Utrecht University

STACKS AND GIT

THESIS MATHEMATICS MSC

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Chapter 0

Introduction

A moduli problem can often be translated to the problem of constructing a certain type of quotient. In his book [DK94] David Mumford uses ideas from classical invariant theory to construct these quotients. In particular Mumford shows that for the action of a reductive group on a scheme X we can construct quotients for the locus of (semi-)stable points $X^{s(s)} \subset X$. Moreover, Mumford provides a numerical criterion for identifying these (semi-)stable points.

In the paper [Alp13] Jarod Alper introduces the notion of a good moduli space which generalizes ideas from geometric invariant theory to the setting of algebraic stacks. One of the main objectives of this text is to introduce the notion of a good moduli space and discuss its properties.

In the first chapter we will start by giving some background on moduli spaces and invariant theory, then we introduce some of the main results in geometric invariant theory. In the second chapter we will introduce the notion of an algebraic stack and develop the theory necessary for the chapters that follow. In chapter three we will introduce the notion of a good moduli space, state some of its properties and work out an example. In particular we will show that there is a correspondence between good moduli for certain quotient stacks and good quotients. Chapter four will revolve around a recent existence criterion for good moduli spaces given by Jarod Alper, Daniel Halpern-Leistner and Jochen Heinloth in the article [AHH23]. In this article, Alper, Halpern-Leistner and Heinloth show that the existence of a good moduli space for an algebraic stack depends on the algebraic stack satisfying a pair of valuative criteria. We start the chapter off by discussing so called filterations and a result on these filtrations given by Daniel Halpern-Leistner in his paper [Hal22], this result will prove to be a useful in the rest of the text. Then we introduce the valuative criteria and we conclude the chapter by discussing the main ideas used in proving the existence result and give an outline of its proof. For our last chapter we return to geometric invariant theory, namely we discuss a generalization to algebraic stacks of Mumfords numerical criterion for determining the (semi-)stability of points, given by Jochen Heinloth in his article [Hei18]. Notably, we will show that for a quotient stack, the stability of its points corresponds to the classical notion of stability.

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Chapter 1

Moduli spaces and GIT

Throughout this chapter the word scheme will mean an algebraic scheme over a field $k = \overline{k}$, i.e., a finite type scheme over an algebraically closed field k, and point will mean closed point, or equivalently k-point, i.e., read mSpec instead of Spec whenever this is convenient. An algebraic group will be a finite type group scheme. The main references for this chapter are [Hos16] and [DK94] but there are many others which are cited throughout the text and included in the bibliography.

1.1 Moduli spaces

In the first section of this chapter we will introduce some of the basic notions in moduli theory and explain the connection between moduli spaces and quotients.

1.1.1 Moduli problems

In this first subsection we want to introduce some basic concepts in moduli theory. To give a moduli problem classifying algebro-geometric objects of our choice we need to give a notion of families of such objects over a scheme and a notion of equivalence of such families. These notions have to be tailored to the moduli problem at hand, we do however have the following general requirements.

1. The equivalence classes of families over the ground field correspond to the equivalence classes of the objects of interest;

2. There is a notion of pullback, i.e., for every morphism of schemes $\varphi : S' \to S$ there is a map sending the class of a family over S, $[F/S]$ to the class of a family over S', $[(\varphi^* F)/S']$; we require this map to be a functor.

Definition 1.1.1. A moduli problem is a presheaf $\mathcal{M} : \text{Sch}^{op}/k \to \text{Set}$, defined by:

1. $S \mapsto \mathcal{M}(S) := \{\text{Families over } S\}/\sim_S$;

2. $(\varphi : S' \to S) \mapsto \mathcal{M}(\varphi) : \mathcal{M}(S) \to \mathcal{M}(S'),\ \mathcal{M}(\varphi)[F/S] = [(\varphi^* F)/S']$.

Definition 1.1.2. We say that a scheme M is a fine moduli space for a moduli problem $\mathcal{M}$ if it represents the moduli problem. If such a fine moduli space exists the family over M corresponding to the identity morphism id_M is called the universal family.

A fine moduli space as a solution for our moduli problem is the ideal scenario, this however is often too much to ask for. One possible obstruction to the existence of such a solution is that the moduli problem simply has no family over some scheme which parametrizes all the objects, if this is the case we say that our moduli problem is unbounded.

Another example of an obstruction to the existence of a fine moduli space is the existence of a non trivial family which is fiberwise trivial, what this means is explained in the following lemma.

Lemma 1.1.3. *Let $\mathcal{M} : Sch^{op}/k \to Set$ be a moduli problem and S a variety with structure map $\pi : S \to \text{Spec}(k)$ and a family $\mathcal{F} \in \mathcal{M}(S)$. Assume*

1. *$\mathcal{F}$ is non-trivial, i.e., $\mathcal{M}$ is not equal to $\pi^* F$ for some $F \in \mathcal{M}(k)$; and*

2. *$\mathcal{F}$ is fiberwise trivial, i.e., $\mathcal{F}_s \simeq \mathcal{F}_{s'}$ for all $s, s' \in S$.*

Then $\mathcal{M}$ does not admit a fine moduli space.

Proof. We argue by contradiction, let $M \in \text{Sch}/\mathbb{C}$ and suppose that $\eta : \mathcal{M} \to M$ is a natural isomorphism and denote the universal family by $\mathcal{U}$. Then since $\mathcal{F}$ is fiberwise trivial and η is an isomorphism there exists a unique point $m \in M(k)$ such that for all $s \in S(k)$, $\eta(s^*\mathcal{F}) = m$. Since S is a variety and therefore reduced the map $\eta(\mathcal{F}) : S \to M$ has the following factorization

$$\begin{array}{ccc} S & \xrightarrow{\eta(\mathcal{F})} & M \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle m} & \\ \text{Spec}(k) & & \end{array}$$

where $\pi : \mathcal{S} \to \text{Spec}(k)$ denotes the structure morphism. This follows from the fact that $\eta(\mathcal{F})$ is constant and equal to m so that we can apply the basic result which says that if $f : X \to Y$ is a morphism of schemes, X is reduced and $f(X) \subset Z \subset Y$ where $Z \subset Y$ is a closed subscheme then f factors through Z, see [Stacks, Tag 0356]. Now we get the following identifications

$$[\mathcal{F}] = [\eta(\mathcal{F})^*\mathcal{U}] = [(m \circ \pi)^*\mathcal{U}] = [\pi^*(m^*\mathcal{U})],$$

this contradicts $\mathcal{F}$ being non-trivial. □

As noted and illustrated by the above obstructions, (interesting) moduli problems are often not represented in the category of schemes. In order to deal with this unfortunate reality two possible strategies are

1. Instead of requiring our moduli problem to be equivalent to a scheme, ask for some weaker form of equivalence;

2. Ask for representability in a larger category.

This first chapter revolves around the first strategy, we will now clarify what we mean by "weaker form of equivalence".

Definition 1.1.4. We say that a scheme M is a coarse moduli space for a moduli problem $\mathcal{M}$ if there exists a natural transformation $\eta : \mathcal{M} \to M$ such that

1. for every algebraically closed field k the families over $\mathrm{Spec}(k)$, $\mathcal{M}(k)$ are in bijection with the k−points in $M(\mathrm{Spec}(k))$;

2. for any scheme M' and natural transformation $\psi : \mathcal{M} \to M'$ there exists a unique morphism of schemes $f : M \to M'$ filling in the commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\psi} & M' \\ \downarrow{\scriptstyle \eta} & \nearrow_{\exists! f} & \\ M & & \end{array}$$

A family F/M such that for every point $m \in M$ the fiber F_m is in the class of $\mathcal{M}$ corresponding to the point m is called a tautological family.

1.1.2 Moduli spaces and quotients

In this subsection we will introduce different notions for quotients of schemes and show that the question of constructing a coarse moduli space for some moduli problem can under some conditions be translated to a question about the construction of such quotients.

Definition 1.1.5. Let G be an algebraic group acting on a scheme X. We say that a morphism of schemes $f : X \to Y$ is a categorical quotient if

1. f is G-invariant;
2. for any G-invariant morphism $g : X \to Z$ there exists a unique morphism $h : Y \to Z$ filling in the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle f} & \nearrow{\scriptstyle \exists! h} & \\ Y & & \end{array}$$

If the preimage of each k-point in Y is a single orbit then f is called an orbit space.

A stronger notion of quotient is the so called good quotient introduced by Seshadri.

Definition 1.1.6. [Ses72, Def. 1.5, 1.6] Let G be an algebraic group acting on a scheme X. We say that a morphism of schemes $\varphi : X \to Y$ is a good quotient if

1. φ is a surjective, affine G-invariant morphism.
2. $\varphi_*(\mathcal{O}_X)^G \simeq \mathcal{O}_Y$.
3. For any pair of closed, disjoint G-invariant subsets $X_1, X_2 \subset X$ we have that $\overline{\varphi(X_1)} \cap \overline{\varphi(X_2)} = \emptyset$.

If φ is a good quotient and for all $x_1, x_2 \in X$

$$f(x_1) = f(x_2) \iff G \cdot x_1 = G \cdot x_2$$

we say that φ is a geometric quotient.

We often use the term good quotient for the codomain of a good quotient as in the above definition.

Remark. It is not too difficult to prove that given the surjectivity of φ the third condition in the definition of a good quotient is equivalent to requiring that φ is closed on invariant subsets and that the images of disjoint invariant subsets are disjoint. The only part of the argument that requires a little work is showing that condition 3 implies that φ is closed on G-invariant closed subsets. The argument for this part goes as follows: assume that $W \subset X$ is a G-invariant closed subset such that $\varphi(W) \subset Y$ is not closed. Then $\overline{\varphi(W)} \setminus \varphi(W)$ contains a closed point y and by surjectivity of φ, the closed G-invariant subset $\varphi^{-1}(x)$ is non-empty, this however is in contradiction with our assumed condition

3 since W and $\varphi^{-1}(x)$ are G-invariant, closed and disjoint but the closures of their images are not as they both contain x.

Another important observation is that the definition of good and geometric quotients is local in the sense that if φ is a good (resp. geometric quotient) then for any open subset $U \subset Y$ the restriction $\varphi|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \to U$ is as well and if φ is G-invariant, $\{U_i\}_{i\in I}$ is an open cover for Y and the restrictions $\varphi|_{\varphi^{-1}(U_i)} : \varphi^{-1}(U_i) \to U_i$ are good (resp. geometric) quotients then φ is one as well.

Proposition 1.1.7. [Hos16, Prop. 3.30] *If $\varphi : X \to Y$ is a good quotient for the action of an algebraic group on a scheme X then it is a categorical quotient.*

Lemma 1.1.8. *Let G be an affine algebraic group acting on a scheme X and $\varphi : X \to Y$ a good quotient. Then for every $x_1, x_2 \in X$*

$$\varphi(x_1) = \varphi(x_2) \iff \overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset.$$

Proof. If $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$ then since φ is G-invariant it is constant on orbits and therefore by continuity also on orbit closures, it follows that $\varphi(x_1) = \varphi(x_2)$. On the other hand if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} = \emptyset$ then by the third property in the definition of a good quotient the images of these sets must be disjoint so that the opposite claim holds as well. □

Motivated by the above lemma, for a good quotient $\varphi : X \to Y$ we can introduce an equivalence relation on the set of k-points of X so that the equivalence classes of $X(k)$ under this relation are in bijective correspondence with the k-points of our good quotient, this relation between geometric points is called S-equivalence. To make this more precise the S-equivalence relation denoted $\sim_S$ is defined by

$$x_1 \sim_S x_2 \iff \overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset \text{ for } x_1, x_2 \in X(k).$$

It is easily verified using the previous lemma that for a good quotient this defines an equivalence relation on the k-points in the domain and that the induced map on equivalence classes of points is well-defined and a bijection. After reading the second section in this chapter also see [Ses77, Prop. 9] and [SS10, Cor. 3.4.6.].

Before we give the relation between quotients and coarse moduli, we first need to define a certain type of family for a moduli problem.

Definition 1.1.9. Let $\mathcal{M}$ denote a moduli problem. We say that a family F/S has the local universal property if for any other family F' over a scheme S' and for any point $s' \in S'$ there exists a neighborhood U of s' in S' and a morphism $\varphi : U \to S$ such that $[F'|_U] = [\varphi^* F] \in \mathcal{M}(U)$.

Proposition 1.1.10. [Hos16, Prop 3.35],[New78, Prop. 2.13] *Let $\mathcal{M}$ denote a moduli problem and F/S a family with the local universal property. Suppose that G is an algebraic group acting on S such that two points $p_1, p_2 \in S$ lie in the same $G-$orbit if and only if the fibers F_{p_1} and F_{p_2} are equivalent. Then*

1. *any coarse moduli space induces a categorical quotient;*

2. *a categorical quotient is a coarse moduli space if and only if it is an orbit space.*

Proof. To prove this we start by showing that for a scheme M there is a bijective correspondence between natural transformations $\eta : \mathcal{M} \to M$ and $G-$invariant morphisms $\varphi : S \to M$. We define this correspondence as follows: given a natural transformation η then $\eta_S([F]) \in M(S)$ defines a $G-$invariant morphism since for any two points $s, s' \in S$ which lie in the same $G-$orbit, by naturality of η, we have

$$\eta_S([F])(s) = \eta_k([F_s]) = \eta_k([F'_s]) = \eta_S([F])(s').$$

On the other hand given a $G-$invariant morphism $\varphi : S \to M$ and a family H/S' by the local universal property there exists an open cover $\{U_i\}$ of S' and morphisms $\varphi_i : U_i \to S$ such that $[H|_{U_i}] = [\varphi_i^* F]$ for every i. For points in the intersection $u \in U_i \cap U_j$, we have that

$$[F_{\varphi_i(u)}] = [(\varphi_i^* F)_u] = [H_u] = [(\varphi_j^* F)_u] = [F_{\varphi_j(u)}]$$

and therefore that $\varphi_i(u)$ and $\varphi_j(u)$ lie in the same $G-$orbit. Now since f is $G-$invariant it follows that $f \circ \varphi_i$ and $f \circ \varphi_j$ agree on the intersections $U_i \cap U_j$ and therefore can be glued to form a morphism $\eta'_S([H]) : S' \to M$. The natural transformation associated to these morphisms defines an inverse.

Now for the first item if we are given a coarse moduli space $\eta : \mathcal{M} \to M$, then $\eta_S([F]) : S \to M$ is $G-$invariant and by the second defining property for coarse moduli spaces, given another $G-$invariant morphism $\eta'_S([F]) : S \to M'$ associated to a natural transformation $\eta' : \mathcal{M} \to M'$ there exists a unique morphism $f : M \to M'$ such that $\eta'_S([F]) = f \circ \eta_S([F])$, thus $\eta_S([F])$ is a categorical quotient.

For the second statement we claim that a categorical quotient $\eta_S([F])$ is an orbit space if and only if it is a bijection on k-points.

Namely, suppose that $\eta_S([F])$ is an orbit space, then for all $m \in M(k)$ there exists a unique orbit $G \cdot s \subset S(k)$ such that $\eta_S([F])(G \cdot s) = m$. It follows that for all $m \in M(k)$ and $\tilde{s} \in S(k)$

$$\eta_k([\tilde{s}^*F]) = m \iff \tilde{s} \in G \cdot s.$$

Now since for any pair $s_1, s_2 \in G \cdot s$ we have that $[s_1^*F] = [s_2^*F]$ we can conclude that for all $m \in M(k)$ there exists a unique family $[s^*F] \in \mathcal{M}(k)$ such that $\eta_k([s^*F]) = x$.

For the opposite implication suppose that $\eta_k : \mathcal{M}(k) \to M(k)$ is a bijection then for all $m \in M(k)$ there exists a unique family $[H] \in \mathcal{M}(k)$ such that $\eta_k([H/k]) = m$. Now let $s_1, s_2 \in \eta_S([F])^{-1}(m)$ then

$$m = \eta_S([F])(s_1) = \eta_k([s_1^*F]) = \eta_k([s_2^*F]) = \eta_S([F])(s_2).$$

It follows that $[s_1^*F] = [H] = [s_2^*F]$ and therefore that s_1 and s_2 lie in the same G-orbit. □

The above result gives us a straightforward way to construct moduli spaces and allows us to take a moduli problem and translate it to a question about quotients. To sum up these first two sections, given a class of algebro-geometric objects, e.g., curves, abelian varieties etc., we can attempt to answer the question of classification by applying the following steps:

1. Formulate the moduli problem, i.e., give a well-defined notion of families of objects, pullbacks of families and equivalence of families.

2. Restrict the problem to a solvable subclass of objects.

3. Find a scheme S which has a family over it satisfying the local universal property and a group action on S satisfying the requirements of the proposition.

4. Construct a geometric quotient of S.

In the next section of the first chapter of this text we will shift our attention to constructing quotients.

1.2 Geometric Invariant Theory

As explained in the previous chapter one strategy for constructing moduli spaces is to convert it to a question about quotients. This latter object is where one of the key concepts in this paper comes into play, namely David Mumford's geometric invariant theory. Simply put geometric invariant theory tells us that for the action of a certain type of group good and geometric quotients exist as well as gives us a method for constructing these quotients.

The "invariant theory" in GIT comes from the fact that on affine patches quotients are the spectra of invariants of the coordinate ring of the affine patches and the study of invariant rings is what the classical subject, invariant theory is about. In the upcoming section we will introduce the reader to the types of groups that are used for constructing GIT quotients and give some important results concerning invariant rings under the action of these groups.

After having introduced these basics about groups and invariants, we will give the main results in GIT starting with the affine case for constructing quotients, which essentially form the building blocks for the more general cases.

1.2.1 The Invariant Theory of Reductive Groups

In this section we will introduce a class of groups whose actions are suitable for constructing quotients, we will give an important, perhaps the most important result, in classical invariant theory which says that under a suitable action of such a group the algebra of invariants for a finitely generated algebra over a field is finitely generated.

Definition 1.2.1. Let G be an affine algebraic group over an algebraically closed field k. We say that

- G is **linearly reductive** if the functor $\mathrm{Rep}(G) \to \mathrm{Vect}_k,\ V \to V^G$ taking a G-representation to its G-invariants is exact.
- G is **reductive** if G is smooth and every smooth, connected, unipotent, normal, algebraic subgroup of G is trivial.
- G is **geometrically reductive** if for every finite dimensional linear representation $G \to \mathrm{GL}_n(V)$ and every non-zero invariant $v \in V^G$ there exists a non-constant G-invariant homogeneous polynomial $f \in \mathcal{O}(V)(:= \mathrm{Sym}^*(V))$ such that $f(v) \neq 0$.

Remark. These definitions can be adapted to more general settings, in the third chapter we will see one such generalization for the notion of linear reductivity.

Linearly reductive groups can be characterized in many equivalent ways, we will now list some of these characterizations.

Proposition 1.2.2. [Hos16, Prop. 4.14.],[Alp24, App. B.1.34.] *Let G be an affine algebraic group over an algebraically closed field. Then the following are equivalent:*

1. *G is linearly reductive.*
2. *Every finite dimensional linear representation decomposes as a direct sum of irreducible representations.*
3. *The functor $Rep^{fd}(G) \to Vect_k$, $V \to V^G$ taking a finite dimensional G-representation to its G-invariants is (right-)exact.*
4. *Given a finite dimensional linear representation $G \to GL(V)$, for any G-invariant subspace $V' \subset V$ there is a subrepresentation $V'' \subset V$ such that $V = V' \oplus V''$.*
5. *For any finite dimensional linear representation $G \to GL(V)$ and every $0 \neq v \in V^G$, there exists a G-invariant linear form $f : V \to k$ such that $f(v) \neq 0$.*

Moreover, for a smooth affine algebraic group over an algebraically closed field the different notions of reductivity are connected in the following way, in char. $p > 0$ we have the implications

$$\text{linearly reductive} \implies \text{reductive} \iff \text{geometrically reductive}$$

and in char. 0 the three notions are equivalent. See [DK94, App. A] for a short history of these results and a proof of the second equivalence, note that for Mumford an algebraic group is by definition smooth. Some examples are the following.

Example 1.2.3.

1. The algebraic groups $(\mathbb{G}_m)^n$ are linearly reductive for every n.
2. Any finite group of order not divisible by the characteristic of k is linearly reductive.
3. For $n > 1$ GL_n, $\mathrm{PGL}_n, \mathrm{SL}_n$ and SP_{2n} are reductive but not linearly reductive in characteristic $p > 0$.
4. The additive group $\mathbb{G}_a$ is not reductive.

Remark. In fact, Nagata showed in his 1961 paper on the complete reducibility of rational representations of a matrix group [Nag61, Thm. 2], that in characteristic $p > 0$ linearly reductive smooth algebraic groups are precisely the groups for which the identity component G_0 is a torus and the quotient G/G_0 is of finite order not divisible by p.

We will now state the result about finite generation of invariant algebras.

Theorem 1.2.4. *Let G be a linearly reductive group acting rationally on a finitely generated k-algebra A. Then A^G is finitely generated.*

Remark. It has been known for a long time that this result is also true in the case of a reductive group in positive characteristic, see [DK94, Thm. A.1.1]. In the 1964 paper titled Invariants of a group in an affine ring [Nag64] Nagata showed that this result holds in the semi-reductive case, semi-reductive is the terminology used by Nagata for what we call geometrically reductive, and therefore by the statement above in the reductive case. Then in the 1979 article on Hilbert's theorem on invariants [Pop79] Popov goes on to show that reductivity is in fact equivalent to the finite generation of invariant algebras.

We will not give a rigorous proof of this theorem, instead we will discuss some of the ideas used. The main ingredient in the proof of the theorem is the following notion.

Definition 1.2.5. For the action of a group on a k-algebra A, a Reynolds operator is a G-invariant linear map $R_A : A \to A^G$ which is a projection, i.e., the restriction of this map to the invariant algebra is the identity.

One of the properties of linearly reductive groups is that if such a group is acting on an affine variety then there exists a unique Reynolds operator for the action on the coordinate ring [DK15, Thm. 2.2.5], the following is a concrete example in the case of the action of a one-dimensional torus, i.e., the multiplicative group.

Example 1.2.6. [DK15, Example 2.2.4] Consider the action of $\mathbb{G}_m$ on an affine algebraic group $\mathrm{Spec}(A)$. There is a coaction $\rho^\# : A \to A[y, y^{-1}]$, $f \mapsto \sum_i f_i y^i$ and the map $R_A : A \to A^G$, $f \mapsto f_0$ defines a Reynolds operator.

We will now discuss the general structure of the argument. First we reduce to the case of a linear action of a polynomial algebra. Since the action of the algebra is assumed to be rational there exists a G-invariant vector subspace V of A containing a set of algebra generators for A. Thus there is a surjection $k[V] \simeq k[x_1, \ldots, x_n] \to A$ and by exactness of taking invariants for linearly reductive groups the map $k[V]^G \to A^G$ is surjective as well so that it is sufficient to prove the finite generation of $k[V]^G$.

Now let I denote the ideal in $k[V]$ generated by all homogeneous invariants of positive degree (> 0), then by Hilbert's basis theorem this ideal is finitely generated, thus we can write $I = (f_1, \ldots, f_r)$ with $f_1, \ldots, f_r \in k[V]^G$. Then the theorem follows from the claim that $k[V]^G$ is generated by the f_i as a k-algebra. Proving the claim is done by an induction argument on the degree and use of the Reynolds operator, see [DK15, Thm. 2.2.10] for details.

Moreover if $f_1, \dots, f_r \in k[V]$ is any set of homogeneous generators for the ideal I then $R_A(f_1), \dots, R_A(f_r)$ form a set of generators for the invariant algebra, see [DK15, Prop. 4.1.1]. We will use this statement later in the construction of an algorithm for computing invariant rings.

1.2.2 Affine GIT quotient

In this subsection we will prove that the action of a reductive group on an affine scheme has a good quotient. In preparation of the proof we will state two results.

Lemma 1.2.7. [Hos16, Lemma 4.29] *Let G be a reductive group acting on an affine scheme X. If W_1 and W_2 are disjoint G-invariant closed subsets of X, then there is an invariant function $f \in \mathcal{O}_X(X)^G$ which separates these sets, i.e.,*

$$f(W_1) = 0 \text{ and } f(W_2) = 1.$$

Lemma 1.2.8. [New78, Lemma 3.4.2] *Let G be a reductive group acting rationally on a finitely generated k-algebra R. If $f_1, \dots, f_n \in (\sum_{i=1}^n f_i R) \cap R^G$, then $f^r \in \sum_{i=1}^n f_i R^G$ for some positive integer r.*

In addition to there being a good quotient for the action of a reductive group on an affine scheme, there also is a subset of so called stable points of the scheme for which there is a geometric quotient.

Definition 1.2.9. Let G be a reductive group acting on an affine scheme X. We say that a point $x \in X(k)$ is stable if its G-orbit is closed in X and $\dim G \cdot x = \dim G$.

We will now give the main result for this subsection.

Theorem 1.2.10. *Let G/k be a reductive group acting (rationally) on an affine algebraic scheme X/k, over an algebraically closed field k. Consider the map $\varphi : X \to X//G :=$ $\operatorname{Spec} \mathcal{O}_X(X)^G$ induced by the inclusion $\varphi^\# : \mathcal{O}_X(X)^G \hookrightarrow \mathcal{O}_X(X)$. Then*

1. *$X//G$ is an affine scheme of finite type over k.*
2. *φ is a good quotient.*
3. *$X^s \subset X$ is an open $G-$invariant subset and $X^s/G := \varphi(X^s)$ is an open subset of $X//G$ such that $\varphi^{-1}(\varphi(X^s)) = X^s$.*
4. *The restriction $\varphi|_{X^s} \to X^s/G$ is a geometric quotient.*

Proof. Since G is a reductive group it follows from our earlier discussion that $\mathcal{O}_X(X)^G$ is a finitely generated k-algebra and therefore that $X//G$ is an affine scheme of finite type over k.

To prove the second item we will show that φ satisfies the defining properties of a good quotient. Since φ is induced by the inclusion map it is G-invariant and affine, and since we are working with a finite type affine scheme over an algebraically closed field, by Chevalley's theorem it is sufficient to show surjectivity on k-points.

Let m_y be the maximal ideal in $\mathcal{O}_X(X)^G$ corresponding to a point $y \in X//G(k)$. By Hilbert's basis theorem the algebra $\mathcal{O}_X(X)^G$ is noetherian, thus we can choose a finite number of generators $f_1, \ldots, f_n$ for m_y.

Since G is reductive it follows from [New78, Lemma 3.4.2] that

$$\sum_1^n f_i \mathcal{O}_X(X) \neq \mathcal{O}_X(X),$$

namely the lemma tells us that if $f \in (\sum_{i=1}^n f_i \mathcal{O}_X(X)) \cap \mathcal{O}_X(X)^G$, then $f^r \in m_y$ for some positive integer r, but since m_y is a maximal ideal there exists some element in its complement such that all powers of this element are also in the complement. Now by the lemma this is an invariant element that is not contained in the ideal $\sum_{i=1}^n f_i \mathcal{O}_X(X)$.

We see that the ideal $(f_1, \ldots, f_n) \in \mathcal{O}_X(X)$ does not generate the ring itself and therefore must be contained in some maximal ideal $m_x \in \mathcal{O}_X(X)$ associated to a point $x \in X(k)$. In particular the f_i are contained in the maximal ideal associated to the point x, therefore $f_i(x) = 0$ for $i = 1, \ldots, n$; it follows that $m_y \subset \varphi(m_x)$ and hence that $f(x) = y$.

Next we want to show that the map $\mathcal{O}_{X//G} \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism. Note that it suffices to show this on the basis of distinguished open subsets

$$\{(X//G)_f : f \in \mathcal{O}_{X//G}(X//G) = \mathcal{O}_X(X)^G\}.$$

Using that localization by G-invariant functions commutes with taking G-invariants, this is Exercise 5.12 in [M F69], we obtain the sequence of isomorphisms

$$\mathcal{O}_{X//G}((X//G)_f) \simeq (\mathcal{O}_X(X)^G)_f \simeq (\mathcal{O}_X(X)_f)^G \simeq \mathcal{O}_X(X_f)^G \simeq \mathcal{O}_X(\varphi^{-1}((X//G)_f))^G.$$

It follows that the map $\varphi^{\#}_{U_f}$ is a surjection for every $f \in \mathcal{O}_X(X)^G$, and therefore an isomorphism.

Lastly we want to show that for a pair of disjoint G-invariant closed subsets $V_1, V_2 \subset X$ the closures of the images $\varphi(V_1)$ and $\varphi(V_2)$ are disjoint. By [Hos16, Lemma 4.29] there exists a $G-$invariant global section $f \in \mathcal{O}_X(X)^G$ such that $f(V_1) = 0$ and $f(V_2) \neq 0$. Since $\mathcal{O}_X(X)^G = \mathcal{O}_{X//G}(X//G)$, f defines a regular function on $X//G$ which separates the subspaces $\varphi(V_1), \varphi(V_2) \subset X//G$. Now it follows from basic topology that $\overline{\varphi(V_1)} \cap \overline{\varphi(V_2)} = \emptyset$

Restricting our attention to the stable subset we first want to show that $X^s \subset X$ is an open $G-$invariant subset. Since we are working with a finite type scheme over an algebraically closed field the set of closed points is a dense subset, therefore it is sufficient to show that for every point $x \in X^s(k)$ there is an open neighborhood of x in X which lies in X^s. By semicontinuity of the dimension of stabilizers [Hos16, Prop.3.21] we have that the set $V := \{x \in X : \dim G_x > 0\}$ is a closed subset of X. Let $x \in X^s$ then as before there exists a $G-$invariant function $f \in \mathcal{O}_X(X)^G$ separating the $G-$invariant closed subsets V and $G \cdot x$, such that $f(V) = 0$ and $f(G \cdot x) \neq 0$. It follows that x is contained in the distinguished open subset X_f. Now we want to show that $X_f \subset X^s$, i.e., that its points have closed orbits and zero dimensional stabilizers. By construction $X_f \cap V = \emptyset$, thus we only have to check that orbits are closed. We argue by contradiction. Let $y \in X_f(k)$ be a point with non-closed orbit and $z \in \overline{G \cdot y} \setminus G \cdot y$. Then by G-invariance of f, we have that $z \in X_f \subset X \setminus V$. However since by [Hos16, Prop. 3.15], $\overline{G \cdot y} \setminus G \cdot y$ is the union or orbits of strictly lower dimension we must have that $\dim G \cdot z < \dim G \cdot y$, this is contradictory because

$$\dim G \cdot z = \dim(G) - \dim(G_z) = \dim(G) = \dim(G \cdot y).$$

Consequently X^s is an open subset of X which can be covered by subsets of the form X_f. Again using the fact that localization by $G-$invariant functions commutes with taking invariants we have that $\varphi(X_f) = (X//G)_f$ is open in $X//G$ and therefore that $\varphi(X^s) \subset X//G$ is open and that $X_f = \varphi^{-1}((X//G)_f)$.

