Master Thesis

The Moduli Space of Abelian Varieties and Its Compactifications

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Introduction

The purpose of this thesis is to study the toroidal compactification of the moduli space of principally polarized abelian varieties.
Any complex abelian variety can be regarded as a complex torus. If a complex torus admits a holomorphic embedding into $\mathbb{P}^N_{\mathbb{C}}$ for some positive integer N, then it is an algebraic variety by the theorem of Chow and thus an abelian variety. It turns out that if the torus V/L admits a Riemann form, i.e, an alternating bilinear form $E : \Lambda \times \Lambda \to \mathbb{Z}$ such that the $\mathbb{R}$-linear extension $E : V \times V \to \mathbb{R}$ satisfies $E(iv, iw) = E(v, w)$ and the associated hermitian form $H(x, y) = E(ix, y) + iE(x, y)$ is positive definite, it is an abelian variety. Using this result, we can construct the moduli space of polarized abelian varieties analytically. Let

$$\mathbb{H}_g = \{Z \in M_g(\mathbb{C}) | {}^tZ = Z, \mathrm{Im}Z > 0\}$$

be the Siegel upper half space. The symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on $\mathbb{H}_g$ biholomorphically, the quotient $A_g = \mathrm{Sp}(2g, \mathbb{Z})\backslash\mathbb{H}_g$ by $\mathrm{Sp}(2g, \mathbb{Z})$ turns out to be the moduli space of principally polarised abelian varieties over $\mathbb{C}$.
This space A_g is not compact. The first compactification is given by Satake [24]. The Satake compactification A_g^{Sat} of A_g can be written as a disjoint union

$$A_g^{\mathrm{Sat}} = A_g \amalg A_{g-1} \amalg \ldots \amalg A_0.$$

It admits a structure of projective variety and contains A_g as a Zariski open subset. However, the boundary $A_g^{\mathrm{Sat}} - A_g$ has codimension g and is highly singular.
The moduli space is a special case of locally symmetric varieties. In [3], Mumford and his collaborators constructed toroidal compactifications of locally symmetric varieties. We will use their method to construct the toroidal compactification of A_g. The boundary of the toroidal compactification $\overline{A_g}$ has codimension 1 and at worst quotient singularities.

In the first section, we discuss complex abelian varieties and construct its analytic moduli space. In the second section, we give all the necessary knowledge of toric varieties for the construction of the toroidal compactification of A_g and provide an application of a compactification of a Kummer modular surface, which is also related to the boundary of toroidal compactification of A_2. In the third section, we introduce all the background for the toroidal compactification, including the boundary components of $\mathbb{H}_g$, the structure of the stabilizing subgroups of the boundary components and a short introduction to the Satake compactification of A_g. In the final section, we first introduce the general steps of the toroidal compactification of A_g. The toroidal compactification has a stratification using the orbit-cone correspondence, which allows us to describe the toroidal compactification of A_g easily. We provide a toy example of $g = 1$ case to help understand the toroidal compactification. Then we work on the case of $g = 2$ in detail.

Contents

1 Abelian Varieties and Moduli

1.1 Abelian Varieties

In this subsection, we provide a brief introduction to abelian varieties over an arbitrary field k. The main reference for this section is Milne's notes [15]. We take the convention that an algebraic variety over k is a geometrically reduced separated scheme of finite type over k.

Definition 1.1.1. A group variety over k is an algebraic variety V over k with morphisms

$$
\begin{aligned}
m : V \times_k V &\to V && \text{(multiplication)} \\
\mathrm{inv} : V &\to V && \text{(inverse)}
\end{aligned}
$$

and an element $e \in V(k)$ such that the structure on $V(\overline{k})$ defined by m and inv is a group with identity element e.

Remark 1.1.2. We list a few properties of group varieties.

- Group varieties are nonsingular. It suffices to prove this over the algebraic closure $\overline{k}$ of k. Note that any variety contains a nonsingular dense open subvariety U. We can define the right translation $t_a : V \to V$ for $a \in V(k)$ by $x \to m(x, a)$. We can translate U such that $\cup_{a \in V(k)} t_a(U)$ covers V.
- Connected group varieties are geometrically connected. Actually, the connectedness of varieties and the existence of a k-rational point are sufficient to show geometric connectedness, see [22, tag 04KV].

Definition 1.1.3. A proper connected group variety is called an abelian variety.

Theorem 1.1.4 ([15],Corollary 1.4)**.** *The group law on an abelian variety is commutative.*

To prove this, we need an important theorem - the Rigidity Theorem:

Theorem 1.1.5 ([15], Theorem 1.1)**.** *Consider a morphism of algebraic varieties $\alpha : V \times W \to U$ and assume that V is complete and $V \times W$ is geometrically irreducible. If there are $u_0 \in U(k)$, $v_0 \in V(k)$ and $w_0 \in W(k)$ such that*

$$\alpha(V \times \{w_0\}) = \{u_0\} = \alpha(\{v_0\} \times W)$$

then $\alpha(V \times W) = \{u_0\}$.

With this theorem, we have

Corollary 1.1.6 ([15],Corollary 1.2)**.** *Every morphism of abelian varieties $\alpha : A \to B$ is the compositition of a homomorphism with a translation.*

Proof. Write the group additively, and denote e as 0. The morphism will send the k-rational point 0 of A to a k-rational point b of B. After postcomposing α with the translation t_{-b}, we may assume $\alpha(0) = 0$. Consider the map $\varphi : A \times A \to B$ given by $\varphi(a, a') = \alpha(a + a') - \alpha(a) - \alpha(a')$. It suffices to show that $\varphi = 0$. Note that φ is the difference of $\alpha \circ m$ and $m \circ (\alpha \times \alpha)$, the map φ is a morphism. Then $\varphi(A \times 0) = 0 = \varphi(0 \times A)$. By the rigidity theorem, $\varphi = 0$. $\square$

Proof of Theorem 1.1.4. Use the corollary above and consider the morphism $A \to A$ given by $a \mapsto -a$. $\square$

Another important property of abelian varieties is:

Theorem 1.1.7 ([15],Theorem 6.4)**.** *Abelian varieties are projective.*

1.2 Complex Abelian Varieties

In this section, we focus on complex abelian varieties. We will show that any complex abelian variety is a torus. And any torus which satisfies certain conditions is an abelian variety.

Definition 1.2.1. A lattice L in a complex vector space V of dimension g is a discrete subgroup of V such that the quotient group V/L is compact.

REMARK 1.2.2. A lattice L is a free abelian group of rank $2\dim_{\mathbb{C}}(V)$ and we have an isomorphism $L \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} V$.

Definition 1.2.3. A complex torus is the quotient V/Λ where V is a complex vector space and Λ is a lattice in V.

Suppose we have an abelian variety X over $\mathbb{C}$. Since it is a proper connected group variety, X_h is a compact connected complex Lie group, i.e., a compact connected complex manifold with a group structure on the underlying set such that the maps $X_h \times X_h \to X_h$ and $X_h \to X_h$ defined by $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are holomorphic. One can show that the group structure on X_h is commutative.

Theorem 1.2.4 ([18], p.1)**.** *Any compact complex connected Lie group is commutative.*

We can show that X_h is a complex torus.

Theorem 1.2.5 ([18], p.2)**.** *Let X be a compact connected Lie group. The exponential map* $\exp : V = Lie(X) \to X$ *is a surjective homomorphism of complex Lie groups with kernel a lattice* Λ*. We have an induced isomorphism* $V/\Lambda \xrightarrow{\sim} X$*, i.e., X is a complex torus.*

Conversely, we want to investigate under what conditions a complex torus can be an abelian variety over $\mathbb{C}$.

If a complex torus admits a holomorphic embedding into $\mathbb{P}^N_{\mathbb{C}}$ for some positive integer N, then it is an algebraic variety by the Theorem of Chow, see the Appendix. In particular, it is an abelian variety.

Example 1.2.6 ([25],Theorem 3.5 & Proposition 3.6)**.** Recall elliptic curves over $\mathbb{C}$. Let $V = \mathbb{C}$ and Λ a lattice in V. We can define the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

so that $\wp'(z) = \sum_{\omega \in \Lambda} -2(z-\omega)^{-3}$. Note that $\wp(z)$ and $\wp'(z)$ are both elliptic functions, i.e. meromorphic functions on $\mathbb{C}$ that satisfy $f(z+\lambda) = f(z)$ for all $z \in \mathbb{C}$ and $\lambda \in \Lambda$. For all $z \in \mathbb{C}\setminus\Lambda$, the Weierstrass $\wp$-function and its derivative $\wp'$ satisfy the relation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where $g_2 = g_2(\Lambda) = 60\sum_{\omega\in\Lambda\backslash\{0\}} \omega^{-4}$ and $g_3 = g_3(\Lambda) = 140\sum_{\omega\in\Lambda\backslash\{0\}} \omega^{-6}$. Note that $\Delta(\Lambda) = g_2^3 - 27g_3^2$ is nonzero. Then the map

$$\phi : \mathbb{C}\backslash\Lambda \to E(\mathbb{C}) \subset \mathbb{P}^2_{\mathbb{C}}, \quad z \mapsto [\wp(z) : \wp'(z) : 1]$$

maps the torus V/Λ to the elliptic curve $E = \{y^2 = 4x^3 - g_2x - g_3\} \subset \mathbb{P}^2_{\mathbb{C}}$. And ϕ is holomorphic.

By the example above, we know that every complex torus of dimension 1 is an abelian variety. It's not true for higher dimensions. We will investigate what condition makes a complex torus an abelian variety. We are now going to find the analogue of the Weierstrass $\wp$-function by relaxing the condition of the elliptic function and requiring that $f(z+\lambda) = (\text{some factor})f(z)$. These are so-called theta functions, which will be formally introduced later. Note that we can get elliptic functions by taking the ratio of two such functions with the same factor.

Recall the definition of a holomorphic line bundle:

Definition 1.2.7. Let X be a complex manifold. A (holomorphic) line bundle on X is a complex manifold $\mathcal{L}$ together with a surjective holomorphic map $\pi : \mathcal{L} \to X$ such that

(a) $\pi^{-1}(x) \cong \mathbb{C}$ for any $x \in X$;

(b) (locally trivial of rank 1): there is an open covering $X = \cup_{\alpha\in A} U_\alpha$ such that there is a biholomorphic map $\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{C}$ such that $\mathrm{pr}_1 \circ \phi_\alpha = \pi|_{\pi^{-1}(U_\alpha)}$ and the transition maps

$$\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}, \quad (x, z) \mapsto (x, f_{\alpha\beta}(x)z)$$

are given by $f_{\alpha\beta}(x) \in \mathrm{GL}(\mathbb{C}) = \mathbb{C}^*$.

Definition 1.2.8. A global section of a line bundle $\mathcal{L}$ is a map $s : X \to \mathcal{L}$ such that $\pi \circ s = \mathrm{id}$.

Remark 1.2.9. The space of global sections is a $\mathbb{C}$-vector space. We denote it as $\Gamma(X, \mathcal{L})$ or $H^0(X, \mathcal{L})$.

If the line bundle $\mathcal{L}$ is basepoint-free and there are linearly independent global sections $s_0, \cdots s_n \in H^0(X, \mathcal{L})$ where $n = \dim H^0(X, \mathcal{L}) - 1$, we obtain a morphism from X to $\mathbb{P}^n$ given by $(s_0 : \cdots : s_n)$. Hence it is meaningful to look at line bundles. We will also discuss the ampleness conditions on line bundles making the morphism a closed immersion.

Definition 1.2.10. A line bundle $\mathcal{L}$ is trivial if it is biholomorphic to $\mathbb{C} \times X$.

Proposition 1.2.11 ([7], Lemma 2.1.1)**.** *All holomorphic line bundles on $\mathbb{C}^N$ are trivial.*

If we have a holomorphic line bundle on a torus $X = V/\Lambda$, then the pullback $\pi^*(\mathcal{L})$ where π is the projection $V \to X = V/\Lambda$ is trivial by the above proposition.

Consider the canonical action of Λ on $\pi^*(\mathcal{L})$, which is an action such that the quotient $\pi^*(\mathcal{L})/\Lambda \cong V \times \mathbb{C}/\Lambda = \mathcal{L}$. Then λ has to act on V by translation and act on $\mathbb{C}$ by a nowhere vanishing holomorphic function $f(\lambda, -) \in H^0(V, \mathcal{O}_V^*)$ to fix the base. The action is given by

$$\lambda : V \times \mathbb{C} \to V \times \mathbb{C}, \qquad (v, z) \to (v + \lambda, f(\lambda, v)z).$$

The function f has to satisfy the 1-cocycle condition to make the action well-defined, i.e.,

$$f(\lambda+\mu, v) = f(\lambda, v+\mu)f(\mu, v)$$

These functions form an abelian group under multiplication $Z^1(\Lambda, H^0(V, \mathcal{O}_V^*))$ and we call them **factors of automorphy**. If we choose another trivialization $\pi^*(\mathcal{L}) \to V \times \mathbb{C}$ by multiplying with nowhere vanishing holomorphic function h on V, the change to $f(\lambda, z)$ is that it is multiplied by a coboundary $h(\lambda+v)h(v)^{-1} \in B^1(\Lambda, H^0(V, \mathcal{O}_V^*))$. Thus we defined a map from $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ to $H^1(\Lambda, H^0(V, \mathcal{O}_V^*))$. Conversely, if we have a cocycle $f \in H^1(\Lambda, H^0(V, \mathcal{O}_V^*))$, we can define a line bundle $\mathcal{L}$ on X as the quotient of $V \times \mathbb{C}$ by the action of Λ given by $(v, z) \mapsto (v+\lambda, f(\lambda, v)z)$. Therefore, we have an isomorphism

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) \cong H^1(\Lambda, H^0(V, \mathcal{O}_V^*)).$$

This is saying that any holomorphic line bundle can be described by means of factors of automorphy.

Next, we will introduce the first Chern class. On any complex analytic space $X = V/\Lambda$, we have a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{e^{2\pi i -}} \mathcal{O}_X^* \longrightarrow 0$$

It has an associated long exact sequence, partially given by

$$0 \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow \cdots$$

Definition 1.2.12. Let $\mathcal{L}$ be a line bundle. The first Chern class of $\mathcal{L}$ is the image $c_1(\mathcal{L})$ of $\mathcal{L} \in H^1(X, \mathcal{O}_X^*)$ in $H^2(X, \mathbb{Z})$.

Remark 1.2.13. We define the Neron-Severi group $\mathrm{NS}(X)$ as the image $c_1(H^1(X, \mathcal{O}_X^*))$ and $Pic^0(X)$ as the kernel of c_1.

The following theorem shows how to calculate the first Chern class of line bundle in terms of a factor of automorphy associated to that line bundle.

Theorem 1.2.14 ([7], Theorem 2.1.2)**.** *There is a canonical isomorphism $H^2(X, \mathbb{Z}) \to Alt^2(\Lambda, \mathbb{Z})$, which maps the first Chern class $c_1(\mathcal{L})$ of a line bundle $\mathcal{L}$ on X with a factor of automorphy $f = exp(2\pi i g)$ to the alternating form*

$$E_{\mathcal{L}}(\lambda, \mu) = g(\mu, v+\lambda) + g(\lambda, v) - g(\lambda, v+\mu) - g(\mu, v)$$

for all $\lambda, \mu \in \Lambda$ and $v \in V$.

We can extend E $\mathbb{R}$-linearly to a map $V \times V \to \mathbb{R}$:

Proposition 1.2.15 ([18], p.18)**.** *If we extend E $\mathbb{R}$-linearly to a map $V \times V \to \mathbb{R}$, then E satisfies $E(ix, iy) = E(x, y)$ for all $x, y \in V$.*

The following lemma shows that the alternating form corresponding to the first Chern class of a line bundle is just the imaginary part of a hermitian form. Recall a hermitian form on a complex vector space V is a function $H : V \times V \to \mathbb{C}$ such that H is $\mathbb{C}$-linear in the first argument and $H(u, v) = \overline{H(v, u)}$ for all $v, u \in V$.

Lemma 1.2.16. *There is 1-1 correspondence between the hermitian forms H on a complex vector space V and the real alternating forms E on V satisfying the identity $E(ix, iy) = E(x, y)$ given by*

$$E(x, y) = ImH(x, y), \quad H(x, y) = E(ix, y) + iE(x, y).$$

Proof. Given E, we have

$$H(x, y) = E(ix, y) + iE(x, y) = -E(iy, -x) - iE(y, x) = \overline{H(y, x)}$$

hence H is hermitian. On the other hand, given H, the form $E = \mathrm{Im}H$ is alternating and $E(ix, iy) = \mathrm{Im}H(ix, iy) = \mathrm{Im}H(x, y) = E(x, y)$. □

Conversely, given an alternating form E with integral values on $\Lambda \times \Lambda$ that satisfies $E(ix, iy) = E(x, y)$, we seek to find the corresponding factor of automorphy as explicitly as possible.

Lemma 1.2.17 ([18],p.20)**.** *Let H be a hermitian form on V such that $E = im\ H$ has integral values on $\Lambda \times \Lambda$. There exists a map $\chi : \Lambda \to U(1) = \{z \in \mathbb{C}^* : |z| = 1\}$, called a semicharacter for H, which satisfies*

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu) \exp(i\pi ImH(\lambda, \mu)) = \chi(\lambda)\chi(\mu)(-1)^{ImH(\lambda,\mu)}$$

for $\lambda, \mu \in \Lambda$. If we put

$$a(\lambda, v) := \chi(\lambda) \exp(\pi H(v, \lambda) + \frac{\pi}{2}H(\lambda, \lambda)),$$

then a is a factor of automorphy.

Remark 1.2.18. We call this factor of automorphy associated to (H, χ) the canonical factor for $L(H, \chi)$.

Definition 1.2.19. $L(H, \chi)$ is the quotient of $V \times \mathbb{C}$ for the action of Λ given by $\lambda : (v, z) \mapsto (v + \lambda, \chi(\lambda) \exp(\pi H(v, \lambda) + \frac{\pi}{2}H(\lambda, \lambda))z)$.

Remark 1.2.20. We denote the set of all pairs (H, χ) as $\mathcal{P}(\Lambda)$. Note that if a_i corresponds to the pair (H_i, χ_i), then a_1a_2 corresponds to the pair $(H_1 + H_2, \chi_1\chi_2)$. Hence we have an isomorphism

$$L(H_1, \chi_1) \otimes L(H_2, \chi_2) \cong L(H_1 + H_2, \chi_1\chi_2).$$

Theorem 1.2.21 (Appell-Humbert,cf. [18, p.20])**.** *There is a canonical isomorphism of exact sequences*

$$\begin{tikzcd}
1 \arrow[r] & Hom(\Lambda, U(1)) \arrow[r, "\iota"] \arrow[d, "L,\cong"] & \mathcal{P}(\Lambda) \arrow[r, "pr"] \arrow[d, "L,\cong"] & NS(X) \arrow[r] \arrow[d, "="] & 0 \\
1 \arrow[r] & Pic^0(X) \arrow[r] & Pic(X) \arrow[r, "c_1"] & NS(X) \arrow[r] & 0
\end{tikzcd}$$

where $NS(X)$ on the top row is regarded as the group of hermitian forms H such that $Im\ H$ has integral values on $\Lambda \times \Lambda$.

The significance of the Appell-Humbert Theorem is that instead of thinking of line bundles, we can think of hermitian forms on complex vector space and semicharacters, which is easy to deal with.

We now focus on the global sections of the line bundle $\mathcal{L}$ of a torus $X = V/\Lambda$. For any line bundle $\mathcal{L}$, we have a natural isomorphism $H^0(X, \mathcal{L}) \xrightarrow{\sim} H^0(V, \pi^*(\mathcal{L}))^\Lambda$. For the trivialization $\alpha : \pi^*(\mathcal{L}) \xrightarrow{\sim} V \times \mathbb{C}$, we have an induced isomorphism: $H^0(V, \pi^*(\mathcal{L}))^\Lambda \to H^0(V, V \times \mathbb{C})^\Lambda$. As we have discussed before, for any trivialization, we have an associated factor of automorphy f, the elements in $H^0(V, V \times \mathbb{C})^\Lambda$ are the holomorphic functions θ on V invariant under the action of Λ, i.e.,

$$\theta(v + \lambda) = f(\lambda, v)\theta(v).$$

Such a function is called a **theta function**. If $\mathcal{L} = L(H, \chi)$ where H is a hermitian form and χ a semicharacter, then we have

$$\theta(v + \lambda) = \chi(\lambda) \exp(\pi H(v, \lambda) + \frac{\pi}{2} H(\lambda, \lambda))\theta(z), \qquad z \in V, \lambda \in \Lambda.$$

We call it a **canonical theta function**.

By studying theta functions, we can prove that

Lemma 1.2.22 ([18], p.25)**.** *If H is degenerate, the line bundle $L(H, \chi)$ cannot be ample.*

Proof. Suppose H is degenerate. Then $N = \{x \in V | H(x, y) = 0, \forall y \in V\} = \{x \in V | E(x, y) = 0, \forall y \in V\}$ is nonempty and is a complex subspace of V, and $N \cap \Lambda$ is a lattice in N. If θ is a canonical theta function, then

$$\theta(v + \lambda) = \chi(\lambda)\theta(v), \qquad \forall \lambda \in N \cap U.$$

If K is a compact subset of N such that $N = K + N \cap U$, we have

$$|\theta(v + w)| \leq \sup_{\zeta \in K} |\theta(v + \zeta)| = c$$

for all $w \in N$. By the Maximum modulus principle, we have $\theta(v + w) = \theta(v)$ for $w \in N$ and θ is constant modulo N. This implies that $\theta(v + \lambda) = \theta(v)$ for all $\lambda \in N \cap U$, hence $\chi(\lambda) = 1$. If $s \in H^0(X, \mathcal{L})$ is the corresponding section for the line bundle $\mathcal{L}$, then $s(x) = s(x + y)$ if $y \in N/(N \cap \Lambda)$, which means that s doesn't separate points. Since N is the same for $\mathcal{L}^{\otimes k}$, the line bundle $\mathcal{L}$ is not ample. □

Lemma 1.2.23 ([18], p.26)**.** *If H is not positive definite, the line bundle $L(H, \chi)$ cannot be ample.*

Proof. Let W be a complex subspace in V such that $H(w, w) < 0$ for all $w \in W$ and $w \neq 0$. Let K be a compact subset of V with $V = \Lambda + K$. Let $v_0 \in V$ and $w \in W$, and write $w = k + \lambda, k \in K, \lambda \in \Lambda$. Then we have

$$|\theta(v_0 + w)| = |\theta(v_0 + k + \lambda)| = |\theta(v_0 + k)| \exp(\pi \mathrm{Re} H(v_0 + k, \lambda) + \frac{1}{2}\pi H(\lambda, \lambda)).$$

And

$$\begin{aligned}
&\mathrm{Re}H(v_0+k,\lambda)+\frac{1}{2}H(\lambda,\lambda)\\
=&\mathrm{Re}H(v_0+k,w)-\mathrm{Re}H(v_0+k,k)+\frac{1}{2}H(w,w)+\frac{1}{2}H(k,k)-\mathrm{Re}H(w,k)\\
=&\frac{1}{2}H(w,w)+\mathrm{Re}H(v_0,w)+\text{a function with respect to } k \text{ and } v_0.
\end{aligned}$$

Fix v_0. Since $\frac{1}{2}H(w,w)$ is a real negative definite quadratic form in w, $\mathrm{Re}H(v_0,w)$ is linear in w and the function with respect to k and v_0 is bounded, we have that $\mathrm{Re}H(v_0+k,\lambda)+\frac{1}{2}H(\lambda,\lambda)$ tends to $-\infty$ as w tends to ∞. By the Maximum modulus principle, we have $\theta(v_0+w)=0$ hence $\theta\equiv 0$. Then $L(H,\chi)$ has no non-zero section. $\square$

Proposition 1.2.24 ([18], p.26)**.** *When H is positive definite and $E = Im\ H$, then*

$$\dim H^0(X, L(H,\chi)) = \dim[\textit{space of theta-functions with respect to } (H,\chi)] = \sqrt{\det E}.$$

Theorem 1.2.25 (Lefschetz,[18], p.26)**.** *Let $\mathcal{L} = L(H,\chi)$ be a line bundle on a complex torus $X = V/\Lambda$. The following statements are equivalent:*

1. *The hermitian form H is positive definite.*
2. *The line bundle $\mathcal{L} = L(H,\chi)$ is ample, and $\mathcal{L}^{\otimes n}$ is very ample for each $n \geq 3$.*

Definition 1.2.26. A polarization or a Riemann form on a complex torus $X = V/\Lambda$ is a positive definite hermitian form H on V such that $\mathrm{Im}H$ has integral values on $\Lambda\times\Lambda$. The pair (X,H) is called a polarized abelian variety.

Remark 1.2.27. The above theorem shows that if a polarization on a complex torus $X = V/\Lambda$ exists, then the complex torus is an abelian variety.

1.3 Moduli

In this section, we discuss the moduli spaces of polarized abelian varieties. Here we only adopt a naive interpretation of moduli spaces - a moduli space is a complex analytic space or complex manifold where there is a one-to-one correspondence between the points in this space and isomorphism classes of certain types of objects.

To describe a complex torus, we introduce the concept of a period matrix.

Definition 1.3.1. Let $e_1,\cdots,e_g$ be a basis of V and $\lambda_1,\cdots\lambda_{2g}$ be a basis of Λ. We can write λ in terms of the basis $e_1,\cdots e_g:\lambda_i=\sum_{j=1}^{g}\lambda_{ji}e_j$. The matrix

$$\Pi = \begin{pmatrix} \lambda_{11} & \cdots & \cdots & \lambda_{1,2g}\\ \vdots & & & \vdots \\ \lambda_{g1} & \cdots & \cdots & \lambda_{g,2g}\end{pmatrix}$$

is called the period matrix.

Not all matrices $\Pi \in M_{g\times 2g}(\mathbb{C})$ are period matrices for some complex tori:

Proposition 1.3.2 ([7], Proposition 1.1.2)**.** *$\Pi \in M_{g\times 2g}(\mathbb{C})$ is the period matrix of a complex torus if and only if the matrix $P = \begin{pmatrix}\Pi \\ \bar{\Pi}\end{pmatrix} \in M_{2g}(\mathbb{C})$ is nonsingular, where $\bar{\Pi}$ denotes the complex conjugate matrix.*

Proof. Π is a period matrix if and only if the column vectors of Π are linearly independent over $\mathbb{R}$.
Suppose the columns of Π are linearly dependent over $\mathbb{R}$. Then there is a nonzero $x \in \mathbb{R}^{2g}$ and $\Pi x = 0$, and so $Px = 0$. This implies that $\det P = 0$. Conversely, suppose P is singular, then there exist some $x, y \in \mathbb{R}^{2g}$, not both zero, such that $P(x+iy) = 0$. But then we have $\Pi(x+iy) = 0$ and $\Pi(x-iy) = \overline{\bar{\Pi}(x+iy)} = \Pi(x-iy) = 0$ which implies that $\Pi x = \Pi y = 0$. Hence the columns of Π are linearly dependent over $\mathbb{R}$. □

We can write the complex torus $X = V/\Lambda$ as $X = \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$. Recall a complex torus is an abelian variety if it admits a polarization. We start with an arbitrary nondegenerate alternating form E on Λ. Denote its matrix with respect to the basis $\lambda_1, ..., \lambda_{2g}$ as A. Extend E $\mathbb{R}$-linear to $\Lambda \otimes \mathbb{R} = \mathbb{C}^g$. Define $H : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ by

$$H(u,v) = E(iu,v) + iE(u,v).$$

We want to check if the form H is a hermitian form and positive definite.

Theorem 1.3.3 ([7], Theorem 4.2.1)**.** *A nondegenerate alternating matrix $A \in M_{2g}(\mathbb{Z})$ determines a polarization if and only if*

i) $\Pi A^{-1} {}^t\Pi = 0$,

ii) $i\Pi A^{-1} {}^t\bar{\Pi} > 0$.

Remark 1.3.4. The conditions *i)* and *ii)* are called Riemann relations.

Lemma 1.3.5 ([7], Lemma 4.2.2)**.** *The form H is a hermitian form on $\mathbb{C}^g$ if and only if $\Pi A^{-1} {}^t\Pi = 0$.*

Proof. By Theorem 1.2.14, the form H is hermitian form if and only if $E(iu, iv) = E(u, v)$ for all $u, v \in \mathbb{C}^g$. Let

$$I = \begin{pmatrix}\Pi \\ \bar{\Pi}\end{pmatrix}^{-1} \begin{pmatrix} i1 & 0 \\ 0 & -i1\end{pmatrix} \begin{pmatrix}\Pi \\ \bar{\Pi}\end{pmatrix}.$$

The matrix satisfies $i\Pi = \Pi I$. Since $E(\Pi x, \Pi y) = {}^t x A y$ for all $x, y \in \mathbb{R}^{2g}$, the form H is hermitian if and only if

$${}^t x A y = E(\Pi x, \Pi y) = E(i\Pi x, i\Pi y) = E(\Pi I x, \Pi I y) = {}^t x {}^t I A I y.$$

or equivalently $A = {}^t I A I$. This says

$$\begin{pmatrix} i1 & 0 \\ 0 & -i1\end{pmatrix} \left(\begin{pmatrix}\Pi \\ \bar{\Pi}\end{pmatrix} A^{-1} \begin{pmatrix} {}^t\Pi & {}^t\bar{\Pi}\end{pmatrix} \right)^{-1} \begin{pmatrix} i1 & 0 \\ 0 & -i1\end{pmatrix} = \left(\begin{pmatrix}\Pi \\ \bar{\Pi}\end{pmatrix} A^{-1} \begin{pmatrix} {}^t\Pi & {}^t\bar{\Pi}\end{pmatrix} \right)^{-1}$$

and hence $\Pi A^{-1} {}^t\Pi = -\Pi A^{-1} {}^t\Pi$ which implies that $\Pi A^{-1} {}^t\Pi = 0$. □

Lemma 1.3.6 ([7], Lemma 4.2.3). *Suppose the form H is hermitian. Then $2i(\overline{\Pi}A^{-1t}\Pi)^{-1}$ is the matrix of H with respect to the basis $\lambda_1, ..., \lambda_{2g}$. In particular H is positive definite if and only if $i\Pi A^{-1t}\overline{\Pi} > 0$.*

Proof. Let $u = \Pi x$ and $v = \Pi y$ with $x, y \in \mathbb{R}^{2g}$. Use the same notation as above. Note that we have $\Pi A^{-1t}\Pi = 0$. Now compute

$$
\begin{aligned}
E(iu, v) &= E(i\Pi x, y) = E(\Pi I x, y) = {}^t x^t I A y \\
&= {}^t\begin{pmatrix} u \\ \overline{u} \end{pmatrix} {}^tP^{-1}\, {}^tIAP^{-1} \begin{pmatrix} v \\ \overline{v} \end{pmatrix} = {}^t\begin{pmatrix} u \\ \overline{u} \end{pmatrix} \begin{pmatrix} i1 & \\ & -i1 \end{pmatrix} \left(\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} A^{-1} \begin{pmatrix} {}^t\Pi & {}^t\overline{\Pi} \end{pmatrix} \right)^{-1} \begin{pmatrix} v \\ \overline{v} \end{pmatrix} \\
&= {}^t\begin{pmatrix} u \\ \overline{u} \end{pmatrix} \begin{pmatrix} i1 & \\ & -i1 \end{pmatrix} \begin{pmatrix} \Pi A^{-1t}\Pi & \Pi A^{-1t}\overline{\Pi} \\ \overline{\Pi}A^{-1t}\Pi & \overline{\Pi}A^{-1t}\overline{\Pi} \end{pmatrix}^{-1} \begin{pmatrix} v \\ \overline{v} \end{pmatrix} \\
&= {}^t\begin{pmatrix} u \\ \overline{u} \end{pmatrix} \begin{pmatrix} & i\left(\overline{\Pi}A^{-1t}\Pi\right)^{-1} \\ -i\left(\Pi A^{-1t}\overline{\Pi}\right)^{-1} & \end{pmatrix} \begin{pmatrix} v \\ \overline{v} \end{pmatrix} \\
&= i^t u \left(\overline{\Pi}A^{-1t}\Pi\right)^{-1} \overline{v} - i^t\overline{u} \left(\Pi A^{-1t}\overline{\Pi}\right)^{-1} v.
\end{aligned}
$$

Similarly, we can compute

$$
\begin{aligned}
E(u, v) &= E(\Pi x, \Pi y) = {}^t x^t I A y \\
&= {}^t\begin{pmatrix} u \\ \overline{u} \end{pmatrix} {}^tP^{-1} A P^{-1} \begin{pmatrix} v \\ \overline{v} \end{pmatrix} = {}^t\begin{pmatrix} u \\ \overline{u} \end{pmatrix} \left(\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} A^{-1} \begin{pmatrix} {}^t\Pi & {}^t\overline{\Pi} \end{pmatrix} \right)^{-1} \begin{pmatrix} v \\ \overline{v} \end{pmatrix} \\
&= {}^t\begin{pmatrix} u \\ \overline{u} \end{pmatrix} \begin{pmatrix} & \left(\overline{\Pi}A^{-1t}\Pi\right)^{-1} \\ \left(\Pi A^{-1t}\overline{\Pi}\right)^{-1} & \end{pmatrix} \begin{pmatrix} v \\ \overline{v} \end{pmatrix} \\
&= {}^t u \left(\overline{\Pi}A^{-1t}\Pi\right)^{-1} \overline{v} + {}^t\overline{u} \left(\Pi A^{-1t}\overline{\Pi}\right)^{-1} v.
\end{aligned}
$$

so

$$H(u, v) = E(iu, v) + iE(u, v) = 2i^t u(\overline{\Pi}A^{-1t}\Pi)^{-1}\overline{v}.$$

Also note that the inverse and the transpose of a positive definite matrix is still positive definite and ${}^tA = -A$, hence H is positive definite if and only if $i\Pi A^{-1t}\overline{\Pi}$ is positive definite. □

To simplify our problem, we can choose a basis such that E has a simple form and write the period matrix with respect to the basis. We now need a result from Frobenius.

