R}_{\geq 0}$, we have*

$$\#\{J \in \mathcal{I}_K \setminus \{0\} : N(J) \leq y\} = Z_K y + O\left(y^{\frac{d_K-1}{d_K+1}} (\log(y+1))^{d_K-1} \right).$$

Putting this into our expression for $N_{K,h}(x)$, we find that

$$N_{K,h}(x) = \sum_{\substack{N(I)\le x \\ I\in D_{K,h}}} \# \left\{ J \in \mathcal{I}_K \setminus \{0\} : N(J) \le \left(\frac{x}{N(I)}\right)^{\frac{1}{h}} \right\}$$

$$= Z_K x^{\frac{1}{h}} \sum_{\substack{N(I)\le x \\ I\in D_{K,h}}} N(I)^{-\frac{1}{h}} + \sum_{\substack{N(I)\le x \\ I\in D_{K,h}}} O\left(\left(\frac{x}{N(I)}\right)^{\frac{d_K-1}{h(d_K+1)}} \left(\log\left(\left(\frac{x}{N(I)}\right)^{\frac{1}{h}} + 1 \right)\right)^{d_K-1} \right)$$

$$= Z_K x^{\frac{1}{h}} \sum_{\substack{N(I)\le x \\ I\in D_{K,h}}} N(I)^{-\frac{1}{h}} + O\left(x^{\frac{d_K-1}{h(d_K+1)}} \left(\log(x+1)\right)^{d_K-1} \sum_{\substack{N(I)\le x \\ I\in D_{K,h}}} N(I)^{-\frac{d_K-1}{h(d_K+1)}} \right).$$

This is not yet in a form that is useful to us, as the main term still has a sum that depends on x of which we do not know the behavior. We could prove that

$$\sum_{I\in D_{K,h}} N(I)^{-\frac{1}{h}}$$

converges, which would eventually give us a nice main term of order $x^{\frac{1}{h}}$. However, the error term is still tricky to deal with, due to the sum that is included. The infinite sum

$$\sum_{I\in D_{K,h}} N(I)^{-\frac{d_K-1}{h(d_K+1)}}$$

does not necessarily converge.

We can find a bound for the sum

$$\sum_{\substack{N(I)\le x \\ I\in D_{K,h}}} N(I)^{-\frac{d_K-1}{h(d_K+1)}}$$

in the error term, but this requires some effort. In order to find such a bound, we will use the methods from [BG58]. They rephrase the problem of finding $N_{K,h}(x)$ in a clearer and more generalizable way, namely as the sum of coefficients of a Dirichlet series, which we will introduce in the next subsection.

4.2 $N_{K,h}(x)$ and Dirichlet series

We must first define what a Dirichlet series in a number field is.

Definition 4.3 Let K be a number field with ring of integers $\mathcal{O}_K$. Then a **Dirichlet series** on $\mathcal{O}_K$ is a sum of the form

$$\sum_{I\neq 0} a(I)N(I)^{-s},$$

where $s \in \mathbb{R}$ and for each ideal $I \in \mathcal{I}_K \setminus \{0\}$, $a(I)$ is a given complex number.

Remark 4.4 If we consider $\mathcal{O}_K = \mathbb{Z}$ and take the positive representative for each ideal, then we get

$$\sum_{I\neq 0} a(I)N(I)^{-s} = \sum_{n=1}^{\infty} a_n n^{-s},$$

so this reduces to the standard definition on $\mathbb{Z}$.

Lemma 4.5 *Let K be a number field with ring of integers $\mathcal{O}_K$. Let $a(I) : \mathcal{I}_K \setminus \{0\} \to \mathbb{C}$ be a function. If $a(I)$ is multiplicative on comaximal ideals, in other words, $a(I)a(J) = a(IJ)$ if $I + J = \mathcal{O}_K$, then on the abscissa of absolute convergence of $\sum_{I\neq 0} a(I)N(I)^{-s}$, we have*

$$\sum_{I\neq 0} a(I)N(I)^{-s} = \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} a(\mathfrak{p}^j)N(\mathfrak{p})^{-js}.$$

Proof. Due to the norm being multiplicative by Lemma 3.12 as well as $a(I)$ being multiplicative on comaximal ideals, it is clear that for every tuple $(e_{\mathfrak{p}})_{\mathfrak{p}}$ where only finitely many elements are nonzero, we have

$$\prod_{\mathfrak{p}} a(\mathfrak{p}^{e_{\mathfrak{p}}})N(\mathfrak{p})^{-e_{\mathfrak{p}}s} = a(I)N(I)^{-s},$$

where

$$I = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}.$$

Due to unique prime factorization of nonzero ideals, comaximality of distinct nonzero prime powers, and the ability to rearrange terms due to absolute convergence, we find that the claim indeed holds. □

Remark 4.6 Since there can be multiple ideals with the same norm in a number ring, we can have

$$\sum_{I\neq 0} a(I)N(I)^{-s} = \sum_{I\neq 0} b(I)N(I)^{-s}$$

even if we do not have that $a(I) = b(I)$ for all I. This is a problem especially when multiplying two Dirichlet series. Hence, we specifically define multiplication in the following way.

Definition 4.7 Let K be a number field with ring of integers $\mathcal{O}_K$. Let $\sum_{I\neq 0} a(I)N(I)^{-s}$ and $\sum_{I\neq 0} b(I)N(I)^{-s}$ be two Dirichlet series. Then define their product

$$\sum_{I\neq 0} c(I)N(I)^{-s} := \sum_{I\neq 0} a(I)N(I)^{-s} \cdot \sum_{I\neq 0} b(I)N(I)^{-s}$$

specifically with

$$c(I) = \sum_{J_1 J_2 = I} a(J_1)b(J_2).$$

Definition 4.8 Let K be a number field with ring of integers $\mathcal{O}_K$. For all prime ideals $\mathfrak{p}$ and all $j \in \mathbb{Z}_{\geq 0}$, let $a(\mathfrak{p}^j)$ be some complex number. Let

$$\sum_{I\neq 0} b(I)N(I)^{-s} := \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} a(\mathfrak{p}^j)N(\mathfrak{p})^{-js}$$

be a Dirichlet series, where $a(\mathcal{O}_K) = 1$. Then for a given $I = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}$, we define $b(I)$ specifically as

$$b(I) = \prod_{\mathfrak{p}} a(\mathfrak{p}^{e_{\mathfrak{p}}}).$$

Remark 4.9 Note that in the above definition, the numbers $b(I)$ are well-defined, since only finitely many $e_{\mathfrak{p}}$ are nonzero, and for all $\mathfrak{p}$ where $e_{\mathfrak{p}} = 0$, we have $a(\mathfrak{p}^{e_{\mathfrak{p}}}) = a(\mathcal{O}_K) = 1$. Therefore, the numbers $b(I)$ are actually finite products.

Also note that $b(\mathfrak{p}^j) = a(\mathfrak{p}^j)$ for all $\mathfrak{p}$ and j, so we may as well write

$$\sum_{I\neq 0} a(I)N(I)^{-s} := \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} a(\mathfrak{p}^j)N(\mathfrak{p})^{-js},$$

and extend the $a(\mathfrak{p}^j)$ to $a(I)$ defined for all ideals $I \neq 0$.

Definition 4.10 Since we only really care about the coefficients of Dirichlet series and not about the convergence, we will define all further equalities of Dirichlet series to be equalities of coefficients.

Let $c(I)$ be equal to 1 if I is an h-ful ideal (so if $I \in S_{K,h}$), and 0 otherwise. Let $b_h(I)$ be equal to 1 if $I \in D_{K,h}$ and 0 otherwise. Then we can consider the Dirichlet series

$$\sum_{I \neq 0} c(I) N(I)^{-s},$$

and by the reasoning in the previous subsection, this is equal to

$$\sum_{I \neq 0} N(I)^{-hs} \sum_{I \neq 0} b_h(I) N(I)^{-s}.$$

The numbers $b_h(I)$ are multiplicative on comaximal ideals, hence, Lemma 4.5 applies. So we can write

$$\sum_{I \neq 0} c(I) N(I)^{-s} = \zeta_K(hs) \prod_{\mathfrak{p}} \left(1 + \sum_{j=h+1}^{2h-1} N(\mathfrak{p})^{-js} \right).$$

Note that the right side consists of two Dirichlet series, with the second written as a product over all primes. But if we multiply this out, we see that it is actually equal to a Dirichlet series (at least on the abscissa of absolute convergence). As it turns out, if we can show that both of these Dirichlet series have certain properties, we can estimate $N_{K,h}(s) = \sum_{N(I) \leq x} c(I)$. This can be seen in the following lemma, which is adapted from Lemma 1 in [BG58]:

Proposition 4.11 (Generalization of Bateman and Grosswald) *Suppose that*

$$\sum_{I \neq 0} a(I) N(I)^{-s} \cdot \sum_{I \neq 0} b(I) N(I)^{-s} = \sum_{I \neq 0} c(I) N(I)^{-s},$$

with the $c(I)$ defined as in Definition 4.7. Let $A(x) = \sum_{N(I) \leq x} a(I)$ and similarly for $B(x)$ and $C(x)$.

Suppose

$$A(x) = \alpha_0 x^{\lambda_0} + \cdots + \alpha_r x^{\lambda_r} + O(x^{\lambda} (\log(x+1))^{\mu})$$

and

$$\sum_{N(I) \leq x} |b(I)| = O(x^{\nu}),$$

with $\lambda, \mu, \nu \in \mathbb{R}_{\geq 0}$ and with all other constants arbitrary complex numbers.

Then

$$C(x) = \gamma_0 x^{\lambda_0} + \cdots + \gamma_r x^{\lambda_r} + O(x^{\max(\lambda, \nu)} (\log(x+1))^{\mu'}),$$

where

$$\gamma_j = \begin{cases} 0 & \text{if } \mathrm{Re}\lambda_j \leq \nu, \\ \alpha_j \sum_{I \neq 0} b(I) N(I)^{-\lambda_j} & \text{if } \mathrm{Re}\lambda_j > \nu, \end{cases}$$

$$\mu' = \begin{cases} \mu & \text{if } \lambda > \nu, \\ \mu + 1 & \text{if } \lambda = \nu, \\ 0 & \text{if } \lambda < \nu \text{ and for all } j, \mathrm{Re}\lambda_j \neq \nu, \\ 1 & \text{if } \lambda < \nu \text{ and there is some } j \text{ with } \mathrm{Re}\lambda_j = \nu. \end{cases}$$

Before we can complete this proof, though, we will need to generalize partial summation to number fields, and use it to get some basic results.

Proposition 4.12 *Let K be a number field with ring of integers $\mathcal{O}_K$. Let $\mathcal{I}_K$ be the set of ideals of $\mathcal{O}_K$, let $\mathcal{J} \subset \mathcal{I}_K$ be a subset of the set of all ideals, and let $f : \mathcal{J} \to \mathbb{Z}_{>0}$ be a function such that $f^{-1}(\{k\})$ is finite for all $k \in \mathbb{Z}_{>0}$. Let $g : [1, x] \to \mathbb{C}$ be a continuously differentiable function. For each ideal $I \in \mathcal{J}$, let $a(I)$ be a complex number, and for $x \in \mathbb{R}$, define*

$$A(x) = \sum_{f(I) \leq x} a(I).$$

Then for $x \geq 1$, we have

$$\sum_{f(I) \leq x} a(I) g(f(I)) = A(x) g(x) - \int_1^x A(t) g'(t) \mathrm{d}t.$$

Proof. This proof is based on the one in [Eve20, §2.1]. Set $y = \lfloor x \rfloor$. Then

$$\sum_{f(I) \leq x} a(I) g(f(I)) = \sum_{k=1}^{y} \sum_{f(I)=k} a(I) g(k) = \sum_{k=1}^{y} g(k) \sum_{f(I)=k} a(I)$$

$$= \sum_{k=1}^{y} g(k)(A(k) - A(k-1)) = \sum_{k=1}^{y} g(k) A(k) - \sum_{k=1}^{y} g(k) A(k-1).$$

Note that $A(0) = 0$, since the minimum value f can attain is 1. Thus, we can rewrite as follows:

$$\sum_{k=1}^{y} g(k) A(k) - \sum_{k=1}^{y} g(k) A(k-1) = \sum_{k=1}^{y} g(k) A(k) - \sum_{k=1}^{y-1} g(k+1) A(k)$$

$$= g(y) A(y) - \sum_{k=1}^{y-1} A(k)(g(k+1) - g(k)).$$

Since $A(t)$ is constant on the interval $[k, k+1)$ with k integer, we know that

$$\int_k^{k+1} A(t) g'(t) \mathrm{d}t = A(k) \int_k^{k+1} g'(t) \mathrm{d}t = A(k)(g(k+1) - g(k)).$$

Hence,

$$\sum_{k=1}^{y-1} A(k)(g(k+1) - g(k)) = \sum_{k=1}^{y-1} \int_k^{k+1} A(t) g'(t) \mathrm{d}t = \int_1^y A(t) g'(t) \mathrm{d}t.$$

Finally, note that $A(y) = A(x)$ and that

$$\int_y^x A(t) g'(t) \mathrm{d}t = A(y) \int_y^x g'(t) \mathrm{d}t = A(x) g(x) - A(y) g(y).$$

It follows that

$$\sum_{f(I) \leq x} a(I) g(f(I)) = g(y) A(y) - \sum_{k=1}^{y-1} A(k)(g(k+1) - g(k))$$

$$= g(y) A(y) - \int_1^y A(t) g'(t) \mathrm{d}t = g(x) A(x) - \int_1^x A(t) g'(t) \mathrm{d}t.$$

This proves the statement. □

A direct corollary of this proposition is the following result, where we take $f(I) = N(I)$ and $\mathcal{J} = \mathcal{I}_K \setminus \{0\}$.

