The fruit of heuristic trees

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Abstract

This mixed methods study examines the impact of the use of heuristic trees on the problem-solving abilities of students in the upper secondary school. During six weeks two groups of students had to solve mathematical problem tasks. The experimental group had access to heuristic trees and the control group had access to worked-out solutions. After every task the students filled in a questionnaire. The solutions of the students were assessed on control of the solving process and the questionnaire determined the students' sense of ownership of the solution and the students' engagement with the task. After the intervention the students took a post-test and filled in a part of the questionnaire. On the post-test the experimental group scored significantly higher than the control group. Moreover, the heuristic trees gave students a greater sense of ownership than the worked-out solutions and the heuristic trees nourished engagement more than the worked-out solutions did.

Introduction

One of the main goals of mathematical education in secondary school is to foster students' mathematical thinking. According to Drijvers (2015), there are three core aspects of mathematical thinking: problem-solving, modeling, and abstracting. In this study, the focus is on problem-solving. To succeed in problem-solving, students need a solid mathematical basis. Practicing routine exercises provides a basis for mathematical thinking, so students learn to perform certain mathematical processes and procedures routinely. It is also important that mathematical processes and procedures are automated when engaging with a mathematical problem task. It is valuable to engage with these mathematical problems since they are more representative of the mathematics that students will encounter outside of school. Furthermore, solving mathematical problems helps to achieve certain abstraction within mathematical concepts.

In this study we are interested in ways to support problem-solving. When working on a problem task the student might get stuck. A teacher can help by determining what solution path a student is attempting to follow and give a hint in the right direction. If the teacher is not available to help, students often reach for the worked-out solution, which can halt the students' own thinking. In the Dutch classroom, the teacher mostly does not have time to help each student individually. Also, when a student wants to work on a problem task at home there is no teacher available to help.

Heuristic trees, as developed by Bos (Bos, 2017; Bos & van den Bogaart, 2022), are digital tools to support problem-solving in the absence of a teacher. For a problem task a unique heuristic tree is designed. The heuristic tree provides hints ordered in a structured way. In their research, Bos and van den Bogaart found that students are able to use heuristic trees to receive help (Bos & van den Bogaart, 2022). The students do have to learn though, how to self-diagnose in which phase they need help. Moreover, below-average students have difficulty with the inflexibility of the heuristic trees. The heuristic trees are inflexible since they do not adapt to the work of a student. An advantage of heuristic trees is that students keep a sense of ownership of their answer: instead of looking at the answer model when stuck, the students have hints to guide them. In this study heuristic trees play a key role: two groups of students will be compared, one of which uses heuristic trees.

Bos and van den Bogaart's research showed that the use of heuristic trees influences the help-seeking behaviour of the students as well as their problem-solving abilities (Bos & van den Bogaart, 2022). However, quantitative research of what the long-term learning outcomes are, when heuristic trees are used regularly for a few months, is still lacking. Will regular use make students quantifiable 'better' problem solvers?

The aim of this study is to examine whether heuristic trees are an effective method to improve students' problem-solving abilities within a subject, as well as to research how heuristic trees affect the approaches and learning processes of students. More specifically we want to examine the impact of the use of heuristic trees on the ability to solve non-routine tasks, compared to the impact of the use of worked-out solutions. Moreover, we want to examine the extent of engagement when using the heuristic tree or worked-out solution, as well as the extent of control of the student during the solving process and the extent of ownership of the solution.

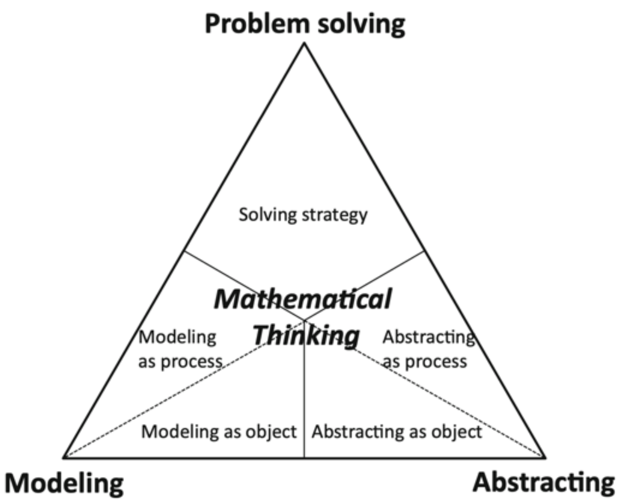
The research for this study was done with an experimental group and a control group at a Dutch secondary school in Amsterdam. During a six week period, the two groups made problem tasks. The experimental group had access to the heuristic trees as support, the control group could use worked-out solutions as support. After the intervention, the students took a post-test.

Theoretical Background

Mathematical thinking

As mentioned in the introduction, one goal of mathematics education is to nurture mathematical thinking. The three core aspects of mathematical thinking are problem-solving, modeling, and abstracting (Drijvers, 2015). These aspects are connected, as shown below in the model made by Kodde-Buitenhuis (2015). While procedures are a part of these aspects, procedural thinking can be overrepresented in mathematics education. An example is fractions. The procedure they represent is division, but seeing fractions as a procedure only, makes the idea of adding them seem impossible (Tall & Gray, 2007). Mathematical ideas must be compressed into thinkable concepts to be able to achieve thinking on a more sophisticated level (Tall & Gray, 2007). Problem-solving helps to go beyond procedural thinking.

Figure 1.
Model of the three core aspects of mathematical thinking by Kodde-Buitenhuis (2015).



This study focuses on problem-solving and more specifically on how to support students in learning problem-solving strategies, the upper part of the triangle in Figure 1. Doorman et al. (2007) formulate problem-solving as follows: "[…] the 'art' of dealing with non-trivial problems which do not yet have a known, routine solution strategy to the student, but which provide opportunities for the students to develop new solution strategies" (p. 405). The way problem-solving is taught is important. The teacher must restrain from telling the students how to solve a problem and instead try to guide them. When a teacher is telling how to solve a problem, this can lead to procedural thinking, rather than the flexible thinking one wants to achieve with problem-solving (Tall, 2013).

Problem-solving

In his landmark book on the education of mathematical problem-solving, 'How to Solve It' (1945), Georg Pólya describes four phases a student should go through when solving a problem. First, a student needs to understand the problem – the concepts as well as the question at hand. Second, the student needs to think of a plan – what is the connection between the known and unknown. Third, the student needs to carry out the plan and carefully check if every step is legit. Lastly, the student should examine the solution. Is it an answer to the problem? Is there an alternative solution? Is generalization of the solution possible? Support for problem-solving should thus be structured in a way that a student can go through these phases.

