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Tropical Curve Counting

Master Thesis

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Abstract

Tropical geometry is the area of research that studies polyhedral-structured varieties over the max-plus algebra, which turns out to have surprising connections with classical algebraic geometry. One such connection is the Mikhalkin Correspondence, which states that the enumeration of nodal curves of fixed degree and number of nodes through a generic configuration of points in a toric surface is equivalent with counting analogous tropical curves. These enumerative invariants are called Severi degrees. Lothar Göttsche conjectured that for smooth surfaces these become polynomial if the number of nodes stay constant and the degree is sufficiently large. This has since been proven by Yu-jong Tzeng. In this dissertation, we give a detailed proof of the Mikhalkin Correspondence and implement a tropical algorithm that computes the Severi degrees of any toric surface by counting lattice paths. The data that this code yields is then used to confirm Göttsche's conjecture and explore the case when the surfaces admit a singularity.

List of Notation

Enumerative Geometry

$N'^{(X,L),\delta}$ Degree of the Severi variety $\mathrm{Sev}'_\delta(X, L)$. 9
$N^{(X,L),\delta}$ Degree of the Severi variety $\mathrm{Sev}_\delta(X, L)$. 9
$N^{\Delta,\delta}$ Abbreviation for the Severi degree $N^{(X(\Delta)L(\Delta)),\delta}$. 10
$N^{d,\delta}$ Degree of the Severi variety over $\mathbb{P}^2$. 9
$\mathrm{Sev}'_\delta(X, L)$ Severi variety parametrizing irreducible δ-nodal curves in the linear system $|L|$. 9
$\mathrm{Sev}_\delta(X, L)$ Severi variety parametrizing δ-nodal curves in the linear system $|L|$. 9

Other

$V(I)$ Closed subvariety of an affine space $\mathrm{Spec}\, R$ given by the ideal $I \subset R$.
$\mathbb{C}\{\{t\}\}$ Field of locally convergent Puiseux series. 20
$\mathrm{cl}_X(S)$ Closure of the subset $S \subset X$ in the topological space X.

Polyhedral Geometry

$C^\vee$ The normal cone of the cone $C \subset \mathbb{R}^n$. 64
$\Delta(f)$ Newton polytope of the (tropical) polynomial f. 64
$\Sigma(\Delta)$ Normal fan of polyhedron Δ. 65
$|\Sigma|$ The support of the polyhedral complex Σ. 64
$\mathrm{aff}(S)$ Affine hull of the set $S \subset \mathbb{R}^n$. 64
$\mathrm{cone}(S)$ The cone spanned by the set $S \subset \mathbb{R}^n$. 64
$\mathrm{conv}(S)$ Convex hull of the set $S \subset \mathbb{R}^n$. 64
$\mathrm{face}_{\mathbf{w}}(\Delta)$ Face of the polyhedron Δ determined by the normal vector $\mathbf{w}$. 64
$\preceq$ Face relationship between polyhedra. 64
$\mathrm{star}_\Sigma(\sigma)$ The star of the polyhedron σ with respect to the complex Σ. 65

Toric Geometry

D_ρ The divisor $D_\rho := \overline{O(\rho)}$ in $X(\Sigma)$ corresponding to a ray $\rho \in \Sigma$. 73
$L(\Delta)$ The divisor of $X(\Delta)$ associated to the polytope Δ. 73
$O(\sigma)$ The orbit in $X(\Sigma)$ corresponding to a cone $\sigma \in \Sigma$. 72
S_σ Semigroup $\sigma^\vee \cap \mathbb{Z}^n$ for $\sigma \subset \mathbb{R}^n$ a rational cone. 67
$X_K(\Delta)$ Toric variety over the field K associated to a rational polyhedron Δ. 73
$X_K(\Sigma)$ Toric variety over the field K associated to a rational polyehdral fan Σ. 69
$\mathbb{H}_r$ Hirzebruch surface. 71
$\mathcal{U}_\sigma$ Affine toric variety associated to a rational cone σ. 68

Tropical Geometry

$V(f)$ Corner locus of the tropical polynomial f. 14
Σ_f Polyhedral complex associated to tropical hypersurface defined by the polynomial f. 16
$\mathcal{T}(\Delta, S)$ Space of tropical curves with Newton polytope Δ and dual subdivision S. 22
$\mathcal{T}_{\mathcal{U},k}(\Delta, S)$ Space of tropical curves $T \in \mathcal{T}(\Delta, S)$ passing through the points in $\mathcal{U}$, where at least k of these points are vertices of T. 24
ν_f Legendre transform of tropicalization $\mathrm{trop}(f)$ of f. 16
$\mathrm{rk}(T)$ The rank of a tropical curve T. 23
$\mathrm{rk}_{\exp}(T)$ The expected rank of a tropical curve T. 23
$\mathrm{trop}(f)$ Tropicalization of Laurent polynomial f. 21

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1 Introduction

The enumeration of algebraic curves is an old and well-studied problem. For instance, in the 19th century, Jakob Steiner determined that the number of degree d curves with 1 node through $\frac{d(d+3)}{2} - 1$ generic points in the projective plane $\mathbb{P}^2$ is $3(d-1)^2$. This is a partial answer to the following central question:

How many δ-nodal curves of degree d exist through $\frac{d(d+3)}{2} - \delta$ generic points in $\mathbb{P}^2$?

The answer to this question is called the *Severi degree* of the projective plane. Other examples include the fact that there exists a unique line through two given points or that 5 generic points determine a conic.

More generally, one can investigate similar research questions for any projective surface X. In this case, instead of counting curves of fixed degree, one considers curves from some linear system $|L|$ given by a line bundle L on X. Throughout this thesis, we identify divisors with their corresponding line bundles.

How many δ-nodal curves $C \in |L|$ exist through $\dim |L| - \delta$ *generic points in X?*

The resulting number is called the *Severi degree* of X and was first introduced by Enriques [9] and Severi [24]. As this will allow for the interplay with tropical geometry, this thesis will specifically focus on toric surfaces. Roughly speaking, this means that the surface X and divisor L are given by some 2-dimensional lattice polytope $\Delta \in \mathbb{R}^2$. The Severi degree of the surface X of δ-nodal curves from $|L|$ is then denoted by $N^{\Delta,\delta}$.

Meanwhile, research in tropical geometry has quickly risen in popularity during the last few decades. This subject studies varieties over the max-plus algebra, where ordinary addition is replaced by taking the maximum and ordinary multiplication has become normal addition. The varieties one then obtains are weighted polyhedral complexes. In the two-dimensional case, tropical curves look like weighted graphs that satisfy a certain balancing condition.

A standard operation in tropical geometry degenerates any algebraic variety to a tropical one. This process, called *tropicalization*, forgets many of the features of the original variety, although some traits are unaffected. For instance, concepts such as the degree, genus, reducibility and number of nodes of a curve all carry over, under some mild conditions, to the tropical setting.

Given the similarities between the two fields, a natural question to ask would be if a tropical version of enumeration of curves exists. Grigory Mikhalkin answered this question in his celebrated paper [21] by showing that the Severi degrees can be computed by counting tropical curves.

Theorem 1.1 (Mikhakin)**.** *There exist positive integral weights $\mu(T)$ for each tropical plane curve T such that the following holds. Fix $\delta \in \mathbb{Z}_{\geq 0}$ and $\Delta \subset \mathbb{R}^2$ a two-dimensional polytope. Let $\mathcal{T}$ denote the set of δ-nodal tropical curves with Newton polytope Δ through* $\dim |L(\Delta)| - \delta$ *generic points in $\mathbb{R}^2$. Then,*

$$N^{\Delta,\delta} = \sum_{T \in \mathcal{T}} \mu(T).$$

Furthermore, Mikhalkin proposed an algorithm for enumerating tropical curves by counting lattice paths in the polytope Δ.

A different perspective on Severi degrees has been offered by Lothar Göttsche in [11]. He proposed that for a smooth projective surface X, the numbers $N^{\Delta,\delta}$ become polynomial in the intersection numbers L^2, LK_X, K_X^2 and the holomorphic Euler characteristic $\chi(\mathcal{O}_X)$ if L is sufficiently ample. Moreover, these polynomials are identical for each smooth projective surface X and can be generated from universal power series. Göttsche's original conjecture has since been proven and generalized to different settings via several methods, including tropical ones. For instnace, in [18] and [12] Göttsche's conjecture is generalized and proven for possibly singular projective toric surfaces.

In this thesis, we give a detailed proof of Mikhalkin's correspondence due to Eugenii Shustin [25]. Furthermore, we elaborate on the lattice path algorithm, and use our own implementation to compute the Severi degrees of several toric varieties. This data is then used to determine the node polynomials predicted by Göttsche's conjecture for several toric varieties, including singular ones. We then confirm Göttsche's predictions for these cases, find bounds where the polynomiality fails and explore the singular case.

No prior knowledge on tropical geometry is assumed, and a chapter is present to introduce the reader with the foundational building blocks required to understand the basics. However, an elementary understanding of algebraic geometry, and preferably of toric geometry as well, is required. Nonetheless, a brief introduction on toric geometry has been added to the appendix to guide the reader unfamiliar with this subject.

2 Counting Curves

This chapter introduces the main objects of study in this dissertation: the Severi degrees. Roughly speaking, these are enumerative invariants of projective surfaces that count curves satisfying certain constraints. In the first section, we define this notion more precisely, and elaborate on the toric case. The second section tells the story of the Göttsche Conjecture, which claims that these enumerative invariants enjoy a polynomial relationship.

2.1 Severi Degrees

Let X be a projective surface over $\mathbb{C}$, L a line bundle on X and $\delta \geq 0$ an integer. Note that, for now, we do not require X to be smooth. Throughout this thesis, we identify the line bundle L with its corresponding divisor. The Severi variety $\mathrm{Sev}_\delta(X, L)$ is the subvariety of the linear system $|L|$ that consists of all curves $C \in |L|$ containing δ nodes as only singularities. A node is a singularity that is analytically isomprhic to $\operatorname{Spec} \mathbb{C}[x, y]/(xy)$. That is, a node conists of two smooth branches intersecting each other transversally. The Severi variety is a quasi-projective variety of dimension $\dim |L| - \delta$. Moreover, the subset $\mathrm{Sev}'_\delta(X, L) \subset \mathrm{Sev}_\delta(X, L)$ containing the irreducible curves forms a subvariety of equal dimension.

Definition 2.1. The degrees of the varieties $\mathrm{Sev}_\delta(X, L), \mathrm{Sev}'_\delta(X, L)$ are known as *Severi degrees*, and are denoted by $N^{(X,L),\delta}$, $N'^{(X,L),\delta}$ respectively.

An alternative description of Severi degrees is given by counting curves passing a generic configuration of points. Let $x_1, \ldots, x_{\dim|L|-\delta} \in X$ be such a set of points. Being generic means that the point $(x_1, \ldots, x_{\dim|L|-\delta})$ lies in an appropriate Zariski open subset of $X^{\dim|L|-\delta}$. This assures that the count is independent from the choice of points. The Severi degree $N^{(X,L),\delta}$ then equals the number of δ-nodal curves from $|L|$ that pass the points $x_1, \ldots, x_{\dim|L|-\delta}$. Furthermore, $N'^{(X,L),\delta}$ equals the number of irreducible curves meeting these conditions.

Example 2.2. Let d be a positive integer and $H \subset \mathbb{P}^2$ a line in the projective plane. The linear system $|dH|$ contains all plane curves of degree d. Given a curve $C \in |dH|$, its arithmetic genus is $(d-1)(d-2)/2$. So, let $0 \leq \delta \leq (d-1)(d-2)/2$ be an integer. The Severi degrees $N^{d,\delta} := N^{(\mathbb{P}^2,dH),\delta}$ and $N'^{d,\delta} := N'^{(\mathbb{P}^2,dH),\delta}$ are equal to the number of (irreducible) degree d curves having δ nodes as only singularities and passing $\dim |dH| - \delta = d(d+3)/2 - \delta$ generic points. We list a couple of instances of Severi degrees.

- $N^{1,0} = N'^{1,0} = 1$. There exists a unique line through each pair of points.
- $N^{2,0} = N'^{2,0} = 1$. Five points define a quadric.
- $N^{3,0} = N'^{3,0} = 1$. Nine points define a cubic.
- $N^{2,1} = 3$ and $N'^{2,1} = 0$. There are three combinations for two lines to pass four points and there exists no singular irreducible quadrics.
- $N^{3,1} = N'^{3,1} = 12$. The closure of $\mathrm{Sev}_1(\mathbb{P}^2, 3H)$ in $|3H|$ equals the zero locus of the discriminant of a general cubic. This is a polynomial of degree 12. For more information, see for instance [19, Example 1.7.2].

  Moreover, reducible degree 3 curves are either the product of three lines or the product of a quadric and a line. The first case has 3 nodes, while the second contains 2. Hence, each cubic with one node is automatically irreducible.

△

In the previous example we have seen a couple of instances where the reducible and irreducible Severi degrees agree. This turns out to be true as long as the degree is sufficiently large, which the following lemma illustrates.

Lemma 2.3. *Let $\delta \geq 0$ and $d \geq \delta + 2$ two integers. Then, each δ-nodal plane curve $C \subset \mathbb{P}^2$ of degree d is irreducible. In particular, $N^{d,\delta} = N'^{d,\delta}$.*

Proof. Assume to the contrary that C is reducible and let $C_1, \dots, C_m$ be the irreducible components of C. Denote the degree of C_i by d_i and the cogenus by δ_i. Since C is nodal, the components $C_1, \dots, C_m$ intersect each other transversally. Because of Bézout's theorem, the cogenus of C is then given by

$$\delta = \sum_{i=1}^{m} \delta_i + \sum_{1 \leq i < j \leq m} d_i d_j \geq \sum_{1 \leq i < j \leq m} d_i d_j.$$

Here, d_i denotes the degree of the curve C_i.

If $m = 2$, then

$$\delta \geq d_1 d_2 = d_1(d - d_1) \geq d - 1 \geq \delta + 1,$$

which is a contradiction.

For $m \geq 3$, we have

$$\begin{aligned} 2 \sum_{1 \leq i < j \leq m} d_i d_j &= d^2 - \sum_{i=1}^{m} d_i^2 \\ &\geq d^2 - (d-2) \sum_{i=1}^{m} d_i \\ &= d^2 - d(d-2) = 2d \geq 2\delta + 4. \end{aligned}$$

Again, this implies that $\delta > \delta$, yielding a contradiction. □

In order to reach the tropical island, we restrict our attention to projective toric surfaces. These arise from 2-dimensional lattice polytopes $\Delta \subset \mathbb{R}^2$, and are denoted by $X(\Delta)$. Moreover, Δ also induces an ample divisor on $X(\Delta)$, which we denote by $L(\Delta)$. The reader unfamiliar with toric geometry may consult Appendix B, in particular Section B.3. To ease notation, we write $N^{\Delta,\delta}$ for $N^{(X(\Delta),L(\Delta)),\delta}$ and $N'^{\Delta,\delta}$ for $N'^{(X(\Delta),L(\Delta)),\delta}$.

The toric context allows for a more explicit description of the Severi degrees. By Corollary B.25, the linear system $|L(\Delta)|$ consists of curves $\mathrm{cl}_{X(\Delta)}\{f = 0\}$ for which $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ is a Laurent polynomial with Newton polytope contained inside Δ. Since $(\mathbb{C}^\times)^2$ is an open and dense subset of $X(\Delta)$, we may choose the configuration of generic points inside $(\mathbb{C}^\times)^2$. Consequently, the count $N^{\Delta,\delta}$ equals the number of δ-nodal curves $V(f) \subset (\mathbb{C}^\times)^2$, where $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ is a Laurent polynomial with Newton polytope Δ, such that $V(f)$ passes $\left|\Delta \cap \mathbb{Z}^2\right| - 1 - \delta$ generic points in $(\mathbb{C}^\times)^2$.

2.2 The Göttsche Conjecture

In [8], Di Francesco and Itzykson conjectured that the Severi degrees of the plane, $N^{d,\delta}$, are eventually polynomial in d. That is, there exists for each nonnegative integer δ a polynomial $n_\delta(d)$ and a lower bound $d_0(\delta) \in \mathbb{Z}_{\geq 1}$ such that $N^{d,\delta} = n_\delta(d)$ for all $d \geq d_0(\delta)$. The functions $n_\delta(d)$ are called *node polynomials*.

An additional observation by Göttsche in [11] generalizes this conjecture to any smooth projective surface. Göttsche's original hypothesis, which has since been proven, consists of the following powerful and surprising theorem.

Theorem 2.4 (Göttsche Conjecture)**.** *[15] [17] Fix $\delta \in \mathbb{Z}_{\geq 0}$. There exists a polynomial $t_\delta(x, y, z, w) \in \mathbb{Q}[x, y, z, w]$ such that if X is a smooth projective surface and L a sufficiently ample divisor with respect to δ, then*

$$N^{(X,L),\delta} = n_\delta(X, L) := t_\delta(L^2, LK_X, K_X^2, \chi(\mathcal{O}_X)).$$

Here, DE denotes the intersection number of two divisors D, E. Also, K_X denotes the canonical divisor of X and $\chi(\mathcal{O}_X)$ is the holomorphic Euler characteristic of X.

Furthermore, there exist formal power series $A_i \in \mathbb{Q}[[t]], i = 1, 2, 3, 4$ such that the generating function

$$n(X, L) := \sum_{\delta \geq 0} n_\delta(X, L) t^\delta$$

can be expressed as

$$n(X, L) = A_1(t)^{L^2} A_2(t)^{LK_X} A_3(t)^{K_X^2} A_4(t)^{\chi(\mathcal{O}_X)}.$$

Also, explicit formulas for $A_1(t)$ and $A_4(t)$ exist in terms of modular forms.

Example 2.5. Let $H \subset \mathbb{P}^2$ be a line in the plane and $d \geq 1$ an integer. The canonical divisor class of $\mathbb{P}^2$ is represented by $-3H$ and the holomorphic Euler characteristic of $\mathbb{P}^2$ equals 1. Hence,

$$(dH)^2 = d^2, \qquad dH \cdot K_{\mathbb{P}^2} = -3d, \qquad K_{\mathbb{P}^2}^2 = 9, \qquad \chi(\mathcal{O}_X) = 1.$$

Therefore, the generating function of the plane is given by

$$n(\mathbb{P}^2, dH) = A_1(t)^{d^2} A_2(t)^{-3d} A_3(t)^9 A_4(t),$$

and so, the polynomial $n_\delta(d)$ is of degree 2δ. △

The Göttsche conjecture has first been proven by Tzeng in [15] and later by Kool, Shende and Thomas in [17] using algebro-geometric techniques. However, we will focus on the tropical story. In [22], Fomin and Mikhalkin confirmed the Göttsche conjecture for $\mathbb{P}^2$ using a combinatorial tool named *floor diagrams*, which are decorated graphs that represent tropical curves. If one counts these objects correctly, with certain multiplicities, then the number agrees with the tropical curve count. Using this technique, they prove polynomiality with a threshold of $d_0(\delta) = 2\,\delta$.

In [2], Block improved the techniques invented by Fomin and Mikhalkin, and established polynomiality for $d_0(\delta) = \delta$. Moreover, optimizations in the algorithm allowed for the computation of all node polynomials with $0 \leq \delta \leq 14$. For these, he confirms that the polynomiality threshold lies at $d_0(\delta) = \lceil \frac{\delta}{2} \rceil + 1$.

The use of floor diagrams is not exclusive to $\mathbb{P}^2$ but can be extended to any projective toric surface $X(\Delta)$ for which Δ is an *h-transverse polygon*. This means that the slopes of edges of Δ equal $0, \infty$ or $1/k$ for $k \in \mathbb{Z}, k \neq 0$. Examples of such surfaces include Hirzebruch surfaces and weighted projective planes $\mathbb{P}(1, 1, m), m \in \mathbb{Z}_{\geq 1}$. In [1], Ardila and Block parametrize h-transverse polygons by the slopes $\mathbf{c}$ and lattice lengths $\mathbf{d}$ of their edges. They prove that $N^{\Delta,\delta}$ is given by universal polynomials in $\mathbf{c}$ and $\mathbf{d}$, whenever these vectors are large enough with respect to δ. Their results do not fully overlap with the Göttsche Conjecture, as toric surfaces $X(\Delta)$ given by h-transverse polygons Δ are not in general smooth. Furthermore, they prove polynomiality in the combinatorial data $\mathbf{c}, \mathbf{d}$ instead of the geometric data $L^2, K_X L, K_X^2, \chi(\mathcal{O}_X)$ of the surface. Liu and Osserman remedy this shortcoming in [18] by again using similar techniques on floor diagrams. There, they prove a version of Göttsche's Conjecture for projective toric surfaces with cyclic quotient singularities.

These results will come back at the end of Chapter 5.1, once we have established an algorithm for computing Severi degrees. Using the data retrieved from this algorithm, we confirm the various variations on the Göttsche Conjecture for specific cases, and compute numerous node polynomials and universal power series.

3 Tropical Geometry

Tropical geometry has become a vast and swiftly growing field of study over the last decades. Therefore, out of necessity we restrict ourselves to a small portion of this subject. Particularly, we introduce only those concepts necessary for the understanding of Mikhalkin's correspondence.

For instance, the tropical semiring and tropical polynomials are studied in Section 3.1. Next, it is shown how a tropical polynomial defines a tropical hypersurface in Section 3.2. In addition, we prove how these objects form polyhedral complexes that are dual to subdivisions of the Newton polytope. Thereafter, we restrict our attention to planar curves in 3.4. In particular, nodal and simple curves, along with their properties, are studied. Lastly, we define general position of points in the tropical setting in 3.5.

For a more wide-scoped introduction to tropical geometry, the reader may consult [19].

3.1 Tropical Arithmetic

In the tropical world, regular addition is replaced by taking the maximum and regular multiplication is replaced by regular addition. The resulting algebraic structure is called the tropical semiring, which is the main subject of study this section. We are mainly interested in polynomials over this semiring, which are certain piecewise-linear functions on $\mathbb{R}^n$. In turn, we prove sufficient conditions for any piecewise-linear function to be a tropical polynomial.

Definition 3.1. The *tropical semiring*, or *max-plus algebra*, is the set $\mathbb{R}\cup\{-\infty\}$ equipped with two binary operations, $\oplus, \odot$, which are given by

$$a \oplus b := \max\{a, b\}, \qquad\qquad a \odot b := a + b,$$

for all $a, b \in \mathbb{R}\cup\{-\infty\}$. By convention, we set $a \oplus -\infty = a$ and $a \odot -\infty = -\infty$ for all $a \in \mathbb{R}\cup\{-\infty\}$. Tropical exponentiation is defined via repeated multiplication. To be more precise, for an integer n and $x \in \mathbb{R}\cup\{-\infty\}$, we set

$$x^{\odot n} := \begin{cases} \underbrace{x \odot x \odot \ldots \odot x}_{n \text{ times}} & \text{if } n > 0, \\ -\underbrace{x \odot x \odot \ldots \odot x}_{-n \text{ times}} & \text{if } n < 0, \\ 0 & \text{if } n = 0. \end{cases}$$

In other words, $x^{\odot n} = n \cdot x$. In the future, we might drop the $\odot$ to ease notation. The tropical semiring is denoted by $\mathbb{R}_{\text{trop}}$.

Note that the max-plus algebra satisfies all axioms of a commutative ring with 1, except for the possession of an additive inverse. Hence, $\mathbb{R}_{\text{trop}}$ is indeed a semiring. The additive and multiplicative identities are $-\infty$ and 0 respectively.

In some of the literature, such as in [19], the min-plus algebra is used instead. This is the set $\mathbb{R}\cup\{\infty\}$ along with the minimum operator and regular addition. The min-plus and max-plus algebras are isomorphic via the map $x \mapsto -x$, and so this choice is merely one of convention.

In order to study geometry over the tropical semiring, it is natural to introduce tropical polynomials.

Definition 3.2. Let n be a positive integer. A *tropical polynomial* in n variables is a real function on $\mathbb{R}^n$ given by an expression of the form

$$f(x_1, \ldots, x_n) = \bigoplus_{\mathbf{u}\in\mathbb{Z}^n} a_{\mathbf{u}} \odot \mathbf{x}^{\odot\mathbf{u}} = \max_{\mathbf{u}\in\mathbb{Z}^n}(a_{\mathbf{u}} + \mathbf{u}\cdot\mathbf{x}),$$

where finitely many of the $a_{\mathbf{u}} \in \mathbb{R}_{\text{trop}}$ are finite.

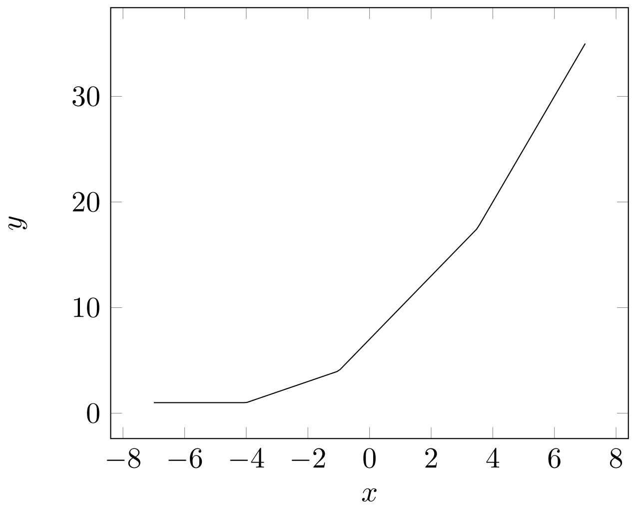


Figure 1: The graph of a tropical polynomial in one variable.

It may be fruitful to stress that a tropical polynomial is a function instead of a formal expression. Therefore, two tropical polynomials are equal whenever they are equal as functions, which allows for multiple expressions to represent the same tropical function, as is showcased in the following example.

Example 3.3. Consider the tropical polynomial

$$f(x) = x^5 \oplus 7x^3 \oplus -2x^2 \oplus 5x \oplus 1.$$

Figure 1 illustrates the graph of f. From this, it is clear that f is a piecewise-linear, convex and continuous function with a finite number of linearity domains. Moreover, we have $f(x) \neq 2x - 2$ for any $x \in \mathbb{R}$. Hence, f may also be written as

$$f(x) = x^5 \oplus 7x^3 \oplus 5x \oplus 1.$$

△

Example 3.4. Consider the tropical polynomial in two variables

$$f(x, y) = x \oplus y \oplus 0.$$

Its graph is plotted in Figure 2. Note that the graph is again continuous, piecewise-linear and convex. Moreover, the domains of linearity are finite and are given by the polyhedra $\{x, y \leq 0\}, \{x \geq 0, y\}, \{y \geq 0, x\}$. △

The observations from the latter two examples hold in more generality, as the following lemma illustrates.

Lemma 3.5. *[19, Lem. 1.1.2] Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function. Then, f is a tropical polynomial in n variables if and only if it is piecewise-linear, continuous, and convex with a finite number of linearity domains such that each of its linear subfunctions can be written as*

$$\mathbf{x} \mapsto \mathbf{w} \cdot \mathbf{x} + c$$

for $\mathbf{w} \in \mathbb{Z}^n$ and $c \in \mathbb{R}$.

Proof. The implication from left to right is clear. So, assume that f is piecewise-linear, continuous, and convex with a finite number of linearity domains and integer coefficients in the linear functions. Let $g_1, \ldots, g_N$ be the linear functions whose values f attains. Then, there exist $a_i \in \mathbb{R}, \mathbf{u}_i \in \mathbb{Z}^n$ such that

$$g_i(\mathbf{x}) = a_i + \mathbf{u}_i \cdot \mathbf{x},$$

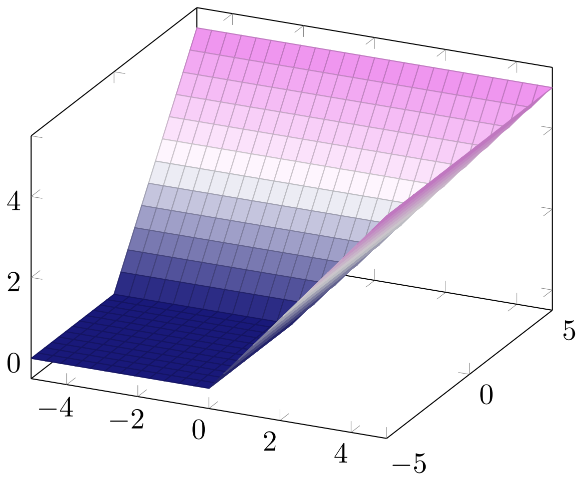


Figure 2: The graph of a tropical polynomial in two variables.

for all $\mathbf{x} \in \mathbb{R}^n$, $1 \leq i \leq N$.

Next, let $\mathbf{x} \in \mathbb{R}^n$ be given and $1 \leq i \leq N$ an index such that $f(\mathbf{x}) = g_i(\mathbf{x})$. Also let $j \neq i$ be another index and $\mathbf{y} \in \mathbb{R}^n$ an interior point of the domain of g_j. By continuity, such an interior points exists, since otherwise for each $\mathbf{y}$ in the domain of $g_j(\mathbf{y})$ there exists a k for which $g_j(\mathbf{y}) = g_k(\mathbf{y})$, and so, g_j would be superfluous.

As $\mathbf{y}$ is an interior point of the domain of g_j, there exists a $\lambda \in (0, 1)$ such that

$$f(\lambda\,\mathbf{x} + (1-\lambda)\mathbf{y}) = g_j(\lambda\,\mathbf{x} + (1-\lambda)\mathbf{y})$$

as well. By linearity of g_j and convexity of f, this means that

$$\lambda\,g_j(\mathbf{x}) + (1-\lambda)g_j(\mathbf{y}) = f(\lambda\,\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda\,g_i(\mathbf{x}) + (1-\lambda)g_j(\mathbf{y}).$$

Consequently, $g_i(\mathbf{x}) \geq g_j(\mathbf{x})$. Because this holds for all $1 \leq j \leq N$, we deduce that

$$f(\mathbf{x}) = g_i(\mathbf{x}) = \max_{1 \leq j \leq N} g_j(\mathbf{x}),$$

and by varying $\mathbf{x}$, we conclude that

$$f = \max_{1 \leq i \leq N} g_i = \bigoplus_{1 \leq i \leq N} a_i \odot \mathbf{x}^{\odot \mathbf{u}_i}.$$

□

3.2 Tropical Hypersurfaces

Analagous to the algebro-geometric case, a tropical hypersurface is a subset of $\mathbb{R}^n$ cut out by a tropical polynomial. One could define varieties determined by multiple tropical polynomials, but this requires an extensive introduction to Gröbner and tropical bases. This is due to the fact that the intersection of two tropical varieties is not necessarily a variety. Since the main results of this thesis require only the study of planar curves, a discussion on general tropical varieties is omitted. However, the interested reader may consult Chapters 2 and 3 of [19].

This section starts off by defining tropical hypersurfaces as subsets of $\mathbb{R}^n$. However, these varieties come with considerable more structure than just sets. In the next section it will be shown that tropical hypersurfaces are a type of polyhedral complexes. The reader unfamiliar with this concept may consult Appendix A. Thereafter, integral weights are added to the maximal cells of the complex in such a way that the tropical hypersurface becomes balanced.

Definition 3.6. Let f be a tropical polynomial in n variables. We define the *hypersurface cut out by* or *corner locus of* f to be the set of points $\mathbf{x} \in \mathbb{R}^n$ such that $f(\mathbf{x})$ attains its maximum twice. This set is denoted by $V(f)$. We say that $V(f)$ is a (planar) curve if $n = 2$.

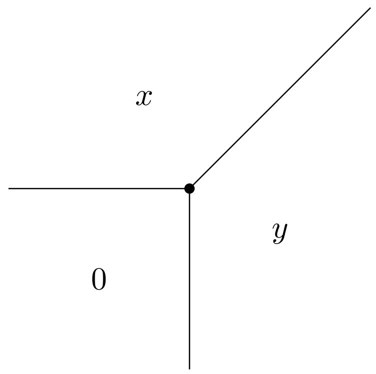
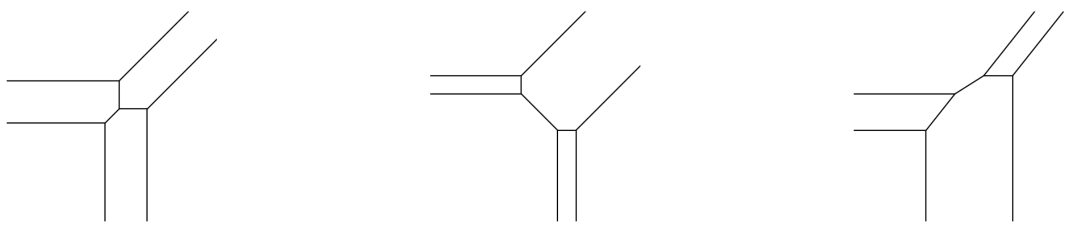


Figure 3: A tropical line. The symbols $0, x, y$ denote the value of the tropical polynomial $x \oplus y \oplus 0$ in the respective domain.

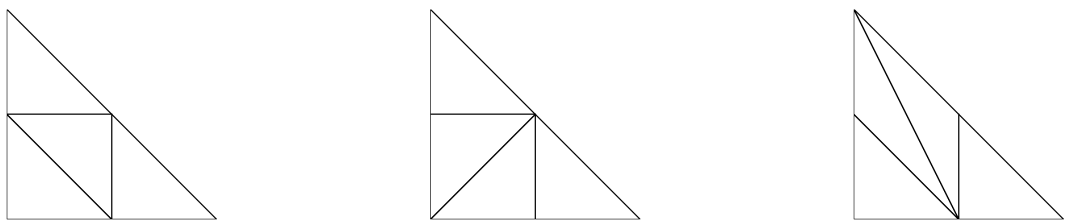
Example 3.7. Reconsider the binary tropical polynomial $f(x, y) = x \oplus y \oplus 0$ from Example 3.4. In Figure 3, we see that its corner locus is given by the union of three half-lines, which meet at the origin. These half-lines are given by $\{x = y \geq 0\}, \{x = 0 \geq y\}$ and $\{y = 0 \geq x\}$.

Changing the coefficients of f effectively translates this line. For instance, the corner locus of $-ax \oplus -by \oplus 0$ yields a tropical line with vertex (a, b). △



(a) $3x^2 \oplus 1xy \oplus 3y^2 \oplus x \oplus y \oplus 0$ (b) $3x^2 \oplus xy \oplus 3y^2 \oplus 1x \oplus 1y \oplus 0$ (c) $1x^2 \oplus 3xy \oplus 1y^2 \oplus x \oplus y \oplus 0$

Figure 4: Three tropical curves.



(a) $3x^2 \oplus 1xy \oplus 3y^2 \oplus x \oplus y \oplus 0$ (b) $3x^2 \oplus xy \oplus 3y^2 \oplus 1x \oplus 1y \oplus 0$ (c) $1x^2 \oplus 3xy \oplus 1y^2 \oplus x \oplus y \oplus 0$

Figure 5: Convex subdivisions dual to the curves of Figure 4.

Example 3.8. See Figure 4 for three examples of tropical curves. Each curve is cut out by a quadratic tropical polynomial, has two infinite rays in the horizontal, vertical and diagonal direction, and possesses 4 vertices. Also, each vertex has a valency of 3, i.e., there are 3 edges or rays adjacent to each vertex. This is a consequence of the *smoothness* of these curves. Smoothness in the tropical setting will be further discussed in Section 3.4. △

3.2.1 Structure of a Tropical Hypersurface

This section adds onto $V(f)$ the structure of a polyhedral complex S_f. First, we define the Legendre transform ν_f of f. This is a convex, piecewise-linear function on the Newton polytope $\Delta(f)$ that induces a regular subdivision of this polytope. For a definition of $\Delta(f)$, see Appendix A. Via the map ν_f, we then define a polyhedral complex Σ_f, whilst showing the connection between the cells of Σ_f and the cells of the subdivision induced by ν_f. Next, we prove that Σ_f is indeed a polyhedral complex, and conclude this section by proving that the support of Σ_f equals the tropical hypersurface $V(f)$.

A reference for the Legendre transform, subdivision of $\Delta(f)$ and the polyhedral complex can be found in [23, Sec. 2.3]. An alternative reference, which does not use the Legendre transform explicitly, is [19, Sec. 3.1].

Definition 3.9 (Legendre Transform)**.** Let $A \subset \mathbb{R}^n$ be any subset and $f\colon A \to \mathbb{R}$ a function. The Legendre transform ν_f is the function given by

$$\nu_f(\mathbf{w}) := \sup_{\mathbf{x}\in\mathbb{R}^n} \{\mathbf{w}\cdot\mathbf{x} - f(\mathbf{x})\},$$

for all $\mathbf{w} \in \mathbb{R}^n$.

It can be shown that the subset of $\mathbb{R}^n$ on which ν_f converges is a (possibly empty) convex set, see also [23, Sec. 2.3]. Furthermore, ν_f is a convex function. In case f is a tropical polynomial, there is an explicit description of its Legendre transform given by the following lemma.

Lemma 3.10. *[23, Lem. 2.3.10] Let $f = \bigoplus_{\mathbf{u}\in I} a_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$ be a tropical polynomial in n variables. The Legendre transform ν_f of f is the convex function $\Delta(f) \to \mathbb{R}$ whose graph is equal to the lower convex hull of*

$$\{(\mathbf{u}, -a_{\mathbf{u}}) \mid \mathbf{u} \in I\}.$$

Recall here that, if $S \subset \mathbb{R}^n$ is a set, then its lower convex hull consists of the union of lower faces of $\operatorname{conv} S$. A lower face is any face $\operatorname{face}_{\mathbf{w}}(\operatorname{conv} S)$, where $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$ and $w_n > 0$. For the meaning behind the notation $\operatorname{face}_{\mathbf{w}}(\operatorname{conv} S)$, see Appendix A.

By the above lemma, ν_f is a piecewise-linear, convex and continuous function on the Newton polytope $\Delta(f)$ of f. Therefore, the linearity domains induce a regular subdivision of $\Delta(f)$.

Definition 3.11 (Regular Subdivision)**.** A regular subdivision S of a polytope $\Delta \subset \mathbb{R}^n$ is called *regular* if it is given by the linearity domains of a piecewise-linear, convex and continuous function on Δ.

Example 3.12. Reconsider the curves from Example 3.12. The regular subdivisions induced by these curves are illustrated in Figure 5. Note the connection between the curves and their regular subdivisions: each vertex of the curve corresponds to a triangle in the subdivision, edges correspond to orthogonal edges and the polyhedra cut out by the curve give vertices in the subdivision. △

The remaining part of this section is dedicated to formalizing the observations made in the example above. To be more precise, we investigate the polyhedral structure of a tropical hypersurface, and show how this corresponds to the subdivision of its Newton polytope.

Definition 3.13. Let $f = \bigoplus_{\mathbf{u}} a_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$ be a tropical polynomial. We define the *polyhedral complex Σ_f dual to the subdivision induced by ν_f* as follows. Let $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection onto the first n coordinates and

$$\Delta_{\text{val}} := \operatorname{conv}\{(\mathbf{u}, -a_{\mathbf{u}}) \mid \mathbf{u} \in \mathbb{Z}^n, a_{\mathbf{u}} \neq -\infty\}.$$

For a lower face F of Δ_{val} we denote the outer normal cone by

$$\mathcal{N}(F) = -C_F(\Delta_{\text{val}})^{\vee} = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \mathbf{x}\cdot(\mathbf{u}-\mathbf{v}) \leq 0,\ \text{for all } \mathbf{u} \in \Delta_{\text{val}}, \mathbf{v} \in F\},$$

and by $\widetilde{\pi}(\mathcal{N}(F))$ we mean the restricted projection $\{\mathbf{w} \in \mathbb{R}^n \mid (\mathbf{w}, -1) \in \mathcal{N}(F)\}$. We then define the polyhedral complex Σ as the collection of sets $\widetilde{\pi}(\mathcal{N}(F))$ where F ranges over all lower faces of Δ_{val}.

