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Removal of singularities through hierarchies of singularities

Master's thesis

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Abstract

In this thesis we consider a removal of singularities proof for the Smale-Hirsch theorem about immersions in the non-critical dimension case. This proof was suggested by M. Gromov in his book *Partial Differential Relations*. In this thesis we present a reconstruction of the argument described by Gromov and provide full details for it. In particular, we present a general construction of hierarchies of singularities in jet spaces of sections, which are based on the Thom-Boardman singularities for jet spaces of functions.

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Contents

Introduction

The main context of this thesis is the *homotopy principle*, commonly abbreviated to the h-principle. In this thesis we will consider a technique to prove certain h-principles, called *removal of singularities*. Our working example to consider this technique will be the h-principle for immersions. The main purpose of this thesis is the construction of certain *hierarchies of singularities*, one of which we will need to prove the h-principle for immersions by removal of singularities.

Background

The h-principle (originally introduced as the *weak homotopy principle*, see for example [6]) is an umbrella term used to describe a number of results in differential topology. Such results are often concerned with the study of a certain type of (smooth) maps between two manifolds. Another type of objects that can be studied using the h-principle are geometric structures on manifolds. Results that consider an h-principle often answer questions of the following form:

- What are sufficient and necessary conditions for the existence of such objects?
- What does the topological space of such objects look like?

As mentioned above, in this thesis the main category of objects we will consider are *immersions*. An example of a question one then might want to answer is, 'Can we classify the space of immersions up to homotopy?'. By such a classification, one indeed gets sufficient and necessary conditions for the existence of immersions and insight in what the topological space of immersions looks like.

The h-principle often considers, besides the actual objects of interest, a set of *formal* objects. These formal objects mimic some of the properties of the actual objects, but not all. This means that we can include the space of the objects we are interested in, into the space of their formal counterparts. We then say that 'the h-principle holds for a certain class of objects', if this inclusion is a weak homotopy equivalence. The upshot of the h-principle is that the topological space of the formal objects is often better understood than the topological space of objects we started with.

M. Gromov is often seen as the founder of the h-principle. Much of his work in the 1970's considered types of problems as described above, see for example [11], [6] and [12]. As described in [7, p. XII], Gromov was inspired by the topological advances that had taken place during the decades before, which can be found in the works of e.g. H. Whitney [31], J. Nash [24], S. Smale [26] and M. Hirsch [16]. With his work, Gromov took a more general approach to the type of problems they had considered, which he then dubbed the h-principle.

In his 1986 book, *Partial Differential Relations* [13], Gromov gives many examples of the h-principle. He also describes some general methods to prove h-principles, that he had explored in his previous papers. These methods are

(i) Removal of singularities
(ii) Continuous sheaves (later also known as holonomic approximation [7, Chapter 3])
(iii) Convex integration

In an attempt to make Gromov's work more accessible, Y. Èliašberg and N. Mischachev wrote a book in 2002 called *Introduction to the h-principle* [7]. In this book they extensively consider the methods of holonomic approximation and convex integration, since those turned out to be powerful general methods to tackle h-principles.[7, p. XII]

However, these methods can, at least as phrased in [13] and [7], mostly be applied to problems of a local nature. By a local nature, we mean problems that can be phrased in terms of the *jet space* of a bundle. Examples of such problems are immersions and submersions. Other problems, such as embeddings, have a non-local nature, meaning that they need to be phrased in terms of *multijet space*.[1] While we will not cover these latter type of problems in this thesis, they do form a motivation for this thesis, especially the example of embeddings.

Removal of singularities

The method 'removal of singularities' was first introduced by Gromov and Èliašberg in their 1971 paper, *removal of singularities* [6].The term singularity is used here to denote any points at which a given map does not satisfy some prescribed properties. For example, when considering immersions between manifolds M and N, the singularity of a map f is given by those points $x \in M$, where $(df)_x$ is not injective. The idea of removal of singularities, is that one can transform a map that has some singularity, into one without any singularity, by a slight perturbation.

More specifically, in a removal of singularities argument, one often tries to avoid singularities by a so-called *general position* or *transversality* argument, meaning that 'most' maps do not have that singularity. For the singularities that cannot be removed in such a way, a similar transversality argument can sometimes be used to make sure that the singularities at least avoid certain 'bad' properties, which ensures that the singularity can still be removed by deforming the map. Such a notion of singularities with extra 'bad/singular' properties is what we describe by a *hierarchy of singularities*. In his book [13], Gromov described how a removal of singularities argument could be applied to prove multiple h-principles, amongst which are immersions.

It turns out that a removal of singularities argument can also be applied to some properties that are of a non-local nature. In his 1982 paper [28], A. Szücs gave a new proof of a theorem from 1962 by Haefliger [15]. This theorem is stated as follows

Theorem 0.1. *Let M be a closed manifold of dimension m and N an arbitrary manifold of dimension n with $n \geq \frac{3}{2}m + 2$. Assume that there exists a continuous map $F : M^2 \to N^2$ which satisfies the following two conditions:*

(i) The following diagram commutes

[1] For an introduction to multijet spaces, we refer the reader to [9, pp. 57-59]

$$\begin{array}{ccccc} (x_1,x_2) & & M^2 \xrightarrow{F} N^2 & & (y_1,y_2) \\ \downarrow & & \downarrow \qquad \downarrow & & \downarrow \\ (x_2,x_1) & & M^2 \xrightarrow[F]{} N^2 & & (y_2,y_1) \end{array}$$

(ii) $F^{-1}(\Delta(N)) = \Delta(M)$, *where* $\Delta(-)$ *denotes the respective diagonals.*

Then there exists a differentiable embedding from M *to* N*.*

This theorem can be seen as an h-principle result, where F is the formal map and the embedding is the map we are actually interested in. The approach Szücs takes was communicated to him by Èliašberg and Gromov [28, p.303] and can be seen as a removal of singularities argument. A later result by Goodwillie on concordance embeddings described in [10] also makes use of (amongst others) some similar techniques.

Thom-Boardman singularities

In a paper from 1956 [30], R. Thom described singularities, which considered the rank of $(df)_x$ for a certain (at least) C^1-map $f : M \to N$ and $x \in M$. Note that this notion of singularity matches the one described above, since the rank of $(df)_x$ exactly tells us what points are a singularity for the immersion property. These singularities of maps are described using subsets of the so-called *first jet space*. By showing that these subsets of jet spaces are in fact submanifolds, Thom shows that for 'most' maps the singularity also admits a manifold structure.

Inspired by Thom's work, J. Boardman published a paper in 1967 [2], that further distinguished these singularities. In particular, he used an iterative construction by restricting a map to its previous singularity and then reconsidering the rank. In this way, he constructed *hierarchies of the singularities*, i.e. he distinguished 'how singular a map was at a given point. His construction made use of hierarchies of subsets of the k^{th} *jet space*, i.e. the lowest level of singularity was determined in the first jet space, the second level of singularities in the 2^{nd} jet space and so on. This means that the higher levels of singularity thus depend on the higher order derivatives of the maps. This construction came to be known as the *Thom-Boardman singularities.* In his paper, Boardman also showed that these subsets form submanifolds of a certain codimension, which is needed when one wants to apply a transversality-type argument.

These Thom-Boardman singularities were the first type of *hierarchies of singularities* to be introduced. In his 1986 book, Gromov refers back to Boardman's paper [13, p35]. He does so in the context of describing how to avoid certain 'bad properties' of singularities of sections when proving the h-principle for immersions. However, the Thom-Boardman singularities only consider singularities of maps and cannot be directly applied to sections of a bundle. Gromov remarks that a similar construction can be made for sections.

Contents of this thesis

In this thesis we will explore the technique of removal of singularities. We will do so by applying it to the case of immersions into Euclidean space, as described by Gromov. In order to provide full details of Gromov's argument, we will need to construct a hierarchy of singularities, which will allow us to remove the singularity. As was suggested by Gromov, these singularities will be based on the construction of the Thom-Boardman singularities, but then translated to a setting where they are applicable to sections. More specifically, in definition 6.3, we will construct increasing levels of singularities of sections of a bundle, based on the higher order derivatives of those sections.

To be able to use these higher singularities for the transversality argument we need, we will need to show that the hierarchy is a collection of submanifolds of jet space that are of increasing codimension. This will be a corollary of the main result of this thesis, namely theorem 6.12. This result is an original contribution to the research field. However, the reason we study these singularities is to use the specific hierarchies mentioned above for a transversality argument in the removal of singularities proof for the h-principle for immersions. This thesis is thus structured accordingly, by first proving theorem 6.12 after which we apply it in the h-principle proof.

In chapter 1 we will start by introducing the notion of *(infinite) jet space*. Similarly to the construction of the Thom-Boardman singularities, this space will form the space in which we can define our hierarchies of singularities. While finite jet spaces admit (finite) manifold structures, this is not the case for infinite jet space. However, it turns out that we can endow it with a so-called *pro-finite manifold structure*. We will use this structure to also construct *the tangent space* and *vector fields* of infinite jet space. We do so mainly, because we are interested in a special subset of this tangent space, namely *the Cartan distribution* and its smooth sections. We will extensively discuss the Cartan distribution and its properties, which we will need for the construction of the hierarchies later in the thesis. We will finish this chapter by discussing a certain type of 'vertical' tangent vectors to infinite jet space, which will play an important role in the proof of theorem 6.12.

In chapter 2 we will discuss two topologies on spaces of sections (of a fibre bundle $\pi : E \to M$), called the *weak* and *strong Whitney topology*. After introducing these topologies we will discuss certain properties they possess. In particular, we will discuss how homotopies between sections induce continuous paths between those sections in the weak Whitney topology. This will make this topology the logical one to use when defining the h-principle. We will also discuss the fact that endowing $\Gamma(E)$ with the strong Whitney topology makes it into a Baire space, which will be a key property when discussing the notion of transversality.

In chapter 3 we will discuss the concept of the h-principle. We will do so by first defining the notion of a differential relation, discuss what its solutions and formal solutions are and then introduce the notion of the h-principle. We will also specifically consider the h-principle for immersions from M to $\mathbb{R}^n$ and discuss the idea of the removal of singularities argument needed to prove this h-principle. In this chapter we will only focus on how we can homotope a formal solution into an actual one, to get a feeling of what the full argument will require. In this chapter we will prove proposition 3.23, which will be used in chapter 7 to make a formal solution componentwise into an actual solution, which then proves lemma 7.5. The next three chapters will then develop the theory needed to reduce the proof of lemma 7.5 to proposition 3.23.

In chapter 4 we will introduce the notion of transversality and discuss the celebrated Thom-transversality theorem (theorem 4.10). Originally stated for the space of maps, $C^\infty(M, N)$, we will also discuss an equivalent result for spaces of sections $\Gamma(E)$. In the proof of the h-principle for immersions in chapter 7, we will need a slightly stronger argument than the Thom-transversality theorem. Namely, when considering a section of a product of *jet spaces*, we will need to ensure that certain components stay *holonomic*.[2] We will show that one can indeed apply such a transversality argument, by proving theorem 4.17.

In chapter 5 we will discuss the Thom-Boardman singularities defined in Boardman's paper [2]. We will start the chapter by discussing some of the intuition behind the construction

[2] All these concepts will be defined in chapter 1

of these singularities (section 5.1). This is followed by some technical notions and results inspired by similar results in Boardman's paper (sections 5.2 and 5.3). In section 5.4, these notions will be used to state some of the main results from Boardman's paper (theorem 5.21 and theorem 5.35), which characterise his singularities. We will also give our own definition of the Thom-Boardman singularities, which incorporates the (hopefully) intuitive ideas discussed at the start of the chapter. This construction is an inductive one, by lifting a given manifold to a higher singularity. The earlier-mentioned technical results are then used to prove lemma 5.39, which shows that our definition matches Boardman's characterisation of the Thom-Boardman singularities.

In chapter 6 we generalise the definition of the Thom-Boardman singularities from the previous chapter, to define other lifts of singularities in definition 6.3. We will then use the same technical results as discussed in the previous chapter, to prove that under certain conditions, these lifts of manifolds stay manifolds of a certain codimension. This result is contained in theorem 6.12. We will finish this chapter by using theorem 6.12 to construct a specific hierarchy of singularities and show that this hierarchy consists of closed manifolds of a certain codimension (theorem 6.2 and corollary 6.21).

In the final chapter, chapter 7, we will specify how the construction from chapter 6 can be used in the proof of the h-principle of immersions. We will discuss both the non-parametric and parametric case and see that with the use of the hierarchies, these cases are actually not too different.

To conclude this thesis, we will give a summary of the results of this thesis and discuss some insights about open questions and potential further research. This thesis also contains an appendix, in which we discuss some elementary definitions and results about *stratifications*. Some of these notions will be used throughout this thesis. To maintain the flow of the story, we have decided not to include this chapter in the main text.

1 | Jet Spaces

When studying smooth maps between manifolds, often one is interested in properties of their (higher order) derivatives. In this chapter we will discuss the notion of jet spaces, which are spaces that contain the information about the (higher order) derivatives of maps in $C^\infty(M,N)$.

Throughout this thesis, jet spaces are used in many different ways. In chapter 2 they are used to endow the spaces of functions with the so-called *Whitney topologies*. In chapter 3 they will be used to define the notion of a *differential relation*. In chapter 4 they are an essential part of the Thom-transversality theorem. Furthermore, in chapter 5 and chapter 6 they will be the ambient space in which we define the hierarchies of singularities.

In this chapter we will define what jet spaces are, see that finite jet spaces admit smooth structures and endow the infinite jet space with a so-called *pro-finite manifold structure*. We will then use this extra structure to define the tangent space of infinite jet space, and an important subset of this tangent space, the *Cartan distribution*. We will finish this chapter with a section containing some technical results regarding a certain type of tangent vectors to the inifinite jet space, which will be used in chapter 6.

1.1 Jet spaces

In this section we will give the definition of jet spaces and state some basic facts about them. We will also give a more general definition of the jet space of a fibre bundle, and show that this definition is indeed a generalisation of jet spaces of function spaces. Finally we will see that finite jet spaces are manifolds and even admit a fibre bundle structure. This section is based on section 2.2 of [9, pp. 37-42].

1.1.1 Order k contact between smooth maps

Definition 1.1. *Let M and N be smooth manifolds and let $x \in M$. Let $f,g : M \to N$ be smooth maps and $0 \leq k \leq \infty$. Then we say that*

- *f and g have **contact of order 0** at x if $f(x) = g(x)$.*
- *f and g have **first order contact** at x if $(df)_x = (dg)_x$*
- *For $k < \infty$, f and g have $\mathbf{k^{th}}$ **order contact** at x if the map $(df) : TM \to TN$ has $(k-1)^{st}$ order contact with the map (dg) at every point in T_xM.*
- *f and g have **contact of order** ∞ at x if they have contact of order l at x for every finite $l \geq 0$.*

If f and g have k^{th} order contact at x we will denote this by $\mathbf{f \sim_k g}$ ***at*** x.

Remark 1.2. Note that it is immediate from the definition that two smooth maps $f, g : M \to N$ which have contact of order k at x, must also have contact of order l for all $0 \le l \le k$. ⋄

Essentially, this definition is a generalisation of the notion of the k^{th} order Taylor polynomial of maps between Euclidean spaces. Two maps have contact of order k at $x \in M$, if the partial derivatives in a local representation match up to order k. This is specified in the following lemmas.

For a proof of this lemma we refer the reader to [9, pp. 37-38].

Lemma 1.3. *Let $U \subset \mathbb{R}^m$ be an open subset and let $x \in U$. Let $f, g : U \to \mathbb{R}^n$ be smooth maps. Then $f \sim_k g$ at x iff*

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(x) = \frac{\partial^{|\alpha|} g_i}{\partial x^\alpha}(x)$$

for every multi-index α with $|\alpha| \le k$ and $1 \le i \le n$.

Thus for maps between Euclidean spaces, having contact of order k is the same as having matching k^{th} order polynomials. We can of course consider any smooth map between manifolds as a map between Euclidean spaces by looking at local representations. The following statement shows that having k^{th} order contact is the same as having matching k^{th} order polynomials, when maps are viewed in local charts.

Lemma 1.4. *Let $f, g : M \to N$ be smooth maps between manifolds. Let (U, φ) and (V, χ) be charts around x and $y = f(x) = g(x)$ respectively. Then $f \sim_k g$ at $x \in M$ if and only if $\chi \circ f \circ \varphi^{-1} \sim_k \chi \circ g \circ \varphi^{-1}$ at $\varphi(x) \in \mathbb{R}^m$.*

Proof. We will prove this statement inductively. The statement is immediate for $k = 0$. For $k = 1$, the statement follows from the chain rule.

We assume that the statement is true for all smooth maps between manifolds for some $k \ge 1$. From the fact that $\varphi : U \to \mathbb{R}^m$ is a chart it follows that $(d\varphi) : TU \to T\mathbb{R}^m \cong \mathbb{R}^{2m}$ is also a chart. Similarly $(d\chi) : TV \to T\mathbb{R}^n \cong \mathbb{R}^{2n}$ defines a chart. Thus by the induction hypothesis we know that $(df) \sim_k (dg)$ at all points in T_xM iff $(d\chi) \circ (df) \circ (d\varphi)^{-1} \sim_k (d\chi) \circ (dg) \circ (d\varphi)^{-1}$ at all points in $T_{\varphi(x)}U$. By the chain rule we get that

$$(d\chi) \circ (df) \circ (d\varphi)^{-1} = \left(d\left(\chi \circ f \circ \varphi^{-1}\right)\right)$$

and similarly

$$(d\chi) \circ (dg) \circ (d\varphi)^{-1} = \left(d\left(\chi \circ g \circ \varphi^{-1}\right)\right)$$

Thus it follows that $(df) \sim_k (dg)$ at all points in T_xM iff $\left(d\left(\chi \circ f \circ \varphi^{-1}\right)\right) \sim_k \left(d\left(\chi \circ g \circ \varphi^{-1}\right)\right)$ at all points in $T_{\varphi(x)}U$. By definition it then follows that $f \sim_{k+1} g$ at x iff $\chi \circ f \circ \varphi^{-1} \sim_{k+1} \chi \circ g \circ \varphi^{-1}$ at $\varphi(x)$.

This proves the statement for all finite k. Thus the statement also follows for $k = \infty$. □

1.1.2 Jet spaces of smooth maps

As discussed in the introduction, jet spaces are spaces that contain the information about (higher order) derivatives of smooth maps. Note that the relation $\sim_k$ at x for some $x \in M$ is an equivalence relation. This allows us to give the following definition.

Definition 1.5. *Let M and N be manifolds. By $\sim_{x,k}$ we denote the equivalence relation of having contact of order k at x. For $0 \le k \le \infty$, we define the* $\mathbf{k^{th}}$ ***jet space of*** $\mathbf{C^\infty(M,N)}$ ***at*** $\mathbf{x \in M}$ *as follows*

$$J^k_x(M,N) = C^\infty(M,N)/\sim_{x,k}$$

We then define the $\mathbf{k^{th}}$ ***jet space of*** $\mathbf{C^\infty(M,N)}$ *as the following disjoint union*

$$J^k(M,N) = \bigsqcup_{x\in M} J^k_x(M,N)$$

We will denote the element $[f] \in J^k_x(M,N)$ by $j^k_x f$. This is called the **k*-jet of*** $\mathbf{f}$ ***at*** $\mathbf{x}$.

