Utrecht University
Department of Physics

Theoretical Physics master thesis

D-brane gauge theories with spontaneous supersymmetry breaking through freely acting orbifolds

Supervisor:
Stefan Vandoren

Candidate:
Hugo Calvo Castro

Second examiner:
Thomas Grimm

Co-Supervisor:
George Gkountoumis

July 22, 2024

Acknowledgements

I want to thank Stefan Vandoren for the great effort that was supervising my master thesis. His guidance led me to understanding how to learn and make science the right way and will keep being invaluable for the rest of my career. Major thanks also go to George Gkountoumis, who sorted my doubts and made this thesis possible through extensive feedback on my drafts.

I also want to thank Raúl, my friend and fellow thesis student, who listened to all my frustrations and complaints very diligently, and gave me confidence in the moments I needed it most.

Finally, I want to thank my family and friends for supporting my academic life, pushing my limits and always believing in me.

Abstract

In the context of String Theory, orbifold compactification has proven to be an effective method for breaking supersymmetry (SUSY) [1]. Specifically, freely acting orbifolds can be tied to spontaneous SUSY breaking, and the description on the closed string sector has been thoroughly studied [2]. This thesis aims to explore the effects of freely acting orbifold compactification in the open string spectrum. We first show that the orbifold compactification works as a projection on the spectrum of the non-orbifolded theory, where projection makes the spectrum invariant under the orbifold group action. Then, we link this result to a Scherk-Schwarz dimensional reduction that shifts the masses of all charged fields in accordance with the projected spectrum.

The main examples will be a D9-brane, for its simplicity, and the D1/D5 brane system. This system is closely linked to black hole solutions of the low energy supergravity, and in the last section we give predictions as to how the orbifold projection acts on the low energy worldvolume CFT and thus the black hole theormodynamics in the system with broken supersymmetry.

Contents

1. Introduction

Physics aims to describe the dynamics between all the fundamental constituents of nature. In the one hand, we can use Quantum Field Theory to describe Particle Physics, and in the other we can use General Relativity to describe gravity. These two theories are fundamentally different in the sense that the first is a quantum theory, while the second one is not, and the most naive attempts to convert it to quantum language fail fundamentally.

String Theory is a paradigm change to the way Particle Physics is built, because the fundamental objects are no longer point-like, but extended one dimensional *strings*. This theory has provided a rich framework which has helped understand very complex physical systems, i.e., quark-gluon plasma [3], black holes [4], but the most notable result might be that it is a theory of Quantum Gravity [5]. Thus, being a promising candidate for a unifying theory of physics.

Bosonic String Theory has a tachyonic vacuum, implying an instability in the theory, and also does not predict fermionic particles. These two features can be solved by considering Superstring Theory, adding fermionic excitations to the world-sheet of the strings. This theory has two features that make conflict with the physics we can observe at current experiments. Firstly, the theory predicts supersymmetry in space-time, but the spectrum of the Standard Model is not supersymmetric. Secondly, the theory is defined in 10 dimensions for internal consistency, but we have only ever observed 4.

These two problems can be approached at the same time through a technique called orbifold compactification. We will consider compactifications of type IIB String Theory via freely acting toroidal orbifolds of the type $(S^1 \times T^4)/Z_p$. This construction will spontaneously break supersymmetry, meaning that some particles of the spectrum acquire a mass, thus breaking the original supermultiplet they belonged to.

The effect of freely acting toroidal orbifolds in type IIB String Theory with a focus on the closed string spectrum has been studied in the past [2]. The orbifold group action charges some fields, and in the low energy effective SUGRA this charge can be identified with a duality twist in a Scherk-Schwarz (SS) dimensional reduction that gives masses to the fields proportional to their charges.

In this thesis we will focus on the open string spectrum, which has not yet beef fully understood in orbifold backgrounds. This work will be in close contact with the developments in the closed string sector. First, we will identify the charges of all relevant particles according to their representations under the local symmetry group of the orbifold. This process will define a projection in the spectrum because ultimately the orbifold group has to be gauged. Then, we will propose a SS dimensional reduction, following the orbifold charges of the fields that will grant them mass in agreement with the spectrum.

The motivation for this thesis comes from the original black hole found in String Theory, which is described microscopically by a D1/D5 brane system [4]. The system was first proposed in a flat background String Theory compactified on a T^5, but we would like to extend the results to more realistic String Theories with less supersymmetry. The first step towards this goal is to understand the high energy dynamics of D-brane world-volumes in orbifold backgrounds, which will be achieved in 3 and 5.

To properly understand the thermodynamics of black holes coming from D-brane systems, one has to be able to take the infrared (IR) limit of the respective world-volume gauge theory and extract the central charge of the resulting CFT. This step will be left for future research.

1.1 Outline

This thesis is organized as follows. In Chapter 2 we briefly describe general aspects of String Theory relevant for developing the latter calculations. In

Chapter 3 we describe the massless spectrum of the open string sector of type IIB String Theory from a group theoretical point of view. We use this spectrum to find the orbifold charges and projection in 4.

Lastly, in Chapter 5 we give a dynamical description to the spectrum found in previous chapters, and understand the orbifold projection as a SS dimensional reduction.

1.2 Conventions

The Minkowski metric in any dimension is mostly positive, i.e., $\eta_{\mu\nu} = \text{diag}\{-1, 1, \ldots, 1\}$.

Representations of $SO(N)$ or $SO(1, N-1)$ for $N \in \mathbb{N}$ will be labeled by the number of components (highest weight) in boldface numbers, i.e., vector representation $\mathbf{N}$, Dirac spinor $\mathbf{2}^{\lfloor \mathbf{N/2} \rfloor}$, etc.

In the case of Weyl spinors, both chiralities have the same number of components, so we will denote $+$ chirality with an s subscript, and $-$ with a c subscript.

Often times, we will need to discuss representations of direct products of Lie groups of the type $SO(1, N-1) \otimes SO(N')$. In that case, the irreducible representations will be labeled by irreducible representations (irreps) of the individual products in parentheses in the same order as the direct product, i.e., $(\mathbf{N}, \mathbf{N'})$, $(\mathbf{2^{N/2-1}}_s, \mathbf{2^{N'/2-1}}_c)$, etc.

2. Preliminaries

In this chapter we will present some basic concepts of String Theory to give a foundation to this thesis. Only the necessary steps will be presented in order to give context for future chapters. An extensive review on Superstring Theory is contained in [6], [7].

2.1 Type IIB String Theory

String Theory is, in its most simple realization, one of the most straightforward generalizations of the quantum theory of Particle Physics. The fundamental object is a 2 dimensional object that sweeps a world-sheet instead of a 1 dimensional one that traces a world-line. This world-sheet is parametrized by coordinates $\sigma^\alpha = (\sigma, \tau)$, through the embeddings $X^\mu(\sigma, \tau)$, $\mu = 0, \ldots, 9$.

Superstring Theory, as the name implies, also has fermionic degrees of freedom, ψ^μ and $\tilde{\psi}^\mu$, that give rise to world-sheet SUSY. This symmetry, as it turns out, gives rise to space-time SUSY when the String Theory is treated carefully. This procedure is known as the GSO projection, and will be an integral part in future chapters.

Consider the following action [8],

$$S = \frac{1}{4\pi} \int_{\mathcal{M}} d^2\sigma \left\{ \frac{1}{\alpha'} \partial^\alpha X^\mu \partial_\alpha X_\mu + i\psi^\mu (\partial_\tau - \partial_\sigma) \psi_\mu + i\tilde{\psi}^\mu (\partial_\tau + \partial_\sigma) \tilde{\psi}_\mu \right\}, \quad (2.1)$$

with $\mathcal{M}$ is $\sigma \in [0, 2\pi)$ and $\tau \in (-\infty, \infty)$.

We can start by solving the classical equations of motion, which can be read as $\partial^\alpha \partial_\alpha X^\mu = (\partial_\tau - \partial_\sigma)\psi^\mu = (\partial_\tau + \partial_\sigma)\tilde{\psi}^\mu = 0$. These imply the fields can be written in terms of right moving and left moving functions,

$$
\begin{aligned}
X^\mu &= X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma),\\
\psi^\mu &= \psi^\mu(\tau+\sigma),\\
\tilde{\psi}^\mu &= \tilde{\psi}^\mu(\tau-\sigma).
\end{aligned}
$$

To find a mode expansion we have to impose boundary conditions for these fields. In order to find the closed string spectrum, we impose periodicity conditions. For the bosonic fields these are $X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)$, while the fermions can close up to a $\pm$ sign. This allows for two sectors in the spectrum, called Rammond (R) and Neveu-Schwarz (NS), given by the periodicity conditions,

$$
\begin{aligned}
\text{R} : \psi^\mu(\tau,\sigma+2\pi) &= +\psi^\mu(\tau,\sigma),\\
\text{NS} : \psi^\mu(\tau,\sigma+2\pi) &= -\psi^\mu(\tau,\sigma),
\end{aligned}
$$

and the same for $\tilde{\psi}^\mu$. All fields can then be expressed in terms of Fourier modes. The bosonic modes will be called a^μ_n, $\tilde{a}^\mu_n$, $n \in \mathbb{Z}$, while the fermionic modes are b^μ_r, $\tilde{b}^\mu_r$, with $r \in \mathbb{Z}$ in the R sector, or $r \in \mathbb{Z}+1/2$ in the NS sector. Let us write the full expansion for the bosonic fields for future reference,

$$
\begin{aligned}
X^\mu_L(\tau+\sigma) &= \frac{x^\mu}{2} + \frac{\alpha' p^\mu}{2}(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{a^\mu_n}{n}e^{-in(\tau+\sigma)}\\
X^\mu_R(\tau-\sigma) &= \frac{x^\mu}{2} - \frac{\alpha' p^\mu}{2}(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{a}^\mu_n}{n}e^{-in(\tau-\sigma)}
\end{aligned}
\tag{2.2}
$$

Focusing on the left movers, we identify the NS vacuum to be a space-time scalar $|0\rangle_{NS}$, while the R vacuum is degenerate under the action of b^μ_0. The zero-modes of the R sector generate the $D = 10$ Clifford algebra,

$$
\{b^\mu_0, b^\nu_0\} = \eta^{\mu\nu}, \tag{2.3}
$$

so the R vacuum is a priori a Dirac spinor, $|a\rangle_R$. This spinor can be characterized by the eigenvalues of the Cartan subalgebra generators[1] $|s_0, s_1, s_2, s_3, s_4\rangle$,

[1] We identify the gamma matrices as $\Gamma^\mu = \sqrt{2}b^\mu_0$, and s_i are eigenvalues of the Cartan

$s_i = \pm 1/2$, spanning a 32 dimensional complex space. This is a Dirac spinor **32** of $SO(1,9)$.

The closed string spectrum will consist of both the right and left moving spectrum along with the level matching condition. This can be summarized in the mass-shell formula of a string,

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n,r} a_{-n}\cdot a_n + r b_{-r}\cdot b_r - a\right) = \frac{1}{\alpha'}\left(\sum_{n,r} \tilde{a}_{-n}\cdot \tilde{a}_n + r \tilde{b}_{-r}\cdot \tilde{b}_r - a\right), \tag{2.4}$$

with $a = -1/2$ in the NS sector and $a = 0$ in the R sector. Notice that in the NS sector the zero-point energy is negative. This means that the NS vacuum $|0\rangle_{NS}$ has $M^2 = -1/2\alpha'$, so it is a tachyonic state. Having a tachyon in the spectrum leads to instabilities in the vacuum, so in an attempt to remove it, and fix many other issues[2], we introduce the GSO projection. This projection keeps states with an odd number of fermions. In the NS sector this can be achieved by the following operators,

$$\begin{gathered} P_{GSO} = \frac{1}{2}\left(1-(-1)^F\right), \quad F = \sum_{r>0} b_{-r}\cdot b_r, \\ \left\{(-1)^F, b_r^\mu\right\} = 0, \quad r \in \mathbb{Z}+1/2. \end{gathered} \tag{2.5}$$

Notice that the operator $(-1)^F$ has eigenvalue $+1$ when acting on a state with an even number of fermionic excitations, and -1 on states with odd fermionic excitations. Then, the vacuum $|0\rangle_{NS}$, has $(-1)^F|0\rangle_{NS} = |0\rangle_{NS}$, so it is projected out of the spectrum. Moreover, all states with an even number of fermion excitations, i.e. $F \in 2\mathbb{Z}$, are projected out. In particular, the massless vector $b^\mu_{-1/2}|0\rangle_{NS}$ survives.

The R sector is a bit more complicated since the vacuum is a chiral space-time spinor. We need to generalize the operator $(-1)^F$ to $\Gamma_{11}(-1)^F$, where $\Gamma_{11} = -i\Gamma_0 \ldots \Gamma_9$ is the 10D chirality matrix, in order to preserve the anti-commutation with all fermionic modes. Thus, in the R sector the projection

subalgebra generators $S_i = -i/4[\Gamma^{2i}, \Gamma^{2i+1}]$. A full derivation is contained in Appendix A.

[2]Modular invariant partition function, space-time supersymmetry, etc.

is defined as follows,

$$P_{GSO} = \frac{1}{2}\left(1 \pm \Gamma_{11}(-1)^F\right), \quad F = \sum_{r>0} b_{-r} \cdot b_r, \\ \left\{\Gamma_{11}(-1)^F, b_r^\mu\right\} = 0, \quad r \in \mathbb{Z}. \tag{2.6}$$

In this case, the projection chooses a chirality for the R vacuum. Precisely, the projection is equivalent to the equation $\Gamma_{11}(-1)^F|a\rangle_R = \pm|a\rangle_R$. Since there are no fermionic excitations on this vacuum, $F = 0$, and the condition reduces to $\Gamma_{11}|a\rangle_R = \pm|a\rangle_R$, meaning that $|a\rangle_R$ has to be a chiral spinor in 10D. Choosing the $+$ or $-$ in the projection leads to different chiralities of the R vacuum. The only physically meaningful difference comes when there is a relative sign difference between the left and right moving vacuum.

In the end, the full spectrum of the closed string is the product of the right and left moving spectrum, which have independent GSO projections. Since the physical differences come only from the relative sign, we can fix the left moving R vacuum to be $+$ chiral, and we arrive at the two string theories,

$$\text{IIA: } (NS, NS), (NS, R-), (R+, NS), (R+, R-) \\ \text{IIB: } (NS, NS), (NS, R+), (R+, NS), (R+, R+) \tag{2.7}$$

Where we denote the sectors by (left, right), their boundary conditions, and the chirality coming from the GSO projection.