Remark 3.14. Let S denote the subdivision of $\Delta(f)$ induced by ν_f and let $F \in S$ be a polytope of dimension r. This corresponds to a unique lower face $\widetilde{F}$ of Δ_{val} such that $\pi(\widetilde{F}) = F$, which in turn corresponds to a polyhedron $\widetilde{\pi}(\mathcal{N}(\widetilde{F})) \in \Sigma_f$ of codimension r.

Moreover, let $\mathbf{x}, \mathbf{y} \in \widetilde{\pi}(\mathcal{N}(\widetilde{F}))$ and $\mathbf{u}, \mathbf{v} \in F$. Then, $(\mathbf{x}, -1), (\mathbf{y}, -1)$ are orthogonal to $\widetilde{F}$, and it follows that

$$(\mathbf{x}, -1) \cdot ((\mathbf{u}, \nu_f(\mathbf{u})) - (\mathbf{v}, \nu_f(\mathbf{v}))) = (\mathbf{y}, -1) \cdot ((\mathbf{u}, \nu_f(\mathbf{u})) - (\mathbf{v}, \nu_f(\mathbf{v}))) = \nu_f(\mathbf{v}) - \nu_f(\mathbf{u}).$$

After rewriting, we obtain $(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{u} - \mathbf{v}) = 0$. Hence, each polytope F in the subdivision of $\Delta(f)$ is orthogonal to its counterpart in the complex Σ_f. This justifies the description of Σ_f as the complex *dual* to the subdivision S.

Lemma 3.15. *The collection Σ_f is a polyhedral complex.*

Proof. Let F be a lower face of Δ_{val}. As $\mathcal{N}(F)$ is a cone, there exists a matrix $A \in \mathbb{R}^{m \cdot (n+1)}$ such that

$$\mathcal{N}(F) = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid A\mathbf{x} \leq 0\}.$$

Consequently, if $A_1, \ldots, A_{n+1}$ denote the columns of A, we have

$$\widetilde{\pi}(\mathcal{N}(F)) = \{\mathbf{w} \in \mathbb{R}^n \mid \begin{pmatrix} A_1 \cdots A_n \end{pmatrix} \mathbf{w} \leq -A_{n+1}\}.$$

Therefore, $\widetilde{\pi}(\mathcal{N}(F))$ is a polyhedron for each lower face F of Δ_{val}.

Now let F' be a face of $\widetilde{\pi}(\mathcal{N}(F))$. This means that there exists a vector $\mathbf{w} \in \mathbb{R}^n$ and scalar $b \in \mathbb{R}$ such that the linear function $\mathbf{x} \mapsto \mathbf{w} \cdot \mathbf{x}$ attains a minimum on $\widetilde{\pi}(\mathcal{N}(F))$ that has a value of b. Then, F' equals the intersection of $\widetilde{\pi}(\mathcal{N}(F))$ and the hyperplane $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{w} \cdot \mathbf{x} = b\}$. Since F is a lower face of Δ_{val}, the outer normal cone $\mathcal{N}(F)$ only contains vectors from $\mathbb{R}^{n+1}$ whose $(n+1)$-th coordinate is negative. Let $\mathbf{y} \in \mathcal{N}(F)$ with nonzero last coordinate be given. We may then write $\mathbf{y} = (\lambda\, \mathbf{x}, -\lambda)$ for some vector $\mathbf{x} \in \mathbb{R}^n$ and positive scalar $\lambda > 0$. As $\mathcal{N}(F)$ is a cone, we also have $(\mathbf{x}, -1) \in \mathcal{N}(F)$. Therefore, $(\mathbf{w}, b) \cdot \mathbf{y} \geq 0$ and there is an equality for at least one $\mathbf{y} \in \mathcal{N}(F)$. We conclude that the function $\mathbf{y} \mapsto (\mathbf{w}, b) \cdot \mathbf{y}$ attains a minimum on the restriction of the cone $\{\mathbf{y} \in \mathcal{N}(F) \mid y_{n+1} > 0\}$ and by continuity, this minimum is also attained on the whole cone $\mathcal{N}(F)$. Therefore,

$$F' = \widetilde{\pi}(\text{face}_{(\mathbf{w}, b)}(\mathcal{N}(F))).$$

Also, a face of $\mathcal{N}(F)$ is the normal cone of a face of Δ_{val} that contains F. Hence, $F' \in \Sigma$ for each face F' of $\widetilde{\pi}(\mathcal{N}(F))$.

To finish the proof, consider two faces F_1, F_2 of Δ_{val}. Then,

$$\widetilde{\pi}(\mathcal{N}(F_1)) \cap \widetilde{\pi}(\mathcal{N}(F_2)) = \widetilde{\pi}(\mathcal{N}(F_1) \cap \mathcal{N}(F_2)).$$

The intersection $\mathcal{N}(F_1) \cap \mathcal{N}(F_2)$ is either empty or equals a face of both cones, since the collection $\mathcal{N}(F)$, for F a face of Δ_{val}, is a fan. See also Appendix A. Consequently, the intersection $\widetilde{\pi}(\mathcal{N}(F_1)) \cap \widetilde{\pi}(\mathcal{N}(F_2))$ is a face of both polyhedra and we thus conclude that Σ_f is indeed a polyhedral complex. □

Theorem 3.16. *Let $f = \bigoplus_{\mathbf{u}} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ be a tropical polynomial. Then, $V(f)$ equals the support of the $(n-1)$-skeleton of Σ_f.*

Proof. We mostly follow the proof as presented in [19, Prop. 3.1.6]. However, slight adjustments are made to accomodate for the max-plus convention used in this thesis, contrary to the min-plus convention used in that source.

Let $\mathbf{w} \in \mathbb{R}^n$ be given. Then, $\mathbf{w}$ lies in $V(f)$ if and only if $f(\mathbf{w})$ attains its maximum at least twice, say in $a_{\mathbf{u}_i} + \mathbf{u}_i \cdot \mathbf{x}$ for $\mathbf{u}_i \in \mathbb{Z}^n$, $i = 1, 2$. This is equivalent with

$$(\mathbf{w}, -1) \cdot (\mathbf{u}_i, -a_{\mathbf{u}_i}) \geq (\mathbf{w}, -1) \cdot (\mathbf{u}, -c_{\mathbf{u}}), \quad \text{for all } \mathbf{u} \in \mathbb{Z}^n, i = 1, 2.$$

In other words, the vector $(\mathbf{w}, -1)$ lies in the outer normal cone of a face F of Δ_{val}, which contains at least the two points $(\mathbf{u}_i, -a_{\mathbf{u}_i}), i = 1, 2$. In particular, the face F has positive dimension, meaning that $\mathcal{N}(F)$ and $\widetilde{\pi}(\mathcal{N}(F))$ are of codimension at least 1. We conclude that $\mathbf{w} \in V(f)$ if and only if $\mathbf{w}$ lies in a polyhedron of Σ_f that is not full-dimensional. □

Corollary 3.17. *Let f be a tropical polynomial in 2 variables. Then, $V(f)$ is a graph dual to the subdivision of $\Delta(f)$ induced by ν_f.*

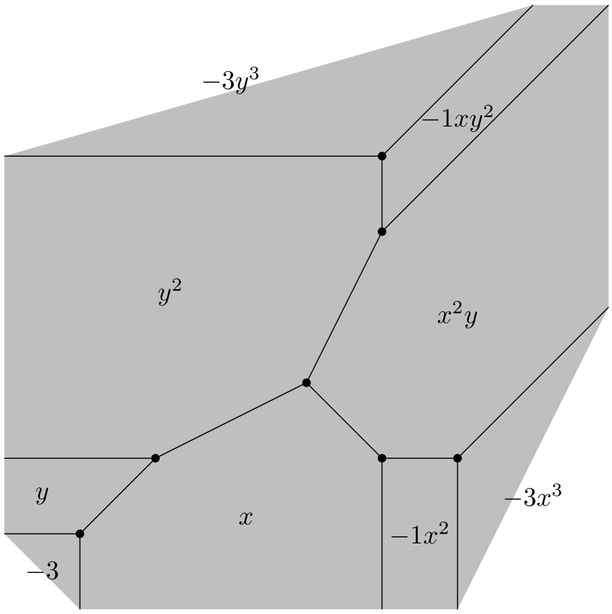


Figure 6

Example 3.18. The proof of Theorem 3.16 gives an alternative description of the complex Σ_f. Instead of the usual definition, Σ_f can also be characterized as follows. If $\mathbf{u}_1, \ldots, \mathbf{u}_r \in \mathbb{Z}^n$ is a collection of integral vectors such that $a_i \odot \mathbf{x}^{\odot \mathbf{u}_i}$ is a term of f, then

$$\{\mathbf{w} \in \mathbb{R}^n \mid a_i \odot \mathbf{w}^{\odot \mathbf{u}_i} = f(\mathbf{w}), \text{ for all } 1 \leq i \leq r\}$$

is a (possibly empty) cell of Σ_f.

For example, consider the cubic

$$f = -3 \oplus x \oplus -1y \oplus -1x^2 \oplus xy \oplus y^2 \oplus -3x^3 \oplus x^2y \oplus -1xy^2 \oplus -3y^3.$$

In Figure 6, the polyhedral complex Σ_f is depicted. In each 2-dimensional cell the tropical monomial is shown in which f attains its maximum. △

3.2.2 Weights and Balances

So far, we defined tropical hypersurfaces $V(f)$ and endowed them with the structure of a polyhedral complex Σ_f, which turns out to be dual to the subdivision of $\Delta(f)$ induced by ν_f. The complex itself may enjoy additional structure, specifically an integral weighting on the codimension 1 cells of Σ_f. Along with these weights, the tropical hypersurface $V(f)$ becomes a *balanced* polyhedral complex. What this means will become clear in the next definition.

Definition 3.19. Let Σ be a rational fan in $\mathbb{R}^n$, pure of dimension d with weights $m(\sigma) \in \mathbb{Z}_{\geq 1}$ for each cell $\sigma \in \Sigma$ of dimension d. Also fix a cell $\tau \in \Sigma$ of dimension $d-1$ and let L be the linear space parallel to the affine span of τ. Since τ is rational, the abelian group $L_{\mathbb{Z}} = L \cap \mathbb{Z}^n$ is free of rank $d-1$ and so, $N(\tau) := \mathbb{Z}^n / L_{\mathbb{Z}} \cong \mathbb{Z}^{n-d+1}$. For each cone $\sigma \subset \Sigma$ containing τ as a proper face, the set $(\sigma + L)/L$ is a one-dimensional rational cone in $N(\tau) \otimes_{\mathbb{Z}} \mathbb{R}$. We denote the primitive vector of this ray by v_σ. The fan Σ is now called *balanced at* τ if

$$\sum_{\sigma \in \Sigma, \tau \subsetneq \sigma} m(\sigma) v_\sigma = 0.$$

We say that the fan Σ is *balanced* if it is balanced at all cells $\tau \in \Sigma$ of dimension $d-1$.

More generally, if Σ is a rational polyhedral complex, pure of dimension d and with weights $m(\sigma) \in \mathbb{Z}_{\geq 1}, \dim(\sigma) = d$, then $\operatorname{star}_\Sigma(\tau)$ inherits the weighting function m for each cell $\tau \in \Sigma$. In this case, we say that Σ is *balanced* if the fan $\operatorname{star}_\Sigma(\tau)$ is balanced for all $\tau \in \Sigma$ with $\dim(\tau) = d-1$.

Now, consider a tropical hypersurface $V(f)$ in $\mathbb{R}^n$ and its polyhedral complex Σ_f. Each $(n-1)$-dimensional cell $\sigma \in \Sigma_f$ is dual to an edge τ in the subdivision of $\Delta(f)$. We define the weight $m(\sigma)$ to be the lattice length of τ, i.e.,

$$m(\sigma) := |\tau \cap \mathbb{Z}^n| - 1.$$

Proposition 3.20. *[19, Prop. 3.3.2] The polyhedral complex of a tropical hypersurface is balanced.*

Proof. Note that for $n = 1$, the condition for Σ_f to be balanced is vacuous. For $n \geq 2$, let τ be an $(n-2)$-dimensional cell of $V(f)$ and L the linear space parallel to the affine span of τ. The cell τ is dual to a 2-dimensional polygon $\overline{\tau}$ in the subdivision of $\Delta(f)$, while each maximal cell $\sigma \in \operatorname{star}_{V(f)}(\tau)$ that contains τ is dual to an edge $\overline{\sigma}$ of $\Delta(f)$. In particular, L is orthogonal to $\overline{\tau}$ and the primitive vectors $u_\sigma \in N(\tau) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2$ are 90 degree rotations of the primitive directional vectors of the edges $\overline{\sigma}$. Because the sum of edges of a polygon is always zero, we conclude that

$$\sum_{\sigma \in \Sigma, \tau \subsetneq \sigma} m(\sigma) v_\sigma = 0.$$

□

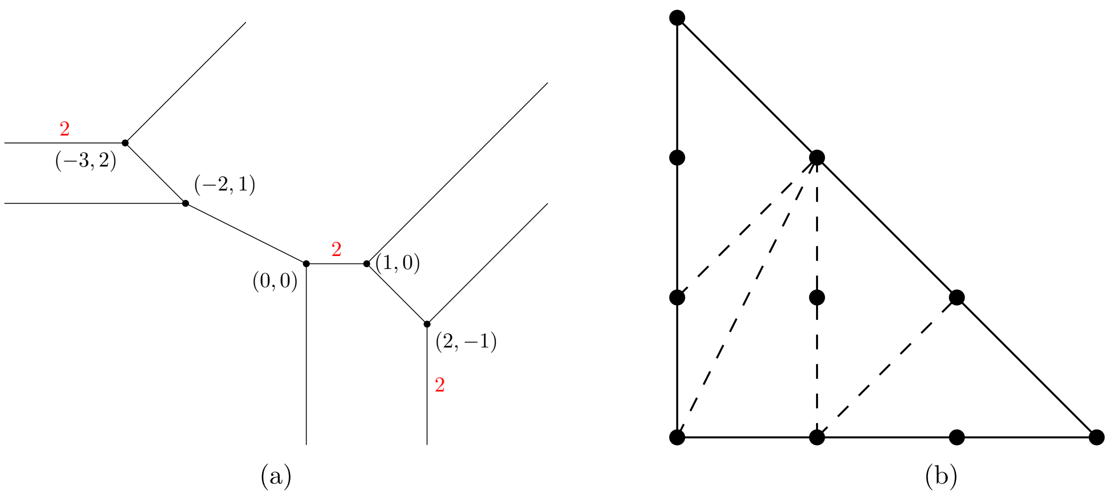


Figure 7

Example 3.21. Consider the corner locus of the tropical polynomial

$$f = 0 \oplus x \oplus -4x^3 \oplus -1y \oplus -1x^2y \oplus xy^2 \oplus -5y^3.$$

This curve, along with its dual subdivision, is depicted in Figure 7. There are 3 edges of $V(f)$ that have a weight equal to 2, while all other edges have weight 1. Note how these edges are dual to edges in the subdivision with lattice length 2. Furthermore, it is easy to check that each vertex of $V(f)$ satisfies the balacing condition. △

3.3 Amoebas and Tropicalization

So far, tropical geometry has been discussed as an isolated topic, while much of its value is derived from the deep connections with algebraic geometry. In this section, a first connection is established by showing how families of algebraic hypersurfaces may yield a tropical curve. This process is called tropicalization and utilizes the logarithmic moment map to transform algebraic varieties into so-called amoebas, which in turn can converge to tropical structures. The main result of this section is Kapranov's theorem, which states that the tropicalization of a family of hypersurfaces depends only on the coefficients of the defining polynomial.

To start off, we introduce the field of locally convergent Puiseux series, which we denote by $\mathbb{K}$ or $\mathbb{C}\{\{t\}\}$. This field consists of all series $\sum_{p=-m}^{\infty} c_p t^{p/q}$ in an indeterminate t, where $q \in \mathbb{Z}_{\geq 1}, m \in \mathbb{Z}_{\geq 0}$ and $c_p \in \mathbb{C}, n \in \mathbb{Z}_{\geq 1}$ such that this series converges for sufficiently small real and positive $t \in \mathbb{R}_{>0}$. That is, there exists a positive $t_0 > 0$ such that $\sum_{p=-m}^{\infty} c_p t^{p/q}$ converges for all $0 < t < t_0$. This is an algebraically closed field of characteristic 0.

Moreover, $\mathbb{K}$ possesses a valuation val: $\mathbb{K} \to \mathbb{R} \cup \{-\infty\}$ that is given by

$$\operatorname{val}\left(\sum_{p=-m}^{\infty} a_p t^{p/q}\right) = -\min\{p/q \mid a_p \neq 0\}.$$

In case $a_p = 0$ for all $p \in \mathbb{Z}_{\geq -m}$, we set $\operatorname{val}(0) = -\infty$. The value group is $\operatorname{val}(\mathbb{K}^\times) = \mathbb{Q}$. The valuation ring is the ring

$$R_{\text{val}} := \{a \in \mathbb{K} \mid \operatorname{val}(a) \leq 0\} = \left\{\sum_{p=-m}^{\infty} a_p t^{p/q} \in \mathbb{K} \mid m \geq 0\right\},$$

which has maximal ideal

$$m_{\text{val}} = \{\sum_{p=-m}^{\infty} a_p t^{p/q} \mid m > 0\},$$

and residue field $R_{\text{val}}/m_{\text{val}} \cong \mathbb{C}$. The valuation also induces a non-Archimedean norm $|\cdot| : \mathbb{K} \to \mathbb{R}$, defined by

$$|a|_{\text{val}} = e^{\operatorname{val}(a)}, \quad \text{for all } a \in \mathbb{K}.$$

Now, consider a family of complex Laurent polynomials in n variables, whose coefficients are locally convergent Puiseux series,

$$f_t(\mathbf{x}) = \sum_{\mathbf{u} \in \mathbb{Z}^n} a_{\mathbf{u}}(t)\mathbf{x}^{\mathbf{u}} \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Then, for $t > 0$ sufficiently small, the function f_t determines an algebraic subvariety

$$X_t := \{\mathbf{x} \in (\mathbb{C}^\times)^n \mid f_t(\mathbf{x}) = 0\}$$

of the algebraic torus $(\mathbb{C}^\times)^n$. The amoeba of X_t is defined to be the image under the logarithmic moment map

$$\operatorname{Log}_t \colon (\mathbb{C}^\times)^n \to \mathbb{R}^n, \ (x_1, \ldots, x_n) \mapsto (\log_t |x_1|, \ldots, \log_t |x_n|),$$

and is denoted by $\mathcal{A}_t(X_t) := \operatorname{Log}_t(X_t)$.

Alternatively, the family of varieties $(X_t)_t$ over $\mathbb{C}$ may be understood as a single variety $X_{\mathbb{K}}$ over $\mathbb{K}$. In this case, $X_{\mathbb{K}}$ is defined as

$$X_{\mathbb{K}} := \{\mathbf{x} \in (\mathbb{K}^\times)^n \mid f_t(\mathbf{x}) = 0\}.$$

In turn, the non-Archimedean amoeba $\mathcal{A}(X_{\mathbb{K}})$ is given to be the image of $X_{\mathbb{K}}$ under the map

$$\mathrm{Log}_{\mathbb{K}} \colon (\mathbb{K}^*)^n \to \mathbb{R}^n,\ (a_1, \ldots, a_n) \mapsto (\log |a_1|_{\mathrm{val}}, \ldots, \log |a_n|_{\mathrm{val}}) = (\mathrm{val}(a_1), \ldots, \mathrm{val}(a_n)).$$

The next two theorems make the connection between the amoeba's $\mathcal{A}(X_t)$ and $\mathcal{A}(X_{\mathbb{K}})$ explicit, and showcase the first connection with tropical geometry.

Theorem 3.22. *[26, Thm. 1.4] Let $f_t(\mathbf{x}) \in \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial over $\mathbb{K}$ in n variables. Then, the amoebas of the family of varieties $X_t := \{f_t = 0\}$ converge with respect to the Hausdorff metric on compacts to $\mathcal{A}(X_{\mathbb{K}})$ as $t \to \infty$.*

Theorem 3.23 (Kapranov)**.** *[25, Thm. 1.5] [19, Thm. 3.1.3] Let $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in \mathbb{K}[x^{\pm 1}]$ be a Laurent polynomial that determines a hypersurface $X_{\mathbb{K}} := V(f) \subset (\mathbb{K}^\times)^n$. Then, the closure of $\mathcal{A}(X_{\mathbb{K}})$ with respect to the Euclidean metric equals the corner locus of the tropical map*

$$\mathrm{trop}(f) := \bigoplus_{\mathbf{u} \in \mathbb{Z}^n} \mathrm{val}(a_{\mathbf{u}}) \mathbf{x}^{\mathbf{u}} = \max_{\mathbf{u} \in \mathbb{Z}^n} (\mathrm{val}(a_{\mathbf{u}}) + \mathbf{u} \cdot \mathbf{x}).$$

Definition 3.24. The tropical map $\mathrm{trop}(f)$ associated to a Laurent polynomial f over $\mathbb{K}$ and defined in the previous theorem, is known as the *tropicalization* of f.

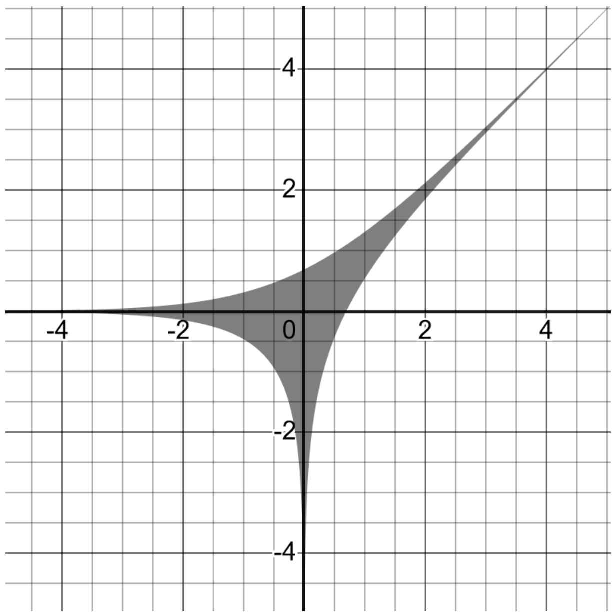


Figure 8: An amoeba of the complex line $z + w + 1 = 0$.

Example 3.25. Let $t > 0$ be given and consider the constant family

$$X_t := \{(z, w) \in (\mathbb{C}^\times)^2 \mid z + w + 1 = 0\}.$$

Let $(z, w) \in X$ a point and denote $x := \log_t |z|, y := \log_t |w|$. We distinguish two cases. First, assume that $x \geq 0$. Then, by the triangle inequality,

$$t^x - 1 = ||-z| - 1| \leq |-z - 1| = t^y \leq |z| + 1 = t^x + 1.$$

Consequently,

$$\log_t(t^x - 1) \leq y \leq \log_t(t^x + 1), \quad \text{for } x \geq 0. \tag{3.1}$$

Second, assume that $x \leq 0$. Then, the triangle inequality implies that

$$1 - t^x = |1 - |-z|| \leq |-z - 1| = t^y \leq |z| + 1 = t^x + 1.$$

Consequently,

$$\log_t(1 - t^x) \leq y \leq \log_t(t^x + 1), \quad \text{for } x \leq 0. \tag{3.2}$$

Hence, the amoeba $\mathcal{A}_t(X)$ is given by all real points $(x, y) \in \mathbb{R}^2$ satisfying the inequalities (3.1) and (3.2). The case $t = e$ has been plotted in Figure 8. Taking the limit for $t \to \infty$ in equations (3.1) and (3.2) gives the set of inequalities

$$-\infty \leq y \leq 0, \text{ for } x = 0, \qquad x \leq y \leq x, \text{ for } x > 0, \qquad 0 \leq y \leq 0, \text{ for } x < 0.$$

We thus conclude that the limit $\lim_{t\to\infty} \mathcal{A}_t(X)$ equals the corner locus of the tropical function $x \oplus y \oplus 0$, which confirms Kapranov's theorem. △

3.4 Nodal and Simple Curves

Recall that a nodal algebraic curve is a curve whose only singularities are nodes. In this section, the analogous tropical concept is introduced, along with the more restrictive family of *simple* curves. The Mikhalkin Correspondence, which is of central importance in this thesis, establishes a deep connection between nodal algebraic curves and these simple tropical curves.

Furthermore, we introduce polyhedral spaces that parametrize tropical curves with given dual subdivisions, or number of nodes. This is analogous to Severi varieties in the algebraic case. In the case of nodal curves, the dimensions of these spaces are directly computable from the cogenus of the curve.

Definition 3.26. A tropical plane curve is *smooth* if the dual subdivision contains only triangles with an area of $\frac{1}{2}$. Equivalently, all vertices of the curve have valency 3 and are maximal given the Newton polytope of the curve.

A tropical curve is *nodal* if each of its vertices is either 3- or 4-valent, and the 4-valent vertices are locally given by the intersection of 2 straight lines. Equivalently, the dual subdivision of the curve consists of triangles and parallelograms. The *cogenus* of such a curve is defined as the sum of the number of parallelograms and the number of lattice points $\Delta \cap \mathbb{Z}^2$ that are not vertices in the subdivision. For a tropical curve C, this is denoted by $\delta(C)$.

A nodal tropical curve is called *simple* if all the lattice points on the boundary of the Newton polytope $\partial\Delta \cap \mathbb{Z}^2$ are vertices of the subdivision.

A tropical curve is called *irreducible* if it cannot be written as the union $T_1 \cup T_2$ of two distinct tropical curves T_1, T_2.

Definition 3.27. Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon and S a regular subdivision of Δ. We define $\mathcal{T}(\Delta, S)$ to be the set of all tropical plane curves with Newton polygon Δ that are dual to S.

Recall from Example 3.3 that two tropical polynomials may be equal even if their coefficients do not coincide. However, by restricting to a fixed combinatorial type we can parametrize a tropical curve by the coefficients of its defining polynomial. If a tropical curve T is given by some polynomial f with dual subdivision S, then the only terms of f that contribute are those corresponding to the vertices of S. Hence, we may write f as

$$f = f_{\mathbf{a}} = \bigoplus_{\mathbf{u} \in \mathrm{Vert}(S)} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}},$$

where $\mathbf{a} = (a_{\mathbf{u}})_{\mathbf{u}} \in \mathbb{R}^{\mathrm{Vert}(S)}$. Moreover, $V(f_{\mathbf{a}})$ is uniquely determined by $f_{\mathbf{a}}$ up to a translation $\mathbf{a} \mapsto \mathbf{a} + \lambda\mathbf{1}$, $\lambda \in \mathbb{R}$. Therefore, there is a bijection

$$\{\mathbf{a} \in \mathbb{R}^{|\mathrm{Vert}(S)|} \mid V(f_{\mathbf{a}}) \in \mathcal{T}(\Delta, S)\} \big/ \mathbb{R} \cdot \mathbf{1} \to \mathcal{T}(\Delta, S), \qquad \mathbf{a} \mapsto V(f_{\mathbf{a}}). \tag{3.3}$$

The following lemma shows that $\mathcal{T}(\Delta, S)$, under this bijection, is the relative interior of a polyhedron. In particular, we may talk about the dimension of $\mathcal{T}(\Delta, S)$.

Lemma 3.28. *Under the bijection of (3.3), the set $\mathcal{T}(\Delta, S)$ is the relative interior of some polyhedron. That is, there exist matrices B, C and vectors $\mathbf{b}, \mathbf{c}$ such that*

$$\mathcal{T}(\Delta, S) = \{\mathbf{a} \in \mathbb{R}^{\mathrm{Vert}(S)} \mid B\mathbf{a} = \mathbf{b}, C\mathbf{a} > \mathbf{c}\}.$$

Proof. Let $\mathbf{a} \in \mathcal{T}(\Delta, S)$ be given. For each polytope $\Delta' \in S$ with vertices $\mathbf{u}_1, \ldots, \mathbf{u}_r$, the points $(\mathbf{u}_1, a_{\mathbf{u}_1}), \ldots, (\mathbf{u}_r, a_{\mathbf{u}_r}) \in \mathbb{R}^3$ lie on a plane. This induces a linear equation on each subset of four vertices from $a_{\mathbf{u}_1}, \ldots a_{\mathbf{u}_r}$. Moreover, for each pair of polytopes $\Delta_1, \Delta_2 \in S$, let $\nu_1, \nu_2 \colon \mathbb{R}^3 \to \mathbb{R}$ be the linear functions with graphs equal to the planes above Δ_1 and Δ_2. The conditions that $\nu_1\big|_{\mathrm{int}(\Delta_1)} > \nu_2\big|_{\mathrm{int}(\Delta_1)}$ and $\nu_2\big|_{\mathrm{int}(\Delta_2)} > \nu_1\big|_{\mathrm{int}(\Delta_2)}$ can be written in terms of strict linear inequalities.

Furthermore, for general $\mathbf{a} \in \mathbb{R}^{\mathrm{Vert}(S)}$ note that the above conditions ensure that $\mathbf{a} \in \mathcal{T}(\Delta, S)$. Indeed, if the above conditions hold, then the piecewise-linear function $\nu \colon \Delta \to \mathbb{R}$ dual to $f_{\mathbf{a}}$ has as linearity domains the subdivision S.

□

Example 3.29. Let Δ be the parallelogram spanned by the vertices $(0,0), (1,0), (0,1), (1,1)$ and $S_1 = \{\Delta\}, S_2 = \{\Delta_1, \Delta_2\}$ subdivisions where

$$\Delta_1 = \mathrm{conv}((0,0),(1,0),(1,1)) \quad \text{and} \quad \Delta_2 = \mathrm{conv}((0,0),(0,1),(1,1)).$$

The family of tropical polynomials T with Newton polygon Δ consists of expressions of the form

$$a \oplus bx \oplus cy \oplus dxy, \quad a, b, c, d \in \mathbb{R}.$$

For (a,b,c,d) to lie in $\mathcal{T}(\Delta, S_1)$ means that the vectors $(0,0,a), (1,0,b), (0,1,c), (1,1,d)$ all lie on a plane. This is equivalent to the condition

$$a + d = b + c.$$

On the other hand, we have

$$\mathcal{T}(\Delta, S_2) = \{(a,b,c,d) \in \mathbb{R}^4 \colon a + d > c + b\}/\mathbb{R}\cdot\mathbf{1}.$$

△

The proof of Lemma 3.28 showed that for any polytope $\Delta' \in S$, each group of 4 vertices of Δ' induces a linear restriction on the coefficients of T. So, one may reasonably suspect that the dimension of $\mathcal{T}(\Delta, S)$ is given by

$$|\mathrm{Vert}(S)| - 1 - \sum_{\Delta' \in S} (|V(S)| - 3).$$

However, this is not true in general, as we will see in Proposition 3.32. Yet, it motivates the following definition.

Definition 3.30. Let T be a tropical plane curve with Newton polygon Δ and dual subdivision S. The rank of T is defined to be

$$\mathrm{rk}(T) := \dim \mathcal{T}(\Delta, S).$$

Furthermore, we define the expected rank of T by

$$\mathrm{rk}_{\exp}(T) := |\mathrm{Vert}(S)| - 1 - \sum_{\Delta' \in S} (\left|\mathrm{Vert}(\Delta')\right| - 3).$$

Remark 3.31. Note that, if T is nodal, then $\mathrm{rk}_{\exp}(T) = \left|\Delta \cap \mathbb{Z}^2\right| - 1 - \delta(T)$.

Proposition 3.32. *[25, Lem. 2.2] Let T be a tropical plane curve. Then, T is nodal if and only if $\mathrm{rk}_{\exp}(T) = \mathrm{rk}(T)$. Otherwise, we have the inequalities*

$$0 \leq 2(\mathrm{rk}(T) - \mathrm{rk}_{\exp}(T)) \leq -1 + \sum_{\mathfrak{m} \geq 2} \left[((2m-3)N_{2m} - N'_{2m}) + (2m-2)N_{2m+1}\right],$$

where $N_m, m \in \mathbb{Z}_{\geq 1}$ denotes the number of m-valent vertices of T and N'_{2m} is the number of $2m$-valent vertices that are locally the intersection of m straight lines.

3.5 General Position

This section introduces the notion of general position in the tropical settting, along with two properties of this concept. First, the description *general* is justified by showing that the set of generic points is dense. Second, we prove that if a tropical curve passes sufficiently many generic points, then these must all lie in the interior of the edges of this curve.

Definition 3.33. Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon, S a regular subdivision and $\mathcal{U} = \{\mathbf{x}_1, \dots, \mathbf{x}_\zeta\} \subset \mathbb{R}^2$ a configuration of points. For all integers $0 \leq k \leq \zeta$, define $\mathcal{T}_{\mathcal{U},k}(\Delta, S)$ to be the space of tropical curves $T \in \mathcal{T}(\Delta, S)$ that pass through $\mathbf{x}_1, \dots, \mathbf{x}_\zeta$, where at least k of these points are vertices of T. We say that the points $\mathbf{x}_1, \dots, \mathbf{x}_\zeta$ are (Δ, S)-*generic* if for all $0 \leq k \leq \zeta$, the space $\mathcal{T}_{\mathcal{U},k}(\Delta, S)$ is either empty or of codimension $k+\zeta$ in $\mathcal{T}(\Delta, S)$. Furthermore, we call these points Δ-*generic* if they are (Δ, S)-generic for each subdivision S of Δ.

Remark 3.34. If the number of generic points equals $\zeta = \dim \mathcal{T}(\Delta, S)$, then there are a finite number of curves $T \in \mathcal{T}(\Delta, S)$ that contain these points. The count of these curves may vary, depending on the configuration of generic points. However, when enumerated with the correct multiplicity, these numbers are invariants under this choice. This follows either from the Mikhalkin Correspondence, see Theorem 4.1, or can be proven directly as done in [13].

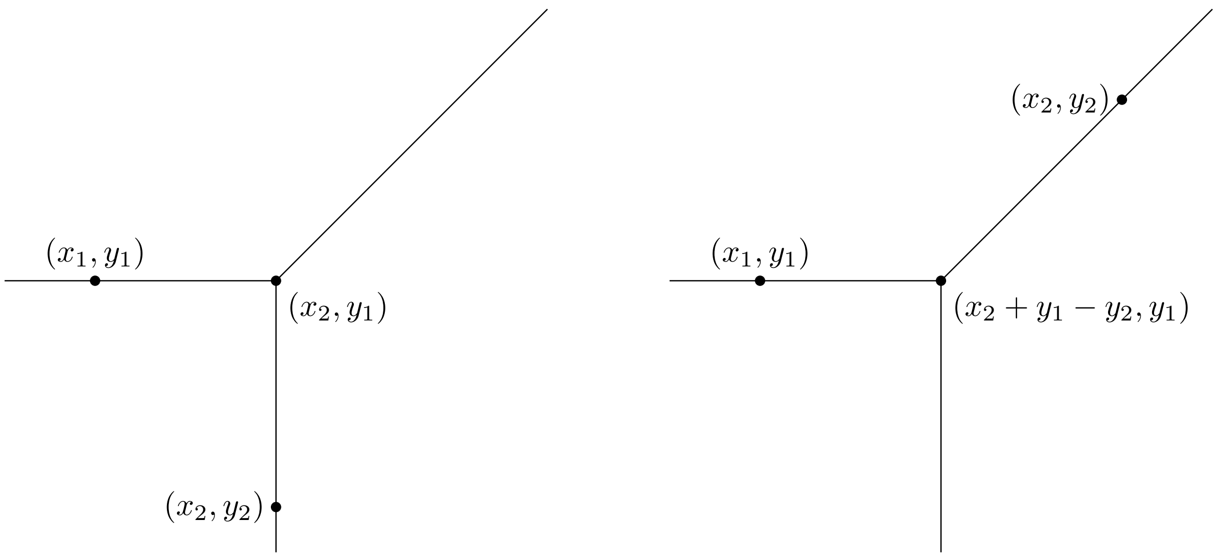


Figure 9: The unique tropical line through two generic points.

Example 3.35. Let Δ be the triangle with vertices $(0, 0), (1, 0), (0, 1)$. Two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ are Δ-generic if and only if both points do not lie on the same horizontal, vertical or diagonal line. If this is not the case, then the spaces $\mathcal{T}_{\mathcal{U}}(\Delta, \{\Delta\}), \mathcal{T}_{\mathcal{U},1}(\Delta, \{\Delta\})$ are respectively 1- and 0-dimensional.

If $(x_1, y_1), (x_2, y_2)$ are Δ-generic, then there exists a unique tropical line passing the two given points. Without loss of generality, we may assume that $x_1 < x_2$. The unique tropical line through $(x_1, y_1), (x_2, y_2)$ is determined by the polynomial

$$0 \oplus ((y_2 - y_1 - x_2) \oplus -x_2) \odot x \oplus -y_1 \odot y.$$

See also Figure 9. △

Lemma 3.36. *The set of all Δ-generic points is a Zariski open (and hence dense) subset of $\mathbb{R}^{2\zeta}$.*

Proof. The space of all Δ-generic points equals the finite intersection of the spaces of (Δ, S)-generic points, where S ranges over all regular subdivisions of Δ. In turn, the space of (Δ, S)-generic points equals the finite intersection of points $\mathcal{U} \in \mathbb{R}^{2\zeta}$ for which $\mathcal{T}_{\mathcal{U},k}(\Delta, S)$ is either empty or of codimension $k + \zeta$, where k ranges over $0, 1, \ldots, \zeta$. We say that the points are (Δ, S, k)-generic if they satisfy the latter condition. Hence, it suffices to show that the space of (Δ, S, k)-generic points is Zariski open.

Let $\mathcal{U} \in \mathbb{R}^{2\zeta}$. We may write

$$\mathcal{T}_{\mathcal{U},k}(\Delta, S) = \bigcup_{e_1,\ldots,e_\zeta} \bigcup_{e'_1,\ldots,e'_k} \bigcap_{i=1}^{\zeta} \bigcap_{j=1}^{k} \left\{ T \in \mathcal{T}(\Delta, S) \middle| \begin{array}{c} \mathbf{x}_i \text{ lies on the edge dual to } e_i \text{ and} \\ \mathbf{x}_j \text{ lies on the edge dual to } e'_j \end{array} \right\},$$

where the first union ranges over all ζ-tuples of edges of S and the second union ranges over all k-tuples of edges of S. The dimension of $\mathcal{T}_{\mathcal{U},k}(\Delta, S)$ is equal to the maximum of the dimensions of the sets

$$\bigcap_{i=1}^{\zeta} \bigcap_{j=1}^{k} \left\{ T \in \mathcal{T}(\Delta, S) \middle| \begin{array}{c} \mathbf{x}_i \text{ lies on the edge dual to } e_i \text{ and} \\ \mathbf{x}_j \text{ lies on the edge dual to } e'_j \end{array} \right\}.$$

The condition for x_i to lie on the edge dual to e_i is a linear condition on the coefficients of T. Therefore, the space

$$\bigcap_{i=1}^{\zeta} \bigcap_{j=1}^{k} \left\{ T \in \mathcal{T}(\Delta, S) \middle| \begin{array}{c} \mathbf{x}_i \text{ lies on the edge dual to } e_i \text{ and} \\ \mathbf{x}_j \text{ lies on the edge dual to } e'_j \end{array} \right\}$$

is described by a system of $k + \zeta$ linear equations. This space is empty or of codimension $k + \zeta$ if and only if this system is linearly independent, which is a Zariski open condition on the coefficients of $T \in \mathcal{T}(\Delta, S)$, as this can be described by the complement of zero loci of subdeterminants of the system. □

Lemma 3.37. *Let T be a tropical plane curve of rank ζ with Newton polytope Δ. Assume that T passes through the Δ-generic points $\mathbf{x}_1, \ldots, \mathbf{x}_\zeta \in \mathbb{R}^2$. Then, each of these points lies in the interior of a line segment of T.*

Proof. Write $\mathcal{U} := (\mathbf{x}_1, \ldots, \mathbf{x}_\zeta)$ and assume for the sake of contradiction that one of the points is a vertex of T. Then, the space $\mathcal{T}_{\mathcal{U},1}(\Delta, s)$ is non-empty and cannot be of codimension $\zeta + 1$, as the dimension of $\mathcal{T}(\Delta, S)$ is ζ. Therefore, each of the points of T lies in the interior of an edge. □

4 The Mikhalkin Correspondence

The Mikhalkin Correspondence establishes a bijection between plane tropical curves and families of complex curves embedded in some toric surface. Consequently, Severi degrees can be computed by counting tropical curves. This result has first appeared in [21] and uses techniques from symplectic geometry. Soon after, a different proof by Shustin, which is more algebro-geometric in nature, appeared in [25], [26]. Here, we follow the approach as presented in these two papers.

