On the Tameness of Perturbative Expansions

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Abstract

The notion of tameness is believed to bear a significant meaning for fundamental physics: according to the Tameness Conjecture, partition functions and correlation functions for UV-consistent theories are tame functions. In this thesis we first introduce the notion of tame structures and functions and illustrate their properties. Secondly, we present the theory of Borel summability and how it is necessary to understand the non-analytic behaviour of path integrals in the weak-coupling limit. These methods will be then applied to the study of partition and correlation functions of certain quantum field theories on a point-like spacetime: by viewing them as the Borel sums of their asymptotic expansions, we will show explicitly that they are tame functions of the tame structure known as $\mathbb{R}_{\mathscr{G}}$. Thereafter, we expound the basics of constructive field theory and, relying on results obtained thereby, we extend our arguments for the tameness of partition functions to more general theories. Finally, we explain how asymptotic expansions appear in quantum mechanics when treated with the exact WKB method; we discuss how the occurrence of Stokes phenomenon prevents us from establishing the tameness of the energy eigenvalues.

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Contents

Introduction

The aim of this thesis is to study the properties of partition functions and correlation functions non-perturbatively. The property at the focus of our interest is that of *tameness*, or *o-minimality*: we will be concerned with proving that, in certain quantum field theories, such physically-meaningful functions are tame. In this introduction, we present the main subjects of study of this thesis and outline the thread of reasoning which motivates each step and explains the meaning of our results.

Tameness

The origin of the concept of tameness, or o-minimality, lies in model theory and logic. Logic deals with formal systems, which are built out of a set of inference rules – formally establishing how to reach a conclusion by a train of deductions – and a *language*. A language should be thought as nothing short of a formalization of the human language: it consists of a set of variables (about which we want to make statements, or *formulas*) and a basic set of 'words', collected in a 'dictionary', by whose combinations a large set of statements can be expressed to predicate the variables.
Of course, the larger is the 'dictionary', the more powerful is the language. In practice, we are interested in languages which predicate numbers and, therefore, can specify to which set a number belongs; yet, due to the different extension of their dictionaries, not all languages are able to define all numerical sets. For instance, a statement like

$$n \in \mathbb{N}$$

can be expressed only by a language which can define the set of integers $\mathbb{N}$. If, moreover, that same language allows for operations such like sums and products, it is said to be a language that *contains arithmetic*. These apparently innocent and mild assumptions bring about several serious consequences, rigorously expressed by theorems known under the name of their discoverers: Kurt Gödel, Alfred Tarski and Alonzo Church. These theorems establish that formal systems containing arithmetic are afflicted by some severe limitations: thus, it is sensible to distinguish formal systems of logic between those which contain arithmetic and those which do not. The latter systems are instances of *o-minimal* (or *tame*) *structures*.

A tame structure can in fact be viewed as a logic language; however, the underlying logic interpretation can be substituted with a more intuitive and pictorial geometric perspective. Adopting the latter, o-minimal structures are then viewed as families of sets (*tame* sets), closed under certain set-theoretic operations (such as union, intersection...) which have a direct logic counterpart. The great advantage of this approach is that, in a geometrical framework, the absence of arithmetic can be straightforwardly formulated: every subset of $\mathbb{R}$ residing in an o-minimal structure is always a *finite union of points and intervals*, which the set of natural numbers $\mathbb{N}$ clearly is not. Moreover, functions can be associated with sets by their graphs: a *tame function* will be then a function whose graph is a tame set. The concept of tame function will be at the focus of our interest throughout this thesis.

Tame structures are conjectured to bear a profound physical meaning in fundamental physics. In particular, according to the *Tameness Conjecture*, it is expected that partition functions and other physical observables,

such as correlation functions, should be tame functions if they result from a valid low-energy effective field theory. The purpose of this thesis is then to test this conjecture and to understand under which conditions such functions can indeed be proved to be tame. A thorough explanation of the concepts anticipated hitherto will be the content of Chapter 1.

Partition functions and correlation functions

Perturbative QFT

Having clarified our goal, the next step to take is to understand in depth the properties of partition functions and correlation functions, found by field insertions in the path integrals. The foremost difficulty soon arises: it is almost always impossible to compute a path integral exactly, so that one has to proceed perturbatively via Feynman diagrams. A m-point correlation function is usually expressed by a path integral like

$$\mathcal{A}(\lambda) = \int \mathcal{D}\phi \; e^{-S_0[\phi]-\lambda S_{int}[\phi]}\phi(x_1)...\phi(x_m)$$

where we can assume for the moment that the path integral measure $\mathcal{D}\phi$ is normalized. If there are no field insertions, i.e. $m = 0$, the path integral simply describes the partition function $Z(\lambda)$. The perturbative approach consists in Taylor-expanding the interaction term $e^{-\lambda S_{int}[\phi]}$ and then *commuting the sum with the path integral*; after that, the remaining integrals are Gaussian and can be computed, at every order n, as a finite sum over Feynman diagrams with n vertices. Yet, let us emphasise from now that the commutation of the sum with the path integral is *not* allowed. The sanction for this transgression is that, at every order n of perturbation, the number of diagrams will grow roughly as $(pn)!$, where p is the order of the interaction; because of such a rapid growth, Feynman diagrams are often said to *proliferate*. As a consequence, the perturbative expansion

$$\mathcal{A}(\lambda) \sim a_0 + a_1\lambda + a_2\lambda^2 + ...$$

is *divergent* for every non-vanishing value of the coupling λ. Therefore, perturbative power series are not convergent series, but rather divergent, *asymptotic* series. This feature has been known since the arguments put forward by Dyson [Dys52].

Unlike convergent series, divergent series cannot be summed in the usual sense. Nevertheless, asymptotic series can still capture the behaviour of the original function $\mathcal{A}(\lambda)$, if properly treated. The method that allows to *resum* divergent series is known as *Borel resummation*, which will be extensively dealt with in Chapter 2. Once this procedure is effected, the asymptotic series is turned back into the function expressed by the path integral, which we were unable to compute explicitly. As it will be clarified, this function is smooth, but *non-analytic in the weak-coupling limit* $\lambda = 0$. Indeed, the Taylor series at $\lambda = 0$ of the Borel sum will be nothing else than the asymptotic expansion: in other words, the derivatives of the Borel sum at $\lambda = 0$ are

$$\frac{d^n}{d\lambda^n}\mathcal{A}(\lambda)\Big|_{\lambda=0} = n!a_n$$

and thus, since the Taylor series is divergent, $\mathcal{A}(\lambda)$ is not analytic at 0.

The Borel resummation method, in its most elementary form, only applies to the so-called *Borel-summable* power series. Those power series which fail to satisfy certain constraints are not Borel-summable and must be treated with more refined techniques, known as *resurgence* and *alien calculus*. The basics of these methods will be presented in Chapter 2. Still, our main focus will be on Borel-summable power series: as it will be explained at the end of the same Chapter, once Borel-resummed these series give rise to tame functions, hosted in the o-minimal structure known as $\mathbb{R}_{\mathscr{g}}$. In light of this realisation, we will be able to present our first results in Chapter 3. We will select some very simple quantum field theories on a point-like spacetime and show explicitly that their partition and correlation functions are tame, as it should be expected according to the Tameness Conjecture.

Constructive QFT

Perturbative series in powers of the coupling are by far the most standard approach to the solution of path integrals. The brilliant idea underlying this method is the possibility of solving Gaussian integrals by Wick's theorem, and to pictorially represent every type of contraction by a Feynman diagram. Amplitudes are often computed by adding up all the *connected* Feynman diagrams, generated by the logarithm of the partition function. As argued earlier though, the number of Feynman diagrams, (even if only the connected ones are considered) grows too quickly and the perturbative series is divergent, rendering the expansion asymptotic.

That perturbative expansions are only asymptotic can be regarded as a serious flaw. Indeed, an asymptotic expansion is asymptotic to infinitely many functions, in analogy with the fact that a convergent series is the Taylor series of infinitely many smooth functions. Therefore, if we want a quantum field theory (QFT) to be truly predictive in the strictest of senses, we should be able to express observables by convergent series, rather than perturbative asymptotic series. This task requires us to forsake Feynman diagrams, whose sums lead to series in powers of the coupling, and substitute them with more sophisticated expansions. This is the achievement of *constructive QFT*.
Constructive QFT consists in computing the equivalent of the sum over connected Feynman diagrams by replacing them by *spanning trees*. Intuitively, spanning trees are graphs that touch all the vertices of Feynman diagrams but have no loops. This feature turns out to be crucial, as it can be proved to ensure the convergence of the sum over spanning trees; the downside though, is that such a sum no longer groups together different powers of the coupling λ. Yet, the main purpose is fulfilled: the convergent sum over spanning trees directly performs the Borel resummation of the perturbative series, so that no ambiguity due to the divergence mars the predictive power of the theory. By constructive methods, a unique, well-defined function of the coupling λ is produced for both the partition function and the correlation functions.

We will provide in Chapter 4 a brief introduction to constructive QFT and to the related technique known as *Loop Vertex Expansion* (LVE). We will then rely on results obtained by these methods to argue that more general partition functions are Borel-summable and therefore tame. Thus, further endorsement to the Tameness Conjecture will be provided.

The eigenvalue problem in quantum mechanics

The Borel resummation procedure, jointly with alien calculus, enables us to treat any kind of divergent power series. It is therefore interesting to explore other scenarios in which perturbation theory yields divergent series which must be Borel-resummed; moreover, thanks to our understanding of Borel sums as tame functions of the o-minimal structure $\mathbb{R}_{\mathscr{G}}$, we aim at testing the Tameness Conjecture on other observables than QFT partition and correlation functions.

Quantum mechanics is another field of Physics where perturbative power series are ubiquitous. Very few time-independent Hamiltonians admit an exact solution, the most notable of which being the harmonic oscillator and the hydrogen atom. In the remaining cases, the wavefunction solutions to Shrödinger's equation and the energy eigenvalues need to be expressed by a perturbative series in powers of a small parameter, tuning the anharmonic part of the potential and usually reabsorbed into $\hbar$. Such series are again divergent and must be Borel-resummed. An explicit computation of the perturbative expansion for the energy eigenvalues can be found by means of the *exact WKB method*: thereby, exact quantisation conditions can be formulated which fix the perturbative expansion at every order.

As it will be explained, the quantum tunnelling between different potential wells spoils the Borel summability of the perturbative expansion, causing the occurrence of the *Stokes phenomenon*. The Stokes phenomenon demands a more careful treatment: it needs to be understood in terms of the aforementioned alien calculus, and requires the perturbative power series to be promoted to *transseries*. In this context, we will be unable to argue the tameness of the energy eigenvalues. When, on the contrary, the potential will be

convex (so that only one potential well is present and no quantum tunnelling occurs) the energy eigenvalues will be again tame functions of $\mathbb{R}_{\mathscr{G}}$.

To investigate the tameness of the energy as function of $\hbar$ (rescaled by the absorption of the coupling constant) in presence of the Stokes phenomenon, we were led to search for a differential equation formally obeyed by its asymptotic expansion. This challenging question was addressed by a direct numerical computation illustrated in Chapter 5, after an introduction to the exact WKB method and an explanation of how to calculate an explicit transseries solution for the energy, in the case of a cubic potential.

Chapter 1

O-minimal structures

In this chapter, we present the basics of the theory of o-minimal structures. O-minimal structures have been introduced and studied at length within the purview of logic and model theory; however, as it will be clarified in this chapter, they can be regarded as purely geometric objects. In fact, a structure is essentially a infinitely large family of sets, closed under several set-theoretic operations: namely union, intersection, complement, Cartesian products and projections. These operations are mirrored by the logic operators $\wedge, \vee, \neg, \exists$ and $\forall$: a structure can therefore be viewed as the geometric counterpart of first-order logic, created by assigning to each logic formula a set where it holds true.

O-minimal structures are special structures which correspond to logic formal systems where arithmetic is not defined. As it is well-known, formal systems strong enough to define arithmetic are affected by Gödel's incompleteness theorems, Church's undecidability theorem and Tarski's theorem of undefinability of truth. All these theorems will not hold in the logic system described by an o-minimal structure. Moreover, when viewed as geometric objects, o-minimal structures are an instance of the *topologie modérée*, envisioned by Groethendick in his *Esquisse d'un programme* [Gro97]: a *tame* topology where some pathological sets (e.g. those with a boundary of their same dimension) are ruled out.

After a brief review of the logic viewpoint on o-minimal structures in the first section, we adopt the geometric approach and, mostly following [Dri98], we study the properties of o-minimal structures and the functions that they host, i.e. the *tame functions*. We then give an extensive (though incomplete) list of known o-minimal structures: in especial, the structure known as $\mathbb{R}_{\mathscr{G}}$, which will be at the core of the results presented in Chapters 3 and 4, is introduced. In section 1.6 we deal with subspecies of o-minimal structures, known as sharply o-minimal structures, which allow for a notion of finite complexity to be defined. Finally, in section 1.8, we explain how o-minimality is conjectured to be a key concept in fundamental physics, motivating thus our interest in the properties of o-minimal structures and our endeavours to establish that partition functions and correlation functions are tame.

1.1 First-order logic and structures

O-minimal structures are geometrical objects intended to mirror the properties of first-order logic systems without arithmetic. Let us then review some basic notions of first-order logic before delving into their geometrical and more pictorial counterparts.

1.1.1 First-order logic

Let us assume that some variables x, y range over non-empty sets X, Y. We can then express statements concerning these variables by specifying relations $\phi(x,y), \psi(x,y), ...$ In logic, these are often referred to as *predicates* or *formulas*. Such statements can be either true or not, and their validity will define other sets, such as

$$\Phi \subseteq X \times Y := \{(x,y) \in X \times Y : \quad \phi(x,y) \quad \text{is true}\}. \tag{1.1}$$

Albeit the approach may look already pretty geometrical, it should not be understood as such yet. Despite their familiar interpretation, the variables x, y, need not stand for numbers, nor need the sets X, Y be subsets of a geometric set. For example, you can think of the following scenario:

$$\begin{aligned} X &:= \text{European countries} \\ Y &:= \text{The alphabet} \\ \phi(x,y) &:= x \ \text{ starts with letter } \ y \ . \end{aligned}$$

By these definitions, one has that the set $\Phi \subseteq X \times Y$ as above comprises the element $(\text{France}, \text{F})$ but not the element $(\text{Germany}, \text{B})$.

There are several operations which can be performed on logic formulas such as ϕ and ψ. As shown in Table 1.1, all of these are mirrored by an operation on the sets Φ, Ψ defined as the sets where the respective formulas hold.

Name in logic	**Logical operation**	**Set-theoretic operation**	**Name in set theory**
Conjunction	$\phi(x,y) \wedge \psi(x,y)$	$\Phi \cup \Psi$	Union
Disjunction	$\phi(x,y) \vee \psi(x,y)$	$\Phi \cap \Psi$	Intersection
Negation	$\neg\phi(x,y)$	Φ^c	Complementary
Existential Quantification	$\exists y \phi(x,y)$	$\pi_X(\Phi)$	Projection

Table 1.1: Correspondence between logic and set-theoretic operations

In the last line projection onto X acts intuitively as

$$\pi_X : \Phi \to X \tag{1.2}$$
$$\pi_X((x,y)) = x \tag{1.3}$$

so that, to be more precise, the set $\{x \in X : \exists y \phi(x,y)\}$ can be found as $\pi_X(\Phi)$. The universal quantification $\forall$ can be expressed by combining the aforementioned operations. Indeed, for a given formula ϕ, one has

$$\forall y \phi(x,y) = \neg(\exists y(\neg\phi(x,y))). \tag{1.4}$$

Therefore, the set $\{x \in X : \exists y \phi(x,y)\}$ can be found as $(\pi_X(\Phi^c))^c$. Observe finally that the set-theoretic operation of the Cartesian product is also needed to represent logic operations in set theory. The Cartesian

product is needed in (1.1) to define the sets Φ and Ψ and, moreover, given two statements $a(x)$ and $b(y)$ valid on $A \subseteq X$ and $B \subseteq Y$, the conjunction $\wedge$ reduces to a Cartesian product:

$$\{(x, y) : a(x) \wedge b(y)\} = A \times B. \tag{1.5}$$

The first three operations in Table 1.1 are special. The absence of the quantifiers defines the so-called **propositional logic**: in this kind of logic the truth or falsehood of formulas depend in an easy way on the truths of its atomic formulas. In fact, in the mid XIX century George Boole realised that upon assigning the numbers 1 and 0 to true and false formulas respectively, it is possible to compute the truth of a compound formula arithmetically. In particular, it is easy to convince oneself that, if the truths of ϕ and ψ are respectively a and b, under the three operations the truth of the compound statement is given by Table 1.2:

Operation	**Truth**
$\phi \wedge \psi$	ab
$\neg\phi$	$1 - a$
$\phi \vee \psi$	$a + b - ab$

Table 1.2: Truth under the operations of a Boolean algebra.

The last row is easily derived as a consequence of the first two by expressing $\phi \vee \phi = \neg((\neg\phi) \wedge (\neg\psi))$, whence $1 - (1 - a)(1 - b) = a + b - ab$. The above table implies that the truth of any logic formula, however complicated, can be computed by simple arithmetic operations on the truths of the elementary formulas it contains.

The set-theoretic counterpart of propositional logic deserves the special name of *Boolean algebra*.

Definition 1.1 (Boolean Algebra)**.** A **Boolean algebra** of subsets of a set X is a nonempty collection $\mathcal{C}$ of subsets of X which is closed under the operations of Union, Intersection and Complementary.

Note that the subtraction between sets $A \backslash B$ can be written as $A \cap B^c$, thus a Boolean algebra is closed also under subtractions. Furthermore, the logic implication $\phi \to \psi$ can be also written in terms of the operations of a Boolean algebra. Indeed it can be written as

$$\phi \to \psi \qquad \Leftrightarrow \qquad \neg(\phi \wedge \neg\psi)$$

where the right-hand side has clearly a set-theoretic counterpart, according to Table 1.1.

As soon as the quantifiers $\exists, \forall$ are allowed for, no such simple table as 1.2 can be written down; we are no longer dealing with propositional logic, but rather with **predicative logic**. Under the assumption that quantifiers are only allowed to act on variables $x, y, ...$ but not on predicates $\phi(x, y), \psi(x, y), ...$ the predicative logic is said to be **first-order**.

1.1.2 Languages and model-theoretic structures

Model-theoretic structures provide a bridge between logic and geometry. The former discussion is way too general, as it puts no constraints on the nature of variables and their relations. Languages and model-theoretic structures can rigorously narrow the range in which the variables x, y can vary, and consequently constrain the formulas $\phi(x, y, ...)$. They are defined as follows:

Definition 1.2 (Model-Theoretic Structure)**.** A **model-theoretic structure** $\mathcal{R} = (R, (S_i)_{i \in I}, (f_j)_{j \in J})$ is a tuple consisting of a set R, called the **underlying set**, a set of relations $S_i \subseteq R^{m(i)}, m(i) \in \mathbb{N}$, called **primitive relations**, and a set of functions $f_j : R^{n(j)} \to R, n(j) \in \mathbb{N}$, called **primitive functions**.

For instance, an abelian group A generates a model-theoretic structure $\mathcal{A} = (A, 0, +, -)$, with 0 being the identity, $+$ the group operation and $-$ the operation of taking the inverse (both are primitive functions). An ordered ring[1] R instead will generate $\mathcal{R} = (R, <, 1, 0, +, -, \cdot)$ where now to the functions $+, \cdot : R^2 \to R$ and $- : R \to R$ the relation $<$ has been added. Thus, we can see that now we have a logical framework where our statements are much more 'mathematical' and, more precisely, geometrical. All variables must now take values in a unique set R and their relations and functions can always be written as combinations of well-specified primitive relations and functions (in simplest cases mentioned above there is but a finite number of them).

Another useful notion is that of *language*. This is closely related to that of model-theoretic structure by the notion of L-structure.

Definition 1.3 (Language, L-structure)**.**

1. A **language** L is a disjoint union of two sets: a set of relation symbols and a set of function symbols. Each relation symbol S and function symbol f is equipped with an integer called -arity, which refers to the number of variables they can predicate. 0-ary symbols are called constants; 1-ary and 2-ary symbols are said unary and binary respectively.

2. A **L-structure** is a model-theoretic structure $\mathcal{R} = (R, (S^{\mathcal{R}}), (f^{\mathcal{R}}))$. $S^{\mathcal{R}}$ and $f^{\mathcal{R}}$ are the interpretations of the relation symbols and the function symbols of the language L: if S is a m-ary relation symbol then $S^{\mathcal{R}} \subseteq R^m$; if f is a n-ary function symbol then $f^{\mathcal{R}} : R^n \to R$.

For example, the language of ordered abelian groups is $L_{\mathrm{ab}}(<) = (<, 0, -, +)$ has a constant symbol 0, a unary function symbol $-$, a binary function symbol $+$, and a binary relation symbol $<$.

Constant symbols can be thought of as functions $f : R^0 \to R$. However, we can also add them directly by declaring that, given a model-theoretic structure $\mathcal{R} = (R, (S_i)_{i\in I}, (f_j)_{j\in J})$, we define $\mathcal{R}_C$ to be an extended model-theoretic structure $\mathcal{R}_C := (R, (S_i)_{i\in I}, (f_j)_{j\in J}, (c)_{c\in C})$ with $C \subseteq R$.

Let us summarise the thread of our reasoning. One can define a language L where all the statements of one's interest are expressible. The language L predicates variables taking values in a set R by means of relations S_i and functions f_j. The statements produced thus by our language can then be readily translated into a model-theoretic structure $\mathcal{R}$ over the set R, which is called L-structure after the language whose properties it inherits. Here, the relations S_i and the functions f_j are 'interpreted' as $S_i^{\mathcal{R}}$ and $f_j^{\mathcal{R}}$.

[1] A brief recap of the terminology might be expedient.

- A **ring** is an abelian group $(R, 0, +, -)$ equipped with an associative multiplication $\cdot$ such that $+$ is distributive with respect to the multiplication $\cdot$. $-$ is the inverse operation of the group operation $+$, while and 0 is the identity thereof.
- An **ordered ring** is a ring equipped with a binary order relation $<$ such that:
  - $0 < 1$;
  - $x < y \implies x + a < y + a$ (translational invariance);
  - If $0 < a$, then $x < y \implies xa < ya$ (invariance under multiplication by positive numbers).
- A **division ring** is a ring such that the multiplication $\cdot$ has an inverse for every $x \neq 0$: thus it is a group also with respect to the multiplication, whose identity is denoted by 1.
- An **ordered field** is an ordered division ring which is abelian with respect to the multiplication $\cdot$. We denote it then by $(R, <, 0, 1, +, -, \cdot)$: note that the inverse of the multiplication is not written to remind ourselves, that, as opposed to $-$, it is not defined for every element in R.
- An **real closed field** is an ordered field with the intermediate value property. Given $a < b$ and a polynomial f, then if $f(a) < f(b)$ there must exist a c such that $a < c < b$ and $f(c) = 0$.

1.1.3 Structures

A model-theoretic structure is an instance of a more general object, called (unsurprisingly) structure. We can now turn to their definition and see how model-theoretic structures belong to their kind.

Definition 1.4 (Structure)**.** A **structure** on a nonempty set R is a sequence $(\mathcal{S}_m)_{m\in\mathbb{N}}$ such that:

1. For every m, $\mathcal{S}_m$ is a Boolean algebra of R^m;
2. If $A \in \mathcal{S}_{m+1}$ then $\pi(A) \in \mathcal{S}_m$ where $\pi : R^{m+1} \to R^m$ is the projection onto the first m coordinates;
3. If $A \in \mathcal{S}_m$ then $R \times A$ and $A \times R$ belong to $\mathcal{S}_{m+1}$.

The first axiom guarantees that each element of the sequence $\mathcal{S}_m$ be a Boolean algebra (or, if you will, a propositional logic system). The second axiom instead corresponds to the fourth row of Table 1.1: it implements the existential and the universal quantifiers respectively. Thus they extend propositional logic to the more general predicative logic.

As the name suggests, structures can be naturally connected to model-theoretic structures: given $\mathcal{R} = (R, (S_i)_{i\in I}, (f_j)_{j\in J})$ it suffices to state that $\mathrm{Def}(\mathcal{R})$ be the smallest structure such that every set defined by the truth of a logical statement (expressed through the relations S_i and functions f_j) belongs to a Boolean algebra $\mathcal{S}_n$, for some $n \in \mathbb{N}$. In the following we will be mostly concerned with the structure $\mathrm{Def}(\mathcal{R}_R)$, namely a structure where all the items in R are considered as acceptable constant symbols (0-ary functions).

We are now ready to deal with notion of o-minimality, which will be at the focus of our interest throughout this thesis. First, the "o" stands for order: we need to require that R be a dense, linearly ordered set. By dense, we mean that the order relation obeys Archimedes's axiom: for any a, b such that $a < b$ there exists c such that $a < c < b$. Having fixed this convention, we can finally state the definition of the subclass of structures known as o-minimal structures.

Definition 1.5 (O-Minimal Structure)**.** Given a dense linearly ordered nonempty set without endpoints $(R, <)$, an **o-minimal structure** on $(R, <)$ is by definition a structure over R such that

the sets in $\mathcal{S}_1$ are finite unions of intervals and points.

Here intervals can also include the endpoints $\pm\infty$: these are defined such that $-\infty < a < \infty$, $\forall a \in R$.

Hence, hearkening back to our previous discussions, we can say now that a model-theoretic structure $\mathcal{R} = (R, <, ...)$ is o-minimal if $\mathrm{Def}(\mathcal{R}_R)$is an o-minimal structure[2]. Remark that the $\mathcal{R}$ must include the order binary relation $<$ for the notion of o-minimality to be well-defined.
As an example, if $\mathcal{G} = (R, <, 0, -, +)$ is an ordered abelian group and $\mathrm{Def}(\mathcal{G}_R)$ (or, for brevity, simply $\mathcal{G}$) is an o-minimal structure, we will say that $\mathrm{Def}(\mathcal{G}_R)$ is an **o-minimal expansion** of the the abelian group $\mathcal{G}$; similarly, if $\mathcal{G}' = (R, <, 0, 1, -, +, \cdot)$ is an ordered ring, we will say that $\mathrm{Def}(\mathcal{G}'_R)$ is an o-minimal expansion of $\mathcal{G}'$ if it is o-minimal. Intuitively, the o-minimal expansion of an ordered group provides the sets that can be defined by only using the group operations and the first-order logic operators $\wedge, \vee\neg, \forall, \exists$.

Before moving on to the next section and explore in detail the properties of o-minimal structures, let us summarise the relationship between logic and structures. A given first-order logic formal system **F**, can be turned into a geometric structure in the following steps:

- Consider the language L which the formal system **F** employs.

[2]By a slight abuse of notation, we will often refer to $\mathrm{Def}(\mathcal{R}_R)$ as simply $\mathcal{R}$ and use the latter symbol for (o-minimal) structures.

- Consider the L-structure associated with L: this is a model-theoretic structure $\mathcal{R}$ which contains the interpretations of the relation and function symbols of the considered language; moreover, it specifies the set R where the variables predicated by L live.
- Consider the structure $\mathrm{Def}(\mathcal{R}_R)$. If R is a dense linearly-ordered set, we may inquire as to the o-minimality of such structure. All the operations in set theory of the sets that belong to this structure are the exact mirror image of logic operations that can be taken within the initial formal system **F** (which includes the language L), according to Table 1.1.

Suppose now that, as we shall consistently assume, $R = \mathbb{R}$ is the real line, i.e. the real closed field of the real numbers. Then we can observe that, if $\mathrm{Def}(\mathcal{R}_R)$ is o-minimal according to Definition 1.5, and $\mathcal{R}_R$ is a L-structure for some language L in a formal system **F**, then **F** cannot be 'strong enough' to include arithmetic. Indeed, arithmetic requires that all the integers be correctly defined, which is exactly what o-minimality (1.5) prevents us from doing: a definition of the whole set of integers $\mathbb{N}$ is altogether lacking. Therefore, there can be in **F** no such expression as "$n \in \mathbb{N}$...", for there is no definition of what is meant by $\mathbb{N}$. As a consequence, the formal system **F** is unaffected by such theorems as Gödel's, Tarski's, and Church's theorems, for which we refer to Appendix A.

1.2 Properties of o-minimal structures

In the previous section we have introduced o-minimal structures from both a model-theoretic and a geometrical point of view. In this and the following sections, we will mostly adopt the latter; however, the logic interpretation of structures will always lie in the background, and we shall occasionally resort to it when need be. As it will become clear, geometrical facts about o-minimal structures can be proved easily by exploiting the closure properties of the structure under first-order logic operations. Let us then list the most significant properties of a fixed o-minimal structure on $(R, <)$, denoted explicitly by $\mathcal{R} = (R, <, \mathcal{S})$, where $\mathcal{S} = (\mathcal{S}_m)_{m\in\mathbb{N}}$ is the sequence of Boolean algebras.

1.2.1 Elementary properties

In order to describe the properties of o-minimal structures, let us first fix some terminology. If a set A belongs to $\mathcal{S}_m$ for some integer m, then we will say that A is **$\mathcal{R}$-definable** or more simply **definable** (when it is clear from context to which structure $\mathcal{R}$ we are referring). Furthermore, a function $f : R^n \to R$ is said to be a ($\mathcal{R}$-)**definable function** if its graph $\Gamma(f)$ is a definable set. Remark that, if $(R, <, \mathcal{S})$ is a L-structure for some language L, all the function symbols $(f_j)_{j\in J}$ of L (or, more precisely, all their interpretations as primitive functions in the corresponding L-structure) are definable functions; however, they are by no means *all* the definable functions: there are many more.

Let us then state some properties obeyed by definable sets.

Lemma 1.1. If $A \subset R$, is a definable set, then:

1. $\inf(A)$ and $\sup(A)$ exist in $R_\infty = R \cup \{\pm\infty\}$ (Dedekind completeness) and are definable.
2. ∂A, i.e. the boundary of A, is definable, finite and if $\partial A = \{a_1, .., a_N\}$, with $a_1 < a_2 < ... < a_N$, then every open interval (a_i, a_{i+1}) either a part of A or it is disjoint from A.
3. More generally, if $X \subset R^m$ is definable, the closure and the interior of X, $\mathrm{cl}(X)$ and $\mathrm{int}(X)$, are definable.

The proof of the statements above displays all the power of the correspondence between logic formulas and

operations in set theory. Indeed, if we manage to *define* a set Ω_A by using first-order logic and starting from a given set A which we know to be definable in a structure (o-minimality is not yet needed), we automatically guarantee that the set Ω_A can be retrieved by an appropriate sequence of set-theoretic operations on A; the translation between the logic formulas and set-theoretic operations is again given by the Table 1.1. The axioms stated in Definition 1.4 will then ensure that, upon taking such operations, the resulting set will still lie within the same structure (even at every in-between step). This is why definability is called so: it is equivalent to the property of being defined by first-order logic.

Proof.

1. We can write the definition of $\inf(A)$ as follows
$$\inf(A) = \{x \in R_\infty : \ \forall y \in R \ \exists a \in A(x < y \ \rightarrow \ x < a < y)\} \tag{1.6}$$
which ensures it to be definable (similar for $\sup(A)$). Furthermore, as $A \in \mathcal{S}_1$, it must be given by a union of intervals and points. It is easy to convince ourselves that the addition of $\{\pm\infty\}$ is in general necessary.

2. The finiteness of ∂A is again a direct consequence of the fact that $A \in \mathcal{S}_1$. The second claim is also due to the fact that $A \in \mathcal{S}_1$ plus the very definition of boundary in first-order logic terms, which ensures definability:
$$\begin{aligned}\partial A = \{x \in R : \ \forall y \in R \, \forall z \in R \, (\exists a \in A) \wedge (\exists b \notin A \ ) \\ \wedge \, [z < x < y \rightarrow (z < a < x < b < y) \vee (z < b < x < a < y)]\}.\end{aligned} \tag{1.7}$$

3. As previously argued, we merely have to write the definition of closure and interior in first-order logic to claim that $\mathrm{cl}(X)$ and $\mathrm{int}(X)$ belong to the same structure as X: o-minimality will be inherited directly from X. We have that
$$\begin{aligned}\mathrm{cl(X)} =&\{(x_1, ..., x_m) \in R^m : \ \forall (y_1, ..., y_m) \in R^m, \ \forall (z_1, ..., z_m) \in R^m \\ &[((z_i < x_i < y_i) \forall i (1 \leq i \leq m)) \rightarrow \exists (a_1, ..., a_m) \in X \wedge ((z_i < a_i < y_i) \forall i (1 \leq i \leq m))]\}\end{aligned} \tag{1.8}$$
$$\mathrm{int}(X) = \{(x_1, ..., x_m) \in X : \exists (a_1, ..., a_m) \in X \ \exists (b_1, ..., b_m) \in X \ (a_i < x_i < b_i) \forall i (1 \leq i \leq m)\} \tag{1.9}$$

whence it follows that $\partial X = \mathrm{cl}(X) \backslash \mathrm{int}(X)$ is also definable. □

Despite the above logical definitions being cumbersome, they are actually much more easily wielded than any sequence whatsoever of operations on sets that we may try to write down (and much more readable).

1.2.2 Definable functions

Let us state here some basic properties of definable functions that will prove to be useful later. First, the definability of a function f is related to that of its domain and codomain.

> **Lemma 1.2.** If a function $f : A \rightarrow R$, with $A \subseteq R^m$ is definable, then A is definable and the codomain $f(A)$ is also definable.

Proof. For the first part, it suffices to observe that A can be written as the projection $\pi_m : R^{m+1} \rightarrow R^m$ onto R^m of the graph of f, $\Gamma(f)$: namely, $A = \pi_m(\Gamma(f))$. Similarly, the codomain $f(A)$ can be viewed as $\pi_1(\Gamma(f))$, where now $\pi_1 : R^{m+1} \rightarrow R$ is the projection onto the last coordinate. □

In order to prove that certain functions are definable in an o-minimal structure, which will be the main purpose of Chapter 3, it will be important to know that definability is preserved under several operations. In the following, let us fix an o-minimal structure $\mathcal{R}$ where addition and multiplication are defined. For instance, as it will be mostly assumed later, this can be a real closed field $\mathcal{R} = (R, <, 0, 1, +, -, \cdot)$.

Lemma 1.3. Given two definable functions $f, g : A \to R$, with $A \subseteq R^m$ a definable set, one has

1. $(f + g) : A \to R$ is definable;
2. $(fg) : A \to R$ is definable;
3. If $f : A \to f(A)$ is injective, then $f^{-1} : f(A) \to A$ is definable.

If, on the other hand, $h : R \to R$ is definable, one also has

4. $(h \circ f) : A \to R$ is definable.

The first two statements refer to the sum and the product of the functions: thus the proof of definability will require the operations $+, \cdot$ to be binary functions of the o-minimal structure $\mathcal{R}$. On the other hand, the third and fourth statements do not require these assumptions and hold in any structure.

Proof.

1. The graph of $(f + g)$ can be written as
$$\Gamma(f + g) = \{(x, y) \in A \times R : \ f(x) + g(x) - y = 0\} \tag{1.10}$$
which must be a definable set as the operations of $+$ and $-$ belong to $\mathcal{R}$.
2. Similarly, the set
$$\Gamma(fg) = \{(x, y) \in A \times R : \ f(x) \cdot g(x) - y = 0\} \tag{1.11}$$
is a definable set as the operations $\cdot$ and $-$ are binary functions included in the definition of $\mathcal{R}$.
3. If f^{-1} exists, which is guaranteed by injectivity, its graph must be the same as that of f:
$$\Gamma(f^{-1}) = \{(x, y) \in A \times R : \ x = f^{-1}(y)\} = \{(x, y) \in A \times R : \ y = f(x)\} = \Gamma(f) \tag{1.12}$$
which automatically gives the definability of f^{-1}.
4. The graph of the composition can be written as
$$\Gamma(h \circ f) = \{(x, y) \in A \times R : \ \forall z \in f(A) \ (y = h(z)) \wedge (z = f(x))\}, \tag{1.13}$$
which is a definable set as it is described by first-order logic statements $f(A)$ is definable.

□

1.2.3 The Monotonicity Theorem

Definable functions in an o-minimal structure are subject to some regularity conditions, as their graphs must be definable sets in that same structure. One of such regularities is expressed for one-variable functions by the following

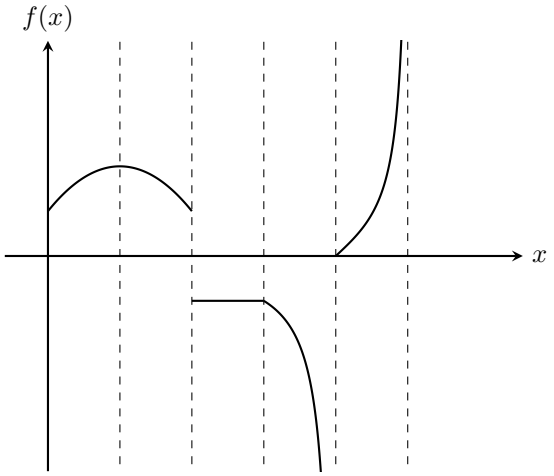


Figure 1.1: A tame function $f(x)$ is always piece-wise monotonic and continuous or constant.

Theorem 1.1 (Monotonicity thoerem)**.** Given an o-minimal structure $(R, <, \mathcal{S})$, let (a, b) be an open interval in R_∞ and $f : (a, b) \to R$ a definable function. Then there exists a finite sequence of points $a =: a_0 < a_1 < ... < a_k < a_{k+1} := b$ in (a, b) such that in every subinterval (a_j, a_{j+1}) f is either constant or strictly monotone and continuous.

This theorem captures an essential feature of one-variable functions: they have a finite number of discontinuities. This is quite remarkable, considering that we have not made any explicit request on the functions themselves; the only constraint is that their graphs can be fitted in a given o-minimal structure. Furthermore, let us assume now that $R = \mathbb{R}$ is the real line, as we shall eventually need to do to interpret definable functions as partition functions. Then the same theorem can be proved by substituting continuity (namely, belonging to the class C^0) with any C^m. The higher m, the finer the the division of the interval (a, b) will be (and thus the larger the integer k), but it will always be possible to partition the original interval into a finite number of subsets where the function is either constant or monotone and continuous together with its first m derivatives. It is worthwhile to observe that the set where f is differentiable must be a definable set. The reason is always the same: it is possible to define such a set with first-order logic. Indeed the set Df where the first derivative can be defined is:

$$\begin{aligned} Df := \{x \in (a, b) :\ & \forall \epsilon \in \mathbb{R}\backslash\{0\}, \exists \delta \in \mathbb{R}\backslash\{0\} \\ & [((x - y)^2 \leq \delta^2) \wedge (y \in (a, b)) \to (f(x) - f(y))^2 \leq \epsilon(x - y)^2] \ \}, \end{aligned} \tag{1.14}$$

and therefore is definable (we had to set $R = \mathbb{R}$ as R is not *a priori* a metric space). Being definable, it consists of a finite number of intervals or points, which guarantees that a partition such as that in Theorem 1.1 must exist. Note though that the foregoing argument only holds for finite m. The impossibility for o-minimal structures to define the integers makes any statement about functions in C^∞ ill-defined.

An important consequence of this theorem is that the function f, unless it is locally constant, is *locally invertible*: indeed the continuity and the monotonicity are sufficient assumptions to guarantee invertibility. Plus, the locality is, roughly said, 'not too small': the intervals have a finite, non-infinitesimal width. The monotonicity theorem ensures that the one-variable functions which we may encounter in o-minimal structures have a in-built regularity. For instance, functions like the Dirichlet function or Thomae function cannot be definable in any o-minimal structure[3]. The Dirichlet function is nowhere continuous: the set of

[3]Recall that the Dirichlet function is a function $D : [0, 1] \to [0, 1]$ defined by

$$\begin{cases} D(x) = 1 & \text{if } x \in \mathbb{Q} \\ D(x) = 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

its discontinuities is its own domain (usually taken to be $[0,1]$. The Thomae function is continuous over the irrational numbers and discontinuous over the rationals $\mathbb{Q}$: as these populate densely the domain $[0,1]$, they again impair the finiteness of the discontinuities which the monotonicity theorem would ensure.

1.2.4 The Cell Decomposition Theorem

The cell decomposition theorem is at the core of the theory of o-minimal structures. In the words of the authors of [BN23], it is the *raison d'etre* of the axioms of o-minimality. In a way, a generalisation of the previous theorem to an arbitrary number of variables.

Consider a definable set X in a given o-minimal structure. Given a generic function $f : X \to R$, we will call $C(X)$ the family of functions f that are both definable and continuous on X. We will call $C_\infty(X)$ the set $C(X) \cup \{\pm\infty\}$.

Definition 1.6 (Cell)**.** Given $(i_1, i_2, ...i_n)$ a sequence of ones and zeroes, a $(\mathbf{i_1}, ..., \mathbf{i_n})$**-cell** is defined inductively by:

1. A 0-cell is a set $\{r\} \subset R$; a 1-cell is an interval $(a, b) \subset R$;
2. A $(i_1, ..., i_m, 0)$-cell is the graph $\Gamma(f)$ for $f \in C(X)$, where X is a $(i_1, ..., i_m)$- cell; a $(i_1, ..., i_m, 1)$-cell is the interval $(f, g)_X$, where f and g belong to $C_\infty(X)$ and
$$(f, g)_X := \{(x, r) \in X \times R : \ f(x) < r < g(x)\}.$$

Intuitively, every entry $i_1, i_2, ...$ tells us whether along any given direction, the cell in question looks like a thin layer (given by the graph of a definable function), or by a thick bulk, delimited by the graphs of *two* definable functions.
A **cell** in R^m is then a a $(i_1, ..., i_m)$-cell for a necessarily unique sequence of $i_1, ..., i_m$; moreover a $(1, ..., 1)$-cell is said to be an **open cell** of R^m, since these are indeed the cells that are open in the ambient space R^m.
By their definition, it is quite clear that a cell C is always a definable set: again, it suffices to define it by using first-order logic and exploiting the assumption that the boundary functions are definable. Furthermore, cells are **definably connected** sets. A set $X \subseteq R^m$ is said to be definably connected if it is a definable set which cannot be written as the disjoint union of two nonempty definable open subsets of X.
Another property of cells is that of being homeomorphic under coordinate projection to an open cell. Intuitively, the desired homeomorphism is determined by the composition of projections along the directions in which a cell C is 'a thin layer'. More rigorously, if C is a $(i_1, ..., i_m)$-cell, we can call $\lambda(1), ..., \lambda(k)$ the indices that are equal to 1 (so that $k = i_1 + ... + i_m$). Then the projection
$$(x_1, ..., x_m) \to (x_{\lambda(1)}, ..., x_{\lambda(k)})$$
maps the original cell C into an open cell of R^k.

As the name might suggest, cells are intended to partition a given ambient space. This intuitive idea is expressed rigorously first by the following

Definition 1.7 (Cell Decomposition)**.** A **decomposition** of R^m is a finite partition of R^m defined inductively as follows:

1. A decomposition of R^1 is a collection
$$\mathcal{D}^1 = \{(-\infty, a_1), (a_1, a_2), ..., (a_k, +\infty), \{a_1\}, ..., \{a_k\}\};$$

The Thomae function is similarly $T : [0,1] \to [0,1]$ with
$$\begin{cases} T(x) = 1/m & \text{if } x \in \mathbb{Q} \text{ and } x = n/m \text{ with } n, m \text{ coprime} \\ T(x) = 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

2. A decomposition of R^{m+1} is a finite partition of R^m into cells $C_1, ..., C_N$ such that the set of projections onto R^m
$$\mathcal{D}^m = \{\pi(C_j), j = 1, ..., N\}$$
is a decomposition of R^m.

Before stating the theorem, we need one last definition. It is intended to capture how cells can partition not only the ambient space R^m, but also any subset thereof.

Definition 1.8 (Partition)**.** A decomposition $\mathcal{D}^m$ of R^m is said to be a **partition** of a given set $A \subseteq R^m$ if each cell $C \in \mathcal{D}^m$ is either a subset of A or not. Equivalently, $\mathcal{D}^m$ is a partition of A if $A = \cup_{i=1}^k C_i$ for some integer k and with $C_i \in \mathcal{D}^m$ for all i.

Finally, we are ready to state the very cell decomposition theorem.

> **Theorem 1.2** (Cell Decomposition Theorem)**.** Let us fix an o-minimal structure $(R, <, \mathcal{S})$. Then the following hold:
>
> 1. Given any finite sequence of definable sets $A_1, ..., A_N$ subsets of R^m, there exists a decomposition $\mathcal{D}^m$ of R^m which is a partition for each $A_1, ..., A_N$;
> 2. Given any definable function $f : A \to R$ with $A \subseteq R^m$, there exists a cell decomposition $\mathcal{D}^m$ of R^m such that for any cell $C \in \mathcal{D}^m$, the restriction of f to C $f|_C : C \to R$ is continuous.

The cell decomposition theorem lies at the foundations of many other results in o-minimality. One example is the following

> **Proposition 1.1.** If X is a nonempty definable set, then X only has finitely many definably connected components. These are both open and closed in X and they form a partition of X.

A **definably connected component** is a maximal definable connected subset of X. This proposition shows that the finiteness notion given in the Definition 1.5 for the sets contained in the first element of the sequence $\mathcal{S}_1$ carries over to all definable sets, although in a quite generalised way.

1.2.5 Definable families

Definable families are families of definable sets labelled by a parameter which varies in a definable set of its own. Let $X \subseteq R^{m+n}$ be a definable set. Then for every $a \in R^m$ we define
$$X_a := \{x \in R^n : \ (a, x) \in X\}, \tag{1.15}$$
which is a subset of R^n. Then we have the following

Definition 1.9. A **definable family** of subsets of R^n, with parameter space R^m is the family of sets $(X_a)_{a \in R^m}$, with X_a as in (1.15). These are called the fibres of the family.

Note that if X is definable X_a is also definable, as it can be written as the intersection of X with the sets
$$\begin{aligned} A_1 &:= \{a_1\} \times R \times .... \times R \\ &\vdots \\ A_i &:= R \times ... \times R \times \{a_i\} \times R \times ... \times R \\ &\vdots \\ A_m &:= R \times ... \times R \times \{a_m\} \end{aligned}$$

where each product has exactly n terms and a_i is the i-th component of $a \in R^m$. Each of the sets is definable by the axioms in Definition 1.4, as they are the products of the ambient space R with one-point sets $\{a_i\}$ which are by definition definable. The intersection operation preserves definability as it is an operation allowed in Boolean algebras (propositional logic). As regards definable families, we have the following:

Proposition 1.2. Let $\pi : R^{m+n} \to R^m$ be the projection onto the first m components of any $x \in R^{m+n}$. Then

1. If C is a cell in R^{m+n} and $a \in \pi(C)$, then C_a is a cell in R^n.
2. If $\mathcal{D}^{m+n}$ is a decomposition of R^{m+n} and $a \in R^m$, the collection
$$\mathcal{D}^n_a := \{C_a : \ C \in \mathcal{D}^{m+n}, a \in \pi(C)\}$$
is a decomposition of R^n.

1.2.6 The Trivialization Theorem

Here we review another important result regarding definable sets and functions. The trivialization theorem provides another way in which the regularity of definable sets manifests itself.

Having fixed an o-minimal expansion $(R, <, \mathcal{S})$ of a real closed field $(R, <, 0, 1, +, -, \cdot)$, consider a definable function $f : X \to A$, where X and A are definable. We can then view the sequence $(f^{-1}(a))_{a \in A}$ as a definable family of definable fibres $f^{-1}(a)$. Indeed, to match the previous Definition 1.9 of definable families, we can call $B := X \times A$ and claim that $B_a := \{x \in X : \ (x, a) = (x, f(x)) \in B\}$ is exactly the same as $f^{-1}(a)$. Then we introduce the following

Definition 1.10. A **definable trivialization** of a definable function $f : X \to A$ is a pair (F, λ), where $F \subseteq R^n$ is a definable set and $\lambda : X \to F$ is a definable function, such that the function
$$\begin{aligned}(f, \lambda) : X &\to A \times F \\ x &\to (f(x), \lambda(x))\end{aligned}$$
is a definable homeomorphism. If such F and λ exist, then f is said to be **definably trivial**.

Furthermore, we can consider a subset $A' \subseteq A$ and consider the function f restricted to $f^{-1}(A')$ namely $f|_{f^{-1}(A')} : \ f^{-1}(A') \to A'$. We will then say that f is definably trivial over A' if $f|_{f^{-1}(A')}$ is definably trivial (i.e. admits a definable trivialization). Then the following theorem holds:

Theorem 1.3 (Trivialization Theorem)**.** Given a definable function $f : X \to A$, there exists a finite partition of A, namely $A = A_1 \cup ... \cup A_k$ into k definable sets such that f is definably trivial over each A_i.

Let us investigate the meaning of this theorem. f being definably trivial over each A_i means that $f|_{f^{-1}(A_i)}$ admits a definable trivialization for every i. Therefore, there must exist k pairs of definable functions and sets (F_i, λ_i), $i = 1, ..., k$ such that $(f|_{f^{-1}(A_i)}, \lambda_i)$ are definable homeomorphisms. Thus, we can group the fibres $f^{-1}(a)$ into k subfamilies
$$X_i = \bigcup_{a \in A_i} f^{-1}(a) = f^{-1}(A_i),$$
each of which must be homeomorphic to $A_i \times F_i$. Thus, we can conclude that the (in general infinitely many) fibres $f^{-1}(a)$ of a definable family only come in a finite number types that are not definably homeomorphic among themselves. This is then another manifestation of the admittedly vague notion of regularity that characterises the definable sets.

1.3 Definable invariants

The axioms (1.4) that define structures are similar to those that define topologies. This analogy heuristically justifies the existence of definable invariants: these are quantities that are preserved by definable homeomorphisms, very much like topological invariants are preserved by general homeomorphisms. The definable invariants which we are going to cover in this section are the dimension and Euler characteristic.

1.3.1 Dimension

In topology, dimension is a property of topological manifolds: a m-dimensional manifold M is a an open set (Hausdorff and second-countable) which locally looks like $\mathbb{R}^m$. The latter, fuzzy statement about the similarity to $\mathbb{R}^m$ is made precise by the definition continuous bijections between subsets the manifold and $\mathbb{R}^m$, called charts; these are then continuously sewn together to form an atlas of compatible charts. It is crucial that *all* the subsets of M are locally isomorphic to $\mathbb{R}^m$, so that the dimension is a global property of M. Definable sets, on the contrary, can consist of patches which are isomorphic to R^i for different values of i: therefore, in general, a definable set is not a manifold. Nevertheless, a notion of dimension can still be formulated by exploiting the cell-decomposition theorem. Indeed, albeit a general definable set X may not a manifold, its cells, by construction, certainly are: a $(i_1, ..., i_m)$-cell has dimension $d = \sum_{k=1}^{m} i_k$. Hence, we can attempt the definition:

Definition 1.11 (Dimension)**.** The **dimension** of a definable set X is given by the maximal dimension of the the cells contained in one of its cell decompositions. The dimension of the empty set is conventionally chosen to be $-\infty$.

At this stage, this definition is admittedly not very useful, as the cell decomposition of a given definable set is not unique. However we have the following

> **Proposition 1.3.** If $X \subseteq R^m$ and $Y \subseteq R^n$ are definable sets, and there exists a definable bijection between them, then $\dim(X) = \dim(Y)$.

The latter statement essentially asserts that the dimension is a definable invariant: it is preserved by definable bijections, just like topological properties are preserved by homeomorphisms, i.e. continuous bijections with continuous inverse. Recall that, as stated in Lemma 1.3, there is no need to specify the definability of the inverse of a definable bijection: given f definable, the graph of f^{-1}, if it exists, is the same as that of f, $\Gamma(f) = \Gamma(f^{-1})$, thus f^{-1} is automatically definable. The dimension of a definable set behaves 'nicely' (that is, as intuition would suggest) under many respects. Some of the main facts are listed in the following

> **Proposition 1.4.** Given two definable sets $X \subseteq Y \subseteq R^m$ one has
>
> 1. $\dim(X) \leq \dim(Y) \leq m$;
> 2. $\dim(X \cup Y) = \max(\dim(Y), \dim(Y))$;
> 3. $\dim(X \times Y) = \dim(X) + \dim(Y)$;
> 4. The dimension of X is strictly greater than that of its boundary and equal to that of its closure, namely
>
> $$\dim(\partial X) < \dim(X), \qquad \dim(\mathrm{cl}(X)) = \dim(X). \tag{1.16}$$

The last property captures an essential feature of definable sets. Although they are not manifolds, definable sets are made up by a finite number of manifolds (namely their cells): this ensures a high degree of regularity

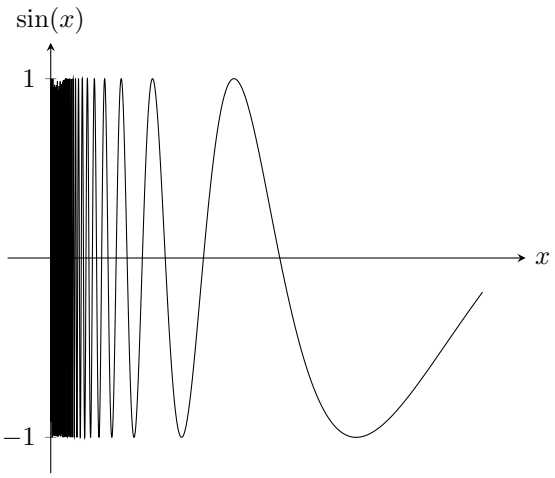


Figure 1.2: Topologist's sine curve. The boundary of the graph is $[-1, 1]$ on the y axis.

and prevents them from being infinitely finely warped or discontinuous. A familiar example of a set which is certainly not definable in any o-minimal structure is the graph of the *topologist's sine curve* $y = \sin(1/x)$. As it can be seen from Figure 1.2, the boundary of the graph Γ, is the interval $[-1, 1]$ on the y axis[4] , which is obviously a definable set of dimension 1. On the other hand, viewed as a submanifold of $\mathbb{R}^2$, Γ is clearly 1-dimensional, as it is the image of the real line. Thus, we see that the graph Γ of the topologist's sine has the same dimension as its boundary and therefore it cannot be a definable set. Of course, this could have been more straightforwardly inferred by observing that a function f having infinitely many zeros cannot be definable in any o-minimal structure, as the set of its zeros $\Gamma(f) \cap \mathbb{R}$, being an infinite union of points, does not abide by Definition 1.5. In this perspective though, it is more clearly illustrated how o-minimal structures are instances of a *tame geometry*, which banishes pathological manifolds such as the graph in Figure 1.2.

As a final remark, we note that the dimension is well-behaved also for definable families. We have the following theorem:

Theorem 1.4. Let $X \subseteq R^{m+n}$ be definable, and for $d \in \{-\infty, 0, 1, ...\}$ let

$$X(d) := \{a \in R^m : \dim(X_a) = d\} \tag{1.17}$$

where X_a is defined in 1.15. Then $X(d)$ is definable and

$$\dim\left(\bigcup_{a \in X(d)} \{a\} \times X_a\right) = \dim(X(d)) + d. \tag{1.18}$$

In the above theorem, $X(d)$ is the locus in the parameter space R^m of parameters whose fibres have dimension d. Hence it is intuitive to conclude that the dimension of the (trivial) bundle given by the Cartesian product of the subset $X(d)$ with its fibres be the dimension of $X(d)$ plus d (in the spirit of point 3 of Proposition 1.4). The theorem guarantees that our intuition's guess is indeed correct.

[4]Explicitly, one has $\text{int}(\Gamma) = \Gamma, \text{cl}(\Gamma) = \Gamma \cup [-1, 1], \partial(\Gamma) = [-1, 1]$.

1.3.2 Euler Characteristic

Another definable invariant is the Euler characteristic, which is related to the previously defined concept of dimension.
If we were to find a cell decomposition for an open interval (a, b), we may choose to do so with an arbitrary integer number k of open intervals. To fill in the gaps, we will have to add the $k-1$ points in-between. Thus the number of points minus the number of intervals gives consistently -1, regardless the decomposition we have chosen. More generally then, let us define the **Euler characteristic**, for a given cell C, to be

$$E(C) = (-1)^d, \tag{1.19}$$

where d is the dimension of the cell. Then, given a finite partition $\mathcal{P}$ into cells we define

$$E_{\mathcal{P}} := \sum_{C \in \mathcal{P}} E(C). \tag{1.20}$$

Again, this definition is not very useful so far, as the partition $\mathcal{P}$ is not unique; however, we have the following

> **Proposition 1.5.** For any given two partitions $\mathcal{P}$ and $\mathcal{P}'$ of a definable set S, one has $E_{\mathcal{P}} = E_{\mathcal{P}'}$.

Thus, the Euler characteristic is well-defined. Furthermore, it can be shown to be a definable invariant. Indeed the following also holds:

> **Proposition 1.6.** The Euler characteristic $E(X)$ for a definable set X is preserved under the action of definable injective map $f : X \to R^n$. Namely
>
> $$E(f(X)) = E(X).$$

1.4 Triangulation

In this section, we describe an important property of definable sets in an o-minimal structure $(R, < \mathcal{S})$ expanding an ordered real closed field $(R, <, 0, 1, +, -, \cdot)$, which we will think of as a vector space over itself. The triangulation property of definable sets in $\mathcal{S}$, expressed by the Triangulation Theorem, elucidates from a different viewpoint how these sets carry a notion of finiteness and regularity.

1.4.1 Simplices and Complexes

Simplices and complexes are a formal way of dealing with polyhedra in an arbitrary number of dimensions. In the following treatment, we shall view simplices and complexes will as subsets of general ordered field R^n rather than the more familiar euclidean space $\mathbb{R}^n$, although we can of course help our intuition by restricting ourselves to this case.
First, let us state that an **affine function** on an ordered real, closed field R is defined as a function $f : R^n \to R$ which can be written as

$$f(x_1, ..., x_n) = \lambda_1 x_1 + ... + \lambda_n x_n + a. \tag{1.21}$$

Given $k+1$ points $a_0, ..., a_k$ in R^n, an affine subspace is given by the **affine span** of $a_0, ..., a_k$, namely by

$$\left\{ x \in R^n : \ x = \sum_{j=1}^{k} t_j a_j, \ t_j \in R, \sum_{i=1}^{k} t_i = 1 \right\}. \tag{1.22}$$

If such a span has dimension k, then the tuple $a_0, ..., a_k$ is said to be **affine independent**[5]. A simplex can then be defined as bounded subspace of an affine subspace of R^n according to the following

Definition 1.12. Given an affine independent tuple $a_0, ..., a_k$, we define the k**-simplex** as

$$(a_0, ..., a_k) := \left\{ x \in R^n : \ x = \sum_{i=0}^{k} t_i a_i, \ \sum_{i=0}^{k} t_i^k = 1 \ , t_i \in R, t_i > 0 \ \forall i \right\}. \tag{1.23}$$

We will say then that the tuple $a_0, ..., a_k$ spans the simplex $(a_0, ..., a_k)$. The points $a_0, ..., a_k$ are called the **vertices** of the simplex $(a_0, ..., a_k)$.

The additional constraint that all the t_i be positive restricts the affine space into a bounded subset thereof, yielding an open simplex $(a_0, ..., a_k)$; we will denote its closure by $[a_0, ..., a_k]$. Then, if we fix for a moment $R = \mathbb{R}$, we can quickly realise that $[a_0, a_1, a_2]$ is a triangle in the Euclidean space, while $[a_0, a_1, a_2, a_3]$ can be viewed as a tetrahedron. Inspired by our discovery, we can safely say that the simplex spanned by any subset of $\{a_0, ..., a_k\}$ is a **face** of the the simplex $(a_0, ..., a_k)$. Note though, that a simplex σ, according to this definition, is a face of itself.

Having thus reviewed simplices, we may now turn to complexes. As the name might suggest, they are defined as follows:

Definition 1.13. A **complex** in R^n is a finite collection K of simplices in R^n, $K = \{\sigma_1, ..., \sigma_N\}$, such that for any two simplices σ_1, σ_2 in K, one of the following is true:

1. $\mathrm{cl}(\sigma_1) \cap \mathrm{cl}(\sigma_2) = \emptyset$;
2. $\mathrm{cl}(\sigma_1) \cap \mathrm{cl}(\sigma_2) = \mathrm{cl}(\tau)$;

where τ is a common face of σ_1 and σ_2.

A complex is then defined as a collection of simplices. It is then natural to consider their union, which is indicated with $|K|$. This is called the **polyhedron** spanned by K. Furthermore, the union of all the vertices of the simplices in K amounts to the set of vertices $\mathrm{Vert}(K)$.

1.4.2 The Triangulation Theorem

Polyhedra are a very restricted class of sets, as they are defined by affine functions. Nevertheless, they play an important rôle in o-minimality not only due to their simplicity, but also because they can be related to any other definable sets by definable homeomorphism. This fact is summarised by the Triangulation Theorem.

> **Theorem 1.5.** Each definable set $X \subseteq R^m$ is definably homeomorphic to a polyhedron $|K|$ for some complex K in R^m.

This theorem states that the topology of a definable set, which is insensitive to definable homeomorphisms (i.e. definable change of coordinates) is the same as that of an object so simple as a polyhedron. A polyhedron can be called simple in the sense that it is described by a combinatorial object, called *scheme*:

$$(\mathrm{Vert}(K), \{S \subseteq \mathrm{Vert}(K) : \ S \ \text{ spans a simplex of } \ K\}). \tag{1.24}$$

[5]Admittedly, this is not quite rigorous at this point, as we do not know yet whether these sets are definable. We can assume for the moment that we can talk about dimension as these sets are clearly manifolds. Later we will argue why they are actually definable sets in a precise o-minimal structure, and thus we are allowed to assign to them a dimension in the sense of the previous section.

Roughly, the set of vertices locates the polyhedron in space, while the second entry of the scheme, S, tells us which subsets are the actual simplices that K consists of and which are not. Then the scheme is given by a set plus a subset of the set of its parts: it is a combinatorial object.

To appreciate the Triangulation Theorem fully, we shall anticipate the content of the next section by describing one of the simplest o-minimal structures. Having defined affine functions in (1.21), we can also define the **semilinear sets** as finite unions of sets which can be expressed as

$$X = \{x \in R^m : \ f_1(x) = ... = f_p(x) = 0, g_1(x) < 0, ...g_q(x) < 0\} \tag{1.25}$$

where $f_1(x), ..., f_p(x)$ and $g_1(x), ..., g_q(x)$ are affine functions on R^m. It is easy to show that these sets form a structure in the sense of (1.4). Then one has the following:

> **Theorem 1.6.** Let $\mathcal{S}_m$ be the Boolean algebra of semilinear sets of m variables. Then the sequence $\mathcal{S} = (\mathcal{S}_m)_{m\in\mathbb{N}}$ is an o-minimal structure.

With this new intelligence, we can think anew of simplices and polyhedra. Simplices are essentially semilinear sets with the addition of a constraint of each t_i, which is again expressible by an affine function. Therefore, simplices are semilinear sets and thus so are polyhedra, which are finite unions of simplices. We can therefore argue that the Cell Decomposition Theorem 1.2 will ensure the existence of a finite number of cells that partition a polyhedron $|K|$. Furthermore, the cells must be definable in the same o-minimal structure where the polyhedron is defined: they must be therefore semilinear sets. The very construction of polyhedra then suggests that a very natural cell decomposition is that of all the simplices that appear in K. Moving back to euclidean spaces, $R = \mathbb{R}$, let us consider closed polyhedra: these are polyhedra that contain all the faces of their simplices. As all faces are cells (other than simplices), we can assign a dimension of each of them by (1.11); furthermore, we can also compute their Euler characteristic according to (1.20). For a closed polyhedron $|K|$ we will have

$$E(|K|) = \sum_{i=0}^{d} (-1)^i N_i \tag{1.26}$$

where N_i is the number of cells – but in this case, also of simplices – of dimension i.
Consider now a closed tetrahedron $|T|$. Here we do not view it as a 3-dimensional simplex $[a_0, a_1, a_2, a_3]$ as before: rather, we view it as a polyhedron given by the union of four closed triangles, so that

$$|T| = [a_0, a_1, a_2] \cup [a_0, a_1, a_3] \cup [a_0, a_2, a_3] \cup [a_1, a_2, a_3] \tag{1.27}$$

So, $|T|$ is actually 2-dimensional because all the simplices (and cells) it is made of are of dimension 2 or lower; more colloquially, it has no 'interior'.
Computing the N_is for $|T|$ is rather simple. There are 4 vertices: these are 0-dimensional cells. Thus $N_0 = 4$. Then, there are 6 edges: these are 1-dimensional cells, so $N_1 = 6$. There are also 4 sides: these are 2-dimensional cells, thus $N_2 = 4$. Plugging in the numbers in (1.26), we retrieve a well-known result from Euclidean geometry:

$$E(|T|) = \#\text{vertices} - \#\text{edges} + \#\text{faces} = 4 - 6 + 4 = 2 \tag{1.28}$$

where by # we mean "the number of".

As a final remark, we hearken back to Proposition 1.6. It tells us that the Euler characteristic is invariant under definable homeomorphisms. Besides, the Triangulation Theorem 1.5 guarantees that definable sets can be definably mapped to polyhedra, whose Euler characteristic we have shown to be easy to compute. Thus, the combination of these gives us a recipe to compute the Euler characteristic of any definable set: as the Euler characteristic will be unchanged, it suffices to turn it into a polyhedron by an appropriate definable homeomorphism; we can then compute the Euler characteristic of the polyhedron in a similar fashion to what we did in (1.28).

1.5 Examples of o-minimal structures

Hitherto, we have always fixed a general o-minimal structure $\mathcal{R}$ to study its properties and how the notion of finiteness encapsulated in the axiom of Definition 1.5 manifests itself in a variety of ways. However, the hardest question to tackle is actually to determine whether a given structure is o-minimal or not; nay, it may not even be clear whether a structure is indeed such. We have already encountered one such statement in 1.6: the structure given by the sequence of the Boolean algebras of the semilinear sets is o-minimal. There are much more general structures whose o-minimality we may want to inquire about; referring to the literature for the proofs, we will now provide some examples of structures that are known to be o-minimal.

Before starting, it is worthwhile to understand more deeply the question that we are posing in terms of logic. We should then remind ourselves that structures can be viewed as L-structures, namely the geometrical realisations of a language L which consists in a set R, a set of primitive relations $(S_i)_{i\in I}$ and a set of primitive functions $(f_j)_{j\in J}$ of different -arity. If we want our structure to be o-minimal, there must be an order relation: therefore one of the relations S_j must be $<$. As far as we are concerned, this will be the unique binary relation that our language includes. If we fix, as we shall do henceforth, the real closed ordered field $(R, <, 0, 1, +, -, \cdot)$ to be the real line $\mathbb{R}$, where the constants and the operations bear their usual meaning, the primitive functions that we are allowing for are the ordinary addition and multiplication among the real numbers. As we will discuss below, the L-structure of this language is the structure of semialgebraic sets and it turns out to be o-minimal, extending thus the previously mentioned o-minimal structure of semilinear sets. This result is already quite non-trivial, but we may want to be even more ambitious. We may want, in other words, to start adding more and more functions f_j to our language L and wonder whether, at every addition, the relative L-structure exits the realm of o-minimality or not. This is roughly the spirit in which research into this field has been led until today; some remarkable results, such as [RSS23] were only attained very recently.

In all the examples below, we will collect the functions f_j in a generating family $\mathcal{F} = \cup_{j\in J} f_j$. Availing ourselves of the close relationship between languages and structures, we like thinking of $\mathcal{F}$ as a 'dictionary': it contains all the elementary 'words' (i.e., primitive functions) wherewith all the other possible statements (i.e., sets) of a language L (i.e., the associated L-structure) can be expressed. The admittedly fancy metaphors used at the beginning of the Introduction are now hopefully unravelled.
With some abuse of notation we will not distinguish between f_j and its interpretation $f_j^{\mathcal{R}}$; thus the structures that will be described below will be $(\mathbb{R}, <, 0, 1, +, -, \cdot, \mathcal{F})$, which we will indicate for brevity $\mathbb{R}_{\mathcal{F}}$. As the ring operations of $\mathbb{R}$ are preserved in $\mathbb{R}_{\mathcal{F}}$, all of the following structures are also to be considered o-minimal expansions of the real field $\mathbb{R}$.

1.5.1 $\mathbb{R}_{\mathbf{alg}}$: Structure of semialgebraic sets

The first case to consider is naturally $\mathcal{F} = \emptyset$. We consider then the structure generated by $(\mathbb{R}, <, 0, 1, +, -, \cdot)$ where the only allowed operations, other than the order relation $<$, sums and products between the coordinates $x_1, ..., x_m \in \mathbb{R}$. It is clear then that the only functions that we will be able to define in this structure (or language) are algebraic, whence the name $\mathbb{R}_{\text{alg}}$ (rather than $\mathbb{R}_{\emptyset}$). Let us now provide a more geometric perspective: the **semialgebraic** sets are then defined as those subsets of $\mathbb{R}^n$ which can be written as finite unions of sets like

$$X := \{x \in \mathbb{R}^n : \ P(x) = 0 \wedge (Q_1(x) < 0) \wedge ... \wedge (Q_s(x) < 0)\} \tag{1.29}$$

where P and $Q_1, ..., Q_s$ are polynomials in $x \in \mathbb{R}^n$ (i.e., of n variables). Note that the condition $P_1(x) = 0 \wedge P_2(x) = 0$ is equivalent to $P_1^2(x) + P_2^2(x) = 0$, which is a single polynomial equality: this is why, in the above definition, we can insert only one polynomial equality $P(x) = 0$. Let us now show explicitly that

Proposition 1.7. For every n, the collection $\mathcal{S}_n \subseteq \mathbb{R}^n$ of semialgebraic sets is a Boolean algebra.

