The derivative through nomograms

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Author:
Laura Brinks

Supervisor:
Dr. R. D. Bos
University Utrecht

Second examiner:
Prof. dr. P. H. M. Drijvers
University Utrecht

Utrecht University

Abstract
When teaching the derivative, geometric meaning-making is usually supported by the visual context of graphs and tangent lines. However, this way of meaning-making is somewhat indirect, easily forgotten, and not always meaningful in certain contexts. To address the challenge of fostering meaning-making of the derivative, our research focuses on how nomograms can enhance this. We developed a dynamic digital learning environment that introduces nomograms as an additional geometry context to provide meaning-making to the instantaneous rate of change as an enlargement factor concerning a local focus. Our pilot study, involving twenty-five 10th-grade pre-university students, provides insights into their thought processes and shows the challenges of this approach, suggesting several ways to improve the design for the next design cycle.

Keywords*: nomograms, dynamic digital environment, calculus education.*

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Introduction

“Math is too difficult”, is probably the most expressed sentiment in a Mathematics class. Secondary school students often perceive mathematics as complex and intricate, particularly in the formal manipulation of relationships, such as in Calculus. Conceptual thinking around relationships and functions is also a significant hurdle for many students.

However, it is important to recognize that the formal handling of relations, particularly in Calculus, grows increasingly important as students progress through upper-secondary education. Calculus serves as a defining feature of modern mathematics, showcasing its strength and adaptability in reducing complex problems into manageable rules and procedures. Its applications span a wide range of knowledge domains, including mathematics, physics, engineering, social sciences, and biology (Berry & Nyman, 2003; Kleiner, 2001).

One of the central concepts within Calculus is the derivative. However, certain difficulties arise in understanding this concept due to teaching methods that prioritize algorithmic approaches. This emphasis on algorithms often leads to significant challenges and errors for students when they encounter tasks that require a deeper understanding of the meaning behind derivatives (Fuentealba et al., 2018). In the Netherlands, the derivative is usually introduced geometrically as the slope of the tangent to the curve. While this provides a meaningful visualization of the concept, we believe that this perspective does not provide the most appropriate picture for understanding the derivative as an instantaneous rate of change.

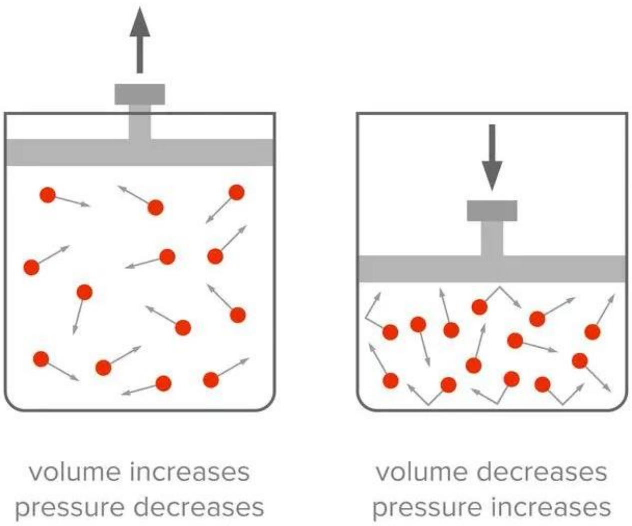


Figure 1 Relationship between volume and pressure in a gas barrel

In a closed gas barrel with a pressure regulating valve, the volume can be described by the function $V(p) = \frac{c}{p}$. The derivative $V'(p) = -\frac{c}{p^2}$ expresses the sensitivity of the volume as a function of pressure. For example, when the pressure is low, the effect on the volume of an increase is large and negative. This can be observed in the graph of V because the slope is steep and negative for small values of p. However, the sense of rate is not easily associated with the steepness of the graph (Thompson, 1994). We hypothesize that the sense of rate can potentially be more naturally understood in terms of enlargement.

An alternative representation, known as a nomogram, will be proposed as a solution to this context. Nomograms, also called arrow graphs (Bos, 2024) or parallel axes representations, have been previously investigated by Nachmias and Arcavi (1990). In our current research, we use the term ‘nomogram’ to describe a function as a family of arrows from input values to corresponding output values (Figure 2).

Nachmias and Arcavi emphasize that linear functions correspond to nomograms in which the arrows intersect at a focal point, extending the arrows into lines if necessary. This point of intersection is called the focus. When the rate of change is equal to 1, the arrows are parallel, and the focus point is ‘at infinity’.

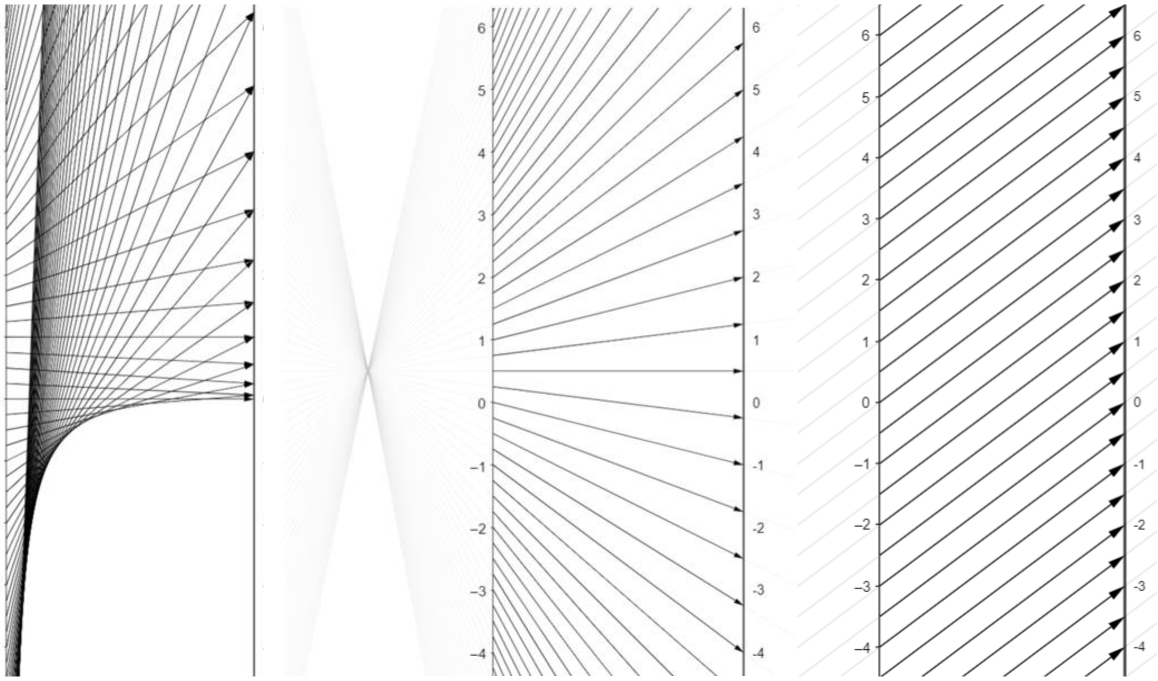


Figure 2 ***a*** $f(x) = x^2$ ***b*** $g(x) = 3x - 1$ *with a focus.* ***c*** $h(x) = x + 3$ *with a focus on infinity.*

As a result, the rate of change of a linear function can be interpreted geometrically as a factor of enlargement in the nomogram. An interval on the input axis is enlarged to an interval on the output axis concerning the focus (Figure 3a). The rate of change corresponds exactly to the enlargement factor between these intervals. The main goal of our study is to investigate whether teaching this new geometric interpretation of the rate of change, in addition to the usual interpretation as slope, supports students' understanding of the rate of change.