Example 1.6. Consider $M = N = \mathbb{R}$. Then its (finite) jet spaces are given as follows.

- For $k = 0$, we see that $f \sim_0 g$ at x iff $f(x) = g(x)$. Thus it follows that

$$J^0_x(\mathbb{R},\mathbb{R}) = \left\{f(x) \mid f \in C^\infty(\mathbb{R},\mathbb{R})\right\} \cong \mathbb{R}$$

  Hence we also get the following bijection

$$\begin{aligned} J^0(\mathbb{R},\mathbb{R}) &\cong \mathbb{R}^2 \\ j^0_x f &\mapsto \left(x, f(x)\right) \end{aligned}$$

  where $\cong$ (for now) means isomorphic as sets.
- For finite $k \ge 1$, we can similarly see that the following map is a bijection

$$\begin{aligned} J^k(\mathbb{R},\mathbb{R}) &\mapsto \mathbb{R}^{k+2} \\ j^k_x f &\mapsto \left(x, f(x), f^{(1)}(x), \ldots, f^{(k)}(x)\right) \end{aligned}$$

△

So far this is only a set-theoretical definition of $J^k(M,N)$. For trivial neighbourhoods U and V we can consider jets of $J^k(U,V)$ as k^{th} order polynomials. The coefficients of these polynomials then give bijections to Euclidean space. Thus we can endow $J^k(M,N)$ with a manifold structure, which is specified below.

Notation 1.7. *(i) Let $f : U \to \mathbb{R}$ be a smooth map with $U \subset \mathbb{R}^m$ open. Then for finite k we denote the k^{th} order Taylor polynomial of f at x by $P^k_x(f)$. We will use $\mathscr{P}^k_x(f)$ for the k^{th} order Taylor polynomial at x without the constant term, i.e. $\mathscr{P}^k_x(f) = P^k_x(f) - f(x)$.*
(ii) By P^k_m we denote the (real) vector space of polynomials in m variables of degree $\le k$.
(iii) We define the linear subspace $(P^k_m)_0 \subset P^k_m$ by

$$\left(P^k_m\right)_0 = \left\{p \in P^k_m \mid p(0) = 0\right\}$$

$(P^k_m)_0$ thus contains the polynomials with constant term 0.

Let (U,φ) and (V,χ) be charts around $x \in M$ and $y \in N$ respectively. Note that we have a canonical inclusion $J^k(U,V) \hookrightarrow J^k(M,N)$. Also note that the following map

$$\begin{aligned} C_{\varphi,\chi} : J^k(U,V) &\to U \times V \times \bigoplus_{i=1}^n \left(P^k_m\right)_0 \\ j^k_x f &\mapsto \left(x, f(x), \mathscr{P}^k_x\left(\chi\circ f_1 \circ \varphi^{-1}\right), \ldots, \mathscr{P}^k_x\left(\chi\circ f_n\circ\varphi^{-1}\right)\right) \end{aligned}$$

is a bijection. Surjectivity of this map follows from the fact that a local polynomial can be represented by a global function through the use of a bump funtion. By lemma 1.4 the map is also well-defined and injective. This $C_{\varphi,\chi}$ will be a chart on the subset $J^k(U,V) \subset J^k(M,N)$. Since we can cover both M and N with charts, we can do the same with $J^k(M,N)$. In fact, this endows $J^k(M,N)$ with a (Hausdorff and second countable) smooth manifold structure, as shown in [9, Theorem 2.7, p. 40].

Theorem 1.8. *Equipped with the atlas as described above, $J^k(M,N)$ is a manifold of dimension $m+n+n\dim\left((P_m^k)_0\right)$.*

Remark 1.9. Note that for $M = N = \mathbb{R}$ one can simply consider the charts $U = V = \mathbb{R}$ and $\varphi = \chi = \mathrm{id}_{\mathbb{R}}$. Furthermore, note that for $m = 1$, we get that $(P_m^k) \cong \mathbb{R}^k$. Thus it follows that we get a bijection $U \times V \times \bigoplus_{i=1}^n \left(P_m^k\right)_0 \cong \mathbb{R}^{k+2}$. Also, the chart on $J^k(\mathbb{R},\mathbb{R})$, $C_{\mathrm{id}_{\mathbb{R}},\mathrm{id}_{\mathbb{R}}}$, is exactly the bijection described in example 1.6.

We will denote the coordinate functions of $J^k(\mathbb{R},\mathbb{R})$ by x, y and $y^{(k)}$ respectively. ◇

We will state the following lemma as a fact. For a proof we refer the reader to [9].

Lemma 1.10. *Let M, N and K be smooth manifolds and let $0 \leq k < \infty$.*

(i) The maps

$$\begin{aligned} \pi_M : J^k(M,N) &\to M \\ j_x^k f &\mapsto x \end{aligned}$$

and

$$\begin{aligned} \pi_N : J^k(M,N) &\to N \\ j_x^k f &\mapsto f(x) \end{aligned}$$

are smooth submersions. So is the map $(\pi_M, \pi_N) : J^k(M,N) \to M \times N$.

(ii) Let $h : N \to K$ be a smooth map. Then

$$\begin{aligned} h_* : J^k(M,N) &\to J^k(M,K) \\ j_x^k f &\mapsto j_x^k(h \circ f) \end{aligned}$$

is a well-defined smooth map.

(iii) For $f \in C^\infty(M,N)$, the map $j^k f : M \to J^k(M,N)$ is a smooth map.

Throughout the rest of this thesis, we will also encounter the jet space $J^k(S \times M, N)$. S will then play the role of a parameter space. One of the parameter spaces used (e.g. in chapter 2) will be $S = [0,1]$. We must therefore also consider S to be a manifold with (perhaps empty) boundary. Note that the definition of jet spaces of manifolds easily extends to manifolds with boundary.

When studying this bundle we will often make use of the following lemma.

Lemma 1.11. *Let S, M and N be manifolds with (perhaps empty) boundary. Then the projection map*

$$\begin{aligned} p_{S,k} : J^k(S \times M, N) &\to J^k(M,N) \\ j_{(t,x)}^k f &\mapsto j_x^k f(t,-) \end{aligned}$$

is a (smooth) submersion.

Proof. Note that the local coordinates of $J^k(S\times M,N)$, are given by the coordinates of the Taylor polynomial of the local representation of f. Recall that for local charts φ_S and φ_M of S and M respectively, the map $\varphi_S\times\varphi_M$ is a local chart of $S\times M$. Since $S\times M$ can be covered by charts of this form, $J^k(S\times M,N)$ can be covered by charts induced by charts of such form. In those charts the map $p_{S,k}$ is an actual projection of coordinates, since it 'forgets' the derivatives that have a t direction. Thus it follows that $p_{S,k}$ is indeed a (smooth) submersion. □

1.1.3 Jet spaces of a fibre bundle

We have now defined jet spaces that store information of maps in $C^\infty(M,N)$. However, throughout this thesis we are also interested in sections of a given fibre bundle $\pi:E\to M$. In this subsection we will define jet spaces for sections of fibre bundles. The definitions and results in this subsection are based on those in section 2.2 of [9, pp. 37-42] as presented above. Alternate proofs of the statements given below can also be found in [21, pp. 13-17].

Definition 1.12. *Let $\pi:E\to M$ be a smooth fibre bundle and let $0\le k\le\infty$. We define the k^{th} jet space of E as follows*

$$J^k(E)=\left\{j^k_x s\mid x\in M,\ \textit{there exists some open } U \textit{ around } x\textit{, such that } s|_U\in\Gamma(E|_U)\right\}\subset J^k(M,E)$$

Lemma 1.13. *Let $\pi:E\to M$ be a fibre bundle and $0\le k<\infty$. Then $J^k(E)\subset J^k(M,E)$ is a submanifold.*

Proof. From lemma 1.10 (ii) we know that the map $\pi_*:J^k(M,E)\to J^k(M,M)$ is smooth. Furthermore, note π_* is a submersion. Also a map $s:M\to E$ is a local section around x if and only if there is an open U around x, such that $(\pi\circ s)_U=\mathrm{id}_U$. Hence it follows that

$$J^k(E)=(\pi_*)^{-1}\left(j^k\mathrm{id}_M(M)\right)$$

In the local charts of theorem 1.8 it is easy to see that $j^k\mathrm{id}_M(M)\subset J^k(M,M)$ is a submanifold. Thus it follows that $J^k(E)\subset J^k(M,E)$ is a submanifold as well. □

We will now look at equivalent properties of $J^k(E)$ as described in lemma 1.10 of $J^k(M,N)$.

Lemma 1.14. *Let $\pi:E\to M$ and $\pi':E'\to M$ be fibre bundles and let $0\le k<\infty$.*

(i) The map $p_k:J^k(E)\to M$, $j^k_x s\mapsto x$ is a smooth submersion.
(ii) Let $h:E\to E'$ be a fibre bundle morphism over the identity. Then $h_*:J^k(E)\to J^k(E')$ is a bundle morphism. Here h_* is the restriction of the map h_* as defined in lemma 1.10.
(iii) For $s\in\Gamma(E)$, the map $j^k s:M\to J^k(E)$ is a smooth map.

Proof. Note that for (ii), since h is a bundle morphism over the identity, h_* as given is both well-defined and fibre preserving. Properties (ii) and (iii) then follow immediately from lemma 1.10.

For property (i), recall the submersion $\pi_*:J^k(M,E)\to J^k(M,M)$. Since $J^k(E)=(\pi_*)^{-1}\left(j^k\mathrm{id}_M(M)\right)$, it follows that $\pi_*|_{J^k(E)}:J^k(E)\to j^k\mathrm{id}_M(M)$ is a submersion. The projection of $j^k\mathrm{id}_M(M)$ onto M is a diffeomorphism, thus $p_k:J^k(E)\to M$ is a submersion. □

Lemma 1.15. *Let $\pi : E \to M$ be a fibre bundle. Then $J^0(E) \cong E$.*

Proof. Note that the map

$$\pi_E : J^0(E) \to E$$
$$j_x^0 s \mapsto s(x)$$

is clearly a bijection. Let W be a trivializing chart of the bundle containing $U \times V$, where U and V are trivializing charts of M and the fibre of the bundle F respectively. Then a local chart of $J^0(M,E)$ can be described by $U \times U \times V$. In that chart $J^0(E)$ is then given by $\Delta(U) \times V \cong U \times V$. The map π_E above gives exactly this correspondence. Therefore it is a local diffeomorphism and also a bijection, hence it is a diffeomorphism. □

Lemma 1.16. *Let M and N be smooth manifolds and let $\pi : M \times N \to M$ be the product fibre bundle. Then for any $k \geq 0$, $J^k(M,N) \cong J^k(M \times N)$ as sets. For k finite, these spaces are diffeomorphic to each other.*

Proof. First of all note that we can define the inclusion

$$\iota : J^k(M,N) \to J^k(M, M \times N)$$
$$j_x^k f \mapsto j_x^k(\mathrm{id}_M, f)$$

Note that for any k this map is injective. Writing this map in the local charts from theorem 1.8 it follows immediately that this is a smooth embedding if k is finite. Furthermore, from the definition it is also immediate that $\iota(J^k(M,N)) = J^k(M \times N)$. This concludes the proof. □

Remark 1.17. From the lemma above it follows that jet spaces of fibre bundles are a generalisation of jet spaces of smooth maps. Furthermore, note that for $U \times V$ a fibred neighbourhood of the bundle, we get $J^k(U \times V) \cong J^k(U,V)$. The induced chart on $J^k(U \times V) \subset J^k(U, U \times V)$ then matches the chart on $J^k(U,V)$ as described in theorem 1.8. This thus gives us explicit local charts on $J^k(E)$. ⋄

To finish this section we will discuss one last important property of finite jet spaces that will be instrumental in defining a structure on the infinite jet space $J^\infty(E)$. A proof of this lemma can be found in [21, p.15].

Lemma 1.18. *Let $\pi : E \to M$ be a fibre bundle and $k \geq 1$. The projection map*

$$\pi_k : J^k(E) \to J^{k-1}(E)$$
$$j_x^k s \mapsto j_x^{k-1} s$$

defines a fibre bundle.

Corollary 1.19. *For $\pi : E \to M$ a fibre bundle, the projection map $p_k : J^k(E) \to M$ defines a fibre bundle structure.*

Example 1.20. The most important example of such a jet bundle for this thesis is the bundle $J^1(M,N) \to M$. In fact, this bundle is isomorphic to the bundle $\mathrm{Hom}(TM, TN) \to M$. △

Notation 1.21. *Let $s \in \Gamma\left(J^1(M,N)\right)$. Then we define the smooth functions*

$$y_s = \pi_N \circ s : M \to N$$

and

$$\begin{aligned} z_s : M &\to \mathrm{Hom}(TM, TN) \\ x &\mapsto (df)_x \end{aligned}$$

where $s(x) = j^1_x f$. Then we can write $s(x) = (x, y_s(x), z_s(x))$. Note that this is slight abuse of notation, since in fact one could argue that $s \sim z_s$.

Remark 1.22. Note that in general the fibres of the bundle $p_k : J^k(M,N) \to M$ do not have a vector structure. However, if $\pi : E \to M$ was a vector bundle to begin with, then $\Gamma(E)$ becomes a vector space. This vector structure can then be transferred to the fibres $J^k_x(M,N)$ as follows:

$$\begin{aligned} j^k_x f + j^k_x g &:= j^k_x(f+g) \\ \lambda\left(j^k_x f\right) &:= j^k_x \lambda f \end{aligned}$$

Since the transition functions of E are compatible with the vector structure, it can be shown that $p_k : J^k(M,N) \to M$ can also be endowed with a vector bundle structure. ◇

The bundles $p_k : J^k(E) \to M$ will play a key role in the rest of this thesis. This bundle also admits a special kind of sections, called *holonomic sections*, that are the main interest when studying so-called *differential relations* (see chapter 3). Therefore, these sections also play a main role in chapter 4 when studying transversality and in chapter 5 and 6 in the definitions of the hierarchies of singularities.

Definition 1.23. *Let $\pi : E \to M$ be a fibre bundle. A smooth section $\sigma \in \Gamma(J^k(E))$ is called **holonomic** if there exists a section $s \in \Gamma(E)$ such that for all $x \in M$ $\sigma(x) = j^k_x s$. In such cases we write $\sigma =: j^k s$.*

If $E = M \times N \to M$ is the product bundle, then $s = (id, f)$ for some $f \in C^\infty(M,N)$ and we write $\sigma = j^k f$.

We will finish this section with a technical proposition that is similar to lemma 1.11. This proposition will be needed in chapter 7.

Proposition 1.24. *Let M and N be manifolds and S a manifold with (perhaps empty) boundary. Then for any $k \geq 0$, the map*

$$\begin{aligned} p^*_{S,1} : J^k\left(J^1(S \times M, N)\right) &\to J^k\left(J^1(M,N)\right) \\ j^k_{(t,x)} s' &\mapsto j^k_x(s'(t,-)) \end{aligned}$$

is a surjective and submersive bundle morphism.

Proof. First of all note that writing this map in coordinate charts immediately shows that it is a smooth bundle morphism.

Furthermore note that for any $t_0 \in S$, we also get an induced map

$$\begin{aligned} p^*_{t_0} : \Gamma\left(J^1(M \times S, N)\right) &\to \Gamma\left(J^1(M,N)\right) \\ s' &\mapsto s'(t_0, -) \end{aligned}$$

Note that this is a surjective map. This can be seen by using the fact that

$$J^1(M \times S, N) \cong \mathrm{Hom}((T(M \times S), TN)$$

Similarly any $s \in \Gamma(J^1(M,N))$ can be seen as a section of $\mathrm{Hom}(TM, TN))$, and thus induces a section $s' : (x,t) \mapsto \big((x,t), y_s(x), (z_s(x), 0))\big)$. Then it follows that $p_{t_0}^*\left(s'_{t_0}\right) = s$ for all $t_0 \in S$.

From this it clearly follows that $p_{S,1}^*$ is surjective. In fact, it then also follows that in local coordinates the map is a projection and hence submersive. □

1.2 Pro-finite manifolds

While every finite jet space $J^k(E)$ admits the structure of a (finite-dimensional) smooth manifold, this is not the case for infinite jet space $J(E)$. However, in a way its building stones are the finite jet spaces, thus we can use the structures of the $J^k(E)$'s when working with $J(E)$. This gives $J(E)$ the structure of a so-called pro-finite manifold. Our exposition in this section is based on Appendix A of [1].

1.2.1 Towers of manifolds

Definition 1.25. *A* ***tower of manifolds*** $M_\bullet$ *is a sequence*

$$M_\bullet : \qquad \dots \longrightarrow M_k \xrightarrow{\pi_k} M_{k-1} \longrightarrow \dots \longrightarrow M_1 \xrightarrow{\pi_1} M_0$$

where each M_k *is a finite dimensional manifold and each* π_k *is a surjective submersion.*

For any $n \le k$*, we will denote by* $\pi_{k,k-n}$ *the composition*

$$\pi_{k,k-n} := \pi_{k-(n-1)} \circ \pi_{k-(n-2)} \circ \dots \pi_{k-1} \circ \pi_k : M_k \to M_{k-n}$$

Example 1.26. Any finite dimensional manifold admits the trivial tower of manifolds

$$\dots \longrightarrow M \xrightarrow{\mathrm{id}_M} M \longrightarrow \dots \longrightarrow M \xrightarrow{\mathrm{id}_M} M$$

△

Example 1.27. From lemma 1.18 we know that the projection maps $\pi_k : J^k(E) \to J^{k-1}(E)$ are submersions. Note that these maps are also clearly surjective, hence the sequence

$$J(E)_\bullet : \qquad \dots \longrightarrow J^k(E) \xrightarrow{\pi_k} J^{k-1}(E) \longrightarrow \dots \longrightarrow J^1(E) \xrightarrow{\pi_1} J^0(E)$$

forms a tower of manifolds. △

We also want to consider morphisms between such towers of manifolds. Besides being smooth maps between the underlying manifolds, these morphisms should be compatible with the tower structure.

Definition 1.28. *Let* $M_\bullet$ *and* $N_\bullet$ *be two towers of manifolds. A* ***concrete morphism*** *between* $M_\bullet$ *and* $N_\bullet$ *is a collection of maps*

$$f_\bullet = \left\{f_k : M_{m_k} \to N_k\right\}_{k \ge 0}$$

with m_k a strictly increasing sequence. Furthermore, the following (infinite) diagram is commutative

$$\begin{array}{ccccccc} \cdots \longrightarrow & M_{m_k} & \xrightarrow{\pi_{m_k,m_{k-1}}} & M_{m_{k-1}} & \longrightarrow \cdots & & \\ & & \searrow^{f_k} & & \searrow^{f_{k-1}} & & \\ & \cdots \longrightarrow & N_k & \xrightarrow{\pi_k} & N_{k-1} & \longrightarrow \cdots \end{array} \tag{1.1}$$

Example 1.29. The collection of projection maps $\pi_M : J^k(E) \to M$ form a concrete morphism between $J(E)_\bullet$ and the trivial tower induced by M. △

Definition 1.30. *Let $M_\bullet$ and $N_\bullet$ be towers of manifolds and let $f_\bullet, g_\bullet : M_\bullet \to N_\bullet$ be two concrete morphisms. We say that **$f_\bullet$ and $g_\bullet$ are equivalent** if and only if for every $k \geq 0$, there exists a large enough K_k, such that the following diagram commutes*

$$\begin{array}{ccccc} & & M_{m^f_k} & & \\ & \nearrow^{\pi_{K_k,m^f_k}} & & \searrow^{f_k} & \\ M_{K_k} & & & & N_k \\ & \searrow_{\pi_{K_k,m^g_k}} & & \nearrow_{g_k} & \\ & & M_{m^g_k} & & \end{array}$$

The above definition defines, as the name suggests, an equivalence relation, and thus it induces an equivalence class of compatible collections.

