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Opleiding Wiskunde

Coherence theorems for Monoidal Categories

Bachelor Thesis

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Abstract

In this thesis we look at two statements relating to coherence of monoidal categories and give the categorical background needed to formulate it. The first one is Mac Lane's coherence theorem, which states that all diagrams containing only $\alpha, \lambda, \rho, 1, - \otimes -$ commute. The second one is strictification, which says that all monoidal categories are monoidally equivalent to a strict monoidal category. We prove this second theorem using an adapted version of the Yoneda lemma for 2-categories to monoidal categories. We also briefly discuss the possibility of proving coherence from strictification.

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Chapter 1

Introduction

When working with algebraic structures like groups or monoids we have the associativity axiom $a * (b * c) = (a * b) * c$. This axiom gives rise to the general associativity theorem[1], that states that any two bracketings of the same elements are equal. This theorem allows us to forget about the brackets and write our products without them.

We want to extend this theory to structures that don't have strict equality for its associativity axiom, but do have an isomorphism. Take for example the cartesian product with sets, we have

$$A \times (B \times C) \cong (A \times B) \times C.$$

To talk about structures with isomorphisms we need the language of category theory, which we will introduce in chapter 2. After this we will introduce the notion of monoidal categories in chapter 3, there we will also state the coherence theorem, which says that all diagrams containing only the isomorphisms for associativity and unit laws commute. We will also look at Mac Lane's proof[4] of the coherence theorem in this chapter.

In chapter 4 we will discuss a different approach to this problem, namely strictification adapting a proof of the Yoneda lemma for 2-categories as proven in Johnson and Yau's book on 2-dimensional categories[3]. We will prove that all monoidal categories are monoidally equivalent to a strict monoidal category. We will also discuss ways in which strictification may be used to prove the coherence theorem.

Chapter 2

Basic Category theory

2.1 Basic definitions

We start by defining the basic structures we will be working with: categories, functors and natural transformations.

Definition 2.1.1 (Category)**.** A category $\mathcal{C}$ is defined by:

- a class of objects $\mathrm{Ob}(\mathcal{C})$
- for each $x, y \in \mathrm{Ob}(\mathcal{C})$, a class of morphisms $\mathrm{Hom}(x, y)$
- for every $x \in \mathrm{Ob}(\mathcal{C})$ a morphism called the identity morphism $1_x \in \mathrm{Hom}(x, x)$
- for each $x, y, z \in \mathrm{Ob}(\mathcal{C})$, a mapping called composition, given by

$$-\circ- \colon \mathrm{Hom}(y, z) \times \mathrm{Hom}(x, y) \to \mathrm{Hom}(x, z)$$

We also have the following axioms. The left and right unit axioms say that for all $x, y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}(x, y)$, we have

$$1_y \circ f = f = f \circ 1_x.$$

And the associativity axiom, which says that for all $w, x, y, z \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f \in \mathrm{Hom}(y, z)$, $g \in \mathrm{Hom}(x, y)$ and $h \in \mathrm{Hom}(w, x)$, we have

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

Note that we defined Ob and Hom as classes instead of as sets. This is so that we can talk about a category of sets or topological spaces (these collections don't form sets). If the objects and morphisms of a category form a set we call the category a *small category* and if for any two objects $x, y \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}(x, y)$ is a set we call the category *locally small*.

Sometimes we also simplify our notation. We may for example write composition as fg instead of $f \circ g$, we may leave out the subscript of the identity and morphism classes and instead of writing $f \in \mathrm{Hom}(x, y)$ we will often write $f\colon x \to y$. We can also write $x \in \mathcal{C}$ instead of $x \in \mathrm{Ob}(\mathcal{C})$ when talking about objects of $\mathcal{C}$.

Definition 2.1.2 (Isomorphisms)**.** An isomorphism $f\colon x \to y$ is a morphism, together with an inverse morphism $f^{-1}\colon y \to x$, such that $ff^{-1} = 1_y$ and $f^{-1}f = 1_x$.

If there exists an isomorphism $f\colon x \to y$, we call x and y isomorphic and denote this as $x \cong y$.

Definition 2.1.3 (Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, a functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a mapping of objects $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, and for every $a, b \in \mathcal{C}$ a mapping $F\colon \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{D}}(Fa, Fb)$.

These mappings also need to satisfy the following axioms. For all $x \in \mathcal{C}$, $F(1_x) = 1_{Fx}$ and for any composable pair of morphisms f, g we have $F(fg) = (Ff)(Fg)$.

Lemma 2.1.4 (Functors preserve isomorphisms)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F\colon \mathcal{C} \to \mathcal{D}$ a functor. Now let $a, b \in \mathcal{C}$ be isomorphic, then $Fa \cong Fb$.*

Proof. Since $a \cong b$ we have an $f\colon a \to b$ with an inverse $g\colon b \to a$. We will now show that $Ff\colon Fa \to Fb$ is an isomorphism with inverse Fg. Note that $(Fg)(Ff) = F(gf) = F(1_b) = 1_{Fb}$ and $(Ff)(Fg) = F(fg) = F(1_a) = 1_{Fa}$, so Ff is an isomorphism, and we can conclude that $Fa \cong Fb$. □

We will now look at some examples of categories.

2.2 Examples of categories

The first example we will look at is the category **Set**.

Definition 2.2.1. We define the category **Set** as follows

- The objects of **Set** are sets
- The morphisms between two sets A, B are functions $A \to B$
- composition is function composition.
- The identity morphisms are the identity functions on each set.

- The unit and associativity laws can be proven pointwise by noting that $(f \circ \mathrm{id})(x) = f(\mathrm{id}(x)) = f(x) = \mathrm{id}(f(x)) = (\mathrm{id} \circ f)(x)$, and $(f \circ (g \circ h))(x) = f(g(h(x))) = ((f \circ g) \circ h)(x)$.

Another example of a category is the category of (abelian) groups, **Ab** and **Grp**, in which the objects are (abelian) groups and the morphisms are group homomorphisms.

We also have the category **Top** of topological spaces and continuous functions between them.

These are all examples in which the objects are sets with additional structure and the morphisms are a specific set of functions, but not all categories have to be of this form. Take for example the category **Mat**.

Definition 2.2.2 (**Mat**)**.** **Mat** is the category of matrices, defined as follows

- The objects Ob(**Mat**) are natural numbers
- The morphisms $n \to m$ are $m \times n$ matrices
- Composition is matrix multiplication
- Identity morphisms $n \to n$ are given by the $n \times n$ identity matrix

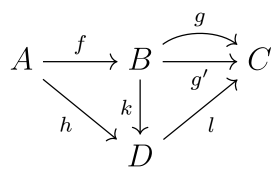
We can also construct categories from already existing categories, for example the opposite category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$.

Definition 2.2.3 (Opposite category)**.** Let $\mathcal{C}$ be a category, we define the opposite category as

- The objects are the same as those of $\mathcal{C}$, $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$
- The morphisms are those of $\mathcal{C}$, but the other way around $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(a, b) = \mathrm{Hom}_{\mathcal{C}}(b, a)$
- Composition is also flipped: If $f^{\mathrm{op}} \colon a \to b$ and $g^{\mathrm{op}} \colon b \to c$, then $g^{\mathrm{op}} \circ_{\mathcal{C}^{\mathrm{op}}} f^{\mathrm{op}} = (f \circ_{\mathcal{C}} g)^{\mathrm{op}}$.
- The identity morphisms are the same as in $\mathcal{C}$. $1_{a^{\mathrm{op}}} = {1_a}^{\mathrm{op}}$.
- The associativity and unit laws follow directly from those of $\mathcal{C}$.

2.3 Diagrams

When reasoning about categories, working with equations can get really cumbersome, that's why category theorists will often draw diagrams and reason about them instead of the equations. For example the diagram below shows us a few morphisms between objects in some category.



In such a diagram compositions of morphisms can be thought of as paths through the diagram. For example the morphism lkf can be thought of as the path that goes from A to B to D and finally to C.

We will now go over the formal definition of a diagram that we will be using for the rest of this thesis.

Definition 2.3.1 (Graphs)**.** A graph $G = (V, E)$ is a collection V of vertices and for every $u, v \in V$ a collection $E(u, v)$ of edges. An edge in $E(u, v)$ is said to go from u to v and will often be denoted by an arrow $u \to v$.

One important notion in graphs is that of paths, probably already have an intuition for the definition of a path, but we will now make the definition rigorous to avoid confusion.

Definition 2.3.2 (Paths in a graph)**.** Let $G = (V, E)$ be a graph, we will now inductively define for every $u, v \in V$, a collection of paths $\mathrm{Path}(u, v)$.

We define for every $u \in V$ a path $() \in \mathrm{Path}(u, u)$ and for every $u, v, w \in V$, $p \in \mathrm{Path}(u, v)$ and $e \in E(v, w)$, a path $(p, e) \in \mathrm{Path}(u, w)$.

From this we can also define concatenation of paths and a length function.

Definition 2.3.3 (Diagram in a category $\mathcal{C}$)**.** Let $\mathcal{C}$ be a category. A diagram in $\mathcal{C}$ is a graph $G = (V, E)$ together with a mapping $\mathcal{C} \colon V \to \mathrm{Ob}(\mathcal{C})$ and for every $u, v \in V$ a mapping $\mathcal{C} \colon E(u, v) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(u), \mathcal{C}(v))$

There also exists a mapping of paths into the category that is defined using recursion, sending the empty paths to the identity morphisms and using composition to add an edge to the path.

Now we can introduce an important notion for working in category theory, namely that of commutative diagrams. A commutative diagram is a diagram $D = (V, E)$ in a category $\mathcal{C}$, such that for every $u, v \in V$ and paths $p, q \in \mathrm{Path}(u, v)$ we have $\mathcal{C}(p) = \mathcal{C}(q)$. So in other words, every path between u, v maps to the same morphism in $\mathcal{C}$.

An important example of the use of commutative diagrams is for commutative squares.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[l]{} & D \end{array}$$

Saying that the previous diagram commutes is equivalent to saying that $h \circ f = l \circ g$.

2.4 Natural transformations

Definition 2.4.1 (Natural transformations)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\theta\colon F \Rightarrow G$ is a mapping from objects in $\mathcal{C}$ to arrows in $\mathcal{D}$, $\theta_x\colon Fx \to Gx$. This mapping also has to satisfy the naturality property: For all $x, y \in \mathcal{C}$ and $f\colon x \to y$, the diagram below has to commute.

$$\begin{array}{ccc} Fx & \xrightarrow{Ff} & Fy \\ {\scriptstyle \theta_x}\downarrow & & \downarrow{\scriptstyle \theta_y} \\ Gx & \xrightarrow[Gf]{} & Gy \end{array}$$

This definition tells us that $\theta_y \circ Ff = Gf \circ \theta_x$.

There are two ways we can compose natural transformations called vertical and horizontal composition. Vertical composition will be used when defining functor categories and horizontal composition will be used later in defining a monoidal structure on certain functor categories.

Definition 2.4.2 (Vertical composition)**.** Let $\mathcal{C}, \mathcal{D}$ be categories, $F, G, H\colon \mathcal{C} \to \mathcal{D}$ be functors and $\alpha\colon F \to G$ and $\beta\colon G \to H$ be natural transformations. Then we define their composition $\beta \circ \alpha\colon F \to H$ by $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$. This is a natural transformation because the following diagram commutes by pasting two naturality squares together.

