Cohomology of Local Systems on the Moduli Space of Curves

MASTER'S THESIS

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Abstract

Let $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ denote the moduli spaces of smooth and stable curves of genus g with n marked points respectively. We introduce and study these spaces. In particular, we are interested in their cohomology groups. There is an action of $\mathbb{S}_n$ on $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ permuting their marked points, which extends to an action on the cohomology, making them $\mathbb{S}_n$-representations. Moreover taking advantage of the fact that -depending on the chosen cohomology theory- the cohomology groups are also mixed Hodge structures or ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we can determine several motivic Euler characteristics using point counts over finite fields. We introduce local systems $\mathbb{V}_\lambda$ on $\mathcal{M}_g$ given by a partition λ of n into at most g parts, and investigate the motivic Euler characteristic given by the sheaf cohomology with coefficients in $\mathbb{V}_\lambda$. Calculating the trace of Frobenius on these Euler characteristics then gives information on the existence of non-tautological cohomology on $\mathcal{M}_g$. For $g = 4$ and $q = 2, 3, 4$, we attempt to detect non-tautological cohomology using these trace calculations.

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Contents

INTRODUCTION

In this thesis we introduce and study the moduli space $\mathcal{M}_{g,n}$ of smooth curves of genus g with n marked points, and its compactification $\overline{\mathcal{M}}_{g,n}$ consisting of stable curves of arithmetic genus g admitting nodal singularities as introduced by Deligne and Mumford in their paper *The irreducibility of the space of curves of given genus* [21].

Over the complex numbers, for low values of g and n, the spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ may be described explicitly. For example, $\mathcal{M}_{0,3}$ is a point, since three distinct points on $\mathbb{P}^1$ can be translated to $0,\ 1,\ \infty$. The moduli space $\mathcal{M}_{1,1}$ parametrizing elliptic curves is isomorphic to the affine line $\mathbb{A}^1$, since elliptic curves are parametrized by the j-line. Curves of genus 2 can always be written as double coverings of the projective line (see for example [32, Exercise IV-2.2]), and can therefore be given as the quotient of $\mathbb{A}^3$ by a finite group.

The spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ were originally investigated as schemes, in which case their $\mathbb{C}$-points correspond bijectively to isomorphism classes of curves over $\mathbb{C}$. Unfortunately, the category of schemes is not flexible enough to guarantee the representability of a moduli functor, so $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ lack certain desirable properties. Most of these problems arise to the existence of curves with non-trivial automorphisms. For example, Rauch proves that $\mathcal{M}_g$ is not smooth in general, with singularities coming from curves which have non-trivial automorphisms, [41].

Many modern authors therefore prefer to work in the category of stacks, where we find the DM stacks $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ which exhibit more desirable properties, such as smoothness. Following other authors in this field, we will occasionaly venture into this category, treating the spaces as schemes as much as possible and adapting our theory to suit stacks whenever needed.

It is hard to determine the precise structure of $\mathcal{M}_{g,n}$ in general for higher values of g and n. To make useful deductions about this space, various authors have studied its Euler characteristic. In 1986, Harer and Zagier [30] find a closed expression for the orbifold Euler characteristic of $\mathcal{M}_{g,n}$ in terms of Bernoulli numbers B_k:

$$\chi(\mathcal{M}_{g,n}) = (-1)^n \frac{(2g-1)B_{2g}}{(2g)!}(2g+n-3)!.$$

Consequently, Bini and Harer [13] find a closed expression for the ordinary Euler characteristic of $\mathcal{M}_{g,n}$ for any g and n non-negative such that $2g-1+n>0$, which agrees with the orbifold Euler characteristic whenever $n \geq 2g+3$, since curves with at least $2g+3$ marked points have no non-trivial automorphisms.

Studying the Euler characteristics of various moduli spaces $\mathcal{M}_{g,n}$ allows us to make useful deductions about their cohomology. For example, in [3], Arbarello and Cornalba prove that $H^1(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$, $H^3(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$ and $H^5(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$ vanish for all values of g and n, and find explicit generators and relations on $H^2(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$ using Euler characteristics.

While knowing the ordinary (numerical) Euler characteristic of $\mathcal{M}_{g,n}$ is certainly useful, we can make stronger deductions about the cohomology groups if we take advantage of some additional structures. As is proved by Deligne in [20], any quasi-projective variety X admits a canonical mixed Hodge structure on its cohomology, so $H^k(\mathcal{M}_{g,n}(\mathbb{C});\mathbb{Q})$ has such a structure. Furthermore, the $\mathbb{S}_n$-action on $\mathcal{M}_{g,n}$ permuting the marked points on a curve extends to $H^k(\mathcal{M}_{g,n}(\mathbb{C});\mathbb{Q})$, making it an
$\mathbb{S}_n$-representation.
We can also study the ℓ-adic cohomology of $\mathcal{M}_{g,n}$. The spaces $H^k(\mathcal{M}_{g,n}\otimes\overline{\mathbb{Q}};\mathbb{Q}_\ell)$ are then ℓ-adic $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations, which we call Galois representations, as well as $\mathbb{S}_n$-representations. Considering $\overline{\mathcal{M}}_g$ as a stack, it is smooth and proper, it satisfies Poincaré duality and its cohomology groups $H^k(\overline{\mathcal{M}_g}(\mathbb{C});\mathbb{Q})$ admit a pure Hodge structure of weight k. This means that we can read off the cohomology from the expression

$$e_c(\overline{\mathcal{M}}_{g,n}) := \sum_{i\geq 0}(-1)^i[H^i_c(\overline{\mathcal{M}}_{g,n}(\mathbb{C});\mathbb{Q})] \in K_0(\mathrm{MHS}_{\mathbb{Q}})$$

taking values in the Grothendieck group of mixed Hodge structures over $\mathbb{Q}$. Since we can write a nodal curve in $\overline{\mathcal{M}}_{g,n}$ in terms of its normalization, given by curves of lower genus, it is possible to define a stratification of $\overline{\mathcal{M}}_{g,n}$ in terms of quotients by products of symmetric groups of products of moduli spaces $\mathcal{M}_{h,m}$. This then means that the expression $e(\overline{\mathcal{M}}_{g,n})$ could be determined if we know all Euler characteristics of the moduli spaces occurring in the stratification. Just the information

$$e_c(\mathcal{M}_{h,m}) := \sum_{i\geq 0}[H^i_c(\mathcal{M}_{h,m}(\mathbb{C});\mathbb{Q})]$$

is not enough, however, since it does not remember the action of $\mathbb{S}_n$ permuting the marked points. To take these structures into account, we will study the $\mathbb{S}_n$-equivariant Euler characteristics

$$e_c^{\mathbb{S}_n}(\mathcal{M}_{g,n}(\mathbb{C});\mathbb{Q}) \in K_0(\mathrm{MHS}_{\mathbb{Q}}) \otimes \Lambda$$

and

$$e_c^{\mathbb{S}_n}(\mathcal{M}_{g,n}\otimes\overline{\mathbb{Q}};\mathbb{Q}_\ell) \in K_0(\mathrm{Gal}_{\mathbb{Q}_\ell}) \otimes \Lambda,$$

where Λ is the ring

$$\varprojlim \mathbb{Q}[x_1,\ldots,x_n]^{\mathbb{S}_n}$$

of symmetric functions, containing all Schur polynomials s_λ. The Euler characteristics $e_c^{\mathbb{S}_n}(\overline{\mathcal{M}}_{g,n}(\mathbb{C});\mathbb{Q})$ and $e_c^{\mathbb{S}_n}(\overline{\mathcal{M}}_{g,n}\otimes\overline{\mathbb{Q}};\mathbb{Q}_\ell)$ are then defined analogously. These expressions take values in the Grothendieck groups of mixed Hodge structures over $\mathbb{Q}$ or Galois representations, both with the additional action of $\mathbb{S}_n$. In their paper titled *Modular Operads* [28], Getzler and Kapranov describe how to determine the Euler characteristic of $\overline{\mathcal{M}}_{g,n}$ in terms of the $\mathbb{S}_n$-equivariant Euler characteristics of certain $\mathcal{M}_{h,m}$'s.

The trace of the Frobenius endomorphism F_q for almost all prime powers q on $\overline{\mathcal{M}}_{g,n}$ then determines the Euler characteristic as an element of $K_0(\mathrm{MHS}_\mathbb{Q})$ or $K_0(\mathrm{Gal}_{\mathbb{Q}_\ell})$ [46]. Using the Lefschetz trace formula on étale cohomology, knowing the trace of Frobenius on the Euler characteristic is essentially reduced to counting points of $\overline{\mathcal{M}}_{g,n}$ defined over a finite field. This is a painstaking exercise of counting all stable curves of genus g and n marked points defined over said finite field, together with the size of their automorphism group. Knowing the trace of F_q on $e_c(\overline{\mathcal{M}}_{g,n})$ for almost all primes is then equivalent to saying that this count is a polynomial in q. We can obtain the $\mathbb{S}_n$-equivariant Euler characteristic in a similar fashion by counting points equivariantly. With an $\mathbb{S}_n$-equivariant point count, we mean the number of fixed points of $F_q\circ\sigma$ on $\mathcal{M}_{g,n}$ for any $\sigma\in\mathbb{S}_n$. If it is also a polynomial in q for all σ, this is sufficient to determine the Euler characteristic as an element of $K_0(\mathrm{MHS}_\mathbb{Q})\otimes\Lambda$ or $K_0(\mathrm{Gal}_{\mathbb{Q}_\ell})\otimes\Lambda$. Equivariant points counts have been determined and are polynomials in q for at least

- $\mathcal{M}_{0,n}$ for $n\leq 11$, found in [34]
- $\mathcal{M}_{1,n}$ for $n\leq 9$, found in [12]
- $\mathcal{M}_{2,n}$ for $n\leq 7$, found in [7]
- $\mathcal{M}_{3,n}$ for $n\leq 5$, found in [7] and [12]
- $\mathcal{M}_{4,n}$ for $n\leq 3$, found in [8].

We can determine the $\mathbb{S}_n$-equivariant point counts for $\overline{\mathcal{M}}_{g,n}$ if we know the equivariant counts for all moduli spaces occurring in the stratification. If they are all polynomial in the sense above, so is the equivariant count of $\overline{\mathcal{M}}_{g,n}$. Finally, this is enough to determine both the equivariant and non-equivariant Euler characteristics of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. These results allow the authors in [8] to extend the above-mentioned results of [3] to conclude that $H^k(\overline{\mathcal{M}}_{g,n})$ vanishes for all odd $k\leq 9$.

Using these methods, we can better understand the cohomology ring $H^*(\mathcal{M}_{g,n};\mathbb{Q}_\ell)$. It contains a subring, the tautological sing

$$RH^*(\overline{\mathcal{M}}_{g,n})\subseteq H^*(\mathcal{M}_{g,n};\mathbb{Q}_\ell)$$

consisting of images of tautological classes from the Chow ring under the cycle class map. This ring has been studies extensively by Mumford [40]. The images of these cycles are sent to cohomology in even degree, so any odd nonzero cohomology must come from non-tautological classes. Conversely, results on this tautological subring give information relevant to the Euler characteristic. In [15], the authors prove that if $H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q}_\ell)=RH^*(\overline{\mathcal{M}}_{g,n})$, then $\overline{\mathcal{M}}_{g,n}$ has polynomial point count. They moreover prove that this condition is satisfied for $g=2,n\leq 9$

and for $g \geq 3$, $2g + n \leq 14$. As a consequence, it is possible to conclude that $\mathcal{M}_{g,n}$ has polynomial point count for all $2g + n \leq 12$. Note that this is a slight improvements on the bounds listed above.

Instead of studying the rational or ℓ-adic cohomology of $\mathcal{M}_{g,n}$, we can also consider the (ℓ-adic) sheaf cohomology of $\mathcal{M}_g$. In particular, there exists a locally constant sheaf $\mathbb{V}$ on $\mathcal{M}_g$ whose stalks $\mathbb{V}_{[C]}$ are given by $H^1(C;\mathbb{Q})$ (or $H^1(C;\mathbb{Q}_\ell)$). Given a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_g \geq 0)$ of n into g parts, we can construct a sheaf $\mathbb{V}_\lambda$ associated to λ, which occurs in the product

$$\mathrm{Sym}^{\lambda_1-\lambda_2}(\wedge^1 \mathbb{V}) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{g-1}-\lambda_g}(\wedge^{g-1} \mathbb{V}) \otimes \mathrm{Sym}^{\lambda_g}(\wedge^g \mathbb{V}).$$

It is then possible to compute the trace of Frobenius at q on the Euler characteristic of $\mathbb{V}_\lambda$

$$e_c(\mathcal{M}_g;\mathbb{V}_\lambda) := \sum_{i\geq 0}(-1)^i[H^i_c(\mathcal{M}_g;\mathbb{V}_\lambda)] \in K_0(\mathrm{MHS}_{\mathbb{Q}}) \text{ (or } \in K_0(\mathrm{Gal}_{\mathbb{Q}_\ell}).$$

There is a close connection between these Euler characteristics and the $\mathbb{S}_n$-equivariant Euler characterstics defined above. For example, it follows from the work of Getzler [27] that making $\mathbb{S}_n$-equivariant point counts of $\mathcal{M}_{g,n}$ over $\mathbb{F}_q$ for all $n \leq N$ is equivalent to computing $\mathrm{Tr}(F_q, e_c(\mathcal{M}_g \otimes \overline{\mathbb{F}}_q);\mathbb{V}_\lambda)$ for all λ such that $|\lambda| = n$. Conversely, the Euler characteristic of $\mathbb{V}_\lambda$ can be expressed in terms of equivariant Euler characteristics of $\mathcal{M}_{g,n}$. Using methods from a recent paper by Chan, Faber and Payne [16] to compute the weight 0 cohomology of $e_c(\mathcal{M}_{g,n}(\mathbb{C});\mathbb{Q})$, we can attempt to detect non-tautological cohomology occurring in $e_c(\mathcal{M}_g;\mathbb{V}_\lambda)$ by testing the polynomiality of this expression for some low values of q.

Chapter I

MODULI SPACE OF SMOOTH CURVES

In this chapter, we will introduce the moduli functor and discuss its representability in the category of sheaves. We then discuss the coarse moduli space $M_{g,n}$, which does not represent the moduli functor, but is useful enough for most of our purposes.

1 Moduli Functors

Broadly speaking, moduli functors occur frequently within the context of classification, or moduli problem. To define a moduli space which parametrizes all isomorphism classes of objects, it might be more natural to first consider a functor which does what we want. The existence of a moduli space is then reduced to showing the representability of this functor.

Definition I.1 (Moduli Functor)**.** [31, Section A] A moduli functor is a functor

$$h\colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Sets},$$

assigning to a scheme B the set of all families of objects with base B.

For example, we can choose the moduli functor sending a scheme B to the set of all families of curves of genus g indexed by B, up to isomorphism. In general the definition of a moduli functor can vary based on the circumstances, like the type of objects indexed by a scheme B, and what it means for two such objects to be isomorphic.

Definition I.2 (Fine Moduli Space)**.** [31, Section A] A fine moduli space is a scheme M representing a moduli functor h. That is, there is a natural isomorphism $\Phi\colon h \to h^M := \mathrm{Mor}_k(-, M)$. Such a scheme M is then called a *fine moduli space* for the functor h.

The fact that h can be represented allows us to define a universal family using the universal element.

Definition I.3. Given h a moduli functor and M a fine moduli space for h, define the *universal family* $U \in h(M)$ to be the element in $h(M)$ which is mapped to the canonical element $\mathrm{Id}_M \in \mathrm{Mor}_k(M, M) = h^M(M)$.

By the Yoneda Lemma, natural transformations of $\mathrm{Mor}(-, M)$ are in one-to-one correspondence with elements of $h(M)$. There is a natural way to define this correspondence. If $\Phi\colon \mathrm{Mor}(-, M) \to h$ is a natural transformation, we can always define $u := \Phi_M(Id_M)$. Conversely, given an element $u \in h(M)$, define the natural transformation $\Phi\colon \mathrm{Mor}(-, M) \to h$ as $\Phi_X(f) = h(f)(u)$. Whenever such a natural transformation is a natural isomorphism, the corresponding element $u \in h(M)$ has an interesting property, which we will characterize in the following definition.

Definition I.4. (Universal Element) A universal element of a functor $F\colon \mathcal{C} \to \mathrm{Sets}$ is a pair (M, u) consisting of $A \in \mathcal{C}$ and $u \in F(M)$ such that for every $X \in \mathcal{C}$ and $v \in F(X)$ there exists a unique morphism $f\colon X \to M$ such that $F(f)(u) = v$.

Given a fine moduli space M for h, we can now deduce the following.

Proposition I.5. *The universal family $U \in h(M)$ is a universal element of the functor h.*

Proof. If $\psi\colon \mathcal{C} \to B$ is any family over B, i.e., some element F of $h(B)$, then $\Phi(F)$ is a morphism $\Phi(F)\colon B \to M$. For U to be a universal element of the functor h, there needs to exist a unique morphism $\varphi\colon B \to M$ such that $h(\varphi)(U) = F$. The natural choice would be $\varphi\colon B \to M$, the image of F under Φ. Indeed, since Φ is a natural isomorphism of functors, we get the following commutative diagram:

$$\begin{array}{ccc} h(M) & \xrightarrow{\Phi(M)} & \mathrm{Mor}(M, M) \\ {\scriptstyle h(\varphi)}\downarrow & & \downarrow{\scriptstyle h^M(\varphi)} \\ h(B) & \xrightarrow[\Phi(B)]{} & \mathrm{Mor}(S, M) \end{array}$$

So $h^M(\varphi) \circ \Phi(M)(U) = h^M(\varphi) \circ \mathrm{Id}_M = \mathrm{Id}_M \circ \varphi = \varphi = \Phi(B)h(\varphi)(U)$. Furthermore, $\Phi(B)(F) = \varphi$ by definition. Since Φ is a natural isomorphism, $\Phi(B)$ is a bijection, so $h(\varphi)(U) = F$. Since Φ is a natural transformation, this map φ is unique. □

Generally, this is what causes obstructions for the existence of a fine moduli space. Simply said, this is caused by the existence of nontrivial automorphisms. The following example is taken from [42, Section 2.3]. Let k be an algebraically closed field and let Y be a scheme over k. Recall the Picard group

$$\mathrm{Pic}(Y) = \{\mathcal{L}\colon \mathcal{L} \text{ an invertible sheaf on } Y\}/\sim .$$

To define a moduli functor using this set, we need to consider what it means to be a family of line bundles over a scheme X. An obvious choice would be to take the product $X \times Y$, the trivial family with fiber Y over X. To this end, consider the following functor.

Definition I.6 (Absolute Picard Functor)**.** Define the *absolute Picard functor* $\mathrm{Pic}_Y \colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$ of the scheme $Y \neq \emptyset$ as

$$\mathrm{Pic}_Y(X) = \mathrm{Pic}(X \times Y).$$

For $f \colon X \to X'$, we define the corresponding morphism $\mathrm{Pic}_Y(X') \to \mathrm{Pic}_Y(X)$ by pulling back an invertible sheaf along $f \times \mathrm{Id}_Y$.

Proposition I.7. *For $Y \neq 0$, the functor Pic_Y is not representable.*

Proof. Let X be any scheme and $\mathcal{M}$ any nontrivial line bundle. Let $\pi_X \colon X \times Y \to X$ be the projection onto X. Now consider $\mathcal{M}_X = \pi_X^* \mathcal{M} \in \mathrm{Pic}(X \times Y) = \mathrm{Pic}_Y(X)$. Since Y is nonempty, it has a k-point. Therefore, we can map $s \colon X \to X \times Y$ such that $\pi_X \circ s$ is the identity. Thus, $s^*\mathcal{M}_X = \mathcal{M}$. This means $\mathcal{M}_X$ is nontrivial. Now assume Pic_Y were representable by some scheme P over k. Let $U \in \mathrm{Pic}_Y(P)$ be the universal family. Then by definition of universal family, there exists a unique morphism $g \colon X \to P$ with $(g \times \mathrm{Id}_Y)^* U = \mathcal{M}_X$. Similarly, there exists a unique morphism $p \colon \operatorname{Spec} k \to P$ corresponding to the trivial line bundle $\mathcal{O}_{\operatorname{Spec} k \times Y}$ under our natural isomorphism of functors. This then satisfies $(p \times \mathrm{Id}_Y)^* U = \mathcal{O}_{\operatorname{Spec} k \times Y}$. Let $\{U_i\}$ be some open affine cover of X trivialising $\mathcal{M}$. This exists by definition of an invertible sheaf. Then the pullbacks $\mathcal{M}_{U_i}$ of $\mathcal{M}_X$ to $U_i \times Y$ are trivial. This means they are pullbacks of $\mathcal{O}_{\operatorname{Spec} k \times Y}$ under the maps $U_i \to \operatorname{Spec} k$ to a point. Therefore, the unique map $U_i \to P$ associated to $\mathcal{M}_{U_i}$ factors through $p \colon \operatorname{Spec} k \to P$. This gives us a diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & P \\ \uparrow & \overset{g'}{\searrow} & \uparrow p \\ \sqcup_i U_i & \longrightarrow & \operatorname{Spec} k \end{array}$$

Since the restrictions of $g \colon X \to P$ to the opens U_i all factor through p, so the map g must also factor through p by some map g'. However,

$$\mathcal{M}_X = (g \times \mathrm{Id}_Y)^* U = (g' \times \mathrm{Id}_Y)^* (p \times \mathrm{Id}_Y)^* U = (g' \times \mathrm{Id}_Y)^* \mathcal{O}_{\operatorname{Spec} k \times Y} = \mathcal{O}_{X \times Y},$$

contradicting the fact that $\mathcal{M}_X$ is nontrivial. □

The idea used in this proof can be generalized to any family over a scheme B in which all fibers are isomorphic, except for one. Take for example the family of curves indexed by $t \in \mathbb{C}$: $E_t : y^2 = x^3 - t$. Then all E_t for $t \neq 0$ are isomorphic as they have the same j-invariant, 0. This is because there is no x-term in E_t. However, for $t = 0$ we get the cuspidal curve $y^2 = x^3$.

Proposition I.8. *There is no fine moduli space for the moduli functor $h \colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$ sending a scheme B over $\mathbb{C}$ to the set of all families of genus 1 curves parametrised by B, up to isomorphism.*

Proof. Consider the family $E_t : y^2 = x^3 - t$ indexed by the scheme $\mathbb{A}^1_{\mathbb{C}}$. Assume there exists a fine moduli space M for the functor h. Then for the nontrivial family indexed by $\mathbb{A}^1_{\mathbb{C}}$, we get a unique morphism $f\colon \mathbb{A}^1_{\mathbb{C}} \to M$. Since M is a fine moduli space, all values of $t \neq 0$ get mapped to the same point $q \in M$, while 0 gets mapped to a different point p. However, since $\mathbb{A}^1_{\mathbb{C}}$ is connected and f is continuous, this map cannot be a morphism of schemes. This is a contradiction, so such a scheme M cannot exist. □

So in the context of families of curves, such a fine moduli space appears to be too much to ask for. Having a functor be representable is certainly a strong condition. However, to find a space that parametrizes curves of a given genus up to isomorphism, it is enough to ask for slightly less. To this end, we define the *coarse moduli space* as follows.

Definition I.9. [31, Section A] Let h be a moduli functor $h\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$. A *coarse moduli space* for the functor h is a scheme M and a natural transformation ψ_M from h to h^M, the functor of points of M such that

1. The map $\psi_{\operatorname{Spec} k}\colon h(\operatorname{Spec} k) \to M(k)$ is a bijection for every algebraically closed field k.
2. Given another scheme M' and a natural transformation $\psi_{M'}\colon h \to h^{M'}$, there exists a unique map $\pi\colon M \to M'$ such that the induced natural transformation $\Pi\colon h^M \to h^{M'}$ satisfies $\psi_{M'} = \Pi \circ \psi_M$.

This definition still preserves the bijection on k-points and if we have any other scheme M' parametrising objects, this parametrisation factors uniquely through M. In fact, it follows from this condition that if a coarse moduli space M exists, it is unique up to canonical isomorphism.