Proposition 1.3.7. *Let Λ be a free finitely generated $\mathbb{Z}$-module and $E : \Lambda \times \Lambda \to \mathbb{Z}$ be a nondegenerate alternating form. Then there exists a basis $\lambda, ..., \lambda_g, \mu_1, ..., \mu_g$ of Λ such that the matrix of E with respect to this basis is*

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = diag(d_1, ..., d_g)$ with integers $d_i \geq 0$ and $d_i \mid d_{i+1}$ for $i = 1, ..., g-1$.

We refer to proposition 3.1 of [6] for a proof of this.

Definition 1.3.8. An abelian variety is called principally polarised if $D = I$.

REMARK 1.3.9. The vector $(d_1, ..., d_g)$ and the matrix D are called the **type** of corresponding line bundle $\mathcal{L}$. The basis $\lambda_1, ..., \lambda_g, \mu_1, ..., \mu_g$ are called a **symplectic basis** of Λ. That the line bundle $\mathcal{L}$ is nondegenerate implies that the form H and thus E, is nondegenerate. This is equivalent to say that $d_i > 0$ for $i = 1, ..., g$.

Define $e_i = \mu_i/d_i$ for $i = 1, ..., g$. The vectors $e_1, ..., e_g$ form a $\mathbb{C}$-basis for V. With respect to the basis, the period matrix is of the form

$$\Pi = (Z, D)$$

for some $Z \in M_g(\mathbb{C})$.

Proposition 1.3.10 ([7], Proposition 8.1.1)**.**

a) ${}^tZ = Z$ and $ImZ > 0$,

b) $(ImZ)^{-1}$ is the matrix of the hermitian form.

Proof. These are just the Riemann Relations. The matrix of H is $2i(\overline{\Pi}\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}{}^t\Pi)^{-1} = (\mathrm{Im}Z)^{-1}$. □

Thus we have seen that a polarized abelian variety of type D with symplectic basis corresponds to a point in the following set:

Definition 1.3.11. The Siegel upper half-plane of degree g is defined to be

$$\mathbb{H}_g := \{Z \in M_g(\mathbb{C}) : {}^tZ = Z, \mathrm{Im}Z > 0\}.$$

REMARK 1.3.12. This is an open subset of the symmetric matrices, hence its dimension is $g(g+1)/2$.

Conversely, given a type D, any Z determines a polarized abelian variety with symplectic basis.

Proposition 1.3.13 ([7], Proposition 8.1.2)**.** *Given a type D, there is 1-1 correspondence between the points on the Siegel upper half plane $\mathbb{H}_g$ and polarized abelian varieties of type D with a symplectic basis.*

Proof. It suffices to show that given a type D and an element $Z \in \mathbb{H}_g$, we can find a polarized abelian variety with a symplectic basis.
Define a hermitian form H_Z by the matrix $(\mathrm{Im}Z)^{-1}$ with respect to the standard basis of $\mathbb{C}^g$ and Λ_Z be the lattice $(Z, D)\mathbb{Z}^{2g}$. Claim that H_Z is a polarization of type D. It is already positive definite. Let $\lambda_1, ..., \lambda_g, \mu_1, ..., \mu_g$ be the columns of (Z, D) and they are a basis of Λ_Z. With respect to this basis, $\mathrm{Im}H_Z|_{\Lambda_Z \times \Lambda_Z}$ is given by the matrix

$$\mathrm{Im}({}^t(Z, D)(\mathrm{Im}Z)^{-1}\overline{(Z, D)}) = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}.$$

□

Now we want to get rid of the choice of basis and construct an analytic moduli space for the polarized abelian varieties.

Definition 1.3.14. $G_D := \{M \in Sp_{2g}(\mathbb{Q}) | {}^tM\Lambda_D \subseteq \Lambda_D\}$ where

$$\Lambda_D = \begin{pmatrix} 1_g & \\ & D \end{pmatrix} \mathbb{Z}^{2g}.$$

Proposition 1.3.15 ([7],Proposition 8.1.2)**.** *Denote* $\Lambda_Z := (Z, D)\mathbb{Z}^{2g}$*,* $X_Z := \mathbb{C}^g/\Lambda_z$ *and* H_Z *a hermitian form whose matrix with respect to the standard basis of* $\mathbb{C}^g$ *is* $(ImZ)^{-1}$*. The polarized abelian varieties* (X_Z, H_Z) *and* $(X_{Z'}, H_{Z'})$ *of type* D *are isomorphic if and only if* $Z' = (\alpha Z + \beta)(\gamma Z + \delta)^{-1}$ *for some* $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_D$*.*

Before proving this, we need a lemma:

Lemma 1.3.16 ([7],Proposition 1.2.1)**.** *Let* $h : X = V/\Lambda \to X' = V'/\Lambda'$ *be a holomorphic map.*

a) There is a unique homomorphism $f : X \to X'$ *such that* $h = t_{h(0)}f$ *where* $t_{h(0)} : X \to X, x \mapsto x + h(0)$*.*

b) There is a unique $\mathbb{C}$*-linear map* $F : V \to V'$ *with* $F(\Lambda) \subset \Lambda'$ *induced from the homomorphicm* f*.*

This lemma gives an injective homomorphism of abelian groups

$$\rho_a : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{C}}(V, V'), f \mapsto F,$$

the analytic representation of $\mathrm{Hom}(X, X')$, and an injective homomorphism (since the restriction $F|_\Lambda$ determines F hence f completely)

$$\rho_r : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda'), f \mapsto F|_\Lambda$$

the rational representation of $\mathrm{Hom}(X, X')$.

Proof of Proposition 1.3.15 . Suppose $Z, Z' \in \mathbb{H}_g$ and there is an isomorphism φ of polarized abelian varieties $\varphi : (X_{Z'}, H_{Z'}) \to (X_Z, H_Z)$ which means ϕ is an isomorphism between the complex tori $X_{Z'}$ and X_Z and $\varphi^*H_Z = H_{Z'}$.
Let $A \in M_{g\times g}(\mathbb{C})$ and $R \in M_{2g\times 2g}(\mathbb{Z})$ denote the matrices of the analytic and rational representation of φ with respect to the standard basis of $\mathbb{C}^g$ and the symplectic bases of $\Lambda_{Z'}$ and Λ_Z. Since $\rho_a(\varphi)(\Lambda') \subset \Lambda$, we have

$$A(Z', D) = (Z, D)R.$$

Define

$$N = \begin{pmatrix} 1_g & \\ & D \end{pmatrix} R \begin{pmatrix} 1_g & \\ & D \end{pmatrix}^{-1} = {}^t\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1}$$

with $\alpha, \beta, \gamma, \delta \in M_{g\times g}(\mathbb{Q})$. Then the relation

$$A(Z', D) = (Z, D)R = (Z, I)\begin{pmatrix} 1_g & \\ & D \end{pmatrix} R$$

is equivalent to

$$AZ' = Z^t\alpha + {}^t\beta \quad \text{and} \quad A = Z\,{}^t\gamma + {}^t\delta$$

Since φ is an isomorphism, the matrix ${}^tA = \gamma Z + \delta$ is invertible. Thus we can write

$$Z' = {}^tZ' = {}^t(Z\,{}^t\alpha + {}^t\beta)\,{}^tA^{-1} = (\alpha Z + \beta)(\gamma Z + \delta)^{-1}.$$

Taking imaginary parts of $\varphi^* H_Z = H_{Z'}$, we have ${}^tR\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} R = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$. In terms of N, this is ${}^tN\begin{pmatrix} 0 & 1_g \\ 1_g & 0 \end{pmatrix} N = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$. Hence N is contained in the symplectic group

$$Sp_{2g}(\mathbb{Q}) = \left\{ M \in M_{2g\times 2g}(\mathbb{Q}) |\, {}^tM \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \right\}.$$

Also note that the construction of N from (1) implies that $N\Lambda_D \subseteq \Lambda_D$ since $R \in M_{2g\times 2g}(\mathbb{Z})$. The group $Sp_{2g}(\mathbb{Q})$ is invariant under transposition, the matrix $M := {}^tN$ is an element of the group

$$G_D = \{M \in Sp_{2g}(\mathbb{Q}) |\, {}^tM\Lambda_D \subseteq \Lambda_D\}.$$

Conversely, if we have $Z' = (\alpha Z+\beta)(\gamma Z+\delta)^{-1}$, then $\begin{pmatrix} 1_g & \\ & D \end{pmatrix}^{-1} {}^tM \begin{pmatrix} 1_g & \\ & D \end{pmatrix}$ is the rational representation of an isomorphism $(X_{Z'}, H_{Z'}) \to (X_Z, H_Z)$. □

We can check the action in the proof above $Z \to (\alpha Z + \beta)(\gamma Z + \delta)^{-1}$ for $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is well-defined.

Lemma 1.3.17 ([7], Lemma 8.2.1)**.** *Let R be a commutative ring with 1.*

a) The group $Sp_{2g}(R)$ is closed under transposition.

b) For a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sp_{2g}(R)$ the following conditions are equivalent:

- *(i) $M \in Sp_{2g}(R)$.*
- *(ii) ${}^t\alpha\gamma$ and ${}^t\beta\delta$ are symmetric and ${}^t\alpha\delta - {}^t\gamma\beta = 1_g$.*
- *(iii) $\alpha\,{}^t\beta$ and $\gamma\,{}^t\delta$ are symmetric and $\alpha\,{}^t\delta - \beta\,{}^t\gamma = 1_g$.*

Proof. If $M \in Sp_{2g}(R)$, then ${}^tM\begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$. Hence $M^{-1} = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} {}^tM \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$. Since $M^{-1} \in Sp_{2g}(R)$, use the ralation ${}^tM^{-1}\begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} M^{-1} = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$, we can get ${}^tM\begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$.
Statement b) follows from the definition and a). □

Proposition 1.3.18 ([7],Proposition 8.2.2)**.** *The group $Sp_{2g}(\mathbb{R})$ acts biholomorphically on $\mathbb{H}_g$ by*

$$Z \mapsto M(Z) = (\alpha Z + \beta)(\gamma Z + \delta)^{-1}$$

for all $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sp_{2g}(\mathbb{R})$.

Proof. Claim that the matrix $\gamma Z + \delta$ is invertible. Note that by Lemma 1.3.17, we can get

$${}^t\overline{(\gamma Z + \delta)}(\alpha Z + \beta) - {}^t\overline{(\alpha Z + \beta)}(\gamma Z + \delta) = Z - \overline{Z} = 2i\mathrm{Im}Z.$$

Suppose $(\gamma z + \delta)v = 0$ for some $v \in \mathbb{C}^g$. Then we have ${}^t\overline{v}(\mathrm{Im}Z)v = 0$ and thus $v = 0$, since $\mathrm{Im}Z > 0$. Hence $(\gamma z + \delta)$ is invertible and $M(Z)$ is well-defined.
Now we check $M(Z) \in \mathbb{H}_g$. We have

$${}^t(\gamma Z + \delta)(M(Z) - {}^tM(Z))(\gamma Z + \delta) = Z - {}^tZ = 0.$$

Thus $M(Z)$ is symmetric.
To check $M(Z)$ is positive definite, note that

$$\begin{aligned} {}^t\overline{(\gamma Z + \delta)}\mathrm{Im}M(Z)(\gamma Z + \delta) &= \frac{1}{2i}\,{}^t\overline{(\gamma Z + \delta)}(M(Z) - \overline{M(Z)})(\gamma Z + \delta) \\ &= \frac{1}{2i}\,{}^t\overline{(\gamma Z + \delta)}(M(Z) - {}^t\overline{M(Z)})(\gamma Z + \delta) \\ &= \mathrm{Im}Z. \end{aligned}$$

By direct calculation, one can show that $M_1(M_2(Z)) = (M_1M_2)(Z)$ for all $M_1, M_2 \in G_D$ and $Z \in \mathbb{H}_g$. □

In addition, we can show that

Proposition 1.3.19 ([7], Proposition 8.2.5)**.** *Any discrete subgroup $G \subset Sp_{2g}(\mathbb{R})$ acts properly and discontinuously on $\mathbb{H}_g$.*

By Theorem A.0.7, we know that $\mathbb{H}_g/G_D$ is a complex analytic space.

Corollary 1.3.20. *The complex analytic space $A_D = \mathbb{H}_g/G_D$ is a moduli space for polarized abelian varieties of type D.*

In this thesis, we mainly focus on the moduli space $\mathbb{H}_g/\mathrm{Sp}(2g, \mathbb{Z})$ of principally polarized abelian varieties. We denote it as A_g. It is not compact and is a quasiprojective variety.

2 Toric Varieties

In this section, we introduce the basic knowledge of toric varieties we need. This section offers a small number of proofs but provides plenty of examples to illustrate the main results. Our main reference for this section is [8] and [14].

2.1 Algebraic Torus over $\mathbb{C}$

Definition 2.1.1. A n-dimensional torus T over $\mathbb{C}$ is an affine variety isomorphic to the affine variety $(\mathbb{C}^*)^n$ with coordinate ring

$$\mathbb{C}[X_1^{\pm 1}, \cdots X_n^{\pm 1}].$$

Elements of this ring are called Laurent polynomials.

The affine variety $(\mathbb{C}^*)^n$ has a group structure under component-wise multiplication and so T inherits a group structure.

Definition 2.1.2. A character of a torus T is a morphism of algebraic groups $\chi : T \to \mathbb{C}^*$.

For example, $r = (r_1, ..., r_n) \in \mathbb{Z}^n$ gives a character $\chi^r : (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by

$$\chi^r : (t_1, \cdots, t_n) \mapsto t_1^{r_1} \cdots t_n^{r_n}. \tag{2}$$

Indeed all characters of $(\mathbb{C}^*)^n$ are given in this way. Thus the characters of $(\mathbb{C}^*)^n$ form a group isomorphic to $\mathbb{Z}^n$.

For an arbitrary torus T, its characters form a free abelian group M of rank equal to the dimension of T. We say $r \in M$ gives the character $\chi^r : T \to \mathbb{C}^*$.

Definition 2.1.3. A one-parameter subgroup of the n-dimensional torus T is a morphism of algebraic groups $\lambda : \mathbb{C}^* \to T$.

For example, $a = (a_1, ..., a_n) \in \mathbb{Z}^n$ gives a one-parameter subgroup $\lambda_a : \mathbb{C}^* \to (\mathbb{C}^*)^n$ defined by

$$\lambda_a : t \mapsto (t^{a_1}, ..., t^{a_n}). \tag{3}$$

Indeed, all one-parameter subgroups of $(\mathbb{C}^*)^n$ are given in this way. Thus the group of one-parameter subgroups of $(\mathbb{C}^*)^n$ is isomorphic to $\mathbb{Z}^n$.

For an arbitrary torus T, the one-parameter subgroups form a free abelian group N of rank equal to the dimension of T. We say $a \in N$ gives the one-parameter subgroup $\lambda_a : \mathbb{C}^* \to T$.

There is a natural pairing $\langle , \rangle : M \times N \to \mathbb{Z}$ defined as follows. Given a character χ^r and a one-parameter subgroup λ_a, the composition $\chi^r \circ \lambda_a : \mathbb{C}^* \to \mathbb{C}^*$ is a character of $\mathbb{C}^*$ given by $t \mapsto t^l$ for some $l \in \mathbb{Z}$. Then $\langle \chi^r, \lambda_a \rangle = l$. If $T = (\mathbb{C}^*)^n$ with $r = (r_1, ..., r_n) \in \mathbb{Z}^n$ and $a = (a_1, ..., a_n) \in \mathbb{Z}^n$, then we have

$$\langle r, a \rangle = \sum_{i=1}^{n} r_i a_i, \tag{4}$$

i.e., the pairing is the usual dot product.

The pairing $\langle , \rangle : M \times N \to \mathbb{Z}$ identifies N with $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and M with $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$.

The torus can be characterized by M and N equally. We have $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong T$ via $u \otimes t \to \lambda_a(t)$. Then it is customary to write a torus as T_N. Picking an isomorphism $T_N \cong (\mathbb{C}^*)^n$ induces isomorphisms $M \cong \mathbb{Z}^n$ and $N \cong \mathbb{Z}^n$ that turn characters into Laurent monomials (2), one-parameter subgroups into monomial curves (3), and the pairing into the dot product (4).

The torus T can also be described analytically as $\tilde{T}/\pi_1(T)$ where $\tilde{T}$ is the universal covering space of T and is a complex vector space. The fundamental group $\pi_1(T)$ is a discrete subgroup, generating $\tilde{T}$ over $\mathbb{C}$, and acts on $\tilde{T}$ via translations. For all $a \in \pi_1(T)$, the map

$$\widetilde{\phi_a} : \mathbb{C} \to \tilde{T}, \quad \lambda \mapsto \lambda \cdot a$$

induces a map

$$\phi_a : \mathbb{C}/\mathbb{Z} \to \tilde{T}/\pi_1(T) = T$$

and we have $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$ via $\lambda \mapsto \exp 2\pi i \lambda$. Thus every $a \in \pi_1(T)$ induces $\phi_a \in N$. Indeed, this is an isomorphism between $\pi_1(T)$ and N. Since $\tilde{T} = \pi_1(T) \otimes \mathbb{C}$, we have

1. $N \otimes \mathbb{C} \cong$ the universal covering space of T.
2. $N \otimes \mathbb{C}/N \cong T$.
3. We have the covering map $\tilde{T} \cong N \otimes \mathbb{C} \cong \mathbb{C}^n \to T = \tilde{T}/\pi_1(T) \cong N \otimes \mathbb{C}/N \cong (\mathbb{C}^*)^n$ given by $z = (z_1, ..., z_n) \in \mathbb{C}^n \mapsto \exp 2\pi i z = (\exp(2\pi i z_1), ..., \exp(2\pi i z_n)) \in (\mathbb{C}^*)^n$.

2.2 Toric Varieties

Definition 2.2.1. A toric variety is an irreducible variety X that contains a torus $T_N \cong (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an action of T_N on X.

Remark 2.2.2. Toric varieties containing a torus T_N were originally known as torus embeddings of T_N.

Example 2.2.3. Two trivial examples are $(\mathbb{C}^*)^n$ and $\mathbb{C}^n$.

Example 2.2.4. The variety $V = \mathbf{V}(XY - ZW) \subset \mathbb{C}^4$ is a 3-dimensional affine toric variety with torus

$$V \cap (\mathbb{C}^*)^4 = \{(t_1, t_2, t_3, t_1 t_2 t_3^{-1}) \mid t_i \in \mathbb{C}^*\} \cong (\mathbb{C}^*)^3.$$

The action $T \times V \to V$ is given by

$$(t_1, t_2, t_3), (x, y, z, w) \mapsto (t_1 x, t_2 y, t_3 z, t_1 t_2 t_3^{-1} w).$$

Example 2.2.5. The projective space $\mathbb{P}^n$ is a toric variety with torus

$$\begin{aligned} T_{\mathbb{P}^n} &= \mathbb{P}^n \backslash \mathbf{V}(X_0 \cdots X_n) = \{(a_0 : ... : a_n) \in \mathbb{P}^n \mid a_0 \cdots a_n \neq 0\} \\ &= \{(1 : t_1 : ... : t_n) \in \mathbb{P}^n \mid t_1, ..., t_n \in \mathbb{C}^*\} \cong (\mathbb{C}^*)^n. \end{aligned}$$

The action $T_{\mathbb{P}^n} \times \mathbb{P}^n \to \mathbb{P}^n$ is

$$(t_1, ..., t_n)(a_0 : a_1 : \cdots a_n) = (a_0 : t_1 a_1 : \cdots t_n a_n).$$

In this subsection, we look only at affine toric varieties.

Now we show how to construct affine toric varieties via lattice points, toric ideals and semigroups.

Let a torus T_N with character lattice M be given. Let $\mathcal{A}$ be a set $\{m_1, ..., m_s\} \subset M$. Consider the map

$$\Phi_{\mathcal{A}} : T_N \mapsto \mathbb{C}^s, \qquad t \mapsto (\chi^{m_1}(t), ..., \chi^{m_s}(t)).$$

Define $Y_{\mathcal{A}}$ to be the Zariski closure of the image of the map $\Phi_{\mathcal{A}}$.

Proposition 2.2.6 ([8, Proposition 1.1.8]). *$Y_{\mathcal{A}}$ is an affine toric variety whose torus has character lattice $\mathbb{Z}\mathcal{A}$, i.e., a lattice generated by $\mathcal{A}$. The action of T_N on $Y_{\mathcal{A}}$ is given by*

$$t \cdot (x_1, ..., x_s) = (\chi^{m_1}(t)x_1, ..., \chi^{m_s}(t)x_n).$$

Proof. The map $\Phi_{\mathcal{A}} : T_N \to (\mathbb{C}^*)^s$ is a map of tori. By [8, Proposition 1.1.1], the image $T = \Phi_{\mathcal{A}}(T_N)$ is a torus that is closed in $(\mathbb{C}^*)^s$. Since $Y_{\mathcal{A}}$ is the Zariski closure of T, the torus T is an open dense torus in $Y_{\mathcal{A}}$. As T is irreducible, its Zariski closure $Y_{\mathcal{A}}$ is also irreducible. Consider the action of T. Since $T \subset \mathbb{C}^s$, an element $t \in T$ acts on $\mathbb{C}^s$. Then $T = t \cdot T \subset t \cdot Y_{\mathcal{A}}$. So $Y_{\mathcal{A}} \subset t \cdot Y_{\mathcal{A}}$. Replacing t with t^{-1}, we have $Y_{\mathcal{A}} = t \cdot Y_{\mathcal{A}}$. So the action of T induces an action on $Y_{\mathcal{A}}$. Hence $Y_{\mathcal{A}}$ is an affine toric variety.
Now compute the character lattice of T, which we denote by M'. Since $T = \Phi_{\mathcal{A}}(T_N)$, we have the following commutative diagram

$$\begin{tikzcd} T_N \arrow[r, "\Phi_{\mathcal{A}}"] \arrow[dr, two heads] & (\mathbb{C}^*)^s \\ & T \arrow[u, hook] \end{tikzcd}.$$

This induces a commutative diagram of character lattices

$$\begin{tikzcd} M & \mathbb{Z}^s \arrow[l, "\hat{\Phi}_{\mathcal{A}}"'] \arrow[d, two heads] \\ & M' \arrow[ul, hook] \end{tikzcd}.$$

Since $\hat{\Phi} : \mathbb{Z}^s \to M$ sends the standard basis $e_1, ..., e_s$ to $m_1, ..., m_s$, the image $\Phi_{\mathcal{A}}$ is $\mathbb{Z}\mathcal{A}$. So we have $M' \cong \mathbb{Z}\mathcal{A}$. ☐

The map $\Phi_{\mathcal{A}} : T_N \to (\mathbb{C}^*)^s$ induces a map of character lattices $\tilde{\Phi}_{\mathcal{A}} : \mathbb{Z}^s \to M$ that maps the standard basis $e_1, ..., e_s$ to $m_1, ..., m_s$. Let L be the kernel of this map, we have an exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^s \longrightarrow M \ .$$

For every $l = (l_1, ..., l_s)$ of L, we have $\sum_{i=1}^s l_i m_i = 0$. Let $l_+ = \sum_{l_i>0} l_i e_i$ and $l_- = -\sum_{l_i<0} l_i e_i$. Then $l = l_+ - l_-$ and $l_+, l_- \in \mathbb{N}^s$. The binomial

$$X^{l_+} - X^{l_-} = \prod_{l_i>0} X_i^{l_i} - \prod_{l_i<0} X_i^{-l_i}$$

vanishes on the image of $\Phi_{\mathcal{A}}$:

$$\prod_{l_i>0}\chi^{l_i m_i}(t) - \prod_{l_i<0}\chi^{l_i m_i}(t) = \chi^{\sum_{l_i>0} l_i m_i}(t) - \chi^{\sum_{l_i>0} l_i m_i}(t) = 0.$$

Since $Y_{\mathcal{A}}$ is the Zariski closure of the image, the binomial also vanishes on $Y_{\mathcal{A}}$.

Proposition 2.2.7 (cf. [8, Proposition 1.1.9])**.** *The ideal of the affine toric variety $Y_{\mathcal{A}} \subset \mathbb{C}^s$ is*

$$I(Y_{\mathcal{A}}) = \langle X^{l_+} - X^{l_-} \mid l \in L\rangle = \langle X^{\alpha} - X^{\beta} \mid \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L\rangle.$$

Let $\{l^1, ..., l^r\}$ be generators of the relations of m_i, i.e., generators of the kernel of the homomorphism $\mathbb{Z}^s \to M$ that maps e_i to m_i. Then the toric ideal $I(Y_{\mathcal{A}})$ is

$$\langle X^{l^i_+} - X^{l^i_-} \mid i = 1, ..., r\rangle.$$

Let $L \subset \mathbb{Z}^s$. We call the ideal $I_L = \{X^{l_+} - X^{l_-} \mid l \in L\}$ the lattice ideal and a toric ideal if it is also prime. Moreover, an ideal is toric if and only if it is prime and generated by binomials (cf. [8, Proposition 1.1.1]).

We now construct toric varieties via semigroups.

Definition 2.2.8. A semigroup is a set S with associative binary operation.

We also assume that a semigroup in our setting contains the identity element.

Given a finitely generated semigroup $S \subset M$, the semigroup algebra $\mathbb{C}[S]$ is the vector space over $\mathbb{C}$ with S as a basis and multiplication induced by the semigroup structure of S. More explicitly, we have

$$\mathbb{C}[S] = \{\sum_{r\in S} c_r\chi^r \mid c_r \in \mathbb{C} \text{ and } c_r = 0 \text{ for all but finitely many } r\}.$$

with multiplication induced by

$$\chi^r \cdot \chi^{r'} = \chi^{r+r'}.$$

If S is generated by $r_1, ..., r_s \in M$, i.e., every element can be written as $\sum_{i=1}^{s} a_i r_i$ for some $a_m \in \mathbb{Z}_{\geq 0}$, then $\mathbb{C}[S] = \mathbb{C}[\chi^{r_1}, ..., \chi^{r_s}]$.

If $e_1, ..., e_n$ is a basis of M, then M is generated by $\{\pm e_1, ..., \pm e_n\}$. Let $T_i = \chi^{e_i}$, we have the ring of Laurent polynomials

$$\mathbb{C}[M] = \mathbb{C}[T_1^{\pm e_1}, ..., T_n^{\pm e_n}],$$

which is the coordinate ring of the torus T_N.

Proposition 2.2.9 ([8, Proposition 1.1.14])**.** *Let $S \subset M$ be finitely generated, then $\mathbb{C}[S]$ is an integral domain and finitely generated as $\mathbb{C}$-algebra. Moreover, Spec$(\mathbb{C}[S])$ is an affine toric variety whose torus has character lattice $\mathbb{Z}S$, the group generated by S. If $S = \mathbb{N}\mathcal{A}$, then Spec$(\mathbb{C}[S]) = Y_{\mathcal{A}}$.*

Proof. If S is generated by $r_1, ..., r_s \in M$, $\mathbb{C}[S] = \mathbb{C}[\chi^{r_1}, \cdots \chi^{r_s}]$ is finitely generated. Since $\mathbb{C}[S] \subset \mathbb{C}[M]$ and $\mathbb{C}[M]$ an integral domain, $\mathbb{C}[S]$ is also an integral domain.
Let $\mathcal{A} = \{r_1, ..., r_s\}$. We have the $\mathbb{C}$-algebra homomorphism

$$\pi : \mathbb{C}[x_1, ..., x_s] \to \mathbb{C}[M]$$

where $x_i \to \chi^{r_i} \in \mathbb{C}[M]$. This corresponds to the morphism

$$\Phi_{\mathcal{A}} : T_N \to (\mathbb{C})^s.$$

The kernel of π is the toric ideal $I(Y_{\mathcal{A}})$. The image of π is $\mathbb{C}[\chi^{m_1}, ..., \chi^{m_s}] = \mathbb{C}[S]$, and then the coordinate ring of $Y_{\mathcal{A}}$ is

$$\mathbb{C}[x_1, ..., x_n]/I(Y_{\mathcal{A}}) = \mathbb{C}[x_1, ..., x_n]/\ker(\pi) \cong \text{Im }(\pi) = \mathbb{C}[S].$$

This proves that $\text{Spec}(\mathbb{C}[S]) = Y_{\mathcal{A}}$. Since $S = \mathbb{N}\mathcal{A}$ implies that $\mathbb{Z}S = \mathbb{Z}\mathcal{A}$, the torus $Y_{\mathcal{A}} = \text{Spec}(\mathbb{C}[S])$ has the character lattice $\mathbb{Z}S$. $\square$

Three constructions are equivalent:

Theorem 2.2.10 ([8, Theorem 1.1.17])**.** *Let V be an affine variety. The following are equivalent:*

a.) V is an affine toric variety.

b.) $V = Y_{\mathcal{A}}$ for a finite set $\mathcal{A}$.

c.) V is an affine variety defined by a toric ideal.

d.) $V =$ Spec$(\mathbb{C}[S])$ for a finitely generated semigroup $S \subset M$.

Proof. All the implications have been proved by Proposition 2.2.6,2.2.7, 2.2.9. It remains to show the implication (a) $\Rightarrow$ (d).
Let V be an affine toric variety containing the torus T_N with character lattice M. Then $\mathbb{C}[V]$ is a subalgebra of $\mathbb{C}[M]$ stable under the action of T_N. Claim that

$$\mathbb{C}[V] = \bigoplus_{\chi^m \in \mathbb{C}[V]} \mathbb{C} \cdot \chi^m$$

so that $\mathbb{C}[V] = \mathbb{C}[S]$ for the semigroup $S = \{m \in M \mid \chi^m \in \mathbb{C}[V]\}$. And since $\mathbb{C}[V]$ is finitely generated, we can find $f_1, ..., f_s \in \mathbb{C}[V]$ such that $\mathbb{C}[v] = \mathbb{C}[f_1, ..., f_s]$. Expressing f_i in terms of characters gives us a finite generating set of S.
Now we restate the claim and prove the claim. Let $A \subset \mathbb{C}[M]$ be a subspace stable under the action of T_N. We want to prove $A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m$. It's easy to see that $\bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m \subset A$. Conversely, we pick $f \neq 0$ in A. Since $A \subset \mathbb{C}[M]$, we can write $f = \sum_{m \in \mathcal{B}} c_m \chi^m$ where $\mathcal{B} \subset \mathcal{M}$ is finite and $c_m \neq 0$ for all $m \in \mathcal{B}$. Then $f \in B \cap A$, where $B = \text{Span}(\chi^m \mid m \in \mathcal{B}) \subseteq \mathbb{C}[M]$. Note that $B \cap A$ is stable under the action of T_N. Since $A \cap B$ is finite-dimensional, $B \cap A$ is spanned by simultaneous eigenvectors of T_N. In $\mathbb{C}[M]$, simultaneous eigenvectors are characters. It follows that $B \cap A$ is spanned by characters. Therefore $f \in \bigoplus_{\chi^m \in A} \mathbb{C}$. $\square$

We have various ways to describe the points of affine toric varieties:

Proposition 2.2.11 ([8, Proposition 1.3.1])**.** *Let $V = Spec(\mathbb{C}[S])$ where S is a finitely generated semigroup in M. There are natural bijections between:*

a) Closed points $p \in V$.

b) Maximal Ideals $\mathfrak{m} \subset \mathbb{C}[S]$.

c) Semigroup homomorphisms $\gamma : S \to \mathbb{C}$, where $\mathbb{C}$ is considered as a semigroup under multiplication.

Proof. The equivalence of a) and b) is standard algebraic geometry.

To show *a*) $\Rightarrow$ *c*), given a point $p \in V$, we define $S \to \mathbb{C}$ by sending $m \in S$ to $\chi^m(p)$.

To show *c*) $\Rightarrow$ *b*), for any semigroup homomorphism $\gamma : S \to \mathbb{C}$, since $\{\chi^m\}_{m\in S}$ is a basis of $\mathbb{C}[S]$, the map γ extends to a surjective linear map $\tilde{\gamma} : \mathbb{C}[S] \to \mathbb{C}$, which is a $\mathbb{C}$-algebra homomorphism. The kernel of the map $\mathbb{C}[S] \to \mathbb{C}$ is a maximal ideal. □

Remark 2.2.12. Given a semigroup homomorphism $\gamma : S \to \mathbb{C}$, we can construct p explicitly. Let $\mathcal{A} = \{m_1, ..., m_s\}$ generate S such that $V = Y_{\mathcal{A}}$. Let $p = (\gamma(m_1), ..., \gamma(m_s)) \in \mathbb{C}^s$. By Proposition 2.2.7, the point p lies in V. Moreover, for $f \in \mathbb{C}[S]$, as we have $\tilde{\gamma}(f) = f(p)$, the maximal ideal $\{f \in \mathbb{C}[S] \mid f(p) = 0\}$ is the kernel of the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[S] \to \mathbb{C}$.