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \\ {\scriptstyle \beta_x}\downarrow & & \downarrow{\scriptstyle \beta_y} \\ H(x) & \xrightarrow[H(f)]{} & H(y) \end{array}$$

Definition 2.4.3 (Horizontal composition)**.** Let $F, F' \colon \mathcal{C} \to \mathcal{D}$, $G, G' \colon \mathcal{B} \to \mathcal{C}$ be functors and $\alpha \colon F \to F'$, $\beta \colon G \to G'$ be natural transformations. Then we define the horizontal composition $\alpha * \beta \colon F \circ G \to F' \circ G'$ as follows

$$(\alpha * \beta)_x = \alpha_{G'(x)} \circ F(\beta_x).$$

This is well-defined because the following diagram commutes by naturality and functoriality.

$$\begin{CD}
F(G(x)) @>{F(\beta_x)}>> F(G'(x)) @>{\alpha_{G'(x)}}>> F'(G'(x)) \\
@V{F(G(f))}VV @VV{F(G'(f))}V @VV{F'(G'(f))}V \\
F(G(y)) @>>{F(\beta_y)}> F(G'(y)) @>>{\alpha_{G'(y)}}> F'(G'(y))
\end{CD}$$

Note that we could have just as well-defined $(\alpha * \beta)_x = F'(\beta_x) \circ \alpha_{G(x)}$, but this does not really matter because by functoriality of α we have that the following square commutes, so these two definitions are actually equal.

$$\begin{CD}
F(G(x)) @>{F(\beta_x)}>> F(G'(x)) \\
@V{\alpha_{G(x)}}VV @VV{\alpha_{G'(x)}}V \\
F'(G(x)) @>>{F'(\beta_x)}> F'(G'(x))
\end{CD}$$

We can now define the functor category for any two categories $\mathcal{C}, \mathcal{D}$.

Definition 2.4.4 (Functor category $\mathcal{D}^{\mathcal{C}}$)**.** The functor category $\mathcal{D}^{\mathcal{C}}$ is defined by the following

- The objects are functors $\mathcal{C} \to \mathcal{D}$.
- The morphisms are natural transformation.
- Composition is given by vertical composition (definition 2.4.2).
- The identity morphisms are given by the identity natural transformations, which has components that are all the identity.

Lemma 2.4.5. *The functor category $\mathcal{D}^{\mathcal{C}}$ is a category.*

Proof. Note that for the unit laws we have $(\alpha \circ 1_G)_x = \alpha_x \circ 1_{G(x)} = \alpha_x$ and similarly $(1_F \circ \alpha)_x = 1_{F(x)} \circ \alpha_x = \alpha_x$.

For associativity, we see that

$$\begin{aligned}((\alpha \circ \beta) \circ \gamma)_x &= (\alpha \circ \beta)_x \circ \gamma_x \\ &= (\alpha_x \circ \beta_x) \circ \gamma_x \\ &= \alpha_x \circ (\beta_x \circ \gamma_x) \\ &= \alpha_x \circ (\beta \circ \gamma)_x \\ &= (\alpha \circ (\beta \circ \gamma))_x. \end{aligned}$$ □

2.5 Equivalence of categories

Now we will introduce the notion of equivalence of categories and its characterization. We will need this later on in the chapter on strictification to talk about monoidal equivalence.

Definition 2.5.1. Let $\mathcal{C}, \mathcal{D}$ be categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. We call F an equivalence of categories, or just an equivalence, iff there exists a $G \colon \mathcal{D} \to \mathcal{C}$, such that there exist natural isomorphisms $F \circ G \cong 1_{\mathcal{D}}$ and $G \circ F \cong 1_{\mathcal{C}}$

Now for the characterization we need to define three more properties, which together will all be equivalent to equivalence of categories.

Definition 2.5.2. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is called essentially surjective iff for every $y \in \mathcal{D}$ there exists an $x \in \mathcal{C}$, such that $F(x) \cong y$

Definition 2.5.3. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. We call F full if for every $x, y \in \mathcal{C}$, the mapping $F \colon \mathrm{Hom}(x, y) \to \mathrm{Hom}(F(x), F(y))$ is surjective. If this mapping is injective we call F faithful. If a functor is full and faithful we call it fully faithful.

Lemma 2.5.4. *A functor $F \colon \mathcal{C} \to \mathcal{D}$ is an equivalence iff F is fully faithful and essentially surjective.*

Proof. First we will prove the implication to the right. Assume $F \colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories. Then it has an inverse $G \colon \mathcal{D} \to \mathcal{C}$, such that $FG \overset{\alpha}{\cong} 1_{\mathcal{D}}$ and $GF \overset{\beta}{\cong} 1_{\mathcal{C}}$. Now we will show that F is essentially surjective. Let $Y \in \mathcal{D}$ be an object. Then by the equivalence property we have $F(G(Y)) \cong Y$, so we are done.

Now we will show that F is faithful. Let $X, Y \in \mathcal{C}$, $f, g \colon X \to Y$ and assume $F(f) = F(g)$, then we also have $G(F(f)) = G(F(g))$. Now by β being an isomorphism and the commutativity of the following diagram we get that $f = g$.

$$\begin{array}{ccc} G(F(X)) & \xrightarrow{\beta_X} & X \\ {\scriptstyle G(F(f))=G(F(g))}\big\downarrow & & \big\downarrow{\scriptstyle f=g} \\ G(F(Y)) & \xrightarrow[\beta_Y]{} & Y \end{array}$$

Now to show that F is full, let $X, Y \in \mathcal{C}$ and $f\colon F(X) \to F(Y)$. We want to show that there is a $g\colon X \to Y$ such that $f = F(g)$.

Let us assume we have such a g. Then we get $G(f) = G(F(g))$, so by the following commutative diagram we get that $g = \beta_Y \circ G(F(g)) \circ \beta_X^{-1}$.

$$\begin{array}{ccc} G(F(X)) & \xrightarrow{G(F(g))} & G(F(Y)) \\ {\scriptstyle \beta_X}\big\downarrow & & \big\downarrow{\scriptstyle \beta_Y} \\ X & \xrightarrow[g]{} & Y \end{array}$$

Now substituting in $G(f) = G(F(g))$ we get that $g = \beta_Y \circ G(f) \circ \beta_X^{-1}$, and we take it as our definition of g. Note that we now have

$$\beta_Y \circ G(f) \circ \beta_X^{-1} = \beta_Y \circ G(F(g)) \circ \beta_X^{-1}.$$

Canceling the isomorphisms we get $G(f) = G(F(g))$, so because G is also an equivalence, and we have already proven that equivalences are faithful, so we can conclude that $f = F(g)$.

Now we assume F is essentially surjective and fully-faithful. We will prove that F is an equivalence of categories assuming the axiom of choice. First we need to construct an inverse of G. Firstly since F is essentially surjective we can choose from every isomorphism class in $\mathcal{D}$ an object in the image of F, $F(X)$. Now for every $Y \in \mathcal{D}$ let $\alpha_Y\colon Y \to F(X_Y)$ be an isomorphism, such that α is a natural isomorphism. We now define G on objects as $G(Y) = X_Y$ and on morphisms $f\colon Y \to Z$ we use the natural isomorphism to bring it to $f'\colon F(X_Y) \to F(X_Z)$ and use the fact that F is fully-faithful to give is the pre-image defined to be $G(f)$. Now we need to show that we have the naturalities necessitated for an equivalence of categories. Note that by definition we have $\alpha\colon F \circ G \to 1$, so we only need to construct a $\beta\colon G \circ F \to 1$. Let $Y \in \mathcal{C}$, then by definition we have $F(Y) \cong F(X_{F(Y)})$, since F is fully faithful we have that $Y \cong X_{F(Y)} = G(F(Y))$. Now we again choose suitable isomorphisms such that they form a natural isomorphism, and we are done. □

Chapter 3

Monoidal Categories

In this chapter we will first define and look at examples of monoidal categories. Then we will formulate and give a proof for the Mac Lane coherence theorem for monoidal categories. In doing this we will closely follow the proof as Mac Lane wrote in his book *Categories for the working mathematician*[4], with some extra details in parts where Mac Lane skipped over them. But first we need to define what a monoidal category is.

3.1 Monoidal Categories

Definition 3.1.1. A monoidal category $\mathcal{M}$ is a category, together with

- A functor $- \otimes - \colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$
- An object $e \in \mathrm{Ob}(\mathcal{M})$
- A natural isomorphism $\alpha_{X,Y,Z} \colon X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ called the associator.
- A natural isomorphism $\lambda_X \colon e \otimes X \to X$ called the left unit law or left unitor
- And a natural isomorphism $\rho_X \colon X \otimes e \to X$ called the left unit law or right unitor

Such that $\rho_e = \lambda_e \colon e \otimes e \to e$ and the pentagon diagram and triangular diagram pictured below commute for all $W, X, Y, Z \in \mathrm{Ob}(\mathcal{M})$.

$$\begin{array}{ccccc}
W \otimes (X \otimes (Y \otimes Z)) & \xrightarrow{\alpha_{W,X,Y\otimes Z}} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{\alpha_{W\otimes X,Y,Z}} & ((W \otimes X) \otimes Y) \otimes Z \\
{\scriptstyle 1\otimes\alpha_{X,Y,Z}}\downarrow & & & & \uparrow{\scriptstyle \alpha_{W,X,Y}\otimes 1} \\
W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow[\alpha_{W,X\otimes Y,Z}]{\qquad\qquad\qquad} & & (W \otimes (X \otimes Y)) \otimes Z
\end{array}$$

$$\begin{array}{ccc}
X \otimes (e \otimes Z) & \xrightarrow{\alpha_{X,e,Z}} & (X \otimes e) \otimes Z \\
& {\scriptstyle 1\otimes\lambda}\searrow & \downarrow{\scriptstyle \rho\otimes 1} \\
& & X \otimes Z
\end{array}$$

An example of a monoidal category is the category **Ab** together with the tensor product or **Set** with the Cartesian product.

We can also create a monoidal category out of another monoidal category, which we call the transpose monoidal category.

Definition 3.1.2. Let $\mathcal{M}$ be a monoidal category, the transpose $\mathcal{M}^\top$ is defined by

- The same underlying category $\mathcal{M}$
- The same unit object e
- A flipped tensor product $X \otimes_{\mathcal{M}^\top} Y = Y \otimes_{\mathcal{M}} X$

The natural isomorphism are defined as follows. $\rho^\top_X \colon X \otimes_{\mathcal{M}^\top} e \to X$ has to be a morphism $e \otimes_{\mathcal{M}} X \to X$, so we can choose $\rho^\top_X = \lambda_X$. Similarly, we can define $\lambda^\top_X = \rho_X$. For the associator, we need a morphism $\alpha^\top_{X,Y,Z} \colon X \otimes_{\mathcal{M}^\top} (Y \otimes_{\mathcal{M}^\top} Z) \to (X \otimes_{\mathcal{M}^\top} Y) \otimes_{\mathcal{M}^\top} Z$, or if we write out the definition $\alpha^\top_{X,Y,Z} \colon (Z \otimes_{\mathcal{M}} Y) \otimes_{\mathcal{M}} X \to Z \otimes_{\mathcal{M}} (Y \otimes_{\mathcal{M}} X)$, so we can choose $\alpha^\top_{X,Y,Z} = \alpha^{-1}_{Z,Y,X}$.

A monoidal category is called strict if its natural isomorphisms are the identity.

We will now look at a few examples of strict monoidal categories. The first example is the endofunctor category $\mathcal{C}^{\mathcal{C}}$ for any category $\mathcal{C}$ as defined in definition 2.4.4, together with functor composition and horizontal composition (definition 2.4.3) as the tensor product and the identity functor as the unit.

Lemma 3.1.3. *The endofunctor category $\mathcal{C}^{\mathcal{C}}$, together with functor composition and the identity functor, is a strict monoidal category.*

Proof. Note that composition of functors is associative because we have for all $X \in \mathcal{C}$ that

$$\begin{aligned}(F \circ (G \circ H))(X) &= F((G \circ H)(X)) \\ &= F(G(H(X))) \\ &= (F \circ G)(H(X)) \\ &= ((F \circ G) \circ H)(X).\end{aligned}$$

The same holds for morphisms, so composition of functors is associative.