First, we state the Mikhalkin correspondence in Section 4.1. Next, the tropical limit is introduced in 4.2. This takes as input a curve $\mathcal{C}$ over the Puiseux series $\mathbb{K}$ and degenerates this into a tropical curve along with a collection of complex curves $C_1, \ldots, C_k$. In the case that the coefficients of the defining polynomial of $\mathcal{C}$ are power series in t, this curve determines a family of complex curves $C^{(t)}$ for $|t| > 0$ sufficiently small. In a sense, we will view the reducible curve $C^{(0)} = C_1 \cup \ldots \cup C_k$ as a limit of the family $C^{(t)}$. In Section 4.3, the limit curve $C^{(0)}$ is reconstructed from the tropical curve. In Section 4.4, additional data is appended to this tropical limit process to add information on the singularities of $C^{(0)}$ along toric divisors. Then, in 4.5 the method of patchworking is introduced. This operation reconstructs the curve $\mathcal{C}$ from the limit curve $C^{(0)}$. Through this, a bijection between tropical curves and curves over $\mathbb{K}$ is achieved. Finally, in 4.6 equivalence of the counts of generic fibres and the curves over $\mathbb{K}$ is established, thus finishing the proof of the Mikhalkin Correspondence.

4.1 Statement of the Mikhalkin Correspondence

Recall from Section 3.4 that $\mathcal{T}(\Delta, S)$ is the set of all tropical curves with Newton polygon Δ and dual subdivision S. For $T \in \mathcal{T}(\Delta, S)$, the rank of T is defined to be the dimension of $\mathcal{T}(\Delta, S)$. For $\mathcal{U} \subset \mathbb{Q}^2$ a finite set of points and $\zeta \in \mathbb{Z}_{\geq 1}$, let $\mathcal{T}_{\Delta,\zeta}(\mathcal{U})$ be the set of tropical curves that have Newton polygon Δ, are of rank ζ and pass through all points of $\mathcal{U}$. The subset $\mathcal{T}^{\mathrm{irr}}_{\Delta,\zeta}(\mathcal{U}) \subset \mathcal{T}_{\Delta,\zeta}(\mathcal{U})$ contains those tropical curves that are also irreducible. Furthermore, let $\mu(T)$ be the product of the normalized areas of the triangles inside the subdivision dual to T. We call this the (complex) multiplicity of T.

Theorem 4.1 (Mikhalkin)**.** *[21, Thm. 1] Let $\Delta \subset \mathbb{R}^2$ be a non-degenerate lattice polygon and $\delta \geq 0$ an integer. Then,*

$$N^{\Delta,\delta} = \sum_{T \in \mathcal{T}_{\Delta,\zeta}(\mathcal{U})} \mu(T), \qquad N'^{\Delta,\delta} = \sum_{T \in \mathcal{T}^{\mathrm{irr}}_{\Delta,\zeta}(\mathcal{U})} \mu(T),$$

where $\mathcal{U} \subset \mathbb{Q}^2$ is a Δ-generic configuration of $\zeta := \left|\Delta \cap \mathbb{Z}^2\right| - 1 - \delta$ points.

The proof of this theorem consists of establishing two bijections. First, a bijection between tropical curves and algebraic curves over $\mathbb{K}$ is proven. The first map in this bijection is called the tropical limit, which degenerates any family of complex algebraic curves to a tropical one. The inverse procedure is called patchworking. Second, a bijection between algebraic curves over $\mathbb{K}$ and $\mathbb{C}$ is established in the last section of this chapter.

Remark 4.2. By choosing different multiplicities for the tropical curves, one may find alternative enumerative invariants. For instance, one can replace $\mu(T)$ by weights ± 1 to compute Welschinger invariants. These are invented by Jean-Yves Welschinger in [29] and count the number of real rational curves passing a generic configuration of real points. However, the regular count is ill-defined as this number depends on the choice of points. To remedy this, Welschinger proposed to count real rational curves with a multipicity ranging over ± 1.

A different approach was pioneered by Florian Block and Lothar Göttsche in [3]. They suggested to replace the multiplicities $\mu(T)$ by symmetric polynomials $[\mu(T)]_y \in \mathbb{Q}[y^{\pm 1/2}]$. By evaluating these polynomials in $1, -1$, one restores the Severi degrees and Welschinger invariant respectively. Therefore, their approach can be viewed as a strict generalization of Mikhalkin's original correspondence.

4.2 The Tropical Limit

The tropical limit of a curve $\mathcal{C}$ over $\mathbb{K}$ is its tropicalization along with a so-called limit curve $C^{(0)}$, which can be viewed as the special fibre of some flat family over $\mathbb{C}$. Section 4.2.1 will be dedicated to the study of this flat family, and will serve as a motivation for the definition of the tropical limit given at the end of this section. Next, we study and prove the properties of the tropical limit inherent to nodal curves in Section 4.2.2. Finally, some examples are given in 4.2.3.

4.2.1 Puiseux Curves as Flat Families

Let $\Delta \subset \mathbb{R}^2$ be a non-degenerate lattice polygon, $\delta \geq 0$ an integer, $\zeta := \left|\Delta \cap \mathbb{Z}^2\right| - 1 - \delta$ the dimension of the Severi variety and $\pi_1, \ldots, \pi_\zeta \in (\mathbb{K}^\times)^2$ a configuration of points in general position. Also consider a Laurent polynomial

$$f(x,y) = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} a_{ij} x^i y^j \in \mathbb{K}[x^{\pm 1}, y^{\pm 1}]$$

such that its zero locus $\mathcal{C} := \mathrm{cl}_{X_{\mathbb{K}}(\Delta)}\{f = 0\}$ is a δ-nodal curve that contains the points $\pi_1, \ldots, \pi_\zeta$.

After a suitable automorphism $\mathbb{K} \to \mathbb{K},\ t \mapsto t^M$ for some appropriate $M \in \mathbb{Z}_{>0}$, the coordinates of the points $\pi_1, \ldots, \pi_\zeta$ and the coefficients $a_{ij}, (i,j) \in \Delta \cap \mathbb{Z}^2$ become Laurent series, i.e., the exponents of t are integral. Moreover, since the Puiseux series from $\mathbb{K}$ are locally convergent, there exists a radius $t_0 > 0$ such that $\pi_s(t), a_{ij}(t), 1 \leq s \leq \zeta, (i,j) \in \Delta \cap \mathbb{Z}^2$ converge for all $t \in D(0;t_0) \setminus \{0\}$. Here, $D(0;t_0) \subset \mathbb{C}$ denotes the open disk centered at 0 with radius t_0.

Consequently, we may define the general curve C as the zero locus of $f_t(x,y)$ in $X_{\mathbb{C}}(\Delta) \times (D(0;t_0) \setminus \{0\})$. This results in a complex-analytic family of curves $C \to D(0;t_0) \setminus \{0\}$ that is given as the vertical map in the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\ \iota\ } & X_{\mathbb{C}}(\Delta) \times (D(0;t_0) \setminus \{0\}) \\ \downarrow & \swarrow_{\pi} & \\ D(0;t_0) \setminus \{0\} & & \end{array}$$

Here, ι is the canonical inclusion and π the projection. The fibres of the family $C \to D(0;t_0)\setminus\{0\}$ are denoted by $C^{(t)}$ for $t \in D(0;t_0) \setminus \{0\}$.

We wish to extend this family to $t = 0$ and call the fibre at this point the tropical limit of C. Let $\nu\colon \Delta \to \mathbb{R}$ be the Legendre transform of $\mathrm{trop}(f)$. Recall from Section 3.2 that the maximal linearity domains of ν give a subdivision $S = \{\Delta_1, \ldots, \Delta_N\}$ of Δ. Now, consider the 3-dimensional polyhedron

$$\widetilde{\Delta} := \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in \Delta \text{ and } z \geq \nu(x,y)\}.$$

The union of its lower faces forms the graph of ν and is therefore given by polytopes $\widetilde{\Delta_1}, \ldots, \widetilde{\Delta_N}$, where the restricted projection $\widetilde{\Delta_j} \to \Delta_j$ is an isomorphism. Hence, the 3-dimensional toric variety $X_{\mathbb{C}}(\widetilde{\Delta})$ contains the surfaces $X_{\mathbb{C}}(\Delta_m), 1 \leq m \leq N$ as closed subvarieties.

Next, consider the map of lattices

$$\widetilde{\varphi}\colon\ \mathbb{Z}^3 \to \mathbb{Z},\ (x,y,t) \mapsto t.$$

This map is compatible with the normal fan of $\widetilde{\Delta}$ and the fan $\Sigma := \{\{0\}, \mathbb{R}_{\geq 0}\}$. Hence, there is a well-defined map of toric varieties

$$\varphi\colon X(\widetilde{\Delta}) \to X(\Sigma) \cong \mathbb{C}\,.$$

Lemma 4.3. *The morphism φ is flat and for $t \in \mathbb{C}^\times$ its fibres are isomorphic to $X(\Delta)$, whereas the special fibre at $t = 0$ is given by*

$$\bigcup_{m=1}^{N} X(\widetilde{\Delta}_m) \cong \bigcup_{m=1}^{N} X(\Delta_m).$$

Proof. We first prove flatness. Let σ be the normal cone of some vertex of $\widetilde{\Delta}$. The morphism φ restricts to

$$\operatorname{Spec}\mathbb{C}[\sigma^\vee\cap\mathbb{Z}^3]\to\operatorname{Spec}\mathbb{C}[t]$$

which is induced by the ring homomorphism

$$\mathbb{C}[t]\to\mathbb{C}[\sigma^\vee\cap\mathbb{Z}^3],\ t\mapsto\mathbf{x}^{(0,0,1)}.$$

By [27, Tag 0AUW], because $\mathbb{C}[t]$ is a Dedekind domain, $\mathbb{C}[\sigma^\vee\cap\mathbb{Z}^2]$ is flat as $\mathbb{C}[t]$-module if and only if it is torsion-free. As $\mathbb{C}[\sigma^\vee\cap\mathbb{Z}^2]$ is an integral domain, it is in particular torsion-free over $\mathbb{C}[t]$. Since the spaces $\operatorname{Spec}\mathbb{C}[\sigma^\vee\cap\mathbb{Z}^3]$, where σ ranges over the normal cones of vertices of $\widetilde{\Delta}$, form an open affine cover of $X(\widetilde{\Delta})$, we conclude that φ is flat.

Next, let N_0 be the kernel of $\widetilde{\varphi}$ and consider the fan

$$\Sigma_0=\{\sigma\in\Sigma(\widetilde{\Delta})\mid\sigma\subset N_0\otimes_{\mathbb{Z}}\mathbb{R}\},$$

that is, Σ_0 is the subfan of $\Sigma(\widetilde{\Delta})$ consisting of all cones that are mapped to 0 by $\widetilde{\varphi}$. Let $X_{\Sigma_0,\mathbb{Z}^3}$ denote the toric variety constructed from the fan F_0 with respect to the lattice $\mathbb{Z}^3$. This is a subvariety of $X(\widetilde{\Delta})$ given by the union of all orbits $O(\sigma)$ where $\sigma\in\Sigma_0$.

We claim that $\varphi(X_{\Sigma_0,\mathbb{Z}^3})=\mathbb{C}^\times$. First, let $\sigma\in\Sigma_0$ be given. We then have

$$\varphi(O(\sigma))\subset O(\widetilde{\varphi}(\sigma))=\mathbb{C}^\times\,.$$

Second, let $\sigma\in\Sigma(\widetilde{\Delta})\setminus\Sigma_0$ be given. Because φ is a toric morphism, there exists a cone $\tau\in\Sigma$ such that $\varphi(O(\sigma))=O(\tau)$. For this, we necessarily have that $\widetilde{\varphi}(\sigma)\subset\tau$ and therefore, $\tau\neq\{0\}$, implying that $\varphi(O(\sigma))\cap\mathbb{C}^\times=\emptyset$. Since $X_{\Sigma_0,\mathbb{Z}^3}=\bigcup_{\sigma\in\Sigma_0}O(\sigma)$, this proves the claim.

Furthermore, by [7, Prop. 3.3.11] there is an isomorphism of toric varieties

$$X_{\Sigma_0,\mathbb{Z}^3}\cong X_{\Sigma_0,N_0}\times T_{\mathbb{Z}^3/N_0},$$

where X_{Σ_0,N_0} is the toric variety corresponding to the cone Σ_0 with respect to the lattice $N_0=\mathbb{Z}^2\times\{0\}$. Also note that Σ_0 equals the normal fan of Δ, and therefore, $X_{\Sigma_0,N_0}\cong X(\Delta)$. Moreover, since $N_0=\mathbb{Z}^2\times\{0\}$, we have $T_{\mathbb{Z}^3/N_0}\cong T_{\mathbb{Z}}\cong\mathbb{C}^\times$.

We thus obtain an isomorphism of toric varieties

$$X_{\Sigma_0,\mathbb{Z}^3}\cong X_{\Sigma_0,N_0}\times\mathbb{C}^\times\,.$$

For each cone $\sigma\in\Sigma_0$, this map is locally given by a ring isomorphism

$$\mathbb{C}[\sigma^\vee\cap\mathbb{Z}^3]\xrightarrow{\cong}\mathbb{C}[\sigma^\vee\cap\mathbb{Z}^2\times\{0\}]\otimes_{\mathbb{C}}\mathbb{C}[\mathbf{x}^{(0,0,1)}].$$

Hence, the composition

$$\mathbb{C}[t^{\pm1}]\to\mathbb{C}[\sigma^\vee\cap\mathbb{Z}^3]\to\mathbb{C}[\sigma^\vee\cap\mathbb{Z}^2\times\{0\}]\otimes_{\mathbb{C}}\mathbb{C}[\mathbf{x}^{(0,0,\pm1)}]$$

is determined by $t\mapsto1\otimes\mathbf{x}^{(0,0,1)}$ and the corresponding toric map

$$X(\Delta)\times\mathbb{C}^\times\xrightarrow{\cong}X_{\Sigma_0,\mathbb{Z}^3}\xrightarrow{\varphi}\mathbb{C}^\times$$

is the projection. In particular, its fibres are isomorphic to $X(\Delta)$.

Since $X_{\Sigma_0,\mathbb{Z}^3}$ consists of the union of orbits corresponding to cones of Σ_0, its complement in $X(\widetilde{\Delta})$ is given by

$$X(\widetilde{\Delta})\setminus X_{\Sigma_0,\mathbb{Z}^3}=\bigcup_{\sigma\in\Sigma(\widetilde{\Delta})\setminus\Sigma_0}O(\sigma)=\bigcup_{m=1}^{N}X(\widetilde{\Delta}_m).$$

Therefore, we conclude that

$$\varphi^{-1}(0)=X(\widetilde{\Delta})\setminus\varphi^{-1}(\mathbb{C}^\times)=\bigcup_{m=1}^{N}X(\widetilde{\Delta}_m).$$

□

Viewing $f_t(x, y)$ as a complex-analytic map on $(\mathbb{C}^\times)^2 \times (D \setminus \{0\})$, let $C \subset X(\widetilde{\Delta})$ denote the complex-analytic closure of $\{f_t(x, y) = 0\}$ in $X(\widetilde{\Delta})$. The flat map $\varphi\colon X(\widetilde{\Delta}) \to \mathbb{C}$ then restricts to a map

$$\psi := \varphi\big|_C \colon C \to D(0; t_0).$$

We denote the fibres of this map by $C^{(t)}$ for $t \in D(0; t_0)$.

Lemma 4.4. *The fibres of $\psi\colon C \to D$ are isomorphic to $V(f_t) \subset X(\Delta)$ for $t \in D \setminus \{0\}$ and the fibre for $t = 0$ is given by $\bigcup_{m=1}^{N} V(f_m)$.*

Proof. Fix $t \in D \setminus \{0\}$. By Lemma 4.3, the fibres of the map $X(\widetilde{\Delta}) \to \mathbb{C}$ are isomorphic to $X(\Delta)$. Under this isomorphism, the fibre $C_t = (X(\widetilde{\Delta}))_t \cap C$ is sent to $V(f_t(x, y)) \subset X(\Delta)$.

For the special fibre, let $k = 1, \dots, N$ and

$$\lambda_k\colon\ \mathbb{R}^2 \to \mathbb{R}, \qquad (x, y) \mapsto \alpha_{k,1}\, x + \alpha_{k,2}\, y + \beta_k$$

the $\mathbb{Z}$-affine map such that $\nu\big|_{\Delta_k} = \lambda_k\big|_{\Delta_k}$. Consider the lattice automorphism $\overline{\chi}\colon\ \mathbb{Z}^3 \to \mathbb{Z}^3$ given by the matrix

$$\begin{pmatrix} 1 & 0 & -\alpha_{k,1} \\ 0 & 1 & -\alpha_{k,2} \\ 0 & 0 & 1 \end{pmatrix}$$

This map is compatible with the cones $\{0\}^2 \times \mathbb{R}_{\geq 0}$ and $C_{\widetilde{\Delta}_k}(\widetilde{\Delta})^\vee = \mathbb{R} \cdot (-\alpha_{k,1}, -\alpha_{k,2}, 1)$, and hence induces an isomorphism of toric varieties

$$\psi\colon (\mathbb{C}^\times)^2 \times \mathbb{C} \cong \operatorname{Spec} \mathbb{C}[x^{\pm 1}, y^{\pm 1}, t] \to \operatorname{Spec} \mathbb{C}[C_{\widetilde{\Delta}_k}(\widetilde{\Delta}) \cap \mathbb{Z}^3].$$

Under this isomorphism, the surface determined by $f_t(xt^{-\alpha_{k,1}}, yt^{-\alpha_{k,2}})$ corresponds to the surface determined by $f_t(x, y)$. Since

$$t^{-b} f_t(xt^{-\alpha_{k,1}}, yt^{-\alpha_{k,2}}) \cong f_k \mod \mathfrak{m}_{\mathbb{K}},$$

where $\mathfrak{m}_{\mathbb{K}}$ denotes the maximal ideal of $\mathbb{K}$, it follows that the special fibre is given by $\operatorname{Spec} \mathbb{C}[x^{\pm 1}, y^{\pm 1}]/(f_k)$. Because this holds for every $k = 1, \dots, N$, the claim follows. □

Definition 4.5 (Tropical Limit)**.** Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon and

$$f(x, y) = \sum_{(i,j) \in \Delta \cap \mathbb{Z}^2} a_{ij}(t) x^i y^j \in \mathbb{K}[x^{\pm 1}, y^{\pm 1}]$$

a Laurent polynomial defining an algebraic curve $\mathcal{C} \subset X_{\mathbb{K}}(\Delta)$. The *tropical limit* of $\mathcal{C}$ is defined to be the tropical curve $V(\operatorname{trop}(f))$ along with its dual subdivision $S = \{\Delta_1, \dots, \Delta_N\}$ and curves $C_m \subset X(\Delta_m)$ defined as follows. For each $1 \leq m \leq N$, the curve C_m is the zero locus in $X(\Delta_m)$ of

$$f_m(x, y) = \sum_{(i,j) \in \Delta_m \cap \mathbb{Z}^2} a_{ij}^{(0)} x^i y^j,$$

where $a_{ij}^{(0)}$ denotes the first nonzero coefficient of $a_{ij}(t)$.

4.2.2 Limits of Nodal Curves

The tropical limit of a nodal curve has a particularly simple form, as the following fundamental proposition illustrates.

Proposition 4.6. *[26] Let $f \in \mathbb{K}[x^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial with Newton polytope Δ such that the curve $V(f) \subset X_{\mathbb{K}}(\Delta)$ has δ nodes, where $\delta \geq 0$ is an integer. Also assume that $V(f)$ passes $\zeta := \left|\Delta \cap \mathbb{Z}^2\right| - 1 - \delta$ points $\pi_1, \dots, \pi_\zeta$ that map to Δ-generic points $p_1 :=$ $\operatorname{val}(\pi_1), \dots, p_\zeta := \operatorname{val}(\pi_\zeta)$. Then, the tropical limit of $V(f)$ consists of*

- *a simple tropical curve of rank ζ;*
- *a collection of curves $C_1, \dots, C_N$ such that for all $1 \leq j \leq N$:*
  - *if Δ_j is a triangle, then C_j is a rational and nodal curve that meets $X(\sigma)$ in precisely one point for each edge $\sigma \subset \Delta_j$;*
  - *if Δ_j is a parallelogram, then C_j is defined by a monomial times the product of two binomials, i.e., the sum of two monomials.*

Remark 4.7. The limit curve $C^{(0)}$ is not necessarily reduced. For instance, say the subdivision contains a parallelogram Δ_j that has an edge $s \subset \Delta_j$ of lattice length $m \geq 2$. Then, Δ_j contains the zero locus of the m-th power of a binomial as component.

The proof of Proposition 4.6. consists of establishing an upper and lower bound on the Euler characteristic of the normalization of generic fibres $C^{(t)}$. We do this by the following two lemmas. We denote $X(\partial\Delta') = \bigcup_{\sigma\subset\partial\Delta'} X(\sigma)$ for a lattice polygon $\Delta' \subset \mathbb{R}^2$, where the union is taken in $X(\Delta')$.

Lemma 4.8. *In the notation of Proposition 4.6, the Euler characteristic $e(C_t)$ of the normalization of $C_t, t \neq 0$ is bounded from above by*

$$e(C_t) \leq \mathrm{Br}(C_0, \partial\Delta) - \sum_{m\geq 2}(N_{2m-1} + N_{2m} - N'_{2m}),$$

where $\mathrm{Br}(C_0, \partial\Delta)$ denotes the number of branches of the limit curve C_0 located on $\cup_{\sigma\subset\partial\Delta} X(\sigma)$, and N_k (resp. N'_k) is the number of polygons with k edges (resp. k edges with parallel opposite side). Furthermore, if this bound is an equality, then

- *for any $2m$-gon with parallel opposite sides, f_k splits into a product of a monomial and two powers of binomials.*
- *for every other polygon Δ_k, there is precisely one irreducible component of C_k that intersects $X(\partial\Delta_k)$ in 3 points, and all other irreducible components intersect $X(\partial\Delta_k)$ in 2 points.*

Proof. Let $U \subset X(\widetilde{\Delta})$ be a small open neighborhood of $\bigcup_{m=1}^N X(\partial\Delta_m)$. By a standard property of the Euler characteristic, we then have

$$e(C^{(t)}) = e(C^{(t)} \setminus U) + e(C^{(t)} \cap U).$$

Pick $z \in C^{(0)} \cap X(\Delta_m) \cap X(\Delta_{m'})$ for a pair $1 \leq m < m' \leq N$. If U is sufficiently small, then the connected component of $C^{(0)} \cap U$ containing z consists of $r + r'$ branches, where $\mathrm{Br}(C_m, z) = r$ and $\mathrm{Br}(C_{m'}, z) = r'$ are the number of branches of C_m respectively $C_{m'}$ at z. By [25, Rmk. 3.4], $C^{(t)}$ consists of at least $\max(r, r')$ handles with $r + r'$ holes. Hence, for a small neighborhood $U_z \subset U$ of z,

$$e(C^{(t)} \cap U_z) \leq r + r' - 2\max\{r, r'\} \leq 0.$$

Therefore,

$$e(C^{(t)} \cap U) \leq \mathrm{Br}(C^{(0)}, \partial\Delta).$$

Moreover, because the Euler characteristic of a flat family is upper semi-continuous, we have

$$e(C^{(t)} \setminus U) \leq e(C^{(0)} \setminus U).$$

Denote the irreducible components of C_k by $\{C_{kj}\}_j$. We then have

$$\begin{aligned} e(C^{(0)} \setminus U) &= \sum_{k=1}^N \sum_j e(C_{kj} \setminus U) \\ &= \sum_{k=1}^N \sum_j (e(C_{kj}) - e(C_{kj} \cap U)). \end{aligned}$$

Because C_{kj} is an irreducible projective curve, its Euler characteristic is bounded from above by 2. Moreover, for U sufficiently small, $C_{kj} \cap U$ is a disjoint union of $\mathrm{Br}(C_{kj}, \partial\Delta_k)$ open disks. Hence, $e(C_{kj} \cap U) = \mathrm{Br}(C_{kj}, \partial\Delta_k)$. We thus obtain

$$e(C^{(0)} \setminus U) \leq \sum_{k=1}^{N} \sum_{j} (2 - \mathrm{Br}(C_{kj}, \partial\Delta_k)).$$

Also, note that each irreducible component C_{kj} intersects $X(\partial\Delta_k)$ in at least two points. The points of intersection of C_k with $X(\partial\Delta_k)$ correspond to the roots of $f_k^\sigma, \sigma \preceq \partial\Delta_k$. Hence, if Δ_k has an odd number of edges, there is at least one irreducible component C_{kj} of C_k for which $\mathrm{Br}(C_{kj}, \partial\mathcal{D}_k) \geq 3$.

Similarly, if Δ_k has an even number of edges but not all opposite sides are parallel, $\mathrm{Br}(C_{kj}, \partial\Delta_k) \geq 3$. Therefore, it follows that

$$e(C^{(0)} \setminus U) \leq \sum_{j\geq 2} (N_{2j-1} + N_{2j} - N'_{2j}),$$

and if there is an equality, we see that

- for any polygon Δ_k with an even number of edges and parallel opposite sides, each irreducible component of C_k intersects $X(\partial\Delta_k)$ in precisely two points;
- for every other polygon Δ_k, there is precisely one irreducible component of C_k that intersects $X(\Delta_k)$ in 3 points, and all other irreducible components intersect $X(\Delta_k)$ in 2 points.

Let $1 \leq k \leq N$ an integer such that Δ_k has an even number of edges, whose opposite sides are parallel and j an index of an irreducible component of C_k. The fact that C_{kj} intersects $X(\sigma)$ for $\sigma \subset \Delta_k$ an edge only for two opposite sides, means that f_j, up to multiplication by a unit in $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, has Newton Polygon σ. Moreover, since the roots of f_{kj}^σ correspond to the points of intersection of C_{kj} with $X(\sigma)$, we conclude that f_{kj} is the power of a binomial. Therefore, f_k is the product of a monomial and binomials. □

Lemma 4.9. *In the notation of Proposition 4.6, the Euler characteristic of $C^{(t)}$ for $t \in \mathbb{C}^\times$ is greater than or equal to*

$$2\left|\partial\Delta \cap \mathbb{Z}^2\right| - |\mathrm{Vert}(S) \cap \partial\Delta| + \sum_{m\geq 3} (m-4)N_m - 2(\mathrm{rk}(T) - \mathrm{rk}_{\exp}(T)).$$

Moreover, if equality holds, then T is of rank ζ.

Proof. Since $\zeta = \dim \mathrm{Sev}_n(\Delta) = \left|\partial\Delta \cap \mathbb{Z}^2\right| + p_g(C^{(t)}) - 1$, we have

$$\begin{aligned} e(C^{(t)}) &= 2 - 2p_g(C^{(t)}) \\ &= 2(1 - (\zeta + 1 - \left|\partial\Delta \cap \mathbb{Z}^2\right|)) \\ &= 2\left|\partial\Delta \cap \mathbb{Z}^2\right| - 2\zeta\,. \end{aligned}$$

Since the points $x_1, \ldots, x_\zeta$ are Δ-generic, the condition to lie on them cuts out of $\mathcal{T}(\Delta, S)$ either an empty set or a subspace of codimension ζ. As T lies in this subspace, we see that it is nonempty and therefore, $\zeta \leq \dim \mathcal{T}(\Delta, S) = \mathrm{rk}(T)$. It follows that

$$e(C^{(t)}) \geq 2(\left|\partial\Delta \cap \mathbb{Z}^2\right| - \mathrm{rk}_{\exp}(T)) - 2(\mathrm{rk}(T) - \mathrm{rk}_{\exp}(T)).$$

Furthermore, recall that by Proposition 3.32,

$$\mathrm{rk}_{\exp}(T) = |V(S)| - \sum_{k=1}^{N} (|V(\Delta_k)| - 3) - 1.$$

For each polytope $\Delta_k \in S$ in the subdivision, the number of edges of Δ_k equals its number of vertices. The sum of all these edges equals the total number of edges in S, where edges in the interior of Δ are counted twice. Hence, we obtain

$$\mathrm{rk}_{\mathrm{exp}}(T) = |V(S)| + |V(S) \cap \partial\Delta| - 2\,|E(S)| + 3N - 1$$

and so,

$$\begin{aligned} e(C^{(t)}) &\geq 2\left|\partial\Delta \cap \mathbb{Z}^2\right| - 2\,|V(S)| + 2 - 2\,|V(S) \cap \partial\Delta| + 4\,|E(S)| - 6N - 2(\mathrm{rk}(T) - \mathrm{rk}_{\mathrm{exp}}(T)) \\ &= 2(\left|\partial\Delta \cap \mathbb{Z}^2\right| - |V(S) \cap \partial\Delta|) + \sum_{m\geq 3}(m-4)N_m - 2(\mathrm{rk}(T) - \mathrm{rk}_{\mathrm{exp}}(T)). \end{aligned}$$

□

Proof of Proposition 4.6. Combining the bounds obtained in Lemma 4.8 and Lemma 4.9, we find

$$\begin{aligned} 0 &\leq 2\left|\partial\Delta \cap \mathbb{Z}^2\right| - |V(S) \cap \partial\Delta| - \mathrm{Br}(C^{(0)}, \partial\Delta) \\ &\leq 2(\mathrm{rk}(T) - \mathrm{rk}_{\mathrm{exp}}(T)) - \sum_{m\geq 2}((2m-3)N_{2m} - N'_{2m}) - \sum_{m\geq 2}(2m-2)N_{2m+1} \leq 0. \end{aligned}$$

Here, the final inequality follows from the bound obtained in Proposition 3.32. Hence, the above inequalities are all equalities, again by Proposition 3.32. In particular, this means by Proposition 3.32 that $\mathrm{rk}(T) = \mathrm{rk}_{\mathrm{exp}}(T)$ and so T is nodal. Also, it follows that

$$2\left|\partial\Delta \cap \mathbb{Z}^2\right| - |V(S) \cap \partial\Delta| = \mathrm{Br}(C^{(0)}, \partial\Delta),$$

implying that T is a simple curve. The remaining claims made in Proposition 4.6 now all follow from Lemmas 4.8, 4.9 and the fact that the inequalities in these lemmas are equalities. □

Lemma 4.10. *[25, Lem. 3.7] In the context of 4.6, the curve $\mathcal{C}$ is irreducible if and only if its tropicalization T is.*

Proof. First off, assume that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ for two curves $\mathcal{C}_1, \mathcal{C}_2 \subset X_{\mathbb{K}}(\Delta)$. Then,

$$\mathrm{trop}(\mathcal{C}) = \mathrm{trop}(\mathcal{C}_1) \cup \mathrm{trop}(\mathcal{C}_2),$$

i.e., T is reducible as well.

Next, assume that $T = T_1 \cup T_2$ for some tropical curves T_1, T_2. As T is simple, T_1, T_2 intersect only in 4-valent vertices. Then the components of $C^{(0)}$ fit in two subsets which intersect each other only in $X(\Delta_k)$ for $\Delta_k \in S$ a parallelogram. In the deformation $C^{(t)}$, these curves do not glue together and so, $C^{(t)}$ is reducible as well. □

4.2.3 Examples

Example 4.11. Let $\Delta \subset \mathbb{R}^2$ be the triangle with vertices $(0,0), (2,0)$ and $(0,2)$. Consider the curve $\mathcal{C} \subset X_{\mathbb{K}}(\Delta) \cong \mathbb{P}^2$ given by

$$f(x,y) = 1 + x + y + xy + tx^2 + ty^2.$$

The associated tropical polynomial is

$$\mathrm{trop}(f)(x,y) = 0 \oplus x \oplus y \oplus xy \oplus -1x^2 \oplus -1y^2,$$

whose corner locus has been plotted in Figure 10, alongside its dual subdivision. The curves C_1, C_2, C_3 are given by the polynomials

$$\begin{aligned} f_1(x,y) &= (1+x)(1+y), \\ f_2(x,y) &= x(1+x+y), \\ f_3(x,y) &= y(1+x+y). \end{aligned}$$

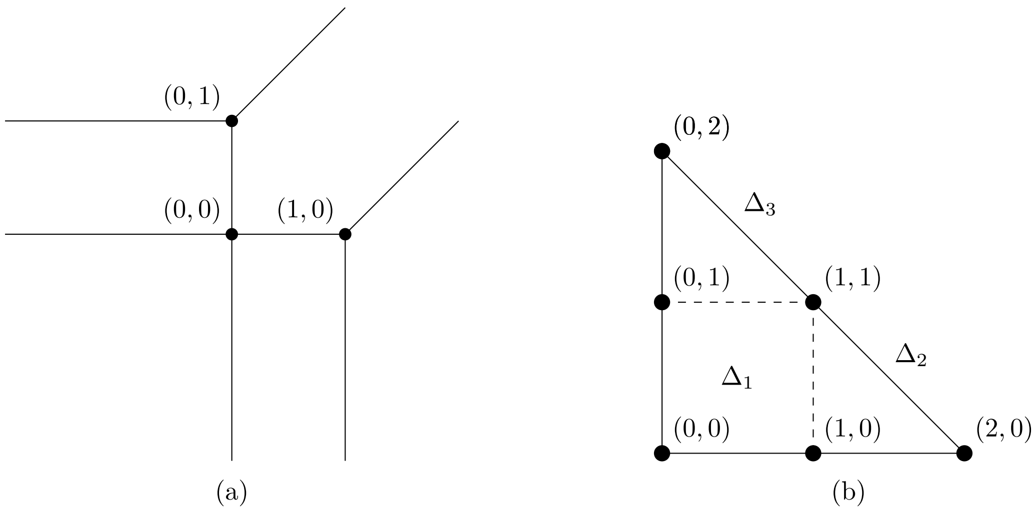


Figure 10

The first polynomial clearly satisfies the results of Proposition 4.6, while the second and third define lines in $X(\Delta_2) \cong X(\Delta_3) \cong \mathbb{P}^2$. In particular, the curves C_2, C_3 are smooth and meet the toric divisors of $X(\Delta_2) \cong X(\Delta_3)$ in one point each.

Furthermore, a simple computation shows that C is smooth, meaning that if the result of Proposition 4.6 were to hold, the rank of $V(\text{trop}(f))$ should be $\dim \text{Sev}_0(\Delta) = 5$. However, by Proposition 3.32 the rank of $V(\text{trop}(f))$ is 4. Hence, no 5 points on $V(\text{trop}(f))$ lie in Δ-generic position. △

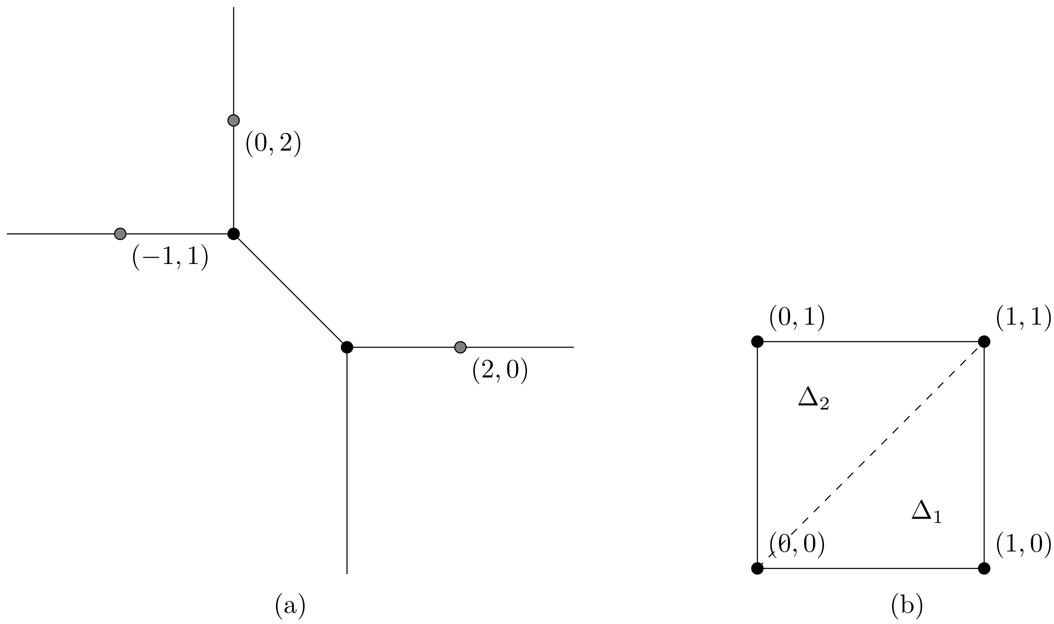


Figure 11

Example 4.12. Let Δ be the square with vertices $(0,0), (1,0), (0,1), (1,1)$. Consider the curve $\mathcal{C} \subset X_{\mathbb{K}}(\Delta)$ given by the polynomial

$$f(x,y) = -(t^{-2}+1) + (t^{-1}+1)x + (t^{-1}+1)y - (t^{-1}+t)xy \in \mathbb{K}[x,y].$$

This is a smooth curve that passes through the $\dim \text{Sev}_0(\Delta) = 3$ points

$$(1, t^{-2}), (t, t^{-1}), (t^{-2}, 1).$$

These map to Δ-generic points

$$(0, 2), (-1, 1), (2, 0).$$

Moreover, the tropical polynomial associated to f is given by

$$\operatorname{trop}(f)(x, y) = 2 \oplus 1x \oplus 1y \oplus 1xy.$$

For its corner locus along with the dual subdivision of Δ see Figure 11. By Proposition 3.32, the rank of $V(\operatorname{trop}(f))$ is 3. Moreover, the curves C_1, C_2 are given by

$$f_1(x, y) = -1 + x - xy,$$
$$f_2(x, y) = -1 + y - xy.$$

These are irreducible quadrics in $X(\Delta_1) \cong X(\Delta_2) \cong \mathbb{P}^2$ and are thus rational curves. Furthermore, they indeed intersect the toric divisors of $X(\Delta_1), X(\Delta_2)$ in a single point each. For instance, for the edge $\sigma = [(0, 0), (1, 0)]$ we have

$$C_1 \cap X(\sigma) = \overline{\{f^\sigma = 0\}} = 1 \in \mathbb{C}^\times \subset X(\sigma).$$

△

Finding curves $\mathcal{C}$ that are both singular and irreducible in the deformation is considerably harder. The following two examples were found with the aid of a computer program. See Appendix C.1 and Appendix C.2 for the relevant SageMath code.