Corollary 4.13 *Let K be a number field with ring of integers $\mathcal{O}_K$. Let $g : [1, x] \to \mathbb{C}$ be a continuously differentiable function. For each ideal $I \in \mathcal{I}_K \setminus \{0\}$, let $a(I)$ be a complex number, and define*

$$A(x) = \sum_{N(I) \leq x} a(I).$$

Then we have

$$\sum_{N(I) \leq x} a(I) g(N(I)) = A(x) g(x) - \int_1^x A(t) g'(t) \mathrm{d}t.$$

We will need to apply partial summation in a few cases which we will then apply in the proof of Proposition 4.11.

Lemma 4.14 *Let K be a field with ring of integers $\mathcal{O}_K$. For each ideal $I \in \mathcal{I}_K \setminus \{0\}$, let $b(I)$ be a complex number, such that for all $x \in \mathbb{R}_{>0}$, we have*

$$\sum_{N(I) \leq x} |b(I)| = O\left(x^{\nu}\right).$$

Let $\eta \in \mathbb{R}$. Then we get the following bounds for $\sum_{N(I) \leq x} |b(I)| N(I)^{-\eta}$.

If $\nu = \eta$, we get

$$\sum_{N(I) \leq x} |b(I)| N(I)^{-\eta} = O(\log(x+1)).$$

If $\nu > \eta$, we get

$$\sum_{N(I) \leq x} |b(I)| N(I)^{-\eta} = O\left(x^{\nu - \eta}\right).$$

And finally, if $\nu < \eta$, we get the following two results:

$$\sum_{N(I) \leq x} |b(I)| N(I)^{-\eta} = O(1),$$

$$\sum_{N(I) > x} |b(I)| N(I)^{-\eta} = O\left(x^{\nu - \eta}\right).$$

Proof. We use Corollary 4.13. This gives

$$\sum_{N(I) \leq x} |b(I)| N(I)^{-\eta} = \sum_{N(I) \leq x} |b(I)| \cdot x^{-\eta} + \eta \int_1^x \sum_{N(I) \leq t} |b(I)| \cdot t^{-\eta - 1} \mathrm{d}t = O\left(x^{\nu - \eta}\right) + \int_1^x O\left(t^{\nu - \eta - 1}\right) \mathrm{d}t.$$

We now have two cases: $\nu = \eta$ and $\nu \neq \eta$. In the first case,

$$O\left(x^{\nu - \eta}\right) + \int_1^x O\left(t^{\nu - \eta - 1}\right) \mathrm{d}t = O(1) + O(\log(x)) = O(\log(x)) = O(\log(x+1)),$$

for $x \geq 1$ (which is what we are interested in). In the second case,

$$O\left(x^{\nu - \eta}\right) + \int_1^x O\left(t^{\nu - \eta - 1}\right) \mathrm{d}t = O\left(x^{\nu - \eta}\right) + O(1),$$

so if $\eta > \nu$, we are only left with $O(1)$ for $x \geq 1$, and if $\eta < \nu$, we are only left with $O(x^{\nu - \eta})$.

Additionally, if $\eta > \nu$, we can use Corollary 4.13 on $\sum_{n > x} |b(I)| n^{-\eta}$ as well, which yields

$$\sum_{N(I) > x} |b(I)| n^{-\eta} = \lim_{R \to \infty} \sum_{x < N(I) \leq R} |b(I)| n^{-\eta}$$

$$= \lim_{R\to\infty}\left(\sum_{N(I)\le R}|b(I)|\cdot R^{-\eta} - \sum_{N(I)\le x}|b(I)|\cdot x^{-\eta} - \eta\int_x^R \sum_{N(I)\le t}|b(I)|t^{-\eta-1}\mathrm{d}t\right)$$

$$= O\left(x^{\nu-\eta}\right) + \lim_{R\to\infty}\left(O\left(R^{\nu-\eta}\right) + \eta\int_x^R O\left(t^{\nu-\eta-1}\right)\mathrm{d}t\right) = O\left(x^{\nu-\eta}\right).$$

This proves the desired statement. □

Now we are ready for the proof of Proposition 4.11

Proof. We know that

$$\sum_{I\neq 0} c(I)N(I)^{-s} = \sum_{I\neq 0} a(I)N(I)^{-s}\cdot\sum_{I\neq 0} b(I)N(I)^{-s},$$

but also

$$\sum_{I\neq 0} c(I)N(I)^{-s} = \sum_{k=1}^{\infty}\sum_{N(I)=k} c(I)k^{-s} = \sum_{k=1}^{\infty} k^{-s}\sum_{N(I)=k} c(I).$$

If we use Lemma 3.12, we find that

$$\sum_{N(I)=k} c(I) = \sum_{d|k}\sum_{N(I)=d}\sum_{N(J)=k/d} a(J)b(I).$$

From there, it follows that

$$C(x) = \sum_{N(I)\le x} c(I) = \sum_{k\le x}\sum_{N(I)=k} c(I)$$

$$= \sum_{k\le x}\sum_{d|k}\sum_{N(I)=d}\sum_{N(J)=k/d} a(J)b(I) = \sum_{d\le x}\sum_{m\le x/d}\sum_{N(I)=d}\sum_{N(J)=m} a(J)b(I)$$

$$= \sum_{d\le x}\sum_{N(I)=d} b(I)\sum_{m\le x/d}\sum_{N(J)=m} a(J) = \sum_{N(I)\le x} b(I)A\left(\frac{x}{N(I)}\right).$$

We find

$$\sum_{N(I)\le x} b(I)A\left(\frac{x}{N(I)}\right) = \sum_{j=0}^{r}\alpha_j x^{\lambda_j}\sum_{N(I)\le x} b(I)N(I)^{-\lambda_j} + \sum_{N(I)\le x} b(I)O\left(x^{\lambda}\left(\log\left(\frac{x}{N(I)}+1\right)\right)^{\mu}N(I)^{-\lambda}\right)$$

$$= \sum_{j=0}^{r}\alpha_j x^{\lambda_j}\sum_{N(I)\le x} b(I)N(I)^{-\lambda_j} + O\left(x^{\lambda}\sum_{N(I)\le x}|b(I)|\left(\log\left(\frac{x}{N(I)}+1\right)\right)^{\mu}N(I)^{-\lambda}\right)$$

$$= \sum_{j=0}^{r}\alpha_j x^{\lambda_j}\sum_{N(I)\le x} b(I)N(I)^{-\lambda_j} + O\left(x^{\lambda}\sum_{N(I)\le x}|b(I)|(\log(x+1))^{\mu}N(I)^{-\lambda}\right)$$

$$= \sum_{j=0}^{r}\alpha_j x^{\lambda_j}\sum_{N(I)\le x} b(I)N(I)^{-\lambda_j} + O\left(x^{\lambda}(\log(x+1))^{\mu}\sum_{N(I)\le x}|b(I)|N(I)^{-\lambda}\right).$$

The rest of the proof follows using Lemma 4.14 and separating into cases. If we have some λ_j with $\mathrm{Re}\lambda_j > \nu$, then use

$$\sum_{N(I)\le x} b(I)N(I)^{-\lambda_j} = \sum_{I\in\mathcal{I}_K\setminus\{0\}} b(I)N(I)^{-\lambda_j} - \sum_{N(I)>x} b(I)N(I)^{-\lambda_j}.$$

If $\lambda < \nu$, then we have to slightly alter the error term in the asymptotic for $A(x)$. Namely, since for all $\varepsilon > 0$, we have $\log(x+1) = O(x^{\varepsilon})$, we can set $O(x^{\lambda}\log^{\mu}(x+1)) = O(x^{\lambda'})$ for some λ' with $\lambda < \lambda' < \nu$. Using this modified error term yields the result we are after. □

With this lemma, we can now start working on the question of estimating $N_{K,h}(x)$. We have already written

$$\sum_{I\neq 0} c(I)N(I)^{-s} = \zeta_K(hs)\prod_{\mathfrak{p}}\left(1+\sum_{j=h+1}^{2h-1} N(\mathfrak{p})^{-js}\right),$$

where each of the two factors on the right are also Dirichlet series. Now we need to write this in the form

$$\sum_{I\neq 0} c(I)N(I)^{-s} = \sum_{I\neq 0} a(I)N(I)^{-s}\sum_{I\neq 0} b(I)N(I)^{-s},$$

to be able to estimate the sums of the $a(I)$ and $b(I)$. We will do this in the next section.

5 Estimating sums of coefficients

For the entirety of this section, for $I \in \mathcal{I}_K \setminus \{0\}$, set $a_h(I)$ and $b_h(I)$ according to Definition 4.8 such that

$$\sum_{I \neq 0} a_h(I) N(I)^{-s} = \zeta_K(hs) = \prod_{\mathfrak{p}} \left(\sum_{j=0}^{\infty} N(\mathfrak{p})^{-jhs} \right)$$

and

$$\sum_{I \neq 0} b_h(I) N(I)^{-s} = \prod_{\mathfrak{p}} \left(1 + \sum_{j=h+1}^{2h-1} N(\mathfrak{p})^{-js} \right)$$

5.1 Dedekind Zeta Functions

We will start with finding the sum of the $a_h(I)$. We know that

$$\zeta_K(hs) = \sum_{J \neq 0} N(J)^{-hs} = \sum_{J \neq 0} \left(N(J)^h \right)^{-s},$$

so

$$\zeta_K(hs) = \sum_{I \neq 0} a_h(I) N(I)^{-s}$$

where $a_h(I) = 1$ if there is some ideal J such that $J^h = I$ and $a_h(I) = 0$ otherwise. It follows that

$$\sum_{N(I) \leq x} a_h(I) = \# \left\{ J \in \mathcal{I}_K \setminus \{0\} : N(J)^h \leq x \right\} = \# \left\{ J \in \mathcal{I}_K \setminus \{0\} : N(J) \leq x^{\frac{1}{h}} \right\}.$$

At this point, we can use Corollary 4.2 again. Hence, we get the following lemma:

Lemma 5.1 *Let K be a number field with ring of integers $\mathcal{O}_K$. Define $a_h(I)$ such that*

$$\zeta_K(hs) = \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} N(I)^{-jhs} = \sum_{I \neq 0} a_h(I) N(I)^{-s},$$

as in Definition 4.8. Then

$$\sum_{N(I) \leq x} a_h(I) = Z_K x^{\frac{1}{h}} + O\left(x^{\frac{d_K - 1}{h(d_K+1)}} (\log(x+1))^{d_K - 1} \right).$$

Proof. As we saw above, we have $a_h(I) = 1$ if there is some ideal J such that $I = J^h$ and $a_h(I) = 0$ otherwise, and hence

$$\sum_{N(I) \leq x} a_h(I) = \#\{J \in \mathcal{I}_K \setminus \{0\} : N(J) \leq x^{\frac{1}{h}}\}.$$

So by Corollary 4.2, we see that

$$\sum_{N(I) \leq x} a_h(I) = Z_K x^{\frac{1}{h}} + O\left(x^{\frac{d_K-1}{h(d_K+1)}} \left(\log\left(x^{\frac{1}{h}} + 1 \right) \right)^{d_K - 1} \right) = Z_K x^{\frac{1}{h}} + O\left(x^{\frac{d_K-1}{h(d_K+1)}} (\log(x+1))^{d_K-1} \right).$$

□

For the $b_h(I)$, we see that when we write $\prod_{\mathfrak{p}} \left(1 + \sum_{j=h+1}^{2h-1} N(\mathfrak{p})^{-js} \right) = \sum_{I \neq 0} b_h(I) N(I)^{-s}$ as in Definition 4.8, then the $b_h(I)$ are bounded by the coefficients of $\zeta_K((h+1)s) \ldots \zeta_K((2h-1)s)$. We will prove this statement in the next two lemmas.