Now we need to check that horizontal composition is associative. Let $\beta\colon F \to F'$, $\gamma\colon G \to G'$ and $\delta\colon H \to H'$ be natural transformations, then

$$\begin{aligned}(\beta * (\gamma * \delta))_X &= \beta_{G'(H'(X))} \circ F((\gamma * \delta)_X) \\ &= \beta_{G'(H'(X))} \circ F(\gamma_{H'(X)} \circ G(\delta_X)) \\ &= \beta_{G'(H'(X))} \circ (F(\gamma_{H'(X)}) \circ F(G(\delta_X))) \\ &= (\beta_{G'(H'(X))} \circ F(\gamma_{H'(X)})) \circ F(G(\delta_X)) \\ &= (\beta * \gamma)_{H'(X)} \circ (F \circ G)(\delta_X) \\ &= ((\beta * \gamma) * \delta)_X.\end{aligned}$$

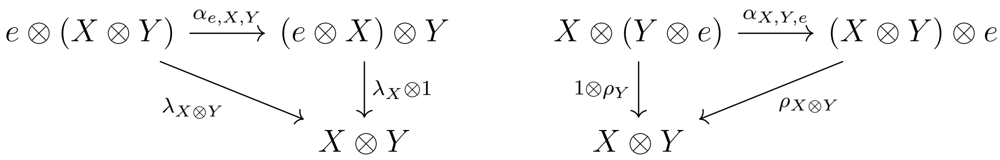
So now we have done associativity of the tensor product, and we go on to the unit laws. We have that $(1 \circ F)(X) = 1(F(X)) = F(X) = F(1(X)) = (F \circ 1)(X)$. Now for the horizontal composition, note that $(1*\gamma)_X = 1 \circ 1(\gamma_X) = \gamma_X = \gamma_{1(X)} \circ F(1) = (\gamma * 1)_X$. □

Another example of a strict monoidal category is the category of matrices **Mat** as defined in definition 2.2.2, where the tensor product is defined as follows

- On objects $n \otimes m = n + m$
- On morphisms $A \otimes B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

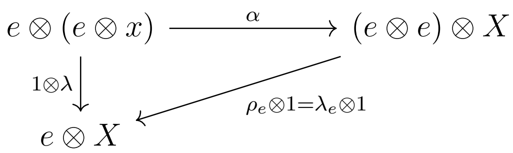
The nice thing about strict monoidal categories and other structures like groups and monoids, is that they let us forget about exactly how we are rearranging the brackets and when we are removing the units. We would now like this to be the same for general monoidal categories. *We want all diagrams consisting only of $\alpha, \lambda, \rho, 1, - \otimes -$ to commute.* This way there is only one way to go between two words. In the next section we will state this theorem in a more rigorous way, but first we will prove that a specific diagram commutes. We will need this lemma when proving the main theorem.

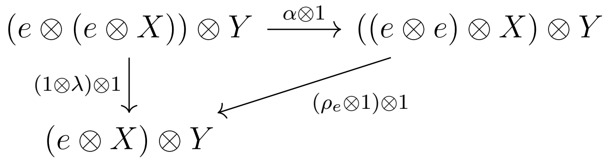
Lemma 3.1.4. *The following diagrams commute*



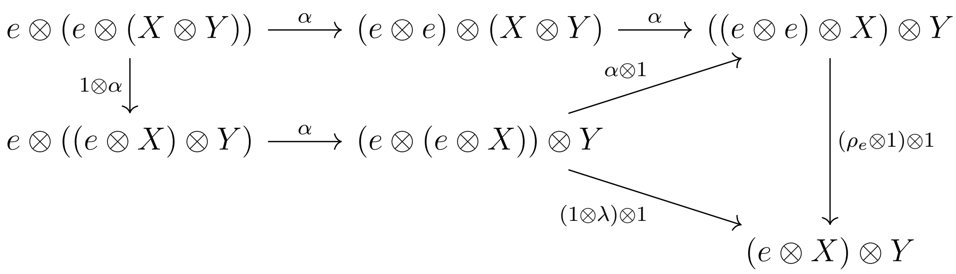
Proof. We will first prove that the diagram on the left commutes and then use a duality argument to show that the one on the right also commutes.

To prove that the diagram commutes we will be pasting together other diagrams we know to commute. We know that in the diagrams pictured below, the diagram on the bottom commutes because applying the functor $- \otimes Y$ on the diagram on the triangle diagram on top conserves commutativity of diagrams.

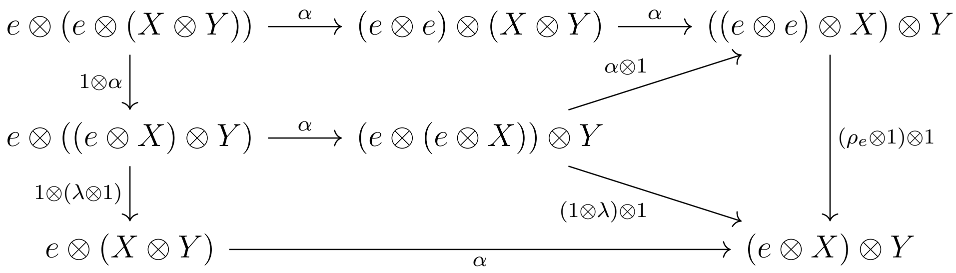




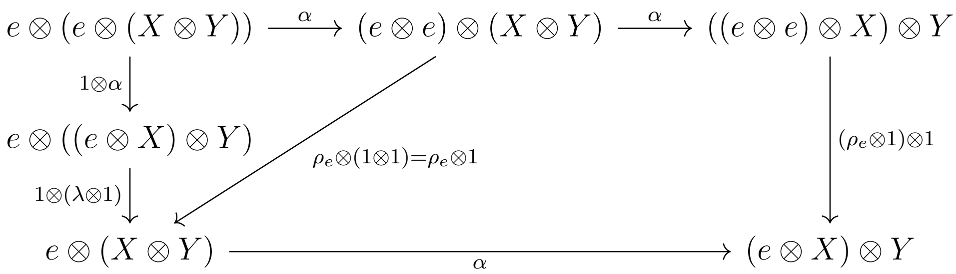
Now we can expand this diagram using a copy of the pentagon diagram, to get the following



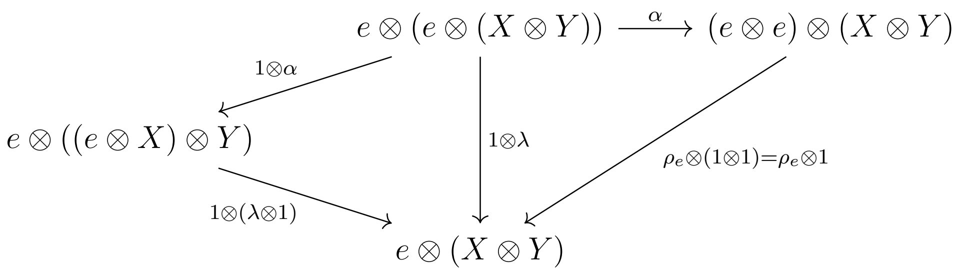
We will now add a naturality square along the bottom



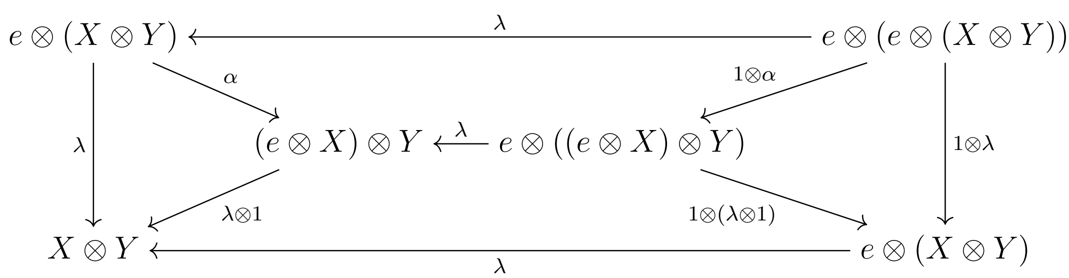
and add α naturality for the arrow $(\rho_e \otimes 1) \otimes 1$. The diagram will get a bit messy, so we will leave out the central vertex because it is not needed anymore. Later we will also remove the vertices on the right for the same reason. Note also that $1 \otimes 1 = 1$ by functoriality of $\otimes$.



Now we add another triangle diagram in the center along the top α edge and the ρ_e edge. We will also remove unneeded vertices on the right.



Now by naturality of λ we get the diagram below. We can find the diagram we wanted to show commutes on the left, and because the entire diagram commutes and all arrows are isomorphisms we get that this triangle commutes by itself.



Now we will prove that the other diagram commutes by duality. Note that the diagram we just proved to commute also commutes in the transposed monoidal category $\mathcal{M}^{\top}$, so from this we get that the second diagram we wanted to commute also commutes. $\square$

3.2 Coherence theorems

To state Mac Lane's coherence theorem we first need to define the free monoidal category on one generator.

Definition 3.2.1. We define the free monoidal category on one generator $\mathcal{W}$ as follows.

We define the objects, or words, $\mathrm{Ob}(\mathcal{W})$ inductively

- $e, (-) \in \mathrm{Ob}(\mathcal{W})$
- if $a, b \in \mathrm{Ob}(\mathcal{W})$ then $(a \otimes b) \in \mathrm{Ob}(\mathcal{W})$

We define the length $\ell(w)$ of a word $w \in \mathrm{Ob}(\mathcal{W})$ inductively by

- $\ell(e) = 0$
- $\ell(-) = 1$
- $\ell(v \otimes w) = \ell(v) + \ell(w)$

Now we define $\mathrm{Hom}(v, w)$ to have a unique element iff $\ell(v) = \ell(w)$.

For example the length of the word $(- \otimes e) \otimes -$ is given by $\ell((- \otimes e) \otimes -) = \ell(- \otimes e) + \ell(-) = \ell(-) + \ell(e) + \ell(-) = 1 + 0 + 1 = 2$. In general, we can find out the length of a word by counting the occurrences of $-$.

Definition 3.2.2. A strict morphism of monoidal categories or strict morphism $F \colon \mathcal{M} \to \mathcal{M}'$ is a functor with the property that $F(e) = e$ and $F(X \otimes Y) = F(X) \otimes F(Y)$.

Theorem 3.2.3 (Mac Lane coherence theorem)**.** *For every monoidal category $\mathcal{M}$ and object $b \in \mathrm{Ob}(\mathcal{M})$ there exists a unique strict morphism of monoidal categories $F \colon \mathcal{W} \to \mathcal{M}$, such that $(-) \mapsto b$.*

The functor F can be thought of as the functor that replaces every $(-)$ with an occurrence of b.

Proof. We will at first define F on the objects by induction, and then show that there is exactly one way we can define F on the morphisms.

We define $F(e) = e$, $F(-) = b$ and $F(v \otimes w) = F(v) \otimes F(w)$. For objects $w \in \mathcal{W}$ we will also denote this mapping by $w \mapsto w_b$.

Now let $n \in \mathbb{N}$, we will define a diagram G_n in $\mathcal{M}$, with as vertices, the words $w \in \mathrm{Ob}(\mathcal{W})$ that don't contain e and which have $\ell(w) = n$. The edges of this diagram are the "basic" arrows, which are defined as follows.

Basic arrows are defined inductively and are either directed or anti-directed. All associativity arrows α are basic and directed and their inverses α^{-1} are basic and anti-directed. If a is basic and (anti-)directed, then $a \otimes 1$ and $1 \otimes a$ are also basic and (anti-)directed. To summarize, the basic arrows are those arrows that have exactly one α or α^{-1} tensored with the identity an arbitrary number of times. The directed arrows are the ones that contain an α and the anti-directed arrows are the ones that contain an α^{-1}.

Paths in G_n from v to w correspond to morphisms $v_b \to w_b$. By showing that G_n commutes we show that for every $v, w \in \mathrm{Ob}(\mathcal{W})$, all arrows that are compositions of basic arrows are equal.