Proof.

1. *Closure under finite intersection:* Consider the sets $A = \bigcup_i^{M_A} X_i$ and $B = \bigcup_j^{M_B} Y_j$ where the set X_i, Y_j are as in (1.29) and M_A and M_B are some integers: then $A, B \in \mathcal{S}_n$. Thus, one has

$$A \cap B = \left(\bigcup_i^{M_A} X_i \right) \cap \left(\bigcup_j^{M_B} Y_j \right) = \bigcup_{i,j} (X_i \cap Y_j) \tag{1.30}$$

   where each $X_i \cap Y_j$ is again of the form (1.29) because one can simply add as many polynomials inequalities as are needed, and merge the polynomial equalities into only one as remarked above. Hence we conclude $A \cap B \in \mathcal{S}_n$.

2. *Closure under complement*: Given $A = \bigcup_i^{M_A} X_i$, one has $A^c = \bigcap_i^{M_A} X_i^c$. Note that if X is in the form (1.29), then

$$\begin{aligned} X_i^c &= \{x \in \mathbb{R}^n : \ P(x) \neq 0 \vee (Q_1(x) \geq 0) \vee ... \vee (Q_s(x) \geq 0\} \\ &= \{x \in \mathbb{R}^n : \ P^2 > 0\} \cup \bigcup_i^s \{x \in \mathbb{R}^n : \ Q_i \geq 0\} \\ &= \{x \in \mathbb{R}^n : \ P^2 > 0\} \cup \bigcup_i^s (\{x \in \mathbb{R}^n : \ Q_i > 0\} \cup \{x \in \mathbb{R}^n : \ Q_i = 0\}) \end{aligned} \tag{1.31}$$

   hence the closure can be expressed as a finite union of sets in the form (1.29) and therefore $X_i^c \in \mathcal{S}_n$. Then from the closure under intersection follows that $A^c \in \mathcal{S}_n$.

3. *Closure under union*: This follows by the definition itself of semialgebraic sets as being a finite union of sets (1.29).

□

It can be shown that the other axioms in (1.4) are satisfied. The Cartesian multiplication by $\mathbb{R}$ clearly maps a semialgebraic set $X \in \mathcal{S}_n$ to $\mathbb{R} \times X \in \mathcal{S}_{n+1}$, as a polynomials in n variables can always be viewed as a special case of a polynomial in $n+1$ variables. The closure property of the sequence of Boolean algebras $(\mathcal{S}_n)_{n\in\mathbb{N}}$ under projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, known as the Tarski-Seidenberg property, is more subtle and we refer to [Dri98] for a detailed proof. Let us merely remark that the sets defined by polynomial inequalities, such as $A = \{x \in \mathbb{R}^n : Q(x) \leq 0\}$ must be included to $\mathcal{S}_n$ to guarantee the closure of the structure under projection: in fact, these sets naturally arise as the projections of sets defined by polynomial equalities in higher dimensions. For example, taking A as above, we can write it:

$$A = \pi\{x \in R^{n+1} : y^2 + Q(x) = 0\} \tag{1.32}$$

where the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates. Hence it can be proved that

> **Proposition 1.8.** The sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ of the Boolean algebras of semialgebraic subsets of $\mathbb{R}^n$ is a structure, which we call $\mathbb{R}_{\text{alg}}$.

Since any polynomial can only have a finite number of roots, intuition may suggest that any set A belonging to $\mathcal{S}_1$ must be a finite collection of points or intervals. This was in fact proved by Tarski in [Tar49]:

> **Theorem 1.7** (Tarski)**.** The structure $\mathbb{R}_{\text{alg}}$ of semialgebraic sets is o-minimal.

Another rigorous and general proof of this important fact can be found in [Dri98], Chapter 2: therein, the semialgebraic sets are more generally regarded as subsets of a topological space which need not be $\mathbb{R}^n$.

1.5.2 Analytic extensions of $\mathbb{R}_{\text{alg}}$

In the previous section, we have found $\mathbb{R}_{\text{alg}}$ to be the o-minimal expansion of the real closed field $(\mathbb{R}, <, 0, 1, +, -, \cdot)$, keeping the generating family $\mathcal{F}$ empty. Here we will enlarge the structure of semialgebraic sets by adding collections $\mathcal{F}$ of *analytic* functions. These will sometimes be taken on the whole real line, but in other cases they will be restricted to bounded subset of $\mathbb{R}^n$. It is worthwhile to remark that the algebraic functions will be always definable in a structure $(\mathbb{R}, <, 0, 1, +, -, \cdot, \mathcal{F})$: although the subscript 'alg' will be omitted from the notation $\mathbb{R}_{\mathcal{F}}$, it should always be understood to be there.

$\mathbb{R}_{\text{exp}}$

A first and very important example is $\mathcal{F} = \{\exp\}$. The structure $\mathbb{R}_{\exp}$, which we can write explicitly as $\mathbb{R}_{\exp} = (\mathbb{R}, <, 0, 1, +, -, \cdot, \exp)$, where of course $\exp : \mathbb{R} \to \mathbb{R}^+$ and $\exp(x) = e^x$, was proved to be o-minimal by Wilkie in [Wil96]. Although only one function has been added to $\mathbb{R}_{\text{alg}}$, this result is considered a real milestone in the theory of o-minimality, as it adds to the generating family $\mathcal{F}$ a transcendental function defined on the whole $\mathbb{R}$. As the next example shows, the latter fact is quite non-trivial.

$\mathbb{R}_{\text{an}}$: Restricted analytic functions

Consider the structure $\mathbb{R}_{\mathcal{F}}$ where $\mathcal{F}$ is the collection of all analytic functions $f : [-1, 1]^n \to \mathbb{R}$ such that, for every f, there exists an analytic function $\tilde{f} : U \to \mathbb{R}$, with $U \supset [-1, 1]^n$, whose restriction to the unit cube is equal to f: $\tilde{f}|_{[-1,1]^n} = f$. Conventionally, the analytic functions f are set to vanish outside the unit cube $[-1, 1]^n$, so that they are more conveniently defined on the whole $\mathbb{R}^n$. This structure, called $\mathbb{R}_{\text{an}}$, was proved to be o-minimal in [DD88]. This result is shows how it is possible even to add an infinite number of functions to $\mathcal{F}$ while preserving the o-minimality of the resulting structure.

It will be expedient to stress here why the analytic functions must be restricted to an interval, and also why the interval must be closed.
The first question is rather straightforward. The most notable example of a function that cannot be fitted in *any* o-minimal structure $\mathbb{R}_{\mathcal{F}}$ is the sine function $\sin(x)$. When taken on the whole real line $\mathbb{R}$, the sine has infinitely many roots, which clearly contradicts the axiom (1.5). The sine is an analytic function, as it is given by a power series with infinite radius of convergence. Thus, if we want to allow for any kind of analytic function on $\mathbb{R}$, we must needs restrict it to an interval so that its zero set is prevented from creating an infinite and discrete sequence of points.
The second issue is slightly more subtle. Consider the plot of the function $\sin(1/x)$ shown in Figure 1.2. $\sin(1/x)$ is analytic at every point of the open interval $(0, 1)$, but clearly has infinitely many zeros in that same interval. The additional requirement that the domain must be closed, however, rules out any such behaviour. In fact, in our example, the function $\sin(1/x)$ clearly has only a finite number of zeros on any closed interval $[\varepsilon, 1]$, however small ε may be; obviously, it is impossible to define it on $[0, 1]$ because it is ill-defined at $x = 0$.

$\mathbb{R}_{\text{an, exp}}$

The exponential is an analytic function, so it is included in $\mathbb{R}_{\text{an}}$ only on restricted closed intervals. Nevertheless, it was proved in [Dri94] that e^x can be added to $\mathbb{R}_{\text{an}}$ also when defined on the whole real line: this gives rise to the o-minimal structure $\mathbb{R}_{\text{an, exp}}$. We remark that this structure has been realised to bear a significant physical significance: all Feynman diagrams, viewed as functions of external momenta and the parameters of the QFT (essentially masses and coupling constants) are definable in this structure [DGS22].

There are more structures in which $\mathcal{F}$ contains analytic functions, one of which, named $\mathbb{R}_{\text{Pfaff}}$, will be

reviewed in the next section in the context of sharply o-minimal structures. Before moving on to structures where $\mathcal{F}$ contains non-analytic functions though, an important remark is due. If $\mathcal{F}$ contains only analytic functions, it does not follow that all the definable functions of $\mathbb{R}_{\mathcal{F}}$ are everywhere analytic. Consider for instance the function $f : \mathbb{R} \to \mathbb{R}$ in Figure 1.3:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases} \tag{1.33}$$

This function is clearly non-analytic at $x = 0$, as its Taylor series is identically vanishing thereat. Nevertheless, it is definable in $\mathbb{R}_{\exp}$, as it can be shown explicitly. Let

$$\begin{aligned} \Gamma_1 &:= \{(x, y) \in \mathbb{R}^2 : \ (y = \exp(z)) \wedge (zx^2 + 1 = 0) \wedge (x > 0)\} \\ \Gamma_2 &:= \{(x, y) \in \mathbb{R}^2 : \ (y = \exp(z)) \wedge (zx^2 + 1 = 0) \wedge (x < 0)\} \\ \Gamma_3 &:= \{(0, 0)\}. \end{aligned} \tag{1.34}$$

Then clearly, $\Gamma(f) := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is the graph of f and is definable in $\mathbb{R}_{\exp}$. So, it is possible to find non-analytic functions even in structures whose generating family $\mathcal{F}$ only contains analytic functions, but this demands an explicit exhibition of the construction of the graph, which may be rather involved in more complicated examples.

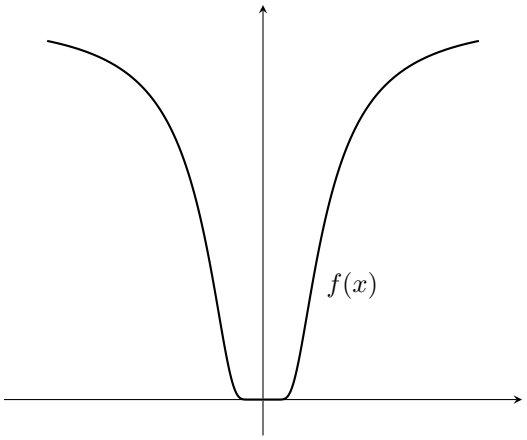


Figure 1.3: A tame function non-analytic at $x = 0$.

1.5.3 Beyond analyticity

Let us now illustrate more sophisticated o-minimal structures $\mathbb{R}_{\mathcal{F}}$ where $\mathcal{F}$ admits functions that are not analytic (at least in one point of their domain).

$\mathbb{R}_{\mathcal{C}(M)}$: Quasi-analytic Denjoy-Carleman classes

This structure is discussed in [RSW03]. It is defined again as $\mathbb{R}_{\mathcal{C}(M)} = (\mathbb{R}, <, 0, 1, +, -, \cdot, \mathcal{F})$, where $\mathcal{F}$ is chosen as follows. Given a sequence of integers $M = (M_0, M_1, ...)$, consider a closed box $B = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n] \subseteq \mathbb{R}^n$. Consider then the functions $f : B \to \mathbb{R}$ which can be extended to an open neighbourhood U of B to a C^∞ function $g : U \to \mathbb{R}$ which has the following property: for every $x \in U$ and $\alpha \in \mathbb{N}^n$ there exist a constant A such that

$$|g^{(\alpha)}(x)| \leq A^{|\alpha|+1} M_{|\alpha|} \tag{1.35}$$

where $|\alpha| = \alpha_1 + ... + \alpha_n$, while $g^{(\alpha)}$ stands for the partial derivative $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} ... \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} g$. All such functions f are grouped in the **Denjoy-Carleman class on B associated with M**, indicated as $\mathcal{C}^0_B(M)$. Because $f \in \mathcal{C}^0_B(M)$ extends to an infinitely differentiable function $g \in C^\infty(U)$, for every $x \in B$ it is possible to define a Taylor map at x_0

$$f \to \hat{f}(x) = \sum_{\alpha \in \mathbb{N}^n} \frac{g^\alpha(x_0)}{\alpha!}(x - x_0)^\alpha$$

where, as it might be intuitive, $\alpha! := \alpha_1! \cdot \alpha_2! \cdot ... \cdot \alpha_n!$, and similarly $x^\alpha := x_1^{\alpha_1} \cdot ... \cdot x_n^{\alpha_n}$. In general, a function is said to be **quasi-analytic** if the Taylor map thus defined is injective. In the specific case of $\mathcal{C}^0_B(M)$, it can be proved that quasi-analyticity is equivalent to the condition

$$\sum_i \frac{M_i}{M_{i+1}} = \infty. \tag{1.36}$$

Note that if $M_i = i!$, then quasi-analyticity reduces to ordinary analyticity.
To build an o-minimal structure, a further constraint on the sequence M must be imposed: it has to be **strongly log-convex**, which means that the rescaled sequence $\tilde{M} := (M_0, \frac{M_1}{1!}, \frac{M_2}{2!}, ...)$ is **log-convex**. This means that

$$\tilde{M}_i^2 \leq \tilde{M}_{i-1}\tilde{M}_{i+1} \Longleftrightarrow M_i^2 \leq M_{i-1}M_{i+1}\frac{i}{i+1}. \tag{1.37}$$

Finally, to ensure closure under differentiation, we define $\mathcal{C}_B(M)$ to be the algebra of functions

$$\mathcal{C}_B(M) := \bigcup_{j=0}^{\infty} \mathcal{C}_B(M^{(j)}) \tag{1.38}$$

where $M^{(j)} := (M_j, M_{j+1}...)$. Having done so, we can then define $\mathcal{F}$ to be the collection of all functions $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that when restricted to the unit cube $B = [-1, 1]^n$ we have $\tilde{f}|_B \in \mathcal{C}_B(M)$, whereas they are identically 0 outside the unit cube. Then, the main result proved in [RSW03] is

> **Theorem 1.8.** Given the structure $\mathbb{R}_{\mathcal{C}(M)}$ built as above, if M is strongly log convex (i.e. obeys (1.37) and obeys the quasi-analyticity condition (1.36), then $\mathbb{R}_{\mathcal{C}(M)}$ is o-minimal, model complete, polynomially bounded and admits a C^∞ cell decomposition[6].

This is a first example where non-analytic functions can be added to an o-minimal structure. Remarkably, it was proved again in [RSW03] that there even exists a function $f \in \mathcal{C}^0_{[-1,1]}(M)$ which is *nowhere analytic* on its domain!

$\mathbb{R}_{\mathbf{an},H}$: Quasi-analytic solutions to ODEs

This structure is described in detail in [RSS07]. As the label suggests, it is an extension of $\mathbb{R}_{\text{an}}$ by the addition of functions $H_1, ..., H_r : (0, \epsilon] \to \mathbb{R}$. These functions are arranged in a vector H which is the solution to the following system of differential equations:

$$x^{p+1}\frac{dy_i}{dx} = A(x, y_1, ..., y_r) \tag{1.39}$$

where p is an integer called **Poincaré rank** and A is a real analytic function at $0 \in \mathbb{R}^{r+1}$ and $A(0, \ \mathbf{0}) = 0$. Furthermore, let us assume that $\lim_{x\to 0} H(x) = \mathbf{0}$ and that H admits an asymptotic expansion

$$\hat{H}(x) = \sum_{k=0}^{\infty} h_k x^k \in \mathbb{R}[[x]]^r \tag{1.40}$$

[6] By model complete, it is meant that the requirement the structure be closed under complements is superfluous (i.e., one can define the structure without this assumption an prove that it is satisfied). By polynomially bounded, it is meant that all the definable functions of the structure diverge at most polynomially. Finally, by C^∞ cell-decomposition it is meant that the Cell Decomposition Theorem 1.2 holds even if the word "continuous" is replaced by "of class C^∞".

as $x \to 0$. Let us now introduce the following notation:

Definition 1.14. Given a power series or a C^∞ function $\phi(x)$ we define:

- $J_k\phi(x) = \sum_{i=0}^{k} \frac{\phi^{(i)}(0)}{i!}x^i$ **(k-th jet)**.
- $T_k\phi(x) = \frac{\phi(x) - J_k\phi(x)}{x^k}$.
- $\mathrm{val}\phi = \min_l\{\phi^{(l)}(0) \neq 0\}$ (the order of the lowest non-vanishing derivative).

Having done so, we can proceed with the following definition:

Definition 1.15 (Strong quasi-analyticity)**.** A solution $H = (H_1, ..., H_r) : (0, \epsilon] \to \mathbb{R}^r$ to equation (1.39) is said to be **strongly quasi-analytic** if it tends to 0 as $x \to 0$, it has an asymptotic expansion (1.40) for $x \to 0$ and moreover, given

- n polynomials $P_1(x), ..., P_n(x)$ with $\mathrm{val}(P_l) > 0$ and $P_l^{\mathrm{val}(P_l)}(0) > 0$ for every $l = 1, .., n$;
- an analytic function $f(x, z_{11}, ..., z_{rn})$ with $f(0) = 0$;

one has that the following implication holds:

$$f(x, T_k\hat{H}_j(P_l)) \equiv 0 \Rightarrow f(x, T_kH_j(P_l)) \equiv 0$$

where by "$\equiv 0$" we mean "identically 0 for every $x \in (0, \epsilon]$".

We then have the following:

> **Theorem 1.9.** Given the differential equation (1.39), assume that it has a strong quasi-analytic solution $H : (0, \epsilon] \to \mathbb{R}^r$. Then the structure $\mathbb{R}_{\mathrm{an},H}$ generated by the restricted analytic functions and H is o-minimal and model complete.

Moreover, the authors of [RSS07] provide sufficient conditions to ensure that H be indeed strongly quasi-analytic:

1. The eigenvalues $\lambda_1, ..., \lambda_r$ of the matrix $\frac{\partial A}{\partial \mathbf{y}}(0, \mathbf{0})$ are non-zero and their arguments are two-by-two distinct. This implies that the formal power series $\hat{H}$ is unique and it is p-summable in every direction except along Stokes lines θ_{lj} (there are pr of them) determined by the equation $p\theta_{lj} = \arg\lambda_j + 2\pi l$. ($1 \leq j \leq r$; $0 \leq l \leq p-1$). With each Stokes line is associated a Stokes coefficient c_{lj}.
2. For every j there is at least one l such that $c_{lj} \neq 0$.

Although we will not deal with this structure in the rest of this thesis, the terminology employed here will be explained at length in Chapter 2, as it is the same needed to understand the o-minimal structure $\mathbb{R}_{\mathscr{G}}$, to be defined later, which will be at the focus of our future discussions.

$\mathbb{R}_{\mathbf{an}^*}$ and $\mathbb{R}_{\mathbf{an}^*,\mathbf{exp}}$: Convergent generalised power series

This structure is a generalisation of $\mathbb{R}_{\mathrm{an}}$ and is described in [vS98].
The primary feature of analytic functions is that they are uniquely determined by a power series with integer powers in every variable. Nevertheless, we may consider a tuple $X = (X_1, ..., X_n)$ and a formal power series

$$F(X) = \sum_{\alpha \in [0,\infty)^m} c_\alpha X^\alpha = \sum_{\alpha \in [0,\infty)^m} c_{\alpha_1,...,\alpha_m} X_1^{\alpha_1} \cdot ... \cdot X_m^{\alpha_m} \tag{1.41}$$

where $\alpha_i \in [0,\infty)$: this is called a **generalised power series**. Just like ordinary power series form an algebra $\mathbb{R}[[X]]$, generalised power series form an algebra closed under sums and Cauchy products[7] denoted by $\mathbb{R}[[X^*]]$. It is possible to endow such an algebra with a norm $||\cdot||_r : \mathbb{R}[[X^*]] \to [0,\infty]$. The norm is determined by a polyradius $r = (r_1, ..., r_m)$, where $r_i \in (0,\infty)$: we then let the a norm be

$$||F||_r = \sum_{\alpha\in[0,\infty)^m} |c_\alpha| r^\alpha = \sum_{\alpha\in[0,\infty)^m} |c_{\alpha_1,...,\alpha_m}| r_1^{\alpha_1} \cdot ... \cdot r_m^{\alpha_m}. \tag{1.42}$$

Given a polyradius r, the subalgebra of generalised power series F with finite norm ($||F||_r < \infty$), is denoted by $\mathbb{R}\{X^*\}_r$.
We can then associate with every $F \in \mathbb{R}\{X*\}_r$ a function $f : \mathbb{R}^m \to \mathbb{R}$ defined as follows

$$f(x) = \begin{cases} \sum_{\alpha\in[0,\infty)^m} c_\alpha x^\alpha & x \in [0, r_1] \times ... \times [0, r_m] \\ 0 & \text{otherwise} \end{cases} \tag{1.43}$$

where the coefficients c_α are those of $F(X) = \sum_{\alpha\in[0,\infty)^m} c_\alpha X^\alpha$. These functions are analytic everywhere in the open box $(0, r_1) \times ... \times (0, r_m)$, while they may be non-analytic on the boundary. After collecting all the functions f as above in the collection $\mathcal{F}$, we can set $\mathbb{R}_{\text{an}^*} = (\mathbb{R}, <, 0, 1, +, -, \cdot, \mathcal{F})$. This structure contains $\mathbb{R}_{\text{an}}$, but many more functions are definable therein. The main result of [vS98] is that $\mathbb{R}_{\text{an}^*}$ is also an o-minimal structure:

> **Theorem 1.10.** The structure $\mathbb{R}_{\text{an}^*}$ is o-minimal and model complete.

In a later work ([DS00]) it was argued that this structure can be enlarged by including the exponential $\exp : \mathbb{R} \to \mathbb{R}$ on the whole real line. The structure $\mathbb{R}_{\text{an}^*,\exp}$ was argued to be o-minimal too in the same paper.

It is worthwhile to remark that a great deal of research in o-minimality has been driven by the search for an o-minimal structure where Euler Gamma function Γ and/or Riemann Zeta function ζ, appropriately restricted on the real line, are definable. The main interest in the structure $\mathbb{R}_{\text{an}^*,\exp}$ is that $\zeta|_{(1,\infty)}$ is definable therein; it was later argued though, that $\Gamma|_{(0,\infty)}$ is not ([RSS23]).

$\mathbb{R}_{\mathscr{G}}$ and $\mathbb{R}_{\mathscr{G},\exp}$: Multisummable series

This structures were introduced in [DS00] and it will be at the core of the results that we shall present in Chapter 3 and 4. The precise definition of the class of functions $\mathscr{G}$ is in general rather involved, since these functions are defined first on the complex plane $\mathbb{C}^n$ and only later are they restricted to $\mathbb{R}^n$. Furthermore, the domains on the complex plane for functions of n variables are not expressed by Cartesian products of the domains of the one variable functions: rather, they are defined by means of inner products (i.e., viewing a complex number $z \in \mathbb{C}^n$ as a vector). We refer to Appendix B for a complete definition of this structure. Subtleties notwithstanding, it can be shown that the functions of one variable in this structure turn out to have a much simpler description; in the following we will merely focus on these.

Consider a sector of the complex plane parametrised as follows

$$S(R, \phi, \kappa) = \{t \in \mathbb{C} \ : \ 0 < |t| < R, |\arg(t)| < \kappa\phi\} \tag{1.44}$$

where $R > 0$, $0 < \kappa \leq 1$ and, most importantly $\phi \in (\frac{\pi}{2}, \pi)$.
Next, consider a holomorphic function $f : S(R, \phi, \kappa) \to \mathbb{C}$ which satisfies the following conditions:

1. $\lim_{t\to 0} f^n(t)$ exists for every $n \in \mathbb{N}$ and $t \in S(R, \phi, \kappa)$;

[7] This is the name that the ordinary product takes when applied to formal power series – more on this in the next chapter.

2. f satisfies the **Gevrey condition**: there exist constants A, B depending on f, such that:

$$\left|\frac{f^{(n)}(t)}{n!}\right| \leq AB^n(n!)^\kappa \tag{1.45}$$

for every $n \in \mathbb{N}$ *and* for every $t \in S(R, \phi, \kappa)$.

The real number κ determines what is known as the *Gevrey class* of f. It is important to observe how the exponential growth of the Taylor coefficients, for every $\kappa > 0$, prevents the convergence of the Taylor series and therefore the analyticity of the function f at 0. Moreover, observe that κ plays the rôle of $1/p$ in the section dedicated to the extension of $\mathbb{R}_{\text{an}}$ by quasi-analytic solutions of ODEs, $\mathbb{R}_{\text{an},H}$.

Since the limit for $t \to 0$ must exist, we are enabled to extend every such function to $S \cup \{0\}$ by setting $f(0) := \lim_{t\to 0} f^n(t)$ and collect all the functions f thus extended in the family $\mathscr{G}(R, \phi, \kappa)$. It was proved that under the above assumptions the Taylor map

$$\begin{aligned} \mathscr{G}(R, \phi, \kappa) &\to \mathbb{C}[[t]] \\ f &\to \hat{f} := \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n \end{aligned} \tag{1.46}$$

is injective, and therefore the ring of functions $\mathscr{G}(R, \phi, \kappa)$ is a ring of quasi-analytic functions.

We must now find an appropriate restriction to real-valued functions on (a subset of) the real line. We define then $\mathscr{G}(R)$ to be the set of all functions $f : [0, R] \to \mathbb{R}$ for which there exist:

- a radius $\tilde{R} \geq R$;
- an angle $\phi \in (\frac{\pi}{2}, \pi)$;
- n constants $\kappa_1, ..., \kappa_n$;
- n functions $f_1, ..., f_n$ with $f_i \in \mathscr{G}(\tilde{R}, \phi, \kappa_i)$ for every $i = 1, ..., n$, such that

$$f(x) = f_1(x) + ... + f_n(x) \ , \qquad \text{for all } \ x \in [0, R]. \tag{1.47}$$

Every f_i is equipped with a different κ, whence we understand the name multisummable: the function f is expressed as the sum of functions f_i which are κ_i-summable (see e.g. [Bal94] for a review of multisummability). The order of summability is what determines the correct form of the Borel-Laplace resummation: this terminology will be explained at length in Chapter 2.

It can be proved that $\mathscr{G}(R)$ is again a ring under point-wise addition and multiplication and that, moreover, it is again quasi-analytic (where now the Taylor map (1.46) takes values in $\mathbb{R}[[x]]$). Finally, the one-variable functions of $\mathscr{G}$ are defined to be $f : \mathbb{R} \to \mathbb{R}$ which are identically 0 outside $[0, 1]$ and their restriction on $[0, 1]$ belongs to $\mathscr{G}(1)$;

$$f \in \mathscr{G} \Rightarrow \begin{cases} f|_{[0,1]} \in \mathscr{G}(1) \\ \\ f(x) = 0 \qquad x \notin [0, 1]. \end{cases} \tag{1.48}$$

We refer to Appendix B and [DS00] for the construction of the functions of $\mathscr{G}$ of multiple variables. We merely state here the main result:

> **Theorem 1.11.** The structure $\mathbb{R}_{\mathscr{G}} = (\mathbb{R}, <, 0, 1, +, -, \cdot, \mathscr{G})$ is o-minimal and model complete . Furthermore, the structure $\mathbb{R}_{\mathscr{G},\exp} = (\mathbb{R}, <, 0, 1, +, -, \cdot, \mathscr{G} \cup \{\exp\})$ is o-minimal and model complete too.

Again, the exponential must be understood as defined on the whole real line (and not restricted to any closed subset).

As it was the case for $\mathbb{R}_{\mathrm{an}^*}$, the interest in this structure originally lay in finding definability statements for the Gamma and the Zeta functions. As we will show in section 2.6.3 it can be argued that $\Gamma|_{(0,+\infty)}$ is definable in $\mathbb{R}_{\mathscr{G},\exp}$. However, it was found that $\zeta|_{(1,+\infty)}$ is not.

$\mathbb{R}_{\mathscr{G}^*}$ and $\mathbb{R}_{\mathscr{G}^*,\exp}$: Generalised multisummable series

These structures were studied very recently in [RSS23]. Their construction is rather involved and we refer to this work for the details. We shall limit ourselves to a qualitative description of these structures.
The rough idea is to extend the class of multisummable functions $\mathscr{G}$ by means of generalised power series, in a similar fashion to that in which ordinary power series with integer exponents, appearing in $\mathbb{R}_{\mathrm{an}}$, were upgraded to power series with real exponents ranging over continuous intervals according to (1.41). To do so, the authors consider an infinite sequence of generalised power series $(F_p)_{p\in\mathbb{N}}$: each of them is the asymptotic expansion of a function f_p, holomorphic on a domain S_p^τ and satisfying Gevrey estimates. Then it is proved that the sum $\sum_{p\in\mathbb{N}} f_p$ converges on a certain domain to a function f, which is then included in the collection $\mathscr{G}^*$. As usual, the functions of this family are non-vanishing only on a closed subset of $\mathbb{R}^n$; moreover, the domains in the complex plane $\mathbb{C}^n$ of functions of n variables of $\mathscr{G}^*$ are not Cartesian products of the domains of one-variable functions, as it is the case for $\mathbb{R}_{\mathscr{G}}$.
In analogy to both the structure of multisummable functions and that of generalised convergent power series ($\mathbb{R}_{\mathscr{G}}$ and $\mathbb{R}_{\mathrm{an}^*}$, respectively) the exponential function can be added to $\mathbb{R}_{\mathscr{G}^*}$, obtaining thus the structure $\mathbb{R}_{\mathscr{G}^*,\exp}$. Both $\mathbb{R}_{\mathscr{G}^*}$ and $\mathbb{R}_{\mathscr{G}^*,\exp}$ are eventually proved to be o-minimal and model complete.

$\mathbb{R}_{\mathscr{G}^*,\exp}$ is a remarkably large structure, as it contains both $\mathbb{R}_{\mathrm{an}^*,\exp}$ and $\mathbb{R}_{\mathscr{G},\exp}$. Furthermore, it enjoys the property that was for long sought by the research community in o-minimality: both the Gamma function $\Gamma|_{(0,+\infty)}$ and the Zeta function $\zeta|_{(1,+\infty)}$ are definable in this o-minimal structure. Besides, it is worth stressing that the proof of o-minimality of $\mathbb{R}_{\mathscr{G}^*}$ and $\mathbb{R}_{\mathscr{G}^*,\exp}$, produced in [RSS23], rests on the earlier results illustrated in [RS15]. In this work, the authors realised that all the previously known proofs of o-minimality could be unified into a unique proof which deals with quasi-analytic algebras of functions and their germs at 0. The structures of generalised multisummable series $\mathbb{R}_{\mathscr{G}^*}$ and $\mathbb{R}_{\mathscr{G}^*,\exp}$ are the first instances of structures whose o-minimality had been previously unknown, and could only be established thanks to the main theorem proved in [RS15].

There are some more o-minimal structures known which we shall describe in the next section within the purview of sharply o-minimal structures. However, we will defer their treatment to the following chapter about sharply o-minimal structures. Rather, in conclusion to this section, we briefly address the following question.

1.5.4 Is there a largest o-minimal structure?

This is an intriguing and far from trivial question. Some structures are naturally nested into each other: $\mathbb{R}_{\mathrm{an}}$ and $\mathbb{R}_{\mathscr{G}}$ are but one of the possible examples. Nevertheless, we may wonder whether there exists a maximal o-minimal structure which encompasses all the others (in the same fashion as $\mathbb{R}_{\mathrm{alg}}$ is contained in all of them). The answer provided in [RSW03] is in the negative. Indeed the authors prove the following:

> **Theorem 1.12.** Given any C^∞ function $f : U \to \mathbb{R}$, where U is an open neighbourhood of the unit cube $[-1,1]^n$, there exist strongly log-convex sequences M and N obeying the quasi-analyticity condition (1.36) and two functions $f_1 \in \mathcal{C}^0_{[-1,1]^n}(M)$, $f_2 \in \mathcal{C}^0_{[-1,1]^n}(N)$ such that, for every $x \in [-1,1]^n$, one has
>
> $$f(x) = f_1(x) + f_2(x).$$

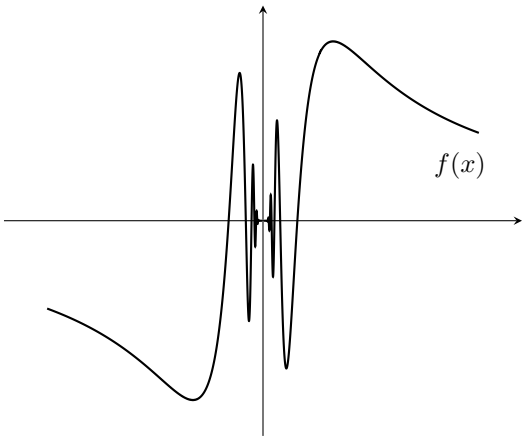


Figure 1.4: A non-tame smooth function.

Combining this theorem with the previous results, it can be argued that it is always possible to find strong log-convex sequences M and N such that $\mathbb{R}_{\mathcal{C}(M)}$ and $\mathbb{R}_{\mathcal{C}(N)}$ are both o-minimal but they are not *both* embedded in a common larger o-minimal structure. Here is an easy example which shows why. Consider the function $f : [-1, 1] \to \mathbb{R}$ of Figure 1.4

$$f(x) = \begin{cases} e^{-1/x^2} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases} \tag{1.49}$$

This function is clearly of class C^∞. Then, by virtue of the previous theorem, there must exist two functions f_1 and f_2, definable in $\mathbb{R}_{\mathcal{C}(M)}$ and $\mathbb{R}_{\mathcal{C}(N)}$ respectively, whose sum is f. If there existed a largest structure $\mathbb{R}_{\text{Max}}$ such that $\mathbb{R}_{\mathcal{C}(M)} \subseteq \mathbb{R}_{\text{Max}}$ as well as $\mathbb{R}_{\mathcal{C}(N)} \subseteq \mathbb{R}_{\text{Max}}$, then it would follow that f is definable in $\mathbb{R}_{\text{Max}}$, being it the sum of two definable functions in that same structure. But this is a patent contradiction: f cannot be definable, since its zero set is clearly an infinite union of points. Therefore, although o-minimal structures can be embedded into each other, there cannot exist a maximally large one that comprises all the o-minimal expansions of the real field $\mathbb{R}$.

1.6 Sharply o-minimal structures

In this section we introduce a refinement of o-minimal structures, called **sharply o-minimal structures**, mostly following the recent work [BNZ22] by G. Binyamini, D. Novikov and B. Zack. While in o-minimal structures $(\mathcal{S}_m)_{m\in\mathbb{N}}$ the definable sets are only distinguished by the collection $\mathcal{S}_m$ in which they are located, sharply o-minimal structure provide a more refined categorisation of the sets, given by the so-called *filtrations*. As it will become clear, filtrations allow us to define a notion of *complexity*, i.e. a quantitative measure of the information carried by a definable set and, as a consequence, by a definable function.

1.6.1 Definitions

Let $\mathcal{S}$ be an o-minimal expansion of the real field $\mathbb{R}$. We can categorise the definable sets living in $\mathcal{S}$ by a **filtration** Ω, which is a sequence of collections of sets $\{\Omega_{F,D}\}_{F,D\in\mathbb{N}}$. The integers F and D are called **format** and **degree**, respectively.

Definition 1.16 (FD-filtrations)**.** We say that $\Omega = \{\Omega_{F,D}\}_{F,D\in\mathbb{N}}$ is a FD**-filtration** on an o-minimal expansion $\mathcal{S}$ of the real field if:

1. Every $\Omega_{F,D}$ is a collection of definable sets in $\mathcal{S}$;
2. For every F and for every D, $\Omega_{F,D} \subset \Omega_{F+1,D} \cap \Omega_{F,D+1}$;
3. Every definable set in $\mathcal{S}$ belongs to a family $\Omega_{F,D}$ for some format F and some degree D.

Note how the second axiom tells us how the families $\Omega_{F,D}$ in the filtration Ω are nested into each other in a non-trivial way. Different filtrations can be related through an operation named *reduction*, which we define precisely in the following

Definition 1.17 (Reduction of filtrations)**.** Let Ω and Ω' be two FD-filtrations on $\mathcal{S}$. Then Ω is said to be **reducible** to Ω' if there exist a function $a : \mathbb{N} \to \mathbb{N}$ such that, for every format F there exists a polynomial p_F , and for every degree D one has that

$$\Omega_{F,D} \subset \Omega'_{a(F),p_F(D)}.$$

We will write then $\Omega \leq \Omega'$.

If two filtrations are each reducible to each other, namely $\Omega \leq \Omega'$ and $\Omega' \leq \Omega$ they are said to be **equivalent**.

By means of filtrations, we can define sharply o-minimal structures, indicated for brevity as #o-minimal structures.

Definition 1.18 (Sharply o-minimal structures)**.** A **#o-minimal structure** is a pair $\Sigma = (\mathcal{S}, \Omega)$ where $\mathcal{S}$ is an o-minimal expansion of $\mathbb{R}$ and $\Omega = \{\Omega_{F,D}\}_{F,D\in\mathbb{N}}$ is a FD-filtration on $\mathcal{S}$ such that, for every F there exist a polynomial P_F with positive coefficients and the following are satisfied. If $A \in \Omega_{F,D}$ then

1. If $A \subset \mathbb{R}$, A has at most $P_F(D)$ connected components;
2. If $A \subset \mathbb{R}^l$, then $F \geq l$;
3. If $A \subset \mathbb{R}^l$, then $\pi_{l-1}(A)$, A^c are in $\Omega_{F,D}$, while $A \times \mathbb{R}$ and $\mathbb{R} \times A$ are in $\Omega_{F+1,D}$.

Furthermore, for any finite number k of subsets of $\mathbb{R}^l$ $A_1, A_2, ..., A_k \subset \mathbb{R}^l$ with $A_i \in \Omega_{F_i,D_i}$, let us fix $F = \max_i\{F_i\}$ and $D = \sum_{i=1}^k D_i$. Then we have

4. $\bigcup_{i=1}^k A_i \in \Omega_{F,D}$;
5. $\bigcap_{i=1}^k A_i \in \Omega_{F,D}$.

Finally, if P is a polynomial of l variables of $\deg(P)$ in the ring $\mathbb{R}[x_1, ..., x_l]$, then

6. The set $\{x \in \mathbb{R}^l : \ P(x) = 0\}$ belongs to $\Omega_{l,\deg(P)}$.

A #o-minimal structure is then characterized by an infinite sequence of polynomials P_F which define the number of connected components of the subsets of the real line; these are forced to be a finite union of points and intervals by the o-minimality axiom 1.5. These structures are actually found by posing stronger constraints on the so called pre-sharp (P#) and weakly-sharp (W#) o-minimal structures. These are defined as follows.

Definition 1.19 (Pre-sharp o-minimal structures)**.** A **P#o-minimal structure** is a pair $\Sigma = (\mathcal{S}, \Omega)$ as above such that if $A \in \Omega_{F,D}$ then

1. If $A \subset \mathbb{R}$, A has at most $P_F(D)$ connected components;
2. If $A \subset \mathbb{R}^l$, then $F \geq l$;
3. If $A \subset \mathbb{R}^l$, then $\pi_{l-1}(A)$, A^c, $A \times \mathbb{R}$ and $\mathbb{R} \times A$ are in $\Omega_{F+1,D}$.

For any $A_1, A_2, ..., A_k \subset \mathbb{R}^l$ with $A_i \in \Omega_{F_i,D_i}$, fixing $F = \max_i\{F_i\}$ and $D = \sum_{i=1}^k D_i$, we have

4. $A_1 \cup A_2 \in \Omega_{F+1,D}$;
5. $A_1 \cap A_2 \in \Omega_{F+1,D}$.

If P is a polynomial in the ring $\mathbb{R}[x_1, ..., x_l]$, then

6. The set $\{x \in \mathbb{R}^l : P(x) = 0\}$ belongs to $\Omega_{l,\deg(P)}$.

By adding some constraints, one has

Definition 1.20 (Weakly sharp o-minimal structures)**.** A **P#o-minimal structure** is a pair $\Sigma = (\mathcal{S}, \Omega)$ as above such that if $A \in \Omega_{F,D}$ then

1. If $A \subset \mathbb{R}$, A has at most $P_F(D)$ connected components;
2. If $A \subset \mathbb{R}^l$, then $F \geq l$;
3. If $A \subset \mathbb{R}^l$, then $\pi_{l-1}(A)$, A^c, $A \times \mathbb{R}$ and $\mathbb{R} \times A$ are in $\Omega_{F+1,D}$.

Given $A_1, A_2 \subset \mathbb{R}^l$ with $A_i \in \Omega_{F_i,D_i}$, and fixing $F = \max_i\{F_i\}$ and $D = D_1 + D_2$ we have

4. $\bigcup_{i=1}^k A_i \in \Omega_{F,D}$;
5. $\bigcap_{i=1}^k A_i \in \Omega_{F+1,D}$.

If P is a polynomial in the ring $\mathbb{R}[x_1, ..., x_l]$, then

6. The set $\{x \in \mathbb{R}^l : P(x) = 0\}$ belongs to $\Omega_{l,\deg(P)}$.

As it can be seen then, pre-sharp, weakly-sharp, and sharply o-minimal structures are subsequent restrictions of each other.

Structures:

$$\#\text{o-minimal} \subset \text{W}\#\text{o-minimal} \subset \text{P}\#\text{o-minimal} \subset \text{o-minimal}$$

Pre-sharpness, weak-sharpness and sharpness all depend on the possibility of finding an appropriate FD-filtration. Finding a filtration on a given o-minimal expansion $\mathcal{S}$ and proving that it is such, is in general rather difficult. In practice, one proceeds by first selecting a collection of definable sets $\{A_\alpha \in \mathcal{S} : A_\alpha \subseteq R^{l_\alpha}\}$ which generate the o-minimal structure $\mathcal{S}$. Each set in this collection can be associated with a format $F_\alpha \geq l_\alpha$ and a degree D_α: then the minimal FD- filtration Ω satisfying all the axioms in (1.18) *except the first*, and such that $A_\alpha \in \Omega_{F_\alpha,D_\alpha}$ is called the FD- filtration *sharply generated* by the collection $\{(A_\alpha, F_\alpha, D_\alpha)\}$. Clearly, only if the axiom 1 in (1.18) is satisfied can $(\mathcal{S}, \Omega)$ be a honest #o-minimal structure. If this were not to be the case, one can always repeat the process by taking all the definable sets A_α of the sharply generated filtration, even though there are now sets associated with more than one (F, D) pair. Thus a new filtration Ω' is found, which is said to be *sharply generated by* Ω. This must again be checked against the first axiom in (1.18) before claiming that $(\mathcal{S}, \Omega')$ is sharply o-minimal.

1.6.2 Properties of sharply o-minimal structures

Sharp cell decomposition

As we have remarked earlier, the Cell Decomposition Theorem (1.2) is perhaps the most essential feature of o-minimal structures, as it captures how definable sets are made up of a finite number of particularly regular

subsets, which we name cells. Sharply o-minimal structures admit of course cell decompositions; in addition to those though, the authors of [BNZ22] advanced a stronger version of the Definition 1.7.

Definition 1.21 (Sharp Cell Decomposition)**.** Let $\mathcal{S}$ be an o-minimal expansion of the real field and Ω a FD-filtration. $(\mathcal{S}, \Omega)$ is said to have a **sharp cell decomposition** (#CD) if, for any collection of k sets $X_1, ..., X_k \subset \mathbb{R}^l$, *each* of format F and D, there exists a cell decomposition, compatible with each X_j, $j = 1, ..., k$, such that:

- There exists a monotone function $a \to C(a)$ such that all the cells have format $C(F)$;
- There exists a polynomial of one variable $P_F(x)$, whose coefficients depend on F, such that all the cells have degree $P_F(D)$;
- There exists a polynomial of two variables $Q_F(x, y)$, whose coefficients depend on F, such that the number of cells is bounded from above by $Q_F(D, k)$.

Note that these functions and polynomials depend on $(\mathcal{S}, \Omega)$, and they are different from the polynomial $P_F(D)$ appearing in the definitions above. We quote here some results from [BNZ22] regarding sharp cell decomposition.

> **Proposition 1.9.** Let $(\mathcal{S}, \Omega)$ be a weakly-sharp (W#) o-minimal structure with sharp cellular decomposition (#CD). Then there exists a filtration Ω' equivalent to Ω such that $(\mathcal{S}, \Omega')$ is #o-minimal with #CD.

Thus, we understand that a sharp cell decomposition guarantees that a weakly-sharp o-minimal structure can indeed obey stronger axioms, once the new (but equivalent) filtration is found. In the previous proposition though, a #CD is taken as an assumption; nevertheless, there exist sufficient conditions that guarantee their existence. First we need the following

Definition 1.22 (*format and * degree)**.** Let $\mathcal{S}$ be an o-minimal expansion of $\mathbb{R}$ and Ω a FD-filtration upon it. Given a set $X \subset \mathbb{R}^l$ definable in $\mathcal{S}$, X is said to belong to $\Omega^*_{F,D}$ if it can be written as a finite union $X = \cup_i \pi^{l_i}_l(X_i^\circ)$, where

- $X_i \subset \mathbb{R}^{l_i}$ belongs to Ω_{F_i,D_i} and X_i° is a connected component thereof;
- $F = \max_i(F_i)$ and $D = \sum_i D_i$;
- $\pi^{l_i}_l : \mathbb{R}^{l_i} \to \mathbb{R}^l$ is the usual projection;

Then X is said to have *format F and *degree D.

It is worth remarking that Ω^*, thus defined, is again a FD-filtration on $\mathcal{S}$. The interest in it lies in then next theorem, which is the main result of [BNZ22]:

> **Theorem 1.13.** Let $(\mathcal{S}, \Omega)$ be a pre-sharp o-minimal structure. Then Ω^* is equivalent to to a filtration Ω' such that $(\mathcal{S}, \Omega')$ is a sharply o-minimal structure with sharp cell decomposition.

The insight of this theorem is clear: it establishes that despite the notions of P#, W# and # o-minimal structures may look different at first sight, they can all be reduced to the strongest case of # o-minimal structures and, furthermore, we can always safely assume a $\#CD$ whenever dealing with these structures. It is also thanks to this result that we can mainly concern ourselves with #o-minimal structures.

Finiteness of connected components

As it has been stressed multiple times, o-minimality is about constraining structures with a notion of finiteness. Such constraint is given by the axiom (1.5) for general o-minimal structures, while for sharp (pre-sharp, weakly-sharp) o-minimal structures, a stronger requirement is given by the first axiom of Definitions 1.18, 1.19 and 1.20, which binds with a polynomial the number of connected components of a subset of $\mathbb{R}$. Nevertheless, the finiteness of connected components and the polynomial bounds carry over to higher dimensions: this was already stated by Proposition 1.1, and the following can be regarded as its enhancement.

> **Proposition 1.10.** Given a pre-sharp structure $(\mathcal{S}, \Omega)$, let $X \subset$ of $\mathbb{R}^l$ be a set of format F and degree D. Then, there exists a polynomial P_F such that X has at most $P_F(D)$ connected components.

Logic of sharply o-minimal structures

Mindful of how geometrical structures reflect the properties of first-order logic, it is worthwhile to spend a few words on how the properties of #o-minimal structures are translated into logic terms.
Let us consider a #o-minimal structure $(\mathcal{S}, \Omega)$, and assume that it corresponds to a language L whose atomic predicates are in the form $x \in X$, with $X \in \Omega_{F,D}$ for some format F and degree D; let us assume further that no constant symbols or function symbols are included in the language. A formula ψ of n variables of such a language will then define a subset of $\mathbb{R}^l$ (the set of points where it holds true). Before taking any truth or falsehood into account though, we can assign to every formula a format and a degree as follows:

Definition 1.23. Given a formula ψ in the language L of n variables, let $X_j \in \Omega_{F_j,D_j}$ be the sets appearing in its atomic predicates. We will then say that ψ has degree $D = \sum_j D_j$ and format $\max(F, n)$, where $F = \max_j(F_j)$.

In this definition, we only look at how ψ is stated, irrespective of whether it never holds true or it is true for every $x_1, ..., x_n$ or all the intermediate cases. The sharp o-minimality of $(\mathcal{S}, \Omega)$ guarantees that:

> **Proposition 1.11.** There exists a function $C : \mathbb{N} \to \mathbb{N}$ and a polynomial P_F such that any formula ψ of format F and degree D defines a set of format $C(F)$ and degree $P_F(D)$.

1.6.3 Examples of sharply o-minimal structures

There are but few examples of structures that are known to be #o-minimal. While many o-minimal structures are conjectured (with some confidence) to be #o-minimal, some others are known instead *not* to be so: a prominent example of a structure which is o-minimal but not #o-minimal is $\mathbb{R}_{\text{an}}$. We refer to [BNZ22] for the proof of this fact; here we rather list some notable #o-minimal structures.

$\mathbb{R}_{\text{alg}}$

The simplest example of o-minimal structure, $\mathbb{R}_{\text{alg}}$, turns out to be also #o-minimal. This fact is quite non-trivial. Indeed, one would naïvely assume that a suitable filtration Ω over $\mathbb{R}_{\text{alg}}$ would be such that an algebraic set X, written as a finite union of basic sets

$$A = \{P_1 = ... = P_r = 0, Q_1, ..., Q_s < 0\} \tag{1.50}$$

with $P_i, Q_j \in \mathbb{R}[x_1, ..., x_l]$, belongs to $\Omega_{l,D}$, where D is the sum of the degrees of all the polynomials P_i and Q_j over all the basic sets A in the form of (1.50). However, it turns out that it is only the filtration Ω' *sharply generated* by Ω that makes $(\mathbb{R}_{\text{alg}}, \Omega')$ a sharply o-minimal structure.

$\mathbb{R}_{\mathbf{rPfaff}}$ and $\mathbb{R}_{\mathbf{Pfaff}}$: Pfaffian functions

Notable examples of #o-minimal structures are those of restricted Pfaffian functions $\mathbb{R}_{\text{rPfaff}}$ and Pfaffian functions $\mathbb{R}_{\text{Pfaff}}$: a brief discussion can be found in [BN23], [BNZ22] and [GSV23]. Given an open set $U \subseteq \mathbb{R}^m$, a **Pfaffian chain** of length n is defined to be a triangular system of n coupled differential equations in m variables, solved by n functions $\zeta_1, ..., \zeta_n : U \to \mathbb{R}$ of the following form:

$$\frac{d\zeta_i}{dx_j} = F_{ij}(x_1, ..., x_m, \zeta_1, ..., \zeta_i), \tag{1.51}$$

where F_{ij} are polynomials in all their variables (that is, in the coordinates x_i and the functions ζ_i). The Pfaffian chain is said to be **restricted** if U is bounded and the functions ζ_i extend analytically to a neighbourhood of $\text{cl}(U)$. These functions are then named (restricted) **Pfaffian functions**. The 'triangularity' lies in the fact that each polynomial F_{ij}, which defines a differential equation for the i-th function, only depends on the previous $(i-1)$-th functions and ζ_i itself, but not on the following functions in the chain. This prevents the building of a chain like

$$\begin{cases} \frac{d\zeta_1}{dx} = \zeta_2 \\ \\ \frac{d\zeta_2}{dx} = -\zeta_1 \end{cases} \tag{1.52}$$

which is obviously solved by $\zeta_1(x) = \sin(x)$ and $\zeta_2(x) = \cos(x)$, which are not definable in any o-minimal structure if their domain is taken to be $U = \mathbb{R}$. The structure of Pfaffian functions $\mathbb{R}_{\text{Pfaff}}$ is then the expansion of the real field $(\mathbb{R}, <, 0, 1, +, \cdot, \mathcal{F})$, with $\mathcal{F}$ being the collection of all functions f that can be expressed as polynomials in the coordinates and Pfaffian functions (1.51) in some Pfaffian chain; if all such Pfaffian chains are restricted the structure is called $\mathbb{R}_{\text{rPfaff}}$. Observe how any such function f, by the chain rule, is a Pfaffian function itself.
The o-minimality of $\mathbb{R}_{\text{Pfaff}}$ was proved in [Wil99]; at the time of writing, the sharp o-minimality is only known for the structure of restricted Pfaffian function $\mathbb{R}_{\text{rPfaff}}$, but it is conjectured that the arguments should carry over to Pfaffian functions defined on the whole real line. Note that, as opposed to the structure $\mathbb{R}_{\text{an}}$ of analytic functions, which are defined on a *closed* box, the Pfaffian functions are defined on open intervals. Roughly, the reason is that the differential equation for Pfaffian functions constrains their behaviours so as to avoid infinitely dense oscillations like those depicted in Figure 1.2.

Let us also remark that not only is $\mathbb{R}_{\text{Pfaff}}$ o-minimal, but taken an o-minimal structure $\mathcal{R}$, its Pfaffian closure $\mathcal{P}(\mathcal{R})$ is o-minimal. By Pfaffian closure, we mean that we can add to the generating family $\mathcal{F}$ of $\mathcal{R}$ all the solutions to the differential equations

$$\frac{\partial}{\partial x_i} f(x_1, ..., x_n) = F_i(x_1, ..., x_n, f(x_1, ..., x_n)) \tag{1.53}$$

where F_i is definable in $\mathcal{R}$. This was proved in [Spe97].

$\mathbb{R}_{\mathbf{rNoether}}$: Noetherian functions

A natural extension to Pfaffian functions is obtained by dropping the requirement that the system be triangular (see for instance [GSV23] or [BN23]). Given an open set $U \subset \mathbb{R}^m$ a collection of n functions $\zeta_1, ..., \zeta_n : \text{cl}(U) \to \mathbb{R}$ obeying polynomial system of differential equations

$$\frac{d\zeta_i}{dx_j} = F_{ij}(x_1, ..., x_m, \zeta_1, ..., \zeta_n) \tag{1.54}$$

where F_{ij} are again polynomials in all the variables, is called a **Noetherian chain**. By the arguments above, it is clear that these functions must needs be restricted to an interval if we want to achieve o-minimality. Moreover, as opposed to $\mathbb{R}_{\text{rPfaff}}$, the domain must be *closed*, essentially for the same reason why the analytic functions of $\mathbb{R}_{\text{an}}$ must be. It is conjectured that the o-minimal structure of restricted Noetherian functions, $\mathbb{R}_{\text{rNoether}}$, be also sharply o-minimal, but there is no proof available yet.

1.6.4 Two exercises

The polynomials P_F of $\mathbb{R}_{\mathbf{alg}}$

It is interesting to note that in the simplest example of a #o-minimal structure, namely $\mathbb{R}_{\text{alg}}$, the polynomials P_F that bound the number of connected components of 1-dimensional sets (as described in Definition 1.18), can be computed directly. Consider first $F = 1$. As a polynomial in one variable of degree D can have at most D roots, the axioms 1 and 6 in Definition 1.18 suggest that $P_1(D) = D$. This must be checked against the other axioms. Consider then n sets $A_i \in \Omega_{1,d_i}$ defined by the zero sets of polynomials Q_i of degree d_i. If each of them has at most $P_1(d_i) = d_i$ connected components, we can note that their union $\bigcup_i^n A_i$ can of course have at most $\sum_i^n d_i$ connected components, which in this case are simply points: they can be then defined as the zero set of a polynomial of degree $D = \sum_i^n d_i$, which matches exactly the degree furnished by the axioms in 1.18. The upper bound $P_1(D) = D = \sum_i^n d_i$ is then still valid. As for the intersection, a very similar argument holds. We can than establish that $P_1(D) = D$.
Let us move to format $F = 2$. It is clear that the number of connected components of an algebraic subset of $\mathbb{R}^2$ is maximised when it is a discrete set. Hence, we must consider intersections between the zero sets of two polynomials without common factors (as opposed to the union). Bézout's theorem guarantees that the number of intersections between the zero sets $\{x \in \mathbb{R}^2 : Q_1(x) = 0\}$ and $\{x \in \mathbb{R}^2 : Q_2(x) = 0\}$ of two polynomials Q_1 and Q_2 of degree d_1 and d_2 is at most $d_1 d_2$. Because the degree of the intersection is fixed to be $D = d_1 + d_2$ by the axioms in 1.18, we have to maximise the value $d_1 d_2$ as a function of D, which is accomplished by $d_1 d_2 \leq D^2/4$ (the problem is that of maximising the area of a rectangle given the semiperimetre). Thus we find $P_2(D) = D^2/4$.
These reasoning can be generalised to any format F, yielding finally

$$P_F(D) = \left(\frac{D}{F}\right)^F . \tag{1.55}$$

The easiest way to convince oneself of this upper bound is to realise that the number of connected components is maximised whenever the polynomials can be factored in a product of degree-one polynomials prior to taking intersections. For instance, for $F = 2$, the intersection between the sets of degree $d_1 = 2$ and $d_2 = 3$, $\{(x, y) \in \mathbb{R}^2 : \ (x-1)(x-2) = 0\} \cap \{(x, y) \in \mathbb{R}^2 : \ (y-1)(y-2)(y-3) = 0\}$ yields the vertices of a grid which are indeed $6 \leq \frac{(2+3)^2}{2^2}$. The equality is obviously achieved when the degrees d_1 and d_2 are equal.

The closure of a set X

In this simple example, we display the power of the interplay between logic and geometry, as applied to sharply o-minimal structures. Given a set $X \subseteq \mathbb{R}^n$ definable in a sharply o-minimal structure $\mathcal{S}$ such that $X \in \Omega_{F,D}$ for a suitable filtration Ω, we want to compute the format and degree of its closure, $\text{cl}(X)$. We first recall that the closure in a metric space can be conveniently defined as:

$$\text{cl}(X) = \left\{ y \in \mathbb{R}^n : \ \forall \epsilon \in \mathbb{R}\backslash\{0\} \ \ \exists x \in X \ \ \left(\sum_{i=1}^n (y_i - x_i)^2 \leq \epsilon^2 \right) \right\}. \tag{1.56}$$

Let us then define the following sets BX_ϵ and A and E:

$$\begin{aligned} BX_\epsilon &:= \left\{ (\epsilon, y, x) \in \mathbb{R}\backslash\{0\} \times \mathbb{R}^n \times X : \ \sum_{i=1}^n (y_i - x_i)^2 \leq \epsilon^2 \right\} \\ &:= \left\{ (\epsilon, y, x) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n : \ \sum_{i=1}^n (y_i - x_i)^2 \leq \epsilon^2 \right\} \cap (\mathbb{R}\backslash\{0\} \times \mathbb{R}^n \times X) \\ &:= A \cap E. \end{aligned} \tag{1.57}$$

The set A defined by the above formulas can be finally written in such a way that we can easily attach to it a format and a degree. Note that this set is definable in $\mathcal{S}$ because it is semialgebraic. By considering the projection π onto the first $2n+1$ coordinates, we have that

$$A = \pi\left\{(\epsilon, y, x, z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \; z^2 + \sum_{i=1}^{n}(y_i - x_i)^2 - \epsilon^2 = 0\right\}. \tag{1.58}$$

The polynomial appearing in the above expression has clearly degree 2: thus given the axioms $A \in \Omega_{2n+2,2}$. All we have to do now is to evaluate the format and the degree of the set E. As $\mathbb{R} \in \Omega_{1,0}$, we have that $\mathbb{R}^n \in \Omega_{n,0}$[8]. Secondly, $\mathbb{R}\backslash\{0\}$ can be viewed as $\{0\}^c$, which clearly belongs to $\Omega_{1,1}$: thus, by the axioms in 1.18, so does $\mathbb{R}\backslash\{0\}$. Referring again to the aforementioned axioms, we can conclude that $E = \big(X \times \mathbb{R}^{n+1}\big) \cap (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\backslash\{0\}) \in \Omega_{F+n+1,D+1}$. Note how the format is determined by the fact that $F \geq n$ as we are assuming that X is a subset of $\mathbb{R}^n$.
Finally, we have but to observe that $\mathrm{cl}(X)$ is simply the projection of BX_ϵ onto the last n coordinates. Thus, again according to the axioms, its format and degree will be the same as those of $BX_\epsilon = A \cap E$. Hence, we can conclude that the format of $\mathrm{cl}(X)$ is $\bar{F} = \max(F + n + 1, 2n + 2)$, while the degree is $\bar{D} = D + 3$. Yet, we might be still dissatisfied with this result as it depends on n. However, if we content ourselves with an upper bound on the format and the degrees, we can safely put $\bar{F} = 2F + 2$, as $F \geq n$. We have thus proved that

> **Proposition 1.12.** Consider a sharply o-minimal structure $\mathcal{S}$, endowed with a filtration $\Omega = \{\Omega_{F,D}\}_{F,D\in\mathbb{N}}$. Given a definable set $X \in \Omega_{F,D}$, the closure of X, $\mathrm{cl}(X)$, belongs to $\Omega_{2F+2,D+3}$.

1.7 Complexity

In this section we wish to illustrate how sharply o-minimal structures, as opposed to ordinary o-minimal structures, can be endowed with a notion of finite complexity. This means that every function or set can be associated with a number (or rather a tuple of numbers) which describes quantitatively the amount of information it carries. It should be noted from the start that there is no unique definition of what complexity is: different definitions can be taken in different fields to formalise the intuitive notion of 'some things being more complex than others'. In the following, we mostly refer to [GSV23].

1.7.1 Complexity of Pfaffian chains

Consider again a Pfaffian system of differential equations (1.51):

$$\frac{d\zeta_i}{dx_j} = F_{ij}(x_1, ..., x_m, \zeta_1, ..., \zeta_i) \qquad i = 1, ..., n \tag{1.59}$$

where we recall that the F_{ij}s are polynomials and the functions $\zeta_1, ..., \zeta_n$ are defined on an open box $U \subseteq \mathbb{R}^m$. A Pfaffian function is then any function f which can be written as a polynomial in the variables $x_1, ..., x_m$ and the functions ζ_i in the Pfaffian chain:

$$f = P(x_1, ..., x_m, \zeta_1, ..., \zeta_n), \tag{1.60}$$

where P is a polynomial. We can then wonder: "how complex is the function f ?". This question is admittedly vague, but we can define a rigorous notion of complexity which allows for a quantitative estimate thereof. It is rather natural to assume that the complexity is increased whenever one of the following is increased:

- The number of variables, m;

[8] Note that $\mathbb{R} = \{x \in \mathbb{R} : 0 = 0\}$: it is the set where a tautology holds!

- The length of the Pfaffian chain (also called the **order**), n;
- The maximal degree of the polynomials F_{ij} defining the Pfaffian chain, which we denote by α;
- The degree of the polynomial P which defines f, which we denote with β.

In order to produce a number for the complexity of the function f a viable course of action would be then combining these quantities into an unique number, according to some formula. Nevertheless, as put forward in [GSV23], there is actually no reason to do this: we can simply claim that the complexity of any given Pfaffian function f is given by the tuple $\mathcal{C}_Z(f) = (m, n, \alpha, \beta)$. The subscript Z refers to the Pfaffian chain of length m: it is needed because a function f could be Pfaffian in several different chains, and the complexity depends on which is chosen.
As pointed out in [GSV23], it is possible to bound the complexity of a Pfaffian function obtained by taking sums, products, compositions or derivatives of Pfaffian functions of known complexity: the bounds are given by the following lemma.

Lemma 1.4. Given two Pfaffian chains Z_1, Z_2 two functions f_1, f_2, of complexity $\mathcal{C}_{Z_1}(f_1) = (m, n_1, \alpha_1, \beta_1)$, $\mathcal{C}_{Z_2}(f_2) = (m, n_2, \alpha_2, \beta_2)$, we have the following:

1. The complexity of the sum $f_1 + f_2$, which is Pfaffian in the chain $Z_1 \cup Z_2$ is bounded by
$$\mathcal{C}_{Z_1 \cup Z_2}(f_1 + f_2) = (m, n_1 + n_2, \max(\alpha_1, \alpha_2), \max(\beta_1, \beta_2)). \tag{1.61}$$
2. The complexity of the product $f_1 f_2$, which is Pfaffian in the chain $Z_1 \cup Z_2$ is bounded by
$$\mathcal{C}_{Z_1 \cup Z_2}(f_1 f_2) = (m, n_1 + n_2, \max(\alpha_1, \alpha_2), \beta_1 + \beta_2). \tag{1.62}$$
3. The complexity of the partial derivative $\frac{\partial f_1}{\partial x_j}$, which is definable in the same Z_1 as f_1, is bounded by
$$\mathcal{C}_{Z_1}\left(\frac{\partial f_1}{\partial x_j}\right) = (m, n_1, \alpha_1, \alpha_1 + \beta_1 - 1). \tag{1.63}$$
4. Assuming now that $m = 1$ and that $f_1 : U_1 \to \mathbb{R}$, $f_2 : U_2 \to \mathbb{R}$ such that $f_1(U_1) \subseteq U_2$, the complexity of the composition $f_2 \circ f_1$ is bounded by
$$\mathcal{C}_{Z_2 \circ Z_1}(f_2 \circ f_1) = (1, n_1 + n_2, \alpha_2\beta_1 + \alpha_1 + \beta_1 - 1, \beta_2), \tag{1.64}$$
where the chain $Z_2 \circ Z_1$ is a composite Pfaffian chain, which consists of the chain Z_1 and the composition of f_1 with all the functions in the chain Z_2.

Proof. Let us fix
$$\begin{aligned} f_1 &= P_1(x_1, ..., x_m, \zeta_1^1, ..., \zeta_{n_1}^1) \\ f_2 &= P_2(x_1, ..., x_m, \zeta_1^2, ..., \zeta_{n_2}^2) \end{aligned} \tag{1.65}$$
where it is understood that the upper index labels the Pfaffian chain (either Z_1 or Z_2) the function belongs to.

1. The number of variables is always the same, m. The length of the Pfaffian chain $Z_1 \cup Z_2$ is of course the sum of the respective lengths, so $n = n_1 + n_2$. The maximal degree of the resulting chain is then obviously the largest between the maximal degrees of the chains Z_1 and Z_2, thus $\alpha = \max(\alpha_1, \alpha_2)$. Finally, because $f_1 + f_2 = P_1 + P_2$, the degree of the polynomial $P \equiv P_1 + P_2$ is the largest between the degrees of P_1 and P_2, so $\beta = \max(\beta_1, \beta_2)$.
2. For m, n, α, the same discussion as above still holds. As for β, it is easy to observe that now $f_1 f_2 = P_1 P_2 =: P$, which is a polynomial of degree given by the sum of the degrees of P_1 and P_2. Hence $\beta = \beta_1 + \beta_2$.

3. By the chain rule, one has:

$$\frac{\partial f_1}{\partial x_j} = \frac{\partial P_1}{\partial x_j} + \sum_{i=1}^{n_1} \frac{\partial P_1}{\partial \zeta_i^1} \frac{\partial \zeta_i^1}{\partial x_j} = \frac{\partial P_1}{\partial x_j} + \sum_{i=1}^{n_1} \frac{\partial P_1}{\partial \zeta_i^1} F_{ij}^1 \tag{1.66}$$

where by F_{ij}^1 we mean the polynomial which equals $\frac{\partial \zeta_i^1}{\partial x_j}$ in the chain Z_1. This shows, in the first place, why the partial derivative of a Pfaffian function is Pfaffian in the same chain: indeed the right-hand side is again a polynomial in the variables $x_1, ..., x_m$ and the functions $\zeta_1^1, ..., \zeta_{n_1}^1$ which solve the original chain ζ_1. Because of this, n and α, which are features of the chain, are unchanged: $n = n_1$ and $\alpha = \alpha_1$; the number of variables is of course also unchanged. Finally, because F_{ij}^1 has degree at most α, while $\frac{\partial P_1}{\partial \zeta_i^1}$ has degree at most $\beta_1 - 1$, the right-hand side must be a polynomial of degree no larger than $\beta = \alpha_1 + \beta_1 - 1$.

4. With the assumptions made, the number of variables is $m = 1$. Secondly, the order of the composite chain $Z_2 \circ Z_1$ is again the sum of the orders of Z_1 and Z_2: $n = n_1 + n_2$. The only difference with the cases 1 and 2 is that now the second chain is no longer Z_2, but rather $Z_2 \circ f_1$ (with some abuse of notation). The degree of this chain can be bounded by writing:

$$\frac{\partial}{\partial x}(\zeta_i^2 \circ f_1) = \sum_{i=1}^{n_1} \frac{\partial \zeta_i^2}{\partial y} \frac{\partial f_1}{\partial x} \tag{1.67}$$

where $y = f_1(x)$. We can then observe that

$$\frac{\partial \zeta_i^2}{\partial y} = F_{i1}^2(y, \zeta_1^2(y), ..., \zeta_i^2(y)) = F_{i1}^2(P_1, \zeta_1^2(P_1), ..., \zeta_i^2(P_1)). \tag{1.68}$$

F_{i1}^2 has degree at most α_2: hence, its composition with P_1, of degree β_1, returns a polynomial of degree $\alpha_2\beta_1$. Therefore, looking back at (1.67) and mindful of the result obtained at point 3, we finally get that the degree of the composite Pfaffian chain is $\alpha = \alpha_2\beta_1 + \alpha_1 + \beta_1 - 1$.
By writing

$$f_2 \circ f_1 = P_2(x, \zeta_1^2(f_1), ..., \zeta_{n_2}^2(f_1)) \tag{1.69}$$

it is clear that the degree of P_2 is unchanged having defined the composite chain $Z_2 \circ Z_1$ appropriately. Hence $\beta = \beta_2$.