Definition 1.31. *Let $M_\bullet$ and $N_\bullet$ be towers of manifolds. We define the **morphisms between $M_\bullet$ and $N_\bullet$** by*

$$Hom(M_\bullet, N_\bullet) = \{[f_\bullet] \mid f_\bullet : M_\bullet \to N_\bullet\}$$

where the equivalence relation $[-]$ is the one defined above.

Let $f_\bullet = \{f_k : M_{m_k} \to N_k\}$ and $g_\bullet = \{g_k : N_{n_k} \to L_k\}$. We can now compose these collections of maps in the following natural way

$$g_\bullet \circ f_\bullet = \left\{g_k \circ f_{n_k} : M_{m_{n_k}} \to L_k\right\}$$

*We denote $f = [f_\bullet] \in Hom(M_\bullet, N_\bullet)$ and $g = [g_\bullet] \in Hom(N_\bullet, L_\bullet)$. We can define the **composition of f and g** by $g \circ f := [g_\bullet \circ f_\bullet]$, since the equivalence class $[g_\bullet \circ f_\bullet]$ is independent of the choice of representatives.*

These morphisms and their composition define the Category of towers of manifolds.

1.2.2 Pf-manifolds

Using the notion of towers of manifolds, we can endow (perhaps infinite-dimensional) spaces with a manifold-like strucutre, by projecting onto each level of a tower.

Definition 1.32. *Let $M_\bullet$ be a tower of manifolds. Then we call the inverse limit*

$$\varprojlim M_k := \{(x_k)_{k\geq 0} \mid x_k \in M_k, \pi_k(x_k) = x_{k-1}\}$$

*the **pro finite (pf) manifold** induced by the tower $M_\bullet$. We will call the collection of projection maps $a_\bullet : \left\{ a_i : \varprojlim M_k \to M_i \right\}$ the **pf atlas** of this pf manifold.*

Remark 1.33. If for any set M and tower $M_\bullet$ we fix a bijection between M and $\varprojlim M_k$, then we can also endow M with a pf-structure. ⋄

Remark 1.34. The definition of a pf-manifold (and pf-atlasses) as defined in [1, Appendix A] is more general than the one we consider here. Our notion of a pf-manifold is equivalent to the notion of *normal pf-manifold* as defined in [1, Appendix A]. However, the notion as introduced above will suffice for the purposes of this thesis. ⋄

Remark 1.35. Let M denote the pf-manifold induced by the tower of manifolds $M_\bullet$. Then for any $k \geq k'$, the following diagram commutes

$$\begin{array}{ccccc} & & M & & \\ & \swarrow^{a_k} & & \searrow^{a_{k'}} & \\ M_k & & \xrightarrow{\pi_{k,k'}} & & M_{k'} \end{array}$$

⋄

Example 1.36. Note that any finite-dimensional manifold M can be equipped with the trivial pf-manifold structure, induced by the trivial manifold tower of M as described in example 1.26. △

Remark 1.37. Let M and N be finite dimensional manifolds. Note that from the definition it follows that as sets $J^\infty(M,N) \cong \varprojlim J^k(M,N)$. Thus the infinite jet space $J^\infty(M,N)$ can be equipped with the pf-structure induced by the tower of manifolds $J^\bullet(M,N)$.

More generally, for any fibre bundle $\pi : E \to M$, the infinite jet space $J^\infty(E) = \varprojlim J^k(E)$ can be equipped with the pf-structure induced by the tower of manifolds $J^\bullet(E)$. ⋄

Remark 1.38. A pf-manifold M naturally carries the limit topology. Thiis is the topology for which the collection

$$\mathscr{B} := \left\{ U \subset M \mid \text{ there exists a } k \geq 0, \text{ such that } U = a_k^{-1}(U_k) \text{ with } U_k \subset M_k \text{ open} \right\}$$

forms a basis. ⋄

The upshot of equipping a manifold with a pf-structure is that we can also define a notion of smooth maps between pf-manifolds.

Definition 1.39. *Let $a : M \to M_\bullet$ and $b : N \to N_\bullet$ be two pf-manifolds with respective pf-atlasses a and b. We say that a set-theoretical map $f : M \to N$ is a **smooth pf-map** if there exists a strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, there exists an $f_k \in C^\infty(M_{m_k}, N_k)$ for which the following diagram commutes*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle a_{m_k}}\downarrow & & \downarrow{\scriptstyle b_k} \\ M_{m_k} & \xrightarrow[f_k]{} & N_k \end{array}$$

We will denote this collection by $f_\bullet = \{f_k : M_{m_k} \to N_k\}_{k\geq 0}$

Remark 1.40. Note that we could have left out the condition that the sequence $(m_k)_{k\in\mathbb{N}}$ is increasing to get an equivalent definition. However, assuming that the sequence is increasing has the advantage that for any smooth $f : M \to N$, the collection $f_\bullet = \{f_k : M_{m_k} \to N_k\}_{k\geq 0}$ is a concrete morphism of towers, i.e. it satisfies the commutative diagram 1.1. Conversely, any concrete morphism of towers gives rise to a smooth pf-map between the corresponding pf-manifolds. ⋄

Example 1.41. Recall from lemma 1.14 that for any $s \in \Gamma(E)$, the map $j^k s : M \to J^k(E)$ is smooth. Furthermore note that the following diagram commutes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & J^k(E) & \xrightarrow{\pi_k} & J^{k-1}(E) & \longrightarrow & \cdots \\ & & \uparrow{\scriptstyle j^k s} & & \uparrow{\scriptstyle j^{k-1} s} & & \\ \cdots & \longrightarrow & M & \xrightarrow{\mathrm{id}_M} & M & \longrightarrow & \cdots \end{array}$$

Thus it follows that any $s \in \Gamma(E)$ induces a smooth pf-map $j^\infty s : M \to J^\infty(E)$, defined by the collection $\{j^k s : M \to J^k(E)\}_{k\geq 0}$. △

Example 1.42. Let M, N and L be finite dimensional manifolds and $f \in C^\infty(N,L)$. Then note that for any $k \geq 0$, we get an induced map

$$\begin{aligned} f_k : J^k(M,N) &\to J^k(M,L) \\ j^k_x g &\mapsto j^k_x (f \circ g) \end{aligned}$$

which is smooth as is shown in [9, p.40-41, Theorem 2.7]. Furthermore, it is easy to check that the following diagram commutes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & J^k(M,N) & \xrightarrow{\pi_k} & J^{k-1}(M,N) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k-1}} & & \\ \cdots & \longrightarrow & J^k(M,L) & \xrightarrow{\pi_k} & J^{k-1}(M,L) & \longrightarrow & \cdots \end{array}$$

Thus f induces the collection $f_\bullet = \{f_k : J^k(M,N) \to J^k(M,L)\}_{k\geq 0}$, which defines the smooth pf-map

$$\begin{aligned} \tilde{f} : J^\infty(M,N) &\to J^\infty(M,L) \\ j^\infty_x g &\mapsto j^\infty_x (f \circ g) \end{aligned}$$

△

Remark 1.43. Let M and N be pf-manifolds with some given towers $M_\bullet$ and $N_\bullet$. Note that smooth maps between M and N are then exactly the morphisms between the towers $M_\bullet$ and $N_\bullet$, i.e. $C^\infty(M,N) \cong \mathrm{Hom}(M_\bullet, N_\bullet)$ [1, p.54]. ⋄

Remark 1.44. Since we have defined the notion of smooth maps between pf-manifolds, we should also consider whether they are compatible with the topology, i.e. whether they are continuous maps. Note that this is indeed the case, since any smooth pf-map gives rise to a collection of smooth maps between manifolds, which are thus also continuous. Using the definition of the limit topology it then easily follows that smooth pf-maps are also continuous maps between the topological spaces. ⋄

Example 1.45. Let M be a pf-manifold with atlas $\{a_k : M \to M_k\}$. Then by equipping $\mathbb{R}$ with the trivial pf-structure, induced by the trivial atlas from example 1.36, we can define $C^\infty(M) := C^\infty(M,\mathbb{R})$. Then a map $f : M \to \mathbb{R}$ defines an element in $C^\infty(M)$ if and only if there exists some m_0 and a $g \in C^\infty\left(M_{m_0}\right)$, such that $f = g \circ a_{m_0}$. We say that f is *defined at the level* m_0 and that g is a *representative* of f. Note that we can write

$$C^\infty(M) = \varinjlim_m C^\infty(M_m)$$

via the pull-back by a. △

One could also wonder what a sensible definition of a *pf-submanifold* would be. The idea is that for a subset $W \subset M$, we want to somehow restrict the pf-structure of M to W. To be able to do this we must impose some condition on W. We have not been able to find a definition of such a concept in literature. However, for the purposes of this thesis, it makes sense to give the following definition.

Definition 1.46. *Let M with $a_\bullet = \{a_k : M \to M_k\}$ be a pf-manifold. We say that a subset $W \subset M$ is a **pf-submanifold** of M, if there exists some finite $l \geq 0$ and submanifold $W_l \subset M_l$, such that*

$$W = (a_l)^{-1}(W_l)$$

*We then say that W is a pf-submanifold of **level** l and **codimension** $\operatorname{codim}(W_l)$.*

Note that the level of such a pf-submanifold is not unique, i.e. one should consider it as a level at which the pf-submanifold is defined as an actual submanifold. Note that this is a similar idea as the level of a smooth function on M. However, as long as $W \neq \emptyset$, the codimension of a pf-submanifold ís uniquely defined.

Remark 1.47. Note that a pf-submanifold $W \subset M$ can then indeed be endowed with a pf-structure. Let (M_k, π_k) be the tower of M and let W be of level l. Then consider the following tower of manifolds. Set for $k \geq 0$, $W^k := \pi^{-1}_{k+l,l}(W_l)$ and consider the following maps

$$\dots \longrightarrow W^k \xrightarrow{\pi_k|_{W^k}} W^{k-1} \longrightarrow \dots \longrightarrow W^1 \xrightarrow{\pi_1|_{W^1}} W^0$$

which are then clearly surjective submersions. We then get that

$$W \cong \varprojlim W^k$$

and the induced atlas is given by

$$a'_\bullet = \left\{a'_k = a_{k+l}|_W : W \to W^k\right\}$$

However, for the purposes of this thesis, it suffices to simply consider them as subsets of the original pf-manifold. ◇

1.2.3 The tangent bundle of a pf-manifold

Definition 1.48. *A **pf-vector bundle** over a pf-manifold M, consists of a pf-manifold E and a smooth pf-map $p : E \to M$, such that for the representative collection of p,*

$$p_\bullet := \left\{p_k : E_k \to M_k \mid k \geq 0\right\}$$

the following diagram commutes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & E_k & \xrightarrow{\pi_k^E} & E_{k-1} & \longrightarrow & \cdots & \longrightarrow & E_1 & \xrightarrow{\pi_1^E} & E_0 \\
 & & \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p_{k-1}} & & & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_0} \\
\cdots & \longrightarrow & M_k & \xrightarrow[\pi_k^M]{} & M_{k-1} & \longrightarrow & \cdots & \longrightarrow & M_1 & \xrightarrow[\pi_1^M]{} & M_0
\end{array}$$

Furthermore, we require each $p_k : E_k \to M_k$ to be a vector bundle and each π_k^E to be a vector bundle morphism over π_k^M.

Remark 1.49. Note that the defintion above can also be extended to a notion of general pf-fibre bundles. However, in this thesis the only examples we will encounter are that of vector bundles, specifically that of the tangent bundle. $\diamond$

Definition 1.50. *Let $p : E \to M$ be a pf-vector bundle. Then a **(pf-)section** of E is any smooth pf-map $\sigma : M \to E$, such that $p \circ \sigma = id_M$.*

Remark 1.51. By remark 1.40 we know that a section consists of a collection of maps

$$\sigma_\bullet = \left\{\sigma_k : M_{m_k} \to E_k\right\}_{k\geq 1}$$

The condition $p \circ \sigma = \mathrm{id}_M$ translates to the following commutative diagram for any k

$$\begin{array}{ccccc}
 & & E_k & & \\
 & \nearrow{\scriptstyle \sigma_k} & & \searrow{\scriptstyle p_k} & \\
M_{m_k} & & \xrightarrow[\pi_{m_k,k}]{} & & M_k
\end{array}$$

Note that σ_k can thus be seen as a section of the pullback bundle $\pi^*_{m_k,k}(E_k) \to M_{m_k}$. $\diamond$

Example 1.52. Let $p : E \to M$ be a vector bundle. Then as described in remark 1.37 we can endow $J^\infty E$ with the structure of a pf-manifold. Furthermore, from remark 1.22 we know that the projections $p_k : J^k E \to M$ are also vector bundles. Hence we get an induced pf-map $p_\bullet$: $E \to M$, where M is endowed with the trivial pf-structure. Note that the projections p_k are compatible with the projections π_k of E, in the sense that the following diagram commutes

$$\begin{array}{cccccccccccc}
J^\infty E: & \dots & \longrightarrow & J^k E & \xrightarrow{\pi_k} & J^{k-1}E & \longrightarrow & \dots & \longrightarrow & J^1 E & \xrightarrow{\pi_1} & J^0 E \cong E \\
 & & & \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p_{k-1}} & & & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_0 = p} \\
M_\bullet : & \dots & \longrightarrow & M & \xrightarrow{\mathrm{id}_M} & M & \longrightarrow & \dots & \longrightarrow & M & \xrightarrow{\mathrm{id}_M} & M
\end{array}$$

Furthermore, since M is endowed with the trivial pf-structure, a section of $J^\infty E$ is defined by a section at some finite level k, $\sigma_k \in \Gamma(J^k E)$. $\triangle$

Example 1.53. Another example of a pf-vector bundle is that of the *tangent bundle* of a pf-manifold. Let $(M, M_\bullet)$ be a pf-manifold equipped with an atlas a and let TM_k denote the tangent space of M_k for all k. Since $\pi_k : M_k \to M_{k-1}$ is a surjective submersion, the same is true for the differential $d\pi_k : TM_k \to TM_{k-1}$. Thus setting $TM = \varprojlim TM_k$, we can endow

$$TM: \quad \dots \longrightarrow TM_k \xrightarrow{d\pi_k} TM_{k-1} \longrightarrow \dots \longrightarrow TM_1 \xrightarrow{d\pi_1} TM_0$$

with a pf-manifold structure. Furthermore, let $p_l : TM_l \to M_l$ denote the usual tangent bundle projection. Then the following diagram commutes

$$
\begin{array}{llllllllll}
TM: & \dots \longrightarrow & TM_k & \xrightarrow{d\pi_k} & TM_{k-1} & \longrightarrow \dots \longrightarrow & TM_1 & \xrightarrow{d\pi_1} & TM_0 \\
 & & \downarrow p_k & & \downarrow p_{k-1} & & \downarrow p_1 & & \downarrow p_0 \\
M: & \dots \longrightarrow & M_k & \xrightarrow{\pi_k} & M_{k-1} & \longrightarrow \dots \longrightarrow & M_1 & \xrightarrow{\pi_1} & M_0
\end{array}
$$

thus the collection $p_\bullet = \{p_k : TM_k \to M_k\}_{k\geq 1}$ induces a vector bundle structure on TM.

It is shown in [14, Lemma 3.15, p.10] that an equivalent definition of TM can be given as follows. One defines $T_xM = \mathrm{Der}_x(C^\infty(M))$ and $TM = \bigcup_{x\in M} T_xM$. Let $TM_\bullet$ be the tower as defined above, then TM can be endowed with the pf-atlas $\tilde{a}_\bullet : TM \to TM_\bullet$, defined by

$$
\begin{aligned}
\tilde{a}_k|_{T_xM} : T_xM &\to T_xM_k \\
V &\mapsto (f_k \mapsto V(f_k \circ a_k))
\end{aligned}
\tag{1.2}
$$

As shown in [14] this atlas is the natural atlas for the inverse limit of the tower of tangent spaces, hence the two definitions of the tangent space of M are indeed equivalent.

The notion of a pf-tangent bundle gives rise to pf-vector fields $\mathfrak{X}(M)$, as sections of the tangent bundle. In [14, Theorem 3.26, p. 17] it is shown that $\mathfrak{X}(M) \cong \mathrm{Der}(C^\infty M)$ and pf-vector fields act on pf-functions in the following way

$$
\begin{aligned}
\mathfrak{X}(M) \times C^\infty(M) &\to C^\infty(M) \\
(X, f) &\mapsto \big(X(f) : p \mapsto X_p(f)\big)
\end{aligned}
$$

In particular, note that for $f \in C^\infty(M)$, there exists a k, such that $f = f_k \circ a_k$, with $f_k \in C^\infty(M_k)$. Then for $X \in \mathfrak{X}(M)$, there exists an m_k, such that we get the following commutative diagram

$$
\begin{array}{ccc}
TM & \xrightarrow{\tilde{a}_k} & TM_k \\
X\uparrow & & \uparrow X_k \\
M & \xrightarrow[a_{m_k}]{} & M_{m_k}
\end{array}
$$

Thus it follows that for $p \in M$

$$
\begin{aligned}
X_p(f) &= X_p(f_k \circ a_k) \\
&\overset{(1.2)}{=} \tilde{a}_k(X)_p(f_k) \\
&= (X_k)_{a_{m_k}(p)}(f_k)
\end{aligned}
$$

where the vector $(X_k)_{a_{m_k}(p)} \in T_{a_{m_k}(p)}M_k$ acts on $f_k \in C^\infty(M_k)$ in the usual sense.

△

This example also allows us to consider the tangent space of pf-submanifolds in their ambient space.

Definition 1.54. *Let $(M, M_\bullet)$ be a pf-manifold with atlas $a_\bullet$. Also, let $\tilde{a}_\bullet$ denote the atlas of its tangent bundle. Let $\Sigma \subset M$ be a pf-submanifold of level k, with $a_k(\Sigma) = \Sigma_k \subset M_k$. Then we define the **tangent space of Σ** at $x \in M$ by*

$$
T_x\Sigma := (\tilde{a}_k)^{-1}\left(T_{a_k(x)}\Sigma_k\right) \subset T_xM
$$

Before we finish this section we make one last observation. For any smooth map $f : M \to N$ between pf-manifolds, generated by the collection $f_\bullet = \{f_k : M_{m_k} \to N_k\}$, the collection of maps $(df)_\bullet = \{df_k : TM_{m_k} \to N_k\}$ is a concrete morphism between the towers $TM_\bullet$ and $TN_\bullet$.