Lemma 3.2.4. *For every $n \in \mathbb{N}$, the diagram G_n commutes.*

Proof. To show that G_n commutes we will first construct a canonical element, then we will define a canonical path to this canonical element, which allows us to define a path between any two elements. We can then prove that any other path is equal to this canonical path.

For the canonical node w^n we choose the one with all the brackets on the left. So recursively we define the canonical node $w^{(1)}$ of G_1 to be $(-)$ and then for G_{n+1}, we define $w^{(n+1)} = (w^{(n)}) \otimes (-)$. For example $w^{(4)}$ is given by $w^{(4)} = (((-) \otimes (-)) \otimes (-)) \otimes (-)$.

Now to define the canonical paths we first need to define a function $r \colon \mathcal{W} \to \mathbb{N}$ called the rank by:

$$r(e) = r(-) = 0$$
$$r(v \otimes w) = r(v) + r(w) + \ell(w) - 1.$$

Lemma 3.2.5. *r has the following properties:*

1. *For all $w \in G_n$, $r(w) = 0$ iff $w = w^{(n)}$.*
2. *If there is a directed basic arrow $a \colon v \to w$, then $r(w) < r(v)$.*

Proof. We will first prove 1. We can show the $\Longleftarrow$ direction by a simple computation. By induction on n we have $r(w^{(1)}) = r(-) = 0$ and

$$\begin{aligned} r(w^{(n+1)}) &= r(w^{(n)} \otimes (-)) \\ &= r(w^{(n)}) + r(-) + \ell(-) - 1 \\ &= 0 + 0 + 1 - 1 \\ &= 0. \end{aligned}$$

Now for the $\implies$ direction, by induction. If $n = 1$, then $G_1 = \{(-)\}$, so we automatically have $w = (-)$ for all $w \in G_1$. Now assume the statement holds for n, then for all $w \in G_{n+1}$ with $r(w) = 0$, we see that $r(w) = r(w_1 \otimes w_2) = r(w_1) + r(w_2) + \ell(w_2) - 1$. Since $\ell(w_2) \geq 1$, we must have $\ell(w_2) = 1$ for $r(w) = 0$ to be possible. Since G_n only contains words without e, we get $w_2 = (-)$ and so $\ell(w_1) = n$. Now by the induction hypothesis we get $w_1 = w^{(n)}$ and so $w = w^{(n)} \otimes (-) = w^{(n+1)}$.

Now we will prove the second statement by induction on directed basic arrows. If $a = \alpha$, we have $v = x \otimes (y \otimes z)$ and $w = (x \otimes y) \otimes z$. We now see that

$$\begin{aligned} r(v) &= r(x \otimes (y \otimes z)) \\ &= r(x) + r(y \otimes z) + \ell(y \otimes z) - 1 \\ &= r(x) + r(y) + r(z) + \ell(z) - 1 + \ell(y) + \ell(z) - 1 \\ &= r(x) + r(y) + r(z) + \ell(y) + 2\ell(z) - 2 \end{aligned}$$

and

$$\begin{aligned} r(w) &= r((x \otimes y) \otimes z) \\ &= r(x \otimes y) + r(z) - \ell(z) - 1 \\ &= r(x) + r(y) + \ell(y) - 1 + r(z) + \ell(z) - 1 \\ &= r(x) + r(y) + r(z) + \ell(y) + \ell(z) - 2. \end{aligned}$$

We see that $r(v) < r(w)$ because $\ell(z) > 0$.

Now assume the statement holds for the arrow $a \colon x \to y$, then $a \otimes 1 \colon x \otimes z \to y \otimes z$ and we see

$$\begin{aligned} r(x \otimes z) &= r(x) + r(z) + \ell(z) - 1 \\ &> r(y) + r(z) + \ell(z) - 1 \\ &= r(y \otimes z) \end{aligned}$$

and similarly for $1 \otimes a\colon z \otimes x \to z \otimes y$, because $\ell(x) = \ell(y)$, we see

$$\begin{aligned} r(z \otimes x) &= r(z) + r(x) + \ell(x) - 1 \\ &= r(z) + r(x) + \ell(y) - 1 \\ &> r(z) + r(y) + \ell(y) - 1 \\ &= r(z \otimes y) \end{aligned}$$

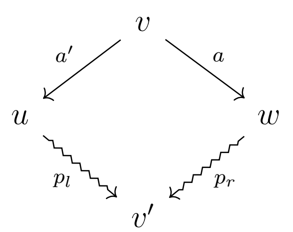
□

We can now define for every $w \in G_n$ a (directed) path $p_w\colon w \to w^{(n)}$, by induction on $r(w)$. If $r(w) = 0$, then $w = w^{(n)}$, so we can take the trivial path. Now assume we have paths for every word $v \in G_n$ with $r(v) < n$ and assume we have a $w \in G_n$ with $r(w) = n$. Since $n \neq 0$ we know that $w \neq w^{(n)}$, so the right side of one of the products in w needs to have another product in it. We can then choose the first from the outer layers that this applies to and start our path with an a on this level. Then by induction we can finish the path because after following a directed edge the rank decreases.

Now since all directed arrows have an anti-directed "inverse" we get a canonical path between any two words $v, w \in G_n$ by first following p_v to $w^{(n)}$ and then following p_w^{-1} to w.

We will now show by induction on the rank and the length of paths that all paths between $v, w \in G_n$ correspond to the same morphism in $\mathcal{M}$, by showing they correspond to the one of the canonical path $p_w^{-1} p_v$. If the length of the path is 0 then $v = w$ and $\mathcal{W}(p_w^{-1} p_v) = \mathcal{W}(p_v^{-1} p_v) = \mathcal{W}(p_v^{-1})\mathcal{W}(p_v) = \mathcal{W}(p_v)^{-1}\mathcal{W}(p_v) = 1 = \mathcal{W}(())$.

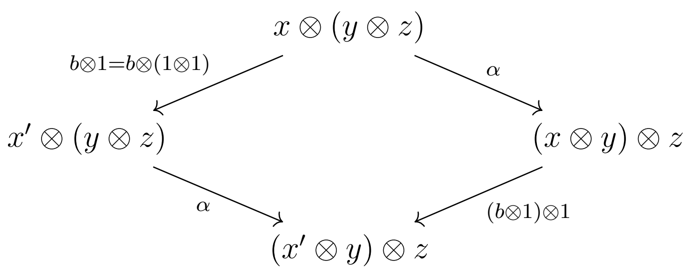
Now assume the statement holds for n in the case that the path has length $n+1$, we see that $p = (a)q$, where q is already shown to be equivalent to the canonical path and $a\colon v \to w$ is either directed or anti-directed, without loss of generality we can however assume a is directed because we can otherwise take the inverse arrow $a^{-1}\colon w \to v$. Now because of the induction on the rank it suffices to construct a commutative 'diamond' diagram with only directed edges as in the diagram below. Note that the squiggly edges are paths instead of only single edges. We now look at the first edge $a'\colon v \to u$ of p_v.



We will now show by cases that we can always construct such a diamond. In the case that $a = a'$, we get that $u = w$, so we can finish the diamond with $v' = u = w$ and $p_l = p_r = ()$. From now on we can assume $a \neq a'$. By induction and without loss of generality it now suffices to look at the following cases.

1. $a = 1 \otimes b$ and $a' = c \otimes 1$
2. $a = 1 \otimes b$ and $a' = 1 \otimes c$
3. $a = b \otimes 1$ and $a' = c \otimes 1$
4. $a = \alpha$ and $a' = b \otimes 1$
5. $a = \alpha$ and $a' = 1 \otimes b$

In the first case we can take $p_l = (1 \otimes b)$ and $p_r = (c \otimes 1)$ because $\mathcal{W}(1 \otimes b)\mathcal{W}(c \otimes 1) = \mathcal{W}(c) \otimes \mathcal{W}(b) = \mathcal{W}(c \otimes 1)\mathcal{W}(1 \otimes b)$ and in the second and third case we can use induction on n.



In the fourth case we can use the naturality of α as is shown the diagram above.

Now we only need to tackle the fifth case, here we again use induction on b. In the cases that $b = 1 \otimes c$ or $b = c \otimes 1$ we can again use naturality. In the case that $b = \alpha$ we can use the pentagon identity from the axioms of monoidal categories definition 3.1.1.

We have now shown that all paths in G_n are equivalent, so G_n commutes. □

For every $n \in \mathbb{N}$, we will now define an extension G'_n of the diagram G_n that also contains words that contain the unit element e. This part of the proof was originally left out by Mac Lane. The basic arrows in this diagram are constructed as follows

- λ, ρ are directed arrows.
- λ^{-1}, ρ^{-1} are anti-directed arrows.
- α, α^{-1} are 'neutral' arrows.

- If a is a (directed/anti-directed/neutral) arrow, then so are $1 \otimes a$ and $a \otimes 1$.

Lemma 3.2.6. *The diagram G'_n commutes.*

Proof. First we will define a new function $\ell_e \colon \mathrm{Ob}(\mathcal{W}) \to \mathbb{N}$ defined similarly to ℓ but counting the number of occurrences of e instead of $(-)$.

We will now prove some properties of the ℓ_e function.

Lemma 3.2.7. *The following properties hold*

1. *If $w \in G'_n$ then $\ell_e(w) = 0$ iff $w \in G_n$.*
2. *If $a \colon v \to w$ is a directed edge, then $\ell_e(v) > \ell_e(w)$.*
3. *If $a \colon v \to w$ is a neutral arrow, $\ell_e(v) = \ell_e(w)$.*
4. *If $w \in G'_n$ and $\ell_e(w) \geq 1$, then there exists a directed arrow $a \colon w \to v$ in G'_n.*

Proof. Statement 1 is just the definition of G_n, because $\ell_e(w) = 0$ precisely means that w has no identities inside.

Now we will prove statement 2. We will do induction on a. If $a = \rho$ or $a = \lambda$, then $v = w \otimes e$ or $v = e \otimes w$ respectively. This means that in both cases $\ell_e(v) = \ell_e(w) + 1 > \ell_e(w)$.

Now assume the statement holds for directed arrow $b \colon v \to w$ and $a = 1 \otimes b \colon x \otimes v \to x \otimes w$ or $a = b \otimes 1 \colon v \otimes x \to w \otimes x$, then $\ell_e(x \otimes v) = \ell_e(x) + \ell_e(v) > \ell(e) + \ell(w) = \ell_e(x \otimes v)$ and similarly for the other case. Statement 2 has now been proven.

We will now prove statement 3. By induction on neutral arrows, for the base case we can without loss of generality assume $a = \alpha$. Then $v = x \otimes (y \otimes z)$ and $w = (x \otimes y) \otimes z$. We see that $\ell_e(v) = \ell_e(x) + \ell_e(y) + \ell_e(z) = \ell_e(w)$. Now assume the statement holds for a neutral arrow $b \colon v \to w$, then $a = 1 \otimes b \colon x \otimes v \to x \otimes w$ and we see that $\ell_e(x \otimes v) = \ell_e(x) \otimes \ell_e(v) = \ell_e(x) \otimes \ell_e(w) = \ell_e(x \otimes w)$. A similar argument can be made for the other inductive case.

Now we will prove the 4th statement. We will prove this by induction on w. Note that we are only looking at words that have $\ell(w) > 0$, so we don't have to consider the case of e. We also don't have to consider $(-)$ because $\ell_e(-) = 0$. Now we look at the case that $v = e \otimes w$ or $v = w \otimes e$, then we can take the $a = \lambda$ or $a = \rho$ respectively. Now the last case $v = w \otimes w'$ with $\ell_e(v) > 0$, then we have that $\ell_e(w) + \ell_e(w') > 0$, so $\ell_e(w) > 0$ or $\ell_e(w') > 0$, in the first case we can use the induction hypothesis on w to get an arrow $a \colon w \to w_0$, so we can use the arrow $a \otimes 1$. It works similarly in the second case. □

For every node $w \in G'_n$ we can now define a canonical path to a node $w' \in G_n$ by induction on $\ell_e(w)$ using the fourth statement of lemma 3.2.7. We can again use these paths to create paths between any two arbitrary nodes $v, w \in G'_n$, by going from v to a node $v' \in G_n$ then to $w' \in G_n$ (using lemma 3.2.4) and then back to w.