Proposition I.10. *If a coarse moduli space (M, ψ) for a moduli functor h exists, it is unique up to canonical isomorphism.*

Proof. Suppose another coarse moduli space (M, ψ') exists. Then by property 2 above, there exists unique morphisms $f\colon M \to M'$ and $f'\colon M' \to M$ such that the induced natural transformations $\Pi\colon h^M \to h^{M'}$ and $\Pi'\colon h^{M'} \to h^M$ satisfy $\psi = \Pi' \circ \psi'$ and $\psi' = \Pi \circ \psi$. Then $\psi = \Pi' \circ \Pi \circ \psi$ and $\psi' = \Pi \circ \Pi' \circ \psi'$. From Yoneda's Lemma it now follows that $f \circ f' = \mathrm{Id}_{M'}$ and $f' \circ f = \mathrm{Id}_M$, so $M \cong M'$ and Π is a natural isomorphism $\psi\colon h^M \to h^{M'}$. □

Example I.11. Let $h\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets}$ be the moduli functor sending a scheme S over an algebraically closed field k to the set of all isomorphism classes of elliptic curves over S with a basepoint. The affine line $\mathbb{A}^1_k$ parametrizes all such elliptic curves according to the j-invariant, [32, IV-4] so $(\mathbb{A}^1_k, \psi)$, where ψ is the natural transformation sending an elliptic curve to its j-invariant is a coarse moduli space for h. Indeed, the first condition is satisfied by definition of the j-invariant. For the second condition, we use [32, IV, Exercise 4.4]. It states that if $k_0 \subseteq k$ is a subfield and X is an elliptic curve, which under its canonical embedding in $\mathbb{P}^2$ has coefficients in k_0, then its j-invariant is also contained in k_0. We have an obvious map $\mathbb{A}^1_k \to \mathbb{A}^1_{k_0}$ induced by the inclusion of subfields which satisfies condition 2, so $\mathbb{A}^1_k$ is a coarse moduli space for h.

2 Coarse Moduli Spaces

Consider the following moduli functor $\mathcal{M}_{g,n} \colon \mathrm{Sch}_k \to \mathrm{Sets}$

$$\mathcal{M}_{g,n}(S) := \{(\pi \colon C \to S; p_1, \ldots, p_n : S \to C) : \text{ smooth curve over } S\}/\sim .$$

In his book on Geometric Invariant Theory, [39], Mumford shows the existence of a coarse moduli space $M_{g,n}$ for $g, n \geq 0$ and $2g-2+n > 0$, for smooth connected curves C of genus g over an algebraically closed field k with n marked points. This space has dimension $3g-3+n$ whenever $g \geq 2$. When $n = 0$, we consider $M_{g,0} = M_g$. In their famous paper [21], Deligne-Mumford show that this space is irreducible for k an algebraically closed field with arbitrary characteristic. It is in general, however, not compact and not projective. If $\{C_t\}$ is a family of curves which becomes a nodal curve in the limit, then clearly this limit cannot be contained in M_g, since it only parametrizes smooth curves. We would like to somehow compactify the space M_g (or $M_{g,n}$) in such a way that it still parametrizes isomorphism classes of curves, albeit not necessarily smooth. It turns out the right class of curves to consider to this end are so-called stable curves, which we will discuss after the following example.

i Curves of Genus 5

We can determine the family of curves of genus 5 whose canonical model in $\mathbb{P}^4$ is the complete intersection of three quadric hypersurfaces. In [32, IV] example 5.5.3, we see that a non-hyperelliptic curve of genus 5 either has a g^1_3 or it is a complete intersection of three quadric hypersurfaces in the canonical embedding as a curve of degree 8. For our conventions on curves and the techniques used in this section, see Appendix C. Using the properties of coarse moduli spaces and some naive techniques from algebraic geometry, it is possible to deduce some results on M_g.

Proposition I.12. *The curves of genus* 5 *whose canonical model in* $\mathbb{P}^4$ *is the complete intersection of three quadric hypersurfaces form a family in* M_5 *of dimension 12.*

Proof. Since M_5 is a coarse moduli space, it suffices to find a variety U parametrising the possible curves C. We then get a morphism $U \to M_5$, and the dimension of the image of U will be given by the dimension of U minus the dimension of its fibres. Let C be a canonically embedded curve of genus 5 which is the complete intersection of three quadric surfaces. To determine the choice of these hypersurfaces, we would like to know how many distinct quadrics can be chosen on C. In general, to choose a quadric in $\mathbb{P}^4$ means choosing some element of $\Gamma(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$. This space has dimension $\binom{6}{2} = 15$. To know which of these vanish on C, consider the exact sequence associated to the closed subscheme C of $\mathbb{P}^4$

$$0 \to I_C \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_C \to 0,$$

where I_C is the ideal sheaf of C. Then we can twist by the sheaf $\mathcal{O}_{\mathbb{P}^4}(2)$, which is exact since it is invertible, to get the sequence

$$0 \to I_C(2) \to \mathcal{O}_{\mathbb{P}^4}(2) \to \mathcal{O}_C(2) \to 0.$$

We can then take its associated long exact cohomology sequence. Since C is a complete intersection, by C.15, the map $\Gamma(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \to \Gamma(C, \mathcal{O}_C(2))$ is surjective. This long exact sequence is therefore

$$0 \to \Gamma(C, I_C(2)) \to \Gamma(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \to \Gamma(C, \mathcal{O}_C(2)) \to 0.$$

As noted above, $\dim_k \Gamma(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = 15$, and $l(2 \cdot K_C) = \dim_k \Gamma(C, \mathcal{O}_C(2))$ can be calculated by using Riemann-Roch. Since C is a complete intersection of three quadrics, $\omega_C \cong \mathcal{O}_C(1)$ by C.16. Hence,

$$l(2K_C) - l(-K_C) = \deg 2K_C + 1 - g = 3g - 3 = 12.$$

Since K_C is effective, $l(-K_C) = 0$. Hence $l(2K_C) = 12$ and $\dim_k \Gamma(C, I_C(2)) = 3$. A curve of this type can therefore be parametrised by opens $U, V, W \subseteq \mathbb{P}\,\Gamma(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$, which each have dimension 14. We therefore get a map

$$U \times V \times W \to M_5.$$

The fibre of an element of M_5 is given by the images of the group PGL(4) which has dimension $(4+1)^2 - 2 = 24$, and since we must choose the quadrics so that they intersect in a curve, the dimension of the fibre increases by the dimension of $\mathbb{P}\,\Gamma(C, I_C(2))$ in each of the factors. Since $\dim_k \Gamma(C, I_C(2)) = 3$, this fibre has dimension $24 + 2 + 2 + 2$. In total the dimension of the image of $U \times V \times W$ is $14 + 14 + 14 - 24 - 2 - 2 - 2 = 12$. □

Proposition I.13. *A non-hyperelliptic curve of genus* 5 *has a* g_3^1 *if and only if it can be represented by a plane quintic with a node or a cusp.*

Proof. Suppose C has a g_3^1. Pick a divisor $D \in g_3^1$. Then $\deg D = 3$ and $\dim |D| = 1$. Consider the divisor $K_C - D$. It has degree $2g - 5 = 5$ and by Riemann-Roch,

$$l(D) - l(K_C - D) = 3 + 1 - 5 = -1.$$

So $\dim |D| + 1 = \dim |K_C - D|$. This divisor therefore induces a map $C \to \mathbb{P}^2$ with image a degree 5 curve of genus 5. By the genus-degree formula, this means we must have one singularity of δ invariant 1, meaning it is either a node or a cusp.
If C admits a map $\varphi\colon C \to \mathbb{P}^2$ with image a curve of degree 5 with a node or a cusp, then this map is induced by an invertible sheaf $\varphi^* \mathcal{O}_{\mathbb{P}^2}(1)$ corresponding to a divisor D which has degree 5 and $\dim |D| = 2$. By Riemann-Roch,

$$l(D) - l(K_C - D) = \deg D + 1 - g = 1.$$

We know $l(D) = 3$, so $l(K_C - D) = 2$ and therefore $\dim |K_C - D| = 1$. So it contains a g_3^1. □

Proposition I.14. *A non-hyperelliptic curve of genus* 5 *cannot have more than one* g^1_3.

Proof. Suppose C is such a curve with two g^1_3's. Then we get a morphism $\varphi\colon C \to \mathbb{P}^1 \times \mathbb{P}^1$ with image a curve C' of degree 6. This means that we get a curve of type $(3,3)$ on Q, the quadric surface which is the image of the Segre embedding. Therefore by C.11, its genus is $3 \cdot 3 - 3 - 3 + 1 = 4$. The degree of φ is 1 if and only if it is an isomorphism, but this is clearly not the case, so $\deg \varphi \geq 2$, so by Hurwitz's Theorem,

$$8 = 2g(C) - 2 = \deg \varphi(2g(C') - 2) + \deg R \geq 12 + \deg R.$$

But $\deg R \geq 0$, so this is not possible. □

Another proof, using the 'basepoint-free pencil trick' is also possible, as suggested by [2], Exercises III-B.

Chapter II

MODULI SPACE OF STABLE CURVES

In the previous section, we have seen the moduli space $M_{g,n}$ for $g, n \geq 0$ such that $2g-2+n > 0$, parametrizing smooth curves of genus g with n marked points. As was briefly mentioned, this space is not compact. The following example illustrates this.

Example II.1. Let k be an algebraically closed field of characteristic not 2 or 3 and consider the family of curves

$$\{x_2^2 x_3 + x_1^3 + x_2^2 t = 0 \colon t \in \mathbb{A}_k^1\}$$

parametrized by the affine line. For all values of $t \neq 0$, this is a curve smooth of genus 1, while for $t = 0$ it is a cuspidal curve. In particular, all curves C_t for $t \neq 0$ in this family are elements of the moduli space $M_{1,1}$, while the limit as t approaches 0 is not.

Moreover, $M_{g,n}$ is general not proper, not complete and not projective. The authors Deligne and Mumford [21] construct a suitable compactification $\overline{M}_{g,n}$ parametrizing stable curves instead. In this section, we will introduce this space and exhibit some of its relevant properties.

1 Stable Curves

In this section, k is algebraically closed and $g \geq 2$ unless otherwise indicated. We follow [31, Chapter C].

Definition II.2. A *stable curve* is a complete connected curve that has only nodes as singularities and has only finitely many automorphisms.

Hurwitz's Theorem tells us that any complete, nonsingular curve of genus at least two over an algebraically closed field has only finitely many automorphisms, at most $84(g-1)$ to be exact. For positive characteristic p, this result holds only whenever p does not divide the order of the automorphism group. It is also true for curves defined over fields that are not algebraically closed, since a base extension of such a curve X to its algebraic closure induces an injection $\mathrm{Aut}(X) \to \mathrm{Aut}(X_{\bar{k}})$. The automorphism group of a complete connected curve can only be infinite if it contains rational components. Therefore, we can reformulate this condition by saying that a complete connected curve C is stable if and only if

- every smooth rational component of C meets the other components of C in at least three points; or,
- every rational component of the normalisation of C has at least 3 points lying over singular points of C.

In short, this is because any automorphism of C preserves the set of singular points and any rational curve with three marked points is fixed.

Proposition II.3. *A complete connected curve C is stable if and only if its dualizing sheaf ω_C is ample.*

Proof. Let C be a stable curve, and let Q be the set of points in the normalisation $\tilde{C}$ of C lying over the nodes of C. Let $\{C_i\}$ be the set of irreducible components of C with normalisations $\tilde{C}_i$. Then

$$\deg \omega|_{C_i} = 2g(\tilde{C}_i) - 2 + |Q \cap \tilde{C}_i|.$$

This can be seen by considering the normalisation as a sequence of blowups at all of its nodes. Then the degree of the canonical divisor is given by this formula, as seen in [32, V]. The invertible sheaf ω_C is ample if and only if its degree is positive on all of its irreducible components, [36, Ch 7, Prop 5.5]. This is true if and only if $|Q \cap \tilde{C}_i| \geq 3$ for all rational components $\tilde{C}_i$ of $\tilde{C}$. □

In fact, in [21], Deligne-Mumford show that for $n \geq 3$, the sheaf $\omega_C^{\otimes n}$ is very ample.

Definition II.4. A *stable n-pointed curve* is a complete connected curve C that has only nodes as singularities, together with an ordered collection $p_1, \ldots, p_n \in C$ of distinct smooth points of C, such that the $(n+1)$-tuple $(C; p_1, \ldots, p_n)$ has only finitely many automorphisms.

Similarly, we can restate this condition by saying that every rational component of the normalisation of C has at least 3 points lying over singular and/or marked points of C.

Consider the following moduli functor $\overline{\mathcal{M}}_{g,n} \colon \mathrm{Sch}_k \to \mathrm{Sets}$:

$$\overline{\mathcal{M}}_{g,n}(S) := \{(\pi \colon C \to S; p_1, \ldots, p_n : S \to C) : \text{ stable } n\text{-pointed curve over } S\} / \sim .$$

Note that the curves we work with are no longer smooth, so from now on, we mean arithmetic genus whenever we say genus. For the arithmetic genus of a singular curve, see Appendix C. It follows from C.14 that if $C_1 \ldots, C_v$ are the irreducible components of C with geometric genera $g_1, \ldots, g_v$ and δ nodes, that

$$g = \sum_{i=1}^{v}(g_i - 1) + \delta + 1 = \sum_{i=1}^{v} g_i + \delta - v + 1.$$

Theorem II.5. *There exist coarse moduli spaces $\overline{M}_g$ and $\overline{M}_{g,n}$ of stable n-pointed curves; and these spaces are projective varieties.*

These spaces are called the *stable compactifications* of M_g and $M_{g,n}$. This theorem gives us an indication that stable curves of genus g indeed form the right class of curves to consider in our moduli problem. This is further strengthened by the *Stable Reduction Theorem* by Deligne-Mumford:

Theorem II.6. *Let R be a discrete valuation ring with quotient field K. Let η and s be the generic and closed points of $\operatorname{Spec} R$ respectively. Let C be a smooth geometrically irreducible curve over K of genus ≥ 2. There exists a finite algebraic extension L of K and a stable curve $C_L \to \operatorname{Spec} R_L$, where R_L is the integral closure of R in L, such that $C_{L,\eta} \cong C \times_K L$.*

We can derive some bounds on the number of nodes and components on a stable curve.

Proposition II.7. *No stable unpointed curve C of genus g can have more than $3g-3$ nodes.*

Proof. We will prove this statement by induction in two different ways. First, we do induction on the number of irreducible components of C. If C has one irreducible component, then C is irreducible, and it is clear that g is an upper bound for the number of singularities, as each singularity in an irreducible component adds to the genus. Suppose that every stable curve with $n-1$ components has no more than $3g-3$ singularities. Let C be a stable curve with n components, and let c be a singular point of C. Consider the normalization $\tilde{C}$ of C at c. We can distinguish two cases: either c lies on a single irreducible component of C, or on the intersection of two irreducible components. In the first case, simply continue the induction on another singularity of C. In the second case, $\tilde{C}$ has two connected components, each with a point marked c in the preimage of c. If the two connected components C_1 and C_2 of $\tilde{C}$ are stable curves of genera g_1, g_2, then we know that $g = g_1 + g_2$. Since C_1 and C_2 both have fewer than n irreducible components, the number of singularities on C_1 and C_2 is bounded by $3g_1 - 3$ and $3g_2 - 3$ respectively. Since the number of singularities of C is the sum of 1 and the number of singularities of C_1 and C_2, we conclude that the number of singularities of C is less than or equal to

$$3g_1 - 3 + 3g_2 - 3 + 1 = 3g - 5 < 3g - 3.$$

Now assume without loss of generality that C_1 is not stable. Then C_1 has some rational irreducible component passing through $c \in C_2$ and exactly two other points on irreducible components of C_1. Consider C_1', the stable curve with this irreducible component contracted. This stable curve has one singularity less than C_1, and one irreducible component of genus 0 less than C_1. This means that the genus g_1' of C_1' equals g_1. The image of $C_1' \cup C_2$ under the normalization map is some stable curve with one rational component removed, and two nodes less. We have that $g = g_1' + g_2$. Moreover, the number of nodes of C is the sum of 2, with the number of nodes of C_1' and C_2. Applying the induction hypothesis gives us the following upper bound on the number of nodes of C. Denote $\#C :=$ number of nodes of C.

$$\#C = \#C_1' + \#C_2 + 2 \leq 3g_1 - 3 + 3g_2 - 3 + 2 = 3g - 4.$$

□

We can see this in a different -although less insightful- way by the following observation: fix a curve C with δ nodes and v irreducible components $C_1, \dots, C_v$ with geometric genera $g_1, \dots, g_v$. To specify such a stable curve we have to specify the normalizations $\tilde{C}_i$ of the C_i and then specify the points on each that will be identified to form the nodes of C. There will be 2δ such points, therefore the family of such curves will have dimension

$$\sum_{i=1}^{v}(3g_i - 3) + 2\delta,$$

which equals

$$3g - 3 - \delta$$

by our genus formula for stable curves. This means precisely that the locus of curves in $\overline{M}_g$ with exactly δ nodes has codimension δ. Therefore there cannot be more than $3g - 3$ nodes.

Proposition II.8. *For every $g \geq 2$ there exists a stable curve with exactly $3g - 3$ nodes.*

Proof. For $g = 2$, take two smooth rational components meeting in three points. For $g > 2$, we can take $2g - 2$ nonsingular lines $L_1, \dots, L_{,2g-2}$ and transversally intersect L_i with L_{i-1}, L_{i+1} and L_{g-1+i} for every $1 \leq i \leq g - 1$ such that $L_0 = L_{g-1}$. This gives for each $1 \leq i \leq g - 1$ two choices L_j, L_k with $1 \leq j, k \leq g - 1$ and one other L_m for $g \leq m \leq 2g - 2$. In total this gives $3g - 3$ singularities. Since every rational component has three special points, this curve is stable. □

Corollary II.9. *A stable curve of genus $g \geq 2$ can have at most $2g - 2$ components. If this maximum is reached, all components are rational.*

Proof. Since the addition of each component adds a node, the maximal number of components is reached whenever we reach the maximal number of nodes, $3g - 3$. Suppose C is a stable curve with $3g - 3$ nodes. Then by the genus formula for stable curves

$$g = \sum_{i=1}^{v}(g_i - 1) + \delta + 1$$

we see that

$$2 - 2g = \sum_{i=1}^{v} (g_i - 1).$$

From this, it follows that there must be at least $2g-2$ rational components. Since all of these rational components must have three special points, there are at least $6g-6$ marked points in the rational components of the normalisation of C as it is a stable curve. Any stable curve with $3g-3$ nodes has exactly $6g-6$ marked points in its normalisation, and every connected component of the normalisation contains a marked point. Therefore these $2g-2$ rational components are all components. □

2 Boundary Divisors

The space M_g of smooth curves of genus g is a dense open in $\overline{M}_g$. Understanding the geometry of $\overline{M}_g$ and M_g in particular means understanding the boundary. This next section and chapter explain some of the phenomena occurring there.

Proposition II.10. *The locus of stable curves of genus g with more than δ nodes lies in the closure of the locus of curves with exactly δ nodes.*

Proof. See [31, Chapter 3C]. □

Since we can always map $\overline{M}_{g,n} \to \overline{M}_g$ by forgetting the marked points, this statement also holds for n-pointed stable curves.
From this result it follows that the boundary $\Delta = \overline{M}_g \setminus M_g$ is a divisor. Each component is the closure of a locus of curves with precisely one node. Stable curves with one node are easy to determine: a stable curve with one node is either irreducible of genus $g-1$ and one node,or the union of two irreducible components with genera i and $g-i$, $i > 0$ meeting at one point. Since this construction is symmetric in i and $g-i$, these determine the prime divisors $\Delta_0, \Delta_1, \ldots, \Delta_{\lfloor g/2 \rfloor}$. We can therefore write

$$\overline{M}_g = M_g \cup \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{\lfloor g/2 \rfloor}.$$

In particular, the curves in Δ_0 with one singularity are precisely those in the image of the map

$$M_{g-1,2} \to \Delta_0$$

identifying the two marked points. Note that since $M_{g-1,2}$ only contains smooth curves, this map is not surjective.

Proposition II.11. *The map*

$$\overline{M}_{g-1,2} \to \Delta_0$$

identifying the two marked points is surjective.

Proof. As mentioned above, restricting this map to $M_{g-1,2}$ yields precisely those curves in Δ_0 with one singularity. Taking the closure, we obtain our result. $\square$

Similarly, the curves in Δ_i for $0 < i \leq \lfloor g/2 \rfloor$ are in the image of the maps

$$\overline{M}_{i,1} \times \overline{M}_{g-i,1} \to \Delta_i,$$

sending two curves to their union at the marked points.

Definition II.12. The universal curve $\overline{\mathcal{C}}_g$ over $\overline{M}_g$ is the space $\overline{M}_{g,1}$. We furthermore define $\mathcal{C}_g = M_{g,1}$.

Even though $\overline{M}_{g,1}$ is only a coarse moduli space, we still refer to this space as the universal curve, which might be slightly misleading.
The boundary $\overline{\mathcal{C}}_g \setminus \mathcal{C}_g$ consists of the closures of those stable 1-pointed curves which have precisely one singularity, which is not equal to the marked point.

Proposition II.13. *We have a decomposition*

$$\overline{\mathcal{C}}_g = \mathcal{C}_g \cup \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_{g-1}$$

where Σ_0 is the closure of the locus of pairs (C, p), C an irreducible marked curve with a single node, and Σ_i is the closure of the locus of pairs (C, p), where C is the union of smooth curves of genera i and $g - i$ meeting at a point such that p lies in the component of genus i.

Proof. By Proposition II.10, it suffices to determine the loci of stable 1-pointed curves with one node, since their closures will then determine the rest of the boundary. If (C, p) is some stable 1-pointed curve with a node, then it is either irreducible, in which case it lands in Σ_0, or it is reducible. If it is reducible, then it must have two smooth irreducible components since there is only one node. Let C' be an irreducible component with genus i. Then the genus of the other irreducible component must be $g - i$. Since the marked point is not in the singular locus, one of the components must contain the marked point p. For every $0 < i \leq g - 1$ we therefore get a unique locus of curves. Since the genus of (C, p) is at least two, it is clear that this construction gives a stable curve. $\square$

i Moduli Space of Rational Curves

In general, two unmarked genus 0 curves are always isomorphic, therefore the moduli space M_0 itself is just a point. Moreover, similarly, $M_{0,n}$ is also a point for $n \leq 3$, since we can always define a projective transformation sending three marked points to $0, 1, \infty$. If $C \in \partial\overline{M}_{0,3}$ is stable, then it has two components of genus 0 meeting at a point. Then in order for this curve to be stable, both components need to contain three special points. Since both components are necessarily smooth outside their intersection, this means not both components can contain three special points, hence such a curve is not stable. Since a rational curve with a node has genus 1, this means the only stable rational curve with three marked points are those in $M_{0,3}$, i.e. $\overline{M}_{0,3} = M_{0,3} = \{*\}$.

For $n = 4$ and marked points P_1, P_2, P_3, P_4, apply the transformation sending these points to $0, 1, \infty, P$. Then the resulting pointed curve is determined up to isomorphism by the choice of $P \neq 0, 1, \infty$, so $M_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$. In the case where such a curve is stable, but not smooth, there can be multiple components. Let $C = C_1 \cup C_2$ be a union of two rational curves at a point. If C_1 and C_2 are nonsingular outside of their intersection, this curve has genus 0. For this curve to be stable, both rational components need to contain at least three special points. Since we have one singularity at their intersection and no others, both components must contain two marked points. Since any automorphism of C must fix the singularity, two such stable curves $(\mathbb{P}^1, 0, 1, \infty, P)$ and $(\mathbb{P}^1, 0, 1, \infty, Q)$ are isomorphic if and only if their marked points of the decompositions agree, not up to ordering.

More generally, we can determine the moduli space $M_{0,n}$ for $n > 4$ in a similar way. If $P_1, \ldots, P_n$ are the marked points of a smooth rational curve, we can always send the first three points to $0, 1, \infty$ to get the marked points $0, 1, \infty, Q_1, \ldots, Q_{n-3}$. Since all these Q_i need to be pairwise distinct, we conclude

$$M_{0,n} \cong (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta,$$

where Δ is the "large diagonal" $\Delta = \{(P_i)_i : \exists i \neq j \ s.t. \ P_i = P_j\}$

Chapter III

STRATIFYING THE MODULI SPACE

In the previous section we have seen that the existence of nontrivial automorphisms prevents the moduli functor from being representable. In particular, this discards the existence of a universal family over the coarse moduli space. In this section, we will see that we can almost, but not quite, identify the space $\overline{M}_{g,n+1}$ as the universal curve over $\overline{M}_{g,n}$. Furthermore, we define some canonical maps between moduli spaces and use them to stratify $\overline{M}_{g,n}$ by dual graphs.
First, we can define a forgetful morphism $\pi\colon M_{g,n+1} \to M_{g,n}$.

Lemma III.1. *There exists a morphism $\pi\colon M_{g,n+1} \to M_{g,n}$ which on $\bar{k}$-points is given by*

$$\begin{aligned} \pi\colon M_{g,n+1}(\bar{k}) &\to M_{g,n}(\bar{k}) \\ [(C; p_1, \dots, p_n, p_{n+1})] &\mapsto [(C; p_1, \dots, p_n)] \end{aligned}$$

by forgetting the last marked point. This is the forgetful morphism.