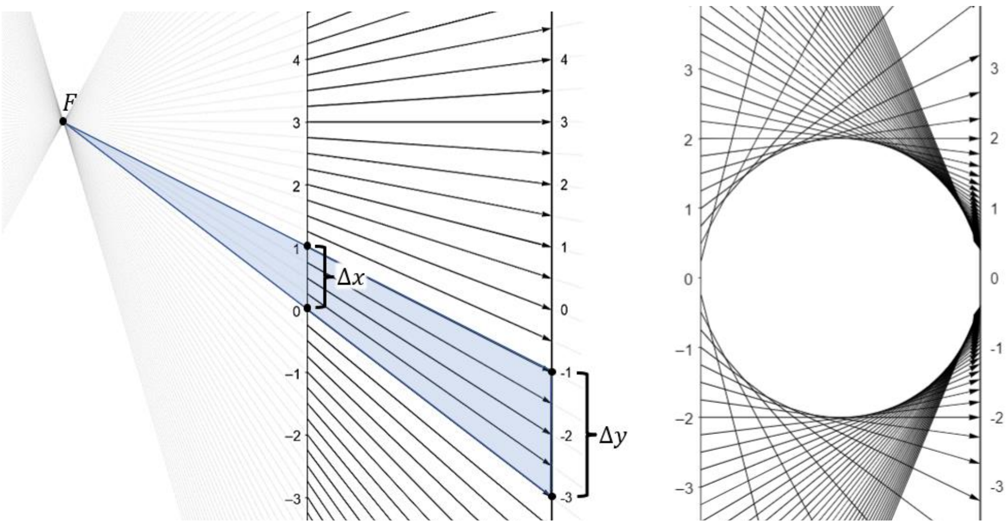


Figure 3 ***a*** *The rate of change of* $f(x) = 2x - 3$ *equals the enlargement factor* $\frac{\Delta y}{\Delta x} = 2$ *of the interval* $[0,1]$ *to* $[-3,-1]$ *with respect to the focus* F *in the nomogram.* ***b*** *The circle as an emerging enveloping curve for the function* $g(x) = \frac{4}{x}$ *in the nomogram.*

Considering the nomogram of a nonlinear function, a certain enveloping curve appears in the nomogram (Figure 3b). These curves in general turn out to be beautifully related to the derivative, as will be shown in a moment.

A function is differentiable only if it is locally linear. In the graph, this means that the function f around a point $(a, f(a))$ can be approximated by a line called the tangent. In the nomogram, it means that on an increasingly small interval around a on the input axis, the arrows approximately pass through one point. The smaller this interval, the sharper and more precise this point becomes visual in the nomogram (Figure 4). The corresponding limit point of the intersection of the arrows is called the local focus. The value of the derivative can be interpreted as the enlargement factor concerning this local focus. Furthermore, the local focus points form the previously mentioned envelope curve. In addition to the previously mentioned purpose, our goal is to investigate how the instantaneous rate of change and derivative can be taught using nomograms.

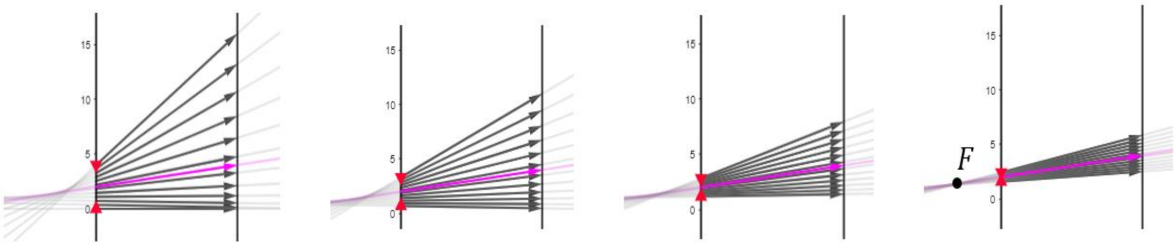


Figure 4 Arrows within an increasingly small interval intersect at an approximate local focus point

This paper presents the findings of the first cycle of design-based research. It will first present a brief outline of the literature related to derivative learning, as well as connecting multiple representations and learning with dynamic digital tools as a basis for inquiry-based learning. Next, the intervention will be shown, which consists of two modules each lasting approximately one hour. Finally, the results of the implementation in a 10th-degree pre-university class will be presented and it will be discussed how the intervention can be improved, based on an analysis of the results and conclusions. Through this research, the aim is to answer the research question: how can interactive tasks in GeoGebra, in which nomograms play a central role, promote the meaning-making of the derivative?

Theoretical framework

The theoretical framework includes notions from offering various representations to the object-process layer model of Zandieh (2000) and insights into dynamic digital geometry environments. These theories form the basis for the design principles that will be applied in developing the intervention for this research.

Various representations

Offering various representations of a specific concept is a crucial element in the teaching and learning of mathematics (Vergnaud, 1987). Each of the representations brings different aspects of the concept to the foreground (Nachmias & Arcavi, 1990). By comprehending different representations and having the ability to translate between them, diverse relationships and processes can be made explicit, facilitating reflection on the concept and potentially fostering further mathematical learning (Kaput, 1987). Also, according to Hiebert and Carpenter (1992), a "mathematical idea or procedure or fact is understood if it is part of an internal network. [...] The degree of understanding is determined by the number and strength of the connections". The better students can make connections between different representations, the better they will understand the concept, in this case, the derivative.

The concept of derivative can be represented in several ways. Firstly, it can be understood graphically as the slope of the tangent to a curve at a given point. Secondly, it can be described verbally as the instantaneous rate of change. Thirdly, physically, it can be thought of as speed or velocity. Finally, it can be represented symbolically as the limit of the difference quotient (Zandieh, 2000). Because of the unique perspective provided by the nomogram, we will extend Zandieh's table for this study, as shown in the table below. In doing so, we also used the already extended version of Roundy et al. (2015).