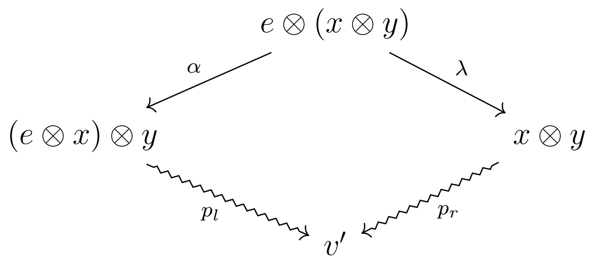
By using the same methods as the proof of lemma 3.2.4 it suffices for us to construct diamond diagrams where one of the top arrows is a directed arrow and the other is a directed or α arrow. So following the conventions of the diamond diagrams, we take a' to be the directed edge and a to be the directed or neutral arrow. We can also assume $a \neq a'$.

If $a = \lambda$, then $v = e \otimes w$. Now we have the following options for a'

1. $a' = \rho$
2. $a' = \alpha$
3. $a' = 1 \otimes b$

In the first case note that $w = u = e$, and by the axioms of monoidal categories definition 3.1.1 we have that $\rho_e = \lambda_e$, so we can take $p_l = p_r = ()$

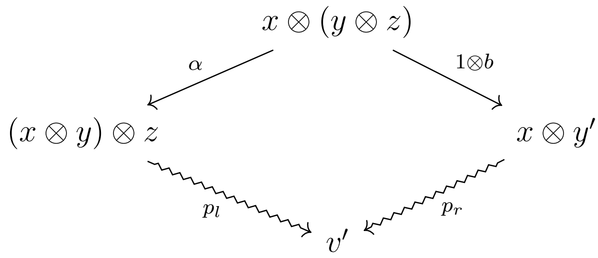
In the second case we have to fill in the diagram below. Which we can fill in with $p_l = \lambda \otimes 1$ and $p_r = ()$ by lemma 3.1.4.



In the third case, the diagram we need to construct is just the naturality diagram of λ.

Now we look at the case where $a = \rho$, so $v = w \otimes e$. Since the case that $a' = \lambda$ is equivalent to the one we proved before we only have to look at the case where $a' = b \otimes 1$, in which we can again take the functoriality of ρ.

We now look at the case that $a = 1 \otimes b$, this gives us that $v = x \otimes y$, $w = x \otimes y'$ and $b\colon y \to y'$. In the case that $a' = 1 \otimes b'$ we can use our induction hypothesis. In the case that $a' = \rho$ or $a' = \lambda$ we can use the cases we proved earlier, so we only need to look at the case where $a' = \alpha$. Now we get that $v = x \otimes (y \otimes z)$, so our situation now looks as follows



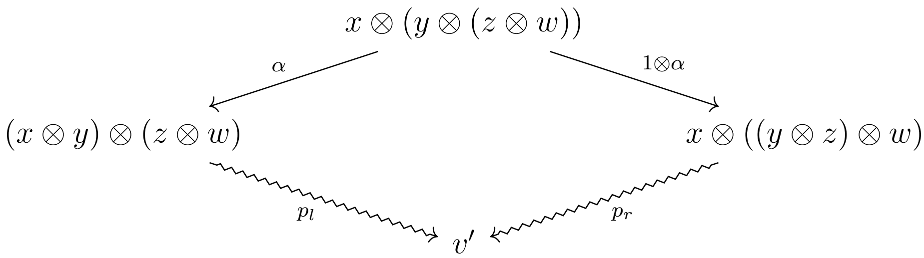
where we need to find paths p_l and p_r. We now distinguish 5 more cases

1. $b = \lambda$
2. $b = \rho$
3. $b = 1 \otimes c$
4. $b = c \otimes 1$
5. $b = \alpha$

In the first case $y = e$ and $y' = z$, so we can fill in $p_l = \rho \otimes 1$ and $p_r = ()$ to get the triangle diagram, which commutes by definition of monoidal categories.

In the second case we can use lemma 3.1.4, and in the 3rd and 4th cases we can use naturality of α.

This only leaves us the following diamond to complete

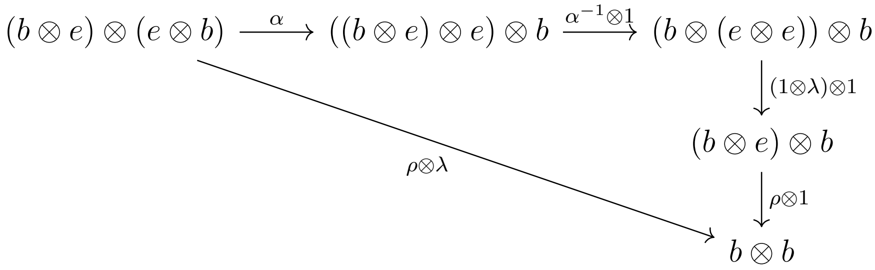


which we can fill in using the pentagon diagram (which commutes by definition) to finish our proof. $\square$

We have now shown that the diagram G'_n is connected and commutes in $\mathcal{M}$, so we can define F to work on morphisms by sending them to the unique morphisms defined by the basic arrows in G'_n. $\square$

3.2.1 Coherence for diagrams with multiple objects

We have now proven that diagrams like the one below commute.



But we can't yet prove that the same diagram with one b changed to a c also commutes. Mac Lane did find a nice trick to also prove this, by constructing a new monoidal category $\mathcal{N}$, which is defined as follows

- The objects are pairs (n, F) consisting of a natural number $n \in \mathbb{N}$ and a functor $F\colon \mathcal{M}^n \to \mathcal{M}$
- As morphisms $(n, F) \to (n', F)$ natural transformations $F \to F'$ (note that then $n = n'$).
- Composition and the identity morphisms are those of natural transformations.
- The tensor product is given by $(n, F) \otimes_{\mathcal{N}} (m, G) = (n + m, (x_0, x_1) \mapsto F(x_0) \otimes_{\mathcal{M}} G(x_1))$, where x_0 are the first n terms of the tuple and x_1 the remaining m.
- The unit object is given by $e_{\mathcal{N}} = (0, () \mapsto e_{\mathcal{M}})$

Note that the definition of the monoidal product makes use of a canonical isomorphism $\mathcal{M}^n \times \mathcal{M}^m \cong \mathcal{M}^{n+m}$, which can be defined by looking at the unique morphism between those object in the image of the Mac Lane functor described in theorem 3.2.3, when looking at the monoidal category **Cat** together with the Cartesian product.

The left and right unit laws and associator are defined by acting on the image of the functors. Then the axioms follow from those of $\mathcal{M}$.

Now we can apply theorem 3.2.3 on the object $(1, \mathrm{id})$ with the identity functor, to get that all diagrams consisting of natural transformations between tensored occurrences of id that are compositions of tensored occurrences of $1, \alpha, \lambda, \rho$ and their inverses commute. So we can conclude that all diagrams in $\mathcal{M}$ consisting of only those arrows commute.

Chapter 4

Strictification and a proof by Abstract Nonsense

In this chapter we will look at another proof of the coherence theorem via a more abstract, categorical method. We will adapt the proof for the 2-categorical Yoneda lemma in Johnson and Yau's book on 2-dimensional categories [3].

4.1 Lax functors and transformations

Definition 4.1.1 (Lax monoidal functor)**.** A lax monoidal functor between monoidal categories $(F, F^2, F^0)\colon \mathcal{M} \to \mathcal{M}'$ is a functor $F\colon \mathcal{M} \to \mathcal{M}'$, together with a natural transformation $F^2\colon (F-) \otimes (F-) \to F(- \otimes -)$, with components $F^2_{X,Y}\colon FX \otimes FY \to F(X \otimes Y)$ and an arrow $F^0\colon e \to Fe$, such that lax associativity and lax unit diagrams pictured below commute.

$$\begin{array}{ccc}
FX \otimes (FY \otimes FZ) & \xrightarrow{\alpha'} & (FX \otimes FY) \otimes FZ \\
{\scriptstyle 1\otimes F^2_{Y,Z}}\downarrow & & \downarrow{\scriptstyle F^2_{X,Y}\otimes 1} \\
FX \otimes F(Y \otimes Z) & & F(X \otimes Y) \otimes FZ \\
{\scriptstyle F^2_{X,Y\otimes Z}}\downarrow & & \downarrow{\scriptstyle F^2_{X\otimes Y,Z}} \\
F(X \otimes (Y \otimes Z)) & \xrightarrow[F\alpha]{} & F((X \otimes Y) \otimes Z)
\end{array}$$

$$\begin{array}{ccc}
e \otimes FX & \xrightarrow{\lambda} & FX \\
{\scriptstyle F^0\otimes 1}\big\downarrow & & \big\uparrow{\scriptstyle F\lambda} \\
Fe \otimes FX & \xrightarrow[F^2]{} & F(e\otimes X)
\end{array}
\qquad\qquad
\begin{array}{ccc}
FX \otimes e & \xrightarrow{\rho} & FX \\
{\scriptstyle 1\otimes F^0}\big\downarrow & & \big\uparrow{\scriptstyle F\rho} \\
FX \otimes Fe & \xrightarrow[F^2]{} & F(X\otimes e)
\end{array}$$

A pseudofunctor is a lax functor in which F^2 is a natural isomorphism and F^0 is also an isomorphism.

Definition 4.1.2 (Lax transformations)**.** Let $(F, F^2, F^0), (G, G^2, G^0)\colon \mathcal{M} \to \mathcal{M}'$ be lax monoidal functors. A lax (monoidal) transformation $\gamma\colon F \to G$ consists of

- An object $\gamma^0 \in \mathcal{M}'$
- A natural transformation $\gamma\colon (G-) \otimes \gamma^0 \to \gamma^0 \otimes (F-)$, with components $\gamma_X\colon GX \otimes \gamma^0 \to \gamma^0 \otimes FX$

Such that the lax unity diagram and the lax naturality diagram pictured below commute for all $X, Y \in \mathrm{Ob}(\mathcal{M})$.

$$\begin{array}{ccccc}
e\otimes\gamma^0 & \xrightarrow{\lambda} & \gamma^0 & \xrightarrow{\rho^{-1}} & \gamma^0\otimes e \\
{\scriptstyle G^0\otimes 1}\big\downarrow & & & & \big\downarrow{\scriptstyle 1\otimes F^0} \\
Ge\otimes\gamma^0 & & \xrightarrow[\gamma_e]{\qquad\qquad} & & \gamma^0\otimes Fe
\end{array}$$

$$\begin{array}{ccccc}
(GX\otimes GY)\otimes\gamma^0 & \xrightarrow{G^2\otimes 1} & G(X\otimes Y)\otimes\gamma^0 & \xrightarrow{\gamma_{X\otimes Y}} & \gamma^0\otimes F(X\otimes Y) \\
{\scriptstyle \alpha^{-1}}\big\downarrow & & & & \big\uparrow{\scriptstyle 1\otimes F^2} \\
GX\otimes(GY\otimes\gamma^0) & & & & \gamma^0\otimes(FX\otimes FY) \\
{\scriptstyle 1\otimes\gamma_Y}\big\downarrow & & & & \big\uparrow{\scriptstyle \alpha^{-1}} \\
GX\otimes(\gamma^0\otimes FY) & \xrightarrow[\alpha]{} & (GX\otimes\gamma^0)\otimes FY & \xrightarrow[\gamma_X\otimes 1]{} & (\gamma^0\otimes FX)\otimes FY
\end{array}$$

A strong transformation is a lax transformation in which γ is a natural isomorphism.