Remark 2.2.13. Viewing a point as a semigroup homomorphism, we can see the toric variety intrinsically without embedding it into affine space. Let $t \in T_N$, $\mathcal{A} = \{m_1, ..., m_s\}$, $p \in V = Y_{\mathcal{A}}$ corresponding to the semigroup homomorphism $m \to \gamma(m)$, the action $t \cdot p$ is given by the semigroup homomorphism $m \mapsto \chi^m(t)\gamma(m)$. This corresponds to the point

$$(\chi^{m_1}(t), ..., \chi^{m_s}(t)) \cdot (\gamma(m_1), ..., \gamma(m_s)) = (\chi^{m_1}(t)\gamma(m_1), ..., \chi^{m_s}\gamma(m_s)).$$

Indeed, we can classify all toric varieties containing a torus T_N as a Zariski open subset.

Definition 2.2.14. Let V_i be the toric variety with torus $T_{N_i}, i = 1, 2$. Then a morphism $\phi : V_1 \to V_2$ is toric if and only if $\phi(T_{N_1}) \subseteq T_{N_2}$ and $\phi|_{T_{N_i}} : T_{N_1} \to T_{N_2}$ is a group homomorphism.

Remark 2.2.15. A toric morphism $\phi : V_1 \to V_2$ is equivariant, i.e., we have a commutative diagram:

$$\begin{array}{ccc} T_{N_1} \times V_1 & \xrightarrow{\Phi_1} & V_1 \\ {\scriptstyle \phi|_{T_{N_1}} \times \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ T_{N_2} \times V_2 & \xrightarrow{\Phi_2} & V_2 \end{array}$$

where Φ_i is the action of T_{N_i} on V_i.

If S generates M, then Proposition 2.2.9 implies that

Proposition 2.2.16 ([14, Proposition 1])**.** *The correspondence $S \mapsto Spec(\mathbb{C}[S])$ defines a bijection between the set of finitely generated semigroups $S \subset M$ which generated M as a group and the set of isomorphic classes of affine toric varieties containing T whose character lattice is M. Moreover, the morphisms of affine toric varieties containing T correspond in a contravariant way to the inclusion between semigroups in M.*

We now discuss the normality of toric varieties.

Definition 2.2.17. Let $S \subset M$ be a semigroup. We say that S is saturated if for all $k \in \mathbb{Z}_{>0}$ and $m \in M$, $km \in S$ implies $m \in S$.

Proposition 2.2.18 ([8, Theorem 1.3.5])**.** *Let X be an affine toric variety with torus T_N corresponding to a finitely generated semigroup S in the character lattice M of T_N, i.e., $X = Spec(\mathbb{C}[S])$. Then X is normal if and only if S is saturated.*

Combining Proposition 2.2.16 and Proposition 2.2.18, we have

Theorem 2.2.19. *The correspondence $S \mapsto Spec(\mathbb{C}[S])$ defines a bijection between the set of finitely generated semigroups $S \subset M$ generating M as a group and saturated in M, and the set of affine normal toric varieties of T_N.*

2.3 Cones

We can associate the semigroups that occur in Theorem 2.2.19 to some combinatorial data.

Extend M and N $\mathbb{R}$-linearly, we have $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.3.1. A convex polyhedral cone in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \text{Cone}(S) = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0\}$$

where $S \subset N_{\mathbb{R}}$ is finite. We say that σ is generated by S.

Remark 2.3.2. A convex polyhedral cone is convex, meaning that $x, y \in \sigma$ implies that $\lambda x + (1-\lambda)y \in \sigma$ for $0 \leq \lambda \leq 1$. As we will only consider convex cones, the cones defined above will be called simply "polyhedral cones".

Definition 2.3.3. Given a polyhedral cone $\sigma \subset N_{\mathbb{R}}$, its dual cone is defined by

$$\sigma^\vee = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}$$

Proposition 2.3.4. *Let $\sigma \subset N_{\mathbb{R}}$ be a polyhedral cone. Then $\sigma^\vee$ is a polyhedral cone in $M_{\mathbb{R}}$ and $(\sigma^\vee)^\vee = \sigma$.*

We can also obtain the polyhedral cone using hyperplanes.

Definition 2.3.5. Given $m \neq 0 \in M_{\mathbb{R}}$, the hyperplane defined by m is

$$H_m = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\} \subseteq N_{\mathbb{R}}.$$

The closed half-space is

$$H_m^+ = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0\} \subseteq N_{\mathbb{R}}.$$

If $\sigma \subset H_m^+$ and σ is a polyhedral cone in $N_{\mathbb{R}}$, then H_m is a supporting hyperplane of σ and H_m^+ is a supporting half-space.

If $m_1, ..., m_s$ generate $\sigma^\vee$, then it is easy to check that

$$\sigma = H_{m_1}^+ \cap \cdots H_{m_s}^+,$$

i.e., every polyhedral cone is an intersection of finitely many closed half-spaces. Note that in [14], there is an equivalent definition of a polyhedral cone by using linear functionals, namely, a polyhedral cone is a set

$$\{x \mid l_i(x) \geq 0 \text{ for all } i = 1, ..., s\}$$

where $l_i, i = 1, ..., s$ are linear functionals. By Rieze's representation theorem, for every linear functional l_i, we have a unique $m_i \in M_\mathbb{R}$ such that $l_i(x) = \langle m_i, x\rangle$. Then each linear function determines a closed half-space $H_{m_i}^+$. The elements satisfying $l_i(x) \geq 0$ for $i = 1, ..., s$ lie in

$$H_{m_1}^+ \cap \cdots H_{m_s}^+.$$

Then it is easy to check Proposition 2.3.4 by noticing that if $\sigma = \{x \mid l_i(x) \geq 0, \quad i = 1, ..., s\}$, then $\sigma^\vee$ is the set of $\sum_{i=1}^s \lambda_i l_i$ where $\lambda_i \geq 0$. So we can write $\sigma^\vee$ as $\mathrm{Cone}(l_1, ..., l_s)$ or $\mathrm{Cone}(m_1, ..., m_s)$.

We can use supporting hyperplanes to define the faces of a polyhedral cone.

Definition 2.3.6. A face of the polyhedral cone σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$. Denote it as $\tau \preceq \sigma$.

Remark 2.3.7. When $m = 0$, then σ is a face of itself. Faces $\tau \neq \sigma$ are called proper faces, written $\tau \prec \sigma$.

The faces of a polyhedral cone have the following properties:

Proposition 2.3.8 ([8, Lemma 1.2.6])**.** *Let σ be a polyhedral cone. Then:*

a.) If $\tau \preceq \sigma$, then τ is a polyhedral cone.

b.) If $\tau_1, \tau_2 \preceq \sigma$, then $\tau_1 \cap \tau_2 \preceq \sigma$.

c.) If $\tau_1 \preceq \tau_2$ and $\tau_2 \preceq \sigma$, then $\tau_1 \preceq \sigma$.

Proposition 2.3.9. *Let $\tau \prec \sigma$. If $v, w \in \sigma$ and $v + w \in \tau$, then $v, w \in \tau$.*

Proof. Since $\tau \prec \sigma$, the face τ equals $H_m \cap \sigma$ for some $m \in \sigma^\vee$. Since $v, w \in \sigma$ and $v + w \in \tau$, we have $\langle v, m\rangle \geq 0$, $\langle w, m\rangle \geq 0$ and $\langle v + w, m\rangle = 0$, this implies that $v, w \in H_m \cap \sigma = \tau$. □

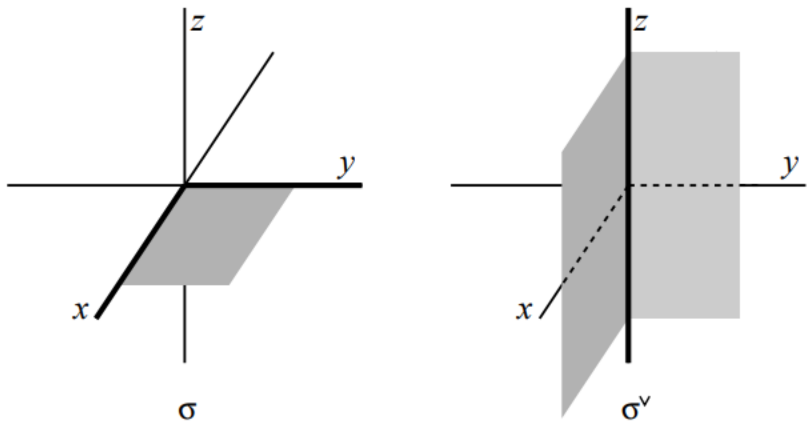
Proposition 2.3.10 ([14, p.7])**.** *Let $\sigma = \{x \mid l_i(x) \geq 0, \quad i = 1, ..., s\}$ be a polyhedral cone and $\tau \preceq \sigma$, then there exists a subset I of $\{1, 2, ..., s\}$ such that $\tau = \sigma \cap \{x \mid l_i(x) = 0$ for all $i \in I\}$. The face τ has codimension $|I|$.*

Example 2.3.11 ([8, Example 1.2.11])**.** Let $N_\mathbb{R} = \mathbb{R}^3$ and $\sigma = \mathrm{Cone}(e_1, e_2) \subset N_\mathbb{R}$.

In the example above, the origin is a face of σ but not a face of $\sigma^\vee$.

Definition 2.3.12. Let $\sigma \subset N_\mathbb{R} \cong \mathbb{R}^n$ be a polyhedral cone. The cone σ is strongly convex if one of the equivalent conditions holds:

Figure 1: σ and its dual cone $\sigma^\vee$



i.) $\{0\}$ is a face of σ;

ii.) σ contains no positive-dimensional subspace of $N_{\mathbb{R}}$;

iii.) $\sigma \cap (-\sigma) = \{0\}$;

iv.) $\dim \sigma^\vee = n$.

To associate a polyhedral cone to a toric variety, we need one more condition:

Definition 2.3.13. A polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is rational if $\sigma = \mathrm{Cone}(S)$ for some finite set $S \subset N$.

Note that faces and duals of rational polyhedral cones are also rational.
Given a rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, the lattice points

$$S_\sigma = \sigma^\vee \cap M$$

form a semigroup.

Proposition 2.3.14 (Gordan's Lemma, cf. [8, Proposition 1.2.17])**.** *S_σ is a finitely generated semigroup.*

Since we can construct affine toric varieties by finitely generated semigroups $S \subset M$, we have

Theorem 2.3.15. *Let $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ be a rational polyhedral cone with semigroup $S_\sigma = \sigma^\vee \cap M$. Then*

$$U_\sigma = \mathit{Spec}(\mathbb{C}[S_\sigma]) = \mathit{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

is an affine toric variety.

Example 2.3.16. Let $0 \leq r \leq n$ and $\sigma = \text{Cone}(e_1, ..., e_r) \subset N_{\mathbb{R}} = \mathbb{R}^n$. Then

$$\sigma^\vee = \text{Cone}(e_1, ..., e_r, \pm e_{r+1}, \pm e_n).$$

The corresponding toric variety is

$$U_\sigma = \text{Spec}(\mathbb{C}[T_1, ..., T_r, T_{r+1}^{\pm 1}, ..., T_n^{\pm 1}]) = \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$$

where $T_i = \chi^{e_i}$. This toric variety is smooth. Indeed a toric variety U_σ is smooth if and only if there is a basis n_1,...,n_r of the lattice N over $\mathbb{Z}$ such that $\sigma = \text{Cone}(n_1, ..., n_r)$ for some $k \leq r$ and we also have $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{r-k}$ (cf. [8, Definition 1.2.16, Theorem 1.3.12]).

Moreover, with the condition of strong convexity, we have

Lemma 2.3.17 ([14, Lemma 2])**.** *The correspondence $\sigma \mapsto \sigma^\vee \cap M$ defines a bijection between the set of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ and the set of finitely generated saturated semigroups $S \subset M$ which generate M as a group.*

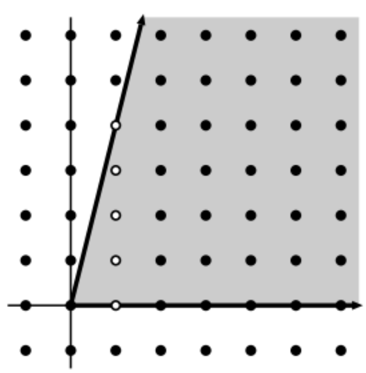
Combining Lemma 2.3.17 and Theorem 2.2.19, we have

Theorem 2.3.18. *The correspondence $\sigma \mapsto Spec(\mathbb{C}[\sigma^\vee \cap M] = U_\sigma$ defines a bijection between the set of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ and the set of affine toric normal varieties with torus T_N.*

Note that cone generators and semigroup generators are not the same.

Example 2.3.19 ([8, Example 1.2.21])**.** Let $N_{\mathbb{R}} = \mathbb{R}^2$ and $\sigma = \text{Cone}(4e_1 - e_2, e_2) \subset \mathbb{R}^2$. One can compute its dual cone is $\sigma^\vee = \text{Cone}(e_1, e_1 + 4e_2)$. The semigroup $\sigma^\vee \cap M$ is generated by the lattice points $(1, i)$ for $0 \leq i \leq 4$.

Figure 2: The cone $\sigma^\vee$



The white dots in Figure 2 are our generators of $\sigma^\vee \cap M$. Let $\mathcal{A}$ be the set containing $(1, i)$ for $0 \leq i \leq 4$. The affine toric variety U_σ is the Zariski closure $Y_{\mathcal{A}}$ of the image of the map $\Phi : (\mathbb{C}^*)^2 \to \mathbb{C}^5$ defined by

$$\Phi(s, t) = (s, st, st^2, st^3, st^4)$$

with the toric ideal

$$I = \langle x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i \leq j \leq 3 \rangle.$$

2.4 Fans

In this subsection, we will use fans to patch affine toric varieties together.

Definition 2.4.1. A fan is a finite collection of strongly convex rational polyhedral cones σ such that

a.) If $\sigma \in \Sigma$ and $\tau \prec \sigma$, then $\tau \in \Sigma$.

b.) If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2 \prec \sigma_1, \sigma_2$.

Recall the gluing procedure (cf. [10, Exercise II 2.12]). If we have a collection of gluing data

$$\{(\{X_i\}_i, \{U_{ij}\}_{i,j}\}, \{g_{ij}\}_{i,j}\}$$

where $\{X_i\}$ is a family of schemes and U_{ij} is Zariski open in X_i and $g_{i,j} : U_{ij} \to U_{ji}$ satisfies:

(1) For each i, j, $g_{ij} = g_{ij}^{-1}$;

(2) Cocycle: For each i, j, k, $g_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and $g_{ik} = g_{jk} \circ g_{ij}$ on $U_{ij} \cap U_{ik}$,

we can glue these X_i along g_{ij}:

$$X = \amalg_i X_i / \sim$$

and for $a \in X_i$ and $b \in X_j$, $a \sim b$ if $a \in U_{ij}$ and $g_{ij}(a) = b$ for some j.

We now show how the fans give the combinatorial data to glue affine toric varieties to yield an abstract toric variety.

If $\tau \prec \sigma$, i.e., $\tau = H_m \cap \sigma$ where $m \in \sigma^\vee \cap M$. We can show that $S_\tau = S_\sigma + \mathbb{Z}(-m)$ (cf. [8, Proposition 1.3.16]). Then the semigroup algebra $\mathbb{C}[S_\tau] = \mathbb{C}[\tau^\vee \cap M]$ is the localization of $\mathbb{C}[S_\sigma] = \mathbb{C}[\sigma^\vee \cap M]$ at $\chi^m \in \mathbb{C}[S_\sigma]$. If $\tau = \sigma_1 \cap \sigma_2$, then by the Separation Lemma (cf. [8, Lemma 1.2.13], we have

$$\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2$$

for some $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$. This implies that

$$U_{\sigma_1} \supseteq (U_{\sigma_1})_{\chi^m} = U_\tau = (U_{\sigma_2})_{\chi^{-m}} \subseteq U_{\sigma_2}.$$

We then have an isomorphism

$$g_{\sigma_2,\sigma_1} : (U_{\sigma_1})_{\chi^m} \cong (U_{\sigma_2})_{\chi^{-m}}$$

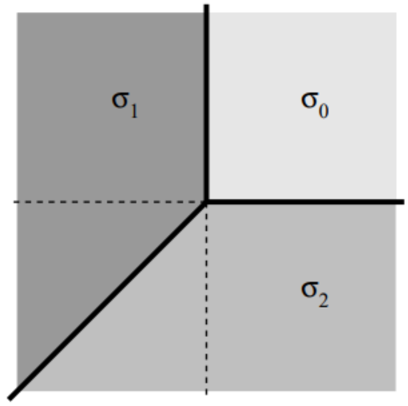
which is the identity on U_τ. Then it is also easy to check the cocycle condition is satisfied.

Theorem 2.4.2 ([8, Theorem 3.1.5])**.** *Let Σ be a fan in $N_\mathbb{R}$. The variety X_Σ obtained by glueing the collection of affine toric varieties $\{U_\sigma\}_{\sigma\in\Sigma}$ is a normal separated toric variety.*

Conversely, any separated normal toric variety comes from a fan:

Theorem 2.4.3 ([8, Corollary 3.1.8])**.** *Let X be a normal separated toric variety with torus T_N. Then there exists a fan Σ in $N_\mathbb{R}$ such that $X \cong X_\Sigma$.*

Figure 3: The fan Σ for $\mathbb{P}^2$



Example 2.4.4. Consider the fan Σ in $N_{\mathbb{R}} = \mathbb{R}^2$. The fan has 3 two-dimensional cones $\sigma_0 = \mathrm{Cone}(e_1, e_2)$, $\sigma_1 = \mathrm{Cone}(-e_1 - e_2, e_2)$ and $\sigma_2 = \mathrm{Cone}(e_1, -e_1 - e_2)$, together with 3 one-dimensional rays and the origin. We can calculate the dual cones $\sigma_0^\vee = \mathrm{Cone}(e_1, e_2)$, $\sigma_1^\vee = \mathrm{Cone}(-e_1 + e_2, -e_1)$ and $\sigma_2^\vee = \mathrm{Cone}(e_1 - e_2, -e_2)$. Then the toric varieties X_Σ is covered by the three affine opens

$$\begin{aligned} U_{\sigma_0} &= \mathrm{Spec}(\mathbb{C}[S_{\sigma_0}]) \cong \mathrm{Spec}(\mathbb{C}[x, y]), \\ U_{\sigma_1} &= \mathrm{Spec}(\mathbb{C}[S_{\sigma_1}]) \cong \mathrm{Spec}(\mathbb{C}[x^{-1}, x^{-1}y]), \\ U_{\sigma_2} &= \mathrm{Spec}(\mathbb{C}[S_{\sigma_2}]) \cong \mathrm{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]). \end{aligned}$$

Moreover, for $\tau_{01} = \sigma_0 \cap \sigma_1$, we have $\tau = H_{e_1} \cap \sigma_1 = H_{e_1} \cap \sigma_2$. Hence we have gluing data on the coordinate rings

$$g^*_{\sigma_1,\sigma_0} : \mathbb{C}[x, y]_x \cong \mathbb{C}[x^{-1}, x^{-1}y^{-1}]_{x^{-1}}.$$

Similarly, we have

$$\begin{aligned} g^*_{\sigma_2,\sigma_0} &: \mathbb{C}[x, y]_y \cong \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}}, \\ g^*_{\sigma_2,\sigma_1} &: \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}y} \cong \mathbb{C}[xy^{-1}, y^{-1}]_{xy^{-1}}. \end{aligned}$$

If we use the homogeneous coordinates (x_0, x_1, x_2) on $\mathbb{P}^2$, then $x \mapsto x_1/x_0$ and $y \mapsto x_2/x_0$ identifies the standard affine open $U_i \subset \mathbb{P}^2$ with U_{σ_i}. We have $X_\Sigma = \mathbb{P}^2$.

Example 2.4.5. We can classify all one-dimensional normal toric varieties as follows. Let $N = \mathbb{Z}$ and $N_{\mathbb{R}} = \mathbb{R}$. The only cones are $\sigma_0 = \mathrm{Cone}(e_1)$, $\sigma_1 = \mathrm{Cone}(-e_1)$ and the trivial cone $\tau = \{0\}$. As all normal varieties arise from some fans, we only need to consider the fans in $\mathbb{R}$. There are only 4 possible fans $\Sigma_1 = \{\tau\}$, $\Sigma_2 = \{\sigma_0, \tau\}$, $\Sigma_1 = \{\sigma_1, \tau\}$ and $\Sigma_4 = \{\sigma_0, \sigma_1, \tau\}$. We have affine toric varieties $U_{\{0\}} \cong \mathrm{Spec}(\mathbb{C}[x, x^{-1}]) \cong \mathbb{C}^*$, $U_{\sigma_0} \cong \mathrm{Spec}(\mathbb{C}[x]) \cong \mathbb{C}$, $U_{\sigma_0} \cong \mathrm{Spec}(\mathbb{C}[x^{-1}]) \cong \mathbb{C}$. Gluing them, we have $X_{\Sigma_1} = \mathbb{C}^*$, $X_{\Sigma_2} \cong \mathbb{C}$, $X_{\Sigma_3} \cong \mathbb{C}$ and $X_\Sigma \cong \mathbb{P}^1$.

We also need to study group actions on normal toric varieties. First note that it is a special case of toric morphisms.

A compatible $\mathbb{Z}$-linear mapping gives rise to a toric morphism of normal toric varieties.

Definition 2.4.6. Let N_1, N_2 be two lattices with Σ_1 a fan in $(N_1)_\mathbb{R}$ and Σ_2 a fan in $(N_2)_\mathbb{R}$. A $\mathbb{Z}$-linear mapping $\overline{\phi} : N_1 \to N_2$ is compatible with the fans Σ_1 and Σ_2 if for every cone $\sigma_1 \in \Sigma_1$ there exists a cone $\sigma_2 \in \Sigma_2$ such that $\overline{\phi}_\mathbb{R}(\sigma_1) \subseteq \sigma_2$.

Theorem 2.4.7 ([8, Theorem 3.3.4])**.** *a) If $\overline{\phi} : N_1 \to N_2$ is a $\mathbb{Z}$-linear map that is compatible with Σ_1 and Σ_2, then there is a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ such that $\phi|_{T_{N_1}}$ is the map*

$$\overline{\phi} \otimes 1 : N_1 \otimes_\mathbb{Z} \mathbb{C}^* \to N_2 \otimes_\mathbb{Z} \mathbb{C}^*.$$

b) Conversely, if $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism, then ϕ induces a $\mathbb{Z}$-linear map $\overline{\phi} : N_1 \to N_2$ that is compatible with the fans Σ_1 and Σ_2.

Remark 2.4.8. We are interested in the case that $N_1 = N_2$.

Suppose a group G acts on an algebraic torus by group homomorphisms. For simplicity, let $M = \mathbb{Z}^r$, $N = \mathbb{Z}^r$ and $T = (\mathbb{C}^*)^r$. For every element $g \in G$, the morphism of tori $g : T \to T$ induces a $\mathbb{Z}$-linear map $g : N \to N$ given by

$$n \in N \mapsto A(g)n$$

for some $r \times r$ matrix $A(g)$. Write $A(g) = (a_{ij})$, this corresponds to the group homomorphism

$$g : (t_1, ..., t_r) \mapsto (t_1^{a^{11}} \cdots t_r^{a_{1r}}, ..., t_r^{a^{r1}} \cdots t_r^{a_{rr}}).$$

From duality, the action of G on the dual lattice M is given by

$$g : M \to M \quad m \mapsto {}^tA(g)^{-1}m$$

by noticing that the pairing is given by $\langle m, n \rangle = {}^tm \cdot n$.

By Proposition 2.4.7, we have

Proposition 2.4.9. *Let a group G act on an algebraic torus T_N by group homomorphisms and Σ a fan in $N_\mathbb{R}$. For every element $g \in G$, if the induced $\mathbb{Z}$-linear map $g : N \to N$ has the property that $g(\sigma) \in \Sigma$ for every $\sigma \in \Sigma$, then the action $G \times T \to T$ extends to an action $G \times X_\Sigma \to X_\Sigma$.*

Remark 2.4.10. If the group action of G on T has the property above, every element $g \in G$ gives a isomorphism of open affine sets U_σ in X_Σ by

$$g \in G : \mathbb{C}[g(\sigma)^\vee \cap M] \xrightarrow{\sim} \mathbb{C}[\sigma^\vee \cap M].$$

2.5 The Orbit-Cone Correspondence

In this subsection, we study the orbits for the action of T_N on the toric variety X_Σ. We will show that there is a one-to-one correspondence between the cones and T_N-orbits.

Definition 2.5.1. Let $\sigma \subset N_\mathbb{R}$, we define

$$\sigma^\perp = \{m \in M_\mathbb{R} \mid \langle m, u \rangle = 0 \text{ for all } u \in \sigma\}.$$

The interior of σ is the set of elements $u \in \sigma$ such that $\langle m, u \rangle > 0$ for all $m \in \sigma^\vee \backslash \sigma^\perp$. We denote it as $\mathrm{Int}(\sigma)$.

Example 2.5.2. Consider the projective space $\mathbb{P}^2$ as we studied in Example 2.2.5 and Example 2.4.4. The torus T_N is $\mathbb{P}^2 \backslash \mathbf{V}(X_0X_1X_2) = \{(1 : t_1 : t_2) \in \mathbb{P}^n \mid t_1, t_2 \in \mathbb{C}^*\}$. For each $u = (a, b) \in N = \mathbb{Z}^2$, we have the corresponding curve in $\mathbb{P}^2$:

$$\lambda_u(t) = (1, t^a, t^b).$$

Take the classical topology, we can consider the limit point of $\lambda_a(t)$ as $t \to 0$. The limit point depends on a as shown in the Figure 4.

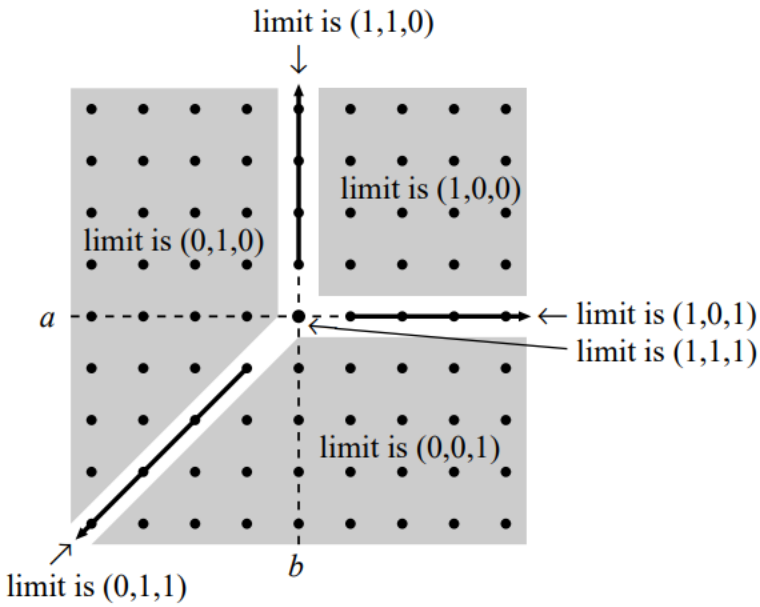


Figure 4: $\lim_{t\to 0} \lambda_u(t)$ for $u = (a, b) \in \mathbb{Z}^2$

The picture shows that given a cone σ, all elements u in $\mathrm{Int}(\sigma) \cap N$ give the limit point $\lim_{t\to 0} \lambda_u(t)$. Different cones give different limit points.

We can relate them to the T_N-orbits. There are seven T_N-orbits in $\mathbb{P}^2$:

1. $O_1 = \{(x_0, x_1, x_2) \mid x_i \neq 0 \text{ for all } i\}$
2. $O_2 = \{(x_0, x_1, x_2) \mid x_2 = 0 \text{ and } x_0, x_1 \neq 0\} \ni (1, 1, 0)$
3. $O_3 = \{(x_0, x_1, x_2) \mid x_1 = 0 \text{ and } x_0, x_2 \neq 0\} \ni (1, 0, 1)$
4. $O_4 = \{(x_0, x_1, x_2) \mid x_0 = 0 \text{ and } x_1, x_2 \neq 0\} \ni (0, 1, 1)$
5. $O_5 = \{(x_0, x_1, x_2) \mid x_1 = x_2 = 0 \text{ and } x_0 \neq 0\} = (1, 0, 0)$
6. $O_6 = \{(x_0, x_1, x_2) \mid x_0 = x_2 = 0 \text{ and } x_1 \neq 0\} = (0, 1, 0)$
7. $O_7 = \{(x_0, x_1, x_2) \mid x_0 = x_1 = 0 \text{ and } x_2 \neq 0\} = (0, 0, 1)$

Each orbit contains one of the unique limit points. Then we have a correspondence between cones σ and orbits O by

$$\sigma \text{ corresponds to } O \Leftrightarrow \lim_{t\to 0} \lambda_u(t) \in O \text{ for all } u \in \operatorname{Int}(\sigma).$$

We wish to generalize these observations on $\mathbb{P}^2$ to all toric varieties.

We first discuss the limit points. They can also be described via semigroup homomorphisms by Proposition 2.2.11. Fix a cone σ in $N_{\mathbb{R}}$. Define $\gamma : S_\sigma = \sigma^\vee \cap M \to \mathbb{C}$ given by

$$m \in S_\sigma \mapsto \begin{cases} 1 & m \in S_\sigma \cap \sigma^\perp = \sigma^\vee \cap \sigma^\perp \cap M = \sigma^\perp \cap M \\ 0 & \text{otherwise} \end{cases}$$

This is a semigroup homomorphism. Indeed $\sigma^\vee \cap \sigma^\perp$ is a face of $\sigma^\vee$. If $m, m' \in S_\sigma$ and $m + m' \in S_\sigma \cap \sigma^\perp$, then $m, m' \in S_\sigma \cap \sigma^\perp$ by Proposition 2.3.9.

We denote this point by γ_σ and call it the **distinguished point corresponding to** σ. The point γ_σ is fixed under the T_N-action if and only if $\dim \sigma = \dim N_{\mathbb{R}}$ (cf. [8, Corollary 1.3.3]).

The distinguished point defined above is exactly the limit point of the one-parameter subgroup $\lambda_u(t)$ for $u \in \sigma$.

Proposition 2.5.3 ([8, Proposition 3.2.2])**.** *Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex polyhedral cone and let $u \in N$. Then*

$$u \in \sigma \Leftrightarrow \lim_{t\mapsto 0} \lambda_u(t) \text{ exists in } U_\sigma.$$

Moreover, if $u \in Int(\sigma)$, then $\lim_{t\to 0} \lambda_u(t) = \gamma_\sigma$.

The proposition indicates that to each cone σ, we can associate a distinguished point γ_σ. This gives a T_N-orbit

$$O(\sigma) = T_N \cdot \gamma_\sigma \subseteq X_\Sigma.$$

Proposition 2.5.4 ([8, Lemma 3.2.5])**.** *Let Σ be a fan in $N_{\mathbb{R}}$ and $\sigma \in \Sigma$. The T_N-orbit $O(\sigma)$ is*

$$O(\sigma) = \{\gamma : S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\} \cong Hom_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*).$$

Proof. Recall T_N acts on semigroup homomorphisms: if $p \in U_\sigma$ is represented by $\gamma : S_\Sigma \to \mathbb{C}$, the point $t \cdot p$ is represented by the semigroup homomorphism:

$$t \cdot \gamma : m \mapsto \chi^m(t)\gamma(m).$$

Let $O' = \{\gamma : S_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\}$. By the definition of distinguished points, the point $\gamma_\sigma \in O'$. Also, O' is invariant under the action of the torus T_N. Hence $O(\sigma) = T_N \cdot \gamma_\sigma \subseteq O'$.

Now we claim that $O' \cong \operatorname{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*)$. Note that $\sigma^\perp$ is the largest vector space contained in $\sigma^\vee$. Hence $\sigma^\perp \cap M$ is a subgroup of $S_\sigma = \sigma^\vee \cap M$. Restricting $\gamma \in O'$ to $\sigma^\perp \cap M$ yields a group homomorphism $\hat{\gamma} : \sigma^\vee \cap M \to \mathbb{C}^*$. Conversely, if $\hat{\gamma} : \sigma^\perp \cap M \to \mathbb{C}^*$, we get a semigroup homomorphism $\gamma : S_\sigma \to$ by seting $\gamma(m) = 0$ for $m \in S_\sigma \backslash \sigma^\perp \cap M$.

Now it remains to show that $O' \subseteq O(\sigma)$. Note that $T_N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$. The inclusion $\sigma^\perp \cap M \subset M$ induces a surjection

$$T_N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \twoheadrightarrow \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \cong O'$$

So T_N acts transitively on O' . Then $O(\sigma) = T_N \cdot \gamma_\sigma = O'$. □

Remark 2.5.5. Note that $O(\sigma)$ is a torus with character group $\sigma^\perp \cap M$. Let N_σ be the sublattice of N spanned by the points in $\sigma \cap N$ and $N(\sigma) = N/N_\sigma$. The orbit is the torus $T_{N(\sigma)}$.