Proof. Consider the natural transformation of moduli functors $\tilde{\pi}\colon \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ defined for every $S \in \mathrm{Sch}_k$ by

$$(\pi\colon C \to S; p_1, \dots, p_n, p_{n+1} : S \to C) \mapsto (\pi\colon C \to S; p_1, \dots, p_n\colon S \to C).$$

Since $M_{g,n}$ is a coarse moduli space for the functor $\mathcal{M}_{g,n}$, we have maps $\mathcal{M}_{g,n} \to h^{M_{g,n}}$ and therefore obtain the diagram

$$\begin{array}{ccc} \mathcal{M}_{g,n+1} & \longrightarrow & h^{M_{g,n+1}} \\ \big\downarrow{\scriptstyle \tilde{\pi}} & & \vdots{\scriptstyle \pi} \\ \mathcal{M}_{g,n} & \longrightarrow & h^{M_{g,n}} \end{array}$$

Since $M_{g,n+1}$ is a coarse moduli space for $\mathcal{M}_{g,n+1}$, the composition $\mathcal{M}_{g,n+1} \to h^{M_{g,n}}$ must factor uniquely through a map $h^{M_{g,n+1}} \to h^{M_{g,n}}$. By definition of a coarse moduli space, this map works as desired on $\bar{k}$-points. □

The construction above does not immediately generalize to the modulis space of stable curves: whenever we omit a marked point on a stable curve, the result need not be stable. Indeed, if $(C; p_1, \dots, p_{n+1})$ of genus g has a component C_v of genus 0 with one node and two marked points, including p_{n+1}, then forgetting p_{n+1} makes C unstable. This can be fixed by contracting the rational component C_v. This removes one genus 0 component and maps the remaining marked point to the node, which becomes a marked point. This operation makes C into a stable n-pointed curve of genus g. Formally, this construction is given as follows:

Definition III.2. [35] A morphism of pointed stable curves C, C' over S with marked points $p_1, \dots, p_{n+1}$ and $p'_1, \dots, p'_n$:

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C' \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} \\ S & \underset{=}{\longrightarrow} & S \end{array}$$

is called a contraction if

1. C is an $(n+1)$-pointed curve, C' is an n-pointed curve and $\varphi \circ p_i = p'_i$ for $1 \leq 1 \leq n$.
2. If we consider the induced morphism on a geometric fibre C_s, we have one of two possible cases:
   (a) $\varphi_s \colon C_s \to C'_s$ is an isomorphism.
   (b) There is a rational component $E \subseteq C_s$ such that $p_{n+1}(s) \in E, \varphi_s(E) = x$ is a closed point of C'_s, and
   $$\varphi_s \colon C_s \setminus E \to C'_s \setminus \{x\}$$
   is an isomorphism.

Why this is the right thing to do is justified by the following theorem. Since we are limited to coarse moduli spaces and moduli functors, we refer to [35, Theorem 2.4] for the (more complete) version for stacks.

Theorem III.3. *There is a unique forgetful morphism $\pi \colon \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ extending $\pi \colon M_{g,n+1} \to M_{g,n}$ which acts on $\bar{k}$-points by forgetting the last marked point.*

Proof. Define a moduli functor $\overline{\mathcal{C}}_{g,n} \colon \mathrm{Sch}_k \to \mathrm{Sets}$ by

$$S \mapsto \{(p \colon C \to S; p_1, \dots, p_n, q \colon S \to C) \colon \text{ stable curve over } C\}/\sim .$$

Here, q can be any point of C. In particular, it can be one of the special points. $(C; p_1, \dots, p_n)$ is an n-pointed stable curve of genus g. We can then define a natural transformation of functors

$$\tilde{\xi}\colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$$
$$(p\colon C \to S; p_1, \dots, p_n, q\colon S \to C) \mapsto (p\colon C \to S; p_1, \dots, p_n\colon S \to C).$$

This is a well-defined map, whence it does not change the curve C nor impact its stability. This natural transformation then induces a map on coarse moduli spaces like in the lemma above. The key of the proof is therefore finding a natural isomorphism of functors $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$. This is the content of Knudsen's paper [35]. The operation φ sends an $(n+1)$-pointed curve $(p\colon C \to S; p_1, \dots, p_n, p_{n+1}\colon S \to C)$ to the n-pointed curve

$$(C \to S; p_1, \dots, p_n, q\colon S \to C),\ \ q = p_{n+1},$$

if this is stable, and otherwise to the n-pointed curve

$$(C'; \varphi \circ p_1, \dots, \varphi \circ p_n, q\colon S \to C),\ \ q = \varphi \circ p_{n+1},$$

where q is now no longer a marked point. The inverse of this morphism is called stabilization. To an n-pointed curve with an extra section

$$(p'\colon C' \to S; p'_1, \dots, p'_n, q\colon S \to C)$$

it assigns the $(n+1)$-pointed curve C with marked points $p'_1, \dots, p'_n, q$ if q is not a special point. If q is one of the special points in the fibre C'_s of $C' \to S$, stabilization modifies C'_s by inserting a rational component at the marking p'_n that contains q. Although formally defining this as a natural transformation of functors is difficult, it is easy to see that stabilization and contraction are inverses when properly defined.
This process yields a natural transformation $\overline{\mathcal{M}}_{g,n+1} \cong \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ which descends to a map of schemes $\xi\colon \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ by the properties of coarse moduli spaces. On $\bar{k}$-points, it is given by

$$(C; p_1, \dots, p_n, p_{n+1}) \mapsto \begin{cases} (C; p_1, \dots, p_n) & \text{if } (C; p_1, \dots, p_n) \text{ is stable} \\ (C'; \varphi \circ p_1, \dots, \varphi \circ p_n) & \text{otherwise.} \end{cases}$$

To prove that this is the unique extension of $\pi\colon M_{g,n+1} \to M_{g,n}$, we can use the projectivity of $\overline{M}_{g,n}$, since this implies it is also separated. Clearly, the restriction of ξ to $M_{g,n+1}$ is just π. Now suppose $\xi'\colon \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is another morphism extending π. Then consider the pullback diagram

$$\begin{array}{ccc} Z & \longrightarrow & \overline{M}_{g,n} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \Delta} \\ \overline{M}_{g,n+1} & \xrightarrow[(\xi,\xi')]{} & \overline{M}_{g,n} \times_{\operatorname{Spec} k} \overline{M}_{g,n} \end{array}$$

By construction, Z is the locus in $\overline{M}_{g,n+1}$ where ξ, ξ' agree. Since $\overline{M}_{g,n}$ is separated, the diagional morphism Δ is a closed immersion. As closed immersions are stable under base change, we may conclude that $i\colon Z \to \overline{M}_{g,n+1}$ is a closed immersion. Having expressed Z as a closed subscheme of $\overline{M}_{g,n+1}$, note that it contains the dense open $M_{g,n+1}$. As this space is dense in the projective variety $\overline{M}_{g,n+1}$ and Z is closed, we conclude $Z = \overline{M}_{g,n+1}$ and therefore $\xi = \xi'$. □

The essence of this proof is essentially the natural isomorphism between the moduli functors $\overline{\mathcal{M}}_{g,n+1}$ and $\overline{\mathcal{C}}_{g,n}$ which would express the equivalence of the universal curve C over the fine moduli space $\overline{\mathcal{M}}_{g,n}$ (in a category where this exists) as the fine moduli space $\overline{\mathcal{M}}_{g,n+1}$. While this does not make sense for our scheme-theoretic interpretation, we can partially justify it. As noted in [31], we can define the moduli functor

$$\begin{aligned} \overline{\mathcal{M}}^0_{g,n}\colon \ &\mathrm{Sch}_k \to \mathrm{Sets} \\ S \mapsto \{(p\colon C \to S; p_1, \ldots, p_n\colon S \to C)\colon &\text{ stable curves } C \\ &\text{admitting no nontrivial automorphisms}\}/\sim, \end{aligned}$$

which is representable in the category of schemes by a space $\overline{M}^0_{g,n}$. Therefore, over this space we do have a universal curve, $\overline{M}^0_{g,n+1}$.

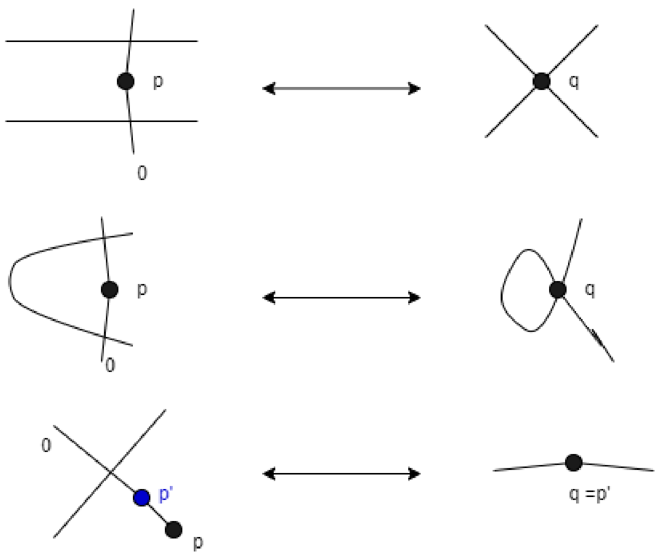

Figure III.1: possible contractions and their inverse stabilizations

1 Gluing Morphisms and Dual Graphs

We have seen that a stable curve can be interpreted as a collection of smooth curves that are attached at certain points to make a nodal curve. This interpretation lends itself nicely to a combinatorial counterpart, the dual graph of a stable curve, which we will introduce in this section. Through this approach, we will formally define what it means to 'glue' curves as a map between moduli spaces. In particular, this theory will allow us to make a stratification of $\overline{M}_{g,n}$ that is essential for the computation of its Euler characteristic.

Definition III.4. [29, A.1] A *stable graph* is a tuple

$$A = (V, H, L,\ g\colon V \to \mathbb{Z}_{\geq 0},\ \gamma\colon H \to V,\ i\colon H \to H)$$

satisfying the following properties:

1. V is a vertex set with genus function g,
2. H is a half-edge set equipped with a vertex assignment γ and an involution i, we denote $H(v) := \{h \in H\colon \gamma(h) = v\}$ the set of vertices incident to h,
3. E, the edge set, is defined by the orbits of i in H (self-edges at vertices are permitted),
4. (V, E) is a connected graph,
5. L is a set of numbered legs attached to the vertices,
6. For each vertex v, the stability condition holds:

   $$2g(v) - 2 + n(v) > 0,$$

   where $n(v)$ is the valence of A at v including both half-edges and legs.

The genus of A is defined as

$$g(A) = \sum_{v \in V} g(v) + h^1(A).$$

Let $(C, p_1, \ldots, p_n)$ be a stable n-pointed curve of genus g and $w\colon \tilde{C}\colon \to C$ its normalization.

Definition III.5. The dual graph Γ_C is given by the following infornmation:

- $V = \{v\colon \tilde{C}_v$ a component of $\tilde{C}\}$, $g\colon V \to \mathbb{Z}_{\geq 0}$ sends a component to its genus,
- $H = \{p_1, \ldots, p_n\} \cup \{q', q''\colon$ preimages $w^{-1}(q),\ q$ a node$\}$
  $\gamma\colon H \to V$ sends a half-edge to the component containing it, $i\colon H \to H$ fixes p_i and exchanges q', q'',
- The legs L are given by $\{p_1, \ldots, p_n\}$, with an ordered enumeration $l(p_i) = i$.

Example III.6. Let $(C, \underline{1}, \underline{2}, \underline{3})$ be the three-pointed genus 5 curve given by three irreducible components C_1, C_2, C_3 of geometric genera $0, 1, 3$ intersecting in the points a, b, c and normalization $w\colon \tilde{C} \to C$ such that

- $\{a', a''\} = w^{-1}(a)$, $\{b', b''\} = w^{-1}(b)$, $\{c', c''\} = w^{-1}(c)$ are the preimages of the nodes,
- $\underline{i} \in C_i$ for $i = 1, 2, 3$.

Then the dual graph Γ_C is given by

- $V = \{v_1, v_2, v_3\}$, $g(v_1) = 0$, $g(v_2) = 1$, $g(v_3) = 3$,
- $H = \{\underline{1}, \underline{2}, \underline{3}\} \cup \{a', a'', b', b'', c', c''\}$,
  $\gamma(\underline{i}) = v_1$, $\gamma(a') = \gamma(c'') = v_2$, $\gamma(c') = \gamma(b') = v_1$, $\gamma(b'') = \gamma(a'')$,
- $L = \{\underline{1}, \underline{2}, \underline{3}\}$.

While this notion of dual graph is heavy in terms of notation, the process is much clearer in terms of pictures:

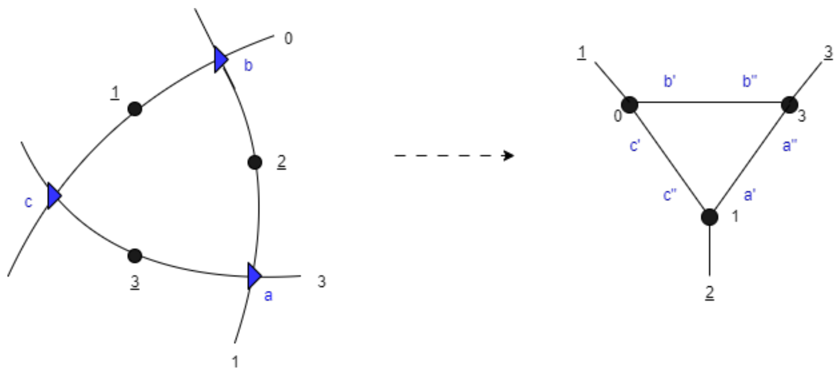

Figure III.2: the curve C and its dual graph

Definition III.7. Let Γ be a stable graph of genus g with $|L| = n$ legs. Define

$$M_\Gamma := \prod_{v \in V(\Gamma)} M_{g(v), n(v)},$$
$$\overline{M_\Gamma} := \prod_{v \in V(\Gamma)} \overline{M}_{g(v), n(v)}.$$

Proposition III.8. *[42, Proposition 4.15] Let Γ be a stable graph of genus g with $|L| = n$ legs. Then there exists a morphism*

$$\xi_\Gamma : \overline{M}_\Gamma \to \overline{M}_{g,n}$$

which acts on $\overline{k}$-points by sending a tuple $(C_v, (q_h)_{h \in H(v)})_{v \in V(\Gamma)}$ to the stable curve $(C, p_1, \ldots, p_n)$ by gluing q_h and $q_{h'}$ if $h = i(h')$. Then p_i is given by the i-th leg of Γ.

Proof. To induce the morphism ξ, we will construct a natural transformation of functors. Define the moduli functor $\overline{\mathcal{M}}_\Gamma \colon \mathrm{Sch}_k \to \mathrm{Sets}$ by

$$S \mapsto \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}(S).$$

This functor by construction has $\overline{M}_\Gamma$ as a coarse moduli space. Consider the natural transformation

$$\begin{gathered}\chi_\Gamma \colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n} \\ (f_v \colon X_v \to S, (\sigma_l \colon S \to X_v)_{l \in H(\Gamma)})_{v \in V(\Gamma)} \mapsto (f' \colon X' \to S,\ \sigma_1', \ldots, \sigma_n' \colon S \to X'),\end{gathered}$$

where the image is a family obtained by identifying the sections $\sigma_l, \sigma_{l'}$ if l, l' form edges of Γ. Why this is a well-defined construction in terms of families is described in [**?**, Section X-7]. In particular, the fiber of $X' \to S$ is precisely the curve obtained from X_s by gluing along its dual curve. This map χ_Γ then descends to a map ξ_Γ of coarse moduli spaces which acts as desired on $\overline{k}$-points. □

Lemma III.9. *Let $g, n \geq 0$ such that $2g - 2 + n > 0$. Let Γ be a stable graph of genus g and n legs, and consider the set*

$$M^\Gamma := \{(C, p_1, \ldots, p_n) \colon \Gamma_C \cong \Gamma\} \subseteq \overline{M}_{g,n}$$

The morphism ξ_Γ is finite, and its image is the closure of M^Γ in $\overline{M}_{g,n}$.

Proof. We first prove that our morphism is finite. Note that the moduli spaces $\overline{M}_{g,n}$ are proper since they are projective varieties and therefore $\overline{M}_\Gamma$ is too. In particular, $\overline{M}_{g,n}$ is separated.

$$\begin{array}{ccc} \overline{M}_\Gamma & \xrightarrow{\ \xi_\Gamma\ } & \overline{M}_{g,n} \\ & {}_{\tau}\searrow \quad \swarrow_{\rho} & \\ & \operatorname{Spec} k & \end{array}$$

where τ is proper, and ρ is separated. Let $G_{\xi_\Gamma} \colon \overline{M}_\Gamma \to \overline{M}_\Gamma \times_k \overline{M}_{g,n}$ be the map induced by $(\mathrm{Id}, \xi_\Gamma)$. Consider the diagram

$$\begin{array}{ccc} \overline{M}_\Gamma & \xrightarrow{G_{\xi_\Gamma}} & \overline{M}_\Gamma \times_k \overline{M}_{g,n} \\ {\scriptstyle \xi_\Gamma}\downarrow & & \downarrow \\ \overline{M}_{g,n} & \xrightarrow[\Delta]{} & \overline{M}_{g,n} \times_k \overline{M}_{g,n} \end{array}$$

We have $\overline{M}_\Gamma \times_k \overline{M}_{g,n} \times_{\overline{M}_{g,n} \times_k \overline{M}_{g,n}} \overline{M}_{g,n} \cong \overline{M}_\Gamma$, so this diagram is cartesian. Since ρ is separated, Δ is a closed immersion, so G_{ξ_Γ} is also a closed immersion. In particular, it is proper. Next, we have the cartesian diagram

$$\begin{array}{ccc} \overline{M}_\Gamma \times_k \overline{M}_{g,n} & \xrightarrow{\tau'} & \overline{M}_{g,n} \\ \downarrow & & \downarrow \\ \overline{M}_\Gamma & \xrightarrow[\tau]{} & \operatorname{Spec} k \end{array}$$

Since τ is proper, so is τ'. We have $\xi_\Gamma = \tau' \circ G_{\xi_\Gamma}$, so ξ_Γ is proper. Since a proper and quasi-finite morphism is finite, we are left with showing that ξ_Γ is quasi-finite, i.e., that its fibers are finite in number. This follows from the fact that our stable curves have only finitely many components, nodes and automorphisms, so the possible number of ways of taking apart a stable curve to glue it back together is finite.
To show $\xi_\Gamma(\overline{M}_\Gamma) = \overline{M}^\Gamma$, we first show that $M^\Gamma = \xi_\Gamma(M_\Gamma)$. Given a curve with dual graph Γ, it can be seen as the gluing of its irreducible components of lower genus by definition. On the other hand, any curve which is in the image of ξ_Γ has dual graph Γ by construction.
To show the image is closed, we use the fact that ξ_Γ can be factored as $\overline{M}_\Gamma \xrightarrow{G_{\xi_\Gamma}} \overline{M}_\Gamma \times_k \overline{M}_{g,n} \to \overline{M}_{g,n}$. We have seen that G_{ξ_Γ} is a closed immmersion, so it is closed, and the second map is closed as $\overline{M}_\Gamma$ is proper. Hence, $\overline{M}^\Gamma \subseteq \xi_\Gamma(\overline{M}_\Gamma)$, but M_Γ is dense in $\overline{M}_\Gamma$, so

$$\overline{M}^\Gamma \subseteq \overline{\xi_\Gamma(M_\Gamma)} \subseteq \xi_\Gamma(\overline{M}_\Gamma) \text{ and } \xi_\Gamma(\overline{M}_\Gamma) \subseteq \overline{\xi_\Gamma(M_\Gamma)} = \overline{M}^\Gamma.$$

□

Corollary III.10. *The set M^Γ is nonempty, irreducible, locally closed and it defines a stratification*

$$\overline{M}_{g,n} = \bigsqcup_\Gamma M^\Gamma,$$

where each stratum has dimension

$$\dim M^\Gamma = \sum_{v \in V(\Gamma)} 3g(v) - 3 + n(v) = \dim \overline{M}_{g,n} - \#E(\Gamma).$$

Proof. The map ξ_Γ is nontrivial and M_Γ is nonempty and irreducible, so the image $\xi_\Gamma(M_\Gamma) = M^\Gamma$ is too as ξ_Γ is closed. It is clear that $\overline{M}^\Gamma \setminus M^\Gamma \subseteq \xi_\Gamma(\overline{M}_\Gamma \setminus M_\Gamma)$ since $\xi_\Gamma(M_\Gamma) = M^\Gamma$. On the other hand, if C is a curve in $\overline{M}_\Gamma \setminus M_\Gamma$, then one of the components C' of C must have a node, and the image of C under ξ_Γ is therefore a curve with more nodes than Γ has edges, so $\xi_\Gamma(C) \notin M^\Gamma$. Now since ξ_Γ is closed and $\overline{M}_\Gamma \setminus M_\Gamma$ is closed, the set $\overline{M}^\Gamma \setminus M^\Gamma$ is closed in $\overline{M}_{g,n}$, so M^Γ is open in $\overline{M}^\Gamma$. Since one curve cannot have multiple dual graphs, the sets M^Γ partition $\overline{M}_{g,n}$. We have that M^Γ is irreducible with closure $\overline{M^\Gamma}$, so it suffices to determine the dimension of $\overline{M}_\Gamma$ since ξ_Γ is a finite morphism. For this we need the identity

$$g = \sum_{v \in V(\Gamma)} g(v) + 1 + \#E(\Gamma) - \#V(\Gamma).$$

Notice that each factor of $\overline{M}_\Gamma$ has dimension $3g(v)-3+n(v)$, so the dimension of $\overline{M}_\Gamma$ is

$$\sum_{v\in V(\Gamma)} 3g(v)-3+n(v).$$

Since $n(v)=\#H(v)$, summing over all valences just gives $\#H(\Gamma)$, but this latter quantity is $n+2\#E(\Gamma)$. Combining these identities, we get

$$\begin{aligned}\dim M^\Gamma = \dim \overline{M}_\Gamma &= \sum_{v\in V(\Gamma)} 3g(v)-3+n(v)\\ &= \sum_{v\in V(\Gamma)} 3g(v)-3\#V(\Gamma)+2\#E(\Gamma)+n\\ &= \sum_{v\in V(\Gamma)} 3g(v)+3\#E(\Gamma)-3\#V(\Gamma)+n-\#E(\Gamma)\\ &= 3g-3+n-\#E(\Gamma)=\dim \overline{M}_{g,n}-\#E(\Gamma).\end{aligned}$$

□

Corollary III.11. *Let $g,n\geq 0$ such that $2g-2+n>0$. The boundary $\partial\overline{M}_{g,n}:=\overline{M}_{g,n}\setminus M_{g,n}$ is a Weil divisor.*

Proof. Let $C\in\partial\overline{M}_{g,n}$ and let $w\colon \tilde{C}\to C$ be its normalization. Let $q\in C$ be a choice of node and let $q',q''\in w^{-1}(q)$. Let C',C'' be the curves obtained by gluing all points except q',q''. This gives us a partial normalization $\hat{w}\colon \hat{C}=C'\sqcup C''\to C$, allowing us to write C as the image of a gluing morphism

$$\overline{M}_{g-i,n-k+1}\times\overline{M}_{i,k+1}\to\overline{M}_{g,n}$$

given by a stable graph with one edge. It is also possible for q' and q'' to be on the same component, in which case C is the image of a curve under the gluing morphism

$$\overline{M}_{g-1,n+2}\to\overline{M}_{g,n}.$$

On the other hand, any stable graph with more than one edge is given by a curve with at least one node. Hence, we have an equality

$$\partial\overline{M}_{g,n}=\bigcup_{\#E(\Gamma)=1}\xi_\Gamma(\overline{M}_\Gamma)=\bigcup_{\#E(\Gamma)=1}\overline{M}^\Gamma.$$

By the preceding corollary, any $\overline{M}^\Gamma$ has codimension 1 in $\overline{M}_{g,n}$, completing the proof. □

Chapter IV

COHOMOLOGY AND DUALITY

In this section, we introduce sheaf cohomology with compact support and highlight some of its important properties. We follow [38]. In this section, X is variety over a k. That is, a geometrically reduced and irreducible scheme, separated and of finite type over a field k and $\mathcal{F}$ a sheaf of abelian groups on X. Note that the cohomology discussed in [38] is based on a treatment of étale cohomology, but as noted in the introduction of Chapter 3, and indeed in [44, Tag 03DW], sheaf cohomology agrees with étale cohomology for quasi-coherent sheaves. We will therefore omit the étale cohomology from our notation.