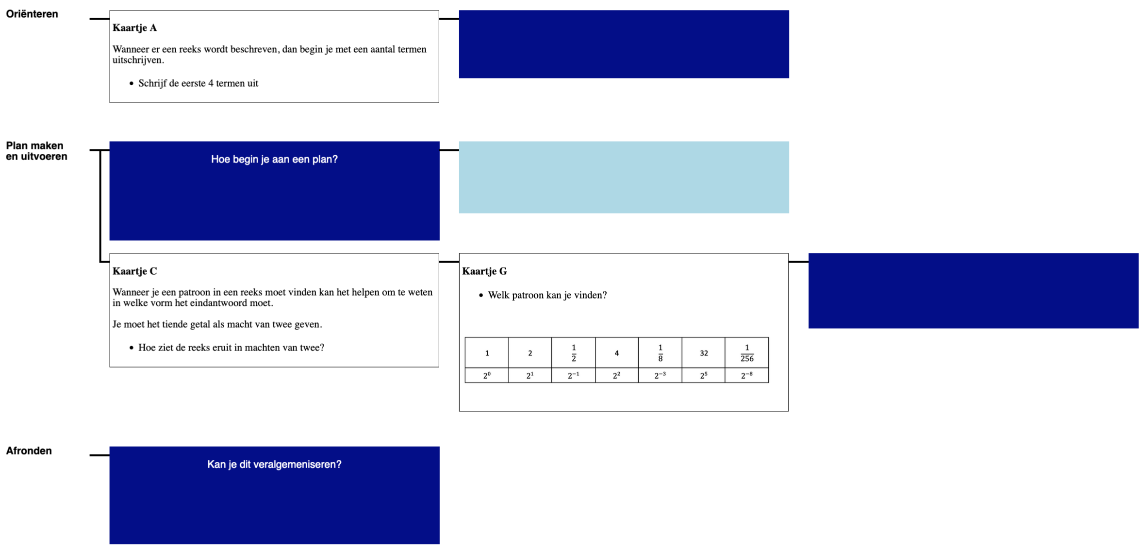
Schoenfeld (1985) continued Pólya's work and instead of four phases, described four categories of knowledge and behavior to explain the process of problem-solving. The first category consists of the resources a student has, the basic mathematical knowledge of concepts and techniques to engage with a mathematical problem. The second category consists of heuristics, the general techniques and strategies used to solve a mathematical problem. The third category Schoenfeld (1985) writes about is control. Schoenfeld states that a student has control on a mathematical problem when they can make a plan to solve the problem and when they are able to ask for help at the right time. The last category is belief system, the state of mind in which a student approaches a problem. In this study we will focus on heuristics and control as important aspects for problem-solving.

Compression and heuristics

Students can find it difficult to engage with mathematical problems since "one needs to combine mathematical skills and activities that have already been mastered" (Bos, 2017, p. 436). To be able to be successful in mathematical problem-solving, the mathematical concepts should be compressed to enable the student to use the concepts flexibly. Compressed mathematical concepts are like the bellows of an accordion: the compressed version is as if the bellows is pressed in, the decompressed version reveals information, like when the bellows is pulled out. For example, when a student recognizes that Pythagoras's theorem can be used, since there is a right-angle triangle, this recognition happens because the theorem is compressed. When students start solving the problem however, they use the decompressed version: they can do the steps necessary to use the theorem.

The reinterpretation of heuristics by Bos & van den Bogaart (2022) is as follows: "A heuristic is a form of help formulated in a compressed language" (p.162). Bos (2017) created the heuristic trees based on Pólya and Schoenfeld's work. The heuristic trees offer the aforementioned heuristics (formulated in compressed language), with the branches helping the student to decompress the heuristic. A heuristic tree is designed for a specific problem task and provides hints ordered in a structured way.

Figure 2.
Example of a heuristic tree.



Note. A student can click on the dark blue boxes, this opens the card (in the example three cards are opened). A heuristic tree has three branches: (1) orientation, (2) plan making and executing, and (3) finalization. Close to the "trunk", the hints are general heuristics (in theory these cards should be able to be reused for another problem task). The further away, the more specific the hints will be to the problem task at hand. Click here to go to this heuristic tree.

Bos & van den Bogaart (2022) found that heuristic trees influenced students' (master's degree in mathematics education) problem-solving positively, and that they could work on the tasks in absence of a teacher while still receiving adequate help when needed. Bos & van den Bogaart (2022) observed that the inflexibility of the heuristic trees can be a problem for below average students, but that using the heuristic trees in the classroom creates time for the teacher to help these students, since other students are assisted by using the heuristic trees.

Aspects of mathematical problem-solving

Mathematical problem-solving has many aspects. This study will focus on three of these aspects: control, ownership, and engagement. Being in control of a mathematical problem can be seen as: "selecting and pursuing the right approaches, recovering from inappropriate choices, and in general monitoring and overseeing the entire problem-solving process" (Schoenfeld, 1985, pp. 98-99). In more recent education literature, the term metacognition is used for control. Another part of metacognition is being able to ask for help – and to know when to ask for it. To help the students with asking for help at the right time and search for it in the right place, Lemmink (2019) developed a help-seeking flowchart. This flowchart can be accessed when using a heuristic tree. We expect that students who use heuristic trees are more in control of their process than students who use worked-out solutions.

Using a worked-out solution when stuck spoils the student's sense of ownership of the answer. As one often sees the whole solution at a single glance when using a worked-out solution, the sense of ownership is dissolved – even if one only wanted to check a small part of the solution. As stated by Bos & van den Bogaart (2022): "By providing support on the heuristic level first, students have the opportunity to maintain ownership of the details of the technique and how they apply to the problem" (p. 162). Sense of ownership is important since it can enhance students' mathematical understanding as well as boost their confidence (Francisco & Maher, 2005). When students use heuristic trees, we hypothesize that students will have a greater sense of ownership over the solution than students who use a worked-out solution.

"In mathematics, behavioral engagement refers to the extent to which students participate, including actual or intended enrolments, and degree of effort applied" (Watt & Goos, 2017, p.135). Engagement in a mathematical problem consists of "continued solving attempts" (Bos & van den Bogaart, 2022, p.163). Engagement is at stake when a student gets stuck; the heuristic trees provide just-in-time help to keep engagement up (provided that the student uses the heuristic trees just in time). For engagement it is important that a problem task is not too easy, so a student does have (or wants to) to try, but also not too difficult as students need to believe they are able to solve the problem task. We expect that heuristic trees will keep engagement up more than worked-out solutions.

Research Questions

In this study we want to answer the research questions: What is the impact of using heuristic trees on students' (15- to 16-year-old, vwo)
(1) problem-solving abilities within a subject,
(2) control of the problem-solving process,
(3) engagement with the problems and,
(4) sense of ownership of the solution of a mathematical problem tasks?
And (5) how do heuristic trees affect the approaches and learning processes of students?

Method

This study will refer to "hints", meaning the support for both the heuristic tree group and the worked-out solution group. The heuristic tree group use the cards in the heuristic tree as hint, while for the worked-out solution group it is not really a hint but a part of the solution.

General approach

This is a mixed methods study. Two kinds of support for problem tasks were compared, heuristic trees and worked-out solutions. Participants were $n = 55$, high school (4 vwo) students (14 male, 41 female; 15 – 16-year-old) at an Montessori secondary school in Amsterdam. The students were from two mathematical (wiskunde A) classes (27 and 28 students) taught by the same teacher-researcher.

Intervention design

Over the course of six weeks the students had to solve a problem task every week. These problems are so-called 'extra' exercises from their mathematics book *Getal & Ruimte*

(2020), chapters three and five, the chapters that the students worked on in their regular lessons during the period of this study. The students had to solve these problems without the help of their teacher. One group ($n = 27$) got access to a heuristic tree for that specific problem, the other group ($n = 28$) had access to the worked-out solution. Every week, the students had 15 minutes to solve the problem task and an additional five minutes to fill in the questionnaire. After the intervention, the students took a post-test. This post-test consisted of three problem tasks, increasing in difficulty (see Table 3). The students had 20 minutes to solve the problem tasks and afterwards they filled in a final questionnaire with questions similar to the regular questionnaire.