In the massless sector we will then have all the massless states corresponding to the different sectors outlined in 2.7,

$$\begin{aligned} &b^\mu_{-1/2}|0\rangle_{NS} \otimes \tilde{b}^\mu_{-1/2}|0\rangle_{NS}, \\ &b^\mu_{-1/2}|0\rangle_{NS} \otimes |\tilde{a}\rangle_R, \\ &|a\rangle_R \otimes \tilde{b}^\mu_{-1/2}|0\rangle_{NS}, \\ &|a\rangle_R \otimes |\tilde{a}\rangle_R, \end{aligned}$$

where $|a\rangle_R$ and $|\tilde{a}\rangle_R$ are the left and right moving R vacua, with no special notation for the chiralities, which will depend on the GSO projection we choose.

Since applying the GSO projection removed the tachyon from the spectrum, these are the lowest energy excitations of the closed string of type IIB. All these states belong to irreducible representations of $SO(1,9) \otimes SO(1,9)$, but it is common to state the spectrum in lightcone gauge. In this gauge we loose vector degrees of freedom corresponding to $\mu = 0, 1$, and the fermions satisfy the massless Dirac equation $p^\mu \Gamma_\mu |a\rangle_R = 0$. Choosing $p^\mu = (-E, E, 0, \ldots)$, the equation reduces to $\Gamma_0^- |a\rangle_R = 0$[3], which amounts to $s_0 = -1/2$. The remaining spinor is spanned by $|-1/2, s_1, s_2, s_3, s_4\rangle$ and has half the degrees of freedom of the original spinor. The right movers also follow the same rules.

To summarize, going to lightcone gauge can be seen as going to the little group $SO(8) \subset SO(1,9)$, where the vectors loose 2 degrees of freedom $\mathbf{10} \to \mathbf{8}$, and the spinors retain their chirality while loosing half the components $\mathbf{16}_s \to \mathbf{8}_s$. In the closed string spectrum, lightcone gauge affects both right and left movers, and we can classify its particles as irreducible representations of the little group $SO(8) \otimes SO(8)$,

$$(\mathbf{8}_v, \mathbf{8}_v) \oplus (\mathbf{8}_v, \mathbf{8}_s) \oplus (\mathbf{8}_s, \mathbf{8}_v) \oplus (\mathbf{8}_s, \mathbf{8}_s).$$

Notice how the number of bosonic degrees of freedom is the same as the fermionic ones in this gauge. This check is a hint that the theory is space-time supersymmetric, and indeed if one calculates the action describing these degrees of freedom, one would find a supersymmetric theory.

2.2 D-branes

In the last section, we used periodic boundary conditions to describe a closed string propagating through space-time. But this choice could be extended to non-periodic boundary conditions. Strings can indeed end on hypersurfaces that are called D-branes.

[3]The operators $\Gamma_i^\pm$ are defined in Appendix A. They raise or lower the eigenvalue s_i of the spinor $|a\rangle_R$.

Consider the bosonic fields of the world-sheet action 2.1. Instead of imposing periodic boundary conditions, we can instead consider the string endpoints fixed on a surface of dimension p, so that $\partial_\sigma X^a(\sigma,\tau)|_{\sigma=0,\pi} = 0$, $\partial_\tau X^i(\sigma,\tau)|_{\sigma=0,\pi} = 0$, with $a = 0,...p$, $i = p+1,...,D$. In this case, the endpoints of the string are free to move in the direction a, which are the Neumann conditions, but are fixed in the i directions, the Dirichlet conditions.

It is crucial to note that the symmetry group of a Dp-brane is no longer the full 10D Lorentz group, but is broken to $SO(1,p) \otimes SO(9-p) \subset SO(1,9)$. This happens because rotations between the N and D directions rotate the brane, thus describiing a different system.

In order to produce the open string spectrum, we start by considering the mode expansion 2.2. If we add the right and left movers, we get

$$X^\mu = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\left(a_n^\mu e^{-in(\tau+\sigma)} + \tilde{a}_n^\mu e^{-in(\tau-\sigma)} \right). \tag{2.8}$$

In the Neumann direction we will have the following condition,

$$\partial_\sigma X^a|_{\sigma=0,\pi} = \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} (a_n^a - \tilde{a}_n^a) = 0, \tag{2.9}$$

which imposes the condition on the modes $a_n^\mu = \tilde{a}_n^\mu$. We can do the same for the Dirichlet directions,

$$\partial_\tau X^i|_{\sigma=0,\pi} = \alpha' p^i + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} (a_n^i + \tilde{a}_n^i) = 0, \tag{2.10}$$

and this is solved by $p^i = 0$, $a_n^i = -\tilde{a}_n^i$. In the end, we can write the full mode expansion for the bosonic fields as,

$$\begin{aligned} \text{D: } X^a &= x^a + \alpha' p^a \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{a_n^a}{n} e^{-i\tau} cos(n\sigma), \\ \text{N: } X^i &= x^i + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{a_n^i}{n} e^{-i\tau} sin(n\sigma). \end{aligned} \tag{2.11}$$

The conclusion is that in the open string there is no such thing as right or left movers. Instead, we have just oscillator modes a_n^a and a_n^i. The process for the fermionic fields is equivalent, and we end up with mode expansions in terms of b_r^a and b_r^i, with $r \in \mathbb{N} + 1/2$ in the NS sector or $r \in \mathbb{N}$ in the R sector.

The construction of the vacua of the open string follows the same steps as in the closed string, leading to the NS scalar vacuum $|0\rangle_{NS}$. The R vacuum has a new feature, because it cannot be a $SO(1,9)$ spinor, since there open string does not in general have $SO(1,9)$ symmetry (unless we only consider the D9 brane). The Clifford algebra 2.3 splits,

$$\begin{aligned} \{b_0^a, b_0^b\} &= \eta^{ab}, \\ \{b_0^i, b_0^j\} &= \delta^{ij}, \end{aligned} \tag{2.12}$$

and this generates the Clifford algebras of $SO(1,p)$ and $SO(9-p)$ independently. The R vacuum is formed by independent spinors $|a_N\rangle$ in the Neumann directions and $|a_D\rangle$ in the Dirichlet directions. The full R vacuum is the tensor product of these two spinors $|a\rangle_R = |a_N\rangle \otimes |a_D\rangle$. The GSO projection in the R vacuum is the same we defined in 2.6, $\Gamma_{11}|a\rangle_R = |a\rangle_R$. The chirality operator splits into subgroup chirality operators as $\Gamma_{11} = \Gamma_c^p \Gamma_c^{9-p}$, which are respectively chirality operators of $SO(1,p)$ and $SO(9-p)$. Thus, the $+$ chirality of the 10D spinor splits into $(+,+)$ and $(-,-)$ chiralities for the $SO(1,p) \otimes SO(9-p)$ spinors.

The excitations of the open string can also carry a mass, following the same mass formula as the closed string 2.4. The detailed calculation of the massless spectrum will be postponed until Chapter 3. We will construct it from the NS and R vacua of the open string with special focus on the irreps of the broken Lorentz group they belong to.

2.3 Supersymmetry

Supersymmetry is a symmetry of systems that exhibit a transformation between bosons and fermions that leaves the theory invariant. As a global symmetry, the existence of a conserved charge, called supercharge, follows from Noether's theorem. The most basic example of supersymmetry can be found in a 2D theory with a massless Majorana-Weyl[4] (MW) spinor and a scalar,

$$S = \int d^2z \left(\partial\phi\bar{\partial}\phi + \psi\partial\psi\right). \tag{2.13}$$

This theory is invariant (on-shell) under the spinor valued transformation,

$$\begin{aligned} \delta_\epsilon\phi &= \bar{\epsilon}\psi \\ \delta_\epsilon\psi &= \epsilon\bar{\partial}\phi \end{aligned} \tag{2.14}$$

In this case, there will be only one supercharge Q in the same representation as the spinor field ψ following the algebra $\{Q, \overline{Q}\} = 2P$, where P is the generator of translations. This illustrative example can be generalized to the case of D dimensions and $\mathcal{N}$ supercharges. These supercharges will follow the algebra,

$$\{Q_i, \overline{Q}_j\} = 2\delta_{ij}\Gamma^\mu P_\mu, \tag{2.15}$$

where $\overline{Q}_j = Q_j^\dagger\Gamma^0$, and Γ^μ are the gamma matrices suitable to the representation of Q_i, and $i, j = 1, \ldots, \mathcal{N}$

The number of components carried by Q_i represents the amount of conserved supercharges of the theory. Assuming these supercharges are Weyl spinors, and $D = 2\nu$, each Q_i would have 2^ν real components, for a total of $2^\nu\mathcal{N}$ supercharges.

There is an extra internal symmetry between the supercharges called R-symmetry. It is given by the index i of the supercharges. In our case, we will make a slight abuse and say that the R-symmetry is $SO(n)$, while in reality it is its double cover $Spin(n)$, and the indices of supercharges in $Spin(n)$ will be spinor indices of the spin group.

Now imagine that we want to write a theory containing a massless vec-

[4]In $D = 2$ and $D = 10$ we can impose both a Majorana and Weyl condition in order to have effectively real Weyl spinors. Details can be found in Appendix A

tor and a massless fermion. Massless vectors in $D = 2\nu$ dimensions have $D - 2$ degrees of freedom, while Weyl fermions have $2^{\nu-1}$. In $D = 10$ it turns out that fermions can be Majorana and Weyl at the same time, so the number of components gets further reduced to $2^{\nu-2}$. Equating these quantities, we find that $\mathcal{N} = 1$ supersymmetry without extra boson fields can only be realized in $D = 10$, and for $D > 10$, we would need spin 2 fields to be able to match degrees of freedom. This is what we will call maximal supersymmetry, the case where only one supercharge pairs a fermion to a boson of a particular spin value supersymmetrically.

Maximally supersymmetric gauge theories will then have 16 supercharges and in this thesis will all come from dimensional reduction of $D = 10$ $\mathcal{N} = 1$ Super Yang-Mills (SYM). Maximal supergravity has 32 supercharges and come naturally from $D = 11$ $\mathcal{N} = 1$ supergravity.

For example, Superstring Theory in a Minkowski background has 2 supercharges Q_+, Q_-, which are MW spinors in $D = 10$ of opposite chiralities. Each of them has 16 real components, for a total of 32 supercharges.

3. Open string spectrum

In this chapter we will describe the spectrum of D-brane systems in the context of type IIB String Theory. In type IIB only odd p branes exist. We start by discussing single brane spectra, and then move on to general brane configurations. We conclude with a D9-brane and the D1/D5 systems as examples.

3.1 Dp-Dp spectrum

Consider a single D9-brane, i.e., strings with Neumann boundary conditions in every direction. The NS vacuum is a scalar $|0\rangle := |0\rangle_{NS}$, while the R vacuum is a 10D MW spinor $|a\rangle := |a\rangle_R$ of $+$ chirality. The massless spectrum can be found from the mass formula 2.4 and setting $M^2 = 0$. This leads to the modes $b^{\mu}_{-1/2}|0\rangle$ and $|a\rangle$, which follow the $\mathbf{10}$ and $\mathbf{16}_s$ of $SO(1,9)$.

As we discussed in the last chapter, a Dp-brane is not symmetric under the full 10D Lorentz group (unless $p = 9$). This breaks the Lorentz group $SO(1,9) \to SO(1,p) \otimes SO(9-p)$. In terms of the representation theory of the spectrum, in the closed string we found that the states arranged themselves into representations of the Lorentz group $SO(1,9)$, but in a Dp-brane, they will be arranged into representations of $SO(1,p) \otimes SO(9-p)$.

The D9 brane spectrum contains a 10D vector and MW spinor. The rest of the single Dp-brane spectra can be computed from the D9 spectrum by dimensional reduction. For the vector, the index $\mu = 0,...,9$ splits into two indices $a = 0,...,p$ and $i = p+1,...,9$, so that a vector $b^{\mu}_{-1/2}|0\rangle \to (b^{a}_{-1/2}|0\rangle,\ b^{i}_{-1/2}|0\rangle)$. The components with an a index are vectors of $SO(1,p)$, but are singlets under $SO(9-p)$, while components with an i index are the inverse. These fall into the $(\mathbf{p+1},\mathbf{1})$ and $(\mathbf{1},\mathbf{9-p})$ respectively.

The dimensional reduction of spinor representations is outlined in detail in Appendix A. In short, the basis for the D9 R vacuum can be written as

$|s_0, s_1, s_2, s_3, s_4\rangle$ with the chirality condition $2^5 \prod s_i = +1$. Upon dimensionally reducing, the index i splits leading to $|s_0, \ldots, s_{(p+1)/2-1}\rangle \otimes |s_{(p+1)/2}, \ldots, s_4\rangle$. The chirality condition for the 10D spinor splits into independent conditions for the $(p+1)$D spinor and the $(9-p)$D, effectively splitting the $+$ chirality into $(+,+)$ and $(-,-)$. The dimensionally reduced spinors fall into the $(\mathbf{2^{(p+1)/2-1}}_s, \mathbf{2^{(9-p)/2-1}}_s)$ and $(\mathbf{2^{(p+1)/2-1}}_c, \mathbf{2^{(9-p)/2-1}}_c)$ of $SO(1,p) \otimes SO(9-p)$.

These expressions are a bit unwieldy, so we will specify the results to the case of a dimensional reduction of the D9 spectrum to the D5 spectrum,

$$\begin{aligned} SO(1,9) &\to SO(1,5) \otimes SO(4), \\ \mathbf{10} &\to (\mathbf{6},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4}), \\ \mathbf{16}_s &\to (\mathbf{4}_s, \mathbf{2}_s) \oplus (\mathbf{4}_c, \mathbf{2}_c). \end{aligned} \tag{3.1}$$

In terms of a 6D theory with $SO(1,p)$ Lorentz group and $SO(4)$ internal symmetry, the 10D vector reduced to a 6D vector, and 4 scalar. The 10D $+$ chiral spinor reduced to 2 $+$ and 2 $-$ chiral 6D spinors.

This concludes the discussion of single Dp-branes, but this analysis can be generalized to a stack of branes in a straightforward manner. Stacking several branes on top of each other adds a Chan-Paton factor to both string endpoints, which labels the adjoint representation of $U(N_p)$, and can be seen as a gauge symmetry. Thus, the full spectrum is classified under the group, schematically $SO(1,p) \otimes SO(9-p) \otimes U(N_p)$.