Lemma 5.2 *Let K be a number field with ring of integers $\mathcal{O}_K$. Suppose that for each prime ideal power $\mathfrak{p}^j$, we have two complex numbers $a(\mathfrak{p}^j), b(\mathfrak{p}^j)$ with $a(\mathfrak{p}^j) \geq b(\mathfrak{p}^j) \geq 0$ for all $j = 0, 1, \ldots$. Define*

$$\sum_{I \neq 0} a(I) N(I)^{-s} := \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} a(\mathfrak{p}^j) N(\mathfrak{p})^{-js},$$

$$\sum_{I \neq 0} b(I) N(I)^{-s} := \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} b(\mathfrak{p}^j) N(\mathfrak{p})^{-js},$$

both according to Definition 4.8. Then we have

$$a(I) \geq b(I), \text{ for all } I.$$

Proof. We know that by our definition concerning Dirichlet series written as products over primes, we have if $I = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}$, then

$$a(I) = \prod_{\mathfrak{p}} a(\mathfrak{p}^{e_{\mathfrak{p}}}), \ b(I) = \prod_{\mathfrak{p}} b(\mathfrak{p}^{e_{\mathfrak{p}}}).$$

Since $a(\mathfrak{p}^j) \geq b(\mathfrak{p}^j) \geq 0$ for all j, we have

$$a(I) = \prod_{\mathfrak{p}} a(\mathfrak{p}^{e_{\mathfrak{p}}}) \geq \prod_{\mathfrak{p}} b(\mathfrak{p}^{e_{\mathfrak{p}}}) = b(I),$$

for all I, which proves the statement. ☐

Lemma 5.3 *Let K be a number field with ring of integers $\mathcal{O}_K$. For all $I \in \mathcal{I}_K \setminus \{0\}$, set $b_h(I)$ and $d_h(I)$ as in Definition 4.8 and Definition 4.7 such that*

$$\sum_{I \neq 0} b_h(I) N(I)^{-s} := \prod_{\mathfrak{p}} \left(1 + \sum_{k=h+1}^{2h-1} N(\mathfrak{p})^{-ks}\right)$$

and

$$\sum_{I \neq 0} d_h(I) N(I)^{-s} = \zeta_K((h+1)s) \ldots \zeta_K((2h-1)s).$$

Then

$$\sum_{N(I)=k} b_h(I) \leq \sum_{N(I)=k} d_h(I).$$

Proof. We know that

$$\zeta_K((h+1)s) \ldots \zeta_K((2h-1)s) = \prod_{k=h+1}^{2h-1} \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} N(\mathfrak{p})^{-jks} = \prod_{\mathfrak{p}} \prod_{k=h+1}^{2h-1} \sum_{j=0}^{\infty} N(\mathfrak{p})^{-jks}$$

$$= \prod_{\mathfrak{p}} \left(1 + \sum_{k=h+1}^{2h-1} N(\mathfrak{p})^{-ks} + \sum_{k=2h+2}^{\infty} f(I) N(\mathfrak{p})^{-ks}\right),$$

where the $f(I)$ are all nonnegative integers, by expanding the product of sums. Clearly,

$$1 + \sum_{k=h+1}^{2h-1} N(\mathfrak{p})^{-ks} + \sum_{k=2h+2}^{\infty} f(I) N(\mathfrak{p})^{-ks} \geq 1 + \sum_{k=h+1}^{2h-1} N(\mathfrak{p})^{-ks} \geq 0,$$

so we can apply Lemma 5.2, which gives us exactly the result we want. ☐

Now that we can bound the coefficients of our Dirichlet series by the coefficients of a product of Dedekind zeta functions, we can use the following result:

Lemma 5.4 *Let K be a number field with ring of integers $\mathcal{O}_K$. Let $s \in \mathbb{R}$, $a \in \mathbb{Z}_{>0}$, and $k \in \mathbb{Z}_{\geq 0}$. For all $I \in \mathcal{I}_K \setminus \{0\}$, set $f(I, a, k)$ as in Definition 4.7 and Definition 4.8 such that*

$$\sum_{I \neq 0} f(I, a, k) N(I)^s := \zeta_K(as)\zeta_K((a+1)s) \dots \zeta_K((a+k)s).$$

Then

$$\sum_{N(I) \leq x} |f(I, a, k)| = O\left(x^{\frac{1}{a}}\right).$$

Proof. Note that

$$\sum_{N(I) \leq x} |f(I, a, k)| = \sum_{N(I) \leq x} f(I, a, k) = \sum_{N(I_0)^a \dots N(I_k)^{a+k} \leq x} 1,$$

since all coefficients are positive.

If $k = 0$, then we find

$$\sum_{N(I) \leq x} |f(I, a, k)| = \sum_{N(I)^a \leq x} 1 = \sum_{N(I) \leq x^{\frac{1}{a}}} 1.$$

From Corollary 4.2, we get that $\sum_{N(I) \leq y} 1 = O(y)$, hence

$$\sum_{N(I) \leq x^{\frac{1}{a}}} 1 = O\left(x^{\frac{1}{a}}\right)$$

as desired.

Suppose that the claim holds for some $k = l - 1$. Then

$$\sum_{N(I) \leq x} |f(n, a, l)| = \sum_{N(I_0)^a \dots N(I_l)^{a+l} \leq x} 1 = \sum_{N(I_l) \leq x^{\frac{1}{a+l}}} \sum_{N(I_0)^a \dots N(I_{l-1})^{a+l-1} \leq x / N(I_l)^{a+l}} 1.$$

By the fact that the claim holds for $k = l - 1$, we find that there exists a constant $C(a, l)$ such that

$$\sum_{N(I_0)^a \dots N(I_{l-1})^{a+l-1} \leq x / N(I_l)^{a+l}} 1 \leq C(a, l) \left(\frac{x}{N(I_l)^{a+l}}\right)^{\frac{1}{a}}.$$

Therefore, we get

$$\sum_{N(I) \leq x} |f(n, a, l)| \leq C(a, l) \sum_{N(I_l) \leq x^{\frac{1}{a+l}}} \left(\frac{x}{N(I_l)^{a+l}}\right)^{\frac{1}{a}}$$

$$= C(a, l) x^{\frac{1}{a}} \sum_{N(I_l) \leq x^{\frac{1}{a+l}}} N(I_l)^{-\frac{a+l}{a}} \leq C(a, l) \zeta\left(\frac{a+l}{a}\right) x^{\frac{1}{a}},$$

where we use that $\frac{a+l}{a} > 1$, which implies that

$$\sum_{I_l \neq 0} N(I_l)^{-\frac{a+l}{a}} = \zeta_K\left(\frac{a+l}{a}\right)$$

converges.

We find that indeed,

$$\sum_{N(I) \leq x} |f(I, a, k)| = O\left(x^{\frac{1}{a}}\right).$$

□

Combining these results, we find the following:

Corollary 5.5 *For all $I \in \mathcal{I}_K \setminus \{0\}$, set $b_h(I)$ as in Definition 4.8 such that*

$$\sum_{I \neq 0} b_h(I) N(I)^{-s} = \prod_{\mathfrak{p}} \left(1 + \sum_{j=h+1}^{2h-1} N(\mathfrak{p})^{-js} \right).$$

Then

$$\sum_{N(I) \leq x} |b(I)| = O\left(x^{\frac{1}{h+1}} \right).$$

Proof. This follows directly from Lemma 5.3 and Lemma 5.4. □

5.2 *h*-ful ideals of bounded norm

Now that we have verified that the necessary conditions are satisfied, we can use Proposition 4.11. This will yield the main theorem:

Theorem 5.6 (Main Theorem) *Let K be a number field of degree d_K with ring of integers $\mathcal{O}_K$. Let Z_K be the residue of $\zeta_K(s)$ at $s = 1$. Then we have*

$$N_{K,h}(x) = Z_K \gamma x^{\frac{1}{h}} + O\left(x^{\alpha} (\log(x+1))^{\beta} \right),$$

where

$$\gamma = \prod_{\mathfrak{p}} \left(1 + \sum_{m=h+1}^{2h-1} N(\mathfrak{p})^{-\frac{m}{h}} \right),$$

where

$$\alpha = \begin{cases} \frac{1}{h+1} & \text{if } 2h+1 \geq d_K \\ \frac{d_K - 1}{h(d_K+1)} & \text{if } 2h+1 < d_K \end{cases},$$

and where

$$\beta = \begin{cases} 0 & \text{if } 2h+1 > d_K \\ d_K & \text{if } 2h+1 = d_K \\ d_K - 1 & \text{if } 2h+1 < d_K \end{cases}.$$

Proof. Let $c(I)$ be 1 if I is h-ful and 0 otherwise. Then

$$N_{K,h}(x) = \sum_{N(I) \leq x} c(I).$$

We know that

$$\sum_{I \neq 0} c(I) N(I)^{-s} = \zeta_K(hs) \prod_{p} \left(1 + \sum_{j=h+1}^{2h-1} p^{-js} \right).$$

Using Definition 4.8, set

$$\sum_{I \neq 0} a_h(I) N(I)^{-s} := \zeta_K(hs)$$

and

$$\sum_{I \neq 0} b_h(I) N(I)^{-s} := \prod_{p} \left(1 + \sum_{j=h+1}^{2h-1} p^{-js} \right)$$

Define $A(x) = \sum_{N(I)\le x} a_h(I)$, $B(x) = \sum_{N(I)\le x} b_h(I)$ and $C(x) = \sum_{N(I)\le x} c(I)$.

Lemma 5.1 gives us

$$A(x) = Z_K x^{\frac{1}{h}} + O\left(x^{\frac{d_K-1}{h(d_K+1)}}(\log(x+1))^{d_K-1}\right).$$

Since all $b_h(I)$ are positive, Corollary 5.5 gives us that

$$\sum_{N(I)\le x} |b_h(I)| = B(x) = O\left(x^{\frac{1}{h+1}}\right).$$

Now, we can apply Proposition 4.11, with $\alpha_0 = Z_K$, $\lambda_0 = \frac{1}{h}$, $\lambda = \frac{d_K-1}{h(d_K+1)}$, $\mu = d_K - 1$, and $\nu = \frac{1}{h+1}$. We remark that $\lambda > \nu$ if and only if

$$(d_K-1)(h+1) > h(d_K+1) \iff -h + d_K - 1 > h \iff d_K > 2h+1.$$

This completes the proof. □

We can now see that the techniques used in this subsection are similar to those in Subsection 4.1. Namely, we have that

$$\gamma = \prod_{\mathfrak{p}}\left(1 + \sum_{m=h+1}^{2h-1} N(\mathfrak{p})^{-\frac{m}{h}}\right) = \sum_{I\in D_{K,h}} N(I)^{-\frac{1}{h}},$$

which is suspiciously similar to the main term we found there, and we could write

$$\sum_{\substack{N(I)\le x\\ I\in D_{K,h}}} N(I)^{-\frac{1}{h}} = \sum_{I\in D_{K,h}} N(I)^{-\frac{1}{h}} - \sum_{\substack{N(I)>x\\ I\in D_{K,h}}} N(I)^{-\frac{1}{h}}$$

to actually get the same main term, with an additional error term. We can use Corollary 4.13 and Corollary 5.5 to bound the error term.

The main difference between these methods is that the method involving the sum of Dirichlet series coefficients can be generalized more easily. Namely, we can factor

$$\sum_{I\ne 0} c(I)N(I)^{-s} = \zeta_K(hs)\zeta_K((h+1)s)\sum_{I\ne 0} d(I)N(I)^{-s},$$

for some Dirichlet series $\sum_{I\ne 0} d(I)N(I)^{-s}$. While finding a bound on $\sum_{N(I)\le x}|d(I)|$ is very possible, it is much more difficult to obtain an asymptotic formula for the sum of the coefficients of $\zeta_K(hs)\zeta_K((h+1)s)$, which is necessary to apply Proposition 4.11. This is why we did not explore this further in this thesis, but it is useful for further exploration, and involves finding a formula for

$$\#\{I, J \in \mathcal{I}_K \setminus \{0\} : N(I^h J^{h+1}) \le x\}.$$

6 Looking at elements

6.1 A new height function

We are also interested in looking at individual elements rather than ideals. However, we cannot use the usual absolute norm of elements. Namely, most number fields have an infinite number of units and the absolute norm is invariant under multiplication by units, and hence there would be an infinite number of elements of bounded norm. Thus, we will need a new norm-like function.

Definition 6.1 Let K be a number field and let Ω_∞ be the set of infinite places of K. Then given an element $\alpha \in K$, define its **height** as

$$H(\alpha) = \prod_{v \in \Omega_\infty} \max(1, ||\alpha||_v).$$

Remark 6.2 We note that this is not too different from the usual norm, since we have that

$$|N(\alpha)| = \prod_{v \in \Omega_\infty} ||\alpha||_v.$$

We first want to check that the number of elements of bounded height is finite, to make sure that this height function fixes the problem of there being an infinite number of elements of bounded norm in most number fields.

Definition 6.3 Let K be a number field as above. Let Ω_∞ be the set of all infinite places of K. Give them some arbitrary order, such that for each $i \in \{1, \dots, r_1 + r_2\}$, we have some place v_i. Then define

$$\varphi_K : K \to \mathbb{R}^{r_1+r_2}, \alpha \mapsto (||\alpha||_{v_i})_i.$$

Definition 6.4 Let K be a number field and let $x \in \mathbb{R}_{>0}$. Then define

$$B_K(x) := \{\alpha \in \mathcal{O}_K : H(\alpha) \leq x\}.$$

Lemma 6.5 *$B_K(x)$ is finite for all number fields K and all $x \in \mathbb{R}_{>0}$.*

Proof. Consider $\tilde{B}_K(x) := \{\alpha \in \mathcal{O}_K : \forall v \in \Omega_\infty,\ \max(1, |\alpha|_v) \leq x\}$. Since $\max(1, |\alpha|_v) \geq 1$ for all $v \in \Omega_\infty$, we find that $\max(1, |\alpha|_{v_i}) \leq \prod_{v \in \Omega_\infty} \max(1, |\alpha|_v)$ for all $v_i \in \Omega_\infty$. It follows that $B_K(x) \subset \tilde{B}_K(x)$, so if $\tilde{B}_K(x)$ is finite, we are done.