Table 1a
An extended theoretical framework for the concept of the derivative

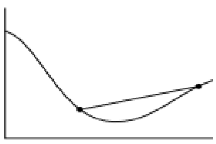
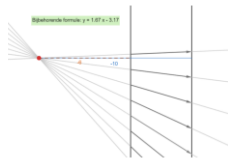
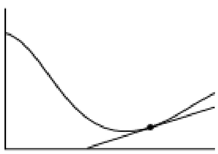
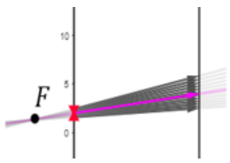
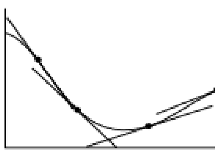
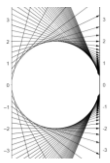
	Contexts			
	Graph		Nomogram	
	Graphical	Verbal	Graphical	Verbal
Process-object layer	Slope in graph	Rate of change	Enlargement	
Ratio		Average rate of change		Enlargement factor w.r.t. focus
Limit		Instantaneous rate of change		Enlargement factor w.r.t. local focus
Function		Rate of change at any point/time		Enveloping curve of local foci

Table 1b

An extended theoretical framework for the concept of the derivative

	Contexts	
	Paradigmatic Physical	Symbolic
Process-object layer	Velocity	Difference quotient
Ratio	The change in distance to the change in time: average velocity.	$\frac{f(x_0+h)-f(x_0)}{h}$
Limit	The average velocity over shorter and shorter intervals of time: instantaneous velocity.	$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$
Function	A function that is associated with each moment in time an instantaneous velocity.	$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Dynamic digital geometry environments

Once students learn to calculate the derivative by applying rules, they often develop a preference for this method, eliminating the geometric interpretation, if it existed at all. Students' preference for computational techniques seems to stem from a daily teaching practice that rapidly shifts from conceptual introduction to computational procedures (Thompson, 1994).

However, for a deeper understanding of derivation, it is desirable to reduce this computational emphasis in students. A dynamic digital geometry environment can help in this regard. Indeed, using interactive tools enables students to engage in higher-level mathematical problems (Nachmias & Arcavi, 1990). One reason is that time-consuming tasks, such as graphing and calculations, need to be performed less frequently. This allows students to focus on higher-level mathematical thinking, which involves being able to think in terms of different representations and intertwine them (Dickson, 1985). This is consistent with and important according to the theory of various representations mentioned earlier.

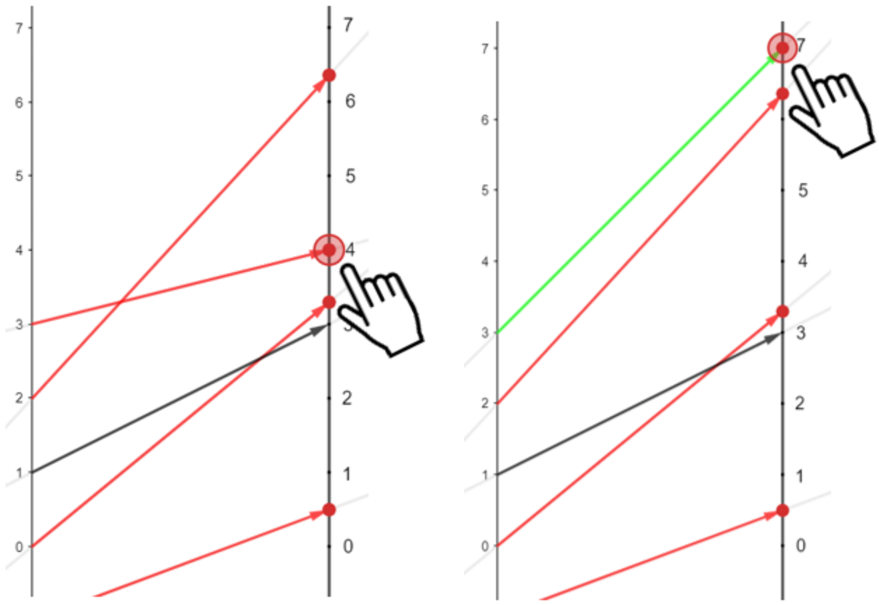


Figure 5 Through interactive learning environments, students receive immediate feedback

Another reason to use a dynamic digital interactive learning environment is that it provides the opportunity for inquiry-based learning (IBL). IBL is a teaching method in which students are encouraged to participate actively in the learning process. Students participate in activities and thought processes used by scientists to generate new knowledge (Maaß & Artigue, 2013). Interactive dragging tasks are a valuable tool within this approach, where students discover and explore concepts through learning-by-doing (Arzarello et al., 2002).

According to Uygun (2020), interactive geometry environments allow students to manipulate geometric figures and find the correct answers through trial-and-error, moving, dragging, animating, and adjusting objects. The first benefit of interactive dragging tasks is that they can provide students with immediate feedback (Figure 5), which helps reinforce their understanding and correct any misconceptions. This promotes deeper understanding and long-term knowledge retention. Moreover, by moving objects, such as the intersection with the y-axis of a linear graph, students can gain deeper insights than would be possible with a static image of a linear graph (example).

Finally, abstract concepts can be offered more intuitively in this way. Leung (2008) states that "dragging is considered a dynamic and powerful tool for acquiring mathematical knowledge through continuous, real-time transformations in which the properties of geometric objects can be maintained or approximately maintained".

Design principles

During the design process of the digital intervention for this research, we focused on two design principles using the above theory. The first design principle is the use of various representations side by side, for clarification and to make connections between each other. We consider this essential for students' knowledge acquisition about the nomogram. Firstly, because students are already familiar with the representation of the function prescription in combination with the graph. By matching the representation of the nomogram to this, we hoped to make the explanation of this more effective. Secondly, the use of different representations is important for understanding the derivative. According to the above the better students can make connections between different representations, the better they will understand the concept, in this case, the derivative.

This connection between different representations also plays an important role in our second design principle, which is to follow the process-object layers of Zandieh's (2000) model. This model, which we discussed in the theoretical framework, provides a clear structure between the different 'layers' of the derivative, starting with the average rate of change and ending with the rate of change at any point/time. Since the understanding of a previous layer is helpful for the understanding of the next layer, we explicitly kept these layers in our design to offer a structural build-up in knowledge about the derivative in the nomogram.

Method

Since the intended learning process about the derivative using nomograms could not be done in a regular educational setting because nomograms are not in regular Mathematics textbooks, we had to create a learning environment to enable this learning process. Therefore, we conducted a design-based study.