To conclude we want to show that $\varphi|_{X^s} \longrightarrow X/G$ is a geometric quotient, i.e., that the preimage of every point is a single orbit. Note that by our earlier discussion the restriction $\tilde{\varphi} := \varphi|_{X^s} \longrightarrow X/G$ is a good quotient and that the action of G on X^s is closed. To conclude we again argue by contradiction, let V_1, V_2 be two distinct closed orbits in the preimage of some point $y \in X/G$, then since $\tilde{\varphi}$ is a good quotient their images must be disjoint. We conclude that $\tilde{\varphi}$ is a geometric quotient. □

1.2.3 Some examples and an algorithm for computing invariants

In this section we want to give some examples of affine GIT-quotients, there are simple cases in which calculating invariant rings is an easy task, but as the variety or scheme with which we are working becomes more complex the task of computing its corresponding invariant ring becomes exponentially more difficult. We are in luck however, since there are mathematicians who have adopted the computational aspect of invariant theory as the subject of their research, we will discuss one of the many results in this area of study and apply it to one of our examples. Throughout this section we work over the field $k := \mathbb{C}$.

We start off this section by computing two simple examples which can easily be done by hand.

Example 1.2.11. For our first example consider the action of the multiplicative group $\mathbb{G}_m$ on the affine line $\mathbb{A}^1$ given by the usual multiplication. Let us first determine its invariant ring, take a function $f(x) := \sum_{i=0}^{n} c_i x^i \in k[x]$ then $g \in \mathbb{G}_m$ acts on f by

$$(g \cdot f)(x) = f(g^{-1}x) = \sum_{i=0}^{n} x c_i g^{-i} x^i,$$

then it is clear that the only invariant polynomials are those of degree 0 so that $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1)^{\mathbb{G}_m} \simeq k$. It follows that there is a good quotient $\mathbb{A}^1 \to k$.

Furthermore, one easily observes that there are precisely two $\mathbb{G}_m$-orbits: the origin and the punctured affine line, as depicted in Figure 1.1.

Figure 1.1: $\mathbb{G}_m$-orbits of $\mathbb{A}^1$

The origin is the only point with a closed orbit, it is not stable however, since its stabilizer is positive-dimensional, therefore the stable locus is empty.

Example 1.2.12. For our next example consider the action of the multiplicative group $\mathbb{G}_m$ on the affine plane $\mathbb{A}^2$ given on points by $g \cdot (x, y) = (gx, g^{-1}y)$. A function in the coordinate ring is of the form $f(x) := \sum_{i,j} c_{ij} x^i y^j \in k[x,y]$, an element $g \in \mathbb{G}_m$ acts on f by

$$(g \cdot f)(x) = f(g^{-1}(x,y)) = \sum_{i=0}^{n} c_{ij} g^{j-i} x^i y^j,$$

thus the polynomial invariants are those of the form $\sum_i c_i (xy)^i$ so that $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)^{\mathbb{G}_m} \simeq k[xy]$. It follows that there is a good quotient $\mathbb{A}^2 \to \mathbb{A}^1$.

In this case there are four types of orbits the punctured axes, the conics and the origin, as depicted in Figure 1.2.

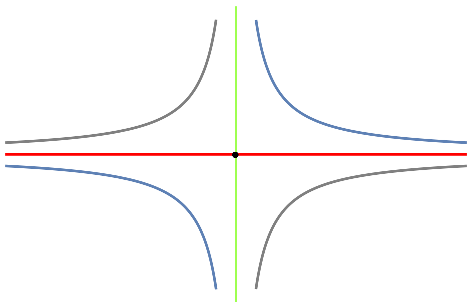


Figure 1.2: $\mathbb{G}_m$-orbits of $\mathbb{A}^2$

Note that the closed orbits are the conics and the origin, the conics are the closed subvarieties given by equations of the form $xy - c$ where $c \in k \setminus 0$. Again, the origin clearly does not have a zero-dimensional stabilizer, the conics however do have zero-dimensional stabilizers therefore the stable locus is given by the open subset $(\mathbb{A}^2)^s = (\mathbb{A}^2)_{xy}$ and its image under the quotient map is isomorphic to $\mathbb{A}^1 \setminus \{0\}$, thus there is a geometric quotient $(\mathbb{A}^2)_{xy} \to \mathbb{A}^1 \setminus \{0\}$.

We will now give Derksen's algorithm for computing invariants for linearly reductive groups [Der99]. Recall from our discussion of the proof for the finite generation of invariant algebras that finding generators for the algebra of invariants can be done by finding a set of generators for the ideal $I_{\mathcal{N}}$ generated by all the homogeneous invariants of (strictly) positive degree and applying the Reynolds operator to the generators, note that if these generators are already invariants then application of the Reynolds operator is not necessary.

Suppose that V is an n-dimensional rational representation for a linearly reductive algebraic group G with coordinate ring $k[x_1, \ldots, x_s]/I_G$ where $I_G := (g_1, \ldots, g_r)$ then the linear action of an element $g \in G$ on the space k^n given by the coordinates $\mathbf{v} = v_1, \ldots, v_n \in V$ is defined

$$g \cdot \mathbf{v} := (a_{ij}(g))_{1 \le i,j \le n} (\mathbf{v})^t$$

where $a_{ij} \in \mathcal{O}_G(G)$. Next consider the closed subset Γ of $G \times V \times V$ with coordinate ring

$$\mathcal{O}_\Gamma(\Gamma) = k[\mathbf{x}, \mathbf{v}, \mathbf{w}]/I_\Gamma, \quad I_\Gamma := (g_1, \ldots, g_r, w_i - \sum_{j=1}^{n} a_{ij} v_{ij})_{1 \le i \le n}$$

where $\mathbf{v} = v_1, \ldots, v_n$ and take $\overline{B}$ to be the closure of the projection of $\Gamma \subset G \times V \times V$ onto $V \times V$. Then $\overline{B}$ is given by the ideal $I_B = I_\Gamma \cap k[\mathbf{v}, \mathbf{w}]$, this intersection can be calculated by using Gröbner bases, namely if S is a Gröbner basis for I_Γ with respect to a monomial ordering on $k[\mathbf{x}, \mathbf{v}, \mathbf{w}]$ such that the x_i are larger than any monomial in $v_1, \ldots, v_n, w_1, \ldots, w_n$, then $S \cap k[\mathbf{v}, \mathbf{w}]$ is a Gröbner basis for I_B. Now suppose that $\{f_1, \ldots, f_t\} \subset k[\mathbf{v}, \mathbf{w}]_{1 \le i \le n}$ is a Gröbner basis for I_B then $\{f_1(\mathbf{v}, \mathbf{0}), \ldots, f_t(\mathbf{v}, \mathbf{0})\}$ generates $I_\mathcal{N}$ (see [Der99, Cor. 3.2]), thus $\{R_{k[V]}(f_1(\mathbf{v}, \mathbf{0})), \ldots, R_{k[V]}(f_t(\mathbf{v}, \mathbf{0}))\}$ are invariant generators for $I_\mathcal{N}$. For a more complete treatment see [Der99] and [DK15, Chapt. 4], the later reference also contains material on Gröbner bases.

This algorithm is implemented in the computer algebra system SINGULAR [Dec+24; Bay24], we will use this in our next example.

Example 1.2.13. Consider the action of $\mathbb{G}_m$, with $\mathcal{O}_{\mathbb{G}_m}(\mathbb{G}_m) = k[s,t]/(st-1)$, on the variety $X := \mathbb{V}(x_3x_4 - x_1x_2x_3 + x_1^3x_2^2) \subset \mathbb{A}^4$ induced by the action given by the diagonal matrix $\operatorname{diag}(s^2, t^3, s, t)$. Then SINGULAR computes that the ideal $I_\mathcal{N}$ for the action on $\mathbb{A}^4$ is generated by the monomials $\{x_3x_4, x_1x_4^2, x_1x_2x_3, x_1^2x_2x_4, x_2x_3^3, x_1^3x_2^2\}$ and since these monomials are invariants there is no need to apply the Reynolds operator.

In order to compute the invariant ring for our variety X we recall that taking invariants by linearly reductive groups is exact, applying this fact to the exact sequence

$$0 \to I \to k[x_1, x_2, x_3, x_4] \to k[x_1, x_2, x_3, x_4]/I \to 0$$

gives us the exact sequence

$$0 \to I^{\mathbb{G}_m} \to k[x_1, x_2, x_3, x_4]^{\mathbb{G}_m} \to (k[x_1, x_2, x_3, x_4]/I)^{\mathbb{G}_m} \to 0$$

where I is the ideal defining X and $I^{\mathbb{G}_m} = I \cap k[\mathbf{x}]^{\mathbb{G}_m}$. It follows that

$$\mathcal{O}_X(X)^{\mathbb{G}_m} \simeq (k[x_1, x_2, x_3, x_4]/I)^{\mathbb{G}_m} \simeq k[x_1, x_2, x_3, x_4]^{\mathbb{G}_m}/I^{\mathbb{G}_m}.$$

Now since I is $\mathbb{G}_m$-invariant we conclude by [Hos16, Lemma 4.24] that $\mathcal{O}_X(X)^{\mathbb{G}_m} \simeq k[x_3x_4, x_1x_4^2, x_1x_2x_3, x_1^2x_2x_4, x_2x_3^3, x_1^3x_2^2]/I$.

1.2.4 Linearizations and the projective GIT quotients

Given the action of a reductive group G on some projective space X the strategy for producing a quotient is the following, for a choice of very ample line bundle $\mathcal{L}$ we can identify our projective space with the projective scheme associated to the algebra $R(X,\mathcal{L}) := \bigoplus_{n\geq 0} H^0(X,\mathcal{L}^n)$, i.e., obtained by the Kodaira embedding. And we want there to be an induced action of G on the algebra, this brings us to the notion of linearization which we will discuss in this section.

Once we have made this identification and produced an action on the algebra there is a natural mapping $i : R(X,\mathcal{L})^G \hookrightarrow R(X,\mathcal{L})$, this however does not necessarily induce a morphism of schemes but rather a rational mapping, see [Stacks, Tag 01RS],

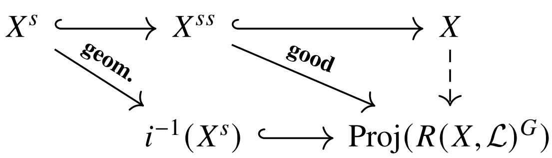
$$\mathrm{Proj}(R(X,\mathcal{L})) \dashrightarrow \mathrm{Proj}(R(X,\mathcal{L})^G).$$

We recall that by definition $\mathrm{Proj}(S)$ for some graded ring S is the set of homogeneous prime ideals of S which do not contain all of $S_+ := \bigoplus_{d>0} S_d$. However, if we take an element of $\mathrm{Proj}(R(X,\mathcal{L}))$ associated to a maximal ideal $m \in R(X,\mathcal{L})$ then the point $i^{-1}(m) \in R(X,\mathcal{L})^G$ might contain $R(X,\mathcal{L})^G_+$, which is inconsistent with the definition we just gave, see [GW20, Rem. 13.7] or [Stacks, Tag 01MX].

To solve this we remove the points for which this is the case so that we are left with the points for which there is some section of $R(X)^G_+$ which does not evaluate to zero on the point. These desirable points are called semi-stable and the restriction to the locus of semi-stable points results in a representative of the rational map which is a good quotient.

We can further restrict to the so called stable locus which corresponds to the semi-stable points with the additional requirement that the points have zero-dimensional stabilizers and the open subsets defined by the non vanishing-locus of G-invariant sections have closed actions. The restriction of the rational map to the stable locus mapping onto its image gives a geometric quotient.

Now the quotient for the semi-stable locus which is the restriction of the rational map we saw above is essentially given by a collection of affine quotients patched together, by definition of semi-stability we can cover the semi-stable locus by the non-vanishing loci of invariant sections, which are affine and whose images are the spectra of invariant rings. In summary we have



In the next section we will make the informal discussion above more precise, in particular we will give a proof for the existence of a general (quasi projective) GIT quotient and also generalize and make more precise the notion of (semi-)stability. We will dedicate the remainder of this section to formalizing the notion of linearization.

Definition 1.2.14. Let G/k be an algebraic group over a field k with multiplication $m : G \times G \to G$ and an action $\rho : G \times X \to X$ on a $k-$scheme X/k. A $G-$linearization is given by an invertible sheaf $\mathcal{L}$ on X together with an isomorphism $\Phi : \rho^*\mathcal{L} \to p_2^*\mathcal{L}$, where $p_2 : G \times X \to X$ is the projection, satisfying the cocycle condition given by the following commuting diagram:

$$\begin{array}{ccccccc} & & (\rho \circ (\mathrm{id}_G \times \rho))^*\mathcal{L} & \xrightarrow{(\mathrm{id}_G \times \rho)^*\Phi} & (p_2 \circ (\mathrm{id}_G \times \rho))^*\mathcal{L} & & \\ & /\!/ & & & & \backslash\!\backslash & \\ (\rho \circ (m \times id_X))^*\mathcal{L} & & & & & & (\rho \circ p_{23})^*\mathcal{L} \\ & \searrow^{(m \times \mathrm{id}_X)^*\Phi} & & & & \swarrow_{p_{23}^*\Phi} & \\ & & (p_2 \circ (m \times \mathrm{id}_X))^*\mathcal{L} & = & (p_2 \circ p_{23})^*\mathcal{L} & & \end{array}$$

where $p_{23} : G \times G \times X \to G \times X$ is the projection onto the last two factors.

Remark. The above definition also has a geometric interpretation, namely a linearization is also given by a geometric line bundle $\pi : L \to X$ together with an isomorphism of line bundles

$$\psi : p_2^* L \to \rho^* L$$

with p_2 and ρ as in the definition, such that the composition

$$\tilde{\rho} := G \times L \xrightarrow{\psi} \rho^* L \xrightarrow{p_2} L$$

where p_2 is the projection $(G \times X) \times_{\rho, X, \pi} L \to L$, is a group action of G on L. In particular one gets linear isomorphisms on the fibers $\psi_{g,x} : L_x \to L_{g \cdot x}$.

One can check that under the equivalence of the categories of invertible sheaves and geometric line bundles this definition corresponds to the definition for invertible sheaves, see [DK94, Chapt. 1.3]. Furthermore the set of G-linearizations on a scheme X modulo G-equivariant isomorphims of line bundles forms an abelian group which is denoted $\mathrm{Pic}^G(X)$ and a G-equivariant morphism of schemes $f : X \to Y$ induces a homomorphism

$$f^* : \mathrm{Pic}^G(Y) \to \mathrm{Pic}^G(X).$$

Example 1.2.15. Consider the trivial action of an affine algebraic group G on $\mathrm{Spec}(k)$ and let $\mathcal{L}$ be the trivial line bundle over $\mathrm{Spec}(k)$. Note that for $\mathrm{Spec}(k)$ there is no other choice of action or line bundle. Then interpreting $\mathcal{L}$ geometrically it is the trivial line bundle $\pi : \mathbb{A}^1 \to \mathrm{Spec}(k)$ and $\mathrm{Pic}^G(\mathrm{Spec}(k)) \simeq Hom(G, \mathbb{G}_m)$. More concretely a character $\alpha : G \to \mathbb{G}_m$ induces an action $G \times \mathbb{A}^1 \to \mathbb{A}^1$, $(g,x) \to \alpha(g) \cdot x$.

1.2.5 General GIT quotient

We begin this section by giving a precise definition of (semi-)stability.

Definition 1.2.16. Let G be a reductive group acting on a quasi-projective scheme X and $\mathcal{L} \in \mathrm{Pic}^G(X)$. Then we say that

1. $x \in X(k)$ is semi-stable with respect to $\mathcal{L}$ if there exists an invariant section $\sigma \in \mathcal{L}^n(X)^G$ for some $n > 0$ such that $\sigma(x) \neq 0$ and X_σ is affine.

2. $x \in X(k)$ is stable with respect to $\mathcal{L}$ if $\dim G_x = 0$ and there exists an invariant section $\sigma \in \mathcal{L}^n(X)^G$ for some $n > 0$ such that $\sigma(x) \neq 0$, X_σ is affine and the action of G on X_σ is closed, i.e., all the G-orbits are closed.

We denote the stable and semi-stable locus of X with respect to $\mathcal{L}$ respectively by $X^s_{\mathcal{L}}$ and $X^{ss}_{\mathcal{L}}$, we will often omit the linearization from the notation.

Remark. Our notion of stability is called proper stability in Mumford's text on geometric invariant theory, in Mumford's text for a point to be stable it does not have to have zero-dimensional stabilizers.

Also note that when X is projective and $\mathcal{L}$ is an ample linearization, then requiring that the X_σ are affine is redundant.

Let us now state and prove one of the main results in GIT.

Theorem 1.2.17. [DK94, Thm. 1.10] *Let G/k be a linearly reductive group acting on a quasi-projective scheme X/k with respect to a linearization L. Then*

1. *There exists a good quotient $\varphi : X^{ss} \to X//_L G$.*

2. *X^{ss} and $X//_L G$ are quasi-projective.*

3. *There exists an open subset $(X//_L G)^s \subset X//_L G$ such that $\tilde{\varphi} : X^s = \varphi^{-1}((X//_L G)^s) \to (X//_L G)^s$ is a geometric quotient.*

Proof. We construct our quotient by gluing affine quotients. By definition of semi-stability with respect to L, for every $x \in X$ there exists a section

$$\sigma_i \in H^0(X, L^{\otimes r_i})^G, \text{ for some } r_i > 0$$

such that $x \in X_\sigma = \{x \in X : \sigma(x) \neq 0\}$ is affine. Hence by quasi-compactness of X we can cover X by a finite number of affines $X_{\sigma_1} \dots X_{\sigma_n}$.

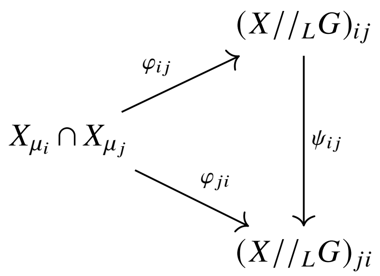
Let $m = \prod_i r_i$ and $m_i = \prod_{j \neq i} r_j$, then $\sigma_1^{m_1}, \dots, \sigma_n^{m_n} \in L^m$ and note that $X_{\sigma_i} \subset X_{\sigma_i^{m_i}}$. Denote the $\sigma_i^{m_i}$ by μ_i and consider the affine covering $X_{\mu_1}, \dots, X_{\mu_n}$. Then by Theorem 1.2.10, there exist good quotients $\varphi_i : X_{\mu_i} \to X_{\mu_i}//_L G$ for the action induced by G.

Now for every $1 \leq i, j \leq n$ consider the fractions $\mu_{ij} := \frac{\mu_j}{\mu_i}$ which induce G−invariant global sections of X_{μ_i} and therefore by construction global sections of $X_{\mu_i}//_L G$, which we will also denote by μ_{ij}. Now consider the sets $(X//_L G)_{ij} := \{y \in X_{\mu_i}//_L G : \mu_{ij}(y) \neq 0\}$. We clearly have

$$\varphi_i^{-1}((X//_L G)_{ij}) = X_{\mu_i} \cap X_{\mu_j} = \varphi_j^{-1}((X//_L G)_{ji});$$

the elements such that the sections μ_i and μ_j are non-zero are precisely the elements in the intersection.

Since good quotients are local the restrictions $\varphi_{ij} : X_{\mu_i} \cap X_{\mu_j} \to (X//_L G)_{ij}$ are good quotients. And since good quotients are in particular categorical quotients there exist unique isomorphisms ψ_{ij} filling in the diagrams



It follows from the uniqueness of these isomorphisms that they must satisfy the conditions $\psi_{ji} = \psi_{ij}^{-1}$ and the cocyle condition. We conclude that $(\{X_{\mu_i}//G\}, \{(X//G)_{ij}\}, \{\psi_{ij}\})$ defines a gluing data which forms a scheme $X//G$, moreover since good quotients are local the morphism φ determined by the φ_i is a good quotient.

It follows immediately from [New78, Lemma 3.20] that L^m defines an ample line bundle on X^{ss} and therefore that X^{ss} is quasi-projective.

Next we want to show that $(X//G)$ is quasi-projective, we do this by constructing an ample line bundle. First note that by construction the sections $\mu_{ij}|_{(X//G)_{ij}}$ form a 1-cocycle and therefore a line bundle on $X//G$. Namely, the $\mu_{ij}|_{(X//G)_{ij}}$ are units, $\mu_{ii}|_{(X//G)_{ij}} = 1$, $\mu_{ij}|_{(X//G)_{ij}} = (\mu_{ji}|_{(X//G)_{ji}})^{-1}$ and $\mu_{ji} \cdot \mu_{kj} \cdot \mu_{ik} = 1$ on $(X//G)_{ji} \cap (X//G)_{kj} \cap (X//G)_{ik}$. We denote this line bundle by $\mathcal{L}$ and claim that it is ample. We use the condition [Stacks, Tag 01PS] for ampleness.

Since the sections $\{\mu_{ij}\}_{i=1,\dots,n}$, for some fixed j define sections on the open cover $\{X_{\mu_i}//_L G\}_{i=1,\dots,n}$ satisfying $\mu_{i_1 j} = \mu_{i_2 j}\mu_{i_1 i_2}$ they induce a global section s_j on $\mathcal{L}$. For every $x \in X^{ss}$ there exists an j such that $x \in X^{ss}_{\mu_j}$ and by construction $x \in X^{ss}_{\mu_j}$ if and only if $\varphi(x) \in (X//_L G)_{t_j} = X_{\mu_j}//_L G$ which is affine. It follows that $\mathcal{L}$ is ample and therefore that $X//_L G$ is quasi-projective.

For the last item we can choose sections $f \in H^0(X, L^r)^G$ for some $r > 0$ such that the action of G on X_f is closed and $X^s \subset \bigcup_f X_f$. Let $(X//_L G)_c := \bigcup_f (X//_L G)_f$. Then we have $X_c = \varphi^{-1}((X//_L G)_c)$ and we can construct a geometric quotient $\tilde{\varphi}_c : X_c \to (X//_L G)_c$ by gluing the geometric quotients $\varphi_f : X_f \to (X//_L G)_f$. Then by semicontinuity the subset of $X^s \subset X_c$ consisting of points which have zero-dimensional stabilizer is open. We define $X/_L G := \varphi(X^s) \subset (X//_L G)_c$ and claim that it is an open subset.

Since $\tilde{\varphi}_c$ is a geometric quotient and $X^s \subset X$ is a G-invariant subset, $\tilde{\varphi}_c^{-1}(X/_L G) = X^s$ and $(X//_L G)_c \setminus X/_L G = \varphi(X_c \setminus X^s)$. Now since $X_c \setminus X^s \subset X_c$ is closed and $\tilde{\varphi}_c$ is a geometric quotient it follows that $(X//_L G)_c \setminus X/_L G \subset (X//_L G)_c$ is closed and therefore that $X/_L G \subset (X//_L G)_c \subset X//_L G$ is open. By locality of geometric quotients we conclude that that the restriction $\tilde{\varphi} := \tilde{\varphi}_c| : X^s \to X/_L G$ is a geometric quotient. □

Remark. Recall from the discussion in the previous section that for the projective case with a very ample linearization there is a morphism of the form

$$X^{ss} \to \mathrm{Proj}(\bigoplus_{n\geq 0} H^0(X, \mathcal{L}^n)^G).$$

Then one can easily show that this defines a good quotient and that $\mathrm{Proj}(\bigoplus_{n\geq 0} H^0(X, \mathcal{L}^n)^G)$ is projective by using the locality of good quotients and the fact that $\bigoplus_{n\geq 0} H^0(X, \mathcal{L}^n)^G$ is a finitely generated k-algebra; see [Hos16, Thm. 5.3].

1.2.6 Hilbert-Mumford Criterion

As has become clear in previous sections being able to determine (semi-)stable points is an important step in the construction of quotients. In this section we state and prove the Hilbert-Mumford criterion for stability, which is a numerical criterion for the stability of points. For its formulation and proof we will mostly be following Mumford's book [DK94, Chapt. 2.1].

Definition 1.2.18. Let G be an algebraic group. A 1-parameter subgroup of G, often abbreviated to 1-PS, is a homomorphism $\lambda : \mathbb{G}_m \to G$.

Consider an algebraic group G with a 1-PS λ and an action ρ on a projective scheme X. Then for every $x \in X(k)$ there is a map $\rho(-,x) \circ \lambda : \mathbb{G}_m \to X$ and since X is proper, by the valuative criterion for properness this uniquely extends to a map $f_{x,\lambda} : \mathbb{A}^1 \to X$ filling in the following diagram

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & X \\ \big\downarrow & \overset{f_{x,\lambda}}{\nearrow} & \big\downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

Now since the point $\lim_{g \to 0} \rho(-,x) \circ \lambda(g) := f_{x,\lambda}(0)$ is fixed under the induced $\mathbb{G}_m$-action, i.e., $\rho(\lambda(g), f_{x,\lambda}(0)) = f_{x,\lambda}(0)$ for all $g \in \mathbb{G}_m$, we have that $f_{x,\lambda}(0)$ corresponds to a $\mathbb{G}_m$-equivariant morphism $\mathrm{Spec}(k) \to X$ and hence for any G-linearization $(\mathcal{L}, \Psi) \in \mathrm{Pic}^G(X)$, its fiber $\mathcal{L}_{f_{x,\lambda}(0)}$ determines a $\mathbb{G}_m$-linearization on $\mathrm{Spec}(k)$ induced by the linear isomorphism $(\psi_{g, f_{x,\lambda}(0)})^{-1} : \mathcal{L}_{g \cdot f_{x,\lambda}(0)} = \mathcal{L}_{f_{x,\lambda}(0)} \to \mathcal{L}_{f_{x,\lambda}(0)}$ discussed in Remark 1.2.4 (here we take the inverse in order to stay consistent with Mumfords sign convention in what follows). By Example 1.2.15 this linearization corresponds to a character of $\mathbb{G}_m$; let d denote the integer corresponding to this character. With this we make the following definition.

Definition 1.2.19. Let G be an algebraic group with an action ρ on a finite type scheme X proper over k. Then the Hilbert-Mumford weight for a triple $x \in X(k)$, $\lambda : \mathbb{G}_m \to G$ and $\mathcal{L} \in \mathrm{Pic}^G(X)$ is denoted $\mu^L(x, \lambda)$ and is equal to $-d$ where d is the integer obtained in the previous paragraph.

We are now in the position to state the Hilbert-Mumford criterion.

Theorem 1.2.20. [DK94, Thm. 2.1] *Let G be a reductive group acting on a projective scheme X/k and $\mathcal{L} \in Pic^G(X)$ an ample linearization. Then for $x \in X(k)$ we have that*

$$x \in X^{ss}_{\mathcal{L}} \iff \mu^{\mathcal{L}}(x, \lambda) \geq 0 \, \textit{for all } \lambda \in Hom(\mathbb{G}_m, G),$$

$$x \in X^{s}_{\mathcal{L}} \iff \mu^{\mathcal{L}}(x, \lambda) > 0 \, \textit{for all } \lambda \in Hom(\mathbb{G}_m, G).$$

Remark. Since $\mu^{\mathcal{L}^n}(x,\lambda) = n\mu^{\mathcal{L}}(x,\lambda)$, by the functorial properties of μ [DK94, p. 49], and $\mathcal{L}$ is ample there exists some positive integer n such that $\mathcal{L}^n$ is very ample, thus we may assume without loss of generality that $\mathcal{L}$ is very ample and therefore induces a G-equivariant embedding $i : X \to \mathbb{P}^{n-1}$. With this assumption we have that $\mathcal{L} \simeq i^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)$, so that

$$\mu^{\mathcal{L}}(x,\lambda) = \mu^{i^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)}(x,\lambda) = \mu^{\mathcal{O}_{\mathbb{P}^{n-1}}(1)}(i(x),\lambda).$$

Now since $X^{s(s)}_{i^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)} = i^{-1}((\mathbb{P}^{n-1})^{s(s)}_{\mathcal{O}_{\mathbb{P}^{n-1}}(1)}$ by the functorial properties given in [DK94, Chapt. 1.5], it is sufficient to prove the Hilbert-Mumford criterion for $X = \mathbb{P}^{n-1}$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

Set $X = \mathbb{P}^{n-1}$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Consider the natural projection from the complement of the origin in the affine cone of $\mathbb{P}^{n-1}$, i.e., $p : \mathbb{A}^n \setminus \{0\} \to \mathbb{P}^{n-1}$, then a point in the affine cone $\tilde{x} \in \mathbb{A}^n(k)$ is said to lie over a point $x \in \mathbb{P}^{n-1}(k)$ if $\tilde{x} \neq 0$ and $p(\tilde{x}) = x$. We also note that the action of G on $\mathbb{P}^{n-1}$ naturally induces a linear action which we will denote by $\tilde{\rho}$ on the affine cone. The action of G on $\mathbb{P}(H^0(\mathbb{P}^{n-1},\mathcal{O}_{\mathbb{P}^{n-1}}(1)))$ with a G-linearization of $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ corresponds to a co-module structure on $H^0(\mathbb{P}^{n-1},\mathcal{O}_{\mathbb{P}^{n-1}}(1)))$ given by the composition

$$H^0(X,\mathcal{L}) \xrightarrow{\rho^\#} H^0(G\times X,\rho^*\mathcal{L}) \xrightarrow{\Phi} H^0(G\times X,p_2^*\mathcal{L}) \simeq H^0(G,\mathcal{O}_G)\otimes H^0(X,\mathcal{L})$$

where the last map is given by the Künneth formula, equivalently if for a section $\sigma \in H^0(X,\mathcal{L})$ its image is given by $\sum_i s_i \otimes \sigma_i \in H^0(G,\mathcal{O}_G)\otimes H^0(X,\mathcal{L})$ we can define a linear representation by $G \to GL(H^0(X,\mathcal{L})),\ g \mapsto (\sigma \mapsto \sum_i s_i(g)\sigma_i)$.