We can see that $\tilde{B}_K(x) = \varphi_K(\mathcal{O}_K) \cap C(x)$, where

$$C(x) := \{(x_1, \dots, x_n) \in \mathbb{R}^{r_1+r_2} : |x_i| \leq x, i = 1, \dots, r_1 + r_2\}$$

and φ_K is defined as in Definition 6.3. Since $\mathcal{O}_K$ is discrete, we must have that $\varphi_K(\mathcal{O}_K)$ is also discrete. We have that $C(x)$ is a bounded region in $\mathbb{R}^{r_1+r_2}$. It follows that their intersection is finite, which completes the proof. □

We can now look at some new quantities.

Definition 6.6 Recall from Definition 3.13 that $S_{K,h}$ is the set of all h-ful ideals. Let

$$\tilde{S}_{K,h} = \{\alpha \in \mathcal{O}_K \setminus \{0\} : (\alpha) \in S_{K,h}\}$$

and

$$\tilde{N}_{K,h}(x) = \#\{\alpha \in \mathcal{O}_K \setminus \{0\} : H(\alpha) \leq x, (\alpha) \in S_{K,h}\}.$$

Similar to earlier, we can wonder how $\tilde{N}_{K,h}(x)$ behaves asymptotically. This new question yields two main differences compared to the original question, where we used h-ful ideals of bounded norm. Firstly, it concerns elements, which are linked to principal ideals specifically, so we can try to consider our original question, but where we restrict ideals to a certain ideal class. Secondly, the height does actually care about units, and we need to figure out how many representatives of ideals have this bounded height.

We will explore various approaches that will deal with at least one of these problems.

6.2 Comparing height to norm

To see if we can use previous results, it makes sense to compare the norm to the height.

Lemma 6.7 *Let K be a number field and let $\alpha \in \mathcal{O}_K$. Then*

$$H(\alpha) \geq |N(\alpha)|.$$

Proof. We have

$$\max(1, ||\alpha||_v) \geq ||\alpha||_v$$

and therefore

$$H(\alpha) = \prod_{v \in \Omega_\infty} \max(1, ||\alpha||_v) \geq \prod_{v \in \Omega_\infty} ||\alpha||_v = |N(\alpha)|.$$

□

We see that we have equality of the height and norm if and only if $||\alpha||_v \geq 1$ for all $v \in \Omega_\infty$. The only number fields where this holds for all nonzero elements of the ring of integers, are those where $\#\Omega_\infty = 1$.

To prove this, we first need an auxiliary lemma.

Lemma 6.8 *Let $\zeta \in \mathbb{C}$ be an algebraic integer of degree n. Let $\zeta^{(i)}$, $i = 1, \ldots, n$ be the roots of the minimal polynomial of ζ. Then ζ is a root of unity if and only if $|\zeta^{(i)}| = 1$ for all i.*

Proof. Note that if ζ is a root of unity, then all $\zeta^{(i)}$ are powers of ζ and thus they all have norm 1. For the converse, see [Hec81, §34, Lemma (a)], and let K be the splitting field of the minimal polynomial of ζ. □

Lemma 6.9 *Let K be a number field. Then we have $H(\alpha) = |N(\alpha)|$ for all $\alpha \in \mathcal{O}_K \setminus \{0\}$ if and only if $r_1 + r_2 = 1$.*

Proof. Suppose that $r_1 + r_2 = 1$. Consider some $\alpha \in \mathcal{O}_K \setminus \{0\}$. Since α is a nonzero algebraic integer, we have $|N(\alpha)| \geq 1$, and since $r_1 + r_2 = 1$, we have $\#\Omega_\infty = 1$. Thus,

$$1 \leq |N(\alpha)| = ||\alpha||_v = \max(1, ||\alpha||_v) = H(\alpha),$$

where v is the one element of Ω_∞.

Suppose that $r_1 + r_1 > 1$. Then we have some fundamental unit $\eta \in \mathcal{O}_K^\times$ that is not a root of unity. Hence, by Lemma 6.8, there is some $v \in \Omega_\infty$ such that $||\eta||_v \neq 1$. Without loss of generality, let $||\eta||_v = y > 1$. Consider some $\alpha \in \mathcal{O}_K \setminus \{0\}$. Then there exists some $m \in \mathbb{Z}_{>0}$ such that $y^m > ||\alpha||_v$, and therefore $||\alpha\eta^{-m}||_v = ||\alpha||_v y^{-m} < 1$. Hence, $H(\alpha\eta^{-m}) > |N(\alpha\eta^{-m})|$, so there exists some $\beta := \alpha\eta^{-m}$ such that $H(\beta) > N(\beta)$. □

So in number fields where $r_1 + r_2 = 1$, the height and the norm are equal. For all other number fields, we can still see that the height is unaffected by multiplication by a root of unity.

Proposition 6.10 *Consider some $\zeta \in \mathcal{O}_K$. Then we have that ζ is a root of unity if and only if for all $\alpha \in \mathcal{O}_K$, we have $H(\alpha) = H(\zeta\alpha)$.*

Proof. Let d_K denote the degree of K over $\mathbb{Q}$. If we embed ζ in $\mathbb{C}$, then it must be an algebraic integer.

If $H(\alpha) = H(\zeta\alpha)$ for all $\alpha \in \mathcal{O}_K$, then also $H(\zeta) = H(1) = 1$. But since each factor in the product in the definition of H is at least 1, we must have that every factor is equal to 1. Hence, $||\zeta||_v \leq 1$ for all $v \in \Omega_\infty$. But since ζ is an algebraic integer, we know that $\prod_{v\in\Omega_\infty} ||\zeta||_v \geq 1$ and thus $||\zeta||_v = 1$ for all $v \in \Omega_\infty$.

Conversely, if $||\zeta||_v = 1$ for all $v \in \Omega_\infty$, then we have that $H(\alpha) = H(\zeta\alpha)$ for all $\alpha \in \mathcal{O}_K$. So $H(\alpha) = H(\zeta\alpha)$ for all $\alpha \in \mathcal{O}_K$ if and only if $||\zeta||_v = 1$ for all $v \in \Omega_\infty$.

By applying Lemma 6.8, we find that $||\zeta||_v = 1$ for all $v \in \Omega_\infty$ if and only if ζ is a root of unity. Combining these two equivalences, we retrieve the desired statement. □

Looking at the number fields with $r_1 + r_2 = 1$ again, consider those with trivial class group. In other words, those where all ideals of $\mathcal{O}_K$ are principal.

If we consider two elements $\alpha, \beta \in \mathcal{O}_K$, then we have $(\alpha) = (\beta)$ if and only if there exists some unit η such that $\alpha = \eta\beta$. But in these number fields, $\mathcal{O}_K$ has only finitely many units, namely only roots of unity.

So we can easily translate our previous result concerning ideals into a result concerning elements and this new height function.

Corollary 6.11 *Let K be either $\mathbb{Q}$ or an imaginary quadratic field with trivial class group. Let w_K be the number of roots of unity of $\mathcal{O}_K$ and let Z_K be the residue of $\zeta_K(s)$ at $s = 1$. Then we have*

$$\tilde{N}_{K,h}(x) = w_K Z_K \gamma x^{\frac{1}{h}} + O\left(x^{\frac{1}{h+1}}\right),$$

where

$$\gamma = \prod_{\mathfrak{p}} \left(1 + \sum_{j=h+1}^{2h-1} N(\mathfrak{p})^{-\frac{j}{h}}\right).$$

Proof. We know that $\mathcal{O}_K$ has a finite number of units, all of which are roots of unity. We can then combine the fact that for two elements $\alpha, \beta \in \mathcal{O}_K$, we have $(\alpha) = (\beta)$ if and only if there exists some unit η such that $\alpha = \eta\beta$ with Proposition 6.10 and Theorem 5.6 to get the desired result. Note that since $d_K = 1$ or $d_K = 2$, we have that $2h + 1 > d_K$ and hence we only have that case of Theorem 5.6. □

6.3 Densities

If we have an imaginary quadratic field that does not have trivial class group, then we can consider the distribution of h-ful ideals over the ideal classes to hopefully find an asymptotic for the principal ideals specifically.

As a start, we can find the density of ideals in a specific class as a subset of all ideals. We find the following lemma in [MVO07].

Lemma 6.12 *Let K be a number field with class number h_K. Recall that Z_K denotes the residue of the Dedekind zeta function $\zeta(s)$ at $s = 1$. Let $N(x, C)$ denote the number of ideals of class C with norm at most x. Then*

$$N(x, C) = \frac{Z_K}{h_K} x + O\left(x^{\frac{d_K - 1}{d_K}}\right).$$

This lemma is true for all number fields, but we are currently mostly interested in imaginary quadratics.

In other words, the ideals are uniformly distributed over the classes of the class group. If a similar thing were true for the h-ful ideals, then we would be able to find an asymptotic for elements that produce h-ful ideals in all imaginary quadratics.

Another possibly useful result is the following:

Lemma 6.13 *Let K be an imaginary quadratic number field. Let $\pi(x, C)$ denote the number of prime ideals of class C with norm at most x. Then*

$$\pi(x, C) = A\frac{x}{\log x} + o\left(\frac{x}{\log x}\right),$$

where $A \in \mathbb{R}_{>0}$ is a constant that does not depend on C.

Proof. Let m be a $\mathcal{O}_K$-submodule of K and let S be a set of primes. Let I^S be the group of fractional ideals of K that are coprime to all primes of S and let $H^{(m)}$ be the subgroup of fractional ideals (α) that are coprime to all primes of S with $\alpha \equiv 1 \pmod{m}$ and where also $\sigma(\alpha) > 0$ for all real embeddings σ. Then set $h_m = \#I^S/H^{(m)}$. Now, we can use [CF67, Ch VIII, §2, Thm 4], which tells us that for all classes $\tilde{C}$ of $I^S/H^{(m)}$, we have

$$\#\left\{\mathfrak{p} \text{ prime} : \mathfrak{p} \in \tilde{C}\right\} = \frac{1}{h_m}\frac{x}{\log x} + o\left(\frac{x}{\log x}\right).$$

If we take the set S to be empty (or in the language of Cassels-Fröhlich, the set of all infinite primes, since they mean valuations when they say primes), then the entire coprimality condition is vacuous. Furthermore, if we take $m = K$, then the modulo condition is trivially satisfied. Finally, since K is imaginary quadratic and thus has no real embeddings, the condition that $\sigma(\alpha) > 0$ for all real embeddings σ (which is usually denoted as α being absolutely positive) is also vacuous.

Hence, in this case, I^S is just the set of all fractional ideals, and $H^{(m)}$ is the set of all principal fractional ideals. It follows that $h_m = h_K$ in this case, and that $I^S/H^{(m)}$ is equal to the usual class group. This proves the statement. □

We cannot easily use this result to show that h-ful ideals also have uniform distribution. We could try to show that tuples of prime exponents yield uniform distributions over classes, but it is difficult to enforce the norm condition.

6.4 h-th powers

We can consider a simplified version of the problem, where we only look at h-th powers. Here, densities are actually useful to get a result. We know there is a bijection between ideals that are h-th powers with norm at most x, and ideals with norm at most $x^{\frac{1}{h}}$, by raising the ideals to the h-th power. We know the number of ideals with norm at most $x^{\frac{1}{h}}$ is given by $Z_K x^{\frac{1}{h}} + O\left(x^{\frac{d_K-1}{h(d_K+1)}} \log(x)^{d_K-1}\right)$ by Corollary 4.2, where Z_K is the residue of the Dedekind zeta function at $s = 1$. We also know that the ideals with norm at most $x^{\frac{1}{h}}$ are uniformly distributed over the ideal classes as x tends to infinity, by Lemma 6.12. Hence, we need to consider the effect of raising ideals to the h-th power on their distribution over the ideal classes.

First, consider a number field with cyclic class group, so it is of the form $\mathbb{Z}/k\mathbb{Z}$ for some $k \in \mathbb{Z}_{>0}$. Raising ideals to the h-th power amounts to multiplying their ideal class by h in this case. We will use the following two lemmas.

Lemma 6.14 *Let G, H be two abelian groups, and let $f : G \to H$ be a group homomorphism. Then for all $h \in \operatorname{im} f$, we have $\#f^{-1}\{h\} = \#\ker f$.*

Proof. Take an arbitrary element $h \in \operatorname{im} f$ and take two arbitrary elements $g_1, g_2 \in f^{-1}\{h\}$. Then we must have $f(g_2 - g_1) = 0$, so that $g_2 - g_1 \in \ker f$. It follows that $f^{-1}\{h\} \subseteq g_1 + \ker f$, where $g_1 + \ker f = \{g_1 + a : a \in \ker f\}$ is a left coset of the kernel. On the other hand, if $a \in \ker f$, then $f(g_1 + a) = h$, and therefore, $g_1 + \ker f \subseteq f^{-1}\{h\}$.