Context

The designed learning environment was piloted in a grade 10 class, with 25 students in a social science pre-university stream. These students attend school at the Jacobus Fruytier in Apeldoorn, where for mathematics the method *Getal en Ruimte* is used. This is important for the study's results, as this method is particularly focused on a more traditional way of teaching, which is in line with the educational approach at Jacobus Fruytier. However, the intervention will be inquiry-based, so it is important to take this into consideration. The students were taught by one of the researchers and during the pilot they were allowed to collaborate.

Hypothetical learning trajectory

For this research, two modules were designed in GeoGebra. The first module begins by activating prior knowledge about linear functions and graphs and then focuses on nomograms of linear functions. The second module builds on this foundation by interpreting the derivative within nomograms of linear and nonlinear functions. The designed tasks in these modules are partly inspired by the designed tasks of Wei et al. (2024).

In the remainder of this section, we describe part of the hypothetical learning trajectory. This includes only tasks that emerged as important in this research or whose results provided interesting insights. For each task, the task description will be given along with an image of the applet that students use in answering the question. The purpose and rationale of the task and design are also outlined. During the design of the modules, Zandieh's (2000) process-object-layer model was explicitly used and employed. Where this emerges and when a transition is made in the model will also be described.

Task 1.4A.

Description task. Drag the red point and find out the relationship between the purple values in the nomogram and the gradient of the graph. Write down your findings below.

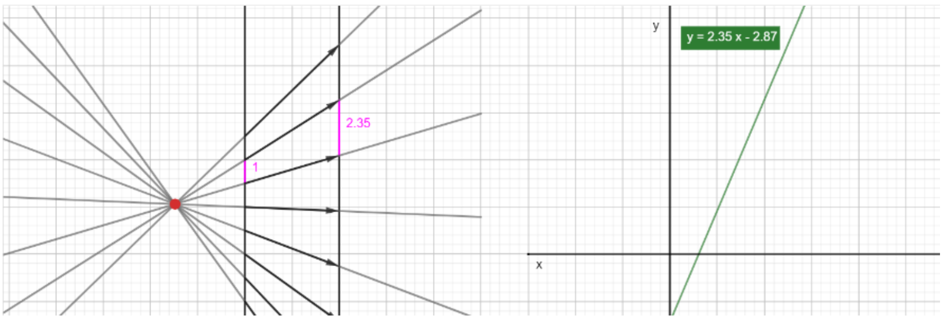


Figure 6 Applet task 1.4A

Purpose and rationale of task and design. The goal of task 1.4 is for students to discover that the enlargement factor in the nomogram corresponds to the gradient of the linear function, already knowing that it corresponds to the slope in the graph. The enlargement factor is visualized in two different ways in the nomogram, which will be explored in two separate subtasks.

In the first subtask, the nomogram, the graph, and the corresponding linear function will be displayed. Within the nomogram, an interval of size 1 on the input axis is highlighted in pink along with its corresponding image and size on the output axis. By dragging the red point in the nomogram, students can adjust the parameters of the linear function. At the same time, they can observe that the size of the enlarged interval corresponds to the gradient in the equation, which represents the slope of the graph.

Process-object layer nomogram. In subtask 1.4A, students explore the first layer of linear functions, namely the focus and enlargement factor in combination with the gradient.

Task 1.4B.
Description task. Drag the red point and find out the relationship between the orange and blue distances and the gradient. Write down your findings below. Use a calculator if necessary.

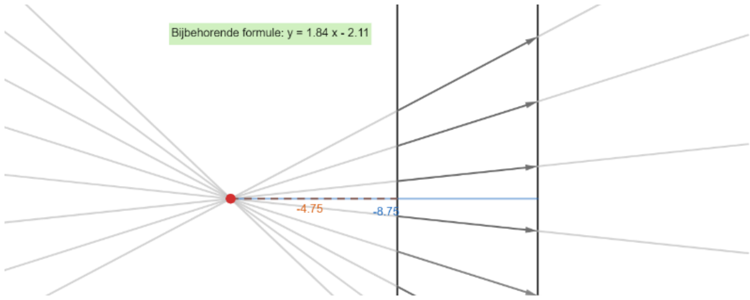


Figure 7 Applet task 1.4B

Purpose and rationale of task and design. In the second subtask, only the nomogram is shown along with the corresponding linear function. The orange and blue values are absolute distances of the focus from respectively the input axis and the output axis. Students again drag the focus. During this task, students are expected to realize that dividing the blue value by the orange value is an alternative method of calculating the enlargement factor, and is therefore equal to the gradient in the function.

Process-object layer nomogram. Students investigate the first layer, just as in the previous subtask.

Task 2.5.
Description task. Below you see the nomogram of $y = x^2$. In the following applet, you can take a smaller and smaller interval around 2 on the input axis. Each time 11 arrows are shown. The pink arrow belongs to the input value 2. The arrows get closer together on the input axis as you continue to reduce the length of the interval from which the arrows start. What do you notice about the light grey lines as the interval on the input axis is reduced?

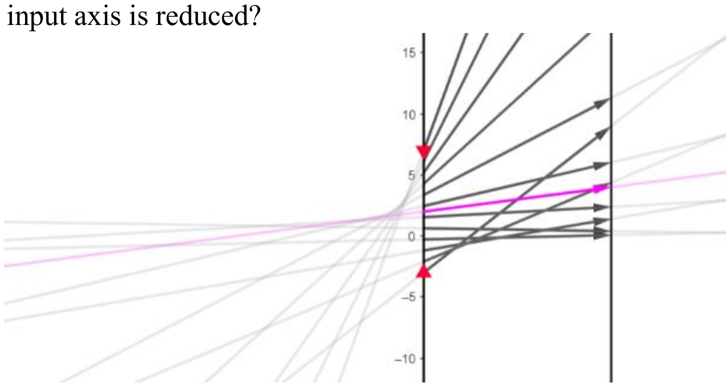


Figure 8 Applet task 2.5

Purpose and rationale of task and design. The goal for students is to gain an understanding of how local linearity is represented in the nomogram. Students can observe that as the interval from which the arrows of the nomogram are shown is reduced, the light grey lines through the arrows intersect at an approximate single point. Students can reduce the interval by using a dragger. By reducing the interval, they are engaging in local linearity in the context of the nomogram.

Process-object layer nomogram. In this task, students go through a transition from the first layer to the second layer. Namely, they are familiar with a linear nomogram and that it has a focal point. Using this knowledge, they can discover in this task that a nonlinear nomogram locally also has a focus point if it is locally linear.