□

1.7.2 Complexity in Pfaffian structures

Khovanskii theorem

The interest in the notion of complexity introduced above resides in some notable results making use of it. It has been pointed out earlier how the number of connected components of an algebraic set is bounded by polynomials $P_F(D)$ of the degree (and labelled by the format) of the set itself. It was also explained how the explicit formula for such polynomials (1.55) relies on Bézout theorem. In some analogy, the following theorem holds:

> **Theorem 1.14** (Khovanskii)**.** Consider a Pfaffian chain Z on an open box $U \subseteq \mathbb{R}^m$ and a set of Pfaffian functions $f_1, ..., f_m$ of complexity $\mathcal{C}(f_i) = (m, n, \alpha, \beta_i)$. The set
>
> $$X = \{x \in \mathbb{R}^m : f_1(x) = ... = f_m(x) = 0\}$$

has cardinality bounded by:

$$|X| = 2^{n(n-1)}\beta_1...\beta_m \left(\min(m,n) + \beta_1 + ... + \beta_m\right)^n$$

The term $\beta_1...\beta_m$ is the same given by Bézout's theorem for polynomials; the correction for Pfaffian functions is given in terms of the complexity of the Pfaffian functions $f_1, ..., f_m$.

Topological complexity

In analogy to semialgebraic sets, a semipfaffian set is the finite union of sets defined by equalities or inequalities of Pfaffian functions, namely of functions definable in $\mathbb{R}_{\text{rPfaff}}$. Taking Pfaffian functions restricted to the open set $U \subseteq \mathbb{R}^m$, we have

$$X = \bigcup_{1 \le i \le M} \{x \in \mathbb{R}^l : f_{i_1}(x) = ... = f_{i_I}(x) = 0, g_{i_1} > 0, ..., g_{i_J} > 0\} \tag{1.70}$$

where all the functions appearing above, for every i_j, are definable in $\mathbb{R}_{\text{rPfaff}}$. A complexity can be then assigned to this set by considering the sum of the Betti numbers of X

$$b(X) = b_1(X) + ... + b_m(X) \tag{1.71}$$

where we recall that the n-th Betti number is the dimension of the n-th homology group of X: $b_n = \dim(H_n(X, \mathbb{R}))$. Suppose that all the functions appearing in (1.70) are all Pfaffian in the same chain and that they all have complexity $\mathcal{C} = (m, n, \alpha, \beta)$. Then it can be shown that, given M the number of unions taken in the definition (1.70), the sum of the Betti numbers is bounded by:

$$b(X) \le M^2 2^{n(n-1)/2} O(\min(m,n)\alpha + m\beta)^{m+n} \tag{1.72}$$

where the notation O indicates an unspecified polynomial. Despite the exact form of this polynomial is not given, it is nevertheless of great interest to know that the complexity grows only polynomially and not faster (e.g. exponentially).

1.7.3 Sharp complexity

A sharply o-minimal structure is an o-minimal structure endowed with a filtration Ω, so that every definable set is associated with infinitely many pairs (F, D). The reason why there are infinitely many lies in the second axiom in (1.16): since $\Omega_{F,D} \subset \Omega_{F+1,D} \cap \Omega_{F,D+1}$, if a set X is definable with format F and degree D, it is also definable with format $F+1$ and degree D or with format F and degree $D+1$. Nevertheless, this is no novelty: it is clear that our previous definition of complexity for Pfaffian chains is also affected by the same redundancy. As put forward in [GSV23], the format and degree (F, D) of a definable set X in a #o-minimal structure can be viewed as a measure of complexity, named sharp complexity. More precisely, the sharp complexity of a set X can be defined as

$$\#\mathcal{C}(X) := \{(F, D) : X \in \Omega_{F,D},\ X \notin \Omega_{F-1,D},\ X \notin \Omega_{F,D+1}\}. \tag{1.73}$$

With this definition, the sharp complexity is defined to be a finite list of 'minimal' format and degree pairs. This is no flaw. On the contrary, it is a feature which captures the fact that the complexity of an object is not purely intrinsic to the object, but depends on how it manifests itself. Observe in conclusion how sharp complexity can enhance Pfaffian complexity: instead of describing the complexity of a restricted Pfaffian function f by a tuple of four integers, we avail ourselves of the fact that $\mathbb{R}_{\text{rPfaff}}$ is sharply o-minimal and assign to it the two-integer sharp complexity of its graph: $\#\mathcal{C}(f) = \#\mathcal{C}(\Gamma(f))$.

1.8 The Tameness Conjecture

In this final section, wish to motivate our interest in o-minimal structures by reviewing the recently advanced conjecture, named *Tameness Conjecture*, which relates o-minimality with fundamental physics.
As it is well known, String Theory is one of the most prominent candidates as a theory of quantum gravity. Disregarding yet-unsolved technical difficulties, which prevent us from claiming that a full-fledged theory of quantum gravity has been attained, String Theory gives rise to the problem of decreeing which effective, low-energy theories are consistent with it and which are not. There are several effective field theories (EFTs) which can result from String Theory: because of the large number of Calabi-Yau manifolds in which the 6 unobserved dimensions may be compactified; because of the different minima of potentials in which highly massive fields take their expectation value at low energies; finally, because there is more than one consistent string theory to begin with. As they originate from a string theory, all such effective theories are said to *admit an UV completion to quantum gravity* or, for the sake of brevity, to constitute the *Landscape*; the Standard Model is then expected to be the needle to be sought in this (potentially infinitely large) haystack. Conversely, theories which do not admit an UV completion to quantum gravity are said to belong to the *Swampland*. The so-called Swampland Program consists then in telling apart effective field theories between those belonging to the Swampland and the physically-meaningful ones, residing in the Landscape. Much endeavour has been dedicated to establish guiding principles for this search, which it is beyond our purposes to review here; we will only mention that one of such guiding principles is the conjecture that the number of EFTs, valid below a certain cut-off energy scale, which admit an UV completion, is finite[9]. The Tameness Conjecture, first formulated in [Gri22] and then elaborated further in [DGS23], enhances this statement by promoting the notion of finiteness to that of tameness, i.e. definability in an o-minimal structure. In order to state it, following mostly [DGS23], we set out to illustrate how a set of quantum field theories can give rise to two structures, named $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$ and $\mathbb{R}_{\mathcal{T},\mathcal{S}}$.

A set of quantum field theories (QFTs) can be parametrised by two sets $\mathcal{T}$ and $\mathcal{S}$. A parameter $\lambda \in \mathcal{T}$ is defined to be a tuple of coupling constants, masses and expectation values of integrated-out fields. Since some of these parameters may be discrete, $\mathcal{T}$ will not, in general, be a manifold, but we will simply view it as a subset of the Euclidean space $\mathbb{R}^k$: more precisely, it can be thought as a moduli space, whose points carry additional information.
Moreover, for each QFT, the fields will be defined on a spacetime Σ with a metric g. We then define $(\Sigma, g)_\rho$ to be a family of spacetimes manifolds Σ endowed with the metric g, labelled by a parameter ρ taking values on a set $\mathcal{S}$. Thus, every point $(\lambda, \rho) \in \mathcal{T} \times \mathcal{S}$ defines a QFT, namely a different low-energy effective theory. The structure $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$ is then defined to be the structure whose associated language allows us to make logical statements about the set of QFTs parametrised by $\mathcal{T}$ and $\mathcal{S}$; in order to make such statements, we will need to require the Lagrangians $\mathcal{L}(\phi, \lambda)$ to be definable in $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$. Observe that Lagrangians can be tame only as functions of the fields ϕ, and not of the spacetime coordinates: since the path integral will sum over all field configurations, including non-tame ones, it will be impossible to ensure the tameness of the Lagrangian, at fixed couplings, for every field configuration.
To provide an example, suppose that the potential $V(\phi, \lambda)$ is an algebraic function of a scalar field ϕ and the coupling constants $\lambda_i \in \mathcal{T} \subseteq \mathbb{R}^k$: it is then certainly definable in the structure $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$ and the set of the minima of the potential

$$m(\lambda) := \{\phi \in \mathbb{R} : \partial^2_\phi V(\phi) > 0 \wedge \partial_\phi V(\phi) = 0\} \tag{1.74}$$

will be definable in $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$ too. Thus, a logic formula such as $\exists \phi \in \mathbb{R} : \ \phi \in m(\lambda) \wedge \lambda > 1$ is well-defined in the language associated with the structure $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$ and they hold true on a definable set of the same structure.

The structure $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$ allows us to make statements about the QFTs, but not about the physical observables, which is what a theory is ultimately concerned about. An observable of an Euclidean quantum field theory over a d-dimesnional spacetime Σ is a correlation function of local operators $\mathcal{O}_i(y_k)$

$$\langle \mathcal{O}_1(y_1)...\mathcal{O}_n(y_n)\rangle := \frac{\int \mathcal{D}\phi \mathcal{O}_1(y_1)...\mathcal{O}_n(y_n) e^{-\int_\Sigma d^d\phi \mathcal{L}(\phi,\lambda)}}{\int \mathcal{D}\phi e^{-\int_\Sigma d^d\phi \mathcal{L}(\phi,\lambda)}} \tag{1.75}$$

[9] More precisely, the space of such EFTs has only finitely many connected components.

where by ϕ we indicate collectively all the fields ϕ_i. The latter is clearly a function

$$O(y_1, ..., y_n; \lambda, \rho) : \Sigma_\rho \times \mathcal{T} \times \mathcal{S} \to \mathbb{R}. \tag{1.76}$$

We can therefore create a new structure $\mathbb{R}_{\mathcal{T},\mathcal{S}}$, the *quantum structure* of the spaces $\mathcal{T}$ and $\mathcal{S}$, by adding all such observables $O(y_1, ..., y_n; \lambda, \rho)$ to the generating family of functions $\mathcal{F}$ of $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$. In this structure, all the first-order logic formulas which predicate the observables are definable.

The Tameness Conjecture can now be stated. We will call $\mathcal{T}_{EFTd}$ the set of effective field theories in d dimensions that can consistently be coupled to quantum gravity.

> **Conjecture 1.1** (Douglas, Grimm, Schlechter)**.** The set $\mathcal{T}_{EFTd}$ of effective field theories that are valid at least below a cut-off energy scale Λ and can be consistently coupled to quantum gravity is tame, i.e definable in some o-minimal structure $\mathbb{R}^{\text{def}}_{EFTd}$. Moreover, the structure $\mathbb{R}_{EFTd}$ built from $\mathbb{R}^{\text{def}}_{EFTd}$ by adding all the correlation functions of each theory in $\mathcal{T}_{EFTd}$ on a tame spacetime manifolds[10] (Σ, g) is o-minimal.

Clearly, this conjecture provides a well-defined guiding principle for the Swampland program. Concretely, the Tameness Conjecture also implies that correlation functions, due to the cell decomposition theorem, 1.2, can only have a finite number of discontinuities. This amounts to the quite reasonable conclusion that UV-completable theories can only undergo a finite number of phase transitions.
But there is more: o-minimal structures are in fact instances of first-order logic formal systems which, containing no notion of arithmetic, are unaffected by Tarski's, Church's and Gödel's theorems (see Appendix A and e.g. [Raa22]). It is very suggestive that the Tameness Conjecture implies a deep link between fundamental physics and logic, and fascinating to speculate that physical observables can be predicated in formal systems simple enough to admit completeness and decidability. A profitable connection between logic and fundamental physics already deserved the attention of the author of [Heu16]: the Tameness Conjecture seems to favour the second of the four options listed therein as to the representation of the fundamental laws of physics in logic formal systems.

A few remarks are due. First, the claim that the addition of the correlation functions preserves tameness is highly non-trivial, since the integrals of tame functions, such as the path integrals (1.75), are not known to be tame in general. Secondly, in the above formulation, the tameness conjecture does not indicate which out of the known o-minimal structures are the correct ones: as already mentioned, there exists no largest o-minimal structure which encloses all the others, so the question is not easily dismissed. In [Gri22], the o-minimal structure $\mathbb{R}^{\text{def}}_{\mathcal{T},\mathcal{S}}$ was conjectured to be $\mathbb{R}_{\text{an,exp}}$, but the addition of the correlation functions is likely to force us to enlarge the structure. Thirdly, it is worthwhile to remind ourselves that correlation functions such as (1.75) are almost always impossible to compute exactly: they are usually computed perturbatively by means of power series in the coupling. Such perturbative expansions, however, have long since been known to be divergent. This feature, as it will be explained in the next chapters, entails that correlation functions are not analytic in the weak-coupling limit. Therefore, in order to satisfy the Tameness Conjecture, we are led to turn our attention to the o-minimal structures whose generating family $\mathcal{F}$ hosts non-analytic functions (at least at one point of the domain) listed in section 1.5.3. Before doing so though, it will be expedient to present the mathematical technique whereby divergent series can be manipulated and *resummed* into a function, which will be ultimately be, to fulfil our purposes, a partition function or a correlation function. Such technique is known as Borel resummation and will be expounded in the next chapter. Having reviewed Borel summability, we will finally show how the o-minimal structures $\mathbb{R}_{\mathscr{G}}$, and $\mathbb{R}_{\mathscr{G},\text{exp}}$ naturally host non-analytic partition and correlation functions, viewed as Borel sums of their divergent asymptotic expansions.

[10] A manifold M is tame if it is endowed with a tame atlas, i.e. an atlas $\mathscr{A} = \{\chi_i : U_i \subseteq M \to V_i \subseteq \mathbb{R}^n\}$ where the V_i are tame sets and the transition functions $\chi_i \circ \chi_j^{-1} : V_j \to V_i$ are definable maps.

Chapter 2

Borel Summability and Resurgence

The purpose of this chapter is to introduce the essential mathematical methods required to deal with divergent power series. As the name suggests, a power series

$$\tilde{\varphi} := a_0 + a_1 x + a_2 x^2 + ...$$

is divergent if its radius of convergence is 0. Unlike convergent series, divergent series cannot be summed for any value of their variable x except for $x = 0$ (or $z = \infty$, for series of powers of z^{-1}).
The radius of convergence of a power series is vanishing if the coefficients a_n grow too fast. In particular, they have to grow faster than x^{-n} for any x, however small: in this way, the limit for large n of $a_n x^n$ will not be 0 and Leibniz criterion for convergence will not be satisfied. The simplest way in which this can happen is if a_n grows factorially like $(n!)^\kappa$ for large n and a positive κ. As it will be illustrated in the next chapters, this is a primary feature of the power series that result from perturbation theory in QFT and in quantum mechanics. Thus, if we want to attach any physical meaning to those power series, we must endeavour to understand how a divergent series can be 'resummed' into a well-defined function. Much in the same way as convergent series are naturally summed, and yield a finite sum for x small enough, we seek a method whereby we can attach a finite 'sum' to a divergent series.

The earliest (and remarkably brilliant) attempts to formulate resummation methods date back to Euler and his seminal *De seriebus divergentibus* [EA18]. Several resummation methods were devised afterwards, including the Borel-Laplace method wherewith we will concern ourselves throughout this thesis. A seminal book on the subject is the very last work of Hardy [Har56], which gathers many relevant results.
Powerful though it is, the Borel-Laplace resummation method is not always applicable in its most elementary form, as it will be expounded in the first two sections of this chapter. A new whole field then opens: that of *alien calculus* and *resurgence*. The thorough study of these methods has only been fulfilled far more recently, mainly thanks to the works by Jean Écalle [Eca81]. Waiving any attempt to give justice to the vastness of this theory, we nonetheless summarise its most salient features in section 2.3. Having done so, we will treat the important examples of the Euler series (section 2.4) and the Stirling series (section 2.5) to show Borel summability and resurgence in action and set the stage for their applications to perturbative power series. Finally, in section 2.6, we will throw a bridge between the theory of o-minimality and Borel summability, showing how divergent power series, once resummed, yield a tame function definable in the o-minimal structure $\mathbb{R}_{\mathscr{G}}$. We will then be fully equipped to present our results in the next chapter.

Throughout this chapter, we mostly follow [Dor19] and, for a more mathematical cut, [Sau14; Bal94; Lod14].

2.1 Divergent series and the Borel transform

In this section we introduce the first tool needed for the resummation of divergent power series: the Borel transform. This operator succeeds in turning a power series into another of increased radius of convergence. If certain conditions are satisfied (established in Proposition 2.1), series with a vanishing radius of convergence have a Borel transform of *finite* radius of convergence, while those with a finite radius of convergence have a Borel transform of *infinite* radius of convergence. We discuss how the formal Borel transform induces a differential algebra isomorphism: the ordinary multiplication between power series is mapped into a convolution product, while the ordinary differentiation is mapped into a multiplication in the usual sense.

2.1.1 Formal power series

For simplicity, it will be convenient in the following to fix a large complex variable z, which we will later interpret, in physical terms, as the inverse of a small coupling constant λ. Consider a **formal power series** without constant term

$$\tilde{\varphi}_1 = \sum_{n=0}^{\infty} a_n \frac{1}{z^{n+1}}. \tag{2.1}$$

Allowing the coefficient a_n to be, in general, complex numbers, we can then say that $\tilde{\varphi}_1 \in z^{-1}\mathbb{C}[[z^{-1}]]$. It should be observed that $z^{-1}\mathbb{C}[[z^{-1}]]$ is a differential algebra. What we mean by this is that, given a second series

$$\tilde{\varphi}_2 = \sum_{n=0}^{\infty} b_n \frac{1}{z^{n+1}}, \tag{2.2}$$

we can take a term-by-term product, known as **Cauchy product**, between the two and find a new power series which belongs to the same family:

$$\tilde{\varphi}_2 \cdot \tilde{\varphi}_1 = \left(\sum_{n=0}^{\infty} a_n \frac{1}{z^{n+1}}\right)\left(\sum_{m=0}^{\infty} b_m \frac{1}{z^{m+1}}\right) = \sum_{n=0}^{\infty} c_n \frac{1}{z^{n+1}} = \tilde{\psi} \in z^{-1}\mathbb{C}[[z^{-1}]], \tag{2.3}$$

where of course

$$c_n = \sum_{p=0}^{n-1} a_p b_{n-p-1}. \tag{2.4}$$

Secondly, given $\partial = \frac{d}{dz}$, the differentiation term-by term $\partial\tilde{\varphi}_1$ of $\tilde{\varphi}_1$ as above still belongs to the same family:

$$\partial\tilde{\varphi}_1 = -\sum_{n=0}^{\infty}(n+1)a_n \frac{1}{z^{n+2}} \in z^{-1}\mathbb{C}[[z^{-1}]], \tag{2.5}$$

whence the claim that $z^{-1}\mathbb{C}[[z^{-1}]]$ is a differential algebra.

A series such as (2.1) must be regarded as a purely algebraic object. It is an element of a differential algebra: as such the composition thereof with another element of $z^{-1}\mathbb{C}[[z^{-1}]]$ by Cauchy product will again yield an element of the same algebra; furthermore, its differentiation term-by-term is still a member of the algebra $z^{-1}\mathbb{C}[[z^{-1}]]$. This is why a series (2.1) is also referred to as a *formal* power series: this is to stress our willingness to remain agnostic as to the possibility to interpret it as a function of z, since no assumption on the radius of convergence has been made.

2.1.2 Gevrey class

Formal power series can be classified according to the behaviour of their coefficients. In particular, we will be interested in whether the coefficients a_n of (2.1) are bounded by some function of n. We can then introduce the following:

Definition 2.1 (Gevrey class)**.** A formal power series

$$\tilde{\varphi} = \sum_{n=0}^{\infty} a_n \frac{1}{z^{n+1}} \tag{2.6}$$

is said to be is said to be of **Gevrey class** $1/\kappa$ if there exist two constants $A, B \in \mathbb{R}$ such that, for every n:

$$|a_n| \leq AB^n (n!)^\kappa. \tag{2.7}$$

It is worth remarking that in the literature different conventions can be found: for instance, [Bal94] trades κ for $1/\kappa$. Moreover, other definitions, such like

$$\begin{aligned} |a_n| &\leq AB^n (n^n)^\kappa, \\ |a_n| &\leq AB^n \Gamma(1+\kappa n) \end{aligned} \tag{2.8}$$

can be easily shown to be equivalent to (2.7) after recalling the Stirling approximation for large n

$$n! \sim \sqrt{\frac{n}{2\pi}} n^n e^{-n} = \frac{1}{\sqrt{2\pi}} e^{n \log n - n + \frac{1}{2}\log n} \tag{2.9}$$

and making some suitable redefinitions of A and B.

The notion of Gevrey class enables us to produce quantitative statements as to the convergence of a formal power series $\tilde{\varphi}$: convergence (more precisely, absolute convergence) is guaranteed only if $\kappa = 0$, namely if the growth of the coefficients is only exponential and not factorial. We shall write that a formal power series without constant term $\tilde{\varphi}$ of Gevrey class $1/\kappa$ belongs to the algebra $z^{-1}\mathbb{C}[[z^{-1}]]_{1/\kappa}$ if $\kappa \neq 0$. If, on the other hand, $\kappa = 0$ the formal power series has a finite radius of convergence and we shall write $\tilde{\varphi} \in z^{-1}\mathbb{C}\{z^{-1}\}$.

2.1.3 The formal Borel transform

The idea underlying the formal Borel transform is that of introducing an operator on the algebra $z^{-1}\mathbb{C}[[z^{-1}]]$ which increases Gevrey class of a formal series or, equivalently, reduces the number κ which determines the exponential growth of its coefficients a_n. In the following, we will mainly focus on formal power series of Gevrey class $\kappa = 1$ and defer the generalisation to other Gevrey classes for later sections.

Definition 2.2 (Formal Borel transform)**.** The **formal Borel transform** is a linear map $\mathcal{B} : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[\zeta]]$ defined by

$$z^{-1}\mathbb{C}[[z^{-1}]] \ni \tilde{\varphi} := \sum_{n=0}^{\infty} a_n \frac{1}{z^{n+1}} \quad \xrightarrow{\mathcal{B}} \quad \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n =: \hat{\varphi} \in \mathbb{C}[[\zeta]]. \tag{2.10}$$

Thus, we shall write $\hat{\varphi} = \mathcal{B}\tilde{\varphi}$.

The idea is simple: by dividing every coefficient by $n!$, the formal Borel transform removes the factorial divergence of the coefficients. Note though that the coefficients of the new power series $\hat{\varphi}$ will still grow factorially if $\tilde{\varphi}$ belongs to a Gevrey class $1/\kappa < 1$. Indeed, we have the following:

> **Proposition 2.1.** Let $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]$. Then $\tilde{\varphi} \in z^{-1}\mathbb{C}\{z^{-1}\}$ if and only if the formal Borel transform $\hat{\varphi} = \mathcal{B}\tilde{\varphi}$ has infinite radius of convergence and it defines an entire function $\hat{\varphi}(\zeta)$ of bounded exponential growth: namely, there exist $A, c \in \mathbb{R}$ such that
>
> $$|\hat{\varphi}(\zeta)| \leq Ae^{c|\zeta|} \qquad \text{for all } \zeta \in \mathbb{C}. \tag{2.11}$$
>
> Moreover, $\hat{\varphi} \in \mathbb{C}\{\zeta\}$ (i.e., $\hat{\varphi}$ has a finite radius of convergence) if and only if $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$. In this

case, $\hat{\varphi}$ defines a holomorphic function $\hat{\varphi}(\zeta)$ in a neighbourhood of the origin.

The change of variable from z to ζ in passing from a formal power series to its Borel transform is of course purely conventional, although quite standard. We will often refer to the the complex plane in which the variable ζ 'lives' as the **Borel plane**.

The formal Borel transform[1] satisfies certain properties which it will be useful to list here. First, let us define the translation operator T_c:

$$\begin{aligned} T_c : z^{-1}\mathbb{C}[[z^{-1}]] &\to z^{-1}\mathbb{C}[[z^{-1}]] \\ \tilde{\varphi}(z) &\to \tilde{\varphi}(z+c). \end{aligned} \tag{2.12}$$

Observe how $T_c\tilde{\varphi}$ belongs to $z^{-1}\mathbb{C}[[z^{-1}]]$ because one has

$$\frac{1}{z+c} = \frac{1}{z}\left(\frac{1}{1+\frac{c}{z}}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{c}{z}\right)^n. \tag{2.13}$$

Thus, T_c is a differential algebra automorphism: in fact, it is an operator which maps (invertibly) the algebra into itself and commutes with the differentiation ∂. We can then state the following:

Lemma 2.1. Let $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]$ and $\hat{\varphi} = \mathcal{B}\tilde{\varphi} \in \mathbb{C}[[\zeta]]$. We then have

1. $\mathcal{B}[\partial\tilde{\varphi}] = -\zeta\hat{\varphi}(\zeta)$;
2. $\mathcal{B}[T_c\tilde{\varphi}](\zeta) = e^{-c\zeta}\hat{\varphi}(\zeta)$;
3. If $\tilde{\varphi} \in z^{-2}\mathbb{C}[[z^{-1}]] \subset z^{-1}\mathbb{C}[[z^{-1}]]$, then $\mathcal{B}[z\tilde{\varphi}] = \frac{d\hat{\varphi}}{d\zeta}$;
4. $\mathcal{B}[z^{-1}\hat{\varphi}] = \int_0^{\zeta} d\zeta'\hat{\varphi}(\zeta')$.

where in the last line the integration is meant term-wise.

Proof. Let $\tilde{\varphi} = \sum_{n=0}^{\infty} a_n \frac{1}{z^{n+1}}$.

1. Differentiating term-by-term, we have that $\partial\tilde{\varphi} = -\sum_{n=0}^{\infty} a_n \frac{n+1}{z^{n+2}}$. Therefore,

$$\mathcal{B}[\partial\tilde{\varphi}] = -\sum_{n=0}^{\infty}\frac{a_n(n+1)}{(n+1)!}\zeta^{n+1} = -\zeta\sum_{n=0}^{\infty}\frac{a_n}{n!}\zeta^n = -\zeta\hat{\varphi} \tag{2.14}$$

   as claimed.

2. It is straightforward to generalise the above result to:

$$\mathcal{B}[\partial^k\tilde{\varphi}] = (-\zeta)^k\hat{\varphi}, \tag{2.15}$$

   whence we can easily derive the desired property by writing the action of the translation operator as $T_c\tilde{\varphi} = e^{c\partial}\tilde{\varphi}$. Then one has:

$$\begin{aligned} \mathcal{B}[T_c\tilde{\varphi}] &= \sum_{k=0}^{\infty}\frac{c^k}{k!}\mathcal{B}[\partial^k\tilde{\varphi}] = \sum_{k=0}^{\infty}\frac{c^k}{k!}(-\zeta)^k\hat{\varphi} \\ &= e^{-c\zeta}\hat{\varphi}. \end{aligned} \tag{2.16}$$

[1] The reason why it is called *formal* is that there also exists an integral representation of the Borel transform, akin to that of the Laplace transform to be introduced in the next section, which acts as the formal Borel transform on polynomials (see e.g. [Bal94]). However, we will not be concerned about this operator and, for the sake of brevity, in later sections we will sometimes refer to the formal Borel transform simply as Borel transform.

3. In this case, we can write $\tilde{\varphi} = \sum_{n=0}^{\infty} a_n z^{-n-2}$. Then one has

$$\hat{\varphi} = \mathcal{B}[\tilde{\varphi}] = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)!}\zeta^{n+1}, \tag{2.17}$$

while

$$\mathcal{B}[z\tilde{\varphi}] = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^{n}, \tag{2.18}$$

which is clearly the term-by-term derivative of (2.17) with respect to ζ.

4. In a very similar fashion, we have

$$\mathcal{B}[z^{-1}\tilde{\varphi}] = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)!}\zeta^{n+1}, \tag{2.19}$$

which is the term-wise integration of $\hat{\varphi}(\zeta') = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta'$, from 0 to ζ.

□

The algebra $z^{-1}\mathbb{C}[[z^{-1}]]$ is closed under the action of both the operators ∂ and T_c. Because both $-\zeta\hat{\varphi}$ and $e^{-c\zeta}\hat{\varphi}$ belong to the same algebra $\mathbb{C}[[\zeta]]$ which $\hat{\varphi}$ belongs to, the above lemma ensures that the formal Borel transform preserves these closure properties on the target algebra. The formal Borel transforms $\hat{\varphi}_1$ and $\hat{\varphi}_2$ of $\tilde{\varphi}_1, \tilde{\varphi}_2 \in z^{-1}\mathbb{C}[[z^{-1}]]$ respectively are again formal power series, and one could still take Cauchy products among them as in (2.3); nevertheless, the formal Borel transform does not map Cauchy products into Cauchy products. To see why, let us consider again $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ as in (2.1) and in (2.2). Their formal Borel transforms are

$$\begin{aligned} \hat{\varphi}_1 &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^{n}, \\ \hat{\varphi}_2 &= \sum_{n=0}^{\infty} \frac{b_n}{n!}\zeta^{n}, \end{aligned} \tag{2.20}$$

but the formal Borel transform of the Cauchy product of $\hat{\varphi}_1$ and $\hat{\varphi}_2$, namely $\mathcal{B}[\tilde{\varphi}_1 \cdot \tilde{\varphi}_2]$, is not $\hat{\varphi}_1 \cdot \hat{\varphi}_2$. Conversely, one has

$$\mathcal{B}[\tilde{\varphi}_1 \cdot \tilde{\varphi}_2] = \sum_{n=0}^{\infty} \frac{c_n}{n!}\zeta^{n} =: \hat{\varphi}_1 * \hat{\varphi}_2, \tag{2.21}$$

where the coefficients c_n are given by (2.4). This product is known as the **convolution product**, and it is with respect to this operation that we make $\mathbb{C}[[\zeta]]$ into an algebra.

We can then define the formal Borel transform of a constant as

$$\mathcal{B}1 = \delta. \tag{2.22}$$

Because 1 is the unity of the algebra $\mathbb{C}[[z^{-1}]]$, δ must be the unity of the convolutive algebra $\mathbb{C}[[\zeta]]$, namely

$$(\delta * \hat{\varphi}) = \hat{\varphi}. \tag{2.23}$$

Thus, we can finally extend the formal Borel transform to the whole differential algebra $\mathbb{C}[[z^{-1}]]$, establishing a linear isomorphism between $\mathbb{C}[[z^{-1}]]$ and $\mathbb{C} \oplus \mathbb{C}[[\zeta]]$. For instance, if we let $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]$, we will have

$$\mathcal{B}(a + \tilde{\varphi}) = a\delta + \mathcal{B}\tilde{\varphi} = a\delta + \hat{\varphi} \in \mathbb{C} \oplus \mathbb{C}[[\zeta]]. \tag{2.24}$$

Furthermore, we can define the derivation

$$\begin{aligned} \hat{\partial} : \mathbb{C}\delta \oplus \mathbb{C}[[\zeta]] &\to \mathbb{C}[[\zeta]] \\ \hat{\partial}(a\delta + \hat{\varphi}) &\to -\zeta\hat{\varphi} \end{aligned} \tag{2.25}$$

so that the formal Borel transform induces the differential algebra isomorphisms:

$$\begin{aligned} \mathcal{B} : (\mathbb{C}[[z^{-1}]], \partial) &\xrightarrow{\sim} (\mathbb{C}\delta \oplus \mathbb{C}[[\zeta]], \hat{\partial}) \\ \mathcal{B} : (\mathbb{C}[[z^{-1}]]_1, \partial) &\xrightarrow{\sim} (\mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}, \hat{\partial}) \end{aligned} \tag{2.26}$$

where in the second line we have restricted the action of the formal Borel transform to the subalgebra of 1-Gevrey formal power series. This case deserves some additional attention as, according to Proposition 2.1 the formal Borel transform of a 1-Gevrey power series has a finite radius of convergence; moreover, the Cauchy product between the Borel transforms takes an explicit form, which is described as follows:

Theorem 2.1. Consider two 1-Gevrey formal power series $\tilde{\varphi}, \tilde{\psi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$ and their Borel transforms $\hat{\varphi}, \hat{\psi} \in \mathbb{C}\{\zeta\}$. Let $R > 0$ be smaller than the radius of convergence of each of them, so that $\hat{\varphi}(\zeta)$ and $\hat{\psi}(\zeta)$ are the corresponding holomorphic functions defined on the disc $D(0, R) := \{\zeta \in \mathbb{C} : |\zeta| < R\}$. Then the convolution product $\hat{\varphi} * \hat{\psi}$ defines the function

$$(\hat{\varphi} * \hat{\psi})(\zeta) = \int_0^\zeta \hat{\varphi}(\zeta')\hat{\psi}(\zeta - \zeta')d\zeta' \tag{2.27}$$

which is holomorphic on $D(0, R)$. This the function defined by the Borel transform of the Cauchy product $\tilde{\varphi} \cdot \tilde{\psi}$.

This theorem allows us to compute convolution products in a straightforward manner. Moreover, we can use it to show that the operator $\hat{\partial}$ introduced earlier is indeed a derivation.

Proposition 2.2. The operator $\hat{\partial} : \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\} \to \mathbb{C}\{\zeta\}$ is a derivation with respect to the convolution product $*$.

Proof. Let us consider $\hat{\varphi}, \hat{\psi} \in \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$, and let us write explicitly $\hat{\varphi}(\zeta) = a\delta + \hat{f}(\zeta)$ and $\hat{\psi}(\zeta) = b\delta + \hat{g}(\zeta)$ where $\hat{f}(\zeta)$ and $\hat{g}(\zeta)$ are holomorphic on a disc $D(0, R)$ as above. We have to prove that the Leibniz rule holds: $\hat{\partial}(\hat{\varphi} * \hat{\psi}) = (\hat{\partial}\hat{\varphi}) * \hat{\psi} + \hat{\varphi} * (\hat{\partial}\hat{\psi})$. We have that the left-hand side is

$$\hat{\partial}(\hat{\varphi} * \hat{\psi}) = -\hat{\partial}(ab\delta + a\hat{g} + b\hat{f} + (\hat{f} * \hat{g})) = -\zeta(a\hat{g} + b\hat{f} + (\hat{f} * \hat{g})), \tag{2.28}$$

while the right-hand side reads

$$\begin{aligned} (\hat{\partial}\hat{\varphi}) * \hat{\psi} + \hat{\varphi} * (\hat{\partial}\hat{\psi}) &= -(\zeta\hat{f}) * (b\delta + \hat{g}) - (a\delta + \hat{f}) * (\zeta\hat{g}) \\ &= -\zeta(b\hat{f} + a\hat{g}) - (\zeta\hat{f}) * \hat{g} - \hat{f} * (\zeta\hat{g}). \end{aligned} \tag{2.29}$$

Therefore, we have to show that $\zeta(\hat{g} * \hat{f}) = (\zeta\hat{f}) * \hat{g} + \hat{f} * (\zeta\hat{g})$, namely that

$$\zeta \int_0^\zeta dx \hat{f}(x)\hat{g}(\zeta - x) = \int_0^\zeta dx\, x\hat{f}(x)\hat{g}(\zeta - x) + \int_0^\zeta dx \hat{f}(x)(\zeta - x)\hat{g}(\zeta - x). \tag{2.30}$$

which clearly holds for every $\zeta \in D(0, R)$. □

2.2 The Borel resummation procedure

In the previous section the formal Borel transform $\mathcal{B}$ has been introduced. As Proposition 2.1 indicates, the idea underlying the definition of the operator $\mathcal{B}$ is to turn a power series $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$, i.e. having a (possibly) vanishing radius of convergence, into a power series $\hat{\varphi} \in \mathbb{C}\{\zeta\}$, which converges to a function $\hat{\varphi}(\zeta)$, holomorphic in a neighbourhood of the origin. Our concern will be now to 'invert'this operation in order to find not so much the initial formal power series $\tilde{\varphi}$, as, on the contrary, a honest holomorphic function $\varphi(z)$. The latter, known as the Borel sum, is closely related to $\tilde{\varphi}$: it admits $\tilde{\varphi}$ as asymptotic expansion. Having described this procedure – known as Borel resummation – for $\kappa = 1$, we generalise it to any κ. We state Nevanlinna-Sokal theorem, which provides a necessary and sufficient condition for Borel summability. Finally, we briefly mention how Borel summability can be understood as a generalisation of analyticity.

2.2.1 The Laplace transform

In the previous section, it has been shown how the Borel operator $\mathcal{B}$ improves the convergence of a formal power series $\tilde{\varphi}$. Yet, the simple sum of the Borel transform $\hat{\varphi}$ is too naïve of an object to represent the sum of $\tilde{\varphi}$, as it would be indistinguishable from a convergent power series which sums to an analytic function. We must then introduce a new operator which 'inverts' the Borel transform by producing a holomorphic function which is not analytic (at infinity). This operator is the Laplace transform.

Consider a continuous function $\hat{\varphi}(\zeta) : \mathbb{R}^+ \cup \{0\} \to \mathbb{C}$ such that, for large enough ζ,

$$|\hat{\varphi}(\zeta)| \leq Ae^{c_0\zeta} \tag{2.31}$$

for some positive constant A. The Laplace transform of $\hat{\varphi}(\zeta)$ is then given by:

Definition 2.3 (Laplace transform)**.** Given a continuous function $\hat{\varphi}(\zeta) : \mathbb{R}^+ \cup \{0\} \to \mathbb{C}$, exponentially bounded (i.e., satisfying (2.31) for all ζ large enough), the **Laplace transform** of $\hat{\varphi}(\zeta)$ is defined as:

$$(\mathcal{L}^0\hat{\varphi})(z) = \int_0^\infty e^{-z\zeta}\hat{\varphi}(\zeta)d\zeta. \tag{2.32}$$

Thanks to the requirement that $\hat{\varphi}(\zeta)$ be exponentially bounded, the Laplace transform $(\mathcal{L}^0\hat{\varphi})(z)$ defines a holomorphic function on the region:

$$\Pi_{c_0} = \{z \in \mathbb{C} : \mathrm{Re}(z) > c_0\} \tag{2.33}$$

which is of course the region where the integrand of (2.32) is exponentially decaying.

As our notation suggests, we want to interpret $\hat{\varphi}(\zeta)$ as the Borel transform $\mathcal{B}\tilde{\varphi}$ of some formal power series $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$. Nevertheless, as indicated by Proposition (2.1), form Borel transforms in general have but a finite radius of convergence. To take care of this, we first introduce the following:

Definition 2.4. Given $c_0 \in \mathbb{R}$ and a half-strip $S_\delta = \{\zeta \in \mathbb{C} : \mathrm{dist}(\zeta, \mathbb{R}^+) < \delta\}$, we define $\mathcal{N}_{c_0}$ to be the family of all convergent power series $\hat{\varphi} \in \mathbb{C}\{\zeta\}$, defining a holomorphic function $\hat{\varphi}(\zeta)$ near 0, which extend analytically to S_δ to an exponentially bounded holomorphic function: namely

$$|\hat{\varphi}(\zeta)| \leq Ae^{c_0|\zeta|} \tag{2.34}$$

for all large enough $\zeta \in S_\delta$.

By a slight abuse of notation, we will denote $\hat{\varphi}(\zeta)$ and its analytic continuation to S_δ with the same symbol.

A strategy now emerges. Let us start with a formal power series $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$, and let us compute its Borel transform $\hat{\varphi} = \mathcal{B}\tilde{\varphi}$. According to Proposition (2.1), $\hat{\varphi}$ will define a holomorphic function $\hat{\varphi}(\zeta)$ in a

neighbourhood of the origin. We can then wonder whether $\hat{\varphi}$ belongs to $\mathcal{N}_{c_0}$ for some real constant c_0. If that happens to be the case, we are then entitled to analytically continue $\hat{\varphi}(\zeta)$ to a strip S_δ and then take its Laplace transform $\varphi(z) = (\mathcal{L}^0\hat{\varphi})(z)$, holomorphic on Π_{c_0} as defined in (2.33). Such a method is dubbed a *resummation* procedure. We will sometimes refer to the complex plane where the variable z 'lives', and which the domain Π_{c_0} is a subset of, as the **Laplace plane**.

That the the function $\varphi(z)$ bears any relationship with the formal power series $\tilde{\varphi}$ is not obvious: we should then investigate whether such relationship exists and, possibly, formulate it precisely. Before addressing this question in general, let us focus our attention to a formal power series $\tilde{\chi} = \sum_{n=0}^{\infty} c_n z^{-n-1}$ having a finite radius of convergence. Since $\tilde{\chi}$ is convergent, it is a very sensible requirement that the resummation procedure return the function defined by the sum of $\tilde{\chi}$, at least in the neighbourhood of infinity where it converges. According to Proposition 2.1, the formal Borel transform $\mathcal{B}\tilde{\chi} = \hat{\chi}(\zeta)$ has an infinite radius convergence and is exponentially bounded: therefore, no analytic continuation is needed and we can compute the Laplace transform straightaway. We have that

$$\hat{\chi}(\zeta) = \sum_{n=0}^{\infty} \frac{c_n}{n!}\zeta^n, \tag{2.35}$$

$$\chi(z) = (\mathcal{L}^0\hat{\chi})(z) = \int_0^{\infty} d\zeta e^{-z\zeta} \sum_{n=0}^{\infty} \frac{c_n}{n!}\zeta^n. \tag{2.36}$$

Now, if $\hat{\chi}$ is convergent for any ζ, then there must exist two real A, B such that $|c_n| \leq AB^n$ for all n. It can then be easily argued that the functions $g_n(\zeta) = e^{-z\zeta}A\frac{B^n}{n!}$ satisfy all the conditions listed in the theorem of Appendix C, for a sufficiently large z. Therefore the sum can be exchanged with the integral to find

$$\chi(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} \int_0^{\infty} d\zeta e^{-z\zeta}\zeta^n = \sum_{n=0}^{\infty} c_n z^{-n-1} = \tilde{\chi}. \tag{2.37}$$

Hearteningly then, the composition of the Borel transform with the Laplace transform returns exactly the initial formal power series if the latter is convergent. This property is known as *regularity*: the resummation procedure resums convergent power series into themselves (see [Har56]). From the previous steps it is clear that this will happen whenever we are allowed to exchange the sum with the Laplace integral. Note though that, had we considered a 1-Gevrey formal power series $z^{-1}\mathbb{C}[[z^{-1}]]_1 \ni \tilde{\varphi} = \sum_{n=0}^{\infty} c_n z^{-n-1}$, following the steps above we would have been unable to apply Theorem C.1. Indeed our only estimate for the functions g_n described therein would be $g_n(\zeta) = e^{-z\zeta}AB^n$, which *do not* satisfy the conditions 2 and 3. This is why, eventually, the Laplace transform of $\hat{\varphi} = \mathcal{B}\tilde{\varphi}$ will yield a function $\varphi(z)$ holomorphic on a neighbourhood of infinity, whereas the initial $\tilde{\varphi}$ is divergent and cannot be summed for any finite z, however large.

Let us now move to the general case. We shall need the following definition:

Definition 2.5 (Uniform asymptotic expansion)**.** Given a function $\varphi : \mathcal{D} \to \mathbb{C}$, with $\mathcal{D} \subset \mathbb{C}^*$, and an asymptotic series $\tilde{\varphi} = \sum_{n=0}^{\infty} c_n z^{-n-1}$, we shall say that φ **admits** $\tilde{\varphi}$ **as uniform asymptotic expansion** if there exists a sequence of positive numbers $(C_N)_{N\in\mathbb{N}}$ such that:

$$\left|\varphi(z) - \sum_{n=0}^{N-1} c_n z^{-n-1}\right| = C_N|z|^{-N} \tag{2.38}$$

for all $z \in \mathcal{D}$ and all $N \in \mathbb{N}$. We shall then write $\varphi \sim \tilde{\varphi}$. Moreover, if $C_N = AB^N(N!)$, we will say that $\varphi(z)$ **admits** $\tilde{\varphi}$ **as uniform 1-Gevrey asymptotic expansion** and we will denote this state of affairs by $\varphi \sim_1 \tilde{\varphi}$.

Note how the more terms are added in the truncated power series *the worse* $\varphi(z)$ is approximated by the sum.

The relationship between a formal power series of Gevrey class 1 $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$ and $\varphi(z)$ obtained as indicated above is expressed by the following theorem.

Theorem 2.2. Let $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$ and let $\hat{\varphi} = \mathcal{B}\hat{\varphi}$ be its formal Borel transform. Assume that $\hat{\varphi} \in \mathcal{N}_{c_0}$ for some $c_0 > 0$. Then the Laplace transform $\varphi(z) = (\mathcal{L}^0\hat{\varphi})(z)$ is holomorphic on Π_{c_0} and, for any $c_1 > c_0$

$$\varphi(z) \sim_1 \tilde{\varphi} \qquad \text{uniformly on } \Pi_{c_1}. \tag{2.39}$$

Remark that the theorem holds only if the non-trivial condition $\hat{\varphi} \in \mathcal{N}_{c_0}$ is satisfied. We can finally understand how the initial information contained in $\tilde{\varphi}$ is transferred to $\varphi(z)$, surviving a long journey through the formal Borel transform and the Laplace transform thereof. Given $\tilde{\varphi} = \sum_{n=0}^{\infty} c_n z^{-n-1}$, then

$$c_n = \frac{1}{(n+1)!} \frac{\partial^{n+1}}{\partial (z^{-1})^{n+1}} \varphi(z) \Big|_{z=\infty}. \tag{2.40}$$

In other words, the c_n *are the coefficients of the Taylor series at infinity.* Intuitively, we can understand this behaviour by observing that the point $z = \infty$ is the only value of z for which the formal power series $\tilde{\varphi}$ is finite: therefore, it is a sensible demand that $\varphi(z)$ and its derivatives behave exactly as $\tilde{\varphi}$ at infinity.

The process of finding a holomorphic function $\varphi(z)$ starting from a 1-Gevrey formal power series $\tilde{\varphi}$ by $\varphi(z) = (\mathcal{L}^0 \circ \mathcal{B})\tilde{\varphi}$ is known as **Borel-Laplace fine summation** or, for shortness, **Borel resummation**. If $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$ is such that its Borel transform $\mathcal{B}\tilde{\varphi} = \hat{\varphi}$ belongs to $\mathcal{N}_{c_0}$ for some constant $c_0 \in \mathbb{R}^+$, we shall say that $\tilde{\varphi}$ is **1-summable** and that $\varphi(z)$ is its **Borel sum**. We also define, for brevity, the **Borel-Laplace operator**

$$\mathscr{S}^0 := \mathcal{L}^0 \circ \mathcal{B}, \tag{2.41}$$

so that we can write $\varphi(z) = (\mathscr{S}^0\tilde{\varphi})(z)$. Recall finally that in the Borel plane we have introduced the unity of convolution products δ, so that the Borel transform is well-defined also for formal power series $\tilde{\varphi} \in \mathbb{C}[[z^{-1}]]_1$. We can then make sense of the Laplace transform on the complete convolutive algebra $\mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$ by simply stating:

$$\mathcal{L}^0(a\delta + \hat{\varphi}) = a + \mathcal{L}^0\hat{\varphi}. \tag{2.42}$$

Clearly, $a\delta + \tilde{\varphi}$ is 1-summable if and only if $\tilde{\varphi}$ is 1-summable.

We conclude the section with a few remarks about algebraic properties of the Borel plane. Upon defining $\mathcal{N} = \cup_{c\in\mathbb{R}}\mathcal{N}_c$ we have the following ([Sau14], Theorem 7.7):

Theorem 2.3. The space $\mathcal{N}$ is a subspace of $\mathbb{C}\{\zeta\}$ which is stable under convolution: namely, given $\hat{\varphi}, \hat{\psi} \in \mathcal{N}$, one has

$$\hat{\varphi} * \hat{\psi} \in \mathcal{N}. \tag{2.43}$$

Moreover, if we further assume $\hat{\varphi}, \hat{\psi} \in \mathcal{N}_{c_0}$, then the convolution product is compatible with the Laplace operator, namely

$$\mathcal{L}^0(\hat{\varphi} * \hat{\psi}) = (\mathcal{L}^0\hat{\varphi})(\mathcal{L}^0\hat{\psi}) \tag{2.44}$$

for every z in Π_{c_1}, for every $c_1 > c_0$.

The important corollary to this theorem is that, given $\tilde{\varphi}_1, \tilde{\varphi}_2$ belonging to the subalgebra $\mathbb{C} \oplus \mathcal{B}^{-1}(\mathcal{N})$ of $\mathbb{C}[[z^{-1}]]$, then the Borel-Laplace operator is compatible with Cauchy products: namely

$$\mathscr{S}(\tilde{\varphi}_1 \cdot \tilde{\varphi}_2) = (\mathscr{S}\tilde{\varphi}_1)(\mathscr{S}\tilde{\varphi}_1). \tag{2.45}$$

Similarly, it can be argued that:

$$(\mathscr{S}^0\partial\tilde{\varphi}_1)(z) = \frac{d}{dz}(\mathscr{S}^0\tilde{\varphi}_1)(z); \tag{2.46}$$

$$(\mathscr{S}^0 T_c\tilde{\varphi}_1)(z) = (\mathscr{S}^0\tilde{\varphi}_1)(z + c). \tag{2.47}$$

Moreover, the algebra Borel-summable formal power series is also closed under non-linear operations. For future reference, we state here the following (Theorem 13.3 of [Sau14]):

Theorem 2.4. Let $\tilde{\varphi} = a + \tilde{\varphi}_0$ and $\tilde{\psi}$ be 1-summable, with $\tilde{\varphi}_0$ in $z^{-1}\mathbb{C}[[z^{-1}]]_1$. Moreover, let $H \in \mathbb{C}\{t\}$ be a convergent power series. Then the power series $\psi \circ (\mathrm{Id} + \tilde{\varphi})$ and $H \circ \tilde{\varphi}_0$ are 1-summable and:

$$\mathscr{S}^0(\psi \circ (\mathrm{Id} + \tilde{\varphi})) = (\mathscr{S}^0\psi) \circ \mathscr{S}^0(\mathrm{Id} + \tilde{\varphi}) \tag{2.48}$$

$$\mathscr{S}^0(H \circ \tilde{\varphi}_0) = H \circ \mathscr{S}^0\tilde{\varphi}_0 \tag{2.49}$$

where the identity Id should be intended as 'z'.

Remark that, as H is convergent, we have $(\mathscr{S}^\theta H)(t) = H(t)$ for any θ, due to the regularity of the Borel resummation, expressed in (2.37). We shall see in section 2.3 a generalisation of this theorem.

2.2.2 Varying the direction of summation

In the foregoing presentation we have fixed the path of the Laplace integration to be the real positive line $\mathbb{R}^+$. Nevertheless, we can generalise it to any given direction in the complex plane, which we denote by the angle θ. Thus, for a holomorphic function $\hat{\varphi}(\zeta)$ which can be analytically continued on the line $e^{i\theta}\mathbb{R}^+$ such that $|\hat{\varphi}(\zeta)| \leq Ae^{c|\zeta|}$ for $\zeta \in e^{i\theta}\mathbb{R}^+$, we can define a generalised Laplace transform

$$(\mathcal{L}^\theta\hat{\varphi})(z) = \int_0^\infty d\zeta e^{i\theta} e^{-z\zeta e^{i\theta}} \hat{\varphi}(\zeta e^{i\theta}) = \int_0^{i\theta\infty} d\zeta\ e^{-z\zeta}\hat{\varphi}(\zeta), \tag{2.50}$$

which is a holomorphic function on $\Pi^\theta_{c_0} = \{z \in \mathbb{C} : \mathrm{Re}(e^{i\theta}z) > c\}$.
Following the lines of Definition 2.4, we can define a half-strip along the direction θ as

$$S^\theta_\delta := \{\zeta \in \mathbb{C} : \mathrm{dist}(\zeta, e^{i\theta}\mathbb{R}^+) < \delta\} \tag{2.51}$$

where $e^{i\theta}\mathbb{R}^+ := \{\zeta \in \mathbb{C} : \arg(\zeta) = \theta\}$ is the ray of angle θ in the complex plane. Next, we can define the family of $\mathcal{N}^\theta_c$ as the family of convergent power series that can be analytically continued on a half-strip S^θ_δ with $|\hat{\varphi}(\zeta)| \leq Ae^{c|\zeta|}$ for every $\zeta \in S^\theta_\delta$. Finally, we also define

$$\mathcal{N}^\theta = \bigcup_{c\in\mathbb{R}} \mathcal{N}^\theta_c \tag{2.52}$$

so that we can finally state the generalisation of Theorem 2.2:

Theorem 2.5. Consider a formal power series $\tilde{\varphi}_0 \in \mathbb{C}[[z^{-1}]]_1$ and assume that its formal Borel transform $\hat{\varphi} = \mathcal{B}\tilde{\varphi}$ can be written as $\hat{\varphi} = a\delta + \hat{\varphi}_0$ with $\hat{\varphi}_0 \in \mathcal{N}^\theta$ for some direction θ. Then the function $\varphi(z) = (\mathcal{L}^\theta \circ \mathcal{B})\tilde{\varphi}$ is holomorphic on Π^θ_c for some $c \in \mathbb{R}$ and

$$\varphi(z) \sim_1 \tilde{\varphi} \qquad \text{uniformly for } z \in \Pi^\theta_c. \tag{2.53}$$

We shall say then that $\tilde{\varphi}$ **is 1-summable in the direction** θ. Furthermore, for brevity, we shall denote by

$$\mathscr{S}^\theta := \mathcal{L}^\theta \circ \mathcal{B} \tag{2.54}$$

the **Borel-Laplace summation operator in the direction** θ.

More often than not, a convergent power series $\hat{\varphi}(\zeta)$ will admit an exponentially bounded analytic continuation along a continuous range of directions. Given an open interval $I \subset \mathbb{R}$, we can consider the family $\mathcal{N}(I)$ of all convergent power series $\hat{\varphi}(\zeta)$ analytically continuable on a strip $S^\theta_{\delta(\theta)}$ for every direction $\theta \in I$ and such that there exist two functions $\alpha, \gamma : I \to \mathbb{R}$ such that

$$|\hat{\varphi}(\zeta)| \leq \alpha(\theta)e^{\gamma(\theta)|\zeta|} \tag{2.55}$$

for all $\theta \in I$ and for all $\zeta \in \cup_{\theta\in I} S^\theta_{\delta(\theta)}$; observe that it is necessary that $\delta(\theta)$ does not vanish for any $\theta \in I$. We shall say then that a formal power series $\tilde\varphi_0 \in \mathbb{C}[[z^{-1}]]_1$ such that its formal Borel transform $\hat\varphi = \mathcal{B}\tilde\varphi$ can be written as $\hat\varphi = a\delta + \hat\varphi_0$ with $\hat\varphi_0 \in \mathcal{N}(I)$ **is 1-summable in the directions of** I. The sum of $\tilde\varphi$, given by the Laplace integrals, will be then well-defined for all $\theta \in I$, since the analytic continuation of $\hat\varphi_0$ is *holomorphic* on the entire sector of directions $\theta \in I$, and therefore there must be no singularities at any ζ for $|\zeta| < \infty$ and $\arg(\zeta) \in I$. It is then sensible to introduce the Borel-Laplace operator in the directions of I

$$\mathscr{S}^I := \mathcal{L}^\theta \circ \mathcal{B} \qquad \text{for every } \theta \in I \tag{2.56}$$

which will return a holomorphic function $\varphi(z)$ defined on

$$\mathscr{D}(I) = \bigcup_{\theta\in I} \Pi^\theta_{\gamma(\theta)} \tag{2.57}$$

where $\gamma : I \to \mathbb{R}$ is the function describing the exponential growth of (the analytic continuation of) $\hat\varphi(\zeta)$, as in (2.55). Uniformly for every $z \in \mathscr{D}(I)$, we shall have again

$$(\mathscr{S}^I\tilde\varphi)(z) \sim_1 \tilde\varphi. \tag{2.58}$$

In conclusion, recall how at the end of the previous section, we explained how the Borel-Laplace operator is compatible with the linear operators T_c, ∂ and the Cauchy product $\cdot$ as a consequence of Theorem 2.3; moreover, Theorem 2.4 regulates the resummation of the composition of 1-summable formal series. All these properties remain valid for any direction θ and in fact for any arc of directions I.

2.2.3 Generalisation to κ-summability

Hitherto, our attention has been focused on 1-summable formal power series, namely those whose formal Borel transform, as defined by (2.2) produces a power series with a finite radius of convergence. Yet, it is evident that the Borel transform of a formal power series whose Gevrey class is smaller than 1 will be still divergent. Indeed if we let e.g. $\kappa = 2$ in (2.7), the formal Borel transform $\mathcal{B}\hat\varphi$ will be a power series $\hat\varphi = \sum_{n=0}^\infty b_n\zeta^n$ where

$$|b_n| = \left|\frac{a_n}{n!}\right| \le AB^n(n!) \tag{2.59}$$

so that now $\hat\varphi$ is only of Gevrey class 1, rather than being convergent. This can also be argued by Proposition 2.1. We can then define the **formal κ-Borel transform** $\mathcal{B}_\kappa$ as the operator $\mathcal{B}_\kappa : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[\zeta]]$ defined by

$$\tilde\varphi := \sum_{n=0}^\infty a_n \frac{1}{z^{n+1}} \quad \xrightarrow{\mathcal{B}_\kappa} \quad \sum_{n=0}^\infty \frac{a_n}{\Gamma(\kappa n+1)}\zeta^n =: \hat\varphi \tag{2.60}$$

where $\Gamma(\kappa n + 1) = (\kappa n)!$ if κ is an integer. As we might expect, if $\tilde\varphi$ is a formal power series of Gevrey class $1/\kappa$, namely $\tilde\varphi \in z^{-1}\mathbb{C}[[z^{-1}]]_{1/\kappa}$, then $\hat\varphi = \mathcal{B}_\kappa\tilde\varphi$ is analytic in a neighbourhood of the origin, namely $\hat\varphi(\zeta) \in \mathbb{C}\{\zeta\}$. For the sake of brevity, in the next chapters we will often refer to the formal κ-Borel transform simply as Borel transform.

A resummation procedure can then be formulated as before. For future reference let us first introduce the following definition, following [Bal94]:

Definition 2.6 (Exponential size)**.** Let $\hat\varphi(\zeta)$ be holomorphic on a strip S^θ_δ defined by (2.51). We will say that $\hat\varphi(\zeta)$ is of **exponential size at most** s if there exist $r > 0$ and two positive constants A and c such that

$$|\hat\varphi(\zeta)| \le Ae^{c|\zeta|^s} \tag{2.61}$$

for every $\zeta \in S^\theta_\delta$ with $|\zeta| > r$.

Let us then consider a formal power series of Gevrey class $1/\kappa$, namely $\tilde\varphi \in z^{-1}\mathbb{C}[[z^{-1}]]_{1/\kappa}$. Then, let us consider the formal κ-Borel transform $\hat\varphi = \mathcal{B}_\kappa\tilde\varphi$. If the following conditions are satisfied:

1. $\hat{\varphi}(\zeta)$ admits analytic continuation on a half-strip S_δ^θ for some angle θ and some non-vanishing width δ;
2. The analytic continuation is of exponential size at most $1/\kappa$;

then $\hat{\varphi}$ is said to be **κ-summable in the direction θ** and a generalised **κ-Laplace transform** can be taken. This is defined by

$$\mathcal{L}_\kappa^\theta \hat{\varphi} := \frac{1}{\kappa} \int_0^{e^{i\theta}\infty} d\zeta e^{-(z\zeta)^{\frac{1}{\kappa}}} (z\zeta)^{\frac{1}{\kappa}-1} \hat{\varphi}(\zeta) \,, \tag{2.62}$$

where again the integration takes place along the line $e^{i\theta}\mathbb{R}^+$. The function $\varphi(z) = \mathcal{L}_\kappa^\theta \hat{\varphi}$ will be then holomorphic on the domain $\Pi_c^\theta(\kappa) := \{z \in \mathbb{C} : \mathrm{Re}((e^{i\theta}z)^{1/\kappa}) > c\}$, where c is the constant appearing in (2.61) which determines the exponential growth of $\hat{\varphi}(\zeta)$ on the half-strip S_δ^θ: we will again call it the Borel sum of $\tilde{\varphi}$. For the sake of brevity, we will also refer to the κ-Laplace transform simply as Laplace transform (but we shall retain the subscript in $\mathcal{L}_\kappa$).

The Borel sum $\varphi(z)$ still bears a close relationship with the formal power series $\tilde{\varphi}$. For future reference, let us generalise the former Definition 2.5:

Definition 2.7 (Uniform asymptotic expansion of order κ)**.** Let $\varphi(z)$ be holomorphic on a domain $\mathcal{D} \subset \mathbb{C}^*$, we will say that $\varphi(z)$ admits the formal power series $\tilde{\varphi} = \sum_{n=0}^\infty a_n z^{-n-1}$ as **uniform asymptotic expansion of order κ** if, for every N,

$$\left| \varphi(z) - \sum_{n=0}^{N-1} a_n z^{-n-1} \right| \leq AB^N (N!)^\kappa |z|^{-N-1} \tag{2.63}$$

for every $z \in \mathcal{D}$ (i.e. uniformly in $\mathcal{D}$). We will denote this state of affairs as $\varphi(z) \sim_\kappa \tilde{\varphi}$.

We remark once again that $(N!)^\kappa$ can be substituted with $\Gamma(\kappa N)$ by an appropriate rescaling of the constants A, B.
In a similar fashion to the $\kappa = 1$ case, the Borel sum $\varphi(z)$ will admit the original formal power series $\tilde{\varphi}$ as uniform asymptotic expansion on $\Pi_c^\theta(\kappa)$. Proceeding along the same lines, we can tilt the line of integration of the Laplace transform to a continuous range of directions I and find a Borel sum φ on the domain

$$\mathscr{D}_\kappa(I) := \bigcup_{\theta \in I} \Pi_{\gamma(\theta)}^\theta(\kappa) \tag{2.64}$$

in analogy to (2.57), with $|\hat{\varphi}(\zeta)| \leq \alpha(\theta) e^{\gamma(\theta)|\zeta|^{1/\kappa}}$ for every $\zeta \in S_{\delta(\theta)}^\theta$ and for every $\theta \in I$ (and the function $\delta(\theta)$ is nowhere vanishing on I).

2.2.4 Nevanlinna-Sokal Theorem and Quasi-Analyticity

The Nevanlinna-Sokal theorem is at the core of the results that we shall present throughout this thesis. Essentially, this theorem establishes an isomorphism between algebras of holomorphic functions and algebras asymptotic expansions. The theorem was first proved in [Sok79], and is also stated in [Fer+24; Riv24; ELT21; LR16]. For our purposes, we will state the theorem for a small variable t, which we can think of the inverse of the large variable z employed hitherto. Moreover, we shall restrict the statement of the theorem to the case $\theta = 0$, as this will be the case of the greatest concern in the following. The generalisation to an arbitrary direction θ is straightforward.

Theorem 2.6 (Nevanlinna-Sokal)**.** Let D_c^κ be a subdomain of the Riemann surface of the logarithm $\tilde{\mathbb{C}}$ defined by:

$$D_c^\kappa := \left\{ t \in \tilde{\mathbb{C}} : \mathrm{Re}\, t^{-1/\kappa} > c \right\} = \left\{ t \in \tilde{\mathbb{C}} : |t| \leq \frac{1}{c^\kappa} \cos^\kappa\left(\frac{\theta}{\kappa}\right) \right\} \tag{2.65}$$

where $\theta = \arg(t)^2$and the second equality only holds for $c \neq 0$. Let $\tilde{\varphi} \in t\mathbb{C}[[t]]_{1/\kappa}$ be a formal power

series of Gevrey class $1/\kappa$, with

$$\tilde{\varphi} := \sum_{n=0}^{\infty} a_n t^{n+1} \tag{2.66}$$

whose κ-th Borel transform

$$\mathcal{B}_\kappa \tilde{\varphi} = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\kappa n + 1)} \zeta^n = \hat{\varphi}(\zeta) \tag{2.67}$$

is convergent, with radius of convergence B. Then we have

1. If $\hat{\varphi}$ admits analytic continuation to a horizontal strip S_B^0, and has exponential size no larger than $1/\kappa$ thereon, with
$$|\hat{\varphi}(\zeta)| \leq A e^{c|\zeta|^{1/\kappa}} \tag{2.68}$$
for every $\zeta \in S_B^0$ and for a constant A, then the Borel sum of $\tilde{\varphi}$, namely
$$\varphi(t) := (\mathcal{L}_\kappa^0 \hat{\varphi})(t) = \frac{1}{\kappa} \int_0^\infty d\zeta e^{-\left(\frac{\zeta}{t}\right)^{\frac{1}{\kappa}}} \left(\frac{\zeta}{t}\right)^{\frac{1}{\kappa}-1} \hat{\varphi}(\zeta) \tag{2.69}$$
is holomorphic on the domain (2.65) and admits $\tilde{\varphi}$ as uniform asymptotic expansion of order κ on the same domain, namely
$$\left| \varphi(t) - \sum_{n=0}^{N-1} a_n t^{n+1} \right| \leq B^{-N} \Gamma(\kappa N + 1) t^N \tag{2.70}$$
for every N and for every $t \in D_c^\kappa$.
2. Conversely, if $\varphi(t)$ is a holomorphic function on a domain D_c^κ, and it admits a formal power series $\tilde{\varphi}$ of Gevrey class $1/\kappa$ as uniform asymptotic expansion of order κ on D_c^κ, i.e. (2.70) holds for every N and for every $t \in D_c^\kappa$, then $\tilde{\varphi}$ is κ-summable. In other words, its Borel transform $\hat{\varphi}$ (2.67) admits analytic continuation on a half-strip S_B^0 and has at most exponential size at most $1/\kappa$ thereon; moreover, $\varphi(t)$ is given by (2.69).

A domain D_c^κ is known as **Sokal disc** (or sometimes as Borel disc). Observe how Sokal discs, for every κ and c are tangent to the origin. The name disc comes from the shape of D_c^1 (Figure 2.1a), which is indeed a disc tangent to the imaginary axis. For larger values of κ, the Sokal disc is also dubbed a **cardioid**, due to the shape displayed in Figure 2.1b. Observe how a Sokal disc D_c^1 is exactly the domain $\Pi_{1/c}^0$ introduced in (2.33) under the map $t = z^{-1}$, and in particular $D_0^1 = \Pi_0^0$.

The first point of this theorem beautifully summarises all the results that we have expounded. But there is more: the second point provides a sufficient condition for a holomorphic function $\varphi(t)$ to be the Borel sum of its asymptotic expansion $\tilde{\varphi}$. Moreover, if $\varphi(t)$ is extended to the origin by fixing

$$\varphi(0) := \lim_{t \to 0} \varphi(t) \tag{2.71}$$

(we will see later in Proposition 2.6 why this limit exists for any direction in D_c^κ) then it will be smooth, but not analytic at $t = 0$. In fact, the Taylor series of $\varphi(t)$ at $t = 0$, namely the asymptotic expansion $\tilde{\varphi} \in t\mathbb{C}[[t]]_{1/\kappa}$, with

$$a_n = \frac{1}{(n+1)!} \frac{d^{n+1}}{dt^{n+1}} \varphi(t) \Big|_{t=0}. \tag{2.72}$$

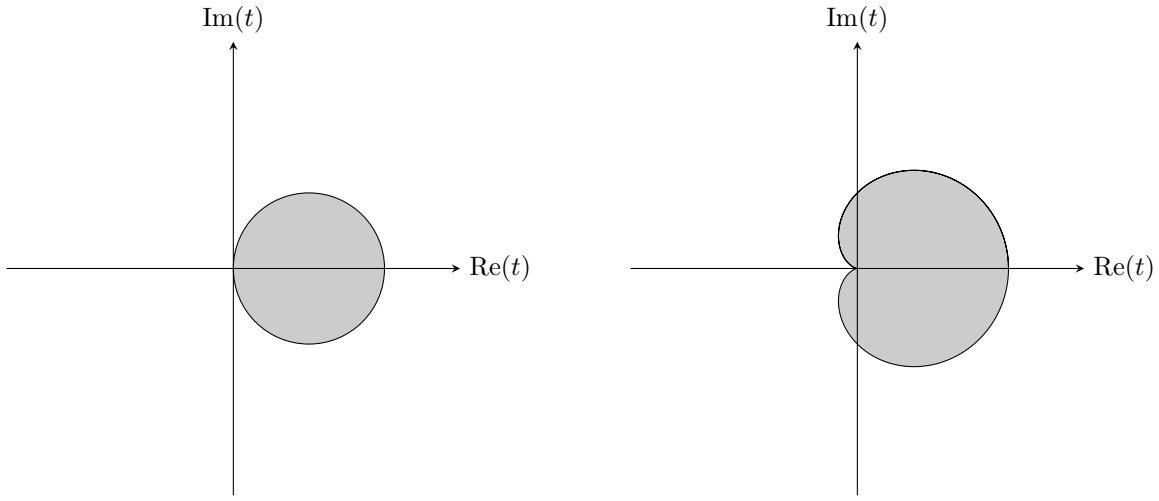
is divergent, the Gevrey class thereof being determined by how quickly the n-th derivatives of $\varphi(t)$ at 0 grow with n. Thus, Borel sums such as $\varphi(t)$ form an algebra of functions holomorphic on the Sokal disc D_c^κ

[2] not to be confused with the line of integration of the Laplace transform.

and non-analytic at the origin. Then Nevanlinna-Sokal theorem ensures that the Taylor map acting on this algebra, is one-to-one. The Taylor map is in general given by

$$\begin{aligned} \mathcal{T} : \tilde{\mathscr{G}}_{1/\kappa} &\to \mathbb{C}[[t]]_{1/\kappa} \\ \varphi(t) &\to \tilde{\varphi} = \sum_{n=0}^{\infty} a_n t^n \end{aligned} \tag{2.73}$$

where by $\tilde{\mathscr{G}}_{1/\kappa}$ we denote the algebra of functions non analytic at the origin admitting a $1/\kappa$-Gevrey asymptotic expansion and a_n is given by (2.72) with $n+1 \to n$ (so that we allow for constant terms in the power series, i.e. for $\varphi(0) \neq 0$). This map is known to be surjective (this is Borel-Ritt theorem: see [Lod14], Theorem 2.4.1). Yet, only when the map is restricted to the subalgebra $\mathscr{G}_{1/\kappa} \subset \tilde{\mathscr{G}}_{1/\kappa}$ of functions holomorphic and allowing uniform asymptotic expansion of order κ on the Sokal disc D_c^κ – thus satisfying Nevanlinna-Sokal theorem – is it also injective and, therefore, invertible. We understand then how Borel summability may be viewed as an extension of analyticity: Borel sums are in one-to-one correspondence with their Taylor series at 0, just like analytic functions; the difference is that the Taylor series are now divergent and thus do not converge to the associated function, but are only their asymptotic expansion. An algebra of functions on which the Taylor map (at 0) is injective is called **quasi-analytic**, since, albeit the functions may not be analytic at 0, they still possess a feature which is characteristic of analytic functions. Therefore, an algebra of functions satisfying Nevanlinna-Sokal theorem is quasi-analytic.