Lastly, we can count on-shell degrees of freedom to make sure the spectrum is supersymmetric as a sanity check. Let us consider a single D5-brane, for which we already know the spectrum from 3.1. By on-shell we refer to adopting light-cone gauge, meaning that the Lorentz group reduces to the little group. For massless particles this is $SO(1,5) \to SO(4)$, effectively identifying 2 degrees of freedom for the vector, and specifying a s_0 eigenvalue for the spinor, as we saw in Section 2.1. Scalars have the same on-shell and off-shell degrees of freedom, thus,

$$\begin{aligned}(\mathbf{6},\mathbf{1}) &\to (\mathbf{4},\mathbf{1}),\\ (\mathbf{1},\mathbf{4}) &\to (\mathbf{1},\mathbf{4}),\\ (\mathbf{4}_s,\mathbf{2}_s) &\to (\mathbf{2}_s,\mathbf{2}_s),\\ (\mathbf{4}_c,\mathbf{2}_c) &\to (\mathbf{2}_c,\mathbf{2}_c).\end{aligned}$$

This spectrum will be on-shell supersymmetric if the number of bosonic degrees of freedom are equal to the number of fermionic degrees of freedom. We count 4 from each the vector and scalar, and 2 from each of the 4 fermions, for a total of 8 bosonic and 8 fermionic degrees of freedom. Indeed, the spectrum is still supersymmetric.

3.2 Dp-Dp′ strings

Now consider two D-branes of dimensions p and p', and a string stretching from one brane to the other. In general, some directions will have the pure boundary conditions, i.e. DD or NN, while some will have mixed boundary conditions, i.e. DN or ND. This is illustrated in Figure 3.1.

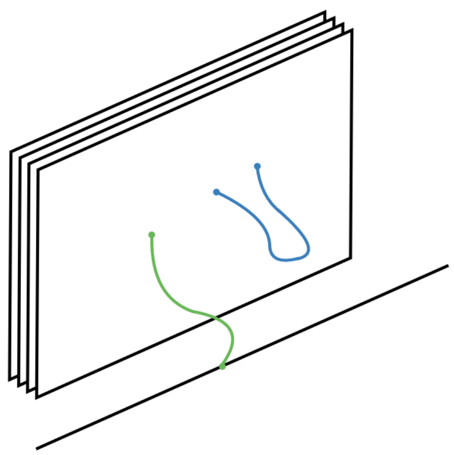

Figure 3.1: Schematic representation of a stack of Dp branes and a single Dp′ brane with strings attached. The blue string will have the same boundary conditions in all directions while the green string will have mixed conditions in some. Figure taken from [9].

We already know what happens in the directions where there are pure boundary conditions. In the DD and NN directions, we will have an expansion in Fourier modes equivalent to 2.11. In the case of mixed boundary conditions, an interesting phenomenon happens. Consider a bosonic degree of freedom, and let us omit the space-time index for brevity. Assume that it has N boundary conditions in one endpoint and D in the other. This is,

$$\begin{aligned} &\text{N: } \partial_\sigma X(\tau,\sigma)|_{\sigma=0} = 0, \\ &\text{D: } \partial_\tau X(\tau,\sigma)|_{\sigma=\pi} = 0. \end{aligned} \tag{3.2}$$

We again start from the Fourier mode expansion of the closed string 2.8, and the boundary conditions read,

$$\begin{aligned} &\text{N: } \sqrt{\frac{\alpha'}{2}} \sum_r (a_r - \tilde{a}_r) = 0, \\ &\text{D: } \alpha' p + \sqrt{\frac{\alpha'}{2}} \sum_r \left(a_r e^{-ir\pi} + \tilde{a}_r e^{ir\pi} \right) = 0. \end{aligned} \tag{3.3}$$

First, from the D condition we see that in mixed directions, the strings cannot carry momentum. The solution to the first equation is the usual N condition $a_r = \tilde{a}_r$. If we apply this condition to the D condition, we arrive at $cos(r\pi) = 0$, which can only be solved if $r \in \mathbb{Z} + 1/2$. So for a bosonic coordinate, the mixed boundary conditions shift the modes by 1/2. The boundary conditions modify the fermionic modes in the same way, so b_r will be $r \in \mathbb{Z}$ in the NS sector and $r \in \mathbb{Z} + 1/2$ in the R sector.

In the end we are most interested in the massless spectrum. The mass formula for any excitation of the string will still be given by 2.4, but with a modified normal ordering constant a. Let us consider the case of ν mixed directions, and $10 - \nu$ pure directions. The zero point contributions from each type of field are [10], where the X row represents the 0 point contribu-

	DD/NN	DN/ND
X	-1/24	1/48
NS	-1/48	1/24
R	1/24	-1/48

tions from bosonic degrees of freedom in directions with either DD/NN or DN/ND conditions. The NS and R rows represent the analogous quantities for fermions of those sectors.

To calculate the zero point energies, we need to fix a gauge. If we go to lightcone gauge using 2 of the NN directions, we would have $8 - \nu$ DD/NN contributions and ν DN/ND contributions. With all this in mind, the calculation of a is straightforward for both the R and NS sector. For the R sector,

$$a_R = \left(-\frac{1}{24} + \frac{1}{24}\right)(8-\nu) + \left(\frac{1}{48} - \frac{1}{48}\right)\nu = 0, \tag{3.4}$$

while for the NS sector,

$$a_{NS} = \left(-\frac{1}{24} - \frac{1}{48}\right)(8-\nu) + \left(\frac{1}{48} + \frac{1}{24}\right)\nu = -\frac{1}{2} + \frac{\nu}{8}. \tag{3.5}$$

We see that there is a special case for $\nu = 4$. In that case, both zero point energies will vanish $a_R = a_{NS} = 0$. Thus, both vacuums are massless, and any excitation of the vacuum will have a positive mass.

3.3 D1/D5 spectrum

We now know the case $\nu = 4$ is a very special case of brane system, so to study it further, we should consider a specific example of such a system. The particular D1/D5 system we consider in this thesis is of great importance because it was the first example of a brane system dual to a black holes. The black hole thermodynamics can be studied through the CFT of the intersection world-volume of the brane system, and a lot can be understood through the study of the spectrum.

First, let us define the system. Consider a stack of N_5 D5 branes covering the x^5, x^6, x^7, x^8, x^9 directions. Next, consider a stack of N_1 D1 branes laying inside the D5 stack in the x^5 direction. Table 3.1 gives a summary of what was described.

	0	1	2	3	4	5	6	7	8	9
D1	-	.	.	.	.	-	.	.	.	.
D5	-	.	.	.	.	-	-	-	-	-

Table 3.1: Schematic representation of the D1/D5 system.

As we can see, the number of mixed directions is $\nu = 4$, so from equation 3.5, the zero point energy of the NS sector of 1-5 and 5-1 string vanishes. Then, the massless spectrum of excitations of these strings will be both the R vacuum and the NS vacuum. Take for instance a 1-5 string, meaning that the endpoint at $\sigma = 0$ lays in the D1 brane, while the one at $\sigma = \pi$ lays in the D5 brane. The massless modes will be generated by,

$$\begin{aligned} \text{NS: } & b_0^i, \quad i = 6,7,8,9 \\ \text{R: } & b_0^M, \quad M = 0,1,2,3,4,5 \end{aligned} \tag{3.6}$$

Both of these sectors generate Clifford algebras separately of the respective subgroups of $SO(1,9)$. At first glance, we could think the important subgroup is $SO(1,5) \otimes SO(4) \subset SO(1,9)$, but the $SO(1,5)$ is not a symmetry of the system, because in x^0, x^5 we have NN conditions, and in x^1, x^2, x^3, x^4, D conditions. Let's define an index $\alpha = 0,5$ and $m = 1,2,3,4$ for the $SO(1,1)$ and $SO(4)_E$ symmetries respectively. The full symmetry group is then $SO(1,1) \otimes SO(4)_E \otimes SO(4)_I \subset SO(1,9)$, where the subindices indicate the directions x^1, x^2, x^3, x^4 for E and x^6, x^7, x^8, x^9 for I.

According to this splitting, the Clifford algebras of Equations 3.6 generate spinors of $SO(1,5)$ and $SO(4)_I$ respectively, let us define them as $|a\rangle_R$ and $|a\rangle_{NS}$ respectively. The $SO(1,5)$ spinor then splits into various $SO(1,1) \otimes SO(4)_E$, which in an abuse of notation[1] we will define as $|a\rangle_{05} \otimes |a\rangle_{1234}$. The representations these spinors follow can be derived from the GSO projection. The projection on either the NS or R sector will have the same shape as outlined in 2.5,

$$P_{GSO} = \frac{1}{2}\left(1 + \overline{\Gamma}\right), \tag{3.7}$$

where this operator $\overline{\Gamma}$ will be defined differently in each sector to respect the condition $\left\{\overline{\Gamma}, b_r^\mu\right\} = 0$, with r in the respective range.

[1] In this section we give the name a to a generic spinor index regardless of the representation. It is enough to know which Clifford algebra, b_0^α, b_0^m or b_0^i generates the spinor, and the GSO projection will tell us which Weyl components of the spinors survive.

In the NS sector, we can start by defining the fermion counting operator,

$$F = \sum_{r \in \mathbb{N}+1/2} (b_{-r})^M (b_r)_M + \sum_{r \in \mathbb{N}+1} (b_{-r})^i (b_r)_i. \tag{3.8}$$

So that $(-1)^F$ anticommutes with all b_r^μ, $r > 0$. For the $r = 0$ modes, there exists an anticommuting operator, $\Gamma_c^4 = (-i)^2 \Gamma_6 \Gamma_7 \Gamma_8 \Gamma_9$, that anticommutes as,

$$\left\{ \Gamma_c^4, b_0^i \right\} = 0. \tag{3.9}$$

This is precisely the chirality operator on $SO(4)_I$. Then, the GSO projection can be defined by the operator $\overline{\Gamma} = \Gamma_c^4 (-1)^F$, and as such, the NS vacuum will satisfy the equation $\overline{\Gamma}|a\rangle_{NS} = \Gamma_c^4 |a\rangle_{NS} = |a\rangle_{NS}$. We read from this that the NS vacuum is a chiral $SO(4)_I$ spinor in the $\mathbf{2}_s$, while being a singlet in $SO(1,1) \otimes SO(4)_E$.

We can repeat the same story for the R sector. In this case, the problem comes from the $M = 0, 1, 2, 3, 4, 5$ directions. The fermion counting operator is now,

$$F = \sum_{r \in \mathbb{N}+1} (b_{-r})^M (b_r)_M + \sum_{r \in \mathbb{N}+1/2} (b_{-r})^i (b_r)_i, \tag{3.10}$$

And to make $\overline{\Gamma}$ anticommute with all fermionic modes we need to multiply $(-1)^F$ by $\Gamma_c^6 = (-i)^3 \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5$. This is the chirality operator of $SO(1,5)$, which can be split as,

$$\Gamma_c^6 = (-i) \left(\Gamma_0 \Gamma_5 \right) (-i)^2 \left(\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \right) = \Gamma_c^2 (\Gamma_c^4)', \tag{3.11}$$

where $\Gamma_c^2 = (-i) \left(\Gamma_0 \Gamma_5 \right)$ is the chirality matrix in the x^0, x^5 directions and $(\Gamma_c^4)'(-i)^2 \left(\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \right)$ is the chirality matrix in the x^1, x^2, x^3, x^4 directions.

Again, Γ_c^6 anticommutes with all the fermionic zero modes of the R sector,

$$\left\{\Gamma_c^6, b_0^M\right\} = 0, \tag{3.12}$$

so the operator $\overline{\Gamma} = \Gamma_c^6(-1)^F$ anticommutes with all the fermionic modes of the R sector. The GSO projection on the R vacuum is equivalent to the equation $\Gamma_c^6|a\rangle_R = |a\rangle_R$, so $\Gamma_c^2|a\rangle_{05} \otimes \Gamma_c^4|a\rangle_{1234} = |a\rangle_{05} \otimes |a\rangle_{1234}$, which defines the possible chiralities of the spinor as (+,+) and (-,-). The R vacuum can be seen as the dimensional reduction of the $\mathbf{4}_s$ of $SO(1,5)$, and follows the $(\mathbf{1}_s, \mathbf{2}_s) + (\mathbf{1}_c, \mathbf{2}_c)$ of $SO(1,1) \otimes SO(4)_E$.

Let us summarize the 1-5 massless spectrum in a few lines. First, both the R and NS sector vacua are massless because their zero point energies vanish when $\nu = 4$. Both vacua are spinors because the fermionic zero modes b_0^i and b_0^M generate Clifford algebras. The GSO projection works similarly as in the single brane, picking chiralities in the vacua. In this case, the chiralities are both positive, but in the R sector, the spinor lives in a space with broken symmetry $SO(1,5) \to SO(1,1) \otimes SO(4)_E$, so it is dimensionally reduced accordingly. The results can be summarized as representations of $SO(1,1) \otimes SO(4)_E \otimes SO(4)_I$,

$$\text{NS: } (\mathbf{1}_s, \mathbf{2}_s, \mathbf{1}) \oplus (\mathbf{1}_c, \mathbf{2}_c, \mathbf{1}) \tag{3.13}$$

$$\text{R: } (\mathbf{1}, \mathbf{1}, \mathbf{2}_s) \tag{3.14}$$

Note that the 5-1 strings will produce the same zero modes, so that the vacuum will be identical.

The only missing piece for a full description of the D1/D5 massless spectrum is the 1-1 and 5-5 massless spectrums. We already know that these come from the dimensional reduction of the **10** and the $\mathbf{16}_s$ of $SO(1,9)$. The only difference with the spectrum discussed in Section 3.1 is that now the symmetry group is broken down from $SO(1,5) \otimes SO(4)_I$ to $SO(1,1) \otimes SO(4)_E \otimes SO(4)_I$.

In the 5-5 NS sector we have bosons that split as,

$$b^{\mu}_{-1/2}|0\rangle_{NS} = \left(b^{\alpha}_{-1/2}|0\rangle_{NS}, b^{m}_{-1/2}|0\rangle_{NS}, b^{i}_{-1/2}|0\rangle_{NS}\right), \tag{3.15}$$

which we can see as 3 components that follow the $(\mathbf{2},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{4},\mathbf{1})$ and $(\mathbf{1},\mathbf{1},\mathbf{4})$ respectively.

In the R sector we have a spinor in the $\mathbf{16}_s$ of $SO(1,9)$ that breaks into the $(\mathbf{1}_s,\mathbf{2}_s,\mathbf{2}_s) + (\mathbf{1}_s,\mathbf{2}_c,\mathbf{2}_c) + (\mathbf{1}_c,\mathbf{2}_s,\mathbf{2}_c) + (\mathbf{1}_c,\mathbf{2}_c,\mathbf{2}_s)$, according to the chirality eigenvalues of $\Gamma_{11} = \Gamma^2_c(\Gamma^4_c)'\Gamma^4_c$. In short, the $+$ chirality turns into all posible combinations of $(\pm,\pm,\pm)$ that multiply up to $+$.