Proposition 1.2.21. *Let $x \in X(k)$. Then*

1. $x \in X^{ss} \iff \exists \tilde{x} \in \mathbb{A}^n$ *that lies over x such that* $0 \notin \overline{G\cdot\tilde{x}}$.
2. $x \in X^{s} \iff \exists \tilde{x} \in \mathbb{A}^n$ *that lies over x such that* $\tilde{\rho}(-,\tilde{x}) : G \to \mathbb{A}^n$ *is proper.*

Proof. ($1, \Longrightarrow$) By definition $x \in X^{ss}$ if and only if there exists an integer $d > 0$ and an invariant section $\sigma \in \mathcal{O}_{\mathbb{P}^{n-1}}(d)(\mathbb{P}^{n-1})^G$ such that $\sigma(x) \neq 0$. This translates to a G-invariant homogeneous polynomial function f of degree d on the affine cone such that $f(\tilde{x}) \neq 0$ for all non-zero lifts of x. Since invariant functions are constant on orbits and their closures, f separates $\overline{G\cdot\tilde{x}}$ and $\{0\}$, hence it follows that $\overline{G\cdot\tilde{x}} \cap \{0\} = \emptyset$.

($1,\Longleftarrow$) For the second implication suppose that $\overline{G\cdot\tilde{x}} \cap \{0\} = \emptyset$. Then since G is reductive by Lemma 1.2.7 there exists a G-invariant global section of the affine cone which separates the two closed sets, i.e., there exists an polynomial $f \in \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n)$ such that $f(\overline{G\cdot\tilde{x}}) = 1$ and $f(0) = 0$. We can decompose f into its homogeneous components so that $f = f_1 + \ldots + f_r$ and each f_i is a homogeneous polynomial of degree $d_i > 0$. In particular there must exists an i such $f_i(\overline{G \cdot \tilde{x}}) \neq 0$, this defines a section in $\mathcal{O}_{\mathbb{P}^{n-1}}(d_i)(\mathbb{P}^{n-1})^G$ which does not vanish at x, thus x is semi-stable.

(2) By definition a point $x \in \mathbb{P}^{n-1}(k)$ is stable with respect to $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ if and only if $\dim G_x = 0$ and there exists an integer $d > 0$ and a G-invariant section $\sigma \in \mathcal{O}_{\mathbb{P}^{n-1}}(d)(\mathbb{P}^{n-1})$ such that $\sigma(x) \neq 0$ and the action of G on $(\mathbb{P}^{n-1})_\sigma$ is closed. This however is equivalent to saying that the map $\rho_x^\sigma : G \to (\mathbb{P}^{n-1})_\sigma$ in the factorization

$$\begin{array}{ccc} G & \xrightarrow{\rho_x^\sigma} & (\mathbb{P}^{n-1})_\sigma \\ & \underset{\rho_x}{\searrow} & \downarrow \\ & & \mathbb{P}^{n-1} \end{array}$$

is proper. With this equivalence to show that (2) holds it suffices to prove that ρ_x^σ is proper if and only if $\tilde{\rho}_{\tilde{x}} := \tilde{\rho}(-, \tilde{x})$ is proper. As before let f denote the homogeneous polynomial function of degree d on the affine cone corresponding to σ. The orbit $G \cdot \tilde{x}$ is contained in the closed subscheme $\mathbb{V}(f - f(\tilde{x})) \subset \mathbb{A}^n$. It follows that we have a factorization

$$\begin{array}{ccc} G & \xrightarrow{\tilde{\rho}_{f,\tilde{x}}} & \mathbb{V}(f - f(\tilde{x})) \\ & \underset{\rho_\sigma^x}{\searrow} & \downarrow \\ & & (\mathbb{P}^{n-1})_\sigma \end{array}$$

where the vertical arrow is induced by the projection $p : \mathbb{A}^n \setminus \{0\} \to \mathbb{P}^{n-1}$, note that this is well-defined since $p(\mathbb{V}(f - f(\tilde{x}))) \subset (\mathbb{P}^{n-1})_\sigma$.

Since the vertical map is proper we can conclude by basic facts about proper morphisms that the composition ρ_σ^x is proper if and only if the map $\tilde{\rho}_{f,\tilde{x}}$ is proper if and only if the map $\tilde{\rho}_{\tilde{x}}$ is proper. □

Now for a $1-PS$ $\lambda : \mathbb{G}_m \to G$ we have an induced $\mathbb{G}_m$-action on the affine cone, which can be diagonalized so that for a suitable choice of basis $e_0, \ldots, e_{n-1}$ for $H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ the action of $\in \mathbb{G}_m$ on $\mathbb{A}^n$ is given by a diagonal matrices $\mathrm{diag}(g^{r_0}, \ldots, g^{r_{n-1}})$ for fixed $r_0, \ldots, r_{n-1} \in \mathbb{Z}$, i.e., $g \cdot e_i = \lambda(g) \cdot e_i = g^{r_i} e_i$ for $i = 0, \ldots, n-1$.

Proposition 1.2.22. *Let $x \in \mathbb{P}^{n-1}$ and $\lambda : \mathbb{G}_m \to G$ a 1-PS. With the above setup we can choose a point $\tilde{x} = (\tilde{x}_0, \ldots, \tilde{x}_{n-1})$ lying over x such that with respect to a suitable basis $\{e_0, \ldots, e_{n-1}\}$,*

$$\tilde{\rho}(\lambda(g),\tilde{x}) = \sum_{i=0}^{n-1} g^{r_i}\tilde{x}_i e_i.$$

Then

$$\mu^{\mathcal{O}_{\mathbb{P}^{n-1}}(1)}(x,\lambda) = \max_{0\leq i\leq n-1}\{-r_i : \tilde{x}_i \neq 0\}$$

Proof. Write $\mu := \max_{0\leq i\leq n-1}\{-r_i : \tilde{x}_i \neq 0\}$. Since $\tilde{\rho}(\lambda(g),x) = (g^{r_0}\tilde{x}_0,\dots,g^{r_{n-1}}\tilde{x}_{n-1})$ we have that $\lim_{g\to 0} g^{\mu}\lambda(g)\cdot\tilde{x} = \tilde{f}_{g^{\mu}\tilde{x},\lambda}(0)$ exists and is non-zero. Furthermore, since the map $p : \mathbb{A}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is equivariant, $\tilde{f}_{g^{\mu}\tilde{x},\lambda}(0)$ lies over the specialization $f_{x,\lambda}(0)$.

Now if we write

$$\tilde{f}_{g^{\mu}\tilde{x},\lambda}(0) = \lim_{g\to 0}(g^{r_0+\mu}\tilde{x}_0,\cdots,g^{r_{n-1}+\mu}\tilde{x}_0) = (s_0,\cdots,s_{n-1})$$

we clearly have that $s_i = 0$ only if $\tilde{x}_i = 0$ or $r_i + \mu \geq 0$, it follows that

$$\tilde{\rho}(\lambda(g),\tilde{f}_{g^{\mu}\tilde{x},\lambda}(0)) = g^{-\mu}\tilde{f}_{g^{\mu}\tilde{x},\lambda}(0).$$

Since the fibers of the tautological bundle correspond to lines in affine space, the torus action on $\mathcal{O}_{\mathbb{P}^{n-1}}(1)_{f_{x,\lambda}(0)}$, induced by the action on the fibre $p^{-1}(f_{x,\lambda}(0))$, is given by $\lambda(g)\cdot s = g^{\mu}s$, thus we can conclude that $\mu^{\mathcal{O}_{\mathbb{P}^{n-1}}(1)}(x,\lambda) = \mu$. □

Corollary 1.2.23. *In the situation of the above proposition:*

1. $\lim_{g\to 0}\tilde{\rho}(\lambda(g),\tilde{x}) = 0 \iff \mu < 0$.
2. $\lim_{g\to 0}\tilde{\rho}(\lambda(g),\tilde{x})$ *exists and is non-zero* $\iff \mu = 0$.
3. $\lim_{g\to 0}\tilde{\rho}(\lambda(g),\tilde{x})$ *does not exist* $\iff \mu > 0$.

Proposition 1.2.24. *With the notations as before:*

1. $0 \in \overline{G\cdot\tilde{x}} \iff$ *there exists a 1-PS* $\lambda : \mathbb{G}_m \to G$ *such that* $\lim_{g\to 0}\tilde{\rho}(\lambda(g),\tilde{x}) = 0$.
2. $\tilde{\rho}_{\tilde{x}}$ *is not proper* $\iff$ *there exists a 1-PS* $\lambda : \mathbb{G}_m \to G$, *such that* $\lim_{g\to 0}\tilde{\rho}(\lambda(g),\tilde{x})$ *exists.*

In order to prove the above proposition we need a result which requires some preparation. Recall that we work over an algebraically closed field k, let $R := k[\![t]\!]$ denote the formal power series ring and $K := k((t))$ its field of fractions and note that $G(R)$ is a subgroup of $G(K)$ via the inclusion $G(R) \hookrightarrow G(K)$ induced by the natural morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$. We also have a mapping $\omega : G(R) \to G(k)$ induced by the natural morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$, and for every 1-PS $\lambda : \mathbb{G}_m \to G$ we have an induced K-point $\langle\lambda\rangle \in G(K)$ given by

$$\langle\lambda\rangle := \mathrm{Spec}(K) \xrightarrow{\varphi} \mathbb{G}_m \xrightarrow{\lambda} G$$

where φ is induced by the map $k[u, u^{-1}] \to k((t))$, $u \mapsto t$. With this notation we are in the position to formulate the result.

Theorem 1.2.25. [DK94, p. 52. Thm. Iwahori] *Let G/k be a reductive group over an algebraically closed field then for any $g \in G(K)$ there exist $h, h' \in G(R)$ and a homomorphism $\lambda : \mathbb{G}_m \to G$ such that*

$$g = h_1 \cdot \langle\lambda\rangle \cdot h_2.$$

Remark. These decompositions are called Cartan-Iwahori-Matsumoto decompositions. We will see in chapter four that the existence of such decompositions is characterized by a stack theoretic condition called **S**-completeness and moreover that it characterizes reductivity.

Now we are ready to prove Proposition 1.2.24.

Proof. ($1, \Longleftarrow$) If there exists a 1-PS λ such that $\lim\limits_{g\to 0} \tilde{\rho}(\lambda(g), \tilde{x}) = 0$ then clearly $0 \in \overline{G \cdot \tilde{x}}$.

($1, \Longrightarrow$) Claim: There exists a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\eta} & G \\ \downarrow & & \downarrow{\scriptstyle \tilde{\rho}_{\tilde{x}}} \\ \mathrm{Spec}(R) & \xrightarrow{\tilde{\eta}} & \mathbb{A}^n \end{array}$$

such that $\eta \in G(K) \setminus G(R)$ and $\lim\limits_{t\to 0} \tilde{\rho}(\eta(t), \tilde{x}) = \tilde{\eta}((t)) = 0$.Then using Theorem 1.2.25 we are able to write

$$\eta = h_1 \cdot \langle\lambda\rangle \cdot h_2$$

for some 1-PS λ and a pair $h_1, h_2 \in G(R)$, note that λ must be non-trivial, as η is not induced by an R-valued point.

Now let $g_i := \omega(h_i)$, then we may choose a basis for the affine cone $\mathbb{A}^n$, with respect to which $g_2^{-1}\lambda g_2$ acts on the affine cone by a diagonal matrix. For $g \in \mathbb{G}_m(k)$ we then have $g_2^{-1}\lambda(g)g_2 = \text{diag}(g^{r_0},\ldots,g^{r_{n-1}})$ for some $r_0,\ldots,r_{n-1} \in \mathbb{Z}$. It follows that

$$\tilde{\rho}_{\tilde{x}} \circ \eta = \rho(\eta,\tilde{x}) = \tilde{\rho}((h_1 \cdot g_2)\cdot(g_2^{-1}\cdot\langle\lambda\rangle\cdot g_2)\cdot(g_2^{-1}\cdot h_2),\tilde{x}),$$

this makes sense if we use the natural identification of the k-valued points $\tilde{x}$ and g_i, with K-valued points given by $k \subset K$.

Using the fact that $\tilde{\rho}$ is a group action we obtain

$$\tilde{\rho}((h_1\cdot g_2)^{-1},\tilde{\rho}_{\tilde{x}}\circ\eta) = \tilde{\rho}((h_1\cdot g_2)^{-1},\tilde{\rho}((h_1\cdot g_2)\cdot(g_2^{-1}\cdot\langle\lambda\rangle\cdot g_2)\cdot(g_2^{-1}\cdot h_2),\tilde{x})$$

$$= \tilde{\rho}(g_2^{-1}\cdot\langle\lambda\rangle\cdot g_2,\tilde{\rho}(g_2^{-1}\cdot h_2,\tilde{x})),$$

note that the term $\tilde{\rho}((h_1\cdot g_2)^{-1},\tilde{\rho}_{\tilde{x}}\circ\eta)$ is an R-valued point of $\mathbb{A}^n$ as $\tilde{\rho}_{\tilde{x}}\circ\eta \in \mathbb{A}^n(R)$ and g_2 can be viewed a K-valued point of G contained in the subgroup of R-valued points hence $(h_1\cdot g_2)^{-1} \in G(R)$. Now as there is an equivalence of ring maps and maps of affine schemes an R-valued point of $\mathbb{A}^n$ can be identified with a point in R^n given by the images of the coordinates for $\mathbb{A}^n$ under the corresponding ring map. Let us denote the i^{th} coordinate corresponding to the R-valued point $\tilde{\rho}((h_1\cdot g_2)^{-1},\tilde{\rho}_{\tilde{x}}\circ\eta)$ by $(\tilde{\rho}((h_1\cdot g_2)^{-1},\tilde{\rho}_{\tilde{x}}\circ\eta))_i$.

Then in the coordinate setting we get

$$(\tilde{\rho}((h_1\cdot g_2)^{-1},\tilde{\rho}_{\tilde{x}}\circ\eta))_i = (\tilde{\rho}(g_2^{-1}\cdot\langle\lambda\rangle\cdot g_2,\tilde{\rho}(g_2^{-1}\cdot h_2,\tilde{x})))_i$$

$$= (\tilde{\rho}(g_2^{-1}\cdot(\lambda\circ\varphi)\cdot g_2,\tilde{\rho}(g_2^{-1}\cdot h_2,\tilde{x})))_i = t^{r_i}(\tilde{\rho}(g_2^{-1}\cdot h_2,\tilde{x}))_i.$$

Now note that since $\lim_{t\to 0}\tilde{\rho}((h_1\cdot g_2)^{-1},\tilde{\rho}_{\tilde{x}}\circ\eta)(t) = \tilde{\rho}((h_1\cdot g_2)^{-1},0) = 0$ the formal power series corresponding to the coordinate $(\tilde{\rho}((h_1\cdot g_2)^{-1},\tilde{\rho}_{\tilde{x}}\circ\eta))_i$ is contained in tR and hence $(\tilde{\rho}(g_2^{-1}\cdot h_2,\tilde{x}))_i$ is contained in $t^{1-r_i}R$.

Since $g_2 = \lim_{t\to 0}h_2(t)$ we have that $\lim_{t\to 0}g_2^{-1}\cdot h_2(t) = e$ where e is the identity element in G so we have that $\lim_{t\to 0}\tilde{\rho}(g_2^{-1}\cdot h_2,\tilde{x}) = \tilde{x}$ so that the i^{th} coordinate is

$$(\tilde{\rho}(g_2^{-1}\cdot h_2,\tilde{x}))_i = (\tilde{x})_i + tf_i$$

for some $f_i \in R$. Now note that if $(\tilde{x})_i \neq 0$ we have

$$(\tilde{\rho}(g_2^{-1}\cdot h_2,\tilde{x}))_i = (\tilde{x})_i + tf_i$$

at the same time we know that

$$(\tilde{\rho}(g_2^{-1} \cdot h_2, \tilde{x}))_i = t^{1-r_i} p$$

for some $p \in R$, thus $p = t^{r_i - 1}(\tilde{x})_i + t^{r_i} f_i$, and we must have that $r_i > 0$ if $(\tilde{x})_i \neq 0$. Using the characterization of the Hilbert-Mumford index given in Proposition 1.2.22 we conclude that the Hilbert-Mumford index corresponding to the 1-PS $g_2^{-1} \cdot \lambda \cdot g_2$ is strictly negative and therefore by Corollary 1.2.23 that

$$\lim_{g \to 0} \tilde{\rho}((g_2^{-1} \lambda g_2)(g), \tilde{x}) = 0.$$

The proof for statement (2) is very similar; we want to note however that for the first part of the proof for the ($\Longrightarrow$) direction we can argue that there exist points η and $\tilde{\eta}$ which make the following diagram commute

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\eta} & G \\ \downarrow & & \downarrow{\scriptstyle \tilde{\rho}_{\tilde{x}}} \\ \mathrm{Spec}(R) & \xrightarrow{\tilde{\eta}} & \mathbb{A}^n \end{array}$$

such that $\eta \in G(K) \setminus G(R)$ by using the fact that morphisms of affine schemes are separated, hence in particular $\tilde{\rho}_{\tilde{x}}$ is separated, so that by using the valuative criteria for properness and separatedness we can argue that since $\tilde{\rho}_{\tilde{x}}$ is separated but not proper there must be points η and $\tilde{\eta}$ which fit into the above diagram such that η does not factor through $\mathrm{Spec}(R)$. □

We conclude this chapter with two examples.

Example 1.2.26. Consider the action of the multiplicative group $\mathbb{G}_m$ on the projective space $X := \mathbb{P}^3$ induced by the action on the affine cone $\mathbb{A}^4$ as in Example 1.2.13, i.e., $s \cdot [x_1 : x_2 : x_3 : x_4] = [s^2 x_1 : s^{-3} x_2 : s x_3 : s^{-1} x_4]$. Recall that we can characterize stability (resp. semi-stability) of a point $x \in \mathbb{P}^3$ by the non-existence (resp. non-vanishing) of the limits

$$\lim_{s \to 0} \lambda(s) \cdot \tilde{x},$$

where $\tilde{x} \in \mathbb{A}^4$ is a point lying over x and the λ are the 1-PS's. Note that since the 1-PS $\mathbb{G}_m \to \mathbb{G}_m$ are of the form $s \mapsto s^n$ for $n \in \mathbb{Z}$ we only have to consider the 1-PS's given by $s \mapsto s$ and $s \mapsto s^{-1}$. We have

$$\lim_{s \to 0} (s^2 x_1, s^{-3} x_2, s x_3, s^{-1} x_4), \quad \text{for} \quad s \mapsto s, \quad \text{and}$$

$$\lim_{s \to 0} (s^{-2} x_1, s^3 x_2, s^{-1} x_3, s x_4), \quad \text{for} \quad s \mapsto s^{-1},$$

both limits do not exist only in the following situation: $x_1x_2 \neq 0,$ or $x_1x_4 \neq 0,$ or $x_2x_3 \neq 0,$ or $x_3x_4 \neq 0$. It follows that

$$X^{ss} = X^s = X_{x_1x_2} \cup X_{x_1x_4} \cup X_{x_2x_3} \cup X_{x_3x_4}.$$

Example 1.2.27. Consider the action of $\mathbb{G}_m$ on $X := \mathbb{P}^2$ induced by the action on the affine cone $\mathbb{A}^3$ given by $s \cdot (x_1, x_2, x_3) = (sx_1, x_2, s^{-1}x_3)$. Following the same reasoning as in the previous example, to determine wether a point is (semi-)stable we have to check for the non-existence and non-vanishing of the limits

$$\lim_{s\to 0}(sx_1, x_2, s^{-1}x_3), \quad \lim_{s\to 0}(s^{-1}x_1, x_2, sx_3).$$

In this case the limits exist and are non-zero only for the points $(0, x_2, 0) \neq 0$ and do not exist if $x_1x_3 \neq 0$, thus we have that

$$X^s = X_{x_1x_3}, \quad X^{ss} = X^s \cup \{[0:1:0]\}.$$

Chapter 2

Algebraic stacks

In the first chapter we have seen that moduli problems often do not admit a fine moduli space in the category of schemes and gave an explicit example of a possible obstruction. The deeper rooted problem which we did not mention then is that many moduli problems have non-trivial automorphisms, roughly speaking this prevents the moduli problem from being a sheaf with respect to a certain notion of topology and therefore in particular cannot be represented by a scheme. We also mentioned that one way to work around this is to ask for representability in some larger category, introducing these larger categories and their geometric structure is what this next chapter is about.

2.1 Basics of stack theory

2.1.1 Prestacks

To start this chapter off we introduce the notion of a prestack, also called categories fibered in groupoids, the reason for this will soon become clear. Essentially a prestack allows a moduli problem to take values in the category of groupoids, which is a category in which all morphisms are isomorphisms. In some sense the notion of a prestack generalizes the notion of a presheaf and with regards to moduli problems it allows us to keep track of automorphisms. Our goal for this section is to make the notion of a prestack precise and give some basic results, constructions and definitions for prestacks.

For a pair of categories $\mathcal{C}$ and $\mathcal{S}$ we say that $\mathcal{C}$ is a category over $\mathcal{S}$ if $\mathcal{C}$ is equipped with a functor $p : \mathcal{C} \to \mathcal{S}$. For a pair of objects $u, v \in \mathcal{C}$, with an arrow $\varphi : u \to v$ we use the notation

$$\begin{array}{ccc} u & \xrightarrow{\varphi} & v \\ \downarrow & & \downarrow \\ p(u) & \xrightarrow{p(\varphi)} & p(v) \end{array}$$

and say that u and v are over $p(u)$ and $p(v)$ and that φ is over $p(\varphi)$.

Definition 2.1.1. A category $\mathcal{X}$ over a category $\mathcal{S}$ equipped with a functor $p : \mathcal{X} \to \mathcal{S}$ is called a prestack over $\mathcal{S}$ if

1. For every object u in $\mathcal{X}$ and object S in $\mathcal{S}$ with an arrow $S \xrightarrow{f} p(u)$ in $\mathcal{S}$ there exists an object over S, which we call a pullback and denote by f^*u, and an arrow $f^*u \to u$ over f filling in the diagram

$$\begin{array}{ccc} f^*u & \overset{\varphi}{\dashrightarrow} & u \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & p(u) \end{array}$$

2. For every triple of objects u, v and w in $\mathcal{X}$ and arrows $u \xrightarrow{\psi} w$, $v \xrightarrow{\eta} w$ and $p(u) \xrightarrow{f} p(v)$ there exists a unique arrow $u \to v$ over f filling in the diagram

$$\begin{array}{ccccc} & & \overset{\psi}{\frown} & & \\ u & \overset{\varphi}{\dashrightarrow} & v & \xrightarrow{\eta} & w \\ \downarrow & & \downarrow & & \downarrow \\ p(u) & \xrightarrow{f} & p(v) & \xrightarrow{p(\eta)} & p(w) \end{array}$$

Remark. A pullback of an object $u \in \mathcal{X}(S)$ by a morphism $f : S' \to S$ is sometimes also denoted $u|_{S'}$.

Definition 2.1.2. If $\mathcal{X}$ is a prestack over $\mathcal{S}$, the fiber category over an object S in $\mathcal{S}$, denoted $\mathcal{X}(S)$ is the category consisting of objects in $\mathcal{X}$ over S and morphisms in $\mathcal{X}$ over the identity id_S in $\mathcal{S}$.

As alluded to at the start of this section the terminology: category fibered in groupoids, is natural, this is the case because the fibered categories of prestacks are groupoids. This follows easily from the second axiom in the definition of a prestack because it ensures the existence of an inverse for every morphism.

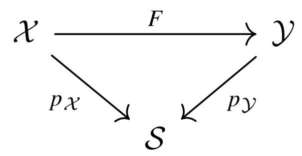
Next we will give our first example of a prestack. We will see that we can take existing moduli problems and identify them with a prestack, moreover we can do this for any presheaf and in particular for any scheme.

Example 2.1.3. Let $\mathcal{S}$ denote some category. To every presheaf $F : \mathcal{S} \to \text{Set}$ we can associate a prestack which we will denote by $\mathcal{X}_F$ over $\mathcal{S}$ in the following way: let $S \in \mathcal{S}$ then an object $(s, S) \in \mathcal{X}_F$ over S is given by an object $s \in F(S)$. Note that $\mathcal{X}_F$ defines a category over $\mathcal{S}$ by sending an object (s, S) to the object $S \in \mathcal{S}$.

A morphism $(s', S') \to (s, S)$ is given by a $f : S' \to S$ such that $F(f)(s) = s'$. In particular by taking the functorial point of view for schemes we can associate to every scheme a prestack. In what follows we will often conflate schemes and presheaves with their associated prestacks.

Now that we have defined prestacks and have given an example, we want to give some basic definitions and constructions, starting with morphisms of prestacks.

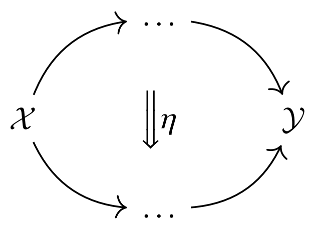
Definition 2.1.4. Let $p_{\mathcal{X}} : \mathcal{X} \to \mathcal{S}$ and $p_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{S}$ be a pair of prestacks over a site $\mathcal{S}$. A morphism of prestacks $F : \mathcal{X} \to \mathcal{Y}$ is a functor such that the following triangle commutes



In addition to morphisms between prestack we can also define certain morphisms between morphisms of prestacks, which will allow us to define the category of morphisms of prestacks.

Definition 2.1.5. Let $F, G : \mathcal{X} \to \mathcal{Y}$ be morphims of prestacks. A natural transformation $\eta : F \to G$ is called a 2-isomorphism if for every object $x \in \mathcal{X}(T)$ the morphism $\eta_x : F(x) \to G(x)$ is contained in the fiber category $\mathcal{Y}(T)$.

We say that a diagram of prestacks



is 2-commutative if the compositions are 2-isomorphic, where $\overset{\eta}{\Longrightarrow}$ is to indicate the 2-isomorphism, we often leave this out of the notation and say commutative instead of

2-commutative when there is no risk of confusion.

For a pair of prestacks $\mathcal{X}$ and $\mathcal{Y}$ we let $\text{Mor}(\mathcal{X}, \mathcal{Y})$ denote the category with objects morphisms of prestacks $\mathcal{X} \to \mathcal{Y}$ and its morphisms 2-isomorphisms.

We say that a morphism of prestacks $f : \mathcal{X} \to \mathcal{Y}$ is a monomorphism if it is fully faithfull and an epimorphism if it is essentially surjective. We say that f is an isomorphism if it is a monomorphism and epimorphism, or equivalently if there exists a morphism $g : \mathcal{Y} \to \mathcal{X}$ and 2-isomorphisms $f \circ g \simeq \text{id}_{\mathcal{Y}}$ and $g \circ f \simeq \text{id}_{\mathcal{X}}$. Moreover, fully faithfullness of f can be checked on fiber categories so that f is an isomorphism if and only if it is an epimorphism and for all $S \in \mathcal{S}$, $f_S : \mathcal{X}(S) \to \mathcal{Y}(S)$ is fully faithfull.

With the above definitions we can give a version of the Yoneda lemma which allows us to identify the fiber categories over objects with the category of morphisms of prestacks given in the previous paragraph. This version of the Yoneda lemma is called the 2-Yoneda lemma due to the fact that it is a 2-categorical analog for the usual result.

Lemma 2.1.6. [Alp24, Lemma 2.4.21] *For a prestack $\mathcal{X} \to \mathcal{S}$ and an object $S \in \mathcal{S}$ there is an equivalence of categories*

$$Mor(S, \mathcal{X}) \to \mathcal{X}(S), \quad F \mapsto F_S(\text{id}_S)$$

Remark. For the remainder of this text we will often not differentiate between objects of prestacks and isomorphisms between objects and their corresponding morphisms and 2-isomorphisms.

An important construction is that of the fiber product we will introduce this next as well as a notion of cartesian diagrams.

Definition 2.1.7. Let $f : \mathcal{X} \to \mathcal{Z}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be a pair of morphisms of prestacks over a category $\mathcal{S}$. Then the 2-fiber product associated to f and g is the prestack over $\mathcal{S}$ denoted $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ consisting of

1. objects which are triples (x, y, α), where $x \in \mathcal{X}(S)$ and $y \in \mathcal{Y}(S)$ are objects over the same object $S \in \mathcal{S}$ and $\alpha : f(x) \to g(y)$ is an isomorphism in $\mathcal{Z}(S)$.
2. For $(x', y', \alpha') \in \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}(S')$ and $(x, y, \alpha) \in \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}(S)$ a morphism $(x', y', \alpha') \to (x, y, \alpha)$ is given by a triple (h, φ, η) where $h : S' \to S$ is a morphism in $\mathcal{S}$, $\varphi : x' \to x$ is a morphism over h in $\mathcal{X}$ and $\eta : y' \to y$ is a morphism in $\mathcal{Y}$ over h such that the following diagram commutes

$$\begin{array}{ccc} f(x') & \xrightarrow{f(\varphi)} & f(x) \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ g(y') & \xrightarrow{g(\eta)} & g(y). \end{array}$$

The fiber category of the prestack $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ over an object $S \in \mathcal{S}$ is the fiber product of groupoids $\mathcal{X}(S) \times_{\mathcal{Z}(S)} \mathcal{Y}(S)$.