(a) A Sokal disc for $\kappa = 1$.

(b) Sokal disc for $\kappa = 2$.

Figure 2.1: Two Sokal discs on the Laplace plane for different values of κ.

2.3 Resurgence and alien calculus

As illustrated by Nevanlinna-Sokal theorem, Borel summability requires that the Borel transform $\hat{\varphi}$ admit analytic continuation on a half-strip S_δ^θ and that it be of the correct exponential size. Only then is the Laplace transform well defined along the line θ. In this section we will discuss how to deal with those cases in which these requirements are not fulfilled. This situation can arise essentially for two reasons:

1. The analytic continuation $\hat{\varphi}(\zeta)$ of $\hat{\varphi} \in \mathbb{C}\{\zeta\}$ exists, but is not exponentially bounded;
2. The analytic continuation $\hat{\varphi}(\zeta)$ has a pole at a finite ζ with $\arg(\zeta) = \theta$: hence, it cannot be analytically continued to the whole line $e^{i\theta}\mathbb{R}^+$.

We will only focus on the latter case, since it is the one which will occur in the following chapters in some of our applications to QFT amplitudes and quantum mechanics. The fields of mathematics which treats this problem are known as Resurgence and Alien Calculus. Developed in the 80s by Jean Écalle in his seminal books [Eca81], these are very broad topics whose exhaustive treatment goes far beyond the scope and purposes of this thesis. We shall limit ourselves to the most essential results. Throughout this section, the Gevrey class of formal power series is fixed to be 1 ($\kappa = 1$). We will mostly follow [Dor19] and [Sau14].

2.3.1 Simple resurgent functions

In this section we will focus on the Borel transforms which fall into the class of simple resurgent functions. These can be viewed as holomorphic germs at the origin whose analytic continuation along any given direction satisfies certain regularity properties, described below.

First, let us fix a discrete set of points Ω on the complex plane: we shall interpret them as the set of the singularities of the analytic continuation of an analytic germ at the origin $\hat{\varphi}(\zeta)$. Then, let us introduce the following definition.

Definition 2.8 (Endless continuability)**.** A germ of analytic functions at the origin $\hat{\varphi}(\zeta) \in \mathbb{C}$ is **endlessly continuable** on $\mathbb{C}$ if for all $R > 0$ there exists a finite set of accessible singularities $\Omega_R \subset \Omega$, such that $\hat{\varphi}(\zeta)$ can be analytically continued along all paths whose length is less then R, avoiding the singularities Ω_R.

This means, roughly, that the singularities cannot be too dense in the Borel plane, and therefore we can successfully dodge them by choosing an appropriate path on $\mathbb{C}\backslash\Omega$. Moreover, we characterise the nature of such singularities by the following:

Definition 2.9 (Simple singularities)**.** A function $\hat{\varphi}(\zeta)$ holomorphic in an open disc $D(\zeta_0, R) \subset \mathbb{C}$ is said to have a **simple singularity** at $\omega \in \mathrm{cl}(D(\zeta_0, R))$ if there exist $r \in \mathbb{C}$ and two germs of analytic functions $\Phi(\zeta), reg(\zeta)$ at the origin such that:

$$\hat{\varphi}(\zeta) = \frac{r}{2\pi i(\zeta - \omega)} + \frac{\hat{\Phi}(\zeta - \omega)}{2\pi i(\zeta - \omega)} \log(\zeta - \omega) + reg(\zeta - \omega) \tag{2.74}$$

where $D(\zeta_0, R)$ is a disc centred at ζ_0 of radius R.

Note how the hypothesis on the discreteness of Ω enables us to consider a disc D of finite radius R where there are no singularities (i.e. $D(\zeta_0, R) \subset \mathbb{C}\backslash\Omega$). Furthermore, $\Phi(\zeta), reg(\zeta)$ being analytic germs at the origin implies that a simple resurgent function can only have either a simple singularity or a logarithmic singularity at Ω, whence the name 'simple'. $r \in \mathbb{C}$ is called the **residue**, while $\hat{\Phi}(\zeta)$ is called the **minor** of $\hat{\varphi}(\zeta)$. Finally, we have:

Definition 2.10 (Simple resurgent function)**.** A **simple resurgent function** $\hat{\psi}(\zeta)$ is such that $\hat{\psi}(\zeta) = c\delta + \hat{\varphi}(\zeta) \in \delta\mathbb{C} \oplus \mathbb{C}\{\zeta\}$ and, for every accessible singularity $\omega \in \Omega$, and for every path γ starting at the origin, dodging Ω and ending close enough to ω (i.e. a disc can be found around the endpoint of γ such that ω is the only encompassed singularity), the analytic continuation of $\hat{\psi}(\zeta)$ along γ has a simple singularity at ω. The subspace of simple resurgent functions is indicated by $\delta\mathbb{C} \oplus \hat{\mathscr{R}}_\Omega^{simp}$.

Hitherto, the only assumption made on the set of singularities Ω is that it be not dense, so that endless continuability of Definition (2.8) is ensured for holomorphic germs $\hat{\varphi}(\zeta)$. In the cases of our concern, Ω will be a very regular set: for example, we shall see that for the Stirling series it is $\Omega = 2\pi i\mathbb{Z}$. Such regularity entails a very desirable closure property of $\delta\mathbb{C} \oplus \hat{\mathscr{R}}_\Omega^{simp}$. Referring to [Sau14], we have the following:

Proposition 2.3. Let Ω be closed under addition. Then the subspace of **simple Ω-resurgent functions** $\delta\mathbb{C} \oplus \hat{\mathscr{R}}_\Omega^{simp} \subset \delta\mathbb{C} \oplus \mathbb{C}\{\zeta\}$ is a differential algebra with the derivation $\hat{\partial}$ in (2.25), the ordinary addition and the convolution product $*$ defined by (2.27). Because $\mathcal{B}$ is an isomorphism (2.26), the space $\mathcal{B}^{-1}(\delta\mathbb{C} \oplus \hat{\mathscr{R}}_\Omega^{simp})$ is also a differential algebra with the derivation ∂, the ordinary addition and the ordinary Cauchy product $\cdot$. The latter algebra is the algebra of **simple Ω-resurgent formal series** and is denoted by $\tilde{\mathscr{R}}_\Omega^{simp}$, which is a subalgebra of $\mathbb{C}[[z^{-1}]]_1$.

Convolution product: an example

Let us digress for a little while to show an illustrative example. Consider the holomorphic germs at the origin

$$\hat{\varphi}_1(\zeta) := \frac{1}{\zeta - \omega_1} \qquad , \qquad \hat{\varphi}_2(\zeta) := \frac{1}{\zeta - \omega_2}. \tag{2.75}$$

These admit a meromorphic continuation to $\mathbb{C}\backslash\{\omega_1, \omega_2\}$, as they have a simple pole at ω_1, ω_2 respectively. Yet, we cannot simply claim that they belong to the convolutive algebra $\delta\mathbb{C} \oplus \hat{\mathscr{R}}_\Omega^{simp}$ with $\Omega = \{\omega_1, \omega_2\}$. Indeed, upon taking their convolution product, we find:

$$\begin{aligned}(\hat{\varphi}_1 * \hat{\varphi}_2)(\zeta) &= \int_0^\zeta d\zeta' \frac{1}{\zeta' - \omega_1} \frac{1}{\zeta - \zeta' - \omega_2} = \frac{1}{\zeta - (\omega_1 + \omega_2)} \int_0^\zeta d\zeta' \left(\frac{1}{\zeta' - \omega_1} + \frac{1}{\zeta - \zeta' - \omega_2} \right) \\ &= \frac{1}{\zeta - (\omega_1 + \omega_2)} \int_0^\zeta d\zeta' \left(\frac{1}{\zeta' - \omega_1} + \frac{1}{\zeta' - \omega_2} \right).\end{aligned} \tag{2.76}$$

The latter expression clearly has a pole at $\omega_1 + \omega_2$. Since $\omega_1 + \omega_2$ does not belong to Ω, Ω is not stable under addition: as a consequence $\delta\mathbb{C} \oplus \hat{\mathscr{R}}_\Omega^{simp}$ is unstable under convolution products and therefore is not an algebra. We need then to ensure that Ω be closed under addition, as assumed in Proposition 2.3, so that convolution products between any two resurgent functions do not generate additional singularities. Notice that we are always allowed to add to Ω a finite set of points, or even a countable infinity of points if they are sufficiently distant from each other so as to comply with the hypothesis made in Definition 2.9. Were we to find a holomorphic germ $\hat{\varphi}(\zeta)$ at the origin that has simple poles located at $2\pi i\mathbb{Z}^*$ in the Borel plane (as we will when dealing with the Stirling series), we can safely claim that it is a simple resurgent function of the convolutive algebra $\delta\mathbb{C} \oplus \hat{\mathscr{R}}_{2\pi i\mathbb{Z}}^{simp}$, although $\hat{\varphi}(\zeta)$ has no pole at the origin (if it had, it would be no holomorphic germ at the origin!).
Similarly, if $\hat{\varphi}(\zeta)$ has only a pole at $\zeta = -1$, we can safely claim that it belongs to the algebra $\delta\mathbb{C} \oplus \hat{\mathscr{R}}_{\mathbb{Z}}^{simp}$. We can *not*, however, set $\Omega = \mathbb{Q} + i\mathbb{Q}$, although $-1 \in \mathbb{Q} + i\mathbb{Q}$, because the rational numbers, as opposed to the integers, are a dense set and therefore it would be impossible to find a disc of finite radius in $\mathbb{C}\backslash\Omega$, as assumed in Definition 2.9.

2.3.2 Alien derivatives and transseries

By now, we are convinced that the divergence of a 1-Gevrey formal power series $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$ is testified by its Borel transform $\hat{\varphi} = \mathcal{B}\tilde{\varphi}$ having one or more poles in the Borel plane (hearken back to Proposition 2.1: if $\tilde{\varphi}$ is divergent, the radius of convergence of $\hat{\varphi}$ is only finite, meaning that there is at least one pole in the Borel plane). If, furthermore, we assume that $\tilde{\varphi} \in \tilde{\mathscr{R}}_\Omega^{simp}$ for some discrete set of points Ω, then all the singularities of $\hat{\varphi} \in \delta\mathbb{C} \oplus \hat{\mathscr{R}}_\Omega^{simp}$ will be simple or logarithmic poles. The idea underlying alien derivatives is that of rigorously defining an object that 'measures' the singularity at any given $\omega \in \Omega$ of the analytic continuation of a resurgent function $\hat{\varphi}(\zeta)$. Note that, by endless continuability (2.8), it is possible to analytically continue $\hat{\varphi}(\zeta)$ to a holomorphic function on a disc D adherent to ω; the rigorous definition of the alien derivative makes use of this important requirement. We refer to [Sau14] for such a definition; opting for a more pedestrian approach, following rather [Dor19], we merely define the alien derivatives by explicitly giving their action on a simple Ω-resurgent function. It must be noted though that this operational definition

is just the result of a more general definition which is valid for any kind of resurgent function (namely having non-simple singularities and where Ω is not necessarily closed under addition).

Definition 2.11 (Alien derivative)**.** Consider a discrete set Ω closed under addition. For any $\omega \in \Omega$, **the alien derivative at** ω is a map

$$\Delta_\omega : \delta\mathbb{C} \oplus \hat{\mathscr{R}}_\Omega^{simp} \to \delta\mathbb{C} \oplus \hat{\mathscr{R}}_\Omega^{simp}$$

such that, if $\hat{\psi} \in \delta\mathbb{C} \oplus \hat{\mathscr{R}}_\Omega^{simp}$ can be written as $a\delta + \hat{\varphi}(\zeta)$, where $\hat{\varphi}(\zeta)$ has residue $r \in \mathbb{C}$ and minor $\hat{\Phi}(\zeta)$ at ω, then

$$\Delta_\omega \hat{\psi} = r\delta + \hat{\Phi}(\zeta). \tag{2.77}$$

By a slight abuse of notation, we shall denote the map $\mathcal{B}^{-1} \circ \Delta_\omega \circ \mathcal{B} : \tilde{\mathscr{R}}_\Omega^{simp} \to \tilde{\mathscr{R}}_\Omega^{simp}$ by the same symbol Δ_ω. In this case, one has

$$\Delta_\omega \tilde{\psi} = r + \tilde{\Phi} \tag{2.78}$$

where $\tilde{\psi} = \mathcal{B}^{-1}\hat{\psi}$ and $\tilde{\Phi} = \mathcal{B}^{-1}\hat{\Phi}$. Remark that the operator $\mathcal{B}^{-1}$ is well defined as the Borel transform is an isomorphism of differential algebras, as expressed by (2.26).

The reason why the alien derivative is given such a name is that it is actually a derivative[3]. More precisely, under the necessary conditions, it obeys the Leibniz rule (Theorem 30.6, [Sau14]):

Theorem 2.7. Let Ω be a discrete set closed under addition. Then for any $\omega \in \Omega$ the alien derivative Δ_ω obeys the Leibniz rule. Namely, given $\hat{\varphi}_1, \hat{\varphi}_2 \in \delta\mathbb{C} \oplus \hat{\mathscr{R}}_\Omega^{simp}$ and $\tilde{\varphi}_1, \tilde{\varphi}_2 \in \tilde{\mathscr{R}}_\Omega^{simp}$

$$\Delta_\omega(\hat{\varphi}_1 * \hat{\varphi}_2) = \hat{\varphi}_1 * (\Delta_\omega \hat{\varphi}_2) + (\Delta_\omega \hat{\varphi}_1) * \hat{\varphi}_2, \tag{2.79}$$

or, equivalently,

$$\Delta_\omega(\tilde{\varphi}_1 \cdot \tilde{\varphi}_2) = \tilde{\varphi}_1 \cdot (\Delta_\omega \tilde{\varphi}_2) + (\Delta_\omega \tilde{\varphi}_1) \cdot \tilde{\varphi}_2. \tag{2.80}$$

Moreover, the alien derivatives are well-behaved also under non-linear operations. Let us state a generalisation to Theorem 2.4 (Theorem 30.9 of [Sau14]).

Theorem 2.8. Let Ω be closed under addition. Let $\tilde{\varphi}, \tilde{\psi}, \tilde{\chi} \in \tilde{\mathscr{R}}_\Omega^{simp}$ be simple Ω-resurgent formal power series, with $\tilde{\chi}$ having no constant term. Let also $H \in \mathbb{C}\{t\}$ be a convergent power series. Then the compositions $\tilde{\psi} \circ (\mathrm{Id} + \tilde{\varphi})$ and $H \circ \tilde{\chi}$ are simple Ω-resurgent formal power series (i.e., they belong to $\tilde{\mathscr{R}}_\Omega^{simp}$) and, for $\omega \neq 0$

$$\Delta_\omega(\tilde{\psi} \circ (\mathrm{Id} + \tilde{\varphi})) = (\partial\tilde{\psi}) \circ (\mathrm{Id} + \tilde{\varphi}) \Delta_\omega \tilde{\varphi} + e^{-\omega\tilde{\varphi}} (\Delta_\omega \tilde{\psi}) \circ (\mathrm{Id} + \tilde{\varphi}); \tag{2.81}$$

$$\Delta_\omega(H \circ \tilde{\chi}) = \left(\frac{dH}{dt} \circ \tilde{\chi} \right) \cdot \Delta_\omega(\tilde{\chi}). \tag{2.82}$$

Observe in especial how the composition of a convergent series with a formal power series does not add any poles on the Borel plane.

Despite being a derivation, the alien derivative (in the multiplicative algebra $\tilde{\mathscr{R}}_\Omega^{simp}$), does not commute with the derivation $\partial = \frac{d}{dz}$. This would be a very desirable property. It can be proved though, that the **dotted alien derivative**, defined as

$$\dot{\Delta}_\omega := e^{-\omega z} \Delta_\omega \ , \tag{2.83}$$

commutes with ∂: namely, $[\dot{\Delta}_\omega, \partial] = 0$. At this stage though, this definition is rather poorly rigorous. Indeed $e^{-\omega z}$ *cannot* be expanded in powers of z^{-1}, so it does not belong to the algebra $\mathbb{C}[[z^{-1}]]$. The action of Δ_ω

[3]In case the reader were rather wondering about the *alien*, we should do Jean Écalle some justice: the original French attribute to these derivatives was, a bit more modestly, *étranger*. Apparently though, in the English translation, it was agreed that these objects are so unusual and unfamiliar that they must be born not simply of a foreign country, but rather of an unearthly, alien world.

returns a (simple) Ω-resurgent formal power series, so that the multiplication by $e^{-\omega z}$ seems to imply that the algebra $\tilde{\mathscr{R}}_\Omega^{simp}$ is not closed under the action of the dotted alien derivatives. Indeed it is not: we must extend it to the algebra of (simple) **Ω-resurgent transseries**, which following [Sau14] we name $\tilde{E}(\Omega)$. A transseries $\tilde{\Phi} \in \tilde{E}(\Omega)$ is a formal sum of formal power series, each equipped with a 'formally multiplicative' *symbol*:

$$\tilde{\Phi}(z, \vec{\sigma}) = \sum_n e^{-\omega_n z} \tilde{\varphi}_n \sigma_n \tag{2.84}$$

where, for every n, $\tilde{\varphi}_n \in \tilde{\mathscr{R}}_\Omega^{simp}$ is simple Ω-resurgent power series. For every n, $\omega_n \in \Omega$, and $e^{-\omega_n z}$ is a symbol called **simple resurgent symbol**. The parameters $\vec{\sigma} = (\sigma_1, ..., \sigma_n, ...)$ are just bookkeeping parameters, which will serve the purpose of selecting a specific $\tilde{\varphi}_n$ (a **sector** of the transseries). This can be done, for instance, by means of differentiation with respect to σ_n and setting all the parameters to 0 to zero afterwards. The dotted alien derivative is then understood to be a derivation on the algebra $\tilde{E}(\Omega)$, which also inherits from $\hat{\mathscr{R}}_\Omega^{simp}$ the derivation $\partial : \tilde{E}(\Omega) \to \tilde{E}(\Omega)$, which acts in the intuitive way:

$$\partial\tilde{\Phi}(z, \vec{\sigma}) = \sum_n \left(e^{-\omega_n z} \partial\tilde{\varphi}_n - \omega_n e^{-\omega_n z} \tilde{\varphi}_n \right) \sigma_n \tag{2.85}$$

and it can be proved to commute with any dotted alien derivative $\dot{\Delta}_\omega$.

The algebra of Ω-resurgent transseries $\tilde{E}(\Omega)$ inherits its algebraic structure from $\tilde{\mathscr{R}}_\Omega^{simp}$ if Ω is closed under addition. In fact, it is sufficient to state

$$\left(e^{-\omega_1 z}\tilde{\varphi}_1\right) \cdot \left(e^{-\omega_2 z}\tilde{\varphi}_2\right) =: e^{-(\omega_1+\omega_2)}\tilde{\varphi}_1 \cdot \tilde{\varphi}_2 \tag{2.86}$$

whence we see the necessity of Ω being closed under addition in order for $\tilde{E}(\Omega)$ to be closed under products and deserve so the name of algebra. Thanks to (2.85), it is actually a differential algebra with the derivative ∂.

A clarification is due. The algebra $\tilde{E}(\Omega)$ is made out of $\tilde{\mathscr{R}}_\Omega^{simp}$ by the addition of the simple resurgent symbols $e^{-\omega z}$ for every $\omega \in \Omega$. Hence, we *must not* think of $e^{-\omega z}$ as a function: actually, they are no more functions than formal power series are. Their suggestive though misleading writing is due to the fact that, upon applying the Borel-Laplace summation operator, the resurgent symbols turn into actual functions (very much like formal power series). By definition, given $\tilde{\varphi} \in \tilde{\mathscr{R}}_\Omega^{simp}$ summable in an arc of directions I, we have

$$\mathscr{S}^I(e^{-\omega z}\tilde{\varphi}) = e^{-\omega z}\mathscr{S}^I\tilde{\varphi}, \tag{2.87}$$

where $e^{-\omega z}$ is *a symbol* on the left hand side, and *a function* on the right.

2.3.3 The Stokes automorphism

Our previous hint at the need to promote formal power series to transseries is admittedly unsatisfactory; yet, by introducing the Stokes automorphism, we shall provide a more convincing evidence for this necessity. The presence of poles in the Borel plane along a certain direction θ prevents us from applying the Borel-Laplace operator $\mathscr{S}^\theta$ correctly. The best we can do is to apply it at angles $\theta+\varepsilon$, or $\theta-\varepsilon$, infinitesimally close to θ. Albeit the lines of integration are infinitesimally close to each other, the two Borel sums $\mathscr{S}^{\theta+\varepsilon}\tilde{\varphi}$ and $\mathscr{S}^{\theta+\varepsilon}\tilde{\varphi}$ (with $\tilde{\varphi} \in \tilde{\mathscr{R}}_\Omega^{simp}$ for a certain Ω some of whose points lie on the direction θ) will yield two different results whose difference is not of the order ε. The Stokes automorphism is the formal way to express such a 'jump': namely we have the following

Definition 2.12 (Stokes Automorphism)**.** Let Ω be a non-empty, discrete set, closed under addition. Let $\tilde{\Phi} \in \tilde{E}(\Omega)$ be a simple Ω-resurgent transseries. Then the **Stokes automorphism** is a map

$$\mathfrak{S}_\theta : \tilde{E}(\Omega) \to \tilde{E}(\Omega) \tag{2.88}$$

such that

$$\mathscr{S}^{\theta-\varepsilon}\tilde{\Phi} = \mathscr{S}^{\theta+\varepsilon} \circ \mathfrak{S}_\theta\tilde{\Phi}. \tag{2.89}$$

Note that the Stokes automorphism is defined on the whole space of transseries $\tilde{E}(\Omega)$ in which $\tilde{\mathscr{R}}_\Omega^{simp}$ is embedded (a resurgent power series can be viewed as a transseries with only one sector and no resurgent symbol). Consider for simplicity $\tilde{\Phi} = \tilde{\varphi} \in \tilde{\mathscr{R}}_\Omega^{simp}$: the Stokes automorphism $\mathfrak{S}_\theta$ will act non-trivially (differently from the identity) on $\tilde{\varphi}$ only if its Borel transform $\hat{\varphi}(\zeta)$ has a pole at $\omega \in \Omega$ with $\arg\omega = \theta$. In this case θ is said to be a **Stokes line**. In fact, it should be clear enough that the difference between two Borel sums $\mathscr{S}^{\theta+\varepsilon}\tilde{\varphi} - \mathscr{S}^{\theta-\varepsilon}\tilde{\varphi}$ can be non-infinitesimal only if $\hat{\varphi}(\zeta)$ has a pole along θ, and thus the Stokes automorphism cannot act as the identity in this case. This fact can be further clarified by the next proposition, which states that the Stokes automorphism can be expressed in terms of the alien derivatives labelled by a pole ω with argument θ.

Proposition 2.4. The Stokes automorphism $\mathfrak{S}_\theta$ at a direction θ can be expressed as

$$\mathfrak{S}_\theta = \exp\left(\sum_{\omega\in\Omega_\theta} e^{-\omega z}\Delta_\omega\right) = \exp\left(\sum_{\omega\in\Omega_\theta} \dot{\Delta}_\omega\right) \tag{2.90}$$

where $\Omega_\theta \subseteq \Omega$ is the subset of singularities $\omega \in \Omega$ lying along the direction θ in the Borel plane.

As an alien derivative Δ_ω, acting on $\tilde{\varphi}$, yields a non-vanishing result only if $\tilde{\varphi}$ has a pole at ω, we clearly see then that the Stokes automorphism $\mathfrak{S}_\theta$ will act on $\tilde{\varphi}$ as the identity unless $\Delta_\omega\tilde{\varphi} \neq 0$ for some $\omega \in \Omega_\theta$.

Although in Definition 2.12 the Stokes automorphism has been defined on the space of Ω-resurgent transseries $\tilde{E}(\Omega)$, we may observe that, since we can make sense of alien derivatives also on $\mathbb{C}\delta \oplus \tilde{\mathscr{R}}_\Omega^{simp}$, it is clear that we can also have the Stokes automorphism act on the image of $\tilde{E}(\Omega)$ under the Borel operator, $\mathcal{B}(\tilde{E}(\Omega))$, which we call $\hat{E}(\Omega)$. This operation is a bit subtle, as $\hat{E}(\Omega)$ must be defined in compliance with (2.87). For this purpose, let $\tilde{\varphi} \in \tilde{\mathscr{R}}_\Omega^{simp}$ with $\mathcal{B}\tilde{\varphi} = a\delta + \hat{\varphi}(\zeta)$, and consider the sector of the transseries $e^{-\omega z}\tilde{\varphi}$. Then, in taking the Borel image of $e^{-\omega z}\tilde{\varphi} \in \tilde{E}(\Omega)$, we must in some way keep track of the resurgent symbol $e^{-\omega z}$. To do so, we introduce a new 'identity' δ_ω for every $\omega \in \Omega$, such that, for every θ,

$$\mathcal{L}^\theta(a\delta_\omega)(z) := ae^{-\omega z} \tag{2.91}$$

in the same spirit as (2.42). Then we let

$$\mathcal{B}\left(e^{-\omega z}\tilde{\varphi}\right) := a\delta_\omega + \hat{\varphi}_\omega(\zeta). \tag{2.92}$$

The Borel transform $\hat{\varphi}(\zeta)$ is now 'dressed' with subscript ω is meant to keep track of the sector of the transseries. If $\tilde{\varphi}$ is 1-summable along θ (i.e $\hat{\varphi}(\zeta)$ admits analytic continuation on this direction) the Laplace operator $\mathcal{L}^\theta$ is then defined to act as

$$\mathcal{L}^\theta(a\delta_\omega + \hat{\varphi}_\omega)(z) = ae^{-\omega z} + e^{-\omega z}\int_0^{e^{i\theta}\infty} d\zeta e^{-z\zeta}\hat{\varphi}(\zeta). \tag{2.93}$$

Remark that again on the right-hand side $e^{-\omega z}$ is now a function and not a symbol. We refer to [Sau14] for further details about this.

It is a general algebraic fact ([Sau14]) that the logarithm of an automorphism is a derivation, or conversely the exponentiation of a derivation yields an automorphism. As the alien derivatives are indeed derivations, this explains why the Stokes automorphism is indeed an algebra automorphism on the space of transseries $\tilde{E}(\Omega)$ (or its image under the isomorphism $\mathcal{B}$), as expressed by the following (Theorem 30.2 [Sau14]):

Theorem 2.9. Let Ω be a discrete non-empty set closed under addition. Then the Stokes automorphism is compatible with the Cauchy product on $\tilde{E}(\Omega)$ and with the convolution product on $\hat{E}(\Omega)$. Namely, given two transseries $\tilde{\Phi}, \tilde{\Psi} \in \tilde{E}(\Omega)$ and their Borel transforms $\hat{\Phi}, \hat{\Psi} \in \hat{E}(\Omega)$ we have

$$\begin{aligned} \mathfrak{S}_\theta(\tilde{\Phi}\cdot\tilde{\Psi}) &= \mathfrak{S}_\theta\tilde{\Phi}\cdot\mathfrak{S}_\theta\tilde{\Psi} \qquad \in \tilde{E}(\Omega), \\ \mathfrak{S}_\theta(\hat{\Phi}*\hat{\Psi}) &= \mathfrak{S}_\theta\hat{\Phi}*\mathfrak{S}_\theta\hat{\Psi} \qquad \in \hat{E}(\Omega), \end{aligned} \tag{2.94}$$

where the convolution product on transseries is defined in analogy to the Cauchy product in (2.86).

The Stokes automorphism is an algebra isomorphism on $\tilde{E}(\Omega)$ or equivalently on its Borel image $\hat{E}(\Omega)$: these algebras are then closed under the action of $\mathfrak{S}_\theta$. Consider the transseries (2.84), which is the most general member of $\tilde{E}(\Omega)$. In light of the previous observation, we must conclude that the action of $\mathfrak{S}_\theta$ upon $\tilde{\Phi}(z,\vec{\sigma})$ will return another transseries in the form

$$\mathfrak{S}_\theta\tilde{\Phi}(z,\vec{\sigma}) = \tilde{\Phi}(z,\vec{\sigma}'). \tag{2.95}$$

The action of the Stokes automorphism can then be viewed as a transformation of the transseries parameters $\vec{\sigma}$

$$\{\sigma_n\} \xrightarrow{\mathfrak{S}_\theta} \{\sigma'_n\} \tag{2.96}$$

which implements correctly the transformation of a transseries into another. The differences $\sigma'_n - \sigma_n = S_n^\theta$ are called the **Stokes constants**. In plenty of physical applications (see for instance [Dor19], [SV23], and the conjectures put forth in [GM23]) there is actually only one Stokes constant, which of course largely simplifies the alien calculus.

2.3.4 Median resummation

In most of the cases of our concern, divergent power series will exhibit poles on the Borel plane on the real line, either at $\theta = 0$ or $\theta = \pi$. Let us focus on the former case: the presence of poles at $\theta = 0$ prevents us from computing the Laplace integral along the positive real line, which is the natural line of integration if we wish to resum the transseries to a real valued function of a positive real variable z (such like an inverse coupling constant). The two lateral summation operators $\mathscr{S}^\varepsilon$ and $\mathscr{S}^{-\varepsilon}$ will then produce two different functions of z differing by an exponentially suppressed and *imaginary* function. The difference is encoded in the Stokes automorphism $\mathfrak{S}_0$, but we still have to understand how to produce a sensible, real-valued and unambiguous function of z. The solution to this problem is provided by the **median resummation** ([Dor19], Appendix C of [SV23]). As the Stokes automorphism is defined through an exponential, one can easily define its powers by

$$\mathfrak{S}_\theta^\nu = \exp\left(\nu \sum_{\omega\in\Gamma_\theta} \dot{\Delta}_\omega\right). \tag{2.97}$$

It can then be proved that the following summations, yield the same result:

$$\mathscr{S}^\theta_{med} := \mathscr{S}^{\theta-\varepsilon} \circ \mathfrak{S}_\theta^{-1/2} = \mathscr{S}^{\theta+\varepsilon} \circ \mathfrak{S}_\theta^{1/2} \tag{2.98}$$

(note how the very definition given by (2.89) strongly suggests that the above indeed holds). In particular, for the case of greatest physical interest $\theta = 0$,

$$\mathscr{S}^0_{med} := \mathscr{S}^{\varepsilon} \circ \mathfrak{S}_0^{1/2} = \mathscr{S}^{-\varepsilon} \circ \mathfrak{S}_0^{-1/2}. \tag{2.99}$$

The operator $\mathscr{S}^0_{med}$ has a very desirable property: it commutes with complex conjugation. Namely, given a transseries $\tilde{\Phi}(z,\sigma)$, we have

$$\mathscr{S}^0_{med}\left(\tilde{\Phi}(z,\sigma)\right)^* = \left(\mathscr{S}^0_{med}\tilde{\Phi}(z,\sigma)\right)^*. \tag{2.100}$$

Thanks to this property, we can claim that the median resummation operator $\mathscr{S}^0_{med}$ will yield a consistent holomorphic function of z even when the asymptotic series in powers of z^{-1} (for large z) has poles on the Borel plane lying on the positive real line. If $z = x$ is taken to be real, then the median resummation will yield a real valued function of x. In fact, as it will be clarified by the example shown i the next section, the median resummation ensures that all the imaginary ambiguities caused by the lateral resummations cancel.

The following sections of this chapter are dedicated to some illustrative examples in which the tools of summability and resurgence are employed in so simple a way, as to allow us to perform explicit computations of Borel transforms and alien algebras.

2.4 The Euler series

In this section we review two simple, and yet very educational, examples: first of a Borel-summable series, secondly of a non Borel-summable series which, as such, exhibits the Stokes phenomenon. In 1760, in an article titled *De seriebus divergentibus* ([EA18]), Euler considered a number of divergent power series for which the limit of the truncated sums $\lim_{N\to\infty}\sum_n^N a_n$ either is divergent or does not exist. At the time, the tools of Borel-Laplace fine summation were yet to come; nevertheless, Euler managed to associate with a number of such divergent series a finite number by employing the so-called Euler transform and Euler resummation (see [Har56] for extensive treatment). Despite Euler's original approach was quite different, we can now understand his results in terms of Borel summability and alien calculus.

2.4.1 The alternate signs Euler series

The series considered by Euler is

$$1 - 1! + 2! - 3! + ... = \sum_{n=0}^{\infty}(-1)^n n! \tag{2.101}$$

which he called the Wallis series, but is now universally renown as the Euler series. In this case, the limit of the partial sum clearly does not exist, as they oscillate with increasing amplitude. The original approach by Euler was to consider the n-th differences

$$D_n = \sum_{k=0}^{n}\binom{n}{k}a_{n-k}(-1)^k \tag{2.102}$$

where $a_k = k!$ in this case (the alternating sign is not taken into account). Then he proposed that the sum should be

$$\frac{a_0}{2} + \frac{D_1}{2^2} + \frac{D_2}{2^3} + ... = \sum_{n=0}^{\infty}\frac{D_n}{2^{n+1}} \tag{2.103}$$

where $D_0 := a_0$. By explicit computation, Euler was able to find the approximate solution $\simeq 0.596$. Let us now retrieve the same result using the machinery of Borel summability and describe how the divergence of the series manifests itself on the Borel plane. An exhaustive treatment of the Euler series can be found in [Lod14].

In order to employ the theory developed in the previous sections, we can define the formal power series of the algebra $z^{-1}\mathbb{C}[[z^{-1}]]_1$

$$\tilde{\varphi}^E = \sum_{n=0}^{\infty}(-1)^n n! z^{-n-1} \tag{2.104}$$

which we aim at evaluating at $z = 1$. This series is clearly 1-Gevrey, and thus its Borel transform can be computed to be:

$$\hat{\varphi}^E = \sum_{n=0}^{\infty}(-1)^n \zeta^n = \frac{1}{1+\zeta} \tag{2.105}$$

which, as we might have expected, has a finite radius of convergence equal to 1, and its sum has a pole at $\zeta = -1$. Except for the direction $\theta = \pi$, which is the unique Stokes line, $\hat{\varphi}^E$ is clearly analytically continuable along any direction $\theta \in I = (-\pi, \pi)$: therefore, in the notation of section 2.2.2, we can safely claim $\hat{\varphi}^E \in \mathcal{N}(I)$, with $\gamma(\theta) = 0$. In fact, $\hat{\varphi}^E(\zeta)$ has no exponential growth, as $\hat{\varphi}^E(\zeta)$ actually decays for large $|\zeta|$, so $\gamma(\theta)$ can be taken to vanish identically. The Laplace transform $\mathcal{L}^\theta\hat{\varphi}^E$ is then well defined for all $\theta \in I$.

As the pole at $\zeta = -1$ is a simple pole, $\hat{\varphi}^E$ is a simple resurgent function in $\hat{\mathscr{R}}_\Omega^{simp}$, and therefore its Borel- pre-image $\tilde{\varphi}^E$ is simple resurgent formal power series. But what is Ω? Certainly, we must ensure

that $\omega = -1$ belongs to Ω: to make it closed under addition, we can simply set $\Omega = \mathbb{Z}$. Although infinitely many discrete points have been added, none of the assumptions made above is violated, as the points are not densely accumulating.
We can now compute the whole alien algebra of $\tilde{\varphi}^E$: namely, for every $\omega \in \Omega$, we compute the action of the (dotted) alien derivatives on $\tilde{\varphi}^E$ or equivalently on $\hat{\varphi}^E$. Since $\hat{\varphi}^E$ is a meromorphic function, the action of the alien derivatives is simply given by the residue of $\hat{\varphi}^E$ at the pole ω: namely, in the multiplicative and in the convolutive model respectively, we have

$$\begin{aligned} \Delta_{-1}\tilde{\varphi}^E &= 2\pi i\delta, \\ \Delta_{-1}\hat{\varphi}^E &= 2\pi i, \end{aligned} \tag{2.106}$$

while all the other alien derivatives simply vanish. The above equation summarises the whole alien algebra of $\tilde{\varphi}^E$ ($\hat{\varphi}^E$ in the convolutive model).

We intend now to find a well-defined function of z which we will eventually compute at $z = 1$. As $\mathrm{Re}(z) > 0$, we must ensure that $\mathrm{Re}(\zeta) > 0$ along the line of integration, so that the exponential $e^{-z\zeta}$ appearing in the Laplace integral is decaying. This leaves us the wide rage of directions $(-\pi/2, \pi/2)$. However, if we want to assign to the series (2.104) a real value when $z = 1$, as it is reasonable, we have to choose the line $\theta = 0$. Since this is not a Stokes line ($\theta \in I$), we can proceed straightforwardly and find the Borel sum $\varphi^E(z)$, holomorphic on the domain $\mathscr{D}(I) = \{z \in \mathbb{C} : -\frac{3}{2}\pi < \arg(z) < \frac{3}{2}\pi\}$ by

$$\varphi^E(z) := (\mathcal{L}^0\hat{\varphi}^E)(z) = \int_0^\infty d\zeta e^{-z\zeta}\frac{1}{1+\zeta}. \tag{2.107}$$

The domain $\mathscr{D}(I)$ should be understood as a subset of the Riemann surface of the logarithm $\tilde{\mathbb{C}}$: the Borel sum $\varphi^E(z)$ is then multi-valued whenever $\mathrm{Re}\, z < 0$. Although the origin is not included in $\mathscr{D}(I)$, we can observe that

$$\lim_{z\to\infty} \varphi^E(z) = 0 \tag{2.108}$$

along any direction $\theta \in I$: the point at the infinity can then be added to $\mathscr{D}(I)$. Slightly more intuitively, we can define $f^E(t) = \varphi^E(1/t)$, namely

$$f^E(t) = \int_0^\infty d\zeta e^{-\frac{\zeta}{t}}\frac{1}{1+\zeta} \tag{2.109}$$

with $f^E : \mathscr{D}(I) \to \mathbb{C}$, as the domain $\mathscr{D}(I)$ is mapped into itself upon letting $z \to 1/z$. Then we can extend the domain of f^E to $\mathscr{D}(I) \cup \{0\}$, by letting

$$f^E(0) := \lim_{t\to 0} f^E(t) = 0. \tag{2.110}$$

f^E is not analytic (holomorphic) at 0, but only C^∞: indeed its Taylor series at 0 is simply the initial formal power series $\tilde{\varphi}^E$ with $z \to 1/t$,

$$\tilde{f}^E = \sum_{n=0}^{\infty}(-1)^n n! t^{n+1}, \tag{2.111}$$

which has vanishing radius of convergence. The non-analyticity at $t = 0$ is then the imprint that the divergence of $\tilde{\varphi}^E$ leaves on its Borel sum. This property is strongly related to another feature of $\tilde{f}^E$, namely its monodromy. Thereby, we simply mean that $\tilde{f}^E$ is not periodic of period 2π, for a fixed radius, on the Riemann surface of the logarithm (for instance, $\tilde{f}^E(e^{i\pi}) \neq \tilde{f}^E(e^{-i\pi})$). Indeed, if that were the case in a neighbourhood of the origin, then $\tilde{f}^E$ would be analytic thereat (see e.g. [Bal94]), which we know it cannot be the case due to the divergence of $\tilde{\varphi}^E$.

Having resummed the initial divergent series (2.104) via the Laplace integral (2.107) , we can evaluate it numerically at $z = 1$ and find $\varphi^E(1) \approx 0.596...$ in agreement with Euler's result, obtained in a much different way. Before ending the section, it is interesting to observe that the Borel resummation can be used to make sense of oscillatory series even if the coefficients do not grow factorially. For instance, the notorious

$$1 - 1 + 1 - 1 + ... \tag{2.112}$$

can be viewed as the Borel sum of the series $\sum_{n=0}^{\infty}(-1)^n z^{-n-1}$ computed at $z=1$. The Borel transform is easily found to be $e^{-\zeta}$: as the formal power series is formally convergent, its Borel transform has infinite radius of convergence, in agreement with Proposition 2.1. Thus, the Borel sum of the above formal power series is given by the Laplace integral (in the direction $\theta = 0$):

$$\int_0^\infty d\zeta e^{-z\zeta-\zeta} = \frac{1}{1+z}, \tag{2.113}$$

which for $z=1$ returns the expected value of $1/2$. Similarly, one can consider

$$1-2+3-4+... \tag{2.114}$$

which can be associated with the series $\sum_{n=0}^{\infty}(-1)^n(n+1)z^{-n-2}$, whose Borel transform is $\zeta e^{-\zeta}$. As expected, this has infinite radius of convergence and exponential size no larger than 1. Then the Borel sum is given by the Laplace transform along $\theta = 0$

$$\int_0^\infty d\zeta e^{-z\zeta-\zeta}\zeta = \frac{1}{z+1}\int_0^\infty d\zeta e^{-z\zeta-\zeta} = \frac{1}{(z+1)^2} \tag{2.115}$$

which evaluates to $1/4$ for $z=1$. Both these examples were also discussed by Euler in [EA18], who found the values $1/2$ and $1/4$ with the method of the n-th differences sketched at the beginning of this section. For a rigorous discussion of the Euler resummation method and its relation with the Borel resummation method, we refer to Hardy's book [Har56].

2.4.2 Modified Euler series

Inspired by the previous results, we could also consider the formal power series

$$\tilde{\varphi} = \sum_{n=0}^{\infty} n! z^{-n-1}. \tag{2.116}$$

Interestingly, this is the formal power series solution to the differential equation

$$f'(z) + f(z) = \frac{1}{z} \tag{2.117}$$

and it is actually the same as the Euler series $\tilde{\varphi}^E$ after switching the sign of z be $z \to -z$. The Borel transform is then

$$\hat{\varphi} = \mathcal{B}\tilde{\varphi} = \sum_{n=0}^{\infty}\zeta^n = \frac{1}{1-\zeta}. \tag{2.118}$$

As before, we can set $\Omega = \mathbb{Z}$ and observe that $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega^{simp}$. Much in analogy to the previous case, we have the alien algebra:

$$\begin{aligned} \Delta_1\tilde{\varphi} &= -2\pi i\delta \\ \Delta_1\hat{\varphi} &= -2\pi i. \end{aligned} \tag{2.119}$$

If we now attempt to take the Laplace-transform along the real line $\mathcal{L}^0\hat{\varphi}$, we encounter a pole at $\zeta = 1$, which prevents the integration. Thus, $\theta = 0$ is a Stokes line, and therefore the two lateral sums $\mathscr{S}^\epsilon$ and $\mathscr{S}^{-\epsilon}$ will yield two different results, differing by an imaginary and exponentially suppressed quantity. Aware of the action of the only non-vanishing alien derivatives (2.119), we readily find the action of the Stokes automorphism on $\tilde{\varphi}$, which we can now view as a transseries of $\tilde{E}(\mathbb{Z})$:

$$\begin{aligned} \mathfrak{S}_0\tilde{\varphi} &= \exp\left(\sum_{\omega\in\Omega_0}\dot{\Delta}_\omega\right)\tilde{\varphi} = \exp\left(e^{-z}\Delta_1\right)\tilde{\varphi} = \left(1 + e^{-z}\Delta_1 + \frac{1}{2!}e^{-2z}\Delta_1^2 + ...\right)\tilde{\varphi} \\ &= \tilde{\varphi} + e^{-z}\Delta_1\tilde{\varphi} = \tilde{\varphi} - 2\pi i e^{-z} \in \tilde{E}(\mathbb{Z}), \end{aligned} \tag{2.120}$$

as all the higher powers of the alien derivatives vanish. Owing to the very definition of the Stokes automorphism (2.89), this means that

$$(\mathscr{S}^{\epsilon} - \mathscr{S}^{-\epsilon})\tilde{\varphi} = 2\pi i e^{-z}. \tag{2.121}$$

It is expedient to stress again that e^{-z} bears two different meanings in the previous equations: it is a *symbol* in the former (2.120), while it is a *function* in the latter (2.121). The difference between the two Laplace integrals can also be found by a contour integral along a path called *Hankel contour* H (see Figure 2.2a), enclosing the pole at $\zeta = 1$. The integral can be solved by the residue theorem, yielding

$$(\mathcal{L}^{\epsilon} - \mathcal{L}^{-\epsilon})\hat{\varphi} = \oint_H d\zeta e^{-z\zeta} \frac{1}{1-\zeta} = 2\pi i e^{-z}. \tag{2.122}$$

It is noteworthy that this result is telling us something meaningful about the solution to the differential equation (2.117): it is actually a solution to the homogeneous equation, which the power series solution could not have captured. Indeed, the exponential term e^{-z} is non-analytic at $z \to \infty$ and falls off faster than any other term in the power series.

How, then, do we find a sensible real-valued function $\varphi(z)$ that can be interpreted as the sum of $\tilde{\varphi}$? As explained in section 2.3.4, we need to resort to the median resummation. This can be written as

$$\begin{aligned} \varphi(z) &= \mathscr{S}^0_{med}\tilde{\varphi} = \mathscr{S}^{\epsilon} \circ \mathfrak{S}_0^{1/2}\tilde{\varphi} \\ &= \mathscr{S}^{\epsilon}\left(\tilde{\varphi} - \pi i e^{-z}\right) = \mathcal{L}^{\epsilon}\hat{\varphi} - \pi i e^{-z}\mathcal{L}^{\epsilon}\delta \\ &= \int_0^{\infty + i\epsilon} d\zeta e^{-z\zeta}\frac{1}{1-\zeta} - \pi i e^{-z}, \end{aligned} \tag{2.123}$$

where $\mathfrak{S}_0^{1/2}\tilde{\varphi}$ was computed by recalling the alien algebra (2.119) and the definition (2.97). The integral in (2.123) will have an imaginary part which can be computed following a half-circle of infinitesimal radius r centred at $\zeta = 1$ and passing *above* it, as shown in Figure 2.2b. Hence, setting $\zeta = 1 + re^{it}$, the imaginary part of the previous integral is

$$\begin{aligned} &\lim_{r\to 0}\int_{\pi}^{0} dt\, rie^{it}e^{-z(1+re^{it})}\frac{1}{1-(1+re^{it})} \\ &= \lim_{r\to 0} \; -ie^{-z}\int_{\pi}^{0} dt\, e^{-z(re^{it})} = i\pi e^{-z}, \end{aligned} \tag{2.124}$$

which, as expected, exactly cancels the transseries contribution added to $\tilde{\varphi}$ by the action of $\mathfrak{S}_0^{1/2}$ in (2.123). The full median resummation (2.123) is then given by the real part of the integral, which is

$$\varphi(z) = \lim_{r\to 0}\left[\left(\int_0^{1-r} d\zeta + \int_{1+r}^{\infty} d\zeta\right)\left(e^{-z\zeta}\frac{1}{1-\zeta}\right)\right]. \tag{2.125}$$

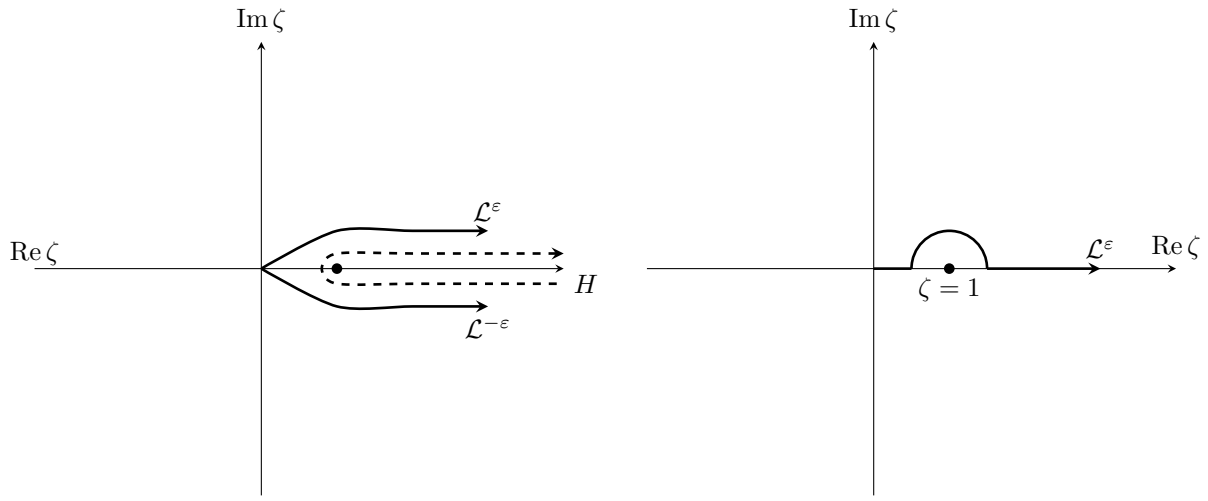
Note that the result is finite: the infinities close to $\zeta = 1$ are of opposite sign and cancel. Upon computing the above integral numerically we can evaluate the series

$$1 + 1! + 2! + 3! + ... \tag{2.126}$$

as $\varphi(1) \approx 2.7317$.

2.5 The Stirling series

The Euler series discussed earlier is arguably the simplest example of a Borel-summable formal power series. In this section, closely following the lines of [Sau14], we will describe in detail a slightly more complicated example: the Stirling series. This series can be understood as the power series providing the subleading corrections by which the Stirling approximation differs from the Gamma function. As it will be clarified in the next section, the Borel sum of the Stirling series is a definable function in the o-minimal structure $\mathbb{R}_{\mathscr{G}}$ described in [DS00] and reviewed in section 1.5.3: this implies that the Gamma function is definable in $\mathbb{R}_{\mathscr{G}}$ on the interval $(0, \infty)$.



(a) Laplace integration paths and the Hankel contour H. (b) Integration contour topologically equivalent to that of $\mathcal{L}^{\epsilon}$ on the left.

Figure 2.2: Borel plane with a pole at $\zeta = 1$ and integration contours for the lateral Laplace operators $\mathcal{L}^{\epsilon}$ and $\mathcal{L}^{-\epsilon}$.

2.5.1 Borel transform for the Stirling series

The Stirling series can be viewed as the formal ratio between the Gamma function and the Stirling approximation. For large integers n the Stirling approximation provides an estimate for $n!$:

$$n! \approx \sqrt{2\pi n}\, n^n e^{-n} \qquad n \to \infty. \tag{2.127}$$

It is also well known that $n! = \Gamma(1+n)$ for every integer n. The Gamma function then can be used to promote the factorial to a continuous function. After rewriting the above as $(n-1)! \approx \sqrt{2\pi} n^{n-1/2} e^{-n}$, we can rephrase it in terms of the continuous variable $x \in \mathbb{R}$ by

$$\Gamma(x) \approx \sqrt{2\pi}\, x^{x-1/2} e^{-x} \qquad x \to \infty. \tag{2.128}$$

The Stirling function $\lambda(z)$ is then defined to be the ratio between the Gamma function and the Stirling approximation function:

$$\lambda(z) = \frac{1}{\sqrt{2\pi}} z^{1/2-z} e^{z} \Gamma(z) \qquad z \in \mathbb{C}\backslash\mathbb{R}^{-} \tag{2.129}$$

where z is now a complex variable which must be understood in the limit $\mathrm{Re}(z) \to \infty$. A property of paramount importance of the Gamma function is its 'factorial' property

$$\Gamma(z+1) = z\Gamma(z), \tag{2.130}$$

from which a similar property will be inherited by the Stirling function $\lambda(z)$. It is straightforward to derive that one has

$$\lambda(z+1) = e\lambda(z)(1+z^{-1})^{-z-1/2}. \tag{2.131}$$

For our purposes, it will be convenient to define $\mu(z) := \log \lambda(z)$. Then from (2.129) it is easily inferred that $\mu(z)$ will obey

$$\mu(z) = -\frac{1}{2}\log(2\pi) - \left(z - \tfrac{1}{2}\right)\log(z) + z + \log\Gamma(z). \tag{2.132}$$

Rather than solving for $\lambda(z)$ and $\mu(z)$, we shall focus on their formal power series solutions of the equations (2.129) and (2.132): these are the Stirling series $\tilde{\lambda}$ and the logarithmic Stirling series $\tilde{\mu}$, respectively. Formally

we still have $\tilde{\mu}(z) = \log(\tilde{\lambda}(z))$: hence, from (2.131) we deduce that

$$\begin{aligned}\tilde{\mu}(z+1) &= \tilde{\mu}(z) + 1 - (z+1/2)\log\big(1+z^{-1}\big) \\ &=: \tilde{\mu}(z) + \tilde{\psi}(z),\end{aligned} \tag{2.133}$$

where $\tilde{\psi}$ can be written conveniently as

$$\tilde{\psi}(z) = -\frac{1}{2}\log\big(1+z^{-1}\big) + z(z^{-1} - \log\big(1+z^{-1}\big)). \tag{2.134}$$

This provides an equation for $\tilde{\mu}(z)$. In order to solve it, we calculate its Borel transform $\hat{\mu}$: recalling property 2 in Lemma 2.1, we have that, for $c = 1$,

$$\mathcal{B}[T_1\tilde{\mu}](\zeta) = e^{-\zeta}\hat{\mu}(\zeta), \tag{2.135}$$

hence we can use (2.133) to write $\hat{\mu}$ as

$$\hat{\mu}(\zeta) = \frac{1}{e^{-\zeta}-1}\hat{\psi}(\zeta), \tag{2.136}$$

where $\hat{\psi} = \mathcal{B}\tilde{\psi}$. The latter is then the object that we must compute in order to find $\hat{\mu}$.

By looking at (2.134), we realise that the first thing to compute is the Borel transform of the logarithm $\log\big(1+z^{-1}\big)$. This is:

$$\mathcal{B}\left(-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{1}{z^n}\right) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\zeta^{n-1} = \frac{1}{\zeta}(1-e^{-\zeta}) =: \hat{L}. \tag{2.137}$$

Recalling now the third point in Lemma 2.1, we are enabled to apply the Borel operator $\mathcal{B}$ to (2.134). We obtain:

$$\begin{aligned}\mathcal{B}\tilde{\psi} &= -\frac{1}{2}\hat{L}(\zeta) + \frac{d}{d\zeta}(1-\hat{L}(\zeta) \\ &= \frac{1}{2}\frac{e^{-\zeta}-1}{\zeta} + \frac{d}{d\zeta}\left(1+\frac{e^{-\zeta}-1}{\zeta}\right) \\ &= \frac{1}{2}\frac{e^{-\zeta}-1}{\zeta} - \frac{e^{-\zeta}-1}{\zeta^2} - \frac{e^{-\zeta}}{\zeta} =: \hat{\psi}(\zeta).\end{aligned} \tag{2.138}$$

Having computed $\hat{\psi}(\zeta)$, we but have to plug it in (2.136) and finally find

$$\hat{\mu}(\zeta) = \frac{1}{\zeta^2}\left\{-1 + \frac{\zeta}{2} + \frac{\zeta}{e^{\zeta}-1}\right\}. \tag{2.139}$$

We can notice that the bracketed quantity can be written in terms of the Bernoulli numbers B_{2k}: this allows us to write

$$\hat{\mu}(\zeta) = \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\zeta^{2k-2} \tag{2.140}$$

whence we can readily infer that $\tilde{\mu}$ is the series in powers of z^{-1} given by

$$\tilde{\mu} = \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}(2k-1)!\frac{1}{z^{2k-1}} = \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}\frac{1}{z^{2k-1}}, \tag{2.141}$$

which amounts to taking the *formal* Laplace transform of $\hat{\mu}$.

2.5.2 Borel plane and Stokes phenomenon

Having computed the Borel transform $\hat{\mu}(\zeta)$, we should now understand whether it has poles or not and, if yes, of what kind they are and where they are located in the Borel plane. A quick glance at (2.139) is sufficient to convince ourselves that the poles are located at $\zeta = 2\pi i m$, with $m \in \mathbb{Z}$. Observe though that, as we should expect of a Borel transform, there is no pole at $\zeta = 0$, as the power series (2.140) also indicates. Indeed, the Laurent series at zero is

$$\begin{aligned}\hat{\mu}(\zeta) &= \frac{-1}{\zeta^2} + \frac{1}{2\zeta} + \frac{1}{\zeta}\frac{1}{(\zeta + \zeta^2/2 + \zeta^3/6 + ...)} \\ &= \frac{-1}{\zeta^2} + \frac{1}{2\zeta} + \frac{1}{\zeta^2}\left(1 - \frac{\zeta}{2} + \frac{\zeta^2}{12} + ..\right) \\ &= \frac{1}{12} + o(\zeta)\end{aligned} \tag{2.142}$$

which is indeed the first term in the expansion (2.139): $B_2 = 1/6$, thus $\frac{B_2}{2!} = \frac{1}{12}$. Hence, we conclude that the poles are located at $2\pi i\mathbb{Z}^*$ and that the directions $\theta = \pm\frac{\pi}{2}$ are the Stokes lines associated with $\tilde{\mu}$. Furthermore, it can be easily argued that the poles are all simple: therefore, $\hat{\mu}(\zeta)$ is a meromorphic simple resurgent function, endlessly continuable in $\mathbb{C}\backslash 2\pi i\mathbb{Z}^*$. Moreover, it is easy to see from (2.139) that $\hat{\mu}(\zeta)$ is bounded along any direction (other than the Stokes lines $i\mathbb{R}^{\pm}$): it follows that the Laplace transform of $\hat{\mu}(\zeta)$, whenever it is defined, will yield a function holomorphic on a non-compact domain. Specifically, we have

Proposition 2.5. The logarithmic Stirling series $\tilde{\mu}$ is 1-summable along $\theta = 0$. Therefore, we have

$$\mu(z) = \int_0^{\infty} d\zeta e^{-z\zeta}\frac{1}{\zeta^2}\left\{-1 + \frac{\zeta}{2} + \frac{\zeta}{e^{\zeta} - 1}\right\} \tag{2.143}$$

and analytic and such that $\mu(z) \sim_1 \tilde{\mu}$ uniformly on a Sokal disc Π_0^0.

In fact, $\mu(z)$ admits $\tilde{\mu}$ as uniform asymptotic expansion on every closed subsector of $\mathbb{C}/\mathbb{R}^-$, which can be seen by rotating the line of integration of the Laplace integral.

The presence of two Stokes lines $i\mathbb{R}^{\pm}$ (at $\theta = \pm\frac{\pi}{2}$), introduces an ambiguity in the Laplace integral. Setting $I^+ = (-\frac{\pi}{2}, \frac{\pi}{2})$ and $I^- = (\frac{\pi}{2}, \frac{3\pi}{2})$, we will have two different types of Laplace integrals $\mathcal{L}^{I^+}$ and $\mathcal{L}^{I^-}$ depending on which interval (either I^+ or I^-) the direction θ of the line of integration belongs to. Thus, we will have two functions $\mu^+(z) = \mathscr{S}^{I^+}\tilde{\mu}$ and $\mu^-(z) = \mathscr{S}^{I^-}\tilde{\mu}$. Since $\hat{\mu}(\zeta)$ has no exponential growth, for every direction θ along which $\hat{\mu}$ is 1-summable, $\mathcal{L}^{\theta}\hat{\mu}$ is defined on $\Pi_0^{\theta} := \{z \in \mathbb{C} : \mathrm{Re}(ze^{i\theta}) > 0\}$: hence we infer that the domains of holomorphicity of μ^+ and μ^- are, respectively:

$$\begin{aligned}\mu^+(z): &\qquad \bigcup_{\theta\in I^+}\Pi_0^{\theta} = \mathbb{C}\backslash\mathbb{R}^-, \\ \mu^-(z): &\qquad \bigcup_{\theta\in I^-}\Pi_0^{\theta} = \mathbb{C}\backslash\mathbb{R}^+.\end{aligned} \tag{2.144}$$

Explicitly, the two Borel sums are given by:

$$\begin{aligned}\mu^+(z) = \mathcal{L}^{\theta}\hat{\mu} &= \int_0^{\infty e^{i\theta}} e^{-z\zeta}\hat{\mu}(\zeta) \qquad \theta \in I^+, \\ \mu^-(z) = \mathcal{L}^{\theta}\hat{\mu} &= \int_0^{\infty e^{i\theta}} e^{-z\zeta}\hat{\mu}(\zeta) \qquad \theta \in I^-.\end{aligned} \tag{2.145}$$

where $\mu^+(z)$ is simply the analytic continuation of the earlier $\mu(z)$, as $0 \in I^+$.

It is then natural to ask what the difference between these solutions is. In order to do so, we must distinguish two cases:

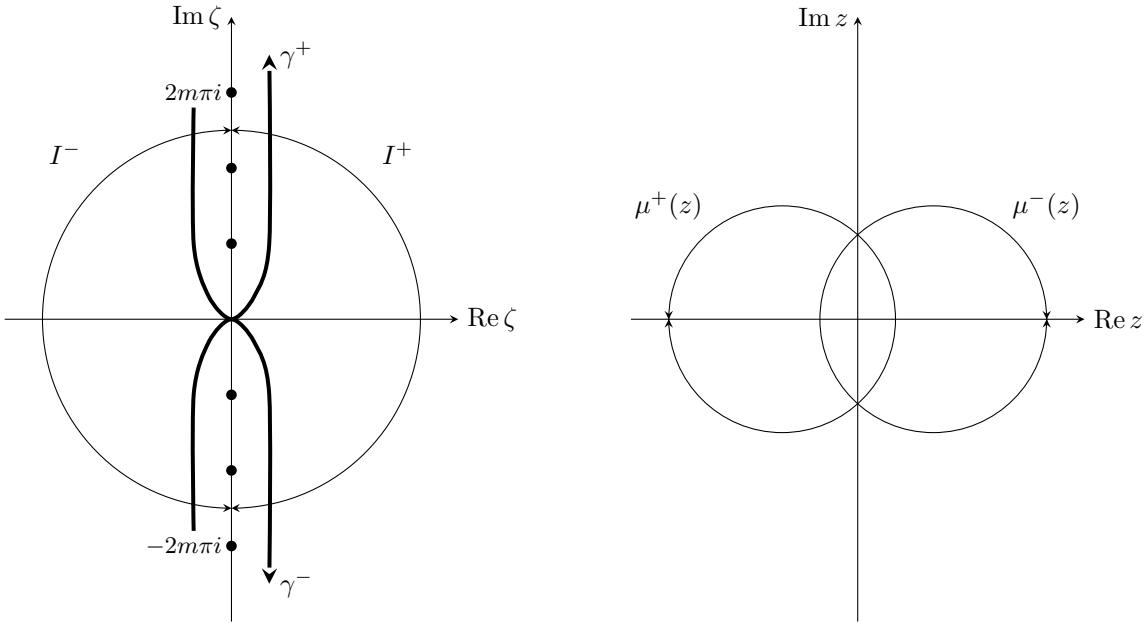
$\mathbf{Im}(z) < 0$: The lower complex plane corresponds to Laplace integrations taken along $\theta \in (0, \frac{\pi}{2}) \subset I^+$ or $\theta \in (\frac{\pi}{2}, \pi) \subset I^-$. Thus, we can compute the difference $\mu^+(z) - \mu^-(z)$ following the integration contour γ^+ in Figure 2.3a. Another way of seeing it is that the exponential $e^{-z\zeta}$ must be decaying: we must then require that $\mathrm{Re}(z\zeta) > 0$, which for $\mathrm{Im}(z) < 0$ enforces $\mathrm{Im}(\zeta) > 0$. Thus the contour γ^+ must be closed in the *upper* part of the Borel plane.
The integral can be computed easily with Cauchy theorem. Recalling that the residue of $\frac{1}{e^\zeta - 1}$ at $\zeta = 2\pi i m$, with $m \in \mathbb{Z}^*$, is 1, we have:

$$\begin{aligned}\mu^+(z) - \mu^-(z) &= \oint_{\gamma^+} d\zeta e^{-z\zeta} \hat{\mu}(\zeta) = \oint_{\gamma^+} d\zeta e^{-z\zeta} \left\{ -1 + \frac{\zeta}{2} + \frac{\zeta}{e^\zeta - 1} \right\} \frac{1}{\zeta^2} \\ &= 2\pi i \sum_{m=1}^{\infty} e^{-2\pi i\, mz} \frac{1}{2\pi i m} = \sum_{m=1}^{\infty} \frac{1}{m} e^{-2\pi i\, mz} \\ &= -\log\bigl(1 - e^{-2\pi i z}\bigr).\end{aligned} \tag{2.146}$$

$\mathbf{Im}(z) > 0$: Following the same thread of reasoning as above, we conclude that the integration must be taken along γ^- as in Figure 2.3a. Hence:

$$\begin{aligned}\mu^+(z) - \mu^-(z) &= \oint_{\gamma^-} d\zeta e^{-z\zeta} \hat{\mu}(\zeta) = \oint_{\gamma^-} d\zeta e^{-z\zeta} \left\{ -1 + \frac{\zeta}{2} + \frac{\zeta}{e^\zeta - 1} \right\} \frac{1}{\zeta^2} \\ &= -2\pi i \sum_{m=-1}^{-\infty} e^{-2\pi i\, mz} \frac{1}{2\pi i m} = \sum_{m=1}^{\infty} \frac{1}{m} e^{2\pi i\, mz} \\ &= -\log\bigl(1 - e^{2\pi i z}\bigr).\end{aligned} \tag{2.147}$$

It should be observed how the exponential corrections in (2.146) and in (2.147) are always decaying exponentials given the assumption made on the imaginary part of z.



(a) Borel plane for the logarithmic Stirling series $\hat{\mu}$. The integration contours γ^+ and γ^- are taken for $\mathrm{Im}(z) < 0$ and $\mathrm{Im}(z) > 0$ respectively.

(b) Domains of holomorphicity of $\mu^+(z)$ and $\mu^-(z)$.

Figure 2.3: Borel plane (left) and Laplace plane (right).

2.5.3 Alien algebra and Stokes automorphism

The computation of (2.146) and (2.147) amounts precisely to computing the Stokes automorphism for $\hat{\mu}$ (or, equivalently, for $\tilde{\mu}$). Since $\hat{\mu}(\zeta)$ is a meromorphic function with only simple poles on the imaginary axis (see Figure 2.3a), the action of the alien derivative $\Delta_{2\pi im}$ simply returns the residue at the pole $\zeta = 2\pi im$, multiplied by $2\pi i$. Hence we can equivalently write (in the convolutive and in the multiplicative model respectively) the full alien algebra of the logarithmic Stirling series:

$$\begin{aligned} \dot{\Delta}_{2\pi im}\hat{\mu} &= \frac{1}{m}\delta_{2\pi im}, \\ \dot{\Delta}_{2\pi im}\tilde{\mu} &= \frac{1}{m}e^{-2\pi i\, mz}. \end{aligned} \tag{2.148}$$

It is then clear that the action of any two alien derivatives on $\hat{\mu}$ or $\tilde{\mu}$ will yield 0. Therefore, the action of the Stokes automorphisms along the two Stokes lines reads:

$$\begin{aligned} \mathfrak{S}_{\frac{\pi}{2}}\hat{\mu} &= \exp\left(\sum_{m=1}^{\infty}\dot{\Delta}_{2\pi im}\right)\hat{\mu} = \hat{\mu} + \sum_{m=1}^{\infty}\frac{1}{m}\delta_{2\pi im}\ , \\ \mathfrak{S}_{\frac{\pi}{2}}\tilde{\mu} &= \exp\left(\sum_{m=1}^{\infty}\dot{\Delta}_{2\pi im}\right)\tilde{\mu} = \tilde{\mu} + \sum_{m=1}^{\infty}\frac{1}{m}e^{-2\pi i\, mz}\ ; \end{aligned} \tag{2.149}$$

and

$$\begin{aligned} \mathfrak{S}_{-\frac{\pi}{2}}\hat{\mu} &= \exp\left(\sum_{m=-1}^{-\infty}\dot{\Delta}_{2\pi im}\right)\hat{\mu} = \hat{\mu} + \sum_{m=-1}^{-\infty}\frac{1}{m}\delta_{-2\pi im} = \hat{\mu} - \sum_{m=1}^{\infty}\frac{1}{m}\delta_{2\pi im}\ , \\ \mathfrak{S}_{-\frac{\pi}{2}}\tilde{\mu} &= \exp\left(\sum_{m=-1}^{-\infty}\dot{\Delta}_{2\pi im}\right)\tilde{\mu} = \tilde{\mu} + \sum_{m=-1}^{-\infty}\frac{1}{m}e^{-2\pi i\, mz} = \tilde{\mu} - \sum_{m=1}^{\infty}\frac{1}{m}e^{2\pi i\, mz}. \end{aligned} \tag{2.150}$$

These are exactly the same equations previously found in (2.146) and (2.147). In fact, recalling the definition of the Stokes automorphism in (2.89) one recovers from (2.149)

$$\begin{aligned} \mu^{+}(z) &= \mathscr{S}^{\frac{\pi}{2}-\varepsilon}\tilde{\mu} = \mathscr{S}^{\frac{\pi}{2}+\varepsilon}\mathfrak{S}_{\frac{\pi}{2}}\tilde{\mu} = \mathscr{S}^{\frac{\pi}{2}+\varepsilon}\tilde{\mu} + \sum_{m=1}^{\infty}\frac{1}{m}e^{-2\pi i\, mz} \\ &= \mu^{-}(z) + \sum_{m=1}^{\infty}\frac{1}{m}e^{-2\pi i\, mz} \end{aligned} \tag{2.151}$$

which is (2.146); similarly from (2.150) we retrieve

$$\begin{aligned} \mu^{-}(z) &= \mathscr{S}^{-\frac{\pi}{2}-\varepsilon}\tilde{\mu} = \mathscr{S}^{-\frac{\pi}{2}+\varepsilon}\mathfrak{S}_{-\frac{\pi}{2}}\tilde{\mu} = \mathscr{S}^{-\frac{\pi}{2}+\varepsilon}\tilde{\mu} - \sum_{m=1}^{\infty}\frac{1}{m}e^{2\pi i\, mz} \\ &= \mu^{+}(z) - \sum_{m=1}^{\infty}\frac{1}{m}e^{2\pi i\, mz} \end{aligned} \tag{2.152}$$

which is (2.147).