We conclude that $f^{-1}\{h\} = g_1 + \ker f$, and hence $\#f^{-1}\{h\} = \#\ker f$ for all $h \in \operatorname{im} f$. □

Lemma 6.15 *Let $h, k \in \mathbb{Z}_{>0}$ and set $d = \gcd(h, k)$. Consider the map*

$$m_h : \mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}, a \mapsto h \cdot a.$$

Then the following statements are true.

1. *m_h is a group homomorphism.*
2. *$\operatorname{im} m_h = \langle d \rangle$.*
3. *$\# \ker m_h = d$.*

Here, we let $\langle d \rangle$ denote the subgroup of $\mathbb{Z}/k\mathbb{Z}$ that is generated by d.

Proof. For Part 1, we see that $m_h(a+b) = m_h(a) + m_h(b)$ for all $a, b \in \mathbb{Z}/k\mathbb{Z}$, which shows that m_h is indeed a group homomorphism.

For Part 2, we see that for all $a \in \mathbb{Z}/k\mathbb{Z}$, we have $m_h(a) = ha$ as element of $\mathbb{Z}/k\mathbb{Z}$. It follows that when we take an arbitrary representative $r \in \mathbb{Z}$ of the equivalence class ha, we have $r \equiv 0 \pmod{k}$ and hence also $r \equiv 0 \pmod{d}$. It follows that $ha \in \langle d \rangle$. This shows that $\operatorname{im} m_h \subseteq \langle d \rangle$.

To show equality, consider the set $S = \left\{0, \ldots, \frac{k}{d} - 1\right\}$. Now apply m_h to S. Suppose that for $a, b \in S$, we have $m_h(a) = m_h(b)$. Then $ha - hb$ must be a multiple of k. Therefore, $\frac{(ha-hb)}{d}$ must be a multiple of $\frac{k}{d}$, and since $\frac{h}{d}$ is coprime to $\frac{k}{d}$, we must even have that $a - b$ is a multiple of $\frac{k}{d}$. But $|a - b| < \frac{k}{d}$, so this is only possible if $a = b$. Hence, m_h is injective on S, which implies that $\# \operatorname{im} m_h \geq \frac{k}{d}$. But since $\operatorname{im} m_h \subseteq \langle d \rangle$ and since $\#\langle d \rangle = \frac{k}{d}$, we find that $\operatorname{im} m_h = \langle d \rangle$ as desired.

For Part 3, we use the fact that by the first group isomorphism theorem, we have $\operatorname{im} m_h \cong (\mathbb{Z}/k\mathbb{Z})/\ker m_h$, which tells us that $\# \ker m_h = \frac{\#(\mathbb{Z}/k\mathbb{Z})}{\operatorname{im} m_h} = \frac{k}{k/d} = d$, as desired. □

We can use these lemmas to find a result about h-th powers of bounded norm in number fields with cyclic class group.

Corollary 6.16 *Let $h, k \in \mathbb{Z}_{>0}$ and set $d = \gcd(h, k)$. Let K be a number field with class group isomorphic to $\mathbb{Z}/k\mathbb{Z}$. Then for all ideal classes a, we have that*

$$\lim_{x \to \infty} \frac{\#\{I \in \mathcal{I}_K : N(I) \leq x, [I] = a, I = J^h \textit{ for some } J\}}{\#\{I \in \mathcal{I}_K : N(I) \leq x, I = J^h \textit{ for some } J\}} = \begin{cases} \frac{d}{k} & \textit{if } d \mid a \\ 0 & \textit{if } d \nmid a \end{cases}.$$

Proof. Define m_h as in Lemma 6.15. This lemma then tells us that $\operatorname{im} m_h = \langle d \rangle$ and hence, $m_h^{-1}\{a\} = \emptyset$ if $d \nmid a$. If we use Lemma 6.14, we find that $\# m_h^{-1}\{a\} = d$ if $d \mid a$.

We remark that

$$\#\{I \in \mathcal{I}_K : N(I) \leq x, I = J^h \text{ for some } J \in \mathcal{I}_K\} = \#\left\{I \in \mathcal{I}_K : N(I) \leq x^{\frac{1}{h}}\right\}$$

and similarly

$$\begin{aligned} &\#\{I \in \mathcal{I}_K : N(I) \leq x, [I] = a, I = J^h \text{ for some } J \in \mathcal{I}_K\} \\ &= \#\left\{I \in \mathcal{I}_K : N(I) \leq x^{\frac{1}{h}}, [I] \in m_h^{-1}\{a\}\right\} \\ &= \#\left(\bigsqcup_{b \in m_h^{-1}\{a\}} \left\{I \in \mathcal{I}_K : N(I) \leq x^{\frac{1}{h}}, [I] = b\right\}\right), \end{aligned}$$

where this final union is a disjoint union.

Now, Lemma 6.12 tells us that

$$\bigsqcup_{b\in m_h^{-1}\{a\}} \#\left\{I \in \mathcal{I}_K : N(I) \le x^{\frac{1}{h}}, [I] = b\right\} = \#m_h^{-1}\{a\} \cdot \frac{Z_K}{k} x^{\frac{1}{h}} + O\left(x^{\frac{d_K-1}{h\cdot d_K}}\right),$$

and Corollary 4.2 tells us that

$$\#\left\{I \in \mathcal{I}_K : N(I) \le x^{\frac{1}{h}}\right\} = Z_K x^{\frac{1}{h}} + O\left(x^{\frac{d_K-1}{h(d_K+1)}}\right).$$

Taking their quotient and taking the limit as x goes to infinity yields the desired result. □

Now that we have a result for number fields with cyclic class groups, we can also try to expand this to number fields with other class groups. Since all class groups are abelian, and since all abelian groups are finite products of cyclic groups, we can expand the previous result without too much effort. We mostly need to generalize Lemma 6.15. The following lemma does this.

Lemma 6.17 *Let $h, t \in \mathbb{Z}_{>0}$. Let $k_1, \ldots, k_t \in \mathbb{Z}_{>0}$ and set $d_i = \gcd(h, k_i)$ for $i = 1, \ldots, t$. Set $G := \prod_{i=1}^t \mathbb{Z}/k_i\mathbb{Z}$. Consider the map*

$$m_h : G \to G, (a_1, \ldots, a_t) \mapsto (h \cdot a_1, \ldots, h \cdot a_t).$$

We find that

1. *m_h is a group homomorphism.*
2. *$\operatorname{im} m_h = \langle (d_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_t)\rangle$.*
3. *$\# \ker m_h = \prod_{i=1}^t d_i$.*

Proof. We see that

$$m_h(a_1, \ldots, a_t) = (m_h(a_1), \ldots, m_h(a_t)).$$

It follows immediately that m_h is a group homomorphism.

For Part 2, we can effectively reuse the proof of Lemma 6.15 on each component separately.

For Part 3, we can also use the same reasoning as in Lemma 6.15, but now $\#\ker m_h = \frac{\#G}{\operatorname{im} m_h} = \frac{\prod_{i=0}^t k_i}{\prod_{i=0}^t k_i/d_i} = \prod_{i=0}^t d_i$. □

Finally, we can apply this to a number field with arbitrary class group.

Theorem 6.18 *Let $h, t \in \mathbb{Z}_{>0}$. Let $k_1, \ldots, k_t \in \mathbb{Z}_{>0}$ and set $d_i = \gcd(h, k_i)$ for $i = 1, \ldots, t$. Let K be a number field with class group isomorphic to $G := \prod_{i=1}^t \mathbb{Z}/k_i\mathbb{Z}$. Then for all ideal classes $(a_1, \ldots, a_t)$, we have that*

$$\lim_{x\to\infty} \frac{\#\{I : N(I) \le x, [I] = (a_1, \ldots, a_t), I = J^h \text{ for some } J \in \mathcal{I}_K\}}{\#\{I : N(I) \le x, I = J^h \text{ for some } J \in \mathcal{I}_K\}} = \begin{cases} \frac{\prod_{i=1}^t d_i}{h_K} & \text{if } d_i \mid a_i \ \forall i = 1, \ldots, t \\ 0 & \text{else} \end{cases}.$$

Proof. Repeat the proof of Corollary 6.16 but with Lemma 6.17 rather than Lemma 6.15. □

This means that for a number field with class group $\prod_{i=1}^t \mathbb{Z}/k_i\mathbb{Z}$ with class number $h_K = \prod_{i=1}^t k_i$, we have

$$\begin{aligned}
&\#\{I \in \mathcal{I}_K : N(I) \le x, I \text{ principal}, I = J^h \text{ for some } J \in \mathcal{I}_K\} \\
&= \#\{I \in \mathcal{I}_K : N(I) \le x^{\frac{1}{h}}, I^h \text{ principal}\} \\
&= \frac{\prod_{i=1}^t d_i}{h_K} Z_K x^{\frac{1}{h}} + O\left(x^{\frac{d_K-1}{h(d_K+1)}} \log(x)^{d_K-1}\right).
\end{aligned}$$

This provides a lower bound on the number of h-ful principal ideals of bounded norm as well, as all h-th powers are h-ful.

6.5 Inversion

Recall from Definition 3.13 that $S_{K,h}$ is the set of all h-ful ideals. Define

$$\lambda_{K,h}(x) := \#\{I \in \mathcal{I}_K : N(I) \leq x, I \text{ principal}, I \in S_{K,h}\}.$$

Let ρ_h be the indicator function of $S_{K,h}$. In other words, for an ideal $I \in \mathcal{I}_K \setminus \{0\}$, we have that $\rho(I) = 1$ if $I \in S_{K,h}$ and $\rho(I) = 0$ else. Then we can express this as

$$\lambda_{K,h}(x) = \sum_{\substack{N(I)\leq x \\ I \text{ principal} \\ I\in S_{K,h}}} 1 = \sum_{\substack{N(I)\leq x \\ I \text{ principal}}} \rho_h(I).$$

We would like to use Möbius inversion here, but we first need to prove that it works on ideals in number fields. For this, we use the techniques from [Apo98, §2.2, §2.6, §2.7] on ideals.

We will first introduce some useful terms.

Definition 6.19 Let K be a number field. We call a function $f : \mathcal{I}_K \setminus \{0\} \to \mathbb{Z}$ an **arithmetic function**.

Definition 6.20 (As in [Apo98, §2.6]) Let K be a number field. Then define the following arithmetic function:

$$E(I) = \begin{cases} 1 & \text{if } I = \mathcal{O}_K \\ 0 & \text{else} \end{cases}.$$

Call this function the **identity function**.

Definition 6.21 (As in [Apo98, §2.7]) Let K be a number field. Then define the arithmetic function $u(I)$ such that for all nonzero ideals I,

$$u(I) = 1.$$

Call this function the **unit function**.

We can define convolutions of arithmetic functions just like in $\mathbb{Z}$.

Definition 6.22 (As in [Apo98, §2.6]) Let K be a number field. Let $f, g : \mathcal{I}_K \setminus \{0\} \to \mathbb{Z}$ be two functions. Then define their **convolution** $f * g$ as follows:

$$(f * g)(I) = \sum_{J_1 J_2 = I} f(J_1)g(J_2).$$

Convolution retains its properties from the standard version in $\mathbb{Z}$.

Lemma 6.23 (As in [Apo98, §2.6, Theorem 2.6]) *Convolutions are commutative and associative.*

Proof. We can simply repeat the proof in Apostol, but with ideals instead of positive integers. This still works due to the unique factorization of ideals. □

The identity function still acts as the identity of convolution.

Lemma 6.24 (As in [Apo98, §2.6, Theorem 2.7]) *Let K be a number field. For all arithmetic functions f, we have*

$$E * f = f * E = f.$$

Proof. This is essentially the same proof as for $\mathbb{Z}$, but Apostol uses the floor function, which makes the notation unclear for number fields. Essentially, we have

$$(f * E)(I) = \sum_{J_1 J_2 = I} f(J_1)E(J_2) = \sum_{\substack{J_1 J_2 = I \\ J_2 = \mathcal{O}_K}} f(J_1)E(J_2) = f(I),$$

since E vanishes on all ideals except for $\mathcal{O}_K$. The commutativity from Lemma 6.23 completes the proof. $\square$

An analogous definition for the Möbius function still works in number fields.

Definition 6.25 (As in [Apo98, §2.2]) Let K be a number field. We define the Möbius function $\mu : \mathcal{I}_K \setminus \{0\} \to \mathbb{Z}$ as follows.

Let $I \in \mathcal{I}_K \setminus \{0\}$ be an ideal. If $I = \mathcal{O}_K$, then set $\mu(I) = 1$. Else, denote its prime factorization as $I = \mathfrak{p}_1^{k_1} \dots \mathfrak{p}_t^{k_t}$ for some integer $t \in \mathbb{Z}_{\geq 0}$, some prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_t$, and some integers $k_1, \dots, k_t \in \mathbb{Z}_{>0}$. If $k_1 = \dots = k_t = 1$, then set $\mu(I) = (-1)^t$, else set $\mu(I) = 0$.

Its main property in $\mathbb{Z}$ also still holds.