Roughly, a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{p} & \mathcal{Y} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Z} \end{array}$$

is called cartesian, or 2-cartesian, if it satisfies the usual universal property for fiber products with statements about commutativity replaced by 2-commutativity, see [Alp24, Thm. 2.4.35, Def. 2.4.36] for a precise definition. We will often place a small square in the center of a diagram to indicate that it is cartesian, as follows

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{p} & \mathcal{Y} \\ \downarrow{\scriptstyle q} & \square & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Z} \end{array}$$

A fiber product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ together with the projections $p_1 : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X},\ (x, y, \alpha) \mapsto x$ and $p_2 : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}, (x, y, \alpha) \to y$ gives rise to a 2-cartesian diagram, see [Alp24, p. 93]. One can analogously define the notion of a cocartesian diagram.

We will conclude this section with two important examples of cartesian diagrams of prestacks.

Example 2.1.8. [Stacks, Tag 04Z1] Our first example is a generalization of the magic diagram in algebraic geometry. Namely for a prestack $\mathcal{X} \to \mathcal{S}$ over a category $\mathcal{S}$ and objects $x_1 \in \mathcal{X}(X_1)$ and $x_2 \in \mathcal{X}(X_2)$ there is a cartesian diagram

$$\begin{array}{ccc} X_1 \times_{\mathcal{X}} X_2 & \longrightarrow & X_1 \times X_2 \\ \downarrow & \square & \downarrow{\scriptstyle x_1 \times x_2} \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X} \end{array}$$

where Δ is the diagonal and x_1, x_2 are the morphisms given by the 2-Yoneda lemma.

For our second example we first must give a definition.

Definition 2.1.9. Let $\mathcal{X} \to \mathcal{S}$ be a prestack over $\mathcal{S}$. Then for an object $S \in \mathcal{S}$ and a pair of objects $x, y \in \mathcal{X}(S)$ we define the presheaf

$$\mathrm{Isom}_{\mathcal{X}(S)}(x,y) : (\mathcal{S}/S)^{op} \to \mathrm{Set}$$

$$(f : T \to S) \to \mathrm{Mor}_{\mathcal{X}(T)}(f^*x, f^*y),$$

where we make a choice of pullbacks f^*x and f^*y. When $x = y$ we also write $\mathrm{Aut}_{\mathcal{X}(S)}(x,x) := \mathrm{Isom}_{\mathcal{X}(S)}(x,x)$. For a description which includes its behaviour on morphisms and a verification that it is indeed a presheaf, see [Stacks, Tag 02Z9].

Example 2.1.10. [Stacks, Tag 04SI] For a prestack $\mathcal{X}$ and a pair of objects $x, y \in \mathcal{X}(S)$ there is a cartesian diagram

$$\begin{array}{ccc} \mathrm{Isom}_{\mathcal{X}(S)}(x,y) & \longrightarrow & S \\ \downarrow & \square & \downarrow{\scriptstyle (x,y)} \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X} \end{array}$$

2.1.2 Sites and sheaves

In our first step towards adding geometric structure to prestacks we need to give a notion of topology on categories and a reasonable notion of a sheaf for these topologies. The notion that gives this generalization of a topological space is that of a Grothendieck topology, we start this section by defining this notion, then we give some examples and give the correct notion of a sheaf.

Definition 2.1.11. Let $\mathcal{C}$ be a category. A Grothendieck topology on $\mathcal{C}$ is given by the following data: for every object $C \in \mathcal{C}$, there is a collection $\mathrm{Cov}(C)$ consisting of sets of morphisms $\{C_i \to C\}_{i \in I}$ in $\mathcal{C}$ called coverings that satisfy the following conditions.

1. (**identity**) If $U \to C$ is an isomorphism, then $\{U \to C\} \in \mathrm{Cov}(C)$.

2. (**restriction**) If $\{C_i \to C\}_i \in \mathrm{Cov}(C)$ and $U \to C$ is a morphism, then the fiber products $C_i \times_C U$ exist in $\mathcal{C}$ and $\{C_i \times_C U \to U\}_{i \in I} \in \mathrm{Cov}(U)$.

3. (**composition**) If $\{C_i \to C\}_{i \in I} \in \mathrm{Cov}(C)$ and for every $i \in I$ there is a covering $\{U_{ij} \to C_i\}_{j \in J_i}$, then $\{U_{ij} \to C_i \to C\}_{i \in I, j \in J_i} \in \mathrm{Cov}(C)$.

A category with a Grothendieck topology is called a site.

We will now give the examples that are most relevant to this text.

Example 2.1.12. Denote the category of schemes by Sch and the category of schemes over some scheme S by Sch/S.

Let X be a scheme. The small étale site on X denoted $X_{ét}$ is the full subcategory of Sch/X whose objects are schemes étale over X with coverings given by collections of étale morphisms $\{U_i \to U\}_{i\in I}$ such that $\coprod_{i\in I} U_i \to U$ is surjective.

The big étale site denoted $\mathrm{Sch}_{ét}$ is the category Sch together with the Grothendieck topology given by coverings $\{X_i \to X\}_{i\in I}$ which consist of étale morphisms such that $\coprod_i X_i \to X$ is surjective.

Replacing étale by open immersions or faithfully flat and locally of finite presentation gives us respectively the notion of the big Zariski site denoted Sch_{Zar} and that of the big fppf site denoted Sch_{fppf}. One can analogously construct the sites $(\mathrm{Sch}/S)_{ét}$, $(\mathrm{Sch}/S)_{Zar}$, $(\mathrm{Sch}/S)_{fppf}$ by replacing schemes by S-schemes and morphisms by S-morphisms.

Another example arises when we take some scheme X and the full subcategory of Sch/X whose objects are schemes smooth over X, denoted $\text{Lis-ÉT}(X)$. Then the lisse-étale site is the category $\text{Lis-ÉT}(X)$ together with the Grothendieck topology given by coverings $\{U_i \to U\}$ which consist of étale morphisms such that $\coprod_i U_i \to U$ is surjective.

Remark. One issue with the above examples is that the coverings may not be sets. This can be resolved by making suitable restrictions as explained in the last paragraph of [Stacks, Tag 00VI]. For the particular example of the big étale site see [Stacks, Tag 0214]. We will not go into further detail on these technicalities, it is however good to keep in mind that throughout this text whenever we encounter a site which has this issue, there is a strategy to resolve it and that the arguments given are mostly not affected by the modification.

Definition 2.1.13. Let $\mathcal{S}$ be a site. A presheaf F on $\mathcal{S}$ is called a sheaf if for every object $U \in \mathcal{S}$ and covering $\{U_i \to U\}_{i\in I}$ the sequence

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j),$$

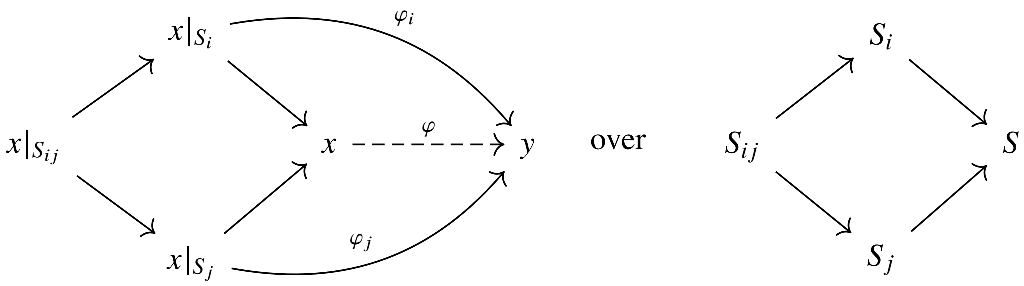
where the first map is induced by the inclusions and the two maps on the right are induced by the projections $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_j$, is exact.

2.1.3 Stacks and stackification

Now that we have given the notion of a site we can give the definition of a stack. In some sense the relation of stack to a prestack can be viewed as analogous to that of a sheaf and a presheaf. Namely a stack over a site is a prestack for which objects and morphisms glue uniquely with respect to the given Grothendieck topology, furthermore in the same way that we can sheafify a presheaf to get a corresponding sheaf we can stackify prestacks to get a stacks. We begin this section by giving the precise definition of a stack.

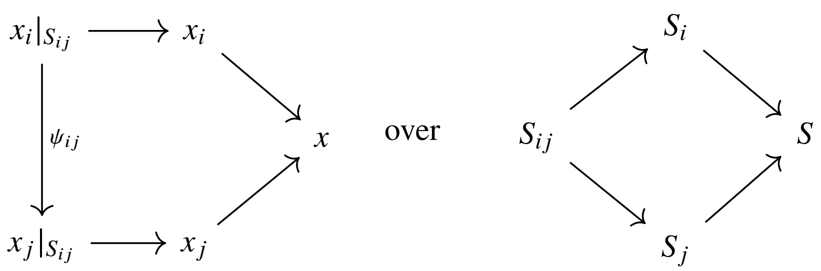
Definition 2.1.14. A stack is a prestack $\mathcal{X}$ over a site $\mathcal{S}$ such that for all coverings $\{S_i \to S\}_{i\in I} \in \mathrm{Cov}(S)$ of objects $S \in \mathcal{S}$ the following conditions are satisfied:

1. For objects $x, y \in \mathcal{X}(S)$ and morphisms $\varphi_i : x|_{S_i} \to y$ such that $\varphi_i|_{S_{ij}} = \varphi_j|_{S_{ij}}$ there exists a unique morphism $\varphi : x \to y$ over id_S making the following diagram commute



   where S_{ij} denotes the fibre product $S_i \times_S S_j$ and the maps $S_{ij} \to S_{i(j)}$ are the projections.

2. For objects $x_i \in \mathcal{X}(S_i)$ and isomorphisms $\psi_{ij} : x_i|_{S_{ij}} \to x_j|_{S_{ij}}$, as displayed in the diagram



   satisfying the cocycle condition $\psi_{jk}|_{S_{ijk}} \circ \psi_{ij}|_{S_{ijk}} = \psi_{ik}|_{S_{ijk}}$ on $S_{ijk} := S_i \times_S S_j \times_S S_k$ there exists an object $x \in \mathcal{X}(S)$ and isomorphisms $\varphi_i : x|_{S_i} \to x_i$ over id_{S_i} such that $\varphi_i|_{S_{ij}} = \varphi_j|_{S_{ij}} \circ \psi_{ij}$ on S_{ij}.

Remark. It is not too difficult to show that the first axiom is equivalent to requiring that for all objects $x, y \in \mathcal{X}(S)$ the presheaf $\mathrm{Isom}_{\mathcal{X}(S)} : \mathcal{S}/S \to \mathrm{Set}$ is a sheaf on $\mathcal{S}/S$.

A morphism of stacks is a morphism of prestacks and a substack of a stack is a strictly full subcategory, see [Stacks, Tag 001D].

We extend the Example 2.1.3 of a prestack, namely if a presheaf is a sheaf then its associated prestack is a stack.

Example 2.1.15. Recall that to every presheaf $F : \mathcal{S} \to \mathrm{Set}$ we can associate a prestack. If we equip $\mathcal{S}$ with a Grothendieck topology then F is a sheaf on the site $\mathcal{S}$ if and only if $\mathcal{X}_F$ is a stack. Since any scheme X is a sheaf on the big étale site its associated prestack is a stack on $\mathrm{Sch}_{ét}$.

As mentioned in the introduction to this subsection we can associate a stack to a prestack through a process called stackification, this is the content of the following proposition.

Proposition 2.1.16. [Alp24, Thm. 2.5.18] *Let $p : \mathcal{X} \to \mathcal{S}$ be a prestack over a site $\mathcal{S}$. There exists a stack $p^{st} : \mathcal{X}^{st} \to \mathcal{S}$ which is called the stackification of $\mathcal{X}$ and a morphism of prestacks $F : \mathcal{X} \to \mathcal{X}^{st}$ over the site $\mathcal{S}$ such that for every stack $\mathcal{Y} \to \mathcal{S}$ the induced functor*

$$Mor(\mathcal{X}^{st}, \mathcal{Y}) \to Mor(\mathcal{X}, \mathcal{Y})$$

is an equivalence of categories.

We will conclude this subsection with some facts on stackification, see [Stacks, Tag 02ZO].

1. The stackification of a prestack is unique up to an equivalence that is unique up to unique 2-isomorphism.
2. For every object $S \in \mathcal{S}$, and any $x' \in \mathcal{X}^{st}(S)$ there exists a covering $\{S_i \to S\}_{i \in I}$ such that for every $i \in I$ the pullback $x'|_{S_i}$ is in the essential image of the functor $F : \mathcal{X}(S_i) \to \mathcal{X}^{st}(S_i)$.
3. Let $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ be a fiber product of prestacks over a site $\mathcal{S}$, then

$$(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})^{st} \simeq \mathcal{X}^{st} \times_{\mathcal{Z}^{st}} \mathcal{Y}^{st}.$$

2.1.4 Algebraic stacks and properties of morphisms

We are now ready to give the definition of an algebraic stack. Furthermore we will extend many properties of morphisms of schemes to morphisms of algebraic stacks, and give an even more general version of the Yoneda Lemma. Before we can define algebraic stacks we first have to define algebraic spaces.

Definition 2.1.17. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of prestacks. We say that f is representable by schemes if for every morphism $S \to \mathcal{Y}$ from a scheme the fiber product $\mathcal{X} \times_{\mathcal{Y}} S$ is a scheme.

Furthermore for a property **P** of morphisms of schemes which is stable under base change, we say that a morphism representable by schemes $f : \mathcal{X} \to \mathcal{Y}$ has the property **P** if for every morphism $S \to \mathcal{Y}$ the projection $p_2 : \mathcal{X} \times_{\mathcal{Y}} S \to S$ has the property **P**.

Definition 2.1.18. Let X be a sheaf over the big étale site. We say that X is an algebraic space if there exists a scheme U and a surjective étale morphism $U \to X$ representable by schemes. Such a morphism is called an étale presentation.

Let **P** be a property of morphisms of schemes. We say that **P** is smooth (resp. étale) local on the source if for every smooth (resp. étale) surjection of schemes $X \to Y$, a morphism $Y \to Z$ satisfies the property **P** if and only if $X \to Y \to Z$ does, and we say that **P** is smooth (resp. étale) local on the target if for every smooth (resp. étale) surjection of schemes $Z \to Y$, a morphism $X \to Y$ satisfies the property **P** if and only if $X \times_Y Z \to Z$ does.

Definition 2.1.19. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of prestacks. We say that f is representable if for every morphism $S \to \mathcal{Y}$ from a scheme the fiber product $\mathcal{X} \times_{\mathcal{Y}} S$ is an algebraic space.

Furthermore for a property **P** of morphisms of schemes which is stable under base change and étale local the on source we say that a representable morphism $f : \mathcal{X} \to \mathcal{Y}$ has the property **P** if for every morphism $S \to \mathcal{Y}$ and every étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} S$ the composition

$$U \to \mathcal{X} \times_{\mathcal{Y}} S \to S$$

has the property **P**.

We are now ready to give the definition of an algebraic stack.

Definition 2.1.20. Let $\mathcal{X} \to \mathrm{Sch}_{ét}$ be a stack over the big étale site. We say that $\mathcal{X}$ is an algebraic stack if there exists a scheme U and a representable smooth surjection, $U \to \mathcal{X}$. We call such morphisms smooth presentations. If the morphism is also étale we say that

$\mathcal{X}$ is a Deligne-Mumford stack, this is often abbreviated to DM-stack. In this case we also say that the morphism is an étale presentation.

Definition 2.1.21. Let $\mathcal{X} \to \mathrm{Sch}_{ét}$ be a stack over the big étale site. A substack $\mathcal{Z}$ of $\mathcal{X}$ is called an open (resp. closed) substack if the inclusion $\mathcal{Z} \hookrightarrow \mathcal{X}$ is representable by schemes and an open (resp. closed) immersion.

A property **P** of schemes is called smooth (resp. étale) local if for any smooth (resp. étale) surjection of schemes $X \to Y$, X has the property **P** if and only if Y has **P**.

Definition 2.1.22. Let **P** be a smooth (resp. étale) local property of schemes, e.g., locally noetherian, reduced or regular, then we say that an algebraic stack (resp. DM-stack) has the property **P** if there is a smooth (resp. étale) presentation from a scheme with the property **P**.

For a stack $\mathcal{Y}$ over $\mathrm{Sch}_{ét}$ and an algebraic stack $\mathcal{X}$ there is a further generalization of the Yoneda lemma giving us a useful description for the category $\mathrm{Mor}(\mathcal{X}, \mathcal{Y})$. Namely, for a smooth presentation $X \to \mathcal{X}$, denote the projections by $p_i : X \times_{\mathcal{X}} X \to X$ and $p_{ij} : X \times_{\mathcal{X}} X \times_{\mathcal{X}} X \to X \times_{\mathcal{X}} X$, then we define $\mathcal{Y}(\mathcal{X})$ to be the category consisting of

- (Objects): An object is a pair (y, ψ) where $y \in \mathcal{Y}(X)$ and $\psi : p_1^* x \xrightarrow{\sim} p_2^* x$ is an isomorphism which satisfies the cocycle condition $p_{23}^* \psi \circ p_{12}^* \psi = p_{13}^* \psi$.
- (Morphisms): A morphism $(y, \psi) \to (y', \psi')$ is given by a morphism $\eta : y \to y'$ such that $p_2^* \eta \circ \psi = \psi' \circ p_1^* \eta$.

i.e., the category of descent data corresponding to the sequence

$$\mathcal{Y}(X) \rightrightarrows \mathcal{Y}(X \times_{\mathcal{X}} X) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \mathcal{Y}(X \times_{\mathcal{X}} X \times_{\mathcal{X}} X) \ ,$$

see [Stacks, Tag 026B]. We note that $\mathcal{Y}(\mathcal{X})$ is independent of the smooth presentation. This generalization of the Yoneda lemma then claims that there is an equivalence of categories [Alp24, Lemma 3.1.24]

$$\mathcal{Y}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Mor}(\mathcal{X}, \mathcal{Y}).$$

Also see [Hal21, Sect. 6.4.3].

Next we want to extend the above definitions for properties of representable morphisms, to properties of morphisms of algebraic stacks which are not necessarily representable, by using presentations.

Definition 2.1.23. Let **P** be a property for morphisms of schemes which is stable under composition and base change and is smooth local on the source and target, e.g., flatness, smoothness, surjectivity, locally of finite presentation and locally of finite type. We say that a morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ has the property **P** if there exist smooth presentations $Y \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} Y$ such that the composition of the latter smooth presentation with the projection map

$$U \to \mathcal{X} \times_{\mathcal{Y}} Y \to Y$$

has the property **P**.

For a property of morphisms of schemes **P** that is stable under base change and a morphism of algebraic stacks representable by schemes $\mathcal{X} \to \mathcal{Y}$, we say that the morphism has the property **P** if for any morphism from a scheme $S \to \mathcal{Y}$ the projection $\mathcal{X} \times_{\mathcal{Y}} S \to S$ has the property **P**.

Remark. For a morphism of algebraic stacks to have a property **P** $\in\{$isomorphism, open immersion, (locally) closed immersion, (quasi-)affine$\}$ it has to be representable by schemes.

Moreover if we replace smooth by étale in the first part of the definition we can make a similar definition for DM-stacks, in particular this allows us to define unramifiedness and étaleness for morphisms of DM-stacks.

We define étale and unramified morphisms separately for a class of morphisms of algebraic stacks which are called DM morphisms.

Definition 2.1.24. A morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is called DM or relatively Deligne-Mumford if for every morphism from a scheme $X \to \mathcal{Y}$ the fiber product $\mathcal{X} \times_{\mathcal{Y}} X$ is a Deligne-Mumford stack.

Definition 2.1.25. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is étale (resp. unramified) if it is DM and for every smooth presentation $U \to \mathcal{Y}$ and étale presentation $V \to \mathcal{X} \times_{\mathcal{Y}} U$ the composition $V \to \mathcal{X} \times_{\mathcal{Y}} U \to U$ is étale (resp. unramified).

We conclude this subsection with a useful result.

Proposition 2.1.26. [Alp24, Prop. 3.3.4] *Consider a property of morphisms of algebraic stacks* **P** $\in\{$*representable, isomorphism, open immersion, closed immersion, (quasi-)affine*$\}$*. Then if we have a cartesian diagram of algebraic stacks*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\ \downarrow & \square & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

and $\mathcal{Y}' \to \mathcal{Y}$ is smooth and surjective, the morphism $\mathcal{X}' \to \mathcal{Y}'$ has the property **P** *if and only if $\mathcal{X} \to \mathcal{Y}$ has the property* **P**.

2.1.5 Topology

In this subsection we generalize the notions of points and topology to algebraic stacks.

Definition 2.1.27. Let $\mathcal{X}$ be an algebraic stack. The topological space of $\mathcal{X}$, denoted $|\mathcal{X}|$, is the set of field valued points $x : \mathrm{Spec}(k) \to \mathcal{X}$ modulo the relation which identifies two points $x_1 : \mathrm{Spec}(k_1) \to \mathcal{X}$ and $x_2 : \mathrm{Spec}(k_2) \to \mathcal{X}$ if and only if there exist field extensions $k_1 \to k$ and $k_2 \to k$ such that the diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k_1) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k_2) & \longrightarrow & \mathcal{X} \end{array}$$

is 2-commutative.

Now the topology on this set is defined as follows: a subset $U \subset \mathcal{X}$ is open if there exists an open substack $\mathcal{U} \subset \mathcal{X}$ such that $|\mathcal{U}| = U$. In particular a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks induces a continuous map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$.

We will now list some properties of algebraic stacks and morphisms thereof which can be defined using the underlying topological space.

Definition 2.1.28. Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks and $f : \mathcal{X} \to \mathcal{Y}$ a morphism, then

- $\mathcal{X}$ is quasi-compact, connected or irreducible if $|\mathcal{X}|$ is.
- f is quasi-separated if its diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and second diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ are quasi-compact.
- $\mathcal{X}$ is noetherian if it is locally noetherian, quasi-compact and quasi-separated.
- f is quasi-compact if for every morphism $\mathrm{Spec}(A) \to \mathcal{Y}$ from an affine scheme the fiber product $\mathcal{X} \times_{\mathcal{Y}} \mathrm{Spec}(A)$ is quasi-compact.
- f is of finite type if it is locally of finite type and quasi-compact.
- f is universally closed if for every morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ induces a closed map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}'|$.
- [Stacks, Tag 04XI] f is surjective if and only if $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is.

- [Stacks, Tag 04XH] For a pair of morphisms $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ the morphism $|\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}| \to |\mathcal{X}| \times_{|\mathcal{Z}|} |\mathcal{Y}|$ is surjective.

Remark. [Stacks, Tag 04YC] For an algebraic stack (resp. quasi-separated algebraic stack) $\mathcal{X}$ quasi-compactness (resp. noetherianness) is equivalent to the existence of a smooth presentation $\mathrm{Spec}(A) \to \mathcal{X}$ (resp. a smooth presentation $\mathrm{Spec}(A) \to \mathcal{X}$ with A a noetherian ring).

2.1.6 Properness, separatedness and valuative criteria

In this section we will introduce the notions of properness and separatedness for algebraic stacks and give valuative criteria for these properties.

Before giving the definitions of separatedness and properness we first have to discuss diagonal morphisms. Namely for algebraic stacks and morphisms of algebraic stacks we have the following results.

Proposition 2.1.29. [Alp24, Chapt. 3.2.1]

1. *The diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ of an algebraic stack (resp. algebraic space) $\mathcal{X}$ is representable (resp. representable by schemes).*
2. *If $\mathcal{X} \to \mathcal{Y}$ is a morphisms (resp. representable morphism) of algebraic stacks, then $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable (resp. representable by schemes).*

Giving the definitions of separatedess and properness has to be done by first giving the definitions for representable morphisms and then building the general definition on the representable case.

Definition 2.1.30. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphisms of algebraic stacks.

1. If f is representable then we say that f is separated if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, which is representable by schemes, is proper.
2. If f is representable we say that f is proper if it is universally closed, separated and of finite type.
3. f is separated if the diagonal morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, which is representable, is proper.
4. f is proper if it is universally closed, separated and of finite type.

We conclude this subsection by stating the valuative cirteria.

Theorem 2.1.31. [Alp24, Thm. 3.8.2] *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of locally noetherian algebraic stacks. Consider a 2-commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec}(K) & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec}(R) & \longrightarrow & \mathcal{Y} \end{array} \tag{2.1.1}$$

where R is a discrete valuation ring with fraction field K. Then

1. *f is proper if and only if f is of finite type and for every diagram 2.1.1, there exists an extension $R \to R'$ of discrete valuation rings with the induced map $K \to K'$ on fraction fields having finite transcendence degree and a lifting of the form*

$$\begin{tikzcd} \operatorname{Spec}(K') \arrow[r] \arrow[d] & \operatorname{Spec}(K) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec}(R') \arrow[r] \arrow[urr, dashed] & \operatorname{Spec}(R) \arrow[r] & \mathcal{Y} \end{tikzcd} \tag{2.1.2}$$

   which is unique up to unique isomorphism.

2. *f is separated if and only if every two liftings of a diagram 2.1.2 are uniquely isomorphic.*

2.2 Examples of algebraic stacks

2.2.1 Quotient Stacks

The leading example in the chapters that follow will be that of the quotient stack, in this section we will define this stack, give some of its properties and show that it is an algebraic stack.

Definition 2.2.1. A groupoid in algebraic spaces consists of the data (X,R,s,t,c,e,i) where X and R are algebraic spaces, $s,t : R \to X$, $c : R\times_X R \to R$, $e : X \to R$ and $i : R \to R$ are morphisms, which respectively are called the source, target, composition, identity and inverse, such that the following diagrams commute:

1. (**Associativity**)

$$\begin{array}{ccc} R\times_X \times R\times_X R & \xrightarrow{c\times id} & R\times_X R \\ \downarrow{\scriptstyle id\times c} & & \downarrow{\scriptstyle c} \\ R\times_X R & \xrightarrow{\quad c \quad} & R \end{array}$$

2. (**Identity**)

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{id} & \downarrow{\scriptstyle e} & \searrow^{id} & \\ X & \xleftarrow{s} & R & \xrightarrow{t} & X \end{array} \qquad \begin{array}{ccccc} R & \xrightarrow{(e\circ s,id)} & R\times_X R & \xleftarrow{(e\circ t,id)} & R \\ & \searrow_{id} & \downarrow{\scriptstyle c} & \swarrow_{id} & \\ & & R & & \end{array}$$

3. (**Inverse**)

$$\begin{array}{ccccc} R & \xrightarrow{i} & R & \xrightarrow{i} & R \\ & \searrow_{s} & \downarrow{\scriptstyle t} & \swarrow_{s} & \\ & & X & & \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{s} & X \\ \downarrow{\scriptstyle (id,c)} & & \downarrow{\scriptstyle e} \\ R\times_X R & \xrightarrow{c} & R \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{t} & X \\ \downarrow{\scriptstyle (id,i)} & & \downarrow{\scriptstyle e} \\ R\times_X R & \xrightarrow{c} & R \end{array}$$

If in addition the source and target morphisms are étale (resp. smooth) we say that it is an étale (resp. smooth) groupoid in algebraic spaces. We often denote this data as $s,t : R \rightrightarrows X$.

Remark. This definition can be given for an arbitrary category with finite fiber products, for our purposes however the only relevant categories are schemes and algebraic spaces. If $(s,t) : R \to U \times U$ is a monomorphism we say that $s,t : R \rightrightarrows U \times U$ is an étale (resp. smooth) equivalence relation of algebraic spaces. We also want to note that the identity and inverse are uniquely determined by the other data and therefore often omitted.

For our first example we consider the action of a group scheme on a scheme, this is also the example that we will use the most throughout this text.

Example 2.2.2. Let S be a scheme. Consider the data $(X, G\times X, p_2, \rho, c, e, i)$ where

1. G/S is an étale (resp. smooth) group scheme with multiplication m, inverse $(-)^{-1} : G \to G$, identity $e_G : S \to G$ and an action $\rho : G \times X \to X$ on the S-scheme X.
2. The source $p_2 : G \times X \to X$ is the second projection.
3. The composition $c : ((G\times X)\times_X (G\times X) \to G\times X) \simeq G\times G\times X \to G\times X$ is the morphism induced by the multiplication $m : G\times G \to G$, on points this is given by $((g', g, x)) \mapsto (g'g, x)$.
4. The identity $e : G\times X \to X$ is the morphism induced by the identity $e_G : S \to G$, on points this is given by $x \mapsto (e_G, x)$.
5. The inverse $i : G\times X \to G\times X$ is the morphism which on points is given by $(g,x) \mapsto (g^{-1}, gx)$. Then $p_2, \rho : G\times X \rightrightarrows X$ is an étale (resp. smooth) groupoid in schemes.

We will now give the definition of a quotient stack.

Definition 2.2.3. Let $s,t : R \rightrightarrows X$ be a smooth groupoid in algebraic spaces. The quotient prestack $[X/R]^{pre}$ associated to $s,t : R \rightrightarrows X$ is the category fibered in groupoids with:

- **Objects**: For every scheme T an object over T is given by a morphism $T \to X$.
- **Morphisms**: A morphism $(T' \xrightarrow{x} X) \to (T \xrightarrow{y} X)$ is given by the data of a morphism $\varphi : T' \to T$ and an element $r \in R(T)$ such that $s(r) = x$ and $t(r) = \varphi^* y$.

The stackification of the quotient prestack is called the quotient stack and is denoted $[X/R]$.

For our second example we will see that algebraic stacks have an associated groupoid, and that algebraic stacks can be identified with the quotient stacks associated to these groupoids.