2.5.4 Analytic continuation and poles of the Stirling function λ

We now turn our attention back to $\lambda(z)$. First, let us note that, since $\hat{\mu}(\zeta)$ is an even function, as it is clear from (2.140), the substitution $\zeta \to -\zeta$ defines the following relation. For $\theta \in I^{-}$ we have that:

$$\mu^{-}(z) = \int_0^{\infty e^{i\theta}} d\zeta e^{-z\zeta}\hat{\mu}(\zeta) = -\int_0^{\infty e^{i\theta+\pi}} d\zeta e^{z\zeta}\hat{\mu}(\zeta) = -\mu^{+}(-z), \tag{2.153}$$

which holds for $z \in \mathbb{C}\backslash\mathbb{R}^+$, namely on the domain of holomorphicity of the $\mu^-(z)$.

We can now move back to $\lambda(z) = e^{\mu^+(z)}$ for $z \in \mathbb{C}\backslash\mathbb{R}^-$. The reason why we choose to exponentiate $\mu^+(z)$ rather than $\mu^-(z)$ is that we want to interpret $\lambda(z)$ as the analytic continuation to the complex plane of a real valued function $\lambda(x) = (2\pi)^{-1/2}x^{-x+1/2}e^x\Gamma(x)$, which, for x integer, has the desired interpretation as the ratio between $(n-1)!$ and the Stirling approximation thereof $\sqrt{2\pi}\, n^{n-1/2}e^{-n}$. We then rule out $\mu^-(z)$, which is not defined on the positive real line.
We show now that we can find an analytic continuation of $\lambda(z)$ on $\mathbb{R}^-$. Consider $\mathrm{Im}(z) < 0$. Then, according to (2.146), and availing ourselves of the above result (2.153), we have

$$\lambda(z)\lambda(-z) = e^{\mu^+(z)+\mu^+(-z)} = e^{\mu^+(z)-\mu^-(z)} = e^{-\log\left(1-e^{-2\pi iz}\right)} = \frac{1}{1-e^{-2\pi iz}}. \tag{2.154}$$

Notice that both the second and the third equality signs are justified on the lower complex plane: the former is valid on $\mathbb{C}\backslash\mathbb{R}^+$, the latter again on $\mathrm{Im}(z) < 0$. Similarly, one has for $\mathrm{Im}(z) > 0$:

$$\lambda(z)\lambda(-z) = e^{\mu^+(z)+\mu^+(-z)} = e^{\mu^+(z)-\mu^-(z)} = e^{-\log\left(1-e^{2\pi iz}\right)} = \frac{1}{1-e^{2\pi iz}}. \tag{2.155}$$

So, it is possible to analytically continue $\lambda(z)$ from the lower complex plane into the upper and *vice versa*; yet, the two analytic continuations are different. Nevertheless, as the above formulas indicate, they both have simple poles at the negative integers; we will now to calculate the residues at such poles and thereby infer those of the Gamma function. Let us start again from $\mathrm{Im}(z) < 0$. For an integer $k \in \mathbb{N}$, we have

$$\begin{aligned}\mathrm{Res}_{z=-k}(\lambda(z)) = \mathrm{Res}_{z=k}(\lambda(-z)) &= \mathrm{Res}_{z=k}\left(\frac{1}{1-e^{-2\pi iz}}\frac{1}{\lambda(z)}\right) \\ &= \frac{1}{2\pi i}\left(\frac{1}{\sqrt{2\pi}}k^{-k+1/2}\Gamma(k)e^k\right)^{-1} = \frac{-i}{\sqrt{2\pi}}\frac{k^{k+1/2}}{k!}e^{-k}\end{aligned} \tag{2.156}$$

where in the last line we have argued that the residue of $\frac{1}{1-e^{-2\pi iz}}$ at k is $\frac{1}{2\pi i}$ and then we have substituted $\Gamma(k) = (k-1)!$, since k is a positive integer. Similarly, for $\mathrm{Im}(z) > 0$, by following the same computations we find

$$\mathrm{Res}_{z=-k}(\lambda(z)) = \frac{i}{\sqrt{2\pi}}\frac{k^{k+1/2}}{k!}e^{-k}, \tag{2.157}$$

since now the residue of $\frac{1}{1-e^{2\pi iz}}$ is $\frac{-1}{2\pi i}$. This results allow us to obtain the residue of the Gamma function at the negative integers. Indeed, we must have

$$\mathrm{Res}_{z=-k}(\lambda(z)) = \frac{1}{\sqrt{2\pi}}(-k)^{k+1/2}e^{-k}\,\mathrm{Res}_{z=-k}(\Gamma(z)) \tag{2.158}$$

which, by comparison with (2.156) and (2.157), tells us that

$$\mathrm{Res}_{z=-k}(\Gamma(z)) = \frac{(-1)^k}{k!}. \tag{2.159}$$

Notice how, to obtain a unique result for the poles of $\Gamma(z)$, we must choose for $\mathrm{Im}(z) < 0$ the branch of the square root $(-1)^{1/2} = -i$, while for $\mathrm{Im}(z) > 0$ we must have $(-1)^{1/2} = i$. Only then will (2.158) match both (2.156) and (2.157) while at the same time fixing the correct vale for the residue of Γ at the negative integers, reported in (2.159).

It is quite remarkable how alien calculus enables us to gain information about the poles of the Gamma function at the negative integers. The only information about the Gamma function that we have introduced into our formulas are the property (2.130) and the asymptotic behaviour of $\Gamma(x)$ for *large positive* x: yet, such information proves to be sufficient to know the behaviour of $\Gamma(x)$ at its poles, located at *arbitrarily large negative integers*.

2.5.5 Reflection formula

For the sake of completeness, we provide a proof of the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{2.160}$$

by availing ourselves of the previous results. We can start by rewriting the left-hand side by using the definition of $\lambda(z)$ as given in (2.129), obtaining

$$\Gamma(z)\Gamma(1-z) = \frac{2\pi}{e} z^{z-1/2}(1-z)^{-z+1/2}\lambda(z)\lambda(1-z). \tag{2.161}$$

Hearkening back to (2.131), we can write $\lambda(z-1)$ in terms of $\lambda(z)$ by

$$\lambda(1-z) = e(1-z^{-1})^{z-1/2}\lambda(-z). \tag{2.162}$$

We can then exploit either (2.154) or (2.155). Let us choose the former, so $\mathrm{Im}(z) < 0$. Plugging everything into (2.161), we eventually have:

$$\Gamma(z)\Gamma(1-z) = \frac{2\pi}{1-e^{-2\pi i z}}(-1)^{z-1/2}. \tag{2.163}$$

Because we are dealing with $\mathrm{Im}(z) < 0$, we choose $(-1)^{z-1/2} = e^{-\pi i(z-1/2)} = ie^{-\pi i z}$. So, finally,

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= \frac{2\pi i}{1-e^{-2\pi i z}} e^{-\pi i z} = \frac{\pi}{\frac{e^{i\pi z}-e^{-i\pi z}}{2i}} \\ &= \frac{\pi}{\sin(\pi z)}.\end{aligned} \tag{2.164}$$

Had we chosen $\mathrm{Im}(z) > 0$ we would have found instead

$$\Gamma(z)\Gamma(1-z) = \frac{2\pi}{1-e^{2\pi i z}}(-1)^{z-1/2}, \tag{2.165}$$

which gives consistently the same result only if we take $(-1)^{z-1/2} = e^{\pi i(z-1/2)} = -ie^{-\pi i z}$.

2.6 Tameness and Borel summability

In this final section, we aim at showing how Borel summability is connected with o-minimality. More precisely, we will show that the o-minimal structures $\mathbb{R}_{\mathscr{G}}$ and $\mathbb{R}_{\mathscr{G},\exp}$ described in section 1.5.3 are the natural o-minimal structures that can host the Borel sums of Borel-summable formal power series. The Gevrey functions described therein, as the name might suggest, are nothing else than the Borel sums of Borel-summable formal power series. This idea will be key to show, in the following chapters, the tameness of some partition functions: as for now though, we shall limit ourselves to elaborate further on the examples of the Euler series and the Stirling series, introduced in the previous sections.

2.6.1 The quasi analytic family of Gevrey functions

Here we wish to match the notations introduced in section 1.5.3 with those of section 2.2 about Borel summability and motivate the assumptions made about the definable functions of $\mathbb{R}_{\mathscr{G}}$. Consider $f(x) = f_1(x)$ in (1.47). Then $f(t)$ is defined on a sector $S(R,\phi,\kappa)$ for some radius R and $f(x)$ is its real-valued restriction onto $[0,R]$. We want now to argue why the assumptions made in section 1.5.3 imply that $f(x)$ is the Borel sum of a Borel-summable formal power series in the direction $\theta = 0$.

First, following [Lod14], we generalise the notion of uniform asymptotic expansion to a non-uniform expansion.

Definition 2.13 (Asymptotic expansion)**.** Consider a function $f(t)$ holomorphic on an open sector $S(R,\alpha,\beta) = \{t \in \mathbb{C}^* : 0 < |t| < R,\ \alpha < \arg(t) < \beta\}$. A function $f(t)$ is said to admit an asymptotic expansion $\sum_{n=0} a_n t^n$ in S if, for every closed[4] subsector of S, namely for every $S' = \{t \in \mathbb{C}^* : 0 < |t| \leq R',\ \alpha' \leq \arg(t) \leq \beta'\}$ with $\alpha < \alpha'$, $\beta' < \beta$, $R' < R$, one has

$$\left| f(t) - \sum_{n=0}^{N} a_n t^n \right| \leq C_{S'} t^{N+1} \tag{2.166}$$

where C'_S is a constant which depends on N, α', β' and R'.

This definition is clearly a generalisation of Definition 2.5. The asymptotic expansion is uniform if the above bounds hold on the whole sector S (with a constant C_N depending only on N). This definition can be specialised to the case of a Gevrey class $1/\kappa$.

Definition 2.14. The asymptotic expansion in (2.166) is of Gevrey class $1/\kappa$ if we can write

$$C_{S'} = A_{S'} B_{S'}^N (N!)^\kappa \tag{2.167}$$

for two constants $A_{S'}, B_{S'}$ depending on α', β', R'.

Again, a uniform asymptotic expansion of order κ, as in Definition 2.7, is found when we can find two constants A, B for which the above bound is valid on the whole $S(R,\alpha,\beta)$.

We now recall Propositions 2.2.11 and 2.3.11 of [Lod14]. The former provides a necessary and sufficient condition for a holomorphic function on a sector S to admit an asymptotic expansion; the latter, on the other hand, specialises the condition to an asymptotic expansion of Gevrey class $1/\kappa$.

Proposition 2.6. Given $f(t)$ holomorphic on S, (where S is an open sector $S(R,\alpha,\beta) = \{t \in \mathbb{C}^* : 0 < |t| < R,\ \alpha < \arg(t) < \beta\}$ as above), $f(t)$ admits an asymptotic expansion on S if and only if, for every closed subsector S' of S, and for every $n \in \mathbb{N}$ there exist a_n such that

$$\lim_{t \to 0} \frac{f^{(n)}(t)}{n!} = a_n \tag{2.168}$$

where the limit is intended on all the possible directions of S'.

Proposition 2.7. Let $f(t)$ as in Proposition 2.6. Then $f(t)$ admits an asymptotic expansion of Gevrey class $1/\kappa$ (of order κ) on a sector S if and only if, for every closed subset S' of S as above, there exist $A'_{S'}, B'_{S'}$ such that, for every $n \in \mathbb{N}$

$$\left| \frac{f^{(n)}(t)}{n!} \right| \leq A'_{S'} (B'_{S'})^n (n!)^\kappa. \tag{2.169}$$

where again $A'_{S'}, B'_{S'}$ depend on n, α', β', R'.

We then understand that the first requirement of Section 1.5.3 intends to guarantee the existence of an asymptotic expansion for the holomorphic function $f(t)$ on the sector $S(R,\phi,\kappa)$ bisected by the direction $\theta = 0$; the second requirement, on the other hand, ensures that the asymptotic series of $f(t)$ is actually of Gevrey class $1/\kappa$. Moreover, the asymptotic expansions are uniform, as the constants mentioned in section 1.5.3 are valid on the whole sector $S(R,\phi,\kappa)$.

These two requirements are still insufficient to ensure that $f(t)$ is Borel-summable along the line $\theta = 0$. The key point is the further demand that ϕ be larger that $\pi/2$. Together with the previous assumptions,

[4] Closed in $\mathbb{C}^*$.

$\phi > \pi/2$ ensures that Nevanlinna-Sokal theorem 2.6 applies. This can be simply argued by observing that the sector $S(R, \phi, \kappa)$ contains the Sokal disc D_c^κ with $c = R^{-1/\kappa}$:

$$D_c^\kappa := \left\{ t \in \mathbb{C} : \operatorname{Re} t^{-1/\kappa} > R^{-1/\kappa} \right\} = \left\{ t \in \mathbb{C} : 0 < |t| < R \cos^\kappa \left(\frac{\theta}{\kappa} \right) \right\}, \tag{2.170}$$

which finally ensures that $f(t)$ is κ-summable in the direction $\theta = 0$ and by Nevanlinna-Sokal theorem, is associated with a unique asymptotic expansion whose Borel sum yields $f(t)$. The algebra of functions $f(t)$ satisfying the requirements of section 1.5.3 is then quasi-analytic.
Requiring that $f(t)$ be analytic and satisfy uniform Gevrey constraints on the domain $S(R, \phi, \kappa)$, larger than the minimal domain given by Sokal disc D_c^κ (with $c = R^{-1/\kappa}$) is not restrictive. This can be understood in terms of Lemma 1 of Section 3.2 in [Bal94]. For simplicity, let us fix $\kappa = 1$. Then:

> **Lemma 2.2.** Given an asymptotic series $\tilde{\varphi}$ is 1-summable in a given direction d, it is also be 1-summable in the directions comprised between $d + \varepsilon$ and $d - \varepsilon$, for $\varepsilon > 0$ small enough.

This means that, if $f(t)$ is the Borel sum of $\tilde{\varphi} \in \mathbb{C}[[t]]_1$ along $d = 0$, and therefore is analytic on the Sokal disc $D^1_{c(\theta)}(\theta) = \{t \in \mathbb{C} : \operatorname{Re}(e^{i\theta} t^{-1}) < c(\theta)\}$ with $\theta = 0$, it will also be analytic on

$$\mathscr{D}(\varepsilon) = \bigcup_{\theta \in (-\varepsilon, \varepsilon)} D^1_{c(\theta)}(\theta) \tag{2.171}$$

where $c(\theta)$ is such that the Borel transform $\hat{\varphi} = \mathcal{B}\tilde{\varphi}$ is bounded by $|\hat{\varphi}(\zeta e^{i\theta})| \leq C(\theta) e^{c(\theta)|\zeta|}$ for every θ in the open arc of directions $I := (-\varepsilon, \varepsilon)$. We use here a similar notation to that of (2.57) but observe that $\mathscr{D}(\epsilon)$ is the same as $\mathscr{D}(I)$, with I as above, only after mapping $z \to 1/t$; it will be clear from context whether we are working in neighbourhoods of ∞ or of 0. The domain $\mathscr{D}(\varepsilon)$ clearly contains a sector $S(R, \phi, 1)$ of opening larger than π for appropriate R.

Definable functions always take values on the real line $\mathbb{R}$ and map them to real values. To ensure that the Borel sum in the direction $\theta = 0$, $f(x)$, take real values when $x \in \mathbb{R}$, we can simply request that the associated formal power series $\tilde{\varphi}$ have real coefficients, namely $\tilde{\varphi} \in \mathbb{R}[[x]]_{1/\kappa}$. Thus, recalling that in section 1.5.3 we have also requested that $0 \leq \kappa \leq 1$, we are led to state a key fact, upon which all the results presented in the following chapters will depend on:

> **Observation 2.1.** *The Borel sum $f(x)$ of a formal power power series $\tilde{\varphi} \in \mathbb{R}[[x]]_1$ of the small variable x of Gevrey class 1 and 1-summable in the direction $\theta = 0$, restricted on a closed interval $[0, R]$, is a tame function, definable in the o-minimal structure $\mathbb{R}_{\mathscr{G}}$.*

As an example, let us hearken back to section 2.4 about the Euler series. The Euler function $f^E(t)$ defined in (2.109) for a small complex variable t can be turned into a definable function of $\mathbb{R}_{\mathscr{G}}$, as it is the Borel sum of a 1-summable formal power series, i.e. the Euler series, with real coefficients $a_n = (-1)^n n!$. As stated explicitly in [DS00], $f^E(t)$ satisfies the assumptions of section 1.5.3: more precisely, it belongs to $\mathscr{G}(R, \phi, 1)$ for every $R \geq 0$ and for every $\phi \in (0, \pi)$. Therefore $f^E(x) : [0, R] \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathscr{G}}$.
In the next section, we generalise our Observation 2.1 to κ larger than 1.

2.6.2 Reduction to 1-summability

The theorem in [DS00] requires $0 < \kappa \leq 1$, apparently limiting the choice of Gevrey class. We illustrate here how this is not quite restrictive. A similar and more extensive treatment can be found in [Lod14], Section 2.3.

Consider an asymptotic series $\tilde{\varphi} = \sum_{n=0}^{\infty} a_n t^{n+1}$ of Gevrey class $1/\kappa$, with $\kappa > 1$. Then we have $|a_n| \leq AB^n(n!)^\kappa$. Yet, consider now a new formal series $\tilde{\chi} = t^{1-\kappa}\tilde{\varphi} \circ t^\kappa$. Then we can write

$$\tilde{\chi} = \sum_{m=0}^{\infty} b_m t^{m+1} \tag{2.172}$$

where now $b_m = 0$ if $m \neq \kappa n$ for all n, and $b_m = a_n$ when $m = \kappa n$ for some n. Then

$$|b_m| \leq AB^{\frac{m}{\kappa}} \left(\left(\frac{m}{\kappa}\right)!\right)^\kappa \tag{2.173}$$

which for the appropriate rescaled constants $\tilde{A}$, $\tilde{B}$, is equivalent to $|b_m| \leq \tilde{A}\tilde{B}^m(m!)$.
It can easily be seen that the Borel transforms $\hat{\chi}(\zeta) = \mathcal{B}_1\tilde{\chi}$ and $\hat{\varphi}(\zeta) = \mathcal{B}_\kappa\tilde{\varphi}$ are related by $\hat{\chi}(\zeta) = \hat{\varphi}(\zeta^\kappa)$. The pole structure in the Borel plane, then, *does indeed* change: every Stokes line of $\tilde{\chi}$ spawns κ new equally spaced Stokes lines for $\tilde{\varphi}$. Nevertheless, if $\hat{\chi}(\zeta)$ does not have poles on $\mathbb{R}^+$, certainly $\hat{\varphi}(\zeta)$ will also have no poles on $\mathbb{R}^+$. Furthermore, if $\hat{\varphi}(\zeta)$ has exponential size no larger than $1/\kappa$ on $\mathbb{R}^+$, then $\hat{\chi}(\zeta)$ has exponential size no larger than 1 on $\mathbb{R}^+$. More precisely, if there exist two positive constants A, c and a half-strip $S_\delta := S_\delta^0$ as given by (2.51) where, for every $\zeta \in S_\delta$, $|\hat{\varphi}(\zeta)| \leq A\exp\left(c|\zeta|^{1/\kappa}\right)$, then clearly $|\hat{\varphi}(\zeta)| \leq A\exp(c|\zeta|)$ for every $\zeta \in S_\delta$. Thus, we conclude then that a κ-summable formal power series $\tilde{\varphi}$ can be turned by an appropriate algebraic change of variable, into a 1-summable formal power series $\tilde{\chi}$.
If $\tilde{\chi}$ is 1-summable, we can compute its Borel sum $\chi(t)$ and claim that it is definable in $\mathbb{R}_{\mathscr{G}}$, by virtue of the Observation 2.1. But the Borel sum of $\tilde{\varphi}$, $\varphi(t)$, is nothing else than $t^{1-p}\chi(t^p)$, and because the change of variable is algebraic, $\varphi(t)$ is also definable in $\mathbb{R}_{\mathscr{G}}$. Indeed, algebraic functions are definable in any o-minimal structure and, recalling Lemma 1.3, the composition of definable functions is still definable. The conclusion of this section is then:

Observation 2.2. *The Borel sum $f(x)$ of a formal power power series $\tilde{\varphi} \in \mathbb{R}[[x]]_{1/\kappa}$ of the small variable x of Gevrey class $1/\kappa$ and κ-summable in the direction $\theta = 0$, restricted on a closed interval $[0, R]$, is a tame function, definable in the o-minimal structure $\mathbb{R}_{\mathscr{G}}$.*

2.6.3 Tameness of the Stirling function

The Borel sum $\mu(z)$ and the Gamma function

Let us now explore the consequences of Observation 2.1 on the Stirling series. Having argued that the poles in the Borel plane for $\tilde{\mu}$ (2.141) lie on the imaginary axis, and that the the Borel transform (2.139) has no exponential growth, we concluded that $\tilde{\mu}$ is 1-summable: its Borel sum $\mu(z)$, computed by taking the Laplace integral for $\theta = 0$, is given in Proposition 2.5 and is analytic on Π_0^0, whereon $\tilde{\mu}$ is its uniform asymptotic expansion. Recall though that, as stated in (2.145), this domain can be enlarged to $\mathbb{C}\backslash\mathbb{R}^-$ by rotating the line of integration of the Laplace transform in the domain $I^+ := (-\frac{\pi}{2}, \frac{\pi}{2})$. By virtue of Proposition 2.6, the limits

$$\lim_{z\to\infty} \mu^{(n)}(z) \tag{2.174}$$

are known to exist for every n and are given by the coefficients in (2.141) multiplied by the appropriate factorial factor. The Gevrey functions of $\mathscr{G}(R, \phi, 1)$ though, are defined by their analytic continuations on neighbourhoods of the origin: we cannot yet claim directly that $\mu(x)$ is definable in $\mathbb{R}_{\mathscr{G}}$. Still, the function

$$f : [0, 1] \to \mathbb{R} \tag{2.175}$$
$$f(x) = \begin{cases} \mu(1/x) & x \neq 0 \\ \lim_{x\to 0} \mu(1/x) & x = 0 \end{cases}$$

must be definable in $\mathbb{R}_{\mathscr{G}}$, as now the limit of each n-th derivative for $x \to 0$ exists, and it obeys uniform 1-Gevrey bounds on every closed subsector of the neighbourhood of 0

$$\mathscr{D}(\pi) = \bigcup_{\theta\in I^+} D_1^1(\theta) = \{x \in \mathbb{C}^*\backslash\mathbb{R}^- : |x| \leq 1\}, \tag{2.176}$$

with the notations of the previous sections. Hence, being related to the definable function f algebraically, $\mu(x)$ must be also definable in $\mathbb{R}_{\mathscr{G}}$ on to $[1,\infty)$. Moreover, $\mu(x)$ is simply analytic on $(0,1)$ and analytically continuable through 0. Hence $\mu(x)$ restricted to $(0,1)$ is definable in $\mathbb{R}_{\rm an}$, which is a substructure of $\mathbb{R}_{\mathscr{G}}$. Recalling now (2.132), we will have that

$$\Gamma(x) = \exp\circ\big((x-\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log(2\pi) + \mu(x)\big) \tag{2.177}$$

which implies that $\Gamma : (0,\infty) \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathscr{G},\exp}$, as stated in [DS00] (recall how compositions and products preserve definability by virtue of Lemma 1.3).

Stokes lines of $\tilde{\mu}$

Here we wish to use the logarithmic Stirling series $\tilde{\mu}$ to show why Stokes lines impair the hypothesis made in section 1.5.3. For instance, let us focus on the Stokes line $\theta = \pi/2$ and convince ourselves that we cannot find a sector of opening larger than π and centred on the line $i\mathbb{R}^+$, for which the hypothesis of section $\mathbb{R}_{\mathscr{G}}$ are satisfied for either Borel sum $\mu^+(z)$ or $\mu^-(z)$ in (2.145). Before moving on, let us sum up some facts that we learnt in section 2.5.

1. $\tilde{\mu}$ is not 1-summable along $\theta = \pi/2$ as the Borel transform $\hat{\mu}$ (2.139) has poles on this line.
2. The Laplace integrals can be taken either side of the Stokes line, either for $\theta \in I^+$ or $\theta \in I^-$. The Borel sums, $\mu^+(z)$ and $\mu^-(z)$ respectively, are then holomorphic respectively on $\mathbb{C}\backslash\mathbb{R}^-$ and $\mathbb{C}\backslash\mathbb{R}^+$: hence they are holomorphic on a sectors centred on $i\mathbb{R}^\pm$ of opening angle *infinitesimally smaller* than π.
3. We could then wonder if we can analytically continue them through the real lines $\mathbb{R}^+$ and $\mathbb{R}^-$. Equations (2.146) and (2.147) teach us that *we can*: there are two ways to do it, depending on whether we start from the upper or the lower complex plane. If we mean to find an analytic continuation for $\mu^+(z)$ through the negative real line $\mathbb{R}^-$, we have an ambiguity: we can either choose it to be $\mu^-(z) - \log\big(1-e^{-2\pi i z}\big)$ or $\mu^-(z) - \log\big(1-e^{2\pi i z}\big)$, depending on whether the path along which the analytic continuation is performed starts, respectively, on the upper plane $\mathrm{Im}(z) < 0$ or on $\mathrm{Im}(z) > 0$. This is in fact a monodromy caused by a branch cut: the analytic continuation of $\mu^+(z)$ starting from $\mathrm{Im}(z) < 0$ and crossing $\mathbb{R}^-$ does not match the value it takes at $\mathrm{Im}(z) > 0$. The same applies to the analytic continuation of $\mu^-(z)$ through $\mathbb{R}^+$.
4. The corrections $\log\big(1-e^{-2\pi i z}\big)$ and $\log\big(1-e^{2\pi i z}\big)$ are decaying exponentials on $\mathrm{Im}(z) < 0$ and $\mathrm{Im}(z) > 0$ respectively, whose derivatives are identically vanishing at $z \to \infty$: this means that both analytic continuations of $\mu^+(z)$ to $\mathbb{R}^-$ have the same Taylor series at infinity.
5. The Taylor series at infinity of both $\mu^+(z)$ and $\mu^-(z)$ (and of all their analytic continuations) are the same: they are nothing else than the asymptotic series $\tilde{\mu}$ given by (2.141).

To compare with o-minimality results in section 1.5.3, let $t = 1/z$. Since the domains of $\mu^+(z)$ and $\mu^-(z)$ are not bounded, the domains of $\mu^+(1/t)$ and $\mu^-(1/t)$ are again exactly those shown in (2.3b).
Consider now an algebra $\mathcal{A}$ of one-variable holomorphic functions defined on a sector centred on $i\mathbb{R}^+$

$$S(R,\phi) = \Big\{t \in \mathbb{C} : 0 < |t| < R, -\phi + \frac{\pi}{2} < \arg(t) < \phi + \frac{\pi}{2}\Big\} \tag{2.178}$$

for some $R > 0$, in analogy to the algebra of holomorphic functions $\mathscr{G}(R,\phi,1)$, holomorphic on a sector $S(R,\phi,1) = \{t \in \mathbb{C} : 0 < |t| < R, |\arg t| < \phi\}$. We also assume that, for any function $f \in \mathcal{A}$, $\lim_{t\to 0} f^{(n)}(t)$ exists and we declare that:

$$f(0) = \lim_{t\to 0} f(t). \tag{2.179}$$

f is then defined on $S(R,\phi) \cup \{0\}$, smooth at 0 but not necessarily analytic there. According to Watson-Nevanlinna theorem (see the introduction of [DS00] and [Bal94]: it is a very close relative of Nevanlinna-Sokal theorem 2.6 once Lemma 2.2 is taken into account), an algebra of functions with the aforementioned

properties can be quasi-analytic only if the angle ϕ is larger than $\pi/2$ (so that the overall opening of the sector is larger than π). Our detailed knowledge of the Stirling series then throws light on why it must be the case. Were we allowed to consider holomorphic functions on a sector of opening *smaller* than π, then our algebra would contain *both* $\mu^+(1/t)$ and $\mu^-(1/t)$: as pointed out earlier, these have the same Taylor series at 0 (namely, at $z = \infty$) and therefore the Taylor map *would not* be injective on the algebra $\mathcal{A}$. The opening angle of the sector being larger than π ensures then that the bisecting line $i\mathbb{R}^+$ of the sector $S(R, \phi)$ is not a Stokes line for any function in $\mathcal{A}$: this would impair quasi-analyticity, as the Stokes phenomenon implies the existence of two holomorphic functions with the same Taylor series.

We might now wonder why we cannot analytically continue $\mu^+(1/t)$ to $\mathbb{R}^-$ and $\mu^-(1/t)$ to $\mathbb{R}^+$, in order to obtain holomorphic functions on a larger sector of opening larger than π. To see this, let us rewrite the equations (2.146) and (2.147) :

$$\mu^+(1/t) - \mu^-(1/t) = -\log\Big(1 - e^{-2\pi i m/t}\Big) := \Delta^+ \qquad \mathrm{Im}(t) > 0, \tag{2.180}$$

$$\mu^+(1/t) - \mu^-(1/t) = -\log\Big(1 - e^{2\pi i m/t}\Big) \qquad \mathrm{Im}(t) < 0. \tag{2.181}$$

Remark how the sign of the imaginary part is now switched due to $z \to 1/t$. Having fixed the sector $S(R, \phi)$ to lie at $\mathrm{Im}(t) > 0$, there is now only one way to analytically continue them: we have to use (2.180). If we were now to analytically continue $\mu^+(1/t)$ through $\mathbb{R}^-$, we would introduce a line where:

$$\lim_{t\to 0} \log\Big(1 - e^{-2\pi i m/t}\Big) \qquad \text{does not exist} \tag{2.182}$$

which would not satisfy the first requirement of section 1.5.3. The reason is of course that the imaginary part of t vanishes and thus the exponential is purely imaginary. These considerations are depicted in Figure (2.4).

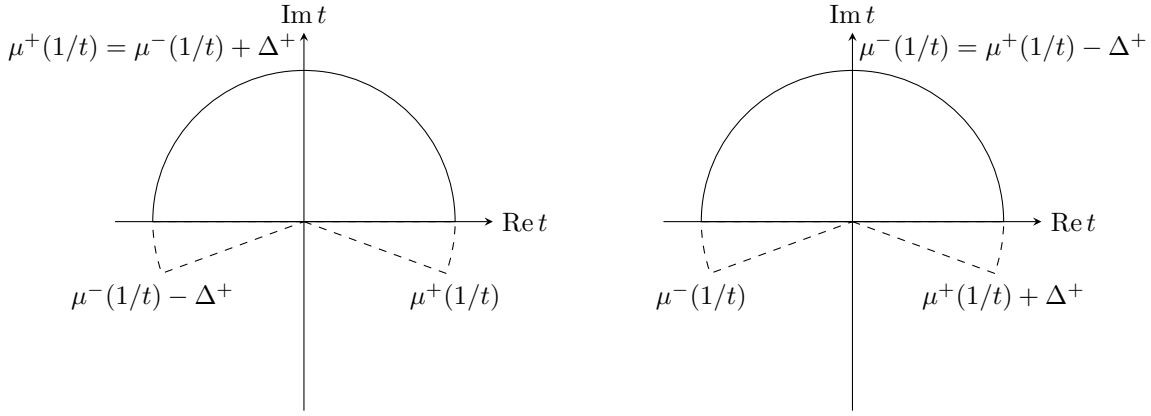


Figure 2.4: Analytic continuation of $\mu^+(1/t)$ (right) and $\mu^-(1/t)$ (left). The analytic continuations are now unique, but the limit for $t \to 0$ is now ill-defined.

Chapter 3

Tame partition functions and correlation functions

In this chapter we present our first results, bringing together the theory o-minimality and Borel summability. First, we will explain in section 3.1 how perturbative series in powers of a small coupling constant, arising in quantum field theories from the sum over Feynman diagrams, are in fact divergent and must be Borel-resummed. As a consequence, amplitudes and partition functions are non-analytic in the weak-coupling limit and, moreover, in absence of the Stokes phenomenon, they can be viewed as Borel sums of their perturbative asymptotic expansions. Under this assumption, it will follow that they are tame functions, definable in the o-minimal structure $\mathbb{R}_{\mathscr{G}}$. Thus, we will be able to provide support to the Tameness Conjecture, indicating a strategy to prove the tameness of observable functions.

In section 3.2 , we will focus on quantum field theories on point-like spacetime, so that path integrals reduce to ordinary integrals. In these theories, only one, self-interacting boson field ϕ will be present; furthermore, there will be only one small, real coupling constant λ, which we will nonetheless have to promote to a complex parameter for our purposes. In the simplest examples, we will show how perturbative expansions can be computed explicitly and proved to be Borel-summable by analysing their Borel transforms . In section 3.3, we will consider a more general theory with a polynomial interaction. In this case, computing Borel transforms explicitly will no longer be feasible and we will then resort to Nevanlinna-Sokal theorem to prove Borel summability. Finally, in section 3.4 we will show how the Stokes phenomenon can affect even very simple theories on a point: we will be then compelled to take a different approach to prove the tameness of the partition function.

3.1 Perturbative QFT

Perturbation theory is an essential tool to obtain predictions from any quantum field theory. As it is well-known, path integrals cannot be solved exactly, and we are forced to settle for approximate solutions, given by perturbative power series. In this section we elucidate some properties of such series.

3.1.1 Divergence of perturbative series

In this chapter we shall be concerned with partition functions and correlation functions depending on a single small coupling constant λ. The coupling constant measures the strength of the interaction between the fields, so that when it is set to zero the theory is free, i.e. the path integral is Gaussian. As λ is thought

to be small, the most common approach to solving a path integral

$$\mathcal{A}(\lambda) = \int \mathcal{D}\phi \; e^{-S_0[\phi]-\lambda S_{int}[\phi]}\phi(x_1)...\phi(x_n) \tag{3.1}$$

is the perturbative one: after Taylor expanding the interaction term $e^{-\lambda S_{\rm int}[\phi]}$, the path integral is split on an infinite sum of Gaussian integrals, which can be computed exactly. Such procedure yields a series in powers of λ, so that an amplitude function $\mathcal{A}(\lambda)$ (or the partition function Z) is represented by a power series

$$a_0 + a_1\lambda + a_2\lambda^2 + ... \; . \tag{3.2}$$

Every coefficient a_n is a number which is usually expressed as a finite sum over Feynman diagrams with the same number n of vertices: indeed, the Feynman rules associate with every vertex a factor of λ. Of course, for the purposes of phenomenological applications, the series must be truncated at some power of $n = n_{\max}$. Notwithstanding the unquestionable success of this approach in making predictions for experiments, it is essential to inquire whether the exact value of $\mathcal{A}(\lambda)$ (as might be measured by an ideal experiment) would match the sum of all the whole power series (3.2), were it possible to perform it exactly. Only then would the claim that our theory is really predictive be entirely justified. Unfortunately, in many cases (especially in bosonic field theories) the answer is in the negative: a power series such as (3.2) is divergent. Roughly, the reason is that Feynman diagrams stem from illegally commuting a sum, arising from the Taylor expansion of the exponential of interaction term, $e^{-\lambda S_{\rm int}[\phi]}$, with the path integral $\int \mathcal{D}\phi$. The consequence of this violation is that the number of Feynman diagrams at every order, i.e. the number of possible Wick contractions, grows faster than $n!$: Feynman diagrams are then said to *proliferate*. Being a series like (3.2) divergent, we are compelled to resort to the tools introduced in the previous chapter to resum the divergent series into a function, $\mathcal{A}(\lambda)$, which could be in principle measured by the experimentalist with arbitrary precision. As already remarked, the prominent feature of Borel sums $\mathcal{A}(\lambda)$ is that they are not analytic at the weak coupling limit $\lambda = 0$, namely when the theory is free. This feature was guessed long ago by F. Dyson in [Dys52] in the case of QED: we briefly present here his arguments.

In QED amplitudes are computed as perturbative power series in the coupling constant e^2 of the fermion field $\psi(x)$ with the abelian gauge field $A_\mu(x)$. If these power series were convergent, they would sum into a function analytic at $e^2 = 0$. Therefore, it would be possible to analytically continue the amplitude function to *negative* values of e^2. These values would describe a theory in which the sign of the Coulomb potential is reversed and, as a consequence, charges of the same sign attract, rather than repel, each other. Therefore, in this 'analytically continued world' an electron-positron pair would separate, rather than annihilate, in order to reach the lowest energy state. Although there is a high potential barrier to overcome, there will be always a finite probability for an electron-positron pair to be created due to quantum tunnelling; when the pair is created though, due to the reversed sign of the interaction, it does not annihilate. It follows that, in the analytically continued theory, every physical state is unstable against the production of large numbers of particles. Note that this theory would be described by any point of our hypothetical function infinitesimally close to 0: this is the contradiction which leads to the conclusion that perturbative power series are divergent, so that their Borel sums are not analytic at 0.

Satisfied that perturbative power series are divergent, we appreciate the importance of the resummation methods expounded in the previous chapter to uphold that, despite their divergence, perturbative power series describe a well-defined function and thus carry a physical meaning. Motivated by the Tameness Conjecture of section 1.8, we then ask ourselves if the amplitudes, viewed as the Borel sums of their asymptotic, perturbative power series, are tame functions.

3.1.2 Taming perturbative power series

The answer to this question lies in section 2.6: we argued there that Borel sums of κ-summable power series at $\theta = 0$ and with real coefficients are definable in the o-minimal structure $\mathbb{R}_{\mathscr{G}}$. Our strategy is then clear. Given an amplitude $\mathcal{A}(\lambda)$ – or a partition function $Z(\lambda)$, which can be viewed as an amplitude with no field

insertions – we will perform a perturbative expansion such as (3.2). We will then find a formal, divergent power series $\tilde{\varphi}$. If we manage to identify its Gevrey class to be $1/\kappa$, we then have

$$\text{The power series } \tilde{\varphi} \text{ is } \kappa\text{-summable along } \mathbb{R}^+ \implies \text{The Borel sum } \varphi(\lambda) \text{ is definable in } \mathbb{R}_{\mathscr{G}}.$$

This is essentially the content of Observations 2.1 and 2.2. In turn, we remind ourselves that, given an asymptotic power series $\tilde{\varphi}$, its summability is determined by the properties of its formal κ-Borel transform $\hat{\varphi}(\zeta) := \mathcal{B}_\kappa \tilde{\varphi}$: namely, if $\hat{\varphi}(\zeta)$

$$\begin{cases} \text{has no poles on } \mathbb{R}^+ \\ \text{has exponential size } \leq 1/\kappa \text{ on a horizontal strip } S_\delta \end{cases} \implies \tilde{\varphi} \text{ is } \kappa\text{-summable along } \mathbb{R}^+. \tag{3.3}$$

It is worthwhile to stress why we are interested in Borel summability along $\theta = 0$. As we want to interpret λ as a real number, we are forced us to take the Laplace transform (2.62) on a line where $\operatorname{Re} \zeta^{1/\kappa} > 0$. This gives an ample range of directions; yet, if the formal power series $\tilde{\varphi}$ has real coefficients, the only Laplace integral that will yield a real-valued Borel sum $\varphi(\lambda)$ is $\theta = 0$. It is essential that the Borel sum take only real values, as tame functions only take values in $\mathbb{R}$.

This is the basic idea that we shall use in the next section to prove the tameness of some partition functions for Euclidean, bosonic theories on a point: on a point-like spacetime, the path integral reduces to an ordinary integral, simplifying the treatment immensely. In later sections though, upon considering even but slightly more complicated models, this approach will no longer be feasible, as the explicit computation of the perturbative expansion $\tilde{\varphi}$ will no longer be possible; nor, as a consequence, that of the Borel transform $\hat{\varphi}$. Nevertheless, Borel summability can be proved by resorting the second statement of Nevanlinna-Sokal theorem 2.6, applied directly to the partition function expressed in its path integral representation. Definability in $\mathbb{R}_{\mathscr{G}}$ will then follow similarly.

3.2 Theories on a point with monomial interaction

In this section we deal with Euclidean theories on a point of a scalar field with a monomial interaction. This will allow us to express the Borel transforms of the asymptotic expansions in a closed form, in terms of known functions.

3.2.1 Real ϕ^4 theory on a point

We begin with the simplest possible example of a quartic interaction. We will establish that the related partition function is a tame function, definable in $\mathbb{R}_{\mathscr{G}}$. Following [GSV23], consider the Euclidean action

$$S(\phi; \lambda) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{3.4}$$

for a positive coupling λ and a real bosonic field ϕ. After the substitutions

$$\phi \to \sqrt{\frac{3}{2\lambda}}\phi, \qquad\qquad g = \frac{3m^4}{4\lambda}, \tag{3.5}$$

we have

$$S(\phi; g) = g\phi^2 + \frac{g}{8}\phi^4, \tag{3.6}$$

whence the partition function to be computed is given by the path integral (which is in fact an ordinary integral)

$$Z(g) = \int_{-\infty}^{\infty} d\phi e^{-g\left(\phi^2 + \frac{\phi^4}{8}\right)}. \tag{3.7}$$

Remarkably, the partition function for this theory can be computed exactly. Recall that the modified Bessel function $K_{1/4}(z)$ is defined by the integral

$$K_{1/4}(z) = \int_0^\infty dt \cosh\left(\frac{t}{4}\right) e^{-z\cosh(t)} \qquad |\arg(z)| < \frac{\pi}{2}. \tag{3.8}$$

If we introduce the change of variable

$$x = \sqrt{2}\sinh\left(\frac{t}{4}\right), \tag{3.9}$$

we will have that, by a twofold application of the duplication formulas

$$\cosh(t) = 2\cosh^2\left(\frac{t}{2}\right) - 1 = 2\left(2\sinh^2\left(\frac{t}{4}\right) + 1\right)^2 - 1 = 2(x^2+1)^2 - 1, \tag{3.10}$$

while

$$dt\cosh\left(\frac{t}{4}\right) = \frac{4}{\sqrt{2}}dx. \tag{3.11}$$

Hence we can rewrite (3.8) as

$$\begin{aligned} K_{1/4}(z) &= \frac{4}{\sqrt{2}}\int_0^\infty dx e^{-z(2x^4+4x^2+1)} = \sqrt{2}e^{-z}\int_{-\infty}^\infty dx e^{-z(4x^2+2x^4)} \\ &= \frac{e^{-z}}{\sqrt{2}}\int_{-\infty}^\infty dx e^{-z\left(x^2+\frac{1}{8}x^4\right)}, \end{aligned} \tag{3.12}$$

where in the last integral we recognise the partition function associated with the action (3.6), with $z = g$. Thus, we have proved that:

$$Z(g) = \sqrt{2}e^g K_{1/4}(g). \tag{3.13}$$

Notice that, despite the integral representation is only valid in the right-half of the complex plane, $K_{1/4}(z)$ only has one singularity at $z = \infty$, so the above formula holds for all $g \in \mathbb{C}$. The singularity at ∞ is essential: therefore, as foreseen by our earlier arguments, $Z(g)$ is non-analytic in the weak coupling limit $g \to \infty$ $(\lambda \to 0)$. Because the Bessel differential equation equation is a second order ODE, it was argued in [GSV23] that $Z(g)$ is definable in the sharply o-minimal structure $\mathbb{R}_{\text{Pfaff}}$, which we described in section 1.6.3. Yet, these observations are rather case-specific and it is not clear how to extend them to more general cases. We will prove now that $Z(g)$ is naturally definable in $\mathbb{R}_{\mathscr{G}}$ as well; to do so, we have to analyse it perturbatively.

Let us then pretend that we are not skilled enough to recast the 'path' integral $\int d\phi e^{-S(\phi;g)}$ into the form of (3.8). We must then proceed perturbatively by computing the Taylor series of $Z(g)$ at $g = \infty$. This can be done in practice by Taylor-expanding the quartic interaction and then exchanging the sum with the integral:

$$\begin{aligned} Z(g) &= \int_{-\infty}^\infty d\phi e^{-g\phi^2}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\left(\frac{g\phi^4}{8}\right)^n \\ &\sim \sum_{n=0}^\infty \frac{(-1)^n}{n!}\left(\frac{g}{8}\right)^n \int_{-\infty}^\infty d\phi e^{-g\phi^2}\phi^{4n}. \end{aligned} \tag{3.14}$$

It is of the utmost importance to highlight that the exchange of the sum with the integral that we used to pass from the first to the second line is *not* allowed: we refer to Appendix C for the conditions under which the exchange is legal. As a consequence, the power series on the right-hand side is divergent: we can no longer claim that it is equal to the function on left-hand side, but merely that it is its asymptotic expansion at $g = \infty$.

The Gaussian integral can be expressed as a Gamma function, yielding the formal power series

$$Z(g) \sim \sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{1}{8^n g^{n+1/2}}\Gamma\left(2n+\frac{1}{2}\right) =: \sqrt{\pi g}\tilde{\varphi}. \tag{3.15}$$

Setting $\tilde{\varphi} := \sum_{n=0}^{\infty} a_n g^{-n-1}$, and recalling that $\Gamma(n/2) = \sqrt{\pi} 2^{-(n-1)/2}(n-2)!!$ for any odd n, we can write explicitly

$$a_n = \frac{(-1)^n}{n!} \frac{(4n-1)!!}{32^n}, \tag{3.16}$$

which diverges factorially as $\sim n!$. Therefore, $\tilde{\varphi}$ belongs to the Gevrey class $1/1 = 1$. Observe how $(4n-1)!!$ is indeed the number of Wick contractions among $4n$ fields under Gaussian integration, i.e. the number of Feynman diagrams: it can now be seen explicitly that they indeed proliferate, namely their number grows like $2n!$ (up to exponentials B^n), causing the perturbative power series $\tilde{\varphi}$ to be divergent.
The Borel transform of $\tilde{\varphi}$, $\hat{\varphi} = \mathcal{B}\tilde{\varphi}$, can be computed with Mathematica: it yields

$$\hat{\varphi}(\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{32^n} \frac{(4n-1)!!}{(n!)^2} \zeta^n = \frac{2}{\pi} \frac{K\left(\frac{1}{2} - \frac{1}{2\sqrt{1+\zeta/2}}\right)}{(1+\zeta/2)^{1/4}}, \tag{3.17}$$

where $K(z)$ is the complete elliptic integral of the first kind. The only pole of $\hat{\varphi}(\zeta)$ is $\zeta = -2$, and in the large ζ limit, $\hat{\varphi}(\zeta)$ is decaying: thus, as there are no poles upon the positive real axis, we conclude that $\tilde{\varphi}$ is 1-summable. Therefore, the formal power series is the asymptotic expansion of order 1 of $Z(g)$, i.e. $Z(g) \sim_1 \sqrt{\pi g}\tilde{\varphi}(g)$, uniformly on the domain Π_0^0 (but actually, on every closed subsector of the larger $\mathbb{C}\backslash\mathbb{R}^-$, very much in analogy to the Euler series discussed in section 2.4). Explicitly, we will have

$$Z(g) \sim_1 \sqrt{\frac{\pi}{g}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(4n-1)!!}{(8g)^n 4^n} \tag{3.18}$$

uniformly on Π_0^0. Upon comparison with (3.13), the latter expression yields an asymptotic expansion for the modified Bessel function (3.8):

$$\begin{aligned} K_{1/4}(z) &\sim_1 \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(4n-1)!!}{(8z)^n 4^n} \\ &\sim_1 \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{(\frac{1}{4} - 1^2)}{1!8z} + \frac{(\frac{1}{4} - 1^2)(\frac{1}{4} - 3^2)}{2!(8z)^2} + ...\right) \end{aligned} \tag{3.19}$$

for $|z| \to \infty$, which matches the expansion stated in [AW95] , Chapter 11. By this example, we can clearly see how, as anticipated earlier, the illegal exchange of the sum with the integral in (3.14) produces an asymptotic expansion for $Z(g)$, rather than a convergent series.

Hearkening back to section 2.2, we compute the Borel sum of $\tilde{\varphi}$ by the Laplace integral

$$\varphi(g) := \mathscr{S}^0 \tilde{\varphi} = \int_0^{\infty} d\zeta e^{-g\zeta} \hat{\varphi}(\zeta). \tag{3.20}$$

By virtue of Observation 2.1, we are entitled to conclude that $\varphi(g)$ is definable in $\mathbb{R}_{\mathscr{g}}$. Hence, $Z(g) = \sqrt{\pi g}\varphi(g)$ is also definable in $\mathbb{R}_{\mathscr{g}}$, being related to $\varphi(g)$ algebraically (recall Lemma 1.3). By numerical integration, it can be checked that, as we should expect, $\sqrt{\pi g}\varphi(g) = \sqrt{2}e^g K_{1/4}(g)$. It is important to realise that the reason for which the poles in Borel plane lie on the negative axis ultimately narrows down to the asymptotic series $\tilde{\varphi}$, as defined by 3.18, having alternating signs.

The Borel summability of the partition function in this very simple example is also proved in [Riv09] by means of Loop Vertex Expansion: we will explain the basics of this theory and its application to this very example in Chapter 4. This proof is rather indirect, and exploits Nevanlinna-Sokal theorem rather than computing the Borel transform of the perturbative power series. Although our approach is in this case much simpler, it lacks the generality which enables the author [Riv09] to extend his results to far more general models, as we shall review in the later sections.

3.2.2 Real ϕ^{2p} theory on a point

Let us now extend the previous result to a more general monomial interaction. We then consider the Euclidean action

$$S(\phi;\lambda) = \frac{m^2}{2}\phi^2 + \lambda\phi^{2p} \tag{3.21}$$

which defines a QFT on a point and where $p > 1$ is an integer. Upon setting $y^2 = \frac{m^2}{2}\phi^2$ and redefining the coupling with $\lambda\left(\frac{2}{m^2}\right)^p \to \lambda$, the partition function reads

$$\int_{-\infty}^{\infty} d\phi e^{-\frac{m^2}{2}\phi^2 - \lambda\phi^{2p}} = \frac{\sqrt{2}}{m}\int_{-\infty}^{\infty} d\phi e^{-\phi^2 - \lambda\phi^{2p}} =: Z(\lambda). \tag{3.22}$$

Again proceeding perturbatively, in analogy to the previous case, we find

$$\begin{aligned} Z(\lambda) &= \frac{\sqrt{2}}{m}\int_{-\infty}^{\infty} d\phi e^{-\phi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\lambda\phi^{2p}\right)^n \\ &\sim_{p-1} \frac{\sqrt{2}}{m}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\lambda^n\int_{-\infty}^{\infty} d\phi e^{-\phi^2}\phi^{2pn} = \frac{\sqrt{2}}{m}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\lambda^n\Gamma\left(np+\frac{1}{2}\right) =: \frac{\sqrt{2}}{\lambda m}\tilde{\varphi}. \end{aligned} \tag{3.23}$$

It is easy to convince oneself that now $\tilde{\varphi}$ is a formal power series (without constant term) of Gevrey class $1/(p-1)$: in fact the Gamma function $\Gamma(np+1/2)$ grows like $(np)!$ for large n, which in turn grows like $(n!)^p$ up to exponentials. In the above equation, we have introduced the sign $\sim_{p-1}$ as, in analogy to the previous case, we expect the right-hand side to be the asymptotic expansion of $Z(\lambda)$: because of the Gevrey class of $\tilde{\varphi}$, the order of the asymptotic expansion must be $p-1$. However, we still do not know on what domain of the complex plane the asymptotic expansion is valid, as we have not proved the Borel summability of $\tilde{\varphi}$ yet. In order to do so, we now compute its Borel transform explicitly.

Lemma 3.1. The Borel transform of $\tilde{\varphi}$, $\mathcal{B}_{p-1}\tilde{\varphi} = \hat{\varphi}(\zeta)$ is

$$\hat{\varphi}(\zeta) = \sqrt{\pi}\,{}_aF_b\left[\left.\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\right| -\frac{p^p}{(p-1)^{p-1}}\zeta\right], \tag{3.24}$$

where $a = |\vec{a}|$, $b = |\vec{b}|$ with $a = p$, $b = p-1$ and

$$\begin{cases}\vec{a} = \left(\frac{1}{2p}, \frac{3}{2p}, ..., \frac{2p-1}{2p}\right)\\ \vec{b} = \left(\frac{1}{p-1}, \frac{2}{p-1}, ..., 1\right).\end{cases} \tag{3.25}$$

Proof. We begin by recalling the most general definition of the hypergeometric function ${}_aF_b\left[\left.\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\right| x\right]$, where $a = |\vec{a}|$, $b = |\vec{b}|$:

$${}_aF_b\left[\left.\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\right| x\right] = \frac{\prod_{i=1}^{b}\Gamma(b_i)}{\prod_{i=1}^{a}\Gamma(a_i)}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{a}\Gamma(a_i+n)}{\prod_{i=1}^{b}\Gamma(b_i+n)}\frac{x^n}{\Gamma(n+1)}. \tag{3.26}$$

We then observe that, for any integer n, recalling that $x\Gamma(x) = \Gamma(x+1)$

$$\begin{aligned}\frac{\Gamma(x+n)}{\Gamma(x)} &= \frac{\Gamma(x+n)}{\Gamma(x+(n-1))}\frac{\Gamma(x+n-1)}{\Gamma(x+(n-2))}\cdots\frac{\Gamma(x+1)}{\Gamma(x)} \\ &= (x+n-1)(x+n-2)\cdots x.\end{aligned} \tag{3.27}$$

By comparing (3.24) with (3.26), we realize that it suffices to prove that

$$\frac{\Gamma\left(np+\frac{1}{2}\right)}{\Gamma\left(n(p-1)+1\right)} = \sqrt{\pi}\prod_{i=1}^{a}\frac{\Gamma(a_i+n)}{\Gamma(a_i)}\left(\prod_{i=1}^{b}\frac{\Gamma(b_i+n)}{\Gamma(b_i)}\right)^{-1}\left(\frac{(p-1)^{p-1}}{p^p}\right)^n. \tag{3.28}$$

Let us focus on the first product. Plugging in the values in (3.25) and resorting to (3.27) we have

$$\prod_{i=1}^{a}\frac{\Gamma(a_i+n)}{\Gamma(a_i)}= \begin{matrix} \left(\frac{1}{2p}+n-1\right) & \dots & \left(\frac{1}{2p}+1\right)\left(\frac{1}{2p}\right)\\ \left(\frac{3}{2p}+n-1\right) & \dots & \left(\frac{3}{2p}\right)\\ \vdots & & \vdots\\ \left(\frac{2p-1}{2p}+n-1\right) & \dots & \left(\frac{2p-1}{2p}\right). \end{matrix}$$

The rows have n terms and the columns have p terms. By taking products starting from the top right corner and moving first down the column and then moving to the column to the left, we have

$$\prod_{i=1}^{a}\frac{\Gamma(a_i+n)}{\Gamma(a_i)}=\frac{(2np-1)!!}{(2p)^{np}}=\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{2np+1}{2}\right)\frac{1}{p^{np}}, \tag{3.29}$$

where we have recalled that $\Gamma(n/2)=\sqrt{\pi}2^{-(n-1)/2}(n-2)!!$ for any odd n. Similarly, we can find

$$\begin{aligned}\prod_{i=1}^{b}\frac{\Gamma(b_i+n)}{\Gamma(b_i)}= &\begin{matrix} \left(\frac{1}{p-1}+n-1\right) & \dots & \left(\frac{1}{p-1}+1\right)\left(\frac{1}{p-1}\right)\\ \left(\frac{2}{p-1}+n-1\right) & \dots & \left(\frac{2}{p-1}\right)\\ \vdots & & \vdots\\ \left(\frac{p-1}{p-1}+n-1\right) & \dots & \left(\frac{p-1}{p-1}\right) \end{matrix}\\ =&\frac{(n(p-1))!}{(p-1)^{n(p-1)}}=\frac{\Gamma(n(p-1)+1)}{(p-1)^{n(p-1)}},\end{aligned}$$

which together with the earlier (3.29) yields the desired identity. □

The poles of the hypergeometric function ${}_aF_b\left[\left.\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\right|z\right]$ lie at $z=1$ whenever $a=b+1$ (see the appendix of [DGS22]). Hence the unique pole in the Borel plane of $\hat{\varphi}(\zeta)$ lies at $\zeta=-\frac{(p-1)^{p-1}}{p^p}$. Therefore $\hat{\varphi}(\zeta)$ is analytic in a horizontal strip S_δ including the positive real axis; it can also be argued that it has no exponential growth thereon. We conclude that $\tilde{\varphi}$ is $(p-1)$-summable: thus the Borel sum $\varphi(\lambda)=\mathcal{L}_{p-1}\hat{\varphi}$ admits $\tilde{\varphi}$ as uniform $1/(p-1)$-Gevrey asymptotic expansion, namely $\varphi(\lambda)\sim_{p-1}\tilde{\varphi}$, on the Sokal disc $D_0^{p-1}=\{\lambda\in\tilde{\mathbb{C}}:|\arg(\lambda)|<(p-1)\frac{\pi}{2}\}$. Moreover, by rotating the line of integration of the Laplace integral from $\theta=-\pi$ to $\theta=\pi$, it can be easily argued that $\varphi(\lambda)$ admits $\tilde{\varphi}$ as uniform Gevrey asymptotic expansion of order $p-1$ on every closed subsector of the enlarged Sokal disc $\{\lambda\in\tilde{\mathbb{C}}:|\arg(\lambda)|<(p+1)\frac{\pi}{2}\}$. We will elaborate further on this feature in the next section.
By virtue of Observation 2.2, the Borel sum $\varphi(\lambda)$ must be a definable function of $\mathbb{R}_{\mathscr{G}}$. Hence, we infer that the partition function for the action (3.21), $Z(\lambda)=\frac{\sqrt{2}}{m\lambda}\varphi(\lambda)$, is definable in $\mathbb{R}_{\mathscr{G}}$ for any integer p.

We remark that these partition functions were studied extensively in [FMQ20]. Therein, the resurgence properties of the partition functions, and in particular the location of the poles in the Borel plane, were determined from the differential equation obeyed by the partition function by means of the Newton polygon. The ODE obeyed by the partition function is

$$\left[\prod_{j=0}^{p-1}(2px\partial_x+2j+1)+\partial_x\right]Z(x)=0, \tag{3.30}$$

where for convenience we have set $x = \frac{\lambda}{2^p}$. Observe that only for $p \leq 2$ the ODE is of order less than 2 and can then be recast into a Pfaffian chain: we cannot therefore argue that these partition functions are definable in $\mathbb{R}_{\text{Pfaff}}$ when $p > 2$. Although the differential equation could be turned in a Noetherian chain, this would force us to restrict $Z(x)$ to a closed domain of the type $[\varepsilon, r]$ for some finite $\varepsilon > 0$; the reason being that we would need the function $f(x) = 1/x$ to appear in the chain. In this way we would not include the weak coupling limit $\lambda = 0$ where the partition function is not analytic (and therefore not trivially definable in $\mathbb{R}_{\text{an}}$).

The connected correlation functions (also known as *cumulants*) can be proved to be Borel-summable in a very similar fashion. Let the j-th moment $G_j(\lambda)$ be:

$$G_j(\lambda) := \int_{-\infty}^{\infty} d\phi \; e^{-\phi^2 - \lambda\phi^{2p}} \phi^{2j}. \tag{3.31}$$

As before, expanding the exponential yields the formal power series of Gevrey class $1/(p-1)$, asymptotic to $G_j(\lambda)$

$$G_j(\lambda) \sim_{p-1} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \Gamma\left(np + j + \tfrac{1}{2}\right) =: \frac{1}{\lambda} \tilde{\varphi}_j. \tag{3.32}$$

The Borel transform of $\tilde{\varphi}_j$ is then

$$\mathcal{B}_{p-1}\tilde{\varphi}_j = \sum_{n=0}^{\infty} \frac{(-\zeta)^n}{n!} \frac{\Gamma\left(np + j + \frac{1}{2}\right)}{\Gamma\left(n(p-1) + 1\right)} =: \hat{\varphi}_j \; , \tag{3.33}$$

which can be computed in a similar fashion as before.

Lemma 3.2. The Borel transform of $\tilde{\varphi}_j$, $\mathcal{B}_{p-1}\tilde{\varphi}_j = \hat{\varphi}_j(\zeta)$ is

$$\hat{\varphi}_j(\zeta) = \Gamma\left(\tfrac{1}{2} + j\right) {}_aF_b \left[\left. \begin{matrix} \vec{a} \\ \vec{b} \end{matrix} \right| - \frac{p^p}{(p-1)^{p-1}} \zeta \right] \tag{3.34}$$

where $a = |\vec{a}|$, $b = |\vec{b}|$ with $a = p$, $b = p - 1$ and

$$\begin{cases} \vec{a} = \left(\frac{1+2j}{2p}, \frac{3+2j}{2p}, ..., \frac{2p-1+2j}{2p} \right) \\ \vec{b} = \left(\frac{1}{p-1}, \frac{2}{p-1}, ..., 1 \right). \end{cases} \tag{3.35}$$

Proof. The proof is quite analogous to the previous case. The only difference is that, in this case, we have to show that

$$\prod_{i=1}^{a} \frac{\Gamma(a_i + n)}{\Gamma(a_i)} = \frac{\Gamma\left(n(p-1) + j + \frac{1}{2}\right)}{\Gamma\left(j + \frac{1}{2}\right)}. \tag{3.36}$$

The left-hand side can be expanded as before, yielding the np products

$$\prod_{i=1}^{a} \frac{\Gamma(a_i + n)}{\Gamma(a_i)} = \begin{matrix} \left(\frac{1+2j}{2p} + n - 1\right) & ... & \left(\frac{1+2j}{2p} + 1\right)\left(\frac{1+2j}{2p}\right) \\ \left(\frac{3+2j}{2p} + n - 1\right) & ... & \left(\frac{3+2j}{2p}\right) \\ \vdots & & \vdots \\ \left(\frac{2p-1+2j}{2p} + n - 1\right) & ... & \left(\frac{2p-1+2j}{2p}\right). \end{matrix}$$

Reading the products from right to left and from the top to the bottom, we realize that the products at the numerator run from $1+2j$ up to $2(np+j)-1$ (the product being only over the odd integers). So we conclude that the above product is

$$\prod_{i=1}^{a}\frac{\Gamma(a_i+n)}{\Gamma(a_i)}=\frac{(2(np+j)-1)!!}{(2j-1)!!}\frac{1}{2^{np}}=\frac{(2(np+j)-1)!!}{2^{np+j}}\frac{2^j}{(2j-1)!!}=\frac{\Gamma\left(np+j+\frac{1}{2}\right)}{\Gamma\left(j+\frac{1}{2}\right)}, \tag{3.37}$$

as desired. □

We can observe how the unique pole of $\hat{\varphi}_j$ in (3.34) is the same as that of the earlier $\hat{\varphi}$ in (3.24): the location of the pole is therefore independent of j, and Borel summability follows from the properties of the hypergeometric function as before. Thus, the function $\varphi_j(\lambda)=\mathcal{L}_{p-1}\hat{\varphi}_j$ is definable in $\mathbb{R}_{\mathscr{G}}$ and so is the moment $G_j(\lambda)=\frac{1}{\lambda}\varphi_j(\lambda)$, being related to it algebraically. Then the j-th cumulant $\mathfrak{K}_j(\lambda)$, defined as

$$\mathfrak{K}_j(\lambda):=\langle\phi^{2j}\rangle=\frac{\int_{-\infty}^{\infty}d\phi e^{-\phi^2-\lambda\phi^{2p}}\phi^{2j}}{\int_{-\infty}^{\infty}d\phi e^{-\phi^2-\lambda\phi^{2p}}} \tag{3.38}$$

is also definable in $\mathbb{R}_{\mathscr{G}}$, as it is the ratio between two definable functions of the same structure.

3.2.3 Complex $(\phi\bar{\phi})^p$ theory on a point

In this section, we will consider again a monomial potential, but for a complex field ϕ. This model was discussed already [Riv17] by the more powerful methods of constructive field theory. The partition function of our interest is that of a complex field in 0 dimensions with an interaction $(\phi\bar{\phi})^p$, namely

$$Z(\lambda)=\int\tfrac{d\phi d\bar{\phi}}{2\pi i}e^{-\phi\bar{\phi}-\lambda(\phi\bar{\phi})^p}. \tag{3.39}$$

The integral measure $\frac{d\phi d\bar{\phi}}{2\pi i}$ can be rewritten in polar coordinates as $\frac{2ir}{2\pi i}d\theta\wedge dr$. Hence the partition function rewrites

$$Z(\lambda)=\int_0^{\infty}dxe^{-x-\lambda x^p}. \tag{3.40}$$

Let us proceed perturbatively once more: upon Taylor-expanding the interaction term $e^{-\lambda x^p}$, we find a formal power series $\tilde{\varphi}$ in λ given by

$$\frac{1}{\lambda}\tilde{\varphi}:=\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n}{n!}\int_0^{\infty}dxe^{-x}x^{pn}=\frac{1}{\lambda}\sum_{n=0}^{\infty}(-1)^n\lambda^{n+1}\frac{(pn)!}{n!}. \tag{3.41}$$

In analogy to the case dealt with in the previous section, $\tilde{\varphi}$ clearly belongs to the Gevrey class $1/(p-1)$, so we can write $\tilde{\varphi}\in\lambda\mathbb{R}[[\lambda]]_{1/(p-1)}$. The Borel transform of $\tilde{\varphi}$ is therefore $\hat{\varphi}=\mathcal{B}_{p-1}\hat{\varphi}$, which we can write as

$$\hat{\varphi}(\zeta)=\sum_{n=0}^{\infty}(-1)^n\zeta^n\frac{pn!}{n!((p-1)n)!}=\sum_{n=0}^{\infty}(-1)^n\zeta^n\binom{pn}{n}, \tag{3.42}$$

which is exactly the function $F_p(-z)$ in the notations of [Riv17]. As observed therein, the binomial coefficient $\binom{pn}{n}$ is closely related to the n-th Fuss-Catalan number

$$C_n^{(p)}=\frac{1}{pn+1}\binom{pn+1}{n}=\frac{1}{(p-1)n+1}\binom{pn}{n}. \tag{3.43}$$

Fuss-Catalan numbers are associated with the generating function

$$T_p(z)=\sum_{n=0}^{\infty}C_n^{(p)}z^n, \tag{3.44}$$

which satisfies the algebraic relation $zT_p^p(z)+1=T_p(z)$. Observing that

$$\hat{\varphi}(\zeta)=-\zeta(p-1)T_p'(-\zeta)+T_p(-\zeta) \tag{3.45}$$

where the prime denotes the derivative with respect to ζ, it is easy to prove that

$$\hat{\varphi}(\zeta)=\frac{1}{1+p\zeta T_p^{p-1}(-\zeta)}. \tag{3.46}$$

According to Theorem III.1 in [Riv17], the radius of convergence of the Borel transform $\hat{\varphi}(\zeta)$ is the same of that of the Fuss-Catalan generating function (3.44) , which is $R_p=\frac{(p-1)^{(p-1)}}{p^p}$. As it might have been expected, this is the same radius of convergence of the Borel transform in the case of the real field dealt with in the previous section. Again according to Theorem III.1, $\hat{\varphi}(\zeta)$ is holomorphic on $\mathbb{C}\backslash[-R_p,-\infty]$ and its growth for $\mathrm{Re}(\zeta)\to\infty$ is polynomially bounded. Therefore, $\tilde{\varphi}$ is $(p-1)$-summable. The partition function $Z(\lambda)$ admits $\tilde{\varphi}$ as uniform asymptotic expansion of order $p-1$: namely, we have $Z(\lambda)\sim_{p-1}\frac{1}{\lambda}\tilde{\varphi}(\lambda)$ on a Sokal disc D_0^{p-1} and, as before, the uniform asymptotic expansion is actually valid on any closed subsector of D_0^{p+1}. The latter conclusion was also reached in [Riv17] by finding an analyticity domain for $Z_p(\lambda)$ which contains a Sokal disc D_0^{p-1}: then Borel summability ensues from Nevanlinna-Sokal theorem, after proving that uniform Gevrey constraints (like in Definition 2.7, with $\kappa=p-1$) hold on the disc. Again, we remark that Borel summability is ultimately owed to the alternate signs in the Borel transform (3.42), which cause the pole to lie on the negative real axis.