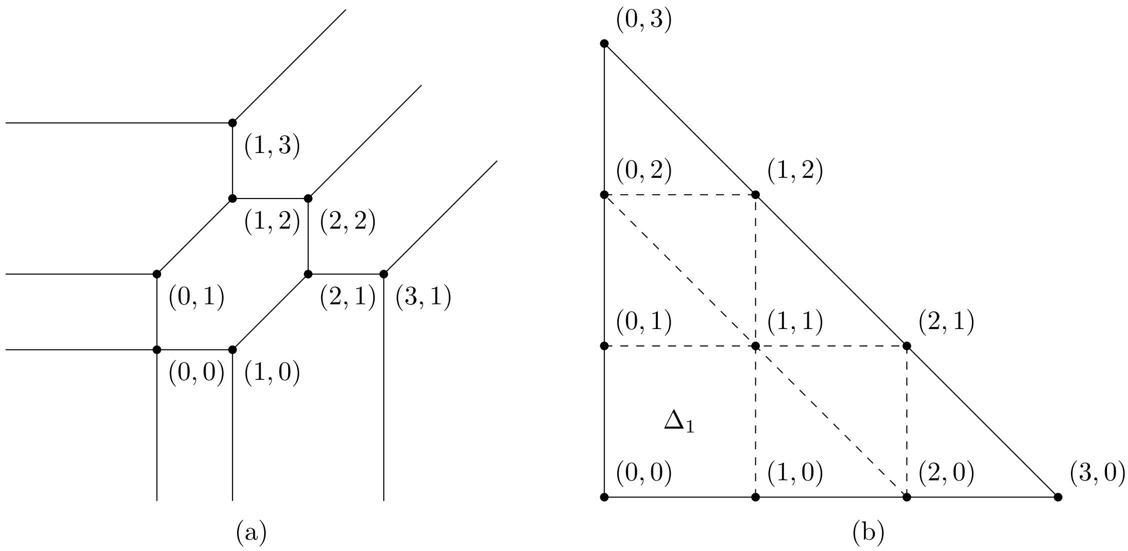


Figure 12

Example 4.13. Next, let Δ be the triangle with vertices $(0, 0), (3, 0), (0, 3)$ and consider the cure $\mathcal{C} \subset X_{\mathbb{K}}(\Delta)$ given by the polynomial

$$f(x, y) = 1 + ax + ay + bxy + tx^2 + ty^2 + t^2x^2y + t^2xy^2 + t^4x^3 + t^4y^3,$$

where

$$a = \frac{t^7 - 3t^6 + 4t^5 - 4t^4 + 3t^3 - t^2 - 1}{t - 1}, \quad b = \frac{-4t^7 + 12t^6 - 16t^5 + 16t^4 - 14t^3 + 8t^2 - 2t + 1}{t^2 - 2t + 1}.$$

This gives a nodal cubic with a single singularity, which is located in the point $(1 - t, 1 - t)$.

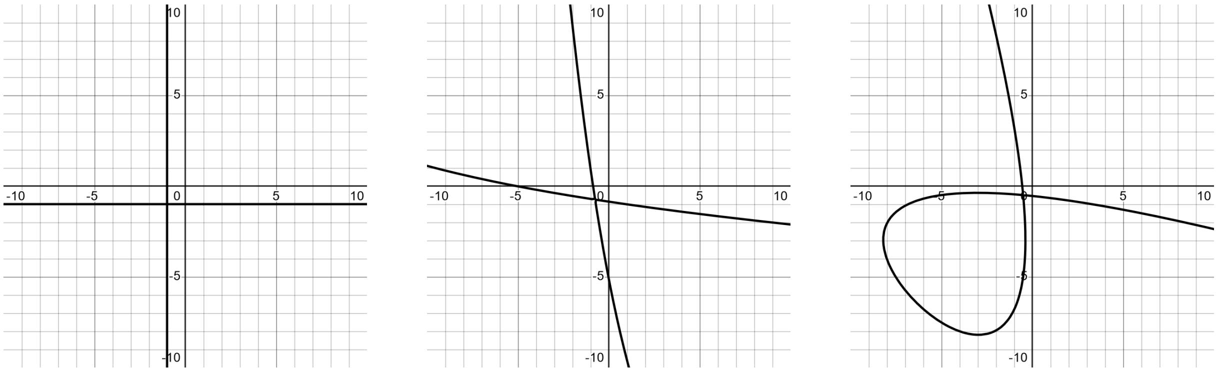
The tropicalization of $\mathcal{C}$ is the tropical curve T given by

$$\operatorname{trop}(f) = 0 \oplus x \oplus y \oplus xy \oplus -1x^2 \oplus -1y^2 \oplus -2x^2y \oplus -2xy^2 \oplus -4x^3 \oplus -4y^3.$$

This curve, along with its subdivision is depicted in Figure 12. The limit curve of $\mathcal{C}$ factors in 9 irreducible components, seven of which are lines in $X(\Delta') \cong \mathbb{P}^2$ for Δ' a triangle in the subdivision. The other two components appear in $X(\Delta_1)$, where Δ_1 is the parallelogram with vertices $(0,0),(1,0),(0,1),(0,0)$. The curve $C^{(0)} \cap X(\Delta_1)$ is given by the polynomial

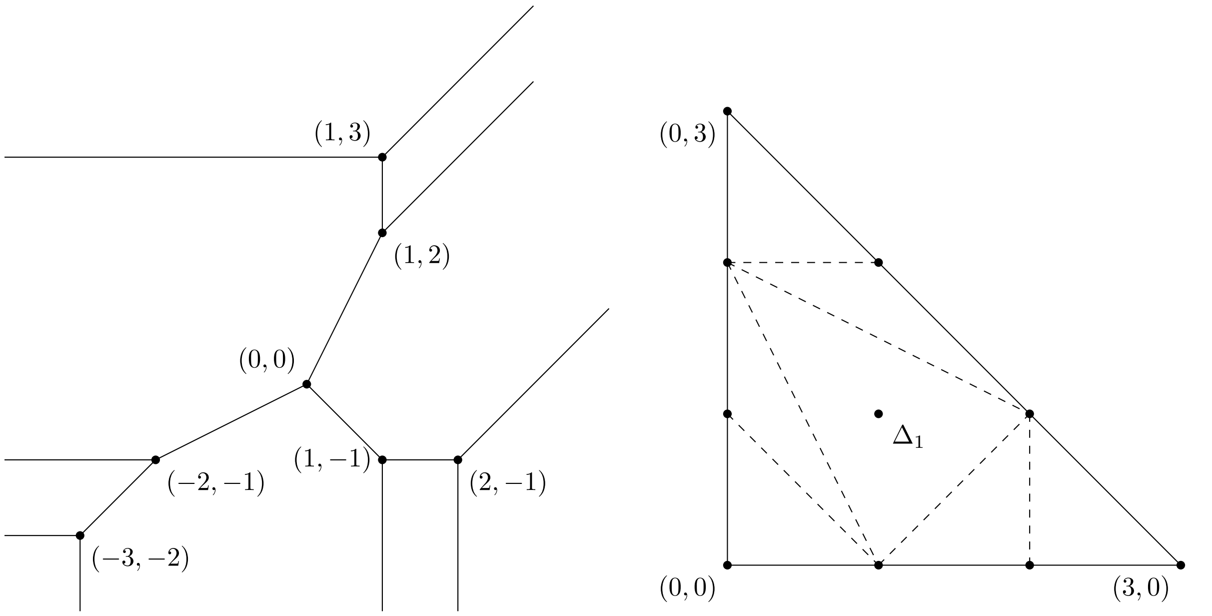
$$f_1 = (1+x)(1+y).$$

For small t, we can view $f_{(t)}$ as a small deformation of f_1. Figure 13 contains three fibres of this deformation. $\triangle$



(a) t=0 (b) t=0.25 (c) t=0.5

Figure 13



(a) (b)

Figure 14

Example 4.14. Once more, let Δ be the triangle with vertices $(0,0),(3,0),(0,3)$. The curve $\mathcal{C} \subset X_{\mathbb{K}}(\Delta)$ given by the polynomial

$$f = t^3 + x + ty + tx^2 + axy + by^2 + t^3x^3 + cx^2y + txy^2 + t^3y^3,$$

where

$$a = \frac{t^6 - 3t^5 + 4t^4 - 10t^3 + 5t^2 - 4t + 3}{t^2 - 2t + 1},$$
$$b = \frac{-t^6 + 3t^5 - 4t^4 + 6t^3 - 5t^2 + 3t - 1}{t^2 - 2t + 1},$$
$$c = \frac{-2t^6 + 6t^5 - 7t^4 + 7t^3 - 2t^2 + t - 1}{t^3 - 3t^2 + 3t - 1},$$

is a nodal plane curve with a single singularity in the point $(t-1, t-1)$.

The tropicalization T of $\mathcal{C}$ is the corner locus of the tropical polynomial

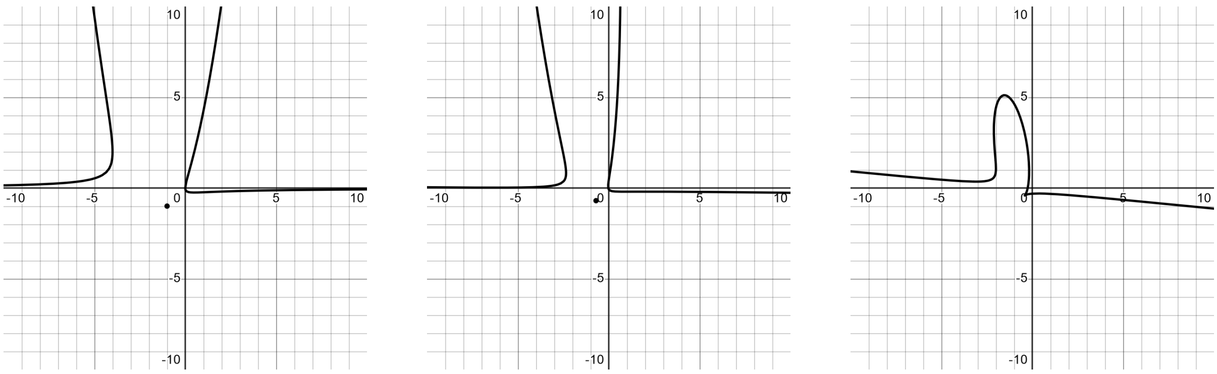
$$\text{trop}(f) = -3 \oplus x \oplus -1y \oplus -1x^2 \oplus xy \oplus y^2 \oplus -3x^3 \oplus x^2y \oplus -1xy^2 \oplus -3y^3.$$

This curve, along with its dual subdivision, has been plotted in Figure 14. Note that T is a simple nodal curve of rank 8.

The limit curve $C^{(0)}$ has 7 irreducible components, 6 of which are lines in $\mathbb{P}^2$. The seventh component is the curve in $X(\Delta_1)$ given by the equation

$$f_1 = x + 3xy + x^2y - y^2.$$

This is a nodal curve with a single singularity in the point $(-1,-1)$. We can view $f_{(t)}$ as a small deformation of this curve. Three of the fibres of this deformation are plotted in Figure 15. For $t = 0$ and $t = 0.3$, the branches of the singularity at $(1-t, 1-t)$ lie in the imaginary plane, so we can only see the single point $(1-t, 1-t)$. For $t = 0.6$, the node is visible in the real plot. △



(a) t=0 (b) t=0.3 (c) t=0.6

Figure 15

4.3 Recovering the Limit Curve

In the previous section we gave a process of degenerating a family of complex algebraic curves $C^{(t)}$ into a tropical curve T and a limit curve $C^{(0)} = C_1 \cup \ldots \cup C_N$. In the context of the Mikhalkin correspondence, where $C^{(t)}$ and T pass a sufficient number of generic points and $C^{(t)}$ is nodal, Proposition 4.6 showed us that this tropical limit has a particularly simple form. Our next step in proving the Mikhalkin correspondence will be to recover the limit curves $C^{(0)}$ from the tropical curve T. The results from this section can also be found in [25] and [14].

Lemma 4.15. *Let $\Delta' \subset \mathbb{R}^2$ be a lattice parallelogram. There is a unique curve C in $X(\Delta')$ given by a Laurent polynomial $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ that equals the product of a monomial and binomials, and whose truncations $f^{\sigma_1}, f^{\sigma_2}$ to two non-parallel edges σ_1, σ_2 of Δ' are fixed.*

Proof. Let $(a, b) \in \mathbb{Z}^2$ be a vertex of Δ' and $(u_1, v_1), (u_2, v_2) \in \mathbb{Z}^2$ two vectors such that the edges σ_1, σ_2 are given by $[(a, b), (a+u_1, b+v_1)], [(a, b), (a+u_2, b+v_2)]$. Also let $d_1 = \gcd(u_1, v_1)$ and $d_2 = \gcd(u_2, v_2)$. Then, f is necessarily given by

$$f(x,y) = Cx^a y^b \prod_{i=1}^{d_1}(1 + \alpha_i\, x^{u_1/d_1} y^{v_1/d_1}) \prod_{i=1}^{d_2}(1 + \beta_i\, x^{u_2/d_2} y^{v_2/d_2}),$$

for some $C, \alpha_i, \beta_i \in \mathbb{C}$. Consequently, the truncations of f to the edges σ_1, σ_2 are

$$f^{\sigma_1}(x,y) = Cx^a y^b \prod_{i=1}^{d_1}(1 + \alpha_i\, x^{u_1/d_1} y^{v_1/d_1}),$$
$$f^{\sigma_2}(x,y) = Cx^a y^b \prod_{i=1}^{d_2}(1 + \beta_i\, x^{u_2/d_2} y^{v_2/d_2}),$$

which clearly fix f. □

Lemma 4.16. *[25, Lem. 3.5] Let $\Delta' \subset \mathbb{R}^2$ be a lattice triangle. Then, there are precisely $2 \cdot \operatorname{Area} \Delta'$ curves in $X(\Delta')$, given by a Laurent polynomial $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$, that satisfy the following conditions:*

(1) the curve C is rational and nodal, and intersects each of the toric divisors $X(\sigma), \sigma \subset \Delta'$ of $X(\Delta')$ in precisely one point, where it is smooth;

(2) the Newton polygon of f is Δ' and the coefficients of f at the vertices of Δ' are predetermined.

Furthermore, if the coefficients of f at one or two edges of Δ' are also given, then the number of curves is divided by the lattice length of the given edges.

Proof. First off, we claim that after some lattice automorphism $\mathbb{Z}^2 \to \mathbb{Z}^2$, we may assume that Δ' is the triangle with vertices $(0, m), (p, 0), (q, 0)$ where $0 \le p < q \le m$. Label the vertices of Δ' by A, B, C such that the sides of Δ' satisfy $AB \le AC \le BC$. Then, we first translate A to the origin. By possibly reflecting in the x- and/or y-axis, we may then assume that B lies in the upper right quadrant. Write $B = (b_1, b_2)$ for $b_1, b_2 \in \mathbb{Z}_{\ge 0}$ and let u, v be integers such that

$$ub_1 + vb_2 = d,$$

where d is the greatest common divisor of b_1, b_2. Next, we apply the linear automorphism given by the matrix

$$\begin{pmatrix} u & v \\ -\frac{b_2}{e} & \frac{b_1}{e} \end{pmatrix}.$$

This maps the point B to $(d, 0)$ and leaves $A = (0, 0)$ fixed. Finally, we apply the horizontal translation such that C is mapped onto the y-axis. This maps the points A, B, C to $(p, 0), (q, 0), (0, m)$ respectively for some $p, q, m \in \mathbb{Z}$ with $q > p$ and $m \ge 0$. The inequalities $0 \le p < q \le m$ can now be easily deduced from the fact that $AB \le AC \le BC$.

Next, let C be a curve satisfying all the conditions from the lemma. Denote the intersection of C with $X([(p, 0), (q, 0)])$ by P_1, the intersection of C with $X([(q, 0), (0, m)])$ by P_2 and the intersection of C with $X([(p, 0), (0, m)])$ by P_3. Since C is rational, there exists a birational map $\mathbb{P}^1 \dashrightarrow C$. Since $\mathbb{P}^1$ is a smooth curve, we may extend this rational map to a surjective morphism $\mathbb{P}^1 \to C$. Let θ be a parameter of $\mathbb{P}^1$. After precomposition by an appropriate automorphism of $\mathbb{P}^1$, we may assume that the map $\mathbb{P}^1 \to C$ sends 0 to P_3 , 1 to P_1, and ∞ to P_2. Consequently, $\theta, \theta - 1, \theta^{-1}$ are uniformizers of the points P_3, P_1, P_2 respectively.

We aim to compute $x(\theta), y(\theta)$ as functions of θ. First, we look in a neighborhood of P_1. Let $\sigma_1 = \operatorname{cone}((0, 1))$ be the normal cone of the edge $[(p, 0), (q, 0)]$. The corresponding affine open of $X(\Delta')$ is given by

$$U_{\sigma_1} = \operatorname{Spec} \mathbb{C}[x^{\pm 1}, y].$$

Let $z_1 \in \mathbb{C}^\times$ be such that $(x - z_1, y)$ is the maximal ideal corresponding to the point $P_1 \in C$. Since C intersects $y = 0$ only in this point, we have

$$f = x^p(x - z_1)^{q-p} + yh,$$

for some $h \in \mathbb{C}[x, y]$. We distinguish the two cases that $q - p = 1$ and $q - p > 1$. For $q - p = 1$, note that $\frac{\partial f}{\partial x}(z_1, 0) = 1$, which means that y is a uniformizer and therefore, $\nu_{P_1}(y) = 1$. Next, assume that $q - p > 1$. In that case, we have $\frac{\partial f}{\partial x}(z_1, 0) = 0$ and $\frac{\partial f}{\partial y}(z_1, 0) = h(z_1, 0)$. Because C is smooth in P_1, we must have $h(z_1, 0) \neq 0$, implying that $h(z_1, 0)$ is invertible in $\mathcal{O}_{C,P_1}$ and $x - z_1$ is a uniformizer at P_1. We thus conclude that $\nu_{P_1}(y) = \nu_{P_1}((x - z_1)^{q-p}/h) = q - p$. Moreover, it is clear that $\nu_{P_1}(x) = 0$.

Because $\theta - 1$ is a uniformizer at P_1, this implies that $y = \beta(\theta)(\theta - 1)^{q-p}$ and $x = \alpha(\theta)$ for some $\alpha, \beta \in (\mathbb{C}[\theta]_{(\theta-1)})^\times$.

Analogously, by turning to the affine opens corresponding to the normal cones of the sides $[(0, m), (p, 0)], [(0, m), (q, 0)]$, one may compute that

$$\nu_{P_2}(x) = -\nu_{P_3}(x) = -m, \ \nu_{P_2}(y) = -\nu_{P_3}(y) = -p.$$

As θ, θ^{-1} are uniformizers at these points, this in turn implies that

$$x = \alpha\, \theta^m, \quad y = \beta\, \theta^p(\theta - 1)^{q-p},$$

for some $\alpha, \beta \in \mathbb{C}(\theta)^\times$ that are units in the local rings of P_1, P_2, P_3. Also, because C intersects the toric divisors $X(\partial\Delta')$ each in a single point, we necessarily have that α and β are constant.

Let d be the greatest common divisor of m and p. The truncations of f to the edges σ_1, σ_3 are then given by

$$f^{\sigma_1} = ux^p(x + z_1)^{q-p},$$
$$f^{\sigma_2} = v(y^{m/d} + z_2 x^{p/d})^d,$$

for some predetermined coefficients $u, v \in \mathbb{C}^\times$. Moreover, we may assume that $uz_1^{q-p} = vz_2^d = 1$. In particular, there are precisely $q - p$ choices for z_1 and d solutions for z_2. Substituting the above-found parametrization and setting $\theta = 1$ for the truncation f^{σ_1} and $\theta = 0$ for f^{σ_3} now gives

$$0 = \alpha + z_1,$$
$$0 = \beta^{m/d} \cdot (-1)^{m(q-p)/d} + z_2\, \alpha^{p/d}.$$

Hence, there are $q - p$ solutions for α, and $m/d \cdot d = m$ solutions for β. In total, we conclude that there are at most $2\,\mathrm{Area}(\Delta') = m(q - p)$ curves C satisfying the given conditions. In order to show that there are precisely $2\,\mathrm{Area}(\Delta')$ such curves it only remains to prove that C is a nodal curve.

For this, consider the map $\theta \to x(\theta), y(\theta)$ as a map between smooth complex manifolds $\mathbb{C}^\times \setminus \{1\} \to (\mathbb{C}^\times)^2$. Since $\frac{dx}{d\theta}$ vanishes nowhere for $\theta \in \mathbb{C}^\times \setminus \{1\}$, this is a smooth immersion. Because an immersion of manifolds is locally an embedding, this implies that C has no singular branches, i.e., the only singularities C may have are given by the transverse intersection of its branches. Therefore, for C to be nodal it suffices to show each point of C has at most two branches.

We prove this statement by contradiction. Say there exist distinct $\theta_1, \theta_2, \theta_3 \in \mathbb{C}^\times$ such that $\alpha\, \theta_1^m = \alpha\, \theta_2^m = \alpha\, \theta_3^m$ and $\beta\, \theta_1^p(\theta_1 - 1)^{q-p} = \beta\, \theta_2^p(\theta_2 - 1)^{q-p} = \beta\, \theta_3^p(\theta_3 - 1)^{q-p}$. The first two equations imply that there are m-th roots of unity $\varepsilon_1, \varepsilon_2$ such that

$$\theta_2 = \theta_1\, \varepsilon_1, \qquad \theta_3 = \theta_1\, \varepsilon_1\, .$$

Substituting these into the second set of equations gives

$$\theta_1 - 1 = \varepsilon_3(\theta_1\, \varepsilon_1 - 1) = \varepsilon_4(\theta_1\, \varepsilon_2 - 1),$$

for some $\varepsilon_3, \varepsilon_4 \in \mathbb{C}$ satisfying $\varepsilon_3^{q-p} = \varepsilon_1^p, \varepsilon_4^{q-p} = \varepsilon_2^p$. Now pick $m(q-p)$-th roots of unity η_1, η_2 satisfying

$$\varepsilon_1 = \eta_1^{q-p} \qquad \varepsilon_3 = \eta_1^p \qquad \varepsilon_2 = \eta_2^p \qquad \varepsilon_4 = \eta_2^p.$$

We then find that

$$\theta_1 = \frac{1-\eta_1^p}{1-\eta_1^q} = \frac{1-\eta_2^p}{1-\eta_2^q}.$$

Substituting

$$\eta_1 = \cos\omega_1 + i\sin\omega_1, \quad \eta_2 = \cos\omega_2 + i\sin\omega_2$$

into this equation gives

$$\frac{\cos(p\omega_1/2)\cos(q\omega_2/2)}{\cos(q\omega_1/2)\cos(p\omega_2/2)} = \cos\frac{(q-p)(\omega_2-\omega_1)}{2} + i\sin\frac{(q-p)(\omega_2-\omega_1)}{2}$$

and hence, $(q-p)(\omega_2-\omega_1) \in 2\pi\mathbb{Z}$. From this we conclude that $\varepsilon_1 = \eta_1^{q-p} = \eta_2^{q-p} = \varepsilon_2$ and therefore the $\theta_1, \theta_2, \theta_3$ are not all distinct, meaning that each point of C has at most two branches. ☐

Proposition 4.17. *Let $\Delta \subset \mathbb{R}^2$ be a non-degenerate lattice polytope, $1 \leq \zeta \leq \left|\Delta \cap \mathbb{Z}^2\right| - 1$ an integer and $\pi_1, \ldots, \pi_\zeta \in (\mathbb{K}^\times)^2$ points that map to Δ-generic points $p_1, \ldots, p_\zeta \in \mathbb{R}^2$. Also let T be a simple tropical curve with Newton polytope Δ that passes the points $p_1, \ldots, p_\zeta$. Denote the edges in the dual subdivision that are dual to the edges which pass $p_1, \ldots, p_\zeta$ by $\sigma_1, \ldots, \sigma_\zeta$. Then, there are*

$$\frac{\mu(T)}{\prod_{[\sigma]}|\sigma|\prod_{i=1}^{\zeta}|\sigma_i|}$$

different ways to reconstruct the limit curve $C^{(0)}$ such that it is compatible, in the sense of Proposition 4.6, with the points $\pi_1, \ldots, \pi_\zeta$ and the tropical curve T.

Proof. Let $1 \leq k \leq \zeta$ an integer. Since p_k lies on the edge of T dual to σ_k, we have

$$0 = f(\xi_k, \eta_k) = t^{-\operatorname{trop}(f)(x_k,y_k)}\left(\sum_{(i,j)\in\sigma_k\cap\mathbb{Z}^2} a_{ij}^{(0)}(\xi_k^{(0)})^i(\eta_k^{(0)})^j + \mathcal{O}(t)\right). \tag{4.1}$$

Moreover, by Proposition 4.6, the polynomial

$$\sum_{(i,j)\in\sigma_k\cap\mathbb{Z}^2} a_{ij}^{(0)} x^i y^j$$

is the power of a binomial. Hence, (4.1) uniquely determines, up to a constant factor, the coefficients $a_{ij}^{(0)}, (i,j) \in \sigma_k \cap \mathbb{Z}^2$. We now retrieve the remaining coefficients by reconstructing the tropical curve T from the edges $\sigma_1, \ldots, \sigma_\zeta$.

An extended edge of T is a maximal line interval contained in T. We denote the set of extended edges of T by $E^*(T)$. Because they are generic, the points $p_1, \ldots, p_\zeta$ lie on distinct extended edges $e_1, \ldots, e_\zeta$ of T. Let G_0 be the union of $e_1, \ldots, e_\zeta$. We inductively construct sets $G_n, 1 \leq n \leq m$ by appending extended edges of T until $G_m = T$.

Let $n \geq 0$ an integer such that the set G_n has been constructed. If $T \setminus G_n \neq \emptyset$, then there exists a vertex v_n of G_n that is 2-valent. Indeed, if this would not be the case, then G_n would contain all 3-valent vertices of T but would miss at least one 4-valent vertex. This would mean that there is an infinite family of tropical curves containing G_n obtained by slightly perturbing the missing extended edge through this vertex. However, this contradicts the generality of the points $p_1, \ldots, p_\zeta$. Let $\varepsilon_n \in E^*(T)$ be the extended edge in $T \setminus G_n$ which has v_n as vertex. We define $G_{n+1} := G_n \cup \{\varepsilon_n\}$.

Let $\Delta_0, \dots, \Delta_{m-1} \in S$ be the triangles corresponding to the vertices $v_0, \dots, v_{m-1}$. At each step, by Lemma 4.16, there are

$$\frac{2\,\mathrm{Area}(\Delta_n)}{|\sigma| \cdot |\sigma'|} \tag{4.2}$$

choices for the curve C_n, where σ, σ' are the edges in S that correspond to the extended edges of G_n adjacent to v_n.

The total number of ways to reconstruct the limit curve $C^{(0)}$ is now equal to the product of (4.2) over all triangles from S. In this product, each edge σ of S appears either once, twice or not at all, depending on whether σ is a bounded or unbouded edge, and whether one of the points $p_1, \dots, p_\zeta$ lies on σ.

It follows that there are precisely

$$\prod_{\text{triangle } \Delta' \in S} 2\,\mathrm{Area}(\Delta') \prod_{1 \le k \le \zeta} |\sigma_k|^{-1} \prod_{\substack{[\sigma] \\ \sigma \subsetneq \partial\Delta}} |\sigma|^{-1}$$

ways to reconstruct the limit curve $C^{(0)}$. Since T is a simple curve, the lattice lengths of edges on the boundary of Δ equal 1. Therefore, the above quantity is also equal to

$$\frac{\mu(T)}{\prod_{k=1}^{r} |\sigma| \prod_{[\sigma]} |\sigma|}.$$

□

4.4 Refinement of the Tropical Limit

In general, the tropical limit as defined above does not uniquely determine the family of curves $\mathcal{C}$. In this section, we therefore append additional data to the tropicalization process obtained from the points of intersection of the limit curve C with the toric divisors $X(\sigma)$ with σ an edge in S.

4.4.1 Refinement at Isolated Singularity

We recall the setting from Proposition 4.6. Let $f \in \mathbb{K}[x^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial with Newton polytope Δ such that the closure of its zero locus $\mathcal{C} := \mathrm{cl}_{X(\Delta)}\{f = 0\}$ has δ nodes as only singularities, where $d \ge 0$ is some integer. Also take $\zeta := \left|\Delta \cap \mathbb{Z}^2\right| - 1 - \delta$ points $\pi_1, \dots, \pi_\zeta \in (\mathbb{K}^\times)^2$ on $\mathcal{C}$ that map to Δ-generic points $p_1 := \mathrm{val}(\pi_1), \dots, p_\zeta := \mathrm{val}(\pi_\zeta)$. Denote the tropicalization of $\mathcal{C}$ by T and the regular subdivision of Δ dual to T by $S = \{\Delta_1, \dots, \Delta_N\}$.

Let σ be an edge of S with lattice length $m := |\sigma| \ge 2$. Since the tropical curve dual to S is simple, σ does not lie on the boundary of Δ and so, there exist two polygons $\Delta_k, \Delta_l \in S$ such that $\sigma = \Delta_k \cap \Delta_l$. For now, we assume that Δ_k, Δ_l are triangles. The case that one of the polygons is a parallelogram will be handled in the next section.

First, pick a lattice transformation $\mathbb{Z}^2 \to \mathbb{Z}^2$ that sends σ to the segment $\sigma' := [(0,0),(m,0)]$. Denote the image of Δ under this map by Δ'. The lattice map induces a monomial transformation that sends the polynomial f to f'. Also, after another monomial transformation of the form $x \mapsto xt^\alpha, y \mapsto yt^\beta$ for some $\alpha, \beta \in \mathbb{Z}$, we may assume that the Legendre transform $\nu_{f'}$ of the tropical polynomial associated to f' is zero along σ'.

Next, consider the tropical limit of f'. This produces polynomials f'_k, f'_l with Newton polygons Δ'_k, Δ'_l. Without loss of generality we assume that Δ'_k lies above σ' and Δ'_l beneath it. The corresponding curves C'_k, C'_l intersect at a point $p_{\sigma'} \in X(\sigma')$ that corresponds to the unique root ξ of $(f'_k)^{\sigma'} = (f'_l)^{\sigma'}$. We then apply a shift $x \mapsto x + \xi$ by introducing the polynomial

$$f''(x,y) := f'(x+\xi, y).$$

Let $\nu_{f''}$ be the Legendre transform of the tropical polynomial corresponding to f''. Then $\nu_{f''}$ is linear on the Newton polygons Δ''_k, Δ''_l of $f'_k(x+\xi,y), f'_l(x+\xi,y)$. Let $\lambda''_k, \lambda''_l \colon \mathbb{R}^2 \to \mathbb{R}$ be the linear functions such that $\lambda''_k\big|_{\Delta_k} = \nu_{f''}\big|_{\Delta_k}$, $\lambda''_l\big|_{\Delta_l} = \nu_{f''}\big|_{\Delta_l}$.

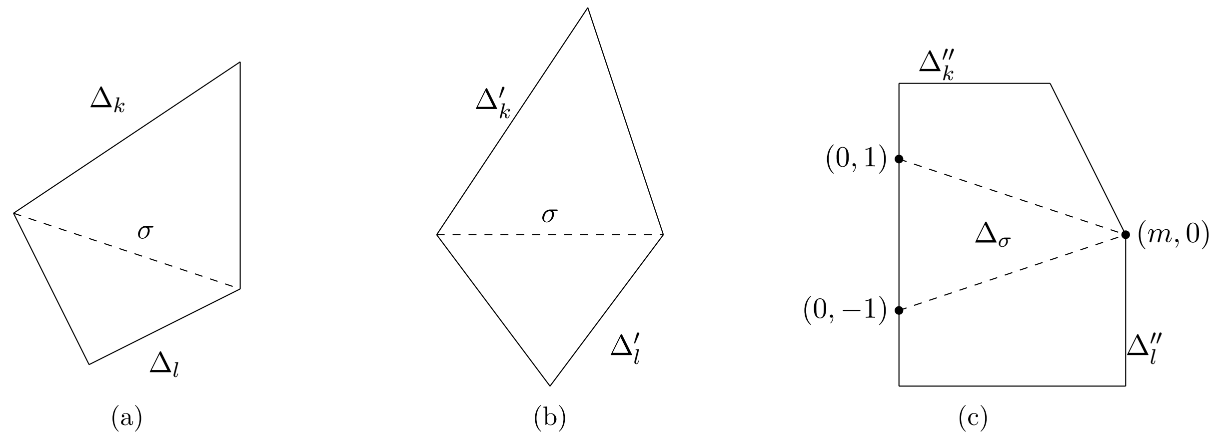


Figure 16: Refinement of the tropical limit at a singularity $p_\sigma \in X(\sigma)$.

We may write

$$\begin{aligned} f'_k(x+\xi, y) &= (f'_k)^{\sigma'}(x+\xi, y) + \sum_{(i,j)\in\Delta'_k\cap\mathbb{Z}^2, j>0} a^{(0)}_{ij}(x+\xi)^i y^j \\ &= Cx^m + \sum_{(i,j)\in\Delta'_k\cap\mathbb{Z}^2,\ j>0} a^{(0)}_{ij}(x+\xi)^i y^j \end{aligned}$$

for some constant $C \in \mathbb{C}^\times$. Moreover, the coefficient of f'_k at y equals

$$\sum_{i\colon (i,1)\in\Delta'_k} a^{(0)}_{i1}\xi^i = \frac{d}{dx} f'_k(\xi, 0).$$

Since $m \geq 2$, we have $\frac{d}{dx} f'_k(\xi, 0) = 0$ and because C'_k is nonsingular at $p_{\sigma'}$, we deduce that the coefficient of $f'_k(x+\xi, y)$ at y is nonzero. Therefore, one of the edges of the polygon Δ''_k is $[(0,1),(m,0)]$. Similarly, Δ''_l contains the edge $[(0,-1),(m,0)]$.

Consequently, the tropicalization of f'' induces a subdivision of the triangle

$$\Delta_\sigma := \operatorname{conv}((0,1),(0,-1),(m,0)).$$

Now, since $\nu_{f''}(i,j) > \max\{\lambda''_k(i,j), \lambda''_l(i,j)\}$ for all $(i,j) \in \operatorname{int}(\Delta_\sigma)$, the map $\nu_{f''}$ is convex and $\nu_{f''}$ vanishes along the edges $[(0,1),(m,0)], [(0,-1),(m,0)]$, we may conclude that $\nu_{f''}(i,j) > 0$ for all $(i,j) \in \operatorname{int}(\Delta_\sigma)$. Therefore, the coefficients a''_{ij} of f'' at $x^i y^j$ satisfy

$$\begin{aligned} a''_{ij}(0) &= 0, \quad \text{for } (i,j) \in \operatorname{int}(\Delta_\sigma), \\ a''_{m,0}(0), a''_{0,1}(0), a''_{0,-1}(0) &\neq 0. \end{aligned}$$

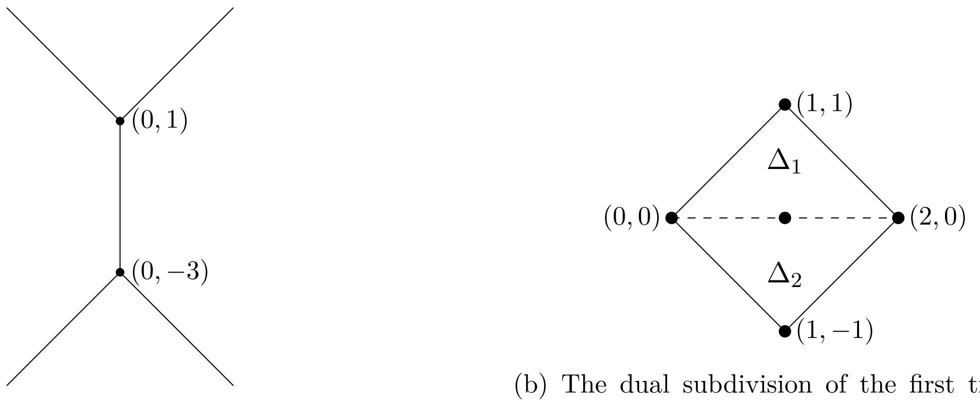
This means in particular that the subdivision of the tropicalization of f'' contains the triangle Δ_σ. Also, there is a unique $\tau \in \mathbb{K}$ with $\tau(0) = 0$ such that the polynomial

$$\widetilde{f}(x,y) := f(x+\tau(t), y)$$

does not contain the monomial x^{m-1}.

Definition 4.18. The tropicalization of $\widetilde{f}$ restricted to the triangle Δ_σ is called the *σ-refinement of the tropicalization of f* and we denote the corresponding polynomial by $\mathcal{T}_\sigma(f)$.

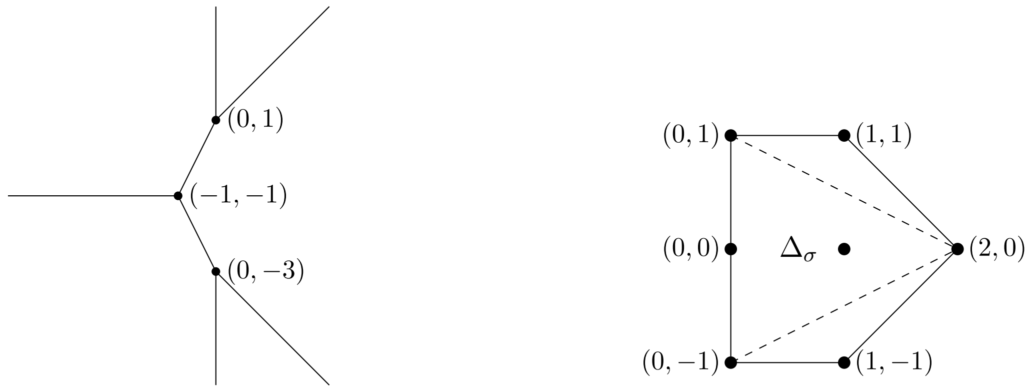
Any polynomial with Newton polygon Δ_σ whose truncations to the edges $[(0,1),(m,0)]$ and $[(0,-1),(m,0)]$ equal $\widetilde{f}^{[(0,1),(m,0)]}$ and $\widetilde{f}^{[(0,-1),(m,0)]}$ respectively, whose coefficient at x^{m-1} vanishes, and which defines a rational curve on $X(\Delta_\sigma)$, is called a *deformation pattern compatible with f_k, f_l and p_σ*.



(a) The first tropical curve from Example 4.19.

(b) The dual subdivision of the first tropical curve from Example 4.19.

Figure 17



(a) The second tropical curve from Example 4.19.

(b) The dual subdivision of the second tropical curve from Eample 4.19.

Figure 18

Example 4.19. Consider the Laurent polynomial

$$f(x, y) = 1 - 2(1 + t^2)x + x^2 + txy + t^3xy^{-1} \in \mathbb{K}[x, y].$$

Its Newton polygon, Δ, is the parallelogram spanned by the vertices $(0, 0), (1, 1), (1, -1), (2, 0)$. The corresponding curve $V(f)$ in $X_{\mathbb{K}}(\Delta)$ has a node at the point $(1, t)$ and is nonsingular at all other points. Its tropical limit consists of the tropical polynomial

$$\operatorname{trop}(f) = 0 \oplus x \oplus x^2 \oplus -1xy \oplus -3xy^{-1}$$

along with the polynomials

$$f_1(x, y) = (x - 1)^2 + xy,$$
$$f_2(x, y) = (x - 1)^2 + xy^{-1},$$

which define smooth, rational curves on the toric surfaces that are associated with the triangles $\operatorname{conv}((0, 0), (0, 1), (2, 0))$ and $\operatorname{conv}((0, 0), (0, -1), (2, 0))$. See also figure 17 for the corner locus of $\operatorname{trop}(f)$ and its dual subdivision.