Lemma 6.26 (As in [Apo98, §2.2, Theorem 2.1]) *Let K be a number field and let $I \in \mathcal{I}_K \setminus \{0\}$ be an ideal. Then*

$$\sum_{J|I} \mu(J) = E(I).$$

Proof. The proof is identical to the one used in Apostol for the normal Möbius function, except we now use ideals instead of positive integers. Since they still have unique prime factorizarion, all steps of the proof are still valid. $\square$

Finally, we get to Möbius inversion for ideals.

Proposition 6.27 (As in [Apo98, §2.7, Theorem 2.9]) *Let K be a number field and let f and g be arithmetic functions. Then, the following two equations imply each other.*

$$f(I) = \sum_{J|I} g(J)$$

and

$$g(I) = \sum_{J_1 J_2 = I} f(J_1)\mu(J_2).$$

Proof. The proof from Apostol applies directly, since it only uses Definition 6.21, Definition 6.22, Lemma 6.23, and Lemma 6.26. $\square$

We would like to find an arithmetic function μ_h such that

$$\sum_{J|I} \mu_h(J) = \rho_h(I).$$

Using Proposition 6.27 gives us that this works, so long as we define

$$\mu_h(I) = \sum_{J_1 J_2 = I} \rho_h(J_1)\mu(J_2).$$

Definition 6.28 Let μ_h be an arithmetic function such that

$$\mu_h(I) = \sum_{J_1 J_2 = I} \rho_h(J_1)\mu(J_2).$$

With some effort, we can prove that μ_h is multiplicative. We first need an auxiliary lemma.

Lemma 6.29 *Let K be a number field, let $I_1, I_2 \in \mathcal{I}_K \setminus \{0\}$ be two coprime ideals and set $I = I_1 I_2$. Then we have a bijection between*

$$A := \{(J_1, J_2, J_3, J_4) \in \left(\mathcal{I}_K \setminus \{0\}\right)^4 : J_1 J_2 = I_1, J_3 J_4 = I_2\}$$

and

$$B := \{(J_5, J_6) \in \left(\mathcal{I}_K \setminus \{0\}\right)^2 : J_5 J_6 = I\}$$

in the form of

$$f : A \to B, (J_1, J_2, J_3, J_4) \mapsto (J_1 J_3, J_2 J_4).$$

Proof. It is clear that if $(J_1, J_2, J_3, J_4) \in \left(\mathcal{I}_K \setminus \{0\}\right)^4$ such that $J_1 J_2 = I_1$ and $J_3 J_4 = I_2$, then we have that $(J_1 J_3, J_2 J_4) \in \left(\mathcal{I}_K \setminus \{0\}\right)^2$ with $(J_1 J_3) \cdot (J_2 J_4) = I$. So f has the correct range.

For surjectivity, take two arbitrary ideals $J_5, J_6 \in \mathcal{I}_K \setminus \{0\}$ so that $J_5 J_6 = I$. Thus, $(J_5, J_6) \in B$. Now, because I_1 and I_2 are coprime, and because $J_5 \mid I_1 I_2$, we must have that every prime factor of J_5 occurs in exactly one of I_1 and I_2. Set J_1 to be the ideal formed by the prime factors of J_5 that also occur in I_1, and set J_3 to be the ideal formed by the prime factors of J_5 that also occur in I_2. We must have $J_1 J_3 = J_5$. We can define J_2 and J_4 similarly based on J_6, so that $J_2 J_4 = J_6$. We hence must have that $J_1 J_2 J_3 J_4 = I_1 I_2$, and since J_1 and J_2 are both coprime to I_2 and J_3 and J_4 are both coprime to I_1, it follows that $J_1 J_2 = I_1$ and $J_3 J_4 = I_2$. We conclude that $(J_1, J_2, J_3, J_4) \in A$ and that $f(J_1, J_2, J_3, J_4) = (J_5, J_6)$, which proves surjectivity.

For injectivity, suppose we have $f(J_1, J_2, J_3, J_4) = f(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3, \tilde{J}_4)$. Then we know

- $J_1 J_2 = I_1 = \tilde{J}_1 \tilde{J}_2$,
- $J_3 J_4 = I_2 = \tilde{J}_3 \tilde{J}_4$,
- $J_1 J_3 = \tilde{J}_1 \tilde{J}_3$,
- $J_2 J_4 = \tilde{J}_2 \tilde{J}_4$.

Since $\tilde{J}_1$ divides I_1, and J_3 divides I_2, we see that $\tilde{J}_1$ must be coprime to J_3. Then it follows that $\tilde{J}_1$ divides J_1 from $J_1 J_3 = \tilde{J}_1 \tilde{J}_3$. But similarly, J_1 divides $\tilde{J}_1$, hence $J_1 = \tilde{J}_1$. We can repeat an analogous argument for the other ideals, which shows that $(J_1, J_2, J_3, J_4) = (\tilde{J}_1, \tilde{J}_2, \tilde{J}_3, \tilde{J}_4)$. We conclude that f is indeed injective.

This proves the statement. ☐

Corollary 6.30 *μ_h is multiplicative.*

Proof. Let I_1 and I_2 be two coprime ideals. Then

$$\mu_h(I_1)\mu_h(I_2) = \sum_{J_1 J_2 = I_1} \rho_h(J_1)\mu(J_2) \sum_{J_3 J_4 = I_2} \rho_h(J_3)\mu(J_4) = \sum_{\substack{J_1 J_2 = I_1 \\ J_3 J_4 = I_2}} \rho_h(J_1 J_3)\mu(J_2 J_4).$$

By Lemma 6.29, we find

$$\sum_{\substack{J_1 J_2 = I_1 \\ J_3 J_4 = I_2}} \rho_h(J_1 J_3)\mu(J_2 J_4) = \sum_{J_5 J_6 = I_1 I_2} \rho_h(J_5)\mu(J_6) = \mu_h(I_1 I_2).$$

Therefore,

$$\mu_h(I_1)\mu_h(I_2) = \mu_h(I_1 I_2),$$

which proves the desired statement. ☐

This allows us to compute μ_h for just the powers of primes, and still know the whole function.

Let $\mathfrak{p}$ be a prime ideal, and let $k \in \mathbb{Z}_{\geq 0}$ be an integer. Consider $\mu_h(\mathfrak{p}^k)$. Assume that $h \geq 2$, since $S_{K,1} = \mathcal{I}_K \setminus \{0\}$, which is not very interesting.

If $k = 0$, then $\mathfrak{p}^k = \mathcal{O}_K$, and by its definition, we have

$$\mu(\mathcal{O}_K) = \sum_{J_1 J_2 = \mathcal{O}_K} \rho_h(J_1)\mu(J_2) = \rho_h(\mathcal{O}_K)\mu(\mathcal{O}_K) = 1.$$

If $k = 1$, then

$$\mu(\mathcal{O}_K) = \sum_{J_1 J_2 = \mathfrak{p}} \rho_h(J_1)\mu(J_2) = \rho_h(\mathfrak{p})\mu(\mathcal{O}_K) + \rho_h(\mathcal{O}_K)\mu(\mathfrak{p}) = -1.$$

If $1 < k < h$, then

$$\mu(\mathcal{O}_K) = \sum_{J_1 J_2 = \mathfrak{p}} \rho_h(J_1)\mu(J_2) = \sum_{j=0}^{k} \rho_h\left(\mathfrak{p}^j\right)\mu\left(\mathfrak{p}^{k-j}\right) = 0,$$

as ρ_h vanishes for all $j \neq 0$, and there, μ vanishes. If $k = h$, then

$$\mu(\mathcal{O}_K) = \sum_{J_1 J_2 = \mathfrak{p}} \rho_h(J_1)\mu(J_2) = \sum_{j=0}^{h} \rho_h\left(\mathfrak{p}^j\right)\mu\left(\mathfrak{p}^{k-j}\right) = 1,$$

as μ vanishes for all $j < h - 1$, and for $j = h - 1$, ρ_h vanishes. Finally, if $k > h$, then

$$\mu(\mathcal{O}_K) = \sum_{J_1 J_2 = \mathfrak{p}} \rho_h(J_1)\mu(J_2) = \sum_{j=0}^{h} \rho_h\left(\mathfrak{p}^j\right)\mu\left(\mathfrak{p}^{k-j}\right) = -\rho_h\left(\mathfrak{p}^{k-1}\right) + \rho_h\left(\mathfrak{p}^k\right) = 0.$$

Hence, we see, for the non-trivial case of $h \geq 2$, that

$$\mu(\mathfrak{p}^k) = \begin{cases} 1 & \text{if } k \in \{0, h\}, \\ -1 & \text{if } k = 1, \\ 0 & \text{else.} \end{cases}$$

We can now use μ_h to manipulate our expression for $\lambda_{K,h}(x)$. We see

$$\lambda_{K,h}(x) = \sum_{\substack{N(I) \leq x \\ I \text{ principal}}} \rho_h(I) = \sum_{\substack{N(I) \leq x \\ I \text{ principal}}} \sum_{J|I} \mu_h(J) = \sum_{N(J) \leq x} \mu_h(J) \sum_{\substack{N(I) \leq x \\ I \text{ principal} \\ J|I}} 1 = \sum_{N(J) \leq x} \mu_h(J) \sum_{\substack{N(\tilde{I}) \leq x/N(J) \\ \tilde{I}J \text{ principal}}} 1.$$

While this formula is exact, we cannot do much with it, so we attempt to find an approximation of the inner sum. Denote

$$\nu(J, x) := \sum_{\substack{N(\tilde{I}) \leq x/N(J) \\ \tilde{I}J \text{ principal}}} 1.$$

Let Z_K denote the residue of the Dedekind zeta function at $s = 1$ and let h_K denote the class number of K. We are looking for the number of ideals in a certain ideal class, namely $[J^{-1}]$, with bounded norm, so we can use Lemma 6.12. We find

$$\nu(J, x) = \frac{Z_K}{h_K} \frac{x}{N(J)} + O\left(\left(\frac{x}{N(J)}\right)^{\frac{d_K - 1}{d_K}}\right).$$

It follows that

$$\lambda_{h,K}(x) = \sum_{N(J)\leq x} \mu_h(J) \frac{Z_K}{h_K} \frac{x}{N(J)} + \sum_{N(J)\leq x} \mu_h(J) \cdot O\left(\left(\frac{x}{N(J)} \right)^{\frac{d_K-1}{d_K}} \right)$$

$$= \frac{Z_K}{h_K} x \sum_{N(J)\leq x} \frac{\mu_h(J)}{N(J)} + O\left(x^{\frac{d_K-1}{d_K}} \cdot \sum_{\substack{N(J)\leq x \\ \mu_h(J)\neq 0}} N(J)^{-\frac{d_K-1}{d_K}} \right).$$

This is problematic, since we expect the main term to be of order $x^{\frac{1}{h}}$, which is not yet clear. The error term is also at the very least $O\left(x^{\frac{d_K-1}{d_K}}\right)$, which is much larger than what we want the main term to be. Therefore, this method does not seem to be very useful.

6.6 Representatives of ideals

To attempt to find $\tilde{N}_{K,h}(x)$, we can express it in the following way.

$$\tilde{N}_{K,h}(x) = \sum_{\substack{N(I)\leq x \\ I \text{ principal} \\ I\in S_{K,h}}} \#\{\alpha \in \mathcal{O}_K \setminus \{0\} : H(\alpha) \leq x, (\alpha) = I\}.$$

The summand can be found by considering the number of units with bounded height. In a specific example, we see the following.

Example 6.31 Suppose we have a number field K with $r_1 + r_2 = 2$, so with a single fundamental unit. Call this fundamental unit η. We also have valuations $||.||_1$ and $||.||_2$. Without loss of generality, assume $||\eta||_1 > 1$ and set $y := ||\eta||_1$. Let w_K be the number of roots of unity in $\mathcal{O}_K$.

Let $x \in \mathbb{R}, x \geq 1$ and let I be a principal ideal with $N(I) \leq x$. Let α be a generator of I and define $\alpha_k := \eta^k\alpha$. Then we must have

$$||\alpha_k||_1||\alpha_k||_2 = N(I).$$

Note that we also have $||\eta||_1||\eta||_2 = 1$, so that $||\eta||_2 = y^{-1}$. Consider the quantity

$$H(\alpha_k) = \prod_{i=1}^{2} \max(1, ||\alpha_k||_i).$$

Then we know $H(\alpha_k) \leq x$ if and only if $\max(1, ||\alpha_k||_i) \leq x$ for $i = 1, 2$. Namely, we know $||\alpha_k||_1||\alpha_k||_2 = N(I) \leq x$. Hence, if $||\alpha_k||_1 > x$, then $||\alpha_k||_2 < 1$ and vice versa, in which case $H(\alpha_k) > x$. And if $||\alpha_k||_i \leq x$ for $i = 1, 2$, then there are two options. If $||\alpha_k||_1 \leq 1$ or $||\alpha_k||_2 \leq 1$, then $H(\alpha_k) \leq x$, and if both are greater than 1, then $H(\alpha_k) = N(I) \leq x$.