Task 2.6A.
Description task. Below is the nomogram and graph of the height in meters of a falling object expressed in time as: $\text{height} = 18 - \frac{1}{2}(\text{time})^2$. Now the enlargement factor of height concerning the time at a given time is the object's velocity at that time. So an enlargement factor of 5 at time = 3 means that after 3 seconds the object falls at a speed of 5 m/s. When you drag the red points in the applet, you see that a pink interval varies with them. So you can determine the size of this interval by dragging the red points. The purple interval, which also varies is the distance that the object falls during the pink interval's time. For example, if you put the red points corresponding to the interval [0,5], you can read that the object falls 12.5 meters in the first five seconds. The average fall velocity is thus 12.5 meters ÷ 5 seconds = 2.5 m/s. Calculate the average speed on the interval [1,3]. Then verify your answer with the applet.

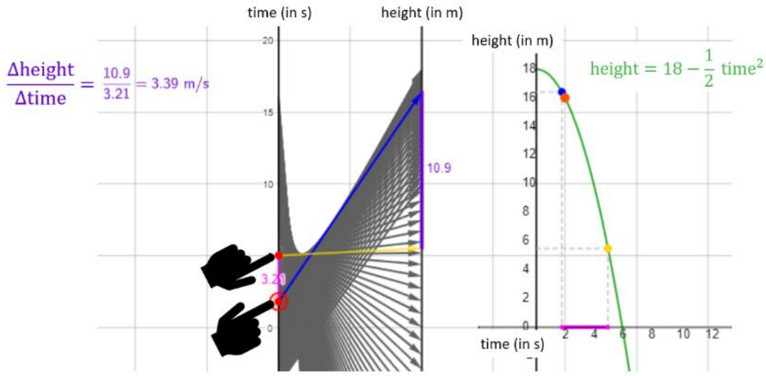


Figure 9 Applet task 2.6

Purpose and rationale of task and design. In this task, the goal is to establish a closer relationship between two different representations, where in this task the data from both representations are needed to perform the correct calculation. Students can reduce the interval on the input axis by dragging the two red input values, between which the distance between these values is given just like the enlargement of the interval on the output axis. The point in the coordinate system corresponding to the blue arrow in the nomogram is blue and the yellow arrow to the yellow point, respectively.

Process-object layer nomogram. This task clarifies once again the transition between the first layer and the second layer, but now in combination with the differential quotient. The transition from the first layer to the second layer when considering the differential quotient is to take the limit. This step is also followed in task 2.6 because, in the subsequent subtasks, the average speed must be calculated on an increasingly smaller interval to determine finally the speed at a point. In the first subtask, they operate in the first layer.

Task 2.6B.
Description task. Determine the average speed on a smaller interval: [1.5,2.5].

Purpose and rationale of task and design. In this task, the goal is to establish a closer relationship between two different representations, where the data from both representations are needed to perform the correct calculation. In this subtask, the students have to calculate the average speed on a smaller interval. The purpose for students is to get a better feel for how to take the limit and know what it means to take the limit on an increasingly smaller interval.

Process-object layer nomogram. From the beginning of the task to the end, the transition from the first to the second layer will be made. In this second subtask, students are still working in the first layer, but are already working toward the third layer.

Task 2.6C.

Description task. If you were to keep repeating this [determine the average speed on an interval] for a smaller and smaller interval, the average speed on the interval will become increasingly accurate to the speed at time is 2. This is exactly in words what the following means: enlargement factor = $\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right)$. Δx indicates how large you choose the interval. Then reduce this interval until you can determine the enlargement factor in the point. $\lim_{\Delta x \to 0}$ means that you choose the interval smaller and smaller until the length of the interval is almost zero. Estimate as accurately as possible the speed at time = 2.

Purpose and rationale of task and design. To give more words to the limit, the explicit example explains what the limit means. Combined with the previous tasks, the goal is for students to get a better feel for taking the limit and know what it means to take the limit on an increasingly smaller interval.

Process-object layer nomogram. In this final subtask, students operate in the second layer because they are working on the instantaneous rate of change in the nomogram.

Task 2.9A.

Description task. Below you see the nomogram of $y = x^2$. The green lines are two (continuous) arrows of the nomogram. Drag the red point. What do you notice about the light green points?

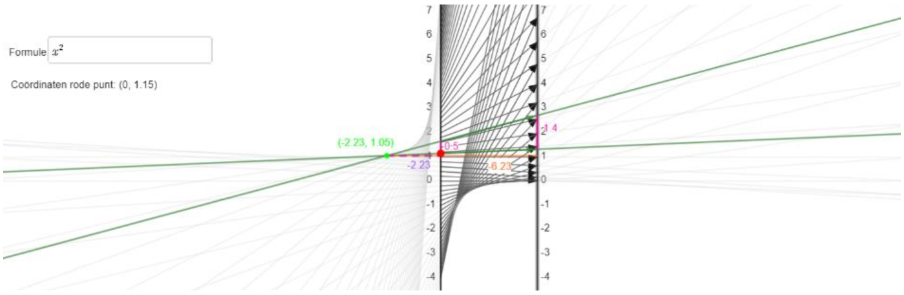


Figure 10 Applet tasks 2.9A and 2.9B

Purpose and rationale of task and design. The enveloping curve of the nomogram can be supportive in calculating the enlargement factor. Only the horizontal location of the focus is needed. The purpose of this task is to point this out to students. The first subtask is intended to draw attention to the enveloping curve.

Process-object layer nomogram. In task 2.9, students act at both the second and third layer levels. In this subtask, students can identify the enveloping curve (third layer) using local foci (second layer). Therefore, this subtask marks a transition from the level of the second layer to the third layer.

Task 2.9B.

Description task. What is the light green point anyway?

Purpose and rationale of task and design. The enveloping curve of the nomogram can be rather helpful in calculating the enlargement factor. In fact, only the horizontal location of the focus is needed. The purpose of this task is to point this out to students. Indeed, the intersection of the line through the arrow of an input value with the enveloping curve is the local focus relative to that input value. Students can observe this in the second task of this assignment and this is explained next.

Process-object layer nomogram. In task 2.9, students act at both the second and third layer levels. In this subtask, the level returns to that of the second layer, in which a controlling question is asked, from which the teacher might note whether students actually understand how the enveloping curve is constructed.

Task 2.9C.

Description task. For each input value, there is an enlargement factor. This is precisely the enlargement factor of the distance between the axes to the focus, belonging to that input value. When we connect all the foci (light green points), which we get by dragging the red point, we get a curve on which all the foci lie. Using this curve, we can thus determine the enlargement factor of an input value. What is the approximate enlargement factor associated with input value 2? Describe how you got your answer.