Note that the segment $\sigma := \Delta_1 \cap \Delta_2 = [(0, 0), (2, 0)]$ already has the required form and ν_f is zero along it. That is, we have $f = f'$. We perform the translation $x \mapsto x + 1 + t^2$ to obtain the polynomial

$$\widetilde{f}(x, y) = f''(x + t^2, y) = f(x + t^2 + 1, y) =$$
$$(-2t^2 - 3t^4) + x^2 + (t + t^3)y + (t^3 + t^5)y^{-1} + txy + t^3xy^{-1}.$$

The corresponding tropical polynomial is given by

$$\mathrm{trop}(\widetilde{f}) = -2 \oplus x^2 \oplus -1y \oplus -3y^{-1} \oplus -1xy \oplus -3xy^{-1}.$$

See also Figure 18 for the corner locus of $\mathrm{trop}(\widetilde{f})$ along with its dual subdivision. The tropical limit of $\widetilde{f}$ restricted to the triangle $\Delta_\sigma = \mathrm{conv}((0,1),(0,-1),(2,0))$ is given by

$$\mathcal{T}_\sigma(f)(x,y) = -2 \oplus x^2 \oplus y \oplus y^{-1}.$$

This defines a rational curve on $X(\mathrm{conv}((0,1),(0,-1),(2,0)))$ with a single singularity at the point $(0,1)$, which is a node. There is one more deformation pattern compatible with f_1, f_2, which is given by the Laurent polynomial

$$g(x,y) = 2 + x^2 + y + y^{-1}.$$

△

Lemma 4.20. *[25, Lemma 3.9] There are precisely $m = |\sigma|$ deformation patterns compatible with f_k, f_l and p_σ.*

Proof. Consider the set of polynomials

$$F(x,y) = ay + bg(x) + cy^{-1},$$
$$a,b,c \in \mathbb{C}^\times, \qquad g(x) = x^m + \dots, \qquad \deg g = m$$

that form compatible deformation patterns. The condition for the truncation of F on $[(0,1),(m,0)]$, $[(0,-1),(m,0)]$ to equal those of f_k, f_l means that a,b,c are fixed. The arithmetic genus of F equals the number of interior lattice points of Δ_σ, which is $m-1$. For $V(F)$ to be rational, we thus require that the total delta invariant of $V(F)$ is $m-1$.

Since the lattice lengths of $[(0,1),(m,0)]$ and $[(0,-1),(m,0)]$ equal 1, it follows that the multiplicities of $V(F)$ at its points of intersection with $X([(0,1),(m,0)])$ and $X([(0,-1),(m,0)])$ are also 1. Hence, the singular locus of $V(F)$ is contained inside the affine open

$$U_{[(0,1),(0,-1)]} = \mathrm{Spec}\,\mathbb{C}[x,y^{\pm 1}] \cong \mathbb{C} \times \mathbb{C}^\times.$$

Consequently, the singular locus of $V(F)$ is given by solutions $(x,y) \in \mathbb{C} \times \mathbb{C}^\times$ to the equations

$$\begin{cases} yF(x,y) = ay^2 + bg(x)y + c &= 0, \\ \frac{d}{dx}yF(x,y) = bg'(x)y &= 0, \\ \frac{d}{dy}yF(x,y) = 2ay + bg(x) &= 0. \end{cases}$$

This system is equivalent to the system of equations

$$\begin{cases} g^2(x) &= \frac{4ac}{b^2}, \\ g'(x) &= 0, \\ y + \frac{b}{2a}g(x) &= 0. \end{cases} \tag{4.3}$$

Let $(x_0,y_0) \in \mathbb{C} \times \mathbb{C}^\times$ be a singular point of $V(F)$. Its Milnor number is given by the dimension of

$$\mathbb{C}[[x,y]]/(\nabla(yF(x+x_0,x+y_0))) = \mathbb{C}[[x,y]]/(bg'(x+x_0)(y+y_0), 2a(y+y_0)+bg(x+x_0))$$
$$\cong \mathbb{C}[[x]]/(g'(x+x_0)).$$

Hence, the Milnor number of $V(F)$ at (x_0,y_0) equals the multiplicity of g at x_0. Consequently, the total Milnor number of $V(F)$ is at most $\deg g' = m-1$. As the Milnor number is also at least equal to the δ-invariant, we conclude that (4.1) has $m-1$ distinct solutions. This means that, up to a shift $x \mapsto x+a$, the polynomial g equals one of the Chebyshev polynomials

$$\cos(m \cdot \arccos(2^{-(m-1)/m}\,\varepsilon\, x)), \qquad \text{for } \varepsilon^m = 1.$$

Since the coefficient of g at x^{m-1} must also vanish, we conclude that there are precisely m deformation patterns compatible with f_k, f_l and p_σ. □

4.4.2 Refinement at Non-Isolated Singularity

Recall from Remark 4.7 that the limit curve $C^{(0)}$ is non-reduced precisely if there is a parallelogram in the subdivision S with one edge of lattice length strictly greater than 1. In this case, the points lying on this non-reduced locus are non-isolated singularities. In the last section, we defined the refinement of the tropical limit at an isolated singularity $p_\sigma = X(\sigma) \cap C^{(0)}$, where σ is the common edge of two triangles Δ_k, Δ_l in the subdivision of Δ. In this section, we generalize this construction to points p_σ for which at least one of Δ_k, Δ_l is a parallelogram.

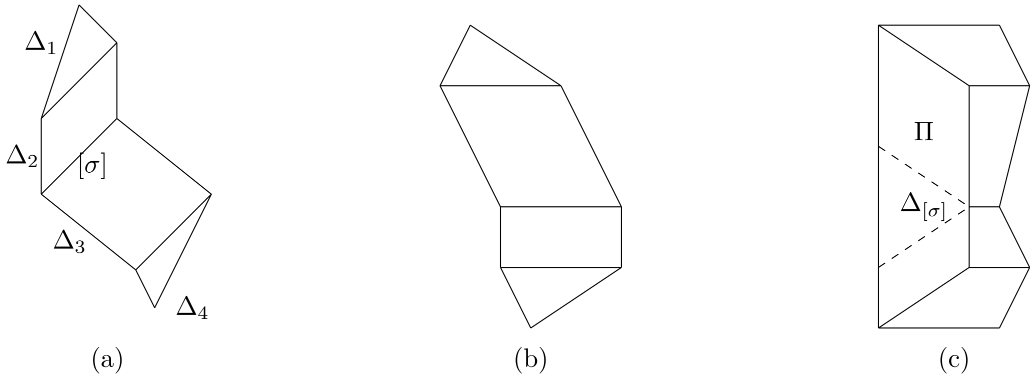


Figure 19

In this case, consider the maximal chain of distinct polygons $\Delta_1, \ldots, \Delta_r$ of S such that each consecutive pair Δ_i, Δ_{i+1} is connected by a common edge parallel to σ. Since the tropical curve T is simple and we assume that the lattice length $|\sigma| \geq 2$, we know that Δ_1, Δ_r must be triangles. See for instance Figure 19a.

Similar to the case of an isolated singularity, we multiply f by a constant from $\mathbb{K}$ such that ν_f is zero along σ and apply a lattice automorphism such that the edges in the class $[\sigma]$ become parallel. Denote the new polynomial obtained by applying these transformations to f by f'.

The points of intersection of $C^{(0)}$ with $X(\tau), \tau \in [\sigma]$ correspond to the same root $\xi \in \mathbb{C}^\times$ of ${f'}^\tau(x, 0)$. We then apply a shift $x \mapsto x + \xi$ to obtain the polynomial f''. The subdivision induced by $\nu_{f''}$ has an additional trapeze Π, which is divided into parallelograms and one triangle $\Delta_{[\sigma]}$. This is again an isosceles triangle of height m and base 2. After an appropriate transformation $x \mapsto x + \tau(t)$, the new polynomial $\widetilde{f}$ has the property that its x^{m-1}-th coefficient is zero, similar to the case of the previous section. We define the tropical limit of $\widetilde{f}$ restricted to $\widetilde{f}$ as the refinement at $[\sigma]$. Similarly, we define a deformation pattern compatible with the equivalence class $[\sigma]$. The polynomial defining the curve in $X(\Delta_{[\sigma]})$ we denote by $\mathcal{T}_{[\sigma]}$. The following is a direct corollary of Lemma 4.20.

Corollary 4.21. *For each edge σ in S of lattice length $m \geq 2$, there are precisely m deformation patterns compatible with the equivalence class of σ.*

In addition, we define the refined tropical limit of a curve $\mathcal{C} = V(f) \in |L_{\mathbb{K}}(\Delta)|$ as its tropical limit along with the refinements $T_{[\sigma]}(f)$ for each equivalence class of an edge σ of S. In view of Proposition 4.17 and Corollary 4.21, the following statement becomes clear.

Corollary 4.22. *The number of refined tropical limits associated with a curve $\mathcal{C}$ is equal to*

$$\frac{\mu(T)}{\prod_{k=1}^{\zeta} |\sigma_k|}.$$

4.4.3 Refined Conditions to Pass through Fixed Points

Consider a point p_m such that the corresponding edge σ_m has lattice length $d \geq 2$. We use the same transformations as sections 4.4.2 and 4.4.1 to derive additional conditions on the coefficients of f using an approximation of the equation $f(\xi_m, \eta_m) = 0$.

First, apply a monomial transformation that sends σ_m to the edge $[(0,0),(d,0)]$ and such that $\nu_{f'}\big|_{\sigma_m} = 0$. Next, apply the shift $x \mapsto x + \xi_m^{(0)}$ and define

$$f''(x,y) := f'(x + \xi^{(0)}, 0).$$

We may then write

$$f''(x,y) = \sum_{k=0}^{d-1} c'_{k,0} x^k + a_{d,0}^{(0)}(x - \xi_m^{(0)})^d + \mathcal{O}(t),$$

where $a_{k,0}^{(0)} \in \mathbb{C}$ and $c''_{k,0} \in \mathbb{K}$ with $c''_{k,0}(0) = 0$. Finally, we apply a shift $x \mapsto x - \tau$, where $\tau \in \mathbb{K}$ is the unique value for which $\widetilde{f}(x,y) := f''(x - \tau, y)$ has vanishing coefficient at x^{d-1}. One can compute that the lowest order terms of τ are given by

$$\tau = \frac{c''_{d-1,0}(t)}{d a_{d,0}^{(0)}} + \text{h.o.t.}. \tag{4.4}$$

Recall from Section 4.4.1 that the triangle $\Delta_\sigma = \operatorname{conv}\{(0,1),(0,-1),(d,0)\}$ lies in the subdivision induced by $\nu_{\widetilde{f}}$. Therefore, the lowest order terms of $\widetilde{f}$ are given by

$$\widetilde{f}(x,y) = \sum_{k=0}^{d-2} \widetilde{c}_{k,0} x^k + (a_{d,0}^{(0)} + \widetilde{c}_{d,0}(t))x^d + (a_{0,1}^{(0)} + c_{0,1}(t))t^r y + (a_{0,-1}^{(0)} + c_{0,-1}(t))t^s + \text{h.o.t.}.$$

Here, $r \neq s$, since otherwise Δ_σ would not be a triangle in the subdivision induced by $\nu_{\widetilde{f}}$. Assume without loss of generality that $r < s$.

By plugging in the new coordinates

$$(t\xi_m^1 - \tau + \text{h.o.t.}, \eta_m^{(0)} + \eta_m^{(1)} t + \text{h.o.t.})$$

of π_m into the equation $\widetilde{f}(x,y) = 0$, we then obtain

$$\eta_m^{(0)} a_{0,1}^{(0)} t^r + a_{d,0}^{(0)}(\xi_m^{(1)} t - \tau)^d + \text{h.o.t.} = 0.$$

Substituting (4.4) into this equation gives a degree d equation in the coefficients of f. The condition for the coefficients of f to satisfy this equation is called the *refined condition to pass through the point* p_m. In light of Corollary 4.22, the following result immediately follows.

Corollary 4.23. *Given a tropical curve T, there are precisely $\mu(T)$ choices of refined tropical limits with refined conditions to pass through the given points $p_1, \ldots, p_\zeta$.*

4.5 Patchworking

Patchworking is a technique developed by Oleg Viro in 1979-1981, which was originally used to construct smooth, real plane curves with complicated prescribed topology. This procedure patches together simpler curves via a one-parameter family, hence the name patchworking. For a thorough introduction to this subject, see also [28].

Eugenii Shustin generalized Viro's results to complex curves which may also contain singularities. The following theorem, due to Shustin, will be an important step in proving the Mikhalkin correspondence.

Theorem 4.24 (Refined patchworking theorem)**.** *[25, Lem. 3.12] Let $\Delta \subset \mathbb{R}^2$ be a non-degenerate lattice polygon and $\pi_1, \ldots, \pi_\zeta \in (\mathbb{K}^\times)^2$ a configuration of points that map to a Δ-generic configuration $p_1 := \operatorname{val}(\pi_1), \ldots, p_\zeta := \operatorname{val}(\pi_\zeta)$. Let us be given*

- *a simple tropical curve T of rank ζ with Newton polygon Δ that passes the points $p_1, \ldots, p_\zeta$ and is dual to a subdivision $S = \Delta_1 \cup \ldots \cup \Delta_N$;*

- *curves $C_m \in |L(\Delta_m)|$ that satisfy the conditions of Proposition 4.6, and have $C_m \cap X(\sigma) = C_l \cap X(\sigma)$ for $\sigma = \Delta_m \cap \Delta_l$;*
- *deformation patterns $\mathcal{T}_{[\sigma]}$ compatible with $C_1, \dots, C_N$ and associated with each equivalence class of edges from S;*
- *refined conditions to pass through the points $p_1, \dots p_\zeta$.*

Then, there exists a unique curve $\mathcal{C} \in |L_{\mathbb{K}}(\Delta)|$ with $\delta := \left|\Delta \cap \mathbb{Z}^2\right| - 1 - \zeta$ nodes as only singularities, passing through $\pi_1, \dots, \pi_\zeta$ such that its refined tropical limit and refined conditions to pass through the points $p_1, \dots, p_\zeta$ fit the given data.

Proof Sketch. As a full proof will require much more background on deformation theory, we will sketch the proof. For a complete proof, see [25, Lem. 3.12] or see [26, Thm. 2.51] for a slightly less complete proof.

For $t \in \mathbb{C}^\times$, consider the family of polynomials

$$f_{(t)} = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} A_{ij} t^{\nu(i,j)} x^i y^j \in \mathbb{C}[\{A_{ij}\}, x, y].$$

We wish to find complex analytic maps $A_{ij}(t)$ satisfying $A_{ij}(0) = a_{ij}$ such that the curve in $X_{\mathbb{K}}(\Delta)$ defined by $f_{(t)}$ matches the tropical data. Without going too much into technical detail, we will only show how to preserve the singularites of $f_1, \dots, f_N$ in the deformation $f_{(t)}$, by means of the implicit function theorem. The remaining conditions require a similar argument, yet also depend on many more technicalities to be checked.

For each integer $1 \le k \le N$, let

$$\lambda_k \colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto \alpha_{0,k} + \alpha_{1,k}\, x + \alpha_{2,k}\, y$$

be the unique linear map such that $\lambda_k \big|_{\Delta_k} = \nu\big|_{\Delta_k}$ and define $\nu_k := \nu - \lambda_k \big|_{\Delta}$. The transformations

$$T_k^{(t)} \colon (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2,\ (x, y) \mapsto (x \cdot t^{\alpha_{1,k}}, y \cdot t^{\alpha_{2,k}})$$

are biholomorphic maps for each $t \in \mathbb{C}^\times$ and $k = 1, \dots, N$. Using these maps, we define

$$\begin{aligned} f_{(t),k}(x, y) &:= f_{(t)}\left(\left(T_k^{(t)}\right)^{-1}(x, y)\right) t^{-\alpha_{0,k}} \\ &= \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} A_{ij} t^{\nu_k(i,j)} x^i y^j \\ &= f_k(x, y) + \mathcal{O}(t), \end{aligned}$$

for all $t \in \mathbb{C}^\times$ and $k = 1, \dots, N$.

Next, since the singularities of $C_1 \cap (\mathbb{C}^\times)^2, \dots, C_N \cap (\mathbb{C}^\times)^2$ are finite in number, there exists a compact set $Q \subset (\mathbb{C}^\times)^2$ such that these singularities are contained inside $\operatorname{int}(Q)$. We claim that there exists a positive number $t_0 > 0$ such that for all $t \in \mathbb{C}, 0 < |t| < t_0$, the compact sets

$$\left(T_1^{(t)}\right)^{-1}(Q), \dots, \left(T_N^{(t)}\right)^{-1}(Q)$$

are pairwise disjoint.

To prove the claim, let $1 \le k < l \le N$ be two integers. Since $Q \subset (\mathbb{C}^\times)^2$ there exist positive $\varepsilon_2 > \varepsilon_1 > 0$ such that

$$Q \subset \{(x, y) \in \mathbb{C}^2 \mid \varepsilon_1 \le |x|, |y| \le \varepsilon_2\}.$$

For $(x, y) \in (T_k^{(t)})^{-1}(Q) \cap (T_l^{(t)})^{-1}(Q)$ it then follows that

$$\begin{cases} \varepsilon_1 \le \left|x t^{\alpha_{1,k}}\right|, \left|y t^{\alpha_{2,k}}\right| \le \varepsilon_2, \\ \varepsilon_1 \le \left|x t^{\alpha_{1,l}}\right|, \left|y t^{\alpha_{2,l}}\right| \le \varepsilon_2\,. \end{cases}$$

Without loss of generality, we may assume that $\alpha_{1,k} < \alpha_{1,l}$. Consequently,

$$|t|^{\alpha_{1,l} - \alpha_{1,k}} = \left|\frac{xt^{\alpha_{1,l}}}{xt^{\alpha_{1,k}}}\right| \geq \frac{\varepsilon_1}{\varepsilon_2}.$$

However, since $\alpha_{1,l} - \alpha_{1,k}$ is positive, there exists a $t_{k,l} > 0$ such that $|t|^{\alpha_{1,l} - \alpha_{1,k}} < \frac{\varepsilon_1}{\varepsilon_2}$ for all $0 < |t| < t_{k,l}$, implying that the intersection $(T_k^{(t)})^{-1}(Q) \cap (T_l^{(t)})^{-1}(Q)$ is empty for such t. To conclude the claim, we choose t_0 as the minimum of all $t_{k,l}, 1 \leq k < l \leq N$.

Since the compacts are disjoint, it follows that the singularities of $f_{(t)}$ in

$$\widetilde{Q} := \bigcup_{k=1}^{N} \left(T_k^{(t)}\right)^{-1} (Q)$$

are in a 1-to-1 correspondence with the singularities of $f_{(t),1}, \ldots f_{(t),k}$ in Q.

Denote the singularities of f_k by $w_{k,1}, \ldots, w_{k,n_k}$. For each $j = 1, \ldots, n_k$, the complex germ $(\{f_{(t),k} = 0\}, (w_j, a))$ forms a deformation of the germ $(\{f_k = 0\}, w_j)$. That is, there is a Cartesian diagram

$$\begin{array}{ccc} (\{f_k = 0\}, w_j) & \longrightarrow & (\{f_{(t),k} = 0\}, (a, w_j)) \\ \downarrow & & \downarrow \\ a & \longrightarrow & (\mathbb{C}^{|\Delta \cap \mathbb{Z}^2|}, a) \end{array}$$

where the map $(\{f_{(t),k} = 0\}, (a, w_j)) \to (\mathbb{C}^{|\Delta \cap \mathbb{Z}^2|}, a)$ is flat and given by the restriction of the projection $\mathbb{C}^{|\Delta \cap \mathbb{Z}^2|} \times \mathbb{C}^2 \to \mathbb{C}^{|\Delta \cap \mathbb{Z}^2|}$. In $(\mathbb{C}^{|\Delta \cap \mathbb{Z}^2|}, a)$ there exists a complex subgerm $(M(f_k, w_j), a)$, called the equisingular stratum, such that all fibres above this germ contain a singularity of the same analytic type in a neighborhood of w_j. Moreover, this is smooth and of codimension 1 for nodal singularities. Hence, the germ $(M(f_k, w_j), a)$ is given as the zero locus of a holomorphic map $\varphi_{k,j}$, defined in a neighborhood of a.

The germs $M(f_k, w_j), 1 \leq k \leq N, 1 \leq j \leq n_k$ all meet transversally in a. This implies the existence of a subset $\Lambda \subset \Delta \cap \mathbb{Z}^2$ with $|\Lambda| = \sum_{k=1}^{N} n_k$ such that the total derivative

$$\left(\frac{\partial\, \varphi_l^{(k)}(A_{ij} t^{\nu_k(i,j)}) \mid (i,j) \in \Delta \cap \mathbb{Z}^2)}{\partial A_{ij}}\right)_{\substack{1 \leq k \leq N,\ 1 \leq l \leq n_k \\ (i,j) \in \Lambda}}$$

evaluated in $A = a, t = 0$ is invertible. Therefore, by the implicit function theorem, there exist holomorphic maps $A_{ij}(t)$, defined in a neighborhood of 0, such that $A_{ij}(0) = a_{ij}$ and

$$\varphi_j^{(k)}(A_{ij}(t) t^{\nu_k(i,j)} \mid (i,j) \in \Delta \cap \mathbb{Z}^2) = 0,$$
$$j = 1, \ldots, n_k,\ k = 1, \ldots, N.$$

Similarly, the deformation patterns and refined conditions to meet a point each correspond to a smooth, complex subgerm of the base $(\mathbb{C}^{|\Delta \cap \mathbb{Z}^2|}, a)$. It can be shown that the intersection of all these germs is one dimensional and transversal. Hence, the implicit function theorem implies that there is a unique parametrization $A(t)$ of this intersection in a neighborhood of a, which in turn determines a unique curve $\{f_{(t)} = 0\}$ satisfying all the requirements. □

4.6 Counting Generic Fibres

The Mikhalkin Correspondence is proven by establishing a bijection between curves over the field of Puiseux series and tropical plane curves. In order to relate this to the count of complex curves, we prove the following auxiliary result, which roughly states that the Severi degrees are equivalent over $\mathbb{K}$ and $\mathbb{C}$.

Theorem 4.25. *Let $\Delta \subset \mathbb{R}^2$ be a non-degenerate lattice polygon, $\delta \geq 0$ an integer, $\zeta := \left|\Delta \cap \mathbb{Z}^2\right| - 1 - \delta$ the dimension of the Severi variety and $\pi_1, \ldots, \pi_\zeta \in (\mathbb{K}^\times)^2$ a configuration of points in general position. Then, there exist generic points $p_1, \ldots, p_\zeta \in (\mathbb{C}^\times)^2$ and a bijective map*

$$\left\{\begin{matrix}\delta\text{-nodal curves } \mathcal{C} \subset X_{\mathbb{K}}(\Delta) \text{ which} \\ \text{contain } \pi_1, \ldots, \pi_\zeta\end{matrix}\right\} \xrightarrow{\text{1-to-1}} \left\{\begin{matrix}\delta\text{-nodal curves } C \subset X_{\mathbb{C}}(\Delta) \text{ which} \\ \text{contain } p_1, \ldots, p_\zeta\end{matrix}\right\}. \tag{4.5}$$

In other words, the Severi degrees over the field of Puiseux series and the field of complex numbers coincide.

Proof. After an automorphism $\mathbb{K} \to \mathbb{K}, t \mapsto t^M$ for an appropriate $M \in \mathbb{Z}$, we may assume that the coordinates of $\pi_1, \ldots, \pi_\zeta$ are Laurent series, i.e., the exponents of t are integral. Note also that the domain of (4.5) is finite, as it equals the intersection of $\dim \mathrm{Sev}_{\mathbb{K},\delta}(\Delta)$ general hyperplane sections with the Severi variety. Hence, we may additionally choose M in such a way that the coefficients $a_{ij}, (i,j) \in \Delta \cap \mathbb{Z}^2$ of each Laurent polynomial $f \in \mathbb{K}[x^{\pm 1}, y^{\pm 1}]$ defining a curve $V(f) \subset X_{\mathbb{K}}(\Delta)$ in the domain of (4.5) are Laurent series as well.

Since $\mathbb{K}$ consists of locally convergent series, there exists a positive $t_0 > 0$ such that $\pi_i(t), 1 \leq i \leq r$ and $a_{ij}(t), (i,j) \in \Delta \cap \mathbb{Z}^2$ converge for $t \in D(0; t_0) \setminus \{0\}$, where $D(0; t_0) \subset \mathbb{C}$ is the open disk centered at 0 with radius t_0.

Consequently, for $t' \in D(0; t_0) \setminus \{0\}$ there are well-defined maps

$$\varphi_{t'} \colon \left\{\begin{matrix}\delta\text{-nodal curves } \mathcal{C} \subset X_{\mathbb{K}}(\Delta) \text{ which} \\ \text{contain } \pi_1, \ldots, \pi_\zeta\end{matrix}\right\} \to \left\{\begin{matrix}\delta\text{-nodal curves } C \subset X_{\mathbb{C}}(\Delta) \text{ which} \\ \text{contain } \pi_1(t'), \ldots, \pi_\zeta(t')\end{matrix}\right\}$$
$$V(f) \mapsto V(f_{t'}).$$

We now claim that there exists a positive $t_1 > 0$ such that $\pi_1(t'), \ldots, \pi_\zeta(t')$ are generic for all $t' \in D(0; t_1) \setminus \{0\}$. The condition for a set of points $p_1, \ldots, p_\zeta$ to be generic means that $(p_1, \ldots, p_\zeta) \in (\mathbb{C}^\times)^{2r} \setminus Z$ for some Zariski closed subset $Z \subset (\mathbb{C}^\times)^{2r}$. Consider the base change $Z_{\mathbb{K}}$ of Z over $\mathbb{K}$, which is a closed subset of $(\mathbb{K}^\times)^{2r}$. If $I \subset \mathbb{C}[x_1^{\pm 1}, y_1^{\pm 1}, \ldots, x_\zeta^{\pm 1}, y_\zeta^{\pm 1}]$ is an ideal whose zero locus $V(I)$ equals X, then $Z_{\mathbb{K}} = V(I \cdot \mathbb{K}[x_1^{\pm 1}, y_1^{\pm 1}, \ldots, x_\zeta^{\pm 1}, y_\zeta^{\pm 1}])$. Since $\pi_1, \ldots, \pi_\zeta$ are generic, we may assume that $(\pi_1, \ldots, \pi_\zeta) \notin Z_{\mathbb{K}}$. Now, for the sake of contradiction, assume that there exists no $t_1 > 0$ such that $(\pi_1(t'), \ldots, \pi_\zeta(t')) \notin Z$ for all $t' \in D(0; t_1)$. In that case, there exists a sequence $(t_n)_{n \in \mathbb{N}} \in (D(0; t_0) \setminus \{0\})^{\mathbb{N}}$ converging to 0 such that

$$f(\pi_1(t_n), \ldots, \pi_\zeta(t_n)) = 0 \quad \text{for all } f \in I \text{ and } n \in \mathbb{Z}_{\geq 1}\,.$$

By the identity theorem from complex analysis, this implies the functions $t \mapsto f(\pi_1(t), \ldots, \pi_\zeta(t))$ are identically zero, and hence, $(\pi_1, \ldots \pi_\zeta) \in Z_{\mathbb{K}}$, a contradiction.

Next, we show that there exists a positive $t_2 > 0$ such that $\varphi_{t'}$ is injective for all $t' \in D(0; t_2) \setminus \{0\}$. Let $\mathcal{C} = V(f), \mathcal{C}' = V(f') \subset X_{\mathbb{K}}(\Delta)$ be two δ-nodal curves passing $\pi_1, \ldots, \pi_\zeta$. The set

$$\{t' \in D(0; t_1) \mid \mathcal{C} = \mathcal{C}'_{t'}\} = \{t' \in D(0; t_1) \mid a_{ij}(t') = a'_{ij}(t')\}$$

is discrete, which follows again from the identity theorem. Since there is a finite number of curves passing $\pi_1, \ldots, \pi_\zeta$, having only δ nodes as singularities, this means that we can always find a sufficiently small $t_2 > 0$ satisfying the requirement.

Finally, for surjectivity, fix $t' \in D(0; t_2) \setminus \{0\}$ and let $C \subset X_{\mathbb{C}}(\Delta)$ be a δ-nodal curve which contains $\pi_1(t'), \ldots, \pi_\zeta(t')$. Then, C is given as the closure of the zero locus of some Laurent polynomial f with Newton polytope Δ. Let $\nu \colon \Delta \to \mathbb{R}$ be the zero map. Then, the patchworking data $\nu, \{\Delta\}, f, \pi_1, \ldots, \pi_\zeta$ and patchworking theorem of Shustin [25, Thm. 5] give a δ-nodal curve $V(f_{(t)}) \subset X_{\mathbb{K}}(\Delta)$ which contains $\pi_1, \ldots, \pi_\zeta$ and whose tropicalization equals the patchworking data. We need Shustin's theorem here instead of Theorem 4.24, as the latter is too restrictive to apply for this case. However, the proof of [25, Thm. 5] is similar to that of Theorem 4.24. The curve $\mathcal{C} := V(f_{t-t'})$ now lies in the preimage $\varphi_{t'}^{-1}(C)$. □

By putting everything together, we may now finish the proof of the Mikahlkin Correspondence.

Proof of Theorem 4.1. Let $\pi_1, \dots, \pi_\zeta \in (\mathbb{K}^\times)^2$ be a generic configuration of points that map to Δ-generic points $p_1 := \mathrm{val}(\pi_1), \dots, p_\zeta := \mathrm{val}(\pi_\zeta) \in \mathbb{R}^2$. By 4.25, the Severi degree $N^{\Delta,\delta}$ equals the number of δ-nodal curves $\mathcal{C} := \mathrm{cl}_{X(\Delta)}\{f = 0\}$ through $\pi_1, \dots, \pi_\zeta$, that are given as zero loci of Laurent polynomials $f \in \mathbb{K}[x^{\pm 1}, y^{\pm 1}]$ with Newton polytope Δ. Each of these curves can be associated with a refined tropical limit and refined conditions to pass through the points. By Theorem 4.24, there is a one-to-one correspondence between curves $\mathcal{C}$ and this tropical data. Furthermore, by Corollary 4.23 there are precisely $\mu(T)$ choices of refined tropical limits with refined conditions to pass through the given points $p_1, \dots, p_\zeta$, for each tropical curve T. This proves that

$$N^{\Delta,\delta} = \sum_{T \in \mathcal{T}_{\Delta,\zeta}(\mathcal{U})} \mu(T).$$

Moreover, by Lemma 4.10, the curve $\mathcal{C}$ is irreducible if and only if its tropicalization is. We may thus also conclude that

$$N'^{\Delta,\delta} = \sum_{T \in \mathcal{T}^{\mathrm{irr}}_{\Delta,\zeta}(\mathcal{U})} .$$

□

5 Lattice Path Algorithm

In Mikhalkin's original paper [21], besides proving the deep connection between algebraic and tropical enumerative geometry, he also gave a combinatorial algorithm for counting tropical curves. This is based on enumerating lattice paths in the polytope Δ. The fact that these two counts concur follows from choosing a specific configuration of generic points. This will first be explained in Section 5.1. The resulting algorithm is then discussed in 5.2 and we finish this chapter off with computations of node polynomials in 5.3.

5.1 Lattice Paths

Definition 5.1. Let $\lambda \in \mathbb{R}^2 \setminus \{0\}$ be a nontrivial vector and $\Delta \subset \mathbb{R}^2$ a nondegenerate lattice polygon such that the minimum and maximum of the function

$$\mathbb{R}^2 \to \mathbb{R}, \qquad \mathbf{x} \mapsto \lambda \cdot \mathbf{x}$$

are uniquely attained in Δ. A function $\gamma\colon [1, n] \to \Delta$ is called a *λ-compatible path of length $n-1$* if it satisfies the following three conditions.

- The map γ is linear on the intervals $[i, i+1]$ for all $1 \leq i < n$.
- The values $\gamma(1)$ respectively $\gamma(n)$ are the minimum respectively maximum of the linear map $\mathbf{x} \mapsto \lambda \cdot \mathbf{x}$.
- The function $[1, n] \to \mathbb{R},\ t \mapsto \lambda \cdot \gamma(t)$ is strictly increasing.

Note that a λ-compatible path is uniquely determined by its vertices. Consequently, when we refer to γ as the path given by the points

$$\mathbf{x}_1, \cdots, \mathbf{x}_n \in \Delta \cap \mathbb{Z}^2$$

we mean that $\gamma\colon [1, n] \to \Delta$ is the unique λ-compatible path such that

$$\gamma(i) = \mathbf{x}_i, \quad \text{for all } i \in [1, n] \cap \mathbb{Z}\,.$$

Remark 5.2. Note that the points $\gamma(1), \gamma(n)$ are necessarily vertices of Δ, as they are the unique optimal solutions of a linear function. Moreover, a λ-compatible path splits the polygon Δ into two, with a part $\Delta_+(\gamma)$ above the graph of γ and a part $\Delta_-(\gamma)$ below it. Two special λ-compatible paths are the paths γ_+, γ_- that traverse the vertices on the boundaries $\Delta_+(\gamma) \cap \partial\Delta$ and $\Delta_-(\gamma) \cap \partial\Delta$. That is, γ_+ traverses the points on $\Delta_+(\gamma) \cap \partial\Delta \cap \mathbb{Z}^2$ in between $\gamma(1)$ and $\gamma(n)$ clockwise, while γ_- traverses the points $\Delta_-(\gamma) \cap \partial\Delta \cap \mathbb{Z}^2$ counter-clockwise. Note that the paths γ_+, γ_- do not depend on the choice of γ, but do on the choice of λ.

Example 5.3. Let $d \geq 1$ be an integer and consider the triangle Δ_d with vertices $(0,0), (d,0), (0,d)$. Also let $\lambda = (1, -\varepsilon)$ for $\varepsilon > 0$ small. Then, λ induces a lexicographical order on the lattice points of Δ_d given by

$$\lambda \cdot (i_1, j_1) \leq \lambda \cdot (i_2, j_2) \iff i_1 < i_2 \text{ or } (i_1 = i_2 \text{ and } j_2 \leq j_1).$$

The λ-compatible lattice paths of length $\left|\Delta \cap \mathbb{Z}^2\right| - 2 = \frac{(d+1)(d+2)}{2} - 2$ are uniquely determined by the one lattice point of Δ they do not contain. Therefore, there is a total of $\left|\Delta_d \cap \mathbb{Z}^2\right| - 2 = \frac{(d+1)(d+2)}{2} - 2$ of such paths. See also Figure 20 for the case that $d = 3$. There, γ_+ is the path traversing the lattice points

$$(0,3), (1,2), (2,1), (3,0),$$

while γ_- is the path traversing

$$(0,3), (0,2), (0,1), (0,0), (1,0), (2,0), (3,0).$$

$\triangle$

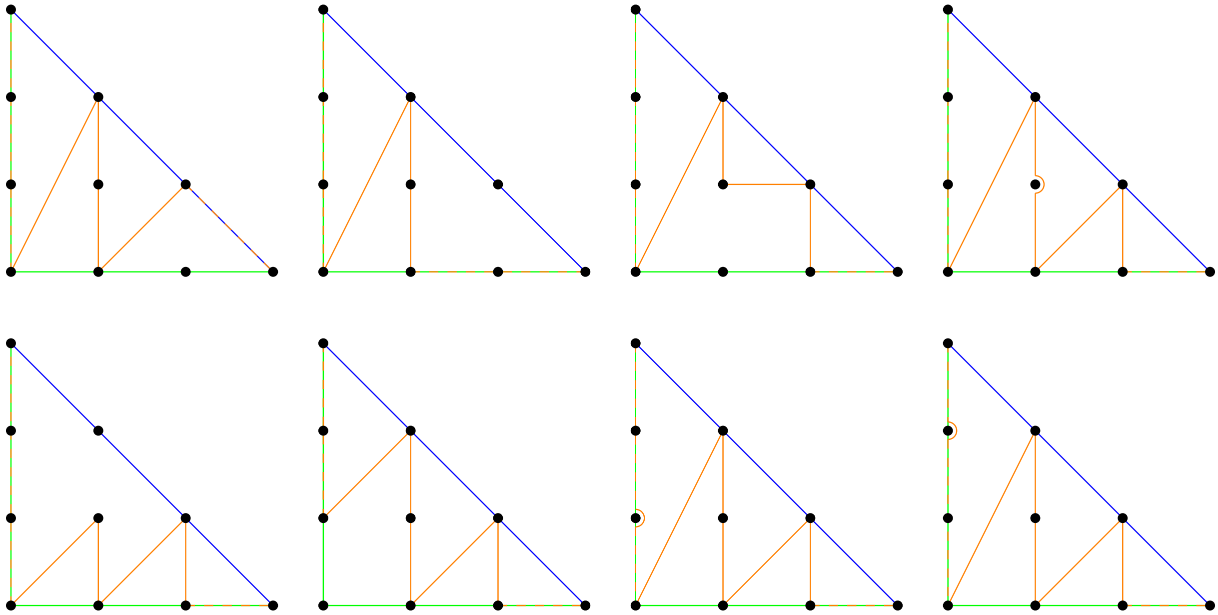


Figure 20: All λ-compatible lattice paths of length 8 for the triangle $\text{conv}((0,0),(3,0),(0,3))$. The paths of length 8 are colored orange. The paths γ_+ resp. γ_- are colored blue resp. green.

Definition 5.4. Let $\gamma\colon [1,n] \to \Delta$ be a λ-compatible path and $s \in \{+,-\}$. Let $2 \le i \le n-1$ be the smallest index, if it exists, for which the angle of $\Delta_s(\gamma)$ at the vertex $\gamma(i)$ is smaller than π radians. A *compression with respect to* $\Delta_s(\gamma)$ of γ is a path γ' defined in either of the following two ways.

- The path $\gamma'\colon [1,n-1] \to \Delta$ is the unique continuous map with

$$\gamma'(j) = \begin{cases} \gamma(j) & \text{if } j \le i, \\ \gamma(j+1) & \text{if } j \ge i, \end{cases}$$

  for integral $1 \le j \le n-1$ such that γ' is linear on the intervals $[j, j+1], 1 \le j \le n-2$.

- If $\gamma(i+1)+\gamma(i-1)-\gamma(i) \in \Delta$, then γ' may also be given as follows. The path $\gamma'\colon [1,n] \to \Delta$ is the unique continuous map with

$$\gamma'(j) = \begin{cases} \gamma(j) & \text{if } j \ne i, \\ \gamma(i+1)+\gamma(i-1)-\gamma(i) & \text{if } j = i, \end{cases}$$

  for integral $1 \le j \le n$ such that γ' is linear on the intervals $[j, j+1], 1 \le j \le n-1$.

Note that a compression of a λ-compatible path is again λ-compatible. This justifies the following definition.

Definition 5.5. Let $\gamma\colon [1,n] \to \Delta$ be a λ-compatible path and $s \in \{+,-\}$. A *compressing procedure of* $\Delta_s(\gamma)$ is a sequence of compressions $\gamma = \gamma_0, \gamma_1, \cdots, \gamma_m = \gamma_s$. We denote the set of pairs $(S_+(\gamma), S_-(\gamma))$ where $S_s(\gamma)$ is a compressing procedure of $\Delta_s(\gamma)$ by $\mathcal{N}_\lambda(\gamma)$.

Example 5.6. Figure 21 illustrates a compressing procedure of $\Delta_+(\gamma)$ for $\Delta = \text{conv}\{(0,0),(3,0),(0,3)\}$ and γ the $(1,-1)$-compatible path traversing the points $(0,3),(0,2),(0,1),(0,0),(1,1),(1,0),(2,0),(3,0)$. Note that each compression either removes a vertex from the path, or replaces it with a new vertex, which is the opposite point in the parallelogram spanned by its adjacent edges. △

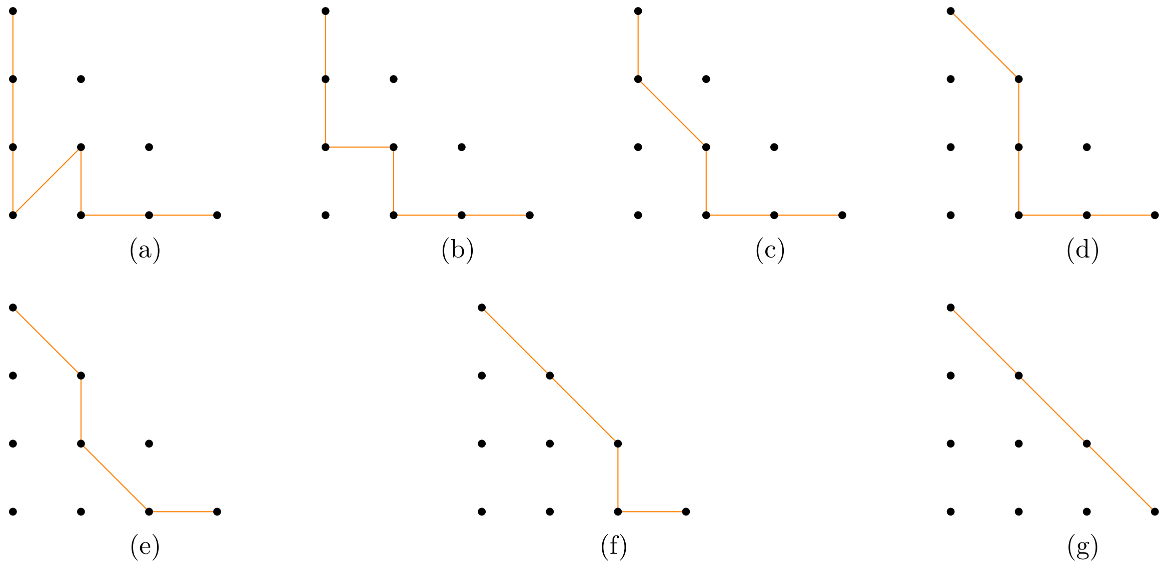


Figure 21: An example of a compressing procedure.