Example 2.2.4. Let $\mathcal{X}$ be an algebraic stack with a smooth presentation $\varphi : U \to \mathcal{X}$. Then we can define a smooth groupoid in algebraic spaces given by the data $(U, U \times_{\mathcal{X}} U, p_1, p_2, c)$:

1. The source and target are given respectively by the projections $p_1 : U \times_{\mathcal{X}} U \to U$ and $p_2 : U \times_{\mathcal{X}} U \to U$.
2. The composition is given by $c := (p_1 \circ p_1', p_2 \circ p_2') : (U \times_{\mathcal{X}} U) \times_U (U \times_{\mathcal{X}} U) \to U \times_{\mathcal{X}} U$. Recall the definition of a fiber product, explicitly on T-points the composition acts by

$$((u_1, u_2, \eta : \varphi(u_1) \xrightarrow{\sim} \varphi(u_2)), (u_1', u_2', \eta' : \varphi(u_1') \xrightarrow{\sim} \varphi(u_2')), f : u_1 \xrightarrow{\sim} u_2')$$

$$\mapsto (u_1, u_2', \varphi(f) : \varphi(u_1) \xrightarrow{\sim} \varphi(u_2')).$$

Moreover the morphism φ induces an equivalence $\varphi_{can} : [U/U \times_{\mathcal{X}} U] \xrightarrow{\sim} \mathcal{X}$, see [Stacks, Tag 04T5]. A smooth groupoid (X, R, s, t, c) together with an equivalence $\varphi : [U/R] \to \mathcal{X}$ is called a presentation of $\mathcal{X}$. Note that analogous constructions can be made if we replace an algebraic stack with a smooth presentation by a DM-stack or algebraic space with an étale presentation.

In our next example we will see that applying the quotients stack construction to a groupoid induced by the action of a smooth affine algebraic group has a particularly nice description.

Example 2.2.5. In the situation of Example 2.2.2, the quotient prestack associated to $p_2, \rho : G \times X \rightrightarrows X$ is denoted by $[X/G]^{pre}$, its objects are the points of X and a morphism $(T' \xrightarrow{x} X) \to (T \xrightarrow{y} X)$ consists of the data of a morphism $\varphi : T' \to T$ and an element $g \in G(T')$ such that $g \cdot x = \varphi^* y$, its stackification is denoted by $[X/G]$. If G/S is smooth and affine a principal G-bundle over an S-scheme T is a morphism of schemes $P \to T$ with an action of G on P via $\sigma : G \times_S P \to P$ such that $P \to T$ is a G-invariant smooth morphism and

$$(\sigma, p_2) : G \times_S P \to P \times_T P, \quad (g, p) \mapsto (gp, p)$$

is an isomorphism. For G/S smooth and affine the stack $[X/G]$ has the following explicit description:

1. For every scheme T an object over T is given by a pair $(P \to T, P \to X)$ where $P \to T$ is a principal $G-$bundle and $P \to X$ is a $G-$equivariant morphism.

2. A morphism $(P' \to T', P' \to X) \to (P \to T, P \to X)$ is given by the data of a morphism $T' \to T$ and a $G-$equivariant morphism $P' \to P$ such that the following diagram commutes

$$\begin{array}{ccccc} P' & \longrightarrow & P & \longrightarrow & X \\ \downarrow & \square & \downarrow & & \\ T' & \longrightarrow & T & & \end{array}$$

and the left square is cartesian.

Remark. The above example can be easily generalized to the situation of the action of a group scheme on an algebraic space. Namely an action of a group scheme G on an algebraic space X is a morphism $G \times X \to X$ satisfying the same axioms as in the scheme theoretic definition of a group action. Moreover similar explicit descriptions can also be given for the actions of group schemes, or more generally group algebraic spaces, which are smooth or even fppf, dropping the affineness assumption, see [Alp24, Chapt. 6.3.2].

Suppose that we are in the following situation, G/k is a smooth algebraic group over a field k acting on an algebraic space X locally of finite type over k. Let $H \hookrightarrow G$ be a subgroup scheme with the free action on $G \times X$ given by $h \cdot (g, x) = (gh^{-1}, hx)$ which is induced by the action of G on X and the group multiplication of G. Then we define $G \times^H X := [G \times X/H]$. In this setting we have the following result.

Lemma 2.2.6. *[Hal21, Lemma 7.2.3.2],[Alp24, Sect. 3.4.2] The quotient $G \times^H X$ is an algebraic space and for the action of G on the algebraic space $G \times^H X$, induced by $g \cdot (g', x) = (gg', x)$, there is an isomorphism*

$$[X/H] \simeq [(G \times^H X)/G].$$

We will now state two properties of quotient stacks. First we look at a useful cartesian diagram.

Lemma 2.2.7. *Let $s, t : R \rightrightarrows X$ be a smooth groupoid in algebraic spaces. There exists a morphism $p : X \to [X/R]$ such that the following diagram is cartesian.*

$$\begin{array}{ccc} R & \xrightarrow{s} & X \\ \downarrow{\scriptstyle t} & \square & \downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & [X/R] \end{array}$$

Proof. Define $p^{pre} : X \to [X/R]^{pre}$ to be the morphism that sends an object $(x,T) \in X(T)$ to the corresponding morphism obtained by the 2-Yoneda lemma $(T \to X) \in [X/R]^{pre}(T)$, and a morphisms $f : (x',T') \to (x,T)$ in X to the morphism $(f, e_{T'}(x')) : x' \to x$ where $e_{T'}$ is the identity of the groupoid over T'. This makes sense because $s(e_T(x')) = x'$ and $f^*x = x' = t(e_{T'}(x'))$. We obtain the morphism p by composing p^{pre} with the stackification functor.

In order to show that the diagram is cartesian we first want to show that it is 2-commutative, that is: there exists a natural transformation $\alpha : p \circ s \to p \circ t$ which for every scheme T and object $r \in R(T)$ defines an isomorphism

$$\alpha_r : p \circ s(r) \to p \circ t(r) \in [X/R](T).$$

We do this by first constructing a 2-isomorphism $\alpha^{pre} : p^{pre} \circ s \to p^{pre} \circ t$. An element $r \in R(T)$ defines an isomorphism $(id_T, r) : s \circ r \to t \circ r$ in $[U/R]^{pre}(T)$, using this we construct the isomorphism α^{pre} which for every scheme T and $r \in R(T)$ is given by $\alpha_r^{pre} := (id_T, r) : (p^{pre} \circ s)(r) \to (p^{pre} \circ t)(r)$. To show that the transformation defined in this way is natural, take a pair of objects $r' \in R(T')$ and $r \in R(T)$ and a morphism $f : (r',T') \to (r,T)$, then using the defining properties of groupoids we have that

$$(id_T, r) \circ ((p^{pre} \circ s)(f)) = (id_T, r) \circ (f, e_{T'}(s_{T'}(r')))$$

$$= (id_T \circ f, c_{T'}(R(f)(r), e_{T'}(s_{T'}(r')))) = (f, c_{T'}(r, e_{T'}(s_{T'}(r'))))$$

$$= (f \circ id_{T'}, c_{T'}(e_{T'}(t_{T'}(r')), r')) = (f \circ id_{T'}, c_{T'}(R(id_{T'})(e_{T'}(t_{T'}(r'))), r'))$$

$$= (f, e_{T'}(t(r'))) \circ (id'_T, r') = ((p^{pre} \circ t)(f)) \circ (id_{T'}, r').$$

Hence α^{pre} is a natural transformation and composing with the stackification gives us the desired natural transformation α, proving the 2-commutativity of the diagram.

Next we want to show that there is an isomorphism $R \to X \times_{[X/R]} X$. Consider the morphism $\varphi : R \to X \times_{[X/R]} X$, $r \mapsto (s(r), t(r), \alpha_r)$ for $r \in R(T)$, we claim that this is an isomorphism.

This follows from the fact that objects in the fiber product are determined by the objects $r \in R(T)$. Namely, consider a pair of objects $y', y \in [X/R]$ and objects $x', x \in [X/R]^{pre}$ which under stackification are respectively mapped to y' and y. The sheaf $Isom(y', y)$ is isomorphic to the sheafification of the presheaf $Isom(x', x)$, the latter however is an algebraic space, it is not difficult to see that it is isomorphic to the fiber product $T \times_{U\times U} R$ given by the morphisms (x,x') and (s,t), see [Stacks, Tag 044V], and therefore in particular a sheaf. Now our claim follows because isomorphisms $x' \to x$ over id_T are, by definition, given by an $r \in R(T)$ such that $s(r) = x'$ and $t(r) = x$. $\square$

Our second result is the statement that a quotient stack is a categorical quotient among stacks.

Lemma 2.2.8. *Let $p_2, \rho : G \times X \rightrightarrows X$ be the smooth groupoid in schemes from Example 2.2.2, then the morphism $p : X \to [X/G]$ is a categorical quotient among stacks, that is, for a 2-commutative diagram:*

$$\begin{array}{ccccc}
G\times X & \xrightarrow{\rho} & X & & \\
\downarrow{\scriptstyle p_2} & \swarrow\alpha & \downarrow{\scriptstyle p} & \searrow{\scriptstyle q} & \\
X & \xrightarrow{p} & [X/G] & & \\
 & \searrow{\scriptstyle q} & & \swarrow\beta & \downarrow \\
 & & & & \mathcal{X}
\end{array}$$

there exists a morphism $\varphi : [X/G] \to \mathcal{X}$ filling in the diagram and a pair of 2-isomorphisms $\tau : q \to \varphi \circ p$ and $\eta : \varphi \circ p \to q$ making the diagram commute.

Proof. We construct the morphism $\varphi^{pre} : [X/G]^{pre} \to \mathcal{X}$ as follows, since p^{pre} is the identity on objects, φ^{pre} sends an object $x \in [X/G]^{pre}$ to $q(x)$. A morphism $(f : T' \to T, g) : x' \to x$ is sent to the morphism

$$\begin{array}{ccccccccc}
q(x') & = & q(\rho(g^{-1},f^*x)) & = & q\circ\rho(g^{-1},f^*x) & \xrightarrow{\beta} & q\circ p_2(g^{-1},f^*x) & = & q(f^*x) \\
 & & & & {\scriptstyle \varphi((f,g))}\searrow & & & & \downarrow{\scriptstyle q(\tilde{f})} \\
 & & & & & & & & q(x)
\end{array}$$

Now by stackification we obtain the desired morphism $\varphi : [X/G] \to \mathcal{X}$. For more details, see [Stacks, Tag 044U]. $\square$

Finally, we want to conclude this section by showing that quotient stacks are algebraic.

Theorem 2.2.9. *If $s,t : R \rightrightarrows X$ is a smooth groupoid of algebraic spaces, then $[X/R]$ is an algebraic stack and the natural map $X \to [X/R]$ is a smooth, surjective and representable morphism.*

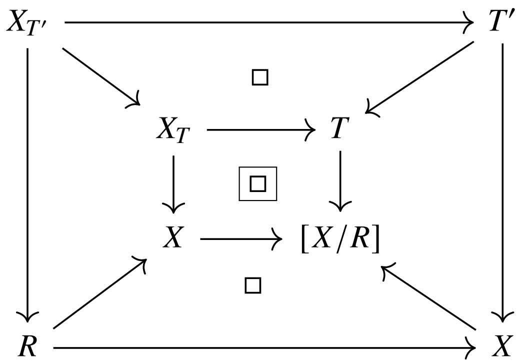
Proof. Let $T \to [X/R]$ be a morphism from a scheme T. Note that by definition $[X/R]$ is the stackification of the prestack $[X/R]^{pre}$, and that the morphism $T \to [X/R]$ can be identified with an object in the fiber category $u \in [X/R](T)$. Hence by properties of stackification there exists an étale cover $T' \to T$ such that the pullback $x|_{T'}$ lies in the essential image of the stackification $[X/R]^{pre} \to [X/R]$, i.e., there exists an object $x^{pre} \in [X/R]^{pre}(T) = X(T)$ which has image x' isomorphic to $x|_{T'}$. Now by applying the Yoneda lemma to the isomorphism $x' \simeq x|_{T'}$ we obtain a 2-commutative diagram

$$\begin{array}{ccc} T' & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & [X/R] \end{array}$$

And since there is a cartesian diagram

$$\begin{array}{ccc} R & \xrightarrow{s} & X \\ {\scriptstyle t}\downarrow & \square & \downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & [X/R] \end{array}$$

we can construct the following cube



The bottom, top, inside and outside squares are cartesian, X_T is a sheaf [Stacks, Tag 05UJ] and $X_{T'}$ is an algebraic space. Since $T' \to T$ is surjective, étale and representable by schemes the base change $X_{T'} \to X_T$ is as well. Now by composing $X_{T'} \to X_T$ with an étale presentation $U \to X_{T'}$ we obtain an étale presentation for X_T. It follows that X_T is an algebraic space, thus $X \to [X/R]$ is representable.

Since $R \to X$ is smooth and surjective the base change $X_{T'} \to T'$ is smooth and surjective and therefore $X_T \to T$ is smooth and surjective. It follows that $X \to [X/R]$ is smooth and surjective. Now by composing the morphism $X \to [X/R]$ with an étale presentation for the algebraic space X_T we get a smooth presentation for $[X/R]$. □

Corollary 2.2.10. *Let G/S be a smooth affine group scheme acting on an algebraic space X/S. The quotient stack $[X/G]$ is an algebraic stack over S and the natural map $p : X \to [X/G]$ is a principal G-bundle, i.e., $X \to [X/G]$ is representable by schemes and for every morphism from a scheme $T \to [X/G]$ the projection*

$$X \times_{[X/G]} T \to T$$

is a principal G-bundle over T.

2.2.2 Orbits, stabilizers and residual gerbes

Before moving on to our next example we will introduce three important notions in the theory of algebraic stacks. The first notion is that of a stabilizer.

Definition 2.2.11. Let $\mathcal{X}$ be an algebraic stack and $x : T \to \mathcal{X}$ a T-point, then the stabilizer of x, often denoted G_x, is the fiber product

$$\begin{array}{ccc} G_x := \mathrm{Aut}_{\mathcal{X}(T)}(x) & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle (x,x)} \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X} \end{array}$$

Remark. For field valued points the stabilizers are group algebraic spaces. Furthermore if $\mathcal{X}$ has a quasi-separated diagonal, these stabilizers are group schemes locally of finite type [Alp24, p. 117].

An important class of morphisms of algebraic stacks are the morphisms which preserve stabilizers in the following sense.

Definition 2.2.12. [AHH23, Sect. 4.1] Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then for a point $x \in |X|$ we stay that f is stabilizer preserving at x if there exists a representative $\tilde{x}$, or equivalently for all representatives, of x such that the natural map $G_{\tilde{x}} \to G_{f(\tilde{x})}$ is an isomorphism of group algebraic spaces.

Remark. In this text we will say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ is stabilizer preserving if it is so for all points $x \in |\mathcal{X}|$. However, in for example the article [Alp17, Sect. 2.1], a more general definition is given where morphisms satisfying the above definition at all points are called pointwise stabilizer preserving and a morphism is called stabilizer preserving if it induces isomorphisms of stabilizers at all T-valued points for schemes T.

For a verification of the fact that the property has to hold only for one representative in order for it to hold for all representatives of x see [Stacks, Tag 0DTV]. Moreover it is useful to note that this property is stable under base change [Stacks, Tag 0DUB].

In addition to stabilizers we can also define orbits of T-valued points.

Definition 2.2.13. Let $\mathcal{X}$ be an algebraic stack, $x : T \to \mathcal{X}$ a T-point and $X \to \mathcal{X}$ an fppf presentation, i.e., a locally of finite presentation, faithfully flat and representable morphism from a scheme. Then we define the orbit of x in X denoted $o_X(x)$, set-theoretically as the image of $X \times_{\mathcal{X}} T \to X \times T$. Moreover if the stabilizer is an fppf group scheme, i.e., fppf over its base scheme, then the orbit is given by the cartesian diagram

$$\begin{array}{ccc} o_X(x) & \longrightarrow & X \times T \\ \downarrow & & \downarrow \\ BG_x & \longrightarrow & \mathcal{X} \times T \end{array}$$

In particular if $x : \mathrm{Spec}(k) \to \mathcal{X}$ is a geometric point then its orbit is given by the fiber product

$$\begin{array}{ccc} o_X(x) & \longrightarrow & X_k \\ \downarrow & & \downarrow \\ BG_x & \longrightarrow & \mathcal{X}_k \end{array}$$

and we say that the point $x : \mathrm{Spec}(k) \to \mathcal{X}$ has closed orbit if $BG_x \to \mathcal{X} \to \mathcal{X}_k$ is a closed immersion. An algebraic stack is said to have closed orbits if every geometric point has a closed orbit.

Another important concept in the theory of algebraic stacks is that of a residual gerbe, this concept is meant to mimic the fact that for points in schemes we can construct monomorphisms from their residue fields. This notion is made precise in the following definition.

Definition 2.2.14. Let $\mathcal{X}$ be an algebraic stack and $x \in |\mathcal{X}|$ a point. Then a residual gerbe of $\mathcal{X}$ at x is an algebraic stack $\mathcal{G}_x$ together with a monomorphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}_x|$ is a singleton and the image of $|\mathcal{G}_x|$ in $|\mathcal{X}|$ is x.

Remark. If a residual gerbe $\mathcal{G}_x$ at a point $x \in |\mathcal{X}|$ exists, then it is unique and $\mathcal{G}_x$ is a reduced and locally noetherian substack of $\mathcal{X}$, moreover by [Stacks, Tag 06MV] it is a regular algebraic stack.

Proposition 2.2.15. *Let $\mathcal{X}$ be an algebraic stack. Then the following are true.*

1. *[Stacks, Tag 06RD] If $\mathcal{X}$ is quasi-separated, then residual gerbes exist at all points $x \in |\mathcal{X}|$.*

2. *[Alp24, Prop. 3.5.17] If $\mathcal{X}$ is quasi-separated and of finite type over an algebraically closed field k, then for every point $x \in |\mathcal{X}|$ with a representative $\tilde{x} \in \mathcal{X}(k)$,*

$$\mathcal{G}_x \simeq BG_{\tilde{x}}.$$

Using the notion of a residual gerbe we can also give a definition for orbits of points $x \in |\mathcal{X}|$.

Definition 2.2.16. Let $\mathcal{X}$ be an algebraic stack, $x \in |\mathcal{X}|$ a point and $X \to \mathcal{X}$ an fppf presentation. The orbit of x in X, denoted $O_X(x)$, is defined as the fiber product

$$\begin{array}{ccc} O_X(x) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{G}_x & \longrightarrow & \mathcal{X} \end{array}$$

For a representative $\tilde{x} : \mathrm{Spec}(k) \to \mathcal{X}$ of x, set-theoretically $O_X(x)$ is the image of $\mathrm{Spec}(k) \times_{\mathcal{X}} X \to X$. If $s, t : X \times_{\mathcal{X}} X \rightrightarrows X$ is the induced groupoid, see Example 2.2.4, and $x' \in |X|$ is a lift of x, then $O_X(x) = s(t^{-1}(x))$ set-theoretically.

2.2.3 Stack of vector bundles over a curve

One of the fundamental examples in moduli theory is the moduli space of vector bundles over a curve. In this section we define the stack and sketch the proof for its algebraicity. We will work over a smooth connected and projective curve C over an algebraically closed field k. For this section we mainly follow Alper's notes [Alp24], also see [Neu09], [Gro14] and [Wan11].

Definition 2.2.17. The moduli stack of vector bundles of rank r and degree d over C, denoted $\underline{\mathrm{Bun}}_{r,d}(C)$ is defined as follows.

- An object $E \in \underline{\mathrm{Bun}}_{r,d}(C)(S)$ is a vector bundle of rank r over C_S which is flat over S and of relative degree d.
- A morphism $(E', S') \to (E, S)$ is given by a morphisms of schemes $f : S' \to S$ with a morphism $E \to (\mathrm{id}_C \times f)_* E'$ of $\mathcal{O}_{C_S}$-modules whose adjoint is an isomorphism.

Our strategy for proving the algebraicity of this stack is to show that it is isomorphic to a quotient stack. We do this by using the following general fact, let E denote a coherent sheaf of rank r and degree d over the curve C, and let $\mathcal{O}_C(1)$ be an ample invertible sheaf on C. Then by a result of Serre [Har77, Thm. II.5.17, Prop. III.5.3], there exists an integer n_0 such that for all $n \geq n_0$ the sheaf $E(n) = E \otimes_{\mathcal{O}_C} \mathcal{O}_C(n)$ is finitely globally generated and for all $i > 0$, $H^i(C, E(n)) = 0$. It follows that the canonical evaluation map $\Gamma(C, E(n)) \otimes_k O_C \to E(n)$, which for sections on some open U is defined $s \otimes f \to f \cdot s|_U$, is a surjection which induces an isomorphism on global sections.

Furthermore, to prove algebraicity we will also use quot schemes, which we will define next. Let F be a coherent sheaf on C/k and $P \in \mathbb{Q}[x]$, consider the moduli problem

$$\mathrm{Quot}^P_{C/k}(F) : \mathrm{Sch/k} \to \mathrm{Set},$$

where $\mathrm{Quot}^P_{C/k}(F)(S)$ consists of families of quasi-coherent and finitely presented quotients $F_S \twoheadrightarrow Q$ on C_S, i.e., surjecive $\mathcal{O}_{C_S}$-linear homomorphisms of sheaves over C_S, such that Q is flat over S and Q_s has Hilbert polynomial P for all $s \in S$. Furthermore two S-families $q_S : F_S \twoheadrightarrow Q$ and $q'_S : F_S \twoheadrightarrow Q'$ are equivalent if $\ker(q_S) = \ker(q'_S)$.

It has long been known that this moduli problem is representable by a projective scheme, see [Alp24, Chapt. 1], the representing scheme is called the quot scheme, we will however also call the moduli problem itself the quot scheme.

It turns out that the the stack of vector bundles of given rank and degree can be realized as the quotient of some subscheme of the quot scheme. Note that the Hilbert polynomial of a coherent sheaf of rank r and degree d on a curve C of genus g is given by

$$P_{r,d}(n) = d + rn + r(1-g).$$

Let $P := P_{r,d}$ and for each integer N consider the open subscheme

$$U_N \subset \mathrm{Quot}^P_{C/k,r,d}(\mathcal{O}_C(-N)^{P(N)})$$

which corresponds to the sheaf which sends a scheme S/k to the set of families of locally free quotient sheaves $pr_1^*\mathcal{O}_C(-N)^{P(N)} \twoheadrightarrow F$, where $pr_1 : C \times S \to C$ is the projection, of rank r and degree d such that for every $s \in S$, $F(N)_s$ is globally generated and $H^1(C_s, F(N)|_{C_s}) = 0$.

The algebraic group $\mathrm{GL}_{P(N)}$ acts on the quot scheme via precomposition, that is, for S−points $g \in \mathrm{GL}_{P(N)}(S)$ and $[q : pr_1^*\mathcal{O}_C(-N)^{P(N)} \to F] \in \mathrm{Quot}^P_{C/k,r,d}(\mathcal{O}_C(-N)^{P(N)})(S)$,

$$g \cdot [q] := [q \circ pr_2^* g^{-1} : pr_1^*\mathcal{O}_C(-N)^{P(N)} \xrightarrow{pr_2^* g^{-1}} pr_1^*\mathcal{O}_C(-N)^{P(N)} \twoheadrightarrow F].$$

and the open subschemes U_N are invariant under this action. We are now prepared to sketch a proof for the algebraicity of the stack of vector bundles over C of given rank and degree.

Theorem 2.2.18. *Let C be a smooth, connected and projective curve over an algebraically closed field k. Then for $r, d \in \mathbb{Z}_{\geq 0}$ the stack $\underline{\mathrm{Bun}}_{r,d}(C)$ is algebraic.*

Proof. Consider the morphism $U_N \to \underline{\mathrm{Bun}}_{r,d}(C)$ which over a scheme S/k is defined by

$$[q : pr_1^*\mathcal{O}_C(-N)^{P(N)} \twoheadrightarrow F] \to F.$$

This map is clearly $\mathrm{GL}_{P(N)}$ invariant, since the natural morphism $U_N \to [U_N/\mathrm{GL}_{P(N)}]^{pre}$ is a categorical quotient among prestacks, the morphism $U_N \to \underline{\mathrm{Bun}}_{r,d}(C)$ factors through $[U_N/GL_{P(N)}]^{pre}$ giving us a morphism $\Psi_N^{pre} : [U_N/\mathrm{GL}_{P(N)}]^{pre} \to \underline{\mathrm{Bun}}_{r,d}(C)$.

This map is fully faithfull, namely the set of morphisms $\mathrm{Mor}_{[U_N/GL_{P(N)}](S)}([p_G], [q_F])$, for objects $[p_G], [q_F]$ in the fiber category $[U_N/GL_{P(N)}](S)$, consists of elements $g \in GL_{P(N)}(S)$ such that $id_S^* F = g \cdot G$, for some choice of pullback $id_S^* F$. Hence showing that the map is fully faithfull comes down to checking that the map from the set $\{g \in GL_{P(N)}(S) : g \cdot q_F = F\}$ to $\mathrm{Mor}_{\underline{\mathrm{Bun}}_{r,d}(C)(S)}(F, F)$ sending g to the automorphism of sheaves $\Psi^{pre}((id_S, g)) : F \to F$ is bijection.

We can construct an inverse by composing an automorphism $F \to F$ with the isomorphism $\mathcal{O}_S^{P(N)} \simeq pr_{2,*} pr_1^* \mathcal{O}_C \simeq F(N)$, this then gives us an automorphism $\mathcal{O}_S^{P(N)} \to \mathcal{O}_S^{P(N)}$

which uniquely corresponds to an element in $G(S)$.

Now stackification gives us a fully faithfull map $\Psi_N : [U_N/GL_{P(N)}] \to \underline{\text{Bun}}_{r,d}(C)$. Taking the disjoint union of these maps gives us a morphism $\Psi := \coprod_N \Psi_N : [U_N/GL_{P(N)}] \to \underline{\text{Bun}}_{r,d}(C)$.

It follows from the result by Serre which we discussed earlier, that the map Ψ is a surjection. Let $[E] \in \underline{\text{Bun}}_{r,d}(k)$, then for some $N >> 0$, $E(N)$ is a quotient $\mathcal{O}_C^{P(N)} \simeq \Gamma(C, E(n)) \otimes_k \mathcal{O}_C \twoheadrightarrow E(N)$ hence in particular E is a quotient of $\mathcal{O}^{P(N)}(-N)$ such that $H^1(C, E(N)) = 0$. □

2.3 Sheaves on algebraic stacks

In this section we will give basic definitions and constructions for sheaves on algebraic stacks. For this section algebraic stacks are assumed to have quasi-compact and quasi-separated diagonal, note that with this assumption, by [Stacks, Tag 050N], morphisms of algebraic stacks are quasi-separated. Some good sources on the subject of sheaves on algebraic stacks are Alper's notes [Alp24] and the book [Ols16] and article [Ols07] by Olsson, also see [LM00], [Alo+15] and [Nit08].

In order to define sheaves on algebraic stacks we first have to give a generalization of the lisse-étale site introduced in the first section.

Definition 2.3.1. Let $\mathcal{X}/S$ be an algebraic stack over a scheme S. The lisse-étale site on $\mathcal{X}$, denoted Lis-Ét($\mathcal{X}$), is the site given by the full subcategory of $\mathcal{X}-$schemes whose objects are smooth $\mathcal{X}-$schemes and has coverings given by collections of étale morphisms $\{(U_i,\varphi_i) \to (U,\varphi)\}_{i\in I}$ such that $\coprod_i (U_i,\varphi_i) \to (U,\varphi)$ is surjective. We will often leave the structure morphism out of the notation, writing U instead of (U,φ). The category of sheaves on Lis-Ét($\mathcal{X}$) is denoted $\mathcal{X}_{lis-ét}$.

Remark. Note that the notion of a scheme over a stack is slightly different than that of a scheme over a scheme, in that the compatibility of morphisms with the structure maps results in a 2-commutative diagram, rather than a commutative diagram.

Using the above definition we define a sheaf on an algebraic stack $\mathcal{X}$ to be a sheaf on the lisse-étale site. The structure sheaf of an algebraic stack $\mathcal{X}$ is the sheaf denoted $\mathcal{O}_{\mathcal{X}}$ whose sections over an $\mathcal{X}-$scheme U in Lis-Ét($\mathcal{X}$) are given by $\mathcal{O}_{\mathcal{X}}(U) = \mathcal{O}_U(U)$. The structure sheaf defined in this way is a ring object in the category of abelian sheaves on Lis-Ét($\mathcal{X}$), denoted Ab($\mathcal{X}$), this allows us to define the category Mod($\mathcal{O}_{\mathcal{X}}$) of $\mathcal{O}_{\mathcal{X}}$-modules. To be precise an $\mathcal{O}_X-$module $\mathcal{F}$ is a sheaf on Lis-Ét($\mathcal{X}$) which to every object $U \in$ Lis-Ét($\mathcal{X}$) associates an $\mathcal{O}_{\mathcal{X}}(U)$-module $\mathcal{F}(U)$ whose module structure is compatible with taking restrictions.

An alternative characterization of $\mathcal{X}_{lis-ét}$ is the following, let $\mathcal{C}$ denote the category consisting of

1. For each $(U,u) \in$ Lis-Ét($\mathcal{X}$) a sheaf $F_{(U,u)}$ on $U_{ét}$.
2. For each morphism $(\varphi,\alpha) : (U',u') \to (U,u)$, where α is the 2-isomorphism giving compatibility of the structure morphisms, a pair of adjoint morphisms of sheaves $\theta_{\varphi,\alpha} : F_{(U,u)} \to \varphi_* F_{U',u'}$ and $\theta^\#_{\varphi,\alpha} : \varphi^{-1} F_{(U,u)} \to F_{U',u'}$.
   Furthermore, this data has to satisfy the following conditions
   - For any composition $(U'',u'') \overset{(\varphi_1,\alpha_1)}{\longrightarrow} (U',u') \overset{(\varphi_2,\alpha_2)}{\longrightarrow} (U,u)$ in Lis-Ét($\mathcal{X}$) the following diagram commutes

$$\begin{array}{ccc}
\varphi_1^{-1}\varphi_2^{-1}F_{(U,u)} & \xrightarrow{\varphi_1^{-1}\theta_{(\varphi_2,\alpha_2)}} & F_{(U',u')} \\
\downarrow{\simeq} & & \downarrow{\theta_{(\varphi_1,\alpha_1)}} \\
(\varphi_2\varphi_1)^{-1}F_{(U,u)} & \xrightarrow{\theta_{(\varphi_2,\alpha_2)\circ(\varphi_1,\alpha_1)}} & F_{(U'',u'')}
\end{array}$$

- For any étale $\mathcal{X}$-morphism $(\varphi,\alpha) : (U',u') \to (U,u)$ the induced morphism $\theta^{\#}_{(\varphi,\alpha)}$ is an isomorphism.