Procedure

Per problem task (see Table 1) a heuristic tree was designed. The students accessed the heuristic trees with their phones. The control group received a piece of paper with the worked-out solution on it, which was handed to them face-down so as not to give away the clue immediately.

Table 1.

Problem tasks used in the study.

3.6	Calculate $\left(1+\frac{1}{2}\right)*\left(1+\frac{1}{3}\right)*\left(1+\frac{1}{4}\right)*\left(1+\frac{1}{5}\right)*\cdots*\left(1+\frac{1}{1001}\right)$
3.14	Red and yellow carps swim in a pond. Two fifths of the carps are yellow, the rest is red. Three quarters of the yellow carps is female. In total there is the same amount of female and male carps. Which part of the total carp-population consists of red males?
3.40	René buys fruit for exactly 10 euros. He buys some apples for 40 cents apiece, three times as many pears for 50 cents apiece, and with the rest of the money he buys bananas for 80 cents apiece. How many pieces of fruit did René buy?
3.72	Sandra runs from A to B and immediately back to A. On the way to A she runs at a speed of 14 km/h and on the way back she runs at a speed of 12 km/h. In total she runs the total distance in 1 hour and 5 minutes. What is the distance between A and B in kilometers?
5.5	The number $2^{59} \cdot 3^{4} \cdot 5^{53}$ ends on several zeros. What is the last number before all these zeros?
5.11	Harry writes down a sequence of numbers. He starts with 1 and the second number is 2. Every next number he determines as follows: he divides the second last number by the last number. Write down the 10^{th} number in the sequence as a power of 2.

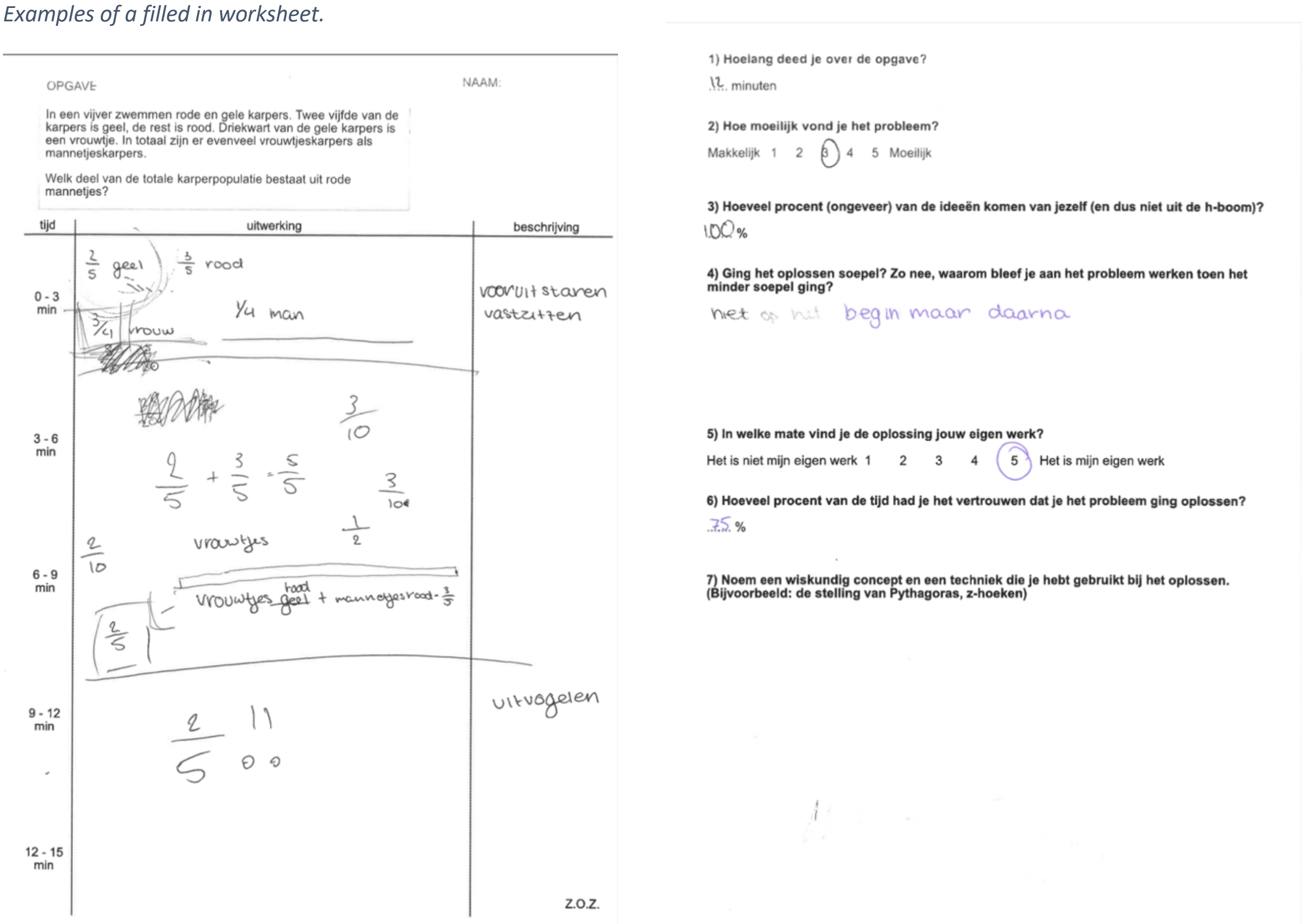
Note. Translated exercises from Getal & Ruimte 4 vwo A (2020). The numbers refer to the chapter and then the exercise number in the book.

Both groups had to write down their solution on a worksheet, which had the problem task printed on it. Every three minutes, the teacher-researcher asked the students to draw a line under their work and to write down which phase of the solution process they were in. While the students worked on their solution, they could see a timer as well as a list of words to help them describe their solution process. Every card in the heuristic tree was lettered so that students could write down which hint(s) they had used. For the worked-out solution it was similar, here every line was lettered, so the students could write down the hint they used on the worksheet.

Questionnaire

The back of the worksheet contained the questionnaire which students were asked to fill in once they finished the problem task (or when the fifteen minutes had passed). They answered seven questions about their solving process. This was used to determine their ownership, engagement and control. The questionnaire was intentionally kept short so as not to overload the students. After all the worksheets were handed in, there was a plenary discussion of the problem task.

Figure 3.
Examples of a filled in worksheet.


OPGAVE

NAAM:

In een vijver zwemmen rode en gele karpers. Twee vijfde van de karpers is geel, de rest is rood. Driekwart van de gele karpers is een vrouwtje. In totaal zijn er evenveel vrouwtjeskarpers als mannetjeskarpers.

Welk deel van de totale karperpopulatie bestaat uit rode mannetjes?

tijd | uitwerking | beschrijving

0 - 3 min | 2/5 geel, 3/5 rood, 3/4 vrouw, 1/4 man | vooruitstaren vastzitten

3 - 6 min | 3/10, 2/5 + 3/5 = 5/5, 3/10

6 - 9 min | 2/10, vrouwtjes, 1/2, vrouwtjes geel + mannetjes rood = 3/5, 2/5

9 - 12 min | 2/5 | uitvogelen

12 - 15 min

z.o.z.

1) Hoelang deed je over de opgave?

12 minuten

2) Hoe moeilijk vond je het probleem?