Theorem 2.5.6 (Orbit-Cone Correspondence, cf. [8, Theorem 3.2.6]). *Let $\Sigma \subset N_{\mathbb{R}}$, and X_Σ the corresponding toric variety.*

a) There is a one-to-one correspondence between

$$\{\sigma \in \Sigma\} \longleftrightarrow \{T_N - \textit{orbits in } X_\Sigma\}$$
$$\sigma \longleftrightarrow O(\sigma) \cong \mathit{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) = T_{N(\sigma)}$$

b) $\dim O(\sigma) = \dim N_{\mathbb{R}} - \dim \sigma$

c) The affine open subset U_σ is the union of orbits

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau)$$

d) $\tau \prec \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, and

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$$

where $\overline{O}(\tau)$ denotes the closure in both the classical and Zariski topology.

Example 2.5.7. We continue our discussion of $\mathbb{P}^2$ in Example 2.5.2. There are three types of cones classified by their dimensions.

- Consider the trivial cone $\sigma = \{(0,0)\}$. The cone corresponds to $O(\sigma) = T_{N(\sigma)} = T_N$, which has dimension $\dim(O_\sigma) = \dim N_{\mathbb{R}} - \dim \sigma = 2$. This is a face of all other cones, hence $U_\sigma = O(\sigma) = T_N$ and all other orbits are contained in the closure of $O(\sigma)$.
- Consider the three 1-dimensional cones τ. For example, we can consider the cone $\tau = \mathrm{Cone}(e_2)$. Then $\tau^\perp \cap M = \mathbb{N}e_1 + \mathbb{N}(-e_1)$. Hence $O(\tau)$ consists of all elements $(1 : x_1 : x_2)$ with $x_1, x_1^{-1} \neq 0$ and $x_2 = 0$ by Proposition 2.5.4. The orbit $O(\tau)$ has dimension 1 and is isomorphic to $\mathbb{C}^*$. The closure of $O(\tau)$ is the coordinate axis $V(x_2)$ in $\mathbb{P}^2$, which is isomorphic to $\mathbb{P}^1$.
- The three maximal cones σ_i in the fan correspond to the three fixed points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ of the torus action of $\mathbb{P}^2$.

Let $V(\tau) = \overline{O(\tau)}$. The torus $O(\tau) = T_{N(\tau)}$ is an open subset of $V(\tau)$. We will show that $V(\tau)$ is a normal toric variety with torus $O(\tau) = T_{N(\tau)}$. Now we want to construct a fan. For each cone $\sigma \in \Sigma$ containing τ, let $\overline{\sigma}$ be the image cone in $N(\tau)_{\mathbb{R}}$ under the quotient map

$$N_{\mathbb{R}} \to N(\tau)_{\mathbb{R}}.$$

Then

$$\operatorname{Star}(\tau) = \{\overline{\sigma} \subseteq N(\tau)_{\mathbb{R}} \mid \tau \preceq \sigma \in \Sigma\}$$

is a fan in $N(\tau)_{\mathbb{R}}$. The orbit closure $V(\tau)$ is isomorphic to $X_{\operatorname{Star}(\tau)}$ by Theorem 2.5.6 *a*) and *d*).

2.6 Applications on Kummer Modular Surfaces

In this subsection, we will introduce Shioda and Kummer Modular Surfaces, which is generalization of the construction of universal elliptic curves over certain modular curves. We use toric varieties to construct a compactification of a Kummer Modular Surface. It turns out that the compactification of these surfaces has relation with the toroidal compactification of A_g, which we will show in the later section.

Let H be an arithmetic subgroup, i.e., commensurable with $\mathrm{SL}(2,\mathbb{Z})$, which means that $H \cap \mathrm{SL}(2,\mathbb{Z})$ has finite index in both H and $\mathrm{SL}(2,\mathbb{Z})$. For example, the group H can be the principal congruence group.

$$\Gamma(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{k}, c \equiv d \equiv 0 \pmod{k} \right\}.$$

Let $X^\circ(H)$ denote $H \backslash \mathbb{H}_1$ and $X^\circ(k)$ when $H = \Gamma(k)$.

Note that $X^\circ(H)$ is a Riemann surface and non-compact. It can be compactified by adding finitely many points to it. We denote $X(H)$ as the compactified Riemann surface and call the points added to $X(H)$ the cusps of $X(H)$.

Consider the lattice $L \subset \mathbb{Q}^2$ of rank 2 which is invariant under the action of H on $\mathbb{Q}^2$ by action

$$(m,n) \mapsto (am + cn, bm + dn).$$

Let

$$H_L = L \rtimes H.$$

The group H_L can be considered as the matrix group

$$\left\{ \begin{pmatrix} 1 & m & n \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H, (m,n) \in L \right\}.$$

Regarding $\mathbb{C} \times \mathbb{H}_1$ as a subset of $\mathbb{P}^2$ by $(z,\tau) \mapsto (z : \tau : 1)$. Then the group action of H_L on homogeneous coordinates of $\mathbb{C} \times \mathbb{H}_1$ is

$$\begin{pmatrix} 1 & m & n \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{bmatrix} z \\ \tau \\ 1 \end{bmatrix} = \begin{bmatrix} z + m\tau + n \\ a\tau + b \\ c\tau + d \end{bmatrix}.$$

Denote $D^\circ(H_L)$ as $H_L \backslash \mathbb{C} \times \mathbb{H}_1$ and $D^\circ(k)$ when $H = \Gamma(k)$ and $\{L = (m,n) \in \mathbb{Q}^2 \mid m, n \in k\mathbb{Z}\}$. The surface $D^\circ(k)$ are known as the **(open) Shioda modular surface of level k**.

Proposition 2.6.1 ([12, Proposition 2.16]). *(i) The space $D^\circ(H_L)$ is a non-compact complex surface which is a holomorphic fiber space over the modular curve $X^\circ(H)$. If $-1 \notin H$, then the fibers are elliptic curves. If $-1 \in H$, then the fibers are rational curves arising as quotients of elliptic curves by their natural involutions $x \mapsto -x$. Such fibers are called* ***Kummer curves*** *since they are 1-dimensional Kummer varieties.*

(ii) If the only elements of finite order in H are ± 1, then $D^\circ(H_L)$ is non-singular. In general, $D^\circ(H_L)$ contains at worst finite quotient singularities.

Sketch of Proof. We only prove the first statement, readers may refer to [12, Proposition 2.16].

The action of H_L on $\mathbb{C} \times \mathbb{H}_1$ and of $\mathbb{H}_1$ are related by the following commutative diagram

$$\begin{array}{ccc} H_L \times (\mathbb{C} \times \mathbb{H}_1) & \longrightarrow & \mathbb{C} \times \mathbb{H}_1 \\ {\scriptstyle q \times \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ H \times \mathbb{H}_1 & \longrightarrow & \mathbb{H}_1 \end{array} .$$

Hence there is a surjective map $\pi : D^\circ(H_L) \to X^\circ(H)$.

If $-1 \notin H$, then a generic point $\tau \in \mathbb{H}_1$ has only $\{1\}$ as its isotropy group in H. The fibre of $D^\circ(H_L)$ over τ is obtained as the quotient of the fibre of $\mathbb{C} \times \mathbb{H}_1 \to \mathbb{H}_1$ over τ by its stablizer group in H_L, which is

$$\left\{ \begin{pmatrix} 1 & m & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| (m,n) \in L \right\}.$$

It acts by $[z : \tau : 1] \mapsto [z + m\tau + n : \tau : 1]$ on $\mathbb{C} \times \{\tau\}$. Thus the fibre of $D^\circ(H_L)$ over τ is an elliptic curve, isomorphic to $\mathbb{C}$ modulo the lattice $\{m\tau + n \mid (m,n) \in L\}$.

If $-1 \in H_L$, then a genetic point $\tau \in \mathbb{H}_1$ has $\{\pm 1\}$ as its isotropy group in H. The fibre of $D^\circ(H_L)$ over τ is obtained as the quotient of $\mathbb{C} \times \{\tau\}$ over τ by its stabilizing subgroup

$$L \rtimes \{\pm 1\} = \left\{ \begin{pmatrix} 1 & m & n \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \middle| (m,n) \in L, \epsilon = \pm 1 \right\}$$

The subgroup L acts on $\mathbb{C} \times \{\tau\}$ as before, giving an elliptic curve E. The element

$$\iota = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

acts on $\mathbb{C} \times \{\tau\}$ by $[z : \tau : 1] \mapsto [-z : \tau : 1]$ which induces on the natural involution on E. The quotient $\mathbb{C} \times \tau$ by $L \rtimes \{\pm 1\}$ is hence a Kummer curve C. Note that the four fixed points on E under the involution are two-torsion points. There are 4 two-torsion points (cf. [25, Theorem 6.1]). Then consider the map $E \to C$ and apply Riemann-Hurwitz theorem, we get $g(C) = 0$ and hence isomorphic to $\mathbb{P}^1$. $\square$

Remark 2.6.2. (i) When $-1 \notin H$, the surface $D^\circ(H_L)$ is called an **(open) elliptic modular surface** and is denoted by $S^\circ(H_L)$. Note that $-1 \notin \Gamma(k)$ when $k \geq 3$, the surface $D^\circ(k)$ are elliptic modular surfaces denoted by $S^\circ(k)$.

(ii) If $-1 \in H$, then $D^\circ(H_L)$ is called **(open) Kummer modular surface** and is denote $K^\circ(H_L)$. Note that $-1 \in \Gamma(k)$ when $k = 1, 2$, the surface $D^\circ(k)$ are Kummer modular surface denoted by $K^\circ(k)$.

(iii) Note that the principal congruence group is torsion-free when $k \geq 3$ (cf.[1, Lemma 1.4]), which indicates that $D^\circ(k)$ is non-singular when $k \geq 3$.

We now focus on compactifying the Kummer modular surface $K^\circ(1)$ to the space $K(1)$ such that the map $\pi : K^\circ(1) \to X^\circ(1) = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}_1$ extends to the map $\pi : K(1) \to X(1)$.

For the compactification of $X^\circ(1) = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}_1$, we simply add this cusp $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{Q} \cup \{\infty\}$. It's easy to see that the stabilizing subgroup $P(\infty)$ of ∞ in $\mathrm{SL}(2,\mathbb{Z})$ is

$$P(\infty) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \middle| \frac{a}{c} = \infty \right\} = \left\{ \begin{pmatrix} \epsilon & b \\ 0 & \epsilon \end{pmatrix} \middle| \epsilon = \pm 1, b \in \mathbb{Z} \right\}.$$

We will construct a partial compactification of $K^\circ(1)$ by "adding a fiber" over the cusp ∞ of $X(H)$. The action of $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ on $K^\circ(1)$ extends to $\mathbb{C} \times (\mathbb{H}_1 \cup \mathbb{Q} \cup \{\infty\})$. The stabilizer $P = P(\mathbb{C} \times \{\infty\})$ is

$$P = L \rtimes P(\infty) = \left\{ \begin{pmatrix} 1 & m & n \\ 0 & \epsilon & b \\ 0 & 0 & \epsilon \end{pmatrix} \middle| m, n \in \mathbb{Z}, \begin{pmatrix} \epsilon & b \\ 0 & \epsilon \end{pmatrix} \in P(\infty) \right\}$$

We can choose a sufficiently small neighbourhood N of $\mathbb{C} \times \{\infty\}$ in $\mathbb{C} \times \mathbb{H}_1$ which is invariant under P, e.g.,

$$N = \{(z, \tau) \in \mathbb{C} \times \mathbb{H}_1 \mid \mathrm{Im}\, \tau > t_0\}, \qquad t_0 \gg 0.$$

For a sufficiently large t_0, if the intersection of $g(N)$ and N is non-empty for all g in $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$, then g must belong to P. So we can choose t_0 large enough (indeed $t_0 = 1$ is sufficient) such that $P\backslash N$ embeds into $(\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z}))\backslash(\mathbb{C} \times \mathbb{H}_1)$. Consider the following normal subgroup of P,

$$P' = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| n, b \in \mathbb{Z} \right\} \tag{5}$$

The quotient $P'' = P/P'$ can be identified with

$$P'' = \left\{ \begin{pmatrix} 1 & m & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \middle| m \in \mathbb{Z}, \epsilon = \pm 1 \right\}. \tag{6}$$

The group P' acts on the neighbourhood N of $\mathbb{C} \times \{\infty\}$ and the corresponding "partial" quotient map can be given:

$$e : N \to (\mathbb{C}^*)^2 \quad e(z, \tau) = (\exp(2\pi i z), \exp(2\pi i \tau)).$$

Hence, $P'\backslash N \cong e(N) \cong \mathbb{C}^* \times D'$ where D' is the punctured disk of radius $\exp(-2\pi t_0)$. The group P'' acts on $e(N)$ and extends to an action on the torus $T = (\mathbb{C}^*)^2$. More precisely, the induced action of $g \in P''$ on T is defined by the requirement that the diagram

$$\begin{tikzcd} N \arrow[r, "g"] \arrow[d, "e"'] & N \arrow[d, "e"] \\ T \arrow[r, "g"'] & T \end{tikzcd}$$

commutes. Note that the two matrices

$$h = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \iota = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

generates the group P''. It suffices to study h and ι. Let $(u, v) = e(z, \tau)$. Then

$$e(h(z,\tau)) = e(z+\tau,\tau) = (\exp(2\pi i(z+\tau)), \exp(2\pi i\tau)) = (uv, v).$$

Similarly,

$$e(\iota(z,\tau)) = e(-z,\tau) = (\exp(-2\pi iz), \exp(2\pi i\tau)) = (u^{-1}, v)$$

We will now construct a normal toric variety X_Σ with the torus T and Σ a P''- compatible fan. Let Y_Σ be the interior of the closure $e(N)$ in X_Σ. In this case, it is $e(N) \cup (X_\Sigma - T)$. The partial compactification is $P''\backslash Y_\Sigma$. As we have only one cusp, then the identification space

$$P''\backslash Y_\Sigma \cup_{P\backslash N} K^\circ(1)$$

is the compactification $K(1)$ of $K^\circ(1)$ we need.

Denote U and V as the two generators of the character group M of the torus T corresponding to the two coordinates u and v. Denote U^* and V^* as the corresponding dual basis. As $M \cong \mathbb{Z}^2$, we may identify $U = (1,0)$ and $V = (1,0)$ and consequently $U^* = (1,0)$ and $V^* = (0,1)$. The induced $\mathbb{Z}$-linear maps h and l on the group N of the one-parameter subgroups of the torus T can be characterized by the matrix $A(h) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A(l) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Consider the fan in $N_\mathbb{R}$ which consists of the 2-dimension cones $\sigma_k = \mathbb{R}_\geq(k,1) + \mathbb{R}_{\geq 0}(k+1,1)$, for all $k \in \mathbb{Z}$ and their one-dimensional faces $\xi_k = \mathbb{R}_\geq(k,1), k \in \mathbb{Z}$ and the vertex $\{(0,0)\}$, see the following figure.

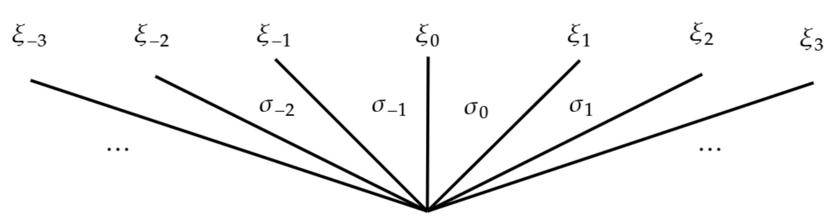


Figure 5: A Decomposition of the upper half plane

By Orbit-Cone correspondence, we have the following correspondence:

(i) the 0-dimensional cone $\{(0,0)\} \Leftrightarrow T \cong (\mathbb{C}^*)^2$;

(ii) the 1-dimensional cone $\xi_k \Leftrightarrow O(\xi_k) \cong \mathbb{C}^*$;

(iii) the 2-dimension cone $\sigma_k \Leftrightarrow O(\sigma_k) \cong (C^*)^0$.

For each orbit $O(\xi_k)$, the closure is a rational curve

$$\overline{O(\xi_k)} = O(\xi_k) \cup O(\sigma_{k-1}) \cup O(\sigma_k) \cong \mathbb{P}^1.$$

Denote it as C_k. We have an infinite string of rational curves, see Figure 6.

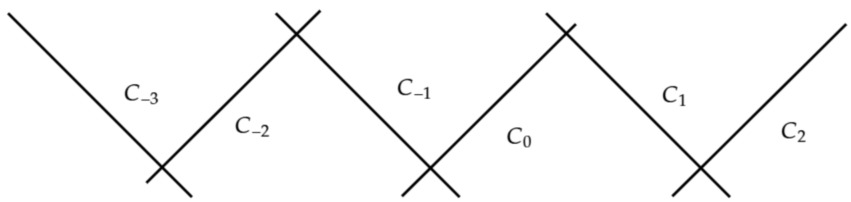


Figure 6: Infinite String of rational curves

As $P''\backslash e(N) = P\backslash N$ embeds into $K^0(1)$, we are now interested in the action of P'' on the infinity string $X_\Sigma - T$ of rational curves.

Consider the action of the element $h \in P''$. We have $A(h)(\xi_k) = \xi_{k+1}$ and $A(h)(\sigma_k) = \sigma_{k+1}$. Hence we can identify the 1-dimensional orbits $O(\xi_k)$'s and identify the 0-dimensional orbits $O(\sigma_k)$'s. Hence $\langle h\rangle\backslash(X_\Sigma - T)$ is a single rational curve $\bar{C}$ with two points identified.

Consider the action of the element $\iota \in P''$. We have $A(\iota)(\xi_k) = \xi_{-k}$ and $A(\iota)(\sigma_k) = \sigma_{-k-1}$. In particular, $A(h)(\xi_0) = \xi_0$ and $A(h)(\sigma_0) = \sigma_{-1}$. We now want to know how ι acts on the torus $O(\sigma_0)$. The torus has character lattice $M' = \sigma_0^\perp \cap M = \mathbb{Z}(1,0)$ and cocharacter lattice $N' = \mathbb{Z}(1,0)$. As $A(l)(1,0) = (-1,0)$. Hence by Remark 2.4.8, the action on $O(\zeta)$ is given by $z \mapsto z^{-1}$. And $P''\backslash X_\Sigma - T$ is $\mathbb{P}^1/z \sim z^{-1}$. Note that the quotient of a non-singular curve by a finite group action is still non-singular. Hence we can apply Riemann-Hurwitzm to get the genus of this curve is 0 and hence it is isomorphic to $\mathbb{P}^1$.

To summarize, the Kummer modular surface $K^\circ(1)$ can be compactified to a surface $K(1)$ over $X(1)$ whose fiber over the cusp ∞ is $\mathbb{P}^1$.

3 The Satake Compactification

In this section, we provide all necessary backgrounds for toroidal compactifications and give a short introduction to the Satake Compactification.

3.1 Boundary Components

In this subsection, we wish to extend $\mathbb{H}_g$. We will show that as a hermitian symmetric domain, it can be realized as a bounded symmetric domain D in $\mathbb{C}^{g(g+1)/2}$ so that we can take the closure $\overline{D}$. We will also classify points and pick those that are "rational". Our main reference for this section is [20] and [12].

We summarize a few results of hermitian symmetric domains from [16].

A manifold is **homogeneous** if its automorphism group acts transitively, **symmetric** if it is homogeneous and at some point p, there is an automorphism s_p such that $s_p^2 = 1$ and p is the only fixed point of s_p in some neighborhood of p. A **riemannian manifold** (M^∞, g) is a smooth manifold M with a riemannian metric g. A **complex manifold** is a smooth complex analytic space. An **almost-complex structure** on a smooth manifold is a smooth tensor field $J = (J_p)_{p \in M}$,

$$J_p : \mathrm{Tgt}_p(M) \to \mathrm{Tgt}_p(M)$$

such that $J_p^2 = -1$. In terms of local coordinates $z^1, ..., z^n$ in a neighbourhood of a point p on a complex manifold and the corresponding real local coordinates $x^1, y^1, ..., x^n, y^n$, J_p acts by

$$\frac{\partial}{\partial x^r} \mapsto \frac{\partial}{\partial y^r}, \quad \frac{\partial}{\partial y^r} \mapsto -\frac{\partial}{\partial x^r}.$$

A hermitian metric on a complex manifold M is a riemnanian metric g such that

$$g(JX, JY) = g(X, Y) \text{ for all vector fields } X, Y.$$

A **hermitian manifold** (M, g) is a complex manifold M with a hermitian metric and so it is also a riemannian manifold. A connected symmetric hermitian manifold is called **hermitian symmetric space**.

Any hermitian symmetric space M decomposes into a product

$$M_0 \times M_1 \times \cdots M_n$$

where M_0 is a hermitian symmetric space of euclidean type and $M_i, i > 0$ is an irreducible and non-euclidean hermitian symmetric space (cf. [3, p.105]). The hermitian symmetric spaces of **euclidean type** are quotients of a complex space $\mathbb{C}^g$ by a discrete subgroup of translations. For example, a torus $\mathbb{C}/\Lambda$ is a hermitian symmetric space of euclidean type. A non-euclidean irreducible hermitian symmetric space is of **compact type** (resp. of **non-compact type**) if it is compact (resp. not compact). The Siegel upper half spaces are hermitian symmetric spaces of non-compact type. The projective space $\mathbb{P}(\mathbb{C})$ is a hermitian symmetric space of compact type. The space M is called **non-euclidean**, resp. **symmetric domain**, resp. **of compact type**, if, the factor M_0 is absent in the decomposition, or all the M_i are of non-compact type, or all M_i are of compact type.

A **domain** D in $\mathbb{C}^n$ is a nonempty open connected subset. Every bounded domain has a hermitian metric, called the Bergman metric. The Bergman metric is invariant under the action of the group $\mathrm{Hol}(D)$ of automorphisms of D as a complex manifold. Hence, a bounded symmetric domain is a hermitian symmetric domain. Conversely, every hermitian symmetric domain can be embedded into $\mathbb{C}^n$ as a bounded symmetric domain.

Back to our example of $\mathbb{H}_g$.

Proposition 3.1.1 (cf. [20, p.2-3])**.**

a) The group $Sp(2g, \mathbb{R})$ acts transitively on $\mathbb{H}_g$.

b) The stabilizer of iI_g is the compact group

$$\left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in Sp(2g, \mathbb{R}) \right\} \cong U(g), \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mapsto \alpha + i\beta$$

where $U(g)$ is the unitary group.

Consequently, the map

$$\mathrm{Sp}(2g, \mathbb{R}) \to \mathbb{H}_g, \quad M \mapsto M(iI_g)$$

is surjective with kernel $U(g)$. Hence we have $\mathbb{H}_g = \mathrm{Sp}(2g, \mathbb{R})/U(g)$ as smooth manifolds. All holomorphic automorphisms of $\mathbb{H}_g$ arise from the action of $\mathrm{Sp}(2g, \mathbb{R})$. The automorphism group $\mathrm{Hol}(\mathbb{H}_g)$ is $Sp(2g, \mathbb{R})/\{\pm I\}$. Moreover, the matrix $s = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ has the property that $s^2 = I_g$ and has iI_g as its only fixed point. Hence $\mathbb{H}_g$ is homogeneous and symmetric as a complex manifold.

We now realize $\mathbb{H}_g$ as a bounded symmetric domain.

Proposition 3.1.2. *The map*

$$\Phi : \mathbb{H}_g \to D_g = \{Z \in Sym(g, \mathbb{C}) \mid I_g - Z\overline{Z} > 0\}$$

given by

$$\tau \mapsto (\tau - iI_g)(\tau + iI_g)^{-1}$$

is isomorphic with the inverse

$$\Phi^{-1} : D_g \to \mathbb{H}_g \textit{ given by } Z \mapsto i(Z + I_g)(-Z + I_g)^{-1}.$$

Remark 3.1.3. The map Φ is called a **Cayley transformation**. Sometimes we write Φ_g to indicate the dimension g. When $g = 1$, the map takes the half-plane $\mathbb{H}_1 = \{z \in \mathbb{C}; \mathrm{Im} z > 0\}$ to the complex unit disc $\{z \in \mathbb{C}; |z| < 1\}$.

Proof. This is a specific instance of the Borel and Harish-Chandra embedding, as discussed in [20, p.118]. $\square$

Note that D_g is indeed bounded in $\mathrm{Sym}(g, \mathbb{C})$. Since $I_g - Z\overline{Z}$ is positive definite, we have $x^H I_g x - x^H Z\overline{Z}x = ||x||_2^2 - ||\overline{Z}x||_2^2 > 0$ for all $x \in \mathbb{C}^g$ where x^H denotes the conjugate transpose of x. Then $||\overline{Z}||_2 = \sup_{x \neq 0} \frac{||\overline{Z}x||_2}{||x||_2} < 1$ for all Z in D_g. By equivalence of norms, the maximal norm is also bounded in D_g. The bounded domain is also symmetric with a symmetry at $0 = \Phi(iI_g)$. The Siegel upper half space $\mathbb{H}_g$ has a hermitian metric from the Bergman metric of D_g, invariant under the action of $\mathrm{Hol}(\mathbb{H}_g)$. Hence $\mathbb{H}_g$ is a hermitian symmetric domain.

Since D_g is a bounded domain in $\mathrm{Sym}(g, \mathbb{C})$, compactify it and we get

$$\overline{D_g} = \{Z \in M(g, \mathbb{C}); {}^tZ = Z, I_g - Z\overline{Z} \geq 0\}.$$

Hence the boundary consists of symmetric matrices Z such that $I_g - Z\overline{Z}$ is positive semi-definite but not positive definite.

Proposition 3.1.4. *The action of $Sp(2g, \mathbb{R})$ on D_g induced from the Cayley transformation*

$$M \in \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : Z \mapsto ((\alpha - i\gamma)(Z + I_g) + i(\beta - i\delta)(Z - I_g))((\alpha + i\gamma)(Z + I_g) + i(\beta + i\delta)(Z - I_g))^{-1}$$

extends to the closure of D_g.

Proof. It suffices to show that $(\alpha + i\gamma)(Z + I_g) + i(\beta + i\delta)(Z - I_g)$ is invertible. For the whole proof, we refer to [12, Proposition 3.3] or [20, Proposition 4.3]. □

We can decompose $\overline{D_g}$ into a disjoint union of components.

Definition 3.1.5. Two points $p, q \in \overline{D_g}$ are equivalent, denoted as $p \sim q$, if and only if they can be connected by finitely many holomorphic curves, i.e., there are holomorphic maps $\phi_i : D = \{z \in \mathbb{C}; |z| < 1\} \to \overline{D_g}$, $i = 1, ..., m$, such that $p \in \phi_1(D), q \in \phi_m(D)$ and $\phi_i(D) \cap \phi_{i+1}(D) \neq 0$ for $i = 1, ..., m - 1$.

Definition 3.1.6. The equivalence classes of points of $\overline{D_g}$ with respect to $\sim$ are called boundary components.

There are a few properties we can get immediately from the definition. Boundary components are connected. If F is a boundary component and $h \in \mathrm{Sp}(2g, \mathbb{R})$, then $h(F)$ is a boundary component as h also acts holomorphically on $\overline{D_g}$. We will also show that D_g is a boundary component. This boundary component is called improper. Boundary components in $\overline{D_g} \backslash D_g$ are called proper.

The key to the classification of boundary components is to associate them with isotropic subspaces in $\mathbb{R}^{2g}$. A subspace $U \subset \mathbb{R}^{2g}$ is called isotropic if $\langle u, v \rangle = uJ^tv$ holds for all $u, v \in U$ and $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

Proposition 3.1.7. *i) For $Z \in \overline{D_g}$, denote by $U(Z)$ the real subspace* $\ker(\Psi_Z)$ *of* $\mathbb{R}^{2g}$ *where*

$$\Psi_Z : \mathbb{R}^{2g} \to \mathbb{C}^g, \quad v \mapsto v \cdot \begin{pmatrix} i(I_g + Z) \\ I_g - Z \end{pmatrix}.$$

Then $U(Z)$ is an isotropic subspace of $\mathbb{R}^{2g}$.

ii) If $Z \in \overline{D_g}$, then $U(Z) \neq 0$ if and only if Z lies on the boundary $\overline{D_g} \backslash D_g$.

iii) If $Z \in \overline{D_g}$ and $h \in Sp(2g, \mathbb{R})$, then $U(h(Z)) = U(Z)h^{-1}$.

Proof. i) Identify $v = (v_1, v_2) \in \mathbb{R}^{2g}$ where $v_1 \in \mathbb{R}^g$ and $v_2 \in \mathbb{R}^g$ with $w = v_1 + iv_2 \in \mathbb{C}^g$. We can calculate

$$\Psi_Z(w) = \Psi_Z(v) = v \cdot \begin{pmatrix} i(I_g + Z) \\ I_g - Z \end{pmatrix} = v \cdot \begin{pmatrix} iI_g & iI_g \\ -I_g & I_g \end{pmatrix} \begin{pmatrix} Z \\ I_g \end{pmatrix} = i(wZ + \overline{w}).$$

If $w, w' \in \mathbb{C}^g$ correspond to $v, v' \in \mathbb{R}^{2g}$, then

$$\langle v, v' \rangle = vJ^t v' = -v_2{}^t v_1^{'} + v_1{}^t v_2^{'} = \mathrm{Im}(\overline{w}^t w').$$

Moreover, if $v, v' \in U(Z)$ and $\overline{w} = -wZ$ and $\overline{w}' = -w'Z$, then

$$\langle v, v' \rangle = \mathrm{Im}(\overline{w}^t w') = \frac{1}{2}(\overline{w}^t w' - w^t \overline{w}') = \frac{1}{2}(-wZ^t w' + wZ^t w') = 0.$$

ii) Suppose $U(Z) \neq 0$. Then there exists a $w \in \mathbb{C}^g$, $w \neq 0$, such that $\overline{w} = -wZ$. Then $w = -\overline{w}\overline{Z} = wZ\overline{Z}$, so $I_g - Z\overline{Z}$ cannot be positive definite.

Suppose $Z \in \overline{D_g} \backslash D_g$. Then 1 is an eigenvalue of $Z\overline{Z}$. Consider an eigenvector $\overline{v} \in \mathbb{C}^g$, $\overline{v} = Z\overline{Z}\overline{v}, v \neq 0$. If $\overline{v} + vZ = 0$, then $v \in U(Z)$. Otherwise, let $w = iv + i\overline{v}\overline{Z}$. Then $w \neq 0$ and $wZ = i(vZ + \overline{v}\overline{Z}Z) = i(vZ + \overline{v}) = -\overline{w}$, so $w \in U(Z)$.

iii) One can compute for $h \in \mathrm{Sp}(2g, \mathbb{R})$ that the matrix multiplication $h \cdot \begin{pmatrix} i(I_g + Z) \\ I_g - Z \end{pmatrix}$ and the h action $\begin{pmatrix} i(I_g + h \cdot Z) \\ I_g - h \cdot Z \end{pmatrix}$ give the same kernel. Then

$$U(h(Z)) = \ker(v \mapsto v \cdot h \cdot \Psi(Z)) = \ker(v \mapsto v \cdot \Psi(z))h^{-1} = U(Z)h^{-1}.$$

□

Proposition 3.1.8. *If $Z_1, Z_2 \in \overline{D_g}$ and $Z_1 \sim Z_2$, then $U(Z_1) = U(Z_2)$.*

Proof. It suffices to show that if $\phi : D \to \mathrm{Sym}(g, \mathbb{C})$ is holomorphic with $\phi(D) \subset \overline{D_g}$ and $v \in U(\phi(0))$, $v \neq 0$, then $v \in U(\phi(t))$ for all $t \in D$. Identify v with $w \in \mathbb{C}^g$. Then we have $\overline{w} = -w\phi(0)$. In particular, $||w\phi(0)|| = ||w|| \neq 0$. Moreover, $||w\phi(t)||^2 = w\phi(t)\overline{\phi(t)}^t\overline{w} \leq w^t\overline{w} = ||w||^2$. By the maximum modulus principle, $w\phi(t)$ is constant. Hence $\overline{w} = -w\phi(t)$. □

If F is a boundary component of D_g, we can denote by $U(F)$ the isotropic subspace $U(Z) \subset \mathbb{R}^{2g}$ for a $Z \in F$. By the proposition above, $U(F)$ is independent of the choice of Z and hence well defined.

Proposition 3.1.9. *$F_{g'} = \left\{ \begin{pmatrix} Z' & 0 \\ 0 & I_{g-g'} \end{pmatrix}; Z' \in D_{g'} \right\} \cong D_{g'}$ is a boundary component with corresponding isotropic subspace*

$$U(F_{g'}) = \{(v_1, ..., v_{2g}) \in \mathbb{R}^{2g} \mid v_1 = ... = v_{g+g'} = 0\}.$$

In particular D_g is a boundary component.

Proof. We refer to the proof in [12, Lemma 3.10]. □

Remark 3.1.10. Boundary components of such form play an important role in compactification. We call them **standard boundary components** and the associated isotropic subspaces **standard isotropic subspaces**.

Proposition 3.1.11. *The map Ψ defined in Proposition 3.1.7 induces a one-to-one correspondence between the set of boundary components F and the set of isotropic subspaces of $\mathbb{R}^{2g}$ with respect to the symplectic form $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$. This correspondence is $Sp(2g,\mathbb{Q})$-equivariant in the sense that $U(h(F)) = U(F)h^{-1}$.*

Before proving the proposition, we need a lemma that claims that any flag of isotropic subspaces can be transformed into a flag of standard isotropic subspaces by an element in $\mathrm{Sp}(2g,\mathbb{R})$.