1 Cohomology with Compact Support

Definition IV.1. Let $f\colon X \to Y$ be a morphism of varieties. The *direct image with compact support* is the functor

$$f_!\colon \operatorname{Sh}(X) \to \operatorname{Sh}(Y)$$

sending the sheaf $\mathcal{F}$ to the sheaf $f_! \mathcal{F}$ given by

$$U \mapsto \{s \in \mathcal{F}(f^{-1}U)\colon f\big|_{\operatorname{supp}(s)}\colon \operatorname{supp}(s) \to U \text{ is proper}\}.$$

This defines $f_! \mathcal{F}$ as a subsheaf of the direct image sheaf $f_* \mathcal{F}$. Whenever f is the embedding of X as an open subvariety of Y, this sheaf is just extension by zero.

Definition IV.2. The *group of sections with compact support* of $\mathcal{F}$ is defined as

$$\Gamma_c(X, \mathcal{F}) = \bigcup \ker(\Gamma(X, \mathcal{F}) \to \Gamma(X \setminus Z, \mathcal{F}))$$

where Z runs through the complete subvarieties of X.

Since the global sections functor is exact, so is the global sections functor $\Gamma_c(X, -)$. However, we do not define the cohomology groups of $\mathcal{F}$ with compact support as the right derived functors of $\Gamma_c(X, -)$. Instead, note that by Nagata's Compactification Theorem, there exists an embedding $j\colon X \to \overline{X}$ of X into a complete subvariety $\overline{X}$ as an open subvariety. This leads naturally to the following definition:

Definition IV.3. Let $j\colon X \to \overline{X}$ be the open embedding of X in a complete subvariety $\overline{X}$. Then the p-th *cohomology with compact support* of $\mathcal{F}$ is

$$H^p_c(X, \mathcal{F}) := H^p(\overline{X}, j_!\,\mathcal{F}).$$

Proposition IV.4. *Let $\mathcal{F}$, $j\colon X \to \overline{X}$ be as above. Then*

1. $H^0_c(X, \mathcal{F}) = \Gamma_c(X, \mathcal{F})$.
2. *The functors $H^p_c(X, -)$ functorially associate long exact sequences of abelian groups to short exact sequences of sheaves.*

Proof. See [38, III-Proposition 1.29]. □

Remark. Note that in the definition of cohomology with compact support, we have made a choice of embedding $j\colon X \to \overline{X}$, but a different choice of embedding does not change the cohomology groups $H^p_c(X, \mathcal{F})$. See [38, VI-Proposition 3.1].

One of the advantages of working with cohomology with compact support is that it has more than one canonical long exact sequence associated to it.

Proposition IV.5. *Let $Z \subseteq X$ be a closed subscheme. Then there exists a long exact sequence of cohomology with compact support*

$$\cdots \to H^p_c(X \setminus Z, \mathcal{F}) \to H^p_c(X, \mathcal{F}) \to H^p_c(Z, \mathcal{F}) \to H^{p+1}_c(X \setminus Z, \mathcal{F}) \to \cdots$$

Proof. As $Z \subseteq X$ is a closed subscheme, we have the closed immersion $i\colon Z \to X$ and the open immersion $k\colon X \setminus Z \to X$ fitting in the short exact sequence of sheaves

$$0 \to k_!\mathcal{F}\big|_{X\setminus Z} \to \mathcal{F} \to i_*\mathcal{F}\big|_Z \to 0.$$

Indeed, $k_!\mathcal{F}|_{X\setminus Z} \to \mathcal{F}$ is just the inclusion of the sheaf $\mathcal{F}|_{X\setminus Z}$ extended by zero, which is clearly injective. The map $\mathcal{F} \to i_*\mathcal{F}|_Z$ is the map restricting $\mathcal{F}$ to Z, which is clearly surjective. We are left with showing exactness at $\mathcal{F}$. Let $s \in \mathcal{F}(U)$ be in the image of $k_!\mathcal{F}|_{X\setminus Z} \to \mathcal{F}$, then it is zero on Z by construction. On the other hand, let $t \in \mathcal{F}(U)$ be a nonzero section which maps to zero under restriction to Z. This must mean it is nonzero on $X \setminus Z$ and can be given as a section on $k_!F|_{X\setminus Z}(U) = \mathcal{F}(X \setminus Z \cap U)$ extended by zero. Let $j\colon X \to \overline{X}$ be the open embedding of X into a complete variety. Now this embeds $j \circ i\colon Z \to \overline{X}$ into $\overline{Z} \subseteq \overline{X}$ and

$j \circ k\colon X \setminus Z \to \overline{X}$ as $\overline{X} \setminus \overline{Z}$. The functor $j_!$ is exact by [38, II-Proposition 3.13] and therefore we obtain

$$0 \to j_! k_! \mathcal{F}\big|_{X \setminus Z} \to j_!\, \mathcal{F} \to j_! i_* \mathcal{F}\big|_{Z} \to 0,$$

hence there is a long exact sequence

$$\cdots \to H^p(\overline{X} \setminus \overline{Z}, j_! k_! \mathcal{F}\big|_{X \setminus Z}) \to H^p(\overline{X}, j_!\, \mathcal{F}) \to H^p(Z, j_! i_* \mathcal{F}\big|_{Z}) \to \cdots$$

yielding the desired long exact sequence of cohomology with compact support. □

Definition IV.6. We define the *numerical Euler characteristic with compact support* of X and $\mathcal{F}$ as

$$e_c(X, \mathcal{F}) := \sum_{i \geq 0} (-1)^i \dim H^i_c(X, \mathcal{F}).$$

The long exact sequence above in particular implies that the Euler characteristic with compact support is additive:

Corollary IV.7. *Let $U \subseteq X$ be an open subscheme and Z its complement. Then*

$$e_c(X, \mathcal{F}) = e_c(U, \mathcal{F}) + e_c(Z, \mathcal{F}).$$

Proof. This is a direct consequence of the existence of the long exact sequence

$$\cdots \to H^p_c(U, \mathcal{F}) \to H^p_c(X, \mathcal{F}) \to H^p_c(Z, \mathcal{F}) \to \cdots$$

□

Corollary IV.8 ([3])**.** *Let X be a quasi-projective variety and let*

$$X = \overline{X}_d \supset \overline{X}_{d-1} \supset \cdots \supset \overline{X}_0 \not\supset \emptyset$$

a filtration by closed subvarieties such that $X_i = \overline{X}_i \setminus \overline{X}_{i-1}$ is empty or of pure dimension i. Then

$$e_c(X, \mathcal{F}) = \sum_{i \leq d} e_c(X_i, \mathcal{F}).$$

Proof. We will do induction on d. If $d = 0$, then the statement is trivially true. Now assume the statement holds for a filtration by $d-1$ subvarieties. Then

$$e_c(\overline{X}_{d-1}, \mathcal{F}) = \sum_{i<d} e_c(X_i, \mathcal{F}).$$

Furthermore, we can write $X = \overline{X}_d = X_d \cup \overline{X}_{d-1}$ since $X_d = \overline{X}_d \setminus \overline{X}_{d-1}$. We therefore have a long exact sequence of cohomology with compact support:

$$\cdots \to H^i_c(X_d, \mathcal{F}) \to H^i_c(X, \mathcal{F}) \to H^i_c(\overline{X}_{d-1}, \mathcal{F}) \to \ldots$$

and hence $e_c(X, \mathcal{F}) = e_c(X_d, \mathcal{F}) + e_c(\overline{X}_{d-1}, \mathcal{F}) = e_c(X_d, \mathcal{F}) + \sum_{i<d} e_c(X_i, \mathcal{F})$. □

As an example, we can use this filtration method to compute the numerical Euler characteristics of the spaces $M_{0,n}$ for some low values of n. By a slight abuse of notation, let $\mathbb{Q}$ be the constant sheaf with value $\mathbb{Q}$.

Proposition IV.9. *$e_c(M_{0,3}, \mathbb{Q}) = 1$, $e_c(M_{0,4}, \mathbb{Q}) = -1$ and $e_c(M_{0,5}, \mathbb{Q}) = 2$.*

Proof. Since three points fix a rational curve, $M_{0,3}$ is just a point and $e_c(\{*\}, \mathbb{Q}) = 1$. We can identify $M_{0,4}$ with the complex plane without two points, so it directly follows that

$$e_c(M_{0,4}, \mathbb{Q}) = e_c(\mathbb{C}^1, \mathbb{Q}) - e_c(\{0, 1\}, \mathbb{Q}) = 1 - 2 = -1.$$

The space $M_{0,5}$ can be identified with the complement of the lines $x_1 = 0, x_1 = 1, x_2 = 0, x_2 = 1, x_1 = x_2$ in $\mathbb{C}^2$. We can see this as the complement of the six projective lines in $\mathbb{P}^2$ with homogeneous coordinates x_0, x_1, x_2:

$$M_{0,5} \cong \mathbb{P}^2 \setminus \{x_0 = 0, x_1 = 0, x_2 = 0, x_0 = x_1, x_0 = x_2, x_1 = x_2\}.$$

Out of the six lines $L_1, \ldots, L_6$ that are omitted, two or more meet in seven distinct points $P_1, \ldots, P_7$. We therefore obtain a filtration

$$\mathbb{P}^2 \supset \bigcup_{i \leq 6} L_i \supset \bigcup_{j \leq 7} P_j$$

such that $\mathbb{P}^2 \setminus \bigcup_{i \leq 6} L_i \cong M_{0,5}$ and for each L_i such that $P_k, P_l, P_m \in L_i$ we have $L_i \setminus \{P_k, P_l, P_m\} \cong M_{0,4}$. Therefore

$$e_c(M_{0,5}, \mathbb{Q}) + 6e_c(M_{0,4}, \mathbb{Q}) + 7 = e_c(\mathbb{P}^2, \mathbb{Q}).$$

Using the fact that $e_c(\mathbb{P}^2, \mathbb{Q}) = 3$ and the results above, we conclude that $e_c(M_{0,5}, \mathbb{Q}) = 2$. □

2 ℓ-adic Cohomology

As we will see later, $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are smooth stacks, and therefore they satisfy a form of Poincaré Duality.

Definition IV.10. Let X be a scheme of finite type over an algebraically closed field k. Let $\mathbb{Z}_\ell$ be the ring of ℓ-adic integers and $\mathbb{Q}_\ell$ its quotient field. We define the ℓ-adic cohomology for a prime $\ell \neq \operatorname{char} k$ by

$$H^i_c(X, \mathbb{Q}_\ell) = \varprojlim_r (H^i_c(X, \mathbb{Z} / \ell^r \mathbb{Z})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

This definition is given in [32, Appendix C] for regular sheaf cohomology, but works in the same way for cohomology with compact support. More generally, we can define a form of ℓ-adic sheaf cohomology.

Definition IV.11. [38, V-1] Let ℓ be a prime number. An ℓ-adic sheaf on X is given by an inverse system of sheaves $F = (F_n)_{n\in\mathbb{N}}$ such that for any n, the given map $F_{n+1} \to F_n$ is isomorphic to the canonical map $F_{n+1} \to F_{n+1} \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z}$, that is, $F_{n+1} \to F_n$ induces an isomorphism $F_{n+1}/\ell^n F_{n+1} \to F_n$. The ℓ-adic cohomology groups are then defined as

$$H^r(X, F) := \varprojlim_n H^r(X, F_n).$$

Choosing an open embedding $X \to \overline{X}$, we can define cohomology with compact support as before via an open embedding $j\colon X \to \overline{X}$. If $X \to S$ is a separated morphism of finite type, define the higher direct image sheaf

$$R^q f_!(F) := R^q g_*(j_! F)$$

where $g\colon \overline{X} \to S$ is a proper morphism such that $g \circ j = f$. If F is a constructible ℓ-adic sheaf, we can make it a $\mathbb{Q}_\ell$-sheaf by setting $F' := F \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. We then set

$$H^r(X, F') := H^r(X, F) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

Let Λ denote the constant sheaf $\mathbb{Z}/(n)$, where n is coprime to the characteristic of our base field k. Then we can express a form of Poincaré Duality for the cohomology with compact support:

Proposition IV.12. *Let X be a smooth scheme of finite type of dimension d over a separably closed field k. Let F be a constructible sheaf of Λ-modules on X. Then there is a perfect pairing*

$$H^r(X, F) \times H_c^{2d-r}(X, F^\vee(d)) \to H_c^{2d}(X, \Lambda(d)) \cong \Lambda,$$

where $F^\vee(d) = \mathrm{Hom}(F, \mu_n^{\otimes d})$ denotes the d-th Tate-twist of the dual of F by μ_n, the group of roots of unity.

Proof. See [18, VI-3]. □

Corollary IV.13. *Let X be a smooth scheme of finite type of dimension d over a separably closed field k and let ℓ a prime not equal to the characteristic p of k. Then for an ℓ-adic sheaf $F = (F_n)_{n\in\mathbb{N}}$, there is a perfect pairing*

$$H^r(X, F) \times H_c^{2d-r}(X, F^\vee(d)) \to \mathbb{Q}_\ell\,.$$

Proof. Consider the perfect pairings for the sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules F_n and take the inverse limit:

$$\varprojlim(H^r(X, F_n) \times \varprojlim(H_c^{2d-r}(X, F_n^\vee(d)) \to \varprojlim(\mathbb{Z}/\ell^i\mathbb{Z}) \cong \mathbb{Z}_\ell\,.$$

Now tensoring by $\mathbb{Q}_\ell$ yields the desired formula. □

Let $G := \mathrm{Gal}(\overline{k}/k)$, then the cohomology groups $H_c^i(X, F^\vee(d))$ are ℓ-adic Galois representations of G. Identifying $\mathbb{Q}_\ell$ with $\mathbb{Q}_\ell(1)$, the ℓ-adic cyclotomic character, we have the isomorphism of ℓ-adic Galois representations

$$H_c^i(X, F)^\vee(d) \cong H_c^{2d-i}(X, F),$$

where $H_c^i(X, F)^\vee(d)$ denotes the d-th Tate twist.

i Lefschetz Fixed-Point Formula

Let X be a smooth scheme of finite type and dimension d that is defined over a finite field $k = \mathbb{F}_q$, and let $\overline{X} = X \times_k \overline{k}$ be the extension to the algebraic closure. We can count the number of k-points on X using étale cohomology. The following is taken from [32, Appendix C].

Definition IV.14. Let X and k be as above. We define the *Frobenius morphism* $f \colon \overline{X} \to \overline{X}$ via the morphism of rings

$$\begin{aligned} f^\# \colon \mathcal{O}_{\overline{X}} &\to \mathcal{O}_{\overline{X}} \\ x &\mapsto x^q. \end{aligned}$$

Furthermore, a point P of X is an $\mathbb{F}_{q^r}$-point if and only if it is fixed under the Frobenius morphism f^r. Denote $N_r :=$ Number of fixed points of f^r.

Proposition IV.15. *Let X and k be as above. Then*

$$N_r = \sum_{i=0}^{d} (-i)^i \operatorname{Tr}((f^r)^*; H^i(\overline{X}, \mathbb{Q}_\ell)).$$

This is known as the Lefschetz Fixed-Point Formula.

In [4], Behrend generalizes this formula to the category of algebraic stacks. We discuss this in a later section, but provisionally take the point of view of schemes.

INTERLUDE

In the first chapters we have introduced the moduli functors $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$; the coarse moduli spaces $M_{g,n}$ and $\overline{M}_{g,n}$ almost representing them and determined some basic facts about their structure. These have been interesting objects of study, but do not satisfy certain properties. Crucially, these schemes are not smooth in general, and do not admit a universal family $\pi\colon M_{g,n+1} \to M_{g,n}$, even though we pretend like they do.

In the coming chapters, we will introduce some objects and techniques which require our spaces to be smooth and have a universal family over them. To be able to apply this to our study of the moduli space of curves, we will follow the example of many other authors in this field and promote to the moduli stacks $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$.
These are smooth (in the appropriate sense) Deligne Mumford (DM) stacks, and come with projection morphisms

$$\begin{aligned} \overline{\rho}\colon \overline{\mathcal{M}}_{g,n} &\to \overline{M}_{g,n} \\ \rho\colon \mathcal{M}_{g,n} &\to M_{g,n}, \end{aligned}$$

allowing us to translate results about the moduli stacks to the coarse moduli spaces. These latter objects are also coarse moduli spaces in the sense of stacks: every morphism from the stack to a scheme must factor through the coarse moduli space via the projection morphism, and this map induces a bijection on geometric points.

All objects used henceforth, like sheaves and cohomology, will need to be redefined for stacks. This is needed for an appropriate justification, but we omit these details here, for they will not be of much use. Indeed, even the definition of a Deligne Mumford stack is outside of the scope of this thesis. Whenever needed, we will highlight the difference between moduli stacks and their coarse spaces, suitably adapting our theory, how unsatisfactory it may be. For more details on the moduli stacks $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ and why their use is justified, we refer to [31, 3D] and [**?**], in particular section 12.5.

Chapter V

LOCAL SYSTEMS

In this section, we introduce the sheaves ${}_\ell\mathbb{V}$, which we will use in the context of sheaf cohomology on the moduli space. These sheaves arise naturally in the sense that they only remember the most important information about the cohomology of a curve. Following [32, Appendix C], we see that the trace of Frobenius on the cohomology of a curve C is most interesting on its first cohomology $H^1(C;\mathbb{Q}_\ell)$. We would like to use this local information to describe the moduli space globally. The sheaves ${}_\ell\mathbb{V}$ turn out to be the right tool for this: the trace of Frobenius on the cohomology of $\mathcal{M}_g$ can then be described using the stalks of ${}_\ell\mathbb{V}$, which are given by $H^1(C;\mathbb{Q}_\ell)$.

Definition V.1. [37] Let X be a topological space. A local system on X is a sheaf of abelian groups $\mathcal{L}$ such that for every $x \in X$ there exists an open neighbourhood $U \subseteq X$ such that $\mathcal{L}|_U$ is isomorphic to a constant sheaf.

If $\mathcal{F}$ is a sheaf of abelian groups on a scheme X and $f\colon X \to Y$ is a morphism of schemes, then we can associate to Y the direct image sheaves $R^i f_*$, which allows us to study the cohomology of X relative to Y. Let $\mathcal{A}(X)$ be the category of sheaves of abelian groups on a scheme X.

Definition V.2. [32, III-8] Let $f\colon X \to Y$ be a continuous map of topological spaces. We define the *higher direct image* functors $R^i f_*\colon \mathcal{A}(X) \to \mathcal{A}(Y)$ to be the right derived functors of the direct image functor f_*.
For each $i \geq 0$ and each $\mathcal{F} \in \mathcal{A}(X)$, $R^i f_*(\mathcal{F})$ is the sheaf associated to the presheaf

$$V \mapsto H^i(f^{-1}V, \mathcal{F}|_{f^{-1}V}).$$

We will apply this construction to the forgetful morphism $\mathcal{M}_{g,1} \to \mathcal{M}_g$ for $g \geq 2$ and $\mathcal{M}_{1,2} \to \mathcal{M}_{1,1}$ for $g = 1$ since elliptic curves require a choice of basepoint.

Definition V.3. Let $\mathbb{Q}$ be the constant sheaf with coefficients in $\mathbb{Q}$ on $\mathcal{M}_{g,1}$ and $\mathcal{M}_{1,2}$ and $\mathbb{Q}_\ell$ the constant sheaf with coefficients in $\mathbb{Q}_\ell$ on $\mathcal{M}_{g,1} \otimes \mathbb{Z}[1/\ell]$ and $\mathcal{M}_{1,1} \otimes \mathbb{Z}[1/\ell]$. Then set

$$\begin{aligned} \mathbb{V} &:= R^1\pi_{*]}\,\mathbb{Q} \\ {}_\ell\mathbb{V} &:= R^1\pi_*\,\mathbb{Q}_\ell\,. \end{aligned}$$

In order to properly define the local system ${}_\ell\mathbb{V}$, we need to define the higher direct image sheaves $R^i\pi_*(\mathcal{F})$ for $\mathcal{F}$ an ℓ-adic sheaf. For details on this construction, see [38]. Here, we moreover find a proof that higher direct images of constructible sheaves are again constructible, which in this case is enough to conclude that the sheaves $\mathbb{V}$ and ${}_\ell\mathbb{V}$ are indeed local systems on $\mathcal{M}_g$ or $\mathcal{M}_{1,1}$.
The sheaves $\mathbb{V}$ and ${}_\ell\mathbb{V}$ are natural objects of interest since they have well-behaved stalks, as is illustrated by the following proposition:

Proposition V.4. *Let* $[C] \in \mathcal{M}_g$*, then* $\mathbb{V}_{[C]} \cong H^1(\mathcal{C}_C, \mathbb{Q})$.

Proof. Let $k([C])$ be the residue field at $[C] \in \mathcal{M}_g$ over the field k. Consider the scheme-theoretical fiber $\mathcal{C}_C$ of the universal family $\pi\colon \mathcal{C}_g \to \mathcal{M}_g$ given by the cartesian square

$$\begin{array}{ccc} \mathcal{C}_C = \mathcal{C}_g \times_{\mathcal{M}_g} \operatorname{Spec} k([C]) & \xrightarrow{p_2} & \operatorname{Spec} k([C]) \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ \mathcal{C}_g & \xrightarrow[\pi]{\qquad\qquad} & \mathcal{M}_g \end{array}$$

where $s \in \operatorname{Spec} k([C]) \to \mathcal{M}_g$ has image $[C]$. We have that the local ring of $\operatorname{Spec} k([C])$ at its unique point is $k([C])$, while the local ring A of $\mathcal{M}_g$ at $[C]$ satisfies $Q(A) = k([C])$. The morphism s is therefore flat, and we can apply [32, III] Proposition 9.3 to the constant sheaf $\mathbb{Q}$ on $\mathcal{C}_g$. We get that

$$s^*R^1\pi_*\mathbb{Q} \cong R^1(p_2)_*(p_1^*\mathbb{Q}).$$

The left hand side of this isomorphism is just the stalk of $\mathbb{V}$ at $[C]$. On the other hand, the pullback of a constant sheaf is constant. Hence, the right hand side is the sheafification of the presheaf

$$\operatorname{Spec} k([C]) \supset U \mapsto H^1(p_2^{-1}(U), \mathbb{Q}),$$

but since p_2 is a constant map on the level of spaces, its preimage on a nonempty open is just $\mathcal{C}_C$. □

In [38], we find an ℓ-adic equivalent of [32, III] Proposition 9.3 for ℓ-adic sheaves. In particular, the proof above extends to the local system ${}_\ell\mathbb{V}$. That is, for $[C] \in \mathcal{M}_g$,

$${}_\ell\mathbb{V}_{[C]} \cong H^1(\mathcal{C}_C; \mathbb{Q}_\ell).$$

Note that the choice of local system ${}_\ell\mathbb{V}$ or $\mathbb{V}$ is based on the field over which $\mathcal{M}_g$ is defined. We could choose $k = \mathbb{C}$ or $k = \overline{\mathbb{F}}_p$ for any $p \neq \ell$. The local systems ${}_\ell\mathbb{V}$ are shown to commute with base-change and are therefore independent of this choice.

Proposition V.5. *Let $\rho\colon \mathcal{M}_g \to \operatorname{Spec}\mathbb{Z}[1/\ell]$ be the structure map. If $\mathbb{V}$ is a locally constant system on $\mathcal{M}_g$, then $R^i\rho_!\,\mathbb{V}$ is a locally constant system over $\operatorname{Spec}\mathbb{Z}[1/\ell]$ that commutes with base change.*

Furthermore, local systems occur quite naturally in topology. If $p\colon \tilde{X} \to X$ is a universal covering space of a topological space X with fundamental group G, then representations of G determine local systems on X in the context described above. In fact, if X is a locally path connected paracompact Hausdorff space, there is a one-to-one correspondence between locally constant sheaves on X and representations of its fundamental group. See [43, Chapter 6]. Given a representation V of G, we equip it with the discrete topology and consider the product

$$\tilde{X} \times V/G,$$

where we quotient out by the relations $(xg, v) = (x, gv)$. Then consider the projection map

$$\tilde{X} \times V/G \to \tilde{X}/G \cong X.$$

We can then define a local system $\mathbb{V}$ corresponding to the representation V by taking the space of continuous sections of this projection. For any $x \in \tilde{X}$, we can find a neighbourhood U such that $gU \cap U = \emptyset$ for any $g \in G$ nontrivial, since $\tilde{X}$ is the universal cover of X. Then on this subset U, all fibers are isomorphic to V, and hence $\mathbb{V}|_U$ is just the constant sheaf associated to V.

In Appendix B, we define the Schur functors assigning to the standard representation S of $\mathrm{Sp}(2g, \mathbb{C})$ an irreducible representation $\mathbb{S}_{\langle\lambda\rangle}\, S$ indexed by a partition λ of n into g parts. To extend these constructions to $\mathrm{GSp}(2g, \mathbb{C})$, let $W(\cong \mathbb{C}^{2g})$ be the standard representation of $\mathrm{GSp}(2g, \mathbb{C})$ and set $V := \eta^{\otimes -1} \otimes W$ [22, p. 224]. Now for a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_g \geq 0)$ of n, we can define an irreducible representation V_λ, occurring with highest weight in

$$\mathrm{Sym}^{\lambda_1-\lambda_2}(\wedge^1 V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{g-1}-\lambda_g}(\wedge^{g-1} V) \otimes \mathrm{Sym}^{\lambda_g}(\wedge^g V),$$

which is the image of the Young symmetrizer on $V^{\langle n\rangle}$. In this case, however, the highest weight is given by $(\lambda_1 - \lambda_2)\gamma_1 + \cdots + \lambda_g\gamma_g - |\lambda|\eta$ for γ_i suitable fundamental roots. This construction does not yield all irreducible representations of $\mathrm{GSp}(2g, \mathbb{C})$, however, since we can still tensor with η.