Note that a compressing procedure for $\Delta_s(\gamma)$ induces a subdivision of $\Delta_s(\gamma)$ into triangles and parallelograms, where all of the lattice points $\Delta_s(\gamma) \cap \partial\Delta$ are vertices of the subdivision. Indeed, each compression adds either a triangle with vertices $\gamma(i-1), \gamma(i), \gamma(i+1)$ to the subdivision, or a paralellogram with vertices $\gamma(i-1), \gamma(i), \gamma(i+1), \gamma(i-1)+\gamma(i+1)-\gamma(i)$. Together, compressing procedures for $\Delta_+(\gamma)$ and $\Delta_-(\gamma)$ induce a convex subdivision of Δ. This subdivision is dual to a simple tropical curve with rank equal to the length of the path γ. To understand this latter point, note that each compression either adds a triangle, and keeps the number of vertices constant, or adds a parallelogram, and then the numbers of vertices and parallelograms are both increased by 1. Hence, if $S = (S_+(\gamma), S_-(\gamma))$ are compressing procedures for γ, then the path length of γ, i.e., the number of edges γ has, is equal to

$$\#\text{vertices of S} + \#\text{parallelograms of S} - 1.$$

By Proposition 3.32, this is also the rank of a tropical curve dual to S.

Denote by $\mathcal{P}(\lambda)$ (resp. $\mathcal{P}_\zeta(\lambda)$) the set of λ-compatible paths (resp. the set of λ-compatible paths of length ζ). Each pair $S = (S_+(\lambda), S_-(\lambda))$ has a (complex) multiplicity $\mu(S)$ equal to the product of the normalized triangle areas of S. This is also the multiplicity of any tropical curve dual to S.

Example 5.7. We continue in the setting of Example 5.3. For $d = 3$, six out of eight lattice paths have compressing procedures, which are illustrated in Figure 22. The compressing procedures of $\Delta_{d,+}(\gamma)$ are shaded blue while the compressing procedures of $\Delta_{d,-}(\gamma)$ are shaded green. Three of the lattice paths have two possible compressing procedures and one subdivision has a multiplicity of 4. All other lattice paths have a single compressing procedure of multiplicity 1. We conclude that

$$\sum_{\gamma \in \mathcal{P}_8(\lambda)} \sum_{S \in \mathcal{N}_\lambda(\gamma)} \mu(S) = 1 + 1 + 2 + 2 + 2 + 4 = 12.$$

△

Theorem 5.8. *[21, Thm. 2] Let $\Delta \subset \mathbb{R}^2$ be a nondegenerate lattice polygon, $\delta \geq 0$ an integer and $\lambda \in \mathbb{Q}^2$ a vector with irrational slope. Choose $\zeta := \left|\Delta \cap \mathbb{Z}^2\right| - 1 - \delta$. Then,*

$$N^{\Delta,\delta} = \sum_{\gamma \in \mathcal{P}_\zeta(\lambda)} \sum_{S \in \mathcal{N}_\lambda(\gamma)} \mu(S).$$

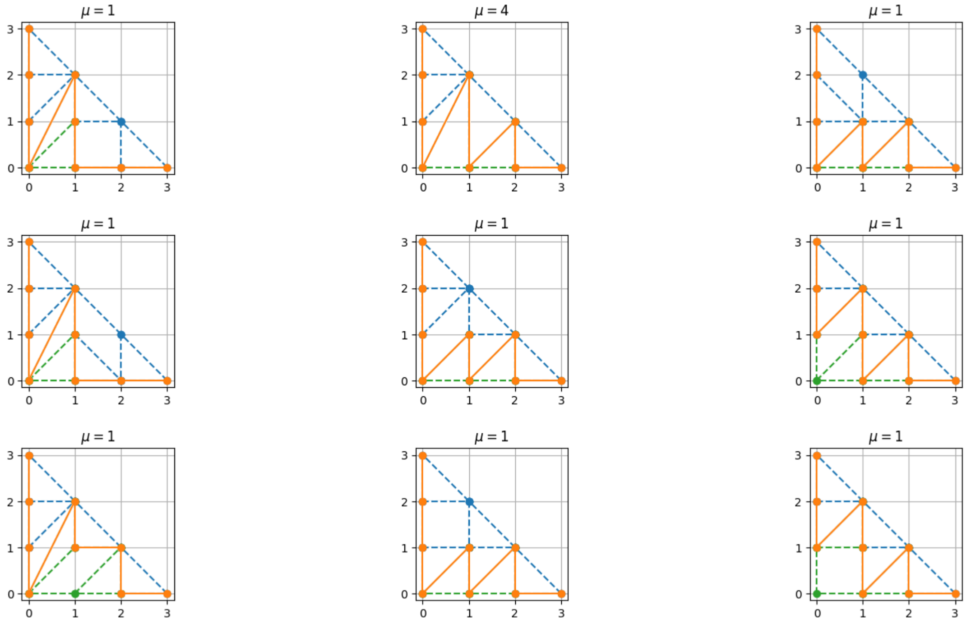


Figure 22: All compressing procedures for lattice paths in Δ_3 of length 8.

Proof. Pick points $\mathbf{p}_1, \cdots, \mathbf{p}_\zeta \in \mathbb{Q}^2$ that lie on a line l with direction λ and satisfy

$$\lambda \cdot \mathbf{p}_1 < \lambda \cdot \mathbf{p}_2 < \cdots < \lambda \cdot \mathbf{p}_\zeta, \quad |\mathbf{p}_{i+1} - \mathbf{p}_i| \ll |\mathbf{p}_{i+2} - \mathbf{p}_{i+1}|, \text{for all } 1 \leq i \leq \zeta - 2. \tag{5.1}$$

We count the number of nodal tropical curves of rank ζ that contain the points $\mathbf{p}_1, \cdots, \mathbf{p}_\zeta$. Let T be such a curve. Because the slope of l is irrational, the points $\mathbf{p}_1, \ldots, \mathbf{p}_\zeta$ are Δ-generic. Lemma 3.37 now implies that each point must lie in the interior of an edge of T. Moreover, $\mathbf{p}_1, \cdots, \mathbf{p}_\zeta$ are the only points of intersection of T with l. Otherwise, there would exist $\zeta + 1$ generic points lying on the rank ζ curve T.

By duality, each point $\mathbf{p}_i$, which lies on a unique line segment of T, corresponds to a unique line segment σ_i in the dual subdivision S of T. Furthermore, as the line l does not intersect the curve T in any other points, the intervals $l \setminus \{\mathbf{p}_1, \ldots, \mathbf{p}_\zeta\}$ are contained inside the complement of T. Hence, they correspond to vertices $\mathbf{u}_1, \ldots, \mathbf{u}_{\zeta+1}$ of S that are connected by the edges $\sigma_1, \ldots, \sigma_\zeta$ and are naturally ordered by λ. Therefore, these vertices form a λ-compatible lattice path γ.

We conclude that the path γ uniquely corresponds to an initial choice of line segments through the points $\mathbf{p}_1, \ldots, \mathbf{p}_{\zeta+1}$. Next, we show how to reconstruct the curve T from this initial configuration. Condition (5.1) ensures that the first pair of points $\mathbf{p}_i, \mathbf{p}_{i+1}$, for which the two line segments of T passing these points are non-parallel, will intersect in a vertex of T. We thus continue the germs through $\mathbf{p}_i, \mathbf{p}_{i+1}$ until they meet, and either draw a third germ, whose direction is uniquely determined by the balancing condition, or continue both germs to form a 4-valent vertex. The first case implies that the polygon in S with edges σ_i, σ_{i+1} is a triangle, the second implies a parallelogram. Therefore, continuing the germs through $\mathbf{p}_i, \mathbf{p}_{i+1}$ is equivalent with compressing the path γ. By continuing this process we reconstruct both the curve T and its subdivision S. Furthermore, this yields that each nodal curve T of rank ζ through the points $\mathbf{p}_1, \ldots, \mathbf{p}_\zeta$ is in a 1-to-1 correspondence with λ-compatible lattice paths γ and their compressing procedures. Since the multiplicity of a curve and its dual subdivision are by definition equal, this shows that the tropical curve and lattice path counts are equal. The claim now follows from the Mikhalkin Correspondence. □

Example 5.9. For fixed $\delta = 1$, the Severi degrees of the plane are given by the polynomial

$$N^{d,1} = 3d^2 - 6d + 3.$$

To see this, we compute the number of λ-compatible lattice paths and their compressions with multiplicities in Δ_d, where λ, Δ_d are as in Example 5.3. Recall that each lattice point $v \in \Delta \cap \mathbb{Z}^2 \setminus \{(0,d),(d,0)\}$ corresponds to a unique λ-compatible lattice path γ_v of length $\left|\Delta_d \cap \mathbb{Z}^2\right| - 2$, given by traversing all lattice points of Δ_d in order, except for v. We categorize all of these paths and compute their weighted sum.

- First, consider the lattice points $(0,i)$ for $i \in \mathbb{Z}, 1 \leq i \leq d-1$. There exists no sequence of compressions $\gamma_{(0,i)} = \gamma_0, \gamma_1, \cdots, \gamma_m$ of $\Delta_-(\gamma_{(0,i)})$ such that $\gamma_m(k) = (0,i)$ for some $k \in \mathbb{Z}_{\geq 0}$.
- Second, consider the points $(i,0)$ for $i \in \mathbb{Z}, 0 \leq i \leq d-1$. There is a unique compressing procedure for $\Delta_+(\gamma_{(i,0)})$ and there are $d-i-1$ compressing procedures for $\Delta_-(\gamma_{(i,0)})$, each with multiplicity 1.
- Third, consider the points $(i, d-i)$ for $i \in \mathbb{Z}, 1 \leq i \leq d-1$. There is a unique compressing procedure for $\Delta_-(\gamma_{(i,d-i)})$ and there are $d+1-i$ compressing procedures for $\Delta_+(\gamma_{(i,d-i)})$, each with multiplicity 1.
- Fourth, consider the points (i,j) for $i,j \in \mathbb{Z}_{\geq 0}, 1 \leq i+j \leq d-1$. There is a unique compressing procedure for both $\Delta_+(\gamma_{(i,j)})$ and $\Delta_-(\gamma_{(i,j)})$ and each procedure has a multiplicity of 2.

We may thus conclude that

$$\begin{aligned} N^{d,1} &= 0 \cdot (d-2) + \sum_{i=0}^{d-1}(d-i-1) + \sum_{i=1}^{d-1}(d+1-i) + 4\left|\operatorname{int}(\Delta) \cap \mathbb{Z}^2\right| \\ &= 3(d-1)^2. \end{aligned}$$

△

Example 5.10. An argument similar to the one presented in Example 5.9 shows that the Severi degrees of $\mathbb{P}(1,1,2)$ of cogenus $\delta = 1$ are given by

$$N^{(\mathbb{P}(1,1,2),dH),1} = 2(d-1)(3d-1),$$

for all $d \in \mathbb{Z}_{>0}$. △

5.2 Algorithm

Theorem 5.8 along with the Mikhalkin Correspondence presents an opportunity for computing Severi degrees $N^{\Delta,\delta}$ of toric surfaces via lattice paths. In this section we discuss an algorithm to perform this computation.

Given a full-dimensional lattice polytope $\Delta \subset \mathbb{R}^2$ and cogenus δ, the objective is to calculate

$$\sum_{\gamma \in \mathcal{P}_\zeta(\Delta)} \sum_{S \in \mathcal{N}_\lambda(\gamma)} \mu(S),$$

where $\zeta := \left|\Delta \cap \mathbb{Z}^2\right| - 1 - \delta$. Note that, if $\mathcal{N}_{\lambda,+}(\gamma)$ respectively $\mathcal{N}_{\lambda,-}(\gamma)$ denotes the set of all compressions of $\Delta_+(\gamma)$ respectively $\Delta_-(\gamma)$, then we may rewrite this sum as

$$\sum_{\gamma \in \mathcal{P}_\zeta(\Delta)} \sum_{S_+ \in \mathcal{N}_{\lambda,+}(\gamma)} \mu(S_+) \sum_{S_- \in \mathcal{N}_{\lambda,-}(\gamma)} \mu(S_-).$$

Our strategy involves calculating the right-most sums independently. This is achieved via Algorithm 1, which takes as input a lattice polygon Δ, a $(1, -\varepsilon)$-compatible path γ, a sign $s \in \{+,-\}$ and a multiplicity $\mu \geq 0$. This is a recursive algorithm that at each step updates the lattice path

γ by removing a vertex or forming a parallelogram. The method then calls itself, now with the new lattice path as input and an updated multiplicity. The new multiplicity is either equal to the old multiplicity or to the old multiplicity times the normalised area of the triangles formed by removing a vertex.

Once the path γ is equal to the boundary path γ_+, the method terminates and returns the multiplicity μ. If the path is not equal to the boundary path and no more compressions are possible, then the method returns 0. At the initial step, we set $\mu = 0$.

Algorithm 1 Compress

```
given lattice polygon Δ ⊂ ℝ², λ-compatible path γ, sign s ∈ {+, −}, and multiplicity μ ≥ 0
return number of compressing procedures, counted with multiplicity, of Δ_s(γ)

N ← 0
u ← left-most vertex of Δ_s(γ) with angle < π.

if u does not exist then
    if γ = γ_s then
        return μ
    else
        return 0
    end if
end if

γ_1 ← path obtained from γ by removing u
μ_1 ← μ times normalised area of triangle formed by removing u
N ← N + Compress(Δ, γ_1, s, μ_1)

γ_2 ← path obtained by forming parallelogram with vertex u
if γ_2 is contained inside Δ then
    N ← N + Compress(Δ, γ_2, s, μ)
end if

return N
```

Example 5.11. Say we run the compression algorithm for $\Delta = \text{conv}((0,0),(2,0),(0,2))$, path γ given by $(0,2),(0,1),(0,0),(1,0),(2,0)$, sign $s = +$ and $\mu = 1$. At the first step of the algorithm, the vertex $(0,0)$ is removed, and possibly the vertex $(1,1)$ is added. The algorithm is called again with the updated lattice paths as new input. This process repeats until the new path is either the boundary path $(0,2),(1,1),(2,0)$, or no more compressions are possible. The final output of the algorithm is 2. See also Figure 23 for a visualization of this process. At each step the multiplicity μ and count N of the compression is given. △

To obtain the final count, we loop over all $(1,\varepsilon)$-compatible lattice paths γ of length $|\Delta \cap \mathbb{Z}^2| - 1 - \delta$, call the recursive method *Compress* that was just described and add $\text{Compress}(\Delta,\gamma,+,1) \cdot \text{Compress}(\Delta,\gamma,-,1)$ to our count. Note that the compression method can be performed independently for each lattice path, which allows for running the code in parallel, resulting in a significant speed boost. The implementation designed and used for this thesis can be found in Appendix C.3.

5.3 Computations of Node Polynomials

Via the lattice path algorithm, the Severi degrees of several toric varieties have been computed and can be found in Appendix D. Using these results, the Göttsche conjecture and polynomial

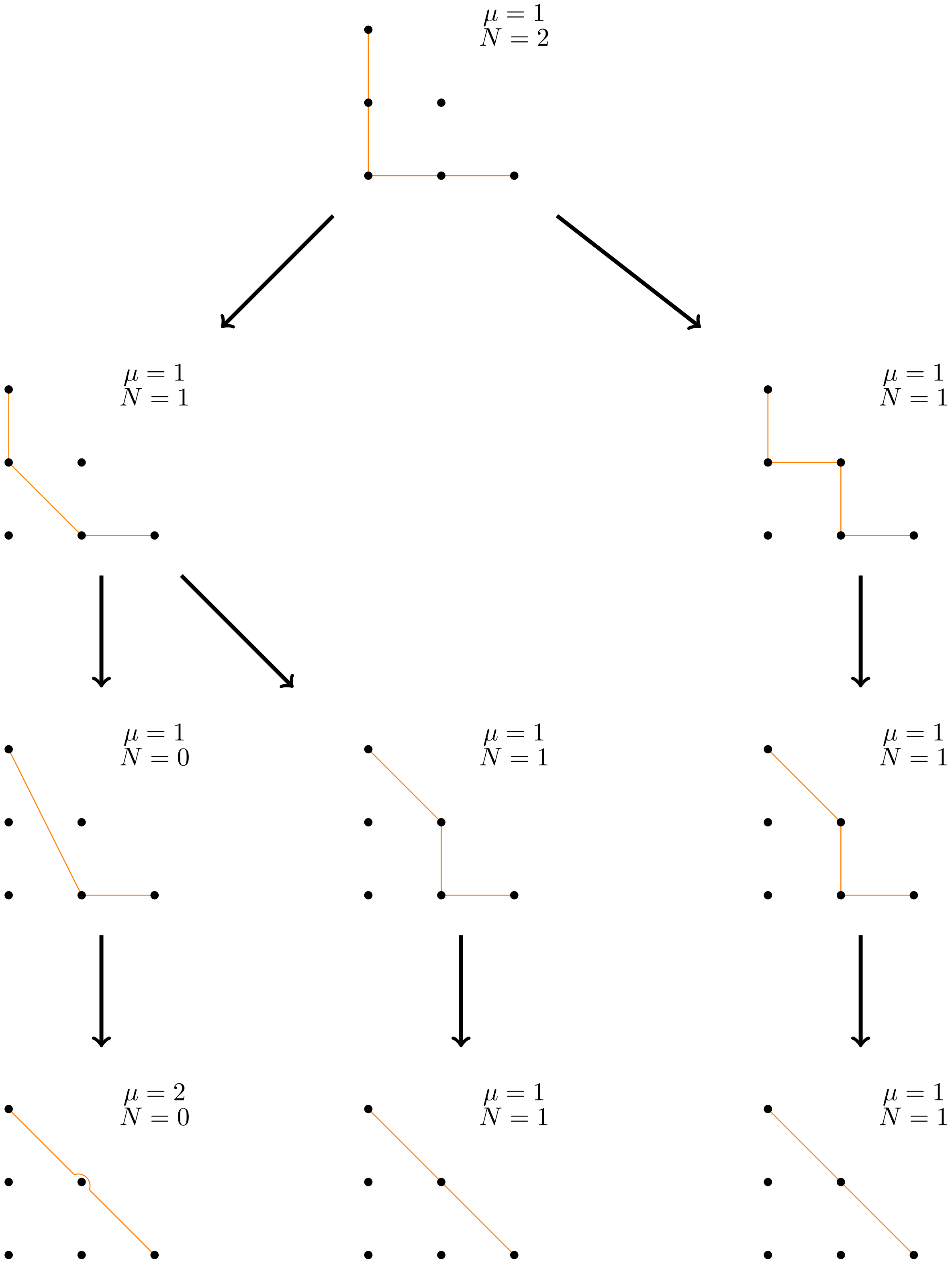


Figure 23: Example of the Compression Algorithm

Algorithm 2 Counting Lattice Paths

```
given lattice polygon Δ ⊂ ℝ², and integer δ ≥ 0
returns N^{Δ,δ}

N ← 0
for all λ-compatible lattice paths γ of length |Δ ∩ ℤ²| − 1 − δ do
    N ← N + Compress(Δ, γ, +, 1) · Compress(Δ, γ, −, 1)
end for

return N
```

interpolation, we find the node polynomials of these varieties. The subject varieties this section are ordered in increasing generality. First, the projective plane is handled, followed by other smooth varieties such as $\mathbb{P}^1 \times \mathbb{P}^1$ and the blown-up plane. At last, we consider singular varieties as well, for which the Göttsche conjecture does not hold, so we require a stronger result from Liu and Osserman [18].

5.3.1 Projective Plane

Fix $\delta \in \mathbb{Z}_{\geq 0}$. The Göttsche conjecture implies that the Severi degrees $N^{d,\delta}$ become polynomial in d of degree 2δ for d sufficiently large. Originally, Fomin-Mikhalin [22, Thm. 5.1] proved this using tropical methods for a bound of $d \geq 2\delta$ and Block [2, Thm. 1.3] improved this bound by showing that polynomiality holds for $d \geq \delta$. However, this bound is far from sharp, and Kleiman and Shende proved in [16, Cor. 3] that polynomiality holds for all $d \geq \lfloor \frac{\delta}{2} \rfloor + 1$. Block [2, Prop. 1.4] has confirmed this bound with tropical compputations for all $d \leq 14$.

Using the Severi degrees computed via the lattice path algorithm, one can determine the node polynomials $N^\delta(d)$. Naively, this can be done by solving for $\mathbf{a} = (a_0, a_1, \cdots, a_{2\delta}) \in \mathbb{Q}^{2\delta+1}$ in the system of linear equations

$$a_{2\delta}\mathbf{x}^{2\delta} + a_{2\delta-1}\mathbf{x}^{2\delta-1} + \cdots + a_1\mathbf{x} + a_0 = N^{d,\delta}, \quad \text{for } \delta \leq x \leq 2\delta\,.$$

Using this method, we have enough data to compute $N^\delta(d)$ for $1 \leq \delta \leq 3$. However, by exploiting the following method, we can determine $N^4(d)$ and $N^5(d)$ as well.

Let X be a smooth projective surface and L a sufficiently ample divisor on X. Recall that the Göttsche conjecture implies that there exist universal power series $A_1, A_2, A_3, A_4 \in \mathbb{Q}[[t]]$ such that

$$N^{(X,L)}(t) = A_1^{L^2} A_2^{K_X L} A_3^{K_X^2} A_4^{\chi(\mathcal{O}_X)}.$$

By taking the formal logarithm $Q^{(X,L)}(t) := \log(N^{(X,L),\delta}(t))$, this is equivalent to the existence of some constants $a_{\delta,1}, a_{\delta,2}, a_{\delta,3}, a_{\delta,4} \in \mathbb{Q}, \delta \in \mathbb{Z}_{\geq 0}$ such that the δ-th coefficient of $Q^{(X,L)}(t)$ is equal to

$$Q^{(X,L),\delta}(t) = a_{\delta,1}L^2 + a_{\delta,2}LK_X + a_{\delta,3}K_X^2 + a_{\delta,4}\chi(\mathcal{O}_X). \tag{5.2}$$

Specifically for $X = \mathbb{P}^2, L = dH$ this turns out to be

$$Q^{(\mathbb{P}^2,dH),\delta}(t) = a_{\delta,1}d^2 - 3a_{\delta,2}d + 9a_{\delta,3} + a_{\delta,4}.$$

We can solve for $a_{\delta,1}, a_{\delta,2}$ and $9a_{\delta,3} + a_{\delta,4}$ once we know $Q^{(\mathbb{P}^2,dH),\delta}$ for $\delta \leq d \leq \delta+2$. By taking the exponential $N^{(\mathbb{P}^2,dH)}(t) = \exp(Q^{(\mathbb{P}^2,dH)}(t))$ we retrieve the node polynomials.

By this method, we obtain

$$
\begin{aligned}
N^0(d) &= 1,\\
N^1(d) &= 3d^2 - 6d + 3,\\
N^2(d) &= \frac{9}{2}d^4 - 18d^3 + 6d^2 + \frac{81}{2}d - 33,\\
N^3(d) &= \frac{9}{2}d^6 - 27d^5 + \frac{9}{2}d^4 + \frac{423}{2}d^3 - 229d^2 - \frac{829}{2}d + 525,\\
N^4(d) &= \frac{27}{8}d^8 - 27d^7 + \frac{1809}{4}d^5 - 642d^4 - 2529d^3 + \frac{37881}{8}d^2 + \frac{18057}{4}d - 8865,\\
N^5(d) &= \frac{81}{40}d^{10} - \frac{81}{4}d^9 - \frac{27}{8}d^8 + \frac{2349}{4}d^7 - 1044d^6 - \frac{127071}{20}d^5\\
&\quad + \frac{128859}{8}d^4 + \frac{59097}{2}d^3 - \frac{3528381}{40}d^2 - \frac{946929}{20}d + 153513.
\end{aligned}
$$

This agrees with Block's computations [2].

Furthermore, by comparing the values of these polynomials with the actual Severi degrees we find that the lower bounds $\delta_0(\delta)$ for which $N^{d,\delta}$ becomes polynomial are given by the following table.

δ	0	1	2	3	4	5
$d_0(\delta)$	1	1	1	3	3	4

Figure 24: The lower bounds $d_0(\delta) \leq d$ for which the Severi degrees $N^{d,\delta}$ become polynomial.

5.3.2 Smooth Varieties

In the last section, we computed the node polynomials of $\mathbb{P}^2$ for cogenus $1 \leq \delta \leq 5$ by solving for $a_{\delta,1}, a_{\delta,2}, 9a_{\delta,3} + 4a_{\delta,4}$ in (5.2). In this section, we additionally solve for $a_{\delta,3}, a_{\delta,4}$ by using the Severi degrees of the surface $\mathbb{P}^1 \times \mathbb{P}^1$.

Let H respectively G be line $\{0\} \times \mathbb{P}^1$ respectively $\mathbb{P}^1 \times \{0\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The (self)-intersection numbers of the pair G, H are

$$G^2 = H^2 = 0 \quad \text{and} \quad G \cdot H = 1.$$

Moreover, the canonical divisor class of $\mathbb{P}^1 \times \mathbb{P}^1$ is represented by $-2G - 2H$. Let d, e be two positive integers. The divisor $dG + eH = L(\Delta)$ arises from the polygon

$$\Delta := \operatorname{conv}\{(0,0), (d,0), (0,e), (d,e)\},$$

and so we can easily compute the Severi degrees $N^{(\mathbb{P}^1 \times \mathbb{P}^1, dG+eH)\,\delta} = N^{\Delta,\delta}$ for small d, e, δ by the lattice path algorithm. See also tables ...

By the Göttsche conjecture, if d, e are big enough with respect to δ, (in this case, $d, e \geq \delta$ suffices) then

$$Q^{(\mathbb{P}^1 \times \mathbb{P}^1, dG+eH),\delta} = 2a_{\delta,1}de - 2a_{\delta,2}(d+e) + 8a_{\delta,3} + a_{\delta,4}.$$

From the Severi degrees of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ we then solve for $a_{\delta,k}, 1 \leq k \leq 4$. This gives

$$
\begin{aligned}
Q^{(X,L),0} &= 0,\\
Q^{(X,L),1} &= 3L^2 + 2LK_X + -1K_X^2 + 12\chi(\mathcal{O}_X),\\
Q^{(X,L),2} &= -21L^2 + -\frac{39}{2}LK_X + \frac{1}{2}K_X^2 + -42\chi(\mathcal{O}_X),\\
Q^{(X,L),3} &= 230L^2 + \frac{788}{3}LK_X + \frac{119}{3}K_X^2 + 276\chi(\mathcal{O}_X),\\
Q^{(X,L),4} &= -3015L^2 + -\frac{15945}{4}LK_X + -\frac{4159}{4}K_X^2 + -1944\chi(\mathcal{O}_X),\\
Q^{(X,L),5} &= \frac{217728}{5}L^2 + \frac{321882}{5}LK_X + \frac{109959}{5}K_X^2 + \frac{42192}{5}\chi(\mathcal{O}_X),
\end{aligned}
$$

for any smooth projective surface X and sufficiently ample divisor L.

In turn, by taking the exponential $N^{(X,L)}(t) = \exp(Q^{X,L}(t))$, we compute the universal node polynomials for $0 \leq \delta \leq 5$. These are given by

$$\begin{aligned}
n_0(x,y,z,w) &= 1,\\
n_1(x,y,z,w) &= 3x + 2y - z + 12w,\\
n_2(x,y,z,w) &= \frac{9}{2}x^2 + 6xy + 2y^2 - 3xz - 2yz + \frac{1}{2}z^2 + 36xw + 24yw - 12zw\\
&\quad + 72w^2 - 21x - \frac{39}{2}y + \frac{1}{2}z - 42w,\\
n_3(x,y,z,w) &= \frac{9}{2}x^3 + 9x^2y + 6xy^2 + \frac{4}{3}y^3 - \frac{9}{2}x^2z - 6xyz - 2y^2z + \frac{3}{2}xz^2\\
&\quad + yz^2 - \frac{1}{6}z^3 + 54x^2w + 72xyw + 24y^2w - 36xzw - 24yzw\\
&\quad + 6z^2w + 216xw^2 + 144yw^2 - 72zw^2 + 288w^3 - 63x^2 - \frac{201}{2}xy\\
&\quad - 39y^2 + \frac{45}{2}xz + \frac{41}{2}yz - \frac{1}{2}z^2 - 378xw - 318yw + 48zw - 504w^2\\
&\quad + 230x + \frac{788}{3}y + \frac{119}{3}z + 276w,\\
n_4(x,y,z,w) &= \frac{27}{8}x^4 + 9x^3y + 9x^2y^2 + 4xy^3 + \frac{2}{3}y^4 - \frac{9}{2}x^3z - 9x^2yz - 6xy^2z\\
&\quad - \frac{4}{3}y^3z + \frac{9}{4}x^2z^2 + 3xyz^2 + y^2z^2 - \frac{1}{2}xz^3 - \frac{1}{3}yz^3 + \frac{1}{24}z^4 + 54x^3w\\
&\quad + 108x^2yw + 72xy^2w + 16y^3w - 54x^2zw - 72xyzw - 24y^2zw\\
&\quad + 18xz^2w + 12yz^2w - 2z^3w + 324x^2w^2 + 432xyw^2 + 144y^2w^2\\
&\quad - 216xzw^2 - 144yzw^2 + 36z^2w^2 + 864xw^3 + 576yw^3 - 288zw^3\\
&\quad + 864w^4 - \frac{189}{2}x^3 - \frac{855}{4}x^2y - 159xy^2 - 39y^3 + \frac{261}{4}x^2z + \frac{207}{2}xyz\\
&\quad + 40y^2z - 12xz^2 - \frac{43}{4}yz^2 + \frac{1}{4}z^3 - 945x^2w - 1458xyw - 552y^2w\\
&\quad + 396xzw + 330yzw - 27z^2w - 3024xw^2 - 2412yw^2 + 540zw^2\\
&\quad - 3024w^3 + \frac{1821}{2}x^2 + \frac{3315}{2}xy + \frac{17171}{24}y^2 - \frac{243}{2}xz - \frac{2317}{12}yz\\
&\quad - \frac{949}{24}z^2 + 4470xw + 4523yw + 179zw + 4194w^2 - 3015x\\
&\quad - \frac{15945}{4}y - \frac{4159}{4}z - 1944w,\\
n_5(x,y,z,w) &= \frac{81}{40}x^5 + \frac{27}{4}x^4y + 9x^3y^2 + 6x^2y^3 + 2xy^4 + \frac{4}{15}y^5 - \frac{27}{8}x^4z\\
&\quad - 9x^3yz - 9x^2y^2z - 4xy^3z - \frac{2}{3}y^4z + \frac{9}{4}x^3z^2 + \frac{9}{2}x^2yz^2 + 3xy^2z^2\\
&\quad + \frac{2}{3}y^3z^2 - \frac{3}{4}x^2z^3 - xyz^3 - \frac{1}{3}y^2z^3 + \frac{1}{8}xz^4 + \frac{1}{12}yz^4 - \frac{1}{120}z^5\\
&\quad + \frac{81}{2}x^4w + 108x^3yw + 108x^2y^2w + 48xy^3w + 8y^4w - 54x^3zw\\
&\quad - 108x^2yzw - 72xy^2zw - 16y^3zw + 27x^2z^2w + 36xyz^2w + 12y^2z^2w\\
&\quad - 6xz^3w - 4yz^3w + \frac{1}{2}z^4w + 324x^3w^2 + 648x^2yw^2 + 432xy^2w^2 + 96y^3w^2\\
&\quad - 324x^2zw^2 - 432xyzw^2 - 144y^2zw^2 + 108xz^2w^2 + 72yz^2w^2 - 12z^3w^2\\
&\quad + 1296x^2w^3 + 1728xyw^3 + 576y^2w^3 - 864xzw^3 - 576yzw^3 + 144z^2w^3\\
&\quad + 2592xw^4 + 1728yw^4 - 864zw^4 + \frac{10368}{5}w^5 - \frac{189}{2}x^4 - \frac{1107}{4}x^3y - \frac{603}{2}x^2y^2
\end{aligned}$$

$$
\begin{aligned}
&- 145xy^3 - 26y^4 + \frac{387}{4}x^3z + \frac{873}{4}x^2yz + 162xy^2z + \frac{119}{3}y^3z - \frac{135}{4}x^2z^2 \\
&- \frac{213}{4}xyz^2 - \frac{41}{2}y^2z^2 + \frac{17}{4}xz^3 + \frac{15}{4}yz^3 - \frac{1}{12}z^4 - 1323x^3w - 2943x^2yw \\
&- 2160xy^2w - 524y^3w + 972x^2zw + 1494xyzw + 564y^2zw - 207xz^2w \\
&- 171yz^2w + 10z^3w - 6804x^2w^2 - 10260xyw^2 - 3816y^2w^2 \\
&+ 3132xzw^2 + 2484yzw^2 - 288z^2w^2 - 15120xw^3 - 11664yw^3 + 3168zw^3 \\
&- 12096w^4 + \frac{3393}{2}x^3 + \frac{8463}{2}x^2y + \frac{27403}{8}xy^2 + \frac{10867}{12}y^3 - \frac{1527}{2}x^2z \\
&- \frac{5879}{4}xyz - \frac{5245}{8}y^2z + \frac{55}{8}xz^2 + 62yz^2 + \frac{473}{24}z^3 + 14814x^2w + 25767xyw \\
&+ \frac{21551}{2}y^2w - 3231xzw - 3730yzw - \frac{631}{2}z^2w + 39726xw^2 + 37128yw^2 \\
&- 1590zw^2 + 30456w^3 - 13875x^2 - \frac{111959}{4}xy - \frac{26189}{2}y^2 - \frac{3289}{4}xz \\
&+ \frac{15175}{12}yz + \frac{12715}{12}z^2 - 57468xw - 68137yw - 12061zw - 34920w^2 \\
&+ \frac{217728}{5}x + \frac{321882}{5}y + \frac{109959}{5}z + \frac{42192}{5}w.
\end{aligned}
$$

To check the correctness of our methods, we compute the Severi degrees of the blow-up $\widetilde{\mathbb{P}^2}$ of $\mathbb{P}^2$ in a point. For this, let d, e be two positive integers and consider the polygon

$$\Delta := \operatorname{conv}\{(0,0),(0,d),(e,d),(d+e,0)\}.$$

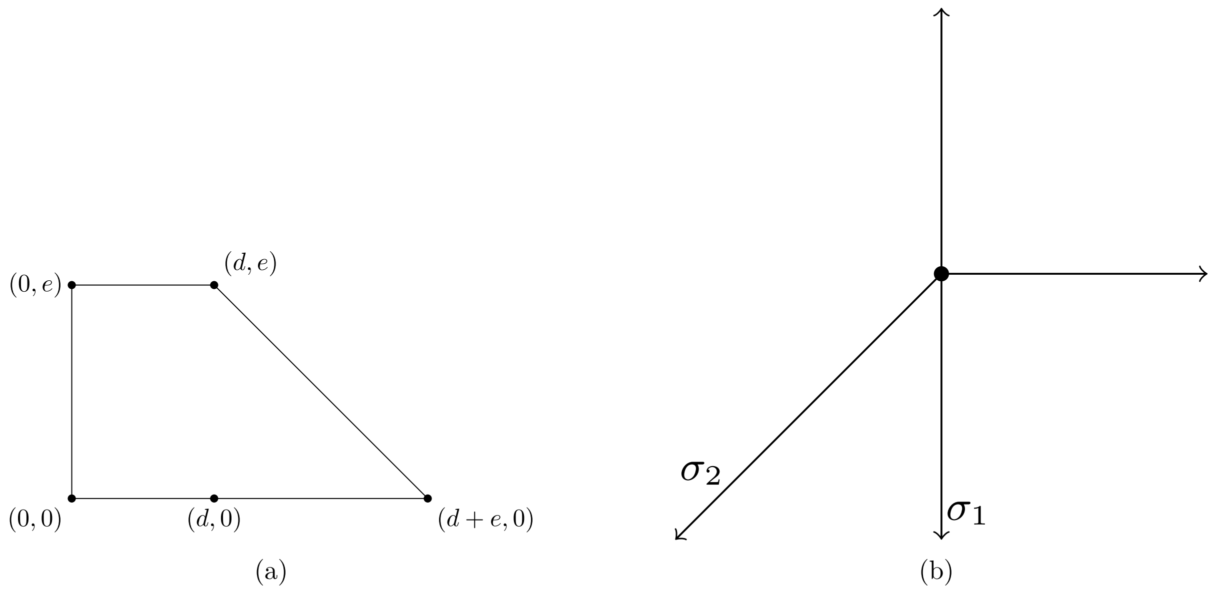


Figure 25: Polygon and fan of the plane blown up in a point

The polygon Δ, along with its dual subdivision is shown in Figure 25. The surface $X(\Delta)$ equals the blow-up of $\mathbb{P}^2$ in a point. Moreover, let E be the closure of the orbit $O(\sigma_1)$ and F the closure of the orbit $O(\sigma_2)$. It turns out that E is the exceptional divisor while F is the pull-back of a line in $\mathbb{P}^2$ passing the blown up point along the blow-up map $\pi\colon \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$. The divisor corresponding to Δ is $(d+e)F + eE$. Moroever, the canonical divisor class of $\widetilde{\mathbb{P}^2}$ is $-3F - 2E$. The (self)-intersections of the pair (E, F) are

$$E^2 = -1, \quad F^2 = 0, \quad E \cdot F = 1.$$

Furthermore, since blowing up respects the Euler characteristic, we have $\chi(\mathcal{O}_{\widetilde{\mathbb{P}^2}}) = 1$. We conclude that, for sufficiently large d, e, we have

$$Q^{(\widetilde{\mathbb{P}^2},(d+e)F+eE),\delta} = a_{\delta,1}(e^2 + 2de) - a_{\delta,2}(2d + 3e) + 8a_{\delta,3} + a_{\delta,4}.$$

By taking the exponential of the generating function $Q^{\widetilde{\mathbb{P}^2},(7)d+e)FeE}(t)$, we then compute the node polynomials of $\widetilde{\mathbb{P}^2}$ for $0 \leq \delta \leq 5$. Comparing these with the Severi degrees computed via the lattice path algorithm, we conclude that these agree for d, e sufficiently large. See also Appendix D for more information.

5.3.3 Singular Varieties

Traditionally, the Göttsche Conjecture strictly treats smooth surfaces. However, in the case of toric varieties, a more general result can be proven. Ardila and Block showed in [1] that the Severi degrees $N^{\Delta,\delta}$ are polynomial in the combinatorial data of Δ, as long as Δ is an h-transversal polytope. This means that the slopes of Δ are either $0, \infty$ or $\frac{1}{k}, k \in \mathbb{Z} \setminus \{0\}$, allowing for the use of floor diagrams to replace the tropical curve count. A limitation of this method is that it is not clear how the node polynomials depend on the topological data of the surface $X(\Delta)$ and its divisor $L(\Delta)$, and the singularity types of $X(\Delta)$. This restriction is partially remedied by a result of Liu and Osserman [18], who proved a Göttsche-like universality for a specific family of singular toric surfaces.