Lemma 3.1.12 ([12, Lemma 3.11])**.** *Let $Sp(2g,\mathbb{R})$ be the symplectic group acting on $\mathbb{R}^{2g}$. If $U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_l$ is any flag of isotropic subspaces in $\mathbb{R}^{2g}$ with $\dim U_i = g - k(i)$, then there exists a $h \in Sp(2g,\mathbb{R})$ such that*

$$h(U_i) = U_i h^{-1} = \{(v_1, ..., v_{2g}) \in \mathbb{R}^{2g} \mid v_1 = ... = v_{g+k(i)} = 0\}.$$

Proof of Proposition 3.1.11 . The well-definedness and equivariance have been established, it remains to show that it is a one-to-one correspondence. Let U be an arbitrary isotropic subspace in $\mathbb{R}^{2g}$. By the above lemma, there is a k, $0 \le k \le g$ and $h \in \mathrm{Sp}(2g,\mathbb{R})$ such that $U = U_k h^{-1}$ and hence $U = U(h(F_{g-k}))$. If $U = U(F') = U(F'')$ for two boundary components F' and F'', then by the lemma above, $U = U_k h^{-1}$ and so $U(h^{-1}F') = U(F_{g-k}) = U(h^{-1}F'')$. By Proposition 3.1.9, we have $h^{-1}F' = F_{g-k} = h^{-1}F''$. □

Definition 3.1.13. A boundary component F of D_g is called degree $g-i$ or corank i boundary component if $\dim_{\mathbb{R}}(U(F)) = i$.

Lemma 3.1.12 also indicates that any boundary component can be transformed to a standard boundary component of the same degree. We can write $\overline{D_g}$ as a union of boundary components as follows:

$$\overline{D_g} = \bigcup_{h \in Sp(2g,\mathbb{R}), 0 \le k \le g} h(F_k).$$

Now we discuss the adjacency of boundary components.

Definition 3.1.14. A boundary component F is said to be adjacent to a secondary component F' if $F \neq F'$ and $F \subset \overline{F'}$. We denote this by $F \prec F'$.

Example 3.1.15. Any proper boundary component is adjacent to the boundary component D_g. For standard boundary components, we have $F^i \prec F^j$ whenever $0 \le i < j \le g$.

Proposition 3.1.16. *i) The correspondence between boundary components of D_g and isotropic subspaces of $\mathbb{R}^{2g}$ gives rise to a one-to-one correspondence between*

a) pairs of adjacent boundary components $F \prec F'$;

b) pairs of isotropic subspaces $U' \subsetneq U$ in $\mathbb{R}^4$.

ii) The group Sp$(2g, \mathbb{R})$ acts on pairs of adjacent boundary components. Moreover, every pair $F \prec F'$ is equivalent under Sp$(2g, \mathbb{R})$ to one of the pairs $F_i \prec F_j$ where $0 \leq i < j \leq g$.

Proof. Suppose $u \in U(F')$. Identify u with $w \in \mathbb{C}^g$. Then $wZ' + \overline{w} = 0$ for all $Z' \in F'$. As $F \subset \overline{F'}$, then $wZ + \overline{w} = 0$ for $Z \in F$ as limit of $Z' \in F'$. This implies that $U(F') \subset U(F)$. Because $F \neq F'$, we have $U(F') \subsetneq U(F)$.

Let $U' \subsetneq U$ be a pair of isotropic subspaces of $\mathbb{R}^4$. Then by Lemma 3.1.12, there exists $h \in \mathrm{Sp}(2g, \mathbb{R})$ such that $U'h \subsetneq Uh$ is one of the pairs of $U_i \subsetneq U_j$ where $i < j$. This implies that the boundary components corresponding to U and U' are $h(F_{g-j})$ and $h(F_{g-i})$. Since we have $F_{g-j} \prec F_{g-i}$, we have $h(F_{g-j}) \prec h(F_{g-i})$.

Part (ii) easily follows from the discussion above.

□

Via Lemma 3.1.12 and Proposition 3.1.16, whenever we need to consider boundary components, we can pass them to standard boundary components and work on standard boundary components.

We have decomposed $\overline{D_g}$. We now look at the rationality of the boundary components. Recall the process of compactifying a modular curve, i.e., $g = 1$, we only focus on the "rational points" $\{\mathbb{Q} \cup \infty\}$. For general g, we also focus only on those "rational boundary components" when compactifying.

Definition 3.1.17. Let F be a boundary component, we define its stabilizing subgroup in $\mathrm{Sp}(2g, \mathbb{R})$ by

$$\mathcal{P}(F) = \{h \in \mathrm{Sp}(2g, \mathbb{R}) \mid h \cdot F = F\}.$$

Remark 3.1.18.

i) If $U = U(F)$ is the corresponding isotropic subspace, then by Proposition 3.1.11,

$$\mathcal{P}(F) = \mathcal{P}(U) = \{h \in \mathrm{Sp}(2g, \mathbb{R}) \mid Uh^{-1} = U\}.$$

ii) Let F_1 and F_2 be two boundary components and $h \cdot F_1 = F_2$ for some $h \in \mathrm{Sp}(2g, \mathbb{R})$, then $\mathcal{P}(F_2) = h\mathcal{P}(F_1)h^{-1}$.

iii) $\mathcal{P}(F)$ is a maximal parabolic subgroup of $\mathrm{Sp}(2g, \mathbb{R})$. Indeed, there is a one-to-one correspondence between proper parabolic subgroups of $\mathrm{Sp}(2g, \mathbb{R})$ and flags of nontrivial isotropic subspaces in $\mathbb{R}^{2g}$(cf. [12, Remark 3.45]). The parabolic subgroup corresponding to a given flag $0 = U_0 \subsetneq U_1 \subsetneq \cdots U_k$ is the stabilizing subgroup $\mathcal{P}$ of this flag, i.e., a group that consists of all elements h such that $h \cdot U_i = U_i$ for $i = 0, ..., k$. The maximal parabolic subgroups correspond to flags of isotropic subspaces which have length 1.

Definition 3.1.19 ([20, Definition 4.15])**.** A boundary component F of D_g is called **rational** if one of the following equivalent conditions holds:

i) The stabilizing subgroup $\mathcal{P}(F)$ is defined over $\mathbb{Q}$, i.e., there is a subgroup $\mathcal{P}_{\mathbb{Q}} \subset \mathrm{Sp}(2g, \mathbb{Q})$ such that $\mathcal{P}(F) = \mathcal{P}_{\mathbb{Q}}(\mathbb{R})$, the $\mathbb{R}$-valued points of the algebraic group of $\mathcal{P}_{\mathbb{Q}}$.

ii) The isotropic subspace $U(F)$ is defined over $\mathbb{Q}$, i.e., it has a basis chosen in $\mathbb{Q}^{2g}$.

iii) There exists $h \in \mathrm{Sp}(2g, \mathbb{Q})$ such that $h \cdot F = F_{g'}$.

Remark 3.1.20. The correspondence in Remark 3.1.18 restricts to a bijective correspondence between parabolic subgroups defined over $\mathbb{Q}$ and flags of isotropic subspaces of $\mathbb{Q}^{2g}$ (cf. [12, Remark 3.45]).

When $g = 1$, we have one standard proper boundary component $F_0 = \{1\}$ which is $\{\infty\}$ under the inverse of the Cayley transformation. The rational proper boundary components are $h \cdot F_0$ for $h \in \mathrm{Sp}(2g, \mathbb{Q})$, which is $\mathbb{Q}$ under the inverse of the Cayley transformation. As $\mathrm{SL}(2, \mathbb{Z})$ acts transitively on the rational boundary components $\mathbb{Q} \cup \{\infty\}$, for higher dimensions we also have

Proposition 3.1.21. *If F is a rational boundary component of $D_{g'}$, then there exists $h \in Sp(2g, \mathbb{Z})$ such that $h \cdot F = F_{g'}$.*

Proof. See [20, Remark 4.16]. ☐

Hence, we can consider the union of all rational boundary components:

Definition 3.1.22. We define the rational closure of D_g as

$$D_g^{\mathrm{rc}} = \bigcup_{F:\text{rational}} F = \bigcup_{h \in \mathrm{Sp}(2g,\mathbb{Z}), 0 \leq k \leq g} h(F_k).$$

3.2 Structure of Stabilizing Subgroups

In this subsection, we discuss the structure of stabilizing subgroups of boundary components. Readers may skip this subsection and revisit it when needed.

As shown in Remark 3.1.18, if $h \cdot F_1 = F_2$ for F_1, F_2 two boundary components and $h \in \mathrm{Sp}(2g, \mathbb{R})$, we have $\mathcal{P}(F_2) = h\mathcal{P}(F_1)h^{-1}$. Hence it is important to know the stabilizing subgroups of standard boundary components.

For a standard boundary component, we have its stabilizing subgroup:

Proposition 3.2.1 ([12, Proposition 3.87])**.**

$$\mathcal{P}_{g'} = \mathcal{P}(F_{g'}) = \left\{ \begin{pmatrix} A & 0 & B & * \\ * & {}^tQ^{-1} & * & * \\ C & 0 & D & * \\ 0 & 0 & 0 & Q \end{pmatrix} \middle| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g', \mathbb{R}), Q \in GL(g - g', \mathbb{R}) \right\}.$$

Definition 3.2.2.
- $R_u(\mathcal{P}(F)) :=$ the unipotent radical of $\mathcal{P}(F)$;
- $\mathcal{P}'(F) :=$ the center of $R_u(\mathcal{P}(F))$;
- $V(F) := R_u(\mathcal{P}(F))/\mathcal{P}'(F)$.

Remark 3.2.3. We have inclusions of normal subgroups $\mathcal{P}'(F) \trianglelefteq R_u(F) \trianglelefteq \mathcal{P}(F)$.

Remark 3.2.4. $R_u(\mathcal{P}(h \cdot F)) = hR_u(\mathcal{P}(F))h^{-1}$ and $\mathcal{P}'(F) = h\mathcal{P}'(F)h^{-1}$ for $h \in \mathrm{Sp}(2g, \mathbb{R})$.

For standard boundary components, we can give more explicit matrix forms of these subgroups.

Proposition 3.2.5 (cf. [20, Proposition 4.8 and 5.4] and [12, Proposition])**.** *For a standard boundary component $F_{g'}$*

$$R_u(\mathcal{P}(F_{g'})) = \left\{ \begin{pmatrix} I_{g'} & 0 & 0 & {}^tN \\ M & I_{g-g'} & N & S \\ 0 & 0 & I_{g'} & -{}^tM \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| M^tN + S = N^tM + {}^tS \right\}.$$

$$\mathcal{P}'(F_{g'}) = \left\{ \begin{pmatrix} I_{g'} & 0 & 0 & 0 \\ 0 & I_{g-g'} & 0 & S \\ 0 & 0 & I_{g'} & 0 \\ 0 & 0 & 0 & I_{g-g'} \end{pmatrix} \middle| S = {}^tS \right\} \cong Sym(g-g', \mathbb{R}).$$

Moreover, $R_u(\mathcal{P}(F_{g'}))/\mathcal{P}'(F_{g'})$ is isomorphic to $Mat(g-g', g'; \mathbb{C})$ by

$$\begin{pmatrix} I_{g'} & 0 & 0 & {}^tN \\ M & I_{g-g'} & N & S \\ 0 & 0 & I_{g'} & -{}^tM \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto iM + N.$$

For a standard boundary component $F_{g'}$, we consider two more subgroups of $\mathcal{P}(F_{g'})$:

$$G_h(F_{g'}) := \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & I_{g-g'} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & I_{g-g'} \end{pmatrix} \middle| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g', \mathbb{R}) \right\} \cong \mathrm{Sp}(2g', \mathbb{R}), \tag{7}$$

$$G_l(F_{g'}) := \left\{ \begin{pmatrix} I_{g'} & 0 & 0 & 0 \\ 0 & {}^tQ^{-1} & 0 & 0 \\ 0 & 0 & I_{g'} & 0 \\ 0 & 0 & 0 & Q \end{pmatrix} \middle| Q \in \mathrm{GL}(g-g', \mathbb{R}) \right\} \cong \mathrm{GL}(g-g', \mathbb{R}). \tag{8}$$

Definition 3.2.6. For a boundary component F of D_g, i.e. $F = h(F_g')$ for some $h \in \mathrm{Sp}(2g, \mathbb{R})$, define

$$G_h(F) = G_h(h(F_{g'})) := hG_h(F_{g'})h^{-1}; \quad G_l(F) = G_l(h(F_{g'})) := hG_l(F_{g'})h^{-1}.$$

Proposition 3.2.7 (cf. [20, Proposition 4.10] and [3, Theorem 3.10])**.**

i) $\mathcal{P}(F) = (G_h(F) \times G_l(F)) \ltimes R_u(\mathcal{P}(F))$.

ii) Let $Z(F)$ be the centralizer of a boundary component F, i.e., $Z(F) = \{g \in G \mid gx = x$ for all $x \in F\}$. Then we have $Z(F) = G_l(F) \ltimes R_u(\mathcal{P}(F))$.

iii) $G_h(F) = Aut(F)$.

To know how these groups act on the Siegel upper space $\mathbb{H}_g$ the following is also important:

Proposition 3.2.8 ([12, Proposition 3.91])**.** *Let* $\tau = \begin{pmatrix} \tau_1 & {}^t\tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_g$ *where the rows and columns are divided into ranges of* g' *and* $g - g'$ *rows resp. columns.Then the groups defined above act as follows:*

$$\begin{aligned}
g_1 \in G_h(F_{g'}) : \quad & g_1(\tau) = \begin{pmatrix} (A\tau_1 + B)(C\tau_1 + D)^{-1} & * \\ \tau_2(C\tau_1 + D)^{-1} & \tau_3 - \tau_2(C\tau_1 + D)^{-1}C^t\tau_2 \end{pmatrix}, \\
g_2 \in G_l(F_{g'}) : \quad & g_2(\tau) = \begin{pmatrix} \tau_1 & * \\ {}^tQ^{-1}\tau_2 & {}^tQ^{-1}\tau_3 Q^{-1} \end{pmatrix}, \\
g_3 \in R_u(\mathcal{P}(F_{g'})) : \quad & g_3(\tau) = \begin{pmatrix} \tau_1 & * \\ \tau_2 + M\tau_1 + N & \tau_3' \end{pmatrix}, \\
& \tau_3' = \tau_3 + M\tau_1{}^tM + M^t\tau_2 + {}^t(M^t\tau_2) + N^tM + S, \\
g_4 \in \mathcal{P}'(F_{g'}) : \quad & g_4(\tau) = \begin{pmatrix} \tau_1 & * \\ \tau_2 & \tau_3 + S \end{pmatrix},
\end{aligned}$$

where the entries $*$ *are determined by symmetry.*

Now we identify the quotient group $\mathcal{P}''(F_g') = \mathcal{P}(F_{g'})/\mathcal{P}''(F_{g'})$.

Proposition 3.2.9 ([12, Proposition 3.90])**.** *The quotient group* $\mathcal{P}''(F_g') = \mathcal{P}(F_{g'})/\mathcal{P}''(F_{g'})$ *in the short exact sequence*

$$1 \to \mathcal{P}'(F_{g'}) \to \mathcal{P}(F_{g'}) \xrightarrow{\pi} \mathcal{P}''(F_{g'}) \to 1$$

can be identified as the group consisting of the block matrices.

$$\begin{pmatrix} {}^tQ^{-1} & M & N \\ 0 & A & B \\ 0 & C & D \end{pmatrix}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g', \mathbb{R})$*,* $Q \in GL(g - g', \mathbb{R})$ *and* $M, N \in Mat(g - g', g', \mathbb{R})$ *satisfying* $M^tN = N^tM$*. Furthermore, the natural quotient map* π *is given by*

$$\pi(g_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix}; \quad \pi(g_2) = \begin{pmatrix} {}^tQ^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \pi(g_3) = \begin{pmatrix} 1 & M & N \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.3 The Satake Compactification

In this subsection, we give a short introduction to the construction of the Satake compactification of the quotient of the Siegel upper half space $\mathbb{H}_g$ by an arithmetic group. By arithmetic subgroup of $\mathrm{Sp}(2g, \mathbb{R})$, we mean a subgroup Γ in $\mathrm{Sp}(2g, \mathbb{Q})$ that is commensurable with $\mathrm{Sp}(2g, \mathbb{Z})$, i.e., the group $H \cap \mathrm{Sp}(2g, \mathbb{Z})$ having finite index in both H and $\mathrm{Sp}(2g, \mathbb{Z})$.

We recall the case of the upper half plane $\mathbb{H}_1$: We consider Γ a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, it is obviously an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{Q})$. Let $\mathbb{H}_1^{\mathrm{rc}} = \mathbb{H}_1 \cup \mathbb{Q} \cup \{\infty\}$. The action of Γ on $\mathbb{H}_1$ extends to $\mathbb{H}_1^{\mathrm{rc}}$ and $\Gamma\backslash\mathbb{H}_1^{\mathrm{rc}}$ is a compactification of $\mathbb{H}_1$.

We topologize $\mathbb{H}_1^{rc}$ as follows. The set $\mathbb{H}_1$ is set to be an open set with its original topology. If $\tau = \infty$, then a neighbourhood basis is given by

$$N_t \cup \{\infty\} = \{\tau' \in \mathbb{H}_1 \mid \mathrm{Im}(\tau') > t\} \cup \{\infty\}, \quad t > 0.$$

If $\tau \in \mathbb{Q}$, there exists $h \in \mathrm{SL}(2,\mathbb{Z})$ such that $g(\infty) = \tau$. For example, if $\tau = a/c$ with $a, c \in \mathbb{Z}$ and $\gcd(a,c) = 1$, meaning that $ad - bc = 1$ for some $b, d \in \mathbb{Z}$. So we have $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $h(\infty) = \tau$. Then τ has a neighbourhood basis $\{h(N_t \cup \{\infty\})\}$. Since fractional linear transformations are conformal and take circles to circles, $h(N_t \cup \infty)$ is a horoball, i.e., an open disk tangent to the real axis at τ with τ itself as shown at the left side of Figure 7.

The Cayley transformation $\Phi : \mathbb{H}_1 \to D_1 = \{z \in \mathbb{C} \mid |z| < 1\}$ extends to a bijection $\mathbb{H}_1^{\mathrm{rc}} \to D_1^{\mathrm{rc}}$. We give $D_1^{rc} = \Phi(\mathbb{H}_1^{\mathrm{rc}})$ a topology by requiring that Φ be a homeomorphism. The Cayley transformation Φ is also a conformal map. Hence Φ takes the open disk $h(N_t)$ to the open disk $\Phi(h(N_t \cup \{\infty\}))$. Specifically, we have $\Phi(\infty) = 1$, the image $\Phi(N_t \cup \{\infty\})$ is a horoball in $\overline{D_1}$ as shown at the right side of Figure 7.

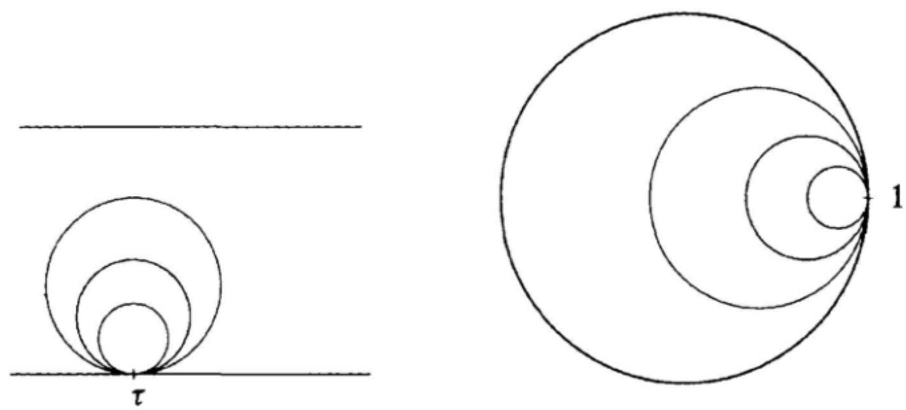


Figure 7: Neighbourhoods of $\tau \in \mathbb{Q}$ in $\mathbb{H}_1^{\mathrm{rc}}$ and neighbourhoods of 1 in D_1^{rc}

We now consider the situation of D_g^{rc}. Consider the standard boundary component F_i. Let $Z \in F_i$, then

$$Z = \begin{pmatrix} Z_i & 0 \\ 0 & I_{g-i} \end{pmatrix}, \qquad Z_i \in D_i.$$

Let τ be the inverse Φ_i^{-1} of the Cayley transformation of Z_i, i.e., $\tau = \Phi_i^{-1}(Z_i)$. We can regard Z as the image under Φ of

$$\begin{pmatrix} \tau & 0 & 0 & 0 \\ 0 & i\infty & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & i\infty \end{pmatrix}.$$

Regarding $F_i \cong \mathbb{H}_i$ as a boundary component of those $F_j \cong \mathbb{H}_j$ where $i \leq j \leq g$, we want to associate a neighbourhood of Z in $\mathbb{H}_j$ as a higher-dimensional generalization of N_t.

Definition 3.3.1. Let $\mathrm{Sym}_+(n, \mathbb{R})$ denote the set of positive definite symmetric $n \times n$ matrices. Let $i \leq j$. Define the maps $\pi_{j,i} : \mathbb{H}_j \to \mathbb{H}_i$ and $\rho_{j,i} : \mathbb{H}_j \to \mathrm{Sym}_+(j-i, \mathbb{R})$:

$$\pi_{j,i}\begin{pmatrix} \tau_1 & {}^t\tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} = \tau_1, \qquad \rho_{j,i}\begin{pmatrix} \tau_1 & {}^t\tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} = \mathrm{Im}\ (\tau_3) - \mathrm{Im}\ (\tau_2)\mathrm{Im}\ (\tau_1)^{-1}\mathrm{Im}\ ({}^t\tau_2).$$

For U an open set in $\mathbb{H}_i$ and $S_{j-i} \in \mathrm{Sym}(j-i, \mathbb{R})$, define the set

$$N_j(U, S_{j-i}) = \{\tau \in \mathbb{H}_j \mid \pi_{j,i}(\tau) \in U, \rho_{j,i}(\tau) - S_{j-i} > 0\}.$$

Example 3.3.2. Note that for a real symmetric matrix $M = \begin{pmatrix} A & {}^tC \\ C & B \end{pmatrix}$ with A positive definite, we have M positive definite if and only if $B - CA^{-1t}C$ is positive definite since

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & {}^tC \\ C & B \end{pmatrix}\begin{pmatrix} I & -A^{-1t}C \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B - CA^{-1t}C \end{pmatrix}.$$

Hence $N_g(\mathbb{H}_{g'}, 0_{g-g'}) = \{\tau \in \mathbb{H}_g \mid \pi_{g,g'}(\tau) \in \mathbb{H}_{g'}, \rho_{g,g'}(\tau) > 0\} = \mathbb{H}_g$.

Example 3.3.3. $N_i(U, S_0) = U$ for U an open set in $\mathbb{H}_i$.

Let z be an element lying in the standard boundary component F_i. We now want to define a neighbourhood basis of z in D_g^{rc}. Let U be a neighborhood of z in $\mathbb{H}_i \cong F_i$. We may view $N_j(U, S_{j-i})$ as a subset of F_j via the identification $\mathbb{H}_j \cong F_j$. As the action of $\mathrm{Sp}(2g, \mathbb{Q})$ extends to D_g^{rc}, we define a subset of D_g^{rc} by

$$\tilde{N}_g(U, S, i) = \tilde{N}_g(U, S_{g-i}, S_{g-1-i}, \cdots, S_{i-i}) = \mathrm{Sp}(2g, \mathbb{Z}) \cap (G_l(F_i) \ltimes R_u(\mathcal{P}(F_i)))(\bigcup_{g \geq j \geq i} N_j(U, S_{j-i})).$$

Definition 3.3.4. We define the **cylindrical topology** on D_g^{rc} as the weakest topology generated by

$$\mathcal{B} = \{N_g(U, S, i)\}$$

and its translates under the action of $\mathrm{Sp}(2g, \mathbb{Q})$.

Remark 3.3.5. i) As for an open subset U in $\mathbb{H}_i \cong F_i$, we have $N_i(U, S_0) = U$, and $Z(F_i) = G_l(F_i) \ltimes R_u(\mathcal{P}(F_i))$ acts trivially on F_i (see Proposition 3.2.7), the topology on F_i is just its usual topology. In particular, as a standard boundary component D_g is open in D_g^{rc}.

ii) The cylindrical topology makes D_g^{rc} a Hausdorff space.

The following proposition shows D_g is dense in D_g^{rc}.

Proposition 3.3.6 ([20, Scholie 5.9])**.** *Let $\tau_n = \begin{pmatrix} \tau_n' & {}^t\tau_n''' \\ \tau_n''' & {}^t\tau_n'' \end{pmatrix}$ be a sequence in $\mathbb{H}_g \cong D_g$ with $\tau_n' \in \mathbb{H}_{g'}$. Let $\tau' \in \mathbb{H}_{g'} \cong F_{g'}$, then τ_n converges to the boundary point τ' in D_g^{rc} with the cylinder topology if and only if*

$$\tau_n' \to \tau', \quad \rho_{g,g'}(\tau_n') \to \infty$$

or under the assumption that $\{\tau_n'''\}$ is bounded, if and only Im $\tau_n''' \to \infty$.

We are now ready to introduce the Satake compactification.

Theorem 3.3.7 ([12, Theorem 6.5])**.** *Let Γ be an arithmetic subgroup of $Sp(2g,\mathbb{R})$ and consider the rational closure D_g^{rc} of D_g.*

a) The action of Γ on D_g extends continuously to a properly discontinuous action of Γ on D_g^{rc}. In the quotient topology, the quotient $\Gamma\backslash D_g^{rc}$ is a compact Hausdorff space containing $\Gamma\backslash D_g$ as a dense open subset.

b) The quotient $\Gamma\backslash D_g^{rc}$ is a normal projective algebraic variety containing $\Gamma\backslash D_g$ as a Zariski open subset (a quasi-projective variety).

Definition 3.3.8. We call $(\Gamma\backslash D_g)^* := \Gamma\backslash D_g^{\mathrm{rc}}$ the **Satake compactification**.

Example 3.3.9. Let $\Gamma = \mathrm{Sp}(2g,\mathbb{Z})$, we see as a set

$$A_g^* = \mathrm{Sp}(2g,\mathbb{Z})\backslash D_g^{rc} = A_g \amalg A_{g-1} \amalg \cdots \amalg A_0$$

where $A_{g'} = \mathcal{P}(F_{g'}) \cap \Gamma\backslash F_{g'} = G_h(F_{g'}) \cap \Gamma\backslash F_{g'} \cong \mathrm{Sp}(2g',\mathbb{Z})\backslash \mathbb{H}_{g'}$. The boundary $A_g^*\backslash A_g$ has codimension

$$g(g+1)/2 - (g-1)g/2 = g.$$

Part a) of Theorem 3.3.7 is due to Satake [24] in the case $\Gamma = \mathrm{Sp}(2g,\mathbb{Z})$, and part b) is due to Baily [4]. Later Baily and Borel [5] proved a generalization of this theorem, constructing compactifications of quotients of bounded symmetric domains (or hermitian symmetric domains by realizing them as bounded symmetric domains) by arithmetic groups. Note that the topology on D_g^{rc} defined by Satake - Baily - Borel (called Satake topology) is different from the cylindrical topology, but they define the same topology in the quotient (cf. [20, Remark 5.11]).

We end this section by stating some general important results of the Baily-Borel compactification.

Theorem 3.3.10 ([16, Theorem 3.12])**.** *Let $D(\Gamma) = \Gamma\backslash D$ be the quotient of a hermitian symmetric domain D by an arithmetic subgroup Γ of the identity component of Hol (D). Then $D(\Gamma)$ has a canonical realization as a Zariski-open subset of a projective algebraic variety $D(\Gamma)^*$.*

Remark 3.3.11. By GAGA, the compactification $D(\Gamma)^*$ is a compact Hausdorff space containing $D(\Gamma)$ as an open dense subset. The compactification $D(\Gamma)^*$ is a finite union of subspaces of the form

$$\Gamma_F\backslash F$$

where F is a rational boundary component and Γ_F is the intersection of the stabilizing group of F and Γ. The closure of $\Gamma_F\backslash F$ in $D(\Gamma)^*$ is the union of $\Gamma_F\backslash F$ and $\Gamma_{F'}\backslash F'$ with F' of smaller dimension (cf. [3, Theorem 6.2]).

Definition 3.3.12. An algebraic variety $D(\Gamma)$ arising from Theorem 3.3.10 is called a **locally symmetric variety**.

4 Toroidal Compactification

In the previous section, we discussed the Satake Compactification of A_g. The boundary is of codimension g. It turns out that it has very complicated singularities

In [2], Mumford and his collaborators developed the toroidal compactification of locally symmetric varieties. In this section, we apply Mumford's toroidal compactification method to the moduli space of principally polarized abelian varieties. The general strategy is that we first construct all the partial compactifications and glue these partial compactifications based on some equivalence relations. We will show that the boundary is of codimension 1 and it is almost smooth, i.e., it contains at worst finite quotient singularities. Our main reference for this section is [12] and [20].

4.1 General Steps

To construct a toroidal compactification, one of the important things is to compactify principal bundles. The definitions we use are based on [17].

Let G be a topological group.

Definition 4.1.1. A left (resp. right) G**-space** Z is a space with a continuous left (resp. right) G-action.

Note that any left G-action on a space X can be converted to a right action by setting $xg = g^{-1}x$ for $x \in X$ and $g \in G$. Similar for a right G-action.

Definition 4.1.2. Suppose that $\mathcal{X}$ is a right G-space equipped with a G-map $\pi : \mathcal{X} \to S$, where G acts trivially on $\mathcal{X}$. We say that $\pi : \mathcal{X} \to S$ is a principal G-bundle if $\pi : \mathcal{X} \to S$ satisfies the following the locally triviality condition:

S has a covering by open sets U such that there exist G-equivariant homeomorphisms $\phi_U : \pi^{-1}U \to U \times G$ commuting in the diagram

$$\begin{tikzcd} \pi^{-1}U \arrow[r, "\phi_U"] \arrow[d] & U \times G \arrow[dl] \\ U & \end{tikzcd}$$

Here $U \times G$ has the right G-action $(u, g)h = (u, gh)$.

Remark 4.1.3. The condition implies that G acts freely on $\mathcal{X}$ and the fibers are isomorphic to G.

Let $\pi : \mathcal{X} \to S$ be a principal G-fiber bundle and Z is a right G-space. We can form the accociated fiber bundle $\mathcal{X} \times_G Z$ over S with fiber Z. We can regard $\mathcal{X} \times_G Z$ as the orbit space $\mathcal{X} \times_G Z$ of the action $(x, z)g = (xg, g^{-1}z)$.

Definition 4.1.4. A (trivial) **torus bundle** of rank r is a (trivial) principal fiber bundle $\pi : \mathcal{X} \to S$ over a complex manifold S whose fiber is an algebraic torus $T \cong (\mathbb{C}^*)^r$.

Remark 4.1.5. We also allow torus bundles to have rank 0. They have fibers $(\mathbb{C}^*)^0$, which are points.

Let $\pi : \mathcal{X} \to S$ be a torus bundle with fiber T_N and Σ be a fan in $N_{\mathbb{R}}$ where N is the group of one-parameter subgroups. Let X_Σ be the normal toric variety with fan Σ. We can form the associated bundle $\mathcal{X}_\Sigma = \mathcal{X} \times_{T_N} X_\Sigma$. If $\bar{\pi} : \mathcal{X}_\Sigma \to S$ is the projection, then we have the commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{i} & \mathcal{X}_\Sigma \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\bar{\pi}} \\ S & \xrightarrow[=]{} & S \end{array}$$

We now outline the steps of the partial compactification in the direction of a rational boundary component F. For more explanations, we refer to [3, III].

Let F be a rational boundary component and $\mathcal{P}'(F)$ be the center of the unipotent radical of the stabilizing group $\mathcal{P}(F)$. Let

$$D(F) = \mathcal{P}'(F)_{\mathbb{C}} \cdot D_g = \bigcup_{h \in \mathcal{P}'(F)_{\mathbb{C}}} h \cdot D_g.$$

It can be shown that $D(F)$ is isomorphic to $F \times V(F) \times \mathcal{P}'(F)_{\mathbb{C}}$ where $V(F)$ is $R_u(\mathcal{P}(F))/\mathcal{P}'(F)$. When F_i is the standard boundary component, we have

$$D(F_i) \cong \left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ {}^t\tau_2 & \tau_3 \end{pmatrix} \mid \tau_1 \in \mathbb{H}_i \cong F_i; \tau_2 \in V(F_i) \cong \mathbb{C}^{i(g-i)}; \tau_3 \in \mathcal{P}'(F_i)_{\mathbb{C}} = \mathrm{Sym}(g-i, \mathbb{C}) \right\}.$$

In general $\mathcal{P}'(F)$ is a real vector space and $P'(F) = \mathcal{P}(F) \cap \Gamma$ for a arithmetic subgroup Γ of $\mathrm{Sp}(2g, \mathbb{R})$ is a lattice in $\mathcal{P}'(F)$. The quotient $\mathcal{P}'_{\mathbb{C}}(F)/P'(F)$ is a complex torus with rank r, where r is the rank of $P'(F)$. We have a trivial torus bundle

$$\mathcal{X}(F) = P'(F) \backslash D(F) \cong F \times V(F) \times (\mathcal{P}'_{\mathbb{C}}(F)/P'(F)) \to F \times V(F).$$

The image $X(F)$ of D_g under the partial quotient

$$e(F) : D_g \to P'(F) \backslash D_g$$

is isomorphic to an open subset of $\mathcal{X}(F)$. The induced action of the quotient group $P''(F) = P(F)/P'(F)$ on the partial quotient $P'(F) \backslash D_g \cong X(F)$ extends to an action of $P''(F)$ on $\mathcal{X}(F)$.