Makkelijk 1 2 (3) 4 5 Moeilijk

3) Hoeveel procent (ongeveer) van de ideeën komen van jezelf (en dus niet uit de h-boom)?

100 %

4) Ging het oplossen soepel? Zo nee, waarom bleef je aan het probleem werken toen het minder soepel ging?

niet op het begin maar daarna

5) In welke mate vind je de oplossing jouw eigen werk?

Het is niet mijn eigen werk 1 2 3 4 (5) Het is mijn eigen werk

6) Hoeveel procent van de tijd had je het vertrouwen dat je het probleem ging oplossen?

75 %

7) Noem een wiskundig concept en een techniek die je hebt gebruikt bij het oplossen. (Bijvoorbeeld: de stelling van Pythagoras, z-hoeken)

Table 2.
Questionnaire.

1	How long did you work on the task?
2	How difficult did you find the task?
3	What percentage of the ideas are yours?
4	Did the solving process go smoothly? If not, why did you keep on working on the task?
5	To what degree would you say the solution is your own work?
6	What percentage of time did you have the belief that you would solve the task?
7	Name a mathematical concept and a technique you used?

The variable ownership is determined by self-reporting of the students and coded by the teacher on a scale of -1, 0 or 1, based on the solution on the worksheet. The intention was to measure the ownership the students feel over their solution, questions three and five were used to determine this. The sense of ownership can be spoiled when using a worked-out solution, since all the steps are written down. In a heuristic tree, you open one card at a time. That is why we distinguished between idea and work. Perhaps they got a key idea from the hints, but they worked out the rest of the solution themselves and thus feel ownership of the work. The means of both questions answered by the students were calculated and compared using the Mann-Whitney-test.

The level of engagement is measured in questions two and four. If the student did not find the process smooth, they were asked to write down why they kept working on the problem. The first part of the answer “no, a bit, yes” was converted to “-1, 0, 1”. The last question to test for engagement was question six. Just like the answers for ownership, the means were calculated and compared with the Mann-Whitney-test.

For the aspect control, the worksheets were assessed and coded. By inductive coding, six categories of use of hints were named, based on the worksheets of the first week (see the results). The other weeks the worksheets were coded on the same day as the students had made the problem task, using the established categories.

Post-test

Table 3.

Problem tasks of the post-test.

1	Anna and Bob walk a relay race together, they divide the distance in the proportion 1:3. Bob walked 30 km on the last day of training, half of the total distance. How many kilometers is Anna going to walk?
2	Mr. van Dijk is 61 years old. He has four sons. Son Jan is twice the age of son Piet. Piet is twice the age of son Theo. Son Wim is five years older than Theo. Mr. van Dijk is the same age as the ages of his sons together. What is the age of Wim?
3	Solve the following equation: $36 \cdot 2^x \cdot 3^x = 6^{x^2}$

The post-test consisted of three problem tasks (Table 3), increasing in difficulty and had similarities to tasks of the intervention (Table 1). The first task had an element of speed, because exercise 3.72 also contained speed as variable. The second task was more worded as a riddle, having similarities to exercise 3.40. The third task had to be solved using the properties of exponentiation (like exercise 5.5) and knowledge about how to solve equations. The students had 20 minutes to make the post-test. The students wrote down their solutions on a worksheet, similar to the worksheets of the previous six weeks. After the post-test, the students filled in a questionnaire (on the back of the worksheet) with similar questions as during the intervention.

The post-tests were graded by the teacher-researcher. The grading was done by using an answer model. The second grader graded 20% of the worksheets of the post-test using the same answer model. The grades were calculated with the standard formula: (number of points / 15) * 9 + 1. Their work was compared using Spearman’s correlation. The mean of each class was calculated and compared using the Mann-Whitney-test. The post-test was conducted in the final week before the Christmas break. Two weeks after the Christmas break, the two groups made a routine test covering the same chapters. The means of the grades scored on this test were also compared using the Mann-Whitney-test.

Results and Analysis

The results are divided into five parts. We start with the results of the post-test, then move on to ownership, engagement, and control. The section concludes with some remarks. As students sometimes get ill, not every student participated every week. For the heuristic tree group, the numbers of students were: 24, 19, 22, 22, 23, and 24. The numbers of the worked-out solution group are: 26, 24, 25, 25, 25, and 26. The post-test was taken by 19 and 20 students respectively.

Post-test

On the post-test the heuristic tree students (HT- students) scored higher, M = 6.3, SD = 1.2, than the worked-out solution students (WS-students), M = 4.6, SD = 2.2. This is a statistically significant difference, M-W = 91.5, p = .008. A second coder graded 20% of both groups. A Spearman's correlation was conducted to evaluate the relationship between the first and second grader. There was a significant positive relationship between the graders: $r_s(13) = .94$, p = < .001, for the HT-group and $r_s(13) = .91$, p = < .001, for the WS-group. As mentioned, the students also made a routine test about the same chapters, here the groups scored the same. The HT-students scored the same, M = 5.2, SD = 1.7, as the WS-students M = 5.2, SD = 1.5. (M-W = 349.5, p = .490).

Table 4.

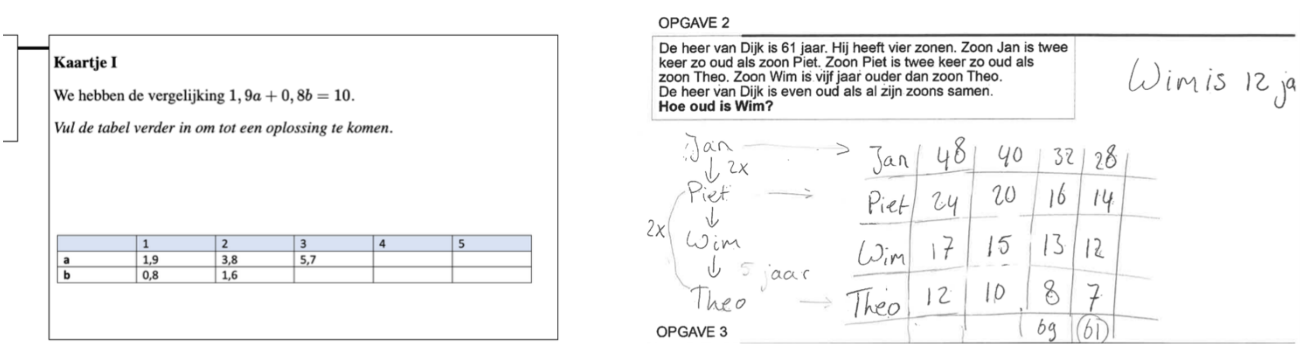
Means of the tasks of the post-test.

	HT	WS
Task 1	4.1	2.8
Task 2	4.3	3.0
Task 3	0.3	0.1

The better results by the HT-students on the post-test can be explained by the fact that the HT-students used the heuristics they had learned. For exercise 3.40 there was the heuristic to make an educated guess based on a table, see Figure 4. The post-test also contained an exercise where the students could apply this heuristic, which a couple of the HT-students did.