In this subsection, we will first explain Liu and Osserman's theorem. Then, we use this result for computations of $\mathbb{P}(1, 1, 2)$.

Definition 5.12. A polygon Δ is called *strongly h-transverse* if either there is a nonzero horizontal edge at the top of Δ, or the vertex v at the top has $\det(v) \in \{1, 2\}$, and the same holds for the bottom of Δ.

Although this condition may seem odd, Liu and Osserman show in [1, Prop. A.1] that a toric surface $X(\Delta)$ associated to an h-transversal Δ is Gorenstein if and only if Δ is strongly h-transverse.

In order to give enough context for Liu-Osserman's theorem, it will be necessary to give a brief introduction on *quotient singularities*. These are singularities that are analytically isomorphic to $\mathbb{A}^2/G$ for G a finite group acting on $\mathbb{C}^2$. Here, the quotient $\mathbb{A}^2/G$ is defined as $\operatorname{Spec} \mathbb{C}[x, y]^G$, where $\mathbb{C}[x, y]^G$ is the subalgebra of $\mathbb{C}[x, y]$ consisting of all polynomials that are invariant under the action of G. If G is cyclic and of degree d, we say that $\mathbb{A}^2/G$ is a *cyclic quotient singularity* of *index d*.

Example 5.13. Consider the matrix group

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

which acts on $\mathbb{C}^2$ by usual matrix multiplication. A polynomial $f(x, y) \in \mathbb{C}[x, y]$ is invariant under G if and only if $f(x, y) = f(-x, -y)$, which in turn means that it is generated by the monomials x^2, xy, y^2. Therefore, the invariant ring is given by $\mathbb{C}[x, y]^G = \mathbb{C}[x^2, xy, y^2]$ and hence,

$$\mathbb{A}^2/G = \operatorname{Spec} \mathbb{C}[x^2, y^2, xy] \cong \operatorname{Spec} \mathbb{C}[u, v, w]/(uv - w^2).$$

This is precisely the singularity of $\mathbb{P}(1, 1, 2) \cong X(\Delta)$, where Δ is the convex hull of $(0, 0), (2, 0), (0, 1)$. △

Theorem 5.14 (Liu-Osserman)**.** *[18, Thm. 1.8] Let $\delta > 0$. Then there exist constants*

$$a'_{\delta,1}, a'_{\delta,2}, a'_{\delta,3}, a'_{\delta,4}, b_\delta, b_{\delta,1}, \ldots, b_{\delta,\delta-1} \in \mathbb{Q}$$

such that if Δ is a strongly h-transverse polygon with all edges having at least length δ, then

$$Q^{\Delta,\delta} = a'_{\delta,1}L^2 + a'_{\delta,2}LK + a'_{\delta,3}K^2 + a'_{\delta,4}\widetilde{c}_2 + Sb_\delta + S_1 b_{\delta,1} + \dots S_{\delta-1}b_{\delta,\delta-1}.$$

Here, $L = L(\Delta)$ is the divisor associated to Δ, $K = K_{X(\Delta)}$ the canonical divisor of $X(\Delta)$, S_i the number of singularities of $X(\Delta)$ of index $i+1$, $\widetilde{c}_2 = c_2 + \sum_{i\geq 2} iS_i$, where c_2 is the second Chern class of $X(\Delta)$ and $S = \sum_{i\geq 1}(i+1)S_i$.

By choosing Δ such that $X(\Delta)$ is smooth, the constants $a'_{\delta,i}$ are related to the constants $a_{\delta,i}$ from (5.2) in the following way.

Lemma 5.15. *We have*

$$a'_{\delta,1} = a_{\delta,1} \qquad a'_{\delta,2} = a_{\delta,2}$$
$$a'_{\delta,3} = a_{\delta,3} - 12a_{\delta,4} \qquad a'_{\delta,4} = 12a_{\delta,4}.$$

Proof. If $X(\Delta)$ is smooth, the Göttsche Conjecture and Theorem 5.14 imply that

$$a'_{\delta,1}L^2 + a'_{\delta,2}LK + a'_{\delta,3}K^2 + a'_{\delta,4}c_2 = a_{\delta,1}L^2 + a_{\delta,2}LK + a_{\delta,3}K^2 + a_{\delta,4}\chi(\mathcal{O}_X).$$

Combining this with Noether's formula,

$$\chi(\mathcal{O}_X) = \frac{K^2 + c_2}{12},$$

gives the desired result. □

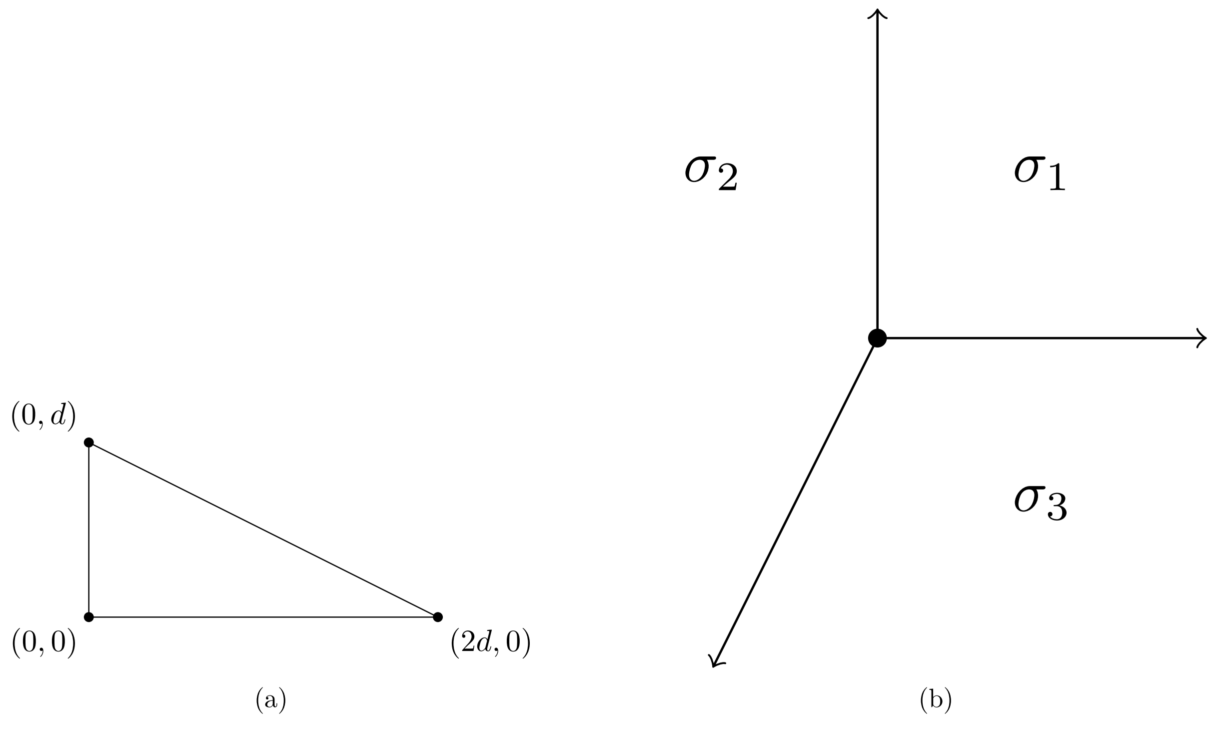


Figure 26: The polytope Δ and its dual fan associated to the weighted projective plane $X(\Delta) = \mathbb{P}(1, 1, 2)$.

Fix an integer $d > 0$. We now focus on the surface $X(\Delta)$ associated to the polygon

$$\Delta = \text{conv}\{(0, 0), (2d, 0), (0, d)\}.$$

This polygon along with its dual fan has been plotted in Figure 26. The 2-dimensional cones $\sigma_1, \sigma_2, \sigma_3$ of this fan give an open affine cover $\mathcal{U}_{\sigma_1}, \mathcal{U}_{\sigma_2}, \mathcal{U}_{\sigma_3}$ where

$$\begin{aligned}
\mathcal{U}_{\sigma_1} &= \operatorname{Spec}\mathbb{C}[x,y] \cong \mathbb{A}^2, \\
\mathcal{U}_{\sigma_2} &= \operatorname{Spec}\mathbb{C}[x^{-1}, x^{-2}y] \cong \mathbb{A}^2, \\
\mathcal{U}_{\sigma_2} &= \operatorname{Spec}\mathbb{C}[y^{-1}, xy^{-1}, x^2y^{-1}] \cong \operatorname{Spec}\mathbb{C}[u,v,w]/(uw - v^2).
\end{aligned}$$

The latter chart contains a cyclic quotient singularity of degree 2. See also Example 5.13. Moreover, if $H \subset X(\Delta)$ is a plane section, the divisor $L(\Delta)$ is linearly equivalent with dH and the canonical divisor class of $X(\Delta)$ is represented by $K_X = -3H$. The self-intersection of H equals 2 and therefore it follows from Theorem 5.14 that

$$Q^{\Delta,\delta} = 2a_{\delta,1}d^2 - 4a_{\delta,2}d + a'_{\delta,3}K^2 + a'_{\delta,4}\widetilde{c}_2 + 2b_\delta + b_{\delta,1}.$$

In the preceding section we have computed $a_{\delta,1}, \alpha_{\delta,2}$ for $0 \le \delta \le 5$. Using the Severi degrees of $\mathbb{P}(1,1,2)$, see also Figure 37, we can now solve for $a'_{\delta,3}K^2 + a'_{\delta,4}\widetilde{c}_2 + 2b_\delta + b_{\delta,1}$. By in turn taking the exponential of the generating series $Q(t)$, we obtain the following node polynomials:

$$\begin{aligned}
n^{\mathbb{P}(1,1,2),0}(d) &= 1, \\
n^{\mathbb{P}(1,1,2),1}(d) &= 6d^2 - 8d + 2, \\
n^{\mathbb{P}(1,1,2),2}(d) &= 18d^4 - 48d^3 + 2d^2 + 62d - 26, \\
n^{\mathbb{P}(1,1,2),3}(d) &= 36d^6 - 144d^5 - 24d^4 + \frac{1868}{3}d^3 - 340d^2 - \frac{2060}{3}d + 440, \\
n^{\mathbb{P}(1,1,2),4}(d) &= 54d^8 - 288d^7 - 108d^6 + 2620d^5 - \frac{5590}{3}d^4 - \frac{24236}{3}d^3 \\
&\quad + \frac{23968}{3}d^2 + \frac{24887}{3}d - 7706, \\
n^{\mathbb{P}(1,1,2),5}(d) &= \frac{324}{5}d^{10} - 432d^9 - 252d^8 + 6744d^7 - 5996d^6 - \frac{597856}{15}d^5 \\
&\quad + \frac{163156}{3}d^4 + \frac{313882}{3}d^3 - \frac{735148}{3}d^2 + \frac{465526}{3}d - \frac{150892}{5}.
\end{aligned}$$

Similarly to the case of $\mathbb{P}^2, \mathbb{P}(1,1,2)$ and $\widetilde{\mathbb{P}}^2$, the values of the node polynomials agree with the actual Severi degrees for all $d \ge \lfloor \frac{\delta}{2} \rfloor + 1, 0 \le \delta \le 5$. However, contrary to the former cases, this bound is not sharp. For instance, $n^{\mathbb{P}(1,1,2),5}(2) = N^{\mathbb{P}(1,1,2),5}(2)$ holds as well.

A Polyhedral Geometry

In this chapter, we introduce the necessary terminology and results from polyhedral geometry necessary for the study of toric and tropical geometry.

A *polyhedron* Δ is a finite intersection of half-spaces. To be more precise, a polyhedron is a subset of $\mathbb{R}^n$ of the form

$$\Delta = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$$

for some matrix $A \in \mathbb{R}^{m\times n}$ and vector $\mathbf{b} \in \mathbb{R}^m$. A *face* of a polyhedron is determined by a vector $\mathbf{w} \in \mathbb{R}^n$ via

$$\mathrm{face}_{\mathbf{w}}(\Delta) := \{\mathbf{x} \in \Delta \mid \mathbf{w}\cdot\mathbf{x} \leq \mathbf{w}\cdot\mathbf{y} \text{ for all } \mathbf{y} \in \Delta\}.$$

If Δ, Δ' are both polyhedra, then the notation $\Delta' \preceq \Delta$ is shorthand for "Δ' is a face of Δ". A *facet* is a face that is not contained in any larger proper face. A face that consists of a single point is called a *vertex*. The *affine span* of Δ is the smallest affine space that contains Δ. This can be described as

$$\mathrm{aff}(S) = \left\{\sum_{i=1}^{r} \lambda_i\,\mathbf{v}_i \mid \lambda_i \in \mathbb{R}, v_i \in S, \sum_{i=1}^{r} \lambda_i = 1, r \in \mathbb{Z}_{\geq 1}\right\}.$$

The *dimension* of Δ is the same as the dimension of its affine span.

Two specific kinds of polyhedra are *polytopes* and *cones.* A *polytope* is a bounded polyhedron. It can alternatively be given as the convex hull of its vertices. That is, if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is the set of vertices of Δ, then

$$\Delta = \mathrm{conv}(S) := \{\lambda_1\,\mathbf{v}_1 + \ldots + \lambda_r\,\mathbf{v}_r \mid \lambda_1, \ldots, \lambda_r \geq 0,\ \lambda_1 + \ldots + \lambda_r = 1\}.$$

A popular instance of polytopes in this thesis are *Newton polytopes.* If $f(\mathbf{x}) = \sum_{\mathbf{u}} a_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$ is a regular or tropical (Laurent) polynomial, then its Newton polytope $\Delta(f)$ is the convex hull of all points $\mathbf{u}$ for which $a_{\mathbf{u}}$ is not the additive identity in $\mathbb{R}$ or $\mathbb{R}_{\mathrm{trop}}$.

A *cone* is another special type of polyhedron, which is of the form

$$C = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{0}\},$$

for some matrix $A \in \mathbb{R}^{m\times n}$. We say that C is *rational* if $A \in \mathbb{Q}^{m\times n}$. Moreover, a *strongly convex* cone is a cone that contains no linear subspaces of positive dimension. Equivalently, the origin $\mathbf{0}$ is a vertex of the cone. Alternatively, a cone can be written as the positive hull of a finite subspace $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subset \mathbb{R}^n$:

$$C = \mathrm{cone}(S) := \{\lambda_1\,\mathbf{v}_1 + \ldots + \lambda_r\,\mathbf{v}_r \mid \lambda_1, \ldots, \lambda_r \geq 0\}.$$

Each cone has a dual, also known as the *inner normal cone*, which is given by

$$C^\vee := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}\cdot\mathbf{y} \geq 0, \text{ for all } \mathbf{y} \in C\}.$$

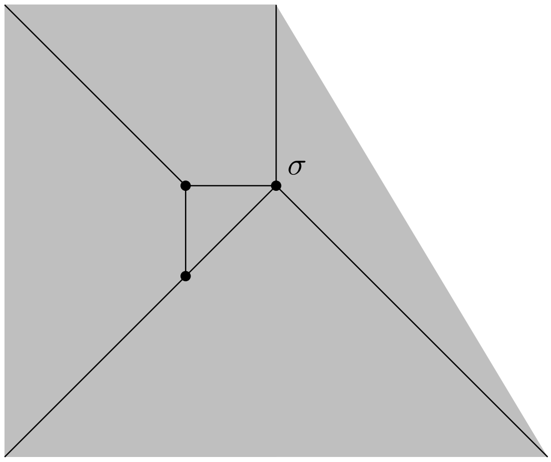
A *polyhedral complex* is a collection Σ of polyhedra satisfying the following two conditions:

(1) if P is in Σ, then so is any face of P;

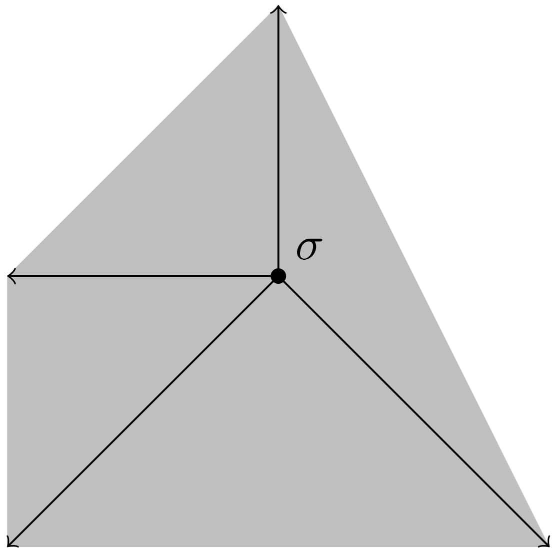
(2) if P and Q lie in Σ, then $P \cap Q$ is either empty or a face of both P and Q.

The polyhedra $P \in \Sigma$ are also called the *cells* of Σ. The *support* $|\Sigma|$ of Σ is the union of all cells contained in Σ. For k a positive integer, the *k-skeleton* of Σ is the polyhedral complex containing all cells of Σ of dimension smaller than or equal to k. A complex Σ is called pure of dimension d if its maximal cells are all of dimension d.

A specific case of polyhedral complexes are *fans*, which are complexes that consist of only cones. We give two constructions of fans.



(a) An example of a polyhedral complex, pure of dimension 2



(b) The star of the 0-dimensional cell σ

Figure 27

First, let Σ be a polyhedral complex in $\mathbb{R}^n$ and let σ be a cell of Σ. The *star* of σ in Σ is a fan in $\mathbb{R}^n$, denoted by $\text{star}_\Sigma(\sigma)$. It consists of all subsets

$$C_\sigma(\tau) := \text{cone}\{\mathbf{u} - \mathbf{v} \mid \mathbf{u} \in \tau, \mathbf{v} \in \sigma\}$$

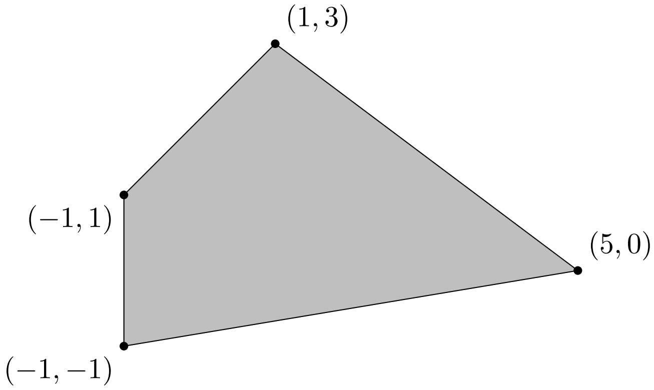
where τ ranges over all cells in Σ that contain σ as face.

Second, each polyhedron Δ has a particular fan $\Sigma(\Delta)$ associated to it, called the *normal fan* of Δ. To construct this fan, we associate to each face σ of Δ the cone spanned by the difference of vectors in σ and Δ:

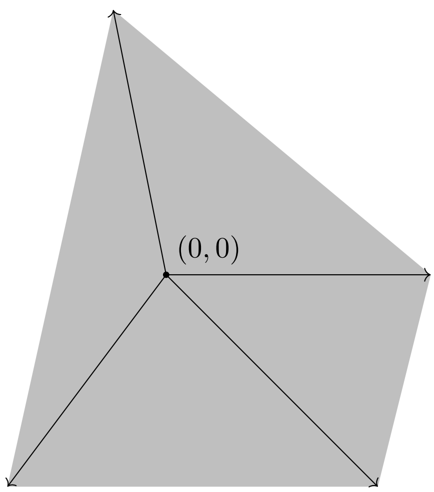
$$C_\sigma(\Delta) := \text{cone}\{\mathbf{u} - \mathbf{v} \mid \mathbf{u} \in \Delta, \mathbf{v} \in \sigma\}.$$

The dual fan of Δ is then defined by

$$\Sigma(\Delta) := \{C_\sigma(\Delta)^\vee \mid \sigma \text{ a face of } \Delta\}.$$



(a)



(b)

Figure 28: The Newton polytope of $f = 17x^{-1}y + 2x^{-1}y^{-1} + 5xy^3 + 22x^5$ and its normal fan.

Example A.1. Consider the Laurent polynomial

$$f = 17x^{-1}y + 2x^{-1}y^{-1} + 5xy^3 + 22x^5.$$

Its Newton polytope $\Delta(f)$ is the convex hull of the points $(-1,1),(1,3),(5,0),(-1,-1)$. See also Figure 28. It has 9 faces, namely the vertices $(1,3),(5,0),(-1,-1),(-1,1)$, the line segments between these vertices and the whole polytope itself. The cone associated to the whole polygon is the zero cone $\{(0,0)\}$. The cones associated to the line segments are the rays spanned by $(-1,5),(5,0),(1,-1),(-3,-4)$, which are normal to the sides of $\mathrm{NP}(f)$. The final four cones in the normal fan are given by

$$\mathrm{cone}\{(-1,5),(5,0)\}, \qquad \mathrm{cone}\{(5,0),(1,-1)\},$$
$$\mathrm{cone}\{(1,-1),(-3,-4)\}, \qquad \mathrm{cone}\{(-3,-4),(-1,5)\},$$

where each 2-dimensional cone corresponds to a unique vertex of $\Delta(f)$. △

B Toric Geometry

In this section all prerequisites on toric geometry assumed throughout the main text of this thesis are presented. By no means is this a complete introduction on toric geometry; proofs are omitted and only the material necessary for the remainder of this dissertation is presented. For a more thorough discussion on the subject, see for instance [7], which will be our main reference this chapter, or [10].

Toric geometry can be described as a mix of algebraic and polyhedral geometry. For those less familiar with the latter subject, we refer to Appendix A. The theme of this chapter will be to construct increasingly sophisticated toric objects from increasingly sophisticated polyhedral ones. In Section B.1, we associate affine toric varieties to rational cones. Building on this construction, we glue these varieties together using fans in Section B.2. Next, we show how an integral polytope can give a projective variety along with an ample divisor in Section B.3. Finally, we discuss hypersurfaces of projective toric varieties in B.4.

However, before doing this, we first define the main objects of study in this appendix: the toric variety. For this, fix an algebraically closed field K. For our purposes, the field K will either be the field of complex numbers, or the field of locally convergent Puiseux series over $\mathbb{C}$. Toric varieties are characterised by a group action of the algebraic torus, which we define shortly.

Definition B.1 (Algebraic Torus)**.** Let N be a lattice, i.e., a finitely generated, abelian, free group. In other words, N is a group that is isomorphic to $\mathbb{Z}^n$ for some non-negative integer n, with n being the rank of N. An *algebraic torus* is a variety isomorphic to

$$T_N := K^\times \otimes_{\mathbb{Z}} N \cong (K^\times)^n,$$

and which inherits the action

$$T_N \times T_N \colon ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto (x_1 y_1, \ldots, x_n y_n)$$

via the isomorphism.

Definition B.2 (Toric Variety)**.** Let n be a non-negative integer and N a lattice of rank n. A *toric variety* over K is an irreducible algebraic variety X over K with a dense open subset isomorphic to the algebraic torus T_N along with a group action of T_N on X that is algebraic and extends the action of T_N on itself. With an algebraic action we mean that the map

$$T_N \times X \to X$$

is a morphism of varieties.

B.1 Affine Varieties

First, we shall associate to a rational cone an affine, toric variety. Let

$$\sigma = \left\{ \sum_{i \in I} \lambda_i v_i \mid \lambda_i \geq 0, i \in I \right\}$$

be such a cone, where I is a finite index set and $v_i \in \mathbb{Z}^n$ for all $i \in I$. To this cone we associate the semigroup $S_\sigma := \sigma^\vee \cap \mathbb{Z}^n$, where $\sigma^\vee$ is the dual cone of σ, see also Appendix A.

According to Gordan's lemma [7, Prop. 1.2.117], the semigroup S_σ is finitely generated. Consequently, the polynomial ring

$$K[S_\sigma] := \left\{ \sum_{\mathbf{u} \in S} a_{\mathbf{u}} x^{\mathbf{u}} \mid S \subset S_\sigma \text{ finite and } a_{\mathbf{u}} \in K \text{ for all } \mathbf{u} \in S \right\}$$

is a finitely generated algebra over K. Furthermore, note that it is an integral domain.

Definition B.3. Let σ be a rational cone. Then, the variety

$$\mathcal{U}_\sigma := \operatorname{Spec} K[S_\sigma]$$

is the toric variety associated with σ.

We check that the scheme $\mathcal{U}_\sigma$ is indeed a toric variety. Since $K[S_\sigma]$ is finitely generated and an integral domain, $\mathcal{U}_\sigma$ is an irreducible variety. Moreover, by $S_\sigma \subset \mathbb{Z}\, S_\sigma$, there is an inclusion $K[S_\sigma] \subset K[\mathbb{Z}\, S_\sigma]$ into the field of Laurent polynomials. Consequently, there is an inclusion $T_{\mathbb{Z}\, S_\sigma} \hookrightarrow \mathcal{U}_\sigma$ of the algebraic torus into the constructed toric variety $\mathcal{U}_\sigma$. If $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are generators of S_σ, then $T_{\mathbb{Z}\, S_\sigma}$ is the complement of $V(\mathbf{x}^{\mathbf{u}_1} \cdots \mathbf{x}^{\mathbf{u}_r})$ in $\mathcal{U}_\sigma$. Therefore, $T_{\mathbb{Z}\, S_\sigma}$ is open in $\mathcal{U}_\sigma$ and as the latter is irreducible, the former is also dense in $\mathcal{U}_\sigma$.

Furthermore, each $\lambda \in (K^\times)^n$ gives a ring homomorphism $K[S_\sigma] \to K[S_\sigma]$ determined by sending x_i to $\lambda_i^{-1} x_i$ for each $i = 1, \ldots, n$. By the contrapositive equivalence of the category of affine varieties over K and the category of finitely generated, integral K-algebras, this gives a morphism of varieties $\operatorname{Spec} K[S_\sigma] \to \operatorname{Spec} K[S_\sigma]$ for each $\lambda \in (K^\times)^n$. In turn, we obtain a group action of $(K^\times)^n$ on $\operatorname{Spec} K[S_\sigma]$ that extends the action of the torus on itself.

Example B.4. Let n be a positive integer and consider the cone $\sigma = (\mathbb{R}_{\geq 0})^n$. Its dual is σ and the standard basis of $\mathbb{R}^n$ generates the semigroup $S_\sigma = (\mathbb{Z}_{\geq 0})^n$. Therefore,

$$\mathcal{U}_\sigma = \operatorname{Spec} K[x_1, \ldots, x_n]$$

equals affine n-space. △

Example B.5. Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$. The zero cone $\sigma = \{0\} \subset \mathbb{R}^n$ is dual to all of $\mathbb{R}^n$ and $\pm e_1, \ldots, \pm e_n$ generates the semigroup $S_\sigma = \mathbb{Z}^n$. So,

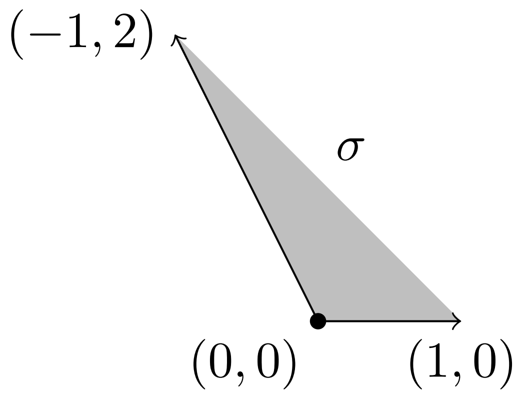
$$\mathcal{U}_\sigma = \operatorname{Spec} K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

equals the algebraic torus $(K^\times)^n$. △

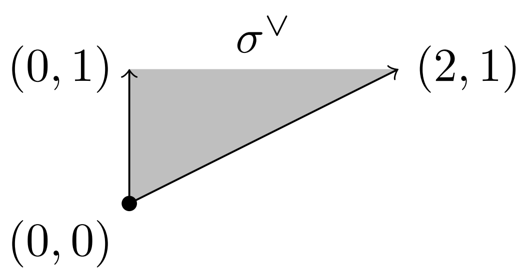
Example B.6. Conversely, the cone $\sigma = \mathbb{R}^n$ is dual to the zero cone and so the semigroup S_σ is trivial. Hence, the space

$$\mathcal{U}_\sigma = \operatorname{Spec} K$$

is a point. △



(a) Cone from Example B.7



(b) Dual cone from Example B.7

Figure 29

Example B.7. Let σ be the cone from Figure 29a, i.e., the rational cone spanned by the vectors $(1,0),(-1,2)$. Its dual is the rational cone spanned by $(0,1),(2,1)$. However, $(0,1),(2,1)$ do not generate the semigroup $\sigma^\vee \cap \mathbb{Z}^2$, while $(0,1),(1,1),(2,1)$ do. The associated affine variety is given by

$$\mathcal{U}_\sigma = \operatorname{Spec} K[y, xy, x^2y] \cong \operatorname{Spec} K[u,v,w]/(v^2 - uw).$$

This is the first example of a singular toric variety, where the singularity is given by the point $(0,0,0)$, in the coordinates u, v, w. △

Definition B.8 (Toric Morphism)**.** Let V_i be a toric variety with torus T_{N_i} for $i = 1,2$. A morphism of varieties $\varphi\colon V_1 \to V_2$ is called *toric* if $\varphi(T_{N_1}) \subset \varphi(T_{N_2})$ and the restriction

$$\varphi\big|_{T_{N_1}} : T_{N_1} \to T_{N_2}$$

is a group homomorphism.

The final part of this section will be on open subsets of $\mathcal{U}_\sigma$ determined by the faces of σ. Say $\sigma \subset \mathbb{R}^n$ is a cone and $\tau \subset \sigma$ a face. That is, there exists a vector $\mathbf{w} \in \sigma^\vee$ such that $\sigma \cap H_{\mathbf{w}} = \tau$, where

$$H_{\mathbf{w}} := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{w} \cdot \mathbf{x} = 0\}$$

is the half-plane orthogonal to $\mathbf{w}$. The following lemma demonstrates how $\mathcal{U}_\tau$ can be viewed as an open affine subset of $\mathcal{U}_\sigma$.

Lemma B.9. *[7, Prop. 1.3.16] Let σ, τ and $\mathbf{w}$ as above. Then, $K[S_\tau]$ is the localization of $K[S_\sigma]$ at $\mathbf{x}^{\mathbf{w}}$. Consequently, $\mathcal{U}_\tau$ is the open affine subset of $\mathcal{U}_\sigma$ given as the complement of $V(\mathbf{x}^{\mathbf{w}})$.*

B.2 Normal Varieties

In the last section we have seen how to associate an affine toric variety $\mathcal{U}_\sigma$ to a lattice cone σ. In general, algebraic varieties are formed by gluing a bunch of affine varieties. One way of obtaining appropriate gluing data is via polyhedral fans. In this section, we first show how to construct a normal, toric variety from a rational fan. What is more, it turns out that any normal, toric variety arises in such a way. Next, the relationship between cones on the one hand and toric orbits on the other is explored. This paramount relationship is known as the Orbit-Cone Correspondence, which is handled in Theorem B.15.

Definition B.10. Let N be a lattice and Σ a rational fan in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. The associated toric variety $X_K(\Sigma)$ is defined as the quotient

$$X_K(\Sigma) := \bigcup_{\sigma \in \Sigma} \mathcal{U}_\sigma \,/ \sim,$$

where the equivalence relation $\sim$ identifies two varieties $\mathcal{U}_{\sigma_1}, \mathcal{U}_{\sigma_2}$ along $\mathcal{U}_{\sigma_1 \cap \sigma_2}$, for each pair of cones $\sigma_1, \sigma_2 \in \Sigma$. The toric action $T_N \times X_K(\Sigma) \to X_K(\Sigma)$ is obtained by gluing the actions $T_N \times \mathcal{U}_\sigma \to \mathcal{U}_\sigma$, $\sigma \in \Sigma$. If the choice of base field is clear, we will also write $X(\Sigma)$ instead of $X_K(\Sigma)$.

Theorem B.11 (Sumihiro)**.** *[7, Thm. 3.1.7] Let Σ be a fan. Then, $X(\Sigma)$ is a normal, separated toric variety. In fact, each normal, separated toric variety arises in this way.*

Example B.12. Let n be a positive integer and $e_1, \ldots e_n$ the standard basis of $\mathbb{R}^n$. Also denote $e_0 := -e_1 - \ldots - e_n$ and consider the fan

$$\Sigma := \{\operatorname{cone}(S) \mid S \subsetneq \{e_0, e_1, \ldots, e_n\}\}.$$

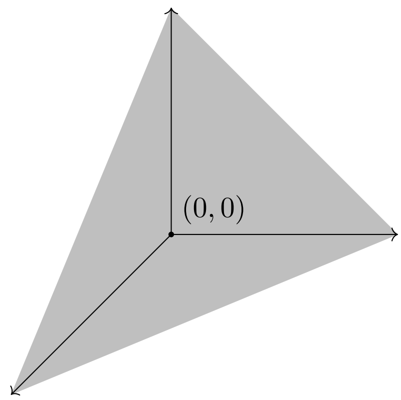


Figure 30: The fan from Example B.12.

See Figure 30 for an illustration of this fan for $n = 2$. The maximal cones of Σ are

$$\sigma_i := \operatorname{cone}(e_0, e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n), \quad i = 0, 1, \dots, n$$

and so $X(\Sigma)$ is covered by the affine open subsets $U_i := \mathcal{U}_{\sigma_i}$. A brief computation shows that

$$\begin{aligned}
\sigma_0^\vee &= \sigma_0, \\
\sigma_i^\vee &= \operatorname{cone}(-e_i, e_1 - e_i, e_2 - e_i, \dots, e_n - e_i), && \text{for } i = 1, 2, \dots, n.
\end{aligned}$$

Hence,

$$\begin{aligned}
U_0 &= \operatorname{Spec} K[x_1, \dots, x_n], \\
U_i &= \operatorname{Spec} K\left[x_i^{-1}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}\right], && \text{for } i = 1, 2, \dots, n.
\end{aligned}$$

Let $\mathbb{P}^n$ be projective n-space with homogeneous coordinates $y_0, y_1, \dots, y_n$. The localizations $\mathbb{P}^n_{y_i}$ of $\mathbb{P}^n$ at $y_i, i = 0, 1, \dots, n$ give an affine open cover of $\mathbb{P}^n$, where

$$\mathbb{P}^n_{y_i} \cong \operatorname{Spec} K\left[\frac{y_0}{y_i}, \dots, \frac{y_n}{y_i}\right].$$

For each $i = 1, \dots, n$, consider the morphism $\varphi_i \colon U_i \to \mathbb{P}^n$ induced by the isomorphism of K-algebras

$$\begin{aligned}
\varphi_i^* \colon K\left[\frac{y_0}{y_1}, \dots, \frac{y_n}{y_i}\right] &\to K\left[x_i^{-1}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}\right], \\
\frac{y_0}{y_i} &\mapsto x_i^{-1}, \\
\frac{y_j}{y_i} &\mapsto \frac{x_j}{x_i}, \quad \text{for } j = 1, 2, \dots, n.
\end{aligned}$$

For $i = 0$ define the map

$$\begin{aligned}
\varphi_0^* \colon K\left[\frac{y_1}{y_0}, \dots, \frac{y_n}{y_0}\right] &\mapsto K\left[x_1, \dots, x_n\right] \\
\frac{y_j}{y_0} &\mapsto x_j, \quad \text{for } j = 1, 2, \dots, n.
\end{aligned}$$

It is easy to check that the maps φ_i, φ_j agree on the overlap

$$U_i \cap U_j = U_{\sigma_i \cap \sigma_j} = \operatorname{Spec} K\left[x_i^{-1}, \frac{x_1}{x_i}, \dots, \frac{x_{j-1}}{x_i}, \left(\frac{x_j}{x_i}\right)^{\pm 1}, \frac{x_{j+1}}{x_i}, \dots, \frac{x_n}{x_i}\right]$$

for all $i, j = 0, 1, \dots, n$ and so, these maps glue to a single morphism $\varphi \colon X(\Sigma) \to \mathbb{P}^n$. This is in fact an isomorphism as it is an isomorphism on an open affine cover. △

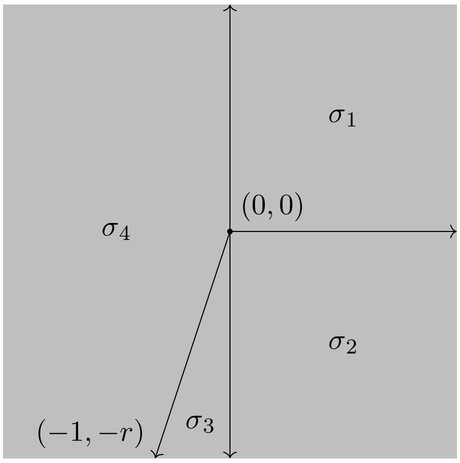


Figure 31: The fan from Example B.13.

Example B.13. Let r be a non-negative integer and Σ_r the fan consisting of the cones $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ shown in Figure 31 and their faces. The associated toric variety $X(\Sigma_r)$ is covered by the affine open subsets

$$\begin{aligned}
\mathcal{U}_{\sigma_1} &= \operatorname{Spec} K[x,y] \cong K^2,\\
\mathcal{U}_{\sigma_2} &= \operatorname{Spec} K[x,y^{-1}] \cong K^2,\\
\mathcal{U}_{\sigma_3} &= \operatorname{Spec} K[x^{-1}y^{-r},y^{-1}] \cong K^2,\\
\mathcal{U}_{\sigma_4} &= \operatorname{Spec} K[x^{-1}y^{-r},y] \cong K^2.
\end{aligned}$$

We call $X(\Sigma_r)$ the *Hirzebruch surface* $\mathbb{H}_r$. The special cases $r \in \{0,1\}$ give $\mathbb{H}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{H}_1 \cong \widetilde{\mathbb{P}^2}$, the plane blown up in a point. △

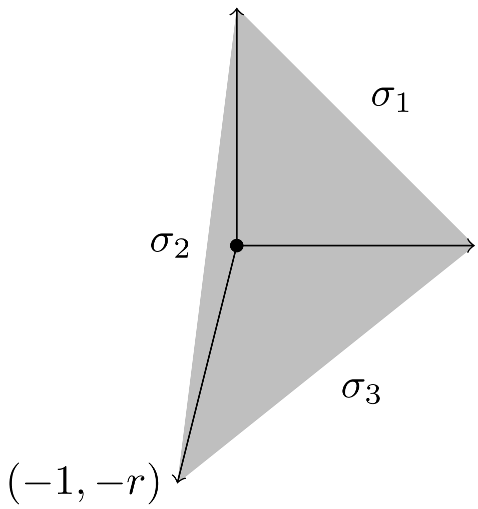


Figure 32: The fan from Example B.14.

Example B.14. Let Σ be the fan consisting of the cones $\sigma_1, \sigma_2, \sigma_3$ from Figure 32 and all their faces. The corresponding toric variety $X(\Sigma)$ is covered by the affine open subsets

$$\begin{aligned}
\mathcal{U}_{\sigma_1} &= \operatorname{Spec} K[x,y] \cong K^2,\\
\mathcal{U}_{\sigma_2} &= \operatorname{Spec} K[x^{-1},x^{-r}y] \cong K^2,\\
\mathcal{U}_{\sigma_3} &= \operatorname{Spec} K[y^{-1},xy^{-1},\dots,x^r y^{-1}].
\end{aligned}$$

It can be shown that $X(\Sigma)$ is isomorphic to the weighted projective plane $\mathbb{P}(1,1,r)$. △

As we have seen, rational fans give rise to normal toric varieties. In this construction, the cones from the fan correspond to dense, open and affine subsets of the entire variety. However, there is another link between the cones and a particular class of subsets, namely the orbits. Recall that if N is a lattice of rank n and $X(\Sigma)$ a toric variety arising from a rational fan Σ with torus $T_N \cong (K^\times)^n$, then there is an algebraic action $T_N \times X(\Sigma) \to X(\Sigma)$. The following paramount result establishes the relationship between the cones of Σ and the orbits of $X(\Sigma)$.