Definition 4.1.3 (Modifications)**.** Let $F, G\colon \mathcal{M} \to \mathcal{M}'$ be lax functors and let $\beta, \gamma\colon F \to G$ be lax transformations. A modification is a morphism $\Gamma\colon \beta^0 \to \gamma^0$, such that the modification diagram pictured below commutes for all $X \in \mathcal{M}$.

$$\begin{array}{ccc} GX \otimes \beta^0 & \xrightarrow{1\otimes\Gamma} & GX \otimes \gamma^0 \\ {\scriptstyle \beta_X}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_X} \\ \beta^0 \otimes FX & \xrightarrow[\Gamma\otimes 1]{} & \gamma^0 \otimes FX \end{array}$$

Definition 4.1.4 (Composition of strong transformations)**.** Let $\beta, \gamma, \delta \colon F \to G$ be strong transformations and let $\Gamma \colon \beta \to \gamma$ and $\Delta \colon \gamma \to \delta$ be modifications. We now need to define a modification $\Delta \circ \Gamma \colon \beta \to \delta$.

By the definition of modifications, we know that $\Gamma \colon \beta^0 \to \gamma^0$ and $\Delta \colon \gamma^0 \to \delta^0$, so we define the composition of modifications to be the composition in $\mathcal{M}'$.

Lemma 4.1.5. *The composition of modifications is well-defined and satisfies the unit laws and associativity law.*

Proof. We need to check that the following diagram commutes

$$\begin{array}{ccc} GX \otimes \beta^0 & \xrightarrow{1\otimes(\Gamma\circ\Delta)} & GX \otimes \delta^0 \\ {\scriptstyle \beta_X}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_X} \\ \beta^0 \otimes FX & \xrightarrow[(\Gamma\circ\Delta)\otimes 1]{} & \delta^0 \otimes FX \end{array}$$

We can see that it commutes because by functoriality of $- \otimes -$, we have that $1 \otimes (\Gamma \circ \Delta) = (1 \otimes \Gamma) \circ (1 \otimes \Delta)$, so by pasting the two modification diagrams together we prove that $\Gamma \circ \Delta$ is indeed a modification.

$$\begin{array}{ccccc} GX \otimes \beta^0 & \xrightarrow{1\otimes\Gamma} & GX \otimes \gamma^0 & \xrightarrow{1\otimes\Delta} & GX \otimes \delta^0 \\ {\scriptstyle \beta_X}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_X} & & \big\downarrow{\scriptstyle \delta_X} \\ \beta^0 \otimes FX & \xrightarrow[\Gamma\otimes 1]{} & \gamma^0 \otimes FX & \xrightarrow[\Delta\otimes 1]{} & \delta^0 \otimes FX \end{array}$$

The identity modification is the identity morphism in $\mathcal{M}'$, so because the composition is also the same as $\mathcal{M}'$ we get that the unit laws hold. Associativity also follows from associativity in $\mathcal{M}'$. □

Using the previous lemma we can define a category of strong transformations $\mathrm{Str}(F, G)$.

Definition 4.1.6. Let $F, G \colon \mathcal{M} \to \mathcal{M}'$ be pseudofunctors. We define the category of strong transformations $\mathrm{Str}(F, G)$ as follows

- The objects are strong transformations $F \to G$.
- The morphisms $\gamma \to \beta$ are modifications.
- Composition is given by composition of modifications.
- The identity modifications are defined to be the identity morphisms in $\mathcal{M}'$.

4.2 The Yoneda lemma for monoidal categories

We will define the Yoneda pseudofunctor $\mathcal{Y}\colon \mathcal{M} \to \mathcal{M}^{\mathcal{M}}$ as follows. Note that $\mathcal{M}^{\mathcal{M}}$ is the monoidal category defined in lemma 3.1.3, and it's monoidal structure solely depends on the underlying category of $\mathcal{M}$.

Definition 4.2.1 (The Yoneda pseudofunctor)**.** On objects $X \in M$ we define $\mathcal{Y}_X = X \otimes -$ and on morphisms $f\colon X \to Y$ we define $\mathcal{Y}_f = f \otimes -\colon (X \otimes - \to Y \otimes -)$ as a natural transformation with components $(\mathcal{Y}_f)_Z = f \otimes 1_Z$.

For the pseudofunctor we also need a natural transformation $\mathcal{Y}^2$, with components $\mathcal{Y}^2_{X,Y}\colon \mathcal{Y}_X\mathcal{Y}_Y \to \mathcal{Y}_{X\otimes Y}$. Note that $\mathcal{Y}_X\mathcal{Y}_Y = (X \otimes -) \circ (Y \otimes -) = X \otimes (Y \otimes -)$ and $\mathcal{Y}_{X\otimes Y} = (X \otimes Y) \otimes -$, so we can choose $\mathcal{Y}^2_{X,Y} = \alpha_{X,Y,-}$, which is a morphism in $\mathcal{M}^{\mathcal{M}}$ because α is a natural isomorphism.

We also need an arrow $\mathcal{Y}^0\colon \mathrm{id} \to \mathcal{Y}_e$. Note that $\mathcal{Y}_e = e \otimes -$, so we can choose $(\mathcal{Y}^0)_X = \ell_X^{-1}\colon X \to e \otimes X$.

To actually show that $(\mathcal{Y}, \mathcal{Y}^2, \mathcal{Y}^0)$ is a pseudofunctor we need to verify that a few diagrams commute.

Lemma 4.2.2. *$\mathcal{Y}$ is a pseudofunctor.*

Proof. We need to show that the following diagrams commute.

$$
\begin{array}{ccc}
\mathcal{Y}_X \circ (\mathcal{Y}_Y \circ \mathcal{Y}_Z) & \xrightarrow{\alpha'=\mathrm{id}} & (\mathcal{Y}_X \circ \mathcal{Y}_Y) \circ \mathcal{Y}_Z \\
{\scriptstyle 1*\mathcal{Y}^2_{Y,Z}}\downarrow & & \downarrow{\scriptstyle \mathcal{Y}^2_{X,Y}*1} \\
\mathcal{Y}_X \circ \mathcal{Y}_{Y\otimes Z} & & \mathcal{Y}_{X\otimes Y} \circ \mathcal{Y}_Z \\
{\scriptstyle \mathcal{Y}^2_{X,Y\otimes Z}}\downarrow & & \downarrow{\scriptstyle \mathcal{Y}^2_{X\otimes Y,Z}} \\
\mathcal{Y}_{X\otimes(Y\otimes Z)} & \xrightarrow[\mathcal{Y}_\alpha]{} & \mathcal{Y}_{(X\otimes Y)\otimes Z}
\end{array}
$$

We can see that this diagram commutes by evaluating the natural transformations and functors at some object $W \in \mathcal{M}$. This gives us the pentagon diagram which we know commutes.

$$\begin{array}{ccc}
X \otimes (Y \otimes (Z \otimes W)) & = & X \otimes (Y \otimes (Z \otimes W)) \\
\downarrow{\scriptstyle 1\otimes\alpha} & & \downarrow{\scriptstyle \alpha} \\
X \otimes ((Y \otimes Z) \otimes W) & & (X \otimes Y) \otimes (Z \otimes W) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
(X \otimes (Y \otimes Z)) \otimes W & \xrightarrow[\alpha\otimes 1]{} & ((X \otimes Y) \otimes Z) \otimes W
\end{array}$$

The next diagrams we need to check are the left and right unit diagrams.

$$\begin{array}{ccc}
1 \circ \mathcal{Y}_X & \xrightarrow{\lambda'=\mathrm{id}} & \mathcal{Y}_X \\
\downarrow{\scriptstyle \mathcal{Y}^0 * 1} & & \uparrow{\scriptstyle \mathcal{Y}_\lambda} \\
\mathcal{Y}_e \circ \mathcal{Y}_X & \xrightarrow[\mathcal{Y}^2]{} & \mathcal{Y}_{e\otimes X}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{Y}_X \circ 1 & \xrightarrow{\rho'=\mathrm{id}} & \mathcal{Y}_X \\
\downarrow{\scriptstyle 1 * \mathcal{Y}^0} & & \uparrow{\scriptstyle \mathcal{Y}_\rho} \\
\mathcal{Y}_X \circ \mathcal{Y}_e & \xrightarrow[\mathcal{Y}^2]{} & \mathcal{Y}_{X\otimes e}
\end{array}$$

We can again see that these diagrams commute because if we evaluate them at some $Y \in \mathcal{M}$, we get the diagrams below. The one on the left we have proven to commute in lemma 3.1.4 and the one on the right commutes by the axioms of monoidal categories.

$$\begin{array}{ccc}
X \otimes Y & = & X \otimes Y \\
\downarrow{\scriptstyle \lambda^{-1}} & & \uparrow{\scriptstyle \lambda\otimes 1} \\
e \otimes (X \otimes Y) & \xrightarrow[\alpha]{} & (e \otimes X) \otimes Y
\end{array}
\qquad
\begin{array}{ccc}
X \otimes Y & = & X \otimes Y \\
\downarrow{\scriptstyle 1\otimes\lambda^{-1}} & & \uparrow{\scriptstyle \rho\otimes 1} \\
X \otimes (e \otimes Y) & \xrightarrow[\alpha]{} & (X \otimes e) \otimes Y
\end{array}$$

We have now proven that $\mathcal{Y}$ is indeed a pseudofunctor. □

By duality, we can also define a pseudofunctor $\mathcal{Y}_*\colon \mathcal{M}^\top \to \mathcal{M}^\mathcal{M}$, with $\mathcal{Y}_*(X) = - \otimes X$, because the underlying categories of $\mathcal{M}^\top$ and $\mathcal{M}$ are the same, so $(\mathcal{M}^\top)^{\mathcal{M}^\top} = \mathcal{M}^\mathcal{M}$.

We will need one final ingredient before we can formulate the Yoneda lemma, namely the evaluation functor.

Definition 4.2.3 (The evaluation functor)**.** Let $F\colon \mathcal{M}^\top \to \mathcal{M}^\mathcal{M}$ be a pseudofunctor, then the evaluation functor is defined by

$$e_\mathcal{M}\colon \mathrm{Str}(\mathcal{Y}_*, F) \to \mathcal{M}$$

- Sending strong transformations $\gamma\colon \mathcal{Y}_* \to F$ to $e_\mathcal{M}(\gamma) = \gamma^0(e)$

- For modifications $\Gamma\colon \gamma \to \delta$, we see that $\Gamma\colon \gamma^0 \to \delta^0$ is a natural transformation, so we choose $e_{\mathcal{M}}(\Gamma) = \Gamma_e$

Lemma 4.2.4. *$e_{\mathcal{M}}$ is a functor.*

Proof. We need to verify that $e_{\mathcal{M}}(1_\gamma) = 1_{e_{\mathcal{M}}(\gamma)}$ and $e_{\mathcal{M}}(\Gamma \circ \Delta) = e_{\mathcal{M}}(\Gamma) \circ e_{\mathcal{M}}(\Delta)$ for all strong transformations $\gamma \in \mathrm{Str}(\mathcal{Y}_*, F)$ and composable pairs of modifications Γ, Δ.

For the identity law, note that $e_{\mathcal{M}}(\gamma) = \gamma^0(e)$ and 1_γ is the identity natural transformation $\gamma^0 \to \gamma^0$, so $(1_\gamma)_e = 1_{\gamma^0(e)}$, so we are done because that are the definitions of the functor.

Now let $\Delta\colon \gamma \to \delta$ and $\Gamma\colon \beta \to \gamma$, then $e_{\mathcal{M}}(\Gamma \circ \Delta) = (\Gamma \circ \Delta)_e = \Gamma_e \circ \Delta_e$, so we are done. ☐

Now that we have constructed the Yoneda pseudofunctor, we can go on to formulate the special case of the Yoneda lemma for monoidal categories and then the strictification theorem.