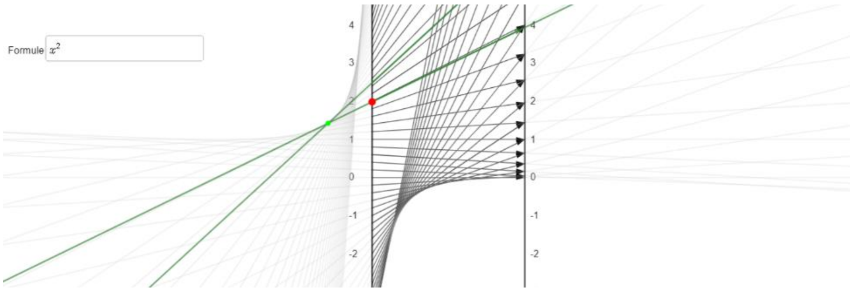


Figure 11 Applet task 2.9C

Purpose and rationale of task and design. In the last subtask, students are asked to determine the enlargement factor for input value = 2. Students can drag on the red dot for this purpose, to move it to input value =2. They can then determine (approximate) the enlargement factor in two ways, namely by using either the horizontal distances from the focus to the axes or the vertical distances on the axes.

Process-object layer nomogram. In this subtask, the enveloping curve is used to determine the enlargement factor in a point. However, this focuses more on the local focus associated with this point than the entire enveloping curve, making this subtask better suited to the level of the second layer, rather than that of the third layer.

Data collection and analysis

To analyze the results of this study, students' answers in the online learning environment were collected and scored. The answers were scored from 0 (not according to HLT) to 5 (completely according to HLT). To be able to make Table 2, we merged scores 4 and 5 as 'according to HLT.' This is justified because the answers of score 4 always contained the correct answer, although, for example, a unit may be missing or the answer may not be fully formulated. The remaining answers were merged as 'not according to HLT.

In addition, some students were interviewed individually during the pilot, and after the pilot, a group interview with five randomly selected students was conducted. These interviews were word-for-word transcribed and will be cited as quotes in the results.

Results, analysis, and local conclusions

The designed modules have produced compelling lessons and results, which we will discuss in this section. We first present a table that shows the extent to which the actual learning trajectory (ALT) is consistent with the hypothetical learning trajectory (HLT). Next, we will analyze these results. Only results from tasks relevant to the purpose of this study, namely whether teaching the geometric interpretation of rate of change as the magnification factor, in addition to the usual interpretation as slope, supports students' understanding, will be discussed. When discussing the results in the remainder of this section, local conclusions and suggestions for redesign will also be mentioned to make reading more continuous for the reader. Overall conclusions and discussion will be addressed in the next section.

Table 2

ALT compared with HLT conjectures for the tasks

+				x				x	
±	x						x		
−		x	x		x	x			x
Task	1.4A	1.4B	1.5	2.3A	2.3B	2.5	2.6A	2.6B	2.6C

+	x	x	x				
±				x			
−					x	x	x
Task	2.7A	2.7B	2.8A	2.8B	2.9A	2.9B	2.9C

Note: an x signifies how well the conjecture accompanying that task matched the observed learning (- refers to up to 1/3 of student answers are in the 'according to HLT' category, and + to at least 2/3 of student answers are in the 'according to HLT' category).

Task 1.4B.

Observations. Students were challenged to discover the relationship between the distances in the nomogram and the gradient. However, only seven of the twenty-five students managed to identify this relationship correctly. The remaining students stuck to superficial observations, such as "there is a relationship", or came to incorrect conclusions, such as "as the gradient gets lower, the numbers also get lower", and "the lower the gradient, the greater the distances of the blue and orange values [distances in the nomogram]". We believe that students may be not accustomed to inquiry tasks, because they are taught traditionally, and lack the experience to delve into the problem situation. We recognize this also in the answers to task 1.5. The task there was to say something about the gradient based on the location and displacement of the focus. Most students formulate a partially correct answer: "The gradient changes with horizontal displacement and remains the same with vertical displacement". Only two of the twenty-five indicated more precisely, what changes the gradient then goes through: "As you move [with the focus] closer to the nomogram, the gradient increases".

Suggestions for redesign. To ensure that student observations are less superficial, the teacher can help by applying discuss-the-screen in class after each student has completed the specific task (Drijvers et. al., 2010). This leads to a classroom discussion of what is happening in the task, on a classroom screen. The superficial observations will be mentioned first, but the teacher can subsequently direct to more in-depth observations. Suggestions from students can then be easily tried out, upon which quick and dynamic feedback can be given. It can be immediately found out whether the student's suggestion is true in its 'generality'. It can also then be addressed as to why the observation "As the focus in the nomogram approaches the input axis from the left, the gradient increases", is true. If so, the moment of discuss-the-screen not only enriches the mathematical content of the task or problem but also gives the learner tools on how to tackle such an 'open' task.

Task 2.3A.

Observations. Contrasting the answers from task 2.3A, where students had to identify which of the two graphs shown was linear, the results from task 2.3B show limited similarity to the HLT. In this task, students had to explain why a specific nomogram shown was or was not linear. Most students indicated that the nomogram was linear, which is correct. However, the reasoning of most students was "because the arrows are straight" or "they are straight lines". This suggests a misconception or a misunderstanding from another representation. The correct reasoning is: "The nomogram is linear because all the (continuous) arrows intersect in one point".

Suggestions for redesign. An intervention to debunk the just mentioned misconception can be by the method of spot-and-show (Drijvers et. al., 2010). In this form of orchestration, students' reasoning is brought to the forefront, which is exactly what is needed in this situation. Students must become aware of their answers and the reasoning behind them. Asking students in class about their answers and then having their classmates reflect and respond to them allows any misconceptions to be addressed. This is important for the continuing progress of the module.

Task 2.5.

Observations. As mentioned earlier, the modules follow Zandieh's (2000) process-object-layer model. This is exemplified in task 2.5, which focuses on discovering a local focus at a fixed input value by reducing the interval of departing arrows around this input value. Despite this setup, many students struggle to recognize this relationship. Instead, when asked what stands out as the interval on the input axis is reduced, they formulate answers such as: "The arrows get closer to the pink line [arrow belonging to input value] and to each other" or "The lines get straighter and straighter."