Theorem B.15 (Orbit-Cone Correspondence)**.** *[7, Thm. 3.2.6] Let Σ be a rational fan in $N_\mathbb{R}$ with associated toric variety $X(\Sigma)$. Then, there is a bijective correspondence*

$$\begin{aligned} \Sigma &\xrightarrow{1-1} \{T_N\text{-orbits in } X(\Sigma)\} \\ \sigma &\mapsto O(\sigma), \end{aligned}$$

between the cones from Σ and the orbits of $X(\Sigma)$, such that

- $\dim \mathcal{O}(\sigma) = \dim N_\mathbb{R} - \dim \sigma$*;*
- *for each cone $\sigma \in \Sigma$ we have*
$$\mathcal{U}_\sigma = \bigcup_{\tau \subset \sigma} O(\tau),$$
where the union is over all faces of σ;
- *for two cones $\tau, \sigma \in \Sigma$, τ is a face of σ if and only if $O(\sigma) \subset \overline{O(\tau)}$, and*
$$\overline{O(\tau)} = \bigcup_{\tau \subset \sigma} O(\sigma),$$
where $\overline{O(\tau)}$ denotes the closure in the Zariski topology.

Example B.16. The projective plane $\mathbb{P}^2$ contains the toric action

$$\begin{aligned} (K^\times)^2 \times \mathbb{P}^2 &\to \mathbb{P}^2 \\ ((u,v),[x:y:z]) &\mapsto [x:uy:vz]. \end{aligned}$$

The orbits of this action are given by

$$\begin{array}{lll} \{[1:0:0]\}, & \{[0:1:0]\}, & \{[0:0:1]\}, \\ \{[x:y:0] \in \mathbb{P}^2 \mid x,y \neq 0\}, & \{[x:0:z] \in \mathbb{P}^2 \mid x,z \neq 0\}, & \{[0:y:z] \in \mathbb{P}^2 \mid y,z \neq 0\}, \\ \{[x:y:z] \in \mathbb{P}^2 \mid x,y,z \neq 0\}. & & \end{array}$$

The first triple of orbits, which are zero-dimensional, correspond to the two-dimensional cones in Figure 30. The second triple of orbits correspond to the one-dimensional cones in Figure 30. The final orbit corresponds to the trivial cone $\{(0,0)\}$. △

Besides the toric varieties themselves, it will be essential to know the behavior of the maps in between. It turns out that toric morphisms $X(\Sigma_1) \to X(\Sigma_2)$ between varieties arising from rational fans can be naturally described by maps between the fans themselves. To understand this description, we need the following definition and theorem.

Definition B.17. Let N_1, N_2 be two lattices, and Σ_1 a fan in $(N_1)_\mathbb{R}$ and Σ_2 a fan in $(N_2)_\mathbb{R}$. A $\mathbb{Z}$-linear map $\overline{\varphi}\colon N_1 \to N_2$ is called *compatible* with the fans Σ_1 and Σ_2 if for each cone $\sigma_1 \in \Sigma_1$, there exists a cone $\sigma_2 \in \Sigma_2$ such that $(\overline{\varphi} \otimes 1)(\sigma_1) \subset \sigma_2$.

Theorem B.18. *[7, Thm. 3.3.4] Let N_i be a lattice and Σ_i a fan in $(N_i)_\mathbb{R}$ for $i = 1,2$. If $\overline{\varphi}\colon N_1 \to N_2$ is a $\mathbb{Z}$-linear map that is compatible with Σ_1 and Σ_2, then there exists a toric morphism $\varphi\colon X(\Sigma_1) \to X(\Sigma_2)$ such that $\varphi_{T_{N_1}}$ is the map*

$$\overline{\varphi} \otimes 1 \colon N_1 \otimes_\mathbb{Z} K^\times \to N_2 \otimes_\mathbb{Z} K^\times.$$

Conversely, each toric morphism $\varphi\colon X(\Sigma_1) \to X(\Sigma_2)$ arises in such a way.

Example B.19. Consider the Hirzebruch surface $\mathbb{H}_r = X(\Sigma_r)$ as defined in Example B.13. Let Σ be the fan in $\mathbb{R}$ consisting of the cones $\mathbb{R}_{\geq 0}, \mathbb{R}_{\leq 0}, \{0\}$. From Example B.12 we know that $X(\Sigma) = \mathbb{P}^1$. The $\mathbb{Z}$-linear map

$$\overline{\varphi}\colon \mathbb{Z}^2 \to \mathbb{Z},$$
$$(x, y) \mapsto x$$

is compatible with Σ_r, Σ since $\overline{\varphi}(\sigma_1), \overline{\varphi}(\sigma_2) \subset \mathbb{R}_{\geq 0}$ and $\overline{\varphi}(\sigma_3), \overline{\varphi}(\sigma_4) \subset \mathbb{R}_{\leq 0}$. Therefore, this induces a toric morphism $\varphi\colon \mathbb{H}_r \to \mathbb{P}^1$. For the cases $r = 0, 1$, this gives the natural projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and the blow-up map $\widetilde{\mathbb{P}^2} \to \mathbb{P}^1$. △

B.3 Projective Varieties and Divisors

In the previous section we have seen how to construct toric varieties from polyhedral fans. Coincidentally, in all examples we saw that the toric variety was a projective one. This is because each fan in the previous section is dual to a lattice polytope. We will see that this implies that there exists an ample divisor, thus giving an embedding into projective space.

Definition B.20. Let $\Delta \subset \mathbb{R}^n$ be an n-dimensional lattice polytope with normal fan $\Sigma(\Delta)$. The projective toric variety associated with Δ is defined to be $X_K(\Delta) := X(\Sigma(\Delta))$.

However, it turns out that a polytope can carry a lot more information than just a toric variety. The remainder of this section depicts how Δ induces a divisor $L(\Delta)$ on $X(\Delta)$.

Let Σ be the normal fan of Δ and denote the set of 1-dimensional cones, or rays, of Σ by $\Sigma(1)$. For each ray $\rho \in \Sigma(1)$, let $\mathbf{u}_\rho$ be the primitive vector that spans ρ. That is, $\mathbf{u}_\rho$ is an integral vector of minimal length that spans ρ. Then, there exist integers $a_\rho \in \mathbb{Z}$ for each ray $\rho \in \Sigma(1)$ such that Δ can be written as

$$\Delta = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{u}_\rho \geq -a_\rho, \rho \in \Sigma(1)\}.$$

Note that, by the Orbit-Cone Correspondence, $D_\rho := \overline{O(\rho)}$ are prime divisors of X. We call these the *toric divisors* of X.

Definition B.21. In the notation above, we associate to the polytope Δ the divisor

$$L(\Delta) := \sum_{\rho \in \Sigma(1)} a_\rho D_\rho.$$

Example B.22. Let d be a positive integer and

$$\Delta := \text{conv}\{(0,0), (1,0), (0,1)\}.$$

We may rewrite $d\Delta$ in terms of its faces as

$$d\Delta = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, -x - y \geq -d\}.$$

The primitive vectors spanning the rays of $\Sigma := \Sigma(d\Delta)$ are $(1,0), (0,1)$ and $(-1,-1)$. So, we have

$$L(d\Delta) = 0 \cdot D_{(0,1)} + 0 \cdot D_{(1,0)} + d \cdot D_{(-1,-1)}.$$

Moreover, note that the normal fan of Δ equals the fan in Example B.12 and so, $X(d\Delta) \cong \mathbb{P}^2$. Also, $D_{(-1,-1)}$ is the prime divisor of a line in $\mathbb{P}^2$ and therefore, $L(d\Delta)$ is linearly equivalent to dH, for H a line in $\mathbb{P}^2$. △

Lemma B.23. *[7, Prop. 6.1.4] Let $\Delta \subset \mathbb{R}^n$ be a full-dimensional polytope. The divisor $L(\Delta)$ is ample and basepoint free.*

The main text deals with counting curves coming from the linear system $L(\Delta)$, when $\Delta \subset \mathbb{R}^2$ is a 2-dimensional polytope. The following proposition and its corollary give a convenient description of these curves via their defining equations and their Newton polytopes.

Proposition B.24. *[7, Prop. 4.3.3] Let $\Delta \subset \mathbb{R}^n$ be a full-dimensional lattice polytope. The global sections of the invertible sheaf $\mathcal{O}_{X(\Delta)}(L(\Delta))$ are given by*

$$\Gamma(X(\Delta), \mathcal{O}_{X(\Delta)}(L(\Delta))) = \bigoplus_{\mathbf{u} \in \Delta \cap \mathbb{Z}^n} K \cdot \mathbf{x}^{\mathbf{u}}.$$

The following corollary easily follows from the previous proposition and the standard bijection

$$\frac{\Gamma(X, \mathcal{O}_X(L)) \setminus \{0\}}{K^\times} \to |L|,$$

which sends the class of an $f \in \Gamma(\mathcal{O}_X(L))$ to the divisor $L + \operatorname{div} f$.

Corollary B.25. *Let $\Delta \subset \mathbb{R}^n$ be a full-dimensional lattice polytope. The linear system $|L(\Delta)|$ consists of all curves $\mathrm{cl}_{X(\Delta)}\{f = 0\}$ for which $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a non-zero Laurent polynomial with Newton polytope contained inside Δ.*

B.4 Hypersurfaces of Toric Varieties

Let $\Delta \subset \mathbb{R}^n$ be a full-dimensional lattice polytope. Then, a hypersurface V of $X = X(\Delta)$, which does not contain the toric divisors, is uniquely determined by its intersection with the torus $(K^\times)^n$. Hence, there exists some Laurent polynomial

$$f(\mathbf{x}) = \sum_{\mathbf{u} \in I} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

such that V is given by the closure of $\{f = 0\} \subset (K^\times)^n$ in X.

Note that, since Δ is full-dimensional, $N\Delta$ becomes unbounded in each direction for $N \in \mathbb{Z}_{\geq 1}$. Therefore, for large enough N, there exists a translation vector $\mathbf{u_0} \in \mathbb{Z}^n$ such that the Newton polytope of $\mathbf{x}^{\mathbf{u_0}} f(\mathbf{x})$ is contained inside $N\Delta$. As Δ and $N\Delta$ have equal normal fans, $X(\Delta) = X(N\Delta)$, and as the zero loci $\{f = 0\} = \{\mathbf{x}^{\mathbf{u_0}} f = 0\}$ are equal, we may assume without loss of generality that the Newton polytope of f is contained inside Δ.

Definition B.26. Let $f = \sum_{\mathbf{u} \in \Delta \cap \mathbb{Z}^n} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial with Newton polytope Δ. If $\sigma \preceq \Delta$ is a face of f, then we define the truncation of f at σ to be

$$f^\sigma(\mathbf{x}) = \sum_{\mathbf{u} \in \sigma \cap \mathbb{Z}^n} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}.$$

Lemma B.27. *Let $\Delta \subset \mathbb{R}^n$ be a full-dimensional lattice polytope and*

$$f(\mathbf{x}) = \sum_{\mathbf{u} \in \Delta' \cap \mathbb{Z}^n} a_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$$

a Laurent polynomial with Newton polytope $\Delta' \subset \Delta$. Also, let σ be a facet of Δ, $\mathbf{w} \in \mathbb{Z}^n$ an inner normal vector of σ and $\sigma' := \mathrm{face}_{\mathbf{w}}(\Delta')$. Then,

$$\mathrm{cl}_X\{f = 0\} \cap X(\sigma) = \mathrm{cl}_{X(\sigma')}\{f^{\sigma'} = 0\}.$$

In particular, if $\Delta' = \Delta$, we have

$$\mathrm{cl}_X\{f = 0\} \cap X(\sigma) = \mathrm{cl}_{X(\sigma)}\{f^{\sigma} = 0\}.$$

Proof. Consider the open affine subset $\mathcal{U}_\sigma$ of X. The intersection $\mathcal{U}_\sigma \cap X(\sigma)$ is then a dense, open and affine subset of $X(\sigma)$, which is given by $V(\mathbf{x}^{\mathbf{w}})$ in $\mathcal{U}_\sigma$. Also, let $k \in \mathbb{Z}$ be the minimal integer such that the Newton polytope of $g := \mathbf{x}^{k\mathbf{w}} f$ is contained inside $C_\sigma(\Delta)$, and denote the Newton polytope of g by Δ''. Then, the face $\sigma'' := \text{face}_{\mathbf{w}}(\Delta'')$ is equal to the polytope $\sigma' - k\mathbf{w}$ and therefore,

$$\begin{aligned}\mathcal{U}_\sigma \cap X(\sigma) \cap \mathrm{cl}_X\{f = 0\} &= V(\mathbf{x}^{k\mathbf{w}} f, \mathbf{x}^{\mathbf{w}})\\ &= V((\mathbf{x}^{k\mathbf{w}} f)^{\sigma''}, \mathbf{x}^{\mathbf{w}})\\ &= \mathcal{U}_\sigma \cap \mathrm{cl}_{X(\sigma)}\{(\mathbf{x}^{k\mathbf{w}} f)^{\sigma''} = 0\}\\ &= \mathcal{U}_\sigma \cap \mathrm{cl}_{X(\sigma)}\{f^{\sigma'} = 0\}.\end{aligned}$$

Since $\mathcal{U}_\sigma \cap X(\sigma)$ is dense inside $X(\sigma)$, this finishes the proof. □

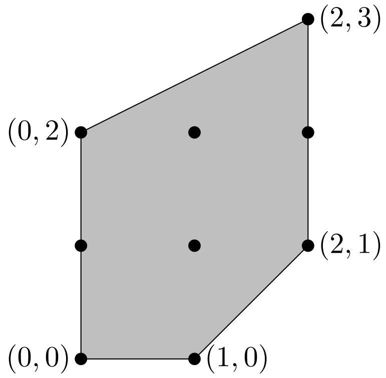


Figure 33: The Newton polygon from Example B.28.

Example B.28. Let f be the Laurent polynomial

$$f(x, y) = (1 + y)^2 + 2x + 5x^2y + 17xy - xy^2 + 3x^2y^2 + 8x^2y^3.$$

Its Newton polytope $\Delta \subset \mathbb{R}^2$ is given by the edges

$$[(0, 0), (1, 0)], [(1, 0), (2, 1)], [(2, 1), (2, 3)], [(2, 3), (0, 2)], [(0, 2), (0, 0)].$$

In turn, each edge σ corresponds to a toric divisor $X(\sigma) \subset X(\Delta)$. Since the toric divisors and $\mathrm{cl}_X\{f = 0\}$ are 1-dimensional, we expect the intersections of $\mathrm{cl}_X\{f = 0\}$ with the toric divisors to be a finite set of points.

Indeed, if $\sigma \preceq \Delta$ is an edge, we may write the integral points of σ by $\mathbf{u} + k\mathbf{v}, 0 \leq k \leq m$ for some $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^2$ and $m := \left|\sigma \cap \mathbb{Z}^2\right| - 1$ the lattice length of σ. The truncation f^σ is then a univariate polynomial

$$f^\sigma = x^{\mathbf{u}} \sum_{k=0}^{m} a_{\mathbf{u}+k\mathbf{v}} \mathbf{x}^{\mathbf{v}}$$

in the indeterminate $\mathbf{x}^{\mathbf{v}}$. In particular, the number of points of intersection of $X(\sigma)$ with $\mathrm{cl}_X\{f = 0\}$ is smaller or equal to m.

Consequently, we conclude that $\mathrm{cl}_X\{f = 0\}$ intersects $X(\sigma)$ in precisely 1 point for the faces

$$\sigma \in \{[(0, 0), (1, 0)], [(1, 0), (2, 1)], [(2, 3), (0, 2)]\}.$$

For the face $\sigma = [(0, 0), (0, 2)]$ we have

$$f^\sigma = (1 + y)^2$$

and so $\mathrm{cl}_X\{f = 0\}$ intersects $X(\sigma)$ in one point with multiplicity 2.

For the face $\sigma = [(2,1),(2,3)]$ we have

$$f^\sigma = x^2y(5 + 3y + 8y^2).$$

As $5 + 3y + 8y^2$ factors in two distinct linear components, $\mathrm{cl}_X\{f = 0\}$ intersects $X(\sigma)$ in two distinct points. △

C Code

C.1 Code used in Example 4.13

The idea of the following SageMath code is to find rational functions $A_{ij}(t) \in \mathbb{Q}(t), i, j \geq 0, i + j \leq 3$ such that the polynomial

$$f(x, y) = A_{00} + A_{10}x + A_{01}y + A_{20}x^2 + A_{11}xy + A_{02}y^2 + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}xy^3$$

defines a cubic curve in $\mathbb{P}^2_{\mathbb{K}^2}$ with a node in $(t-1, t-1)$.

This is achieved by substituting

$$\begin{aligned} &x = 1 - t, && y = 1 - t, && A_{00} = 1, && A_{01} = A_{10}, && A_{20} = t, \\ &A_{02} = t, && A_{30} = t^4, && A_{21} = t^2, && A_{12} = t^2, && A_{03} = t^4. \end{aligned}$$

into the singular ideal (f, f_x, f_y). This gives an ideal in the ring $\mathbb{Q}(t)[A_{10}, A_{11}]$. By computing a Gröbner basis, we find solutions for A_{10}, A_{11}. A Gröbner basis of this ideal is

$$\begin{aligned} &A_{10} + \frac{-t^7 + 3t^6 - 4t^5 + 4t^4 - 3t^3 + t^2 + 1}{t - 1}, \\ &A_{11} + \frac{4t^7 - 12t^6 + 16t^5 - 16t^4 + 14t^3 - 8t^2 + 2t - 1}{t^2 - 2t + 1}. \end{aligned}$$

```
degree = 3 # Degree of polynomial

# Create list of variable names
var_names = ['x', 'y', 't']
for i in range(degree + 1):
    for j in range(degree + 1 - i):
        var_names.append('A{0}{1}'.format(i, j))

K.<t> = FunctionField(QQ) # Function field Q(t)
R = PolynomialRing(K, var_names) # Polynomial ring Q(t)[{A_ij}, x, y]

# Variables
x, y = R('x'), R('y')
A = [[ R('A{0}{1}'.format(i, j)) for j in range(degree + 1 - i)] for i in range(degree + 1)]

# Create general polynomial f = A_00 + A_10 x + ... + A_30 x^3
f = 0
for i in range(degree + 1):
    for j in range(degree + 1 - i):
        f += A[i][j] * x**i * y**j

# Make list of generators of the singular ideal (f, \partial_x f, \partial_y f)
gens = []
for g in [f.subs({x : -1+t, y : -1+t}), diff(f, x).subs(({x : -1+t, y : -1+t})), diff(f, y).subs(({x : -1+t, y : -1+t}))]:
    gens.append(g.subs({A[0][0] : 1, A[0][1] : A[1][0], A[0][2] : t, A[2][0] : t, A[2][1] : t^2, A[3][0] : t^4, A[0][3] : t^4, A[1][2] : t^2}))

# Create singular ideal and print its Groebner basis
J = ideal(gens)
for g in J.groebner_basis():
    print(latex(g))
```

C.2 Code used in Example 4.14

The idea of the following SageMath code is to find rational functions $A_{ij}(t) \in \mathbb{Q}(t), i, j \geq 0, i + j \leq 3$ such that the polynomial

$$f(x, y) = A_{00} + A_{10}x + A_{01}y + A_{20}x^2 + A_{11}xy + A_{02}y^2 + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}xy^3$$

defines a cubic curve in $\mathbb{P}^2_{\mathbb{K}^2}$ with a node in $(t-1, t-1)$.

This is achieved by substituting

$$\begin{aligned} x &= 1-t, & y &= 1-t, & A_{00} &= t^3, \\ A_{10} &= 1, & A_{01} &= t, & A_{20} &= t, \\ A_{30} &= t^3, & A_{12} &= t, & A_{03} &= t^3. \end{aligned}$$

into the singular ideal (f, f_x, f_y). This gives an ideal in the ring $\mathbb{Q}(t)[A_{10}, A_{11}]$. By computing a Gröbner basis, we find solutions for A_{10}, A_{11}. A Gröbner basis of this ideal is

$$\begin{aligned} &A_{02} + \frac{t^6 - 3t^5 + 4t^4 - 6t^3 + 5t^2 - 3t + 1}{t^2 - 2t + 1}, \\ &A_{11} + \frac{-t^6 + 3t^5 - 4t^4 + 10t^3 - 5t^2 + 4t - 3}{t^2 - 2t + 1}, \\ &A_{21} + \frac{2t^6 - 6t^5 + 7t^4 - 7t^3 + 2t^2 - t + 1}{t^3 - 3t^2 + 3t - 1}. \end{aligned}$$

```
degree = 3 # Degree of polynomial

# Create list of variable names
var_names = ['x', 'y', 't']
for i in range(degree + 1):
    for j in range(degree + 1 - i):
        var_names.append('A{0}{1}'.format(i, j))

K.<t> = FunctionField(QQ) # Function field Q(t)
R = PolynomialRing(K, var_names) # Polynomial ring Q(t)[{A_ij}, x, y]

# Variables
x, y = R('x'), R('y')
A = [[ R('A{0}{1}'.format(i, j)) for j in range(degree + 1 - i)] for i in range(degree + 1)]

# Create general polynomial f = A_00 + A_10 x + ... + A_30 x^3
f = 0
for i in range(degree + 1):
    for j in range(degree + 1 - i):
        f += A[i][j] * x**i * y**j

# Make list of generators of the singular ideal (f, \partial_x f, \partial_y f)
gens = []
for g in [f.subs({x : -1+t, y : -1+t}), diff(f, x).subs(({x : -1+t, y : -1+t})), diff(f, y).subs(({x : -1+t, y : -1+t}))]:
    gens.append(g.subs({A[0][0] : t^3, A[3][0] : t^3, A[0][3] : t^3, A[1][0] : 1, A[2][0] : t, A[0][1] : t, A[1][2] : t}))

# Create singular ideal and print its Groebner basis
J = ideal(gens)
for g in J.groebner_basis():
    print(latex(g))
```

C.3 Implementation of the Lattice Path Algorithm

C.3.1 LatticePoint Class

```python
class LatticePoint:
    '''
        Class of a point in Z^2
        Supports lexicographic ordering
    '''

    def __init__(self, x, y):
        """ Initialization method

            Keyword arguments:
            x -- int representing the abscissa
            y -- int representing the ordinate
        """
        self.x = x
        self.y = y

    def __eq__(self, other):
        '''Return if two lattice points are equal.'''
        return self.x == other.x and self.y == other.y

    def __lt__(self, other):
        '''Return whether this lattice point is smaller than other.'''
        return self.x < other.x or (self.x == other.x and self.y > other.y)

    def __gt__(self, other):
        '''Return whether this lattice point is greater than other.'''
        return self.x > other.x or (self.x == other.x and self.y < other.y)

    def __le__(self, other):
        '''Return whether this lattice point is lsmaller or equal to other.'''
        return self < other or self == other

    def __ge__(self, other):
        '''Return whether this lattice point is greater or equal to other.'''
        return self > other or self == other

    def __str__(self):
        '''Return string representation.'''
        return "({0}, {1})".format(self.x, self.y)

    def __repr__(self):
        '''Return string representation.'''
        return str(self)

    def __add__(self, other):
        '''Add two lattice points coordinate-wise.'''
        return LatticePoint(self.x + other.x, self.y + other.y)

    def __sub__(self, other):
        '''Subtract two lattice points coordinate-wise.'''
        return LatticePoint(self.x - other.x, self.y - other.y)

    def __rmul__(self, other):
        '''Multiply two lattice points coordinate-wise.'''
        return LatticePoint(other * self.x, other * self.y)

    def __floordiv__(self, other):
        '''Floor divide two lattice points coordinate-wise.'''
        return LatticePoint(self.x // other, self.y // other)
```

C.3.2 ConvexHull2D Class

```python
from LatticePoint import *
from math import gcd, ceil
import itertools as it

def get_boundary_path(lattice_points, start_point, end_point, upper=True):
    ''' Compute upper or lower boundary path of the convex hull of lattice_points.

    Keyword arguments:
    lattice_points -- list of LatticePoint objects
    start_point -- LatticePoint object that represents the starting point of the path.
    end_point -- LatticePoint object that represents the end point of the path.
    upper -- boolean whose value is True if we want the upper path and False for lower.

    returns a list of LatticePoint objects representing the boundary path,
    along with the inner normal vectors of the edges of this path.
    '''

    # Form boundary path
    boundary_path = [start_point]
    while boundary_path[-1] != end_point:
        new_point = end_point
        max_slope = float('-inf') if upper else float('inf')
        q = boundary_path[-1]

        for p in lattice_points:
            if p == boundary_path[-1] or p.x < q.x:
                continue

            try:
                slope = (p.y - q.y) / (p.x - q.x)
            except ZeroDivisionError:
                slope = float('inf') if p.y > q.y else float('-inf')

            if (slope > max_slope and upper) or (slope < max_slope and not upper):
                max_slope = slope
                new_point = p

        boundary_path.append(new_point)

    # Compute inner normal vectors
    sgn = 1 if upper else -1
    normal_vecs = []
    for i in range(len(boundary_path) - 1):
        u = boundary_path[i + 1] - boundary_path[i]
        n = (sgn * -u.y, sgn * u.x, sgn * (-u.y * boundary_path[i].x + u.x * boundary_path[i].y))

        if len(normal_vecs) > 0:
            if normal_vecs[-1][0] * n[1] == n[0] * normal_vecs[-1][1]:
                continue

        normal_vecs.append(n)

    # Add all lattice points on the boundary path
    new_path = []
    for i in range(len(boundary_path) - 1):
        u = boundary_path[i + 1] - boundary_path[i]
        d = gcd(u.x, u.y)
        new_path = new_path + [boundary_path[i] + j * u // d for j in range(d)]
```

```
    new_path.append(boundary_path[-1])

    return new_path, normal_vecs


class ConvexHull2D:
    '''
    Class of a 2D integral convex hull
    '''

    def __init__(self, lattice_points):
        '''Initialize convex hull

        Keyword arguments:
        lattice_points -- list of LatticePoint objects whose convex hull we take.
        '''

        lattice_points.sort() # Make sure the lattice points are lexicographically ordered
        self.lattice_points = lattice_points
        self.start_point = lattice_points[0]
        self.end_point = lattice_points[-1]

        # Compute upper and lower boundary paths of this polygon
        b1, n1 = get_boundary_path(lattice_points, self.start_point, self.end_point)
        b2, n2 = get_boundary_path(lattice_points, self.start_point, self.end_point, upper=False)

        self.boundary = [b1, b2]
        self.normal_vecs = n1 + n2

    def get_lattice_paths(self, length):
        '''
        Keyword arguments:
        length -- a positive integer denoting the length of the lattice path

        returns the set of all lambda-compatible lattice paths of given length,
            where lambda = (1, -epsilon) for epsilon > 0 small
        '''
        if length <= 0:
            return []

        paths = []
        for path in it.combinations(self.lattice_points[1:-1], length - 1):
            yield [self.lattice_points[0]] + list(path) + [self.lattice_points[-1]]

    def compress(self, lattice_path, mu=1, upper=True):
        if upper and len(lattice_path) < len(self.boundary[1]):
            return 0
        elif not upper and len(lattice_path) < len(self.boundary[0]):
            return 0

        # Find first vertex of lattice_path whose angle is positively oriented
        index = 0
        sgn = 1 if upper else -1
        for i in range(1, len(lattice_path) - 1):
            u, v = lattice_path[i + 1] - lattice_path[i], lattice_path[i - 1] - lattice_path[i]

            # Check if angle is correctly oriented
            if sgn * u.x * v.y < sgn * v.x * u.y:
                index = i
```

```
                break
        else:
            # Check if lattice path contains all points on the boundary
            index = 1 if upper else 0
            return mu if self.boundary[index] == lattice_path else 0

        # Add new compressions
        N = 0
        p = lattice_path[index - 1] + lattice_path[index + 1] - lattice_path[index]
        if p in self.lattice_points:
            new_path = lattice_path[:index] + [p] + lattice_path[index + 1:]
            N += self.compress(new_path, mu, upper)

        u = lattice_path[index + 1] - lattice_path[index - 1]
        # Compute area of the triangle added
        u = lattice_path[index] - lattice_path[index - 1]
        v = lattice_path[index] - lattice_path[index + 1]
        area = abs(u.x * v.y - v.x * u.y)
        N += self.compress(lattice_path[:index] + lattice_path[index + 1:], mu * area, upper)

        return N

    def get_path_multiplicity(self, path):
        '''
        :param path: a list of lattice points representing a lattice path
        :return: an integer representing the multiplicity of the lattice path
        '''
        N = self.compress(path)
        if N == 0:
            return 0
        return N * self.compress(path, upper=False)


class PlanePolytope(ConvexHull2D):
    '''
    Class of the polytope conv((0, 0), (d,0), (0, d)), where d is the degree
    '''

    def __init__(self, degree):
        lattice_points = []
        for x in range(degree + 1):
            for y in range(degree + 1 - x):
                lattice_points.append(LatticePoint(x, y))
        super().__init__(lattice_points)


class WeightedPolytope(ConvexHull2D):
    '''
    Class of the polytope conv((0,0), (d m, 0), (0, d)) for some positive integers d and m
    The associated toric variety is P(1, 1, m) and the divisor is dH for H some hyperplane section
    '''

    def __init__(self, weight, degree):
        lattice_points = []
        for x in range(weight * degree + 1):
            for y in range(degree + 1 - ceil(x / weight)):
                lattice_points.append(LatticePoint(x, y))
        super().__init__(lattice_points)


class ProductPolytope(ConvexHull2D):
```

```
    '''
    Class of the polytope conv((0,0), (d, 0), (0, e), (d, e)) for some positive integers d, e
    The associated toric variety is P^1 \times P^1 and the divisor is dH + eE for some non-parallel lines H and E
    '''

    def __init__(self, d, e):
        lattice_points = []
        for x in range(d + 1):
            for y in range(e + 1):
                lattice_points.append(LatticePoint(x, y))
        super().__init__(lattice_points)


class BlowUpPolytope(ConvexHull2D):
    '''
    Class of the polytope conv((0, 0), (d + e, 0), (0, d), (e, d)) for some positive integers d, e
    The associateed toric variety is the blow-up of P^2 in a point
    The divisor is dH + eE, for H the strict transform of a line in P^2 that does not intersect the blown-up point
    and E the exceptional divisor
    '''

    def __init__(self, d, e):
        lattice_points = []

        for x in range(d + 1):
            for y in range(e + 1):
                lattice_points.append(LatticePoint(x, y))

        for x in range(d + 1, d + e + 1):
            for y in range(d + e + 1 - x):
                lattice_points.append(LatticePoint(x, y))

        super().__init__(lattice_points)
```

C.3.3 Main

```python
import pandas as pd
import numpy as np
from time import perf_counter
from multiprocessing import Pool
from LatticePoint import *
from ConvexHull2D import *


def count_curves(polygon, delta, multi_processing=False):
    path_length = len(polygon.lattice_points) - 1 - delta
    paths = polygon.get_lattice_paths(path_length)

    if multi_processing:
        return sum(pool.map(polygon.get_path_multiplicity, paths))
    else:
        count = 0
        for path in paths:
            count += polygon.get_path_multiplicity(path)
        return count


if __name__ == "__main__":
    # Ask for input from user
    #weight = int(input("Weight: "))
    min_degree = int(input("Min degree: "))
    max_degree = int(input("Max degree: "))
    min_delta = int(input("Min delta invariant: "))
    max_delta = int(input("Max delta invariant: "))
    multi_processing = True if input("Multi processing (True or False): ") == "True" else False
    print("")

    if multi_processing:
        pool = Pool()

    # Start time of computations
    start_time = perf_counter()
    temp_time = perf_counter()

    table = [] # Table of Severi degrees
    for delta in range(min_delta, max_delta + 1): # Compute Severi degrees per delta invariant
        row = [] # Row of table

        for degree in range(min_degree, max_degree + 1): # Compute Severi degrees per degree
            row.append(count_curves(WeightedPolytope(2, degree), delta, multi_processing))

            dt = perf_counter() - temp_time

            # Percentage completed
            perc = 100 * (degree - min_degree + 1 + (max_degree - min_degree + 1) * (delta - min_delta)) / \
                    (max_delta - min_delta + 1) / (max_degree - min_degree + 1)

            print("Computed Severi degree for degree={0} and delta={1}. ".
                  format(degree, delta) + \
                 "Last computation took {0:.2e}s. Completed for {1:.2f}%.".
                  format(dt, perc))
            temp_time = perf_counter()

        table.append(row)
```

```

    # End time of computations
    finish_time = perf_counter()
    print("")
    print("Program finished in {0} seconds.\n".format(finish_time - start_time))

    # Output table of Severi degrees, as readable and LaTeX versions
    df = pd.DataFrame(table, columns=['\\(d={}\\)'.format(i) for i in range(
   min_degree, max_degree + 1)],
                       index=['\\(e={}\\)'.format(i) for i in range(min_delta,
   max_delta + 1)])
    print(df)
    print("")
    print(df.to_latex())
```

D Data

D.1 Severi degrees of the projective plane

The following table contains all Severi degrees $N^{d,\delta}$ of the projective plane for $0 \leq \delta \leq 5$ and $1 \leq d \leq 6$. Values above and to the right of the dashed line agree with the values of the node polynomials.

δ \ d	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1
1	0	3	12	27	48	75	108
2	0	0	21	225	882	2370	5175
3	0	0	15	675	7915	41310	145383
4	0	0	0	666	36975	437517	2667375
5	0	0	0	378	90027	2931831	33720354
6	0	0	0	105	109781	12597900	302280963
7	0	0	0	0	65949	34602705	1950179922

Figure 34: The Severi degrees $N^{d,\delta} = N^{(\mathbb{P}^2,L),\delta}$ of the plane for small degrees d and number of nodes δ.

D.2 Severi degrees of plane blown up in a point

Consider the blow-up $\pi \colon \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ of $\mathbb{P}^2$ in a point. Let d, e be two positive integers, F the pull-back of a line in $\mathbb{P}^2$ that passes the blown up point and E the exceptional divisor. The following data contain the Severi degrees $N^{(\widetilde{\mathbb{P}^2},(d+e)F+eE),\delta}$ for $0 \leq \delta \leq 5$ and $1 \leq d, e \leq 6$. Values below and to the right of the dashed line agree with the values of the node polynomials.

e \ d	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	1	1	1	1	1
4	1	1	1	1	1	1
5	1	1	1	1	1	1
6	1	1	1	1	1	1

(a) $\delta = 0$

e \ d	1	2	3	4	5	6
1	3	12	27	48	75	108
2	5	20	41	68	101	140
3	7	28	55	88	127	172
4	9	36	69	108	153	204
5	11	44	83	128	179	236
6	13	52	97	148	205	268

(b) $\delta = 1$

e \ d	1	2	3	4	5	6
1	0	21	225	882	2370	5175
2	6	105	615	1914	4488	8931
3	15	252	1200	3345	7281	13710
4	28	463	1981	5176	10750	19513
5	45	738	2958	7407	14895	26340
6	66	1077	4131	10038	19716	34191

(c) $\delta = 2$

e \ d	1	2	3	4	5	6
1	0	15	675	7915	41310	145383
2	0	159	4249	29091	115806	344287
3	10	860	13405	72268	249620	672820
4	35	2632	30904	145484	460393	1163848
5	84	5988	59491	256740	765702	1850140
6	165	11440	101910	414036	1183123	2764464

(d) $\delta = 3$

e \ d	1	2	3	4	5	6
1	0	0	666	36975	437517	2667375
2	0	126	14204	261945	1930305	8970509
3	0	1260	83057	983690	5716755	22782041
4	15	7038	287805	2671863	13455396	48563553
5	70	25208	747445	5955716	27260165	91821217
6	210	67611	1619366	11624450	49701891	159109658

(e) $\delta = 4$

e \ d	1	2	3	4	5
1	0	0	378	90027	2931831
2	0	45	22647	1448496	21895974
3	0	999	291612	8851035	92574315
4	0	10023	1661553	33835795	284757831
5	21	58044	6145719	97893432	714235476

(f) $\delta = 5$

Figure 35: Severi degrees $N^{(\widetilde{\mathbb{P}^2},(d+e)F+eE),\delta}$ for $0 \leq \delta \leq 5$ and $1 \leq d, e \leq 6$

D.3 Severi degrees of $\mathbb{P}^1 \times \mathbb{P}^1$

Let d, e be two positive integers, G the line $\{0\} \times \mathbb{P}^1$ and H the line $\mathbb{P}^1 \times \{0\}$. The following data contain the Severi degrees $N^{(\mathbb{P}^1 \times \mathbb{P}^1, dG+eH),\delta}$ for $0 \leq \delta \leq 5$ and $1 \leq d, e \leq 6$. Values below and to the right of the dashed line agree with the values of the node polynomials.

e \ d	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	1	1	1	1	1
4	1	1	1	1	1	1
5	1	1	1	1	1	1
6	1	1	1	1	1	1

(a) $\delta = 0$

e \ d	1	2	3	4	5	6
1	2	4	6	8	10	12
2	4	12	20	28	36	44
3	6	20	34	48	62	76
4	8	28	48	68	88	108
5	10	36	62	88	114	140
6	12	44	76	108	140	172

(b) $\delta = 1$

e \ d	1	2	3	4	5	6
1	0	3	10	21	36	55
2	3	22	105	252	463	738
3	10	105	396	883	1566	2445
4	21	252	883	1914	3345	5176
5	36	463	1566	3345	5800	8931
6	55	738	2445	5176	8931	13710

(c) $\delta = 2$

e \ d	1	2	3	4	5	6
1	0	0	4	20	56	120
2	0	20	160	860	2632	5988
3	4	160	1944	7956	20940	43640
4	20	860	7956	29092	72268	145484
5	56	2632	20940	72268	174192	344288
6	120	5988	43640	145484	344288	672820

(d) $\delta = 3$

e \ d	1	2	3	4	5	6
1	0	0	0	5	35	126
2	0	6	133	1261	7038	25208
3	0	133	4115	37590	161906	476827
4	5	1261	37590	262000	983691	2671863
5	35	7038	161906	983691	3443805	8970636
6	126	25208	476827	2671863	8970636	22782042

(e) $\delta = 4$

e \ d	1	2	3	4	5	6
1	0	0	0	0	6	56
2	0	0	60	1008	10024	58044
3	0	60	3702	93912	746580	3328596
4	0	1008	93912	1449696	8851104	33835796
5	6	10024	746580	8851104	47348298	167458656
6	56	58044	3328596	33835796	167458656	565408920

(f) $\delta = 5$

Figure 36: Severi degrees $N^{(\mathbb{P}^1\times\mathbb{P}^1, dG+eH),\delta}$ for $0 \leq \delta \leq 5$ and $1 \leq d, e \leq 6$

D.4 Severi degrees of $\mathbb{P}(1,1,2)$

Let H be a plane section of $\mathbb{P}(1,1,2)$. The following table contains all Severi degrees $N^{(\mathbb{P}(1,1,2),dH),\delta}$ for $0 \leq \delta \leq 5$ and $1 \leq d \leq 5$. Values above and to the right of the dashed line agree with the values of the node polynomials.

δ \ d	1	2	3	4	5	6
0	1	1	1	1	1	1
1	0	10	32	66	112	170
2	0	10	340	1790	5584	13378
3	0	0	1440	25960	163840	647344
4	0	0	2397	220122	3152839	21581780
5	0	0	1200	1125972	42004752	526510120

Figure 37: Severi degrees $N^{(\mathbb{P}(1,1,2),dH),\delta}$ for $0 \leq \delta \leq 5$ and $1 \leq d \leq 5$

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