Lemma 4.2.5 (Yoneda lemma for monoidal categories)**.** *The functor $e_{\mathcal{M}}\colon \mathrm{Str}(\mathcal{Y}_*, F) \to \mathcal{M}$ is an equivalence of categories.*

Proof. Suppose $\gamma \in \mathrm{Str}(\mathcal{Y}_*, F)$ is a strong transformation. Then we know it consists of the following data. An object $\gamma^0 \in \mathcal{M}^{\mathcal{M}}$ and a natural transformation $\gamma\colon (F-) \circ \gamma^0 \to \gamma^0 \circ (\mathcal{Y}_*-)$, such that for every $X \in \mathcal{M}$ we get a natural transformation $\gamma_X\colon FX \circ \gamma^0 \to \gamma^0 \circ \mathcal{Y}_X$, or in other words $\gamma_X\colon (FX)(\gamma^0(-)) \to \gamma^0(\mathcal{Y}_*(X)(-)) = \gamma^0(- \otimes X)$. We can evaluate this at $e \in \mathcal{M}$ to get an isomorphism

$$\gamma_{X;e}\colon (FX)(\gamma^0(e)) \to \gamma^0(e \otimes X)$$

Now assume $\beta, \gamma \in \mathrm{Str}(\mathcal{Y}_*, F)$ are strong transformations and $\Gamma\colon \beta \to \gamma$ is a modification. The modification axiom says that the following diagram commutes in $\mathcal{M}^{\mathcal{M}}$

$$\begin{array}{ccc} F_X \circ \beta^0 & \xrightarrow{1*\Gamma} & F_X \circ \gamma^0 \\ {\scriptstyle \beta_X}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_X} \\ \beta^0 \circ \mathcal{Y}_*(X) & \xrightarrow[\Gamma*1]{} & \gamma^0 \circ \mathcal{Y}_*(X) \end{array}$$

If we now evaluate this diagram in $e \in \mathcal{M}$ we get the following diagram

$$\begin{array}{ccc} F(X)(\beta^0 e) & \xrightarrow{F(X)(\Gamma_e)} & F(X)(\gamma^0 e) \\ {\scriptstyle \beta_{X;e}}\downarrow & & \downarrow{\scriptstyle \gamma_{X;e}} \\ \beta^0(e \otimes X) & \xrightarrow[\Gamma_{e\otimes X}]{} & \gamma^0(e \otimes X) \end{array}$$

Now since Γ is a natural transformation we get that the naturality diagram given by $\lambda_X \colon e \otimes X \to X$ commutes:

$$\begin{array}{ccc} \beta^0(e \otimes X) & \xrightarrow{\beta^0(\lambda_X)} & \beta^0(X) \\ {\scriptstyle \Gamma_{X\otimes e}}\downarrow & & \downarrow{\scriptstyle \Gamma_X} \\ \gamma^0(e \otimes X) & \xrightarrow[\gamma^0(\lambda_X)]{} & \gamma^0(X) \end{array}$$

Note that the horizontal arrows in this diagram are isomorphisms because ρ is a natural isomorphism.

If we put the previous two diagrams together we see that there is Γ_X is fully determined by Γ_e. Now since $e_{\mathcal{M}}(\Gamma) = \Gamma_e$ we can conclude that $e_{\mathcal{M}}$ is fully faithful.

Now we will show that $e_{\mathcal{M}}$ is essentially surjective. Let $X \in \mathcal{M}$ be an object. We will now define a strong transformation $\overline{X} \colon \mathcal{Y}_* \to F$. First we define $\overline{X}^0$

- For each $Y \in \mathcal{M}$ we define $\overline{X}^0(Y) = (FY)(X)$
- For every $f \colon Y \to Z$ in $\mathcal{M}$ we define $\overline{X}^0(f) = (Ff)_X$.

And now for each $Y \in \mathcal{M}$ we define a natural transformation $\overline{X}_Y \colon (FY) \circ \overline{X}^0 \to \overline{X}^0 \circ \mathcal{Y}_*(Y)$, with the following components for $Z \in \mathcal{M}$

$$(\overline{X}_Y)_Z \colon (FY)(FZ(X)) \to F(Z \otimes Y)(X)$$

This is equivalent to $(\overline{X}_Y)_Z \colon (FY \circ FZ)(X) \to F(Z \otimes Y)(X)$, so we can choose $(\overline{X}_Y)_Z = (F^2_{Y,Z})_X$.

Now we need to check the lax unity and lax naturality axioms. First we will look at the lax unity axiom. We need to show that the following diagram commutes.

$$\begin{array}{ccccc} 1 \circ \overline{X}^0 & \xrightarrow{\lambda=1} & \overline{X}^0 & \xrightarrow{\rho^{-1}=1} & \overline{X}^0 \circ 1 \\ {\scriptstyle F^0 * 1}\downarrow & & & & \downarrow{\scriptstyle 1 * \mathcal{Y}^0_*} \\ Fe \circ \overline{X}^0 & & \xrightarrow[\overline{X}_e]{} & & \overline{X}^0 \circ \mathcal{Y}_*(e) \end{array}$$

We can do this by looking at the components at $Y \in \mathcal{M}$ and writing out the definitions. First we look at the case for the objects, we get the diagram below

$$\begin{array}{ccccc}
(1 \circ \overline{X}^0)(Y) & \xrightarrow{\lambda=1} & \overline{X}^0(Y) & \xrightarrow{\rho^{-1}=1} & (\overline{X}^0 \circ 1)(Y) \\
{\scriptstyle (F^0 * 1)_Y}\downarrow & & & & \downarrow{\scriptstyle (1 * \mathcal{Y}^0_*)_Y} \\
(Fe \circ \overline{X}^0)(Y) & & \xrightarrow[(\overline{X}_e)_Y]{} & & (\overline{X}^0 \circ \mathcal{Y}_*(e))(Y)
\end{array}$$

If we now work out the definitions we get that we need to make the following diagram commute

$$\begin{array}{ccccc}
(FY)(X) & = & (FY)(X) & = & (FY)(X) \\
{\scriptstyle F^0}\downarrow & & & & \downarrow{\scriptstyle (F\rho_Y^{-1})_X} \\
(Fe)((FY)(X)) & & \xrightarrow[(F^2_{e,Y})_X]{} & & F(Y \otimes e)(X)
\end{array}$$

This diagram commutes because of the right unity axiom of pseudofunctors for F.

Now we take a look at the lax naturality axiom. We need to show that the following diagram commutes for all $Y, Z \in \mathcal{M}$

$$\begin{array}{ccccc}
(FY \circ FZ) \circ \overline{X}^0 & \xrightarrow{F^2_{Y,Z} * 1} & F(Z \otimes Y) \circ \overline{X}^0 & \xrightarrow{\overline{X}_{Z \otimes Y}} & \overline{X}^0 \circ \mathcal{Y}_*(Z \otimes Y) \\
{\scriptstyle \alpha^{-1}=1}\downarrow & & & & \uparrow{\scriptstyle 1 * \mathcal{Y}^2_{Y,Z}} \\
FY \circ (FZ \circ \overline{X}^0) & & & & \overline{X}^0 \circ (\mathcal{Y}_*(Y) \circ \mathcal{Y}_*(Z)) \\
{\scriptstyle 1 * \overline{X}_Z}\downarrow & & & & \uparrow{\scriptstyle \alpha^{-1}=1} \\
FY \circ (\overline{X}^0 \circ \mathcal{Y}_*(Z)) & \xrightarrow[\alpha=1]{} & (FY \circ \overline{X}^0) \circ \mathcal{Y}_*(Z) & \xrightarrow[\overline{X}_Y * 1]{} & (\overline{X}^0 \circ \mathcal{Y}_*(Y)) \circ \mathcal{Y}_*(Z)
\end{array}$$

We will do this by looking at the components at $W \in \mathcal{M}$. First we look at the case for objects. We will leave out the arrows that are the identity.

$$\begin{array}{ccccc}
(FY)((FZ)(\overline{X}^0(W))) & \xrightarrow{(F^2_{Y,Z})_{\overline{X}^0(W)}} & (F(Z \otimes Y))(\overline{X}^0(W)) & \xrightarrow{(\overline{X}_{Z \otimes Y})_W} & \overline{X}^0(\mathcal{Y}_*(Z \otimes Y)(W)) \\
{\scriptstyle (FY)((\overline{X}_Z)_W)}\downarrow & & & & \uparrow{\scriptstyle \overline{X}^0((\mathcal{Y}^2_{0,Y,Z})_W)} \\
(FY)(\overline{X}^0(\mathcal{Y}_*(Z)(W)) & & \xrightarrow[(\overline{X}_Y)_{\mathcal{Y}_*(Z)(W)}]{} & & \overline{X}^0(\mathcal{Y}_*(Y)(\mathcal{Y}_*(Z)(W)))
\end{array}$$

After working out the definitions we get the following diagram

$$\begin{array}{ccc}
(FY)((FZ)((FW)(X))) & \xrightarrow{(FY)((F^2_{Z,W})_X)} & (FY)((F(W \otimes Z)(X)) \\
{\scriptstyle (F^2_{Y,Z})_{(FW)(X)}}\Big\downarrow & & \\
(F(Z \otimes Y))((FW)(X)) & & \Big\downarrow{\scriptstyle (F^2_{Y,Z\otimes W})_X} \\
{\scriptstyle (F^2_{Y\otimes Z,W})_X}\Big\downarrow & & \\
(F(W \otimes (Z \otimes Y)))(X) & \xleftarrow[(F\alpha^{-1}_{Y,Z,W})(X)]{} & (F((W \otimes Z) \otimes Y))(X)
\end{array}$$

This diagram commutes because of the lax associativity law for F.

We have now proven that $\overline{X}$ is a strong transformation. Now we look at $e_{\mathcal{M}}(\overline{X})$

$$e_{\mathcal{M}}(\overline{X}) = \overline{X}^0(e) = (Fe)(X) \overset{F^0_X}{\cong} 1(X) = X.$$

So we see that $e_{\mathcal{M}}$ is essentially surjective. Now we can conclude that $e_{\mathcal{M}}$ is an equivalence of categories. □

Now we will take a look at the special case that $F = \mathcal{Y}_*$. Then strong transformations $\theta \in \mathrm{Str}(\mathcal{Y}_*, \mathcal{Y}_*)$ are endofunctors $\theta^0 \in \mathcal{M}^{\mathcal{M}}$, together with a natural transformation with components

$$\theta_X \colon \mathcal{Y}_*(X) \circ \theta^0 \to \theta^0 \circ \mathcal{Y}_*(X),$$

such that the lax transformation axioms hold. The component of this natural transformation at $Y \in \mathcal{M}$ is given by

$$(\theta_X)_Y \colon \theta^0(Y) \otimes X \to \theta^0(Y \otimes X).$$

For the morphisms in $\mathrm{Str}(\mathcal{Y}_*, \mathcal{Y}_*)$, we have modifications $\Gamma \colon \theta \to \gamma$, which are morphisms (so natural transformations) $\Gamma \colon \theta^0 \to \gamma^0$, such that the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{Y}_*(X) \circ \theta^0 & \xrightarrow{1*\Gamma} & \mathcal{Y}_*(X) \circ \gamma^0 \\
{\scriptstyle \theta_X}\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma_X} \\
\theta^0 \circ \mathcal{Y}_*(X) & \xrightarrow[\Gamma*1]{} & \gamma^0 \circ \mathcal{Y}_*(X)
\end{array}$$

Evaluating this diagram at its components at $Y \in \mathcal{M}$ gives us the following diagram.

$$\begin{array}{ccc}
\theta^0(Y) \otimes X & \xrightarrow{\Gamma_Y \otimes 1} & \gamma^0(Y) \otimes X \\
{\scriptstyle (\theta_X)_Y}\Big\downarrow & & \Big\downarrow{\scriptstyle (\gamma_X)_Y} \\
\theta^0(Y \otimes X) & \xrightarrow[\Gamma_{Y \otimes X}]{} & \gamma^0(Y \otimes X)
\end{array}$$

We see that $\mathrm{Str}(\mathcal{Y}_*, \mathcal{Y}_*)$ is actually just $\mathcal{M}^{\mathcal{M}}$ with some extra requirements

Lemma 4.2.6. *The image of* $\mathcal{Y}$ *lies inside* $\mathrm{Str}(\mathcal{Y}_*, \mathcal{Y}_*)$

Proof. We need to show that for all $X, X' \in \mathcal{M}$, $\mathcal{Y}_X$ is a strong transformation $\mathcal{Y}_* \to \mathcal{Y}_*$ and that for all $f \colon X \to X'$, $\mathcal{Y}_f$ is a modification $\mathcal{Y}_X \to \mathcal{Y}_{X'}$.