The 1-1 massless spectrum is perfectly equivalent. In the end, the full D1/D5 spectrum can be written as,

	1-1 strings	5-5 strings	1-5 strings
Bosonic	$(2,1,1)+(1,4,1)+(1,1,4)$	$(2,1,1)+(1,4,1)+(1,1,4)$	$2(1,1,2_s)$
Fermionic	$(1_s,2_s,2_s)+(1_s,2_c,2_c)+$ $(1_c,2_s,2_c)+(1_c,2_c,2_s)$	$(1_s,2_s,2_s)+(1_s,2_c,2_c)+$ $(1_c,2_s,2_c)+(1_c,2_c,2_s)$	$2(1_s,2_s,1)+2(1_c,2_c,1)$

Table 3.2: Massless spectrum of the D1/D5 system before projecting charged states.

As a last comment, we should calculate how much supersymmetry this spectrum has. We count the supercharges in terms of right or left moving spinors of $SO(1,1)$. These supercharges are carried by the fermionic representations of the spectrum, and the R-symmetry group will necessarily be a subgroup of $SO(4)_E \otimes SO(4)_I$.

Just considering the 1-1 string, and counting the number of fermions, we see that this subsector of the spectrum allows for the full $SO(4)_E \otimes SO(4)_I$ to be the R-symmetry. Thus, we count 8 left moving supercharges following $(\mathbf{1}_s,\mathbf{2}_s,\mathbf{2}_s) + (\mathbf{1}_s,\mathbf{2}_c,\mathbf{2}_c)$, and 8 right moving supercharges following $(\mathbf{1}_c,\mathbf{2}_c,\mathbf{2}_s) + (\mathbf{1}_c,\mathbf{2}_s,\mathbf{2}_c)$. So in total, we say this system has $\mathcal{N} = (8,8)$[2], for a total of 16 supercharges, as it should be since this is the same as the spectrum of a single D1 brane, which we know to be maximally supersymmetric SYM.

We could consider the same case for just the 5-5 sector, but the conclu-

[2] This notation means 8 right moving and 8 left moving supercharges.

sion is the same, it has 16 supercharges and is a multiplet of maximal SYM. The interesting bit comes from the 1-5 and 5-1 strings, which cannot carry as many supercharges since there are not enough degrees of freedom. Instead, this sector only allows for supercharges following the $(\mathbf{1}_s, \mathbf{2}_s, \mathbf{2}_s)$ and $(\mathbf{1}_c, \mathbf{2}_c, \mathbf{2}_s)$ for a total of 8 supercharges, or $\mathcal{N} = (4, 4)$.

A heuristic explanation of these 8 supercharges can be made if we think of the representations as indices. If we want to transform the $(\mathbf{1}_s, \mathbf{2}_s, \mathbf{1})$ into the $(\mathbf{1}, \mathbf{1}, \mathbf{2}_s)$, we first need a spinor index in the $\mathbf{1}_s$ in the x^0, x^5 directions to contract with the first $\mathbf{1}_s$ of the spinor to make it a scalar. Then we need a $\mathbf{2}_s$ index in the x^1, x^2, x^3, x^4 directions to contract it with the second component $\mathbf{2}_s$ to make it another scalar. Lastly, we need a free $\mathbf{2}_s$ index in the x^6, x^7, x^8, x^9 directions to match the free $\mathbf{2}_s$ index of the boson. This very wordy explanation can be summarized as,

$$
\begin{array}{ccc}
(\mathbf{1}_s, & \mathbf{2}_s, & \mathbf{1}) \\
\mathbf{1}_s \big\downarrow & \mathbf{2}_s \big\downarrow & \mathbf{2}_s \big\downarrow \\
(\mathbf{1}, & \mathbf{1}, & \mathbf{2}_s)
\end{array}
$$

For the other fermion of this sector, the transformation works similarly,

$$
\begin{array}{ccc}
(\mathbf{1}_c, & \mathbf{2}_c, & \mathbf{1}) \\
\mathbf{1}_c \big\downarrow & \mathbf{2}_c \big\downarrow & \mathbf{2}_s \big\downarrow \\
(\mathbf{1}, & \mathbf{1}, & \mathbf{2}_s)
\end{array}
$$

Thus, the full D1/D5 system only preserves 8 supercharges in the form $\mathcal{N} = (4, 4)$, or more explicitly, $(\mathbf{1}_s, \mathbf{2}_s, \mathbf{2}_s) + (\mathbf{1}_c, \mathbf{2}_c, \mathbf{2}_s)$.

4. Orbifolds

String Theory can be defined on a vast number of different target spaces, that can be classified as Calabi-Yau spaces [11]. Smooth Calabi-Yau manifolds are a good tool, but since we don't have a metric in general, the best we can do is obtain topological results. We can consider toroidal compactification, which is a special case of Calabi-Yau manifold, and String Theory on these target spaces is an exactly solvable theory [12]. The problem now is that toroidal compactification does not produce phenomenologically plausible results, because tori preserve all supercharges of the flat non-compact background [13].

The next thing to consider is singular Calabi-Yau spaces for compactification because they can break supersymmetry [14], but again we run into the issue of not knowing the metric for these spaces. Toroidal orbifolds are to singular Calabi-Yau spaces the equivalent of tori to smooth Calabi-Yau spaces. Their metric is flat, and their construction requires only knowledge about the discrete symmetry group of tori.

The effect of orbifold compactification is to project states that are not invariant under the discrete group with which we construct the orbifold. This process can break supersymmety, and was first introduced in [15] for toroidal orbifolds.

In order to understand singular Calabi-Yau spaces as a compactification scheme in general, it is a good idea to focus our effort first in the case of toroidal orbifolds.

4.1 Orbifold compactification

We are interested in target spaces of the type,

$$\mathbb{R}^{1,4} \times S^1 \times T^4. \tag{4.1}$$

Consider the world-sheet scalars $X^M = (X^\mu, Z, Y^m)$ with indices according to the decomposition of the target space above $M = 0, \ldots, 9$, $\mu = 0, 1, 2, 3, 4$, $m = 1, 2, 3, 4$. We can arrange the coordinates along the torus directions as,

$$W^i = \frac{1}{\sqrt{2}}(Y^{2i-1} + iY^{2i}), \quad i = 1, 2. \tag{4.2}$$

The torus structure is contained in each complex torus coordinate as an identification of the type $W^i \sim W^i + 1 \sim W^i + \tau^i$, where $\tau \in \mathbb{C}$ is the complex structure of the torus. The scalars split on-shell into left and right movers as,

$$W^i(\tau, \sigma) = W^i_L(\tau + \sigma) + W^i_R(\tau - \sigma). \tag{4.3}$$

It is now apparent that there can be 4 independent $\mathbb{Z}_p$ rotations acting on each of the torus coordinates that we have split into right and left movers.

$$\begin{aligned} W^1_L &\to e^{2\pi i \tilde{u}_3/p} W^1_L \\ W^1_R &\to e^{2\pi i u_3/p} W^1_R \\ W^2_L &\to e^{2\pi i \tilde{u}_4/p} W^2_L \\ W^2_R &\to e^{2\pi i u_4/p} W^2_R, \end{aligned} \tag{4.4}$$

The S^1 coordinate will be identified with a shift, $Z \sim Z + 2\pi r/p$, making the orbifold freely acting. A freely acting orbifold is an orbifold with no fixed points, so it has no singularities. In this case, even if the $T^4/\mathbb{Z}_p$ has singular points, for example at the origin, the $S^1/\mathbf{Z}_p$ coordinate always shifts, leaving no point in $(S^1 \times T^4)/\mathbb{Z}_p$ invariant.

This story is not complete, as we should have introduced the discrete symmetry as a subgroup of the T-duality group of the 4-torus $SO(4, 4; \mathbb{Z})$. This T-duality group rotates between left and right moving coordinates of the torus, but in this thesis we focus on the open string, which has no concept of right/left movers. The rotations $SO(4; \mathbb{Z}) \subset SO(4, 4; \mathbb{Z})$ suffice. The detailed derivation of the allowed values of the parameters u_i and $\tilde{u}_i$ are contained in [2], [16] and the integers are characterized by mass parameter

m_i,

$$\frac{2\pi\tilde{u}_3}{p} = m_1 + m_3, \quad \frac{2\pi u_3}{p} = m_2 + m_4,$$
$$\frac{2\pi\tilde{u}_4}{p} = m_1 - m_3, \quad \frac{2\pi u_4}{p} = m_2 - m_4. \tag{4.5}$$

Notice that the mass parameters may not be equal in certain cases. When the right movers and left movers are rotated in equivalent ways, i.e. $m_1 = m_2$ and $m_3 = m_4$, we refer to the orbifold as a symmetric orbifold. Whereas if they rotate inequivalently, we say it is a symmetric orifold.

Only some values of p are actually allowed for 4-tori, because not every rotation leads to the same complex structure. In our case, we want to restrict to orbifolds that can preserve some supercharges, so we restrict to $p = 2, 3, 4, 6, 12$.

In order to fully characterize the action of the orbifold on the massless spectrum of the superstring, we still need to know the charges the R vacuum (the NS vacuum is a space-time scalar, so it is uncharged by definition, and the vectors are charged according to 4.4).

Take a basis element $|s_0, s_1, s_2, s_3, s_4\rangle_{L/R}$ of the R vacuum. The eigenvalues $s_i = \pm 1/2$, $i = 1, 2, 3, 4$, by construction, are eigenvalues of the $SO(2)$ rotations of the plane $(2i, 2i+1)$. Since the orbifold acts precisely as separate discrete subgroups of these $SO(2)$ rotations for both left and right movers, we can conclude that the orbifold action on spinors is,

$$|s_0, s_1, s_2, s_3, s_4\rangle_L \to e^{2\pi i(\tilde{u}_1 s_3 + \tilde{u}_2 s_4)}|s_0, s_1, s_2, s_3, s_4\rangle_L, \tag{4.6}$$

$$|s_0, s_1, s_2, s_3, s_4\rangle_R \to e^{2\pi i(\tilde{u}_3 s_3 + \tilde{u}_4 s_4)}|s_0, s_1, s_2, s_3, s_4\rangle_R. \tag{4.7}$$

By specifying values for s_3 and s_4 we can find the charges of the different spinor components. For the left moving R vacuum the charges are,

s_3	s_4	Charge
1/2	1/2	m_1
1/2	-1/2	$-m_3$
-1/2	1/2	$-m_3$
-1/2	-1/2	$-m_1$

Table 4.1: Charges under the orbifold group action of the left moving R vacuum depending on the values of s_3 and s_4.

while for the right moving R vacuum they are,

s_3	s_4	Charge
1/2	1/2	m_2
1/2	-1/2	$-m_4$
-1/2	1/2	$-m_4$
-1/2	-1/2	$-m_2$

Table 4.2: Charges under the orbifold group action of the left moving R vacuum depending on the values of s_3 and s_4.

We are ready to give all the charges of the NS and R massless states,

Sector	State	L charge	R charge
NS	$b^1_{-1/2}\|0\rangle$	$m_1 + m_3$	$m_2 + m_4$
	$\bar{b}^1_{-1/2}\|0\rangle$	$-(m_1 + m_3)$	$-(m_2 + m_4)$
	$b^2_{-1/2}\|0\rangle$	$m_1 - m_3$	$m_2 - m_4$
	$\bar{b}^2_{-1/2}\|0\rangle$	$-(m_1 - m_3)$	$-(m_2 - m_4)$
R	$\|s_0, s_1, s_2, +1/2, +1/2\rangle$	m_1	m_2
	$\|s_0, s_1, s_2, -1/2, -1/2\rangle$	$-m_1$	$-m_2$
	$\|s_0, s_1, s_2, -1/2, +1/2\rangle$	m_3	m_4
	$\|s_0, s_1, s_2, -1/2, -1/2\rangle$	$-m_3$	$-m_4$

Table 4.3: Here we outline all the massless states of the NS and R sector that are charges non-trivially under the orbifold action. The NS vector modes $b^i_{-1/2}$ and $\bar{b}^i_{-1/2}$ come from the expansion of $W^i_{L/R}$, thus they are charges in the same way as the torus coordinates. The R vacuum is charged according to Tables 4.1 and 4.2.

We can notice that only 2 aspects mattered to find out how all these objects were charged by the orbifold action. First, we need to know if the state is left or right moving. Second, since the orbifold action acts as discrete

$SO(4)$ rotations in the torus directions of the Lorentz group $SO(1,9)$, we only need to know which representation of the Lorentz group the state follows, and we will automatically know how it is charged under the orbifold action.

To make contact with representation theory, we can take the states from Table 4.3, and analyze which representations they follow. The vector components come from W^i rotate naturally under $SO(2) \times SO(2) \subset SO(4)$, in which each $SO(2) = U(1)$ component rotates W^1 or W^2 independently through a phase. As such, we say that W^1 follows the $(\mathbf{2},\mathbf{1})$ and W^2 follows the $(\mathbf{1},\mathbf{2})$.

The R vacuum states have already been identified in 3.1. Half of the components of the R vacuum follow the $\mathbf{2}_s$ of $SO(4)$, and the other half the $\mathbf{2}_c$. The first kind comes from a $+$ chiral spinor in the torus directions, thus $s_3 = s_4 = \pm 1/2$, and the second comes from a $-$ chiral spinor, $s_3 = -s_4 = \pm 1/2$. From this description, we can read the charges of the different $SO(4)$ representations, which can be summarized as,

SO(4) rep	L charge	R charge
$(\mathbf{2},\mathbf{1})$	$m_1 + m_3$	$m_2 + m_4$
$(\mathbf{1},\mathbf{2})$	$m_1 - m_3$	$m_2 - m_4$
$\mathbf{2}_s$	m_1	m_2
$\mathbf{2}_c$	m_3	m_4

Table 4.4: Orbifold charges for relevant representations of the subgroup of the Lorentz group corresponding to the torus $SO(4) \subset SO(1,9)$.

4.2 Branes in orbifold backgrounds

Building D-branes on orbifold backgrounds is not as easy as one could have expected. The orbifold action may break the boundary conditions of the open strings attached to the D-brane, forbidding it from existing in the theory. As we will see in the following, both the D1 and D5 branes we are interested in can survive if we restrict our study to symmetric orbifolds.

First, consider the general scenario of a single Dp-brane as discussed in 2.2, that is extended in the x^a, $a = 0, \ldots, p$ directions and point-like in the x^i,

$i = p+1, \ldots, 9$ directions. We introduced the conditions on the open string modes that enable us to define Dp-branes,

$$a^a = \tilde{a}^a, \tag{4.8}$$

if the boundary conditions were Neumann, or

$$a^i = -\tilde{a}^i, \tag{4.9}$$

if they were Dirichlet. The a and i indices are in the vector representation of the corresponding subgroups $SO(1,p)$ and $SO(9-p)$ of $SO(1,9)$. The Dp-brane corresponding to these boundary conditions will be able to exist in the orbifold background if the boundary conditions are respected by the orbifold group action.