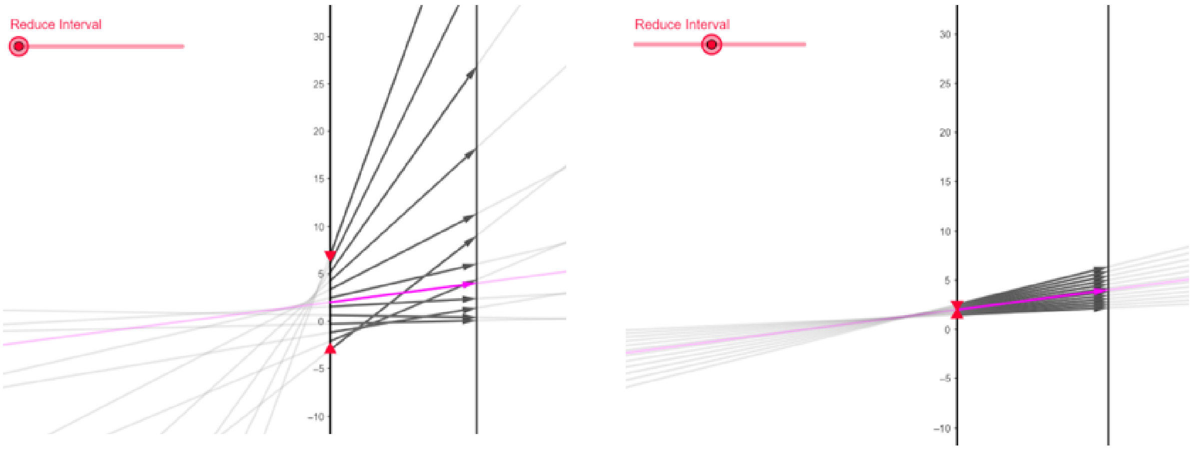


Figure 12a Current design task 2.5

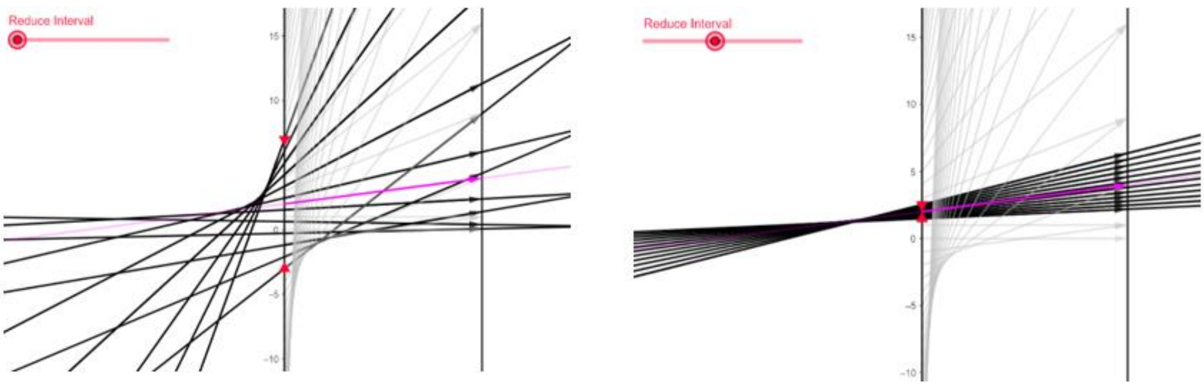


Figure 12b Proposed redesign task 2.5

Suggestions for redesign. This task was intended to be a discovery learning experience. In the theory that follows completing this task, local linearity is discussed from linearity onward. To bring the results of this assignment more in line with the HLT, we believe it is mainly important for students to have tools to deal with an open and discovering task. The intervention for this has been mentioned above, namely through discuss-the-screen, where a classroom conversation takes place about what is happening on the classroom screen and students can exchange their ideas and reflect on each other's ideas. The teacher then could guide this conversation and possibly take it to the next level. Furthermore, we would not want to intervene in this assignment, precisely because it is a discovery task, it should remain open-ended. Something that might be helpful here, though, is an adjustment in the visual design of this task. In the current situation, the interval on which the arrows are shown changes, which gives two very different pictures (Figure 12a). A possible improvement would be to show continuously the whole nomogram and just change which arrows are highlighted (Figure 12b). This way, students realize that considering a smaller interval does not change the function.

Task 2.6.
Observations. Task 2.5 is followed by a theoretical section on local focus in a nonlinear nomogram. Then task 2.6 serves as a reproduction task to this theory. In this task, given a given distance-time relationship, students are expected to estimate the speed at a specific time using average speeds at an ever-decreasing interval around that time. Both the formula, graph, and nomogram are shown here. The task is structured in several steps, with subtasks A and B as preparatory steps for subtask C. In the first subtask, many answers do not match the HLT because a concrete calculation to support the answer is missing. The transition from subtask B to C precisely marks the transition to the next layer, according to Zandieh's model (2000). The results suggest that this transition is not understood by most students.

Suggestions for redesign. To ensure that students write down a calculation on the first subtask, a small suggestion for redesign is to state explicitly in the task: "Write down your calculation". The low score of task 2.6C compared to the HLT may have several causes, two of which we would like to name. First, it may be due to the formulation of the question. It reads as follows: "Estimate as accurately as possible the speed at time = 2". During observation and guidance, we noticed that many students were unsure of their answers because of the words "as accurately as possible". This misled them and they thought they had to do something difficult because it had to be as accurate as possible. A possible adjustment in formulation to remedy this could be: "Estimate speed at time = 2". Another cause could be the transition from linearity to local linearity. The result then confirms that students are not yet ready for this step. To guide students in this, a reproduction task could be added after the theory, which students first make on their own but can then check against answers. The purpose of this task is then to apply the theory in a similar situation as in the theory to let the theory sink in. This could make the abstract theory more concrete for students, allowing them to then work with this theory independently.

Task 2.7.
Observations. Task 2.7 shows only the representation of the nomogram and omits the graph. Students are now asked not to estimate the velocity at a point, but to determine the enlargement factor corresponding to a point. As shown in Table 1, this task was performed in accordance with the HLT by most students. It may be because the applet in which the nomogram and values are shown is completely consistent with the applet shown in the theory. As a result, the enlargement factor could easily be determined by dividing an orange value by a blue value, a trick that could literally be applied in task 2.7.

Suggestions for redesign. Because the color formatting in this task is the same as the color formatting in the explanation of calculating the enlargement factor, we could not find out whether this trick actually caused the many correct answers. This points us to the next problem with this task: there is no way for the teacher (or in this case, the researcher) to determine whether a student truly understands the concept and understands why something works or is correct, or whether the student is just applying a trick. Therefore, conceptual learning must be made more visible. We will return to this later in the overall conclusion.

Task 2.8.
Observations. Task 2.8 goes beyond mere reproduction and application. The task focuses on identifying the acceleration of a falling object, both in the graph (2.8A) and the nomogram (2.8B). For task 2.8A, almost all answers are correct. However, the answers to task 2.8B vary more, from: “The line goes up” and “More lines in the nomogram” to the almost correct “Between the arrows is more and more distance”.