3. A morphism $(\{F_{(-)}\},\{\theta_{(-)}\}) \to (\{G_{(-)}\},\{\eta_{(-)}\})$ in $\mathcal{C}$ is a collection of morphisms $\psi_{(U,u)} : F_{(U,u)} \to G_{(U,u)}$ for each $(U,u) \in$ Lis-Ét$(\mathcal{X})$, such that for any morphism $(\varphi,\alpha) : (U',u') \to (U,u)$ the following diagram commutes

$$\begin{array}{ccc}
\varphi^{-1}F_{(U,u)} & \xrightarrow{\varphi^{-1}\psi_{(U,u)}} & \varphi^{-1}G_{(U,u)} \\
\downarrow{\theta_{(\varphi,\alpha)}} & & \downarrow{\eta_{(\varphi,\alpha)}} \\
F_{(U',u')} & \xrightarrow{\psi_{(U',u')}} & G_{(U',u')}
\end{array}$$

There is an equivalence of categories $\mathcal{C} \to \mathcal{X}_{lis-ét}$, see [Ols16, Prop. 9.1.12].

Remark. For a sheaf $\mathcal{F} \in \mathcal{X}_{lis-ét}$ and an object $(U,u) \in$ Lis-Ét$(\mathcal{X})$ we will use the notiation $\mathcal{F}_{(U,u)}$ to mean the restriction of $\mathcal{F}$ to $U_{ét}$.

Given a morphism of algebraic stacks $F : \mathcal{X} \to \mathcal{Y}$, there are adjoint functors

$$\mathcal{X}_{lis-ét} \underset{F^{-1}}{\overset{F_*}{\rightleftarrows}} \mathcal{Y}_{lis-ét} \qquad \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \underset{F^*}{\overset{F_*}{\rightleftarrows}} \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}})$$

Let (Y,y) be an object in $\mathcal{Y}_{lis-ét}$. We define a category denoted $\mathcal{I}(Y,y)$ which has as its objects triples $(x : X \to \mathcal{X}, f : X \to Y, \alpha : y \circ f \xrightarrow{\sim} F \circ x)$, where $(X,x) \in$ Lis-Ét$(\mathcal{X})$, these triples correspond to 2-commutative diagrams

$$\begin{array}{ccc}
X & \xrightarrow{x} & \mathcal{X} \\
f\downarrow & \swarrow \alpha & \downarrow F \\
Y & \xrightarrow{y} & \mathcal{Y}
\end{array}$$

A morphism is given by a triple $(h,\beta,\gamma) : (x',f',\alpha') \to (x,f,\alpha)$ consisting of a morphism $h : X' \to X$ and 2-isomorphisms $\beta : x' \xrightarrow{\sim} x \circ h$ and $\gamma : f' \xrightarrow{\sim} f \circ h$.

Now consider the functor $\eta_{(Y,y)} : \mathcal{I}(Y,y)^{op} \to Y_{ét}$ which sends

- an object (x, f, α) to the sheaf $f_*\mathcal{F}_{(X,x)}$ which is defined by $(Y' \to Y) \mapsto F_{(X,x)}(X \times_Y Y' \to X)$, where $X \times_Y Y' \to X$ is the projection, which is étale since étale morphisms are stable under base change.
- A morphism (h, β, γ) is sent to the morphism $f_*\theta_{h,\beta} : f_*\mathcal{F}_{(X,x)} \to f'_*\mathcal{F}_{(X',x')}$, this makes sense because of the identifications $\gamma : f' \xrightarrow{\sim} f \circ h$ and $f_*h_* \simeq (f \circ h)_*$. On (Y, y) we define the pushforward of $\mathcal{F}$ by F to be the limit $(F_*\mathcal{F})_{(Y,y)} = \lim_{\mathcal{I}(Y,y)} \eta_{(Y,y)}$.

If F is representable by schemes the pushforward is simply the sheaf

$$(Y, y) \mapsto \mathcal{F}(Y \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X})$$

where $Y \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is the projection, which is clearly in Lis-Ét$(\mathcal{X})$ by smoothness of y. In a similar fashion one can define the inverse image functor $F^{-1}\mathcal{G}$ for a sheaf $\mathcal{G} \in \mathcal{Y}_{lis-ét}$. In particular if F is smooth it induces a functor Lis-Ét$(\mathcal{X}) \to$ Lis-Ét$(\mathcal{Y})$, $(X, x) \mapsto (Y, F \circ x)$ and we denote the direct image of $\mathcal{G}$ by F as $\mathcal{G}_{\mathcal{Y},F}$ which is defined via $(X, x) \mapsto \mathcal{G}(F \circ x)$.

Furthermore the functor F^* is defined by sending an $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{M}$ to the $\mathcal{O}_{\mathcal{X}}$-module $F^*\mathcal{M} := F^{-1}\mathcal{M} \otimes_{F^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}$, where the tensor product is the sheafification of $U \mapsto F^{-1}\mathcal{M}(U) \otimes_{F^{-1}\mathcal{O}_{\mathcal{Y}}(U)} \mathcal{O}_{\mathcal{X}}(U)$.

For a sheaf $\mathcal{F}$ on an algebraic stack $\mathcal{X}$ over a scheme S with structure morphism $\varphi : \mathcal{X} \to S$ we define its global sections by $\mathcal{F}(\mathcal{X}) := (\varphi_*\mathcal{F})_{(S,\mathrm{id}_S)}(S))$. In addition for a morphism $F : \mathcal{X} \to \mathcal{Y}$ the sections of the pushforward over an object $(Y, y) \in$ Lis-Ét$(\mathcal{Y})$ are given by $F_*\mathcal{F}(Y, y) := (p_{2,*}(\mathcal{F}_{\mathcal{X} \times_{\mathcal{Y}} Y, p_1}))_{Y,\mathrm{id}_Y}(Y)$.

Definition 2.3.2. Let $\mathcal{X}$ be an algebraic stack. A sheaf $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ is quasi-coherent if

1. for every $U \in$ Lis-Ét$(\mathcal{X})$, the restriction of $\mathcal{F}$ to the small Zariski site on U denoted $\mathcal{F}|_{U_{Zar}}$ is a quasi-coherent $\mathcal{O}_{U_{Zar}}$−module.
2. For every morphism $f : U \to V \in$ Lis-Ét$(\mathcal{X})$, the induced morphism $f^*(\mathcal{F}|_{V_{Zar}}) \to \mathcal{F}|_{U_{Zar}}$ is an isomorphism.

We denote the category of quasi-coherent sheaves on $\mathcal{X}$ by QCoh$(\mathcal{X})$.

Remark. A quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module with a compatible structure as a ring object.

Lemma 2.3.3. [Ols07, Lemma 6.5] *Let $F : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks, then F^* and F_* preserve quasi-coherence.*

Example 2.3.4. [Kha22, Cor. 6.13],[Hei18, 1.A.a.] Let G be an algebraic group acting on an algebraic space X, then

$$\mathrm{QCoh}([X/G]) \simeq \mathrm{QCoh}^G(X),$$

where $\mathrm{QCoh}^G(X)$ is the category of G-equivariant quasi-coherent sheaves on X.

In particular, consider the multiplicative group $\mathbb{G}_m$ over an algebraically closed field k acting on the affine line $\mathbb{A}^1$ over k by the usual multiplication and a linearization $\mathcal{L} \in \mathrm{Pic}^{G_m}(\mathbb{A}^1)$. Since $\mathcal{L}$ is a quasi-coherent rank one $\mathcal{O}_{\mathbb{A}^1}$-module we have that $\mathcal{L}(\mathbb{A}^1) = k[x] \cdot e$ where e is some section which is unique up to scalar multiple.

We define an integer called the weight of $\mathcal{L}$ denoted $\mathrm{wt}(\mathcal{L})$, which is given by the weight of the $\mathbb{G}_m$-coaction on the section e, i.e., $wt(\mathcal{L}) := d$ with $\rho^\# : \mathcal{L}(\mathbb{A}^1) \to k[y, y^{-1}] \otimes \mathcal{L}(\mathbb{A}^1)$, $\rho^\#(e) = y^d \otimes e$. Then

$$\mathcal{L}([\mathbb{A}^1/\mathbb{G}_m]) = \mathcal{L}(\mathbb{A}^1)^{\mathbb{G}_m} = \begin{cases} k \cdot x^{-d}e & \text{if } \mathrm{wt}(\mathcal{L}) = d \leq 0 \\ 0 & \text{if } \mathrm{wt}(\mathcal{L}) = d > 0 \end{cases},$$

see [Hei18, 1.A.a.].

Now we want to introduce the stack theoretic versions of relative spectra and Proj constructions.

Definition 2.3.5. Let $\mathcal{A}$ be an quasi-coherent $\mathcal{O}_{\mathcal{X}}-$algebra. The relative spectrum of $\mathcal{A}$ denoted $\underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A})$ is the algebraic stack consisting of

1. objects which are pairs $(x : T \to \mathcal{X}, \varphi)$ where $x \in \mathcal{X}(T)$ and $\varphi : x^*\mathcal{A} \to \mathcal{O}_T$ is a map of $\mathcal{O}_T-$algebras.
2. A morphism $(x' : T' \to \mathcal{X}, \varphi') \to (x : T \to \mathcal{X}, \varphi)$ is given by a pair of morphisms $(f : T' \to T, f^b : x' \to x)$ where f^b is a morphism over f such that the following diagram commutes

$$\begin{array}{ccc} x'^*\mathcal{A} & \xrightarrow{\quad f^b \quad} & f^*x^*\mathcal{A} \\ & {}_{\varphi'}\searrow \quad \swarrow_{f^*\varphi} & \\ & \mathcal{O}_{T'} & \end{array}$$

Definition 2.3.6. Let $\mathcal{A}$ be an quasi-coherent graded $\mathcal{O}_{\mathcal{X}}-$algebra. The relative Proj of $\mathcal{A}$ denoted $\underline{\mathrm{Proj}}_{\mathcal{X}}(\mathcal{A})$ is the algebraic stack consisting of

1. objects which are pairs $(x : T \to \mathcal{X}, \varphi)$ where $x \in \mathcal{X}(T)$ and $\varphi : T \to \mathrm{Proj}_T(x^*\mathcal{A})$ is a section of the T−scheme $\underline{\mathrm{Proj}}_T(x^*\mathcal{A})$.
2. A morphism $(x' : T' \to \mathcal{X}, \varphi') \to (x : T \to \mathcal{X}, \varphi)$ is given by a pair of morphisms $(f : T' \to T, \tilde{f} : x' \to x)$ where $\tilde{f}$ is a morphism over f such that the following diagram commutes

$$\begin{array}{ccc} T' & \xrightarrow{\quad f \quad} & T \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} \\ \underline{\mathrm{Proj}}_{T'}(x'^*\mathcal{A}) & \xrightarrow{\tilde{f}} & \underline{\mathrm{Proj}}_T(x^*\mathcal{A}) \end{array}$$

Remark. We have sneakily inserted stackyness and algebraicity into the above definitions, this however is not too difficult to prove, we refer the reader to [Ols16, Sect. 10.2].

Next we introduce a variant of the stack theoretic relative Proj, following [QR22], which is described by a quotient stack. We will work on schemes as this is sufficient for our purposes, it is however possible to work on general algebraic stacks using the notions of actions on stacks and quotients of such actions introduced in the article [Rom05], alternatively there is also a discussion of this construction in Olsson's book [Ols16, Sect. 10.2] which is not directly identified with a quotient stack.

Let $A = \bigoplus_{n \geq 0} A_n$ be a quasi-coherent graded $\mathcal{O}_X$-algebra and write A_+ for the ideal of A generated by $\bigoplus_{n>0} A_n$. There is a natural action of the multiplicative group scheme $\mathbb{G}_m$ on $\underline{\mathrm{Spec}}_X(A)$ induced by the coaction $A \to A \otimes_{\mathcal{O}_X} \mathcal{O}_X[x, x^{-1}]$ that sends a degree d section a to $x^d a$. Let $\mathbb{V}(A)^\circ$ denote the complement of the closed subscheme of $\underline{\mathrm{Spec}}_X(A)$ defined by the ideal A_+, then there is an induced action of $\mathbb{G}_m$ on $\mathbb{V}(A)^\circ$ and we define this version of the relative Proj denoted $\mathcal{P}\mathrm{roj}_X(A)$ to be the quotient stack $[\mathbb{V}(A)^\circ / \mathbb{G}_m]$.

This construction comes with a universal property [QR22, Prop. 1.5.1], which we will not write out here, more important to us is the following consequence [QR22, Prop. 1.7.1]: Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks and $\mathcal{L}$ an invertible sheaf on $\mathcal{X}$. Then if for every $x \in X$, there exists an integer $N > 0$ such that $f^* f_* \mathcal{L}^{\otimes N} \to \mathcal{L}^{\otimes N}$ is surjective at x there exists a morphism $\varphi_{\mathcal{L}} : \mathcal{X} \to \mathcal{P}\mathrm{roj}_{\mathcal{Y}}(A)$.

Chapter 3

Good Moduli spaces

A good moduli space for an algebraic stack generalizes the notion of a good quotient to a stack theoretic setting. In this chapter we will introduce the notion of a good moduli space, we will see that it shares many of the properties of good quotients and that important results from Mumford's geometric invariant theory can be generalized for these good moduli spaces. Throughout this chapter schemes are assumed to be quasi-separated, an algebraic space over a scheme S is assumed to be quasi-separated over S and an algebraic stack over S is assumed to have separated and quasi-compact diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$. The main references for this chapter are [Alp13], [AE12] and [Alp24].

3.1 Cohomological affineness

In this section we will introduce the notion of cohomological affineness, which is one of the defining properties of good moduli spaces. We list some of its features and work out an example. Furthermore, we will also generalize the notion of linear reductivity introduced in chapter one.

Definition 3.1.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say that f is cohomologically affine if f is quasi-compact and the induced functor

$$f_* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$$

is exact.

We will now list some useful results for cohomological affineness.

Proposition 3.1.2. [Alp13, Prop. 3.3, 3.10, Lemma 4.9]

1. *Cohomologically affine morphisms are stable under composition.*
2. *Affine morphisms are cohomologically affine.*
3. *[Alp13, Prop. 3.3](Serre's criterion) A quasi-compact morphism of algebraic spaces is affine if and only if it is cohomologically affine.*
4. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks over a scheme S, and suppose that g is quasi affine or $\mathcal{Z}$ has quasi affine diagonal over S. Then if $g \circ f$ is cohomologically affine and g has affine diagonal, f is cohomologically affine.*
5. [Alp13, Lemma 4.9] *(Analogue of Nagata's fundamental lemmas) If $f : \mathcal{X} \to \mathcal{Y}$ is cohomologically affine and $\mathcal{I}$ and $\mathcal{J}$ are a pair of quasi-coherent ideal sheaves on $\mathcal{X}$ then*

   (i) $f_*\mathcal{O}_{\mathcal{X}}/f_*\mathcal{I} \simeq f_*(\mathcal{O}_{\mathcal{X}}/\mathcal{I})$

   (ii) $f_*\mathcal{I} + f_*\mathcal{J} \simeq f_*(\mathcal{I} + \mathcal{J})$.

   Consider the 2-cartesian diagram of algebraic stacks

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ \downarrow{\scriptstyle g'} & \square & \downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

6. *If g is faithfully flat and f' is cohomologically affine, then f is cohomologically affine.*
7. *If f is cohomologically affine and g is quasi-affine, then f' is cohomologically affine.*
8. *If f is cohomologically affine and $\mathcal{Y}$ has quasi-affine diagonal, then f' is cohomologically affine.*

We will now give a generalization of linear reductivity.

Definition 3.1.3. Let G/S be a faithfully flat, finitely presented and separated group scheme over a scheme S. We say that G/S is linearly reductive if $BG \to S$ is cohomologically affine.

Remark. In the case that S is the spectrum of some algebraically closed field, we can use the fact that the category of quasi-coherent $\mathcal{O}_{BG}$-modules is equivalent to the category of G-representations, see [Sch18, Thm. 2.1]. The characterization of a linearly reductive group scheme as a group scheme for which the functor $V \mapsto V^G$ taking a G-representation to its G-invariants is exact then implies that the above definition is equivalent to the classical definition from chapter one.

We conclude this section with an example.

Example 3.1.4. Let G/S be a linearly reductive group scheme with an action $\rho : G \times X \to X$ on a scheme X/S affine over S. Then the induced map $\varphi : [X/G] \to S$ is cohomologically affine. Since the square

$$\begin{array}{ccc} G \times X & \longrightarrow & G \\ \downarrow & \square & \downarrow \\ X & \longrightarrow & S \end{array}$$

is cartesian, it follows from [Stacks, Tag 04ZN] that the square

$$\begin{array}{ccc} X & \longrightarrow & [X/G] \\ \downarrow & \square & \downarrow \\ S & \longrightarrow & BG \end{array}$$

is cartesian. Since $X \to S$ is affine and $S \to BG$ is smooth and surjective it follows from Proposition 2.1.26 that $[X/G] \to BG$ is affine and since G is linearly reductive $BG \to S$ is cohomologically affine, thus the composition $[X/G] \to BG \to S$ is cohomologically affine.

3.2 Good moduli spaces and the GIT quotient

We will now finally give the definition of a good moduli space.

Definition 3.2.1. Let $\pi : \mathcal{X} \to X$ be a quasi-compact and quasi-separated morphism of algebraic stacks where X is an algebraic space. We say that f is a good moduli space if

1. π is cohomologically affine, and
2. $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.

Remark. There also is the notion of a tame moduli space, see [Alp13, Sect. 7], which is a good moduli space that also is a bijection of sets on geometric points. In particular for a locally noetherian algebraic stack a tame moduli space is a good moduli space as well as a coarse moduli space. In some sense the notion of a tame moduli space generalizes geometric quotients. Also see [AOV08] for more background on the theory of tame stacks.

The earlier mentioned relation with the notion of a good quotient will become clear once we list some properties of these good moduli.

Proposition 3.2.2. [Alp13, Thm. 4.16, 6.6, Prop. 4.7] *Let $\pi : \mathcal{X} \to X$ be a good moduli space, then*

1. *π is surjective.*
2. *π is universally closed.*
3. *If $\mathcal{Z}_1, \mathcal{Z}_2$ are closed substacks of $\mathcal{X}$, then*

$$im(\mathcal{Z}_1) \cap im(\mathcal{Z}_2) = \mathrm{im}(\mathcal{Z}_1 \cap \mathcal{Z}_2)$$

   where the intersections and images are scheme-theoretic.

4. *Let k be an algebraically closed field, there is an equivalence relation on the isomorphism classes of geometric points in $\mathcal{X}(k)$ given by $x_1 \sim x_2$ if $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$ in the base change $|\mathcal{X}_k|$. This relation induces a bijection $[\mathcal{X}(k)]/\!\sim \longrightarrow X(k)$. In particular π induces a bijection on closed points.*
5. [Alp13, Thm. 4.16, 6.6] *If $\mathcal{X}$ is locally noetherian then π is universal for maps to algebraic spaces, i.e., the natural map $Mor(X,Y) \to Mor(\mathcal{X},Y)$ is a bijection of sets. If Y is a scheme we can drop the locally noetherian assumption.*
6. [Alp13, Prop. 4.7] *Suppose*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ \downarrow{\scriptstyle \pi'} & \square & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{g} & X \end{array}$$

is a cartesian diagram of algebraic stacks with X' and X algebraic spaces. Then
(i) If π is a good moduli space, then π' is a good moduli space.
(ii) If g is fpqc and π' is a good moduli space, then π is a good moduli space.

Once one looks at some of the proofs of some of these results it starts to become clear why we require cohomological affineness in the definition of a good moduli space. Property 3 for example follows from Proposition 3.1.2.5 and 4 is a consequence of 3, in particular we see that 4 is a clear analogue for our discussion of good quotients in chapter one where we introduced the notion of S-equivalence.

Example 3.2.3. Let G/S be a linearly reductive group scheme with an action $\rho : G \times X \to X$ on a scheme X/S affine over S. Then the map $\pi : [X/G] \to \underline{\mathrm{Spec}}_S(\varphi_* \mathcal{O}_{[X/G]})$, obtained from the factorization

$$\begin{array}{ccc} [X/G] & \xrightarrow{\varphi} & S \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle f} & \\ \underline{\mathrm{Spec}}_S(\varphi_* \mathcal{O}_{[X/G]}) & & \end{array}$$

where $\varphi : [X/G] \to S$ is the cohomologically affine map from Example 3.1.4, f is affine and by construction $\mathcal{O}_{\underline{\mathrm{Spec}}_S(\varphi_* \mathcal{O}_{[X/G]})} \simeq \pi_* \mathcal{O}_{[X/G]}$, is a good moduli space.
For cohomological affineness of π write $\mathcal{X} := [X/G]$, $X := \underline{\mathrm{Spec}}_S(\varphi_* \mathcal{O}_{[X/G]})$ and consider the pair of 2-cartesian diagrams

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{(\mathrm{id}_{\mathcal{X}},\pi)} & \mathcal{X} \times_S X \\ \downarrow & \square & \downarrow \\ X & \xrightarrow{\Delta} & X \times_S X \end{array} \qquad \begin{array}{ccc} \mathcal{X} \times_S X & \xrightarrow{p_2} & X \\ \downarrow & \square & \downarrow \\ \mathcal{X} & \longrightarrow & S \end{array}$$

In the left diagram, since affine morphisms are stable under base change and $X \to S$ is affine, the diagonal $X \xrightarrow{\Delta} X \times_S X$ is affine and therefore by Proposition 2.1.26 the map $(\mathrm{id}_{\mathcal{X}}, \pi) : \mathcal{X} \to \mathcal{X} \times_X X$ is affine, hence also cohomologically affine. In the righthand diagram the morphism $\mathcal{X} \to S$ is cohomologically affine and S is a scheme and therefore has quasi-affine diagonal, thus by Proposition 3.1.2.8 $\mathcal{X} \times_S X \xrightarrow{p_2} X$ is cohomologically affine. It follows that the composition $p_2 \circ (\mathrm{id}_{\mathcal{X}}, \pi) = \pi$ is cohomologically affine, hence a good moduli space.

Remark. In fact if G/S is a smooth affine linearly reductive group scheme acting on a scheme, or more generally an algebraic space X/S, and $[X/G] \to S$ is cohomologically affine, then $X \to S$ is affine. This follows from Proposition 3.1.2.3. Since $X \to S$ is the composition $X \to [X/G] \to S$, by Proposition 2.1.26 and affineness of $G \to S$ we can deduce from the cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & [X/G] \\ \uparrow & \square & \uparrow \\ G \times_S X & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ G & \longrightarrow & S \end{array}$$

that $X \to [X/G]$ is affine and by assumption $[X/G] \to S$ is cohomologically affine, thus the composition $X \to S$ is affine.

Extending this further, we can say that if there exists a G-invariant morphism $\varphi : X \to Y$, then the induced morphism $\pi : [X/G] \to Y$ is a good moduli space if and only if

1. φ is affine.
2. $\mathcal{O}_Y \to \varphi_*(\mathcal{O}_X)^G$ is an isomorphism.

Once we observe that $\pi_*\mathcal{O}_{[X/G]} \simeq \varphi_*(\mathcal{O}_X)^G$ this follows from the above.

Recalling the definition of a good quotient from chapter one, if we take S to be the spectrum of an algebraically closed field, this last result gives us a clear correspondence between good quotients and good moduli, namely $\varphi : X \to Y$ is a good quotient if and only if $\pi : [X/G] \to Y$ is a good moduli space.

The above discussion can be further generalized for arbitrary stacks, via the following generalization of stability for algebraic stacks.

Definition 3.2.4. Let $p : \mathcal{X} \to S$ be an algebraic stack, quasi compact over S, $\mathcal{L}$ an invertible sheaf on $\mathcal{X}$ and $x \in \mathcal{X}(k)$ a geometric point with $p(x) = s \in S(k)$. Then

1. x is pre-stable if there exists an open substack $\mathcal{U} \subset \mathcal{X}$ containing x which is cohomologically affine over S and has closed orbits. We denote the pre-stable locus by $\mathcal{X}^s_{pre}$.
2. x is semi-stable with respect to $\mathcal{L}$ if there is an open subscheme $U \subset S$ containing s and a section $\sigma \in \mathcal{L}^n(p^{-1}(U))$ for some $n > 0$ such that $\sigma(x) \neq 0$ and $p^{-1}(U)_\sigma \to U$ is cohomologically affine. We denote the semi-stable locus by $\mathcal{X}^{ss}_{\mathcal{L}}$.
3. x is stable with respect to $\mathcal{L}$ if there is an open subscheme $U \subset S$ containing s and a section $\sigma \in \mathcal{L}^n(p^{-1}(U))$ for some $n > 0$ such that $\sigma(x) \neq 0$, $p^{-1}(U)_\sigma \to U$ is cohomologically affine and $p^{-1}(U)_\sigma$ has closed orbits. We denote the stable locus by $\mathcal{X}^s_{\mathcal{L}}$.

Example 3.2.5. [Alp13, Chapt. 13.5] Consider the action of a linearly reductive group scheme over an algebraically closed field k on a quasi-compact k-scheme X and let L be a G-linearization on X. If $\mathcal{L}$ is the corresponding line bundle on $[X/G]$, then scheme theoretic GIT stability with respect to the linearization L and the above notion of stability coincide in the sense that $[X/G]_{\mathcal{L}}^{(s)s} = [X_L^{(s)s}/G]$.

Remark. [ER21, Rem. 3.16] An alternative characterization of the semistable locus $\mathcal{X}_{\mathcal{L}}^{ss}$ is the following. Let $\mathcal{U} \subset \mathcal{X}$ be the open locus where $p^*p_*\mathcal{L}^N \to \mathcal{L}^N$ is surjective for some positive integer N and let $V \subset \mathcal{P}\mathrm{roj}_S(\otimes_{n\geq 0} p_*\mathcal{L}^n)$ be the largest open subset over which the induced morphism $\varphi_{\mathcal{L}} : \mathcal{U} \to \mathcal{P}\mathrm{roj}_S(\otimes_{n\geq 0} p_*\mathcal{L}^n)$ is cohomologically affine. Then we have $\mathcal{X}_{\mathcal{L}}^{ss} = \varphi_{\mathcal{L}}^{-1}(V)$ so that $\mathcal{X}_{\mathcal{L}}^{ss} \to V$ is a good moduli space.

The above remark gives us the following analogue of Theorem 1.2.17.

Theorem 3.2.6. [Alp13, Thm. 11.5] *Let $p : \mathcal{X} \to S$ be an algebraic stack, quasi-compact over S, and $\mathcal{L}$ an invertible sheaf on $\mathcal{X}$. Then*

1. *There exists a good moduli space $\pi : \mathcal{X}_{\mathcal{L}}^{ss} \to X$ with X an open subscheme of $\mathcal{P}roj_S(\bigoplus_{n\geq 0} p_*\mathcal{L}^n)$.*
2. *There is an open subscheme $U \subset X$ such that $\pi^{-1}(U) = \mathcal{X}_{\mathcal{L}}^s$ and $\pi|_{\mathcal{X}_{\mathcal{L}}^s} : \mathcal{X}_{\mathcal{L}}^s \to U$ is a tame moduli space.*

Chapter 4

Existence criterion for good moduli spaces

In this chapter we will introduce a recent result giving us an existence criterion for good moduli spaces. Namely in the article, [AHH23] Alper, Halpern-Leistner and Heinloth prove that the existence of a good moduli space for a certain class of algebraic stacks depends on the fulfillment of two valuative criteria called Θ-reductivity and S-completeness. We will start this chapter off with a different result given by Halpern-Leistner in his article [Hal22], which gives us a description of the so called filtrations of an algebraic stack, which are maps from a certain quotient stack. This latter result will prove to be a useful tool in this chapter as well as the next chapter where we introduce a numerical criterion for the stability of points in algebraic stacks.

4.1 Filtrations

As noted in the introduction to this chapter we will dedicate this section to introducing the notion of filtrations and give a description of these filtrations.

Definition 4.1.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be a pair of stacks over a scheme S. We define the mapping stack denoted $\underline{\mathrm{Map}}_S(\mathcal{Y},\mathcal{X})$ to be the prestack with fiber category over an S-scheme T given by $\mathrm{Mor}_S(\mathcal{Y}\times T,\mathcal{X})$.

Over $\mathrm{Spec}(\mathbb{Z})$ we write $\Theta := [\mathbb{A}^1/\mathbb{G}_m]$ where the multiplicative group acts on the affine line by the standard multiplication, and we let $B\mathbb{G}_m := [\mathrm{Spec}(\mathbb{Z})/\mathbb{G}_m]$ denote the classifying stack of the multiplicative group. For a scheme S we write $\Theta_S := \Theta\times S$ and $B\mathbb{G}_{m,S} := B\mathbb{G}_m\times S$.