The local system $\mathbb{V}$ under this construction will now correspond to the representation V of $\mathrm{GSp}(2g, \mathbb{C})$ given above. By applying a Schur functor we then obtain the local system $\mathbb{V}_\lambda$ with highest weight $(\lambda_1 - \lambda_2)\gamma_1 + \cdots + \lambda_g\gamma_g - |\lambda|\eta$.

We obtain other irreducible representations by tensoring V_λ with η^k for $k \in \mathbb{Z}$. As noted in [22, p. 238], these will then correspond to Tate-twists of $\mathbb{V}_\lambda$. These are the local systems $\mathbb{V}_\lambda(k)$. For example, [9] if $g = 2$,

$$\wedge^2 V \cong V_{1,1} \oplus V_{0,0} \otimes \eta^{-1}$$

and hence

$$\wedge^2\, \mathbb{V} \cong \mathbb{V}_{1,1} \oplus \mathbb{V}_{0,0}(-1).$$

i The Moduli Space of Abelian Varieties

We have seen that the local systems $\mathbb{V}_\lambda$ arise naturally via representations of the symplectic group, and can be expressed as a tensor product of the local system $\mathbb{V}$ whose stalks are given by cohomology groups $H^1(C;\mathbb{Q})$. To see why these local systems occur naturally on $\mathcal{M}_g$, we will need to go through the moduli space of principally polarized abelian varieties of dimension g, commonly denoted $\mathcal{A}_g$. Since a proper treatment of this space is outside the scope of this thesis, we will restrict ourselves to a simple overview of the most relevant facts, summarized in the following theorem. For details and proofs of these statements, see [47] and [22].

Theorem V.6. *There exists a Deligne-Mumford stack $\mathcal{A}_g$, which is smooth over* $\operatorname{Spec}\mathbb{Z}$ *and of relative dimension $g(g+1)/2$. Over the complex numbers, this space parametrizes all isomorphism classes of principally polarized abelian varieties of dimension g, and can be identified as the arithmetic quotient*

$$[\mathbb{H}_g/\operatorname{Sp}(2g,\mathbb{Z})],$$

where $\mathbb{H}_g$ denotes the Siegel upper half space. See [47].

Note that this generalizes the construction of $\mathcal{M}_{1,1}$ as the arithmetic quotient $[\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})]$. Indeed, for $g=1$, we would expect an isomorphism $\mathcal{M}_{1,1}\to A_1$, since abelian varieties of dimension 1 are simply elliptic curves. More generally, there is a morphism

$$\begin{aligned} t_g\colon \mathcal{M}_g &\to \mathcal{A}_g \\ [C] &\mapsto [J(C)], \end{aligned}$$

sending a curve to its jacobian. See [32, Chapter 4]. As a consequence of the Torelli Theorem, this map is injective on coarse moduli spaces, and we often call it the Torelli morphism.
For $g=2$, it is an open dense embedding, but for $g\geq 3$, $t_g\colon \mathcal{M}_g\to\mathcal{A}_g$ has degree 2 as a morphism of stacks, since any jacobian variety admits an automorphism of order 2, while generally, curves of genus $g\geq 3$ do not.

The space $\mathcal{A}_g$ also comes with a universal family

$$\pi\colon \Xi_g\to\mathcal{A}_g,$$

which allows us to define the local system ${}_\ell\mathbb{V}' := R^1\pi_*\,\mathbb{Q}_\ell$ on $\mathcal{A}_g[1/\ell]$ or $\mathbb{V}' := R^1\pi_*\,\mathbb{Q}$ on $\mathcal{A}_g(\mathbb{C})$ like we did for $\mathcal{M}_g$. These local systems then come with a symplectic pairing

$${}_\ell\mathbb{V}\times{}_\ell\mathbb{V}\to\mathbb{Q}_\ell(-1).$$

As is then noted in [22, p. 238], local systems on $\mathcal{A}_g$ correspond to representations of the group $\mathrm{GSp}(2g,\mathbb{Z})$, defining the local systems $\mathbb{V}'_\lambda$ as before. As is noted by various authors [26] [11], the pullback $t_g^*\,\mathbb{V}'_\lambda$ to $\mathcal{M}_g$ is just the local system $\mathbb{V}_\lambda$ defined previously, so we will simply write $\mathbb{V}_\lambda$ instead of $\mathbb{V}'_\lambda$ to indicate a local system on either $\mathcal{M}_g$ or $\mathcal{A}_g$.

Chapter VI

EQUIVARIANT EULER CHARACTERISTICS

In a previous chapter we have discussed the stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type, expressing the disjoint strata $\mathcal{M}^\Gamma$ as a product of certain moduli spaces $\mathcal{M}_{h,m}$ for $h \leq g$, $m \leq n$ and how these appear in the computation of the numerical Euler characteristic. In their paper *Modular Operads*, Getzler and Kapranov [28] generalize this procedure and give a generating function for the $\mathbb{S}_n$ equivariant Ruler characteristic of $\overline{\mathcal{M}}_{g,n}$. In this chapter, following [11], we will introduce some of their results and define the $\mathbb{S}_n$-equivariant Euler characteristic of $\mathcal{M}_{g,n}$ taking values in different Grothendieck groups. We first summarize some important information about counting points on stacks.

1 Point Counts on Stacks

To adjust our point counts for the context of stacks, we need to include the information on the automorphism groups of all $\mathbb{F}_q$-points. We take the following definition from [8].

Definition VI.1. If $\mathcal{X}$ is an algebraic stack defined over $\mathbb{Z}$ so that the reduction to a finite field $\mathbb{F}_q$ with q elements is defined, then $\mathcal{X}(\mathbb{F}_q)$ is a finite groupoid and by definition

$$\#\mathcal{X}(\mathbb{F}_q) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)} \frac{1}{\# \operatorname{Aut}_{\mathbb{F}_q} x}.$$

Proposition VI.2 ([5] [4])**.** *Let $\mathcal{X}$ be an algebraic stack.*

1. *If $\mathcal{X}$ is stratified by locally closed substacks $\mathcal{X} = \mathcal{X}_0 \sqcup \cdots \sqcup \mathcal{X}_n$, then*

$$\#\mathcal{X}(\mathbb{F}_p) = \#\mathcal{X}_0(\mathbb{F}_p) + \cdots + \#\mathcal{X}_n(\mathbb{F}_p).$$

2. If $\mathcal{X}$ is a quotient $[X/G]$ for X a scheme and G a connected linear algebraic group, then

$$\#\mathcal{X}(\mathbb{F}_q) = \#X(\mathbb{F}_q)/\#G(\mathbb{F}_q).$$

3. If $\mathcal{X}$ is a DM stack with course moduli space X, then $\#\mathcal{X}(\mathbb{F}_q) = \#X(\mathbb{F}_q)$.

4. If $\mathcal{X}$ is a DM stack over $\mathbb{F}_q$ and l is prime to q, then

$$\#\mathcal{X}(\mathbb{F}_q) = \mathrm{Tr}(F_q, H_c^\bullet(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell))$$

where F_q denotes the geometric Frobenius endomorphism and Tr *the graded trace.*

The final identity is commonly known as Behrend's Trace Formula, generalizing the Grothendieck Trace Formula to stacks. In fact, we can also express a more general version of this same statement using ℓ-adic sheaf cohomology.

Proposition VI.3 ([24])**.** *Let $\mathcal{X}$ be a smooth DM stack of dimension n over $\mathbb{F}_q$ of finite type and $\mathcal{F}$ a constructible $\mathbb{Q}_\ell$-sheaf for $\ell \neq \mathrm{char}\,\mathbb{F}_q$. Then*

$$\sum_{x\in\mathcal{X}(\mathbb{F}_q)} \mathrm{Tr}(F_q, \mathcal{F}_x) = \sum_{i=0}^{2n}(-1)^i \,\mathrm{Tr}(F_q, H_c^i(\mathcal{X}; \mathcal{F}))$$

Indeed, the stacks we are interested in satisfy all of these properties. The following theorem justifies much of what we do.

Theorem VI.4 ([44, Section 0E9C])**.** *Let $g \geq 2$. The algebraic stack $\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack, proper and smooth over* Spec $\mathbb{Z}$ *of pure relative dimension. Moreover, the locus $\mathcal{M}_g$ parametrizing smooth curves is a dense open substack.*

In the proposition above, we can see that counting points over $\mathbb{F}_q$ can be done by taking traces of the Frobenius over the ℓ-adic cohomology of $\mathcal{X}_{\overline{\mathbb{F}}_q}$. The following two Lemmas (3.1 and 3.2 from [46]) show that we can also work over $\mathbb{Q}_p$, and that for our purposes, the coarse moduli space of a DM stack provides enough information to do work in the context of schemes, given the information above.

Lemma VI.5. *Let $\mathcal{X}$ be a Deligne-Mumford stack which is smooth and proper over $\mathbb{Z}_p$. For every prime $\ell \neq p$ and every $i \geq 0$, the canonical map of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$*-representations*

$$H_c^i(\mathcal{X}_{\overline{\mathbb{F}}_p}; \mathbb{Q}_\ell) \to H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}_p}; \mathbb{Q}_\ell)$$

is an isomorphism. In particular, $H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}_p}; \mathbb{Q}_\ell)$ is unramified.

Lemma VI.6. *Let $\mathcal{X}$ be a separated DM stack of finite type over an algebraically closed field of characteristic* 0*. Let X denote its coarse moduli space and $q\colon \mathcal{X} \to X$ the canonical projection. Then the pullback maps*

$$q^*\colon H_c^i(X, \mathbb{Q}_\ell) \to H_c^i(\mathcal{X}, \mathbb{Q}_\ell)$$

are isomorphisms for all $i \geq 0$.

The proofs of these lemmas are in [46], in our case adapted to cohomology with compact support by Poincaré duality.

i Grothendieck Groups and the Lefschetz Motive

To define an Euler characteristic which is the sum of cohomology groups, we first need to find a space where these sums are well-defined. We can take the direct sum of two cohomology groups as mixed Hodge structures for example, defining a commutative monoid of mixed Hodge structures. This can then be turned into an abelian group, called the Grothendieck group.

We take the following definition from [1].

Definition VI.7. Given an additive category $\mathcal{C}$, we can define a Grothendiek Group $K_0^s(\mathcal{C})$ by generators and relations as follows. We have one generator $[X]$ for each isomorphism class of objects $X \in \mathcal{C}$, and we impose the relation $[X_3] = [X_1] + [X_2]$ whenever $X_3 \cong X_1 \oplus X_2$. When the category $\mathcal{C}$ possesses exact sequences, we define $K_0(\mathcal{C})$ by imposing the relation $[X_3] = [X_1] + [X_2]$ whenever there is a short exact sequence

$$0 \to X_1 \to X_3 \to X_2 \to 0.$$

Following [27], let C be a symmetric monoidal category that is additive over a field of characteristic 0 and has finite colimits. The exact definition of such a category can be found in [27]. For example, $\mathrm{MHS}_{\mathbb{Q}}$ is an example of such a category. We associate to C the category $C^{\mathbb{S}_n}$ whose objects are objects of C with an action of the symmetric group $\mathbb{S}_n$ by morphisms in C. Let $\mathrm{Gal}_{\mathbb{Q}}$ be the category of $\mathbb{Q}_\ell$-vector spaces equipped with the ℓ-adic topology that have a continuous action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. As noted in [11], this is a symmetric monoidal additive category over $\mathbb{Q}_\ell$ with finite colimits, which we will call the category of Galois representations. For example, if $k = \mathbb{F}_q$, $H_c^i(\mathcal{M}_{g,n} \otimes_k \bar{k}, \mathbb{Q}_\ell)$ is a Galois representation equipped and the action of $\mathbb{S}_n$ permuting the marked points of a $\mathcal{M}_{g,n}$ extends to a morpism of Galois representations by pullback.

Theorem VI.8 ([27, Theorem 4.8])**.** *There is a canonical isomorphism*

$$K_0(C^{\mathbb{S}_n}) \cong K_0(C) \otimes \Lambda_n.$$

One very important class occurring in the Grothendieck groups we will use is the *Lefschetz motive*

Definition VI.9. [11] [33] Denote by $\mathbb{Q}(-1)$ the one-dimensional Hodge Structure of motivic weight 2 [8]. This can be given by the pure Hodge structure on $H_c^2(\mathbb{P}^1, \mathbb{Q})$ and is denoted by $\mathbb{L}$ in $K_0(\mathrm{MHS}_{\mathbb{Q}})$. For any $H \in \mathrm{MHS}_{\mathbb{Q}}$, denote $H(-k) := H \otimes \mathbb{Q}(-1)^{\otimes k}$. Similary, let $\mathbb{Q}_\ell(1)$ denote the ℓ-adic cyclotomic character, that is, the Tate module of the group of roots of unity in $\mathbb{Q}_\ell^s / \mathbb{Q}_\ell$ and $\mathbb{Q}_\ell(-1)$ its dual representation. For $H \in \mathrm{Gal}$, denote $H(-k) := H \otimes \mathbb{Q}_\ell(-1)^{\otimes k}$. In $K_0(\mathrm{Gal})$, $[\mathbb{Q}_\ell(-1)] = [H_c^2(\mathbb{P}^1, \mathbb{Q}_\ell)]$. In both cases, we say $H(-k)$ is the *k-th Tate twist of* H.

2 Boundary Contributions

To determine the Euler characteristic of $\overline{\mathcal{M}}_{g,n}$ from its strata, Getzler and Kapranov [28] develop the following theory.

Definition VI.10. A stable $\mathbb{S}$-module $\mathcal{V}$ in the category C is a collection, for all $g, n \geq 0$ of chain complexes $\{\mathcal{V}((g,n))_i\}$ of objects of $C^{\mathbb{S}_n}$ such that $\mathcal{V}((g,n)) = 0$ if $2g + n - 2 \leq 0$.

Definition VI.11. Let $R := \{\mathbb{R}_i\}$ be a finite chain complex of objects of $C^{\mathbb{S}_n}$ for some $n \geq 0$. The characteristic of R is defined as

$$\mathrm{ch}_n(R) := \sum_i (-1)^i [R_i] \in K_0(C^{\mathbb{S}_n}).$$

For example, if $\mathcal{X}$ is an algebraic stack over $\mathbb{C}$, we can define a chain complex $\{C_i\}$ of objects $C_i := H^i_c(\mathcal{X}(\mathbb{C}); \mathbb{Q})$ in $\mathrm{MHS}_{\mathbb{Q}}$ by setting all differentials to be zero. Define the $\mathbb{S}_n$-*equivariant Hodge Euler characteristic* of $\mathcal{X}$ as

$$e_c^{\mathbb{S}_n}(\mathcal{X}) := \mathrm{ch}_n(H_c^\bullet(\mathcal{X}; \mathbb{Q})) \in K_0(\mathrm{MHS}_{\mathbb{Q}}) \otimes \Lambda_n.$$

Definition VI.12. Let $\mathcal{V}$ be a stable $\mathbb{S}$-module. Then the *characteristic* of $\mathcal{V}$ is defined as

$$\mathbb{C}\mathrm{h}(\mathcal{V}) := \sum_{2g+n-2>0} h^{g-1} \mathrm{ch}_n(\mathcal{V}((g,n))) \in K_0(C) \otimes \Lambda((h)),$$

where $\Lambda((h))$ is the ring of Laurent series with coefficients in Λ.

In [28, Section 6], to an $\mathbb{S}$-module $\mathcal{V}$ in the category C is associated a free modular operad $\mathbb{M}\mathcal{V}$. Considering the stable $\mathbb{S}$-module given by $\mathcal{V}((g,n)) := H_c^\bullet(\mathcal{M}_{g,n} \otimes \bar{\mathbb{Q}}; \mathbb{Q}) \in \mathrm{MHS}_{\mathbb{Q}}$, the free modular operad associated to it is given by $\mathbb{M}\mathcal{V}((g,n)) := H_c^\bullet(\overline{\mathcal{M}}_{g,n} \otimes \bar{\mathbb{Q}}; \mathbb{Q})$ (see [27]).

Definition VI.13. For every pair of non-negative integers g, n such that $2g + n - 2 > 0$, let

$$\mathcal{D}_{g,n} := \{(h,m) : 0 \leq h \leq g, \max\{0, 3-2h\} \leq m \leq 2(g-h) + n\}$$

Theorem VI.14 ([28, Theorem 8.13]). *There exists an operation on $K_0(C) \otimes \Lambda((h))$, which we will denote F, allowing us to express the relationship between the characteristics $\mathbb{C}\mathrm{h}(\mathcal{V})$ and $\mathbb{C}\mathrm{h}(\mathbb{M}\mathcal{V})$;*

$$\mathbb{C}\mathrm{h}(\mathbb{M}\mathcal{V}) = F(\mathbb{C}\mathrm{h}(\mathcal{V})).$$

In particular, to compute $\mathbb{C}\mathrm{h}(\mathbb{M}\mathcal{V}((G,N)))$, it suffices to know $\mathbb{C}\mathrm{h}(\mathcal{V}((g,n)))$ for $(g,n) \in \mathcal{D}_{G,N}$. Whenever $\mathrm{ch}_n(\mathcal{V}((g,n)))$ is an element of $\mathbb{Q}[\mathbb{L}] \otimes \Lambda$ for all $(g,n) \in \mathcal{D}_{G,N}$, so is $\mathbb{C}\mathrm{h}(\mathbb{M}\mathcal{V}((G,N)))$. In this case, the operations in F can be computed by use of Stembridge's Maple package [45] for symmetric functions.

3 Equivariant Point Counts

As mentioned above, the compactly supported ℓ-adic cohomology $H^i_c(\mathcal{M}_{g,n} \otimes \bar{k}; \mathbb{Q}_\ell)$ of $\mathcal{M}_{g,n}$ (defined over $k = \mathbb{F}_q$ or $k = \mathbb{Q}$) has the action of Frobenius $F_q \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which commutes with the action of $\mathbb{S}_n$ permuting marked points. This gives a Galois representation, or when including the $\mathbb{S}_n$-action, a $\mathrm{Gal}(\overline{k}/k) \times \mathbb{S}_n$-representation, i.e. an element of $\mathrm{Gal}^{\mathbb{S}_n}_{\mathbb{Q}}$. We can therefore consider an Euler characteristic of $\mathcal{M}_{g,n}$ taking values in the Grothendieck group $K_0(\mathrm{Gal}^{\mathbb{S}_n}_{\mathbb{Q}})$.

Definition VI.15. Suppose $\mathcal{X}$ is a space defined over k with an action of $\mathbb{S}_n$ commuting with the action of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. Since any representation is a direct sum of irreducible representations, we can write $H^i_c(\mathcal{X} \otimes \overline{k}; \mathbb{Q}_\ell)$ as a direct sum of irreducible representations indexed by partitions of n, so we can write

$$H^i_c(\mathcal{X} \otimes \overline{k}; \mathbb{Q}_\ell) = \sum_{\lambda \vdash n} H^i_{c,\lambda}(\mathcal{X} \otimes \overline{k}; \mathbb{Q}_\ell) s_\lambda \in \mathrm{Gal}^{\mathbb{S}_n}_{\mathbb{Q}},$$

where $H^i_{c,\lambda}(\mathcal{X} \otimes \overline{k}; \mathbb{Q}_\ell)$ is the irreducible $\mathbb{S}_n$ representation indexed by λ, and s_λ is the corresponding Schur polynomial. Now define

$$e_{c,\lambda}(\mathcal{X} \otimes \overline{k}, \mathbb{Q}_\ell) = \sum_{i \geq 0} (-1)^i [H^i_{c,\lambda}(\mathcal{X} \otimes \overline{k}; \mathbb{Q}_\ell)] \in K_0(\mathrm{Gal}_{\mathbb{Q}}).$$

Using the terminology defined earlier, we define a stable $\mathbb{S}$-module $\mathcal{V}$ by

$$\mathcal{V}((g,n)) := H^\bullet_c(\mathcal{M}_{g,n}, \mathbb{Q}_\ell).$$

Since we can write the decomposition

$$H^i_c(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_\ell) = \bigoplus_{\lambda \vdash n} H^i_{c,\lambda}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_\ell) s_\lambda \in \mathrm{Gal}^{\mathbb{S}_n}_{\mathbb{Q}},$$

we have the following expression in the Grothendieck group of Galois representations with $\mathbb{S}_n$-actions:

$$[H^i_c(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_\ell)] = \sum_{\lambda \vdash n} [H^i_{c,\lambda}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_\ell)] s_\lambda$$

and therefore

$$\begin{aligned} e^{\mathbb{S}_n}_c(\mathcal{M}_{g,n}, \mathbb{Q}_\ell) &:= \mathrm{ch}_n(\mathcal{V}((g,n))) = \sum_{i \geq 0} (-1)^i [H^i_c(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_\ell)] \\ &= \sum_{i \geq 0} (-1)^i \sum_{\lambda \vdash n} [H^i_{c,\lambda}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_\ell)] s_\lambda. \end{aligned}$$

This is what we call the $\mathbb{S}_n$-*equivariant Euler characteristic*. Let χ_λ denote the character of the irreducible $\mathbb{S}_n$-representation indexed by λ. Then by definition of this character, (see [11, Section 3.1])

$$e^{\mathbb{S}_n}_c(\mathcal{M}_{g,n}, \mathbb{Q}_\ell) = \sum_{\lambda \vdash n} (\chi_\lambda(\mathrm{id}))^{-1} e_{c,\lambda}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_\ell) s_\lambda \in K_0(\mathrm{Gal}_{\mathbb{Q}}) \otimes \Lambda_n.$$

We have seen that it is possible to express the trace of the Frobenius endomorphism acting on $e_c(\mathcal{M}_{g,n})$ as the point count of $\mathcal{M}_{g,n}$ over a finite field, but this Euler characteristic does not keep track of the additional $\mathbb{S}_n$-action like the $\mathbb{S}_n$-equivariant Euler characteristic $e_c^{\mathbb{S}_n}(\mathcal{M}_{g,n})$ that we have introduced above. To make this work for our new characteristic, we need to count points differently.

Definition VI.16. Let X be a variety over $\mathbb{F}_q$ with $\sigma \in \operatorname{Aut} X$. Then there is a unique *twisted form* of X, denoted X^σ with an isomorphism $X^\sigma_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$ that identifies the geometric Frobenius action on $X^\sigma_{\overline{\mathbb{F}}_q}$ with the action of $F_q \circ \sigma$ on $X_{\overline{\mathbb{F}}_q}$.

We can realize the moduli space $\overline{\mathcal{M}}_{g,n}$ $\mathbb{S}_n$-equivariantly by using the fact that it can be written as the quotient of a smooth and proper variety $X_{g,n}$ by a finite group H, see [14]. Then define the twisted form

$$\overline{\mathcal{M}}_{g,n} := [X^\sigma_{g,n}/H].$$

Definition VI.17. Let F_q be the geometric Frobenius and let $\mathcal{X}_{\mathbb{F}_q}$ be a smooth DM stack of constant dimension and of finite type over $\mathbb{F}_q$ that has an action of $\mathbb{S}_n$. We denote by $X_{\overline{\mathbb{F}}_q}$ the coarse moduli space of $\mathcal{X}_{\overline{\mathbb{F}}_q}$. An $\mathbb{S}_n$-*equivariant count* of the number of points defined over $\mathbb{F}_q$ of $\mathcal{X}_{\overline{\mathbb{F}}_q}$ is the number of fixed points

$$|X^{F_q\circ\sigma}_{\overline{\mathbb{F}}_q}|$$

for all $\sigma \in \mathbb{S}_n$.

By [4, 3.1.2], extending the Lefschetz Trace Formula to stacks,

$$\operatorname{Tr}(F_q \circ \sigma, H^\bullet_c(X_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)) = |X^{F_q\circ\sigma}_{\overline{\mathbb{F}}_q}|.$$

Using our identity for the $\mathbb{S}_n$-equivariant Euler characteristic in terms of virtual representations, we set the $\mathbb{S}_n$-equivariant point count to be

$$\#^{\mathbb{S}_n} \mathcal{M}_{g,n}(\mathbb{F}_q) := \sum_{\lambda \vdash n} \chi_\lambda(\mathrm{id})^{-1} \operatorname{Tr}(F_q, e_{c,\lambda}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_\ell)) s_\lambda.$$

To see why these counts can be useful, we need the following lemma from [25, 2.31].