Having proved that $\tilde{\varphi}$ is Borel-summable, we invoke again Observation 2.2 to conclude that its Borel sum

$$\varphi(\lambda)=(\mathcal{L}_{p-1}^0\hat{\varphi})(\lambda)=\frac{1}{(p-1)}\int_0^\infty d\zeta e^{-(\frac{\zeta}{\lambda})^{1/(p-1)}}\left(\frac{\zeta}{\lambda}\right)^{\frac{1}{p-1}-1}\hat{\varphi}(\zeta) \tag{3.47}$$

is definable in $\mathbb{R}_{\mathscr{G}}$; as a consequence $Z(\lambda)=\lambda^{-1}\varphi(\lambda)$ is definable in $\mathbb{R}_{\mathscr{G}}$ too, for all positive integers p.

It is worthwhile to observe how the Loop Vertex Representation (LVR) II-5 of [Riv17] is indeed equivalent to the Borel-Laplace resummation procedure 'in one stroke'. As opposed to the Loop Vertex Expansion (see section 4.3), in which an auxiliary field is introduced by performing a Hubbard-Stratonovich transformation, the LVR uses as auxiliary field the initial field itself (see also the introduction to [KRS19a]). Setting the sources to 0, indeed, the LVR is

$$\begin{aligned}Z(\lambda)&=\int d\phi d\bar{\phi}e^{-\phi\bar{\phi}}\sum_{n=0}^\infty\binom{pn}{n}(-1)^n\lambda^n(\phi\bar{\phi})^{n(p-1)} \\ &=\int_0^{+\infty}d(r^2)e^{-r^2}\sum_{n=0}^\infty\binom{pn}{n}(-1)^n\lambda^n r^{2n(p-1)}.\end{aligned} \tag{3.48}$$

Upon redefining $\lambda r^{2(p-1)}=\zeta$, one finds

$$Z(\lambda)=\frac{1}{\lambda(p-1)}\int_0^\infty d\zeta e^{-(\frac{\zeta}{\lambda})^{1/p-1}}\left(\frac{\zeta}{\lambda}\right)^{\frac{1}{p-1}-1}\hat{\varphi}(\zeta) \tag{3.49}$$

with $\hat{\varphi}$ given by (3.42). We then readily recognise this integral as the Laplace transform (2.62) $\mathcal{L}_{p-1}^0\hat{\varphi}$.

3.3 Theories on a point with polynomial interaction

In this section we consider a theory on a point with a polynomial interaction. Let us then consider the partition function for a 0-dimensional, scalar, bosonic QFT

$$Z(\lambda)=\int_{-\infty}^\infty d\phi\, e^{-\phi^2-\lambda V(\phi)}\ , \tag{3.50}$$

where we assume that $V(\phi)$ is a polynomial of degree $2p$,

$$V(\phi) = a_{2p}\phi^{2p} + a_{2p-1}\phi^{2p-1} + ... + a_1\phi \tag{3.51}$$

with $a_{2p} > 0$ so that the integral is well-defined for a positive coupling λ. Moreover, under our assumptions, $V(0) = 0$. In this case, the explicit computation of the Borel transform is not possible, and we must avail ourselves of the second point of Nevanlinna-Sokal theorem 2.6 to prove Borel summability of of $Z(\lambda)$. This can be done in two steps: first, by finding an analyticity domain for $Z(\lambda)$ which contains a Sokal disc D_0^{p-1}; second, by proving that it is asymptotic to a $1/(p-1)$-Gevrey formal power series on that domain.

Before proceeding with the proof, let us dwell on an important point. A peculiarity of non-analytic, Borel-summable partition functions is that they admit two different integral representations: one is the usual 'path' integral, as (3.50); the other is obtained by Borel-transforming their asymptotic expansion and then taking the Laplace transform of the resulting function. It will be important to remark that integral representations of functions of a complex variable only hold on a subset of their domain of definitions. For instance, consider the function

$$f(\lambda) = \int_0^\infty e^{-\lambda y} dy = \frac{1}{\lambda}. \tag{3.52}$$

The integral is well defined only for $\operatorname{Re}\lambda > 0$, but obviously the result is analytic on the whole $\mathbb{C}^*$. An important observation for the following calculations is that a change of variable in the integral representation allows us to perform indirectly an analytic continuation. For example, the change of variable $y \to y/\sqrt{\lambda}$ leads to

$$f(\lambda) = \int_0^\infty e^{-\sqrt{\lambda}y} \frac{dy}{\sqrt{\lambda}} \tag{3.53}$$

where now the integral is well defined for $\operatorname{Re}\lambda^{1/2} > 0$, namely for $\lambda \in \mathbb{C}\backslash\mathbb{R}^-$. The change of variable in the integral has then allowed us to analytically continue $f(\lambda)$ to a larger set. We shall use this idea extensively.

3.3.1 Analyticity Domain

To find the analyticity domain of $Z(\lambda)$, let us rewrite (3.50) by making the leading term in the potential explicit, namely by setting $V(\phi) = v(\phi) + \phi^{2p}$ (we set $a_{2p} = 1$ for simplicity). Then

$$Z(\lambda) = \int_{-\infty}^\infty d\phi\, e^{-\phi^2 - \lambda\phi^{2p} - \lambda v(\phi)}. \tag{3.54}$$

This is analytic on $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$, but this domain can be extended by means of a change of variable. We let $\phi^2 = \lambda^{-\frac{q-1}{qp}} x^2$, for a general positive integer q. Since we are eventually interested in a real coupling λ, we make the change of variable thinking of λ as being real and we only afterwards do we look into the analytic continuation to the complex values. We then have

$$Z(\lambda) = \int_{-\infty}^\infty dx\, \lambda^{-\frac{q-1}{2qp}} \exp\left\{-\lambda^{-\frac{q-1}{qp}} x^2 - \lambda^{\frac{1}{q}} x^{2p} - \lambda v\left(\lambda^{-\frac{q-1}{qp}} x\right)\right\}. \tag{3.55}$$

$Z(\lambda)$ is now analytic on the Sokal disc $D_0^q = \{\lambda \in \tilde{\mathbb{C}} : \operatorname{Re}\lambda^{\frac{1}{q}} > 0\}$, namely on $\{\lambda \in \tilde{\mathbb{C}} : |\arg\lambda| < q\frac{\pi}{2}\}$[1]. The analyticity domain can then be extended indefinitely, by letting q be large enough. However, we do not merely seek an analyticity domain, but an analyticity domain whereon the function $Z(\lambda)$ admit an asymptotic expansion. As stated in [Lod14] and reported in Proposition 2.6, this is true if and only if the limit for $|\lambda| \to 0$ of $Z(\lambda)$ and all its derivatives exists on every direction of D_0^q. If we change variable again to $y^2 = |\lambda|^{-\frac{q-1}{qp}} x^2$ we clearly find

$$Z(\lambda) = \int_{-\infty}^\infty dy\, e^{-i\theta\frac{q-1}{2qp}} \exp\left\{-y^2 e^{-i\theta\frac{q-1}{qp}} - |\lambda|\left(e^{-i\frac{\theta}{q}} y^{2p} + v\left(e^{-i\frac{q-1}{2qp}\theta} y\right)\right)\right\} \tag{3.56}$$

[1]Recall that $\tilde{\mathbb{C}}$ is the Riemann surface of the logarithm. We refer to [Sau14] for a rigorous definition and explanation.

where $\lambda = |\lambda| e^{i\theta}$. When $|\lambda| \to 0$, only the first term in the exponential survives. The limit will exist only if $\mathrm{Re}\, e^{-i\theta \frac{q-1}{qp}} > 0$ for every θ such that $|\theta| < q\frac{\pi}{2}$: the limit indeed fails to exist when the term multiplying $-y^2$ at the exponential is purely imaginary. We must then impose

$$\frac{q-1}{qp}\theta < \frac{\pi}{2}, \tag{3.57}$$

which yields $\theta < \frac{\pi}{2}\frac{qp}{q-1}$. By demanding that this hold for every θ, where $|\theta| < q\frac{\pi}{2}$, we obtain $\frac{p}{q-1} \geq 1$, whence $q \leq p+1$. The largest domain of analyticity in which $Z(\lambda)$ admits an asymptotic expansion is then

$$-(p+1)\frac{\pi}{2} < \theta < (p+1)\frac{\pi}{2}. \tag{3.58}$$

This was to be expected. Indeed, in analogy to the monomial potential $V(\phi) = \phi^{2p}$, we expect the asymptotic expansion of $Z(\lambda)$ to be of Gevrey class $1/(p-1)$, and the associated Borel transform $\hat{\varphi}(\zeta)$ to have a unique pole on $\mathbb{R}^-$. Assuming that $\hat{\varphi}(\zeta)$ has the correct exponential size, the Laplace transform $\mathcal{L}^0_{p-1}\hat{\varphi}$ is analytic on a sector of opening $(p-1)\pi$ centred on $\mathbb{R}^+$; however, it can be analytically continued to a sector of opening $\alpha + (p-1)\pi$ by rotating the integration line of the Laplace transform, where α is the opening of the sector whereon $\hat{\varphi}(\zeta)$ can be analytically continued (with exponential size at most $1/(p-1)$). In our case, α is infinitesimally smaller than 2π: therefore the whole opening is $(p+1)\pi$, which is indeed the opening of the open sector in (3.58).

Thus, we infer that $Z(\lambda)$ can be analytically continued on the Sokal disc of infinite radius $D_0^{p-1} = \{\lambda \in \tilde{\mathbb{C}} : \mathrm{Re}\, \lambda^{-\frac{1}{p-1}} > 0\}$. The first assumption of Nevanlinna-Sokal theorem is then satisfied. Note that, for $p \geq 3$, the Sokal disc D_0^{p-1} is a subset of $\tilde{\mathbb{C}}$ but not of $\mathbb{C}$.

As a final remark, observe that the change of variable $\phi^2 = \lambda^{-\frac{q+1}{qp}} x^2$ would have led to the same analyticity domain $|\theta| < q\frac{\pi}{2}$, but in this case, following the same lines as above, we would have found $q < p-1$. This should not be regarded as a contradiction, but simply as an example of how different changes of variables provide different analytic continuations with different ranges of validity.

3.3.2 Gevrey Asymptotics

Next, we need to prove that $Z(\lambda)$ admits an asymptotic expansion of Gevrey class $1/(p-1)$. To do so, we resort to Taylor expansion with integral remainder: given a function $f(x)$ differentiable infinitely many times at 0, one has

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{x^{n+1}}{n!} \int_0^1 du (1-u)^n f^{(n+1)}(xu). \tag{3.59}$$

Hence, after setting

$$k! c_k := \frac{d^k}{d\lambda^k} Z(0) = \int_{-\infty}^{\infty} d\phi e^{-\phi^2} (-V(\phi))^k, \tag{3.60}$$

we have

$$\left| Z(\lambda) - \sum_{k=0}^{n} c_k \lambda^k \right| = \left| \frac{\lambda^{n+1}}{n!} \int_0^1 du (1-u)^n \int_{-\infty}^{\infty} d\phi e^{-\phi^2 - \lambda u V(\phi)} (V(\phi))^{n+1} \right|. \tag{3.61}$$

We now have to prove that the right-hand side can by bounded by $\lambda^{n+1} A B^{n+1} ((n+1)!)^{p-1}$ for two constants A, B independent of λ and n, uniformly on the Sokal disc $D_0^{p-1} = \{\lambda \in \tilde{\mathbb{C}} : \mathrm{Re}\, \lambda^{-\frac{1}{p-1}} > 0\}$. To do so, we perform a sequence of changes of variables like in the previous case. We first let $\phi^2 = \lambda^{-\frac{p+1}{p^2}} x^2$: the integral over ϕ takes then the form

$$I := \int_{-\infty}^{\infty} dx \lambda^{-\frac{p+1}{2p^2}} \exp\left\{ -\lambda^{-\frac{p+1}{p^2}} x^2 - u\lambda^{-\frac{1}{p}} x^{2p} - u\lambda v \left(\lambda^{-\frac{p+1}{2p^2}} x \right) \right\} V^{n+1} \left(\lambda^{-\frac{p+1}{2p^2}} x \right). \tag{3.62}$$

As before, the modulus can be reabsorbed in the integration variable be letting $y^2 = |\lambda|^{-\frac{p+1}{p^2}} x^2$, whence

$$I = \int_{-\infty}^{\infty} dy\, e^{-i\theta\frac{p+1}{2p^2}} \exp\left\{ -e^{-i\theta\frac{p+1}{p^2}} y^2 - u|\lambda| \left[e^{-i\frac{\theta}{p}} y^{2p} + ... + a_1 e^{i\theta\frac{2p^2-p-1}{p^2}} y \right] \right\} V^{n+1}\left(\lambda^{-\frac{p+1}{2p^2}} y \right). \quad (3.63)$$

On the Sokal disc D_0^{p-1}, we have $|\theta| < (p-1)\frac{\pi}{2}$: thus the integral is well defined because $e^{i\frac{\theta}{p}}$ has positive real part for every θ. We now set out to put an upper bound to the integral. We can bound $|I|$ by taking the integral of the modulus of the integrand: this amounts to taking the real part of all the coefficients of the polynomial at the exponential, and also to replacing V^{n+1} by $|V^{n+1}|$. It will be convenient to separate the terms with positive or negative coefficients at the exponential, by collecting them into two polynomials $f(y)$ and $g(y)$. We can then write

$$|I| \leq \int_{-\infty}^{\infty} dy\, e^{-f(y)+g(y)} h(y) \quad (3.64)$$

where $f(y)$ is a polynomial of degree $2p$ whose coefficients are all positive, and whose leading term is $u|\lambda|\cos\left(\frac{\theta}{p}\right) y^{2p}$; $g(y)$ is a polynomial of degree smaller or equal to $2p-1$ whose coefficients are again all positive; finally $h(y) = \left| V^{n+1}\left(y e^{-i\theta\frac{p+1}{2p^2}} \right) \right|$ is function which is positive on the whole domain of integration. We now introduce the following lemma:

> **Lemma 3.3.** Let $f(y)$ and $g(y)$ be two polynomials whose coefficients are all positive, and such that the degree of f is even and $\deg f > \deg g$. Let us split the monomials of even and odd degree of f by writing $f(y) = P(y^2) + yQ(y^2)$, where P and Q are again polynomials of positive coefficients. Let $h(y)$ be a function such that $h(y) \geq 0$ for every y and which grows polynomially for large y. Then there exists a unique positive constant K such that
>
> $$I = \int_{-\infty}^{\infty} dy\, e^{-f(y)+g(y)} h(y) \leq \int_{-\infty}^{\infty} dy\, e^{-cP(y^2)} h(y) \quad (3.65)$$
>
> for every positive $c \leq K$.

Proof. Consider the smooth function

$$F(c) = \int_{-\infty}^{\infty} dy \left[e^{-f(y)+g(y)} - e^{-cP(y^2)} \right] h(y). \quad (3.66)$$

Then we observe that:

1. $F(0) = -\infty$, as
$$F(0) = I - \int_{-\infty}^{\infty} dy h(y), \quad (3.67)$$
and the second term is divergent to $+\infty$ as $h(y)$ grows polynomially for large y;
2. $F(\infty) = I$;
3. $F'(c) > 0$, which can be seen by writing
$$F'(c) = \int_{-\infty}^{\infty} dy \left[e^{-f(y)+g(y)} + P(y^2) e^{-cP(y^2)} \right] h(y) \quad (3.68)$$
and by recalling that $P(y^2)$ is everywhere positive because it has positive coefficients.

Then, by the mean value theorem, there must exist $K > 0$ such that $F(K) = 0$. As $F'(c) > 0$, K is unique and $F(c) < 0$ for every positive $c < K$. □

Thanks to the foregoing result, we are now enabled to write, for some $c < K$,

$$|I| \leq \int_{-\infty}^{\infty} dy\, e^{-cP(y^2)} \left| V^{n+1}\left(y e^{-i\theta\frac{p+1}{2p^2}} \right) \right| \tag{3.69}$$

where $P(y^2) = u|\lambda| \cos\left(\frac{\theta}{p}\right) y^{2p} + \cos\left(\theta\frac{p+1}{p^2}\right) y^2 + \dots$. Indeed, for $|\theta| < (p-1)\frac{\pi}{2}$, we have that $\cos\left(\theta\frac{p+1}{p^2}\right) > 0$. As all the terms in $P(y^2)$ are positive for every y, we can write

$$e^{-cP(y^2)} \leq e^{-c\cos\left(\theta\frac{p+1}{p^2}\right)y^2}. \tag{3.70}$$

This step enables us to split the integrals in (3.61), as now the dependence on u and $|\lambda|$ is dropped. Moreover, as $|\theta| < (p-1)\frac{\pi}{2}$, we have $\cos\left(\theta\frac{p+1}{p^2}\right) > \cos\left(\frac{\pi}{2}\frac{p^2-1}{p^2}\right) =: \varepsilon$. We can then bound $|I|$ with the Gaussian integral

$$|I| \leq \int_{-\infty}^{\infty} dy\, e^{-c\varepsilon y^2} \left| V^{n+1}\left(y e^{-i\theta\frac{p+1}{2p^2}} \right) \right|, \tag{3.71}$$

so that the dependence on θ of the exponential is also dropped.

We turn now our attention to $\left| V\left(y e^{-i\theta\frac{p+1}{2p^2}} \right) \right|$. The absolute value can be bounded by taking the sum of the absolute values of every monomial. Then we will have

$$\left| V\left(y e^{-i\theta\frac{p+1}{2p^2}} \right) \right| \leq b_{2p} y^{2p} + b_{2p-1}|y^{2p-1}| + \dots \tag{3.72}$$

where $b_k = \left| a_k \cos\left(i\theta\left(1 - \frac{k(p+1)}{2p^2}\right)\right)\right| \leq |a_i|$ and we take the absolute value of the odd powers of y. After putting $A := \max\{|a_i|, i = 1, \dots, p\}$, we have

$$\left| V\left(y e^{-i\theta\frac{p+1}{2p^2}} \right) \right|^{n+1} \leq A^{n+1}(y^{2p} + \dots + |y|)^{n+1} = A^{n+1} \sum \frac{(n+1)!}{k_1! \cdot \dots \cdot k_{2p}!} \prod_{i=1}^{2p} (|y|^i)^{k_i} \tag{3.73}$$

where the sum runs over all the integers $k_1, \dots, k_{2p}$ such that $k_1 + \dots + k_{2p} = n+1$. Because these terms are integrated over a Gaussian measure $dy e^{-c\varepsilon y^2}$, we can replace $|y|^j$, for $1 \leq j \leq 2p(n+1)$, with the leading power $y^{2p(n+1)}$ (although it is of course not true that $|y|^j < y^{2p(n+1)}$ for every y). Recalling that the sum over the multinomial factors is bounded exponentially, as

$$\sum \frac{(n+1)!}{k_1! \cdot \dots \cdot k_{2p}!} = (2p)^{n+1}, \tag{3.74}$$

we can finally write

$$|I| \leq (2Ap)^{n+1} \int_{-\infty}^{\infty} dy\, e^{-c\varepsilon\, y^2} y^{2p(n+1)} = \frac{1}{\sqrt{c\varepsilon}} \left(\frac{2Ap}{(c\varepsilon)^p} \right)^{n+1} \Gamma\left(p(n+1) + \tfrac{1}{2} \right). \tag{3.75}$$

The integral over u in (3.61) is now trivial and we finally reach

$$\left| Z(\lambda) - \sum_{k=0}^{n} c_k \lambda^k \right| \leq \frac{\lambda^{n+1}}{(n+1)!} \frac{1}{\sqrt{c\varepsilon}} \left(\frac{2Ap}{(c\varepsilon)^p} \right)^{n+1} \Gamma\left(p(n+1) + \tfrac{1}{2} \right) \tag{3.76}$$

for every n and for every $\lambda \in D_0^{p-1}$. The asymptotic expansion is of Gevrey class $1/(p-1)$ due to the factor of $(n+1)!$ at the denominator. It is instructive to observe that the Gevrey bounds are uniform because ε is finite: tracing back our steps, we realise the importance of our (seemingly prescient) change of variable $\phi^2 = \lambda^{-\frac{p+1}{p^2}} x^2$, as opposed to the earlier $\phi^2 = \lambda^{-\frac{1}{p+1}} x^2$ which led us to the maximal domain

$\{\lambda \in \tilde{\mathbb{C}} : |\arg(\lambda)| < \frac{\pi}{2}(p+1)\}$ whereon $Z(\lambda)$ admits an asymptotic expansion. Indeed, had we chosen the latter, we would have found a vanishing ε and the Gevrey bounds would not have been uniform. This teaches us the rather subtle lesson that a Borel-summable functions admits *uniform* Gevrey asymptotic expansion only in *closed* subsectors of the *maximal* domain whereon it admits an asymptotic expansion.

Using the Stirling approximation, one can then find two positive constants $\tilde{A}, \tilde{B}$ and rephrase the latter equation as

$$\left| Z(\lambda) - \sum_{k=0}^{n} c_n \lambda^k \right| \leq \tilde{A}\tilde{B}^{n+1}((n+1)!)^{p-1}\lambda^{n+1}. \tag{3.77}$$

Thus, Nevanlinna-Sokal theorem 2.6 is satisfied and $Z(\lambda)$ is Borel-summable, and more precisely $(p-1)$-summable: in other words, it is the Borel sum of its $(p-1)$-summable asymptotic expansion. Clearly, this proof is also comprehensive of the monomial interaction cases, i.e. $V(\phi) = \phi^{2p}$, dealt with in the previous sections. Observe that Borel summability of $Z(\lambda)$ is inherited from the Borel summability of the partition function for the monomial potential $V(\phi) = \phi^{2p}$. Indeed, the formal power series

$$\tilde{\varphi} := \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int_{-\infty}^{\infty} d\phi \; e^{-\phi^2} V^n(\phi) \tag{3.78}$$

has alternating signs from n large enough, since eventually the Gaussian integration of leading term $(a_{2p}\phi^{2p})^n$, which grows like $\Gamma(np+1)$, will dominate entirely the subleading terms. This feature pushes the pole in the Borel plane on the negative real axis $\mathbb{R}^-$.

Invoking once more Observation 2.2, we can conclude that $Z(\lambda)$, when restricted on a finite interval $\lambda \in [0, R] \subset \mathbb{R}$, with $R > 0$, is a tame function, definable in the o-minimal structure $\mathbb{R}_{\mathscr{g}}$.

3.3.3 Correlation functions

In analogy to the monomial potential, the correlation functions are also Borel-summable. The j-th moment

$$G_j(\lambda) := \int_{-\infty}^{\infty} d\phi \; e^{-\phi^2 - \lambda V(\phi)} \phi^j \tag{3.79}$$

with $V(\phi)$ a polynomial as above, can be proved to be Borel-summable following the same steps as before. After putting $h(y) = \left| y^j e^{-i\theta j \frac{p+1}{2p^2}} V^{n+1}\left(y e^{-i\theta \frac{p+1}{2p^2}} \right) \right|$, the only difference is that, instead of (3.75), we will get

$$|I| \leq (2Ap)^{n+1} \int_{-\infty}^{\infty} dy \; e^{-c\varepsilon \, y^2} y^{2p(n+1)+j} = \frac{1}{\sqrt{(c\varepsilon)^{j+1}}} \left(\frac{2Ap}{(c\varepsilon)^p} \right)^{n+1} \Gamma\left(p(n+1) + \tfrac{j+1}{2} \right). \tag{3.80}$$

For the Gamma function we can write, if $j = 2k$,

$$\begin{aligned} \Gamma\left(p(n+1) + \tfrac{j+1}{2} \right) &= \left(p(n+1) + \tfrac{2k-1}{2} \right) \left(p(n+1) + \tfrac{2k-3}{2} \right) ... \left(p(n+1) + \tfrac{1}{2} \right) \Gamma\left(p(n+1) + \tfrac{1}{2} \right) \\ &\leq \left(p(n+1) + \tfrac{j-1}{2} \right)^k \Gamma\left(p(n+1) + \tfrac{1}{2} \right) \leq (n+1)^k \left(p + \tfrac{j-1}{2} \right)^k \Gamma\left(p(n+1) + \tfrac{1}{2} \right) \\ &\leq e^{kn} \left(p + \tfrac{j-1}{2} \right)^k \Gamma\left(p(n+1) + \tfrac{1}{2} \right); \end{aligned} \tag{3.81}$$

similarly, when $j = 2k+1$,

$$\Gamma\left(p(n+1) + k + 1 \right) \leq e^{nk} (p+k)^k \Gamma(p(n+1)+1). \tag{3.82}$$

Thus, the Gevrey bounds are also satisfied for $G_j(\lambda)$, which is again $(p-1)$-summable and a tame function in $\mathbb{R}_{\mathscr{g}}$. The cumulants

$$\mathfrak{K}_j(\lambda) := \langle \phi^j \rangle = \frac{G_j(\lambda)}{Z(\lambda)} \tag{3.83}$$

are therefore tame functions of $\mathbb{R}_{\mathscr{g}}$, being the ratio of two definable functions of the same structure.

3.4 Resurgent QFT on a point

All the partition and correlation functions shown hitherto are Borel-summable. In this section, following [CDU15], we show a QFT in zero dimensions whose partition function is affected by the Stokes phenomenon at $\theta = 0$ and therefore is not Borel-summable. Nevertheless, we will manage to argue that it is definable in the o-minimal structure $\mathbb{R}_{\text{Pfaff}}$ described in section 1.6.3.

3.4.1 Perturbative and non-perturbative expansions

Consider the following partition function for a theory on a point:

$$Z(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx e^{-\frac{1}{2\lambda}\sin^2(x)}. \tag{3.84}$$

The potential $\frac{1}{2\lambda}\sin^2(x)$ is a tame function, since the interval is restricted: we should then expect the partition function to be tame. The potential presents now a minimum at $x = 0$ and a saddle at $x = \pm\pi/2$. Hence we perform perturbation theory first around the minimum. Setting $x = \arcsin y$ and $2\lambda t = y^2$, we have

$$Z(\lambda) = \frac{2}{\lambda}\int_0^1 \frac{dy}{\sqrt{1-y^2}} e^{-\frac{y^2}{2\lambda}} = \sqrt{2}\int_0^{\frac{1}{2\lambda}} \frac{dt}{\sqrt{t}} \frac{e^{-t}}{\sqrt{1-2\lambda t}}. \tag{3.85}$$

Expanding the square root, we find a divergent formal power series:

$$\sqrt{2}\sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!}(2\lambda)^n \int_0^{\frac{1}{2\lambda}} \frac{dt}{\sqrt{t}} t^n = \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}(2\lambda)^n \gamma\left(n+\frac{1}{2}, \frac{1}{2\lambda}\right), \tag{3.86}$$

where γ is the incomplete Gamma function. Yet, for fixed n, in the limit $\lambda \to 0$, this differs by the complete Γ only by non-analytic terms, so we find the formal power series

$$\sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma^2\left(n+\frac{1}{2}\right)}{n!}(2\lambda)^n =: \sqrt{\frac{2}{\pi}} \frac{1}{2\lambda} \tilde{\varphi}_0 \tag{3.87}$$

with $\tilde{\varphi}_0 \in \lambda\mathbb{R}[[\lambda]]$, as found in [CDU15]. Note the absence of the alternating signs, which impairs Borel summability. The Borel transform can actually be computed exactly: Mathematica yields

$$\mathcal{B}\tilde{\varphi}_0 = \sum_{n=0}^{\infty} \frac{\Gamma^2\left(n+\frac{1}{2}\right)}{(n!)^2}(2\zeta)^n = 2K(2\zeta) \tag{3.88}$$

where K is the elliptic integral of the first kind. With (3.26), it is also easy to find

$$\mathcal{B}\tilde{\varphi}_0 = 2\pi\ {}_2F_1\left(\left.\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\right| 2\zeta\right) =: \hat{\varphi}_0(\zeta) \tag{3.89}$$

with $\vec{a} = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\vec{b} = (1)$. In either case, there is a pole at $\zeta = 1/2$, so that the series is not 1-summable: we are then unable to invoke Observation 2.1 to argue the tameness of the partition function in $\mathbb{R}_{\mathscr{G}}$.

Roughly, the perturbative power series $\tilde{\varphi}_0$ fails to be Borel-summable because of the non-perturbative effects, which must be computed around the saddle point $x = \pm\pi/2$. The perturbation around these points can be expressed as the perturbation around $x = 0$ by simply reversing the potential by $\lambda \to -\lambda$, and then multiplying by the value of the action at $\pm\pi/2$, namely $e^{-\frac{1}{2\lambda}}$. From a different viewpoint, $e^{-\frac{1}{2\lambda}}$ is the resurgent symbol $e^{-\omega/\lambda}$, where ω is the unique pole in the Borel plane located at $\zeta = 1/2$. This pole however, is neither simple nor logarithmic, so $\hat{\varphi}_0(\zeta)$ is not a simple resurgent function in the sense of

Definition 2.10. We refer to [CDU15] for a careful treatment of this model in terms of the Lefschetz thimbles. The non-perturbative formal expansion is

$$e^{-\frac{1}{2\lambda}}\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma^2\left(n+\frac{1}{2}\right)}{n!}(-2\lambda)^n =: e^{-\frac{1}{2\lambda}}\sqrt{\frac{2}{\pi}}\frac{1}{2\lambda}\tilde{\varphi}_1 \tag{3.90}$$

with $\tilde{\varphi}_1 \in \lambda\mathbb{R}[[\lambda]]$. Its Borel transform is, in analogy to (3.89),

$$\mathcal{B}\tilde{\varphi}_1 = 2\pi\ {}_2F_1\left(\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\middle| -2\zeta\right) =: \hat{\varphi}_1(\zeta). \tag{3.91}$$

where again $\vec{a} = \left(\frac{1}{2},\frac{1}{2}\right)$ and $\vec{b} = (1)$. Although the partition function indeed admits $\tilde{\varphi}_0$ as 1-Gevrey asymptotic expansion, namely $Z(\lambda) \sim_1 \sqrt{\frac{2}{\pi}}\frac{1}{2\lambda}\tilde{\varphi}_0$, the asymptotic expansion will not be uniform on a Sokal disc D_c^1 (for any positive c). In fact in this model, as the pole of $\hat{\varphi}_0$ at $\zeta = 1/2$ indicates, the Stokes phenomenon occurs at $\theta = 0$, and therefore $Z(\lambda)$ is not the Borel sum of either the two formal power series $\tilde{\varphi}_0$ or $\tilde{\varphi}_1$. Rather, the correct object whose median resummation returns the actual function $Z(\lambda)$ is the transseries

$$\mathfrak{Z}(\lambda,\sigma) = \tilde{\Phi}_0(\lambda) + \sigma e^{-\frac{1}{2\lambda}}\tilde{\Phi}_1 \tag{3.92}$$

where $\tilde{\Phi}_0 = \sqrt{\frac{2}{\pi}}\frac{1}{2\lambda}\tilde{\varphi}_0$ is the perturbative sector and $\tilde{\Phi}_1 = \sqrt{\frac{2}{\pi}}\frac{1}{2\lambda}\tilde{\varphi}_1$ is the non perturbative sector, as indicated by the resurgent symbol $e^{-1/2\lambda}$.

3.4.2 Resurgence of the non-perturbative sector

As $\hat{\varphi}_0(\zeta)$ has a pole along the positive real line, namely in the direction $\theta = 0$, the action of the Stokes automorphism $\mathfrak{S}_0$ will be non-trivial on $\tilde{\varphi}_0$. We can compute this action explicitly by exploiting Cauchy theorem. Let us define H to be the Hankel contour in the Borel plane which winds clockwise around the half line $(1/2,\infty)$, exactly as we did for the modified Euler series in Figure 2.2a. Then let $\varepsilon > 0$ to be a small angle: following [CDU15], we compute

$$\begin{aligned}\left(\mathscr{S}^{\varepsilon} - \mathscr{S}^{-\varepsilon}\right)\tilde{\varphi}_0 &= \oint_H d\zeta e^{-\zeta/\lambda}\hat{\varphi}_0(\zeta) \\ &= 2\pi\int_{\frac{1}{2}}^{\infty} d\zeta e^{-\zeta/\lambda}\left[{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\middle| 2(\zeta+i\varepsilon)\right) - {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\middle| 2(\zeta-i\varepsilon)\right)\right].\end{aligned} \tag{3.93}$$

Whenever $a_1 + a_2 = b$ as it is the case here, the discontinuity of the hypergeometric function ${}_2F_1\left(\frac{\vec{a}}{\vec{b}}\middle| t\right)$ is described by the identity

$${}_2F_1\left(\begin{matrix}a_1,a_2\\ b\end{matrix}\middle| t+i\varepsilon\right) - {}_2F_1\left(\begin{matrix}a_1,a_2\\ b\end{matrix}\middle| t-i\varepsilon\right) = \frac{2\pi i\Gamma(b)}{\Gamma(a_1)\Gamma(a_2)}{}_2F_1\left(\begin{matrix}b-a_1,b-a_2\\ 1\end{matrix}\middle| 1-t\right). \tag{3.94}$$

Hence, we find

$$\begin{aligned}\left(\mathscr{S}^{\varepsilon} - \mathscr{S}^{-\varepsilon}\right)\tilde{\varphi}_0 &= 4\pi i\int_{\frac{1}{2}}^{\infty} d\zeta e^{-\zeta/\lambda}\ {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\middle| 1-2\zeta\right) \\ &= 4\pi i e^{-1/2\lambda}\int_0^{\infty} d\zeta e^{-\zeta/\lambda}\ {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\middle| -2\zeta\right)\end{aligned} \tag{3.95}$$

where, quite remarkably, the vectors $\vec{a}$ and $\vec{b}$ are unchanged: we still have $a_1 = a_2 = \frac{1}{2}, b = 1$. Therefore, the last integral is nothing else then the Laplace transform of (3.91). Hence, we can conclude that

$$\left(\mathscr{S}^{\varepsilon} - \mathscr{S}^{-\varepsilon}\right)\tilde{\varphi}_0 = 2ie^{-1/2\lambda}\mathscr{S}^0\tilde{\varphi}_1 = 2ie^{-1/2\lambda}\mathscr{S}^{\varepsilon}\tilde{\varphi}_1 \tag{3.96}$$

where in the last line we have argued that, since $\hat{\varphi}_1(\zeta)$ has no poles on $\mathbb{R}^+$, tilting the line of the Laplace integration by ε does not affect the result. Recalling the definition of Stokes automorphism (2.89), we realise that we have proved that

$$\mathfrak{S}_0\tilde{\varphi}_0 = \tilde{\varphi}_0 - 2ie^{-1/2\lambda}\tilde{\varphi}_1 \tag{3.97}$$

which, recast in terms of the transseries $\mathfrak{Z}$ of (3.92), yields

$$\mathfrak{S}_0\mathfrak{Z}(\lambda,0) = \sqrt{\frac{2}{\pi}}\frac{1}{2\lambda}\mathfrak{S}_0\tilde{\varphi}_0 = \sqrt{\frac{2}{\pi}}\frac{1}{2\lambda}\left[\tilde{\varphi}_0 - 2ie^{-1/2\lambda}\tilde{\varphi}_1\right] = \mathfrak{Z}(\lambda,-2i). \tag{3.98}$$

This means that, as $-2i$ is the Stokes constant associated with the Stokes phenomenon of this problem. As the above equation shows, the Stokes automorphism relates in a very simple way the perturbative sector to the non-perturbative one: in other words, $\tilde{\varphi}_1$ *resurges* from $\tilde{\varphi}_0$, whence the name resurgence.

3.4.3 Tameness of the partition function

As described in the previous sections, the perturbative power series $\tilde{\varphi}_0$ is not 1-summable: we cannot then argue the definability of $Z(\lambda)$ in $\mathbb{R}_{\mathscr{G}}$. Yet, we can observe that

$$Z(\lambda) = \frac{1}{\sqrt{4\lambda}}e^{-\frac{1}{4\lambda}}\int_{-\pi}^{\pi} e^{\frac{\cos y}{4\lambda}} = \frac{\pi}{\sqrt{\lambda}}e^{-\frac{1}{4\lambda}}I_0\left(\frac{1}{4\lambda}\right) \tag{3.99}$$

where I_0 is the modified Bessel function of the first kind. This is very reminiscent of the case treated in section 3.2.1. In analogy to [GSV23], we can exploit this fact to argue the tameness of this partition function in the o-minimal structure $\mathbb{R}_{\text{Pfaff}}$, described in section 1.6.3. Let $g = 1/(4\lambda)$ and let $\tilde{Z}(g) = Z(g)/(2\pi\sqrt{g})$. Then we have

$$\tilde{Z}(g) = e^{-g}I_0(g). \tag{3.100}$$

We can equivalently argue for the tameness of $\tilde{Z}(g)$ rather than of $Z(g)$, as they are related algebraically. The modified Bessel function $I_\alpha(g)$ obeys the differential equation

$$\left[g^2\frac{d^2}{dg^2} + g\frac{d}{dg} - (g^2+\alpha^2)\right]I_\alpha(g) = 0. \tag{3.101}$$

Thus, for $\alpha = 0$ and setting $I_0(\alpha) = e^{-g}\tilde{Z}(g)$ we find:

$$\left[\frac{d^2}{dg^2} + \left(2+\frac{1}{g}\right)\frac{d}{dg} + \frac{1}{g}\right]\tilde{Z}(g) = 0. \tag{3.102}$$

We can then write the following Pfaffian chain:

$$\begin{cases} \zeta_1(g) = 1/g \\ \zeta_2(g) = \frac{\tilde{Z}'(g)}{\tilde{Z}(g)} \\ \zeta_3(g) = \tilde{Z}(g) \end{cases} \qquad \begin{cases} \zeta_1' = -\zeta_1^2 \\ \zeta_2' = -\zeta_2^2 - \zeta_2(\zeta_1+2) - \zeta_1 \\ \zeta_3' = \zeta_3\zeta_2 \end{cases} \tag{3.103}$$

which proves that $Z(g)$ is definable in $\mathbb{R}_{\text{Pfaff}}$. In this case then, the presence of a differential equation ensures the tameness of a non-Borel-summable partition function. The same model has already been considered in [DGS22], where it was argued that the partition function for this model is also definable in $\mathbb{R}_{\text{an, exp}}$.

Although $\mathbb{R}_{\text{Pfaff}}$ is a different structure from $\mathbb{R}_{\mathscr{G}}$, we can recall from section 1.6.3 that the Pfaffian closure of an o-minimal structure remains o-minimal. Therefore, we can argue that the Borel-summable partition and correlation functions presented in the previous sections of this Chapter and the partition function $Z(\lambda)$ presented in this section (which is not Borel-summable) are all definable in a unique o-minimal structure $\mathcal{P}(\mathbb{R}_{\mathscr{G}})$, namely the Pfaffian closure of $\mathbb{R}_{\mathscr{G}}$. However, we can hardly expect that all non-Borel-summable partition and correlation functions be definable in $\mathbb{R}_{\text{Pfaff}}$, so that the problem of establishing the tameness of these functions systematically remains open. It is nonetheless tempting to speculate that some more non-Borel summable partition functions may be definable in the Pfaffian closure of $\mathbb{R}_{\mathscr{G}}$. We shall leave these questions for future research.

Chapter 4

Constructive QFT

In this chapter we aim at extending our previous results to more general theories: to do so, we will need to introduce constructive field theory. Constructive field theory (see e.g. the introduction of [Riv07] and [Riv09] for a brief pedagogical review) provides an alternative to perturbative field theory, avoiding the blemish of divergence that affects perturbative expansions. As explained in section 3.1, perturbative field theory expresses an amplitude $\mathcal{A}(\lambda)$ as series in powers of the coupling λ: since Feynman diagrams proliferate, the sum is divergent, and thus it merely provides an asymptotic expansion to the function $\mathcal{A}(\lambda)$, which is our true purpose to compute. However, a divergent series is always asymptotic to infinitely many functions[1], so that a theory purely based on perturbative expansions is not truly predictive and, according to the stern judgement of J. Magnen and V. Rivasseau in [Riv07], is in fact no theory at all. Constructive field theory aims at expanding the path integral in a such a way that divergences are avoided. In order to fulfil this quite ambitious end, the common notion of sum over Feynman diagrams must be forsaken in favour of a sum over *forests* and *spanning trees*. The deep insight is that it is not necessary to know the full loop structure of Feynman diagrams to capture all the information they carry: the latter is stored in their loop-less subgraphs, namely their spanning trees. This realisation led to the formulation of the 'constructive golden rule' ([MR08; Riv07])

Thou shall not know all the loops, or thou shall diverge!

In the first section of this chapter, 4.1, we will introduce the basic terminology and the BKAR forest formula, which lies at the foundation of constructive QFT. In section 4.2, we will clarify the relationship between constructive and perturbative QFT by explaining how Feynman diagrams are related to spanning trees. We will then show how, when combined with the *Loop Vertex Expansion*, constructive field theory provides a very powerful tool to prove Borel summability of partition functions and amplitudes: we will apply these tools explicitly to the ϕ^4 theory on a point in section 4.3. In the remaining sections, referring to the literature, we will point out how these methods can be used to show the Borel summability of more general partition functions. In light of the observations stated since section 2.6, these partition functions will be tame in $\mathbb{R}_{\mathscr{G}}$.

[1] To understand why, suppose $\tilde{\varphi} \in x\mathbb{R}[[x]]_1$ is a 1-summable formal series, so that on a Sokal disc D_c^1 its Borel sum $\varphi(x) = \mathscr{S}^0\tilde{\varphi}$ satisfies $\varphi(x) \sim_1 \tilde{\varphi}$. Then for any $a \in \mathbb{R}$, the function $\varphi(x) + af(x)$, with $f(x)$ given by our prototypical example of a non-analytic function at the origin with vanishing Taylor series at $x = 0$, namely (1.33), admits the same asymptotic expansion on the same domain. Recall also the discussion about quasi-analytic algebras of functions in section 2.6.

4.1 The BKAR forest formula

The essential ingredient of constructive expansions is the so-called BKAR forest formula, which we introduce in this section from a purely algebraic point of view. In order to do so, we first have to define some terminology of graph theory, following mostly [AR95; Riv09; GRS14] and references therein.

Let us begin with the definition of a graph

Definition 4.1 (Graph)**.** A **graph** $G = (V, E)$ is a set of **vertices** V and a set of **edges** E. An edge is a pair of two (not necessarily distinct) elements of V: namely, if $e \in E$, then $e = (a, b)$ with $a, b \in V$. Such a and b are called the **ends** of e.

Remark how in the set of edges E we may find a pair (a, b) more than once; moreover, we may also find an edge $e = (a, a)$ in which case e is called a loop (a tadpole in physicists' jargon). If $a \neq b$, with $a, b \in V$, we will say that $e = a(a, b)$ is a **link**.

Consider a graph G over n vertices, i.e. for which the cardinality $|V|$ of V is n. It will be often convenient to identify V with the set $\{1, ..., n\}$, so as to label every vertex with a number i, $i \in \{1, ..., n\}$. Let then $\mathcal{P}_n$ be the set of all the unordered pairs (i, j), with $i \neq j$ and $i, j \in \{1, ..., n\}$. The cardinality of $\mathcal{P}_n$ is therefore $\binom{n}{2}$, and an unordered pair $(i, j) \in \mathcal{P}_n$ is a link, which will be indicated with l. We introduce now the following

Definition 4.2 (Unordered Forest)**.** A **u-forest** (unordered forest) $\mathfrak{F}$ over n vertices is a graph (V, E), where $V = \{1, ..., n\}$, with with no cycles (loops in physicists' language). Namely, E is a subset of $\mathcal{P}_n$ such that one cannot find a subset of E of the form $\{(i_1, i_2), (i_2, i_3), ..., (i_k, i_1)\}$.

It will be important in the following to focus on connected forests, namely forests $\mathfrak{F} = (V, E)$ such that, taken any two distinct vertices $i, j \in V$, there exists a path $P \subseteq E \subset \mathcal{P}_n$, where $P = \{(i, i_1), (i_1, i_2), ..., (i_k, j\}$ connecting i and j. Because a forest has no cycles the path P *is unique*. This fact will be crucial to define the forest formula.

Definition 4.3 (Spanning tree)**.** A **spanning tree** $\mathfrak{T}$ is a connected forest (V, E) over n vertices which touches every vertex at least once. Namely, for every vertex $i \in \{1, ..., n\}$, there exists an edge (in fact, a link) $e \in E$ such that one of the ends of e is i: $e = (i, i') = (i', i)$.

The number of links which jut out of a vertex v is called the **coordination number** of v, and is indicated by $c(v)$. A vertex with coordination number equal to 1 is called a **leaf** of the spanning tree.

Let us remark here some results that will be useful later. The number of links of a spanning tree $\mathfrak{T}$ over n vertices is $n - 1$. To convince oneself of this, one can observe that all trees over n vertices can be obtained by rearranging the links of one of the trees which join all the vertices into a unique path (i.e., a tree with only two leaves): these trees have obviously $n-1$ vertices. As a consequence, the sum of all the coordination numbers is

$$\sum_{i=1}^{n} c_i = 2(n - 1), \tag{4.1}$$

because in summing over the coordination numbers of all the n vertices every link is counted twice. Observe, incidentally, that adding one link to any given tree would create a cycle (or loop). Finally, let us also state without proof a very important result which is at the heart of constructive theories:

> **Theorem 4.1** (Cayley)**.** The number of spanning trees over n vertices is n^{n-2} .

The reason of the importance of this result is that, for large n, we can infer from the Stirling approximation (2.127) that $n^{n-2} \sim (n!)e^n n^{-3/2}$. Thus, the number of spanning trees over n vertices grows roughly as $n!$ up to 'convergent' exponential terms.

We are now ready to state the BKAR (Brydges-Kennedy-Abdesselam-Rivasseau) forest formula (see especially [AR95] for extensive treatment and proofs). Consider a smooth function $f(x) = f(x_1, ..., x_N)$ of N variables, where N is a triangular number, namely $N = \binom{n}{2}$ for some integer n. Let us then draw a forest $\mathfrak{F}$ over n vertices: we can relabel each variable of f with one of the $\binom{n}{2}$ links of $\mathfrak{F}$. Note that, to do so, we must fix an ordering of the $\binom{n}{2}$ elements of $\mathcal{P}_n$, which can be done arbitrarily. Moreover, let us assign a weight (or *weakening factor*) w_l to every link l. With a slight abuse of notation, if $\mathfrak{F} = (V, E)$ and $l \in E$, we will write $l \in \mathfrak{F}$ without any risk of confusing links with vertices. Then the forest formula states that

$$f(1, ..., 1) = \sum_{\mathfrak{F}} \int_0^1 \left(\prod_{l \in \mathfrak{F}} dw_l \right) \left(\prod_{l \in \mathfrak{F}} \frac{\partial}{\partial x_l} \right) f(x) \Big|_{x^{\mathfrak{F}}(\boldsymbol{w})} \tag{4.2}$$

where the sum runs over all the unordered forests over n vertices. The point $x^{\mathfrak{F}}(\boldsymbol{w})$ at which the complicated derivative is evaluated is defined as follows:

- If $l = (i, j)$ is such that $l \in \mathfrak{F}$, then $x_l = w_l$.
- If $l = (i, j)$ is such that $l \notin \mathfrak{F}$, but there exists a path $P = \{(i, i_1), (i_1, i_2), ..., (i_k, j)\}$ in the forest which connects i and j, then x_l is defined to be the infimum of the weights w_l of the path P, namely:
$$x_l = \inf_{0 \leq s \leq k} \{w_{l_s}\} \tag{4.3}$$
where we have put $l_s = (i_s, i_{s+1})$, so that $P = \{l_1, l_2, ..., l_k\}$ and we agree that $i_0 := i$, $i_{k+1} := j$. This operation is well-defined because the path P is unique, if it exists.
- Finally, if $l = (i, j)$ is such that $l \notin \mathfrak{F}$ and there is no path P connecting i and j, then $x_l = 0$.

Clearly, the N-tuple $x^{\mathfrak{F}}(\boldsymbol{w})$ depends both on the topology of the forest $\mathfrak{F}$ and on the values of the weights w_l, whence our notation. As a final remark, note that the sum includes also the trivial forest, i.e. the empty forest with no links. In that case, the products over the links $l \in \mathfrak{F}$ are empty and they should be understood as being a multiplication by 1.

To convince ourselves that the formula makes sense, we can test it for $n = 2$, which implies $N = 1$. There are only two forests over 2 vertices, namely the empty forest and the tree whose only link is $(1, 2)$ (Figure 4.1 below). Then, being mindful of the above conventions, we have

$$f(1) = f(0) + \int_0^1 dw_1 \frac{\partial}{\partial x} f(x) \Big|_{w_1} \tag{4.4}$$

which is simply the fundamental theorem of calculus.

Figure 4.1: Forests over $n = 2$ vertices.

4.2 Feynman diagrams and spanning trees

Before setting out to apply the BKAR forest formula to quantum field theories, it will be expedient to discuss how spanning trees in constructive field theory are related to Feynman diagrams and perturbative field theory. An illuminating discussion con be found in [RW13], which we relate here briefly.

To fix ideas, let us quickly recap the example of section 3.2.1. For convenience, let us rewrite here the partition function in a slightly different way:

$$Z(\lambda) = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{\phi^2}{2} - \frac{\lambda}{8}\phi^4}, \tag{4.5}$$

so that the Gaussian measure $d\mu(\phi) := \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2}$ is normalized and with unit covariance. As already observed, the perturbative series obtained by Taylor-expanding the interaction term $e^{-\frac{\lambda}{8}\phi^4}$ and illegally commuting the sum with the integral is divergent. We proved that, on a Sokal disc D_0^1,

$$Z(\lambda) \sim_1 \sum_{n=0}^{\infty} a_n \left(-\frac{\lambda}{8}\right)^n \tag{4.6}$$

where now, since the covariance (or propagator) is unitary, all the Feynman diagrams evaluate to 1 and therefore a_n is simply given by the number of Wick contractions in the Gaussian integral $\int d\mu(\phi)\phi^{4n}$:

$$a_n = \frac{4n!}{2^{2n}(2n)!} = (4n-1)!! \,. \tag{4.7}$$

Thus, as already remarked, Feynman diagrams proliferate. Thus, in general terms, perturbative field theory expresses an amplitude $\mathcal{A}$ as a formal sum over Feynman graphs G, namely

$$\mathcal{A} = \sum_G A_G, \tag{4.8}$$

where A_G is the amplitude of each Feynman graph. Constructive field theory, on the contrary, expresses it as a sum over spanning trees $\mathfrak{T}$. To do this, consider the spanning trees $\mathfrak{T}$ of G: namely, if $G = (V, E)$ the spanning trees $\mathfrak{T} = (V, L)$ with $L \subseteq E$. To each such tree, we can assign a weight $w(G, \mathfrak{T})$ such that

$$\sum_{\mathfrak{T} \subset G} w(G, \mathfrak{T}) = 1. \tag{4.9}$$

Then we will have

$$\mathcal{A} = \sum_G A_G = \sum_G \sum_{\mathfrak{T} \subset G} w(G, \mathfrak{T}) A_G = \sum_{\mathfrak{T}} A_{\mathfrak{T}}, \tag{4.10}$$

where the tree amplitude $A_{\mathfrak{T}}$ is clearly

$$A_{\mathfrak{T}} = \sum_{G \supset \mathfrak{T}} w(G, \mathfrak{T}) A_G. \tag{4.11}$$

The weights can be computed explicitly: it was proved in [RW13] that

$$w(G, \mathfrak{T}) = \int_0^1 \prod_{l \in \mathfrak{T}} dw_l \prod_{l \notin \mathfrak{T}} x^{\mathfrak{T}}(\mathbf{w}) \tag{4.12}$$

where, as before, $x^{\mathfrak{T}}(\mathbf{w})$ is the infimum of the weights $w_{l'}$ along the path that joins the two ends of the link l. There is also an alternative definition, based on the notion of Hepp sector. For a graph $G = (V, E)$, a Hepp sector is simply an ordering $\sigma = \{\sigma(1), ..., \sigma(|E|)\}$ of the edges E, namely a pairing between the set E and the integers $\{1, ..., |E|\}$. For each of these, Kruskal greedy algorithm ([Kru56]) finds a spanning tree $\mathfrak{T}(\sigma)$ which minimizes the sum $\sum_{l \in \mathfrak{T}} \sigma(l)$, then one can actually *define* the weights $w(G, \mathfrak{T})$ as

$$w(G, \mathfrak{T}) = \frac{N(G, \mathfrak{T})}{|E|!} \tag{4.13}$$

where $N(G, \mathfrak{T})$ is the number of Hepp sectors such that $\mathfrak{T}(\sigma) = \mathfrak{T}$.

The idea behind constructive field theory is that the sum in (4.10) is convergent. In practice though (at least for theories with a quartic interaction), instead of repacking the sum over Feynman diagrams as shown before, the Feynman graphs with four-vertices resulting from perturbative expansion of the 'path' integral (4.5) are substituted by graphs with three-vertices. This can be achieved by performing a Hubbard-Stratonovich transformation and integrating out the initial field: the loops in the initial diagrams are then shrunk to *loop vertices* while the initial vertices are replaced by propagators of the new field. Only then is the sum proved to be be convergent and to be the Borel sum of the perturbative asymptotic expansion. This method is dubbed *Loop Vertex Expansion* and we will show the simplest application thereof in the next section.

4.3 Constructive ϕ^4 on a point

In this section, we will illustrate how, with the aid of the BKAR forest formula (4.2), the partition function $Z(\lambda)$ for a ϕ^4 Euclidean theory in zero dimensions dealt with in section 3.2.1 can be expressed constructively. Most interestingly, this approach will enable us to prove the Borel summability of $Z(\lambda)$, this time by exploiting the second point of Nevanlinna-Sokal theorem 2.6. We will follow the lines of [Riv09] and adapt from [ELT21].

4.3.1 Loop Vertex Expansion

Consider again the partition function of a bosonic ϕ^4 model in 0 dimensions (4.5). Recalling the identity

$$e^{-\frac{\lambda}{8}\phi^4} = \int_{-\infty}^{\infty} \frac{d\sigma}{\sqrt{2\pi}} e^{-\frac{\sigma^2}{2} - i\frac{\sqrt{\lambda}}{2}\phi^2\sigma} \tag{4.14}$$

we can perform a Hubbard-Stratonovich transformation, so as to trade the field ϕ for σ: by doing so, the quartic interaction ϕ^4 becomes a cubic mixed interaction $\phi^2\sigma$. The initial field can then be integrated away to find:

$$\begin{aligned} Z(\lambda) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{d\phi}{\sqrt{2\pi}}\tfrac{d\sigma}{\sqrt{2\pi}} e^{-\frac{\phi^2}{2} - \frac{\sigma^2}{2} - i\frac{\sqrt{\lambda}}{2}\sigma\phi^2} = \int_{-\infty}^{\infty} \tfrac{d\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\log\left(1+i\sigma\sqrt{\lambda}\right)} e^{-\frac{\sigma^2}{2}} \\ &= \int d\mu(\sigma) \sum_{n=0}^{\infty} \frac{1}{n!} V^n(\sigma) \end{aligned} \tag{4.15}$$

where $d\mu(\sigma)$ is the Gaussian measure of unit covariance $\frac{d\sigma}{\sqrt{2\pi}} e^{-\sigma^2/2}$ and $V(\sigma)$ is the *loop vertex* $V(\sigma) = -\frac{1}{2}\log\left(1 + i\sqrt{\lambda}\sigma\right)$, whence the name Loop Vertex Expansion (LVE).

We now wish to apply the forest formula (4.2). To do so, we must introduce *replicas* of the auxiliary field σ. To be more precise, we introduce n fields $\sigma_1, ..., \sigma_n$ and make the substitution

$$V^n(\sigma) \to \prod_{i=1}^{n} V(\sigma_i) \tag{4.16}$$

We can think of every replica as 'living' on a vertex i of a graph of n vertices. For consistency, we must now rewrite the integral measure in a similar fashion. We introduce then a normalized Gaussian measure of degenerate covariance $d\mu_C(\{\sigma_i\})$, where the covariance C is a symmetric matrix whose diagonal entries are simply equal to 1: $C_{ii} = 1$. Since there are $\binom{n}{2}$ off-diagonal entries, we can now set them to be equal to the variables x_l of a function f of $\binom{n}{2}$ variables. Then $C_{ij} = x_l$, with $l = (i, j)$ and $i \neq j$. The measure is degenerate because, when all the x_l are equal to 1, all the entries of the n-by-n matrix C are equal to 1 and thus C has rank 1. This divergence is absorbed in the normalization. Explicitly, one can write

$$d\mu_C(\{\sigma_i\}) = \frac{d\sigma_1}{\sqrt{2\pi}} e^{-\frac{\sigma_1^2}{2}} \prod_{i=2}^{n} \delta(\sigma_1 - \sigma_i) d\sigma_i \; ; \tag{4.17}$$

more conveniently though, the action of the Gaussian integration with normalized measure $d\mu_C$ can be expressed as a differential operator

$$\int d\mu_C(\{\sigma_i\}) f(\sigma_1, ..., \sigma_N) = e^{\frac{1}{2}C_{ij}\partial_i\partial_j} f(\sigma_1, ..., \sigma_N)\Big|_{\sigma_i = 0} \; , \tag{4.18}$$

where $N = \binom{n}{2}$, $\partial_i = \frac{d}{d\sigma_i}$ and repeated indices are summed over, each from 1 to n. In this way, the divergences are already taken into account and the result is well-defined. The integral

$$f(C) := \int d\mu_C(\{\sigma_i\}) \prod_{i=1}^{n} V(\sigma_i) \tag{4.19}$$

is now a function of the $\binom{n}{2}$ off diagonal couplings C_{ij}: we can then use the forest formula. Exchanging the sum with the integral, we obtain

$$f(C) = \sum_{\mathfrak{F}} \int_0^1 \left(\prod_{l \in \mathfrak{F}} dw_l \right) \left(\prod_{l \in \mathfrak{F}} \frac{\partial}{\partial C_l} \right) \int d\mu_C(\{\sigma_i\}) \prod_{i=1}^n V(\sigma_i) \Big|_{C = C^{\mathfrak{F}}(\boldsymbol{w})} \tag{4.20}$$

where by C_l we mean $C_{ij} = C_{ji}$, since $l = (i, j)$; moreover, $C^{\mathfrak{F}}(\boldsymbol{w})$ is the matrix whose entries C_l are computed according to the prescription listed above for the vector of $\binom{n}{2}$ components $x^{\mathfrak{F}}(\boldsymbol{w})$. A key observation to make is that the covariance $C^{\mathfrak{F}}(\boldsymbol{w})$ remains positive (see [AR95] for the proof), so that the Gaussian integral remains well-defined. The derivatives can be easily computed using the representation of the Gaussian integral as a differential operator (4.18): they simply bring down a product of pairs of derivatives with respect to the replicas σ_i. We can then write

$$f(C) = \sum_{\mathfrak{F}} \int_0^1 \left(\prod_{l \in \mathfrak{F}} dw_l \right) \int \left[\left(\prod_{l \in \mathfrak{F}} \frac{\partial}{\partial \sigma_{i(l)}} \frac{\partial}{\partial \sigma_{j(l)}} \right) \prod_{i=1}^n V(\sigma_i) \right] d\mu_{C^{\mathfrak{F}}(\boldsymbol{w})}(\{\sigma_i\}) \tag{4.21}$$

where again the integration over the Gaussian measure can be re-expressed as a differential operator. In the above formula, we write $i(l)$ and $j(l)$ for the two ends of the link $l = (i, j)$. Of course, which is which is not important because derivatives commute.

We can now write a constructive expansion for $Z(\lambda)$ and $F(\lambda) = \log Z(\lambda)$. Exchanging the sum with the integral in (4.15) we have

$$Z(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathfrak{F}} \int_0^1 \left(\prod_{l \in \mathfrak{F}} dw_l \right) \int \left[\left(\prod_{l \in \mathfrak{F}} \frac{\partial}{\partial \sigma_{i(l)}} \frac{\partial}{\partial \sigma_{j(l)}} \right) \prod_{i=1}^n V(\sigma_i) \right] d\mu_{C^{\mathfrak{F}}(\boldsymbol{w})}(\{\sigma_i\}) \tag{4.22}$$

where now the sum over the forests over n vertices is repeated for every n. The full power of this machinery though is displayed when we consider $\log Z(\lambda)$. It can be proved that, in analogy to Feynman diagrams, a very similar formula holds: it suffices to substitute for the sum over forests of n vertices a sum over *spanning trees* of n vertices, i.e. connected forests. The sum will now start from $n = 1$ ([Riv09]). Then we have that the free energy $F(\lambda) = \log Z(\lambda)$ is given by

$$F(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathfrak{T}} \int_0^1 \left(\prod_{l \in \mathfrak{T}} dw_l \right) \int \left[\left(\prod_{l \in \mathfrak{T}} \frac{\partial}{\partial \sigma_{i(l)}} \frac{\partial}{\partial \sigma_{j(l)}} \right) \prod_{i=1}^n V(\sigma_i) \right] d\mu_{C^{\mathfrak{T}}(\boldsymbol{w})}(\{\sigma_i\}). \tag{4.23}$$

The prescription to calculate $C^{\mathfrak{T}}(\boldsymbol{w})$ (or $x^{\mathfrak{T}}(\boldsymbol{w})$ in our earlier notation) is the same as for forests: the only observation is that now there always exists a path in $\mathfrak{T}$ connecting any two vertices i, j. The usefulness of the tree formula above is that spanning trees, unlike forests or Feynman diagrams, only grow factorially according to Cayley theorem. As opposed to the perturbative power series, whose coefficients are given by sums over Feynman diagrams with a fixed number of vertices, the sum over trees is convergent, as we will show next; moreover, it converges to the Borel sum of the perturbative power series. In the following sections we sketch the proof of the Borel summability of $F(\lambda)$ resorting to Nevanlinna-Sokal theorem 2.6, point 2.

4.3.2 Analyticity domain

In order to prove Borel summability, we have to find first a domain of analyticity in λ. Let us first observe that the derivatives of the loop vertices read

$$\left(\frac{d}{d\sigma} \right)^k V(\sigma) = -\left(\frac{d}{d\sigma} \right)^k \frac{1}{2} \log\Big(1 + i\sqrt{\lambda}\sigma\Big) = -\frac{1}{2} \frac{(-i\sqrt{\lambda})^k}{(1 + i\sqrt{\lambda}\sigma)^k} (k-1)! \, . \tag{4.24}$$

Since there are n replicas of every loop vertex, we can then rewrite the integrand of (4.23) as

$$-\frac{\lambda^{(n-1)}}{2^n} \prod_{i=1}^n \frac{1}{(1 + i\sqrt{\lambda}\sigma)^{c_i}} (c_i - 1)! \tag{4.25}$$

where c_i is the coordination number of the vertex i and we recall that their sum is $2(n-1)$. We then bound the module of the integral by taking the integral of the module. If we let $\lambda = re^{i\theta}$, we have

$$\left|\frac{1}{1+i\sqrt{\lambda}\sigma}\right|^2 = \left(\left(1 - r^{1/2}\sigma \sin\left(\tfrac{\theta}{2}\right)\right)^2 + r\sigma^2 \cos^2\left(\tfrac{\theta}{2}\right)\right)^{-1} \tag{4.26}$$

$$= \left(1 + r\sigma^2 - 2r^{1/2}\sigma \sin^2\left(\tfrac{\theta}{2}\right)\right)^{-1} = \left(1 - \sin^2\left(\tfrac{\theta}{2}\right) + \left(r^{1/2}\sigma - \sin\left(\tfrac{\theta}{2}\right)^2\right)^2\right)^{-1} \leq \frac{1}{\cos^2\left(\tfrac{\theta}{2}\right)}.$$

This is nothing else than the one-dimensional version of Lemma 1 in [ELT21]. Hence, we can bound the integrand (4.25) by

$$\frac{\lambda^{n-1}}{2^n \left[\cos\left(\tfrac{\theta}{2}\right)\right]^{2(n-1)}} \prod_{i=1}^{n} (c_i - 1)! \tag{4.27}$$

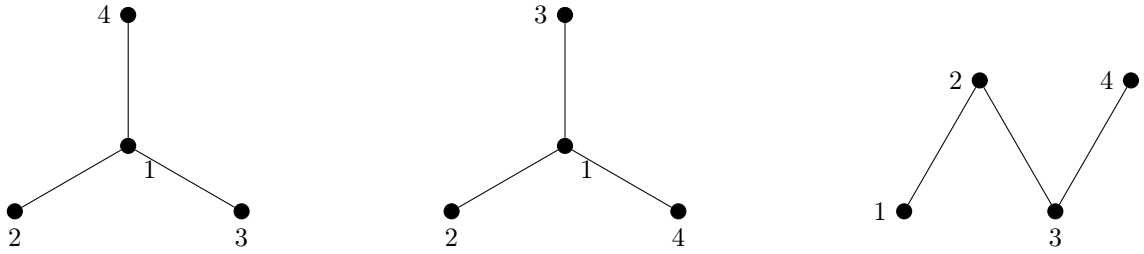
which no longer depends on the auxiliary fields σ_i. Therefore, the Gaussian integral gives simply 1, as the integral measure $d\mu_{C^{\mathfrak{T}}(\boldsymbol{w})}(\{\sigma_i\})$ is normalized. Similarly, the integrals over the weights will also yield 1, as the dependence of the integrand on the weights has been removed. We can then put an upper bound on the coefficient F_n in the sum for $F(\lambda)$ as

$$F_n \leq \frac{1}{n!} \sum_{\mathfrak{T}} \frac{\lambda^{n-1}}{2^n \left[\cos\left(\tfrac{\theta}{2}\right)\right]^{2(n-1)}} \prod_{i=1}^{n} (c_i - 1)! \,. \tag{4.28}$$

The sum over the trees can be estimated by a sum over the trees with a fixed configuration of coordination numbers $\{c_i\}$, multiplied by (an upper bound of) the number of possible configurations. As argued in [ELT21], the number of trees over n vertices with coordination numbers $c_1, ..., c_n$, which we call $\Omega(n\{c_i\})$, can be bounded by:

$$\Omega(n, \{c_i\}) \leq \frac{n!}{\prod_{i=1}^{n}(c_i - 1)!}. \tag{4.29}$$

To understand intuitively why the formula holds, one can first observe that the equality holds for the trees 'without branches', namely with only two leaves, as in Figure 4.2c . Then one simply estimates $\Omega(n\{c_i\})$ by dividing the number of all the possible labelling of the vertices ($n!$) by the number permutations up to overall rotations at every vertex i, namely $(c_i - 1)!$. This still implies overcounting. Consider for instance $n = 4$: the only 'branched' tree is obtained by $\{c_i\} = (3, 1, 1, 1)$ and permutations thereof, which amounts to 4 trees. Yet, the formula yields 12, as, for instance, the branched trees in Figure 4.2a, 4.2b below are counted separately.



(a) Branched tree 1. (b) Branched tree 2. (c) 'Branchless' tree.

Figure 4.2: Three trees over four vertices

The only thing that remains to do is then to put an upper bound to the number $K(n)$ of possible configurations of coordination numbers on a tree of n vertices. Following again [ELT21], we can argue that, since the sum of the n coordination numbers is $2(n-1)$, $K(n)$ is bounded by the area of the sphere S^{n-1} of radius $\sqrt{2(n-1)}$. The measure of the n-dimensional sphere S^n of radius R is

$$A(n; R) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} R^n, \tag{4.30}$$

hence that the area of the sphere of our interest is

$$A\left(n;\sqrt{2(n-1)}\right)=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}(2(n-1))^{\frac{n-1}{2}}. \tag{4.31}$$

Using the Stirling approximation, we realise that the factorial terms cancel and so $K(n)\leq A(n)\leq b^n$, for an appropriate b. Therefore, for $n\geq 1$, we have that

$$F_n\leq \cos^2\left(\tfrac{\theta}{2}\right)\frac{\lambda^{n-1}}{[\cos\left(\frac{\theta}{2}\right)]^{2n}}b^n \tag{4.32}$$

Thus, the tree formula (4.23) is convergent whenever λ is contained in the cardioid domain

$$\mathcal{C}:=\left\{\lambda\in\mathbb{C}:|\lambda|\leq\frac{1}{b}\cos^2\left(\frac{\theta}{2}\right)\right\} \tag{4.33}$$

which contains a Sokal disc D_b^1, as required by Nevanlinna-Sokal theorem 2.6.

To make the proof complete, we also have to deal with the term $n=1$, in which case there are no derivatives acting on the loop vertices. However, we can rewrite it as (see [Riv09]):

$$\int_{-\infty}^{\infty}\frac{d\sigma}{\sqrt{2\pi}}e^{-\frac{\sigma^2}{2}}\log\Big(1+i\sqrt{\lambda}\sigma\Big)=\int_{-\infty}^{\infty}\frac{d\sigma}{\sqrt{2\pi}}e^{-\frac{\sigma^2}{2}}\int_0^1 dt\frac{i\sqrt{\lambda}\sigma}{1+ti\sqrt{\lambda}\sigma}\ ; \tag{4.34}$$

then, by integrating by parts in σ, we find

$$\int_{-\infty}^{\infty}\frac{d\sigma}{\sqrt{2\pi}}e^{-\frac{\sigma^2}{2}}\int_0^1 dt\frac{\lambda t}{[1+i\sqrt{\lambda}t\sigma]^2}. \tag{4.35}$$

In this way, this term can be treated in a similar fashion to the $n\geq 1$ case. As there is of course only one tree over one vertex this term is easily bounded: we have thus proved that the tree formula (4.23) is then convergent for λ in the cardioid (4.33).

4.3.3 Gevrey bounds

Here we want to prove that $F(\lambda)$ admits a 1-Gevrey uniform asymptotic expansion. Namely, for λ in a Sokal disc contained in the cardioid $\mathcal{C}$ found earlier, for every $r\in\mathbb{N}$, there exist $a_1,...,a_r$ such that

$$\left|F(\lambda)-\sum_{n=0}^{r-1}\lambda^n a_n\right|\leq\lambda^r AB^r r! \tag{4.36}$$

for two constants A,B independent of λ and r.