We now want to construct a normal toric variety on the fiber $T = \mathcal{P}'_{\mathbb{C}}(F)/P'(F) \cong P'(F) \otimes \mathbb{C}/P'(F)$. We wish to have a fan such that it is $P''(F)$-compatible (see Proposition 2.4.9). First the group $\mathcal{P}(F)$ acts on $\mathcal{P}'(F)$ by conjugation. It can be shown that only the $G_l(F)$-part of $\mathcal{P}(F)$ (see Proposition 3.2.7) acts non-trivially, i.e.,

$$G_l(F) \cong \mathrm{Aut}(\mathcal{P}'(F)).$$

For example, when F is a standard boundary component, this is easy to verify using the matrix shown in Subsection 3.2. Hence it suffices to construct a $G_l(F) \cap \Gamma$-compatible fan Σ. Let T_Σ be the normal toric variey with fan Σ. Let

$$\mathcal{X}_\Sigma(F) := \mathcal{X}(F) \times_T T_\Sigma$$

be the associated fiber bundle with fiber T_Σ. We use $\bar{P}(F)$ to denote the group $G_l(F) \cap \Gamma$. Then the action of $P''(F)$ on $\mathcal{X}(F)$ extends to an action of $P''(F)$ on $\mathcal{X}_\Sigma(F)$. Define

$$X_\Sigma(F)$$

as the interior of the closure of $X(F) = P'(F)\backslash D_g$ in $\mathcal{X}_\Sigma(F)$. Define

$$Y_\Sigma(F) := P''(F)\backslash X_\Sigma(F)$$

as the partial compactification of $\Gamma\backslash D_g$ in the direction F.

We summarize the key steps for the partial compactification in the direction of a rational boundary component F:

Step 1 Consider the partial quotient $X(F) = P'(F)\backslash D_g$. It is an open subset of the trivial torus bundle $\mathcal{X}(F) = F \times V(F) \times (P'(F) \otimes \mathbb{C}/P'(F)) \to F \times V(F)$.

Step 2 Choose a $\bar{P}(F)$-compatible fan Σ in $\mathcal{P}'(F)$ and do torus embedding on the fiber to get $\mathcal{X}_\Sigma(F)$.

Step 3 Take the interior $X_\Sigma(F)$ of the closure of $X(F)$ in $\mathcal{X}_\Sigma(F)$. The partial compactification of $\Gamma\backslash D_g$ in the direction of F is $Y_\Sigma(F) = P''(F)\backslash X_\Sigma(F)$.

To do toroidal compactifications, the fan needs to be more than $\bar{P}(F)$-compatible. We now describe the so-called $\bar{P}(F)$-admissible fan we need for toroidal compactification. We hope this fan is neither too large nor too small and gives a good quotient $P''(F)\backslash X_\Sigma(F)$.

Definition 4.1.6. (i) The Harish-Chandra map φ of the group $\mathrm{Sp}(2g, \mathbb{R})$ is the homomorphism of Lie groups

$$\varphi : U(1) \times \mathrm{SL}(2, \mathbb{R})^g \to \mathrm{Sp}(2g, \mathbb{R})$$

$$\varphi(\lambda, \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \cdots, \begin{pmatrix} \alpha_g & \beta_g \\ \gamma_g & \delta_g \end{pmatrix}) = \begin{pmatrix} \alpha_1 & & & \beta_1 & & \\ & \ddots & & & \ddots & \\ & & \alpha_g & & & \beta_g \\ \gamma_1 & & & \delta_1 & & \\ & \ddots & & & \ddots & \\ & & \gamma_g & & & \delta_g \end{pmatrix}$$

where $U(1)$ is the unitary group.

(ii) Let F be a boundary component of D_g, i.e., $F = h(F_i)$ for some $h \in \mathrm{Sp}(2g, \mathbb{R})$. Define a map

$$\varphi_F : U(1) \times \mathrm{SL}(2, \mathbb{R}) \to \mathrm{Sp}(2g, \mathbb{R})$$
$$\varphi_F(e^{i\theta}, s) = h\varphi(e^{i\theta}, \underbrace{H(\theta), \cdots, H(\theta)}_{i}, \underbrace{s, \cdots, s}_{g-i})h^{-1}$$

where

$$H(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

(iii) Define
$$\Omega_F = \varphi(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}).$$

Note that for any boundary component F of D_g, we have $F = h(F_i)$ for some $h \in \mathrm{Sp}(2g, \mathbb{R})$ and F_i standard boundary component. By definition (ii) above, $\Omega_F = \Omega_{h(F_i)} = h\Omega_{F_i}h^{-1}$ where
$$\Omega_{F_i} = \begin{pmatrix} I_i & 0 & 0 & 0 \\ 0 & I_{g-i} & 0 & I_{g-i} \\ 0 & 0 & I_i & 0 \\ 0 & 0 & 0 & I_{g-i} \end{pmatrix} \in \mathcal{P}'(F_i).$$
Since $\mathcal{P}'(hF_i) = h\mathcal{P}'(F)h^{-1}$, $\Omega_F = \Omega_{h(F_i)} \in \mathcal{P}'(F)$. Recall $\mathcal{P}(F)$ acts on $\mathcal{P}'(F)$ by conjugation, this leads to the following definition:

Definition 4.1.7. Let F be a boundary component of D_g. Define the $\mathcal{P}(F)$-orbit of Ω_F
$$C(F) := \{g\Omega_F g^{-1} \mid g \in \mathcal{P}(F)\}.$$

REMARK 4.1.8. (i) By our previous discussion, only $G_l(F)$ acts on non-trivially $\mathcal{P}'(F)$. Hence $C(F)$ is also the $G_l(F)$-orbit of Ω_F.

(ii) If $F = h(F')$ for some $h \in \mathrm{Sp}(2g, \mathbb{R})$, we have $C(F) = hC(F')h^{-1}$ from $\mathcal{P}(F) = h(\mathcal{P}(F'))h^{-1}$.

(iii) It can be shown that $C(F)$ is an open homogeneous cone in $\mathcal{P}'(F)$. The centralizer of Ω_F in $G_l(F)$ is the maximal compact subgroup $G_l(F) \cap U(g)$. Hence $C(F) \cong G_l(F)/G_l(F) \cap U(g)$ (cf. [3, Theorem III.4.1]).

Proposition 4.1.9. *Let F be a standard boundary component. The cone $C(F_i)$ in $\mathcal{P}'(F_i)$ is isomorphic to the cone $Sym_+(d-i, \mathbb{R})$ of positive definite symmetric $(d-i)\times(d-i)$ matrices, i.e.,*
$$C(F_i) = \left\{ \begin{pmatrix} 1_i & 0 & 0 & o \\ 0 & I_{g-i} & 0 & S \\ 0 & 0 & I_i & 0 \\ 0 & 0 & 0 & I_i \end{pmatrix} \middle| S \in Sym_+(g-i, \mathbb{R}) \right\}.$$

Proof. Using the matrix form (8) of $G_l(F_i)$ and noting that $G_l(F_i)$ acts on $\mathcal{P}'(F_i)$ by conjugation, we have
$$C(F_i) = \left\{ \begin{pmatrix} 1_i & 0 & 0 & o \\ 0 & I_{g-i} & 0 & S \\ 0 & 0 & I_i & 0 \\ 0 & 0 & 0 & I_i \end{pmatrix} \middle| S = {}^tQ^{-1} I_{g-i} Q^{-1}, Q \in \mathrm{GL}(g-i, \mathbb{R}) \right\}.$$
□

To introduce the fan we are going to use, we give a precise statement of the boundary components of open cones in a real vector space V.

Definition 4.1.10. Suppose that C is an open cone in a real vector space V.

(i) A proper boundary component C' of C is a cone C' of the form $C' = (\bar{C} \cap V')^\circ$ where V' is a linear subspace of V such that $V' \cap C = \emptyset$. The closure $\bar{C}$ of C is taken in V, and the interior of $\bar{C} \cap V'$ is taken in V'. If C' is a proper boundary component of C, then C' is adjacent to C, which we denote $C' \prec C$.

(ii) Now suppose V has an underlying integral structure $V_{\mathbb{Z}}$, i.e., $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. A boundary component C' of C is called rational if it can be written $C' = (\bar{C} \cap V')^\circ$ where V' is a rationally defined subspace of V.

(iii) The union of C with all rational boundary components of C is called the rational closure of C. We denote it by C^{rc}.

The adjacencies of the open cones $C(F)$ have a correspondence with the adjacencies of the boundary components F of D_g.

Proposition 4.1.11 (cf. [12, Proposition 3.60] and [3, Theorem III.4.8])**.** *(i) If $F \prec F'$ is a pair of adjacent boundary components, then $\mathcal{P}'(F')$ is a subspace of $\mathcal{P}'(F)$ and $C(F') = (\overline{C(F)} \cap \mathcal{P}'(F'))^\circ$ where $C(F) \cap C(F') = \emptyset$. Hence $C(F')$ is a proper boundary component of $C(F)$.*

(ii) Fixing F, the map $F' \mapsto C(F')$ is an order-reversing bijection between the set of boundary components F' with $F \prec F'$ of D_g and the set of boundary components of $C(F)$. Moreover, adding the rational condition gives an order-reversing bijection between the set of rational boundary components of D_g and the set of rational boundary components of open cones.

We now define the fan we need for the toroidal compactification.

Definition 4.1.12. Let F be a rational boundary component of D_g. A fan Σ in $\mathcal{P}'(F)$ is called $\bar{P}(F)$**-admissible** for the construction of a partial compactification if it satisfies the following three conditions:

(a) $\bigcup_{\sigma \in \Sigma} \sigma = C(F)^{\mathrm{rc}}$.

(b) $\bar{P}(F)$-compatible: $h(\sigma) \in \Sigma$ for every $\sigma \in \Sigma$ and every $h \in \bar{P}(F)$.

(c) There are only finitely many orbits in Σ under the action of $\bar{P}(F)$ on Σ, i.e., $\bar{P}(F)\backslash\Sigma$ is a finite set.

Proposition 4.1.13 (cf. [12, Proposition 3.62])**.** *Let F be a rational boundary component of D_g, and let Σ be an $\bar{P}(F)$-admissible fan in $\mathcal{P}'(F)$. Then the induced action of $P''(F)$ on $X(F)$ extends in a unique way to a properly discontinuous action of $P''(F)$ on $X_\Sigma(F)$. The quotient space*

$$Y_\Sigma(F) = P''(F)\backslash X_\Sigma(F)$$

is a complex analytic space. If $X_\Sigma(F)$ is smooth, then $Y_\Sigma(F)$ contains at worst finite quotient singularities. The space $Y_\Sigma(F)$ contains $P(F)\backslash D_g$ as an open and dense analytic subspace, whose boundary $Y_\Sigma(F) - (P(F)\backslash D_g)$ is a purely 1-codimensional complex analytic space.

Assume that for every direction of rational boundary components F, we have constructed one partial compactification $Y_{\Sigma(F)}(F)$ and have a collection $\tilde{\Sigma}$ that contains all the fans $\Sigma(F)$. We will now investigate under which conditions on the collection $\tilde{\Sigma}$ the partial compactifications can be glued to form a compactification $\Gamma\backslash D_g$.

There are three conditions we need to consider. First, of course, every fan $\Sigma(F)$ in the collection $\tilde{\Sigma}$ needs to be $\bar{P}(F)$-compatible. Second, we hope for some compatibility of the partial compactifications with the action of Γ on the set of rational boundary components of D_g. The last condition should ensure that the gluing of adjacent partial compactifications possible. These lead to the following definition:

Definition 4.1.14. Let $\tilde{\Sigma} = \{\Sigma(F)\}$ be a collection of fans $\Sigma(F) \subset \mathcal{P}'(F)$, one fan $\Sigma(F)$ in each $\tilde{\Sigma}$ for each rational boundary component F of D_g. The collection $\tilde{\Sigma}$ is called an **Γ-admissible collection** (of fans) if it satisfies the following conditions:

(a) $\Sigma(F) \subset \mathcal{P}'(F)$ is $\bar{P}(F)$-admissble.

(b) If $F = h(F')$ for some $h \in \Gamma$, then $\Sigma(F) = h(\Sigma(F')) = \{h\sigma' h^{-1} \mid \sigma' \in \Sigma(F')\}$ as fans in $\mathcal{P}'(F) = h(\mathcal{P}'(F))$.

(c) If $F \prec F'$, then $\Sigma(F') = \Sigma(F) \cap \mathcal{P}'(F')$ holds as fans in $\mathcal{P}'(F') \subset \mathcal{P}'(F)$.

Remark 4.1.15. (i) Let $\tilde{\Sigma} = \{\Sigma(F)\}$ be an admissible collection of fans and let $F \prec F'$. For every cone $\sigma \in \Sigma(F)$ the intersection $\sigma \cap \mathcal{P}'(F')$ is a face of σ.

(ii) Part (c) implies that an admissible collection is determined by the fans $\Sigma(F)$ associated with minimal rational boundary components. We will discuss this in the case of $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$ in Proposition 4.4.15. A rational boundary component F is called **minimal** if there is no rational boundary component F' such that $F' \prec F$.

For those rational boundary components F, F' on the same Γ-orbit, i.e., $F = h(F')$ for some $h \in \Gamma$, we wish to glue the partial compactifications together. Condition (b) in Definition 4.1.14 is sufficient for the existence of a natural isomorphism between $Y_{\Sigma(F')}(F')$ and $Y_{\Sigma(F)}(F)$ if $F = h(F')$ for some $h \in \Gamma$ so that the two partial compactifications $Y_{\Sigma(F)}(F)$ and $Y_{\Sigma(F')}(F')$ can be identified pointwise.

First note that if $F = h(F')$ for $h \in \Gamma$, then h induces a natural isomorphism from $P(F')\backslash D_g$ to $P(F)\backslash D_g$ by sending $[x]$ to $[h \cdot x]$.

For convenience, we will use T_σ and T_Σ instead of U_σ and X_Σ in Section 2 to denote toric varieties.

Proposition 4.1.16 ([12, Proposition 3.69]). *Let F and F' be rational boundary components of D_g such that $F = h(F')$ for some $h \in \Gamma$. Suppose that $\Sigma(F)$ and $\Sigma'(F)$ are admissible fans in $\mathcal{P}'(F)$ resp. $\mathcal{P}'(F)$ satisfying $\Sigma(F) = h(\Sigma(F'))$. Then*

(i) There exists a natural isomorphism $\tilde{h} : X_{\Sigma(F')}(F') \xrightarrow{\sim} X_{\Sigma(F)}(F)$ such that the following diagram commutes:

$$\begin{tikzcd} X_{\Sigma(F')}(F') \arrow[r, "\tilde{h}"] & X_{\Sigma(F)}(F) \\ P'(F')\backslash D_g \arrow[u] \arrow[r, "h"'] & P'(F)\backslash D_g \arrow[u] \end{tikzcd}$$

(ii) There is a natural isomorphism $\bar{h} : Y_{\Sigma(F')}(F') \xrightarrow{\sim} Y_{\Sigma(F)}(F)$ *induced by the isomorphism* $\tilde{h}$ *of (i) such that the following diagram commutes*

$$\begin{array}{ccc} Y_{\Sigma(F')}(F') & \xrightarrow{\bar{h}} & Y_{\Sigma(F)}(F) \\ \uparrow & & \uparrow \\ P(F')\backslash D_g & \xrightarrow[h]{} & P(F)\backslash D_g \end{array}$$

Proof. The action of h on D_g induces an isomorphism $h : P(F')\backslash D_g \to P(F)\backslash D_g$ which extends to an isomorphism $h_1 : \mathcal{X}(F') \to \mathcal{X}(F)$ of the trivial torus bundles.

There is an isomorphism

$$h : T' = \mathcal{P}'(F) \otimes_{\mathbb{R}} \mathbb{C}/P'(F') \xrightarrow{\sim} \mathcal{P}'(F) \otimes_{\mathbb{R}} \mathbb{C}/P'(F).$$

By Definition 4.1.14 (b), the action of h is compatible with the fans $\Sigma(F)$ and $\Sigma(F')$(see Definition 2.4.6) and thus the map $h : T' \to T$ extends to an isomorphism $h_2 : T'_{\Sigma(F')} \xrightarrow{\sim} T_{\Sigma}(F)$ (see Theorem 2.4.7). The isomorphism of products $h_1 \times h_2 : \mathcal{X}(F') \times T'_{\Sigma(F')} \xrightarrow{\sim} \mathcal{X}(F) \times T_{\Sigma(F)}$ induces a natural isomorphism $\hat{h} : \mathcal{X}_{\Sigma(F')}(F') \to \mathcal{X}_{\Sigma(F)}(F)$. Restricting $\hat{h}$ on $X_{\Sigma(F')}(F')$ yields the isomorphism $\tilde{h}$ which extends $h : P'(F')\backslash D_g \to P'(F)\backslash D_g$. Part (ii) is a trivial consequence of part (i). □

We now consider condition (c) of Definition 4.1.14. If $F \prec F'$ is a pair of adjacent rational boundary components of D_g, we have $\mathcal{P}'(F') \subset \mathcal{P}'(F)$ and hence $P'(F') \subset P'(F)$. There exists a natural quotient map

$$\pi_0(F', F) : P'(F')\backslash D_g \to P'(F)\backslash D_g$$

which is an unramified analytic covering. We will show if condition (c) of Definition 4.1.14 is satisfied, the map $\pi_0(F', F)$ extends to an étale map $\pi(F', F)$ from $X_{\Sigma}(F')$ to $X_{\Sigma(F)}(F)$. Note that a **étale map** here means a smooth map with discrete fibers. And note that $\pi(F', F)$ is not surjective.

Proposition 4.1.17 (cf. [12, Proposition 3.71] and [3, Lemma III.5.4])**.** *Let* $F \prec F'$ *be a pair of adjacent rational boundary components of* D_g *and let* $\Sigma(F)$ *and* $\Sigma(F')$ *be two admissible fans in* $\mathcal{P}'(F)$ *resp.* $\mathcal{P}'(F')$ *satisfying* $\Sigma(F') = \Sigma(F) \cap \mathcal{P}'(F')$. *Consider the complex tori* $T = \mathcal{P}'(F) \otimes_{\mathbb{R}} \mathbb{C}/P'(F)$ *and* $T' = \mathcal{P}'(F') \otimes_{\mathbb{R}} \mathbb{C}/P'(F')$.

(i) The inclusion $\mathcal{P}'(F') \subset \mathcal{P}'(F)$ *gives rise to an inclusion of complex tori* $T' \subset T$ *and hence gives* T *the structure of a principal* T'*-bundle over* T/T'*. Then there is a natural isomorphism*

$$T_{\Sigma(F')} \cong T \times_{T'} T'_{\Sigma(F')}.$$

Moreover, $T_{\Sigma(F')}$ *is an open and dense subvariety of* T_{Σ}.

(ii) The group $P'(F)$ *acts on* $\mathcal{X}(F')$*. The action induces a natural étale map*

$$\pi(F', F) : X_{\Sigma(F')}(F') \to X_{\Sigma(F)}(F)$$

between the subsets $X_{\Sigma(F')}(F')$ *of* $\mathcal{X}_{\Sigma(F')}(F')$ *and* $X_{\Sigma(F)}(F)$ *of* $\mathcal{X}_{\Sigma(F)}(F)$*. The map* $\pi(F', F)$ *extends* $\pi_0(F', F)$.

Proof. (i) Denote N and N' as the groups of one-parameter subgroups $P'(F)$ and $P'(F')$ of T and T' respectively. Denote M and M' as the dual lattices of N and N'. Note that we have $N' = P'(F') \subset P'(F) = N$ and $M' \subset M$.

Regard $T \times_{T'} T'_{\Sigma(F')}$ as the quotient of $T \times T'_{\Sigma(F')}$ by the right T'-action by $t' : (t, x) \mapsto (tt', t'^{-1}x), t' \in T'$. Pick $(t, x) \in T \times T'_{\Sigma(F')}$. Then t and x can be represented by semigroup homomorphisms $\varphi_t^{(1)} : M \to \mathbb{C}$ and $\sigma^{\vee}_{M'_{\mathbb{R}}} \cap M' \to \mathbb{C}$ where $\sigma \in \Sigma(F')$ is chosen such that $x \in T'_\sigma \subset T'_{\Sigma(F')}$ and $\sigma^{\vee}_{M'_{\mathbb{R}}}$ is the dual cone of σ in $M'_{\mathbb{R}}$. The dual cone $\sigma^{\vee}_{M_{\mathbb{R}}}$ of σ in $M_{\mathbb{R}}$ projects onto $\sigma^{\vee}_{M_{\mathbb{R}}}$ via the natural projection $q : M_{\mathbb{R}} \to M'_{\mathbb{R}}$. Hence we have compositions of semigroup homomorphisms:

$$\sigma^{\vee}_{M_{\mathbb{R}}} \cap M \xrightarrow{\hat{i}} M \xrightarrow{\varphi_t^{(1)}} \mathbb{C}$$
$$\sigma^{\vee}_{M_{\mathbb{R}}} \cap M \xrightarrow{\hat{q}} \sigma^{\vee}_{M'_{\mathbb{R}}} \cap M' \xrightarrow{\varphi_x^{(2)}} \mathbb{C}$$

where $\hat{q}$ is the restriction q on $\sigma^{\vee}_{M_{\mathbb{R}}} \cap M$.

Define $\varphi_y : \sigma^{\vee}_{M_{\mathbb{R}}} \cap M \to \mathbb{C}$ to be the product $\varphi_y = (\varphi_t^{(1)} \circ \hat{i}) \cdot (\varphi_x^{(2)} \circ \hat{q})$, so φ_y represents a point $y \in T_\sigma \subset T_{\Sigma(F')}$. This point does not depend on the choice of σ. Two points $(t_1, x_1), (t_2, x_2) \in T \times T'_{\Sigma}(F)$ determine the same point $y \in T_{\Sigma(F')}$ if and only if $(t_2, x_2) = (t_1 t', t'^{-1} x_1)$ for some $t' \in R'$. We can set $f(t, x) = y$ which determines an isomorphism $f : T \times_{T'} T'_{\Sigma(F')} \xrightarrow{\sim} T_{\Sigma(F')}$. As $\Sigma(F')$ is a subfan of $\Sigma(F)$, the toric variety $T_{\Sigma(F')}$ is a union of some open affines and thus $T_{\Sigma(F')}$ is open in $T_{\Sigma(F)}$.

$$\begin{aligned}
P'(F)\backslash \mathcal{X}_{\Sigma(F')}(F') &= P'(F)\backslash(\mathcal{X}(F') \times_{T'} T'_{\Sigma(F')}) \\
&\subset \mathcal{X}(F) \times_{T'} T'_{\Sigma(F')} \\
&\cong (\mathcal{X}(F) \times_T T) \times_{T'} T'_{\Sigma(F')} \\
&\cong \mathcal{X}(F) \times_T (T \times_{T'} T'_{\Sigma(F')}) \\
&\cong \mathcal{X}(F) \times_T T_{\Sigma(F')} \\
&\subset \mathcal{X}(F) \times_T T_{\Sigma(F)}
\end{aligned}$$

where both occurrences of "$\subset$" mean open inclusions here.

The quotient map of part (ii) restricts to a map from $X_{\Sigma(F')}(F')$ to $X_{\Sigma(F)}(F)$. This map is étale because it is an unramified covering on the base spaces of the fiber bundle, and open inclusions of their fibers.

□

Remark 4.1.18. We use the notation $\Pi(F', F)$ to denote the quotient map $\mathcal{X}_{\Sigma(F')}(F') \to P'(F)\backslash \mathcal{X}_{\Sigma(F')}(F') \subset \mathcal{X}(F) \times_T T_{\Sigma(F)} = \mathcal{X}_{\Sigma(F)}(F)$. Note that $\Pi(F', F)|_{X_{\Sigma(F')}(F')} = \pi(F', F)$.

Proposition 4.1.19 ([3, Lemma III.5.5]). *Let $x \in X_{\Sigma(F)}(F)$. Among all rational boundary components F' such that there is some $x' \in X_{\Sigma(F')}(F')$ such that $\pi(F', F)(x') = x$, there is a maximal one F_x. Moreover, $F_{h \cdot x} = h \cdot F_x$ for all $h \in \Gamma$.*

We call this boundary component F_x the boundary component associated with x or say that x belongs to the F_x-stratum.

The following proposition is a consequence of the naturality of the two constructions in Proposition 4.1.17 and Proposition 4.1.16.

Proposition 4.1.20. *Let $F \prec F'$ be a pair of adjacent rational boundary components of D_g and let $\Sigma(F)$ and $\Sigma(F')$ be two admissible fans in $\mathcal{P}'(F)$ resp. $\mathcal{P}'(F')$ satisfying $\Sigma(F') = \Sigma(F) \cap \mathcal{P}'(F')$. For any $h \in \Gamma$, the following diagram commutes:*

$$\begin{tikzcd} X_{\Sigma(F')}(F') \arrow[r, "\pi(F',F)"] \arrow[d, "\tilde{g},\cong"'] & X_{\Sigma(F)}(F) \arrow[d, dashed, "\tilde{g},\cong"'] \\ X_{h(\Sigma(F'))}(h(F')) \arrow[r, "\pi(h(F'),h(F))"] & X_{h(\Sigma(F))}(h(F)) \end{tikzcd}$$

Now we are ready to introduce the toroidal compactification of $\Gamma\backslash D_g$.

Definition 4.1.21. Suppose that $\tilde{\Sigma} = \{\Sigma(F)\}$ is an admissible collection of fans. We define the **toroidal compactification** $\overline{\Gamma\backslash D_g}$ determined by $\tilde{\Sigma}$ to be the quotient space

$$\overline{\Gamma\backslash D_g} = X/\sim$$

where

$$X = \coprod_{\substack{F:\text{rational}\\ \text{boundary component}}} X_{\Sigma(F)}(F).$$

The space $\overline{\Gamma\backslash D_g}$ is given the quotient topology. The equivalence relation $\sim$ is defined as follows: Let $x \in X_{\Sigma(F)}(F)$ and $x' \in X_{\Sigma(F')}(F')$, then

(a) $x \sim x'$ if there is a $h \in \Gamma$ such that $F = h(F')$ and $x = \tilde{h}(x')$ where $\tilde{h}$ is the isomorphism defined in Proposition 4.1.16.

(b) $x \sim x'$ if $F \prec F'$ and $\pi(F', F)(x') = x$, where $\pi(F', F)$ is the étale gluing map defined in Proposition 4.1.17.

Remark 4.1.22. (i) By Proposition 4.1.16, the group acts on X by permuting $X_{\Sigma(F)}(F)$. The equivalence (a) implies that $X \to X/\sim$ factors through $\Gamma\backslash X$. Note that it is possible that $F = F'$, i.e., $h \in P(F)$. The quotient of $X_{\Sigma(F)}(F)$ by $P(F)$ is the partial compactification $Y_{\Sigma(F)}(F)$. Thus $\Gamma\backslash X$ maybe considered as an identification space arising from an equivalence relation on the disjoint union

$$Y = \coprod_{\substack{F:\text{rational}\\ \text{boundary component}}} Y_{\Sigma(F)}(F).$$

We have

$$X \to Y \to \Gamma\backslash X \to X/\sim.$$

(ii) Note that the equivalence relation defined here is the same as the equivalence relation defined in [3, p.163]. Let $x_1 \in X_{\Sigma(F')}(F')$ and $x_2 \in X_{\Sigma(F'')}(F'')$. Then $x_1 \sim x_2$ if and only if

(a) there is a rational boundary component F and $h \in \Gamma$, such that $F' \prec F$ and $h(F'') \prec F$.

(b) there is a point $x \in X_{\Sigma(F)}(F)$ that $\pi(F, F')(x) = x_1$ and $\pi(F, h(F''))(x) = \tilde{h}(x_2)$ where $\tilde{h}$ is the isomorphism $X_{\Sigma(F'')}(F'') \to X_{\Sigma(h(F''))}(h(F''))$.

Remark 4.1.23 ([12, Remark 3.77]). By the remark above, the compositions $X_{\Sigma(F)}(F) \hookrightarrow X \twoheadrightarrow X/\sim = \overline{\Gamma\backslash D_g}$ give rise to natural maps

$$p^*(F) : Y_{\Sigma(F)}(F) \to \overline{\Gamma\backslash D_g}$$

which extend the natural projections

$$p(F) : P(F)\backslash D_g \twoheadrightarrow \Gamma\backslash D_g.$$

Since the maps used to define the equivalence relation $\sim$ are open maps, the maps $p^*(F)$ are open maps. By Proposition 4.1.17, the image of $p^*(F')$ is contained in the image of $p^*(F)$ as a dense open subset whenever $F \prec F'$. Then it follows the sets of images $p^*(F)$, F minimal, form an open cover of $\overline{\Gamma\backslash D_g}$. Moreover the image of $p^*(F)$ is dense in $\overline{\Gamma\backslash D_g}$ for every boundary component F. Finally, it's easy to check that the equivalence relation $\sim$ does not introduce any identifications on $\Gamma\backslash D_g$ and so $p^*(D_g) : \Gamma\backslash D_g \to \overline{\Gamma\backslash D_g}$ is an open inclusion with dense image.

Definition 4.1.24. For a rational boundary component F of D_g, the open boundary component $\partial_F(\overline{\Gamma\backslash D_g})$ is defined to be

$$\partial_F(\overline{\Gamma\backslash D_g}) = p^*(F)(Y_{\Sigma(F)}(F)) - \bigcup_{F \prec F'} p^*(F')(Y_{\Sigma(F')}(F'))$$

and the closed boundary component

$$\bar{\partial}_F(\overline{\Gamma\backslash D_g}) = \partial_F(\overline{\Gamma\backslash D_g}) \cup \bigcup_{F' \prec F} \partial_{F'}(\overline{\Gamma\backslash D_g}).$$

Remark 4.1.25. The closure of $\partial_F(\overline{\Gamma\backslash D_g})$ is $\bar{\partial}_F(\overline{\Gamma\backslash D_g})$ (cf. [12, Remark 3.79 ii.]). Note that if F is minimal, then $\partial_F(\overline{\Gamma\backslash D_g}) = \bar{\partial}_F(\overline{\Gamma\backslash D_g})$.

Hence we have a stratification

$$\overline{\Gamma\backslash D_g} = \coprod_{F \bmod \Gamma} \partial_F(\overline{\Gamma\backslash D_g}).$$

Definition 4.1.26. The boundary of the toroidal compactification of $\Gamma\backslash D_g$ is

$$\partial(\overline{\Gamma\backslash D_g}) = \overline{\Gamma\backslash D_g} - \Gamma\backslash D_g.$$

4.2 Stratification By Torus Orbits

Now we will provide a more detailed description of a toroidal compactification by using torus orbits, which will help us understand the stratification and boundaries of toroidal compactification better and compute the compactly supported cohomology.

For each $\mathcal{X}_{\Sigma(F)}(F)$, we have a fibrewise $T = \mathcal{P}'_{\mathbb{C}}(F)/P'(F)$-orbit decomposition (cf. Theorem 2.5.6):

$$\coprod_{\sigma\in\Sigma(F)} \mathcal{X}(F) \times_T O(\sigma).$$

Denote by $\mathbb{O}_F(\sigma)$ the fibrewise orbit $\mathcal{X}(F) \times_T O(\sigma)$ for convenience.

It's easy to see that the decomposition has the following properties:

Proposition 4.2.1 ([20, p.61])**.**

(i) *Each $\mathbb{O}_F(\sigma)$ is an torus bundle over $F \times V(F)$ (cf. Theorem 2.5.6 (a)).*

(ii) *$\tau \prec \sigma$ if and only if $\mathbb{O}_F(\sigma) \subset \overline{\mathbb{O}_F(\tau)}$. Moreover, $\overline{\mathbb{O}_F(\tau)} = \bigcup_{\tau\prec\sigma} \mathbb{O}_F(\sigma)$ (cf. Theorem 2.5.6 (d)).*

(iii) *If $\sigma = \{0\}$, then $\mathbb{O}_F(\sigma) = \mathcal{X}(F)$.*

(iv) $\dim \sigma + \dim \mathbb{O}_F(\sigma) = \dim D_g$.

The following proposition tells us which fibrewise orbits lie in $X_{\Sigma(F)}(F)$, the interior of the closure of $P'(F)\backslash D_g$ in $\mathcal{X}_{\Sigma(F)}(F)$.