Lemma VI.18. *Let χ_λ be the character of the irreducible representation of $\mathbb{S}_n$ indexed by λ. Then*

$$\pi_\lambda := \frac{1}{n!}\chi_\lambda(\mathrm{id}) \sum_{\sigma\in\mathbb{S}_n} \chi_\lambda(\sigma)\sigma \colon H^i_c(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \to H^i_c(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$$

is the projection of $H^i_c(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ to $H^i_{c,\lambda}(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$.

Corollary VI.19.

$$\operatorname{Tr}(F_q, e_{c,\lambda}(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)) = \frac{1}{n!}\chi_\lambda(\mathrm{id}) \sum_{\sigma\in\mathbb{S}_n} \chi_\lambda(\sigma)|X^{F_q\circ\sigma}_{\overline{\mathbb{F}}_q}|.$$

Proof. We can express $\mathrm{Tr}(F_q, e_{c,\lambda}(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell))$ as a trace on $e_c^{\mathbb{S}_n}(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell))$ by use of π_λ:

$$\mathrm{Tr}(F_q, e_{c,\lambda}(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)) = \mathrm{Tr}(F_q \circ \pi_\lambda, e_c^{\mathbb{S}_n}(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)).$$

We can write $F_q \circ \pi_\lambda = \frac{1}{n!}\chi_\lambda(\mathrm{id}) \sum_{\sigma \in \mathbb{S}_n} \chi_\lambda(\sigma) F_q \circ \sigma$. Since the Trace map is additive, we conclude

$$\mathrm{Tr}(F_q \circ \pi_\lambda, e_c^{\mathbb{S}_n}(\mathcal{X}_{\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)) = \frac{1}{n!}\chi_\lambda(\mathrm{id}) \sum_{\sigma \in \mathbb{S}_n} \chi_\lambda(\sigma) |X_{\overline{\mathbb{F}}_q}^{F_q \circ \sigma}|.$$

□

4 Galois Representations

Note that we have defined the Frobenius endomorphism as a map on cohomology induced by a morphism of schemes defined over a finite field $\mathbb{F}_q$. In the definition of the Euler characteristics above, however, we define our spaces over $\mathbb{Q}$ and $\overline{\mathbb{Q}}$, so to count fixed points under the Frobenius endomorphism, we will need a workaround.

We have seen that for a smooth and proper DM stack over $\mathbb{Z}_p$ and $\ell \neq p$ primes, there is an isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representations

$$H_c^i(\mathcal{X}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell) \to H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$$

which is induced by a natural surjection $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. We can then define the geometric Frobenius map $F_q \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ to be the inverse of $x \mapsto x^q$. To extend this to $\overline{\mathbb{Q}}$, note that we can choose an embedding $\mathbb{Q} \to \mathbb{Q}_p$ for each prime p with dense image, which therefore gives an injection $\iota\colon \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then we can speak of a Frobenius element $F_p' \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ coming from $F_p \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

The map ι moreover makes the cohomology groups $H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}}; \mathbb{Q}_\ell)$ into a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation isomorphic to $H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}_p}; \mathbb{Q}_\ell)$, and hence

$$\mathrm{Tr}(F_p', H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}}; \mathbb{Q}_\ell)) = \mathrm{Tr}(F_p, H_c^i(\mathcal{X}_{\overline{\mathbb{F}}_p}; \mathbb{Q}_\ell)).$$

So from now on we write $F_p' = F_p$ by abuse of notation. The same now holds for F_q for $q = p^k$.

Following [10], when $\mathcal{X}$ is a smooth and proper DM stack over $\mathbb{Z}$ of relative dimension d with an action of $\mathbb{S}_n$, the irreducible pieces of the semisimplification of the Galois representation $H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}}; \mathbb{Q}_\ell)$ will have weight i. Therefore $e_c^{\mathbb{S}_n}(\mathcal{X}_{\overline{\mathbb{Q}}}; \mathbb{Q}_\ell)$ as an element of $K_0(\mathrm{Gal}_{\mathbb{Q}}^{\mathbb{S}_n})$ determines $H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}}; \mathbb{Q}_\ell)$ as an element of $K_0(\mathrm{Gal}_{\mathbb{Q}}^{\mathbb{S}_n})$ for all i. Moreover, Poincaré duality implies that in $K_0(\mathrm{Gal}_{\mathbb{Q}}^{\mathbb{S}_n})$ we have

$$\mathbb{L}^d H_c^i(\mathcal{X}_{\overline{\mathbb{Q}}}; \mathbb{Q}_\ell)^\vee \cong H_c^{2d-i}(\mathcal{X}_{\overline{\mathbb{Q}}}; \mathbb{Q}_\ell)$$

for all $0 \leq i \leq d$.

The following theorem [11, Theorem 6] illustrates when equivariant point counts can be used to determine $e_c^{\mathbb{S}_n}(\mathcal{X}_{\overline{\mathbb{Q}}}; \mathbb{Q}_\ell)$ as an element of $K_0(\mathrm{Gal}_{\mathbb{Q}}^{\mathbb{S}_n})$.

Theorem VI.20. *Let $\mathcal{X}$ be a DM stack defined over $\mathbb{Z}$ which is proper, smooth, of pure relative dimension d and that has an action of $\mathbb{S}_n$. Let $X_{\overline{\mathbb{F}}_p}$ be the coarse moduli space of $\mathcal{X}_{\overline{\mathbb{F}}_p}$. For every partition λ of n, denote by χ_λ the character of the irreducible representation of $\mathbb{S}_n$ indexed by λ. Furthermore, let S be a set of all but finitely many primes.*
Assume that for a partition λ of n there exists a polynomial $P_\lambda(t) \in \mathbb{Q}[t]$ such that

$$\frac{1}{n!}\chi_\lambda(id)\sum_{\sigma\in\mathbb{S}_n}\chi_\lambda(\sigma)|X_{\overline{\mathbb{F}}_p}^{\sigma\circ F_{p^r}}| = P_\lambda(p_r)$$

for all $r \in \mathbb{Z}_{r\geq 1}$ and $p \in S$. Then $P_\lambda(t)$ has degree d and non-negative integer coefficients. Furthermore, if we let b_j be the coefficient of q^j in P_λ, then for all primes ℓ and all $i \geq 0$ there is an isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-representations*

$$H^i_{c,\lambda}(\mathcal{X}_{\overline{\mathbb{Q}}};\mathbb{Q}_\ell) \cong \begin{cases} 0 & \text{if } i \text{ is odd} \\ \mathbb{Q}_\ell(-i/2)^{b_{i/2}} & \text{if } i \text{ is even.} \end{cases}$$

The condition in this theorem is that the trace of the Frobenius element F_{p^r} on $e_{c,\lambda}(\mathcal{X}_{\overline{\mathbb{F}}_{p^r}})$ for all $r \geq 1$ is a polynomial in p^r, or equivalently, that the $\mathbb{S}_n$-equivariant point counts are. This is commonly referred to as the *polynomiality condition*.
Furthermore, note that indeed $\overline{\mathcal{M}}_{g,n}$ is a proper and smooth DM stack over $\mathbb{Z}$ of pure relative dimension, with an action of $\mathbb{S}_n$.

5 Mixed Hodge Structures

In a previous section we have introduced the $\mathbb{S}_n$-equivariant Euler characteristic $e_c^{\mathbb{S}_n}(\mathcal{X})$ taking values in $K_0(\mathrm{MHS}_\mathbb{Q}) \otimes \Lambda_n$ for an algebraic stack (or scheme) $\mathcal{X}$ over $\mathbb{C}$ with an action of $\mathbb{S}_n$. Similarly to the case of Galois representations, we can decompose the groups $H^i_c(\mathcal{X},\mathbb{Q})$ into irreducible $\mathbb{S}_n$-representations indexed by a partition $\lambda \vdash n$.

Definition VI.21. Let $H^i_{c,\lambda}(\mathcal{X},\mathbb{Q})$ be the direct sum of all copies of irreducible representations of $\mathbb{S}_n$ indexed by λ that appear in the rational cohomology groups $H^i_c(\mathcal{X},\mathbb{Q})$. Then set

$$e_{c,\lambda}(\mathcal{X}(\mathbb{C}),\mathbb{Q}) := \sum_{i\geq 0}[H^i_{c,\lambda}(\mathcal{X},\mathbb{Q})] \in K_0(\mathrm{MHS}_\mathbb{Q}).$$

Define a stable $\mathbb{S}$-module $\mathcal{V}$ by

$$\mathcal{V}((g,n)) := H^\bullet_c(\mathcal{M}_{g,n}(\mathbb{C}),\mathbb{Q}).$$

In the Grothendieck group of mixed Hodge structures with $\mathbb{S}_n$-actions, we can write

$$[H^i_c(\mathcal{M}_{g,n}(\mathbb{C}),\mathbb{Q})] = \sum_{\lambda\vdash n}[H^i_{c,\lambda}(\mathcal{M}_{g,n}(\mathbb{C}),\mathbb{Q})]s_\lambda.$$

Again, for χ_λ the character of the irreducible $\mathbb{S}_n$-representation indexed by γ,

$$e_c^{\mathbb{S}_n}(\mathcal{M}_{g,n}(\mathbb{C}),\mathbb{Q}) = \sum_{\lambda\vdash n}(\chi_\lambda(\mathrm{id}))^{-1}e_{c,\lambda}(\mathcal{M}_{g,n}(\mathbb{C}),\mathbb{Q})s_\lambda \in K_0(\mathrm{MHS}_\mathbb{Q})\otimes\Lambda_n.$$

We can now reformulate the previous theorem for mixed Hodge structures.

Theorem VI.22 ([11, Theorem 7])**.** *Let $\mathcal{X}$ be a DM stack defined over $\mathbb{Z}$ which is proper, smooth, of pure relative dimension d and that has an action of $\mathbb{S}_n$. Let $X_{\overline{\mathbb{F}}_p}$ be the coarse moduli space of $\mathcal{X}_{\overline{\mathbb{F}}_p}$. For every partition λ of n, denote by χ_λ the character of the irreducible representation of $\mathbb{S}_n$ indexed by λ. Furthermore, let S be a set of all but finitely many primes. Assume that for a partition λ of n there exists a polynomial $P_\lambda(t)\in\mathbb{Q}[t]$ such that*

$$\frac{1}{n!}\chi_\lambda(id)\sum_{\sigma\in\mathbb{S}_n}\chi_\lambda(\sigma)|X_{\overline{\mathbb{F}}_p}^{\sigma\circ F_{p^r}}| = P_\lambda(p_r)$$

and assume that the coarse moduli space $X_\mathbb{Q}$ of the stack $\mathcal{X}_\mathbb{Q}$ is the quotient of a smooth projective $\mathbb{Q}$-scheme by a finite group. Then for all partitions λ of n and for all $i\geq 0$, there is an isomorphism of pure $\mathbb{Q}$-Hodge structures

$$H^i_{c,\lambda}(\mathcal{X}(\mathbb{C});\mathbb{Q}) \cong \begin{cases} 0 & \text{if } i \text{ is odd} \\ \mathbb{Q}_\ell(-i/2)^{b_{i/2}} & \text{if } i \text{ is even,}\end{cases}$$

where the left hand side is equipped with the canonical Hodge structure of [20].

Note that by [14], the spaces $\overline{\mathcal{M}}_{g,n}$ have a coarse moduli space which over $\mathbb{Q}$ is a quotient of a smooth $\mathbb{Q}$-scheme by a finite group.

Thus, whenever a point count on $\mathcal{X}$ satisfies the polynomiality condition, we can determine the Euler characteristic of $\mathcal{X}$ as an element of the Grothendieck group of Galois representations or mixed Hodge structures.

Given a polynomial point count for every $\mathcal{M}_{g,n}$ such that $(g,n)\in\mathcal{D}_{G,N}$ and every partition λ of n, we can determine $e_{c,\lambda}(\partial\mathcal{M}_{g,n})$ using the methods from [28]. Since $\mathrm{Tr}(F_q, e_{c,\lambda}(\mathcal{M}_{g,n}\otimes\overline{\mathbb{Q}};\mathbb{Q}_\ell))$ is a polynomial in q for all $(g,n)\in\mathcal{D}_{G,N}$, we can conclude the same for $\mathrm{Tr}(F_q, e_{c,\lambda}(\overline{\mathcal{M}}_{g,n}\otimes\overline{\mathbb{Q}};\mathbb{Q}_\ell))$. This allows us to apply the theorems in the sections above. Note that this argument also extends to the category of mixed Hodge structures. We summarize this in the following theorem:

Theorem VI.23 ([11, Theorem 8])**.** *Assume that for all $\mathcal{M}_{g,n}$ with $(g,n)\in\mathcal{D}_{G,N}$ and for all partitions λ of n, the equation*

$$\frac{1}{n!}\chi_\lambda(id)\sum_{\sigma\in\mathbb{S}_n}\chi_\lambda(\sigma)|X_{\overline{\mathbb{F}}_p}^{\sigma\circ F_{p^r}}| = P_{\lambda,g,n}(p^r) \qquad \text{(VI.1)}$$

is fulfilled for some set S and polynomial $P_{\lambda,g,n}(t)$. Then the following holds for all $(g,n) \in \mathcal{D}_{G,N}$ and for all partitions λ of n.
For $\overline{\mathcal{M}}_{g,n}$, equation (1) is fulfilled for some set S and polynomials $Q_\lambda(t)$. Hence both theorems above hold for $\overline{\mathcal{M}}_{g,n}$. Moreover,

1. $e_\lambda(\mathcal{M}_{g,n} \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell) = P_{\lambda,g,n}([\mathbb{Q}_\ell(-1)]) \in K_0(\mathrm{Gal}_{\mathbb{Q}})$, *and*
2. $e_\lambda(\mathcal{M}_{g,n}(\mathbb{C}), \mathbb{Q}) = P_{\lambda,g,n}(\mathbb{L}) \in K_0(\mathrm{MHS}_{\mathbb{Q}})$.

Chapter VII

COHOMOLOGY OF LOCAL SYSTEMS

Let $\mathcal{M}_g$ be defined over a finite field k with characteristic p. Recall that for $\ell \neq p$, the local system ${}_\ell\mathbb{V} = R^1\pi_*\,\mathbb{Q}_\ell$ on $\mathcal{M}_g \otimes \mathbb{Z}[1/\ell]$ and a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_g \geq 0)$ determine a local system appearing in

$$\mathrm{Sym}^{\lambda_1-\lambda_2}(\mathbb{V}) \otimes \mathrm{Sym}^{\lambda_2-\lambda_3}(\wedge^2\mathbb{V}) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{g-1}-\lambda_g}(\wedge^g\mathbb{V})$$

on $\mathcal{M}_g \otimes \mathbb{Z}[1/\ell]$. Since $\mathbb{V}$ is an ℓ-adic sheaf, so is the associated local system $\mathbb{V}_\lambda$. For ease of notation, we denote the sheaf ${}_\ell\mathbb{V}$ as $\mathbb{V}$, since it will be clear from context which sheaf we mean. This leads us to the following definition:

Definition VII.1. Let $\mathbb{V}_\lambda$ be the ℓ-adic local system on $\mathcal{M}_g \otimes \mathbb{Z}[1/\ell]$ defined above. We set

$$e_c(\mathcal{M}_g \otimes \overline{k}; \mathbb{V}_\lambda) := \sum_{i \geq 0} (-1)^i [H^i_c(\mathcal{M}_g; \mathbb{V}_\lambda)],$$

taking values in the Grothendieck group of Galois representations or mixed Hodge structures, depending on the choice of local system $R^1\pi_*\,\mathbb{Q}$ or $R^1\pi_*\,\mathbb{Q}_\ell$. Like before, the cohomology groups $H^i_c(\mathcal{M}_g \otimes \overline{\mathbb{Q}}; \mathbb{V}_\lambda)$ come with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which has a Frobenius element coming from $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ by way of the surjection $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and injection $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, moreover inducing isomorphisms [9]

$$H^i_c(\mathcal{M}_g \otimes \overline{\mathbb{F}}_q; \mathbb{V}_\lambda) \cong H^i_c(\mathcal{M}_g \otimes \overline{\mathbb{Q}}_p; \mathbb{V}_\lambda) \cong H^i_c(\mathcal{M}_g \otimes \overline{\mathbb{Q}}; \mathbb{V}_\lambda)$$

of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representations for all $p \neq \ell$. This gives us an equality of traces

$$\mathrm{Tr}(F_q, e_c(\mathcal{M}_g \otimes \overline{\mathbb{F}}_q; \mathbb{V}_\lambda)) = \mathrm{Tr}(F_q, e_c(\mathcal{M}_g \otimes \overline{\mathbb{Q}}_p; \mathbb{V}_\lambda)) = \mathrm{Tr}(F_q, e_c(\mathcal{M}_g \otimes \overline{\mathbb{Q}}; \mathbb{V}_\lambda)),$$

allowing us to compute traces over finite fields.

The stalks of the sheaf $\mathbb{V}$ are known: for $[C] \in \mathcal{M}_g(\mathbb{F}_q)$ $\mathbb{V}_{[C]} \cong H^1(C;\mathbb{Q}_\ell)$. The trace formula therefore allows us to express the trace of the Frobenius on $e_c(\mathcal{M}_g;\mathbb{V}_\lambda)$ in terms of the eigenvalues of the Frobenius acting on $H^1(C;\mathbb{Q}_\ell)$.

Proposition VII.2. *Let $\alpha_1(C),\dots,\alpha_{2g}(C)$ be the eigenvalues of F_q acting on $H^1(C;\mathbb{Q}_\ell)$ ordered such that $\alpha_i(C)$ is conjugate to $\alpha_{g+i}(C)$ and let $s_{<\lambda>}(x_1,\dots,x_{2g};t) \in \mathbb{Z}[t]$ be the Schur polynomial for* $\mathrm{Sp}(2g,\mathbb{C})$ *associated to λ. Then*

$$\mathrm{Tr}(e_c(\mathcal{M}_g \otimes \overline{\mathbb{F}}_q;\mathbb{V}_\lambda)) = \sum_{[C]\in\mathcal{M}_g(\mathbb{F}_q)} \frac{s_{<\lambda>}(\alpha_1(C),\dots,\alpha_{2g}(C);q)}{|\operatorname{Aut}_{\mathbb{F}_q} C|}.$$

Proof. Since the local system $\mathbb{V}_\lambda$ is induced by an irreducible representation given by a Schur functor, we know that the trace of Frobenius on $(\mathbb{V}_\lambda)_{C\otimes\overline{\mathbb{F}}_q}$ is given by

$$s_{<\lambda>}(\alpha_1(C),\dots,\alpha_g(C),\alpha_{g+1}(C),\dots,\alpha_{2g}(C)).$$

All eigenvalues satisfy $|\alpha_i(C)| = \sqrt{q}$, and the complex conjugate of $\alpha_i(C)$ is also an eigenvalue, which we order such that $\alpha_{g+i}(C)\alpha_i(C) = q$. We can then make the Schur polynomial $s_{\langle\lambda\rangle}$ into a homogenized weighted polynomial in the variables $x_1,\dots,x_g$ of weight 1 and t of weight 2. Then the trace of Frobenius will be given by $s_{\langle\lambda\rangle}(\alpha_1(C),\dots,\alpha_g(C);q)$. Now applying the trace formula for stacks yields the desired result. ☐

1 Local Systems on the Moduli Space of Elliptic Curves

We can see the formula above working for elliptic curves. In this case, the local systems $\mathbb{V}_\lambda$ are given by partitions of size 1, yielding the local systems $\mathbb{V}_m = \operatorname{Sym}^m \mathbb{V}$. The Euler characteristics of these local systems on $\mathcal{M}_{1,1}$ are well-known. In [9], it is recalled that

$$e_c(\mathcal{M}_{1,1},\mathbb{V}_m) = \begin{cases} 0 & m \text{ odd} \\ \mathbb{L} & m = 0 \\ -1 - S[m+2] & m > 0 \text{ even} \end{cases}$$

as motives in the Grothendieck group of mixed Hodge structures. A motivic construction of the spaces $S[a+2]$ can be found in [17]. It is the motive corresponding to the space S_{a+2} of cusp forms of weight $a+2$. In particular, the trace of Frobenius at p on $S[a+2]$ equals the trace of the Hecke operator at p on S_{a+2}. Hence, computing the trace of Frobenius at p of $e_c(\mathcal{M}_{1,1},\mathbb{V}_m)$ gives useful information on traces of Hecke operators on cusp forms.
In order to make the computations more manageable, the following lemma will be of use:

Lemma VII.3. *Let α_1, α_2 be the eigenvalues of the Frobenius at p on $H^1(C;\mathbb{Q}_\ell)$. Let $h_m(x_1, x_2)$ be the complete homogeneous symmetric polynomial of degree m in two variables. Then the eigenvalues of Frobenius at p on $\operatorname{Sym}^m H^1(C;\mathbb{Q}_\ell)$ are given by*

$$h_m(\alpha_1, \alpha_2) = \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i \binom{m-2i}{i} e_1(\alpha_1, \alpha_2) p^i.$$

Proof. Suppose M is a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and a basis of eigenvectors $v_1, \ldots, v_n$. Then the basis of eigenvectors of $\operatorname{Sym}^k M$ is given by the symmetrizations of the tensors

$$v_{i_1} \otimes \cdots \otimes v_{i_k}$$

indexed by integers $i_1 \leq \ldots \leq i_k$. The eigenvalues are then given by $\lambda_{i_1} \cdots \lambda_{i_k}$. And thus $\operatorname{Tr}(F_p, \operatorname{Sym}^k M) = h_k(\lambda_1, \ldots, \lambda_n)$. We can express these complete homogeneous symmetric polynomials in terms of lower degrees and elementary symmetric polynomials since there exists a relation

$$\sum_{i=0}^{m} (-1)^i e_i h_{m-i} = 0,$$

and therefore

$$h_m = \sum_{i=1}^{m} (-1)^{i+1} e_i h_{m-i}.$$

We will use induction. For $m = 1$, we have

$$h_1(\alpha_1, \alpha_2) = \alpha_1 + \alpha_2 = (-1)^0 \binom{m}{0} e_1(\alpha_1, \alpha_2) p^0 = \alpha_1 + \alpha_2,$$

corresponding to the formula above.
Now suppose that for all $m \geq i \geq 1$, $h_{m-i}(\alpha_1, \alpha_2)$ is of the desired form. Then

$$\begin{aligned} h_m(\alpha_1, \alpha_2) &= \sum_{i=1}^{m} (-1)^{i+1} e_i(\alpha_1, \alpha_2) h_{m-i}(\alpha_1, \alpha_2) \\ &= \sum_{i=1}^{m} (-1)^{i+1} e_i(\alpha_1, \alpha_2) \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \binom{m-2i}{i} e_1^{m-3i} p^i. \end{aligned}$$

The eigenvalues α_1 and α_2 satisfy $|\alpha_1| = |\alpha_2| = \sqrt{p}$, so $e_2(\alpha_1, \alpha_2) = p$ and $e_k(\alpha_1, \alpha_2) = 0$ for all $k \geq 3$. Hence,

$$\begin{aligned} h_m(\alpha_1, \alpha_2) &= e_1(\alpha_1, \alpha_2) \cdot \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \binom{m-2i}{i} e_1^{m-3i} p^i - p \cdot \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \binom{m-2i}{i} e_1^{m-3i} p^i \\ &= \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i \binom{m-i}{i} e_1^{m-2i} p^i. \end{aligned}$$

□

Given a list of elliptic curves up to $\mathbb{F}_p$-isomorphism, together with their number of points over $\mathbb{F}_p$ and the order of their automorphism group, we can determine the trace of Frobenius at p on the Euler characteristic $e_c(\mathcal{M}_{1,1}; \mathbb{V}_m)$ for all $m \geq 1$. Carel Faber has determined this list of isomorphism classes, which for $p = 2, 3$ is shown in the following tables:

$\lvert C(\mathbb{F}_2)\rvert$	$\sum \frac{1}{\lvert \mathrm{Aut}_{\mathbb{F}_2} C\rvert}$	$\alpha_1(C) + \alpha_2(C)$
1	1/4	2
2	1/2	1
3	1/2	0
4	1/2	−1
5	1/4	−2

Table VII.1

We also include the polynomial $e_1(\alpha_1(C), \alpha_2(C))$, which we can determine using the Lefschetz fixed-point formula, which yields the identity

$$|C(\mathbb{F}_{p^r})| = 1 + p^r - \alpha_1(C)^r - \alpha_2(C)^r.$$

For reference, see [32, Appendix A]. Moreover, we know that the number of points on an elliptic curve over a finite field is bounded by the Hasse-Weil bound,

$$|C(\mathbb{F}_p) - (p^r + 1)| \leq 2\sqrt{p^r}.$$

In particular, over $\mathbb{F}_2$, an elliptic curve has at most 5 points, while it has at most 7 points over $\mathbb{F}_3$. For $p = 3$, we obtain the following table:

$\lvert C(\mathbb{F}_3)\rvert$	$\sum \frac{1}{\lvert \mathrm{Aut}_{\mathbb{F}_3} C\rvert}$	$\alpha_1(C) + \alpha_2(C)$
1	1/6	3
2	1/2	2
3	1/2	1
4	2/3	0
5	1/2	−1
6	1/2	−2
7	1/6	−3

Table VII.2

Proposition VII.4.

$$\mathrm{Tr}(F_p, e_c(\mathcal{M}_{1,1}; \mathbb{V}_m)) \equiv -1 \pmod{p}$$

for $p = 2, 3$ and $m > 1$ even.