As the orbifold group action only charges the T^4 directions, we can restrict the study of the boundary conditions to these directions. First, taking the complex coordinate definition 4.2, we define the modes,

$$w^i = \frac{1}{\sqrt{2}}(a^{2i-1} + ia^{2i}), \tag{4.10}$$

$$\tilde{w}^i = \frac{1}{\sqrt{2}}(\tilde{a}^{2i-1} + i\tilde{a}^{2i}). \tag{4.11}$$

The orbifold action on these complex modes is as in 4.4. We can use equations 4.5 to translate between the u_i, $\tilde{u}_i$ to m_1, m_2, m_3 and m_4 parameters,

$$\begin{aligned} w^i &\to e^{i(m_1 \pm m_3)} w^i, \\ \tilde{w}^i &\to e^{i(m_2 \pm m_4)} \tilde{w}^i, \end{aligned} \tag{4.12}$$

where the $i = 1$ takes the $+$, and $i = 2$, the $-$ sign.

Now, take for instance the first left moving complex coordinate w^1 and left moving $\tilde{w}^1$. If the brane we wanted to construct had only N or D conditions on this T^2, say $a^1 = \pm\tilde{a}^1$ and $a^2 = \pm\tilde{a}^2$, this translates as $w^1 = \pm\tilde{w}^1$. In the other hand, if we had mixed conditions on these directions, $a^1 = \pm\tilde{a}^1$ and $a^2 = \mp\tilde{a}^2$, they would imply $\overline{w}^1 = \pm\tilde{w}^1$.[1]

[1]The overline represents complex conjugation.

Since the orbifold action is a phase on the complex modes, the only difference between different brane constructions will be if they imply complex conjugation in the boundary conditions. This leads to essentially 4 different cases that we can treat without loss of generality. The cases work in the same way if we exchange D and N conditions, sin it will just imply sign flips.

Case 1: We consider a brane that has N boundary conditions along all directions of the torus. Thus, $w^1 = \tilde{w}^1$ and $w^2 = \tilde{w}^2$. The group action on these conditions is,

$$\begin{cases} e^{i(m_1+m_3)} w^1 = e^{i(m_2+m_4)} \tilde{w}^1 \\ e^{i(m_1-m_3)} w^2 = e^{i(m_2-m_4)} \tilde{w}^2 \end{cases} \tag{4.13}$$

From which we extract the condition on the mass parameters,

$$\begin{cases} m_1 = m_2 \\ m_3 = m_4 \end{cases} \tag{4.14}$$

This is what is known as a symmetric orbifold, one in which the left and right movers are rotated in the same manner. The D1 and D5 branes defined in Table 3.1 are examples that follow this case.

Case 2: We consider a brane with N conditions in the x^6, x^8 directions, and D in the x^7, x^9 directions. In this case $\overline{w}^1 = \tilde{w}^1$ and $\overline{w}^2 = \tilde{w}^2$. The group action on these conditions is,

$$\begin{cases} e^{-i(m_1+m_3)} \overline{w}^1 = e^{i(m_2+m_4)} \tilde{w}^1 \\ e^{-i(m_1-m_3)} \overline{w}^2 = e^{i(m_2-m_4)} \tilde{w}^2 \end{cases} \tag{4.15}$$

From which we extract the condition on the mass parameters,

$$\begin{cases} m_1 = -m_2 \\ m_3 = -m_4 \end{cases} \tag{4.16}$$

This is the definition of an anti-symmetric orbifold. The right movers rotate with opposite phase from the left movers.

The last two cases are not particularly useful for this thesis but are left here for the sake of completeness. There is no special name for the orbifolds that allow these kinds of branes.

Case 3: We consider branes with N conditions in the x^6, x^7 and x^8 directions, and D in the x^9. Thus, $w^1 = \tilde{w}^1$ and $\overline{w}^2 = \tilde{w}^2$. The group action on these conditions is,

$$\begin{cases} e^{i(m_1+m_3)}w^1 = e^{i(m_2+m_4)}\tilde{w}^1 \\ e^{-i(m_1-m_3)}\overline{w}^2 = e^{i(m_2-m_4)}\tilde{w}^2 \end{cases} \tag{4.17}$$

From which we extract the condition on the mass parameters,

$$\begin{cases} m_1 = m_4 \\ m_2 = m_3 \end{cases} \tag{4.18}$$

Case 4: For the last case, we consider branes with N conditions over x^6, x^8 and x^9, and D over x^7. Thus, $\overline{w}^1 = \tilde{w}^1$ and $w^2 = \tilde{w}^2$. The group action on these conditions is,

$$\begin{cases} e^{-i(m_1+m_3)}\overline{w}^1 = e^{i(m_2+m_4)}\tilde{w}^1 \\ e^{i(m_1-m_3)}w^2 = e^{i(m_2-m_4)}\tilde{w}^2 \end{cases} \tag{4.19}$$

From which we extract the condition on the mass parameters,

$$\begin{cases} m_1 = -m_4 \\ m_2 = -m_3 \end{cases} \tag{4.20}$$

In the following section we will study in more detail the D1/D5 system defined in Table 3.1, and as we have seen in **Case 1**, only symmetric orbifolds allow the branes in that system. The D1 brane has D condition on all direction of the torus, while the D5 has N conditions on all. Thus, we will restrict to $m_1 = m_2$ and $m_3 = m_4$ for the rest of the thesis.

4.3 Orbifold group action on the spectrum of the D1/D5 system

The effect of orbifolding on the spectrum of a theory projects out states that are not invariant under the orbifold action, gauging away the global symmetry that defined the orbifold in the first place [14]. It was discovered that for freely acting orbifolds, this process happens through a Higgs-like mechanism that gives mass to states charged under the obrifold action [2] (thus the name *mass parameters*). So in this chapter, whenever we speak of the orbifold projection, we actually mean that the charged states acquire a mass, thus getting effectively projected out of the massless spectrum.

In this section we will study the resulting spectrum of the D1/D5 system defined in Table 3.1 after orbifolding the target space. The field content will lead us to discover the amount of supercharges in the orbifolded theory, from which we will read and classify the remaining supersymmetry for different orbifolds.

First, let us summarize the results of section 3.3. Note that the notation $(\cdot,\cdot,\cdot)$ labels the representations under $SO(1,1)\times SO(4)_E\times SO(4)_I$ in that order.

	1-1 strings	5-5 strings	1-5 strings
Bosonic	$(2,1,1)+(1,4,1)+(1,1,4)$	$(2,1,1)+(1,4,1)+(1,1,4)$	$2(1,1,2_s)$
Fermionic	$(1_s,2_s,2_s)+(1_s,2_c,2_c)+$ $(1_c,2_s,2_c)+(1_c,2_c,2_s)$	$(1_s,2_s,2_s)+(1_s,2_c,2_c)+$ $(1_c,2_s,2_c)+(1_c,2_c,2_s)$	$2(1_s,2_s,1)+2(1_c,2_c,1)$

Table 4.5: Massless spectrum of the D1/D5 system before projecting charged states.

Case 1: $m_i=0$, corresponds to the spectrum in 4.5, so the spectrum has $\mathcal{N}=(4,4)$, or 8 supercharges. The full supersymmetry of the original spectrum is preserved, as expected, since this orbifold is just the plain torus.

Case 2: $m_1 = m_2 = 0$ and $m_3 = m_4 \neq 0$. In this case, all representations containing the $\mathbf{2}_c$ or $\mathbf{4}$ of $SO(4)_I$ are projected out. This projection leads to the following spectrum,

	1-1 strings	5-5 strings	1-5 strings
Bosonic	$(2,1,1)+(1,4,1)$	$(2,1,1)+(1,4,1)$	$2(1,1,2_s)$
Fermionic	$(1_s,2_s,2_s)+(1_c,2_c,2_s)$	$(1_s,2_s,2_s)+(1_c,2_c,2_s)$	$2(1_s,2_s,1)+2(1_c,2_c,1)$

Counting states in each sector of the spectrum we see that supersymmetry can still be present and the supercharges will be in the $(1_s,2_s,2_s)$ and $(1_c,2_c,2_s)$, since all sectors allow for these supercharges. The supersymmetry is then the unbroken $\mathcal{N}=(4,4)$ supersymmetry, or 8 supercharges. The supercharges will be in the $(1_s,2_s,2_s)$ and $(1_c,2_c,2_s)$

Case 3: $m_1 = m_2 \neq 0$ and $m_3 = m_4 = 0$. This case is similar to the previous but now the $\mathbf{2}_c$ survives, and the $\mathbf{2}_s$ is the one projected out.

	1-1 strings	5-5 strings	1-5 strings
Bosonic	$(2,1,1)+(1,4,1)$	$(2,1,1)+(1,4,1)$	-
Fermionic	$(1_s,2_c,2_c)+(1_c,2_c,2_c)$	$(1_s,2_c,2_c)+(1_c,2_c,2_c)$	$2(1_s,2_s,1)+2(1_c,2_c,1)$

In this case, just looking at the 1-5 sector we notice that only fermions survive. This is a clear indication that all supercharges were broken, as the superpartners of the 1-5 fermions are not in the spectrum anymore. The remaining supercharges are $\mathcal{N}=(0,0)$.

Case 4: $m_1 = m_2 \neq 0$ and $m_3 = m_4 \neq 0$, with $m_1 \neq m_3$. In this case, every object that is not a singlet under $SO(4)_I$ is charged under the orbifold action.

	1-1 strings	5-5 strings	1-5 strings
Bosonic	$(2,1,1)+(1,4,1)$	$(2,1,1)+(1,4,1)$	-
Fermionic	-	-	$2(1_s,2_s,1)+2(1_c,2_c,1)$

It is clear again that no supercharges remain in the theory. It was always expected that turning all the mass parameters would break all supersymmetry as it was already known for the closed string.

Case 5: $m_1 = m_2 = m_3 = m_4 \neq 0$. In this special case when all parameters are equal, we see that we should split the representations according to $SO(2) \times SO(2) \subset SO(4)_I$. In this special case, half of the **4** of $SO(4)_I$ is uncharged.

	1-1 strings	5-5 strings	1-5 strings
Bosonic	$(2,1,1)+(1,4,1)+(1,1,(1,2))$	$(2,1,1)+(1,4,(1,2))$	-
Fermionic	-	-	$2(1_s,2_s,1)+2(1_c,2_c,1)$

Although the field content is enlarged in this case compared to the previous, again all supersymmetry is broken to $\mathcal{N}=(0,0)$.

In conclusion, toroidal orbifold compactification can either preserve all supercharges of the D1/D5 spectrum (Case 1 and 2), or break all SUSY (Case 3, 4 and 5). Even then, the spectra in each case are different from each other, so the effects of the orbifold should be seen in the theories describing the system. For instance, we can expect to see the effects of the orbifold in the CFT describing the IR limit of the system, and thus a difference in the central charge which we could compare to the SUGRA black holes dual to this system.

As a last note, we want to remark that partial supersymmetry breaking can happen on the single brane spectrum. If we consider for instance a D1 brane, its spectrum will be given by the first column of Table 4.5. Recall that this system is maximal SYM, so it has 16 supercharges, or $\mathcal{N}=(8,8)$. If we now consider orbifolds of Case 2, we see that the supercharges following the $(\mathbf{1}_s,\mathbf{2}_c,\mathbf{2}_c)$ and $(\mathbf{1}_c,\mathbf{2}_s,\mathbf{2}_c)$ are projected out, so the orbifolded spectrum has half the supercharges of the original one, for a total of 8 supercharges, or $\mathcal{N}=(4,4)$. All the other cases of orbifolded spectrum of single D-branes can be read from the tables of the D1/D5 spectrum if we focus of the 1-1 or 5-5 strings. Only the trivial orbifold preserves all supercharges, Case 2 and Case 3 break half the SUSY, but have different R-symmetry since the $SO(4)_I$ representation of the supercharges is $\mathbf{2_s}$ and $\mathbf{2}_c$ respectively. Case 4 and 5 break all SUSY as expected.

5. D-brane gauge theories

In this chapter we will discuss the actions that correspond to the spectra discussed in Chapter 3. In the context of D-branes, string endpoints can move through the world-volume of the brane. The massless excitations describe p dimensional $U(N)$ maximal SYM for a stack of N Dp-branes. Later on, we will study the world-volume theory of the D1/D5 system, this will be a gauge theory on the intersection of the two branes, leading to a 2D action.

5.1 Gauge theory on a brane stack and dimensional reduction

There is a straight forward way of deriving the SYM action for the world-volume theory of the Dp-brane fields. We can start by introducing the DBI action for a single Dp-brane in a bosonic theory [17],

$$S = -T_p \int d^{p+1}\xi \; e^{-\phi} \sqrt{-\det(G_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})}, \tag{5.1}$$

where the G, B and ϕ are the pullbacks of the 10D metric G, B field and dilaton ϕ to the D-brane world-volume. And the indexes $\alpha, \beta = 0, \ldots, p$ are $SO(1,p)$ vector indices. Meanwhile, F is the field strength tensor associated to the gauge field A_α that represents the excitations of the brane in the transverse direction. Basically, the endpoints of the strings attached to the brane behave like gauge particles. Expanding to first order (and omitting factors irrelevant to this discussion),

$$S = -\frac{1}{4g_{\text{YM}}^2} \int d^{p+1}\xi \left(F_{\alpha\beta}F^{\alpha\beta} + \frac{2}{(2\pi\alpha')^2} \partial_\alpha X^a \partial^\alpha X^a \right) + \mathcal{O}(F^4), \tag{5.2}$$

from which we read a $U(1)$ gauge theory in $p+1$ dimensions with $9-p$ scalars coming from the directions $a = p+1, \ldots, 9$ perpendicular to the brane, with a YM coupling given by,

$$g_{\text{YM}}^2 = \frac{1}{4\pi^2\alpha'^2\tau_p} = \frac{g}{\sqrt{\alpha'}}(2\pi\sqrt{\alpha'})^{p-2}. \tag{5.3}$$

The supersymmetric extension of the DBI action is possible to write but unnecessary for this duscussion, because to first order it is effectively the supersymmetric completion of the previous action. The extension to multiple coincident Dp-branes is also straightforward as we just need to consider $U(N)$ adjoint gauge fields instead of $U(1)$.