Proposition 4.2.2. *[20, Facts 7.9] Let F be a rational boundary component and $C(F)$ be its open cone. If $\sigma \cap C(F) \neq \emptyset$, then $\mathbb{O}_F(\sigma) \subset X_{\Sigma(F)}(F)$.*

Recall in Proposition 4.1.17, for a pair of adjacent boundary boundary components $F \prec F'$, we have a map $\Pi(F', F) : \mathcal{X}_{\Sigma(F')}(F') \to \mathcal{X}_{\Sigma(F)}(F)$. Restrict it to fibrewise torus orbits, we have

$$\mathbb{O}_{F'}(\sigma) \to \mathbb{O}_F(\sigma) \qquad \text{for all } \sigma \in \Sigma(F').$$

Moreover, the disjoint union of fibrewise orbits

$$\mathbb{O}_F := \coprod_{\sigma\cap C(F)\neq\emptyset} \mathbb{O}_F(\sigma)$$

is the complement of the union of the image of $\pi(F', F) : X_{\Sigma(F')}(F') \to X_{\Sigma(F)}(F)$ for all F' such that $F \prec F'$ (cf. [20, p.64]). These are the points essentially added with respect to F. By Proposition 4.2.1 (ii), the disjoint union is closed in $X_{\Sigma(F)}(F)$.

To get the partial compactification $Y_{\Sigma(F)}(F)$ in the direction of a rational boundary component F, we quotient $X_{\Sigma(F)}(F)$ by $P''(F) = P(F)/P'(F)$. Then the quotient $P''(F)\backslash\mathbb{O}_F$ is a closed set in the partial compactification $Y_\Sigma(F)$. By reduction theory, it can be shown that the map $p^*(F) : Y_{\Sigma(F)}(F) \to \overline{\Gamma\backslash D_g}$ is injective near $P''(F)\backslash\mathbb{O}_F$ (cf. [20, p.69]).

The open boundary component $\partial_F(\overline{\Gamma\backslash D_g})$ in the last subsection is

$$\partial_F(\overline{\Gamma\backslash D_g}) = P''(F)\backslash\mathbb{O}_F.$$

When F is a standard boundary component of degree i, we can write

$$\partial_F(\overline{\Gamma\backslash D_g}) = P''(F)\backslash \mathbb{O}_F = \coprod_{\substack{\sigma \bmod \mathrm{GL}(g,\mathbb{Z}) \\ \sigma\cap C(F)\neq\emptyset}} \mathrm{Stab}(\sigma)\backslash \mathcal{T}(\sigma)$$

where $\mathrm{Stab}(\sigma) = \{h \in \mathrm{GL}(g,\mathbb{Z} \mid h\cdot\sigma = \sigma\}$ and $\mathcal{T}(\sigma)$ is the trivial torus bundle over the $g-i$ fold universal family $\mathcal{X}_i^{g-i}$, i.e., the quotient of $\mathbb{H}_i \times \mathbb{C}^{(g-i)i}$ by the group with generators

$$\left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & I_{g-i} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & I_{g-i} \end{pmatrix} \middle| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2i,\mathbb{Z}) \right\}$$

and

$$\left\{ \begin{pmatrix} I_i & 0 & 0 & {}^tN \\ M & I_{g-i} & N & 0 \\ 0 & 0 & I_i & -{}^tM \\ 0 & 0 & 0 & I_{g-i} \end{pmatrix} \middle| M, N \in \mathrm{Mat}(g-i,i;\mathbb{Z}) \right\}.$$

Such groups are called Jacobi groups (cf. [9, p.17]). The toroidal compactification $\overline{\Gamma\backslash D_g}$ has a stratification as a set

$$\overline{\Gamma\backslash D_g} = \coprod_{F \bmod \Gamma} \partial_F(\overline{\Gamma\backslash D_g}) = \coprod_{F \bmod \Gamma} P''(F)\backslash \mathbb{O}_F.$$

Next, we briefly discuss the relation between the toroidal compactification of $\overline{\Gamma\backslash D_g}$ and the Satake compactification $(\Gamma\backslash D_g)^*$. We want to define a map

$$f : X = \coprod_{F:\text{rational}} X_{\Sigma(F)}(F) \to (\Gamma\backslash D_g)^*.$$

Let $x \in X_{\Sigma(F)}(F)$ and let F_x be its associated rational boundary component. By Proposition 4.1.19, there exists a point $x' \in X_{\Sigma(F_x)}(F_x)$ such that $\pi(F_x, F)(x') = x$. Recall $X_{\Sigma(F_x)}(F_x)$ is the interior of the closure of $P'(F)\backslash D_g$ in $F_x \times V(F_x) \times T \times_T T_{\Sigma(F_x)}$, there is a natural projection

$$\mathrm{pr}_{F_x} : X_{\Sigma(F_x)}(F_x) \to F_x.$$

We define the image via f of x as the image x' via the composition

$$X_{\Sigma(F_x)}(F_x) \xrightarrow{\mathrm{pr}_{F_x}} F_x \subset D_g^{\mathrm{rc}} \to (\Gamma\backslash D_g)^*.$$

It's easy to see that

(i) The definition of $f(x)$ is independent of the choice of x'.

(ii) The restriction $f|_{D_g}$ of f on D_g is the natural projection from D to $\Gamma\backslash D_g \subset (\Gamma\backslash D_g)^*$.

(iii) Two points in X that are equivalent under the equivalence relation $\sim$ defining the toroidal compactification have the same image in $(\Gamma\backslash D_g)^*$.

(iv) $f|_{X_{\Sigma(F)}(F)}$ factors as

$$X_{\Sigma(F)}(F) \to P'(F)\backslash D_g^{\mathrm{rc}} \to (\Gamma\backslash D_g)^*.$$

(v) The restriction of f on $\mathbb{O}_F$ is the natural projection from $\mathbb{O}_F \to P(F)\backslash F \subset (\Gamma\backslash D_g)^*$. Moreover $f|^{-1}_{X_{\Sigma(F)}(F)}(P(F)\backslash F) = \mathbb{O}_F$.

Then we get a map

$$\bar{f} : \overline{\Gamma\backslash D_g} = X/\!\sim \to (\Gamma\backslash D_g)^*.$$

It can be shown that the map $\bar{f}$ is continuous (cf. [3, Proposition 6.8]).

As $X \to \overline{\Gamma\backslash D_g}$ factors through $Y \to \overline{\Gamma\backslash D_g}$, we have the following commutative diagram

$$\begin{array}{ccc}
\bar{f} : \overline{\Gamma\backslash D_g} & \longrightarrow & (\Gamma\backslash D_g)^* \\
\uparrow & & \| \\
Y_{\Sigma(F)}(F) & \longrightarrow & (\Gamma\backslash D_g)^* \\
\cup & & \cup \\
\partial_F(\overline{\Gamma\backslash D_g}) = P''(F)\backslash \mathbb{O}_F & \longrightarrow & P(F)\backslash F
\end{array}$$

This gives us a correspondence between the stratifications of the toroidal compactification of $\Gamma\backslash D_g$ and of the Satake compactification of $\Gamma\backslash D_g$.

4.3 g=1

We give a detailed description of the toroidal compactification in the simplest case:

$$\mathbb{H}_1 = \{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}, \quad \Gamma = \mathrm{Sp}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z}).$$

By the Cayley transformation

$$\Psi : \mathbb{H}_1 \xrightarrow{\sim} D_1 = \{z \in \mathbb{C} \mid |z| < 1\}.$$

Take the closure in $\mathbb{C}$, we have $\overline{D_1} = \{z \in \mathbb{C} \mid |z| \leq 1\}$. There are two standard boundary components:

$$F_0 = \{1\} \text{ and } F_1 = D_1.$$

Since all boundary components are either $h \cdot F_0$ or $h \cdot D_1$ for $h \in \mathrm{SL}(2,\mathbb{Z})$ (see Proposition 3.1.21), we only need to consider these two standard boundary components F_0 and F_1. All other partial compactifications are isomorphic to either $Y_{\Sigma(F_0)}(F_0)$ or $Y_{\Sigma(F_1)}(F_1)$ by Proposition 4.1.16. Note that 1 corresponds to ∞ and $h \cdot F_0$ corresponds to $\{\mathbb{Q} \cup \infty\}$ under the inverse of the Cayley transformation.

We first do the partial compactification in the directions of $F_1 = D_1$ and $F_0 = \{1\}$ and glue these partial compactifications.

(a) Consider $F_1 = D_1$. By the matrix forms of different subgroups given in Subsection 3.2, we have

$$\mathcal{P}(F) = \mathrm{SL}(2,\mathbb{R}); \quad R_u(\mathcal{P}(F)) = \mathcal{P}'(F) = \{I\}; \quad V(F) = R_u(\mathcal{P}(F))/\mathcal{P}'(F) = \{I\}$$

and thus

$$P(F) = \mathrm{SL}(2,\mathbb{Z}); \quad P'(F) = \{I\}; \quad P''(F) = P(F)/P'(F) = \mathrm{SL}(2,\mathbb{Z}).$$

Step 1. Take the partial quotient

$$e(F_1) : D_1 \cong \mathbb{H}_1 \xrightarrow{\cong} P'(F)\backslash\mathbb{H}_1 = \mathbb{H}_1 \cong \mathbb{H}_1 \times V(F) \times (\mathcal{P}'_{\mathbb{C}}(F)/P'(F)).$$

Note that in Step 1, we actually consider $\mathbb{H}_g$ instead of D_g as the action of $\mathrm{Sp}(2g,\mathbb{R})$ on $\mathbb{H}_g$ is easier than the action on D_g.

Step 2. The fan $\Sigma(F_1) = \{\{0\}\}$ is the only possible fan in $\mathcal{P}'(F)$. Hence $T_{\Sigma(F_1)} = (\mathbb{C}^*)^0$ and thus $\mathcal{X}_\Sigma(F_1) = \mathbb{H}_1$.

Step 3. Indeed we've changed nothing by the torus embedding in Step 2. Hence we have

$$X_{\Sigma(F_1)}(F_1) = \mathbb{H}_1$$

and the partial compactification in the direction of F_1 is

$$P''(F)\backslash\mathbb{H}_1 = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}_1.$$

(b) Consider $F_0 = \{1\}$. We have

$$\mathcal{P}(F_0) = \left\{ \begin{pmatrix} Q^{-1} & * \\ 0 & Q \end{pmatrix} \middle| Q \in \mathrm{GL}(1,\mathbb{R}) = \mathbb{R}^* \right\}; \quad R_u(\mathcal{P}(F_0)) = \mathcal{P}'(F_0) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}; \quad V(F_0) = \{I\}$$

and thus

$$P(F_0) = \left\{ \begin{pmatrix} Q^{-1} & * \\ 0 & Q \end{pmatrix} \middle| Q = \pm 1 \right\}; \quad P'(F_0) = \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \middle| S \in \mathbb{Z} \right\}; \quad T = \mathcal{P}'_{\mathbb{C}}(F_0)/P'(F_0) \cong (\mathbb{C})^*.$$

Step 1. Take the partial quotient

$$e(F_0) : \mathbb{H}_1 \to P'(F_0)\backslash\mathbb{H}_1 \subset T \cong (\mathbb{C})^*.$$

As $P'(F_0)$ acts on $\mathbb{H}_1$ by

$$\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} : \tau \mapsto \tau + S$$

the partial quotient is given by

$$e(F_0) : \tau \mapsto \exp(2\pi i\tau).$$

The image $X(F_0)$ of $e(F_0)$ is

$$X(F_0) = \{z \in \mathbb{C} \mid 0 < |z| < 1\}.$$

Step 2. We have

$$\bar{P}(F_0) = \mathrm{Aut}(\mathcal{P}'(F_0))\cap\mathrm{SL}(2,\mathbb{Z}) = G_l(F_0)\cap\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \middle| Q = \pm 1 \right\} \cong P''(F_0)$$

Note that $\bar{P}(F_0)$ acts on $\mathcal{P}'(F_0)$ by conjugation, hence it is indeed a trivial action. We have the open homogeneous cone associated to F_0

$$C(F_0) = \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \middle| S > 0 \right\} \cong \mathbb{R}_{>0}.$$

The rational closure $C(F_0)^{\mathrm{rc}}$ is $\mathbb{R}_{\geq 0}$. The fan $\Sigma(F_0) = \{\{0\}, \mathbb{R}_{\geq 0}\}$ is the only possible $\bar{P}(F_0)$-admissible fan in $\mathcal{P}'(F_0)$. Then $T_{\Sigma(F_0)} \cong \mathbb{C}$.

Step 3. Take the interior of the closure of $X(F_0)$ in $\mathbb{C}$, we have

$$X_{\Sigma(F_0)}(F_0) = \{z \in \mathbb{C} \mid |z| < 1\}.$$

Since $P''(F_0) \cong \bar{P}(F_0)$ acts on $T = \mathbb{C}$ trivially, the partial compactification in the direction of F_0 is

$$Y_{\Sigma(F_0)}(F_0) = X_{\Sigma(F_0)}(F_0) = \{z \in \mathbb{C} \mid |z| < 1\}.$$

As $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, we don't need to consider condition (a) of the equivalence relation used to define a toroidal compactification (see Definition 4.1.21) and only need to consider the adjacent pair $F_0 \prec F_1$.

Note that the map

$$\pi_0 : X(F_1) = \mathbb{H}_1 \to X(F_0) = P'(F_0)\backslash\mathbb{H}_1 = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$$

is the same as the map $e(F_0)$. The map extends to a map

$$\pi(F_1, F_0) : X_{\Sigma(F_1)}(F_1) = \mathbb{H}_1 \to X_{\Sigma(F_0)}(F_0) = \{z \in \mathbb{C} \mid |z| < 1\}.$$

The toroidal compactification of $A_1 = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}_1$ is

$$\overline{A_1} = X/\sim = (X_{\Sigma(F_1)}(F_1) \amalg X_{\Sigma(F_0)}(F_0))/\sim'$$

where the equivalence relation $\sim'$ is as follows: Let $x \in X_{\Sigma(F_0)}(F_0)$ and $x' \in X_{\Sigma(F_1)}(F_1)$, then $x \sim' x'$ if $\pi(F_1, F_0)(x') = x$.

We have $\partial_{F_0}(\overline{\Gamma\backslash D_g}) = \{z = 0\}$, which corresponds to the infinity point ∞ as $\infty = \lim_{z\to 0} \log Z$, and $\partial_{F_1}(\overline{\Gamma\backslash D_g}) = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}_1 = A_1$. As a set, we have

$$\overline{A_1} = A_1 \cup \{\infty\}.$$

4.4 g=2

In this subsection, we discuss the toroidal compactification in the simplest nontrivial case, i.e.,

$$D_2 \cong \mathbb{H}_2, \qquad \Gamma = \mathrm{Sp}(4,\mathbb{Z}).$$

Similar to what we did for $g = 1$, we analyze the partial compactification in the direction of the three standard boundary components $F_0 \cong \mathbb{H}_0, F_1 \cong \mathbb{H}_1$ and $D_2 = F_2 \cong \mathbb{H}_2$.

4.4.1 Degree One And Two Boundary Components

Similar to what we get in the case of $g = 1$, we have

Proposition 4.4.1. *For the standard boundary component $F_2 \cong \mathbb{H}_2$, we have $\mathcal{P}'(F_2) = \{0\}$ and $C(F_2) = \{0\}$. The trivial fan $\Sigma(F_2) = \{\{0\}\}$ is the only admissible fan in $\mathcal{P}'(F_2)$. It gives rise to*

$$X_{\Sigma(F_2)}(F_2) = \mathbb{H}_2; \quad Y_{\Sigma(F_2)}(F_2) = Sp(4,\mathbb{Z})\backslash\mathbb{H}_2 = A_2.$$

Now we analyze the standard boundary component $F_1 \cong \mathbb{H}_1$. Again by Subsection 3.2, we have

$$\mathcal{P}'(F_1) = \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & S \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\} \cong \mathbb{R}; \quad V(F_1) \cong \mathbb{C}; \quad \mathcal{P}'_{\mathbb{C}}(F_1)/P'(F_1) = \mathbb{C}^*.$$

As the group $\mathcal{P}'(F_2)$ acts on $\mathbb{H}_2$ by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & S \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 + S \end{pmatrix},$$

the partial quotient $e(F_1) : D_2 \cong \mathbb{H}_2 \to P'(F_1)\backslash\mathbb{H}_2$ is given by

$$e(F_1) : \mathbb{H}_2 \to \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C}^* \quad \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto (\tau_1, \tau_2, \exp(2\pi i \tau_3)).$$

The group $\bar{P}(F_1) = \mathrm{Aut}(\mathcal{P}'(F_1)) \cap \Gamma$ has the form

$$\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & Q \end{pmatrix} \middle| Q = \pm 1\right\}.$$

It acts on $\mathcal{P}'(F_1)$ by conjugation, hence it acts trivially on $\mathcal{P}'(F_1)$ and preserves the cone $C(F_1) = \mathbb{R}_{>0}$ in $\mathcal{P}'(F_1)$. The only admissible fan $\Sigma(F_1)$ in $\mathcal{P}'(F_1)$ with support $C(F_1)^{\mathrm{rc}} = \mathbb{R}_{\geq 0}$ is

$$\Sigma(F_1) = \{\{0\}, \mathbb{R}_{\geq 0}\}.$$

Hence $T_{\Sigma(F_1)} = \mathbb{C}$. This means that we have a principal $\mathbb{C}^*$-bundle $\mathcal{X}(F_1) = \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C}^*$ embedded into its associated bundle $\mathcal{X}_{\Sigma(F_1)}(F_1) = \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C}$ with fibre $\mathbb{C}$.

Note that the map $\pi_0(F_2, F_1) : X(F_2) = P'(F_2)\backslash\mathbb{H}_2 = \mathbb{H}_2 \to X(F_1) = P'(F_1)\backslash\mathbb{H}_2$ coincides with the map $e(F_1)$ and extends to the map $\pi(F_2, F_1) : X_{\Sigma(F_2)}(F_2) = \mathbb{H}_2 \to X_{\Sigma(F_1)}(F_1)$. The complement of Im $\pi(F_2, F_1)$ in $X_{\Sigma(F_1)}(F_1)$ is

$$\mathbb{O}_{F_1} = \mathbb{H}_1 \times \mathbb{C} \times O_{F_1}(\mathbb{R}_{\geq 0}) = \mathbb{H}_1 \times \mathbb{C} \times \{0\}.$$

Since the action of $P''(F_1)$ on the fibre of $\mathcal{X}(F_1)$ is trivial (as $\bar{P}(F_1)$ acts trivially on $P'(F_1)$), it remains to investigate its action on the base $\mathbb{H}_1 \times \mathbb{C}$.

Proposition 4.4.2. *The open boundary component $\partial_{F_1}(\overline{A_2}) = P''(F_1)\backslash \mathbb{O}_{F_1}$ is isomorphic to the open Kummer modular surface $K^0(1) = \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})\backslash(\mathbb{H}_1 \times \mathbb{C})$.*

Proof. By Proposition 3.2.9, we have

$$P''(F) = \left\{ \begin{pmatrix} \epsilon & m & n \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), m,n \in \mathbb{Z}, \epsilon = \pm 1 \right\}$$

with the induced action on $\mathbb{H}_1 \times \mathbb{C}$ given by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : (\tau_1, \tau_2) \mapsto (\tau_1, -\tau_2);$$
$$\begin{pmatrix} 1 & m & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : (\tau_1, \tau_2) \mapsto (\tau_1, \tau_2 + m\tau_1 + n);$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} : (\tau_1, \tau_2) \mapsto ((a\tau_1 + b)(c\tau_1 + d)^{-1}, \tau_2(c\tau_1 + d)^{-1}).$$

This shows that $P''(F_1)\backslash(\mathbb{H}_1 \times \mathbb{C} \times \{0\}) \cong K^\circ(1)$. $\square$

Remark 4.4.3. There is no priori reason to believe that $\bar{\partial}_{F_1}(A_g)$ is isomorphic to the respective compactified Kummer modular surfaces, although it indeed is. We will prove this later.

4.4.2 Degree Zero Boundary Component

Now we consider the standard boundary component F_0.

We have

$$\mathcal{P}'(F_0) = \left\{ \begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix} \middle| {}^tS = S \right\} \cong \mathrm{Sym}(2,\mathbb{R}); \quad V(F) = \{I\}.$$

Hence the partial quotient map $e(F_0) : \mathbb{H}_2 \to P'(F_0)\backslash\mathbb{H}_2 \subset (\mathbb{C}^*)^3 = \mathcal{X}(F_0)$ is given by

$$\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto (\exp(2\pi i\tau_1), \exp(2\pi i\tau_2), \exp(2\pi i\tau_3)).$$

In this case, the partial quotient $X(F_0)$ is contained in a trivial torus bundle whose base is just a point and whose fibre is a 3-dimensional algebraic torus.

Note that

$$G_h(F_0) = \{I\}, \quad G_l(F_0) = \left\{ \begin{pmatrix} {}^tQ^{-1} & 0 \\ 0 & Q \end{pmatrix} \middle| Q \in \mathrm{GL}(2,\mathbb{Z}) \right\}, \quad R_u(\mathcal{P}(F_0)) = \mathcal{P}'(F_0).$$

The group $P(F_0)$ consists of matrices

$$\begin{pmatrix} {}^tQ^{-1} & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix}.$$

Let $\pi : P(F_0) \to \mathrm{GL}(2,\mathbb{Z})$ be defined by

$$\pi\left(\begin{pmatrix} {}^tQ^{-1} & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix}\right) = Q.$$

Then the map π identifies the quotient group $P''(F_0) = P(F_0)/P'(F_0)$ with $\mathrm{GL}(2,\mathbb{Z})$. Let $\psi : \mathcal{P}'(F_0) \to \mathrm{Sym}(2,\mathbb{R})$ be defined by

$$\psi\begin{pmatrix} I_2 & S \\ 0 & I_2 \end{pmatrix} = S.$$

The open cone $C(F_0)$ under the identification of ψ is $\mathrm{Sym}_+(2,\mathbb{R})$. Under the identifications π and ψ defined above, the action of $P''(F_0)$ on $\mathcal{P}'(F_0)$ is given by

$$Q \cdot S = {}^tQ^{-1}SQ^{-1}.$$

In this case, $P''(F_0)$ is isomorphic to $\bar{P}(F_0)$. Thus a $\bar{P}(F_0)$-compatible fan is the same as a $P''(F_0)$-compatible fan.

We now define a fan $\Sigma \in \mathcal{P}'(F_0)$ to construct a normal toric variety $T_\Sigma = \mathcal{X}_\Sigma(F_0)$ with torus $\mathcal{X}(F_0) \cong (\mathbb{C}^*)^3$. The group $P'(F_0)$ of the one-parameter subgroups of $\mathcal{X}(F_0)$ is generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in the lattice $\mathrm{Sym}(2,\mathbb{Z})$ under the identification of ψ.

Definition 4.4.4. Let $\sigma_3 \subset \mathrm{Sym}(2,\mathbb{R})$ be the 3-dimensional simplicial cone

$$\sigma_3 = \mathbb{R}_{\geq 0}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The **Legendre decomposition** Σ_L is the set of cones in $\mathrm{Sym}(2,\mathbb{R})$ which consists of all $\mathrm{GL}(2,\mathbb{Z})$-translates of σ_3 with all their faces.

Our current goal is to show that

Theorem 4.4.5. *The Legendre decomposition Σ_L is a $\bar{P}(F_0)$-admissible fan.*

We can regard $\mathrm{Sym}_+(2,\mathbb{R})$ as the space of positive definite binary quadratic form $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ giving the quadratic forms $\varphi(x_1,x_2) = (x_1,x_2)S^t(x_1,x_2) = ax_1^2+2bx_1x_2+cx_2^2$. Recall $\mathrm{GL}(2,\mathbb{Z})$ acts on $\mathrm{Sym}_+(\mathbb{R})$ by contragredient congruence, i.e., $h \in \mathrm{GL}(2,\mathbb{Z})$ acts by $h(S) = {}^th^{-1}Sh^{-1}$. It is indeed a integral change of basis: If $(x_1,x_2) \mapsto h((x_1,x_2)) = (x_1,x_2)h^{-1}$, then $S \mapsto h(S)$. Sometimes we use quadratic forms to denote cones. For example, the cone σ_3 can be written as $\langle x_1^2, (x_1+x_2)^2, x_2^2\rangle$.

We discuss more general cases. There is a one-to-one correspondence between quadratic forms and symmetric bilinear forms on a real vector space. Regard a quadratic form φ as a symmetric bilinear form and set

$$\ker(\varphi) = \{v \in V \mid \varphi(v,w) = 0 \text{ for all } w \in V\}.$$

It's easy to see the following.

Proposition 4.4.6. *Let V be a finite-dimensional inner product space over $\mathbb{R}$ with a fixed orthonormal basis. Then there is a one-to-one correspondence between $\overline{Sym_+(V)}$, the space of positive semi-definite forms on V, and*

$$\{(W, \varphi') \mid W \text{ is a subspace of } V \text{ and } \varphi' \in Sym_+(W)\}$$

given by

$$\varphi \mapsto (W = \ker(\varphi)^{\perp}, \varphi' = \varphi|_W).$$

Moreover, this correspondence is equivariant with respect to the action of $GL(V)$ given by

$$h(\varphi) = {}^t h^{-1} \varphi h^{-1}, \quad h(W, \varphi') = (W h^{-1}, {}^t h^{-1} \varphi h^{-1}|_{W h^{-1}}).$$

Remark 4.4.7. (i) If φ has rank r, then W has dimension r. Fix a subspace W of V, $\mathrm{Sym}_+(W)$ forms a boundary component of the open homogeneous cone $\mathrm{Sym}_+(V)$. Hence, we can write

$$\overline{\mathrm{Sym}_+(V)} = \coprod_{W\text{: subspace of } V} \mathrm{Sym}_+(W).$$

(ii) Now assume that $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ for some lattice $V_{\mathbb{Z}}$. Then a rationally defined subspace W of V determines a rational boundary component $\mathrm{Sym}_+(W)$ of $\mathrm{Sym}_+^{\mathrm{rc}}(V)$. Hence

$$\mathrm{Sym}_+^{\mathrm{rc}}(V) = \coprod_{\substack{W\text{:rationally defined} \\ \text{subspace of } V}} \mathrm{Sym}_+(W).$$

Return to our situation. Every cone $\sigma \neq 0$ in the Legendre decomposition Σ_L is spanned by one, two, or three rational rays on the boundary of $\mathrm{Sym}_+(2, \mathbb{R})$, that is, by 1-dimensional cones of the form $\mathbb{R}_{\geq 0}\varphi$, where φ is a positive semi-definite, but not definite, symmetric matrix with rational (or integral) coefficients. If W is a 1-dimensional subspace of $\mathbb{R}^2$, then for any two elements in $\mathrm{Sym}_+(W)$, they are proportional by a positive real number. We have a one-to-one correspondence

$$\left\{ \begin{array}{c} \mathbb{R}_{\geq 0}\varphi \text{ (rational) ray of positive} \\ \text{semi-definite forms of rank 1 on } \mathbb{R}^2 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} W \text{ 1-dimensional (rationally defined)} \\ \text{subspace of } \mathbb{R}^2 \end{array} \right\}$$

We can use the 1-dimensional subspaces to index rational rays.

Definition 4.4.8. Given a pair (a, b) of relative prime integers, $(a, b) \neq (0, 0)$. We label by (a, b) the rational ray $\mathbb{R}_{\geq 0}\varphi_{a,b}$ on the boundary of the cone $\mathrm{Sym}_+(2, \mathbb{R})$, where the form $\varphi(a, b)$ corresponds to the 1-dimensional subspace $\ker(\varphi_{a,b}) = (a, b)^{\perp}$. Up to a positive scalar, the form $\varphi_{a,b}$ is

$$\varphi_{a,b} = \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}.$$

Remark 4.4.9. The index is equivariant in the sense that for any $Q \in \mathrm{GL}(2, \mathbb{Z})$, we have $Q(\varphi_{a,b}) = {}^t Q^{-1} \varphi_{a,b} Q^{-1} = \varphi_{a',b'}$, where $(a', b') = (a, b)Q^{-1}$.

Using reduction theory, we can find the fundamental domain in $\mathrm{Sym}_+(2, \mathbb{R})$ with respect to the action of $\mathrm{GL}(2, \mathbb{Z})$.

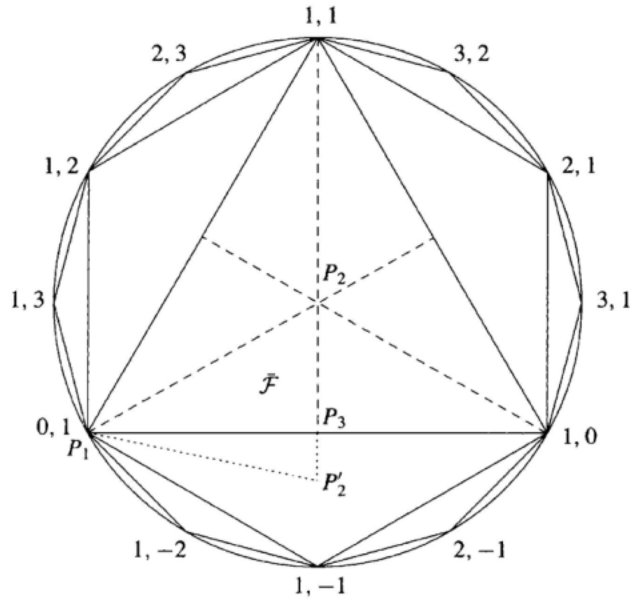


Figure 8: The Legendre Decomposition

Theorem 4.4.10 (Legendre)**.** *Every positive definite binary quadratic form is equivalent under $GL(2,\mathbb{Z})$ to a unique form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, 0 < a \leq c, 0 \leq b \leq a/2.$$

We call such a form reduced.

Proof. See [12, Theorem 3.120]. □

We may represent the Legendre decomposition by taking its intersection with a sphere around the origin of $\mathrm{Sym}(2,\mathbb{R})$ as in Figure 8. The triangle marked as $\bar{\mathcal{F}}$ is the intersection of the fundamental domain with the sphere.

Proof of Theorem 4.4.5. We prove this by means of triangle groups.

Definition 4.4.11. Let l, m, n be integers greater than or equal to 2. A triangle group $T(l,m,n)$ is a group of motions of the hyperbolic plane generated by the reflections along the sides of a triangle with angles π/l, π/m and π/n.

Give the disk the Poincaré metric, substitute geodesics for the straight lines we have drawn and consider the simplicial complex obtained by taking the first barycentric subdivisions of all simplices. We obtain the tesselation of the disk by fundamental domains for the triangle group $T(3,2,\infty)$, see Figure 9.

By Theorem 4.4.10, we have a fundamental domain for the action of $\mathrm{GL}(2,\mathbb{Z})$ on quadratic forms which is indicated as $\bar{\mathcal{F}}$ in Figure 8. We then have an identification of $\mathrm{GL}(2,\mathbb{Z})$ with $T(3,2,\infty)$. Let $\bar{\sigma}_3$ be the intersection of the cone σ_3 with the sphere, which is the triangle with vertices $(0,1)$, $(0,1)$ and $(1,1)$. The group $T(3,2,\infty)$ acts on the disk in such a way that

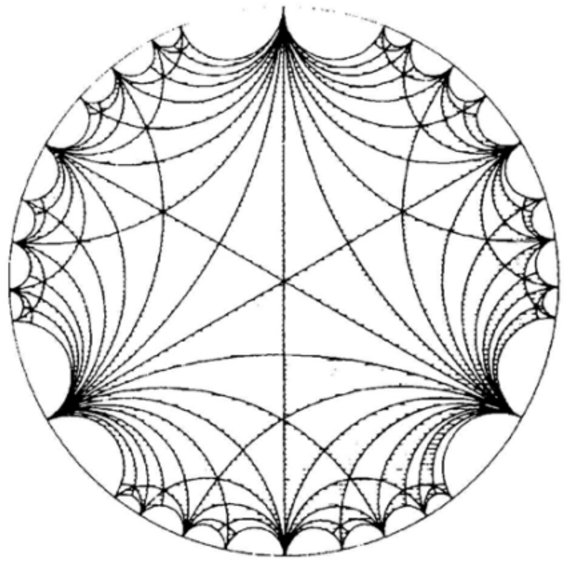


Figure 9: Tesselation for $T(3,2,\infty)$

any translate $g(\overline{\sigma}_3)$, $g \in T(3,2,\infty)$ either coincides with $\bar{\sigma}_3$ (by reflection through the dotted line in $\bar{\sigma}_3$) or has one common edge with $\overline{\sigma}_3$ (by reflection through the edges of $\bar{\sigma}_3$), or has just one common vertex with $\bar{\sigma}_3$. Consequently, each cone $g(\sigma_3) \in \Sigma_L$, $g \in \mathrm{GL}(2,\mathbb{Z}) \cong T(3,2,\infty)$, either coincides with σ_3, or intersects σ_3 in a proper face of σ_3. This suffices to show that Σ_L is a fan.

Now we prove the $\mathrm{GL}(2,\mathbb{Z})$-admissibility. Since $\mathcal{F}$ is a fundamental domain, every point in $C(F_0)^{rc}$ is in the image of some $\mathrm{GL}(2,\mathbb{Z})$-translate of $\mathcal{F}$, and hence of σ_3.