Proof. Let $s_{p,m}(C_i)$ denote the polynomial h_m for the prime p evaluated at $(\alpha_1(C_i), \alpha_2(C_i))$ for C_i an elliptic curve with i points over $\mathbb{F}_p$, the eigenvalues of Frobenius at p on $H^1(C;\mathbb{Q}_\ell)$. Then

$$\mathrm{Tr}(F_p, e_c(\mathcal{M}_{1,1};\mathbb{V}_m)) = \sum_{[C]\in\mathcal{M}_{1,1}} \frac{s_{p,m}(C)}{|\operatorname{Aut}_{\mathbb{F}_p} C|}.$$

Since $m \geq 0$ is even, the sign of $e_1(\alpha_1(C), \alpha_2(C))$ is irrelevant, hence $s_{p,m}(C_i) = s_{p,m}(C_{k-i})$ for $k = 5, 7$ and $p = 2, 3$ respectively. We therefore find that

$$\mathrm{Tr}(F_2, e_c(\mathcal{M}_{1,1};\mathbb{V}_m)) = s_{2,m}(C_1)/2 + s_{2,m}(C_2) + s_{2,m}(C_3)/2.$$

Note that the expressions $s_{2,m}(C_1)$ and $s_{2,m}(C_3)$ both contain a factor of 2 in every term, so clearly $s_(2,m)(C_1)/2 + s_{2,m}(C_2) + s_{2,m}(C_3)/2 \equiv 0 \pmod 2$. On the other hand, the expression $s_{2,m}(C_2)$ contains a factor of 2 in every term, except when $i = 0$, so

$$\mathrm{Tr}(F_p, e_c(\mathcal{M}_{1,1};\mathbb{V}_m)) = s_{2,m}(C_1)/2 + s_{2,m}(C_2) + s_{2,m}(C_3)/2 \equiv -1 \pmod 2.$$

For $p = 3$,

$$\mathrm{Tr}(F_3, e_c(\mathcal{M}_{1,1};\mathbb{V}_m)) = s_{3,m}(C_1)/3 + s_{3,m}(C_2) + s_{3,m}(C_3) + 2s_{3,m}(C_4)/3.$$

The terms $s_{3,m}(C_1)$ and $s_{3,m}(C_4)$ are both clearly divisible by 3, so that $s_{3,m}(C_1)/3+2s_{3,m}(C_4)/3 \equiv 0 \pmod 3$. The terms of $s_{3,m}(C_3)$ are all divisible by 3, except when $i = 0$, we have $(-1)^0\binom{m}{0}1^m = 1$. Lastly,

$$s_{3,m}(C_2) = \sum_{i=0}^{m/2} (-1)^i \binom{m-i}{i} 2^{m-2i} p^i,$$

whose terms are also all divisible by 3, except for the term where $i = 0$, in which case it is of the form 2^{2k} for some integer k, since m is assumed to be odd. Since $4 \equiv 1 \pmod 3$, we conclude that

$$\begin{aligned}\mathrm{Tr}(F_3, e_c(\mathcal{M}_{1,1};\mathbb{V}_m)) &= s_{3,m}(C_1)/3 + s_{3,m}(C_2) + s_{3,m}(C_3) + 2s_{3,m}(C_4)/3 \\ &\equiv 0 + 1 + 1 + 0 \equiv -1 \pmod 3.\end{aligned}$$

□

As a direct consequence, we can now conlude the following:

Corollary VII.5. *Let $T(p)$ be the Hecke operator at the prime p acting on the space of cusp forms S_{m+2} for m even. Then*

$$\mathrm{Tr}(T(p), S_{m+2}) \equiv 0 \pmod p.$$

for $p = 2, 3$.

2 Detecting Non-tautological Cohomology

Recall that the cohomology $H_c^\bullet(\mathcal{M}_{g,n}(\mathbb{C});\mathbb{Q})$ carries a canonical mixed Hodge structure and a weight filtration W. In 2022, Chan, Faber, Galatius and Payne [16] find an expression for the $\mathbb{S}_n$-equivariant top weight Euler characteristic of $\mathcal{M}_{g,n}$ for $g \geq 2$. Their results are stated for the top weight of the weight filtration on $H^i(\mathcal{M}_{g,n}(\mathbb{C});\mathbb{Q})$, which corresponds to weight 0 on compactly supported cohomology by Poincaré Duality. The filtration is supported in degrees from 0 to $2d$ for $d = \dim \mathcal{M}_{g,n}$.
Like $H_c^i(\mathcal{M}_{g,n}(\mathbb{C});\mathbb{Q})$ can be decomposed into irreducible $\mathbb{S}_n$-representations, the weight 0 cohomology can be written as

$$\mathrm{Gr}_0^W H_c^i(\mathcal{M}_{g,n}(\mathbb{C});\mathbb{Q}) \cong \bigoplus_{\lambda \vdash n} c_\lambda^i V_\lambda$$

for V_λ the irreducible $\mathbb{S}_n$-representation indexed by λ, a partition of n. As an expression in the Grothendieck group of mixed Hodge structures with an action of $\mathbb{S}_n$, we can write the generating function of this weight 0 cohomology as

$$z_g = \sum_{i,\lambda} (-1)^i c_\lambda^i s_\lambda.$$

In [16], the authors exhibit a closed expression for z_g given in terms of homogeneous power sums and Bernoulli numbers. Note that this formula holds for the cohomology of $\mathcal{M}_{g,n}$, not the cohomology of $\mathcal{M}_g$ with respect to $\mathbb{V}_\lambda$. We can compute the weight 0 part of this latter cohomology using the former. The Euler characteristics $e_c(\mathcal{M}_g \otimes \overline{\mathbb{Q}}; \mathbb{V}_\lambda)$ can be used to determine the $\mathbb{S}_n$-equivariant Euler characteristic of $\mathcal{M}_{g,n}$ and vice versa. The following lemma illustrates this connection.

Lemma VII.6 ([10, Lemma 4.7]). *For each partition μ of n, there are elements $a_{\mu,\lambda} \in \mathbb{Z}[\mathbb{L}]$ such that*

$$e_{c,\lambda}(\mathcal{M}_{g,n};\mathbb{Q}_\ell) = \sum_{|\lambda| \leq n} a_{\mu,\lambda} e_c(\mathcal{M}_g;\mathbb{V}_\lambda).$$

Conversely, for each λ with $|\lambda| = n$, there are elements $b_{\lambda,\mu} \in \mathbb{Z}[\mathbb{L}]$ such that

$$e_c(\mathcal{M}_g;\mathbb{V}_\lambda) = \sum_{|\mu| \leq n} b_{\lambda,\mu} e_{c,\mu}(\mathcal{M}_{g,n};\mathbb{Q}_\ell).$$

Alternatively, we can formulate this in terms of generating series in $K_0(\mathrm{MHS}_\mathbb{Q}) \otimes \Lambda$ as seen in [27]. We have

$$\sum_{n=0}^{\infty} e_c^{\mathbb{S}_n}(\mathcal{M}_{g,n};\mathbb{Q}) = \prod_{k=1}^{\infty} (1+p_k)^{\frac{1}{k}\sum_{d|k} \mu(k/d)(1-\psi_d(\mathbb{V})-\mathbb{L})},$$

where the p_i are the symmetric polynomials of the power sums, $\mu(k/d)$ is the Mobius function and ψ_d is the d-th Adams operation.

Having expressed $e_c(\mathcal{M}_g;\mathbb{V}_\lambda)$ in terms of $e_c(\mathcal{M}_{g,n}(\mathbb{C});\mathbb{Q})$ for all $n \leq |\lambda|$, we can then determine the weight 0 part of $e_c(\mathcal{M}_g;\mathbb{V}_\lambda)$, which we will denote by $z_{g,\lambda}$. Using the trace formula, we can then compute the expressions

$$\mathrm{Tr}(F_q, e_c(\mathcal{M}_g;\mathbb{V}_\lambda)) - z_{g,\lambda} \pmod q \qquad (\star)$$

using Maple. If the expression $(\star)$ is nonzero, then we know there exists cohomology which is not Tate-twisted, since $\mathrm{Tr}(F_q,\mathbb{L}) = q$.
Using this method, it is possible to detect the motive $S[12]$ on $\mathcal{M}_{1,11}$ at the prime $p = 11$. Consequently, the detection of motives in $\mathcal{M}_{g,n}$ for (g, n) such that $(1, 11) \in \mathcal{D}_{g,n}$ is not very effective at $p = 11$: this could simply be the motive $S[12]$ coming from $\mathcal{M}_{1,11}$. More generally, if the DM stack $\overline{\mathcal{M}}_{g,n}$ has a unirational coarse moduli space, then we expect no non-tautological cohomology [10, Theorem 7.1]. If $\mathcal{M}_{g',n'}$ for $(g', n') \in \mathcal{D}_{g,n}$ has non-tautological cohomology which is detected at the prime p, then we subsequently expect the expression $(\star)$ to be nonzero for p and some λ such that $|\lambda| = n$ on the dense open $\mathcal{M}_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$.

Example VII.7. For example, we can detect non-tautological cohomology for $g = 3$ in weight 17 for the local systems given by $\lambda = (11, 3, 3)$ and $(7, 7, 3)$ ([10]) at $p = 5$. Due to the existence of the gluing morphism identifying two points

$$\mathcal{M}_{3,17} \to \overline{\mathcal{M}}_{4,15},$$

there is a stratum isomorphic to $\mathcal{M}_{3,17} / \mathbb{S}_2$ in the boundary of $\mathcal{M}_{4,15}$. We then expect to find local systems $\mathbb{V}_\lambda$ of weight 15 such that $(\star)$ is nonzero for $p = 5$ since $\mathcal{M}_{4,n}$ is unirational for all $n \leq 15$ [6]. We find four local systems given by the partitions $\lambda =$

$$(11, 2, 2, 0) \quad (10, 3, 2, 0) \quad (7, 6, 2, 0) \quad (6, 6, 3, 0),$$

such that $\mathrm{Tr}(F_5, e_c(\mathcal{M}_4;\mathbb{V}_\lambda)) - z_{4,\lambda} \not\equiv 0 \pmod 5$. Indeed, these are all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0)$ that can be induced by the partitions $(11, 3, 3)$ and $(7, 7, 3)$. The local system $\mathbb{V}_\lambda$ occurs in the Euler characteristic $e_{c,\mu}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_\ell)$ for $|\lambda| = n$ and μ the conjugate partition to λ. The $\mathbb{S}_2$-action permuting two marked points then restricts the partition μ to a partition μ' such that $|\mu'| = |\mu| - 2$. To do this, consider the Young tableaux corresponding to μ, and remove a block at the end of a row and column twice. Since the alternating representation vanishes under an action permuting two points, we cannot remove two blocks from the same column. Now taking the dual partitions of all possible μ' gives the local systems satisfying $\mathrm{Tr}(F_5, e_c(\mathcal{M}_4;\mathbb{V}_\lambda)) - z_{4,\lambda} \not\equiv 0 \pmod 5$.

Detecting new motives in the cohomology of local systems on $\mathcal{M}_4$ is therefore not possible at $p = 5$, since it is likely to come from an already known motive from $\mathcal{M}_3$. We can still use this method at $q = 2,\ 3$ or 4.
For all local systems $\mathbb{V}_\lambda$ up to weight 20, we find that

$$\mathrm{Tr}(F_q, e_c(\mathcal{M}_4;\mathbb{V}_\lambda)) - z_{4,\lambda} \equiv 0 \pmod q,$$

for $q = 2, 3, 4$. These calculations have been performed using Maple, using a program written by Carel Faber.

Appendix A

HODGE STRUCTURES

1 Hodge Structures

In this section we introduce Hodge structures and Mixed Hodge structures. We follow [23].

Definition A.1. A *(pure) Hodge structure of weight* $n \in \mathbb{Z}$, denoted $(H_{\mathbb{Z}}, H^{p,q})$ consists of a finitely generated free abelian group $H_{\mathbb{Z}}$ along with a decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$ of the complexification $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, which satisfies $H^{p,q} = \overline{H^{q,p}}$.

As an example, define $H_{\mathbb{C}} = H^{k,k}$ and $H^{p,q} = 0$ for $(p,q) \neq (k,k)$. This is the trivial Hodge Structure of weight $2k$.
Less trivially, we can take $H_{\mathbb{Z}} = 2\pi i\,\mathbb{Z} \subseteq \mathbb{C}$ and $H_{\mathbb{C}} = H^{-1,-1}$. This is a pure Hodge structure of weight -2. It is the unique 1-dimensional pure Hodge structure of weight -2 up to isomorphism. It is called the *Tate Hodge structure* and is often denoted by $\mathbb{Z}(1)$. Its n-th tensor product, denoted $\mathbb{Z}(n)$, is a Hodge structure of dimension 1 and weight $-2n$.
We can also define a pure Hodge structure by a Hodge filtration.

Definition A.2. Given a complex vector space $H_{\mathbb{C}}$, a *Hodge filtration of degree* $n \in Z$ on $H_{\mathbb{C}}$ is a filtration $\{F^p\}$

$$H_{\mathbb{C}} = F^0 \supset F^1 \supset \cdots \supset F^n \supset \{0\}$$

such that $H_{\mathbb{C}} \cong F^p \oplus \overline{F^{n-p+1}}$.

In fact, this is no different from a pure Hodge structure of weight n: given a decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$, define a filtration by setting

$$F^p := H^{n,0} \oplus \cdots \oplus H^{p,n-p}.$$

Conversely, given a filtration $\{F^p\}$, define a decomposition by setting $H^{p,q} := F^p \cap \overline{F^q}$.

We can take the dual of a Hodge structure $(H_{\mathbb{Z}}, H^{p,q})$ of weight n by setting $H_{\mathbb{Z}}^{\vee} := \mathrm{Hom}(H_{\mathbb{Z}}, \mathbb{Z})$ with the dual Hodge decomposition $(H^{\vee})^{p,q} = (H^{-p,-q})^{\vee}$. This is a pure Hodge structure of weight $-n$. In particular, $\mathbb{Z}(1)^{\vee} \cong \mathbb{Z}(-1)$.

Definition A.3. Given a Hodge structure $(H_{\mathbb{Z}}, H^{p,q})$ of weight n, we define its *r-th Tate twist* to be the Hodge structure $(H(r)_{\mathbb{Z}}, H(r)^{p,q})$ of weight $n-2r$ given by

$$H(r)_Z = H_Z, \quad H(r)^{p,q} = H^{p-r,q-r}.$$

Alternatively, given $(H_{\mathbb{Z}}, H^{p,q})$ and $(H', H'^{p,q})$ two Hodge structures of weight n and n', define their tensor product $(H''_{\mathbb{Z}}, H''^{p,q})$ by setting $H''_Z := H_{\mathbb{Z}} \otimes H'_{\mathbb{Z}}$ and

$$H''^{p,q} := \bigoplus_{r+r'=p,\ s+s'=q} H^{r,s} \otimes H'^{r',s'}.$$

This defines a Hodge structure of weight $n+n'$. We can then see the r-th Tate twist of a Hodge structure $(H_{\mathbb{Z}}, H^{p,q})$ as the tensor $(H_{\mathbb{Z}}, H^{p,q}) \otimes \mathbb{Z}(r)$.
In addition, new Hodge structures can be formed by the following multi-linear algebra constructions:

1. Given two Hodge structures $(H_{\mathbb{Z}}, H^{p,q})$ and $(H'_{\mathbb{Z}}, H'^{p,q})$ of weight n, their direct sum is a Hodge structure of weight n given by the lattice $H_{\mathbb{Z}} \oplus H'_{\mathbb{Z}}$ and setting the (p,q)-components to be the direct sum of the (p,q)-components of each term $H^{p,q} \oplus H'^{p,q}$.
2. Using the dual Hodge structure and tensor product Hodge structure for Hodge structures $(H_{\mathbb{Z}}, H^{p,q})$ and $(H'_{\mathbb{Z}}, H'^{p,q})$ of weights n and n', we obtain a Hodge structure on $\mathrm{Hom}(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) \cong H_{\mathbb{Z}}^{\vee} \otimes H'_{\mathbb{Z}}$ of weight $n'-n$.
3. Given $(H_{\mathbb{Z}}, H^{p,q})$ a Hodge structure of weight n, the symmetric and wedge products $\mathrm{Sym}^k H_{\mathbb{Z}}$ and $\wedge^k H_{\mathbb{Z}}$ are Hodge structures of weight kn.

The cohomology of algebraic varieties provides an ample source of Hodge structures. In particular, recall that a smooth projective variety X over $\mathbb{C}$ admits the structure of a compact Kähler manifold. We have the following

Theorem A.4 ([48, Section 6.1.3]). *Let X be a smooth projective variety over $\mathbb{C}$. Then there exists a decomposition*

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X),$$

where $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$, Ω_X the sheaf of 1-forms on X and $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

If we set $H_{\mathbb{Z}}(X) := H^n(X, \mathbb{Z})/\text{torsion}$, then this theorem shows $(H_{\mathbb{Z}}, H^{p,q}(X))$ defines a pure Hodge structure of weight n.

Example A.5. Let X be a smooth curve of genus g over $\mathbb{C}$. Then H^i for $i = 0, 1, 2$ admits a pure Hodge structure. We have

$$H^0(X, \mathbb{C}) = H^0(X, \mathcal{O}_X)$$
$$H^2(X, \mathbb{C}) = H^1(X, \Omega_X)$$
$$H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \oplus H^1(X, \mathcal{O}_X)$$

2 Mixed Hodge Structures

We have seen that the cohomology groups $H^n(X, \mathbb{C})$ for a smooth projective variety X admit a pure Hodge structure of weight n. This is not true for varieties that are not smooth/projective, however. The cohomology groups of a variety that is not smooth still admit certain filtrations $\{F^p\}$. These induce a pure Hodge structure on the graded pieces of $H_{\mathbb{Q}}$. Using these filtrations, we can generalize the previously defined Hodge structures to mixed Hodge structures. In 1973, Deligne [19] proves that all cohomology groups of smooth varieties over $\mathbb{C}$ carry a natural and functorial mixed Hodge structure. He extends this result in 1974 [20] to arbitrary varieties over $\mathbb{C}$.

Definition A.6. A *mixed Hodge structure* $(H_{\mathbb{Z}}, W, F)$ consists of a $\mathbb{Z}$-module $H_{\mathbb{Z}}$ together with an increasing filtration W

$$\cdots \subseteq W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots$$

of $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and a decreasing filtration F

$$H_{\mathbb{C}} = F^1 \supset F_2 \supset F^3 \supset \cdots$$

of $H_{\mathcal{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ such that F defines a pure Hodge structure of weight k on the graded piece $\mathrm{Gr}_k^W H_{\mathbb{Q}} = W^k H_{\mathbb{Q}} / W^{k-1} H_{\mathbb{Q}}$.

Intuitively, the weight filtration can be interpreted by considering Gr_k as the k-th cohomology group of a smooth projective variety. As an example, let X be a smooth projective variety and set

$$H_{\mathbb{Z}} = \bigoplus_i H^i(X, \mathbb{Z})/\text{torsion}$$
$$W_k H_{\mathbb{Q}} = \bigoplus_{i \leq k} H^i(X, \mathbb{Q})$$
$$F_{\mathbb{C}}^p = \bigoplus_i F^p H^i(X, \mathbb{C})$$

where $F^p H^i(X, \mathbb{C})$ denotes the usual Hodge filtration of $H^i(X, \mathbb{C})$. Then $(H_{\mathbb{Z}}, W, F)$ is a mixed Hodge structure.
After proving that the cohomology of a smooth variety over $\mathbb{C}$ admits a canonical mixed Hodge structure [19], Deligne [20] proves the following:

Theorem A.7. *Let X be a quasi-projective variety over $\mathbb{C}$. For each k, there is an increasing weight filtration*

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2k} = H^k(X, \mathbb{Q})$$

and a decreasing Hodge filtration

$$H^k(X, \mathbb{C}) = F^0 \supset F^1 \supset \cdots \supset F^k \supset F^{k+1} = 0$$

such that the filtration induced by $F^\bullet$ on Gr_k^W gives a pure Hodge structure of weight k.

Moreover, the weight filtration satisfies the following properties for X a variety over $\mathbb{C}$:

1. If X is projective and smooth, the weight filtration is trivial, i.e. for all k,

   $$0 = W_{k-1} \subseteq W_k = H^k(X, \mathbb{Q})$$

   and hence $H^k(X, \mathbb{Q})$ is pure of weight k.

2. If X is projective, then for all k,

   $$0 = W_{-1} \subseteq W_0 \subseteq \cdots W_k = H^k(X, \mathbb{Q}).$$

3. If X is smooth, then for all k,

   $$0 = W_{k-1} \subseteq W_k \subseteq \cdots \subseteq W_{2k} = H^k(X, \mathbb{Q}).$$

4. For any $\varphi \colon X \to Y$ a morphism of algebraic varieties over $\mathbb{C}$,

   $$\varphi^*(W_k) \subseteq W_k.$$

   Moreover, φ^* preserves the pure Hodge structure of weight k on Gr_k^W for each k.

Importantly, this last property implies that an exact sequence of cohomology groups induced by morphisms of varieties remains exact when considering only the graded pieces Gr_k^W.
For our purposes, we can extend this construction to a mixed Hodge structure on the cohomology groups $H_c^k(X, \mathbb{Q})$ by Poincaré duality.

Appendix B

SYMPLECTIC REPRESENTATIONS

Following [22, Page 219], let $\mathrm{GSp}(2n, \mathbb{C})$ be the group of matrices

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that $AB^T = BA^T$, $CD^T = DC^T$, and $AD^T - BC^T = \eta I_n$ for some $\eta \in \mathbb{C}^\times$. The function $\eta\colon \mathrm{GSp}(2n, \mathbb{C}) \to \mathbb{C}$ defines a character, which we will call the *multiplier representation*.
We define $\mathrm{Sp}(2n, \mathbb{C})$ to be the subgroup of matrices $\gamma \in \mathrm{GSp}(2n, \mathbb{C})$ such that $\eta = 1$. These are called the general symplectic group and symplectic group respectively.

1 Symmetric Functions

The following results and definitions are taken from [25, Appendix A].

Definition B.1. Let Λ_n be the subring of $\mathbb{Z}[x_1, \ldots, x_n]$ generated by the monomials that are invariant under permutations. Define the *complete symmetric polynomials*

$$h_j = \sum_{1 \leq i_1 \leq \cdots \leq i_j} x_{i_1} \cdots x_{i_j} \text{ for } 1 \leq j \leq n.$$

This identifies Λ_n with the ring $\mathbb{Z}[h_1, \ldots, h_n]$. Another class of symmetric polynomials are the *elementary symmetric polynomials*

$$e_j = \sum_{1 < i_1 < \cdots < i_j} x_{i_1} \cdots x_{i_j} \text{ for } 1 \leq j \leq n.$$

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition of d. This uniquely defines a *Schur polynomial*, which we will denote by s_λ. These polynomials can be expressed in the h_j as a determinant

$$s_\lambda = \begin{vmatrix} h_{\lambda_1} & \cdots & h_{\lambda_1+n-1} \\ h_{\lambda_2-1} & \cdots & \\ \vdots & & \\ h_{\lambda_n-n+1} & \cdots & h_{\lambda_n} \end{vmatrix}$$

The rings Λ_n consist of homogeneous polynomials of at most degree n, but it would be more convenient not to have to worry about the number of variables or their degree. To this end, set

$$\Lambda := \varprojlim \Lambda_n.$$

The subring $\Lambda_n \subseteq \Lambda$ can be identified with $\mathbb{Z}[h_1, \ldots, h_n]$. The ring Λ can be freely generated as an abelian group by the Schur polynomials or the elementary symmetric polynomials. Extending to $\mathbb{Q}$, the ring $\Lambda \otimes \mathbb{Q}$ is freely generated as an algebra by the *elementary power sums*

$$p_n = \sum_{i \geq 1} x_i^n.$$

2 Schur Functors

Given a representation V of a group G, we can construct new representations by considering its n-th tensor power $V^{\otimes n}$ or its symmetric and antisymmetric powers $\operatorname{Sym}^n V$ and $\wedge^n V$. Here, we will introduce an operation, the Schur functor, which makes a new G-representation V_λ by means of the Young symmetrizer. We follow [25, Sections 4, 6].