To estimate the right-hand side of (4.36), namely the remainder of order r, can be expressed by Taylor's integral remainder theorem:

$$R_r(F(\lambda))=\frac{\lambda^r}{(r-1)!}\int_0^1 du(1-u)^{r-1}F^{(r)}(\lambda u). \tag{4.37}$$

To compute it, we resort to the BKAR formula for $F(\lambda)$ (4.23). Solving for the derivatives of the loop vertices, it reads

$$F(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\mathfrak{T}}\int_0^1\left(\prod_{l\in\mathfrak{T}}dw_l\right)\frac{(-i\sqrt{\lambda})^{2(n-1)}}{(-2)^n}\int\prod_{i=1}^{n}\frac{(c_i-1)!}{(1+i\sqrt{\lambda}\sigma_i)^{c_i}}d\mu_{C^{\mathfrak{T}}(\boldsymbol{w})}(\{\sigma_i\}). \tag{4.38}$$

Because of the factor $(\sqrt{\lambda})^{2(n-1)}$, to find the remainder R_r, we have to expand the resolvent $(1+i\sqrt{\lambda}\sigma_i)^{-c_i}$ up to order $2r-2(n-1)$ for every n. Note though that, since r is fixed, only a finite number of terms in the sum will contribute: the others will be untouched and will converge in the cardioid $\mathcal{C}$ by virtue of the foregoing discussion.

We now perform the expansion in powers of $\sqrt{\lambda}$. To begin with, observe that

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^N \left(\frac{1}{1-ax}\right)^c = \frac{(c+N-1)!}{(c-1)!}\frac{a^N}{(1-ax)^{c+N}}. \tag{4.39}$$

Hence, expanding up to order $2r-2(n-1)$, we find that, for every n the modulus of the integrand of (4.38) has remainder

$$\frac{\lambda^r}{2^n(2r-2n+1)!}\sum_{\vec{N}}\frac{(2r-2n+2)!}{\prod_{i=1}^n N_i!}\int_0^1 du(1-u)^{2r-2n+1}\prod_{i=1}^n\frac{\sigma_i^{N_i}(c_i-1)!}{(1-i\sqrt{\lambda}\sigma_i u)^{c_i+N_i}}\frac{(c_i+N_i-1)!}{(c_i-1)!}. \tag{4.40}$$

In this way, the integrand is correctly multiplied by λ^r for every n. The sum $\sum_{\vec{N}}$ runs over all the partitions of $2r-2(n-1)$ into n *distinguishable* integers $N_1,...,N_n$ and the multinomial factor

$$\frac{(2r-2n+2)!}{\prod_{i=1}^n N_i!} \tag{4.41}$$

guarantees that overcounting is avoided. The number of terms in the sum is

$$\binom{2r-2n+2+n-1}{n} = \binom{2r-n+1}{n} \leq \binom{2r}{r} \leq 2^{2r}. \tag{4.42}$$

We can then bound the sum by multiplying the largest term by an exponential factor 2^{2r}. By similar arguments as before, we can put an upper bound to the integrand by observing that, for every σ_i and u in the respective intervals of integration,

$$\left|\frac{1}{1+i\sqrt{\lambda}\sigma_i u}\right| \leq \frac{1}{\cos\left(\frac{\theta}{2}\right)}. \tag{4.43}$$

If we restrain ourselves to the Sokal disc $\mathrm{Re}\,\lambda^{-1} > R^{-1}$, for an appropriate radius R no larger than the radius b of the cardioid $\mathcal{C}$ found earlier, we shall have $\cos\left(\frac{\theta}{2}\right) \geq 1/2$. Therefore, recalling that

$$\sum_i^n (c_i+N_i) = 2r \tag{4.44}$$

we are enabled to remove the resolvent, at the cost of of another overall exponential factor 2^{2r}: of course one could prove the validity of the Gevrey bounds on a larger subset of the cardioid $\mathcal{C}$ simply by putting a larger exponential bound $B^r \geq 2^{2r}$.

Having bounded the resolvents, the only dependence on the replicas σ_i is polynomial. Before dealing with the Gaussian integral over the replicas, let us first perform the integral over u, which gives a factor of $(2r-2n+2)^{-1}$. In this way, the two terms $(2r-2n+2)!$ cancel each other. Moreover, we have to deal with the term

$$\prod_{i=1}^n \frac{(c_i+N_i-1)!}{(c_i-1)!N_i} = \prod_{i=1}^n \binom{c_i+N_i-1}{N_i}. \tag{4.45}$$

The binomial coefficient is bounded by $2^{c_i+N_i-1}$ (see [ELT21]), so the product is bounded by 2^{2r-n}. There is still one more product $\prod_{i=1}^n (c_i-1)!$, but this only depends on n, and convergence has already been argued earlier.

We can now perform the Gaussian integrals and the integral over the w_l:

$$\int_0^1 \left(\prod_{l\in\mathfrak{T}} dw_l\right) \int d\mu_{C^{\mathfrak{T}}(\boldsymbol{w})}(\{\sigma_i\}) \prod_{i=1}^n \sigma_i^{N_i} \tag{4.46}$$

Since we have $\sum_i N_i = 2r - 2n + 2 \leq 2r$, the Gaussian integrals are certainly bounded by $B^r(r!)$: after putting this upper bound, the integral over the weights only gives 1. Only the terms up to $n = r + 1$ contribute: therefore we can factor out the bound $B^r r!$ and sum over the first $r+1$ terms. The sum has then a bound of the form $k_1 r!$. The remaining (infinitely many) terms are bounded by $AB^n\lambda^n$, as argued before, so that their sum converges to a constant k_2. As clearly $k_1 r! + k_2 \leq (k_1 + k_2)r!$, we find that eventually (4.36) is satisfied. Nevanlinna-Sokal theorem then applies and $F(\lambda)$ is Borel-summable, and in particular 1-summable. As already argued in section 3.2.1, this implies that the partition function (4.5) is definable in $\mathbb{R}_{\mathscr{G}}$.

4.3.4 Alternative viewpoint: composition of series

There is actually a much quicker way to prove the 1-summability of $F(\lambda) = \log Z(\lambda)$. As intuition suggests, $F(\lambda)$ inherits its 1-summability property from $Z(\lambda)$. As shown in section 3.2.1, the 1-summability of $Z(\lambda)$ can be argued by computing explicitly its asymptotic expansion, taking its Borel transform and showing that it has no poles on the positive real axis, nor exponential growth thereon; yet, as already shown in section 3.3, the more refined tool of Nevanlinna-Sokal theorem can also be exploited. Specifically, we can observe first that the integral

$$Z(\lambda) = \int_{-\infty}^{\infty} \tfrac{d\phi}{\sqrt{2\pi}} e^{-\frac{\phi^2}{2} - \frac{\lambda}{8}\phi^4} \tag{4.47}$$

is clearly analytic for $\operatorname{Re}\lambda > 0$, which is nothing but a Sokal disc of infinite radius $D_0^1 = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda^{-1} > 0\}$. Secondly, we can argue that $Z(\lambda)$ admits a 1-Gevrey asymptotic expansion on this domain with Taylor's integral remainder theorem. Indeed, we have:

$$\left| Z(\lambda) - \sum_{n=0}^{r-1} a_n \lambda^n \right| = \frac{\lambda^r}{(r-1)!} \int_0^1 du (1-u)^{r-1} \int_{-\infty}^{\infty} \tfrac{d\phi}{\sqrt{2\pi}} e^{-\frac{\phi^2}{2} - \frac{u\lambda}{8}\phi^4} \phi^{4r}. \tag{4.48}$$

The integral over the field ϕ can be easily bounded by observing that $\left| e^{-\frac{u\lambda}{8}\phi^4} \right| \leq 1$ for every $\lambda \in D_0^1$. Then the integral becomes Gaussian and simply yields:

$$\int_{-\infty}^{\infty} \tfrac{d\phi}{\sqrt{2\pi}} e^{-\frac{\phi^2}{2}} \phi^{4r} = \frac{2^{2r}}{\sqrt{\pi}} \Gamma(2r + \tfrac{1}{2}) = (4r-1)!! = \frac{4r!}{2^{2r}(2r!)}. \tag{4.49}$$

Then the integral over u provides the uniform Gevrey bounds on the Sokal disc D_0^1

$$\left| Z(\lambda) - \sum_{n=0}^{r-1} a_n \lambda^n \right| \leq \lambda^r \frac{4r!}{2^{2r}(2r!)r!} \leq AB^r r! \,, \tag{4.50}$$

where the constants A, B can be found easily by using the Stirling approximation.

To transfer the 1-summability property of $Z(\lambda)$ to its logarithm, observe first that the first coefficient of the asymptotic expansion is simply 1. Then we can write

$$Z(\lambda) \sim_1 1 + \tilde{\varphi}(\lambda) \tag{4.51}$$

where $\tilde{\varphi}(\lambda) \in \lambda\mathbb{C}[[\lambda]]_1$, i.e. it is a formal power series without constant term of Gevrey class 1, which we also know to be 1-summable. We then hearken back to to Theorem 2.4: given $\tilde{\varphi}(\lambda)$ as above, 1-summable in an arc of directions $I \subset (-\pi, \pi)$, we can then simply choose $H(t) = \log(1 + t)$ and argue that $F(\lambda) = H \circ \tilde{\varphi}$ must be 1-summable in the direction of I. Although this approach looks simpler, it relies on showing Borel summability of the partition function starting from its integral representation, which may not be straightforward in more complicated (especially higher-dimensional) models.

4.4 Partition functions for vector and matrix models

In this and the next section we mention some results of Borel summability obtained by means of the forest formula and the Loop Vertex Expansion. By virtue of Observations 2.1 and 2.2, the Borel summability of the partition function $Z(\lambda)$ implies its definability in $\mathbb{R}_{\mathscr{G}}$ when restricted on a closed interval $[0, R]$. We first relate results for matrix models, which extend QFTs on a point to a finite lattice of points; in the next section we will move on to theories in higher dimensions.

4.4.1 Complex matrices on a point

Quartic Interaction

Matrix models with a quartic interaction were studied in [Riv07] by means of Loop Vertex Expansion, extending the method portrayed in [Riv17] to complex N by N matrices Φ. The partition function to be analysed can be written as

$$Z(\lambda) = \int d\mu(\Phi) e^{-\frac{\lambda}{N} \operatorname{Tr}\{\Phi^* \Phi \Phi^* \Phi\}}, \tag{4.52}$$

where $d\mu(\Phi)$ is a normalized Gaussian measure

$$d\mu(\Phi) = \pi^{-N^2} e^{-\frac{1}{2} \operatorname{Tr}\{\Phi^* \Phi\}} \prod_{i,j}^{N} d\mathrm{Re}\Phi_{ij} d\mathrm{Im}\Phi_{ij}. \tag{4.53}$$

Following the method depicted in the previous section, the 1-summability of $Z(\lambda)$ is proved by performing first a Hubbard-Stratonovich transformation, thus trading a quartic interaction for a cubic one; then by integrating out the original field Φ, finding thus the loop vertices; finally, by applying the forest formula. Thus, Theorem 3.1 of [Riv07] states the 1-summability for $Z(\lambda)$, uniformly in the size N of the matrix. Therefore, recalling our Observation 2.1, we are entitled to conclude that the partition function for this model is also definable in $\mathbb{R}_{\mathscr{G}}$.

Higher order interaction

The previous results were extended to higher order interactions in [LR16]. Further improvements were accomplished in [KRS19a; KRS19b] employing the Loop Vertex Representation and the Taylor BKAR forest formula. As already anticipated in section 3.2.3, the Loop Vertex Representation (LVR) is a technique akin to the Loop Vertex Expansion, except that instead of performing a Hubbard-Stratonovich transformation, one uses the initial field itself as auxiliary field (see [Riv17]). The model considered has a partition function

$$Z(\lambda, N) = \int \mathrm{d}M \mathrm{d}M^\dagger \exp\left(-\operatorname{Tr}(MM^\dagger) - \tfrac{\lambda}{N^{p-1}} \operatorname{Tr}(MM^\dagger)^p\right), \tag{4.54}$$

where M is a N-by-N complex matrix and the integral measure can be written explicitly as $\mathrm{d}M\mathrm{d}M^\dagger = \prod_{1 \le i,j \le N} \mathrm{d}M_{ij}\, \mathrm{d}M^\dagger_{ij}$.

The partition function $Z(\lambda, N)$ was proved to be $(p-1)$-summable in Theorem 3 of [LR16]. More precisely, it was proved to be analytic and to admit Gevrey bounds on a Sokal disc

$$D^{p-1}_{\rho(p,N)} := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda^{-1/(p-1)} > \rho(p, N)\} \tag{4.55}$$

where $\rho(p, N) = \rho_p N^{1+2/(p-1)}$. This disc shrinks as N increases, but it was proved in [KRS19a; KRS19b] that the analyticity domain can be extended to a domain $P(\epsilon) := \{\lambda \in \mathbb{C} : 0 < |\lambda| < \eta\ , |\arg \lambda| < \pi - \epsilon\}$

which is uniform in N. We can therefore conclude that $Z(\lambda, N)$, once restricted to an interval including the origin, is definable in $\mathbb{R}_{\mathscr{G}}$, for every N. Four our purposes, it is irrelevant that the Sokal disc (4.55) shrinks as N increases. In fact, because $\mathbb{R}_{\text{an}}$ is a substructure of $\mathbb{R}_{\mathscr{G}}$, given a definable function of $\mathbb{R}_{\mathscr{G}}$ defined on an interval $[0, r_1]$, with $r_1 > 0$, its analytic continuation to an interval $[0, r_2]$ remains definable in $\mathbb{R}_{\mathscr{G}}$ as long as r_2 is finite. A very recent improvement on this result is provided in [Riv24]. By considering a partition function with sources

$$Z(\lambda, N, J, J^\dagger) = \int dM dM^\dagger \exp\left(-N\text{Tr}(MM^\dagger) - \lambda \text{Tr}(MM^\dagger)^p + N\text{Tr}(JM^\dagger) + N\text{Tr}(MJ^\dagger)\right) \tag{4.56}$$

we can compute the cumulant of order $2k$ (namely, a connected $2k$-point correlation function)

$$\mathfrak{K}_k(\lambda, N) := \left[\frac{\partial^2}{\partial J^*_{a_1,b_1}\partial J_{c_1,d_1}} \dots \frac{\partial^2}{\partial J^*_{a_k,b_k}\partial J_{c_k,d_k}}\right] \log Z(\lambda, N, J, J^\dagger)\Big|_{J=J^\dagger=0} \tag{4.57}$$

where $J^*_{a_i,b_i} = J^\dagger_{b_i a_i}$. Theorem 1 guarantees that even the cumulants of order $2k$ are analytic in the cardioid domain

$$\mathcal{C} = \left\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{2(p-1)} \cos^{p-1}\left(\frac{\arg \lambda}{p-1}\right)\right\} \tag{4.58}$$

uniformly in N, provided that $||JJ^\dagger|| < \epsilon_\lambda$ for some ϵ depending on λ. Moreover, the same theorem guarantees that the cumulants admit an asymptotic expansion of Gevrey class $1/(p-1)$. Note that the cardioid domain is in fact a Sokal disc D_c^{p-1}, with $c = c(0) = (2(p-1))^{p-1}$: the theorem then ensures Borel summability. Consequently, the cumulants are also definable in $\mathbb{R}_{\mathscr{G}}$ as functions of the coupling λ.

4.5 Partition functions for higher dimensional theories

In the previous sections, all the examples analysed are 0-dimensional, namely QFTs on a point. Yet, the tools provided by LVE can extend beyond 0-dimensional theories: in this section we relate thus further results concerning theories in higher dimensions. This generalisation will compel us to impose UV and IR cutoffs on the propagators to ensure the finiteness of the path integral: in the following examples, this will be done using a slice of the renormalization group ([GRS14]).

4.5.1 Real ϕ^4 theory in 4 dimensions

The Borel summability of the partition function for a scalar, polynomial theory in two dimensions was studied already in [EMR75], and later in d dimensions in [AR95]. Both these methods involve the so-called *cluster field theory*, which amounts to a discretisation of the spacetime and subsequently a Mayer expansion, which removes the hard-core constraints between the clusters. Only afterwards is the forest formula applied, which can be recast in terms of *jungles* as done in [AR95]. These methods are quite complicated and involve a symmetry breaking due to the discretisation. In [MR08], a much simpler method is proposed, which accomplishes the desired proof of Borel summability without breaking the symmetry.

Consider the field theory in 4 dimensions

$$Z(\lambda, \Lambda) = \int d\mu_{C_j} e^{-\lambda \int_\Lambda d^4x \phi^4}, \tag{4.59}$$

where $d\mu_{C_j}$ is the Gaussian measure of covariance C_j and $\Lambda \subset \mathbb{R}^4$ is a bounded region in which the quartic interaction is localised. We refer to [GR13], section 3.1, for a detailed definition of Gaussian integral measures in higher dimensions. As the problem is now no longer 0-dimensional, for the sake of convergence we must

impose infrared and ultraviolet cutoffs at small and large momenta p, as explained in e.g. [Riv99]. Thus, C_j is defined to be the j-th slice of the renormalization group, namely

$$C_j(p) = \int_{M^{-2j}}^{M^{-2(j-1)}} d\alpha e^{-\alpha(p^2+m^2)} \tag{4.60}$$

or, in coordinate space

$$C_j(x,y) = \int_{M^{-2j}}^{M^{-2(j-1)}} \frac{d\alpha}{(4\pi\alpha)^2} e^{-\alpha m^2} e^{-\frac{(x-y)^2}{4\alpha}}, \tag{4.61}$$

where M is the 'thickness' of the slice (see [GRS14] for more details). With this assumption, by means of the Loop Vertex Expansion and a *rooted* tree formula, the authors of [MR08] prove in Theorem 3.1 the Borel summability of the 'pressure'

$$p(\lambda) = \lim_{\Lambda\to\mathbb{R}^4} \frac{\log Z(\lambda,\Lambda)}{|\Lambda|} \tag{4.62}$$

uniformly in the in index slice j (and where $|\Lambda|$ is the volume of Λ). In this case, it is necessary to take the ratio between the free energy $\log Z(\lambda)$ and the volume $|\Lambda|$ in order to obtain an intensive quantity, which as opposed to extensive quantities do not diverge in the infinite volume limit. We conclude thus that the pressure $p(\lambda)$ defined above must be tame and definable in $\mathbb{R}_{\mathscr{G}}$.

More significantly, in the same work, it was proved that the connected correlation functions (or cumulants)

$$\mathfrak{K}(x_1,...,x_{2p};\lambda) = \lim_{\Lambda\to\mathbb{R}^4} \frac{1}{Z(\lambda,\Lambda)} \int d\mu_{C_j} e^{-\lambda\int_\Lambda d^4x\phi^4} \phi(x_1)...\phi(x_{2p}) \tag{4.63}$$

are similarly Borel-summable and therefore definable in $\mathbb{R}_{\mathscr{G}}$. It is remarkable to notice how these correlation functions are Borel summable irrespective of the choice of the spacetime points $x_1,...,x_{2p}$ at which the field insertions are computed: Borel summability is uniform in these parameters. However, this does not mean that these functions are definable in some o-minimal structure also when viewed as functions of the spacetime coordinates; although this should be expected in light of the Tameness Conjecture 1.1, we are only allowed – to our present understanding – to pronounce ourselves on the tameness of the cumulants $\mathfrak{K}(x_1,...,x_{2p};\lambda)$ as functions of the sole λ.

4.5.2 Higher dimensional vector models

Here we relate the findings of [ELT21], regarding a theory for a vector field in dimension $d \leq 2$ with a quartic interaction. By means of constructive field theory, it was proved that the free energy for this model is also Borel-summable. Consider the Euclidean action for N real bosonic fields $\Phi = (\phi_1,...,\phi_N)$

$$S[\Phi;\lambda] = \int d^dx \left(\frac{1}{2}\sum_{i=1}^N \tilde{C}^{-1}\phi_i^2 + \frac{\lambda}{4!}\sum_{i,j,k,l}^N \mathcal{W}_{ijkl}\phi_i\phi_j\phi_k\phi_l \right) \tag{4.64}$$

where $\mathcal{W}_{ijkl}$ is a completely symmetric tensor. For $d \geq 1$, the integral can be written explicitly as

$$\int d^dx = \int_{-\beta/2}^{\beta/2} dt \int_{-\infty}^{\infty} d^{d-1}x \tag{4.65}$$

where t is the Euclidean time and $\tilde{C}$ is the (diagonal) propagator. For non-relativistic bosons, it reads

$$\tilde{C}^{-1} = \frac{d}{dt} - \frac{\hbar^2}{2m^*}\sum_{i=1}^{d-1}\frac{d^2}{dx_i^2} + m, \tag{4.66}$$

where m^* is the effective mass. For relativistic bosons it reads

$$\tilde{C}^{-1} = -\frac{d^2}{dt^2} - \sum_{i=1}^{d-1} \frac{d^2}{dx_i^2} + m^2. \tag{4.67}$$

The authors of [ELT21] put a UV cutoff to the propagator, regularising it in the Schwinger reparametrisation as in the previous example: explicitly, this amounts to fixing a cutoff j_{max} and writing

$$\tilde{C} = \sum_{j=1}^{j_{max}} \tilde{C}_j + \tilde{C}_0 \tag{4.68}$$

where for non-relativistic bosons in momentum space

$$\tilde{C}_j(\omega, \vec{p}) = (-i\omega + E) \int_{M^{-2j}}^{M^{-2(j-1)}} d\alpha \; e^{-\alpha(\omega^2 + E^2)} \tag{4.69}$$

with $E = m^* + \frac{\hbar^2}{2m^*}\vec{p}^2$. $\tilde{C}_0$ is defined in the same way, but with the integral running from 1 to ∞ (we refer again to [GRS14] for details). For relativistic bosons, instead, one has

$$\tilde{C}_j(\omega, \vec{p}) = \int_{M^{-2j}}^{M^{-2(j-1)}} d\alpha \; e^{-\alpha(\omega^2 + \vec{p}^2 + m^2)}. \tag{4.70}$$

In the above notations, $\vec{p}$ is the spatial momentum and $\omega = \frac{2\pi n}{\beta}$ is the conjugate momentum of the Euclidean time t; as before, M is a constant that fixes the thickness of the slice. Note that, as opposed to the case dealt with in the previous section, there are no IR cutoffs in this regularisation prescription.

The partition function for this model is then

$$Z(\lambda) = \int \prod_{i=1}^{N} d\phi_i \; e^{-S[\Phi;\lambda]}, \tag{4.71}$$

while the free energy is $F(\lambda) = \log Z(\lambda)$. For $d = 2$, the theory requires renormalization, which can be implemented by using the Multiscale Loop Vertex Expansion (MLVE), detailed in [GR13]. This technique features the introduction of auxiliary Grassmann fields, whose integration is treated constructively by an additional sum over forests: this is why the MLVE eventually employs a two-level *jungle* expansion. Jungles are, intuitively, forests nested into each other: we refer to [AR95] for an exhaustive definition. The authors of [ELT21] proved that, under the assumption that $\mathcal{W}_{ijkl}$ has only positive eigenvalues, the largest of which we denote by w_0^2, for non-relativistic bosons in $d \leq 1$ and for relativistic bosons in $d \leq 2$ the free energy $F(\lambda)$ is Borel-summable in a cardioid

$$|\lambda| \leq O(1) \frac{1}{N w_0^2} \cos^2\left(\tfrac{\theta}{2}\right), \tag{4.72}$$

where $\lambda = |\lambda| e^{i\theta}$ and $O(1)$ denotes constants depending on the specific model. While in the assumptions above we have defined the model at a finite temperature, it was argued in the same work that the zero-temperature limit $\beta \to \infty$ can be safely taken, preserving the Borel-summability of $F(\lambda)$. Once more, Borel summability implies that the free energy $F(\lambda)$ is tame in the structure $\mathbb{R}_{\mathscr{G}}$. Moreover, as the exponential function e^x is analytic on $[0, R]$, by composition we infer that the partition function $Z(\lambda) = e^{F(\lambda)}$ must be definable in $\mathbb{R}_{\mathscr{G}}$ as well. Remark that there is no need to extend $\mathbb{R}_{\mathscr{G}}$ to $\mathbb{R}_{\mathscr{G},\exp}$ to perform this operation, as we do not need the exponential to be defined on the whole positive real line $\mathbb{R}^+$.

Chapter 5

Summability and Resurgence in Quantum Mechanics

In the previous chapters, we have illustrated how QFT amplitudes and partition functions can be expressed perturbatively by asymptotic series in the coupling constant, which must be Borel-resummed. Yet, the scope of application of perturbative asymptotic expansions is much broader, and it will be the purpose of this chapter to explore it further.

As it is well known, the time-independent Shrödinger equation can be solved exactly but for very few systems, such as the harmonic oscillator and the hydrogen atom; in all the other cases we must content ourselves with approximate solutions. These usually consist in perturbative power series of a small parameter g, measuring the degree of deviation of the physical potential from an ideal, simpler potential – like that of the aforementioned harmonic oscillator – whose associated Shrödinger equation can be solved exactly. This approach applies to both wavefunctions and energy eigenvalues, but throughout this chapter we will be only concerned with the latter. To fix ideas, let us consider a perturbed harmonic oscillator, and let the parameter g tune the anharmonic part of the potential. With due rescalings, the parameter g can be incorporated in the Planck constant $\hbar$, which becomes then the only free parameter of the system; the perturbative expansion of the ground state energy eigenvalue will then read

$$E(\hbar) = \frac{1}{2}\hbar + E_2\hbar^2 + E_3\hbar^2 + \dots \, .$$

It turns out, however, that such a series is once again divergent and must be Borel-resummed in order to find a well-defined function of $\hbar$. Although the coefficients $E_2, E_3, \dots$ can be computed iteratively, they can be more systematically found by means of the *exact WKB method*, whereby exact quantisation conditions are established. We briefly present this theory in section 5.1. As it will become clear, these series may not Borel-summable: specialising to the case of a cubic potential, in section 5.2 we will therefore employ alien calculus to find the correct transseries whose median resummation yields the desired function.

In the same spirit of the previous chapters, our goal is to establish whether the energy function $E(\hbar)$ is a tame function in an o-minimal structure; when Borel summability fails though, we are unable to claim that it is definable in $\mathbb{R}_{\mathscr{g}}$. A different approach was then attempted, described in section 5.3, based on finding numerically a differential equation obeyed formally by the perturbative asymptotic series. Still, when the potential satisfies certain conditions, the series can be proved to be Borel-summable, whence it can be inferred that the energy eigenvalues are tame functions of the coupling $\hbar$, definable in $\mathbb{R}_{\mathscr{g}}$. It will be the purpose of the last section 5.4 to relate one such example.

5.1 Exact WKB method

The WKB approximation was developed in 1926 by Wentzel, Kramers and Brillouin as an approximation method to solve the Shrödinger equation. In its modern version, called exact WKB method, it provides a systematic way to solve perturbatively for the eigenvalues of the Hamiltonian. In this section we shortly illustrate this technique.

5.1.1 Quantum periods

The exact WKB method is based on the definition of a complex manifold, called the WKB curve, and its *quantum periods*. Here we shall limit ourselves only to the most essential definitions, referring to [SV23] and references therein for further details.

Consider the Shrödinger equation with a polynomial potential $V(x)$ of degree d

$$-\frac{\hbar^2}{2}\psi(x)'' + V(x)\psi(x) = E\psi(x). \tag{5.1}$$

After making the following ansatz

$$\psi(x;\hbar) = \exp\left(\frac{i}{\hbar}\int_0^x S(x';\hbar)dx'\right), \tag{5.2}$$

the Shrödinger equation can be recast as a Riccati equation

$$S^2(x;\hbar) - i\hbar S'(x;\hbar) = 2(E - V(x)). \tag{5.3}$$

We can then expand $S(x;\hbar)$ in power series as

$$S(x;\hbar) = \sum_{n=0}^{\infty} p_n(x)\hbar^n =: P(x;\hbar) + Q(x;\hbar), \tag{5.4}$$

where $P(x;\hbar)$ and $Q(x;\hbar)$ contain all the even and odd powers of $\hbar$ respectively. By plugging back in (5.3) we obtain

$$P^2 + Q^2 + 2PQ - i\hbar(P' + Q') = 2(E - V), \tag{5.5}$$

where the arguments $(x;\hbar)$ have been dropped to avoid cluttering, and the prime denotes a derivative with respect to x. By matching the even and odd powers of $\hbar$ we observe that

$$2PQ - i\hbar P' = 0, \tag{5.6}$$

whence we argue that Q is given by a total derivative $Q = \frac{-\hbar}{2}\frac{d\log P}{dx}$. Hence, we rewrite the WKB ansatz (5.2) as

$$\psi(x;\hbar) = \frac{1}{\sqrt{P(x;\hbar)}}\exp\left(\frac{i}{\hbar}\int_0^x P(x';\hbar)dx'\right). \tag{5.7}$$

As anticipated, the exact WKB method pivots around the definition of a WKB curve. This can be defined as the phase space curve Σ embedded in the plane $\mathbb{R}^2$

$$\Sigma: \quad y^2 = p^2(x) = 2(E - V(x)) \tag{5.8}$$

where $p(x)$ is the classical momentum and it can be readily inferred from (5.3) that $p(x) = p_0(x)$. However, it is expedient to promote x to a complex variable. The right-hand side of the above equation will then be defined on a Riemann sphere $\mathbb{CP}$, which is obtained by the addition of the point at infinity to the

complex plane $\mathbb{C}$[1]. As the function $p(x)$ is defined through a squared root, we must add branch cuts on the real line connecting the roots of the polynomial $E - V(x)$, which we will always assume to be d, and also the point at infinity when d is odd. In fact, there will be two Riemann spheres – one for each choice of sign in $p(x) = \pm\sqrt{2(E - V(x))}$ – which must be glued along the branch cuts, so that one can pass from either sphere to the other by crossing one of them. Indeed, the two spheres (sometimes more generally called *sheets*) must coincide at the roots of the polynomial, where $p(x)$ takes the unique value 0. After these operations, the WKB curve Σ is promoted to the complex manifold of real dimension 2, on which the function $p(x) = \sqrt{2(E - V(x))}$ is holomorphic and whose genus is determined by the degree d of the potential $V(x)$. Under the assumption that the polynomial $E - V(x)$ has d real roots, the number of branch cuts is $(d+1)/2$ or $d/2$ for odd and even degree potentials respectively; the genus of the WKB curve Σ will be therefore $(d-1)/2$ or $d/2-1$ respectively. In the following, we will be mostly concerned with the case $d = 3$, namely with a cubic potential: in this case, the genus of the WKB curve is 1, thus Σ is simply a torus[2], and this construction reduces to the one depicted in Figure 5.2. We refer to [CMP03] for an exhaustive presentation of how elliptic curves are complexified.

We are now ready to define quantum periods. By calling the roots of $E - V(x)$ (namely the classical turning points) $a_1, a_2, ..., a_d$, we assume that the classically allowed regions are $[a_{2n-1}, a_{2n}]$, while the classical potential barriers lie at $[a_{2n}, a_{2n+1}]$. In turn, these will be associated respectively with cycles γ_{A_n} and γ_{B_n} belonging to the homology group $H_1(\Sigma, \mathbb{C})$, of which they form a basis.

Definition 5.1. A **quantum A period** is defined as

$$\Pi_{\gamma_{A_n}}(E;\hbar) = \oint_{\gamma_{A_n}} S(x;\hbar)dx = 2\int_{a_{2n-1}}^{a_{2n}} S(x;\hbar)dx = 2\pi\ t_n(E;\hbar), \tag{5.11}$$

while, similarly, a **quantum B period** is defined as

$$\Pi_{\gamma_{B_n}}(E;\hbar) = \oint_{\gamma_{B_n}} S(x;\hbar)dx = 2\int_{a_{2n}}^{a_{2n+1}} S(x;\hbar)dx = i\ t_n^D(E;\hbar). \tag{5.12}$$

We will also refer to $t_n(E;\hbar)$ simply as a **quantum period**; conversely, we will refer to $t_n^D(E;\hbar)$ as a **dual quantum period**.

The distinction between quantum A and B periods is meant to tell apart the periods by the regions, defined by the roots of $E - V(x)$, whereon the integration takes place. Quantum A periods are defined on *classically allowed* regions: namely, the roots $[a_{2n-1}, a_{2n}]$ enclose a minimum of $E - V(x)$, which is why quantum A periods are also referred to as 'perturbative'. Conversely, quantum B periods are defined by an integration on a *classically prohibited* region: namely, the roots $[a_{2n}, a_{2n+1}]$ enclose a region where the potential $V(x)$ is higher than the energy E and thus they encode information about the tunnelling probability. The quantum periods $t_n(E;\hbar)$ and the dual quantum periods $t_n^D(E;\hbar)$ are simply convenient rescalings: in particular, both $t_n(E;\hbar)$ and $t_n^D(E;\hbar)$ are real.
It is worthwhile to remark that quantum periods are found by integrating along the cycles a meromorphic

[1] To convince oneself that $\mathbb{C} \cup \{\infty\}$ is isomorphic to S^2, it suffices to recall how the stereographic projection maps the plane $\mathbb{R}^2 \cong \mathbb{C}$ to the two-sphere subtracted of the north pole, which can be assigned to the extra point ∞.

[2] There is an alternative view on why the complexified WKB curve for a cubic potential is a torus. When the potential is cubic, the WKB curve is an elliptic curve which can be cast in the form

$$y^2 = 4x^3 - g_2 x - g_3 = p^2(x). \tag{5.9}$$

Suggestively, the Weierstrass function $\wp(z)$ obeys the differential equation (see e.g. [Pas17])

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3, \tag{5.10}$$

for any $z \in \mathbb{C}$. We realise then that $\wp(t)$ with $t \in \mathbb{R}$ must be the solution to the classical equation of motion, namely $x(t) = \wp(t)$ and $p(t) = dx/dt = \wp'(t)$. It turns out that $\wp(z)$ has a double periodicity on the complex values: namely, there exist $\omega_1, \omega_2 \in \mathbb{C}$ depending on g_2 and g_3, such that $\wp(z + \omega_1) = \wp(z)$ and $\wp(z + \omega_2) = \wp(z)$, which means that its fundamental domain is the torus $z \sim z + \omega_1, z \sim \omega_2$. Therefore the WKB curve, which is nothing else than the complexified phase space $\{x, p(x)\}$, must be isomorphic to the torus whereon the complexified time law has its fundamental domain.

differential form $S(x;\hbar)$ having a pole at infinity. Thus, quantum periods are different from the ordinary notion of periods (see e.g. [BDU17]), defined by the integration of the holomorphic differential $dz = dx/p(x)$.[3]

A closed expression for quantum periods will be very hard to come by, as we do not have a closed expression for $S(x;\hbar)$. However, we can expand $S(x;\hbar)$ like in (5.4) and, having argued that only the even powers contribute, we can write the formal power series

$$\begin{aligned} \tilde{t}_n(E) &= \sum_{k=0}^{\infty} t_{n,2k}\hbar^{2k}, \\ \tilde{t}_n^D(E) &= \sum_{k=0}^{\infty} t_{n,2k}^D\hbar^{2k}, \end{aligned} \tag{5.14}$$

where

$$\begin{aligned} t_{n,k}(E) &= \frac{1}{\pi}\int_{a_{2n-1}}^{a_{2n}} p_{2k}(x)dx, \\ t_{n,k}^D(E) &= -2i\int_{a_{2n}}^{a_{2n+1}} p_{2k}(x)dx. \end{aligned} \tag{5.15}$$

It is worthwhile to dwell briefly on the physical significance of the leading term $t_{n,0}$. Dropping for now the subscript n, which simply labels a potential well where the motion is classically allowed, it reads

$$t_0(E) = \frac{1}{\pi}\int_{a_1}^{a_2} dx\sqrt{2(E-V(x))} = \frac{1}{\pi}\int_{a_1}^{a_2} dx p(x) \tag{5.16}$$

where $p(x)$ is the classical momentum, and the one-form $dx\ p(x)$ is known as the Liouville one-form. The leading (and 'classical') term to the quantum period $t(E)$ is therefore a measure of the area of the phase space. We will hearken back to this later, when dealing with quantisation conditions. Let us also remark that the 'quantum corrections', i.e. $t_{n,2k}(E)$ for $k \geq 1$ can be found by means of the differential operator method, for which we refer to [FKN19] and the appendix A of [IMS19]. Moreover, quantum periods and dual quantum periods are related by the Perturbative-Non Perturbative (PNP) relation (see [BDU17]), which can be expressed as a quantum corrected Wronskian (where the derivatives are with respect to $\hbar$), as shown in [SV23].

At this point, the reason for introducing quantum periods may still be obscure. As it will be explained later, the interest in quantum periods resides in the fact that exact quantisation conditions can be expressed in terms of them. More precisely, they are defined in terms of the **Voros symbols**:

$$\mathcal{V}_\gamma = e^{\frac{i}{\hbar}\Pi_\gamma} \tag{5.17}$$

where $\gamma \in H_1(\Sigma)$ is an independent cycle of the WKB curve Σ. We will clarify this later in section 5.1.3.

5.1.2 The DDP formula

In general, the perturbative expansion (5.14) will yield a divergent series. The divergence is caused by the coefficients $t_{n,2k}$ and $t_{n,2k}^D$ growing factorially like $(2k)!$: the power series $\tilde{t}_n(E)$ and $\tilde{t}_n^D(E)$ are therefore of Gevrey class 1. In analogy to the asymptotic series of partition functions discussed earlier in section 3.2, the

[3] We refer again to [CMP03] for a thorough discussion about periods. As for why $dx/p(x)$ is the unique holomorphic one-form on the torus, recall that the WKB curve is isomorphic to a torus with $x = \wp(z)$, $y = p(x) = \wp'(z)$. It is then clear that

$$\frac{dx}{y} = \frac{d\wp(z)}{\wp'(z)} = dz \tag{5.13}$$

is the unique holomorphic one-form on the WKB curve (recall that on a compact complex manifold only the constant function is everywhere holomorphic).

divergence of the power series (5.14) is due to the illegal commutation of the sum and the integral in the definitions

$$t_n(E;\hbar) = \frac{1}{\pi}\int_{a_{2n-1}}^{2n} dx \sum_{k=0}^{\infty} p_{2k}(x)\hbar^{2k}, \tag{5.18}$$
$$t_n^D(E;\hbar) = -2i\int_{a_{2n}}^{2n+1} dx \sum_{k=0}^{\infty} p_{2k}(x)\hbar^{2k}.$$

As we eventually want to interpret $\hbar$ as purely real, and we know that the quantum (dual) periods t_n and t_n^D are real functions of $\hbar$, we are interested in performing a Borel resummation of their perturbative power series along the line $\theta = 0$. However, due to the presence of poles in the Borel plane, not all the formal power series t_n and t_n^D will be 1-summable in this direction: we must then promote them to transseries and compute the full alien algebra of these power series. Luckily, there is a powerful and very general formula, called the Delabaere-Dillinger-Pham (DDP) formula (see e.g. [IN15], [SV23]), which directly provides the action of the Stokes automorphism on the Voros symbols.

Proposition 5.1 (Delabaere-Dillinger-Pham formula)**.** If the period Π_γ, having a Stokes line along θ, is defined on a cycle γ which crosses n cycles $\gamma_1, \gamma_2, ..., \gamma_n$ such that:

$$\frac{i}{\hbar}\Pi_{\gamma_j} < 0 \qquad j = 1, 2, ..., n \tag{5.19}$$

then one has that the following holds[4]:

$$\mathfrak{S}_\theta \mathcal{V}_\gamma = \prod_{j=1}^{n} \mathcal{V}_\gamma (1 + \mathcal{V}_{\gamma_i})^{-\langle \gamma, \gamma_j \rangle}. \tag{5.20}$$

where $\langle \gamma, \gamma' \rangle$ is the intersection number of the cycles γ, γ', defined as the antisymmetric inner product

$$\langle \gamma, \gamma' \rangle = \iint_\Sigma \gamma^* \wedge (\gamma')^* = -(\gamma, \gamma'^*), \tag{5.21}$$

where the star $*$ at the apex indicates the Poincaré dual and Σ is the complexified WKB curve.

The DDP formula is quite too general at this stage to compute the alien algebra of the quantum periods: it will be then expedient to study more specific cases to deploy the power of this formula. Following [SV23], we will discuss in the next section how the quantum periods and, as a consequence, the energy eigenvalues for of a cubic potential, can be expressed by transseries. Before that though, let us explain how quantum periods, through their Voros symbols (5.17), can provide exact quantisation conditions for the bound states of a given one-dimensional potential.

5.1.3 Quantisation conditions

In quantum mechanics, as it is well known, bound states are quantised: they can labelled by a principal quantum number n which only takes discrete values. However, finding directly from Shrödinger equation the explicit dependence on n of the quantised eigenenergies of a time-independent Hamiltonian, even in only one dimension, is seldom possible (the fortunate cases being the harmonic oscillator, the particle in a box and very few others). The exact WKB method, as the name suggests, provides exact quantisation conditions in terms of the Voros symbols (5.17): as it will be explained below, this can be achieved by enforcing that the wavefunction $\psi(x)$ fall off to 0 fast enough for $x \to \pm\infty$, so that it is correctly square-integrable (i.e., belongs to the space $L^2(\mathbb{R})$). In this presentation, we mostly follow [Kam+23] and [IN15].

[4]The minus sign at the exponential is due to our conventions on the Stokes automorphism.

Let us consider here a more general Shrödinger equation

$$\left(\frac{d^2}{dz^2} - \frac{1}{\hbar^2}V(z;\hbar)\right)\psi(z;\hbar) = 0 \tag{5.22}$$

where

$$V(z;\hbar) = V_0(z) + \hbar V_1(z) + ... \tag{5.23}$$

is the term previously written explicitly as $V(x) - E$, enhanced by allowing the potential to have infinitely many corrections in powers of $\hbar$. We can of course retrieve the previously discussed case by letting $V(z;\hbar) = V(z)$. The potential $V(z;\hbar)$ transforms like a meromorphic quadratic differential, i.e, under a coordinate transformation $z \to w(z)$ it transforms according to

$$V(z;\hbar) \to V(z(w),\hbar)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{2}\hbar^2\{z(w),w\} \tag{5.24}$$

where the latter term is the Schwarzian derivative

$$\{z(w),w\} = \frac{\left(\frac{d^3z}{dw^3}\right)^3}{\left(\frac{dz}{dw}\right)} - \frac{3}{2}\left[\frac{\left(\frac{d^2z}{dw^2}\right)}{\left(\frac{dz}{dw}\right)}\right]^2. \tag{5.25}$$

It will be useful to define the meromorphic quadratic differential

$$\phi(z) = V_0(z)dz \otimes dz, \tag{5.26}$$

on which we will later impose some regularity conditions. Moreover, by considering the ansatz

$$\psi_a(z;\hbar) = \exp\left(\frac{i}{\hbar}\int_a^z dz' S(z';\hbar)\right), \tag{5.27}$$

where a is a classical turning point, i.e. a root of $V_0(z)$, we can write the Riccati equation

$$-S^2(z;\hbar) + i\hbar\frac{dS(z;\hbar)}{dz} = V(z;\hbar). \tag{5.28}$$

If we again expand $S(z;\hbar) = \sum_{n=0} p_n(z)\hbar^n$, it will follow that $p_0 = \pm\sqrt{V_0(z)}$, which, as before, leads to an ambiguity in the choice of the sign. As all the subsequent corrections $p_n(z)$ are calculated recursively, we can call $S^+(z;\hbar)$ and $S^-(z;\hbar)$ the two determinations of $S(z;\hbar)$. If we now set

$$P = \frac{S^+ - S^-}{2}, \qquad Q = \frac{S^+ + S^-}{2}, \tag{5.29}$$

by simply plugging S^+ and S^- in (5.28) it can be easily inferred that again only P contributes while Q is equal to a total derivative (in analogy to (5.6), although now P and Q have a different definition and interpretation). Hence we have two wavefunctions $\psi_a^{\pm}(z;\hbar)$ differing by their phase

$$\psi_a^{\pm}(z;\hbar) = \frac{1}{\sqrt{P(z;\hbar)}}\exp\left(\pm\frac{i}{\hbar}\int_a^z dz' P(z';\hbar)\right), \tag{5.30}$$

which for later convenience we will stack into a two-row column vector

$$\psi_a(z;\hbar) = \begin{pmatrix}\psi_a^+(z;\hbar)\\ \psi_a^-(z;\hbar)\end{pmatrix}. \tag{5.31}$$

As the exponential is a power series in $\hbar$, the wavefunction can be itself expanded as a sum in powers of $\hbar$. We can then wonder about the Borel summability of the wavefunction itself (uniformly in z). Following [IN15], we define P_0 and P_∞ to be the locations of the zeros and the poles of ϕ respectively; we then make the following technical assumptions:

1. ϕ has at least 1 pole and 1 zero;
2. All the zeroes of ϕ are simple;
3. The order of any pole is ≥ 2.

Furthermore, to guarantee Borel summability everywhere except on a finite number of Stokes lines, we assume that:

4. If p is a pole for V_n and $n \geq 1$, then $p \in P_\infty$;
5. If ϕ has a pole at p of order $m \geq 3$, then the order of V_n at p is $< 1 + m/2$, for every $n \geq 1$;
6. If ϕ has a pole of order $m = 2$, then V_n has at most a single pole at p for $n = 1, n \geq 3$ *and* $V_2(z)$ has a double pole at p, with
$$V_2(z) = -\frac{1}{4z^2} + o(z) \qquad \text{as } z \to 0. \tag{5.32}$$

We must now describe the complex manifold Σ whereon the wavefunction of the complex variable z is defined. First, we remove from the complex plane the points $P_{tot} = P_0 \cup P_\infty$. Since the perturbative corrections $p_n(z)$ are determined iteratively depending on $p_0(z) = \pm\sqrt{V_0(z)}$, we perform the previous operations twice, once for every Riemann sheet whereon the double-valuedness of p_0 is resolved. As already explained, on every sheet we draw branch cuts which connect the roots of V_0 pairwise; moreover, due to the presence of the factor $1/\sqrt{P(z;\hbar)}$ there will be an additional branch cut along the imaginary axis.

Having drawn the branch cuts, we must now trace the Stokes lines. In general, they may not be straight lines, but they are given by:
$$\mathrm{Im}\left(\frac{i}{\hbar}\int_a^z dz'\sqrt{V_0(z')}\right) = 0 \tag{5.33}$$
which amounts to setting the exponential factor in $\psi_a^\pm(z)$ as purely real. This implies the following:

a) The position of the Stokes lines depends on the phase that is given to $\hbar$ to resolve the ambiguity in Borel resummation at $\theta = 0$.[5]

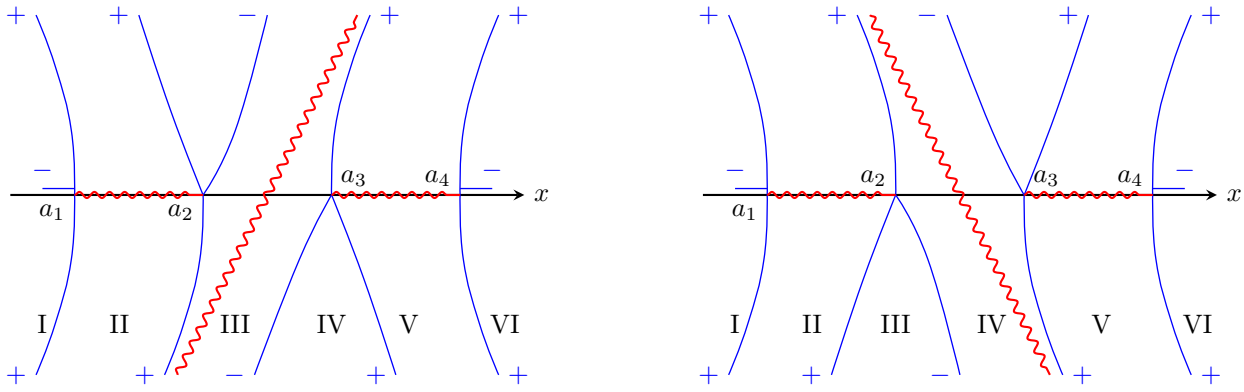
b) Along the Stokes line, the exponential is purely real: therefore $\psi_a^+(z)$ will either tend to 0 or infinity along it. This allows us to establish an orientation on the Stokes line, from $-$ to $+$ (choosing $\psi_a^+(z)$ rather than $\psi_a^-(z)$ for reference is of course purely conventional).

The trajectories defined as above can be classified as:

- *saddle trajectories*: they connect two roots of $V_0(z)$, belonging to P_0;
- *separating trajectories*: they connect a root in P_0 with a pole in P_∞;
- *generic trajectories*: they connect two poles in P_∞;
- *close trajectories*: they form a loop in $\Sigma \backslash P_{tot}$;
- *degenerate trajectories*: they are trajectories whose limiting set is more than one point in at least one direction.

[5]Here we adopt the convention most frequently used in the literature. Suppose that $\theta = 0$ is a Stokes line for a formal 1-Gevrey series $\tilde{\varphi}$ of powers of $\hbar$. Then the resummation along θ is ill-defined, and one has to dodge the poles by either taking the resummation at $\theta + \varepsilon$ or $\theta - \varepsilon$. Equivalently, one can supply $\hbar$ with a small imaginary part ε, so that the poles are lifted by ε in the Borel plane and thus the resummation along $\theta = 0$ is made legitimate. Observe though that, if $\mathrm{Im}\,\hbar = \varepsilon > 0$, then the poles in the Borel plane are pushed *upwards* and thus the resummation $\mathscr{S}^0\tilde{\varphi}$ is equivalent to $\mathscr{S}^{-\varepsilon}\tilde{\varphi}$, as the integration contour of the Laplace transform dodges the pole *from below*; conversely, if $\mathrm{Im}\,\hbar = -\varepsilon < 0$, then the poles in the Borel plane are pushed *downwards* and the resummation $\mathscr{S}^0\tilde{\varphi}$ is equivalent to $\mathscr{S}^{\varepsilon}\tilde{\varphi}$, which now dodges the poles from *above*.
I am grateful to Alexander van Spaendonck for clarifying this subtlety to me.

Finally, we must observe that, if ϕ has a simple zero at p, then there are always 3 Stokes lines flowing out of p; if the pole is double though, there are more complicated shapes. Sticking to the former case, we can observe that the *Stokes graph*, namely the collection of all the Stokes lines, will partition Σ in a number of regions, which we denote as $I, II, \dots$. If the potential defines N potential wells, with the above assumptions (1-6) there will be $3N$ regions. The following picture adapted from [Kam+23] is hopefully clarifying.



(a) $G_+(\mathrm{Im}\,\hbar > 0)$ (b) $G_-(\mathrm{Im}\,\hbar < 0)$

Figure 5.1: Stokes graphs (in blue) and Stokes regions for a double well potential. The wavy red lines indicate the branch cuts.

Because the Stokes lines depend on the phase of $\hbar$, there will be two Stokes graphs, G_+ and G_-, according to the sign of the imaginary part of $\hbar$. We can finally state the following [IN15], [Kam+23], which tells us how the wavefunctions cross the Stokes lines.

Theorem 5.1. Assume that the Stokes graphs G_+, G_- are saddle-free. Assume further that $a \in P_0$ is a simple turning point and that all the assumptions (1-6) are satisfied. Then, if a Stokes line C, issuing from a , separates two regions I and II , the analytic continuation of the wavefunction $\psi_a(z;\hbar)$ from region II to region I, moving clockwise, obeys:

$$\psi_a^I(z;\hbar) = M_\pm \psi_a^{II}(z;\hbar) \tag{5.34}$$

where

$$M_+ = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \quad , \quad M_- = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \tag{5.35}$$

and the lower sign refers to the asymptotic behaviour of $\psi_a(z,\hbar)$ along C, namely is the same label that is attached to C at the endpoint other than a.

Note that the sign of the matrices $M_\pm$ is *not* the same sign labelling the Stokes graphs, which on the contrary refers to the sign of the infinitesimal imaginary part of $\hbar$.

Having learnt how to cross the Stokes lines, we still have to show how the wavefunctions behave at the crossing of a branch cut. Crossing branch cut amounts to choosing the opposite sign in the equation $p_0(z) = \pm\sqrt{V_0(z)}$. This implies that the roles of S^+ and S^- get switched; hence, we argue from (5.29) that P will change sign. Therefore the signs at the exponential of $\psi_a^+(z;\hbar)$ and $\psi_a^-(z;\hbar)$ (we assume here that the Stokes region is fixed) will be exchanged. Because of the normalization term though, which includes a square root, we have to add an extra factor of i. We then have

$$\psi_a'(z;\hbar) = T\psi_a(z;\hbar) \tag{5.36}$$

where the prime indicates the crossing of the branch cut and the matrix T, in light of the previous discussion,

is

$$T = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{5.37}$$

Finally, we have to understand how the difference determinations of the wavefunction change if we trade a root a_1, where the integration begins, for an adjacent a_2. This change of normalization must be performed before crossing every Stokes line. The ansatz (5.27) tells us that

$$\psi_{a_1}(z;\hbar) = N_{a_1,a_2}\psi_{a_2}(z;\hbar), \tag{5.38}$$

where

$$N_{a_1,a_2} = \exp\begin{pmatrix} \frac{i}{\hbar}\int_{a_1}^{a_2} dz' P(z';\hbar) & 0 \\ 0 & -\frac{i}{\hbar}\int_{a_1}^{a_2} dz' P(z';\hbar) \end{pmatrix} \tag{5.39}$$

or, more concisely

$$N_{a_1,a_2} = \begin{pmatrix} \mathcal{V}_{\gamma_{1,2}}^{1/2} & 0 \\ 0 & \left(\mathcal{V}_{\gamma_{1,2}}\right)^{-1/2} \end{pmatrix} \tag{5.40}$$

where by$\gamma_{1,2}$ we indicate the cycle enclosing the region $[a_1, a_2]$.

We are now ready to state the quantisation conditions. Moving clockwise from the rightmost determination of the wavefunction $\psi^{3N}_{a_{2N}}(z;\hbar)$ to the leftmost $\psi^{I}_{a_1}(z;\hbar)$ (where N is the number of potential wells and the apex indicated the Stokes region) upon crossing every line we multiply the wavefunction by the matrices shown above, depending on whether we are crossing a branch cut or a Stokes line and renormalizing before crossing the latter. The process must be repeated twice: once for every Stokes graph or, if you will, for every choice of the phase of $\hbar$. Consider, for example, a double well with four classical turning points, as in Figure 5.1. The method just described yields

$$\begin{cases} \psi^{I}_{a_1}(z;\hbar) = M_+ N_{a_1,a_2} M_+ N_{a_2,a_3} T M_- M_+ N_{a_3,a_4} M_+ N_{a_4,a_1} \psi^{VI}_{a_4}(z,\hbar) := \mathcal{M}^+ \psi^{VI}_{a_4}(z;\hbar) & \mathrm{Im}(\hbar) > 0; \\ \psi^{I}_{a_1}(z;\hbar) = M_+ N_{a_1,a_2} M_+ M_- N_{a_2,a_3} T M_+ N_{a_3,a_4} M_+ N_{a_4,a_1} \psi^{VI}_{a_4}(z,\hbar) := \mathcal{M}^- \psi^{VI}_{a_4}(z;\hbar) & \mathrm{Im}(\hbar) < 0. \end{cases} \tag{5.41}$$

The complex planes represented in Figure 5.1 are understood as the first Riemann sheet: we are then eventually interested in the upper entry of the resulting column vector $\psi^{I}_{a_1}(z;\hbar)$. We can observe that the Stokes line issuing from a_4 at an angle $\theta = 0$ has a $-$ (minus) label: this means that along this direction, $\left(\psi^{VI}_{a_4}(z;\hbar)\right)^+$ (upper entry) falls off to zero, while $\left(\psi^{VI}_{a_4}(z;\hbar)\right)^-$ (lower entry) blows up to infinity. This happens for both Stokes graphs. The quantisation condition, which amounts to requiring the square-integrability of the wavefunction, can then be stated as

$$\mathcal{M}^{\pm}_{1,2} = 0. \tag{5.42}$$

Recalling how the matrices $M_\pm$ and T are defined, we realise that the quantisation condition is just a polynomial in the Voros symbols and their reciprocals. These are the exact quantisation condition provided by the exact WKB method.

Notice how we have not one, but *two* quantisation conditions, depending on whether the imaginary part of $\hbar$ is positive or negative or, in other words, on whether the resummation of the transseries for the Voros symbols is taken respectively below or above the positive real line $\mathbb{R}^+$ by applying (respectively) the Borel-Laplace operators $\mathscr{S}^{-\varepsilon}, \mathscr{S}^{\varepsilon}$, where $\varepsilon > 0$ is a small positive angle. Nevertheless, the ambiguity is resolved once the median resummation (section 2.3.4) of the Voros symbols transseries is taken: only then the imaginary corrections cancel and the quantisation condition is, as it should, unique.

5.2 The cubic potential

Following [SV23], we will show in this section how to exploit the exact WKB method to express the eigenenergies of a one-dimensional quantum system. Consider the time-independent Shrödinger equation for a cubic

potential

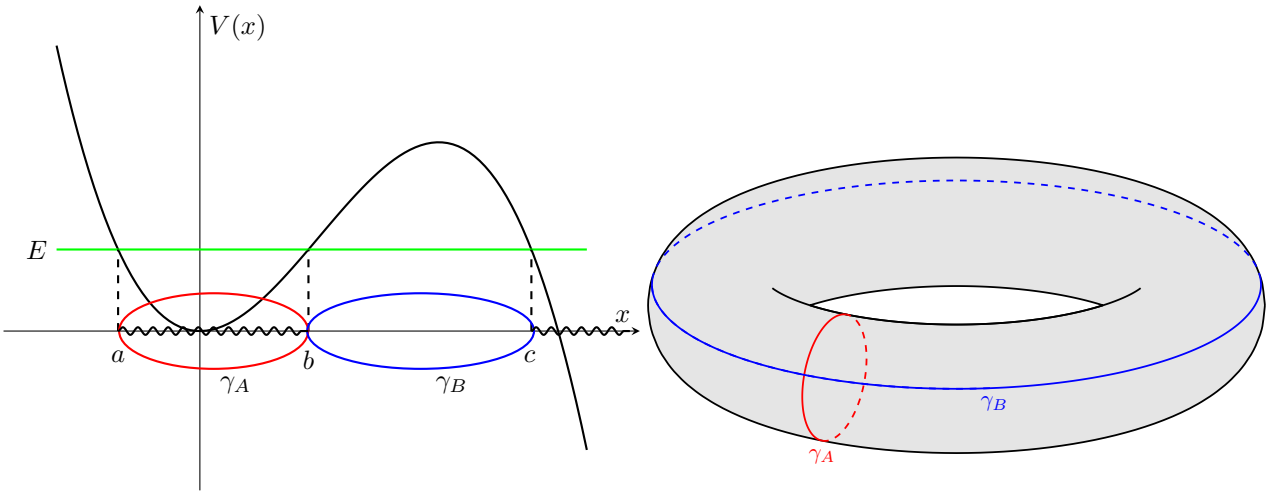
$$\left[-\frac{\hbar^2}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2-gx^3-E\right]\psi(x)=0. \tag{5.43}$$

It will be convenient to rescale $x \to x/g$, $E \to E/g^2$, $\psi(x/g) \to \psi(x)$ and finally $\hbar \to \hbar g^2$, so that the Shrödinger equation will read

$$\left[-\frac{\hbar^2}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2-x^3-E\right]\psi(x)=0. \tag{5.44}$$

We therefore define $V(x) := \frac{1}{2}x^2 - x^3$. Having thus merged the two parameters into the single $\hbar$, we can now use it as a small parameter whereby to expand quantum periods and the energy E perturbatively.

Assuming that $0 < E < 1/54$, the polynomial $E - V(x)$ has 3 real roots a, b, c. As shown in Figure 5.2, this implies that there is only one classically allowed region $[a, b]$ where bound (though unstable) states can exist; and only one tunnelling region $[b, c]$ through which the particle can leak. We will call γ_A, γ_B the cycles encircling the classical and the tunnelling regions, respectively. Since there are only two branch cuts, the complexified WKB curve is simply a torus $\mathbb{T}$, whose homology group $H_1(\mathbb{T}, \mathbb{C})$ admits γ_A and γ_B as a basis of independent cycles. Thus, there are only one perturbative period $t(E;\hbar)$ and one dual period $t^D(E;\hbar)$.



(a) Cycles for the quantum A and B periods in red and blue respectively. The wiggly lines indicate the branch cuts.

(b) Complexified WKB curve.

Figure 5.2: WKB curve and A and B cycles for the polynomial potential.

As foreshadowed in the previous section, the asymptotic series (5.14) for $t_n(E;\hbar) =: t(E;\hbar)$ and $t_n^D(E;\hbar) =: t^D(E;\hbar)$, which we shall denote by $\tilde{t}(E)$ and $\tilde{t}^D(E)$, are divergent series of Gevrey class 1. We must therefore analyse their Borel planes and identify the precise location of the poles. As stated in [SV23], it can be shown that the Borel transform of $\tilde{t}^D(E)$ has infinitely many poles, equally spaced along the the imaginary axis; likewise, the Borel transform of the formal series $\tilde{t}(E)$ has poles at

$$\omega = nt_0^D(E) \qquad n \in \mathbb{Z}^*. \tag{5.45}$$

For brevity, we will set $\mathcal{A} := t_0^D(E)$ to be the **instanton action**, named thus because it encodes information about the instantaneous tunnelling of the particle through the potential barrier. Recall that $t_0^D(E)$ is real: therefore, the previous equation tells us that $\theta = 0$ is a Stokes line for the series $\tilde{t}(E)$. As the quantum periods $t(E;\hbar)$ and $t^D(E;\hbar)$ are real functions of the real variable $\hbar$, we are primarily interested in resumming their asymptotic series along $\theta = 0$: yet, due to the presence of the Stokes line, this is not possible. Hence, the perturbative power series $\tilde{t}(E)$ must be promoted to a transseries, whose non-perturbative sectors are formally multiplied by the resurgent symbols $e^{-\omega/\hbar}$ with $\omega \in \Omega := \mathcal{A}\mathbb{Z}$ (closed under addition). However, it

will be often convenient to consider as resurgent symbols the formal terms $e^{-nt^D(E;\hbar)/\hbar}$, which are nothing else than the proper resurgent symbol $e^{-n\mathcal{A}/\hbar}$ multiplied by a formal power series.

Because the quantum period $t(E;\hbar)$ is expressed by a transseries, the quantised energy $E(\hbar)$ will be also expressed by a transseries. We will derive now the exact transseries solution for the energy function $E(\hbar)$, closely following [SV23]. We will show two equivalent approaches: either exploiting either the quantisation conditions and the Lagrange inversion theorem, or the DDP formula.

5.2.1 Exact transseries solution: Lagrange inversion

The quantisation conditions for the cubic potential can be found with the method described in the previous section as functions of the Voros symbols. As explained earlier, there are in fact two quantisation conditions, depending on sign of the imaginary part of $\hbar$: explicitly, they read (see [SV23])

$$\begin{cases} D^- := 1 + e^{\frac{2\pi i}{\hbar} t(E;\hbar)} + e^{-\frac{1}{\hbar} t^D(E;\hbar)} = 0 & \mathrm{Im}(\hbar) < 0; \\ D^+ := 1 + e^{\frac{2\pi i}{\hbar} t(E;\hbar)} = 0 & \mathrm{Im}(\hbar) > 0. \end{cases} \tag{5.46}$$

Notice that the second equation simply yields the perturbative quantisation condition

$$t(E;\hbar) = \hbar \left(n + \frac{1}{2} \right). \tag{5.47}$$

It is beyond the scope of this illustrative discussion to derive the quantisation conditions (5.46); yet, let us show that the above condition is quite sensible. Indeed, hearkening back to (5.16) we realise that the quantisation condition is, at first order in $\hbar$

$$t_0(E) = \frac{1}{\pi} \int_a^b dx\, p(x) = \frac{1}{\pi} \int_a^b dx \sqrt{2 \left(E - \frac{x^2}{2} + x^3 \right)} = \hbar \left(n + \frac{1}{2} \right), \tag{5.48}$$

which, up to the prefactor, implies that the area in the classical phase space must be quantised! This is known as the Bohr-Sommerfeld quantisation condition, which used to be the guiding principle to the 'old quantum theory'. As the Planck constant has the units of angular momentum, it was a sensible surmise that the area of phase space, which has the same units, should be measured in units thereof: it was then expected that a quantum theory would emerge by imposing that such area take only discrete values in units of $\hbar$. Although the path integral formulation (which is ultimately at the background of this derivation) rather measures the action in units of the Plank constant, at first order, the Bohr-Sommerfeld quantisation condition coincides with the one given by the exact WKB method.

For every n, it is possible to invert (5.47) and derive a perturbative series $\tilde{E}(t)$, in powers of $\hbar$, for the energy E. This can be done numerically with the aid of the Bender-Wu package of Mathematica (see e.g. [SÜ18]). Letting $\nu = n + \frac{1}{2}$, the formal solution reads

$$\tilde{E}(\nu) = \nu\hbar - \left(\frac{7}{16} + \frac{15}{4} \nu^2 \right) \hbar^2 + \dots . \tag{5.49}$$

The series is not 1-summable, and should be regarded as a transseries with only one, perturbative sector. It is then possible to find a transseries, which we will indicate by $\mathfrak{E}$, by imposing that the the resummation of $\mathfrak{E}$ for $\mathrm{Im}(\hbar) < 0$ (i.e. on a line $e^{i\varepsilon}\mathbb{R}$, with $\varepsilon > 0$) yields the same result as the resummation of $\tilde{E}$ for $\mathrm{Im}(\hbar) > 0$ (i.e. on a line $e^{-i\varepsilon}\mathbb{R}$) once the quantisation condition D^- is implemented.

To fulfill this task, let us first promote the asymptotic series $\tilde{t}(E)$ of the quantum period to a transseries $\mathfrak{t}(E)$. We will then write, very generally

$$\mathfrak{t}(E) = \tilde{t}(E) + \widetilde{\Delta t}(E), \tag{5.50}$$

where $\widetilde{\Delta t}(E)$ includes all the non-perturbative instanton sectors. The quantisation conditions (5.46) will now *formally* read (momentarily dropping the argument E to avoid cluttering)

$$\begin{cases} D^- := 1 + e^{2\pi i(\tilde{t}+\widetilde{\Delta t})/\hbar} + e^{-\tilde{t}^D/\hbar} = 0 & \mathrm{Im}(\hbar) < 0, \\ D^+ := 1 + e^{2\pi i\tilde{t}/\hbar} = 0 & \mathrm{Im}(\hbar) > 0, \end{cases} \tag{5.51}$$

whence it can be easily derived that

$$\widetilde{\Delta t}(E) = -\frac{i\hbar}{2\pi}\log\left(1 + e^{-\tilde{t}^D(E)/\hbar}\right). \tag{5.52}$$

In order to solve for the energy, we will use the quantum period $t(E;\hbar)$ (or its formal transseries solution) as independent variable. Having already fixed the perturbative sector to solve the equation $D^+ = 0$ above, namely to be $\tilde{t} = \hbar(n + 1/2)$, the independent variable happens to be in fact $\widetilde{\Delta t}$. Hence we rewrite the above as

$$\begin{aligned} \tilde{\Delta} t &= -\frac{i\hbar}{2\pi}\log\left(1 + e^{-\tilde{t}^D(\tilde{t}+\widetilde{\Delta t})/\hbar}\right) \\ &= -\frac{\hbar}{2\pi i}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\exp\left\{-\tfrac{k}{\hbar}e^{\widetilde{\Delta t}\partial_t}\tilde{t}^D(\tilde{t})\right\}. \end{aligned} \tag{5.53}$$

The formal expression in the second line will now allow us to exploit the Lagrange inversion theorem. In the following, we shall use variables t and Δt which we will later interpret as the perturbative quantum period and the non-perturbative correction, as in (5.50); it should be understood that the following steps also hold formally, i.e. by substituting t and Δt with their formal asymptotic expansions $\tilde{t}$, $\widetilde{\Delta t}$.