Definition 4.1.2. For an algebraic stack $\mathcal{X}\to S$ over a scheme S we define and denote the stack of filtered points in $\mathcal{X}$ to be the following mapping stack

$$\mathrm{Filt}(\mathcal{X}) := \underline{\mathrm{Map}}_S(\Theta_S, \mathcal{X}).$$

Note that for $\mathbb{A}^1 \times T \to \Theta_T$ is a smooth presentation. Then using the generalization of the Yoneda lemma given in section 2.1.4 together with the cartesian diagrams given in section 2.2.1 we can describe the fiber categories $\mathrm{Filt}(\mathcal{X})(T) = \mathrm{Mor}(\Theta_T, \mathcal{X})$ as the categories of descent data associated to the sequences

$$\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}^1 \times T \overset{m}{\underset{a}{\overset{-\rho\to}{\rightrightarrows}}} \mathbb{G}_m \times \mathbb{A}^1 \times T \overset{\rho}{\underset{a}{\rightrightarrows}} \mathbb{A}^1 \times T$$

where m denotes the group multiplication, ρ is the action of $\mathbb{G}_m$ on $\mathbb{A}^1$ and a is the map that forgets the leftmost group element. Thus, the category $\mathrm{Filt}(\mathcal{X})(T)$ consists of

- (Objects): An object is a pair (x, ψ) where $x \in \mathcal{X}(\mathbb{A}^1 \times T)$ and $\psi : a^*x \to \rho^*x$ a morphism satisfying the cocycle condition $\rho^*\psi \circ a^*\psi = m^*\psi$.
- (Morphisms): A morphism $(x, \psi) \to (x', \psi')$ is given by a morphism $\eta : x \to x'$ such that $\psi' \circ a^*(\eta) = \rho^*(\eta) \circ \psi$.

In the paper [Hal22] Halpern-Leistner gives an even nicer description of these filtrations by identifying them with certain quotient stacks. This identification will prove to be useful in upcoming sections, the rest of this section is dedicated to formulating this result.

Let k be an algebraically closed field, G a smooth algebraic group over k acting on a quasi-separated algebraic space over k and $\lambda : \mathbb{G}_m \to G$ a cocharacter. We define the functors

$$L_\lambda := \{l \in G : l = \lambda(t) l \lambda(t)^{-1} \ \forall t\},$$

$$P_\lambda^+ := \{p \in G : \lim_{t \to 0} \lambda(t) p \lambda(t)^{-1} \text{ exists}\},$$

$$X_\lambda^0 := \underline{\mathrm{Map}}_k^{\mathbb{G}_m}(\mathrm{Spec}(k), X),$$

$$X_\lambda^+ := \underline{\mathrm{Map}}_k^{\mathbb{G}_m}(\mathbb{A}^1, X).$$

The first two functors are self explanatory, the latter two functors are $\mathbb{G}_m$−equivariant mapping stacks. The first of the two associates to a k-scheme T, $\mathbb{G}_m$-equivariant maps $T \to X$ where $\mathbb{G}_m$ acts trivially on T and the action on X is induced by the action of the group G and the cocharacter λ. The second functor associates to a k-scheme T a $\mathbb{G}_m$-equivariant map $T \times \mathbb{A}^1 \to X$ where $\mathbb{G}_m$ acts trivially on T and with the scaling action on $\mathbb{A}^1$ and again via the action of G and the map λ on X.

When X is separated there is a particularly nice description for the k-points of X_λ^+, given in [Dri15, Sect. 1.3], namely consider the map

$$ev_1^\lambda : X_\lambda^+(k) \to X(k)$$

given by evaluating morphisms $\mathbb{A}^1 \to X$ at 1, i.e., compose with $1 : \mathrm{Spec}(k) \to \mathbb{A}^1$. The collection of k-points $x := ev_1^\lambda(\mathbb{A}^1 \to X)$ can be identified with the set $\{\lambda(-)\cdot x : \mathbb{G}_m \to X\}$ so that $X_\lambda^+(k)$ surjects onto the set of maps $\lambda(-)\cdot x$ with $x \in X(k)$ which can be extended to $\mathbb{A}^1$, i.e.,

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\lambda(-)\cdot x} & X \\ \big\downarrow & \nearrow_{\exists f_{x,\lambda}} & \\ \mathbb{A}^1 & & \end{array}$$

and since X is separated by the valuative criterion these extensions are unique if they exist. Now using the notation introduced in our discussion of the Hilbert-Mumford criterion we have that $X_\lambda^+(k)$ is in bijective correspondence with the set $\{x \in X(k) : \lim_{g\to 0} \lambda(g)\cdot x \text{ exists}\}$.

We are now ready to formulate the result.

Theorem 4.1.3. [Hal22, Thm. 1.4.8] *Let G be a smooth affine algebraic group over an algebraically closed field k with split maximal torus acting on a quasi-separated algebraic space locally of finite type over k. Then there exists an isomorphism*

$$\mathit{Filt}([X/G]) \simeq \bigsqcup_{\lambda\in\Lambda} [X_\lambda^+/P_\lambda^+]$$

where $\Lambda = Hom(\mathbb{G}_m, G)/(\lambda \sim g\lambda g^{-1},\ \forall g \in G)$.

Remark. The isomorphism in the Theorem 4.1.3 is induced by maps $\Theta \times [X_\lambda^+/P_\lambda^+] \to [X/G]$ which on k-points are given by $\mathbb{A}^1 \times X \ni (a,x) \mapsto \lambda(a)\cdot x \in X$, see [Hal13, Lemma 4.2.1].

In addition to the a stack of filtered points there also is a stack of so called graded points which is denoted and defined by:

$$\mathrm{Grad}(\mathcal{X}) := \underline{\mathrm{Map}}_S(B\mathbb{G}_{m,S}, \mathcal{X})$$

For quotient stacks, as in 4.1.3, Halpern-Leistner ([Hal22, Thm. 1.4.8]) also shows that there are isomorphisms

$$\mathrm{Grad}([X/G]) \simeq \bigsqcup_{\lambda\in\Lambda} [X^0_\lambda / L_\lambda].$$

4.2 Θ-Reductivity

If R is a discrete valuation ring with maximal ideal m_R generated by the uniformizer π, field of fractions $K := \mathrm{Frac}(R)$ and residue field $\kappa := R/m_R$, then we write $\Theta_R := \Theta \times \mathrm{Spec}(R) \simeq [\mathbb{A}^1_R/\mathbb{G}_m]$ and $0 \in \Theta_R$ is the closed point associated to the maximal ideal (x,π).

Definition 4.2.1. A morphism of locally noetherian algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is Θ-reductive if for every discrete valuation ring R there is a unique dotted arrow filling in any commutative diagram of the form

$$\begin{array}{ccc} \Theta_R \setminus 0 & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow \\ \Theta_R & \longrightarrow & \mathcal{Y} \end{array}$$

We say that an algebraic stack $\mathcal{X}$ over S is Θ-reductive if its structure morphism is Θ-reductive.

Remark. As explained in [Hal22, Warning 5.1.4] for an algebraic stack $\mathcal{X}$ over a scheme S, Θ-reductivity can equivalently be defined by the requirement that the evaluation morphism $ev_1 : \mathrm{Filt}_S(\mathcal{X}) \to \mathcal{X}$ given by $f \mapsto f(1)$ satisfies the valuative criterion for properness with respect to the class of all valuation rings.

Proposition 4.2.2. *Let G be a smooth split reductive algebraic group over a field k acting on a quasi-separated algebraic space X locally of finite type over k. Then*

$$[X/G] \textit{ is } \Theta\textit{-reductive} \iff \forall \lambda \in Hom(\mathbb{G}_m, G),\ ev^\lambda_1 : X^+_\lambda \to X \textit{ is proper.}$$

Proof. Under the identification of filtrations of quotient stacks given in 4.1.3, the morphism $ev_1 : \mathrm{Filt}([X/G]) \to [X/G]$ corresponds to the coproduct of morphisms of quotient stacks: $[ev^\lambda_1] : [X^+_\lambda/P^+_\lambda] \to [X/G]$, induced by the $(P^+_\lambda \to G)$-equivariant morphisms $ev^\lambda_1 : X^+_\lambda \to X$. Now since we have equivalences $[X^+_\lambda/P^+_\lambda] \simeq [G \times^{P^+_\lambda} X^+_\lambda/G]$, the morphisms $[ev^\lambda_1]$ can be factored as follows:

$$[G \times^{P^+_\lambda} X^+_\lambda/G] \to [G \times^{P^+_\lambda} X/G] \to [X/G],$$

induced by the morphisms

$$G \times^{P_\lambda^+} X_\lambda^+ \to G \times^{P_\lambda^+} X \to X$$

which on points are defined by $(g, f) \mapsto (g, f(1)) \mapsto \lambda(g) \cdot x$. Since G is reductive, the P_λ^+ are parabolic subgroups of G and therefore the G/P_λ^+ are projective, in particular the structure morphisms $[G/P_\lambda^+] \to \mathrm{Spec}(k)$ of the associated stacks are proper. Since properness is a property that is stable under base change, it follows from the cartesian diagrams

$$\begin{array}{ccc} [X/P_\lambda^+] & \longrightarrow & [X/G] \\ \downarrow & \square & \downarrow \\ [G/P_\lambda^+] & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

that the compositions are proper if and only if the maps $ev_1^\lambda : X_\lambda^+ \to X$ are proper. □

Corollary 4.2.3. *Let G be a smooth split reductive algebraic group over an algebraically closed field k acting on an affine finite type k-scheme $X := \mathrm{Spec}(R)$. Then $[X/G]$ is Θ-reductive.*

Proof. Let $\lambda \in \mathrm{Hom}(\mathbb{G}_m, G)$, then the $\mathbb{G}_m$-action on X induced by λ corresponds to a grading on $\mathcal{O}_X(X)$ such that the nth graded piece is given by functions $f \in \mathcal{O}_X(X)$ with $(g \cdot f)(x) = f(g^{-1} \cdot x) = f(\lambda(g)^{-1} \cdot x) = g^n f(x)$ for all $g \in \mathbb{G}_m$.

Recall from our discussion in the first section of this chapter that since X is affine and therefore separated over $\mathrm{Spec}(k)$ we can view $X_\lambda^+(k)$ as the collection of points in $x \in X(k)$ for which $\lim_{g \to 0} \lambda(-) \cdot x$ exists, now it is clear that these limits only exist for positively graded pieces, thus X_λ^+ can be identified with the closed subscheme of X given by $\mathrm{Spec}(R/I_-)$ where I_- is the ideal generated by the homogeneous elements of strictly negative degree. We conclude that the morphisms $ev_1^\lambda : X_\lambda^+ \to X$ are closed immersions and therefore in particular proper so that by the previous result $[X/G]$ is indeed Θ-reductive. □

4.3 S-completeness

Let R be a discrete valuation ring with maximal ideal m_R generated by the uniformizer π, field of fractions $K := \mathrm{Frac}(R)$ and residue field $\kappa := R/m_R$. For the action of $\mathbb{G}_m$ on $R[x,y]/(xy-\pi)$ given by $g\cdot(x,y) = (gx, g^{-1}y)$ we write $\overline{\mathrm{ST}}_R := [\mathrm{Spec}(R[x,y]/(xy-\pi))/\mathbb{G}_m]$ and $0 \in \overline{\mathrm{ST}}_R$ is the closed point associated to the maximal ideal (x,y).

Definition 4.3.1. A morphism of locally noetherian algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is **S**-complete if for every discrete valuation ring R there is a unique dotted arrow filling in any commutative diagram of the form

$$\begin{array}{ccc} \overline{\mathrm{ST}}_R \setminus 0 & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow \\ \overline{\mathrm{ST}}_R & \longrightarrow & \mathcal{Y} \end{array}$$

We say that an algebraic stack $\mathcal{X}$ over S is **S**-complete if its structure morphism is **S**-complete.

Remark. Note that the complement of the point 0 in $\overline{\mathrm{ST}}_R$ is isomorphic to two copies of $\mathrm{Spec}(R)$ glued along $\mathrm{Spec}(K)$

$$\overline{\mathrm{ST}}_R \setminus 0 \simeq \mathrm{Spec}(R) \bigcup_{\mathrm{Spec}(K)} \mathrm{Spec}(R).$$

Namely we have the following identifications for the open cover given by the non-vanishing locus of the coordinates x and y

$$\overline{\mathrm{ST}}_R \setminus \{x=0\} \simeq [\mathrm{Spec}(R[x,x^{-1},y]/(xy-\pi))/\mathbb{G}_m] \simeq [\mathrm{Spec}(R[x,x^{-1}])/\mathbb{G}_m] \simeq \mathrm{Spec}(R) \text{ and}$$

$$\overline{\mathrm{ST}}_R \setminus \{y=0\} \simeq [\mathrm{Spec}(R[x,y,y^{-1}]/(xy-\pi))/\mathbb{G}_m] \simeq [\mathrm{Spec}(R[y,y^{-1}])/\mathbb{G}_m] \simeq \mathrm{Spec}(R)$$

so that we have a cocartesian diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \hookrightarrow & \mathrm{Spec}(R) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \hookrightarrow & \overline{\mathrm{ST}}_R \setminus 0. \end{array}$$

Moreover we see that a morphism to an algebraic stack $\varphi : \overline{\mathrm{ST}}_R \setminus 0 \to \mathcal{X}$ is given by the pair $\varphi|_{\{x\neq 0\}}, \varphi|_{\{y\neq 0\}} : R \to \mathcal{X}$ together with an isomorphism of the restrictions $(\varphi|_{\{x\neq 0\}})|_{\mathrm{Spec}(K)} \simeq (\varphi|_{\{y\neq 0\}})|_{\mathrm{Spec}(K)}$. In the terminology of the paper [AHH23, Def. 3.36] these two morphisms are a modification of one another.

We will now list some properties of **S**-completeness given in [AHH23, Sect. 3.5].

Proposition 4.3.2.

1. *$\mathbf{S}$-complete morphisms are stable under composition and base change provided that the locally noetherian assumption is preserved.*
2. *An affine morphisms of locally noetherian algebraic stacks is $\mathbf{S}$-complete.*
3. *Let k be an algebraically closed field, then $BGL_{n,k} \to \mathrm{Spec}(k)$ is $\mathbf{S}$-complete.*
4. *If $\mathcal{X}$ and $\mathcal{Y}$ are locally noetherian and $\mathcal{Y}' \to \mathcal{Y}$ is étale representable and surjective, then $\mathcal{X} \to \mathcal{Y}$ is $\mathbf{S}$-complete if and only if $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ is.*

Proposition 4.3.3. *Let k be an algebraically closed field, $\mathcal{X}$ an algebraic stack of finite type over $\mathrm{Spec}(k)$ with affine diagonal and $\pi : \mathcal{X} \to X$ a good moduli space. Then $\mathcal{X}$ is $\mathbf{S}$-complete if and only if X is separated.*

Proof. ($\Longrightarrow$) Suppose that $\mathcal{X}$ is **S**-complete and that we have a (strictly) commutative diagram

$$\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(R) \overset{f_1}{\underset{f_2}{\rightrightarrows}} X.$$

Then by surjectivity of good moduli, after possibly an extension of $\mathrm{Spec}(R)$, we choose a lift $\mathrm{Spec}(K) \to \mathcal{X}$

$$\begin{array}{ccccc} & & & & \mathcal{X} \\ & \dashrightarrow & & & \downarrow \pi \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(R) & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & X. \end{array}$$

Furthermore, since good moduli are universally closed, by the valuative criterion for universal closedness we may, after possibly extending $\mathrm{Spec}(R)$ once more, choose lifts $\tilde{f}_1, \tilde{f}_2 : \mathrm{Spec}(R) \to \mathcal{X}$ yielding a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathcal{X} \\ \downarrow & \overset{\tilde{f}_1,\tilde{f}_2}{\rightrightarrows} & \downarrow \pi \\ \mathrm{Spec}(R) & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & \mathcal{X}. \end{array}$$

Now by our earlier remark the diagram

$$\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(R) \overset{\tilde{f}_1}{\underset{\tilde{f}_2}{\rightrightarrows}} \mathcal{X}.$$

induces a morphism $\overline{\mathrm{ST}}_R \setminus 0 \simeq \mathrm{Spec}(R) \bigcup_{\mathrm{Spec}(K)} \mathrm{Spec}(R) \to \mathcal{X}$ and by $\mathbf{S}$-completeness of $\mathcal{X}$ this morphism can be extended to a morphism $\overline{\mathrm{ST}}_R \to \mathcal{X}$. Now since $\overline{\mathrm{ST}}_R \to \mathrm{Spec}(R)$ is a good moduli space, by universality of good moduli for morphisms to algebraic spaces we obtain a unique morphism $f : \mathrm{Spec}(R) \to X$. It follows from the unicity of this morphism that we must have $f = f_1 = f_2$ so that X satisifies the valuative criterion for separatedness.

For proof of the ($\Leftarrow$) direction, see [AHH23, Prop. 6.8.20]. □

Note that $\mathbf{S}$-completeness of the quotient of an affine scheme by a reductive group directly follows, namely we have the following.

Corollary 4.3.4. *Let G be a smooth affine linearly reductive group over an algebraically closed field k acting on a finite type affine k-scheme* $\mathrm{Spec}(A)$*. Then* $[\mathrm{Spec}(A)/G]$ *is* $\mathbf{S}$*-complete.*

Proof. We have seen in the previous chapter that $[\mathrm{Spec}(A)/G]$ admits a good moduli space $[\mathrm{Spec}(A)/G] \to \mathrm{Spec}(A^G)$. Then since $\mathrm{Spec}(A^G)$ is separated, by the previous result $[\mathrm{Spec}(A)/G]$ is $\mathbf{S}$-complete. □

Note that in particular the classifying stack of a linearly reductive group is $\mathbf{S}$-complete, this however is a specific example of a much stronger result, namely $\mathbf{S}$-completeness characterizes reductivity in the following sense.

Proposition 4.3.5. *Let G be a smooth affine algebraic group over an algebraically closed field k. Then G is reductive if and only if BG is* $\mathbf{S}$*-complete.*

Proof. ($\Longrightarrow$) Suppose that G is reductive. Note that since G is an affine algebraic group it can be realized as a subgroup of GL_n for some n, and since GL_n is reductive as well it follows from Matsushima's theorem, see [Alp24, Thm. B.1.43], that the quotient GL_n/G is affine. Now since there is a cartesian diagram

$$\begin{array}{ccc} GL_n/G & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & \square & \downarrow \\ BG & \longrightarrow & BGL_n \end{array}$$

where $\mathrm{Spec}(k) \to BGL_n$ is smooth and surjective, it follows from Proposition 2.1.26 that $BG \to BGL_n$ is affine and hence by Proposition 4.3.2.2 $\mathbf{S}$-complete, moreover since $BGL_n \to \mathrm{Spec}(k)$ is $\mathbf{S}$-complete by Proposition 4.3.2.3 and $\mathbf{S}$-completeness is stable under composition we can conclude that $BG \to \mathrm{Spec}(k)$ is $\mathbf{S}$-complete.

($\Leftarrow$) Before we prove the second implication we will first give another property of groups that $\mathbf{S}$-completeness characterizes. □

Namely, in the article [AHH21] Alper, Halpern-Leistner and Heinloth prove that **S**-completeness is also equivalent to existence of Cartan-Iwahori-Matsumoto deompositions. Assume that R is a complete discrete valuation ring and note that a formal power series ring over an algebraically closed field is an example of such a ring. Recall the notation from our discussion of the Hilbert-Mumford criterion in chapter one, for a 1-PS $\lambda : \mathbb{G}_m \to G$ of a smooth affine algebraic group G/k over an algebraically closed field k we write

$$\langle \lambda \rangle := \mathrm{Spec}(K) \xrightarrow{\varphi} \mathbb{G}_m \xrightarrow{\lambda} G$$

where φ is induced by the ring map $k[u, u^{-1}] \to K,\ u \mapsto \pi$. Then we have the following result.

Proposition 4.3.6. [AHH21, Lemma 3.6] *Let G/k be a smooth affine algebraic group over an algebraically closed field k. Then the following are equivalent:*

1. *For any $g \in G(K)$ there exist $h_1, h_2 \in G(R)$ and a homomorphism $\lambda : \mathbb{G}_m \to G$ such that*

$$g = h_1 \cdot \langle \lambda \rangle \cdot h_2.$$

2. *There exists a dotted arrow filling in the commutative diagram*

$$\begin{array}{ccc} \overline{ST}_R \setminus 0 & \xrightarrow{\tau_g} & BG \\ \downarrow & \nearrow & \\ \overline{ST}_R & & \end{array}$$

We will now return to our proof of proposition 4.3.5.

Proof of prop 4.3.5($\Leftarrow$) . We will prove the contrapositive statement. Suppose that G is not reductive. Then since G is not reductive its unipotent radical $\mathcal{R}(G)_u$ is non-trivial. Note that since $\mathcal{R}(G)_u$ is non-trivial and unipotent it admits a normal subgroup $H \simeq \mathbb{G}_a$, see [Mil15, Chapt. 15.b]. Then by [Mil15, Thm. 5.21] the quotients $G/\mathcal{R}(G)_u$ and $\mathcal{R}(G)_u/H$ are affine and using the cartesian diagrams

$$\begin{array}{ccc} G/\mathcal{R}(G)_u & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & \square & \downarrow \\ B\mathcal{R}(G)_u & \longrightarrow & BG \end{array} \qquad \begin{array}{ccc} \mathcal{R}(G)_u/H & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & \square & \downarrow \\ BH & \longrightarrow & B\mathcal{R}(G)_u \end{array}$$

as before we have that $BH \to \mathcal{R}(G)_u$ and $\mathcal{R}(G)_u \to BG$ are affine and therefore that the composition $BH \to \mathcal{R}(G)_u \to BG$ is affine, then if $BG \to \mathrm{Spec}(k)$ is **S**-complete we have that $BH \simeq B\mathbb{G}_a \to \mathrm{Spec}(K)$ is **S**-complete. This however cannot be the case as the additive group does not admit a Cartan-Iwahori-Matsumoto decomposition. To see this, first note that the only 1-PS $\mathbb{G}_m \to \mathbb{G}_a$ is the trivial one. Namely, the elements of $\mathbb{G}_a$ are unipotent, as the group can be embedded in the standard unipotent group $\mathbb{U}_2$ via the map

$$g \mapsto \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}$$

and the elements of $\mathbb{G}_m$ can be realized as diagonal matrices via the obvious map. Since by [Mil15, p. 11.20] homomorphisms $\mathbb{G}_m \to \mathbb{G}_a$ preserve unipotent and semi-simple elements and only the identity element is both unipotent and semisimple, the only homomorphism $\lambda : \mathbb{G}_m \to \mathbb{G}_a$ is the trivial one. Now we can simply take $R := k[\![x]\!]$ with $K := k((x))$, then the element $\frac{1}{x} \in \mathbb{G}_m(K)$ does not admit a decomposition. We conclude that BG is not **S**-complete. □

We can conclude by the above that in our setting we have the following equivalences

$$G \text{ is reductive} \iff G \text{ has Cart.-Iwa.-Mats. decomps.} \iff BG \text{ is } \mathbf{S}\text{-complete}$$

this chain of equivalences can be further generalized for the setting of a smooth affine group scheme over a noetherian scheme, see [AHH21, Thm. 1.3].

Another useful conclusion that we can draw from the above is that a closed point of an **S**-complete locally noetherian algebraic stack with smooth and affine stabilizers has a reductive stabilizer.

4.4 Existence result

We will not state the main result for this chapter.

Theorem 4.4.1. [AHH23, Thm. 5.4] *Let $\mathcal{X}$ be an algebraic stack of finite presentation over a quasi separated and locally noeherian algebraic space S, with affine stabilizers and separated diagonal. Then $\mathcal{X}$ admits a good moduli space X separated over S if and only if*

1. *every closed point of $\mathcal{X}$ has linearly reductive stabilizer.*
2. *$\mathcal{X} \to S$ is Θ-reductive.*
3. *$\mathcal{X} \to S$ is $\mathbf{S}$-complete.*

We will not give a proof for this theorem, we will however lay out the main ingredients and sketch the proof for ($\Leftarrow$) in the case that $S := \mathrm{Spec}(k)$ where k is an algebraically closed field of characteristic zero and $\mathcal{X}$ is an algebraic stack of finite type over $\mathrm{Spec}(k)$ with affine diagonal, following [Alp24]. The simplification to characteristic zero allows us to leave out the first condition since **S**-completeness implies that stabilizers at closed points of $\mathcal{X}$ are reductive and hence linearly reductive by our characteristic zero assumption. Besides [Alp24] and [AHH23] another useful reference is [Hal21, Lec. 15], moreover, for a precursor to the current existence result also see the article [AFS17].

When introducing the main results used in the proof we will, in some cases, state versions of these results that are more general than needed for our application, as this might be of independent interest to the reader, and then modify to the version suitable for our application.

Definition 4.4.2. Let $\mathcal{X}$ be an algebraic stack and $x \in |\mathcal{X}|$. A quotient presentation around x is a pointed flat morphism $f : (\mathcal{W}, w) \to (\mathcal{X}, x)$ of algebraic stacks such that the following holds

1. $\mathcal{W} \simeq [\mathrm{Spec}(A)/\mathrm{GL}_N]$ for some N.
2. f induces an isomorphism of stabilizer groups at the point w.

Theorem 4.4.3. *Assume the following*

1. *S is a quasi-separated algebraic space.*
2. *$\mathcal{X}$ is an algebraic stack locally of finite presentation and quasi separated over S, with affine stabilizers.*
3. *$x \in |\mathcal{X}|$ with image $s \in |S|$ such that the residue field extension is finite and the stabilizer of x is linearly reductive.*

Under these assumptions there exists an étale quotient presentation $f : (\mathcal{W}, w) \to (\mathcal{X}, x)$ around x. Moreover if $\mathcal{X}$ has separated (resp. affine) diagonal, then there exists exists a representable (resp. affine) étale quotient presentation $f : (\mathcal{W}, w) \to (\mathcal{X}, x)$.

Remark. We have stated the version of the local structure theorem for algebraic stacks given in [AHH23, Thm. 2.2]. The version that we will use is the following [Alp24, Thm 6.6.1]: if S is the spectrum of an algebraically closed field of characteristic zero and $\mathcal{X}$ is of finite type over S with affine diagonal, there exists an affine étale quotient presentation $f : (\mathcal{W}, w) \to (\mathcal{X}, x)$ such that $\mathcal{W} \simeq [\mathrm{Spec}(A)/G_x]$ where A is a finite type k-algebra.

We will now introduce two notions which motivate the necessity of Θ-reductivity and **S**-completeness. Namely as will become clear later on, we will want to construct an étale quotient presentation which in some neighborhood preserves closed points and stabilizers. These conditions are captured respectively by the notions, Θ-surjectivity and unpunctured inertia. Roughly speaking Θ-reductivity is required to show that Θ-surjectivity holds and Θ-surjectivity ensures that the étale quotient presentation preserves closed points on some open neighborhood. On the other hand **S**-completeness implies the condition, which we have not defined yet, called unpunctured inertia and this condition in turn implies that the étale quotient presentation is stabilizer preserving.

We will now work towards making this vague explanation more precise. For the sake of completeness, we will begin by defining Θ-surjectivity and unpunctured inertia. Let k be a field, then taking the complement of the closed point $0 \in \Theta_k$ we have an open immersion which we denote by $j : \mathrm{Spec}(k) \simeq \Theta_k \setminus 0 \hookrightarrow \Theta_k$.

Definition 4.4.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and $x \in \mathcal{X}(k)$ a geometric point. Then f is said to be Θ-surjective at x if there exists a morphism $\Theta_k \to \mathcal{X}$ filling in any diagram of the form

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{x} & \mathcal{X} \\ \downarrow{\scriptstyle j} & & \downarrow \\ \Theta_k & \longrightarrow & \mathcal{Y} \end{array}$$

We say that f is Θ-surjective if and only if it is Θ-surjective at every geometric point in $\mathcal{X}$.

Definition 4.4.5. A noetherian algebraic stack is said to have unpunctured inertia if for any closed point $x \in |\mathcal{X}|$ and any formally smooth morphism $p : (U, u) \to (\mathcal{X}, x)$, where U is the spectrum of a local ring of a smooth neighborhood of x with closed point u, each connected component of the inertia groupscheme $\mathrm{Aut}_{\mathcal{X}}(p) \to U$ has non-empty intersection with the fiber over u.

We will now give an important result which is essential to the proof of the existence theorem.

Theorem 4.4.6. [Alp24, Prop. 6.9.12, 6.9.19, Thm. 6.9.22] *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field with affine diagonal such that the closed points in $\mathcal{X}$ have linearly reductive stabilizers. Suppose that $x \in \mathcal{X}$ is a closed point and $f : ([\mathrm{Spec}(A)/G_x], w) \to (\mathcal{X}, x)$ is an affine étale quotient presentation and write $\pi : [\mathrm{Spec}(A)/G_x] \to \mathrm{Spec}(A^{G_x})$. Then*

1. *$\mathcal{X}$ is $\mathbf{S}$-complete $\implies$ $\mathcal{X}$ has unpunctured inertia.*
2. *$\mathcal{X}$ has unpunctured inertia $\implies$ there exists an affine open neighborhood $U \subset \mathrm{Spec}(A^{G_x})$ of $\pi(w)$ such that the restriction $f|_{\pi^{-1}(U)}$ induces isomorphisms on the stabililizer groups for all closed points.*
3. *$\mathcal{X}$ is Θ-reductive $\implies$ there exists an affine open neighborhood $U \subset \mathrm{Spec}(A^{G_x})$ of $\pi(w)$ such that the restriction $f|_{\pi^{-1}(U)}$ is Θ-surjective, in particular this restriction preserves closed points.*

The last result that we want to introduce before beginning our proof is Luna's fundamental lemma and one of its corollaries.

Theorem 4.4.7. [Alp24, Thm. 6.4.27, Cor. 6.4.30] *Consider a commutative diagram*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{g} & X \end{array}$$

where $\mathcal{X}$ and $\mathcal{X}'$ are noetherian algebraic stacks with affine diagonal, $f : \mathcal{X}' \to \mathcal{X}$ is a separated and representable morphism, and where $\pi' : \mathcal{X}' \to X'$ and $\pi : \mathcal{X} \to X$ are good moduli spaces. Let $x' \in \mathcal{X}'$ be a point such that

- *f is étale at x'.*
- *f is stabilizer preserving at x'.*
- *$x' \in \mathcal{X}'$ and $f(x') = x \in \mathcal{X}$ are closed points.*

Then there is an open neighborhood $U' \subset X'$ of $\pi'(x')$ such that $U' \to X$ is étale and such that $U' \times_X \mathcal{X} \simeq \pi'^{-1}(U')$.