Definition 1.55. *Let M be a pf-manifold with atlas $a_\bullet$ and let $f \in C^\infty(M)$. Then we define the differential of f, $(df) : TM \to TN$ as the smooth map generated by the concrete morphism $(df)_\bullet$.*

Remark 1.56. Let $\mathbb{R}$ be equipped with the trivial pf-structure and let $f \in C^\infty(M)$ be a map of level k. Then it follows that (df) is also of level k. Take $\mathcal{V} \in T_xM$, then it follows by definition that

$$\begin{aligned}(df)_x(\mathcal{V}) &= \left(df_k\right)_{a_k(x)}(\tilde{a}_k(\mathcal{V}) \\ &= (\mathcal{V}_k)_{a_k(x)}(f_k) = \mathcal{V}(f)\end{aligned}$$

Thus differentials of $C^\infty(M)$ act on TM exactly how we would expect them to. ◇

1.3 The ∞-Cartan distribution

In the previous section we defined the tangent bundle of $J^\infty E$. Of this tangent bundle, we are mostly interested in a special subset, namely the Cartan distribution. One should think about the Cartan distribution as the subset of the tangent bundle containing all holonomic directions, i.e. vectors 'tangent' to a holonomic section. This distribution will therefore play a key role in the definition of the hierarchies of singularities in chapter 6. In this section we will discuss the definition of the (infinite) Cartan distribution and discuss some of its (technical) properties that we will need in chapter 6. This section is based on [20, pp.1636-1637] and Boardman's definition of and results about *total vector fields* in [2, pp. 23-29].

Remark 1.57. Recall from example 1.41 that for any section $s : M \to E$, we get an induced section $j^\infty s \in \Gamma(J^\infty E)$. From definition 1.55, we also get the following map

$$dj^\infty s : TM \to TJ^\infty E$$

defined by the collection of differentials $(dj^\infty s)_\bullet = \{dj^k s : TM \to TJ^k E\}_{k \geq 0}$. ◇

Definition 1.58. *The* ***Cartan distribution of order*** **k**, *$\mathcal{C}_k \subset TJ^kE$, is defined as the smallest subbundle of TJ^kE containing all vectors that are tangent to a holonomic section, i.e. it is defined by*

$$\mathcal{C}_k = \langle (dj^k s)_x(V) \mid j^k_x s \in J^k(E), V \in T_xM \rangle$$

The ***infinite Cartan distribution*** *$\mathcal{C}_\infty \subset TJ^\infty E$ is given by*

$$\mathcal{C}_\infty = \left\{ (dj^\infty s)_x(V) \mid j^\infty_x s \in J^\infty(E), V \in T_xM \right\}$$

In the finite Cartan distribution we need to 'add' elements to make its fibres into vector spaces. Note that in the C^k-Cartan distribution, these are precisely those vectors tangent to the projection $\pi_k : J^k(E) \to J^{k-1}(E)$.

This is not the case for the infinite Cartan distribution. Note that any $k+1$-jet of a given section contains all the information of the k-jet of that section and also its differential. Thus if all k-jets match, then so must their differentials. The infinite Cartan distribution at a given point $j^\infty_x s \in J^\infty(E)$, is thus simply a copy of T_xM. This is made precise in the following propositions. These propositions are based on a remark from [20, p.1636].

Proposition 1.59. *Let $\pi : E \to M$ be a fibre bundle and s, s' local sections of E around $x \in M$ with $j_x^{k+1} s = j_x^{k+1} s'$. Then $(dj^k s)_x = (dj^k s')_x$*

Proof. Note that a trivializing chart U of the bundle, with coordinates $x^1, \dots, x^m$ on M and $y^1, \dots, y^n$ on the fibre of E, induces a trivialisation of $J^k(E)$, in which we can write

$$\begin{aligned} j^k s : \pi(U) &\to J^k(E)|_U \\ x &\mapsto \left(x, ((\partial^{\mathcal{J}} s)(x))_{0 \le |\mathcal{J}| \le k}\right) \end{aligned}$$

Here $\mathcal{J}$ denotes some multi-index, and $|\mathcal{J}|$ is the length of $\mathcal{J}$.

Thus it follows that for any $1 \le i \le m$

$$\begin{aligned} \left(dj^k s\right)_x \left(\frac{\partial}{\partial x_i}\right) &= \left(\left(\frac{\partial}{\partial x_i}\right)_x, \left(\left(\partial_i \partial^{\mathcal{J}} s\right)(x)\right)_{0 \le |\mathcal{J}| \le k}\right) \\ &= \left(\left(\frac{\partial}{\partial x_i}\right)_x, \left(\left(\partial^{(i,\mathcal{J})} s\right)(x)\right)_{0 \le |\mathcal{J}| \le k}\right) \end{aligned}$$

Note that $\partial^{(i,\mathcal{J})}$ is a partial derivative of order at most $k+1$. Since $j^{k+1} s = j^{k+1} s'$, it thus follows from lemma 1.4 that $\left(\left(\partial^{(i,\mathcal{J})} s\right)(x)\right) = \left(\left(\partial^{(i,\mathcal{J})} s'\right)(x)\right)$ and thus $\left(dj^k s\right)_x \left(\frac{\partial}{\partial x_i}\right) = \left(dj^k s'\right)_x \left(\frac{\partial}{\partial x_i}\right)$ for all $1 \le i \le m$. Hence we get indeed that $(dj^k s)_x = (dj^k s')_x$. □

Proposition 1.60. *The fibre $(\mathcal{C}_\infty)_{j_x^\infty s}$ of $\mathcal{C}_\infty$ is isomorphic to $T_x M$.*

Proof. First of all note that for all finite k, $j^k s : M \to J^k E$ is an immersion, thus the map $\left(dj^\infty s\right)_x : T_x M \to (C_\infty)_{j_x^\infty s}$ is injective. We will now show that this map is also surjective. Assume that $\mathcal{V} \in (C^\infty)_{j_x^\infty s}$. Then $\mathcal{V}$ can be written as $(dj^\infty s')_x(V)$, for some $V \in T_x M$ and $j_x^\infty s = j_x^\infty s'$. Then since $j_x^k s = j_x^k s'$ for all k it follows from proposition 1.59 that $(dj^k s)_x = (dj^k s')_x$ for all k. Thus we get that $(dj^\infty s)_x = (dj^\infty s')_x$ and hence $\mathcal{V} = (dj^\infty s)_x(V)$. Therefore it follows that $\left(dj^\infty s\right)_x : T_x M \to (C_\infty)_{j_x^\infty s}$ is surjective. □

Remark 1.61. Note that in fact, we can view the Cartan distribution as a pf-vector bundle over $J^\infty(E)$. Let $p_{*,k} : p_k^*(TM) \to J^k(E)$ denote the pull-back bundle of $TM \to M$ by the map $p_k : J^k(E) \to M$. Note that we then get the following pf-vector bundle

$$\begin{array}{llcccccc} \varprojlim p_k^*(TM): & \dots & \longrightarrow & p_k^*(TM) & \xrightarrow{\tau_k} & p_{k-1}^*(TM) & \longrightarrow & \dots \\ & & & \big\downarrow p_{*,k} & & \big\downarrow p_{*,k-1} & & \\ J^\infty(E): & \dots & \longrightarrow & J^k(E) & \xrightarrow{\pi_k} & J^{k-1}(E) & \longrightarrow & \dots \end{array}$$

We denote the induced atlas of $\varprojlim p_k^*(TM)$ by $a'_\bullet$. Note that we can define the following bijective map

$$\begin{aligned} \theta : \varprojlim p_k^*(TM) &\to \mathcal{C}_\infty \\ (j_x^\infty s, V) &\mapsto (dj^\infty s)_x(V) \end{aligned}$$

Hence, $\mathcal{C}_\infty$ can indeed be endowed with a pf-vector bundle structure over $J^\infty(E)$. ⋄

Lemma 1.62. *The inclusion map*

$$\theta : \varprojlim p_k^*(TM) \to TJ^\infty(E)$$
$$\left(j_x^\infty s, V\right) \mapsto \left(dj^\infty s\right)_x(V)$$

is a smooth pf-map.

Proof. First of all, note that by proposition 1.59, we get well-defined maps

$$\theta_k : p_{k+1}^*(TM) \to TJ^k(E)$$
$$\left(j_x^{k+1} s, V\right) \mapsto \left(dj^k s\right)_x(V)$$

Furthermore, writing this in coordinates shows that it is a smooth map. The collection $\theta_\bullet$ is compatible with the pf-manifold structure and we also have $\tilde{a}_k \circ \theta = \theta_k \circ a'_{k+1}$. Thus θ is indeed a smooth pf-map. □

1.3.1 Sections of the ∞-Cartan distribution

Note that from remark 1.61 we now get two definitions of a smooth section of the ∞-Cartan distribution, namely a smooth section of $\varprojlim p_k^*(TM)$ and a smooth section of $TJ^\infty(E)$ whose image lies in $\mathcal{C}_\infty$. We claim that in fact these two definitions are the same.

To see this we first want to pay attention to certain special sections of the Cartan distribution.

Definition 1.63. *Let $X \in \mathfrak{X}(M)$ be a vector field of M. Then we define its lift $\tilde{X} \in \Gamma(TJ^\infty(E))$ by*

$$\tilde{X}_{j_x^\infty s} := (dj^\infty s)_x(X_x)$$

Note that $\tilde{X}$ indeed maps into the right fibres of the bundle $TJ^\infty(E)$. However, for it to be a section it must also be a smooth pf-map. This follows from the fact that it is generated by the collection of smooth maps $\left(X^k\right)_{k\geq 0}$ defined as

$$X^k : J^{k+1}(E) \to TJ^k(E)$$
$$j_x^{k+1} s \mapsto \left(dj^k s\right)_x(X_x)$$

Similarly it follows that it also maps into the right fibres of $\varprojlim p_k^*(TM)$ and is generated by the collection of smooth maps

$$\left(X^k\right)' : J^k(E) \to p_k^*(TM)$$
$$j_x^k s \mapsto \left(j_x^k s, X_x\right)$$

If we then consider a trivializing neighbourhood U of M and the sections $X_1, \dots X_m$ spanning TU, it is easy to see that the sections $\tilde{X}_1, \dots \tilde{X}_m$ locally (i.e. on $a^{-1}(U) \subset J^\infty(E)$) span the Cartan distribution for both pf-structures.

From the above discussion we get the following lemma.

Lemma 1.64. *Let $\mathscr{X} : J^\infty(E) \to \mathcal{C}_\infty$ be a map that respects the fibre-structure. Then the following are equivalent* *(i)* $\mathscr{X}$ *is a smooth section of* $TJ^\infty(E)$.

(ii) $\mathscr{X}$ *is a smooth section of* $\varprojlim p_k^*(TM)$.

(iii) For $U \subset M$ *a trivializing chart and* $TU = \langle X_1, \dots, X_m \rangle$*, on* $\tilde{U} := a^{-1}(U)$*,* $\mathscr{X}$ *can be written as*

$$\mathscr{X}|_{\tilde{U}} = \sum_{i=1}^{m} \alpha_i \tilde{X}_i$$

where each $\alpha_i \in C^\infty(\tilde{U})$*. These* α_i*'s are called the* ***local coefficients*** *of* $\mathscr{X}$.

This thus gives us a unique description of sections of the Cartan distribution. In fact it matches the definition Boardman gives in his paper [2] of section of the *total tangent bundle*. We will denote the space of such sections by $\Gamma(\mathscr{C}_\infty)$.

Definition 1.65. *We say that an element* $\mathscr{X} \in \Gamma(\mathscr{C}_\infty)$ *is* ***of level k*** *if all of its local coefficients are of level* k.

Note that elements $\mathscr{X} \in \Gamma(\mathscr{C}_\infty)$ of level 0 are exactly the lifts $\tilde{X}$ of vector fields $X \in \mathfrak{X}(M)$. It turns out that these sections of $\mathscr{C}_\infty$ act in a nice way as derivatives on $C^\infty(J^\infty(E))$, in the sense that they lift the level of a given function on $J^\infty(E)$ by 1. This is made precise in the following proposition. The following proposition and its proof are based on [2, Lemma 1.12, p. 27].

Proposition 1.66. *Let* $\tilde{X} \in \mathfrak{X}(J^\infty E)$ *be the lift of a vector field* $X \in \mathfrak{X}(M)$*. Furthermore let* $f \in C^\infty(J^\infty E)$ *be of level* k*, i.e. there exists some* $f_k \in C^\infty(J^k E)$ *such that* $f = f_k \circ a_k$*. Then* $\tilde{X}(f) \in C^\infty(J^\infty E)$ *is of level* $k+1$.

Proof. We want to show that the function $\tilde{X}(f)$ is defined at the level $k+1$. To see this, take two local sections s, s' of E with $j_x^{k+1}s = j_x^{k+1}s'$.

Note that by example 1.53 we see that

$$\begin{aligned}
\tilde{X}(f)(j_x^\infty s) &= \tilde{X}_{j_x^\infty s}(f_k \circ a_k) \\
&= (\tilde{a}_k)\left(\tilde{X}_{j_x^\infty s}\right)(f_k) \\
&= (df_k)_{j_x^k s}\left((X_k)_{a_k(j_x^\infty s)}\right) \\
&= (df_k)_{j_x^k s}\left((X_k)_{j_x^k s}\right)
\end{aligned}$$

Where X_k is the lift of X to a section of TJ^kE, given by

$$(X_k)_{j_x^k s} = (dj^k s)_x(X_x)$$

Note that since $j_x^{k+1}s = j_x^{k+1}s'$ it follows that $(df_k)_{j_x^k s} = (df_k)_{j_x^k s'}$. Furthermore, it follows from proposition 1.59 that

$$(X_k)_{j_x^k s} = (dj^k s)_x(X_x) = (dj^k s')_x(X_x) = (X_k)_{j_x^k s'}$$

Hence it follows that $\tilde{X}_{j_x^\infty s}(f) = \tilde{X}_{j_x^\infty s'}(f)$. Thus $\tilde{X}(f)$ indeed factors through $J^{k+1}E$, i.e. there exists an (a priori not necessarily smooth) map $g : J^{k+1}E \to \mathbb{R}$, such that the following diagram commutes

$$\begin{array}{ccc}
J^\infty E & \xrightarrow{\tilde{X}(f)} & \mathbb{R} \\
{\scriptstyle a_{k+1}}\downarrow & \nearrow{\scriptstyle g} & \\
J^{k+1}E & &
\end{array}$$

By [14, Theorem 3.25, p.17] we know that $\tilde{X}(f) \in C^\infty(J^\infty E)$, hence there exists some $l > 0$ and an $h \in C^\infty(J^l E)$, such that the following diagram commutes

$$\begin{array}{ccc} J^\infty E & \xrightarrow{\tilde{X}(f)} & \mathbb{R} \\ {\scriptstyle a_l}\downarrow & \nearrow_{h} & \\ J^l E & & \end{array}$$

Note that we can choose $l \geq k+1$, such that $a_{k+1} = \pi_{l,k+1} \circ a_l$. Thus it follows that the following diagram commutes

$$\begin{array}{ccc} J^l E & \xrightarrow{h} & \mathbb{R} \\ {\scriptstyle \pi_{l,k+1}}\downarrow & \nearrow_{g} & \\ J^{k+1} E & & \end{array}$$

Since $\pi_{l,k+1}$ is a surjective submersion and h is smooth, it follows from the submersion theorem that g is also smooth. Hence $\tilde{X}(f)$ is indeed defined at the level $k+1$. □

We also get the following stronger result, proven by Boardman in [2, Lemma 1.21, p.28].

Proposition 1.67. *Let $\mathscr{X} \in \Gamma(\mathscr{C}_\infty)$ be of level $k+1$. Furthermore let $f \in C^\infty(J^\infty E)$ be of level k, i.e. there exists some $f_k \in C^\infty(J^k E)$ such that $f = f_k \circ a_k$. Then $\mathscr{X}(f) \in C^\infty(J^\infty E)$ is of level $k+1$.*

The proof of this proposition is completely analogous to the one from proposition 1.66.

Remark 1.68. The conclusions of the previous propositions do not hold for every section of the Cartan distribution $\mathscr{X} \in \Gamma(\mathscr{C}_\infty)$. While any tangent vector of $\mathscr{C}_\infty$ is defined as the lift of a vector V of the base manifold, this vector V might vary as the base point of the vector field, $j^\infty_x s$, varies, which is exactly what is described by the local coefficients.

Take for example the trivial product bundle $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which we denote by E, and define $\mathscr{X} \in \Gamma(C_\infty)$ by

$$\mathscr{X}_{j^\infty_x s} = (d j^\infty s)_x \left(\frac{\partial^2 f}{\partial x^2}(x) \right)$$

where we used the canonical identification $T_x\mathbb{R} \cong \mathbb{R}$ and $f \in C^\infty(M)$ denotes the map for which $s = \big(\mathrm{id}_{\mathbb{R}}, f\big)$. Note that we can write

$$\mathscr{X} = \left(\frac{\partial^2 f}{\partial x^2}(x) \right) \tilde{\partial}_x$$

where the coefficient is simply a coordinate projection of $J^2(E)$, hence it is a smooth map of level 2.

Now define $\varphi \in C^\infty(J^\infty E)$ as the projection map on to $\mathbb{R}$, sending $j^\infty_x s$ to x, which is clearly smooth. Also note that φ is defined at level 0. Then by definition we get

$$\mathscr{X}(\varphi)(j^\infty_x s) = (\mathrm{pr}_1)_{j^0_x s} \circ (d j^0 s)_x \left(\frac{\partial^2 f}{\partial x^2}(x) \right) = \frac{\partial^2 f}{\partial x^2}(x)$$

where $(\mathrm{pr}_1)_{j^0_x s} : T_{j^0_x s}(J^0 E) \cong \mathbb{R}^2 \to \mathbb{R}$ denotes the projection onto the first coordinate. This expression is dependent on the value of the second derivative of f, hence $X(\varphi)$ does not have a factorization through $J^1(E)$. Hence the conclusions of the previous lemma do not apply to this vector field $\mathscr{X}$ and function φ. ⋄

1.3.2 Subbundles of the ∞-Cartan distribution

Now that we have seen that we can view the Cartan distribution as a pf-vector bundle over $J^\infty(E)$, it also makes sense to define subbundles. These subbundles give us a tool to put restrictions on what directions we want to consider in the Cartan distribution, which will make them play a key role in the definition of the hierarchies of singularities in chapters 5 and 6.

Definition 1.69. *Recall that in a local chart U of M, TU is spanned by some sections $X_1, \dots X_m \in \mathfrak{X}(M)$. Let $\mathcal{S} \subset \mathcal{C}_\infty$. We say that $\mathcal{S}$ is a* ***smooth subbundle of rank* l** *if around every point in $J^\infty(E)$, there exists a neighbourhood $\tilde{U}$, contained in a trivial open of $J^\infty(E)$ induced by a trivialization on M, such that*

$$\mathcal{S}|_{\tilde{U}} = \langle Y_1, \dots, Y_l \rangle$$

where each Y_j is a smooth section of the Cartan distribution. Furthermore, let $\alpha_{i,j}$ denote the local coefficients of each Y_j respectively. Then we require the matrix $\left(\alpha_{i,j}(j^\infty_x s)\right)_{i,j}$, to be of rank l for any $j^\infty_x s \in \tilde{U}$.