Since we know $x \geq 1$, we only care about the variable term in the maximum. Hence, we need $||\alpha_k||_i \leq x$ for $i = 1, 2$. This gives us the following conditions:

$$y^k||\alpha||_1 \leq x,$$

$$y^{-k}||\alpha||_1^{-1}N(I) \leq x.$$

Solving for k gives us that

$$k \leq \frac{\log x - \log ||\alpha||_1}{\log y},$$

$$k \geq \frac{\log N(I) - \log ||\alpha||_1 - \log x}{\log y}.$$

The number of integers k such that $H(\alpha_k) \leq x$ is therefore equal to

$$\left\lfloor \frac{\log x - \log ||\alpha||_1}{\log y} \right\rfloor - \left\lceil \frac{\log N(I) - \log ||\alpha||_1 - \log x}{\log y} \right\rceil + 1 = \frac{2\log x - \log N(I)}{\log y} + O(1),$$

where the implied constant of $O(1)$ is 1.

Recall from Proposition 6.10 that we can always multiply by roots of unity without changing the height, and recall that a principal ideal has a generator that is unique up to units. By our previous reasoning, it follows that for this principal ideal I, we have

$$\#\{\alpha \in \mathcal{O}_K \setminus \{0\} : H(\alpha) \leq x, (\alpha) = I\} = w_K \frac{2\log x - \log N(I)}{\log y} + O(1).$$

We can now calculate

$$\tilde{N}_{K,h}(x) = w_K \sum_{\substack{N(I) \leq x \\ I \text{ } h\text{-ful} \\ I \text{ principal}}} \left(\frac{2\log x - \log N(I)}{\log y} + O(1) \right) = w_K \sum_{\substack{N(I) \leq x \\ I \text{ } h\text{-ful} \\ I \text{ principal}}} \frac{2\log x - \log N(I)}{\log y} + O\left(x^{\frac{1}{h}}\right)$$

$$= \frac{2w_K \log x}{\log(y)} \#\{I \in \mathcal{I}_K \setminus \{0\} : N(I) \leq x, I \text{ principal}, I \in S_{K,h}\} - \frac{w_K}{\log y} \sum_{\substack{N(I) \leq x \\ I \text{ principal} \\ I \in S_{K,h}}} \log N(I) + O\left(x^{\frac{1}{h}}\right).$$

We can use partial summation on the sum in the above expression, in other words, Corollary 4.13. For ease of notation, set $N_{K,h,0}(x) := \#\{I \in \mathcal{I}_K \setminus \{0\} : N(I) \leq x, I \text{ principal}, I \in S_{K,h}\}$, and for all $I \in \mathcal{I}_K \setminus \{0\}$, set $\rho(I) = 1$ whenever I principal and $I \in S_{K,h}$, and set $\rho(I) = 0$ otherwise. Hence, $\sum_{N(I) \leq x} \rho(I) = N_{K,h,0}(x)$. Then we see that

$$\sum_{\substack{N(I) \leq x \\ I \text{ principal} \\ I \in S_{K,h}}} \log N(I) = \sum_{N(I) \leq x} \rho(I) \log N(I) = N_{K,h,0}(x) \log x - \int_1^x \frac{N_{K,h,0}(t)}{t} \mathrm{d}t.$$

Since we have that $N_{K,h} = O\left(x^{\frac{1}{h}}\right)$, we also have that $N_{K,h,0} = O\left(x^{\frac{1}{h}}\right)$. This yields that

$$\int_1^x \frac{N_{K,h,0}(t)}{t} \mathrm{d}t = \int_1^x O\left(x^{\frac{1}{h}-1}\right) \mathrm{d}t = O\left(x^{\frac{1}{h}}\right).$$

It follows that

$$\tilde{N}_{K,h}(x) = \frac{2w_K \log x}{\log y} N_{K,h,0}(x) - \frac{w_K}{\log y} \sum_{\substack{N(I) \leq x \\ I \text{ principal} \\ I \in S_{K,h}}} \log N(I) + O\left(x^{\frac{1}{h}}\right) = \frac{w_K \log x}{\log y} N_{K,h,0}(x) + O\left(x^{\frac{1}{h}}\right).$$

Hence, if we can find an asymptotic for $N_{K,h,0}(x)$, then we can also find one for $\tilde{N}_{K,h}(x)$.

We remark that the dependence on $\log y$ may seem curious, since y is chosen seemingly arbitrarily. However, note that $||\eta||_1 ||\eta||_2 = 1$ since η is a unit, and therefore, $\log ||\eta||_1 + \log ||\eta||_2 = 0$. Using one or the other will not end up changing the result, as it will simply flip the sign of all allowed values of k, but not the number of allowed values. We also see that $\log y$ is equal to the regulator R_K of the number field. This allows us to express the formula we have in a form that is more clearly independent from our choices, since we find

$$\tilde{N}_{K,h}(x) = \frac{\log x}{R_K} N_{K,h,0}(x) + O\left(x^{\frac{1}{h}}\right).$$

Based on Subsection 6.4 and on results from the appendix, Subsection 7.2, one could conjecture that there is some density of the set of principal h-ful ideals within the set of h-ful ideals. More specifically, one could conjecture that there exist constants $\theta \in (0,1], \varepsilon \in \mathbb{R}_{>0}$ such that

$$N_{K,h,0}(x) = \theta N_{K,h}(x) + O\left(x^{\frac{1}{h}-\varepsilon}\right).$$

If this were true, then we could use Theorem 5.6 and we would have that

$$\tilde{N}_{K,h}(x) = \frac{\theta w_K Z_K \gamma}{R_K} x^{\frac{1}{h}} \log x + O\left(x^{\frac{1}{h}}\right).$$

7 Appendix

7.1 Results when trying approximation by inversion

In Subsection 6.5, we concluded that the error term was most likely too large compared to the main term. If we do still try the approximation used, we find the following results, with the help of SageMath. This uses the approximation

$$\#\{I \in \mathcal{I}_K : N(I) \leq x, I \text{ principal}, I \in S_{K,h}\} = \frac{Z_K}{h_K} x \sum_{N(J) \leq x} \frac{\mu_h(J)}{N(J)} + O\left(x^{\frac{d_K - 1}{d_K}} \cdot \sum_{\substack{N(J) \leq x \\ \mu_h(J) \neq 0}} N(J)^{-\frac{d_K - 1}{d_K}} \right).$$

The actual values in the following tables are non-integer, but we rounded them to the nearest integer since they are meant to approximate an integer value.

h	**2**		**3**		**4**		**5**	
$\mathbf{log_2(x)}$	Actual	Approx.	Actual	Approx.	Actual	Approx.	Actual	Approx.
0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
2	2	2	1	1	1	1	1	1
3	3	2	2	1	1	0	1	0
4	5	5	3	3	2	2	1	1
5	8	4	5	2	3	0	2	-1
6	11	10	6	5	4	4	3	3
7	18	12	9	7	6	4	4	3
8	26	20	12	6	8	6	6	5
9	38	25	15	7	9	1	7	2
10	55	38	21	12	12	5	9	2
11	80	68	27	24	14	15	10	13
12	116	57	38	5	20	-4	13	-10
13	166	159	49	52	24	34	16	30
14	240	119	63	10	31	-20	19	-24
15	345	281	85	76	39	59	24	41
16	497	355	109	71	49	29	28	28
17	710	522	142	65	62	24	35	-4

Table 1: For various values of x and h, the number of h-ful numbers less than or equal to x, both the actual values and the approximations.

h	**2**		**3**		**4**		**5**	
$\mathbf{log_2(x)}$	Actual	Approx.	Actual	Approx.	Actual	Approx.	Actual	Approx.
0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
2	2	2	1	2	1	2	1	2
3	3	2	2	1	1	1	1	1
4	4	4	3	2	2	1	1	0
5	6	6	4	4	3	2	2	1
6	7	3	5	1	4	0	3	-1
7	11	11	7	7	5	5	4	4
8	14	17	8	9	6	7	5	7
9	19	20	9	7	7	6	6	6
10	27	37	13	9	9	4	7	4
11	36	58	15	21	10	13	8	9
12	50	59	19	12	12	-1	10	-5
13	68	112	25	23	14	4	11	1
14	93	144	30	45	17	25	13	18
15	127	201	38	72	20	53	14	39
16	172	306	48	24	24	-7	16	-8
17	238	502	62	189	30	138	19	128

Table 2: For various values of x and h, the number of principal h-ful ideals in $\mathbb{Q}(i)$ (with ring of integers $\mathbb{Z}[i] = \mathbb{Z}[t]/\mathbb{Z}[t^2+1]$) with norm less than or equal to x, both the actual values and the approximations.

h	**2**		**3**		**4**		**5**	
$\mathbf{log_2(x)}$	Actual	Approx.	Actual	Approx.	Actual	Approx.	Actual	Approx.
0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1
4	3	2	1	1	1	1	1	1
5	4	4	2	1	1	0	1	0
6	6	6	3	3	1	1	1	1
7	7	7	4	3	2	0	1	-1
8	11	13	6	5	4	3	2	2
9	15	22	7	12	4	9	2	6
10	21	21	9	5	6	1	4	-1
11	27	31	10	5	6	2	4	1
12	39	49	14	13	9	5	6	3
13	51	64	18	13	10	3	7	-2
14	72	113	22	40	11	24	8	20
15	99	166	30	60	15	37	10	38

Table 3: For various values of x and h, the number of principal h-ful ideals in $\mathbb{Q}(\sqrt{-3})$ (with ring of integers $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}] = \mathbb{Z}[t]/\mathbb{Z}[t^2+t+1]$) with norm less than or equal to x, both the actual values and the approximations.

h	**2**		**3**		**4**		**5**	
$\mathbf{log_2(x)}$	Actual	Approx.	Actual	Approx.	Actual	Approx.	Actual	Approx.
0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
2	2	0	1	0	1	0	1	0
3	3	0	2	-1	1	-2	1	-2
4	5	5	3	3	2	3	1	2
5	8	3	5	4	3	3	2	3
6	11	1	6	-3	4	-3	3	-2
7	17	5	9	-4	6	-5	4	-5
8	24	19	12	11	8	10	6	9
9	33	20	15	18	9	14	7	12
10	47	-8	21	-22	12	-30	9	-25
11	64	72	26	13	14	12	10	9
12	88	28	36	17	20	6	13	-3
13	119	136	45	62	24	56	16	50
14	165	-6	57	-69	30	-105	19	-80
15	224	180	75	-5	37	5	24	-36

Table 4: For various values of x and h, the number of principal h-ful ideals in $\mathbb{Q}(\sqrt{-5})$ (with ring of integers $\mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[t]/\mathbb{Z}[t^2+5]$) with norm less than or equal to x, both the actual values and the approximations.

h	**2**		**3**		**4**		**5**	
$\mathbf{log_2(x)}$	Actual	Approx.	Actual	Approx.	Actual	Approx.	Actual	Approx.
0	1	1	1	1	1	1	1	1
1	1	0	1	0	1	0	1	0
2	3	0	1	-1	1	-1	1	-1
3	4	3	2	3	1	1	1	1
4	9	3	4	2	3	2	1	0
5	12	-1	7	-2	5	-1	3	-1
6	18	12	8	4	6	0	4	3
7	28	8	12	10	10	10	6	6
8	39	9	18	-3	15	-5	10	-4
9	51	10	23	-5	18	-9	11	-7
10	77	73	31	44	22	31	15	25
11	98	-20	40	-12	28	-7	17	-20
12	137	94	54	-1	39	-10	22	2
13	186	124	69	86	50	49	30	53
14	252	78	84	-12	60	11	35	-42
15	352	94	108	-53	76	-84	44	-45

Table 5: For various values of x and h, the number of principal h-ful ideals in $\mathbb{Q}(\sqrt{-23})$ (with ring of integers $\mathbb{Z}[\frac{1+\sqrt{-23}}{2}] = \mathbb{Z}[t]/\mathbb{Z}[t^2+t+6]$) with norm less than or equal to x, both the actual values and the approximations.

7.2 Density of principal h-ful ideals

Though we could not prove that there was some nice distribution of the h-ful ideals over the class group, we did explore some weaker statements in Subsection 6.3 and Subsection 6.4. With the help of SageMath, we can explore the densities of principal h-ful ideals within the h-ful ideals, in various number fields. Specifically, for various number fields K and for various values of x and h, we calculate

$$\frac{\#\{I \in \mathcal{I}_K \setminus \{0\} : I \in S_{K,h}, I \text{ principal}\}}{N_{K,h}(x)}.$$

Since $N_{K,h}(x)$ is much larger for lower h given the same K and x, we needed to lower the highest value of x for the lower values of h for computational reasons. The script already takes a few hours to come up with these results, so any higher bounds would not be feasible.