Restricting our attention to the solid lines, we obtain the tesselation of the disk by fundamental domains of the triangle group $T(\infty,\infty,\infty)$, see Figure 10. Since $\bar{\mathcal{F}}$ is one-sixth of the fundamental domain $\bar{\sigma}_3$ of $T(\infty,\infty,\infty)$, the group $T(\infty,\infty,\infty)$ is index 6 in $\mathrm{GL}(2,\mathbb{Z})$. The quotient $\mathrm{GL}(2,\mathbb{Z})/T(\infty,\infty,\infty)$ is indeed isomorphic to S_3, which is generated by reflections along the sides of $\bar{\mathcal{F}}$. The triangle group $T(\infty,\infty,\infty)$ also acts freely on Σ_L. Thus the action of $\mathrm{GL}(2,\mathbb{Z})$ preserves Σ_L.

By the definition and the analysis above, we have 4 orbits of Σ_L, which are by

$$\sigma_0 = \{0\}, \qquad \sigma_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \middle| \lambda \geq 0 \right\}, \qquad \sigma_2 = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \geq 0 \right\}, \qquad \sigma_3.$$

□

In the spirit of Proposition 4.1.11, we discuss the correspondence between the adjacencies of the boundary components of $C(F_0)$ and of boundary components of D_2.

Proposition 4.4.12. *Let l be an isotropic line. Denote by $F(l)$ the corank 1 (degree 1) boundary component associated to the line l. The correspondence between the rational rays $\mathbb{R}_{\geq 0}\varphi_{a,b}$ lying on the boundary of $Sym_+(2,\mathbb{R})$ and corank 1 boundary components $F(l)$ to which F_0 is adjacent to is given by*

$$\mathbb{R}_{\geq 0}\varphi_{a,b} \Leftrightarrow l = \mathbb{Q}(0,0,a,b)$$

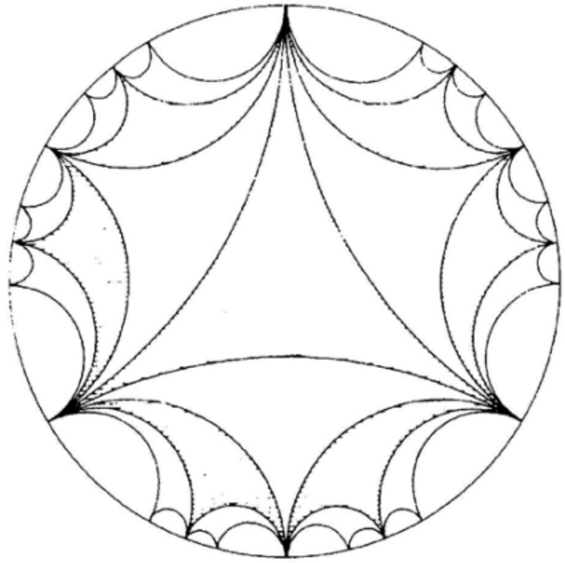


Figure 10: Tesselation for $T(\infty,\infty,\infty)$

and this correspondence is $P(F_0)$-equivariant.

Proof. Regard $C(F_0)$ as $\mathrm{Sym}_+(2,\mathbb{R})$. By Proposition 4.1.11 (ii), every rational ray $\mathbb{R}_{\geq 0}\varphi_{a,b}$, $(a,b)\neq(0,0)$ corresponds to a corank 1 rational boundary component F of D_2 such that $F_0 \prec F$.

First $\mathbb{R}_{\geq 0}\varphi_{0,1} = (C(F_0)^{rc}\cap\mathcal{P}'(F_1))^\circ$ and $l_{(0,1)} = U(F_1) = \mathbb{Q}(0,0,0,1)$. So the ray $\mathbb{R}_{\geq}\varphi_{0,1}$ corresponds to $l_{(0,1)} = \mathbb{Q}(0,0,0,1)$. Now for any pair $(a,b)\in\mathbb{Z}^2$ with $\gcd(a,b)=1$, we can find $Q\in\mathrm{GL}(2,\mathbb{Z})$ such that $(a,b)=(0,1)Q^{-1}$. Then $Q(\mathbb{R}_{\geq 0}\varphi_{0,1}) = \mathbb{R}_{\geq 0}Q(\varphi_{0,1}) = \mathbb{R}_{\geq 0}\varphi_{a,b}$.

Let $h\in P(F_0)$ be such that it has the form

$$\begin{pmatrix} {}^tQ^{-1} & 0 \\ 0 & Q\end{pmatrix}\begin{pmatrix} I_2 & S \\ 0 & I_2\end{pmatrix}$$

where S is chosen arbitrarily such that ${}^tS = S$, then

$$h(C(F_0)^{\mathrm{rc}}\cap\mathcal{P}'(F_1)) = C(F_0)^{\mathrm{rc}}\cap h(\mathcal{P}'(F_1)) = C(F_0)^{\mathrm{rc}}\cap\mathcal{P}'(h(F_1)).$$

Also note that $U(h(F_1)) = U(F_1)h^{-1} = \mathbb{Q}(0,0,0,1)h^{-1} = \mathbb{Q}(0,0,a,b)$ since $(0,0,0,1)h^{-1} = (0,0,a',b')$ where $(a',b') = (0,1)Q^{-1} = (a,b)$. □

Proposition 4.4.13. *(i) The normal toric variety T_{Σ_L} defined by the Legendre decomposition Σ_L is smooth.*

(ii) The partial compactification $Y_{\Sigma_L}(F_0)$ in the direction of F_0 has at worst quotient singularities.

Proof. (i.) The cone σ_3 is generated by the basis

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix},\quad \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix},\quad \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

of the lattice $\mathrm{Sym}(2,\mathbb{Z})$, hence the toric variety T_σ is smooth by the discussion in Example 2.3.16. Since each $h \in \mathrm{GL}(2,\mathbb{Z})$ induces an isomorphism $h : T_{\sigma_3} \to T_{h(\sigma_3)}$, all toric varieties $T_{h(\sigma_3)}$ are smooth, hence the identification space T_{Σ_L} of $T_{h(\sigma_3)}$ is smooth.

(ii.) By Proposition 4.1.13, the group $P''(F)$ acts properly discontinuously on $\mathcal{X}_{\Sigma_L}(F_0) = T_{\Sigma_L}$. And $Y_{\Sigma_L}(F_0)$ is an open subset of the quotient space. Thus $Y_{\Sigma_L}(F_0)$ has at worst quotient singularities.

□

Use m_1, m_2, m_3 to denote the three generators in the character lattice $M \cong \mathrm{Sym}(2,\mathbb{Z})$ of the torus $T = P'(F) \otimes \mathbb{C}/P'(F)$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall the orbit-cone correspondence gives us a stratification of a toric variety. There are four cones up to the permutation of $GL(2,\mathbb{Z})$, which are $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ as discussed in the proof of Theorem 4.4.5. These four cones give four orbits:

(i) $\sigma_0 = \{0\}$, $\sigma_0^\perp \cap M = M$, hence $O(\sigma_0) \cong \mathrm{Hom}(\sigma_0^\perp \cap M, \mathbb{C}^*) \cong \{(t_1, t_2, t_3) \mid t_1, t_2, t_3 \neq 0\} \cong T$.

(ii) $\sigma_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \middle| \lambda > 0 \right\}$, $\sigma_1^\perp \cap M = \mathbb{Z}m_1 + \mathbb{Z}m_2$, hence $O(\sigma_1) \cong \mathrm{Hom}(\sigma_1^\perp \cap M, \mathbb{C}^*) \cong \{(t_1, t_2, 0) \mid t_1, t_2 \neq 0\} \cong (\mathbb{C}^*)^2$.

(iii) $\sigma_2 = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \geq 0 \right\}$, $\sigma_2^\perp \cap M = \mathbb{Z}m_2$, hence $O(\sigma_2) \cong \mathrm{Hom}(\sigma_2^\perp \cap M, \mathbb{C}^*) \cong \{(0, t_2, 0) \mid t_2 \neq 0\} \cong \mathbb{C}^*$.

(iv) σ_3, $\sigma_3^\perp \cap M = \{0\}$, hence $O(\sigma_3) \cong \{(0,0,0)\} \cong (\mathbb{C}^*)^0$.

Among all these four cones, only σ_2 and σ_3 contain positive definite symmetric matrices, hence we have

$$\mathbb{O}_{F_0} = \coprod_{h \in GL(g,\mathbb{Z}), \sigma \in \{\sigma_2, \sigma_3\}} O(h(\sigma)).$$

The open boundary component is

$$\partial_{F_0}(\overline{A}_2) = P''(F) \backslash \mathbb{O}_{F_0} = \mathrm{Stab}(\sigma_3) \backslash O(\sigma_3) \amalg \mathrm{Stab}(\sigma_2) \backslash O(\sigma_2)$$

where $\mathrm{Stab}(\sigma) = \{h \in GL(2,\mathbb{Z}) \mid h(\sigma) = \sigma\}$ for $\sigma = \sigma_2$ or σ_3.

As $O(\sigma_3)$ is just a point, there is nothing to quotient out. We mainly look at $\mathrm{Stab}(\sigma_2) \backslash O(\sigma_2)$. By the analysis of the $\mathrm{GL}(2,\mathbb{Z})$-action on the fan Σ_L in Theorem 4.4.5, we know that $\mathrm{Stab}(\sigma_2)$ contains the elements $\pm I_2$, which act on the fan Σ_L trivially, and the reflection of $\bar{\sigma}_3$ along the edge connecting the vertices $(1,0)$ and $(0,1)$, which is

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 4.4.14. *The elements* $\pm\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ *act on the torus* $O(\sigma_2) =$ *by* $z \mapsto z^{-1}$.

Proof. Let z denote the coordinate corresponding to the generator $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ of the character lattice $\sigma_2^{\perp} \cap M$ of the torus $O(\sigma_2)$. The cocharacter lattice is also generated by $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. The two elements act on the generator of the cocharacter lattice by

$$ {}^t h^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} h^{-1} = -\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad h = \pm\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} $$

which corresponds to the morphism $z \mapsto z^{-1}$ by the discussion in the last part of Subsection 2.4. □

The open boundary component $\partial_{F_0}(\overline{A_2})$ is $\{0\} \cup (\mathbb{C}^*/\sim)$ where $z \sim z^{-1}$. We can also view it as a projective line quotient by its involution $z \mapsto z^{-1}$, which is isomorphic to $\mathbb{P}^1$.

4.4.3 Gluing and Summary

We can define a $\mathrm{Sp}(4,\mathbb{Z})$-admissible collection of fans: if $F = h(F_i)$ for some $h \in \mathrm{Sp}(4,\mathbb{Z})$, then let $\Sigma(F) = h(\Sigma(F_i))$. It is independent of the choice of h. It suffices to check that if $h(F_i) = F_i$ for some $h \in P(F_i)$ and F_i a standard rational boundary component, $h(\Sigma(F_i)) = \Sigma(F_i)$. As $\bar{P}(F_i) = G_l(F_i) \cap \mathrm{Sp}(4,\mathbb{Z}) = \mathrm{Aut}(\mathcal{P}'(F_i)) \cap \mathrm{Sp}(4,\mathbb{Z})$, $h(\Sigma(F_i)) = \Sigma(F_i)$ as $\Sigma(F_i)$ is $\bar{P}(F_i)$-admissible. It's also easy to check the condition (c) in Definition 4.1.14 holds so that the collection of fans is admissible.

We proceed a bit further to more general cases.

Proposition 4.4.15. *To do the toroidal compactification of* D_g *in the case of* $\Gamma = Sp(2g,\mathbb{Z})$, *giving an* $Sp(2g,\mathbb{Z})$*-admissible collection of fans is equivalent to giving a* $GL(g,\mathbb{Z})$*-admissible fan in* $Sym(g,\mathbb{R})$.

Proof. First, for the standard boundary component F_0, we have

$$\mathcal{P}'(F_0) \cong \mathrm{Sym}(g,\mathbb{R}) \text{ and } \bar{P}(F) \cong \mathrm{GL}(g,\mathbb{Z}).$$

Hence a $\bar{P}(F)$-admissible fan in $\mathcal{P}'(F_0)$ is a $\mathrm{GL}(g,\mathbb{Z})$-admissible fan in $\mathrm{Sym}(g,\mathbb{R})$.

Suppose we have an admissible collection $\tilde{\Sigma}$. For any rational boundary component F, there exists an $h \in \mathrm{Sp}(2g,\mathbb{Z})$ such that $F = h(F_i)$. Then

$$\Sigma(F) = \Sigma(h(F_i)) = h(\Sigma(F_i)) = h(\Sigma(F_0) \cap \mathcal{P}'(F_i)).$$

Then a $\bar{P}(F)$-fan in $\mathcal{P}'(F)$ is determined by $\Sigma(F_0)$. Conversely, given a $\mathrm{GL}(G,\mathbb{Z})$-admissible fan in $\Sigma(F_0)$, we have an admissible collection $\tilde{\Sigma}$ by its definition. □

Remark 4.4.16. It is an old problem in reduction theory to find an explicit decomposition of $\mathrm{Sym}_+(n,\mathbb{R})$ invariant under $\mathrm{GL}(n,\mathbb{Z})$. For higher dimensions, it is also important to find a fundamental cone σ like σ_3 in this subsection. When $n \leq 3$, $\mathrm{Sym}_+(n,\mathbb{R})$ is the union of all tranlates $\gamma\sigma$ of σ where $\gamma \in \mathrm{GL}(n,\mathbb{Z})$ and σ is the fundamental cone. For $n \geq 4$, we need more cones. There are at least three widely-used decompositions: Perfect cone decomposition (also known as the 1st Voronoi decomposition), Central cone decomposition, and Second Voronoi decomposition. For more details, we refer to [2, II.6] or [20, §8].

We give a detailed description of the gluing map in Proposition 4.1.17. Note that due to Proposition 4.1.20, it suffices to discuss gluing maps on standard boundary components.

Proposition 4.4.17. *(i.) The gluing maps $\pi(F_2, F)$ where F is a rational boundary component can be described as compositions*

$$\pi(F_2, F) : X_{\Sigma(F_2)}(F_2) = \mathbb{H}_2 \twoheadrightarrow P'(F_2)\backslash\mathbb{H}_2 \hookrightarrow X_{\Sigma(F)}(F)$$

(ii.) The image of $X_{\Sigma(F_1)}(F_1)$ under the gluing map $\pi(F_1, F_0)$ is contained in the affine toric variety $T_\xi \subset \mathcal{X}_{\Sigma_L}(F_0)$ where ξ is the cone $\mathbb{R}_{\geq 0}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(iii.) The gluing map $\pi(F_1, F_0)$ is obtained at the restriction to $X_{\Sigma(F_1)}(F_1)$ of the composition of the exponential map

$$e'(F_1) : \mathcal{X}_{\Sigma(F_1)}(F_1) \cong \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}$$
$$e'(F_1)(\tau_1, \tau_2, t_3) = (\exp(2\pi i \tau_1), \exp(2\pi i \tau_2), t_3)$$

with the natural inclusion $T_\xi \hookrightarrow T_{\Sigma_L}$.

Proof. (i.) Since $\pi(F_2, F)$ extends the map $\pi_0(F_2, F) : \mathbb{H}_2 \to X(U)$, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{H}_2 = X_{\Sigma(F_2)}(F_2) & \xrightarrow{\pi(F_2,F)} & X_{\Sigma(F)}(F) \\ \| & & \uparrow \\ \mathbb{H}_2 = X(F_2) & \xrightarrow{\pi_0(F_2,F)} & X(F) = P'(F)\backslash\mathbb{H}_2 \end{array}$$

and $\pi_0(F_2, F)$ is the quotient by the action of the group $P'(F)$.

(ii.) We are following the construction of Proposition 4.1.17. In this case F' corresponds to F_1 and F corresponds to F_0, so $T' \cong \mathbb{C}^*$ and $T \cong (\mathbb{C}^*)^3$. The fans are $\Sigma(F_0) = \Sigma_L$ and $\Sigma(F_1) = \{\{0\}, \xi\}$ where $\xi = \mathbb{R}_{\geq 0}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
Note that $\mathcal{X}(F_0)$ is a T-bundle over a single point, hence $\mathcal{X}(F_0) = T_{\Sigma_L}$. As a consequence, the quotient map in Proposition 4.1.17 is

$$\Pi(F_1, F_0) : \mathcal{X}_{\Sigma(F_1)}(F_1) \to P'(F_0)\backslash\mathcal{X}_{\Sigma(F_1)}(F_1) \subset T_{\Sigma(F_1)} = T_\xi \subset T_{\Sigma_L}.$$

Thus the image of $\pi(F_1, F_0)$ lies in the open subset T_ξ of T_{Σ_L}.

(iii.) follows easily by the discussion of (ii.).

□

Proposition 4.4.18. *The open boundary component $\partial_{F_1}(\overline{A_2})$ is the image of the open Kummer modular surface $K^\circ(1)$ under an embedding $f^\circ : K^\circ(1) \hookrightarrow \overline{A_2}$. This map f_0° extends to an analytic map $f : K(1) \to \overline{A_2}$*

Proof. The first statement has been proved in Proposition 4.4.2.
The proof of Proposition 4.4.2 shows that the embedding f° is induced by an embedding $\tilde{f}^\circ$ of $\mathbb{H}_1 \times \mathbb{C}$ into $X_{\Sigma(F_1)}(F_1)$, that is, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{H}_1 \times \mathbb{C} & \xrightarrow{\tilde{f}^\circ} & X_{\Sigma(F_1)}(F_1) \\ {\scriptstyle \mathrm{mod}\ SL(2,\mathbb{Z})\ltimes\mathbb{Z}^2}\downarrow & & \downarrow {\scriptstyle \mathrm{mod}\ \sim} \\ K^\circ(1) & \xrightarrow{f^\circ} & \overline{A_2} \end{array} .$$

The map f° is also induced by the embedding of the image $\pi(F_1, F_0)(\tilde{f}^\circ(\mathbb{H}_1 \times \mathbb{C}))$ into $X_{\Sigma_L}(F_0)$. By Proposition 4.4.17, the map from $\mathbb{H}_1 \times \mathbb{C}$ to $\pi(F_1, F_0)(\tilde{f}^\circ(\mathbb{H}_1 \times \mathbb{C}))$ coincides with the quotient map defined by the action of P' on $\mathbb{H}_1 \times \mathbb{C}$, where P' is defined in (5)

$$P' = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| n, b \in \mathbb{Z} \right\}.$$

This gives a natural action of the group

$$P'' = \left\{ \begin{pmatrix} 1 & m & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \middle| m \in \mathbb{Z}, \epsilon = \pm 1 \right\}$$

defined in (6) on $\pi(F_1, F_0)(\tilde{f}^\circ(\mathbb{H}_1 \times \mathbb{C}))$. We can identify $\pi(F_1, F_0)(\tilde{f}^\circ(\mathbb{H}_1 \times \mathbb{C}))$ with $D^* \times \mathbb{C}^*$ where $D^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. We have a commutative diagram

$$\begin{array}{ccc} D^* \times \mathbb{C}^* & \xrightarrow{\hat{f}^\circ} & X_{\Sigma_L}(F_0) \\ \downarrow {\scriptstyle \mathrm{mod}\ P''} & & \downarrow \\ P''\backslash D^* \times \mathbb{C}^* & \xrightarrow{f^\circ} & \overline{A_2} \end{array} .$$

Denote by Y_Σ the interior of the closure of $D^* \times \mathbb{C}^*$ in T_Σ defined by the fan Σ as shown in Figure 5. The partial compactification of $K^\circ(1)$ over the cusp ∞ is $P''\backslash Y_\Sigma$.

We now extend the embedding $\hat{f}^\circ$ to an embedding $\bar{f}^\circ : Y_\Sigma \hookrightarrow X_{\Sigma_L}(F_0)$.

By Proposition 4.4.17, the image $\hat{f}^\circ(D^* \times \mathbb{C}^*)$ lies in $O(\xi)$ contained in the boundary of $X_{\Sigma_L}(F_0) \subset T_{\Sigma_L}$, $T = (\mathbb{C}^*)^3$. Consider the closure of $O(\xi)$ in T_{Σ_L}. It is isomorphic to $T_{\mathrm{Star}(\xi)}$, a nomal toric varieties arising from the fan

$$\mathrm{Star}(\xi) = \{\bar{\sigma} \subset N(\xi)_\mathbb{R} \mid \xi \preceq \sigma \in \Sigma\}.$$

Choose the three matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

as a basis. Then $\mathrm{Star}(\xi)$ is determined by the projections of cones $\sigma \in \Sigma_L$ onto the plane generated by the first two elements in the basis. It is easy to see that $\mathrm{Star}(\xi)$ is the same as the fan constructed in Figure 5. Thus we may extend $\hat{f}^\circ$ to an embedding $\bar{f}^\circ$ as follows:

$$\begin{tikzcd} Y_\Sigma \arrow[r, "\bar{f}^\circ"] \arrow[d, "\bmod P''"'] & X_{\Sigma_L}(F_0) \arrow[d, "\bmod \sim"] \\ P''\backslash Y_\Sigma \arrow[r, "\exists ?"'] & \overline{A_2} \end{tikzcd}$$

It remains to be seen that $\bar{f}^\circ$ is an equivariant map such that the above diagram commutes. For this, we need to verify that the stabilizing subgroup of ξ in $\mathrm{GL}(2, \mathbb{Z})$ is isomorphic to P'' such that $\bar{f}^\circ$ is an equivariant map. The stabilizing subgroup of ξ in $\mathrm{GL}(2, \mathbb{Z})$ is of the form

$$\mathrm{Stab}(\xi) = \left\{ Q \in \mathrm{GL}(2, \mathbb{Z}) \middle| Q \in \begin{pmatrix} \pm 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \right\}$$

from which the claim follows.
Thus we have extended f° to a map $f : K(1) \hookrightarrow \overline{A_2}$. The image $f(K(1))$ is contained and dense in $\bar{\partial}_{F_1}(\overline{A_2}) = \overline{\partial_{F_1}(\overline{A_2})}$. Then we have $f(K(1)) = \bar{\partial}_{F_1}(\overline{A_2})$ since both are closures of $f^\circ(K(1)) = \partial_{F_1}(\overline{A_2})$. $\square$

To summarize, we have the toroidal compactification of A_2 as a set

$$\overline{A_2} = A_2 \cup (K^\circ(1)) \cup \mathbb{P}^1$$

The map $\varphi : \overline{A_2} \to A_2^*$ from the toroidal compactification of A_2 to the Satake compactification of A_2 is given as $\varphi^{-1}(A_0) = \mathbb{P}^1$, $\varphi^{-1}(A_1) = K^0(1)$ and $\varphi^{-1}(A_2) = A_2$ Take the closure of $K^\circ(1)$ in $\overline{A_2}$, we get the boundary of $\overline{A_2}$: $\overline{A_2} - A_2 = K(1)$. Moreover, it can be shown that $\overline{A_2}$ is a projective variety (cf. [12, Proposition 3.151]).

A Complex Analytic Spaces

In this appendix, we compile various results on algebraic varieties over $\mathbb{C}$, complex analytic spaces, and their correspondences as described in [10]. Additionally, we discuss the quotients of complex analytic spaces, referencing the works [7], [21], and [23].

Definition A.0.1. A complex analytic space is a ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ which admits an open covering $\mathcal{U}$ such that each $(U, \mathcal{O}_U)$ is isomorphic, as a locally ringed space, to the following locally ringed space $(Y, \mathcal{O}_Y)$: Let $D \subset \mathbb{C}^n$ be the polydisc $\{|z_i| < 1 | i = 1, ..., n\}$, let $f_1, ..., f_q$ are holomorphic functions on U, let $Y \subseteq D$ be the closed subset with respect to the usual topology consisting of the common zeros of $f_1, ..., f_q$ and take $\mathcal{O}_Y$ to be the sheaf $\mathcal{O}_D/(f_1, ..., f_q)$ where $\mathcal{O}_D$ is the sheaf of holomorphic functions on D.

If X is a scheme of finite type over $\mathbb{C}$, we can define the **associated complex analytic space** X_h. We can cover X by open affines $Y_i = \mathrm{Spec} A_i$. Then A_i is an algebra of finite type over $\mathbb{C}$, hence we can write $A_i = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_q)$ where $f_1, ..., f_q$ are polynomials which can be regarded as holomorphic functions on $\mathbb{C}^n$. Then their set of common zeros is a complex analytic subspace $(Y_i)_h$. Use the same glueing data as we glued Y_i to X, we can glue the analytic spaces $(Y_i)_h$ into an analytic space X_h.

The construction is functorial. We denote the functor from the category of schemes of finite type over $\mathbb{C}$ to the category complex analytic spaces as h.

For the coherent sheaves $\mathcal{F}$ on X, we can also define the associated coherent analytic sheaf $\mathcal{F}_h$. Since the sheaf $\mathcal{F}$ is coherent on a locally noetherian scheme X, it is locally finite presentable

$$\mathcal{O}_U^m \xrightarrow{\varphi} \mathcal{O}_U^n \to \mathcal{F}.$$

Since the topology on X_h is finer than the Zariski topology, U_h is open in X_h. Hence we have

$$\mathcal{O}_{U_h}^m \xrightarrow{\phi_h} \mathcal{O}_{U_h}^n \to \mathcal{F}_h \to 0.$$

We list a few factors about the relationship between a scheme x and its associated analytic space X_h.

- X is separated over $\mathbb{C}$ if and only if X_h is Hausdorff.
- X is connected in the Zariski topology if and only if X_h is connected in the usual topology.
- X is reduced if and only if X_h is reduced.
- X is smooth over $\mathbb{C}$ if and only if X_h is a complex manifold.
- X is proper over $\mathbb{C}$ If and only if X_h is compact.

The main theorem in Serre's paper GAGA is

Theorem A.0.2 (Serre)**.** *Let X be a projective scheme over $\mathbb{C}$. Then the functor h induces an equivalence of categories from the category of coherent sheaves on X to the category of coherent analytic sheaves on X_h. Furthermore, for every coherent sheaf $\mathcal{F}$ on X, the natural maps*

$$\alpha_i : H^i(X, \mathcal{F}) \to H^i(X_h, \mathcal{F}_h)$$

are isomorphisms, for all i.

Serre also obtains a new proof of a theorem of Chow:

Theorem A.0.3 (Chow)**.** *If $\mathfrak{X}$ is a compact analytic subspace of the complex manifold $\mathbb{P}^n_{\mathbb{C}}$, then there is a subscheme $X \subset \mathbb{P}^n$ with $X_h = \mathfrak{X}$.*

Using Theorem A.0.2, one can also prove that

Corollary A.0.4. *If X and X' are two projective scheme schemes such that $X_h \cong X'_h$, then is $X \cong X'$.*

Now we discuss the quotients of complex analytic spaces.

Let G be a group acting on a complex analytic space $\mathfrak{X}$. The quotient $\mathfrak{X}/G$, endowed with the quotient topology, naturally admits the structure of a ringed space. Let $\pi : \mathfrak{X} \to G$ be the canonical projection. Then by definition $\mathcal{O}_{\mathfrak{X}/G}(U)$, for $U \subset \mathfrak{X}/G$ open, is the set of functions $f : U \to \mathbb{C}$, for which $f \circ \pi$ is is an element of $\mathcal{O}_{\mathfrak{X}}(\pi^{-1}U)$.

Definition A.0.5. A group G acts properly discontinuously on a complex analytic space $\mathfrak{X}$ if for all compact sets $K \subset \mathfrak{X}$ the set

$$\{\gamma \in G : \gamma(K) \cap K \neq \emptyset\}$$

is finite.

Remark A.0.6.

(a) The condition implies that the isotropy groups

$$G_x := \{\gamma \in G : \gamma(x) = x\}$$

are finite for all $x \in \mathfrak{X}$.

(b) Replacing K by the union of two compact sets K and L and noting that $\gamma(K) \cap L$ is a subset of $\gamma(K \cup L) \cap (K \cup L)$, this leads to the following equivalent definition: G acts properly discontinuously on $\mathfrak{X}$, if for all compact sets $K, L \subset \mathfrak{X}$, the set

$$\{\gamma \in G : \gamma(K) \cap L \neq \emptyset\}$$

is finite.

(c) Under this condition, one can show that for all $x \in X$, there exist neighborhoods U_x of x such that

$$\gamma(U_x) = U_x, \quad \forall \gamma \in G_x; \qquad \gamma(U_x) \cap U_x = \emptyset, \qquad \forall \gamma \in G/G_x.$$

Theorem A.0.7 (cf. [7, Theorem A.6,Corollary A.7.])**.** *Let $\mathfrak{X}$ be a complex analytic space and G be a group acting properly discontinuously on $\mathfrak{X}$. The orbit space $\mathfrak{X}/G$ is also a complex analytic space. If $\mathfrak{X}$ is normal, so is $\mathfrak{X}/G$. If $\mathfrak{X}$ is a complex manifold and G is a group acting free and properly discontinuously on $\mathfrak{X}$, the quotient $\mathfrak{X}/G$ is also a complex manifold.*

Remark A.0.8. The complex analytic spaces that are locally equivalent to orbit spaces are called "complex V-manifolds"(cf.[21]) .

Definition A.0.9 (cf. [21])**.** A singularity that is isomorphic to a singularity of a quotient $\mathfrak{X}/G$ of a complex manifold $\mathfrak{X}$ by a properly discontinuously action of a group G is called a quotient singularity. Singularities represented by pairs$(\mathfrak{X}, p)$ with $\mathfrak{X}$ a complex V-manifold and $p \in \mathfrak{X}$ are called V-germs.

Theorem A.0.10 (cf. [21, p. 379] and [23, Theorem 8.10])**.** *Each quotient singularity is isomorphic to a quotient $\mathbb{C}^n/G$ at the origin, where G is a finite subgroup of $GL(n, \mathbb{C})$. We call $(\mathbb{C}^n/G, \tau(0))$ where $\tau : \mathbb{C}^n \to \mathbb{C}^n/G$ the standard model of the singularity.*

References

[1] Locally symmetric varieties. Available online at https://wiki.epfl.ch/shimvar/documents/shimvar-locallysymvar.pdf.

[2] Avner Ash, David Mumford, Michael Rapoport, and Yung sheng Tai. *Smooth Compactification of Locally Symmetric Varieties*. Math. Sci. Press, Brookline MA, 1975.

[3] Avner Ash, David Mumford, Michael Rapoport, and Yung sheng Tai with the collaboration of Peter Scholze. *Smooth Compactifications of Locally Symmetric Varieties*. Cambridge Mathematical Library. Cambridge University Press, 2 edition, 2010.

[4] Walter L. Baily. Satake's compactification of V_n. *American Journal of Mathematics*, 80(2):348–364, 1958.

[5] Walter Lewis Baily and Armand Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Annals of Mathematics*, 84(3):442–528, 1966.

[6] Arnaud Beauville. Theta functions, old and new, 2013. Available online at https://math.univ-cotedazur.fr/~beauvill/pubs/thetaon.pdf.

[7] Christina Birkenhake and Herbert Lange. *Complex Abelian Varieties*. Springer, 2004.

[8] John Little David Cox and Hal Schenck. Toric varieties, 2010. Available at https://www.mimuw.edu.pl/~jarekw/pragmatic2010/CoxLittleSchenckJan2010.pdf.

[9] Samuel Grushevsky, Klaus Hulek, and Orsola Tommasi. Stable cohomology of the perfect cone toroidal compactification of A_g. *Journal Für Die Reine Und Angewandte Mathematik (Crelles Journal)*, 2018(741):211–254, 2018.

[10] Robin Hartschorne. *Algebraic Geometry*. Springer, 2000.

[11] Klaus Hulek. Degenerations of abelian varieties, 2000. Available online at https://www.iag.uni-hannover.de/fileadmin/iag/homepages/hulek/Skripten/Pragmatic2000.pdf.

[12] Klaus Hulek, Constantin Kahn, and Steven H. Weintraub. *Moduli Spaces of Abelian Surfaces: Compactification, Degenerations and Theta Functions*. De Gruyter, Berlin, New York, 1993.

[13] Klaus Hulek and G. K. Sankaran. The geometry of siegel modular varieties. *Advances Studies in Pure Mathematics*, 35:89–156, 2002.

[14] George R. Kempf, Finn F. Knudsen, David Mumford, and B. Saint-Donat. *Toroidal Embeddings I*. 1973.

[15] J.S. Milne. Abelian varieties, 2008. Available online at https://math.univ-cotedazur.fr/~beauvill/pubs/thetaon.pdf.

[16] J.S. Milne. Introduction to shimura varieties, 2017. Available at `https://www.jmilne.org/math/xnotes/svi.pdf`.

[17] Stephen A. Mitchell. Notes on principal bundles and classifying spaces, 2011. Available online at `https://sites.math.washington.edu/~mitchell/Notes/prin.pdf`.

[18] David Mumford. *Abelian Varieties*. Tata Inst., 1974.

[19] Yukihiko Namikawa. A new compactification of the siegel space and degeneration of abelian varieties. i. *Mathematische Annalen*, 221:97–141, 1976.

[20] Yukihiko Namikawa. *Toroidal Compactification of Siegel Spaces*. Springer, 1980.

[21] David Prill. Local classification of quotients of complex manifolds by discontinuous groups. *Duke Mathematical Journal*, 34:375–386, 1967.

[22] The Stacks project authors. The stacks project. Available online at `https://stacks.math.columbia.edu/`.

[23] Oswald Riemenschneider. Singular points of complex analytic surfaces, 2011. Available online at `https://www.math.uni-hamburg.de/home/riemenschneider/`.

[24] Ichiro Satake. On the compactification of the siegel space. *The Journal of the Indian Mathematical Society*, 20(1-3):259–281, 1956.

[25] Joseph Silverman. *The Arithmetic of Elliptic Curves*. Springer, 2009.