Consider a function $R(\Delta, z)$ which can be expanded in the following way:

$$R(\Delta, z) = \hbar\sum_{k=1}^{\infty} r_k z^k \exp\{-\tfrac{k}{\hbar}e^{\Delta\partial_t}F(t)\}, \tag{5.54}$$

where z merely serves the purpose of a bookkeeping variable and will be set to 1 at the end of the calculations. If Δt is the solution to the equation

$$\Delta t = R(\Delta t, z), \tag{5.55}$$

the Lagrange inversion theorem ensures that, for a generic smooth function $\varphi(\Delta)$, one has

$$\varphi(\Delta t) = \sum_{m=1}^{\infty}\frac{1}{m}[\Delta^{m-1}]\varphi'(\Delta)R(\Delta, z)^m \tag{5.56}$$

where $[\Delta]^{m-1}$ is an operator that extracts the $(m-1)$-th power of Δ in the expansion of the expression on the right. For our purposes, moreover, it will be convenient to set

$$\varphi(\Delta) := f(t + \Delta) = e^{\Delta\partial_t}f(t) \tag{5.57}$$

and expand $\varphi(\Delta t)$ in powers of z by

$$\varphi(\Delta t) = f(t + \Delta t) = \sum_{k=1}^{\infty} f_n z^n. \tag{5.58}$$

Our aim is then interpreting $f(t + \Delta t)$ as the energy transseries, depending on the full transseries $\mathfrak{t}$ (5.50). Clearly, the coefficient f_n can be extracted by differentiation with respect to z and, recalling (5.56), one finds

$$f_n = \sum_{m=1}^{\infty}\frac{1}{m!n!}\left(\frac{\partial}{\partial z}\right)^n\left(\frac{\partial}{\partial\Delta}\right)^{m-1}\varphi'(\Delta)R(\Delta, z)^m\Big|_{\Delta=z=0}. \tag{5.59}$$

Let us first compute the derivative with respect to z. Using Faà di Bruno's formula, one finds that

$$\left(\frac{\partial}{\partial z}\right)^n R(\Delta,z)^m = \sum_{k=1}^{n}\left(\frac{\partial^k}{\partial R^k}R^m\right)B_{n,k}(R',R'',...,R^{(n-k+1)}) \tag{5.60}$$

where $B_{n,k}$ are the incomplete Bell polynomials. These polynomials are homogeneous of degree n, whence, upon defining $R^{(k)} := \exp\{-\frac{k}{\hbar}e^{\Delta\partial_t}F(t)\}\tilde{R}^{(k)}$, the exponential terms are factored out:

$$B_{n,k}(R',R'',...,R^{(n-k+1)}) = \exp\left\{-\frac{n}{\hbar}e^{\Delta\partial_t}F(t)\right\}B_{n,k}(\tilde{R}^{(1)},\tilde{R}^{(1)},...,\tilde{R}^{(n-k+1)}). \tag{5.61}$$

The only dependence on Δ is now on the exponential term. Therefore, focusing now on the derivatives with respect to Δ, we find

$$\begin{aligned}\left(\frac{\partial}{\partial\Delta}\right)^{m-1}\left[\varphi'(\Delta)\exp\left\{-\frac{n}{\hbar}e^{\Delta\partial_t}F(t)\right\}\right]\Bigg|_{\Delta=0} &= \left(\frac{\partial}{\partial\Delta}\right)^{m-1}e^{\Delta\partial_t}\left[f'(t)\exp\left\{-\frac{n}{\hbar}F(t)\right\}\right]\Bigg|_{\Delta=0} \\ &= \left(\frac{\partial}{\partial t}\right)^{m-1}f'(t)\exp\left\{-\frac{n}{\hbar}F(t)\right\}.\end{aligned} \tag{5.62}$$

We can finally put everything together: observing that $\frac{\partial^k}{\partial R^k}R^m = \delta_{k,m}k!$, we reach

$$f_n = \frac{1}{n!}\sum_{k=1}^{n}B_{n,k}(\tilde{R}^{(1)}(0),\tilde{R}^{(2)}(0),...,\tilde{R}^{(n-k+1)}(0))\left(\frac{\partial}{\partial t}\right)^{k-1}f'(t)\exp\left\{-\frac{n}{\hbar}F(t)\right\} \tag{5.63}$$

where, as it can easily be read from (5.54), $\tilde{R}^{(k)}(0) = \hbar k! r_k$. Recalling the properties of Bell polynomials, a factor of $\hbar^{k+1}$ can be factored out, so that we rewrite

$$f_n = \hbar\sum_{m=0}^{n-1}u_{n,m}\left(\hbar\frac{\partial}{\partial t}\right)^m f'(t)\exp\left\{-\frac{n}{\hbar}F(t)\right\}, \tag{5.64}$$

where

$$u_{n,m} = \frac{1}{n!}B_{n,m+1}(1!r_1,...,(n-m)!r_{n-m}). \tag{5.65}$$

Having done so, we now let $f(t+\Delta t)$ to be, rather than a function, the formal energy transseries $\mathfrak{E}(t)$, depending on the quantum period t, to be resummed with the quantisation condition D^+ along a direction $e^{-i\varepsilon}\mathbb{R}$. Then we reach the exact transseries solution

$$\mathfrak{E}(t) = \tilde{E}(t) + \hbar\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}u_{n,m}\left(\hbar\frac{\partial}{\partial t}\right)^m\frac{\partial\tilde{E}(t)}{\partial t}\exp\left\{-\frac{n}{\hbar}\tilde{t}^D(t)\right\}, \tag{5.66}$$

where, comparing (5.53) with (5.54), we find that

$$r_k = \frac{(-1)^k}{k}\left(-\frac{1}{2\pi i}\right). \tag{5.67}$$

By the properties of incomplete Bell polynomials, we can factor out the terms $(-1)^k$ and $(-2\pi i)^{-1}$. The transseries will then read

$$\mathfrak{E}(t) = \tilde{E}(t) + \hbar\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}(-1)^n c_{n,m}\left(-\frac{1}{2\pi i}\right)^{m+1}\left(\hbar\frac{\partial}{\partial t}\right)^m\frac{\partial\tilde{E}(t)}{\partial t}\exp\left\{-\frac{n}{\hbar}\tilde{t}^D(t)\right\}, \tag{5.68}$$

where now $c_{n,m} = \frac{1}{n!}B_{n,m+1}(0!,...,(n-m-1)!)$. It is now possible to summarise our results by defining the one-parameter transseries

$$\mathfrak{E}(t;\sigma) = \tilde{E}(t) + \hbar\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}(-1)^n c_{n,m}\sigma^{m+1}\left(\hbar\frac{\partial}{\partial t}\right)^m\frac{\partial\tilde{E}(t)}{\partial t}\exp\left\{-\frac{n}{\hbar}\tilde{t}^D(t)\right\}, \tag{5.69}$$

which allows us to state the action of the Stokes automorphism on the perturbative series $\tilde{E}(t) = \mathfrak{E}(t;0)$ as

$$\mathfrak{S}_0\mathfrak{E}(t;0) = \mathfrak{E}\left(t;\left(-\tfrac{1}{2\pi i}\right)\right). \tag{5.70}$$

In fact, the perturbative energy series $\tilde{E}(t)$ is to be resummed below the real axis, while the transseries (5.68) is to be resummed above: thus $\mathscr{S}^{-\varepsilon}\tilde{E}(t) = \mathscr{S}^{\varepsilon}\mathfrak{E}(t) = \mathscr{S}^{\varepsilon}\mathfrak{S}_0\tilde{E}(t)$. In this formulation then, we conclude that $\left(-\frac{1}{2\pi i}\right)$ is the unique Stokes constant.

5.2.2 Exact transseries solution: DDP formula

In the previous derivation we have exploited our knowledge of the quantisation conditions to find the transseries solution (5.69) for the energy. In this section, we provide an alternative viewpoint, based rather on our knowledge of the DDP formula.

The DDP formula (5.20), specialised to the Voros symbols $\mathcal{V}_A$, $\mathcal{V}_B$, with $\langle\gamma_A,\gamma_B\rangle = 1$, reads

$$\mathfrak{S}_0\mathcal{V}_A = \mathcal{V}_A(1+\mathcal{V}_B)^{-1}. \tag{5.71}$$

As the exponential is convergent, we may write $\log\mathfrak{S}_0 e^{\frac{2\pi i}{\hbar}\tilde{t}} = \frac{2\pi i}{\hbar}\mathfrak{S}_0\tilde{t}$, whence it is found that,

$$\mathfrak{S}_0\tilde{t}(E) = \tilde{t}(E) + \frac{i\hbar}{2\pi}\log\left(1+e^{-\tilde{t}^D(E)/\hbar}\right). \tag{5.72}$$

As the poles in the Borel plane for the dual quantum period $\tilde{t}^D(E)$ lie on the imaginary axis, the DDP formula also yields

$$\mathfrak{S}_0\tilde{t}^D(E) = \tilde{t}^D(E) \tag{5.73}$$

implying that all the alien derivatives acting on $\tilde{t}^D(E)$ vanish.

Let us now expand (5.72). We will get

$$\mathfrak{S}_0\tilde{t}(E) = \tilde{t}(E) + \dot{\Delta}_{\mathcal{A}}\tilde{t}(E) + \left(\frac{1}{2}\dot{\Delta}_{\mathcal{A}}^2 + \dot{\Delta}_{2\mathcal{A}}\right)\tilde{t}(E) + ... \tag{5.74}$$

whence we infer that we must have $\dot{\Delta}_{\mathcal{A}}\tilde{t}(E) = \frac{i\hbar}{2\pi}e^{-\tilde{t}^D(E)/\hbar}$, which only depends on $\tilde{t}^D(E)$. Hence, as all the alien derivatives on $\tilde{t}^D(E)$ vanish, it follows that $\dot{\Delta}_{\mathcal{A}}^n\tilde{t}(E) = 0$ for every $n \geq 1$. Hence, by comparison of (5.72) and (5.74), we can easily argue that, for every $l \geq 1$,

$$\dot{\Delta}_{lA}\tilde{t}(E) = \left(-\frac{\hbar}{2\pi i}\right)\frac{(-1)^{l+1}}{l}e^{-\frac{l}{\hbar}\tilde{t}^D(E)}. \tag{5.75}$$

In order to find the action of the dotted alien derivatives on $\tilde{E}(t)$, we need the following formula. Let $\tilde{\chi}(y)$ be a formal power series of $\mathbb{C}[[x]]$, whose coefficients depend on a variable y, namely

$$\tilde{\chi} = \sum_{n=0}^{\infty} a_n(y)x^n. \tag{5.76}$$

Let now $\tilde{\varphi} \in \mathbb{C}[[x]]$ to be a a formal power series $\sum_{n=0}^{\infty} b_n x^n$. We define then $\tilde{\chi}(\tilde{\varphi}) \in \mathbb{C}[[x]]$ as the power series obtained by formally substituting $\tilde{\varphi}$ for y. Then one has (see [SV23])

$$\dot{\Delta}_\omega\tilde{\chi}(\tilde{\varphi}) = \dot{\Delta}_\omega\tilde{\chi}(y)\Big|_{y=\tilde{\varphi}} + \frac{\partial\tilde{\chi}(y)}{\partial y}\Big|_{y=\tilde{\varphi}}\dot{\Delta}_\omega\tilde{\varphi}. \tag{5.77}$$

If we now interpret $\tilde{\chi}(y) := \tilde{E}(t)$, $\tilde{\varphi} = \tilde{t}(E)$ and of course $x = \hbar$, we will have that $\tilde{\chi}(\tilde{\varphi}) = E$, on which all the alien derivatives must vanish. It follows that

$$\dot{\Delta}_{l\mathcal{A}}\tilde{E}(t) = -\frac{\partial\tilde{E}}{\partial t}\dot{\Delta}_{l\mathcal{A}}\tilde{t}(E) = \left(-\frac{\hbar}{2\pi i}\right)\frac{(-1)^l}{l}\frac{\partial\tilde{E}}{\partial t}e^{-\frac{l}{\hbar}\tilde{t}^D(t)}. \tag{5.78}$$

With this formula, we can verify that the transseries (5.69) is indeed the result yielded by the action of the Stokes automorphism on $\tilde{E}(t)$. Let us first check that the transseries (5.69) and (5.74) match order by order. The one-instanton correction is simply $\dot{\Delta}_{\mathcal{A}}\tilde{E}(t)$, which can be rewritten as

$$\dot{\Delta}_{\mathcal{A}}\tilde{E}(t) = \left(\frac{\hbar}{2\pi i}\right)\frac{\partial\tilde{E}}{\partial t}e^{-\frac{1}{\hbar}\tilde{t}^D(t)} = \hbar\left(\frac{1}{2\pi i}\right)(-1)c_{1,1}e^{-\frac{1}{\hbar}\tilde{t}^D(t)}. \tag{5.79}$$

Let us check the order $n=2$ as well. To do so, we will need an expression for

$$\dot{\Delta}^2_{\mathcal{A}}\tilde{E}(t) = \left(\frac{\hbar}{2\pi i}\right)\dot{\Delta}_{\mathcal{A}}\left[\frac{\partial\tilde{E}}{\partial t}e^{-\frac{1}{\hbar}\tilde{t}^D(t)}\right]. \tag{5.80}$$

Recalling that the dotted alien derivatives obey the Leibniz rule (according to Theorem 2.7), let us compute first

$$\begin{aligned}\dot{\Delta}_{\mathcal{A}}e^{-\frac{1}{\hbar}\tilde{t}^D(t)} &= -e^{-\frac{1}{\hbar}\tilde{t}^D(t)}\frac{1}{\hbar}\frac{\partial\tilde{t}^D(E)}{\partial E}\dot{\Delta}_{\mathcal{A}}\tilde{E}(t) = -\frac{1}{\hbar}\frac{\partial\tilde{t}^D(E)}{\partial E}\left(\frac{\hbar}{2\pi i}\right)\frac{\partial\tilde{E}}{\partial t}e^{-\frac{2}{\hbar}\tilde{t}^D(t)} \\ &= -\frac{1}{\hbar}\frac{\partial\tilde{t}^D(t)}{\partial t}\left(\frac{\hbar}{2\pi i}\right)e^{-\frac{2}{\hbar}\tilde{t}^D(t)},\end{aligned} \tag{5.81}$$

where in the first equality we have recalled (2.82) and for the second we have resorted again to (5.77), which is simplified by the fact that all the alien derivatives vanish on $\tilde{t}^D(E)$. In a similar fashion, we compute now

$$\dot{\Delta}_A\frac{\partial\tilde{E}}{\partial t} = \frac{\partial}{\partial t}\dot{\Delta}_A\tilde{E}(t) = \left(\frac{\hbar}{2\pi i}\right)\frac{\partial}{\partial t}\left[\frac{\partial\tilde{E}}{\partial t}e^{-\frac{1}{\hbar}\tilde{t}^D(t)}\right] = \left(\frac{\hbar}{2\pi i}\right)\left[\frac{\partial^2\tilde{E}}{\partial t^2} - \frac{1}{\hbar}\frac{\partial\tilde{t}^D(t)}{\partial t}\right]e^{-\frac{1}{\hbar}\tilde{t}^D(t)}, \tag{5.82}$$

whence, using (5.81) found before, we compute (5.80) to be

$$\dot{\Delta}^2_{\mathcal{A}}\tilde{E}(t) = \left(\frac{\hbar}{2\pi i}\right)^2\left[\frac{\partial^2\tilde{E}}{\partial t^2} - \frac{2}{\hbar}\frac{\partial\tilde{t}^D(t)}{\partial t}\frac{\partial\tilde{E}}{\partial t}\right]e^{-\frac{2}{\hbar}\tilde{t}^D(t)} = \left(\frac{\hbar}{2\pi i}\right)^2\frac{\partial}{\partial t}\left[\frac{\partial\tilde{E}}{\partial t}e^{-\frac{2}{\hbar}\tilde{t}^D(t)}\right]. \tag{5.83}$$

With this result, we are finally enabled to verify that

$$\begin{aligned}\left(\frac{1}{2}\dot{\Delta}^2_{\mathcal{A}} + \dot{\Delta}_{2\mathcal{A}}\right)\tilde{E} &= \hbar\frac{1}{2}\left(-\frac{1}{2\pi i}\right)^2\hbar\frac{\partial}{\partial t}\left[\frac{\partial\tilde{E}}{\partial t}e^{-\frac{1}{\hbar}\tilde{t}^D(t)}\right] + \frac{1}{2}\left(-\frac{\hbar}{2\pi i}\right)\frac{\partial\tilde{E}}{\partial t}e^{-\frac{l}{\hbar}\tilde{t}^D(t)} \\ &= \hbar\left[c_{2,1}\left(-\frac{1}{2\pi i}\right)^2\hbar\frac{\partial}{\partial t} + c_{2,0}\left(-\frac{1}{2\pi i}\right)\right]\frac{\partial\tilde{E}}{\partial t}e^{-\frac{l}{\hbar}\tilde{t}^D(t)},\end{aligned} \tag{5.84}$$

since, as it can be easily checked, $c_{2,1} = c_{2,0} = \frac{1}{2}$.

Convinced to be on the right track, one can generalise (5.83) and find that a product of dotted alien derivatives on $\tilde{E}$ acts like:

$$\dot{\Delta}^{j_1}_{\mathcal{A}}\dot{\Delta}^{j_2}_{2\mathcal{A}}\ldots\dot{\Delta}^{j_m}_{m\mathcal{A}}\tilde{E}(t) = \left(\frac{-\hbar}{2\pi i}\right)^k\prod_{l=1}^{m}\left[\frac{(-1)^l}{l}\right]\left(\frac{\partial}{\partial t}\right)^{k-1}\left[\frac{\partial\tilde{E}}{\partial t}e^{-\frac{n}{\hbar}\tilde{t}^D(t)}\right], \tag{5.85}$$

where we put $k := \sum_{l=1}^m j_l$, $n := \sum_{l=1}^m l j_l$. In this way the action of the Stokes automorphism can be computed as a formal composition of series, involving the incomplete Bell polynomials. Explicitly, we have

$$\mathfrak{S}_0\tilde{E}(t) = \exp\left(\sum_{l=1}^{\infty}\dot{\Delta}_{l\mathcal{A}}\right)\tilde{E}(t) = \left\{1 + \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=1}^{\infty}B_{n,k}(1!\dot{\Delta}_{\mathcal{A}}, ..., (n-k+1)!\dot{\Delta}_{(n-k+1)\mathcal{A}})\right\}\tilde{E}(t). \tag{5.86}$$

The action of the Bell polynomials on $\tilde{E}(t)$ reads

$$B_{n,k}(1!\dot{\Delta}_{\mathcal{A}}, ..., (n-k+1)!\dot{\Delta}_{(n-k+1)\mathcal{A}})\tilde{E}(t) = \sum\frac{n!}{j_1!\ldots j_{n-k+1}!}\dot{\Delta}^{j_1}_{\mathcal{A}}\dot{\Delta}^{j_2}_{2\mathcal{A}}\ldots\dot{\Delta}^{j_{n-k+1}}_{(n-k+1)\mathcal{A}}\tilde{E}(t), \tag{5.87}$$

where the sum runs over all the configurations $j_1, ..., j_{n-k+1}$ such that

$$\begin{cases} j_1 + j_2 + ... + j_{n-k+1} = k \\ j_1 + 2j_2 + ... + (n-k+1)j_{n-k+1} = n. \end{cases} \tag{5.88}$$

Comparing with (5.85), we realise that (5.87) rewrites as

$$(-1)^n \left(\frac{-\hbar}{2\pi i}\right)^k \left(\frac{\partial}{\partial t}\right)^{k-1} \left[\frac{\partial \tilde{E}}{\partial t} e^{-\frac{n}{\hbar}\tilde{t}^D(t)}\right] B_{n,k}\left(\frac{1!}{1}, ..., \frac{(n-k+1)!}{n-k+1}\right), \tag{5.89}$$

where we recognise the coefficients $c_{n,k+1}$: upon shifting $k \to m+1$, we find again

$$\mathfrak{S}_0 \tilde{E}(t) = \tilde{E}(t) + \hbar \sum_{n=1}^{\infty} (-1)^n \sum_{m=0}^{\infty} c_{n,m} \left(\frac{-1}{2\pi i}\right)^{m+1} \left(\hbar \frac{\partial}{\partial t}\right)^m \left[\frac{\partial \tilde{E}}{\partial t} e^{-\frac{n}{\hbar}\tilde{t}^D(t)}\right], \tag{5.90}$$

namely $\mathfrak{E}(t; (-2\pi i)^{-1})$, as in (5.69).

Before ending this section, it is worthwhile to stress that the fact that the transseries to be resummed for $\mathrm{Im}(\hbar) > 0$ only has the perturbative sector $\tilde{E}(t)$ is merely a coincidence: in fact the next-simplest case of the double well no longer shows this feature, as detailed in [SV23].

5.3 Do energy eigenvalues obey a differential equation?

From the foregoing discussion, we learn that any energy eigenvalue $E(\nu, \hbar)$ is a function of the rescaled coupling $\hbar$ yielded by the median resummation of the transseries $\mathfrak{E}(t)$, quantised according to (5.46) in the case of a cubic potential. In the median resummation – which we are enabled to perform thanks to our knowledge of the alien algebra (5.78) – all the imaginary ambiguities cancel and thus the functions $E(\nu, \hbar)$ are real-valued. Still, due to the Stokes phenomenon, Nevanlinna-Sokal theorem 2.6 does not apply: therefore we are not allowed to conclude that the functions $E(\nu, \hbar)$ are definable in the o-minimal structure $\mathbb{R}_{\mathscr{G}}$. Nevertheless, they are a physically-meaningful observables and, prompted by the Tameness Conjecture 1.1, we are led to inquire whether they are tame functions in another o-minimal structure.

Hearkening back to our list of o-minimal structures presented in section 1.5, we observe that there is indeed an o-minimal structure hosting the resummation of formal power series featured by the Stokes phenomenon: it is the structure $\mathbb{R}_{\mathrm{an},H}$ described in section 1.5.3. However, to claim definability in this structure, we need to be in possession of a differential equation obeyed by the function $E(\nu, \hbar)$ and, formally, by its asymptotic expansion $\tilde{E}(\nu)$ (5.49). As mentioned earlier, this power series can be computed at every order by means of the Bender-Wu Mathematica package: such a differential equation can be then sought numerically. Our approach consisted in fixing a value for ν (which was chosen to be the ground state value $\nu = 1/2$), making an ansatz for the differential equation obeyed by $\tilde{E}(1/2)$ and then solving numerically for the free parameters, in the attempt to capture a hidden regularity in the coefficients of the power series $\tilde{E}(1/2)$. The main drawback of this approach is that there is very little we can foresee about the differential equation the eigenvalue $E(\nu, \hbar)$ might satisfy which could help us formulating an ansatz. The only sensible guess is that, since energy is defined up to one additive constant, first-order differential equations suggest themselves as the most promising ones. Nevertheless, in the implementation of the algorithm, we allowed for differential equations of higher order, as this does not imply any insurmountable computational complication.

This numerical approach led to no result. Although the reason may well be that our ansätze were too simple, it is also possible that such a differential equation may not exist at all. In fact, its existence would enable us to find the quantised energy by circumventing the wavefunction solutions to the Shrödinger equations, decoupling the eigenvalue problem from the eigenvector problem. Yet, we have no clear argument as to why it should not exist, which is why we attempted a numerical solution as a first check. Despite our failure, we still briefly relate here the precise algorithm, which was implemented both in Mathematica and in Python, so that the exact form of the ruled-out differential equations is clarified.

5.3.1 An algorithm for differential equations

Ansatz for the differential equation For shortness, let $f = \tilde{E}(1/2)$ be the series in powers of $\hbar$ asymptotic to the ground-state energy, which is provided by the Bender-Wu package at all orders in $\hbar$. Let also $f', f'', ..., f^{(n)}$ be the term-by-term derivatives of f with respect to $\hbar$. We choose the ansatz:

$$\begin{aligned} & p_{0,1} f + p_{1,1} f' + p_{2,1} f'' + ... + p_{N,1} f^{(N)} \\ & + p_{0,2} f^2 + p_{1,2} \left(f'\right)^2 + p_{2,2} \left(f''\right)^2 + ... + p_{N,2} \left(f^{(N)}\right)^2 \\ & \vdots \\ & + p_{0,M} f^M + p_{1,M} \left(f'\right)^M + p_{2,M} \left(f''\right)^2 + ... + p_{N,M} \left(f^{(N)}\right)^M = -p_{0,0} \end{aligned} \tag{5.91}$$

where p_{ij}, with $0 \leq i \leq N$ and $0 \leq j \leq M$, are polynomials of degree d

$$p_{i,j}(\hbar) = a_0^{i,j} + a_1^{i,j} \hbar + a_2^{i,j} \hbar^2 + ... + a_d^{i,j} \hbar^d. \tag{5.92}$$

We can then check more and more general ansätze by iterating over the order of the differential equation N, the degree d of the polynomials and the maximal power M.

Counting degrees of freedom The idea is to solve for the coefficients $a_k^{i,j}$ by solving the ansatz (5.91) order by order. This amounts to simply solving a linear system; to guarantee that it be solvable though, we must ensure that the number of our free parameters matches the number of independent coefficients in f.

The number of free parameters D in the ansatz (5.91) is easily computed:

$$D = (d+1)[(N+1)M + 1] - 1, \tag{5.93}$$

where the subtraction of 1 is due to the fact that one can always factor out an overall multiplicative constant leaving the differential equation unchanged. This means that, in order to solve for the coefficients $a_k^{i,j}$, we must truncate the power series of f *exactly* at order D, namely

$$f_D := \frac{1}{2}\hbar - \frac{22}{16}\hbar^2 + ... + \frac{f^{(D)}(0)}{D!}\hbar^D. \tag{5.94}$$

Observe that, as the energy is always defined only up to an additive arbitrary constant, the number of independent coefficients in this truncated series is indeed D, and not $D+1$.

Iteration and termination condition Let us fix N, M, d. After plugging the truncated power series at order D in the ansatz (5.91), we will obtain a new power series of order $DM+d$ (the term $\hbar^{DM+d}$ originating from the term $p_{0,M} f^M$). This is clearly much larger than the number of free parameters, and therefore we have truncate the result at order $D-1$. We will then obtain a linear system of D equations (one for each order, from 0 to $D-1$) in $D+1$ unknowns, i.e. the free parameters $a_k^{i,j}$. As such, the solution will not be numerical, but rather a one-parameter family of solutions: in other words, all the $D+1$ coefficients will be expressed as functions of one of them, which of course can be chosen arbitrarily among those that are not identically vanishing.

Having found and saved a solution for fixed N, M, d, we now repeat the same procedure to find a solution for $N, M, d+1$. If the solution found at the previous step defines the correct differential equation, then the new solution must be the same: namely, provided that all the new parameters $a_{d+1}^{i,j}$ are set to zero, the remaining coefficients are linked by the same one-parameter family of solutions found at the previous step. This condition terminates the algorithm, which will return the parameters $a_k^{i,j}$ – defined up to an overall multiplicative factor – as a candidate differential equation. It should be understood that this is only

a candidate solution, as the termination condition is certainly necessary, but not sufficient to claim that the solution is the correct one. In fact, as the power series for f must be truncated, it is impossible to decree with absolute certainty that any candidate solution is the correct one: it is always possible for a candidate solution to hold for $d, d+1, d+2, ..., d+n$ (and fixed N, M) and then fail for the next case $d+n+1$, which, being satisfied with the n previous confirmations, we may have chosen not to check.

The termination condition is quite subtle to check computationally, as the solution is not numerical, but a one-parameter family of solutions. It is more convenient, rather, to check that the linear system at order $d+1$ is consistent with the system at order d. To be more precise, let us write schematically the system at order d for fixed N, M as

$$A_d \left|a_d\right\rangle = 0, \tag{5.95}$$

where A_d is a $D(d)$-by-$(D(d)+1)$ matrix, $\left|a_d\right\rangle$ is $(D(d)+1)$-by-1 column vector where we arrange all the parameters a_k^{ij} and $D(d)$ is the number of degrees of freedom (5.93) as a function of d. The solution at order $d+1$ is then consistent with that at order d if the matrix A_d is 'nested' into the matrix A_{d+1}, the precise meaning of nested depending on how the coefficients a_k^{ij} are arranged in $\left|a_d\right\rangle$ and $\left|a_{d+1}\right\rangle$. The advantage of formulating the termination condition in terms of the matrices A_d and A_{d+1}, as opposed to the solutions, is that it is a purely numerical check which can be performed easily by the computer.

This algorithm can be run first by fixing N and M and then by iterating over d from $d=0$ up to $d=d_{\max}$. One can of course iterate over N and M as well, but is should be understood that the termination condition consists in a comparison between two solutions having the same N and M.

5.3.2 Example: the sine function

Let us give a quick example of how the algorithm can work. Switching to a more convenient variable x, suppose we want to find a differential equation for the function whose expansion is given by

$$f = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots . \tag{5.96}$$

The algorithm will then start with $N=1, M=1, d=0$, so that the ansatz is

$$af + bf' + c = 0. \tag{5.97}$$

Clearly, the number of degrees of freedom is $D=2$, so we will truncate f at order 2 with $f_2 = x$. Plugging this in, by setting to zero the zeroth and first order in x we will find a linear system of D equations

$$\begin{cases} b + c &= 0 \\ ax &= 0 \end{cases} \tag{5.98}$$

which yields a one-parameter family of solutions $(a,b,c) = (0,b,-b)$. We then move on to $d=1$, keeping N and M fixed. We can write the new ansatz as

$$(a+\alpha x)f + (b+\beta x)f' + (c+\gamma x) = 0, \tag{5.99}$$

where the number of degrees of freedom is $D=5$: f must therefore be truncated by $f_5 = x - \frac{x^3}{6} + \frac{x^5}{120}$. We find thus the system

$$\begin{cases} b + c &= 0 \\ (a+\beta+\gamma)x &= 0 \\ \left(\alpha - \frac{b}{2}\right)x^2 &= 0 \\ \left(-\frac{a}{6} - \frac{\beta}{2}\right)x^3 &= 0 \\ \left(-\frac{\alpha}{6} + \frac{b}{24}\right)x^4 &= 0 \end{cases} \tag{5.100}$$

which yields again a one-parameter family of solutions $(a, b, c, \alpha, \beta, \gamma) = (-3\beta, 0, 0, 0, \beta, 2\beta)$. This solution is incompatible with the former: in fact, if we set the new coefficients α, β and γ to 0, we find the trivial solution where all the parameters are 0, which is clearly at variance with the previous solution $(0, b, -b)$. Hence our algorithm does not terminate here.

Suppose that we want to iterate over the degree of the polynomials $p_{i,j}$ only up to $d_{\max} = 1$. Hence, we will now go back to $d = 0$ for a different choice of N, M. Let us choose $N = 2, M = 1$. We will have the ansatz

$$af + bf' + cf'' + d = 0. \tag{5.101}$$

Clearly, $D = 3$. With the same considerations as before, we reach the system

$$\begin{cases} d & = 0 \\ (a - c)x & = 0 \\ -\frac{b}{2}x^2 & = 0 \end{cases} \tag{5.102}$$

whose solutions are given by the one-parameter family $(a, b, c, d) = (a, 0, -a, 0)$. Moving on to next degree $d = 1$, we will have the ansatz with $D = 7$

$$(a + \alpha)f + (b + \beta x)f' + (c + \gamma x)f'' + (d + \delta x) = 0. \tag{5.103}$$

After plugging in the truncation f_7, this system is easily shown to admit the family of solutions

$$(a, b, c, d, \alpha, \beta, \gamma, \delta) = (a, 0, -a, 0, \alpha, 0, -\alpha, 0). \tag{5.104}$$

This is a *two*-parameter family of solutions: moreover, setting $\alpha = 0$, the solution reduces to the same family found at the previous step. Therefore, the algorithm terminates and, after setting a to any value (say, $a = 1$), provides the candidate differential equation

$$f + f'' = 0 \tag{5.105}$$

which is indeed the differential equation obeyed by $\sin(x)$, of which f is clearly the Taylor expansion. Had we chosen instead $N = 1, M = 2$, we would have found the equivalent candidate solution $f^2 + (f')^2 - 1 = 0$, again satisfied by the sine function.

5.4 Tame energy eigenvalues

In section 5.2 it was shown how, in the case of a cubic potential, the energy eigenvalues are not 1-summable and are represented by a transseries, whose non-perturbative sectors are labelled by resurgent symbols $e^{-n\mathcal{A}/\hbar}$, where $\mathcal{A} = t_0^D$ is the instanton action. As already observed, the instantons $e^{-n\mathcal{A}/\hbar}$ encode information about the tunnelling phenomenon, since the dual quantum period results from an integration over the classically prohibited region. We might then wonder whether under the assumption that the potential be convex – namely, consisting of a single well through which the particle has no probability to leak – the instantons might disappear, and the perturbative series $\tilde{E}(t)$ might be Borel-summable and, therefore, definable in $\mathbb{R}_{\mathscr{G}}$. After all, in the trivial case of the harmonic potential, the Hamiltonian

$$H(\hbar) = -\frac{1}{\hbar^2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 \tag{5.106}$$

admits the eigenvalues $E(\hbar) = (n + \frac{1}{2})\hbar$: these are polynomials and, as such, definable in every o-minimal structure. Although with other methods than the exact WKB method, which it is beyond our purposes to expound here, the authors of [Gom08] indeed found that a convex potential returns Borel-summable asymptotic expansions for the energy eigenvalues. Therein, the following the time-independent, one-dimensional Hamiltonian was considered

$$H(g) = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \frac{g^{p-1}x^{2p}}{1 + \alpha g x^2}\ , \tag{5.107}$$

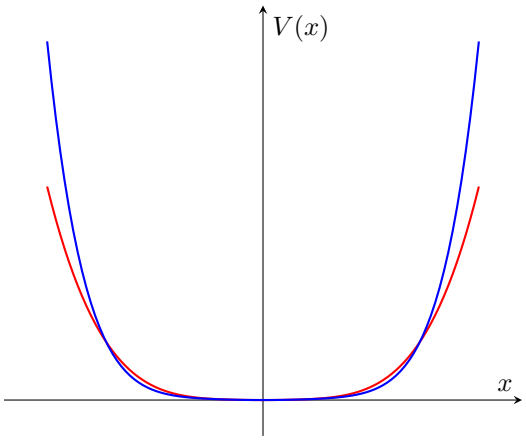


Figure 5.3: Convex potential for $p = 3$ (red) and $p = 4$ (blue), with $g = 1$ and $\alpha = 1, 4$ respectively.

where g is the coupling constant, α is a fixed *positive* constant and $p \geq 3$ is an integer. Note how this potential is convex (see Figure 5.3): there is only one, global minimum at $x = 0$: therefore there are no tunnelling phenomena. Theorem 4.1 of [Gom08] states that each eigenvalue of the Hamiltonian $H(g)$, $E_j(g)$, is analytic in a sector

$$S_j = \left\{ g \in \mathbb{C} : 0 < |g| \leq r_j < 1, |\arg g| < (p-1)\frac{\pi}{2} + \delta \right\} \tag{5.108}$$

where δ is a positive constant. Moreover, every eigenvalue admits an asymptotic expansion

$$E_j(g) \sim_{p-1} \sum_{n=0}^{\infty} E_{j,n} g^n \tag{5.109}$$

namely such that, for every $g \in S_j$, and for every N integer, there exist constants A_j and B such that

$$\left| E_j(g) - \sum_{n=0}^{N-1} E_{j,n} g^n \right| \leq A_j B^N \Gamma\left(N(p-1) + \tfrac{1}{2}\right) |g|^N. \tag{5.110}$$

Thus, by Nevanlinna-Sokal theorem, every eigenvalue $E_j(g)$ is $(p-1)$-summable. The limit for $g \to 0$ of $E_j(g)$ exists along all directions in S_j and it must be the eigenvalue $E_{j,0} = j + \frac{1}{2}$ of the harmonic oscillator. Hence, we conclude that, as opposed to the cubic potential case, every function $E_j(g)$ is a tame function definable in $\mathbb{R}_{\mathscr{G}}$ once restricted on the line $[0, r_j]$.

Conclusions and outlook

After a preliminary introduction to o-minimal structures and Borel summability, in this thesis we have analysed partition and correlation functions arising in several toy models. The key feature of these functions is that they are not analytic in the weak coupling limit, since their Taylor series at 0 are divergent. As the very writing "not analytic" indicates, this is not so much a property as, conversely, the *absence* of a property. Motivated by the Tameness Conjecture expounded in section 1.8, we were led to investigate whether the true mathematical property shared by these quantum field theory observables is rather that of tameness, i.e. definability in an o-minimal structure.
After studying several examples of known o-minimal structures, we realised how the tame structure $\mathbb{R}_{\mathscr{G}}$ is generated by the family of Gevrey functions $\mathcal{F} = \mathscr{G}$ which, in the special case of a single variable, are the Borel sums of Borel-summable power series. We concluded then that any Borel-summable partition or correlation function of a single coupling constant λ is tame in this structure. Having made this observation, we proceeded by showing some examples in zero dimensional bosonic Euclidean theories: in these simple cases, the path integral is reduced to an ordinary integral, simplifying immensely the computations and allowing us to calculate Borel transforms explicitly. Still, with the aid of constructive QFT techniques and the Loop Vertex Expansion, we found that more complicated partition functions are also Borel-summable, and therefore definable in $\mathbb{R}_{\mathscr{G}}$; in particular, we realised that it is possible to extend proofs of Borel summability (and therefore of tameness) to models in higher dimensions.

A question of primary importance, which remains open, is whether non-Borel-summable partition functions and correlation functions, namely those whose asymptotic expansions exhibit the Stokes phenomenon on the positive real line $\mathbb{R}^+$, remain tame. An example of how this instance can occur was provided in section 3.4. Therein the partition function was argued to be definable in $\mathbb{R}_{\text{Pfaff}}$, thanks to the fact that the differential equation it obeys could be turned into a Pfaffian chain; as it is clear though, this treatment lacks generality, and a more systematic answer must be sought in future research. In especial, some efforts could be devoted to understanding whether the knowledge of the non-analytic behaviour of partition and correlation functions (and, in particular, the location of the poles in their Borel planes) can be combined with the knowledge of the differential equation they obey. In admittedly vague and wishful terms, an o-minimal structure lying at the 'intersection' of $\mathbb{R}_{\mathscr{G}}$ and $\mathbb{R}_{\text{Pfaff}}$ or $\mathbb{R}_{\text{Noether}}$, which could combine Borel sums with solutions to differential equations, seems to be needed. It would be certainly a considerable achievement – both for pure model theory and for its applications to fundamental physics – if an o-minimal structure playing this rôle could be identified. As already hinted at in section 3.4, the Pfaffian closure $\mathcal{P}(\mathbb{R}_{\mathscr{G}})$ of $\mathbb{R}_{\mathscr{G}}$ may likely deserve further attention to this end. The o-minimal structure $\mathbb{R}_{\text{an,H}}$ also seems to require studying, as it is explicitly generated by functions H whose asymptotic expansions are affected by the Stokes phenomenon and obey a differential equation.
A related question which remains open is whether QFT amplitudes (Borel summable or not) are definable in a *sharply* o-minimal structure, so as to be endowed with a notion of sharp complexity. Since $\mathbb{R}_{\text{rPfaff}}$ is known to be sharply o-minimal and $\mathbb{R}_{\text{Pfaff}}$ an $\mathbb{R}_{\text{Noether}}$ are conjectured to be, answering this question may likely require a stronger exploitation of the differential equations obeyed by these observables rather than the analysis of their Borel planes, as we have done throughout this thesis. The aforementioned $\mathcal{P}(\mathbb{R}_{\mathscr{G}})$ and $\mathbb{R}_{\text{an,H}}$, unfortunately, fall short of fulfilling this ambitious goal since, containing both $\mathbb{R}_{\text{an}}$, they cannot be sharply o-minimal structures.

Moreover, it should be observed that the full power of the very structure $\mathbb{R}_{\mathscr{G}}$, with which we have concerned ourselves throughout this thesis, has not been exploited to the full. We have only identified the physical functions of our interest with the Borel-summable functions of one variable: yet, the structure also hosts Gevrey functions of multiple variables (whose graphs could be intersected in various ways) and even multisummable functions. It would be then of great interest to investigate whether these functions can be interpreted as path integrals parametrised by one or more couplings.

Assuming now a more physical perspective, it should be noted that all the partition functions analysed and proved to be tame in Chapters 3 and 4 ought to be regarded more as toy models than actual physical theories. It would be of the greatest interest to test the Borel summability – and thus the definability in $\mathbb{R}_{\mathscr{G}}$ – of the partition and correlation functions of theories including fermion fields and gauge fields, possibly in higher dimensions. This could confront us with a twofold challenge: that of proving the Borel summability of partition and correlation functions in more than one coupling parameter, and that of dealing with Lagrangians of several fields invariant under more than one symmetry. These questions have not been addressed in this thesis, but they require investigation in order to fully validate the Tameness Conjecture.

In Chapter 5, the tameness of energy eigenvalues in simple quantum mechanical systems has been studied too. The Stokes phenomenon has in this case a direct physical interpretation: it encodes the corrections due to the tunnelling of the particle through the barrier separating two different potential wells. As shown in section 5.4, only some convex potential configurations allow for Borel-summable energy eigenvalues, since in this case the tunnelling effects are completely suppressed by an infinitely high potential barrier. In this case, the energy eigenvalues (viewed as functions of the rescaled coupling $\hbar$) are definable in $\mathbb{R}_{\mathscr{G}}$. Conversely, when the potential is not convex, such as in the case of the cubic potential detailed in section 5.2, the Stokes phenomenon occurs on the positive real line $\mathbb{R}^+$, manifesting itself in the non-perturbative sectors of the exact transseries solution. In this case, we could not conclude that the energy eigenvalues are tame functions of $\mathbb{R}_{\mathscr{G}}$; moreover, as no differential equation was found by the numerical method described in section 5.3, we are not able to argue definability in $\mathbb{R}_{\mathrm{an,H}}$ or $\mathbb{R}_{\mathrm{Pfaff}}$, as it was the case for the partition function of the sine model in zero dimension (section 3.4). We are then left with two open questions: first, whether the energy eigenvalues remain tame functions of the rescaled coupling $\hbar$ after the median resummation of the full transseries solution is performed; second, whether there exists a differential equation formally obeyed by the perturbative expansion $\tilde{E}(t)$, once the quantum period t is quantised. It is worthwhile to stress that such a differential equation would be of great import: if its dependence on the potential and the quantisation could be understood precisely, one could simply solve the differential equation and find a formal (and divergent) perturbative power series for the energy, circumventing so the solution of the Shödinger equation. One could thus find the eigenvalues of the Hamiltonian without finding the eigenstates, namely the wavefunctions.

Appendix A

A brief in-depth in logic

At the International Congress of Mathematicians in 1900 in Paris, David Hilbert proposed 23 problems to whose solutions mathematicians should commit their efforts in the upcoming century. One of these problems was to establish whether the axioms of arithmetic are consistent. The aim was indeed to found the entire Mathematics on arithmetic, and therefore, by proving the consistency of the latter, demonstrate also that of the former. Another problem posed by Hilbert was that of creating a set of axioms for Physics; as Physics makes use of mathematical tools, there exist, to some extent, connections between the two problems. The second problem, concerning Physics, is somewhat vague and no real solution has been found yet; as to the first though, a satisfactory answer was given in the 30s and 40s by a sequence of results due to K. Gödel, A. Church, A. Tarski and A. Turing. The answer was neither in the positive or the negative: rather, it established the impossibility of an answer altogether. In the next sections we review these theorems briefly, following [Raa22].

A.1 Gödel theorems

Logic mainly deals with formal systems **F**, which consist of a language L, a set of axioms and a set of inference rules. A requirement of the axioms is that they be either finite or at least decidable: namely, there must exist a method, or algorithm, to decree whether any given statement is an axiom or not and, more generally, whether a finite sequence of formulas is a valid derivation. In other words, it must be possible to determine whether a given proof of a theorem is correct or not.

A formal system **F** is said to be *complete* if every statement in **F** can be either proved true or false within **F** itself. Keeping to classical logic, we may also require that **F** be *consistent*: namely if **F** proves A, it must not prove $\neg A$.

There are various ways to lay down axioms for arithmetic. One of the simplest is the Robinson arithmetic:

Definition A.1 (Robinson arithmetic)**.** Robinson arithmetic **Q** consists of the following axioms:

- $\neg(0 = x')$
- $x' = y' \to x = y$
- $\neg(x = 0) \to \exists y\ (x = y')$
- $x + 0 = x$
- $x + y' = (x + y)'$

- $x \times 0 = 0$
- $x \times y' = (x \times y) + x$

where x' is the successor function.

The Robinson arithmetic can be enhanced to the first-order Peano arithmetic **PA** by adding the induction scheme

$$\phi(0) \wedge \forall x[\phi(x) \to \phi(x')] \to \forall x \phi(x)$$

which is the most ordinary notion of arithmetic.

In 1931, Gödel proved two Incompleteness Theorems, which, after an improvement by Rosser in 1936, we can state as follows.

Theorem A.1 (Gödel)**.**

1. Any consistent formal system **F** which contains **Q** contains a statement G_F such that **F** does not prove G_F nor $\neg G_F$.
2. Any consistent formal system **F** which contains a sufficient amount of elementary arithmetic cannot prove its own consistency.

In the second theorem a slightly stronger assumption on the "amount of arithmetic" that **F** must contain is made. However, the assumption is still very mild: Peano arithmetic **PA** is more than sufficient.
Gödel's theorem decreed that no satisfactory answer can be given to Hilbert problem: a formal system **F** cannot prove its own consistency if we demand that it contain arithmetic. Moreover, the first theorem tells us the disturbing fact that there are certain meaningful mathematical statements which we cannot prove either true or false.

A.2 Tarski's and Church's theorems

Another theorem in the same spirit as Gödel's was proved in 1936 by A. Tarski, sometimes referred to as the theorem of the *undefinability of truth*. The basic idea behind the theorem is that truth in a certain language cannot be defined in the same language. The truth statements regarding formulas of a certain language, called object language, cannot belong to that same language; rather, they must belong to another one called meta-language, i.e. a language which refers to another language. Given a statement A, let us call the statement of its truth $Tr(A)$. Then we have:

Theorem A.2 (Tarski)**.** Let F by a formal system which contains a sufficient amount of arithmetic. Then there is no formula $Tr(x)$ in F such that for every statement A in F,

$$F \text{ proves } Tr(A) \longleftrightarrow A$$

Again, the request that F contain a 'sufficient amount of arithmetic', although stated quite vaguely, is to be regarded as rather mild and applying to a usual notion of arithmetic, such as **PA**.

A related result is due to A. Church. There exist in logic different rigorous definitions of *decidability*, produced by Gödel, Church and Turing independently and proved to be equivalent. The most notorious is Turing's, which refers to the ability of a Turing machine (which in contemporary terms is nothing short of a computer) to check the truth of a statement by following a finite number of steps in an algorithm. We can say then that a set X is decidable if it is recursive, namely if there exists an algorithm that establishes whether a given element x belongs to X. The theorem than can be states as:

> **Theorem A.3** (Church)**.** The formal system **F** of first-order arithmetic is undecidable.

This means that there are statements in arithmetic whose truth cannot be checked by an algorithm. A rather illuminating example is a theorem proved in 1961 by J. Robinson, M. Davis and H. Putnam as to exponential Diophantine equations. After an improvement by Y. Matiyasevich in 1970, the theorem can be stated as follows:

> **Theorem A.4** (MRDP)**.** There exists no general method to determine whether a Diophantine equation has solutions or not.

As it may be intuitive, if a formal system **F** is complete, then it is also decidable (and thus an undecidable system is always incomplete). The inverse statement is not true in general, but it does hold in most cases. The inclusion of the Robinson arithmetic **Q**, for instance, is a sufficient requirement for both incompleteness and undecidability. Such logical 'flaws' of mathematics though, only appear when we want to perform arithmetic on the integers; if we consider the real numbers, we have conversely the following result due again to Tarski [Tar49]:

> **Theorem A.5** (Tarski, 1948)**.** The first-order logic theory of arithmetic of real numbers (real closed fields) with both addition and multiplication is complete and decidable.

The latter theorem is essentially the statement that $\mathbb{R}_{\text{alg}}$ is o-minimal. The proof relies on the possibility to eliminate quantifiers from any given formula from the related language: in absence of quantifiers, the logic is propositional and therefore decidable (recall Table 1.2). Yet, the full elimination of quantifiers is not strictly necessary for o-minimality, and indeed it does not hold for certain structures, such as $\mathbb{R}_{\text{exp}}$. We refer to e.g. [Mar96] and references therein for further details about this.

Appendix B

Gevrey functions of several variables

In this appendix we provide the full definition of the family $\mathcal{F} = \mathscr{G}$ of Gevrey functions in m variables, which generate the structure $\mathbb{R}_{\mathscr{G}}$. We will need first to fix some notation. Let us switch to a variable z; with $z \in \mathbb{C}$, we conventionally agree that $\arg(z) \in (-\pi, \pi]$. Let $k = (k_1, .., k_m) \in (0, \infty)^m$, $R = (R_1, ..., R_m) \in (0, \infty)^m$ and $z = (z_1, ..., z_m) \in \mathbb{C}^m$. We will write $z^k = z_1^{k_1} ... z_m^{k_m} \in \mathbb{C}$, and similarly for R. We will write $R < \tilde{R}$ if $R_i < \tilde{R}_i$ for every $i = 1, ..., m$. We will define the product

$$k \cdot |\arg(z)| := k_1 |\arg(z_1)| + ... + k_m |\arg(z_m)| \tag{B.1}$$

and let $|z| = \sup\{|z_i|, i = 1, ..., m\}$. Let also $[0, R] = [0, R_1] \times ... \times [0, R_m]$ and $[0, R)$ similarly. Finally, let $\mathscr{K} = \{0\} \cup [1, \infty)$.

We now define the polydisc $D(R)$ and the polysector $S^k(R, \phi)$, where $\phi \in (0, \pi)$ and $k \in \mathscr{K}^m$, as

$$D(R) = \{z \in \mathbb{C}^m : |z_i| < R_i \text{ for } i = 1, ..., m\}, \tag{B.2}$$

$$S^k(R, \phi) = \{z \in D(R) : k \cdot |\arg(z)| < \phi\}. \tag{B.3}$$

Notice that, if $m = 1$ and $k \geq 1$, one has $S^k(R, \phi) = S(R, \phi, 1/k) \cup \{0\}$ in the notation of section 1.5.3. For any $p \in \mathbb{N}$, let

$$D_p^k(R) = \left\{ z \in D(R) : |z|^k < \frac{R^k}{p+1} \right\}, \tag{B.4}$$

$$S_p^k(R) = S^k(R, \phi) \cup D_p^k(R). \tag{B.5}$$

Finally, for any *finite*, non-empty set $K \subseteq \mathscr{K}$ we define

$$S^K(R, \phi) = \bigcap_{k \in K} S^k(R, \phi), \tag{B.6}$$

$$S_p^K(R, \phi) = \bigcap_{k \in K} S_p^k(R, \phi). \tag{B.7}$$

Given a set $U \subseteq \mathbb{C}^m$ and a function $f : U \to \mathbb{C}$, we define the norm

$$||f||_U = \sup_{z \in U}\{|f(z)|\} \in [0, \infty]. \tag{B.8}$$

It will be useful to define $\tau = (K, R, r, \phi)$, where $K \subseteq \mathscr{K}^m$ is finite and non-empty, $R \in (0, \infty)^m$, $r \in (0, \infty)$, $\phi \in (0, \pi)$. We will then write, for brevity, $S(\tau) = S^K(R, \phi)$ and $S_p(\tau) = S_p^K(R, \phi)$. Let us fix a certain τ. From our definitions, it is clear that $S_{p+1}(\tau) \subset S_p(\tau)$. Then, if for every p, $f_p : S_p(\tau) \to \mathbb{C}$ is a bounded

holomorphic function such that $\sum_p ||f_p||_{S_p(\tau)} r^p < \infty$, it can be shown that $\sum_p f_p$ converges to a bounded and continuous function $f : S(\tau) \to \mathbb{C}$. This state of affairs is denoted by

$$\sum_p f_p =_\tau f. \tag{B.9}$$

Having established this fact, we define $\mathscr{G}_\tau$ as the collection of functions $f : S(\tau) \to \mathbb{C}$ such that there exists a sequence $(f_p)_{p\in\mathbb{N}}$ of functions $S_p(\tau) \to \mathbb{C}$ as above such that $f =_\tau \sum_p f_p$.

Let $\mathscr{T}_m$ to be the collection of all $\tau = (K, R, r, \phi)$, where K and R are as above, while we constrain r and ϕ by $r > 1$ and $\phi \in (\frac{\pi}{2}, \pi)$. A crucial result proved in [DS00] is that if $\tau \in \mathscr{T}_m$, $\mathscr{G}_\tau$ is a differential algebra of quasi-analytic functions: i.e. the Taylor map $\mathscr{G}_\tau \to \mathbb{C}[[Z]]$

$$\mathscr{G}_\tau \ni f \to \sum_{n\in\mathbb{N}^m} \frac{Z^n}{n!} f^{(n)}(0) \in \mathbb{C}[[Z]] \tag{B.10}$$

where $n! = n_1!...n_m!$ and $f^{(n)} = \frac{\partial^{n_1}}{\partial z_1^{n_1}} ... \frac{\partial^{n_m}}{\partial z_m^{n_m}} f$, is injective.

Consider now a polyradius $R = (R_1, .., R_m)$. We define $\mathscr{G}(R)$ to be the ring of functions $f : [0, R] \to \mathbb{R}$ for which there exists $\tilde{\tau} = (\tilde{K}, \tilde{R}, \tilde{r}, \tilde{\phi}) \in \mathscr{T}_m$, with $\tilde{R} > R$, and $\tilde{f} \in \mathscr{G}_{\tilde{\tau}}$ such that $f(x) = \tilde{f}(x)$ for every $x \in [0, R]$. The fact that $\mathscr{G}(R)$ is a ring under point-wise addition and multiplication is non-trivial; we refer to [DS00] for the proof.

Finally we define $\mathscr{G}$ to be the family of functions $f : \mathbb{R}^m \to \mathbb{R}$ which are identically vanishing outside the unit cube $[0, 1]^m$, while

$$f|_{[0,1]^m} \in \mathscr{G}(1, ..., 1). \tag{B.11}$$

The choice of a polyradius with all entries equal to 1 is purely conventional and by no means restrictive. Thus, the o-minimal structure $\mathbb{R}_{\mathscr{G}}$ is the o-minimal expansion

$$\mathbb{R}_{\mathscr{G}} := (\mathbb{R}, <, 0, 1, +, -, \cdot, \mathscr{G}). \tag{B.12}$$

Observe how it is not straightforward to infer that when $m = 1$ the Gevrey functions described here reduce to those of section 1.5.3. We refer once more to [DS00] for the proof that it is indeed the case.

Appendix C

Commuting sums with integrals

We state here the theorem that establishes when the commutation of the integral with an infinite sum is allowed.

> **Theorem C.1.** Consider the functions $f_n : M \to \mathbb{R}$, where f_n are measurable functions according to Lebesgue. Assume that there exists a sequence of measurable functions $g_n : M \to \mathbb{R}$, satisfying the following conditions:
>
> 1. $|f_n(x)| \leq g_n(x)$ for all $n \in \mathbb{N}$ and for almost all $x \in M$;
> 2. The sum $\sum_{n=0}^{\infty} g_n(x)$ converges for almost all $x \in M$;
> 3. $\int_M \sum_{n=0}^{\infty} g_n(x)dx < \infty$
>
> Then, one has
>
> $$\int_M \sum_{n=0}^{\infty} f_n(x)dx = \sum_{n=0}^{\infty} \int_M f_n(x)dx$$
>
> where by "almost all $x \in M$" it is intended "on all $x \in M \backslash S$" where $S \subset M$ is a set of Lebesgue measure 0.

Bibliography

[Tar49] Alfred Tarski. 'A Decision Method for Elementary Algebra and Geometry'. In: *Journal of Symbolic Logic* 14.3 (1949), pp. 188–188. DOI: 10.2307/2267068.

[Dys52] F. J. Dyson. 'Divergence of Perturbation Theory in Quantum Electrodynamics'. In: *Phys. Rev.* 85 (4 Feb. 1952), pp. 631–632. DOI: 10.1103/PhysRev.85.631. URL: https://link.aps.org/doi/10.1103/PhysRev.85.631.

[Har56] G. H. Hardy. *Divergent Series*. Oxford: Clarendon Press, 1956.

[Kru56] Joseph B. Kruskal. 'On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem'. In: *Proceedings of the American Mathematical Society* 7.1 (1956), pp. 48–50. ISSN: 00029939, 10886826. URL: http://www.jstor.org/stable/2033241 (visited on 09/05/2024).

[EMR75] Jean-Pierre Eckmann, Jacques Magnen and Seneor Roland. 'Decay properties and borel summability for the Schwinger functions inP()2 theories'. In: *Communications in Mathematical Physics* 39 (Dec. 1975), pp. 251–271. DOI: 10.1007/BF01705374.

[Sok79] Alan D. Sokal. 'An improvement of Watson's theorem on Borel summability'. English (US). In: *Journal of Mathematical Physics* 21.2 (1979), pp. 261–263. ISSN: 0022-2488. DOI: 10.1063/1.524408.

[Eca81] J. Ecalle. *Les fonctions résurgentes: (en trois parties)*. Les fonctions résurgentes: Université de Paris-Sud, Département de Mathématique, Bât. 425, 1981.

[DD88] J. Denef and L. van den Dries. 'p-Adic and real subanalytic sets'. In: *Annals of Mathematics* 128.1 (1988), pp. 79–138.

[Bal94] Werner Balser. *From Divergent Power Series to Analytic Functions*. Vol. 1582. Lecture Notes in Mathematics. Springer, 1994.

[Dri94] Lou van den Dries. 'The elementary theory of restricted analytic fields with exponentiation'. In: *Annals of Mathematics* 140.1 (1994), pp. 183–205.

[AR95] Abdelmalek Abdesselam and Vincent Rivasseau. 'Trees, Forests and Jungles: A Botanical Garden for Cluster Expansions'. In: *Lecture Notes in Physics* 446 (1995), pp. 7–36.

[AW95] George B Arfken and Hans J Weber. *Mathematical methods for physicists; 4th ed.* San Diego, CA: Academic Press, 1995. URL: https://cds.cern.ch/record/379118.

[Mar96] David Marker. 'Model Theory and Exponentiation'. In: *Notices of the American Mathematical Society* 43.7 (July 1996), pp. 753–759.

[Wil96] A. J. Wilkie. 'Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function'. In: *J. Amer. Math. Soc.* 9.4 (1996), pp. 1051–1094. ISSN: 0894-0347,1088-6834. DOI: 10.1090/S0894-0347-96-00216-0. URL: https://doi.org/10.1090/S0894-0347-96-00216-0.

[Gro97] Alexandre Grothendieck. 'Sketch of a Programme (translation into English)'. In: *Geometric Galois Actions*. Ed. by Leila Schneps and PierreEditors Lochak. London Mathematical Society Lecture Note Series. Cambridge University Press, 1997, pp. 243–284.

[Spe97] Patrick Speissegger. *The Pfaffian closure on an o-minimal structure*. 1997. arXiv: math/9710220 [math.RA].

[Dri98] Lou van den Dries. *Tame topology and o-minimal structures*. Vol. 248. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1998, pp. x+180. ISBN: 0-521-59838-9. DOI: 10.1017/CBO9780511525919. URL: https://doi.org/10.1017/CBO9780511525919.

[vS98] Lou van den Dries and Patrick Speissegger. 'The real field with convergent generalized power series'. English (US). In: *Transactions of the American Mathematical Society* 350.11 (1998), pp. 4377–4421. ISSN: 0002-9947.

[Riv99] V. Rivasseau. *Constructive Renormalization Theory*. 1999. arXiv: math-ph/9902023 [math-ph].

[Wil99] A. J. Wilkie. 'A theorem of the complement and some new o-minimal structures'. In: *Selecta Mathematica* 5 (1999), pp. 397–421. URL: https://api.semanticscholar.org/CorpusID:122959059.

[DS00] LOU VAN DEN DRIES and PATRICK SPEISSEGGER. 'THE FIELD OF REALS WITH MULTISUMMABLE SERIES AND THE EXPONENTIAL FUNCTION'. In: *Proceedings of the London Mathematical Society* 81.3 (2000), pp. 513–565. DOI: 10.1112/S0024611500012648.

[CMP03] James Carlson, Stefan Müller-Stach and Chris Peters. *Period mappings and period domains*. Vol. 85. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2003, pp. xvi+430. ISBN: 0-521-81466-9.

[RSW03] J. Rolin, Patrick Speissegger and Alex Wilkie. 'Quasianalytic Denjoy–Carleman Classes and O-Minimality'. In: *Journal of the American Mathematical Society* 16 (Oct. 2003). DOI: 10.1090/S0894-0347-03-00427-2.

[Riv07] Vincent Rivasseau. 'Constructive matrix theory'. In: *Journal of High Energy Physics* 2007.09 (Sept. 2007), pp. 008–008. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2007/09/008. URL: http://dx.doi.org/10.1088/1126-6708/2007/09/008.

[RSS07] J-P Rolin, Fernando Sanz and Reinhard Schäfke. 'Quasi-analytic solutions of analytic ordinary differential equations and o-minimal structures'. In: *Proceedings of the London Mathematical Society* 95.2 (2007), pp. 413–442.

[Gom08] M. Gomes. 'Borel-Leroy summability of a nonpolynomial potential'. In: *Reports on Mathematical Physics - REP MATH PHYS* 61 (June 2008), pp. 401–415. DOI: 10.1016/S0034-4877(08)80021-3.

[MR08] Jacques Magnen and Vincent Rivasseau. 'Constructive ϕ^4 Field Theory without Tears'. In: *Annales Henri Poincaré* 9.2 (Apr. 2008), pp. 403–424. ISSN: 1424-0661. DOI: 10.1007/s00023-008-0360-1. URL: http://dx.doi.org/10.1007/s00023-008-0360-1.

[Riv09] V. Rivasseau. 'Constructive Field Theory in Zero Dimension'. In: *Advances in Mathematical Physics* 2009 (2009), pp. 1–12. ISSN: 1687-9139. DOI: 10.1155/2009/180159. URL: http://dx.doi.org/10.1155/2009/180159.

[GR13] Razvan Gurau and Vincent Rivasseau. *The Multiscale Loop Vertex Expansion*. 2013. arXiv: 1312.7226 [math-ph].

[RW13] Vincent Rivasseau and Zhituo Wang. 'How to Resum Feynman Graphs'. In: *Annales Henri Poincaré* 15.11 (Dec. 2013), pp. 2069–2083. ISSN: 1424-0661. DOI: 10.1007/s00023-013-0299-8. URL: http://dx.doi.org/10.1007/s00023-013-0299-8.

[GRS14] Razvan Gurau, Vincent Rivasseau and Alessandro Sfondrini. *Renormalization: an advanced overview*. 2014. arXiv: 1401.5003 [hep-th].

[Lod14] Michèle Loday-Richaud. 'Divergent series and differential equations'. Part of a book. Submitted. Jan. 2014. URL: https://hal.science/hal-01011050.

[Sau14] David Sauzin. *Introduction to 1-summability and resurgence*. 2014. arXiv: 1405.0356 [math.DS].

[CDU15] Aleksey Cherman, Daniele Dorigoni and Mithat Unsal. *Decoding perturbation theory using resurgence: Stokes phenomena, new saddle points and Lefschetz thimbles*. 2015. arXiv: 1403.1277 [hep-th].

[IN15] Kohei Iwaki and Tomoki Nakanishi. 'Exact WKB Analysis and Cluster Algebras II: Simple Poles, Orbifold Points, and Generalized Cluster Algebras'. In: *International Mathematics Research Notices* 2016.14 (Oct. 2015), pp. 4375–4417. ISSN: 1687-0247. DOI: 10.1093/imrn/rnv270. URL: http://dx.doi.org/10.1093/imrn/rnv270.

[RS15] J.-P. Rolin and T. Servi. 'Quantifier elimination and rectilinearization theorem for generalized quasianalytic algebras'. In: *Proceedings of the London Mathematical Society* 110.5 (Mar. 2015), pp. 1207–1247. ISSN: 0024-6115. DOI: 10.1112/plms/pdv010. URL: http://dx.doi.org/10.1112/plms/pdv010.

[Heu16] Erik Heusden. 'On Undecidability and the Laws of Physics'. In: *Journal of Physics: Conference Series* 701 (Mar. 2016), p. 012025. DOI: 10.1088/1742-6596/701/1/012025.

[LR16] Luca Lionni and Vincent Rivasseau. *Intermediate Field Representation for Positive Matrix and Tensor Interactions*. 2016. arXiv: 1609.05018 [math-ph].

[BDU17] Gokce Basar, Gerald V. Dunne and Mithat Unsal. 'Quantum Geometry of Resurgent Perturbative/Nonperturbative Relations'. In: *JHEP* 05 (2017), p. 087. DOI: 10.1007/JHEP05(2017)087. arXiv: 1701.06572 [hep-th].

[Pas17] Georgios Pastras. *Four Lectures on Weierstrass Elliptic Function and Applications in Classical and Quantum Mechanics*. 2017. arXiv: 1706.07371 [math-ph].

[Riv17] Vincent Rivasseau. 'Loop vertex expansion for higher-order interactions'. In: *Letters in Mathematical Physics* 108.5 (Dec. 2017), pp. 1147–1162. ISSN: 1573-0530. DOI: 10.1007/s11005-017-1037-9. URL: http://dx.doi.org/10.1007/s11005-017-1037-9.

[EA18] Leonhard Euler and Alexander Aycock. *On divergent Series*. 2018. arXiv: 1808.02841 [math.HO].

[SÜ18] Tin Sulejmanpasic and Mithat Ünsal. 'Aspects of perturbation theory in quantum mechanics: The BenderWu Mathematica ® package'. In: *Comput. Phys. Commun.* 228 (2018), pp. 273–289. DOI: 10.1016/j.cpc.2017.11.018. arXiv: 1608.08256 [hep-th].

[Dor19] Daniele Dorigoni. 'An Introduction to Resurgence, Trans-Series and Alien Calculus'. In: *Annals Phys.* 409 (2019), p. 167914. DOI: 10.1016/j.aop.2019.167914. arXiv: 1411.3585 [hep-th].

[FKN19] Fabian Fischbach, Albrecht Klemm and Christoph Nega. 'WKB Method and Quantum Periods beyond Genus One'. In: *J. Phys. A* 52.7 (2019), p. 075402. DOI: 10.1088/1751-8121/aae8b0. arXiv: 1803.11222 [hep-th].

[IMS19] Katsushi Ito, Marcos Mariño and Hongfei Shu. 'TBA equations and resurgent Quantum Mechanics'. In: *JHEP* 01 (2019), p. 228. DOI: 10.1007/JHEP01(2019)228. arXiv: 1811.04812 [hep-th].

[KRS19a] Thomas Krajewski, Vincent Rivasseau and Vasily Sazonov. *Constructive Matrix Theory for Higher Order Interaction*. 2019. arXiv: 1712.05670 [math-ph].

[KRS19b] Thomas Krajewski, Vincent Rivasseau and Vasily Sazonov. *Constructive Matrix Theory for Higher Order Interaction II: Hermitian and Real Symmetric Cases*. 2019. arXiv: 1910.13261 [math-ph].

[FMQ20] Frédéric Fauvet, Frédéric Menous and Julien Quéva. 'Resurgence and holonomy of the 2k model in zero dimension'. In: *Journal of Mathematical Physics* 61.9 (Sept. 2020). ISSN: 1089-7658. DOI: 10.1063/5.0009292. URL: http://dx.doi.org/10.1063/5.0009292.

[ELT21] Harold Erbin, Vincent Lahoche and Mohamed Tamaazousti. 'Constructive expansion for vector field theories I. Quartic models in low dimensions'. In: *Journal of Mathematical Physics* 62.4 (Apr. 2021). ISSN: 1089-7658. DOI: 10.1063/5.0038599. URL: http://dx.doi.org/10.1063/5.0038599.

[BNZ22] Gal Binyamini, Dmitri Novikov and Benny Zack. *Sharply o-minimal structures and sharp cellular decomposition*. 2022. arXiv: 2209.10972 [math.LO].

[DGS22] Michael R. Douglas, Thomas W. Grimm and Lorenz Schlechter. 'The Tameness of Quantum Field Theory, Part I – Amplitudes'. In: (Oct. 2022). arXiv: 2210.10057 [hep-th].

[Gri22] Thomas W. Grimm. 'Taming the landscape of effective theories'. In: *Journal of High Energy Physics* 2022.11 (Nov. 2022). ISSN: 1029-8479. DOI: 10.1007/jhep11(2022)003. URL: http://dx.doi.org/10.1007/JHEP11(2022)003.

[Raa22] Panu Raatikainen. 'Gödel's Incompleteness Theorems'. In: *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, 2022.

[BN23] Gal Binyamini and Dmitry Novikov. 'Tameness in geometry and arithmetic: beyond o-minimality'. In: Dec. 2023, pp. 1440–1461. DOI: 10.4171/icm2022/117.

[DGS23] Michael R. Douglas, Thomas W. Grimm and Lorenz Schlechter. 'The Tameness of Quantum Field Theory, Part II – Structures and CFTs'. In: (Feb. 2023). arXiv: 2302.04275 [hep-th].

[GSV23] Thomas W. Grimm, Lorenz Schlechter and Mick van Vliet. 'Complexity in Tame Quantum Theories'. In: (Oct. 2023). arXiv: 2310.01484 [hep-th].

[GM23] Jie Gu and Marcos Marino. 'On the resurgent structure of quantum periods'. In: *SciPost Phys.* 15.1 (2023), p. 035. DOI: 10.21468/SciPostPhys.15.1.035. arXiv: 2211.03871 [hep-th].

[Kam+23] Syo Kamata et al. 'Exact WKB analysis for SUSY and quantum deformed potentials: Quantum mechanics with Grassmann fields and Wess-Zumino terms'. In: *Phys. Rev. D* 107.4 (2023), p. 045019. DOI: 10.1103/PhysRevD.107.045019. arXiv: 2111.05922 [hep-th].

[RSS23] Jean-Philippe Rolin, Tamara Servi and Patrick Speissegger. *Multisummability for generalized power series*. 2023. arXiv: 2203.15047 [math.CA].

[SV23] Alexander van Spaendonck and Marcel Vonk. 'Exact instanton transseries for quantum mechanics'. In: (Sept. 2023). arXiv: 2309.05700 [hep-th].

[Fer+24] L. Ferdinand et al. 'Borel Summability of the $1/N$ Expansion in Quartic O(N)-Vector Models'. In: *Annales Henri Poincare* 25.3 (2024), pp. 2037–2064. DOI: 10.1007/s00023-023-01350-w. arXiv: 2209.09045 [math-ph].

[Riv24] Vincent Rivasseau. *Loop Vertex Representation for Cumulants*. 2024. arXiv: 2305.08399 [math-ph].