Suggestions for redesign. A notable aspect of these answers is a certain ‘formula shyness’, where students struggle with finding the appropriate words or sentence structures to phrase their answers accurately. This struggle arises from their unfamiliarity with how to construct sentences on the topic or from not knowing the suitable vocabulary to use. A class discussion may be an appropriate intervention to address this problem. This conversation need not necessarily be about this particular task, but it should be about a situation or problem with a nomogram. During the conversation, silences may fall when students cannot find words to express what they want to say. However, these silences create a need for words, allowing a valuable learning process to take place. In this process, the teacher can play an important role by providing words or sentence structures that can be used.

Task 2.9.
Observations. Finally, task 2.9 acts as a concluding assignment, taking the material of the module to a higher level, namely the final layer in Zandieh’s (2000) model. This task was added to the module because of its ultimate outcome (!): the derivative in a point can be determined in the nomogram by just the horizontal position of the intersection of the corresponding arrow with the emerging enveloping curve. However, the results suggest that this was an overstated goal of the researchers for the modules of this study. None of the students formulated a correct answer, except for a few cases in the first subtasks.

Suggestions for redesign. The conclusion, which will also be discussed more fully in the overall conclusion, is that progressing to the last layer of derivation is still a step too far within the time frame of the modules. The layer before that, the ‘local layer’, requires more practice first, as evidenced by superficial answers to subtasks such as: “It [the arrows] are getting more and more” and “[the light green dots] are making an arc”. While these observations are correct, they are superficial. This degree of superficiality is also evident in the answers to questions 2.9B and 2.9C.

Conclusion and discussion

This first cycle has brought with it conclusions that can be valuable for the further continuation of the research on methods of teaching the derivative using nomograms. Local conclusions have already been given in the results section. Global conclusions will be addressed in this section. We present three final conclusions, with supporting results.

Firstly, we conclude that during the students' learning process, more attention should be given to instrumental orchestration. During the design process, we focused mainly on the visual design of the modules, which resulted in a lack of adequate guidance in developing knowledge about the derivative in the nomogram for learners. This is supported by the aforementioned local conclusions, which show that applying specific forms of orchestration can solve or reduce certain problems. This will also be highlighted in the remainder of this section.

Secondly, we conclude that some tasks should be redesigned, particularly to make conceptual learning more visible. This would promote conceptual discussions among students and between students and teachers and facilitate our analysis as researchers. For example, based on this reason, task 2.7 should be redesigned. Here some students gave correct calculations and answers, but due to the design of the task, they had no opportunity to show why their answer was correct, thereby indicating that they understood what they were doing. Similar occurrences happen more often, including, for example, task 1.4A, where all students see that a value in the nomogram matches the value for the gradient in the formula, but otherwise do not have the opportunity to explain whether they know why these values match. Making conceptual learning more visible can be accomplished in several ways, for example, by adding phrases to the task such as, "Explain in words how you arrived at your answer". Conceptual learning can also be made visible by having the format of the task differ from the format of the theory. This can be achieved, for instance, in task 2.7, by avoiding the use of different colors in the nomogram. This prevents students from simply applying the trick of dividing 'orange by blue' and instead requires them to demonstrate a deeper understanding of calculating the enlargement factor concerning the local foci. A final example of making conceptual learning more visible is choosing one or two tasks in both modules that are explained to the class by a student. This is known as the Sherpa-at-work orchestration (Drijvers et.al., 2010). Having a student explain the task forces them to explain exactly what the answer is. To do this, we would choose a student who has given the correct answer to see if he or she actually understands the answer and can explain why it is correct. If the student cannot do this, it provides insight for the teacher, researcher, and the student themselves. A class moment can then lead to a class discussion that will allow this student, and probably others, to gain understanding. In summary, it can be concluded that both modifications in the design of the learning environment and the orchestration of the classroom situation contribute to better visibility of conceptual learning.

Thirdly, our research shows that learning to work with nomograms and understanding their relationships with functions and graphs is challenging. This challenge includes several aspects that we will outline. Firstly, it appears that some of the theoretical frameworks in the module were too long or inaccessible. This is evidenced by responses collected during interviews during and after making the modules: "A lot of [theory] you didn't know before and then you think: what am I reading now. And then you don't really understand anything". So, first of all, the complexity of learning appears to lie not only in the complexity of the concepts and tasks themselves, but also in the unfamiliarity with those concepts. Moreover, our research shows that in the current modules, too little time was spent on processing and internalizing the concepts within the modules. This is considered a shortcoming of the modules, manifested in the answers the students gave, for example, in the last task of the module. This task was intended to anticipate the next and final step in Zandieh's (2000) model. However, many students did not understand this transition, which the researchers believe is due in part to insufficient processing of the previous, required concepts. Specifically, students have difficulty understanding and articulating fundamental differences, such as those between the focus and local focus, and between the enlargement factor concerning the focus and the enlargement factor concerning the local focus in the nomogram. This is evidenced by student responses in interviews, which reveal that students largely fail to understand and articulate these conceptual differences. Finally, we found that classroom interaction between teacher and student is necessary for guiding the process of learning to work with nomograms. This was discussed earlier in the results section that describing the ways the teacher's role can be used to address certain challenges that emerged during specific tasks or situations. Students also indicated that they valued explanations from the teacher to clarify abstract theories: "I now appreciate explanations from a real teacher more". Although this was the first time students were building knowledge through an inquiry approach, and it is a logical consequence that they struggled with it, we argue that classroom moments should not be used only as traditional moments of explanation, but rather as guided moments, in ways described in the results section. During these moments, students are encouraged to formulate their answers to questions, while the teacher guides class discussion to clarify misconceptions, discuss strategies for approaching open-ended tasks, or share insights. These classroom moments can help students develop their inquiry skills. These skills then enable students to develop more and deeper knowledge. In conclusion, the difficulty of working with nomograms lies not in using the nomograms themselves, but in developing the necessary knowledge through the current module.

How can interactive tasks in GeoGebra, in which nomograms play a central role, promote the meaning-making of the derivative?

In summary, and to answer the research question posed above we can say that further development of the modules is needed. This development should not only emphasize visual aspects but also prioritize effective guidance and orchestration of the learning process, with the aim to optimize the development of students' knowledge and skills. Consequently, we cannot consider the research completed yet and are not ready to give up investigating the role of nomograms in teaching derivation, stimulated by the success of Wei et al. (2024) in using nomograms to foster functional thinking.

Several factors, independent of the module itself, may explain the limited success to date. Firstly, unlike working with graphs, which students are prepared for from an early age, nomograms are new to students and the learning curve in two hours is very steep, as mentioned earlier. Secondly, this research was conducted in a class from the social science stream. However, the nomogram could be more effective for students in the natural science stream, as they generally have more affinity for geometry. Therefore, the next step in the design study will be to test an expanded version of the modules, applying the suggestions for improvement previously raised while discussing the results of the first cycle with students in a natural science stream.

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