If the Y_j's are everywhere of level k, we say that $\mathcal{S}$ is ***of level*** *k.*

Lemma 1.70. *If $\mathcal{S} \subset \mathcal{C}_\infty$ is a smooth subbundle of rank l and level k, then it follows that*

$$\theta^{-1}(\mathcal{S}) = \left(a'_k\right)^{-1}(\mathcal{S}_k)$$

where $\mathcal{S}_k \subset p_k^*(TM)$ is a smooth subbundle of rank l.

Proof. First of all we define $S \subset p_k^*(TM)$ by $S = a'_k\left(\theta^{-1}(\mathcal{S})\right)$. Note that the restriction of a'_k defines an isomorphism between the fibres $\varprojlim p_k^*(TM)\Big|_{j^\infty_x s} \cong p_k^*(TM)|_{j^k_x s}$. Thus it follows immediately that $\theta^{-1}(\mathcal{S}) = \left(a'_k\right)^{-1}(S)$. Thus all that remains to be shown is that S is a smooth subbundle of rank l.

Let $U \subset M$ be a trivializing open and let $\langle X_1, \dots, X_m \rangle$ be a local frame of TU. Then it induces a local frame $\langle X'_1, \dots, X'_m \rangle$ on $p_k^{-1}(U)$, where $p_k : J^k(E) \to M$ denotes the projection. These local sections X'_i are defined as

$$\begin{aligned} X'_i : p_k^{-1}(U) &\to p_k^*(TU) \\ j^k_x f &\mapsto \left(j^k_x f, X_i(x)\right) \end{aligned}$$

Let $\alpha_{i,j}$ denote the local coefficients of Y_i. Since the functions $\alpha_{i,j}$ are of level k, we know that they can be represented by functions $\left(\alpha_{i,j}\right)_k \in C^\infty(J^k(E))$. Define the sections $Y'_i : J^k(E) \to p_k^*(TM)$ by

$$Y'_i := \sum_{i=1}^m \left(\alpha_{i,j}\right)_k X'_i$$

Now note that the restriction of S to $p_k^{-1}(U)$ is given by

$$s|_{p_k^{-1}(U)} = \langle Y'_1, \dots, Y'_l \rangle$$

Furthermore, since the matrix α is everywhere of rank l, it follows that the vector fields $Y'_1, \dots, Y'_l$ are everywhere linearly independent. □

1.4 Pf-tangent vectors of $J^\infty(E)$ induced by homotopies of sections

In the previous section we have considered the Cartan distribution of $TJ^\infty(E)$, which essentially contains all 'holonomic directions'. In this section we will consider tangent vectors that are the exact opposite, namely tangent vectors in the fibre direction of $\pi \circ a_0 : J^\infty(E) \to M$. We can consider such vectors as tangent vectors generated by paths in a fibre of the jet bundle. Since the base point is constant for such a path, we can thus also consider it as being generated by a homotopy of sections. Specifically, we will use a construction as described by Boardman in [2, section 5] to construct such vectors for the bundle $J^\infty(M,N)$.

These tangent vectors will play an important role in the proof of theorem 6.12 in chapter 6, which is the main result of this thesis. They will give us exactly those vectors in $TJ^\infty E$ needed to show that a certain map is a submersion, thus making the vanishing set of that map into a submanifold.

This section is based on [2, pp.42-44]. The definitions and results are slightly generalised to accomodate for general fibre bundles $\pi : E \to M$, but are in essence the same as Boardman described.

Definition 1.71. *Let $s \in \Gamma(E)$ and let $S : I \times M \to E$ be a homotopy of sections such that $S(0) = s$. Here I is an open interval around 0. Then we define the tangent vector $X_{S,x} \in T_{j_x^\infty(s)} J^\infty(E)$ as follows. For $\varphi \in C^\infty(E)$, we set*

$$X_{S,x}(\varphi) := \left.\frac{d}{dt}\right|_{t=0} \left(\varphi\left(j_x^\infty S(t,-)\right)\right)$$

Proposition 1.72. *$X_{S,x}$ as defined above is indeed an element of $T_{j_x^\infty s} J^\infty(E)$.*

Proof. Note that from the discussion in example 1.53 it follows that it suffices to show that $X_{S,x}$ acts as a derivation on $C^\infty(J^\infty(E))$. This follows immediately from the definition, since the operator $\left.\frac{d}{dt}\right|_{t=0}$ is linear and satisfies the Leibniz identity. □

Notation 1.73. *For a homotopy $S : I \times M \to E$, we will use S_t to denote the map $S(-,t)$.*

As mentioned at the start of this section, we want to construct vertical vectors of the bundle $\pi \circ a_0 : J^\infty(E) \to M$, i.e. we want to construct vectors in $\ker(d(\pi \circ a_0))$. We claim that the vectors induced by homotopies as described above satisfy exactly this condition.

Proposition 1.74. *Let $S : I \times M \to E$ be a homotopy of sections with $S(0) = s$. Then it follows that*

$$\tilde{a}_0(X_{S,x}) \in \ker\left((d\pi)_{s(x)}\right) \subset T_{s(x)}E$$

Proof. Let $f \in C^\infty(E)$. Then as described in example 1.53 we see that

$$\begin{aligned} \tilde{a}_0(X_{S,x})(f) &= X_{S,x}(f \circ a_0) \\ &= \left.\frac{d}{dt}\right|_{t=0} \left(f \circ a_0\left(j_x^\infty S_t\right)\right) = \left.\frac{d}{dt}\right|_{t=0} f(S_t(x)) \end{aligned}$$

Thus it follows that

$$\tilde{a}_0(X_{S,x}) = \left.\frac{d}{dt}\right|_{t=0} S_t(x)$$

i.e. it is the derivative of a path in the fibre over x of $\pi : E \to M$. Thus it follows that indeed $\tilde{a}_0\left(X_{S,x}\right) \in \ker\left((d\pi)_{s(x)}\right)$. □

Furthermore, if we require the homotopy S to be constant at some level k, we even get the following stronger result.

Proposition 1.75. *Let $S : I \times M \to E$ be a homotopy of sections with $S(0) = s$. Furthermore, assume that $j^k_x S_t = j^k_x s$ for all $t \in I$. Then it follows that $\tilde{a}_k(X_{S,x}) = 0 \in T_{j^k_x s} J^k(E)$.*

Proof. Let $f \in C^\infty(J^k(E))$ and write $\tilde{f} = f \circ a_k$. Then it follows similarly as in the proof of proposition 1.74 that

$$\tilde{a}_k(X_{S,x})(f) = \left.\frac{d}{dt}\right|_{t=0} f\left(j^k_x S_t\right)$$

Since $j^k_x S_t$ is constant in t, it then thus follows that the expression above vanishes. □

This proposition also has the following corollary, which can tell us that the constructed tangent vectors are tangent to some pf-submanifold of jet space. This is the key observation in using these type of vectors in the proof of theorem 6.12 as described at the beginning of this section.

Corollary 1.76. *Let $\Sigma \subset J^\infty(E)$ be a pf-submanifold of level k. Also, let $S : M \times I \to E$ be a homotopy of sections with $S(0) = s$ and $j^k_x S_t = j^k_x s$ for all $t \in I$. Then $X_{S,x} \in T_{j^\infty_x s}\Sigma$.*

Remark 1.77. Note that the definition and results in this section are given for global sections of E. However, these results also hold for local sections, since any of the properties we discussed are local. ⋄

Example 1.78. Let $E = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the trivial bundle. Furthermore, let $x, y, y^{(1)}, \ldots, y^{(k)}$ denote the coordinates of $J^k(\mathbb{R},\mathbb{R})$. Then for a section $s = (\mathrm{id}, f) \in \Gamma(E)$, we define the homotopy $S_t = (\mathrm{id}, f_t)$, where $f_t(x) = f(x) + tx$. Let us examine how the induced vector X_{S,x_0} acts on the coordinate functions. We know from proposition 1.74 that $X_{S,x_0}(x) = 0$. Furthermore, note that for $k \geq 2$, we see that $f_t^{(k)}(x) = f^{(k)}(x)$, hence $X_{S,x_0}\left(y^{(k)}\right) = 0$. Furthermore we can calculate that $X_{S,x_0}(y) = x$ and $X_{S,x_0}\left(y^{(1)}\right) = 1$. Thus we get that $X_{S,x_0} = x\partial_y + \partial_{y'}$. By this slight abuse of notation we mean that for any $k \geq 1$ we get $\tilde{a}_k\left(X_{S,x_0}\right) = x\partial_y + \partial_{y'}$. △

2 Whitney topologies

In Chapter 1 we have defined the jet space $J^\infty(E)$ of a bundle $\pi : E \to M$, which in essence stores information about the derivatives of sections of E. We have also endowed this jet space with a pro-finite manifold structure, which in turn endows the jet space with a natural topology, namely the limit topology. We can use this structure to endow the space of sections $\Gamma(E)$ with the so-called Whitney topologies. In this section we will discuss both the strong and weak Whitney topologies, which both will be used throughout this thesis. This chapter is based on [23, Chapter 4] and [9, Section 2.3].

2.1 Whitney topologies

In this section we will introduce the definition of the weak and strong Whitney topologies and discuss how they are related to each other. The following definitions are based on definitions of similar topologies on $C^\infty(M, N)$ given by P.W. Michor in [23, section 4].

Definition 2.1. *(i) Let $\pi : E \to M$ be a fibre bundle. Let $0 \le k \le \infty$. On the space of C^k sections, $\Gamma^k(E)$, we define the* ***weak C^k Whitney topology*** *as the topology generated by the following collection*

$$\mathscr{B}_{W,k} = \bigcup_{\substack{U \subset j^k E \text{ open} \\ K \subset M \text{ compact}}} \{\mathscr{O}_{K,U}\}$$

where

$$\mathscr{O}_{K,U} = \left\{ s \in \Gamma(E) \mid j^k s(K) \subset U \right\}$$

(ii) Let $\pi : E \to M$ be a fibre bundle. Let $0 \le k \le \infty$. On the space of C^k sections, $\Gamma^k(E)$, we define the ***strong C^k Whitney topology*** *as the topology generated by the following collection*

$$\mathscr{B}_{S,k} = \bigcup_{U \subset j^k E \text{ open}} \{\mathscr{O}_U\}$$

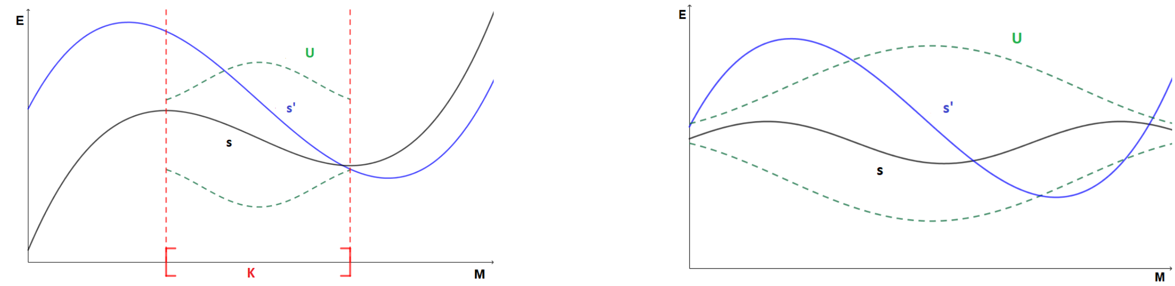
where

$$\mathscr{O}_U = \left\{ s \in \Gamma(E) \mid j^k s(M) \subset U \right\}$$

An illustration of the generating opens of these topologies can be found in figure 2.1. Note that the collections $\mathscr{B}_{W,k}$ and $\mathscr{B}_{S,k}$ form bases of the respective topologies.

Remark 2.2. Note that for any $k' \ge k \ge 0$ we have that $\mathscr{B}_{W,k} \subset \mathscr{B}_{W,k'}$ and $\mathscr{B}_{S,k} \subset \mathscr{B}_{S,k'}$. In particular, this also holds for $k' = \infty$, thus the infinite topologies are the unions of the respective finite topologies. ⋄

Figure 2.1: Illustration of the Whitney topologies



(a) Illustration of the generating open $\mathcal{O}_{K,U}$ of the weak C^0 Whitney topology. The section s (black) is included in the open $\mathcal{O}_{K,U}$, while the section s' (blue) is not.

(b) Illustration of the generating open $\mathcal{O}_U$ of the strong C^0 Whitney topology. The section s (black) is included in the open $\mathcal{O}_U$, while the section s' (blue) is not.

Remark 2.3. Note that as sets we have $C^\infty(M,N) \cong \Gamma(M\times N)$, where by $M\times N$ we mean the product bundle over M. We can therefore also endow $C^\infty(M,N)$ with either the weak or strong topology. ⋄

Notation 2.4. *Throughout this thesis by the strong/weak (Whitney) topology on* $\Gamma(E)$ *we mean respectively the strong/weak* C^∞ *Whitney topology on* $\Gamma(E)$*. Furthermore, we will denote* $\Gamma(E)$ *edowed with the weak and strong topology by* $\Gamma_W(E)$ *and* $\Gamma_S(E)$ *respectively. Similarly we will write* $C^\infty_W(M,N)$ *and* $C^\infty_S(M,N)$.

One might wonder how different these topologies actually are. The following proposition contains the main observations in comparing these topologies.

Proposition 2.5. *Let* $\pi : E \to M$ *be a fibre bundle.*

(i) The strong Whitney topology is stronger than the weak Whitney topology on $\Gamma(E)$.
(ii) If M *is compact, then the strong and weak Whitney topologies agree on* $\Gamma(E)$.

Proof. First of all, note that any compact subset of a manifold is closed. Thus for $K\subset M$ compact, $U\subset J^k(E)$ open, it follows that

$$U' := U \cup p_k^{-1}(M\setminus K) \subset J^k(E)$$

is open. Thus the generating open $\mathcal{O}_{K,U} = \mathcal{O}_{U'}$ is also a generating open of the strong Whitney topology. Thus the strong Whitney topology is indeed stronger than the weak one.

Also, if M is compact, it follows that for any $U\subset J^k(E)$ open, the generating open $\mathcal{O}_U = \mathcal{O}_{M,U}$ of the strong topology is also a generating open for the weak topology. Hence if M is compact the two topologies indeed agree. □

The following natural question is whether these topologies differ in the case that M is noncompact. It turns out that these topologies are in such a case indeed different, as illustrated by the following example.

Example 2.6. Consider the sequence of smooth maps $f_n : \mathbb{R} \to \mathbb{R}$ defined as

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq n \\ e^{-\frac{1}{x-n}} & \text{if } x > n \end{cases}$$

Then for any compact $K \subset \mathbb{R}$ and large enough n we have that $f_n|_K = 0$. Thus in the space $C^\infty_W(\mathbb{R},\mathbb{R})$, we see that the sequence $(f_n)_{n\geq 0}$ converges to the constant function $f = 0$.

However, if we consider the same sequence in the space $C^\infty_S(\mathbb{R},\mathbb{R})$, this convergence does not hold. To see this, consider the open $U \subset J^0(\mathbb{R},\mathbb{R}) \cong \mathbb{R}^2$ given by

$$U := \left\{(x, y) \in \mathbb{R}^2 \mid |y| < e^{-x}\right\}$$

and the corresponding open $\mathcal{O}_U \subset C^\infty_S(\mathbb{R},\mathbb{R})$ around the constant function $f = 0$. Note that any function in $\mathcal{O}_U$ would have to tend to 0 as $y \to \infty$. Therefore it easily follows that $f_n \notin \mathcal{O}_U$ for all $n \geq 0$.

Actually, since the strong topology is stronger than the weak topology, it follows that any limit in $\Gamma_S(E)$ is also a limit in $\Gamma_W(E)$. Since it is easy to check that the weak topology is Hausdorf (incidentally, so is the strong one), it follows that the sequence $(f_n)_{n\geq 0}$ does not have a limit in $C^\infty(\mathbb{R},\mathbb{R})$. △

Before we finish this section we make the following observation. Let $\pi : E \to M$ and $\pi' : E' \to M$ be two fibre bundles. Note that for any smooth bundle morphism $h : E \to E'$ covering the identity map, we get the induced map on section spaces

$$\begin{aligned} h_* : \Gamma(E') &\to \Gamma(E) \\ s &\mapsto h \circ s \end{aligned}$$

Note that from lemma 1.14 we already know that all the induced maps $h_* : J^k(E') \to J^k(E)$ are smooth (and thus continuous) maps. We therefore get the following corollary.

Corollary 2.7. *Let $\pi : E \to M$ and $\pi' : E' \to M$ be two fibre bundles and $h : E \to E'$ a bundle morphism covering the identity map. Then the induced map $h_* : \Gamma(E') \to \Gamma(E)$ is continuous in both Whitney topologies.*[1]

2.2 Smooth homotopies and homotopy groups

In chapter 3 we will introduce the concept of the h-principle. In that context, one is often interested in classifying maps (or other objects) up to homotopy. In this section we will consider (smooth) homotopies between maps and discuss why the weak Whitney topology is the natural one to use when one wants to consider homotopies.

Let I denote the closed interval $[0, 1] \subset \mathbb{R}$. The following definition is taken from [22, section 6.4].

Definition 2.8. *Let M and N be two smooth manifolds and let $f, g \in C^\infty(M, N)$. Then we define a **smooth homotopy** from f to g to be a homotopy $F : M \times I \to N$ from f to g that is also a smooth map. We say that f and g are **smoothly homotopic** if there is a smooth homopty between them.*

[1] The domain and target of h_* are then both equipped with the same kind of Whitney topology.

Let S be some manifold with (perhaps empty) boundary. Note that in either of the Whitney topologies we get the following inclusion

$$\left\{ \substack{\mathscr{F}:S\to C^\infty(M,N) \\ \text{a continuous map}} \right\} \hookrightarrow \left\{ \substack{F:M\times S\to N \\ \text{continuous map}} \right\}$$

where $F(x,t) = \mathscr{F}(t)(x)$.

However, note that since $\mathscr{F}(t) \in C^\infty(M,N)$, certainly not any continous map $F : M \times S \to N$ induces a map $\mathscr{F} : S \to C^\infty(M,N)$. On the other hand, a smooth map F will in fact induce such a map $\mathscr{F}$, which is continuous if we consider the weak topology. I.e. we also get the following inclusion

$$\left\{ \substack{F:M\times S\to N \\ \text{a smooth map}} \right\} \hookrightarrow \left\{ \substack{\mathscr{F}:S\to C^\infty_W(M,N) \\ \text{a homotopy}} \right\}$$

where $\mathscr{F}(t)(x) = F(x,t)$.

In the simple case where $S = I = [0,1]$, this inclusion means that a smooth homotopy $\gamma : M \times I \to N$ induces a path of fucntions $\gamma' : I \to C^\infty_W(M,N)$.

This section is mainly dedicated to showing that this last inclusion is a well-defined map.[2] However, before showing that, we will first consider the following example, which shows that this map is not well-defined in the strong topology.