Moreover, if we assume that f is étale and that for all closed points $x' \in \mathcal{X}'$,

- *$f(x')$ is closed and*
- *f is stabilizer preserving at x',*

then $g : X' \to X$ is étale and the diagram is cartesian.

Now using the above we will outline the proof for the existence result.

Proof of existence theorem in char. 0. **Step 1:** Under the assumption of $\mathcal{S}$-completeness every closed point has reductive stabilizer and since we are working over a field of characteristic zero reductivity of an algebraic group is equivalent to linear reductivity. Combining this observation with the local structure theorem it follows that every closed point has an affine étale quotient presentation of the form $f_x : ([\mathrm{Spec}(A)_x/G_x], w) \to (\mathcal{X}, x)$.

Step 2: Now by the above theorem for every $f_x : ([\mathrm{Spec}(A_x)/G_x], w) \to (\mathcal{X}, x)$ there exists an open substack $[\mathrm{Spec}(\tilde{A}_x)/G_x] \subset [\mathrm{Spec}(A)_x/G_x]$, note that open/closed substacks of quotient stacks are of this form [Hei09, Rem. 2.3], such that the restriction $f_x|_{[\mathrm{Spec}(\tilde{A}_x)/G_x]}$ is Θ-surjective and stabilizer preserving.

Step 3: Since $\mathcal{X}$ is of finite type and hence in particular quasi-compact there exist finitely many closed points $\{x_i\}_{i \in I}$ of $\mathcal{X}$ such that the images of $f_{x_i}|_{[\mathrm{Spec}(\tilde{A}_{x_i})/G_{x_i}]}$ cover $\mathcal{X}$. The stabilizers are linearly reductive, hence we can choose embeddings $G_{x_i} \hookrightarrow \mathrm{GL}_n$ for some n. Note that there are equivalences $[\mathrm{Spec}(\tilde{A}_{x_i})/G_{x_i}] \simeq [\mathrm{Spec}(\tilde{A}_{x_i}) \times^{G_{x_i}} \mathrm{GL}_n/\mathrm{GL}_n]$ for every $i \in I$. Now set $A := \prod_{i \in I}(\tilde{A}_{x_i} \times^{G_{x_i}} \mathrm{GL}_n)$, so that we have constructed a surjective, affine and étale morphism

$$f : \mathcal{X}_1 := [\mathrm{Spec}(A)/\mathrm{GL}_n] \to \mathcal{X}$$

which is Θ-surjective and stabilizer preserving. And $\mathcal{X}_1$ admits a good moduli space $\pi_1 : \mathcal{X}_1 \to X_1 := \mathrm{Spec}(A^{\mathrm{GL}_n})$ since GL_n is linearly reductive in characteristic zero.

Step 4: We can construct a groupoid $p_1, p_2 : \mathcal{X}_2 := \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1 \rightrightarrows \mathcal{X}_1$, where the projections $\mathcal{X}_2 \to \mathcal{X}_1$ are affine, étale, Θ-surjective and stabilizer preserving by the stability of these properties under base change. Moreover $\mathcal{X}_2 \simeq [\mathrm{Spec}(B)/\mathrm{GL}_n]$, This follows from the characterization of global quotient stacks given in [Kha22, Prop. 6.8.(iii)], because the composition $\mathcal{X}_2 \to \mathcal{X}_1 \to B\mathrm{GL}_n$ is affine by simple base change arguments and the fact that the diagrams of the form

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ [X/G] & \longrightarrow & BG, \end{array}$$

where $X \to \mathrm{Spec}(k)$ is the structure morphism, are cartesian. This representation of $\mathcal{X}_2$ implies that there is a good moduli space $\pi_2 : \mathcal{X}_2 \simeq [\mathrm{Spec}(B)/\mathrm{GL}_n] \to X_2 := \mathrm{Spec}(B^{GL_n})$. It follows from universality of good moduli for morphisms to schemes that there is a commutative diagram

$$\begin{tikzcd} [\mathrm{Spec}(B)/\mathrm{GL}_n] \simeq \mathcal{X}_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{X}_1 \arrow[d] \\ \mathrm{Spec}(B^{\mathrm{GL}_n}) := X_2 \arrow[r, shift left] \arrow[r, shift right] & X_1 \end{tikzcd}$$

and by Luna's fundamental lemma this diagram is cartesian and the projection morphisms, which we will denote by $q_1, q_2 : X_2 \rightrightarrows X_1$, are étale. Furthermore, $X_2 \rightrightarrows X_1$ inherits a groupoid structure from $\mathcal{X}_2 \rightrightarrows \mathcal{X}_1$ in the following way

1. (**composition**) First note that since by our previous diagram $\mathcal{X}_2 \simeq X_2 \times_{X_1} \mathcal{X}_1$ the diagram

$$\begin{tikzcd} \mathcal{X}_2 \times_{\mathcal{X}_1} \mathcal{X}_2 \arrow[r] \arrow[d] & X_2 \times_{X_1} X_2 \arrow[d] \\ \mathcal{X}_1 \arrow[r] & X_1 \end{tikzcd}$$

   is cartesian and since good moduli are stable under base change the map $\mathcal{X}_2 \times_{\mathcal{X}_1} \mathcal{X}_2 \to X_2 \times_{X_1} X_2$ is a good moduli space. We obtain our composition map from the universal property for good moduli, namely there exists a map $X_2 \times_{X_1} X_2 \to X_2$ filling in the commutative diagram

$$\begin{tikzcd} \mathcal{X}_2 \times_{\mathcal{X}_1} \mathcal{X}_2 \arrow[r, "c"] \arrow[d] & \mathcal{X}_2 \arrow[r] & X_2 \\ X_2 \times_{X_1} X_2 \arrow[urr, dashed] & & \end{tikzcd}$$

   where $c : \mathcal{X}_2 \times_{\mathcal{X}_1} \mathcal{X}_2 \to \mathcal{X}_2$ denotes the composition map for the groupoid $\mathcal{X}_2 \rightrightarrows \mathcal{X}_1$.

2. (**identity/inverse**) The identity and composition map are also easily obtained by the universality of good moduli. Let $e : \mathcal{X}_1 \to \mathcal{X}_2$ and $i : \mathcal{X}_2 \to \mathcal{X}_2$ denote respectively the identity and inverse morphisms. Then there are morphisms $X_1 \to X_2$ and $X_2 \to X_2$ filling in the diagrams

$$\begin{tikzcd} \mathcal{X}_1 \arrow[r, "e"] \arrow[d] & \mathcal{X}_2 \arrow[r] & X_2 \\ X_1 \arrow[urr, dashed] & & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X}_2 \arrow[r, "i"] \arrow[d] & \mathcal{X}_2 \arrow[r] & X_2 \\ X_2 \arrow[urr, dashed] & & \end{tikzcd}$$

   giving us the groupoid structure for $\mathcal{X}_2 \rightrightarrows \mathcal{X}_1$.

Conclusion: Next we want to show that the induced groupoid structure on $X_2 \rightrightarrows X_1$ defines an algebraic group $[X_1/X_2]$, then we can argue for the existence of a good moduli space $\mathcal{X} \to [X_1/X_2]$ as follows: by étale descent there exists a morphism $\mathcal{X} \to [X_1/X_2]$ such that the diagram

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & [X_1/X_2] \end{array}$$

is cartesian and we can conclude by Proposition 3.2.2.6 that $\mathcal{X} \to [X_1/X_2]$ is a good moduli space, moreover by **S**-completeness of $\mathcal{X}$, $[X_1/X_2]$ is separated.

Step 5: The last step required to reach the above conclusion is showing that $X_2 \rightrightarrows X_1$ is indeed an equivalence relation. In our setting it is sufficient to show that the morphism $(q_1,q_2) : X_2 \to X_1 \times X_1$ is injective on k-points.

With this in mind, let (x_1,x_1) denote a k-point in the image of $(q_1,q_2) : X_2 \to X_1 \times X_1$ and let $x_2, x_2' \in (q_1,q_2)^{-1}((x_1,x_1))$, our objective is to show that $x_2 = x_2'$. Since $\pi_1 : \mathcal{X}_1 \to X_1$ and $\pi_2 : \mathcal{X}_2 \to X_2$ are good moduli spaces, by Proposition 3.2.2.4, there exist unique closed points $\tilde{x}_1 \in \pi_1^{-1}(x_1)$, $\tilde{x}_2 \in \pi_2^{-1}(x_2)$ and $\tilde{x}_2' \in \pi_2^{-1}(x_2')$. By Θ-surjectivity of the $p_i's$ the points $p_i(\tilde{x}_2)$ and $p_i(\tilde{x}_2')$ in $\mathcal{X}_1$ are closed points for $i = 1,2$. Since these points lie over the point x_1 which has the unique closed point $\tilde{x}_1$ in its preimage under the map π_1 it follows that these points are equivalent to $\tilde{x}_1$. Now as f and the projections p_i are stabilizer preserving it follows that

$$G_{\tilde{x}_2} \simeq G_{\tilde{x}_2'} \simeq G_{\tilde{x}_1} \simeq G_{f(\tilde{x}_1)}.$$

Let us denote the group corresponding to these stabilizers by G. We claim that the following diagram is cartesian

$$\begin{array}{ccccc} BG & \longrightarrow & \mathcal{X}_2 & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ BG \times BG & \longrightarrow & \mathcal{X}_1 \times \mathcal{X}_1 & \longrightarrow & \mathcal{X} \times \mathcal{X} \end{array}$$

Using the generalization of the magic diagram we see immediately that the right hand square is cartesian since $\mathcal{X}_2 \simeq \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$ and that

$$BG \times BG \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X} \simeq BG \times_{\mathcal{X}} BG.$$

The natural map $BG \times_{\mathcal{X}} BG \to BG$ is an isomorphism. It is clearly a surjection and since BG is the residual gerbe at $f(\tilde{x}_1)$, $BG \to \mathcal{X}$ is a monomorphism hence by [Stacks, Tag 04ZX] $BG \times_{\mathcal{X}} BG \to BG$ is a monomorphism as well and therefore fully faithful [Stacks, Tag 04ZZ]. Thus, $BG \times_{\mathcal{X}} BG \to BG$ is an isomorphism. It follows that the outer diagram is cartesian, thus the left hand diagram is also cartesian.

Now since $\mathcal{X}_2 \times_{\mathcal{X}_1 \times \mathcal{X}_1} BG \times BG \simeq BG$ and BG consists of a single point, by definition of the fiber product of stacks we have that there exists precisely one point in $\mathcal{X}_2$ whose image under the morphism $\mathcal{X}_2 \to \mathcal{X}_1 \times \mathcal{X}_1$ is equivalent to $\tilde{x}_1$. In conclusion we have that $\tilde{x}_2 = \tilde{x}_2'$ and therefore that $x_2 = x_2'$. □

Chapter 5

A numerical criterion for stability on algebraic stacks

5.1 The criterion

In this section we introduce a numerical criterion for the stability of points in algebraic stacks, which generalizes the Hilbert-Mumford criterion. Our discussion will be based on the work of Jochen Heinloth [Hei18]. Throughout this section we work with an algebraic stack $\mathcal{X}$ which is locally of finite type over an algebraically closed field k and has affine diagonal.

Definition 5.1.1. Let $x \in |\mathcal{X}|$ be a geometric point. We say that a filtration $f : [\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{X}$ is a very close degeneration for x if $f(1) \simeq x$ and $f(0) \not\simeq x$ where 0 and 1 are the geometric points corresponding to 0 and 1 in $\mathbb{A}^1$.

Now for a line bundle $\mathcal{L}$ on an algebraic stack and a very close degeneration f we can generalize the Hilbert-Mumford index introduced in chapter one in the following way. Recall that a line bundle on a quotient stack $[X/G]$ corresponds to a G-linearization on X. With this in mind we define the weight of a line bundle on $\mathcal{X}$ with respect to a very close degeneration f, to be the weight of the $\mathbb{G}_m$-action on the fiber $(f^*\mathcal{L})_0$, we denote the weight of a line bundle with respect to a very close degeneration by $\mathrm{wt}_{\mathbb{G}_m}(\mathcal{L}, f)$.

Definition 5.1.2. Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. A geometric point $x \in |\mathcal{X}|$ is called $\mathcal{L}$-stable if

1. For every very close degeneration $f : [\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{X}$ for x we have

$$\mathrm{wt}_{\mathbb{G}_m}(\mathcal{L}, f) < 0, \text{ and}$$

2. $\dim_k(\mathrm{Aut}_{\mathcal{X}(k)}(x)) = 0$.

A point is $\mathcal{L}$-semi-stable if the first condition holds with $\leq$ instead of the strict inequality.

Throughout this section by point we will mean geometric point.

Recall that a geometric point $x \in \mathcal{X}$ is semi-stable with respect to a line bundle on $\mathcal{X}$, in the sense of Definition 3.2.4, if there exists a section $s \in \mathcal{L}^n(\mathcal{X})$ for some $n > 0$ such that $s(x) \neq 0$ and $\mathcal{X}_s$ is cohomologically affine. It is not too difficult to show that if a point is semi-stable in the usual sense it is also semi-stable according to the notion introduced in this section, this is the content of the following lemma.

Lemma 5.1.3. *[Zha22, Lemma 6.0.2] A geometric point $x \in \mathcal{X}(k)$ is semi-stable with respect to a line bundle $\mathcal{L}$ over $\mathcal{X}$ in the sense of Definition 3.2.4 then it is semi-stable in the sense of Definition 5.1.2.*

Proof. Suppose that for some $n > 0$ there exists a global section $s \in \mathcal{L}^n(\mathcal{X})$ such that $s(x) \neq 0$ and $\mathcal{X}_s$ is cohomologically affine. If $f : [\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{X}$ is a very close degeneration then since $f(1) \simeq x$ and s does not vanish at x we have that the pullback $f^*s \in f^*\mathcal{L}^n(\mathcal{X})$ does not vanish at 1. Now recall from Example 2.3.4 that a line bundle $\mathcal{L}$ on the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ has a non-zero global section if and only if its weight is ≤ 0. We recall the properties of the Hilbert-Mumford weight given in [DK94, p. 49] and conclude that since f^*s is a non-vanishing global section of $f^*\mathcal{L}$ the following holds

$$\mathrm{wt}_{\mathbb{G}_m}(\mathcal{L}, f) \leq n \cdot \mathrm{wt}_{\mathbb{G}_m}(\mathcal{L}, f) = \mathrm{wt}_{\mathbb{G}_m}(\mathcal{L}^n, f) \leq 0.$$

□

5.2 Application to quotient stacks

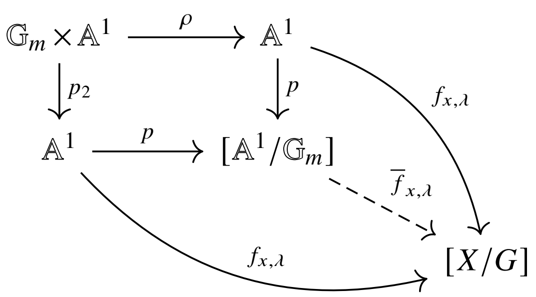
The remainder of this chapter will be dedicated to showing that for a certain class of quotient stacks the semi-stable points are precisely those points corresponding to the semi-stable points of the scheme that we are taking the quotient of.

Theorem 5.2.1. *Let G be a reductive group acting on a projective scheme X over k and $\mathcal{L} \in Pic^G(X)$ an ample linearization. A geometric point in $X(k)$ is semi-stable in the sense of chapter one if and only if the point corresponding to it in $[X/G](k)$ is semi-stable in the sense introduced in this chapter.*

To prove this we want to show that there is a correspondence between very close degenerations for a point x in a quotient stack and the maps $f_{x,\lambda}$ obtained in the section on the classical Hilbert-Mumford criterion by using the valuative criterion for properness. This is made precise in the following lemma.

Proposition 5.2.2. *For any 1-PS $\lambda : \mathbb{G}_m \to G$ and point $x \in X(k)$ that is not a fixed point of λ the equivariant map $f_{x,\lambda}$, as defined in chapter one, defines a very close degeneration $\overline{f}_{x,\lambda} : [\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ for the point in $[X/G]$ corresponding to x. Moreover, any very close degeneration in the stack $[X/G]$ is of the form $\overline{f}_{x,\lambda}$ for some x, λ.*

Proof. This proposition is essentially a corollary of Theorem 4.1.3. First suppose that we have a G-equivariant morphism $f_{x,\lambda}$ corresponding to a point $x \in X(k)$ and a 1-PS λ such that x is not a fixed point of λ. The morphism $f_{x,\lambda} : \mathbb{A}^1 \to X$ induces an equivariant morphism $\mathbb{A}^1 \to [X/G]$ and by Lemma 2.2.8 there is a morphism $\overline{f}_{x,\lambda}$ filling in the 2-commutative diagram



Now since $f_{x,\lambda}(0)$ is a fixed point of λ, in the sense that $\rho(\lambda(g), f_{x,\lambda}(0)) = f_{x,\lambda}(0)$ for all $g \in \mathbb{G}_m$ and x is not, we have that

$$\overline{f}_{x,\lambda}(0) \not\simeq \overline{f}_{x,\lambda}(1) \simeq \overline{p}(\rho(\lambda(1), x)) \simeq \overline{p}(x)$$

where $\overline{p} : X \to [X/G]$ is the natural map. Thus $\overline{f}_{x,\lambda}$ is a very close degeneration for x.

Now suppose that $\overline{f} : [\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ is a very close degeneration. Then $\overline{f}$ is a point in $\mathrm{Filt}([X/G])(k)$ and therefore under the isomorphism given in Theorem 4.1.3 corresponds to a point in $[X^+_\lambda/P^+_\lambda](k)$ with λ the conjugacy class of some 1-PS $\mathbb{G}_m \to G$. In particular we get a pair $(x, \tilde{\lambda})$ where $\tilde{\lambda}$ is some representative for λ, $x \in X(k)$ is a point corresponding to $\overline{f}(1)$ and $\lim_{g\to 0} \lambda(g) \cdot x$ exists and corresponds to $\overline{f}(0)$. This data gives us the desired $\mathbb{G}_m$-equivariant map $f : \mathbb{A}^1 \to X$ which fills in the commutative triangle

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\tilde{\lambda}\cdot x} & X \\ \downarrow & \nearrow_{f} & \\ \mathbb{A}^1 & & \end{array}$$

□

Theorem 5.2.1 is an easy consequence of the above proposition.

Proof of Theorem 5.2.1. First note that by projectivity of X for any point $x \in X(k)$ and 1-PS $\lambda : \mathbb{G}_m \to G$ there is an equivariant map $f_{\lambda,x} : \mathbb{A}^1 \to X$ extending the map $\lambda(-) \cdot X : \mathbb{G}_m \to X$. Now by our proposition this is equivalent to a very close degeneration $\overline{f} : [\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ and for a linearization $\mathcal{L}$ on X, by definition of the Hilbert-Mumford weight, we have that

$$\mu^{f^*\mathcal{L}} = -\mathrm{wt}_{\mathbb{G}_m}(\mathcal{L}, \overline{f}).$$

We conclude that x satisfies the Hilbert-Mumford criterion if and only if the point corresponding to it in the quotient stack is $\mathcal{L}$-semi-stable. □

Bibliography

[AE12] J. D. Alper and R.W. Easton. “Recasting results in equivariant geometry”. In: *Transformation Groups* 17.1 (Jan. 2012), pp. 1–20. ISSN: 1531-586X. DOI: 10.1007/s00031-011-9170-5. URL: https://doi.org/10.1007/s00031-011-9170-5.

[AFS17] Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth. “Second flip in the Hassett–Keel program: existence of good moduli spaces”. In: *Compositio Mathematica* 153.8 (2017), pp. 1584–1609. DOI: 10.1112/S0010437X16008289.

[AHH21] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth. “Cartan-Iwahori-Matsumoto decompositions for reductive groups”. In: *Pure Appl. Math. Q.* 17.2 (Aug. 2021), pp. 593–604. URL: https://dx.doi.org/10.4310/PAMQ.2021.v17.n2.a1.

[AHH23] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth. “Existence of moduli spaces for algebraic stacks”. In: *Inventiones mathematicae* 234.3 (Aug. 2023), pp. 949–1038. ISSN: 1432-1297. DOI: 10.1007/s00222-023-01214-4. URL: http://dx.doi.org/10.1007/s00222-023-01214-4.

[Alo+15] Leovigildo Alonso Tarrío et al. “A functorial formalism for quasi-coherent sheaves on a geometric stack”. In: *Expositiones Mathematicae* 33.4 (2015), pp. 452–501. ISSN: 0723-0869. DOI: 10.1016/j.exmath.2014.12.007. URL: http://dx.doi.org/10.1016/j.exmath.2014.12.007.

[Alp13] Jarod Alper. “Good moduli spaces for Artin stacks”. en. In: *Annales de l’Institut Fourier* 63.6 (2013), pp. 2349–2402. DOI: 10.5802/aif.2833. URL: https://aif.centre-mersenne.org/articles/10.5802/aif.2833/.

[Alp17] Jarod Alper. *On the local quotient structure of Artin stacks*. 2017. arXiv: 0904.2050 [math.AG].

[Alp24] Jarod Alper. *Stacks and Moduli*. https://sites.math.washington.edu/~jarod/moduli.pdf. May 2024.

[AOV08] Dan Abramovich, Martin Olsson, and Angelo Vistoli. "Tame stacks in positive characteristic". en. In: *Annales de l'Institut Fourier* 58.4 (2008), pp. 1057–1091. DOI: 10.5802/aif.2378. URL: https://aif.centre-mersenne.org/articles/10.5802/aif.2378/.

[AS23] Enrico Arbarello and Giulia Saccà. *Singularities of Bridgeland moduli spaces for K3 categories: an update*. 2023. arXiv: 2307.07789 [math.AG].

[Bay24] Thomas Bayer. rinvar.lib. *D* SINGULAR *4-4-0 — library for computing the normalization of affine rings*. 2024.

[Con17] Alberto Cononaco. *Lectures on algebraic stacks*. https://www1.mat.uniroma1.it/ricerca/rendiconti/ARCHIVIO/2017(1)/1-169.pdf. 2017.

[Dec+24] Wolfram Decker et al. SINGULAR *4-4-0 — A computer algebra system for polynomial computations*. http://www.singular.uni-kl.de. 2024.

[Der99] Harm Derksen. "Computation of Invariants for Reductive Groups". In: *Advances in Mathematics* 141.2 (1999), pp. 366–384. ISSN: 0001-8708. DOI: https://doi.org/10.1006/aima.1998.1787. URL: https://www.sciencedirect.com/science/article/pii/S000187089891787X.

[DK15] Harm Derksen and Gregor Kemper. *Computational Invariant Theory*. 2nd edition. Springer, 2015.

[DK94] J. Fogarthy D. Mumford and F. Kirwan. *Geometric Invariant Theory*. 3rd edition. Springer, 1994.

[Dri15] Vladimir Drinfeld. *On algebraic spaces with an action of G_m*. 2015. arXiv: 1308.2604 [math.AG].

[ER21] Dan Edidin and David Rydh. "Canonical reduction of stabilizers for Artin stacks with good moduli spaces". In: *Duke Mathematical Journal* 170.5 (2021), pp. 827–880. DOI: 10.1215/00127094-2020-0050. URL: https://doi.org/10.1215/00127094-2020-0050.

[GHZ23] Tomás L. Gómez, Andres Fernández Herrero, and Alfonso Zamora. *A guide to moduli theory beyond GIT*. 2023. arXiv: 2302.01871 [id='math.AG' full$_name =' AlgebraicGeometry' is_active = True alt_name =' alg-geom' in_archive =' math' is_general = False description =' Algebraic varieties, stacks, sheaves, schemes, mod$

[Gro14] Michael Groechenig. *Algebraic stacks*. http://individual.utoronto.ca/groechenig/stacks.pdf. 2014.

[GW20] Ulrich Görtz and Torsten Werdhorn. *Algebraic Geometry I: Schemes*. 2nd edition. Springer, 2020.

[Hal13] Daniel Halpern-Leistner. *Geometric invariant theory and derived categories of coherent sheaves*. 2013.

[Hal21] Daniel Halpern-Leistner. *Moduli theory*. https://book.themoduli.space/moduli.pdf. Jan. 2021.

[Hal22] Daniel Halpern-Leistner. *On the structure of instability in moduli theory*. 2022. arXiv: 1411.0627 [math.AG].

[Har77] Robin Hartshorne. *Algebraic Geometry*. 1st ed. Springer, 1977.

[Hei09] Jochen Heinloth. *Lectures on the moduli stack of vectorbundles on a curve*. https://webusers.imj-prg.fr/~bernhard.keller/gdtcluster/HeinlothLecturesOnTheModuliStackOfVectorBundlesOnACurve.pdf. Jan. 2009.

[Hei18] Jochen Heinloth. "Hilbert-Mumford stability on algebraic stacks and applications to $\mathcal{G}$-bundles on curves". In: *Épijournal de Géométrie Algébrique* Volume 1 (Jan. 2018). ISSN: 2491-6765. DOI: 10.46298/epiga.2018.volume1.2062. URL: http://dx.doi.org/10.46298/epiga.2018.volume1.2062.

[Hos16] Victoria Hoskins. *Moduli problems and geometric invariant theory*. https://userpage.fu-berlin.de/hoskins/M15_Lecture_notes.pdf. 2016.

[Hos23] Victoria Hoskins. *Moduli spaces and geometric invariant theory: old and new perspectives*. 2023. arXiv: 2302.14499 [math.AG].

[Kha22] Adeel A. Khan. *A modern introduction to algebraic stacks*. https://www.preschema.com/lecture-notes/2022-stacks/stacksncts.pdf. Jan. 2022.

[LM00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*. 1st ed. Springer, 2000.

[M F69] I. G. MacDonald M. F. Atyah. *Introduction to Commutative Algebra*. 1st ed. Addison-Wesley, 1969.

[Mil15] J.S. Milne. *Algebraic Groups*. https://www.jmilne.org/math/CourseNotes/iAG200.pdf. Dec. 2015.

[Nag61] Masayoshi Nagata. "Complete reducibility of rational representations of a matric group". In: *Journal of Mathematics of Kyoto University* 1.1 (1961), pp. 87–99. DOI: 10.1215/kjm/1250525107. URL: https://doi.org/10.1215/kjm/1250525107.

[Nag64] Masayoshi Nagata. "Invariants of group in an affine ring". In: *Journal of Mathematics of Kyoto University* 3.3 (1964), pp. 369–378. DOI: 10.1215/kjm/1250524787. URL: https://doi.org/10.1215/kjm/1250524787.

[Neu09] Frank Neumann. *Algebraic stacks and moduli of vector bundles*. https://www.cimat.mx/~luis/seminarios/Pilas-algebraicas/neumann-Stacks.pdf. 2009.

[New78] P.E. Newstead. *Introduction to moduli problems and orbit spaces*. Springer, 1978.

[Nit08] Nitin Nitsure. *Introduction to the Geometry of Stacks*. https://www.cimat.mx/~luis/seminarios/Pilas-algebraicas/Guana-talks.pdf. October 2008.

[Ols07] Martin Olsson. "Sheaves on Artin stacks". In: *Journal für die reine und angewandte Mathematik* 2007.603 (2007), pp. 55–112. DOI: doi:10.1515/CRELLE.2007.012. URL: https://doi.org/10.1515/CRELLE.2007.012.

[Ols16] Martin Olsson. *Algebraic Spaces and Stacks*. American Mathematical society, 2016.

[Pop79] Vladimir Popov. "On Hilbert's theorem on invariants". In: *Doklady Mathematics* 20 (May 1979), pp. 1318–1322.

[QR22] Ming Hao Queck and David Rydh. *Weighted Blow-ups*. https://people.kth.se/~dary/weighted-blowups20220329.pdf. Mar. 2022.

[Rom05] Matthieu Romagny. "Group actions on stacks and applications". In: *Michigan Mathematical Journal* 53.1 (2005), pp. 209–236. DOI: 10.1307/mmj/1114021093. URL: https://doi.org/10.1307/mmj/1114021093.

[Sch18] R. M. Schwarz. *Gromov-Witten invariants of the classifying stack of principle $\mathbb{G}_m$-bundles*. https://www.universiteitleiden.nl/binaries/content/assets/science/mi/scripties/master/2017-2018/masterthesisrosaschwarz.pdf. August 2018.

[Ses72] C. S. Seshadri. "Quotient Spaces Modulo Reductive Algebraic Groups". In: *Annals of Mathematics* 95.3 (1972), pp. 511–556. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970870 (visited on 05/31/2024).

[Ses77] C.S Seshadri. "Geometric reductivity over arbitrary base". In: *Advances in Mathematics* 26.3 (1977), pp. 225–274. ISSN: 0001-8708. DOI: https://doi.org/10.1016/0001-8708(77)90041-X. URL: https://www.sciencedirect.com/science/article/pii/000187087790041X.

[SS10] Pramathanath Sastry and C. S. Seshadri. *Geometric Reductivity–A Quotient Space Approach*. 2010. arXiv: 1012.0418 [math.AG].

[Stacks] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu. 2018.

[Wan11] Jonathan Wang. *The moduli stack of G-bundles*. 2011. arXiv: 1104.4828 [id='math.AG' full$_name = 'AlgebraicGeometry' is_active = True alt_name = 'alg-geom' in_archive = 'math' is_general = False description = 'Algebraic varieties, stacks,$

[Zha22] Xucheng Zhang. “Characterizing open substacks of algebraic stacks that admit good moduli spaces”. PhD thesis. Sept. 2022. DOI: 10.17185/duepublico/76595. URL: https://doi.org/10.17185/duepublico/76595.