Consider the case of a stack of N D9 branes. To first order, the worldvolume theory is $D = 10$, $\mathcal{N} = 1$, $U(N)$ SYM. We can absorb the coupling g_{YM} by redefinition of the fields so that it appears only in the covariant derivative,

$$S = \int d^{10}\xi \, \left(-\frac{1}{4}\text{Tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\text{Tr}\, \bar{\psi}\Gamma^\mu D_\mu \psi \right). \tag{5.4}$$

Here we see that the field content is a vector in the **10** and a MW spinor in the $\mathbf{16}_s$ of $SO(1,9)$, in agreeance with the spectrum found on Chapter 3. Note that the gauge fields A^μ and the spinors ψ are both in the adjoint representation of $U(N)$, so the field strength will be,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{YM}[A_\mu, A_\nu], \tag{5.5}$$

and the covariant derivative on the adjoint spinors will be,

$$D_\mu \psi = \partial_\mu \psi - ig_{\text{YM}}[A_\mu, \psi]. \tag{5.6}$$

Dimensional reduction of the action 5.4 can be performed by assuming that all coordinate dependence of the fields is on ξ^α with $\alpha = 0, \ldots, p$. In this case the index μ is split into (α, a) with $a = p+1, \ldots, 9$, effectively splitting the Lorentz symmetry group $SO(1,9) \to SO(1,p) \times SO(9-p)$. All derivatives ∂_a drop out of the action, and we are left with,

$$S = \frac{1}{4g_{\text{YM}}^2} \int d^{p+1}\xi \; \text{Tr}\, (-F_{\alpha\beta}F^{\alpha\beta} - 2(D_\alpha X^a)^2 + [X^a, X^b]^2 + \text{fermions}). \tag{5.7}$$

The fermionic portion of the action deserves a careful treatment in order to justify it splits as discussed in Chapter 3. In order to do this, it is convenient to write the Gamma matrices in the chiral basis [18] and Wick rotate the time component to be in Euclidean signature,

$$\begin{aligned}
\Gamma_0 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \\
\Gamma_1 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2, \\
\Gamma_2 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3, \\
\Gamma_3 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \otimes \mathbb{1}, \\
\Gamma_4 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes \mathbb{1}, \\
\Gamma_5 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1}, \\
\Gamma_6 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1}, \\
\Gamma_7 &= \sigma_1 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\
\Gamma_8 &= \sigma_1 \otimes \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\
\Gamma_9 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},
\end{aligned} \tag{5.8}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices, and $\mathbb{1}$ is the 2x2 identity matrix.

Now, for a specific example, let us consider a free 10D Dirac spinor with the action,

$$S = \int d^{10}\xi \bar{\lambda} \partial_{\hat{\mu}} \Gamma^{\hat{\mu}} \lambda, \tag{5.9}$$

with $\hat{\mu} = 0, ..., 9$. Now dimensionally reduce by effectively dropping the ξ^8 and ξ^9 dependence of the spinor. We are left with,

$$S = \int d^8\xi \bar{\lambda} \partial_\mu \Gamma^\mu \lambda, \tag{5.10}$$

with $\mu = 0, ..., 8$. Now, if we take a close look at the first 8 Gamma matrices in the basis written in 5.8, we can see that they contain the full 8D Clifford algebra as a subalgebra in the form,

$$\Gamma_\mu = \sigma_1 \otimes \Gamma^8_\mu = \begin{pmatrix} 0 & \Gamma^8_\mu \\ \Gamma^8_\mu & 0 \end{pmatrix}, \tag{5.11}$$

where Γ^8_μ is the 8D Clifford algebra in Euclidean signature. Now, the 10D chirality matrix is defined as $\Gamma^{10}_c = (-i)\Gamma_0 \dots \Gamma_9$, and it is explicitly,

$$\Gamma^{10}_c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}. \tag{5.12}$$

Thus, it is compatible with 5.11, in the sense that the 10D Dirac spinor splits into its $\mathbf{16}_s$ and $\mathbf{16}_c$ Weyl components, and each follow the 8D free Dirac equation. We can name the Weyl components λ_+ and λ_-, from which we read,

$$S = \int d^8\xi \left(\overline{\lambda}_- \partial^\mu \Gamma^8_\mu \lambda_- + \overline{\lambda}_+ \partial^\mu \Gamma^8_\mu \lambda_+ \right). \tag{5.13}$$

These 8D Dirac spinors can once again be split into their Weyl components using $\Gamma^8_c = \Gamma^8_0 \dots \Gamma^8_8$. The result is analogous to the previous,

$$\lambda_- = \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix}, \quad \lambda_+ = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}. \tag{5.14}$$

What is left is to check their 2D chiralities coming from the 2D subalgebra,

$$\Gamma^2_c = -i\Gamma_8\Gamma_9 = \sigma_3 \otimes \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \tag{5.15}$$

From this we read that the 2D chiralities are $+, -, -, +$ respectively for $\phi_-, \phi_+, \psi_-, \psi_+$. Consider for example ϕ_-, it is by construction $+$ chiral in $SO(1,7)$, and its $SO(2)$ chirality is $-$ coming from equation 5.15. If we multiply these two chiralities we get a $-$ $SO(1,9)$ chirality, which had to be that was because ϕ_- came from λ_-, which is $-$ chiral in $SO(1,9)$.

The spinors $\phi_-, \phi_+, \psi_-, \psi_+$ have chiralities $-, +, -, +$ in $SO(1,7)$ and $+, -, -, +$ in $SO(2)$ respectively, so the representations are indeed $(\mathbf{8}_s, \mathbf{1}_c), (\mathbf{8}_c, \mathbf{1}_s), (\mathbf{8}_c, \mathbf{1}_c), (\mathbf{8}_s, \mathbf{1}_s)$ respectively.

This procedure that was explicitly discussed for the case $SO(1,9) \rightarrow SO(1,7) \times SO(2)$ can be generalized for dimensional reductions of all kinds in a similar manner[1]. Notice how in this case, the 10D Dirac spinor splits into 2 8D Dirac spinors. If we reduced instead to 6D we would have 4 Dirac spinors, and they would carry an $SO(4)$ R-symmetry. The details of this reduction are detailed in Appendix B and the interplay of the $SO(1,5)$ with $SO(4)$ indices is very enlightening.

The agreeance with the bosonic spectrum is easy to check, as the vector A^α is on the $(\mathbf{p}, \mathbf{1})$ and the scalars form the $(\mathbf{1}, \mathbf{9} - \mathbf{p})$ just by looking at the indices α and a. The fermions are not as easy to check from the action, but they also follow the decomposition described in Chapter 3. Since we started with 16 supercharges, 16 supercharges remain regardless of the value of p. For example, in the case of a stack of D4-branes, the world-volume theory would be maximal $\mathcal{N} = 4$ SYM in $D = 4$. If we were to study a stack of D1-branes, the action would be that of $\mathcal{N} = (8,8)$ SYM in $D = 2$. The fact is that since we started with maximally supersymmetric SYM, we always land in a theory with maximal supersymmetry.

5.2 Gauge theory of the D1/D5 system

Now that we know that single brane stacks are described by SYM theories with maximal supersymmetry, we want to discuss the case of the world-volume theory in the intersection of Q_1 D1 and $Q-2$ D5 branes. As hinted by the spectrum of Chapter 3, we will have 3 sectors [19]:

1-1 strings: As discussed in the previous section, the theory of this sector will be the dimensional reduction of $\mathcal{N} = 1$, $U(Q_1)$ SYM from $D = 10$ to 1+1 dimensions. In our case the theory will be defined in the (t, x^5)

[1] For an in depth derivation of the dimensional reduction $SO(1,9) \rightarrow SO(1,7) \times SO(4)$, we refer to Appendix B. All non-abelian terms are treated carefully, and the example is less trivial, with more structure in the R-symmetry group.

directions. The bosonic content of this sector is,

$$\begin{aligned}&\text{Vector multiplet: } A_0^{(1)}, A_5^{(1)}, Y_m^{(1)}, m = 1,2,3,4,\\ &\text{Hypermultiplet: } Y_i^{(1)}, i = 6,7,8,9.\end{aligned} \tag{5.16}$$

5-5 strings: The field content of this sector is essentially the same of the 1-1 string when we dimensionally reduce to the 1+1 theory, but instead having $U(Q_5)$ gauge group.

$$\begin{aligned}&\text{Vector multiplet: } A_0^{(5)}, A_5^{(5)}, Y_m^{(5)}, m = 1,2,3,4\\ &\text{Hypermultiplet: } Y_i^{(5)}, i = 6,7,8,9\end{aligned} \tag{5.17}$$

Up until now, we have essentially two copies of $\mathcal{N} = (8,8)$ SYM in 1+1 dimensions, with R-symmetry group $SO(4)_I$, and all fields in the adjoint representation of $U(Q_1)$ or $U(Q_5)$ depending on if they describe a 1-1 or 5-5 string. The next ingredient in the theory will break supersymmetry by half. The R-symmetry group will be $SU(2)_R$, to be thought of as a subgroup of $SO(4)_I = SU(2)_R \times SU(2)_L$. Basically, the supercharges will have a $\mathbf{2}_s$ index of $SO(4)$ as we saw in Chapter 3.

1-5 and 5-1 strings: In Chapter 3, we argued that the bosonic spectrum of 1-5 and 5-1 follows the $(1, 1, \mathbf{2}_s)$ of $SO(1, 1) \times SO(4)_E \times SO(4)_I$, this is, a chiral spinor of $SO(4)_I$ that is a singlet under $SO(1, 1) \times SO(4)_E$. The fields can be described by χ^1 and χ^2 and can be joined as $\chi = 1/\sqrt{2}(\chi^1 + i\chi^2)$ to form a Weyl spinor of $SO(4)$ with + chirality. It will be useful to split the spinor into its components,

$$\chi = \begin{pmatrix} A \\ B^\dagger \end{pmatrix} \tag{5.18}$$

It is only left to define the gauge indices. Since both the 1-5 and 5-1 string have one end in the D1 and other in the D5 branes, the fields will carry gauge indices in the fundamental representation of $U(Q_1)$ and of $U(Q_5)$. This is commonly called the bi-fundamental representation in the sense that the object has 2 indices, one in each of the fundamentals. The important difference is that now when χ gets coupled to the gauge fields $A^{(1)}$ or $A^{(5)}$ they will do it as a fundamental field instead of an adjoint field.

We are now ready to write the action, we will go term by term,

$$S_{1-1} = k_{11} \int d^2\xi \, \mathrm{Tr}\, (-F^{(1)}_{\alpha\beta} F^{(1)\alpha\beta} - 2(D_\alpha Y^{(1)a})^2 + [Y^{(1)a}, Y^{(1)b}]^2), \quad (5.19)$$

$$S_{5-5} = k_{55} \int d^2\xi \, \mathrm{Tr}\, (-F^{(5)}_{\alpha\beta} F^{(5)\alpha\beta} - 2(D_\alpha Y^{(5)a})^2 + [Y^{(5)a}, Y^{(5)b}]^2), \quad (5.20)$$

these 2 terms are the dimensional reduction of the bosonic part of $D = 10$ SYM.

$$S_{1-5} = \int d^2\xi \, \mathrm{Tr} \left(|D_\alpha \chi|^2 + \frac{\chi^\dagger \chi}{2\pi\alpha'} (Y^{(1)}_m - Y^{(5)}_m)^2 + g \sum_{I=1}^{3} (\chi^\dagger \tau^I \chi)^2 \right). \quad (5.21)$$

In this last action, the first term is just the kinetic term with covariant derivative $D_\alpha \chi = (\partial_\alpha + iA^{(1)}_\alpha - iA^{(5)}_\alpha)\chi$. The last 2 terms come from supersymmetry.

What we should mention at this point, is that the only terms that matter for the discussion on this thesis are the kinetic terms of the action. Keep in mind that the orbifold charges states through rotations in $SO(4)_I$, which has always been a symmetry of our system. Thus, the potential terms, even if not explicitly, are invariant under $SO(4)_I$ rotations, and thus invariant under the orbifold group action. The kinetic terms, although similar in the sense that they are also globally invariant under those $SO(4)_I$ discrete rotations, will have a dependence on the circle component in the next chapter that will be affected by the derivatives of the kinetic terms.

6. Orbifold gauge theory via Sherk-Schwarz reduction

Until now, we discussed the open string spectrum in Chapter 3, where we found how the orbifold acts on the different representations of the internal symmetry $SO(4)$. After successfully finding the massless spectrum in orbifold backgrounds, we presented the world-volume effective actions of D-brane stacks and a very special brane system in Chapter 5, with the caveat that all those theories lived in a flat background.

The goal of this chapter is to merge those two ideas and arrive at an effective theory of brane world-volumes on orbifold backgrounds. We will see that we can impose periodicity conditions on the S^1 coordinate with a duality twist for the charged fields, that will give them masses according to the open string spectrum[2].

6.1 Gauge theory of the D9 brane stack on orbifold backgrounds

The starting point of this section is the D9 effective action in a flat background,

$$S = \int d^{10}\xi \ \mathrm{Tr}\left(-\frac{1}{4}F^2 + \frac{i}{2}\bar{\lambda}D^{\hat{\mu}}\Gamma_{\mu}\lambda\right). \tag{6.1}$$

Next, we are going to compactify on a T^4. If we assume the volume of the torus to be small, $V_4 \approx O(\alpha'^2)$, then we can ignore all the KK momentum modes and pick only the zero mode, leading to the same theory as the one of a D6 described in Appendix B. For the sake of applying the knowledge developed in Chapter 3, we also want to have the T^4 components of the 10D vector in the $(2,1)+(1,2)$ of $SU(2)_L \times SU(2)_R \subset SO(4)$, so we define the complex fields $N_1 = 1/\sqrt{2}\,(A_6 + iA_7)$ and $N_2 = 1/\sqrt{2}\,(A_8 + iA_9)$. In terms of these fields the world-volume action is,

$$S = S_{kin} + S_{pot}, \tag{6.2}$$

where

$$\begin{aligned} S_{kin} &= \int d^6\xi \; \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + 2D_\mu N_i D^\mu \overline{N}^i + \phi_+^{\dagger\alpha} D_\mu \gamma_\mu \phi_+^\alpha + \phi_-^{\dagger\dot{\alpha}} D_\mu \overline{\gamma}_\mu \phi_-^{\dot{\alpha}}\right), \\ S_{pot} &= i\int d^6\xi \; \mathrm{Tr}\left([N_i, N_j][\overline{N}^i, \overline{N}^j] + [N_i, \overline{N}_j][\overline{N}^i, N^j] + \phi_+^{\dagger\alpha} \sigma^i_{\alpha\dot{\beta}} \gamma_5 [\overline{N}_i, \phi_-^{\dot{\beta}}] + \phi_-^{\dagger\dot{\alpha}} \overline{\sigma}^i_{\dot{\alpha}\beta} [N_i, \phi_+^\beta]\right), \end{aligned} \tag{6.3}$$

where $\mu = 0, \ldots, 5$, $i = 1, 2$, α, β are $\mathbf{2}_s$ and $\dot{\alpha}, \dot{\beta}$ are $\mathbf{2}_c$ indices of $SO(4)$. As representations of $SO(1,5) \times SO(4)$, N_1 and N_2 are in the $(\mathbf{1}, (\mathbf{2}, \mathbf{1}))$ and $(\mathbf{1}, (\mathbf{1}, \mathbf{2}))$. Now, in order to give masses to these fields, we resort to the Scherk-Schwarz reduction [1]. Depending on the $SO(4)$ representation the fields belong to, different monodromies (in this case just phases) will be acceptable. The idea is that when compactifying on S^1, fields can be expanded via the Scherk-Schwarz ansatz as,

$$f(x,z) = e^{iMz/2\pi R} \sum_{n=-\infty}^{\infty} f_n(x) e^{inz/R}, \tag{6.4}$$

which is just a generalized Fourier expansion. Now, we can see that the field is not periodic, $f(x, z + 2\pi R) = e^{iM} f(x,z)$, so an expansion of this type is only admissible if the transformation $f \to e^{iM} f$ is a global symmetry.