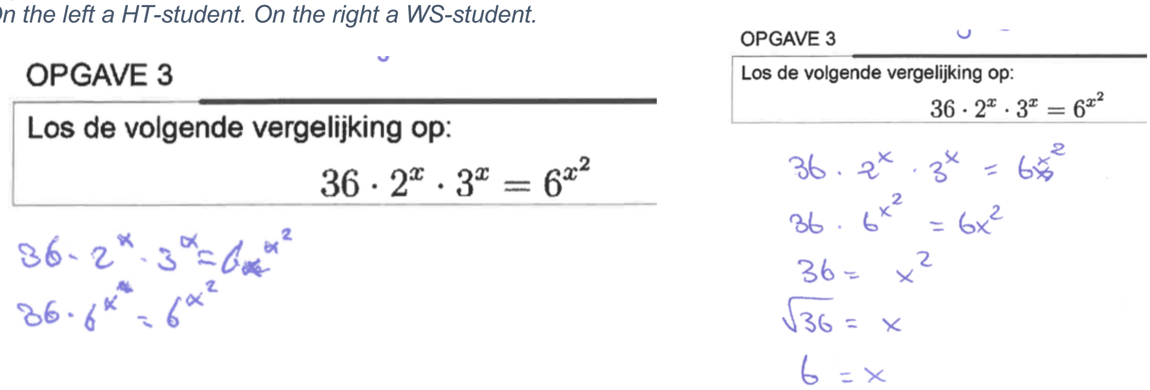
Figure 4.

Example of use of heuristic, on the left a card from the heuristic and on the right a solution by a HT-student using a table.



The post-test also contained a task in which the students had to use the properties of exponentiation. Neither group solved the problem, but the HT-students were a bit more familiar with the properties, so they stopped with their solution when they did not know how to go further, while the WS-students tried something that was incorrect, see Figure 5. As the teacher-researcher, I noticed that the HT-group had paid more attention to the plenary explanation after exercise 5.5 (with the properties of exponentiation), so they had a better understanding when making the post-test

Figure 5.
On the left a HT-student. On the right a WS-student.



Note. The HT-student has corrected themselves. The WS-student has made the same mistake as the HT-student but has not corrected it. Furthermore, the rest of the solution by the WS-student is incorrect.

During the post-test the students did not have access to hints. Therefore, the aspects of ownership and control were not tested in the post-test, but engagement was. For engagement the same questions about difficulty, smoothness and belief were asked. The results in Table 5 show that there is no significant difference (tested with Mann-Whitney) between the two groups. The engagement of the students is the same, so it seems that it is the kind of hint (heuristic tree or worked-out solutions) that can explain the different levels of engagement during the intervention.

Table 5.
Means questionnaire post-test.

	Difficulty			**Smooth**			**Belief**		
	1	2	3	1	2	3	1	2	3
HT	2.35	2.94	4.06	.65	.35	-.24	83.53	75.29	36.29
WS	2.60	3.07	3.87	.33	.20	-.33	63.67	65.67	41.00

Note. Difficulty is the answer on question two of the questionnaire, belief is the answer to question six. For smooth, the answers *yes, a bit, and no* were converted to 1, 0 and -1.

Ownership

Table 6.

Reported sense of ownership.

	Week 1		Week 2		Week 3		Week 4		Week 5		Week 6	
	HT	WS	HT	WS	HT	WS	HT	WS	HT	WS	HT	WS
Ownership	-.08	.08	.33	.08	.36	.04	.23*	-.36	-.15	.00	.26*	-.09
Idea	51.00	22.58	70.38*	48.33	68.14	55.00	64.27*	29.60	40.75	37.05	81.09*	41.00
Work	3.13*	1.96	3.78*	3.04	3.64	3.08	3.50*	2.36	2.60	2.70	4.26*	2.82

Note. Ownership is judged by the teacher-researcher on a scale -1, 0 and 1. Idea and work are the answers to questions three and five respectively. *= $p < .05$ (tested with Mann-Whitney).

Overall, the HT-students report higher scores of ownership of their ideas and their work than the WS-students. In the teacher-researcher assessed ownership, this is not always the case. In week one, the ownership of the heuristic tree group is lower than the worked-out solution group. This may be explained by the fact that the students had to get accustomed to the use of heuristic trees, while the worked-out solutions are something the students already know and use. The HT-students also had difficulty with the content of the hints themselves. After the first week, the teacher-assessed ownership is higher in the heuristic tree group each time, but not always significantly.

In the second half of the intervention, the HT-students obtained higher scores on both the self-reported variables as the teacher assessed ownership. Week five is an exception, there is almost no difference between the groups. In this exercise, students had to use properties of exponentiation, for example: when multiplying power with the same base, you add the exponents. As the students were not familiar enough with those rules yet, they did not have the proficiency to solve the exercise by themselves. Since the HT-students could not ask questions about the hints in the heuristic tree, they had difficulty understanding them. The worked-out solution group also had difficulty understanding the hints, but at a different level, they remarked: “Hoe komen ze erop? [how does one come up with this]?” With the heuristic trees this does not happen, since often the heuristics are structured in a sentence as: when a sequence is described, write the first few terms down.

Figures 6 and 7 shows the teacher-researcher assessed ownership per exercise per group. The judged ownership of the HT-students is higher than that of the WS-students. For tasks 3.6 and 5.5 more students opened all the cards of the heuristic tree. These are also the exercises the students deemed the most difficult. We see that the WS-students often judge their ownership as less than the HT-students. In the case of exercise 5.5, the WS-students had the advantage of understanding their kind of hints better than the HT-students did theirs. The heuristic tree for exercise 5.5 used the scientific notation, something the students had learned, but the HT-students may not have been familiar enough with the concept to use it themselves in a non-routine way.

Figure 6.
Judged ownership by teacher-researcher of the HT-group.

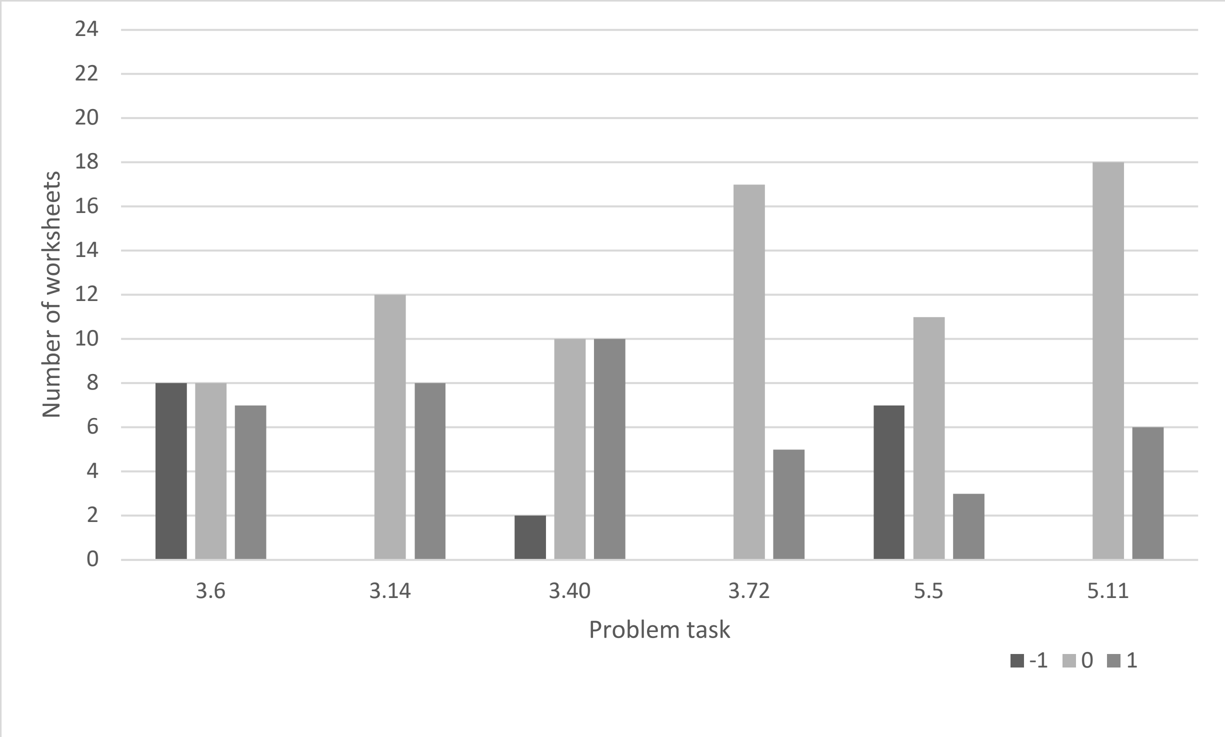


Figure 7.
Judged ownership by teacher-researcher of the WS-group.