Let $X \in \mathcal{M}$. We need to construct a natural transformation $\theta_X \colon \mathcal{Y}_*(-) \circ \mathcal{Y}_X \to \mathcal{Y}_X \circ \mathcal{Y}_*(-)$, so we define it on indices $Y, Z \in \mathcal{M}$, which leaves us with the task to define the following

$$((\theta_X)_Y)_Z \colon \mathcal{Y}_*(Y)(\mathcal{Y}_X(Z)) \to \mathcal{Y}_X(\mathcal{Y}_*(Y)(Z)),$$

which is equivalent to

$$((\theta_X)_Y)_Z \colon (X \otimes Z) \otimes Y \to X \otimes (Z \otimes Y).$$

This means we can choose $((\theta_X)_Y)_Z = \alpha^{-1}_{X,Z,Y}$, which is natural in Y and Z (and X), so we are done.

Now we need to verify that the lax unity and lax naturality diagrams commute. We first look at the lax unity diagram.

$$\begin{array}{ccccc}
1 \circ \mathcal{Y}_X & \xrightarrow{\lambda=1} & \mathcal{Y}_X & \xrightarrow{\rho^{-1}=1} & \mathcal{Y}_X \circ 1 \\
\downarrow{\scriptstyle \mathcal{Y}^0_* * 1} & & & & \downarrow{\scriptstyle 1 * \mathcal{Y}^0_*} \\
\mathcal{Y}_*(e) \circ \mathcal{Y}_X & & \xrightarrow[(\theta_X)_e]{} & & \mathcal{Y}_X \circ \mathcal{Y}_*(e)
\end{array}$$

Evaluating this diagram at $Z \in \mathcal{M}$ we get the following

$$\begin{array}{ccccc}
X \otimes Z & = & X \otimes Z & = & X \otimes Z \\
\downarrow{\scriptstyle \rho^{-1}} & & & & \downarrow{\scriptstyle 1 \otimes \rho^{-1}} \\
(X \otimes Z) \otimes e & & \xrightarrow[\alpha^{-1}]{} & & X \otimes (Z \otimes e)
\end{array}$$

Which commutes by lemma 3.1.4.

Now we take a look at the lax naturality diagram

$$\begin{array}{ccccc}
(\mathcal{Y}_*(Y) \circ \mathcal{Y}_*(Z)) \circ \mathcal{Y}_X & \xrightarrow{\mathcal{Y}^2_* * 1} & \mathcal{Y}_*(Z \otimes Y) \circ \mathcal{Y}_X & \xrightarrow{(\theta_X)_{Y \otimes Z}} & \mathcal{Y}_X \circ \mathcal{Y}_*(Z \otimes Y) \\
\downarrow{\scriptstyle \alpha^{-1}=1} & & & & \uparrow{\scriptstyle 1 * \mathcal{Y}^2_*} \\
\mathcal{Y}_*(Y) \circ (\mathcal{Y}_*(Z) \circ \mathcal{Y}_X) & & & & \mathcal{Y}_X \circ (\mathcal{Y}_*(Y) \circ \mathcal{Y}_*(Z)) \\
\downarrow{\scriptstyle 1 * (\theta_X)_Z} & & & & \uparrow{\scriptstyle \alpha^{-1}=1} \\
\mathcal{Y}_*(Y) \circ (\mathcal{Y}_X \circ \mathcal{Y}_*(Z)) & \xrightarrow[\alpha=1]{} & (\mathcal{Y}_*(Y) \circ \mathcal{Y}_X) \circ \mathcal{Y}_*(Z) & \xrightarrow[(\theta_X)_Y * 1]{} & (\mathcal{Y}_X \circ \mathcal{Y}_*(Y)) \circ \mathcal{Y}_*(Z)
\end{array}$$

Evaluating at $W \in \mathcal{M}$ we get the following diagram

$$\begin{array}{ccccc}
((X \otimes W) \otimes Z) \otimes Y & \xrightarrow{\alpha^{-1}} & (X \otimes W) \otimes (Z \otimes Y) & \xrightarrow{\alpha^{-1}} & X \otimes (W \otimes (Z \otimes Y)) \\
\Vert & & & & \uparrow{\scriptstyle 1\otimes\alpha^{-1}} \\
((X \otimes W) \otimes Z) \otimes Y & & & & X \otimes ((W \otimes Z) \otimes Y) \\
{\scriptstyle \alpha^{-1}\otimes 1}\downarrow & & & & \Vert \\
(X \otimes (W \otimes Z)) \otimes Y & = & (X \otimes (W \otimes Z)) \otimes Y & \xrightarrow[\alpha^{-1}]{} & X \otimes ((W \otimes Z) \otimes Y)
\end{array}$$

which commutes because of the pentagon axiom.

Now we will show that for all $f\colon X \to X'$, $\mathcal{Y}_f$ is a modification. For this we need to show that the following diagram commutes for all $Y \in \mathcal{M}$.

$$\begin{array}{ccc}
\mathcal{Y}_*(Y) \circ \mathcal{Y}_X & \xrightarrow{1*\mathcal{Y}_f} & \mathcal{Y}_*(Y) \circ \mathcal{Y}_{X'} \\
{\scriptstyle (\theta_X)_Y}\downarrow & & \downarrow{\scriptstyle (\theta_{X'})_Y} \\
\mathcal{Y}_X \circ \mathcal{Y}_*(Y) & \xrightarrow[\mathcal{Y}_f*1]{} & \mathcal{Y}_{X'} \circ \mathcal{Y}_*(Y)
\end{array}$$

We do this by checking that the diagram commutes when evaluated for all $Z \in \mathcal{M}$

$$\begin{array}{ccc}
(X \otimes Z) \otimes Y & \xrightarrow{(f\otimes 1)\otimes 1} & (X \otimes Y) \otimes Z \\
{\scriptstyle \alpha^{-1}}\downarrow & & \downarrow{\scriptstyle \alpha^{-1}} \\
X \otimes (Z \otimes Y) & \xrightarrow[\mathcal{Y}_f\otimes 1=\mathcal{Y}_f\otimes(1\otimes 1)]{} & X' \otimes (Z \otimes Y)
\end{array}$$

We see that this diagram indeed commutes because of the naturality of α. $\square$

Now we will show that $\mathcal{Y}$ is actually an inverse of $e_{\mathcal{M}}$

Lemma 4.2.7. $\mathcal{Y}\colon \mathcal{M} \to \mathrm{Str}(\mathcal{Y}_*, \mathcal{Y}_*)$ *is an inverse of* $e_{\mathcal{M}}$

Proof. We need to construct a natural isomorphism $e_{\mathcal{M}} \circ \mathcal{Y} \to 1$. Note that for all objects $X \in \mathcal{M}$ we have $e_{\mathcal{M}}(\mathcal{Y}_X) = X \otimes e \overset{\rho}{\cong} X$ and for all morphisms $f\colon X \to X'$ we have $e_{\mathcal{M}}(\mathcal{Y}_f) = f \otimes 1_e$, so the right unitor ρ is the natural isomorphism we are looking for.

Now we need to construct a natural isomorphism $\mathcal{Y} \circ e_{\mathcal{M}} \to 1$. Let $\theta \in \mathrm{Str}(\mathcal{Y}_*, \mathcal{Y}_*)$, then

$$
\begin{aligned}
\mathcal{Y}_{e_{\mathcal{M}}(\theta)}(X) &= \theta^0(e) \otimes X \\
&= \mathcal{Y}_*(X)(\theta^0(e)) \\
&= (\mathcal{Y}_*(X) \circ \theta^0)(e) \\
&\overset{(\theta_X)_e}{\cong} (\theta^0 \circ \mathcal{Y}_*(X))(e) \\
&= \theta^0(e \otimes X) \\
&\overset{\theta^0(\lambda_X)}{\cong} \theta^0(X)
\end{aligned}
$$

So $\theta^0(\lambda_x) \circ (\theta_X)_e$ suffices on objects. Because of the naturality and modification axiom of modifications we also get that this natural transformation works on the morphisms level, so we have proven that $\mathcal{Y}$ is an inverse of $e_{\mathcal{M}}$. □

Lemma 4.2.8. $\mathrm{Str}(\mathcal{Y}_*, \mathcal{Y}_*)$ *is a strict monoidal category with the same monoidal structure of* $\mathcal{M}^{\mathcal{M}}$.

Proof. Let $\theta, \gamma \in \mathrm{Str}(\mathcal{Y}_*, \mathcal{Y}_*)$ be strong transformations. We will follow the same monoidal structure of $\mathcal{M}^{\mathcal{M}}$, so it is forced that $(\theta \otimes \gamma)^0 = \theta^0 \circ \gamma^0$. Now we need to construct a natural isomorphism with components $(\theta \otimes \gamma)_X : \mathcal{Y}_*(X) \circ (\theta^0 \circ \gamma^0) \to (\theta^0 \circ \gamma^0) \circ \mathcal{Y}_*(X)$. Note that $\theta_X * 1_{\gamma^0} : \mathcal{Y}_*(X) \circ \theta^0 \circ \gamma^0 \to \theta^0 \circ \mathcal{Y}_*(X) \circ \gamma^0$ and $1_{\theta^0} * \gamma_X : \theta^0 \circ \mathcal{Y}_*(X) \circ \gamma^0 \to (\theta^0 \circ \gamma^0) \circ \mathcal{Y}_*(X)$.

This means that we can choose

$$(\theta \otimes \gamma)_X = (1_{\theta^0} * \gamma_X) \circ (\theta_X * 1_{\gamma^0}).$$

For the morphisms we can also copy the monoidal structure because modifications are just natural transformations. □

Corollary 4.2.9. $\mathcal{Y} : \mathcal{M} \to \mathrm{Str}(\mathcal{Y}_*, \mathcal{Y}_*)$ *is a pseudofunctor and a monoidal equivalence.*

4.3 From strictification to coherence

Because of time constraints we were not able to prove coherence from strictification, but we will have a quick discussion about how one might prove it. A naive way to try to prove that coherence follows from strictification is by saying that commutativity of diagrams is reflected by monoidal equivalences. This is not sufficient however because we don't in general have that $F(X \otimes Y) = F(X) \otimes F(Y)$. One idea put forward by Etingof et al.[2] is given a diagram with words $A \otimes (B \otimes C)$

build a diagram around the commutative (because it's in a strict monoidal category) diagram with the words translated to $F(A) \otimes (F(B) \otimes F(C))$ to the same diagram with words like $F(A \otimes (B \otimes C))$ using the axioms of pseudofunctors. We were however not able to make this argument rigorous enough for this thesis. One of the problems might get solved by showing a kind of coherence theorem for pseudofunctors, which says that all diagrams containing only F^2 and F^0 commute.

Another idea that might be work, but may need the full theory of 2-categories, is to strictify the pseudofunctor of the equivalence. This idea comes from the nlab[5], but it needs a 2-categorical pseudofunctor.

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