Example 2.9. Recall the functions f_n from example 2.6. We can similarly define for any (non-integer) $\alpha \geq 1$ the function $f_\alpha : \mathbb{R} \to \mathbb{R}$.

$$f_\alpha(x) = \begin{cases} 0 & \text{if } x \leq \alpha \\ e^{-\frac{1}{x-\alpha}} & \text{if } x > \alpha \end{cases}$$

We can use this to define the smooth homotopy

$$\begin{aligned} H : \mathbb{R} \times [0,1] &\to \mathbb{R} \\ (x,t) &\mapsto \begin{cases} 0 & \text{if } t = 0 \\ f_{\frac{1}{t}}(x) & \text{if } t > 0 \end{cases} \end{aligned}$$

Note that since we already saw in example 2.6 that $\lim_{n\to\infty} f_n \neq H(-,0)$, it follows that H does not induce a continuous map $I \to C^\infty_S(\mathbb{R},\mathbb{R})$. △

Proposition 2.10. *Let M and N be manifolds and let S be a manifold with (perhaps empty) boundary. Let $F : M \times S \to N$ be a smooth map. Then the induced map*

$$\begin{aligned} \mathscr{F} : S &\to C^\infty_W(M,N) \\ t &\mapsto F(-,t) \end{aligned}$$

is continuous

Proof. Consider an open $\mathscr{O}_{K,U} \subset C^\infty_W(M,N)$ with $K \subset M$ compact and $U \subset J^k(M \times N)$ open. Then its inverse image under $\mathscr{F}$ is of the form

$$A := \mathscr{F}^{-1}(\mathscr{O}_{K,U}) := \left\{ t \in S \mid \forall x \in K, j^k_x F(-,t) \in U) \right\}$$

[2] It can be easily shown that this map is then also injective. However, we will not use that fact in this thesis.

From lemma 1.11 it follows that the map

$$\begin{aligned} J^k F : M \times S &\to J^k(M,N) \\ (x,t) &\mapsto j^k_x F(-,t) \end{aligned}$$

is smooth.

Let $t \in A$, then for any $x \in K$, there exists an open U_x around t in S and a neighbourhood V_x of x in K, such that $U_x \times V_x \subset \left(J^k F\right)^{-1}(U)$. Since K is compact, it then follows that there is some open $U_s \subset S$ around s, such that $U_s \times K \subset \left(j^k F'\right)^{-1}(U)$. Thus $U_s \subset A$. Hence A is an open set, which means that $\mathscr{F}$ is continuous. □

Corollary 2.11. *Let M, N and S be manifolds. Let $H : M \times S \times I \to N$ be a smooth homotopy from F_0 to F_1, where F_0 and F_1 are smooth S-families of maps in $C^\infty(M,N)$. Then the induced map $\mathscr{H} : S \times I \to C^\infty_W(M,N)$ is a homotopy from $\mathscr{F}_0$ to $\mathscr{F}_1$.*

In chapter 3 we will be interested in the homotopy groups $\pi_\kappa\left(O,f\right)$ for some subset $O \subset C^\infty_W(M,N)$ and base-point $f \in O$. We will then want to represent any element $[g] \in \pi_n\left(O,f\right)$ by some smooth g. Similarly, for g and h in the same class of $\pi_\kappa\left(O,f\right)$ we will want to represent any homtopy between them as a smooth homotopy. That we can do so follows from the well-known Whitney Approximation Theorem. We will state this theorem as a fact and refer the reader to [22, theorem 6.26] for a proof.

Theorem 2.12 (Whitney approximation theorem). *Let X be a smooth manifold with (perhaps empty) boundary and let Y be a smooth manifold. Furthermore, let $F : X \to Y$ be a continuous map. Then F is homotopic to a smooth map. Furthermore if F is already smooth on a closed subset $A \subset X$, then the homotopy can be taken to be relative to A, i.e. it leaves f fixed on A.*

Actually, in the above theorem, we can choose the smooth map to be arbitrarily close to F, even in the strong topology [18, Theorem 2.2.5, Theorem 2.3.3]. Also remark that the homotopy constructed in the proof of [22, Theorem 6.26] is a type of 'linear interpolation'. This means we also get the following corollary.

Corollary 2.13. *Let M, N and S be manifolds with (perhaps empty) boundary. Also, let $F : M \times S \to N$ be a continuous map, such that for all $t \in S$, the map $F_t : M \to N$ is smooth. Then in any neighbourhood $\mathscr{N} \subset C^0_S(M \times S, N)$ of F, there exists a smooth F', and a homotopy*

$$H : M \times S \times I \to N$$

from F to F', such that $H(-,t_1,t_2) \in \mathscr{N}$ for all $(t_1,t_2) \in S \times I$. Furthermore, if F was already smooth on some closed subset A of $M \times S$, we can take H to be relative to A.

Note that $H(-,t_1,t_2) \in \mathscr{N}$ implies in particular that it is a smooth map from M to N. In summary, we get the following proposition.

Proposition 2.14. *Let M and N be smooth manifolds and $\kappa \geq 0$. Let $O \subset C^\infty_W(M,N)$ be a subset that is open in the strong topology.[3] Then*

(i) Any element of $\pi_\kappa\left(O,f\right)$ has a representative $\mathscr{F}$, such that the associated map $F : \mathbb{S}^n \times M \to N$ is smooth.

[3] We write $O \subset C^\infty_W(M,N)$ to emphasize that this is the topology we endow O with when considering the homotopy groups.

(ii) *Let $\mathscr{F}$ and $\mathscr{G}$ be two representatives of elements in $\pi_\kappa(O, f)$ that satisfy (i). Then they are representatives of the same element, if and only if there exists a smooth homotopy*

$$H : M \times \mathbb{S}^n \times I \to N$$

from F to G, such that
- *$H(x, p, t) = f(x)$ for all $(x, t) \in M \times I$. Here p denotes the basepoint of $\mathbb{S}^n$.*
- *$H(-, t_1, t_2) \in O$ for all $(t_1, t_2) \in \mathbb{S}^n \times I$.*

Note that for any fibre bundle E, the space of sections is locally simply the space of functions. Thus by reasoning in local charts, and applying Whitney approximation relative to the boundary of local charts, the Whitney approximation theorem also has the following corollary.

Corollary 2.15. *Let $\pi : E \to M$ be a smooth fibre bundle. Let $F : M \times S \to E$ be a continuous map, such that for all $t \in S$, $F(-, t) \in \Gamma(E)$. Then in any neighbourhood $\mathcal{N} \subset C^0_S(M \times S, N)$ of F, there exists a smooth F', such that*

(i) *$F'(-, t) \in \Gamma(E)$ for all $t \in S$.*
(ii) *F and F' are homotopic to each other through a homotopy $H : M \times S \times I \to E$, such that*
   (a) *$H(-, t_1, t_2) \in \Gamma(E) \cap \mathcal{N}$ for all $(t_1, t_2) \in S \times I$.*
   (b) *$H(0, 0, t) \in \mathcal{N}$ for all $t \in I$.*

Furthermore, if F was already smooth on some closed subset $A \subset M \times S$, we can take H to be relative to A.

2.3 Baire spaces

We will finish this chapter with a discussion of Baire spaces. In particular, we will see that function spaces endowed with the strong Whitney topology are in fact Baire spaces, which will be used in the Thom-transversality theorem in chapter 4.

Definition 2.16. *Let X be a topological space. We say that a subspace $Y \subset X$ is **residual** if it is the countable intersection of open and dense sets.*

Definition 2.17. *A topological space X is called a **Baire space** if every residual subset is dense.*

Remark 2.18. Note that the intersection of any two residual subsets is once again residual. However, this is a priori not the case for dense sets. Thus, in a Baire space, working with residual sets has the advantage that one can take intersections and still keep residual (and thus dense) sets. ◇

Remark 2.19. In this section we will only focus on showing that $\Gamma_S(E)$ is a Baire space, since that is how the Thom-transversality theorem is phrased. However, the same holds for $\Gamma_W(E)$. From for example [18, Theorem 2.4.4] it follows that $C^\infty_W(M, E)$ is a complete metric space. Since $\Gamma_W(E) \subset C^\infty_W(M, E)$ is a closed subset, $\Gamma_W(E)$ is also a complete metric space. From the Baire category theorem (see for example [25, Theorem 20.6]) it then follows that $\Gamma_W(E)$ is a Baire space. ◇

The rest of this section will be dedicated to showing that, for a fibre bundle $\pi : E \to M$ with connected fibres, the space $\Gamma_S(E)$ is a Baire space. This proof is based on the proof of [9, proposition 3.3, p.44], which states that for any manifolds M and N, the space $C^\infty_S(M, N)$ is a Baire space. This is the particular instance where E is the product bundle $M \times N \to N$. While $\Gamma_S(E)$

might no longer be a complete metric space, we can still endow it with a semi-metric, or semi-distance functions, which can be used to show that $\Gamma_S(E)$ is a Baire space. A different proof can be found for example in [18, pp. 58-64].

Notation 2.20. *For any $k \geq 0$, we pick a complete Riemannian distance function d_k on $J^k(E)$. We define the semi-distance d^k on $\Gamma_S(E)$ as follows. For $s, s' \in \Gamma_S(E)$*

$$d^k(s,s') := \sup_{x\in M} d_k\left(j^k_x s, j^k_x s'\right)$$

Note that d^k indeed defines a semi-distance on $\Gamma_S(E)$.

Lemma 2.21. *Let $k \geq 0$, $\epsilon > 0$ and $s \in \Gamma_S(E)$. Then the set*

$$B^k_\epsilon(s) := \left\{s' \in \Gamma_S(E) \mid d^k\left(s,s'\right) < \epsilon\right\}$$

is open in $\Gamma_S(E)$.

Proof. Define $U \subset J^k(E)$ by

$$U := \left\{j^k_x s' \in J^k(E) \mid d_k\left(j_x ks, j_x ks'\right) < \epsilon\right\}$$

Note that $B^k_\epsilon(s) = \mathcal{O}_U$. Thus all that remains to show is that U is an open subset of $J^k(E)$.

For $j^k_x s' \in U$, set $\delta = d_k\left(j^k_x s, j^k_x s'\right)$. Since $\delta < \epsilon$, it follows that $\epsilon' = \epsilon - \delta > 0$. Furthermore note that for any $\sigma \in J^k(E)$ with $d_k\left(\sigma, j^k_x s'\right) < \epsilon'$, that $d_k\left(\sigma, j^k_x s\right) < \epsilon$. Hence it follows that the open ball with radius ϵ' around $j^k_x s'$ is contained in U. Hence U is indeed an open subset of $J^k(E)$. □

Before we show that $\Gamma_S(E)$ is indeed a Baire space, we first prove the following technical lemma.

Lemma 2.22. *Assume that $\{V_i\}_{i\in I}$ is a countable collection of open and dense subsets of $\Gamma_S(E)$. Also let $U \subset J^\infty(E)$ be open such that $\mathcal{O}_U \neq \emptyset$.*

Then for any $i \in \mathbb{N}$, there exist an $s_i \in \Gamma_S(E)$, $k_i \in \mathbb{N}$ and $U_i \subset J^{k_i}(E)$ open such that the following hold

(i) $s_i \in \mathcal{O}_U$
(ii) $s_i \in \bigcap_{j=1}^{i-1} \mathcal{O}_{U_j} \cap V_i$
(iii) $\mathcal{O}_{\overline{U_i}} \subseteq V_i$ *and* $s_i \in \mathcal{O}_{U_i}$
(iv) for all $x \in M$ and all $0 \leq k \leq i$, $i > 1$, $d_k\left(j^k_x s_i, j^k_x s_{i-1}\right) < \frac{1}{2^i}$

Proof. We will construct this data inductively on i.

First of all note that since $\mathcal{O}_U$ is open and non-empty and V_1 is dense, it follows that we can choose $s_1 \in \mathcal{O}_U \cap V_1$. Then we can find a $k_1 > 0$ and $U_1 \subset J^{k_1}(E)$ open, s.t. $s_1 \in \mathcal{O}_{U_1}$ and $\mathcal{O}_{\overline{U_1}} \subset V_1$. Then by construction, for $i = 1$ conditions (i), (ii) and (iii). Also, for $i = 1$ (iv) is an empty condition.

Next we assume that for some $i \geq 1$, there exist s_j, k_j and U_j for all $1 \leq j \leq i$. Then consider the set

$$D_i = \left\{s' \in \Gamma_S(E) \mid d^k\left(s', s_{i-1}\right) < \frac{1}{2^i} \text{ for } 0 \leq k \leq i \text{ and } x \in X\right\} = \bigcap_{0\leq k\leq i} B^k_{\frac{1}{2^i}}(s_{i-1})$$

From lemma 2.21 we know that these sets on the right side of the equation are open, thus so is D_i. Thus it follows that

$$E_i = D_i \cap \left(\bigcap_{j=1}^{i-1} \mathscr{O}_{U_j} \right) \cap \mathscr{O}_U$$

is open. Furthermore, by the induction hypothesis, E_i contains s_{i-1}, and thus is non-empty. Hence it follows that we can choose $s_i \in E_i \cap V_i$. By the definition of the Whitney topology we can then find some k_i and $U_i' \subset J^{k_i}(E)$, such that $s_i \in \mathscr{O}_{U_i'} \subset V_i$. Since $J^{k_i}(E)$ is a manifold and $j^{k_i} s_i(M)$ is a closed subset, it follows that we can find an open $U_i \subset J^{k_i}(E)$ containing $j^{k_i} s_i(M)$ such that $\overline{U_i} \subset U_i'$. Hence it follows that $s_i \in \mathscr{O}_{\overline{U_i}} \subset V_i$.

Note that by construction all conditions (i)-(iv) are satisfied, which completes the proof. □

Theorem 2.23. $\Gamma_S(E)$ *endowed with the strong Whitney topology is a Baire space.*

Proof. Let $\{V_i\}_{i\in I}$ be a countable collection of dense and open subsets of $\Gamma_S(E)$ and let $V \subset \Gamma_S(E)$ be a non-empty open set. Then there exists an open $U \subset J^{k_0}(E)$, such that $\mathscr{O}_U \neq \emptyset$ and $\mathscr{O}_{\overline{U}} \subset V$. Let $a_\bullet$ be the pf-atlas of $J^\infty(E)$. Then setting $U' = a_{k_0}^{-1}(U) \subset J^\infty(E)$ gives $\mathscr{O}_{U'} = \mathscr{O}_U$.

From lemma 2.22 we know that we can find $s_i \in \Gamma_S(E)$, $k_i \in \mathbb{N}$ and $U_i \subset J^{k_i}(E)$ that satisfy the conditions (i)-(iv) of that lemma. Note that for any $k \geq 0$ and $x \in M$, by condition (iv), the sequence $\left(j^k_x s_i\right)_{i\geq 1}$ is a Cauchy sequence in $J^k(E)$ endowed with distance d_k. Since d_k is complete it follows that this sequence converges. Thus we can now define the set-theoretical map $t_k : M \to J^k(E)$ by

$$t_k(x) = \lim_{i\to\infty} j^k_x s_i$$

Note that by (iv) we also have uniform convergence on M of the sequences $\left(j^k s_i\right)_{i>0}$, thus the maps t_k are continuous. We will now show that the map $t_0 : M \to J^0(E) \cong E$ is a smooth map. Note that $t_0(x) = \lim_{i\to\infty} s_i(x)$. We will also show that this t_0 lies in the intersection of V and $\cap_{i\in I} V_i$.

We will now first show that t_0 is a smooth section of E. For $x \in M$, let $K \subset M$ and $K' \subset E$ be compact neighbourhoods around x and $t_0(x)$ respectively, such that $t_0(K) \subset K'$ and K is contained in a trivializing chart of M. Since for any $k \geq 0$, the maps $j^k s_i$ converge uniformly on K, it follows that for any multi-index β, the maps

$$\frac{\partial^\beta s_i}{\partial x^\beta} \in C^\infty(M)$$

converge uniformly as $i \to \infty$. By inductively applying a classical result of Dieudonné [5, (8.6.3), p.157], it follows that for any multi-index β

$$\frac{\partial^\beta t_0}{\partial x^\beta} = \lim_{i\to\infty} \frac{\partial^\beta s_i}{\partial x^\beta}$$

Thus all partial derivatives of t_0 exist and are continuous. Hence t_0 is indeed a smooth map. Furthermore, we see that $j^k t_0 = t_k$ for all $k \geq 0$.

Furthermore, note that since $\pi : E \to M$ is a continuous map, it follows that

$$\pi \circ t_0 = \pi \circ \lim_{i\to\infty} s_i = \lim_{i\to\infty} \pi \circ s_i = \mathrm{id}_M$$

hence $t_0 \in \Gamma_S(E)$.

Furthermore, note that by property (i) of lemma 2.22, it follows that $s_i \in \mathscr{O}_U$ for all i, thus $j_x^{k_0} s_i \in U$ for all $x \in M$. Since $j_x^{k_0} t_0 = t_{k_0}(x) = \lim_{i \to \infty} j_x^{k_0} s_i$, it then follows that $j_x^{k_0} t_0 \in \overline{U}$ for all $x \in M$. Hence it follows that $t_0 \in \mathscr{O}_{\overline{U}} \subset V$. Furthermore, since $s_j \in \mathscr{O}_{U_i}$ for $j \geq i$, it follows through similar reasoning that $t_0 \in \mathscr{O}_{\overline{U_i}} \subset V_i$ for all i. Hence we have indeed that $t_0 \in V \cap (\bigcap_{i \in I} V_i)$.

Thus it follows that $\bigcap_{i \in I} V_i$ is a dense set, hence $\Gamma_S(E)$ is indeed a Baire Space. □

3 The h-principle and removal of singularities

In this chapter we will discuss the notion of differential relations and the h-principle, the latter of which is extensively discussed in the introduction of this thesis. We will also consider the specific h-principle for immersions. In particular, we will discuss how we could apply a removal of singularities argument to prove this particular h-principle, as described by Gromov in [13, section 2.1.1].

3.1 Differential relations

As discussed in the introduction of this thesis, many types of maps can be characterized by a certain condition posed on its differentials. This information is of course contained in the jet spaces that we defined in chapter 1. These jet spaces are therefore used to formally define a *differential relation*. In this section we will define such a differential relations and discuss some related notions. The main reference for this section is [7, Chapter 5].

Definition 3.1. *Let M and N be manifolds. A* ***differential relation of order k*** *imposed on the maps $C^\infty(M,N)$, is a subset $\mathscr{R} \subset J^k(M,N)$. The set $\Sigma_{\mathscr{R}} := J^k(M,N) \setminus \mathscr{R}$ is called the* ***space of singularities*** *of the differential relation $\mathscr{R}$.*

These singularities will, as the name suggests, play an important role in the technique called *removal of singularities*. We will therefore also introduce the following terminology.

Definition 3.2. *Let $\Sigma_{\mathscr{R}} \subset J^r(M,N)$ be the space of singularities of some differential relation $\mathscr{R}$. Then for a section $s \in \Gamma(J^r(M,N))$, we define the* ***singularity locus of s*** *by*

$$\Sigma_{\mathscr{R}}(s) := s^{-1}(\Sigma_{\mathscr{R}}) \subset M$$

For a map $f \in C^\infty(M,N)$, by the ***singularity locus of f***, *we mean the singularity locus of the section $j^r f \in \Gamma(J^r(M,N))$, which we will denote by $\Sigma_{\mathscr{R}}(f)$.*

Remark 3.3. We will use the notation used in the definition above more generally. Let $\pi : E \to M$ be a fibre bundle. For any manifold $\Sigma \subset J^k(E)$ and $s \in \Gamma(E)$, we will write $\Sigma(s) := j^k s^{-1}(\Sigma)$.

In fact, for any smooth map $F \in C^\infty(M',E)$, and submanifold $\Sigma \subset E$, we will write $W(F) := F^{-1}(W)$. ◇

Example 3.4.