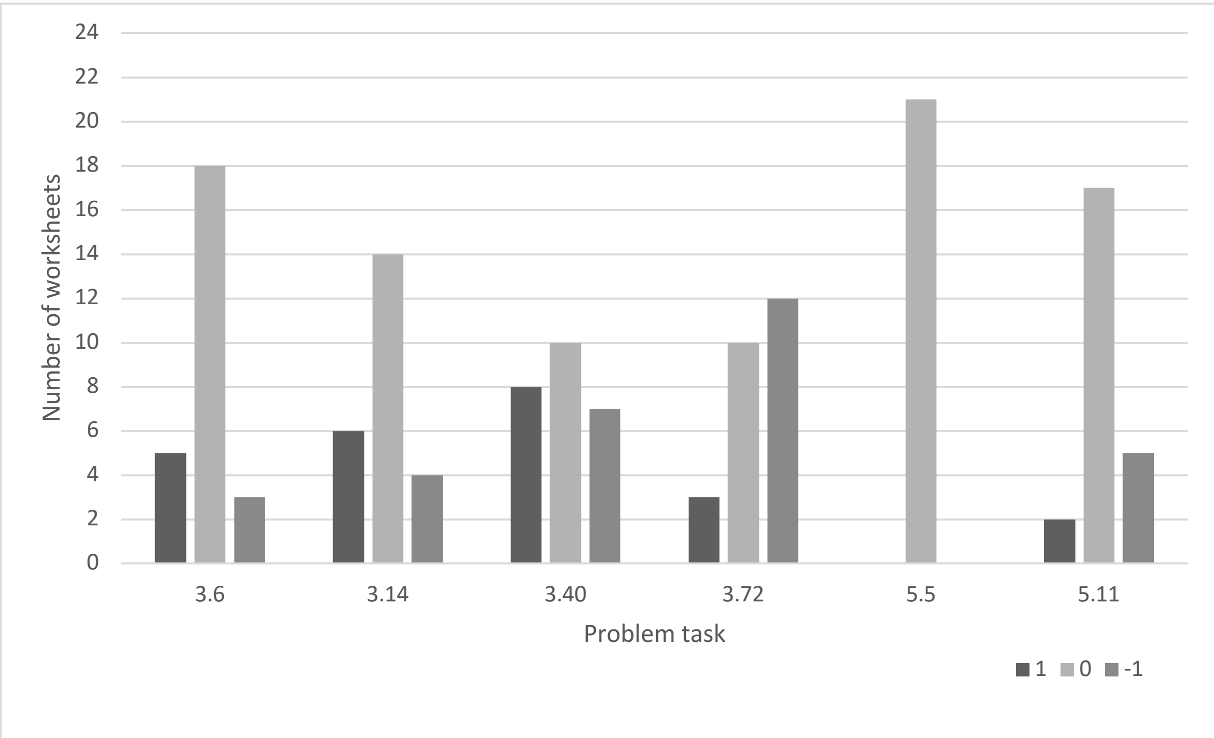


Table 7.
Reported engagement.

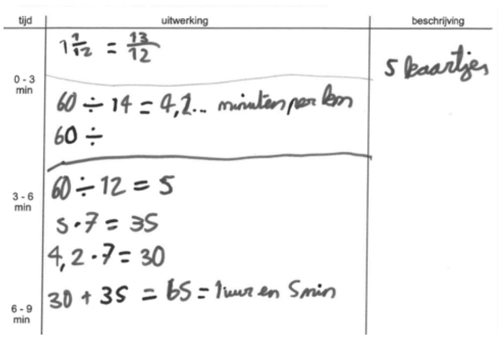
	Week 1		Week 2		Week 3		Week 4		Week 5		Week 6	
	HT	WS	HT	WS	HT	WS	HT	WS	HT	WS	HT	WS
Difficulty	4.63	4.42	3.18	3.58	3.36	3.12	2.28*	4.00	4.73*	3.56	3.17*	4.05
Smooth	-.67	-.92	.24	-.17	.00	.00	.41*	-.48	-.82*	-.28	.30*	-.45
Belief	36.38	23.46	71.41*	51.04	76.36*	58.08	84.09*	44.00	23.86	39.78	72.26*	44.09

Note. Difficulty is the answer on question two of the questionnaire, belief is the answer to question six. For smooth, the answers *yes, a bit, and no* were converted to 1, 0 and -1. *= p < .05 (tested with Mann-Whitney).

The heuristic tree group has more belief in being able to solve the problem task, with week 5 again as an anomaly. The difference between the groups in week 4 can be explained by an overestimation of themselves by the HT-students. Additionally, a heuristic tree is not really a tool to use to check your solution. This might lead students to think they have solved the problem, while it might be incorrect. The WS-students can use the hints to check their solutions. They discover they had an incorrect approach and then just copy the worked-out solution. This could explain the difference in the perceived difficulty between the two groups, because the HT-students are not always confronted with their solution being wrong and that the task is more difficult than they thought.

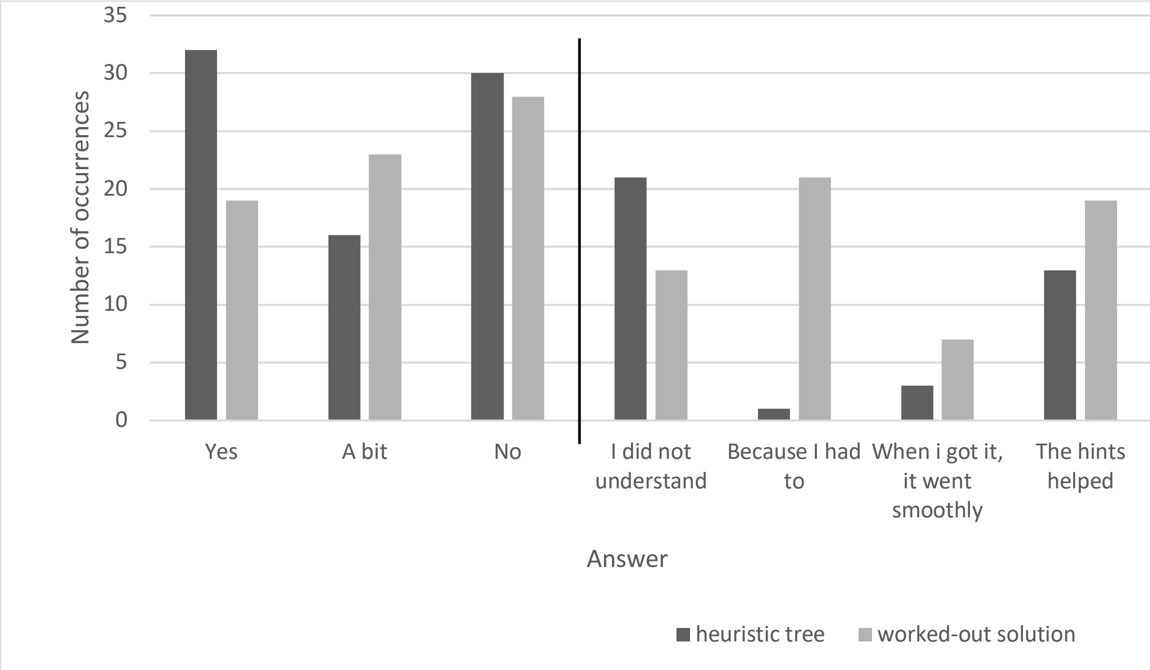
In both groups, most students start with their own idea, and occasionally get stuck. The worked-out solution group can then check their solution and can conclude that they need to change their solution. This may explain their lower percentage of reported belief. The HT-students are not always able to translate the hint to the solution they are working on.