In our case, it's clear that after T^4 compatifications, some discrete subgrups of $SO(4)$ are actually global symmetries of our system, and the possible charges associated to different representations of this group have already been classified. Thus, we can use the SS reduction to spontaneously break supersymmetry in the gauge theory describing the branes. What results is the effective world-volume theory when the String Theory is defined in an orbifolded background.

Now, in order to retrieve the spectrum desired, we need to consider the limit where $R \to 0$, effectively selecting the zeroth KK mode of the SS expansion[1]. Besides that, remember that the action descends from a theory with an $SO(1,9)$ global symmetry, thus $SO(4) \subset SO(1,9)$ will also be a global symmetry, thus any potential term will remain unchanged.

The proposed expansions for the charged fields are,

$$\begin{aligned} N_1(x,z) &= e^{i(m_1+m_3)z} N_1(x) \\ N_2(x,z) &= e^{i(m_1-m_3)z} N_2(x) \\ \phi_+^\alpha(x,z) &= e^{im_1 z} \phi_+^\alpha(x) \\ \phi_-^\alpha(x,z) &= e^{im_3 z} \phi_-^\alpha(x) \end{aligned} \tag{6.5}$$

Now, the only change happens in the kinetic part of the action, where the derivatives ∂_z now turn into masses,

$$\begin{aligned} \partial_z N_1 \partial^z \overline{N}_1 &= (m_1+m_3)^2 |N_1|^2 \\ \partial_z N_2 \partial^z N_2 &= (m_1-m_3)^2 |N_2|^2 \\ \phi_+^{\dagger\alpha} \partial_z \gamma_z \phi_+^\alpha &= \phi_+^{\dagger\alpha} i m_1 \gamma_z \phi_+^\alpha \\ \phi_-^{\dagger\alpha} \partial_z \gamma_z \phi_-^\alpha &= \phi_-^{\dagger\alpha} i m_3 \gamma_z \phi_-^\alpha \end{aligned} \tag{6.6}$$

as we can see, the fields acquire a mass corresponding to their charge under the orbifold action according to their $SO(4)$ representation. Now, depending on the orbifold, the massless fields will correspond to the projected orbifold spectrum described in Section 4.3, and supersymmetry will be broken (partially or completely).

As we can see, the charged fields are not actually projected out from the theory, but they acquire a mass, effectively leaving the massless spectrum. This matches the results from [2] for the closed string.

The case for more complicated actions, like the one describing the D1/D5 system, works similarly. Since the potential terms are $SO(4)_I$ invariant, and don't involve any derivatives, they are unaffected by the orbifold group action. The only affected terms are the kinetic terms, and they also transform

[1]There is an interesting story here about higher momentum modes. It turns out that in certain orbifolds, for specific values of the S^1 radius R, some momentum modes remain massless since the orbifold charge and the KK momentum cancel out.

into a mass term under a SS dimensional reduction.

7. Conclusions

This thesis explored the extension of the well understood orbifold supersymmetry breaking of the closed string spectrum to the open string spectrum. Firstly, we studied the spectrum of the open string, and found the charges of the spectrum classified by the representation of the $SO(4)$ rotations in the torus directions. Secondly, in an attempt to reproduce the spontaneous symmetry breaking known for the closed string effective supergravity through a SS reduction, we proposed an analogous procedure in the effective world-volume theory.

As it turned out, this process seems well-behaved, and it can be tuned to reproduce the spectrum found in the high energy String Theory.

In single brane stacks, the number of supercharges can be broken from 16 to 8 or 0. While for the D1/D5 brane system, it can only be totally broken from 8 to 0 or preserved.

Some questions still linger. At the hearth of this thesis, we wanted to calculate the infrared limit of the D1/D5 orbifold world-volume theory, but due to a lack of tools to describe that IR limit, it was not possible to perform it. Besides, there is a story about theories with an S^1 radius $R > 1$ that can restore the supersymmetry normally broken by the orbifold, which could be worthwhile to study.

Appendices

A. Spinors in various dimensions

In this appendix we will juntify some claims about spinors in general even dimensions that are used throughout the thesis.

It is well known that the Dirac representation is not irreducible in even dimensions, in which a chirality projection exists into the two different Weyl basis. There is also always a Majorana condition that can induces a real structure in the spinor spaces of all dimensions, but is only truncates the degrees of freedom of Weyl spinors in some dimensions. All these topics will be covered in the following sections in detail with the main objective of describing irreducible spinors in all even dimensions $D \leq 10$.

A.1 Weyl Spinors in $D = 2, 4, 6, 8, 10$

We start with the Clifford algebra,

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}, \tag{A.1}$$

and let the metric be $\eta^{\mu\nu} = \text{diag}\left(-1, 1, \ldots, 1\right)$ the $D = 2k + 2$ dimensional Minkowski metric.

We first make a change of basis for the algebra elements so that the fundamental representation can be directly extracted from the algebra. Take the following linear combinations,

$$\Gamma^{0\pm} = \frac{1}{2}\left(\pm\Gamma^0 + \Gamma^1\right), \tag{A.2}$$

$$\Gamma^{a\pm} = \frac{1}{2}\left(\Gamma^{2a} \pm i\Gamma^{2a+1}\right), \quad a = 1, \ldots, k. \tag{A.3}$$

Now the Clifford algebra A.1 can be stated in terms of the new operators as,

$$\begin{aligned} \left\{\Gamma^{a+}, \Gamma^{b-}\right\} &= \delta^{ab}, \\ \left\{\Gamma^{a+}, \Gamma^{b+}\right\} &= \left\{\Gamma^{a-}, \Gamma^{b-}\right\} = 0. \end{aligned} \tag{A.4}$$

We can see that these operators are raising and lowering operators for $k+1$ different eigenvalues. We can write an arbitrary basis element of this representation as $|s_0, \ldots, s_k\rangle$, with $s_i = \pm 1/2$. An arbitrary spinor in this representation is then in the complex span of this basis, which has 2^{k+1} complex components. This is what is called a Dirac spinor, or $\mathbf{2^{k+1}}_{\text{Dirac}}$.

These eigenvalues s_i are actually eigenvalues of rotations in planes given by the grouping of A.2. To see this explicitly, we need to recover the Lorentz algebra from the Clifford algebra. Let

$$\Sigma^{\mu v} = -\frac{i}{4}\left[\Gamma^{\mu}, \Gamma^{v}\right]. \tag{A.5}$$

The elements $\Sigma^{\mu v}$ define the Lorentz algebra os $SO(1, 2k+1)$. The generators $\Sigma^{2a,2a+1}$, $a = 1, \ldots, k$, commute and have eigenvalues proportional to s_a when acting on the basis element $|s_0, \ldots, s_k\rangle$. To be precise, the operator,

$$S_a \equiv i^{\delta_{a,0}} \Sigma^{2a,2a+1} = \Gamma^{a+}\Gamma^{a-} - \frac{1}{2}, \tag{A.6}$$

has eigenvalue s_a.

Next, we can define the chirality operator,

$$\Gamma = i^{-k}\Gamma^0\Gamma^1 \ldots \Gamma^{d-1}, \tag{A.7}$$

which has the properties,

$$(\Gamma)^2 = 1, \quad \{\Gamma, \Gamma^{\mu}\} = 0, \quad [\Gamma, \Sigma^{\mu v}] = 0. \tag{A.8}$$

From the first property, we see that Γ has eigenvalues ± 1. From the rest we see that we can split the basis $|s_0, \ldots, s_k\rangle$ into two subspaces according to the eigenvalues of Γ. We can rewrite the chirality operator in terms of the rotation generators S_a as follows,

$$\Gamma = 2^{k+1} S_0 S_1 \dots S_k, \tag{A.9}$$

which allows us to identify the two chiralities as $+1$ when the product of the s_a is positive and -1 when it is negative. The two subspaces defined by the chirality operator are called Weyl representations and are labeled as $\mathbf{2^k}_s$ and $\mathbf{2^k}_c$ for the $+1$ and -1 subspaces respectively. So we can finally state that the dirac representation splits into Weyl representations as,

$$\mathbf{2^{k+1}}_{\text{Dirac}} = \mathbf{2^k}_s \oplus \mathbf{2^k}_c. \tag{A.10}$$

The classic example in string theory is $D = 10$, where we have the decomposition,

$$\mathbf{32}_{\text{Dirac}} = \mathbf{16}_s \oplus \mathbf{16}_c. \tag{A.11}$$

A.2 Majorana condition

Up until now, we have been able to define Weyl spinors, that have a well defined chirality and have half the degrees of freedom of Dirac spinors. It will be shown in this section that a real structure can be imposed for some values of D, effectively halving the degrees of freedom of a Dirac spinor.

From the definition A.2, and the action on the basis elements $|s_0, \dots, s_k\rangle$, we see that as matrices, $\Gamma^{a\pm}$ are real. For the original gamma matrices defined in A.1, we see that the matrices Γ^{2a} are real, while the matrices Γ^{2a+1} are purely imaginary.

Now, take all of the purely imaginary matrices, and define the operators,

$$B_1 = \Gamma^3 \Gamma^5 \dots \Gamma^{d-1}, \quad B_2 = \Gamma B_1. \tag{A.12}$$

From the commutation relations of the Gamma matrices we have that,

$$B_1 \Gamma^\mu B_1^{-1} = (-1)^k \Gamma^{\mu *}, \quad B_2 \Gamma^\mu B_2^{-1} = (-1)^{k+1} \Gamma^{\mu *}, \tag{A.13}$$

and for the Lorentz generators,

$$B\Sigma^{\mu\nu}B^{-1} = -\Sigma^{\mu\nu *}, \tag{A.14}$$

for either B_1 or B_2.

Now, take a Dirac spinor ξ, and make a change of basis given by $\xi \to B\xi$. Then, by the transformation rules of ξ and the relation A.14, we see that $B\xi$ transforms as a conjugate spinor ξ^*.

The fact that we are able to linearly transform into conjugate spinors means that we may be able to relate the real and imaginary components of a spinor in a way consistent with Lorentz transformations. Concretely, we propose the Majorana condition,

$$\xi^* = B\xi. \tag{A.15}$$

Now, taking the conjugates $(B\xi)^* = \xi = B^*B\xi$, so $B^*B = 1$. Now, we can calculate explicitly using the definitions for B_1 and B_2 that,

$$B_1^*B_1 = (-1)^{k(k+1)/2}, \quad B_2^*B_2 = (-1)^{k(k-1)/2}. \tag{A.16}$$

So the condition $B^*B = 1$ can only be satisfied with $k = 0, 1, 3$ (mod 4). Which means that Majorana spinors can exist in $D = 2, 4, 8$ (mod 8) but not in $D = 6$ (mod 8).

We are ultimately interested in Majorana-Wey spinors, so we need to discuss whether a Majorana condition can be applied to a Weyl spinor. Take the chirality matrix Γ. We need to check if a Majorana change of basis can keep the chirality consistent. Otherwise, there would be mixing between right and left moving spinors, making them inconsistent with Lorentz transofrmations.

From the properties of the chirality matrix A.8, we calculate,

$$B_1\Gamma B_1^{-1} = B_2\Gamma B_2^{-1} = (-1)^k\Gamma^*, \tag{A.17}$$

so when k is even, each Weyl representation transforms as its own conjugate, while for k odd, the transformation rules get exchanged.

This automatically forbids Majorana-Weyl spinors from existing in $D = 4, 6$ (mod 8), so that both conditions are only compatible in $D = 2$ (mod 8). This is a very interesting result, because Supertring Theory is formulated from MW spinors in the world-sheet, this is $D = 2$, and produces MW spinors in space-time, which is necessarily $D = 10$, it is quite a beautiful coincidence.

A.3 Table of irreducible spinors in even dimensions

To wrap up this brief lesson on spinors, we can list all the minimal degrees of freedom of a spinor in any (even) dimension mod 8. For the sake of completeness I will also list the minimal representations in odd dimensions without any derivation (see [6] for details),

d	Majorana	Weyl	Majorana-Weyl	minimal dof
2	yes	self	yes	1
3	yes	-	-	2
4	yes	complex	-	4
5	-	-	-	8
6	-	self	-	8
7	-	-	-	16
8	yes	complex	-	16
9	yes	-	-	16
10	yes	self	yes	16

Table A.1: Summary of spinors in $D \leq 10$. A dash implies that the condition cannot be applied. Under Weyl, self means that the spinor is conjugate to itself, while complex means that conjugation transforms a Weyl spinor of one chirality into the opposite.

B. Dimensional reduction of SYM

Consider the SYM action in $D = 10$ with gauge group $U(N)$, with Euclidean signature (Wick rotation will yield results for other signature). Consider $g_{YM} = 1$ because it is of no relevance for this analysis,

$$S = \int d^{10}\xi \mathrm{Tr}\left(-\frac{1}{4}F_{\hat{\mu}\hat{\nu}}F^{\hat{\mu}\hat{\nu}} + i\overline{\lambda}D_{\hat{\mu}}\Gamma^{\hat{\mu}}\lambda\right), \tag{B.1}$$

where all field content is adjoint, $\hat{\mu} = 0, \ldots, 9$,

$$\begin{aligned} F_{\hat{\mu}\hat{\nu}} &= \partial_{\hat{\mu}}A_{\hat{\nu}} - \partial_{\hat{\nu}}A_{\hat{\mu}} + i[A_{\hat{\mu}}, A_{\hat{\nu}}], \\ D_{\hat{\mu}} &= \partial_{\hat{\mu}} + i[A_{\hat{\mu}}, \cdot], \end{aligned} \tag{B.2}$$

and the fermions are MW in the sense,

$$\begin{aligned} \Gamma_c^{10}\lambda &= \lambda, \\ C\lambda &= \overline{\lambda}^T, \end{aligned} \tag{B.3}$$

with C the charge conjugation matrix defined in Appendix A, so that they have 16 real components.