1. The main example of such a differential relation (of order 1) in this thesis will be the

differential relation of immersions. Let M and N be manifolds with $\dim(M) \leq \dim(N)$. Then the differential relation of immersions from M to N is given by

$$\mathscr{R}_{\mathrm{Imm}(M,N)} := \{j^1_x f \in J^1(M,N) \mid \mathrm{rank}\big((df)_x\big) = \dim(M)\}$$

The singularity of this relation, $\Sigma_{\mathrm{Imm}(M,N)}$, is then given by those jets $j^1_x f$, where $(df)_x$ is not of maximal rank. For any $f \in C^\infty(M,N)$ its singularity locus $\Sigma_{\mathrm{Imm}(M,N)}(f)$ is then given by those points $x \in M$ for which $(df)_x$ is not of maximal rank.

2. Similarly, we can define submersions as a differential relation. Let M and N be manifolds with $\dim(M) \geq \dim(N)$. Then the differential relation of submersions from M to N is given by

$$\mathscr{R}_{\mathrm{subm}(M,N)} := \{j^1_x f \in J^1(M,N) \mid \mathrm{rank}\big((df)_x\big) = \dim(N)\}$$

The singularity of this relation is determined similarly as for immersions.

△

Example 3.5. Another large class of examples of differential relations can be found in (partial) differential equations. Note that a k^{th} order (partial) differential equation on functions $f \in C^\infty(\mathbb{R}^m,\mathbb{R}^n)$ is given by some equation $\Psi(x,f) = 0$, where the map $\Psi : \mathbb{R}^m \times C^\infty(\mathbb{R}^m,\mathbb{R}^n) \to \mathbb{R}^q$ depends on

(i) the element $x \in \mathbb{R}^m$
(ii) the values $f(x) \in \mathbb{R}^n$
(iii) the values of the partial derivatives $\frac{\partial^{|\alpha|} f_j}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}$ with $|\alpha| \leq k$.

Since all of this information is contained in the k^{th} jet space, we can define a differential relation $\mathscr{R}_\Psi \subset J^k(\mathbb{R}^m,\mathbb{R}^n)$ as follows

$$\mathscr{R}_\Psi := \left\{ j^k_x f \in J^k\left(\mathbb{R}^m,\mathbb{R}^n\right) \mid \Psi(x,f) = 0 \right\}$$

△

Definition 3.6. *Let M and N be manifolds and $\mathscr{R}$ a differential relation of order k.*

*(i) A **solution** of $\mathscr{R}$ is a map $f \in C^\infty(M,N)$ such that the jet*

$$j^k f : M \to J^k(M,N)$$

takes its values in $\mathscr{R}$, i.e. $j^k f(M) \subset \mathscr{R}$. We will denote the space of solutions by $Sol(\mathscr{R}) \subset C^\infty_W(M,N)$.

*(ii) A **formal solution** of $\mathscr{R}$ is a section*

$$s : M \to J^k(M,N)$$

which takes its values in $\mathscr{R}$. We will denote the space of formal solutions by $Sol_F(\mathscr{R}) \subset \Gamma_W\left(J^k(M,N)\right)$

The solutions of a differential relation are thus exactly its holonomic formal solutions.

Remark 3.7. Note that we have endowed the spaces of (formal) solutions with the weak Whitney topology. The reason we have chosen the weak instead of the strong Whitney topology is that this is the natural setting in which to consider homotopies, as was discussed in section 2.2. We will consider this in more detail in section 3.2. ◇

Example 3.8. If we reconsider examples 3.4 and 3.5, we get the following (formal) solutions.

- For the relations $\mathscr{R}_{\text{Imm}(M,N)}$ and $\mathscr{R}_{\text{subm}(M,N)}$, the solutions are exactly immersions and submersions respectively. Formal solutions on the other hand are given by bundle morphims $TM \to TN$ that are injective and surjective respectively.
- For some differential equation $\Psi = 0$, the solutions to the associated differential relation $\mathscr{R}_\Psi$ are exactly the solutions of the differential equation.

△

Remark 3.9. Note that we could have defined a differential relation of order k as a subspace of $J^\infty(M,N)$ of order k, i.e. $\tilde{\mathscr{R}} = a_k^{-1}(\mathscr{R})$, where $\mathscr{R} \subset J^k(M,N)$. Note that we can thus also see the singularity of such a relation as a subset of $J^\infty(M,N)$. In the rest of the thesis we will use both definitions interchangeably. ⋄

3.2 The h-principle

From the definition of (formal) solutions of some differential relation $\mathscr{R}$, it immediately follows that the existence of a formal solution is a necessary condition for the existence of an actual solution. Perhaps surprisingly, it turns out that there are actually many interesting differential relations where this is also a sufficient condition. Moreover, in many cases the spaces of solutions and formal solutions are actually very similar. Such a property is described by the h-principle, which was introduced by Gromov (and Èliašberg) in the 1970's [7, p. 60]. The main reference for this section is [7, Chapter 6].

Definition 3.10. *Let $\mathscr{R}$ be a differential relation of order k. Then we say that $\mathscr{R}$ satisfies **the h-principle** if the inclusion*

$$\begin{aligned} \iota : Sol(\mathscr{R}) &\hookrightarrow Sol_F(\mathscr{R}) \\ f &\mapsto j^k f \end{aligned}$$

is a weak homotopy equivalence.

Remark 3.11. There are actually many notions of an h-principle. In certain cases, for example when considering an existence question, one might only be interested in whether the above inclusion induces an isomorphism of the π_0-groups. Actually in his book [13], Gromov defined many types of an h-principle. However, in this thesis, we will consider the above-mentioned notion. ⋄

Let us think a little about what it means for a differential relation to satisfy the h-principle. First of all, the surjectivity of the induced map on the π_0-groups, means that any formal solution is homotopic to an actual solution. Even stronger, the homotopy connecting the two is a formal solution at any $t \in I$. In fact, from proposition 2.14 it follows that for any open $\mathscr{R}$, we can assume this homotopy to be smooth. On the other hand, corollary 2.11 shows that the existence of such smooth homotopies suffices to prove that the induced map on the 0-th homotopy group is a surjection. The injectivity of the induced map on the π_0-groups means exactly that any two solutions of the relation that are homotopic through formal solutions must also be homotopic through solutions.

The intuition for higher homotopy groups is very similar, except that in such cases we talk about $\mathbb{S}^n$ families of (formal) solutions. In particular, if $\mathscr{R}$ is open, then $\mathscr{R}$ satisfies the h-principle if any smooth $\mathbb{S}^n$ -family of formal solutions is smoothly homotopic to a smooth

$\mathbb{S}^n$-family of actual solutions, where the homotopy moves through smooth $\mathbb{S}^n$-families of formal solutions. Furthermore, if two $\mathbb{S}^n$-families of solutions are homotopic to each other in $\mathrm{Sol}_f(\mathscr{R})$, then they are homotopic to each other in $\mathrm{Sol}(\mathscr{R})$.

Example 3.12 (Immersions)**.** Recall from example 3.4 the definition of the immersion relation $\mathscr{R}_{\mathrm{Imm}(M,N)}$. This relation is the classical example of a relation that satisfies the h-principle. In fact, this h-principle was investigated by Smale and Hirsch already a decade before Gromov introduced the notion of the h-principle. In his papers [26] and [27] Smale proved this h-principle for $M = \mathbb{S}^m$ and $N = \mathbb{R}^n$, with $n \geq m+1$. In [26], Smale used this result to show that any two immersions of $\mathbb{S}^2$ into $\mathbb{R}^3$ are 'regularly homotopic' to each other, i.e. homotopic through immersions. He then noted that this implies that reflection of $\mathbb{S}^2$ in $\mathbb{R}^3$ can be achieved through a homotopy of immersions [26, p. 281]. This result has become known as *Smale's sphere eversion* [7, p.34].

Hirsch generalized the results from Smale's papers in [16] and [17] to the following theorem.

Theorem 3.13 (Hirsch, 1959, 1961). *Let M and N be manifolds. Then the immersion relation $\mathscr{R}_{\mathrm{Imm}(M,N)}$ satisfies the h-principle if either of the following hold:*

(i) $\dim(M) < \dim(N)$
(ii) $\dim(M) = \dim(N)$ *and M is an open manifold*

In 1985, Cohen used this theorem to solve the *immersion conjecture*, which gives necessary and sufficient conditions for the existence of immersions with compact domains [3, p.238]. △

In 1969, Gromov proved a result about differential relations that display a certain regularity w.r.t. M, so-called $\mathrm{Diff}(M)$-invariant relations. The following definition is based on [7, p.59].

Definition 3.14. *Let $\mathscr{R} \subset J^k(M,N)$ be a differential relation. Then $\mathscr{R}$ is called $\mathrm{Diff}(M)$-invariant if for all diffeomorphisms $h : M \to M$, the induced map on $J^k(M,N)$*

$$h_* : J^k(M,N) \to J^k(M,N)$$
$$j^k_x f \mapsto j^k_{h(x)} f$$

is $\mathscr{R}$-invariant.

The following theorem by Gromov [11] tells us that there are a lot of such relations for open manifolds M that satisfy the h-principle, thus providing us with a rich class of examples of differential relatios that satisfy the h-principle.

Theorem 3.15 (Gromov, 1969). *Let M and N be manifolds with M open. Furthermore, let $\mathscr{R} \subset J^k(M,N)$ be an open $\mathrm{Diff}(M)$-invariant differential relation. Then $\mathscr{R}$ satisfies the h-principle.*

Note that the above theorem covers immersions, submersions and k-mersions (maps of rank $\leq k$), given that the domain is an open manifold. However, this theorem can not be applied if the manifold M is compact.

3.3 Removal of singularities for immersions

One of the methods introduced to prove h-principles in Gromov's 1986 book [13] is *removal of singularities*. Removal of singularities a priori considers differential relations of the form

$\mathscr{R} \subset J^k(M,\mathbb{R}^n)$. However, it can still be applied to general target manifolds N if one manages to also make some globalisation argument [13, p. 53, Remark B'].

In this section we will focus on the philosophy of the removal of singularities argument, specifically in the case of the immersion relation. Thus the relation we consider here is $\mathscr{R}_{\text{Imm},n} := \mathscr{R}_{\text{Imm}(M,\mathbb{R}^n)} \subset J^1(M,\mathbb{R}^n)$. We will start by introducing the necessary notation. Then we will consider the idea of the removal of singularities strategy used to prove the 'π_0-surjectivity part' of the h-principle, i.e. to prove that the map

$$\iota_* : \pi_0\left(\text{Sol}\left(\mathscr{R}_{\text{Imm},n}\right)\right) \to \pi_0\left(\text{Sol}_f\left(\mathscr{R}_{\text{Imm},n}\right)\right)$$

is surjective. As discussed in the previous section, this means that any formal solution is homotopic to an actual solution through a path of formal solutions. The main reference for this section is [13, section 2.1.1].

When studying the fibre bundle $J^k(M,\mathbb{R}^n)$, the following definition is a useful one to consider. This definition is taken from [19, p. 16].

Definition 3.16. *Let $\pi : E \to M$ and $\pi' : E' \to M$ be two fibre bundles. Then we define the* ***Whitney sum*** *or* ***fibre product*** *$E \oplus E'$ by*

$$E \oplus E' = \left\{(s,s') \in E \times E' \mid \pi(s) = \pi'(s)\right\}$$

Remark 3.17. Note that the fibre product then naturally forms a fibre bundle over M. Also, note that for vector bundles, the fibre product is isomorphic to the direct sum, which is also suggested by the notation.

Furthermore, any sections $s \in \Gamma(E)$ and $s' \in \Gamma(E)$ form a new section in $\Gamma(E \oplus E')$, as follows

$$\begin{aligned} s \oplus s' : M &\to E \oplus E' \\ x &\mapsto (s(x), s'(x)) \end{aligned}$$

◇

Note that the fibre bundle $J^k(M,\mathbb{R}^n)$ naturally admits the structure of a fibre product as follows. For any $f \in C^\infty(M,\mathbb{R}^n)$, we write $f = (f_1,\ldots,f_n)$, where the $f_i \in C^\infty(M,\mathbb{R})$ denote the component functions of f. Note that we then get the following isomorphism of bundles.

$$\begin{aligned} J^k\left(M,\mathbb{R}^n\right) &\cong \bigoplus_{i=1}^{n} J^k(M,\mathbb{R}) \\ j^k_x f &\mapsto \left(j^k_x f_1,\ldots,j^k_x f_n\right) \end{aligned}$$

Hence, we can study the k-jet at x of a function f by studying the jets of its components. Using this correspondence, we get for $1 \le i \le m$ and $n \ge 1$, the following (smooth) projection

$$\begin{aligned} P_i : J^k\left(M,\mathbb{R}^n\right) &\mapsto J^k\left(M,\mathbb{R}^{n-1}\right) \\ j^k_x f &\mapsto \left(j^k_x f_1,\ldots,j^k_x f_{i-1}, j^k_x f_{i+1},\ldots,j^k_x f_n\right) \end{aligned}$$

which deletes the i-th component.

Notation 3.18. *Let s be a section of the bundle $J^k(M,\mathbb{R}^n)$. Then we denote the section $P_i \circ s \in \Gamma\left(J^k\left(M,\mathbb{R}^{n-1}\right)\right)$ by $\hat{s}_i$.*

Notation 3.19. *Let* $s' = (s_1, \dots, s_{i-1}, s_{i+1}, \dots s_n) \in \Gamma\left(J^1\left(M, \mathbb{R}^{n-1}\right)\right)$ *and* $s_i \in \Gamma\left(J^1\left(M, \mathbb{R}^n\right)\right)$. *Then we use the expresion* $s' \oplus_i s_i$ *for the section* $s = (s_1, \dots, s_n)$.

Remark 3.20 (Strategy removal of singularities). Recall that we want to prove that any formal solution of $\mathscr{R}_{\mathrm{Imm},n}$ is homotopic to an actual solution through a path of formal solutions. Concretely, a formal solution is a section

$$\begin{aligned} s : M &\to J^1\left(M, \mathbb{R}^n\right) \\ x &\mapsto (x, y_s(x), z_s(x)) \end{aligned}$$

such that $z_s(x) : T_x M \to \mathbb{R}^n$ is injective for all x. An actual solution is such a section that is also holonomic.[1]

Let us consider the following question. Let $s \in \mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$. For $1 \le i \le n$, can we find a function $f_i \in C^\infty(M, \mathbb{R})$ such that the section $\hat{s}_i \oplus j^1 f_i$ is still a formal immersion?

Note that if we can indeed always find such a function, we can make the given formal solution into an actual one, by making the components holonomic 1 by 1. However, there are certainly formal solutions, for which we can not find such a function f_i, as can be seen in example 3.22.

We must thus tweak the idea a little. While this is not possible for all formal solutions s, it would actually be sufficient if it is possible for 'most' formal solutions s, i.e. for a dense subset of $\mathrm{Sol}_f(\mathscr{R})$. Then by a slight perturbation of s, we can get a formal solution to which we can apply the abovementioned procedure.

This is indeed the argument given by Gromov. However, to apply this argument we need to take the following issues into account.

1. Is there a indeed a dense subset of $\mathrm{Sol}_f(\mathscr{R}_{\mathrm{Imm},n})$ for which we can replace the i-th component by a holonomic component?
2. By applying this procedure repeatedly, how do we make sure we do not lose the holonomicity we had previously achieved during the perturbation?
3. If we can indeed replace the i-th component of s by a holonomic component $j^1 f_i$, how do we make sure that s and $\hat{s}_i \oplus_i j^1 f_i$ are homotopic to each other through a path of formal solutions?

The answer to question 1 is what the main part of the following 3 chapters is about. The main tool to obtain a dense subset of $\Gamma(E)$ with certain properties is through the Thom-transversality theorem, which will be covered in chapter 4. However, to be able to apply the Thom-transversality theorem, we need to be able to describe the desired properties of a given section s in terms of its higher jets $j^k_x s$. In example 3.22 we will already give an idea of what we will want these properties to be. This will be formalized in proposition 3.23. In chapter 6 we will then construct subsets of $J^k\left(J^1\left(M, \mathbb{R}^n\right)\right)$ that exactly contain those jets with properties that we want to avoid.

The second question will be tackled in section 4.3. There we will consider a corollary of the Thom-transversality theorem, which tells us that we can apply tranversality in such a way that we can indeed preserve (partial) holonomicity. The third question will be tackled in the proof of proposition 3.23. When constructing the function f, we will make sure that this condition is satisfied. ◇

[1] The notation we use here was introduced in notation 1.21.

Note that for a given $s \in \mathrm{Sol}_f(\mathscr{R}_f)$, the question whether for some $f \in C^\infty(M,\mathbb{R})$ the new section $\hat{s}_i \oplus_i j^1 f$ is once again a formal solution can be determined by the behaviour of f on the singularity locus $\Sigma_{\mathrm{Imm},n-1}(\hat{s}_i)$. More specifically, we get the following proposition.

Proposition 3.21. *Let $s \in \mathrm{Sol}_f(\mathscr{R}_{\mathrm{Imm},n})$ and let $f_i \in C^\infty(M,\mathbb{R})$. Then the section $\hat{s}_i \oplus_i j^1 f_i \in \Gamma(J^1(M,\mathbb{R}^n))$ is a formal solution of $\mathscr{R}_{\mathrm{Imm},n}$ if and only if for all $x \in \Sigma_{\mathrm{Imm},n-1}(\hat{s}_i)$, there exists a $V \in \ker(z_{\hat{s}_i}(x))$, such that*

$$(df)_x(V) \neq 0$$

Proof. Note that a section of $J^1(M,\mathbb{R}^n)$ is a formal immersion if its z-component is injective for all $x \in M$. Outside of the singularity $\Sigma_{\mathrm{Imm},n-1}(\hat{s}_i)$, the z-component of $\hat{s}_i$ is injective, which guarantees that the same will hold for $\hat{s}_i \oplus_i j^1 f_i$. Thus we indeed only need to consider any points contained in $\Sigma_{\mathrm{Imm},n-1}(\hat{s}_i)$.

Note that since z_s injective everywhere means that if we remove one component the kernel we get is mostly 1-dimensional. In other words, for all $x \in \Sigma_{\mathrm{Imm},n-1}(\hat{s}_i)$, $\ker(z_{\hat{s}_i}(x)) \subset T_xM$ is a 1-dimensional subspace. The condition that $(df_i)_x$ does not vanish on the kernel of $z_{\hat{s}_i}(x)$ then indeed guarantees that the z-component of $\hat{s}_i \oplus_i j^1 f_i$ is injective. □

Example 3.22. In this example we will consider two cases of a section $s \in \Gamma(J^1(M,\mathbb{R}^n))$ and describe why it is or is not possible to replace a component of s by a holonomic one as described in remark 3.20. In particular, we will see that it is not always possible to do so. The setting we will consider is $M = \mathbb{R}^3 \setminus (\{(0,0)\} \times \mathbb{R})$ and $n = i$.

Note that both the singularity $\Sigma_{\mathrm{Imm},n-1} \subset M$ and the kernel $\ker(z_{\hat{s}_i})$ is fully determined by z_s. We will thus only consider that part of the section s.