Figure 8.
Example of a student that opened 5 cards without results.



One of the questions to determine engagement was the following: *Did the task go smoothly? If not, why did you keep on working on it?* Not every student answered the second question (‘If not…’). Figure 9 shows the answers to question four, for all six tasks combined.

Figure 9.
Answers on question four of the questionnaire.



Note. The answers 'yes', 'a bit' and 'no' are answers on the question if the solving process was smooth. The other answers are on the follow up question why they kept working on the problem if the process was not smooth.

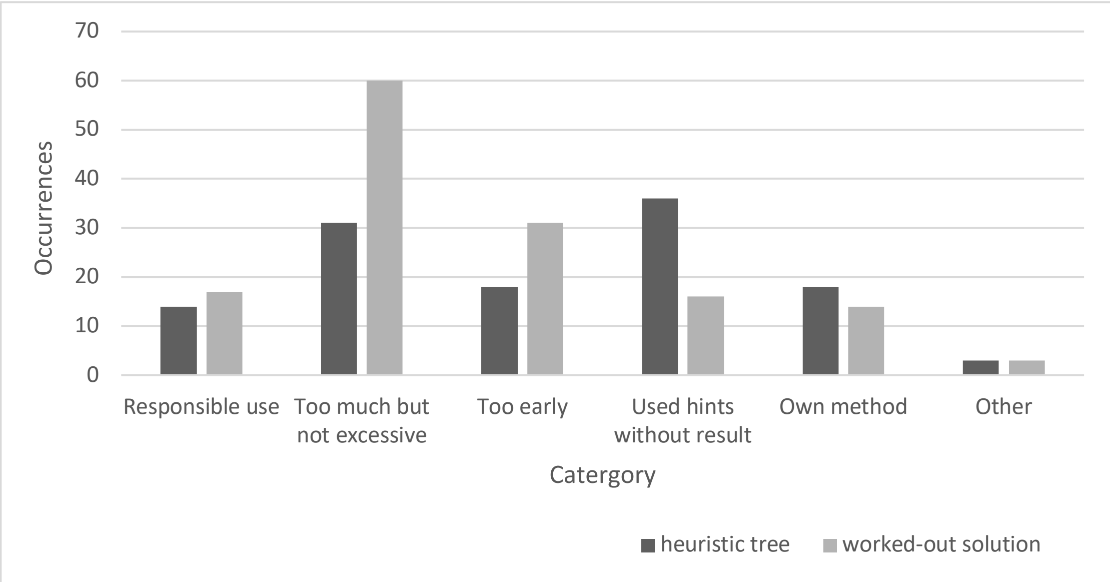
The frequency of the answer 'because I had to' differs greatly between the groups. This indicates that the heuristic trees keep the students engaged, even if they struggle. The instant answer of a worked-out solution does not seem to nourish engagement.

Control

The students' worksheets were categorized in six different categories: (1) responsible use, (2) too much but not excessive, (3) too early, (4) used hints without result, (5) own method, and (6) other. The categories are based on the amount of time passed before the student opened the first hint. Students in the category *responsible use* have used the hints after they tried for themselves, and they did not open all hints simultaneously but only when needed. For the category *too much but not excessive* the students begin with trying for themselves, but then switch to the method of the heuristic tree / worked-out problem. A worksheet is categorized *too early* when students did not try something themselves. The category *used hints without result* speaks for itself, as well as the categories *own method* and *other* (empty worksheet etc.). Figure 10 shows the number of students in the different categories over the whole intervention.

The groups are difficult to compare for control, since the heuristic trees and the worked-out solutions are used in different ways. The WS-students use the worked-out solution to check their work and see the whole solution once they have turned the piece of paper. Heuristic trees offer hints one at the time, but a student can also just click through the cards. Another difference is that the WS-students can just copy the worked-out solution, whereas heuristic trees nudge students to think about the hint. This reason can explain the difference in occurrences in the category *too much but not excessive*. The difference in the category *used hints without result* can be explained by the fact that the HT-students had a learning curve of how to use the heuristic trees, as well as the fact that the content of the hints was not always clear to the students.

Figure 10.
Help-seeking behavior.



Remarks

We conclude the results section with some final remarks. Firstly, the last question of the questionnaire asked the students to write down the mathematical concept(s) and technique(s) they had used to solve the problem task. This question could not be analyzed since a too small number of students answered this question over the course of the intervention.

Secondly, in week four the heuristic tree group used the heuristic tree well or worked on the problem independently. The majority reported that it went smoothly or that the hints helped, while the worked-out solution group is quite divided on how it went. An uncommonly high number of WS-students did not find a solution or used the hints too early.

Thirdly, as stated before, exercise 5.5 is an exception, where the heuristic tree group reports that they had more difficulty with the task than the worked-out solution group. The majority reports that it did not go smoothly or that they did not understand the task or the hints. While the worked-out solution group has a fifty-fifty report to the question of whether the task went smoothly. Looking at the dosage of the hints, the HT-students are too early or have no success with using the heuristic tree. The worked-out solution group is split between *too much but not excessive* and *too early*. The worked-out solution group had more success with the worked-out solutions.

Finally, another difference between the groups is what happened after the 15 minutes they had for the problem task. During the 15 minutes, I did not help the students since the absence of a teacher had to be simulated. After the 15 minutes, when all work was handed in, I would discuss the problem task plenary. Here I noticed that the heuristic tree group was more interested and engaged, especially the students that hadn't finished the problem task, or had almost found the solution. The worked-out solution group had more trouble with seeing value in a plenary discussion since they already ‘knew’ the answer.

Conclusion and Discussion

In this study we aimed to answer the question: What is the impact of using heuristic trees on students’ (15- to 16-year-old, vwo) (1) problem-solving abilities within a subject, (2) control of the problem-solving process (3) engagement with the problems and (4) sense of ownership of the solution of a mathematical problem tasks? And (5) how do heuristic trees affect the approaches and learning processes of students? The hypothesis was that students who used heuristic trees regularly would have more control while problem-solving, as well as having a higher engagement and feel more ownership (Bos & van de Bogaart, 2022).