We choose the chiral basis of the gamma matrices in order to be able to split the spinor via chirality as it was stated in Appendix A,

$$\begin{aligned}
\Gamma_0 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \\
\Gamma_1 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2, \\
\Gamma_2 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3, \\
\Gamma_3 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \otimes \mathbb{1}, \\
\Gamma_4 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes \mathbb{1}, \\
\Gamma_5 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1}, \\
\Gamma_6 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1}, \\
\Gamma_7 &= \sigma_1 \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\
\Gamma_8 &= \sigma_1 \otimes \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\
\Gamma_9 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},
\end{aligned} \tag{B.4}$$

and the chirality and charge conjugation matrices are defined as,

$$\begin{aligned}
\Gamma_c^{10} &= i \prod_{\hat{\mu}} \Gamma_{\hat{\mu}} = \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\
C &= \prod_{i=1}^{5} \Gamma_{2i-1} = -\sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2.
\end{aligned} \tag{B.5}$$

Consider that we dimensionally reduce from $D = 10$ to $D' = 6$ as it is the easiest non-trivial exaple. The indices split into $\hat{\mu} = (\mu, i)$, $\mu = 0, \ldots, 5$, $i = 6, \ldots, 9$. Let's discuss the two parts of the action independently. First the vector goes as follows,

$$\begin{aligned}
S_{\text{boson}} &= -\frac{1}{4} \int d^6\xi \mathrm{Tr} \left(F_{\mu\nu} F^{\mu\nu} + 2F_{\mu i} F^{\mu i} + F_{ij} F^{ij} \right) = \\
&= -\frac{1}{4} \int d^6\xi \left(F_{\mu\nu} F^{\mu\nu} + 2D_\mu A_i D^\mu A^i + i[A_i, A_j][A^i, A^j] \right),
\end{aligned} \tag{B.6}$$

where we can see that A_μ is in the $(\mathbf{6}, \mathbf{1})$, and A_i in the $(\mathbf{1}, \mathbf{4})$.

The fermion part after dimensional reduction is,

$$S_{\text{fermion}} = i \int d^6 \mathrm{Tr} \left(\xi \lambda^\dagger D^\mu \Gamma_0 \Gamma_\mu \lambda + \lambda^\dagger \Gamma_0 \Gamma_i [A^i, \lambda] \right). \tag{B.7}$$

Now, let us define the following lower dimensional gamma matrices,

$$\begin{aligned}
\Gamma^6_0 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \\
\Gamma^6_1 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_2, \\
\Gamma^6_2 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_3, \\
\Gamma^6_3 &= \sigma_1 \otimes \sigma_2 \otimes \mathbb{1}, \\
\Gamma^6_4 &= \sigma_1 \otimes \sigma_3 \otimes \mathbb{1}, \\
\Gamma^6_5 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1}.
\end{aligned} \tag{B.8}$$

We can write,

$$\Gamma_0 \Gamma_\mu = \mathbb{1} \otimes \mathbb{1} \otimes \Gamma^6_0 \Gamma^6_\mu. \tag{B.9}$$

The spinors λ can be split according to their $SO(1,5)$ and $SO(4)$ chirality, as eigenspinors of,

$$\begin{aligned}
\Gamma^6_c &= \mathbb{1} \otimes \mathbb{1} \otimes \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1}, \\
\Gamma^4_c &= \sigma_3 \otimes \mathbb{1} \otimes \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1}.
\end{aligned} \tag{B.10}$$

The projection split the spinor as,

$$\lambda = \begin{pmatrix} \lambda_+ \\ 0 \end{pmatrix}, \quad \lambda_+ = \begin{pmatrix} \phi^\alpha_+ \\ \phi^{\dot\alpha}_- \end{pmatrix}. \tag{B.11}$$

The spinors ϕ^α_+ and $\phi^{\dot\alpha}_-$ belong to the $(\mathbf{4}_s, \mathbf{2}_s)$ and $(\mathbf{4}_c, \mathbf{2}_c)$. With this construction in mind, we can calculate the following matrices,

$$\begin{aligned}
\Gamma^6_0 \Gamma^6_0 &= \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\
\Gamma^6_0 \Gamma^6_1 &= \mathbb{1} \otimes \mathbb{1} \otimes \sigma_3, \\
\Gamma^6_0 \Gamma^6_2 &= \mathbb{1} \otimes \mathbb{1} \otimes \sigma_2, \\
\Gamma^6_0 \Gamma^6_3 &= \mathbb{1} \otimes \sigma_3 \otimes \sigma_1, \\
\Gamma^6_0 \Gamma^6_4 &= \mathbb{1} \otimes \sigma_2 \otimes \sigma_1, \\
\Gamma^6_0 \Gamma^6_5 &= \sigma_3 \otimes \sigma_1 \otimes \sigma_1.
\end{aligned} \tag{B.12}$$

So we see that depending on the 6D chirality of the spinor, they will couple through different vectors, akin to a 6D generalization of the Pauli matrices. Let us define γ_μ and $\overline{\gamma}_\mu$, the 4x4 matrices resulting from,

$$\Gamma_0^6\Gamma_\mu^6 = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & \overline{\gamma}_\mu \end{pmatrix}. \tag{B.13}$$

We are finally ready to write the kinetic term of the fermion action, which results in,

$$S_{\text{kinetic}} = i \int d^6\xi \text{Tr} \left(\phi_+^{\dagger\alpha} D_\mu \gamma_\mu \phi_+^\alpha + \phi_-^{\dagger\dot{\alpha}} D_\mu \overline{\gamma}_\mu \phi_-^{\dot{\alpha}} \right). \tag{B.14}$$

The potential term involves the matrices,

$$\begin{aligned} \Gamma_0\Gamma_6 &= \mathbb{1} \otimes \mathbb{1} \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_1, \\ \Gamma_0\Gamma_7 &= \mathbb{1} \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \\ \Gamma_0\Gamma_8 &= \mathbb{1} \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \\ \Gamma_0\Gamma_9 &= \sigma_3 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1. \end{aligned} \tag{B.15}$$

Taking into account that for the first index, every spinor has eigenvalue +1, we can reduce the problem to,

$$\begin{aligned} \Gamma_0\Gamma_6 &= 1 \otimes \mathbb{1} \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_1, \\ \Gamma_0\Gamma_7 &= 1 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \\ \Gamma_0\Gamma_8 &= 1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \\ \Gamma_0\Gamma_9 &= 1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1. \end{aligned} \tag{B.16}$$

From the discussion of the kinetic term we defined $\gamma_5 = \sigma_1 \otimes \sigma_1$, which we can now use to find,

$$S_{\text{potential}} = i \int d^6\xi \left(\phi_+^{\dagger\alpha} \sigma^i_{\alpha\dot{\beta}} \gamma_5 [A_i, \phi_-^{\dot{\beta}}] + \phi_-^{\dagger\dot{\alpha}} \overline{\sigma}^i_{\dot{\alpha}\beta} [A_i, \phi_+^\beta] \right). \tag{B.17}$$

The matrices $\sigma^i_{\alpha\dot{\beta}}$ and $\overline{\sigma}^i_{\dot{\alpha}\beta}$ are the corresponding intertwiners between the different representation of $SO(4)$ that exist in the action. i is an index in the $\mathbf{4}$ of $SO(4)$, α, β in the $\mathbf{2}_s$ and $\dot{\alpha}, \dot{\beta}$ in the $\mathbf{2}_c$

Further dimensional reductions follow the same procedure described in this appendix. Note that we chose to show the details for $D = 10 \to 6$ because this is the smallest dimension in which the R-symmetry group has indices with more than one component, so intertwiners appear in a non-trivial manner.

B.1 Supercharges under dimensional reduction

The action resulting from the dimensional reduction described in Appendix B does not break any supersymmetry (this is apparent since all fields remain massless), so the total number of supercharges is always 16. Before dimensional reduction, these could be written as,

$$\begin{aligned} \delta_\epsilon A_{\hat{\mu}} &= \bar{\epsilon}\Gamma_{\hat{\mu}}\lambda, \\ \delta_\epsilon \lambda &= -\frac{1}{2}F_{\hat{\mu}\hat{\nu}}\Gamma^{\hat{\mu}\hat{\nu}}\epsilon, \end{aligned} \tag{B.18}$$

where ϵ is A MW spinor in the same representation as λ, the $\mathbf{16}_s$. Since the expression for the fermions get rather cumbersome, we will only derive in detail the dimensional reduction of the vector transformations. The only matrix calculations we need to check for the following are the commutators $\Gamma^{\mu\nu}$.

$$\Gamma_{\mu\nu} = \Gamma_{[\mu}\Gamma_{\nu]} = \Gamma_{[\mu}(\Gamma_0)^2\Gamma_{\nu]} = \mathbb{1} \otimes \mathbb{1} \otimes \Gamma^6_{[\mu}\Gamma^6_{\nu]}. \tag{B.19}$$

Then, the transformation rules are simply,

$$\begin{aligned} \delta_\epsilon A_\mu &= \epsilon_+^{\dagger\alpha}\gamma_\mu\phi_+^\alpha + \epsilon_-^{\dagger\dot{\alpha}}\overline{\gamma}_\mu\phi_-^{\dot{\alpha}}, \\ \delta_\epsilon A^i &= \epsilon_+^{\dagger\alpha}\sigma^i_{\alpha\dot{\beta}}\gamma_5\phi_-^{\dot{\beta}} + \epsilon_-^{\dagger\dot{\alpha}}\overline{\sigma}^i_{\dot{\alpha}\beta}\gamma_5\phi_+^{\beta}, \end{aligned} \tag{B.20}$$

where we have split ϵ into ϵ_+^α and $\epsilon_-^{\dot{\alpha}}$ in the same manner we split λ. As we can see, the transformation $\delta_\epsilon \to \delta_{\epsilon_+} + \delta_{\epsilon_-}$. The 16 spercharges remain, but they are split as $\mathcal{N} = (2,2)$ in the sense that they are carried by 2 $\mathbf{4}_s$ and 2 $\mathbf{4}_c$, with the appropriate R-symmetry group.

Bibliography

[1] J. Scherk and J. H. Schwarz, "Spontaneous breaking of supersymmetry through dimensional reduction," *Physics Letters B*, vol. 82, no. 1, pp. 60–64, 1979, ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(79)90425-8. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0370269379904258.

[2] G. Gkountoumis, C. Hull, K. Stemerdink, and S. Vandoren, *Freely acting orbifolds of type iib string theory on T^5*, 2023. arXiv: 2302.09112 [hep-th].

[3] M. Natsuume, *String theory and quark-gluon plasma*, 2007. arXiv: hep-ph/0701201 [hep-ph]. [Online]. Available: https://arxiv.org/abs/hep-ph/0701201.

[4] A. Strominger and C. Vafa, "Microscopic origin of the bekenstein-hawking entropy," *Physics Letters B*, vol. 379, no. 1–4, pp. 99–104, Jun. 1996, ISSN: 0370-2693. DOI: 10.1016/0370-2693(96)00345-0. [Online]. Available: http://dx.doi.org/10.1016/0370-2693(96)00345-0.

[5] C. Callan, D. Friedan, E. Martinec, and M. Perry, "Strings in background fields," *Nuclear Physics B*, vol. 262, no. 4, pp. 593–609, 1985, ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(85)90506-1. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0550321385905061.

[6] J. Polchinski, *String theory I-II. volume II: Superstring theory and beyond*. Cambridge University Press, 1998.

[7] C. V. Johnson, *D-branes*. Cambridge University Press, 2023.

[8] T. Morris, "The quantum equivalence of nambu and polyakov string actions," *Nuclear Physics B*, vol. 341, no. 2, pp. 443–463, 1990.

[9] K. Mayer, "On Quantum Corrections in String Compactifications: Effective Actions and Black Holes," Ph.D. dissertation, University Utrecht, 2020. DOI: 10.33540/136.

[10] G. Mandal, *A review of the d1/d5 system and five dimensional black hole from supergravity and brane viewpoint*, 2000. arXiv: hep-th/0002184 [hep-th].

[11] B. Greene, *String theory on calabi-yau manifolds*, 1997. arXiv: hep-th/9702155 [hep-th]. [Online]. Available: https://arxiv.org/abs/hep-th/9702155.

[12] M. B. Green, J. H. Schwarz, and L. Brink, "Superfield theory of type (ii) superstrings," *Nuclear Physics B*, vol. 219, no. 2, pp. 437–478, 1983, ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(83)90651-X. [Online]. Available: https://www.sciencedirect.com/science/article/pii/055032138390651X.

[13] M. Graña and H. Triendl, *String Theory Compactifications*. Jan. 2017, ISBN: 978-3-319-54315-4. DOI: 10.1007/978-3-319-54316-1.

[14] S. Giaccari and R. Volpato, "A fresh view on string orbifolds," *Journal of High Energy Physics*, vol. 2023, no. 1, Jan. 2023, ISSN: 1029-8479. DOI: 10.1007/jhep01(2023)173. [Online]. Available: http://dx.doi.org/10.1007/JHEP01(2023)173.

[15] L. Dixon, J. Harvey, C. Vafa, and E. Witten, "Strings on orbifolds," *Nuclear Physics B*, vol. 261, pp. 678–686, 1985, ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(85)90593-0. [Online]. Available: https://www.sciencedirect.com/science/article/pii/0550321385905930.

[16] K. C. Stemerdink, "Black Holes from Branes: Various string theoretical constructions," Ph.D. dissertation, 2022. DOI: 10.33540/1379.

[17] W. Taylor, *Lectures on d-branes, gauge theory and m(atrices)*, 1998. arXiv: hep-th/9801182 [hep-th].

[18] A. Pais, "On Spinors in n Dimensions," *Journal of Mathematical Physics*, vol. 3, no. 6, pp. 1135–1139, Nov. 1962, ISSN: 0022-2488. DOI: 10.1063/1.1703856. eprint: https://pubs.aip.org/aip/jmp/article-pdf/3/6/1135/19064337/1135_1_online.pdf. [Online]. Available: https://doi.org/10.1063/1.1703856.

[19] J. R. David, G. Mandal, and S. R. Wadia, "Microscopic formulation of black holes in string theory," *Physics Reports*, vol. 369, no. 6, pp. 549–686, Nov. 2002, ISSN: 0370-1573. DOI: 10.1016/s0370-1573(02)00271-5. [Online]. Available: http://dx.doi.org/10.1016/S0370-1573(02)00271-5.