Definition B.2. Let V be a representation of a group G and $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of $n > 0$. Then define the operation a_λ as

$$a_\lambda \colon V^{\otimes n} \mapsto \operatorname{Sym}^{\lambda_1} V \otimes \operatorname{Sym}^{\lambda_2} V \otimes \cdots \otimes \operatorname{Sym}^{\lambda_k} V \subseteq V^{\otimes n}.$$

For $\mu = (\mu_1, \ldots, \mu_l)$ the conjugate partition to λ, define the operation b_λ as

$$b_\lambda \colon V^{\otimes n} \mapsto \wedge^{\mu_1} V \otimes \wedge^{\mu_2} V \otimes \cdots \otimes \wedge^{\mu_l} V \subseteq V^{\otimes n}.$$

Finally, let c_λ be the *Young symmetrizer*, defined as

$$c_\lambda = a_\lambda \circ b_\lambda.$$

For example, when $\lambda = (n)$, $c_\lambda(V^{\otimes d}) = \operatorname{Sym}^n V$. For $\lambda = (1, 1, \ldots, 1)$, we get $c_\lambda(V^{\otimes n}) = \wedge^n V$.

The images of the symmetrizers c_λ in $V^{\otimes n}$ provide essentially all the finite-dimensional irreducible representations of $\mathrm{GL}(V)$. In particular, for $\mathbb{S}_n$ we know the following.

Theorem B.3. *Some scalar multiple of c_λ is idempotent, and the image of c_λ is an irreducible representation V_λ of $\mathbb{S}_n$ for all partitions λ of n. Moreover, every irreducible representation of $\mathbb{S}_n$ can be obtained in this way for a unique partition.*

We therefore find a $1-1$ correspondence given by partitions λ of n,

$$\{\text{Schur polynomials } s_\lambda\} \leftrightarrow \{\text{Irreducible representations of } \mathbb{S}_n\}.$$

Since conjugacy classes of $\mathbb{S}_n$ are determined by their cycle types, which in turn determine partitions of n, we also find a correspondence between irreducible representations of $\mathbb{S}_n$ and conjugacy classes.
As an example, the partition (n) gives the trivial representation U of $\mathbb{S}_n$, while $\lambda = (1, 1, \dots, 1)$ gives the alternating representation.

In [25, Section 4], we can find explicit identities for the characters and dimensions of the representations V_λ. In particular, we find that the character of $\mathbb{S}_n$ evaluated at g in the conjugacy class C_i of cycle type $\mathbf{i} = (i_1, \dots, i_n)$ can be directly related to certain Schur polynomials. For a large enough $k \in \mathbb{Z}$ and symmetric polynomials $p_j(x) = x_1^j + x_2^j + \cdots + x_k^j$,

$$\prod_j p_j(x)^{i_j} = \sum \chi_\lambda(C_i) s_\lambda,$$

where we sum over all partitions λ of n in at most k parts. Furthermore, the conjugacy class of the identity corresponds to $\mathbf{i} = (n)$, so

$$\dim V_\lambda = \chi_\lambda(C_{(n)}).$$

If $\lambda = (\lambda_1, \dots, \lambda_k)$, the identity then becomes

$$\dim V_\lambda = \frac{n!}{l_1! \cdots l_k!} \prod_{i<j} (l_i - l_j),$$

with $l_i = \lambda_i + k - i$.
Alternatively, we can compute the dimension using the *hook length formula.* Given a Young diagram of a partition λ, the hook length of a box is given by the number of squares directly below or directly to the right of the box, including the box once. Then,

$$\dim V_\lambda = \frac{n!}{\prod(\text{Hook lengths})}.$$

Definition B.4. Let λ be a partition of $n > 0$ and let V be a representation of a group G, then $V^{\otimes n}$ is also a representation of G. Denote $\mathbb{S}_\lambda V := \operatorname{im}(c_\lambda|_{V^{\otimes n}})$. This is also a representation of G. We call the construction

$$V \mapsto \mathbb{S}_\lambda V$$

the *Schur functor* corresponding to λ.

Note that we can consider the space $V^{\otimes n}$ as an $\mathbb{S}_n$-representation, with structure given by permuting the components. For an arbitrary group G, $\mathbb{S}_\lambda V$ is not necessarily irreducible, but it is for $G = \mathrm{GL}(V)$.
As an example, consider the representation $V \otimes V$ of $\mathrm{GL}(V)$ for V some finite dimensional complex vector space. We have a canonical decomposition

$$V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V.$$

Continuing this process, $V \otimes V \otimes V$ is also a representation of $\mathrm{GL}(V)$, which now has a decomposition into irreducible representations as

$$V \otimes V \otimes V \otimes V = \mathrm{Sym}^3 V \oplus \wedge^3 V \oplus (\mathbb{S}_{(2,1)} V)^{\oplus 2}.$$

In fact, we can generalize this to arbitrary tensor powers. The following theorem summarizes the important information about the representations $\mathbb{S}_\lambda V$.

Theorem B.5 ([25, Theorem 6.3]). *Let $k = \dim V$ and $\lambda = (\lambda_1 \geq \cdots \geq \lambda l \geq 0)$ a partition of n. Then $\mathbb{S}_\lambda V$ is zero if $\lambda_{k+1} \neq 0$. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$, then*

1. $\dim \mathbb{S}_\lambda V = s_\lambda(1, \ldots, 1)$.
2. *Let m_λ be the dimension of the irreducible representation V_λ of $\mathbb{S}_n$ corresponding to λ. Then*
$$V^{\otimes n} \cong \bigoplus_\lambda \mathbb{S}_\lambda V^{\otimes m_\lambda}.$$
3. *For any $g \in \mathrm{GL}(V)$, the trace of g on $\mathbb{S}_\lambda V$ is the value of the Schur polynomial on the eigenvalues $x_1, \ldots, x_k$ of g on V:*
$$\chi_{\mathbb{S}_\lambda V}(g) = s_\lambda(x_1, \ldots, x_k).$$
4. *Each $\mathbb{S}_\lambda V$ is an irreducible representation of $\mathrm{GL}(V)$.*

3 Representations of the Symplectic Group

The Schur functors $\mathbb{S}_\lambda$ also induce irreducible representations of the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$. In this case, we need to adapt the construction slightly. We will follow the outline of Weyl's construction for symplectic groups as seen in [25, Section 17.3]. Let $n \geq 0$ and $V = \mathbb{C}^{2n}$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{2n})$ be a partition of $n = \sum \lambda_i$. The symplectic form Q determines a contraction for each pair of integers $I = \{p < q\}$ of integers between 1 and n:

$$\begin{aligned} \Phi_I \colon V^{\otimes n} &\to V^{\otimes(n-2)}, \\ v_1 \otimes \cdots \otimes v_d &\mapsto Q(v_p, v_q) v_1 \otimes \cdots \otimes \hat{v}_p \otimes \cdots \otimes \hat{v}_q \otimes \cdots \otimes v_n. \end{aligned}$$

Definition B.6. Let $V^{\langle n\rangle} \subseteq V^{\otimes n}$ denote the intersection of the kernels of all of these contractions. We then set

$$\mathbb{S}_{\langle\lambda\rangle} V = V^{\langle n\rangle} \cap \mathbb{S}_\lambda V.$$

We can also see this representation as the image of the Young symmetrizer:

$$\mathbb{S}_{\langle\lambda\rangle} V = V^{\langle n\rangle} c_\lambda = \operatorname{im}(c_\lambda \colon V^{\langle n\rangle} \to V^{\langle n\rangle}).$$

We summarize some of the most useful facts about the representations $\mathbb{S}_{\langle\lambda\rangle} V$ in the following theorem.

Theorem B.7. *The space $\mathbb{S}_{\langle\lambda\rangle} V$ is nonzero if and only if $\lambda_{n+1} = 0$. In this case, $\mathbb{S}_{\langle\lambda\rangle} V$ is an irreducible representation of the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$. It is the irreducible representation of highest weight occuring in*

$$\mathrm{Sym}^{\lambda_1-\lambda_2}(\wedge^1 V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{n-1}-\lambda_g}(\wedge^{g-1} V) \otimes \mathrm{Sym}^{\lambda_g}(\wedge^g V).$$

The characters of these irreducible representations are now now longer directly given by the Schur polynomials. Rather, we need the so-called symplectic Schur polynomials.

Definition B.8. [25, A.45] For $\lambda = (\lambda_1, \dots, \lambda_k)$ and h_j the j-th complete symmetric polynomial in k variables, let $s_{\langle\lambda\rangle}$ be the polynomial given by the determinant whose i-th row is given by

$$|h_{\lambda_i-i+1}\ h_{\lambda_i-i+2} + h_{\lambda_i-i} \cdots h_{\lambda_i-i+k} + h_{\lambda_i-k+2}|.$$

Following [25, Section 24.2], let

$$J_d(x_1, \dots, x_n) = h_d(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}),$$

where h_d is the d-th complete symmetric polynomial in $2n$ variables. That makes J_d the character of the representation $\mathrm{Sym}^d(\mathbb{C}^{2n})$ of $\mathrm{Sp}(2n, \mathbb{C})$. Generalizing this,

Proposition B.9. *The character of the irreducible representation $\mathbb{S}_{\langle\lambda\rangle} V$ is the determinant of the $k \times k$ matrix whose i-th row is*

$$|J_{\lambda_i-i+1}\ J_{\lambda_i-i+2} + J_{\lambda_i-i} \cdots J_{\lambda_i-i+k} + J_{\lambda_i-i-k+2}|.$$

In other words, it is the symplectic Schur polyonmial $s_{\langle\lambda\rangle}$ evaluated in $x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}$. We can use this character formula to determine the local systems $\mathbb{V}_\lambda$ in terms of simpler local systems.

Example B.10. The character of the local system $\mathbb{V}_{1,1}$ is given by the determinant

$$\begin{vmatrix} J_1 & J_2 + J_0 \\ J_0 & J_1 \end{vmatrix}.$$

So even withous computing these polynomials, we know that

$$\mathbb{V}_{1,1} \cong \mathbb{V}_1 \otimes \mathbb{V}_1 - \mathbb{V}_0 \otimes (\mathbb{V}_2 \oplus \mathbb{V}_0),$$

since J_d equals the character of the representation $\mathrm{Sym}^d V$.

Appendix C

CURVES

In this thesis we will study the isomorphism classes of algebraic curves, by which we will mean a complete nonsingular variety over a field k of dimension one. Over the complex numbers, these correspond to Riemann surfaces, for which there is a well-known invariant called the genus, corresponding to its number of handles. Topologically, this is enough to classify these surfaces completely. We introduce the algebraic equivalent, which defines a birational invariant. By a variety, we mean a geometrically reduced and irreducible scheme, separated and of finite type over a field k.

Definition C.1. [32, III] Let X be a smooth variety over k. We define the *canonical sheaf* on X to be $\omega_X = \bigvee^n \Omega_{X/k}$, where n is the dimension of X and $\Omega_{X/k}$ is the sheaf of relative differentials of X. If X is projective and nonsingular, we define the *geometric genus* of X to be $p_g := \dim_k \Gamma(X, \omega_X)$.

This geometric genus is a birational invariant.

Theorem C.2. *[32, III-Theorem 8.19] Let C and C' be two birationally equivalent smooth projective varieties over k. Then $p_g(C) = p_g(C')$.*

Heuristically, this is the algebraic equivalent of the topological genus. Indeed, these concepts overlap when considering a smooth curve C over $\mathbb{C}$ as a Riemann surface. There is another notion of genus, which will be of use later when working with curves with nodal singularities.

Definition C.3. Let C be a curve over k. Then we define the *arithmetic genus* of X as $p_a(g) = \dim_k H^1(X, \mathcal{O}_X)$.

It is possible to define the arithmetic genus in a more general way for varieties using the Hilbert polynomial. For the case of curves, however, this is equal to the dimension of $H^1(X, \mathcal{O}_X)$. Fortunately, these notions of genus overlap whenever C is smooth.

Proposition C.4. *If C is a smooth curve, $p_a = p_g$.*

Proof. This is a consequence of Serre duality; the space $H^1(X, \mathcal{O}_X)$ is dual to $H^0(X, \omega_X)$, hence their dimensions are equal. □

When we say a curve C over the field k, there is a structure morphism $C \to k$. We can generalize this to define a family of curves parametrized by a scheme S

Definition C.5. Given $g, n \geq 0$, an n-pointed familiy of genus g curves over a scheme S is a tuple

$$(\pi\colon C \to S; p_1, \ldots, p_n\colon S \to C)$$

such that

- π is a (smooth/flat), proper, surjective, finitely presented morphism of schemes such that the fibre C_s over any geometric point $s \in S$ is a smooth/stable, projective, connected curve of arithmetic genus g.
- the morphisms $p_1, \ldots, p_n$ are pairwise disjoint sections of π, with image in the smooth locus of π.

Note that when π is smooth, it is also flat, so the flatness condition is required only in the case of stable curves.
Morphisms of families of curves are diagrams of the form

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C' \\ {\scriptstyle p_i}\big\uparrow\big\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi'}\big\uparrow\big\downarrow{\scriptstyle p_i'} \\ S & \xrightarrow[g]{} & T \end{array}$$

where

- $\varphi\, p_i' = p_i g$ for $1 \leq 1 \leq n$
- φ, π' induce an isomorphism $C' \xrightarrow{\sim} C \times_S T$

1 Curves on Quadric Surface

We have determined some results on curves of genus 5 in chapter 1. We use several results and definitions from [32, II-6] and [32, V-1] to compute the genus of curves on a quadric surface using intersection theory. As a bonus, we can use this theory to show that smooth curves of genus g exist for all $g \geq 0$. Following [32] Chapter V, can define an intersection pairing on the divisors of a surface. This unique pairing satisifies the following conditions. [32, V-Theorem 1.1] Let X be a surface, i.e. a nonsingular projective surface over an algebraically closed field k.

Theorem C.6. *There is a unique pairing* $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$*, denoted* $C.D$ *for any two divisors* $C.D$ *such that*

1. *if* C *and* D *are nonsingular curves meeting transversally, then* $C.D = \#(C \cap D)$*, the number of points of* $C \cap D$*,*
2. *it is symmetric:* $C.D = D.C$*,*
3. *it is additive:* $(C_1 + C_2).D = C_1.D + C_2.D$*,*
4. *it depends only on the linear equivalence classes: if* $C_1 \sim C_2$*, then* $C_1.D = C_2.D$*.*

Lemma C.7. *[32, V-Lemma 1.3] Let* C *be an irreducible nonsingular curve on* X *and let* D *be any curve meeting* C *transversally. Then*

$$\#(C \cap D) = \deg_C(\mathcal{L}(D) \otimes \mathcal{O}_C)$$

Proof. See [32]. □

Recall the Segre Embedding $\mathbb{P}^1_k \to \mathbb{P}^3_k$. Its image is a quadric curve defined by $xy = zw$. From this, we gather the following fact.

Lemma C.8. *Let* $Q \subseteq \mathbb{P}^3_k$ *be the quadric surface defined by* $xy = zw$*. Then the class group of* Q*,* $\operatorname{Cl}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$*.*

Proof. See [32], Example 6.6.1 from Chapter II. □

Definition C.9. Let Q be the surface as defined above. Then for any divisor D on Q, define the *type* of D to be the ordered pair of integers (a, b) corresponding to the class of D in $\operatorname{Cl}(Q)$. We say the divisor D is of type (a, b).

So whenever C is a curve on Q, we can see it as a prime divisor and therefore has a type.

Proposition C.10. *[32, V-Proposition 1.5] If* C *is a nonsingular curve of genus* g *on the surface* X*, and if* K *is the canonical divisor on* X*, then*

$$2g - 2 = C.(C + K).$$

Proof. By Proposition II-8.20 from [32], the canonical sheaf $\omega_C \cong \omega_X \otimes \mathcal{L}(C) \otimes \mathcal{O}_C$. We know the canonical divisor of a curve has degree $2g-2$, so the degree of ω_C is $2g-2$. On the other hand, by C.7, we have

$$\deg_C(\omega_X \otimes \mathcal{L}(C) \otimes \mathcal{O}_C) = C.(C+K).$$

This identity is called the *adjunction formula* □

Note that it is in particular possible to consider the self intersection D^2 of a divisor. Since this cannot clearly be stated in terms of intersection points, we use property 4 from the intersection pairing: simply choose a linearly equivalent divisor $D' \sim D$ and consider $D^2 = D.D'$.

Theorem C.11. *Let C be a curve of type (a,b) on Q. Then*

$$g(C) = ab - a - b + 1.$$

Proof. On the surface Q, choose generators $(1,0)$ and $(0,1)$ of $\operatorname{Cl} Q$ corresponding to projective lines $l, m \subseteq Q$. Since $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, we may conclude that $l.m = 1$, while $l^2 = m^2 = 0$. Now by linearity, this extends to divisors D, D' of types (a,b) and (a',b') respectively to the identity $D, D' = ab' + a'b$. Since we have embedded Q in $\mathbb{P}^3$, $\omega_X \cong \mathcal{O}_X(-2)$ by C.16. Therefore, the type of K is $(-2,-2)$, since $\mathcal{O}_X(-2) \cong \mathcal{O}_X(-2H)$ for H some hyperplane section, whose class generates $\operatorname{Cl} Q$, and the class of H is $(1,1)$. Furthermore, the type of $C+K$ is $(a-2, b-2)$. Using the adjunction formula, we get that

$$2g - 2 = a(b-2) + b(a-2), \quad \text{so}$$
$$g = ab - a - b + 1,$$

as desired. □

Corollary C.12. *For any $g \geq 0$, there exist curves of genus g.*

Proof. Let D be a divisor of type $(g+1, 2)$ on Q, the quadric surface above. Then by [32, III], Exercise 5.6, there exists a nonsingular curve Y, which as a divisor is linearly equivalent to D. Then by C.11, the genus of this curve is g. □

2 Arithmetic Genus of Singular Curves

The following result is a combination of [32, IV], exercise 1.8 and [31, 3A], exercise 3.2.

Lemma C.13. *Let X be a curve over k (algebraically closed), and let $\pi \colon \tilde{X} \to X$ be its normalisation. Let $\tilde{\mathcal{O}_P}$ be the integral closure of $\mathcal{O}_P$. Then there exists a short exact sequence of sheaves*

$$0 \to \mathcal{O}_X \to \pi_* \mathcal{O}_{\tilde{X}} \to \sum_{P \in X} \tilde{\mathcal{O}_P} / \mathcal{O}_P \to 0.$$

Proof. We know that for each $U = \operatorname{Spec} A \subseteq X$ open affine, $\pi^{-1}(U) \subseteq \tilde{X}$ is given by $\tilde{U} := \operatorname{Spec} \tilde{A}$, the affine given by the integral closure of A in its fraction field. The canonical map $\mathcal{O}_X \to \pi_* \mathcal{O}_{\tilde{X}}$ induced by the morphism of schemes π defines the first map. The second map can be given on an open $U \subseteq X$ by sending $x \in \mathcal{O}_{\tilde{X}}(\pi^{-1}U)$ to its classes in $\tilde{\mathcal{O}_P}(U)/\mathcal{O}_P(U)$. This then extends to the sheafification. To show this is exact, we can take the stalks. If we take the stalk at a point $Q \in X$, then $(\tilde{\mathcal{O}_P}/\mathcal{O}_P)_Q = 0$ for $P \neq Q$ and is the identity otherwise. Since the normalisation is given on all open affines by taking integral closure, the stalk of $\pi_* \mathcal{O}_{\tilde{X}}$ is given by the integral closure of $\mathcal{O}_Q$. So on stalks, the maps above give the sequence

$$0 \to \mathcal{O}_Q \to \tilde{\mathcal{O}_Q} \to \tilde{\mathcal{O}_Q}/\mathcal{O}_Q \to 0,$$

which is clearly exact. □

Theorem C.14. *Let X be a curve over k (algebraically closed), and let $\pi \colon \tilde{X} \to X$ be its normalisation. Let $\tilde{\mathcal{O}_P}$ be the integral closure of $\mathcal{O}_P$. Define $\delta_P = \operatorname{length}(\tilde{\mathcal{O}_P}/\mathcal{O}_P)$. Then*

$$p_a(X) = p_a(\tilde{X}) + \sum_{P \in X} \delta_P.$$

Proof. By [32, Chapter III], exercise 4.1, we know that $H^i(X, \pi_* \mathcal{O}_{\tilde{X}}) \cong H^i(\tilde{X}, \mathcal{O}_{\tilde{X}})$. Furthermore, since X is a curve, the arithmetic genus is given by $H^1(X, \mathcal{O}_X)$. Since k is algebraically closed, both $H^0(X, \mathcal{O}_X)$ and $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$ are one-dimensional. Since $\tilde{\mathcal{O}_P}/\mathcal{O}_P$ is nontrivial only if P is singular, it is clear that for $V \subseteq U$ opens in X, the restriction map $(\sum_{P \in X} \tilde{\mathcal{O}_P}/\mathcal{O}_P)(U) \to (\sum_{P \in X} \tilde{\mathcal{O}_P}/\mathcal{O}_P)(V)$ is surjective, as V contains possibly less singularities than U. The sheaf $\sum_{P \in X} \tilde{\mathcal{O}_P}/\mathcal{O}_P$ is therefore flasque, hence all its cohomologies H^i vanish for $i > 0$. The long exact sequence of cohomology associated to the short exact sequence in the lemma above is

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to \Gamma(X, \sum_{P \in X} \tilde{\mathcal{O}_P}/\mathcal{O}_P) \to$$
$$H^1(X, \mathcal{O}_X) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^1(X, \sum_{P \in X} \tilde{\mathcal{O}_P}/\mathcal{O}_P) \to \cdots$$

By our computations above, this gives the short exact sequence

$$0 \to \Gamma(X, \sum_{P \in X} \tilde{\mathcal{O}_P}/\mathcal{O}_P) \to H^1(X, \mathcal{O}_X) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to 0.$$

Since there are only finitely many singularities and cohomology commutes with finite direct sums, this given us $p_a(X) = p_a(\tilde{X}) + \sum_P \delta_P$. □

Importantly, δ_P is 1 whenever P is a node. We conclude with some results on complete intersections.

Proposition C.15. *If Y is a complete intersection of codimension at least 1 in $\mathbb{P}^n_k$ that is normal, then for all $l \geq 0$, the natural map $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) \to \Gamma(Y, \mathcal{O}_Y(l))$ is surjective.*

Proof. We will first show that such a Y is projectively normal, i.e. the ring $k[x_0, \ldots, x_n]/(I(Y))$ is integrally closed. Note that if Y is of codimension at least one, then its singular locus must be of codimension at least 2 by applying Theorem II, 8.23b from [32], and therefore the singular locus of the affine cone of Y, $C(Y)$ also has codimension at least 2, hence it is normal. Since $C(Y)$ is affine by Exercise I-2.10 in [32], every localisation at a prime of $k[x_0, \ldots, x_n]/(I(C(Y)))$ is integrally closed by definition of normal for affine schemes. As $I(C(Y))$ and $I(Y)$ are the same ideal, we conclude $k[x_0, \ldots, x_n]/(I(Y))$ is integrally closed. Now apply II-Exercise 5.14a, which says that $k[x_0, \ldots, x_n]/(I(Y)) \cong S' := \bigoplus_{l \geq 0} \Gamma(Y, \mathcal{O}_Y(l))$ since Y is projectively normal. We have that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)) = S_l$, and $\Gamma(Y, \mathcal{O}_Y(l)) = S'_l$. Since S' arises as a quotient of S, $S \to S'$ is surjective, so in particular $S_l \to S'_l$ is surjective. This proves our result □

Proposition C.16. *Let $Y \subseteq \mathbb{P}^n_k$ be a nonsingular complete intersection of hypersurfaces $H_1, \ldots, H_r$, $r < n$ such that $\deg H_i = d_i$. Then the canonical sheaf $\omega_Y \cong \mathcal{O}_Y(\sum_{i=1}^r d_i - n - 1)$.*

Proof. We begin with the case where Y itself is a hypersurface. Suppose $Y = H$ for some hypersurface H of degree d. Then by Proposition II-8.20 in [32], $\omega_Y \cong \omega_{\mathbb{P}^n_k} \otimes \mathcal{L} \otimes \mathcal{O}_Y$, where $\mathcal{L}$ is the invertible sheaf on $\mathbb{P}^n_k$ associated to Y, i.e. the sheaf denoted $\mathcal{L}(Y)$. Since $\operatorname{Pic} \mathbb{P}^n_k \cong \mathbb{Z}$ by the degree map, this sheaf is isomorphic to $\mathcal{L}(d)$. Thus, $\omega_Y \cong \mathcal{O}_{\mathbb{P}^n_k}(-n-1) \otimes \mathcal{L}(d) \otimes \mathcal{O}_Y \cong \mathcal{O}_Y(d - n - 1)$. Now repeat this formula to obtain the desired result. □

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