1) The HT-students scored significantly better on the post-test than the WS-students, so heuristic trees seem to have a positive impact on students’ problem-solving abilities.
2) Control of the problem-solving abilities is the same between the groups. They both use too much help too early in the process (Figure 10). The HT-students click through the cards too fast, while the WS-students see the whole solution at a glance.
3) Engagement of the HT-students was much better than the WS-students (Table 7). For the post-test, when the students had no access to hints, and no difference in engagement between the groups was found.
4) Heuristic trees let students keep a sense of ownership of their solution more than worked-out solutions do (Table 6).
5) The HT-students were more interested during the plenary discussion than the WS-students. The HT-students wanted to learn something from the plenary discussion, while the WS-students saw no value, since they already knew the answer.

There is an indication that using a heuristic tree, without successfully solving a problem, offers more learning outcomes than solving a problem task and writing down an answer with the help of worked-out solutions. This can be underpinned with literature about productive struggle. Kapur (2009) has done notable research to productive failure, by giving students ill-structured (mathematical) problems. Kapur’s reasoning is that when structure is left out, the students have the opportunity to develop their own structure (Kapur, 2009). The difference is that heuristic trees are structured, but they still leave room for failure by the student. Kapur (2009) states: “students from the productive failure condition significantly outperformed their counterparts from the lecture and practice condition on the targeted content in post-test 1” (p. 543). In this study we have found that the HT-students outperformed the WS-students. Having difficulty with the heuristic tree actually may be the reason the HT-students outperformed their counterparts, since “many findings suggest that some form of struggle is a key ingredient in students’ conceptual learning” (Hiebert & Grouws, 2007, p. 388).

Even though for both exercises 3.72 and 5.11 the heuristic tree group has higher ownership and engagement scores, their solutions were not always correct. The heuristic tree is not a tool you can easily use to check whether your solution is correct (in part because it does not offer the final answer). Both exercises were made in the second half of the research period, so one could argue that the heuristic tree group had by then gotten used to making a problem task without knowing the answer immediately, while the worked-out solution group still needed the 'fix' of knowing whether they were correct.

Although the heuristic tree group did not always find a solution to the problem tasks – or opened the hints too soon and/or too often – they scored significantly better on the post-test than the worked-out solution group. An explanation is found in the difference between a heuristic tree and worked-out solutions. When a student uses a heuristic tree 'too early', they do not try their own method but follow the process of the heuristic tree. But when following this process, they are compelled to engage with the hints, since the cards are revealed one by one, and the cards can contain questions as well. A worked-out solution immediately shows answers. A common remark by students when using a worked-out solution as a hint is: "ik snap niet hoe ze hierop komen [I don't understand where this comes from]." This remark was rarely heard in the heuristic tree group since the steps of the process are clearly laid out and solution strategies are offered.

In both groups there were students who could not think of a solution themselves and then used hints. The results do imply that there is a better learning outcome when using a heuristic tree. Since a heuristic tree is made to help in the process of problem-solving, you learn more by clicking through a heuristic tree than by reading a worked-out solution. A heuristic tree fosters a way of thinking, shows the logical steps to take – while the worked-out solutions only give the steps to get to the answer. Traditionally, students develop heuristics based on experience but as the post-test showed: you can be taught heuristics. This is in line with Schoenfeld's (1985) findings.

In line with Bos & van den Bogaart (2022), this study found that a lot of the students followed the approach offered in the heuristic tree. It has to be said that the same effect was seen in the worked-out solution group, where the students followed the approach of the worked-out solution. A new finding was that HT-students switched to the approach of the heuristic tree when seeking help there, sometimes because the heuristic tree was different to their approach, maybe thinking their solution was incorrect. This effect was probably reinforced by the fact that the teacher did not offer help. Bos & van den Bogaart (2022) also concluded that the below average students had difficulties with seeking help in the heuristic tree. In this study, this was also observed. If the teacher is present, they can help these students while other students can use the heuristic trees as support.

Limitations

There are a few limitations to this study. Firstly, this study was conducted in two classrooms. Unfortunately, students get sick, so in the HT-group and WS-group groups there were respectively only 10 and 15 students that did all problem tasks for the six consecutive weeks. Secondly, we used two parallel classes of pre-university social science stream mathematics students, since these were the students that the teacher-researcher teaches. Since there is less focus on problem-solving within social science mathematics, perhaps a group of natural science students would give different outcomes. Thirdly, in some cases, the mathematical ability of the students was not strong enough. The students were able to solve the problem tasks and to use the hints, but often they switched to the method of the heuristic tree / worked-out solution. The students often did not have the mathematical ability to see how to use the hints for their solution. The students also did not always recognize when their own solution was on the same path as the hints. Here one misses a teacher who can translate the hint to the solution the student is working on (given that their own approach is promising).

Final Thoughts

The study was focused on problem-solving in absence of a teacher. One could conclude that using a heuristic tree replaces a teacher more effectively than a worked-out solution, since a heuristic tree guides a student through the solving process, while a worked-out solution only shows the steps to the answer. The heuristic tree could work in combination with a teacher. Firstly, the hints offered in the heuristic trees used in this study were not always clear for the students, a teacher could give an additional explanation. Secondly, a teacher gets more time to help students that have difficulty with solving the problem when a part of the class can solve the problem task by themselves with help of a heuristic tree.

In the study the method to test for compression turned out to be invalid. The questionnaire contained a question to write down which mathematical concept(s) and which technique(s) the students had used when solving the problem task. However, this question was often kept blank by the students, or they only gave one-word answers like: “powers” or “addition”. Perhaps the students do not have the vocabulary to explain their techniques. An additional explanation might lie in the design choice for the heuristic trees. We chose to keep the trees compact, which led to some steps of compression that were done in one step, or were not made explicit. Therefore, the students have not learned the vocabulary from the heuristic tree. As shown in the results the students did pick up techniques, like the systematic guessing with help of a table (Figure 4).

A disadvantage of the heuristic trees is their inflexibility (Bos & van de Bogaart, 2022). The tree leads a student down the path of one specific solution. But the same can be said about worked-out solutions, these often also show only one solution. The advantage of the heuristic tree as opposed to worked-out solutions is that students are learning heuristics when using them. Also – as seen in the results – when both groups had no access to hints, their engagement was the same, but there is a strong indication that the heuristic trees foster more engagement than the worked-out solutions.

Another aspect making a difference between the groups was how accustomed the students were to the method of getting support. The heuristic tree was new and required more work from a student than a worked-out solution. The students in this study are used to accessing worked-out solutions when they get stuck while doing mathematics. The worked-out solution often gives instant answers for the student to continue working. However, many students do not engage with the worked-out solutions other than getting the answer, most times they do not try to understand why this would be the next step. A heuristic tree forces a student to do that thinking, since the cards always have heuristics on them to guide the students through the solving process.

The introduction started with the three main aspects of mathematical thinking. When we focus on the aspect of problem-solving and want to give this a more prominent role in Dutch mathematics education, the use of heuristic trees may be a good solution. In this study we have seen the following: students who use heuristic trees are more engaged with the problem task than students using worked-out solutions. Also, in plenary discussions of problem tasks, HT-students were more interested. The aim of the heuristic trees, to give students heuristics as hints to solve problem tasks, bears fruit. Once the students learn these heuristics, they can grab the fruit of the trees. Finally, the heuristic tree can create time for a teacher to help students who profit more from in-person individual help.

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