We have the following results.

h $\mathbf{log_{10}(x)}$	**2**	**3**	**4**	**5**
0	1.0000	1.0000	1.0000	1.0000
1	0.8000	0.5000	1.0000	1.0000
2	0.7143	0.5556	0.8333	0.6667
3	0.6047	0.5000	0.6429	0.5000
4	0.6020	0.5366	0.6486	0.5714
5	0.5941	0.4842	0.5667	0.5208
6	0.5851	0.5147	0.5665	0.5196
7	0.5843	0.5054	0.5527	0.5174
8	0.5820	0.5025	0.5386	0.5080
9	0.5812	0.5005	0.5272	0.5035
10	0.5805	0.5005	0.5284	0.5026
11		0.4992	0.5223	0.4985
12		0.5007	0.5261	0.5013
13		0.5005	0.5239	0.4988
14			0.5253	0.5030
15			0.5224	0.5020
16			0.5223	0.5020
17				0.5006
18				0.5010
19				0.5007
20				0.5003

Table 6: For various values of x and h, the proportion of h-ful ideals with norm less than or equal to x that are principal, in $\mathbb{Q}(\sqrt{-5})$ (with ring of integers $\mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[t]/\mathbb{Z}[t^2+5]$), which has class number $h_K = 2$.

h	**2**	**3**	**4**	**5**
$\mathbf{log_{10}(x)}$				
0	1.0000	1.0000	1.0000	1.0000
1	0.6667	0.5000	1.0000	1.0000
2	0.7778	0.6000	0.7500	0.6667
3	0.7250	0.4545	0.5714	0.5000
4	0.6812	0.5000	0.6429	0.5000
5	0.6770	0.5244	0.6552	0.5000
6	0.6660	0.5118	0.6429	0.5172
7	0.6594	0.5115	0.6370	0.5738
8	0.6553	0.5063	0.6062	0.5333
9	0.6540	0.5045	0.5926	0.5197
10	0.6529	0.5024	0.5882	0.5143
11		0.5037	0.5802	0.5141
12		0.5037	0.5799	0.5083
13		0.5020	0.5708	0.5031
14			0.5675	0.5039
15			0.5664	0.5071
16			0.5642	0.5078
17				0.5045
18				0.5043
19				0.5044
20				0.5037

Table 7: For various values of x and h, the fraction of h-ful ideals with norm less than or equal to x that are principal, in $\mathbb{Q}(\sqrt{-13})$ (with ring of integers $\mathbb{Z}[\sqrt{-13}] = \mathbb{Z}[t]/\mathbb{Z}[t^2+13]$), which has class number $h_K = 2$.

h	**2**	**3**	**4**	**5**
$\mathbf{log_{10}(x)}$				
0	1.0000	1.0000	1.0000	1.0000
1	0.4286	1.0000	1.0000	1.0000
2	0.3750	0.5714	0.3333	0.6000
3	0.3429	0.4000	0.3636	0.4667
4	0.3333	0.3642	0.3425	0.3659
5	0.3382	0.3630	0.3333	0.3673
6	0.3368	0.3676	0.3318	0.3396
7	0.3362	0.3642	0.3326	0.3349
8	0.3340	0.3621	0.3326	0.3318
9	0.3340	0.3617	0.3352	0.3383
10	0.3339	0.3605	0.3355	0.3369
11		0.3586	0.3326	0.3350
12		0.3578	0.3337	0.3352
13		0.3574	0.3341	0.3359
14			0.3335	0.3344
15			0.3333	0.3341
16			0.3334	0.3346
17				0.3338
18				0.3340
19				0.3340
20				0.3340

Table 8: For various values of x and h, the fraction of h-ful ideals with norm less than or equal to x that are principal, in $\mathbb{Q}(\sqrt{-23})$ (with ring of integers $\mathbb{Z}[\frac{1+\sqrt{-23}}{2}] = \mathbb{Z}[t]/\mathbb{Z}[t^2+t+6]$), which has class number $h_K = 3$.

h $\mathbf{log_{10}}$**(x)**	**2**	**3**	**4**	**5**
0	1.0000	1.0000	1.0000	1.0000
1	0.4000	0.5000	1.0000	1.0000
2	0.4545	0.5556	0.8333	0.6667
3	0.3636	0.2812	0.5333	0.3000
4	0.3389	0.2857	0.5128	0.3182
5	0.3214	0.2627	0.4000	0.2745
6	0.3238	0.2705	0.3668	0.2679
7	0.3210	0.2559	0.3367	0.2723
8	0.3193	0.2538	0.3182	0.2563
9	0.3186	0.2543	0.3145	0.2636
10	0.3176	0.2535	0.3036	0.2599
11		0.2525	0.3005	0.2548
12		0.2522	0.2966	0.2546
13		0.2518	0.2939	0.2538
14			0.2902	0.2507
15			0.2877	0.2510
16			0.2867	0.2516
17				0.2518
18				0.2514
19				0.2506
20				0.2512

Table 9: For various values of x and h, the fraction of h-ful ideals with norm less than or equal to x that are principal, in $\mathbb{Q}(\sqrt{-14})$ (with ring of integers $\mathbb{Z}[\sqrt{-14}] = \mathbb{Z}[t]/\mathbb{Z}[t^2 + 14]$), which has class number $h_K = 4$.

h $\mathbf{log_{10}}$**(x)**	**2**	**3**	**4**	**5**
0	1.0000	1.0000	1.0000	1.0000
1	0.4000	0.5000	1.0000	1.0000
2	0.4737	0.5556	0.8333	0.6667
3	0.3614	0.2593	0.4615	0.3000
4	0.3574	0.3026	0.5000	0.3500
5	0.3447	0.2823	0.4217	0.3182
6	0.3276	0.2641	0.3717	0.2340
7	0.3301	0.2617	0.3614	0.2717
8	0.3286	0.2577	0.3447	0.2663
9	0.3267	0.2547	0.3244	0.2597
10	0.3262	0.2543	0.3189	0.2562
11		0.2537	0.3108	0.2585
12		0.2527	0.3075	0.2548
13		0.2523	0.3048	0.2559
14			0.3003	0.2508
15			0.2978	0.2514
16			0.2962	0.2528
17				0.2519
18				0.2518
19				0.2519
20				0.2514

Table 10: For various values of x and h, the fraction of h-ful ideals with norm less than or equal to x that are principal, in $\mathbb{Q}(\sqrt{-17})$ (with ring of integers $\mathbb{Z}[\sqrt{-17}] = \mathbb{Z}[t]/\mathbb{Z}[t^2 + 17]$), which has class number $h_K = 4$.

It seems like the density of h-ful ideals in K is given by $\frac{1}{h_K}$, where h_K is the class number of K, if h and h_K are coprime. If h and h_K are not coprime, then the density is either something different, or it converges much more slowly.

7.3 Code

Here is the code used to generate the previous results.

```
from time import perf_counter as pf
from functools import lru_cache

def simplify_ideal(ideal):
    # Represent a SageMath ideal as a tuple of its norm, and its ideal class.
    return (ideal.norm(), tuple(ideal.ideal_class_log()))

def change_class(ideal_1, ideal_2, expo, group_struct):
    # Carry out ideal_1 * ideal_2**expo on the ideal class.
    return tuple((k + expo * l) % m for k, l, m in zip(ideal_1, ideal_2, group_struct))

def mu_cases(exp, h):
    if exp == 0 or exp == h:
        return 1
    elif exp == 1:
        return -1
    else:
        return 0

@lru_cache(None)
def mu_h(n, h):
    # The mu function, the Mobius inverse of the h-fulness indicator function.
    return product(mu_cases(exp, h) for prime, exp in factor(n))

def field_prime_range_simple(K, bound):
    # Find the prime ideals in K up to a bound, and represent them
    # as a tuple of their norm and their ideal class.
    all_primes = []
    for p in prime_range(bound + 1):
        primes = K.ideal(p).factor()
        for prime in primes:
            all_primes.append(prime[0])
    all_primes = [simplify_ideal(prime) for prime in all_primes if prime.norm() <= bound]
    all_primes.sort()
    return all_primes

def find_h_ful_ideals_simple(K, h, bound):
    # Find all h-ful ideals of bounded norm. Store ideals
    # as a tuple of their norm and their ideal class.
    group_struct = K.class_group().gens_orders()
    found = {simplify_ideal(K.ideal(1)): 1}

    # We can generate h-ful ideals by looping through the prime ideals in order,
    # and for each prime p, multiplying our existing h-ful ideals by p**h, p**(h + 1), ...
    # until we reach the bound.
    for p in field_prime_range_simple(K, bound ** (1/h)):
        found_new = found.copy()
        # Find which power of p is the greatest we can multiply by while keeping
        # the norm under the bound. By keeping our dictionary found in sorted order
        # (based on norm), we can break as soon as we find an ideal I such that
```

```
        # I * p**h has a norm greater than our bound, since all future ideals J
        # we have yet to loop over, will have a greater or equal norm.
        for pair, count in found.items():
            k = int(log(bound / pair[0]) / log(p[0]))
            if k < h:
                break
            for i in range(h, k + 1):
                new_pair = (pair[0] * p[0]**i, change_class(pair[1], p[1], i, group_struct))
                found_new[new_pair] = found_new.get(new_pair, 0) + count
        found = dict(sorted(found_new.items()))
    return found

def approx_h_ful(K, h, bound):
    # Use the formula we get by using Mobius inversion, to see how good
    # or bad of an approximation this is. We generate all ideals for which
    # the mu function is nonzero, and then calculate the relevant sum.
    # Note that we do not include the residue of the Dedekind zeta function
    # here yet, we will multiply this in later as it is a constant anyway.
    found = {(1, 1): 1}
    for p in field_prime_range_simple(K, bound):
        found_new = found.copy()
        # Use the same technique as for finding the h-ful ideals.
        # Generate the relevant ideals based on their prime factorization,
        # and add on the primes in order of norm.
        for pair, count in found.items():
            if pair[0] * p[0] > bound:
                break
            new_pair_1 = (pair[0] * p[0], pair[1] * -1)
            found_new[new_pair_1] = found_new.get(new_pair_1, 0) + count
            if pair[0] * p[0]**h > bound:
                continue
            new_pair_2 = (pair[0] * p[0]**h, pair[1])
            found_new[new_pair_2] = found_new.get(new_pair_2, 0) + count
        found = dict(sorted(found_new.items()))
    return float(sum(count * pair[1] * bound / pair[0] for pair, count in found.items()))

def approx_h_ful_2(K, h, bound):
    # Use the formula we get by using Mobius inversion, to see how good
    # or bad of an approximation this is. This is mostly useful if K = Q,
    # else the other method is faster.
    # Note that we do not include the residue of the Dedekind zeta function
    # here yet, we will multiply this in later as it is a constant anyway.
    ideals = [ideal for ideals in K.ideals_of_bdd_norm(bound).values() for ideal in ideals]
    return float(bound * sum(mu_h(ideal, h) / ideal.norm() for ideal in ideals))

def h_ful_density_simple(K, h, bound):
    # Find all h-ful ideals up to a bound, and check how many of those are principal.
    b = find_h_ful_ideals_simple(K, h, bound)
    total_count = sum(b.values())
    princ_count = sum(count for ideal, count in b.items() if not(any(ideal[1])))
    return bound, float(princ_count/total_count), total_count

def h_ful_density_main():
    # Find the proportion of principal h-ful ideals of bounded norm
    # compared to all h-ful ideals of bounded norm, for various fields K,
    # and various values of h and the bound.
    t_0 = pf()
    x = polygen(ZZ, 'x')
```

```
    K0.<y0> = NumberField(x)
    K1.<y1> = NumberField(x^2 + 5)
    K2.<y2> = NumberField(x^2 + 13)
    K3.<y3> = NumberField(x^2 + x + 6)
    K4.<y4> = NumberField(x^2 + 14)
    K5.<y5> = NumberField(x^2 + 17)
    fields = [K1, K2, K3, K4, K5]
    for field in fields:
        print()
        print()
        print(field)
        print(f'Class Number: {field.class_number()}')
        print(f'Class Group: {field.class_group()}')
        for h in range(2, 5):
            print()
            print(f'h: {h}')
            # This line has been edited sometimes to calculate
            # results for larger bounds so does not match
            # the bounds in the tables. The core logic is
            # identical, though.
            for i in range(6, 4*h + 1):
                print(h_ful_density_simple(field, h, 10**i))
    print()
    print(pf() - t_0)
    return

def approx_h_ful_main():
    # Use the Mobius inversion formula for various number fields,
    # for various values of x and the bound on the norm.
    t_0 = pf()
    x = polygen(ZZ, 'x')
    # K0.<y0> = NumberField(x)
    K1.<y1> = NumberField(x^2 + 5)
    K2.<y2> = NumberField(x^2 + 13)
    K3.<y3> = NumberField(x^2 + x + 6)
    K4.<y4> = NumberField(x^2 + 14)
    K5.<y5> = NumberField(x^2 + 17)
    fields = [K1, K2, K3, K4, K5]
    for field in fields:
        # Calculate the residue of the Dedekind zeta function here,
        # and multiply it in, since we did not do that before.
        r_1 = len(field.real_embeddings())
        r_2 = (field.degree() - r_1) // 2
        constant = 2**r_1 * (2 * pi)**r_2 * field.regulator() / (field.zeta_order() * sqrt(abs(field.discriminant())))
        print()
        print()
        print(field)
        print(f'Class Number: {field.class_number()}')
        print(f'Class Group: {field.class_group()}')
        for h in range(2, 6):
            print()
            print(f'h: {h}')
            for i in range(16):
                print(f'{i}: {float(approx_h_ful(field, h, 2**i) * constant)}, {len(find_h_ful_ideals_simple(field, h, 2**i))}')
    print()
    print(pf() - t_0)
```

```
return

h_ful_density_main()
approx_h_ful_main()
```

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