1. Let us first consider a section s, for which z_s is given by

$$z_s(x) = \begin{pmatrix} dx_1 + (x_1^2 + x_2^2 - 1)\, dx_3 \\ dx_2 + x_3\, dx_3 \\ dx_1 + x_3\, dx_2 \\ dx_1 - x_3\, dx_2 \\ (x_3+1)\, dx_3 \end{pmatrix} : T_xM \to \mathbb{R}^5$$

   First of all note that $z_s(x)$ is injective for all x, hence s is a formal solution. However, if we consider $z_{\hat{s}_5}$, then we see that it is not injective on $\mathbb{S}^1 \times \{0\} \subset M$. For $x \in \mathbb{S}^1 \times \{0\}$ we see that $\ker(z_{\hat{s}_5}(x)) = \langle \partial_{x_3} \rangle$. The singularity and this kernel are depicted in figure 3.1a. If we thus take $f_5 \in C^\infty(M)$ to be the function $f_5(x) = x_3$, it follows that $\hat{s}_5 \oplus_5 j^1 f_5$ is indeed still a formal solution of $\mathscr{R}_{\mathrm{Imm},n}$.
2. Next, let us consider a section s, with z_s given by

$$z_s(x) = \begin{pmatrix} x_1\, dx_1 + x_2\, dx_2 \\ (x_1^2 + x_2^2 - 1)\, dx_1 + dx_3 \\ x_3\, dx_2 + dx_3 \\ x_2\, dx_1 - x_1\, dx_2 \end{pmatrix} : T_xM \to \mathbb{R}^4$$

   Then s is once again a formal solution of $\mathscr{R}_{\mathrm{Imm},4}$. However, if we consider $z_{\hat{s}_4}$, this is no longer the case. Note that at $\mathbb{S}^1 \times \{0\} \subset M$, $z_{\hat{s}_4}(x)$ is not injective. Hence the submanifold $\mathbb{S}^1 \times \{0\}$ is contained in $\Sigma_{\mathrm{Imm},n}(\hat{S}_1)$. Furthermore, for $x \in \mathbb{S}^1 \times \{0\}$, we get $\ker(z_{\hat{s}_4}(x)) =$

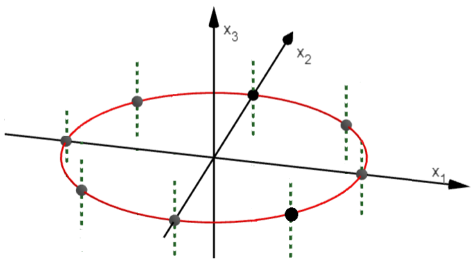
$\langle -x_2\partial_{x_1} + x_1\partial_{x_2}\rangle$, which is exactly the tangent space of $\mathbb{S}^1 \times \{0\}$. This is illustrated in figure 3.1b.
We claim that in this case there exists no $f_4 \in C^\infty(M)$, such that $z_{\hat{s}_4} \oplus_4 j^1 f_4$ is again a formal solution. To see this, we consider the path

$$\begin{aligned}\gamma : [0,1] &\to \mathbb{R}^3\\ t &\mapsto \left(e^{2\pi t}, 0\right)\end{aligned}$$

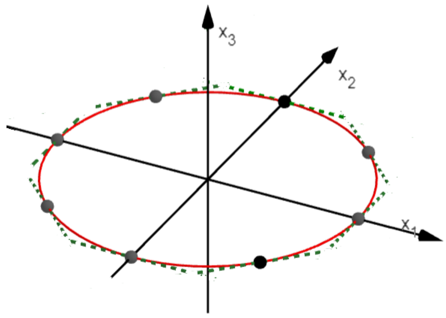
where the (x_1, x_2)-plane is considered in complex coordinates. Note that the image of γ is fully contained in the singularity $\Sigma_{\mathrm{Imm},n}$. Furthermore, its derivative $\gamma'(t)$ exactly spans the tangent space of $\mathbb{S}^1 \times \{0\}$ and thus spans the kernel of $z_{\hat{s}_4}(x)$ for any $\gamma(t) = x$. Thus, if we were to find an $f \in C^\infty(M)$ that would satisfy the given condition, then by proposition 3.21 it would follow that $\left(f \circ \gamma\right)'(t) \neq 0$ for all $t \in [0,1]$. Thus $f \circ \gamma$ would have to be strictly increasing or decreasing on the interval. However, this is impossible since $\gamma(0) = \gamma(1)$.

△

Figure 3.1: Illustrations of $\Sigma_{\mathrm{Imm},n-1}$ and $\ker z_{\hat{s}_i}$ as described in example 3.22.



(a) The singularity $\Sigma_{\mathrm{Imm},n-1}\left(z_{\hat{s}_5}\right)$ is depicted in red. The kernel $\ker\left(z_{\hat{s}_5}\right) = \langle \partial x_3\rangle$ at points in the singularity is depicted in green.



(b) The subset of the singularity $\Sigma_{\mathrm{Imm},n-1}\left(z_{\hat{s}_4}\right)$, $\mathbb{S}^1 \times \{0\}$, is depicted in red. The kernel $\ker\left(z_{\hat{s}_4}\right) = \langle x_1\partial x_1 + x_2\partial x_2\rangle$ at points in the singularity is depicted in green.

Note that in the example above, for the second section the problem arose because the kernel of $z_{\hat{s}_i}$ was tangent to (a manifold contained in) the singularity. On the other hand, in the first example we saw that the singularity was a manifold to which the kernel of $z_{\hat{s}_i}$ was no-where tangent. Unfortunately, we will not be able to reduce the problem to a situation as nice as depicted in figure 3.1a. However, we will be able to reduce to a situation that avoids the problem of 3.1b. Specifically, we will reduce the componentwise step in remark 3.20 to the following proposition. This proposition makes use of the notion of a stratification, which is covered in appendix A. The proposition and its proof are inspired by [13, p.49, Lemma (B)].

Proposition 3.23. *Let $s \in \Gamma\left(J^1(M,\mathbb{R}^n)\right)$ be a formal solution of $\mathscr{R}_{\mathrm{Imm},n}$. Assume that for some $1 \le i \le n$, the singularity $\Sigma_{\mathrm{Imm},n-1}\left(z_{\hat{s}_i}\right)$ is a manifold, that furthermore admits a stratification, such that $\ker\left(z_{\hat{s}_i}\right)$ is no-where tangent to that stratification. Then there exists an $f_i \in C^\infty(M)$, such that*

1. *$\hat{s}_i \oplus_i j^1 f_i$ is once again a formal solution of $\mathscr{R}_{\mathrm{Imm},n}$.*
2. *s is homotopic to $\hat{s}_i \oplus_i j^1 f_i$ through formal solutions.*

An example to keep in mind when considering this proposition is the one depicted in figure 3.2. The stratification of the singularity is then as follows: the smallest stratum is formed by the two points where the green lines are tangent to the singularity. The rest of the singularity forms the other stratum. Note that this stratification indeed satisfies the conditions of proposition 3.23.

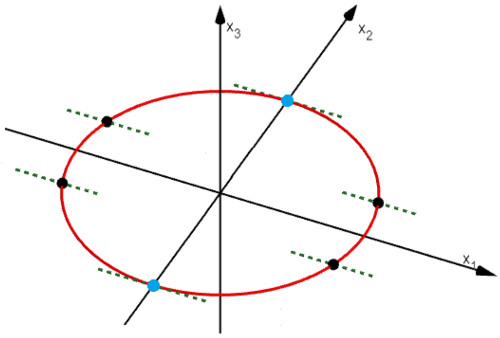


Figure 3.2: A singularity of some reduced formal solution, together with the kernel $\ker(z_{\hat{s}_i}) = \langle \partial x_1 \rangle$, which is depicted in green. The singularity has been stratified into two strata namely S^0 (red) and S^1 (blue).

Remark 3.24. Note that the fact that $\ker(z_{\hat{s}_i})$ is assumed to be no-where tangent to the stratification of $\Sigma_{\mathrm{Imm},n-1}$ ensures that the stratification is of positive codimension. ⋄

Remark 3.25. Note that over the manifold $\Sigma := \Sigma_{\mathrm{Imm},n}(\hat{s}_i)$, we know that $\ker(z_{\hat{s}_i})$ forms a trivial line bundle. Indeed, we know that the kernel must be of rank at least 1 on the singularity, and from the fact that $s \in \mathrm{Sol}_f(\mathscr{R}_{\mathrm{Imm},n})$ it follows that the rank is also not larger than 1. Furthermore, the trivialization is given by the i-th component of z_s

$$\begin{aligned} \ker(z_{\hat{s}_i})\big|_\Sigma &\cong \Sigma \times \mathbb{R} \\ V_x &\mapsto \big(x, (z_s)_i(x)(V_x)\big) \end{aligned}$$

where x is the basepoint of the tangent vector V_x.

From here on out we will use the vector X_x to denote the vector that is mapped to $(x,1)$ under this isomorphism. Then X is a vector field over Σ spanning $\ker(z_{\hat{s}_i})|_\Sigma$. Note that if we can find a function $f_i \in C^\infty(M)$ for which $X_x(f_i) > 0$ for all $x \in \Sigma$, then the conditions 1 and 2 of the proposition are satisfied. Indeed, in this case, the homotopy given by linear interpolation between s and $j^1\hat{s}_i \oplus_i j^1 f_i$ satisfies condition (ii).[2] This is due to the fact that $(z_s)_i(x)$ and $(df_i)_x$ both send X_x to a positive number, hence so does their linear interpolation. All the other z_s components remain the same. Thus combining this with proposition 3.21 exactly gives us the required result. ⋄

Thus to prove proposition 3.23, we first need to prove the following lemmas.

Lemma 3.26. *Let $S \subset M$ be a submanifold of codimension q and X a non-vanishing vector field defined on some open neighbourhood U' of S, such that X is no-where tangent to S. Then for a perhaps smaller neighbourhood of S, $U' \subset U$, there exists an $f \in C^\infty(U')$, such that the following hold:*

(i) $X(f) > 0$ *on all of* U'
(ii) $f(x) = 0$ *for all* $x \in U' \cap S$

[2]The linear interpolation we mean is that of the y and z coordinates of the sections.

Proof. For $x \in S$, let $U_x \subset U$ denote a trivializing neighbourhood around x, with coordinates $(x_1, \dots, x_m)$, such that S is given by the first $m-q$ coordinate functions. On U_x, we can write

$$X = \sum_{j=1}^{m} X_j \partial_{x_j}$$

Note that since X is not tangent to S at x, it follows that there exists some $m-q < j_0 \leq m$, such that $X_{j_0}(x) \neq 0$. Then there exists another neighbourhood $V_x \subset U_x$ of x (with the same coordinate functions), such that X_{j_0} does not vanish on V_x. Since we can assume that V_x is path-connected, it follows that X_{j_0} has the same sign (positive or negative) on all of V_x. We will denote this by $\operatorname{sign}\left(X_{j_0}\right)$. Let us now define the function $f_x := \operatorname{sign}\left(X_{j_0}\right) x^{j_0}$. Then it follows that $X\left(f_x\right) > 0$ on V_x.

$\{V_x\}_{x \in S}$ is an open cover of some neighbourhood V of S. Let $\left\{\eta_x\right\}_{x \in S}$ be a smooth partition of unity subordinated to it. Then we can define the function f on V by

$$f = \sum_{x \in S} \eta_x f_x$$

Note that for $x' \in S \cap V_x$, we get

$$X\left(\eta_x f_x\right)\left(x'\right) = \eta_x\left(x'\right) X_{x'}\left(f_x\right) + X_{x'}\left(\eta_x\right) f_x(x') = \eta_x\left(x'\right) X_{x'}\left(f_x\right)$$

since $f_x\left(x'\right) = 0$. Thus it follows that $X(f)(x) > 0$ for all $x \in S$. Therefore, $X(f) > 0$ on some open neighbourhood $U' \subset V$ of S. Furthermore, all functions f_x are 0 on $S \cap V_x$, hence so is f. □

Lemma 3.27. *Let $\Sigma := \Sigma_{\mathrm{Imm},n-1}\left(z_{\hat{s}_i}\right) \subset M$ be as in proposition 3.23. Let X be the vector field described in remark 3.25. Then for any $x_0 \in \Sigma$, there exists a coordinate neighbourhood $(U, (x_1, \dots, x_m))$ of M around x_0 and a $\mu > 0$, such that*

(i) $X = \partial_{x_m}$.
(ii) $\Sigma \cap U \subset \left\{x \in U \mid |x_m| < \mu\right\}$.

Furthermore, this neighbourhood U can be taken arbitrarily small.

Proof. Note that the first item is a consequence of the flow-box theorem [4, p.45]. We can now take this coordinate chart U to be arbitrarily small.

For the second item, note that by the stratification of Σ, we know that the set

$$\{((x_0)_1, \dots, (x_0)_{m-1}, x_m) \mid x_m \in \mathbb{R}\}$$

is not contained in Σ. In fact, we can find $\epsilon_0, \epsilon_1 > 0$, such that the points $((x_0)_1, \dots, (x_0)_{m-1}, \epsilon_0)$ and $((x_0)_1, \dots, (x_0)_{m-1}, -\epsilon_1)$ are not contained in Σ.

Note that $\Sigma_{\mathrm{Imm},n-1} \subset J^1\left(M, \mathbb{R}^{n-1}\right)$ is closed, thus it follows that $\Sigma \subset M$ is closed. Hence we can find open neighbourhoods of $((x_0)_1, \dots, (x_0)_{m-1}, \epsilon_0)$ and $((x_0)_1, \dots, (x_0)_{m-1}, -\epsilon_1)$ respectively that are disjoint from Σ. The result follows. □

We are now ready to prove proposition 3.23.

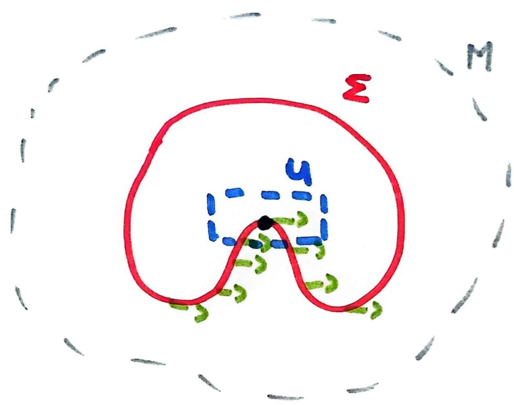


Figure 3.3: Illustration of the open 'box' U constructed in lemma 3.27. The green arrows denote the vector field X.

Proof of proposition 3.23. We will prove proposition 3.23 by applying lemma 3.26 inductively to the strata of $\Sigma := \Sigma_{\mathrm{Imm},n-1}(z_{\hat{s}_i})$. Let $\bigsqcup_{j=0}^{k} S^j$ denote this stratification. From remark 3.25 we know that there exists some smooth vector field of M, X, which is defined on Σ that spans $\ker(z_{\hat{s}_i})|_\Sigma$, such that $(z_s)_i(x)(X_x) > 0$ for all $x \in \Sigma$. Now note that we can extend X to some vector field on a tubular neighbourhood $\mathcal{N}$ of Σ. By lemma 3.26, we can then construct some function f_i^k on an open neighbourhood contained in $\mathcal{N}$, such that $X(f_i^k) > 0$ and which is 0 on S^k.

As induction hypothesis we now assume that we have constructed some f_i^1 on a neighbourhood of $\bigsqcup_{j=1}^{k} S^j$, which we will denote by V_1. Note that S^0 is a submanifold of M over which $\ker(z_{\hat{s}_i})$ forms a trivial subbundle, hence we can apply lemma 3.26 to S^0, to construct a function f_i^0 on an open neighbourhood of U, V_0, such that

- $f_i^0(x) = 0$ for all $x \in U$
- $X_x(f_i^0) > 0$ for all $x \in V_0$

Furthermore, note that we can construct V_0 such that it is disjoint from $\bigsqcup_{j=1}^{k} S^k$.

Then we can use lemma 3.27 to construct neighbourhoods $V_{x'}$ around $x' \in \bigsqcup_{j=1}^{k} S^k$ contained in V_1 that satisfy the conditions on that lemma. Then on such a domain V_x, we can construct a smooth function $\eta_{x'} : V_{x'} \to \mathbb{R}$ that satisfies the following conditions.

- $0 \le \eta_{x'} \le 1$ on all of $V_{x'}$.
- $\eta_{x'}$ is constant in the variable x_m.
- $\eta_{x'}$ is constantly 1 on some small open around x'.
- If we restrict η to $U_0 := \{x \in U \mid x_m = 0\}$, then the support of $\eta_\alpha|_{U_0}$ is contained in U_0.

Then by $U_{x'} \subset V_{x'}$ we denote the subset where $\eta_{x'}$ is non-zero. Then $\{U_{x'}\}_{x' \in \bigsqcup_{j=1}^{k} S^k}$ forms a cover of $\bigsqcup_{j=1}^{k} S^k$. Using the paracompactness of $\Sigma \setminus S^0$, we can refine this cover to some locally finite cover $\{U_\alpha\}_{\alpha \in J}$ over $\Sigma \setminus S^0$. We can then readjust the η_α's, such that the support of $\eta_a|_{U_0}$ is the closure of $U_\alpha \cap U_0$.

Furthermore, let η^0 be a function defined on V^0 that satisfies

- η^0 is constantly 1 on $S^0 \setminus \bigsqcup_{j=1}^k S^j$.
- the support of η^0 is contained in V^0.

Then by perhaps further shrinking the neigbourhood $\mathcal{N} \cup V_0$ of Σ, we can define the function

$$f_i = \sum_{\alpha \in J} \eta_\alpha f_i^1 + \eta_0 f_i^0$$

on it. Furthermore, by construction it follows that $X(f_i) > 0$ on Σ, and thus also on an open neighbourhood of Σ.

Thus it follows that $\hat{s}_i \oplus_i j^1 f_i$ is a formal solution of the relation $\mathscr{R}_{\mathrm{Imm},n}$. Furthermore, as described in remark 3.25, $\hat{s}_i \oplus_i j^1 f_i$ is homotopic to s through formal solutions using linear interpolation. □

Remark 3.28. Note that in the proof of proposition 3.23, we have created a certain 'hierarchy' of the singularity. In a way, the points in the largest stratum S_0 are the least singular and as we go to the smaller strata, the points get more and more singular. Then we remove the singularity by first taking care of the most singular points and step by step work our way back to the less singular points. This strategy comes back more often in 'removal of singularities' arguments, for example in Szücs's proof of Haefliger's theorem [28, definition 4.1]. We will discuss this more extensively in the outlook. ⋄

In the next three chapters we will discuss and develop the tools needed to reduce the componentwise step in remark 3.20 to the proposition above. In chapter 7 we will then use those tools to finish the proof of the h-principle for $\mathscr{R}_{\mathrm{Imm},n}$, including the argument for the injectivity on π_0 and the argument for the higher homotopy groups.

4 Transversality

When studying differential relations we want to know more about the singularity $\Sigma_{\mathscr{R}} \subset J^\infty(E)$ of a certain relation $\mathscr{R}$ and specifically how the jet of a section, $j^\infty s$, interacts with this singularity. An important tool in studying the interaction of a map and such singularities is *transversality*.

First of all, one could ask the question when for a given $s \in \Gamma(E)$ and $\Sigma \subset J^\infty(E)$ a pf-submanifold, the inverse image $\left(j^\infty s\right)^{-1}(\Sigma) \subset M$ is a submanifold. It turns out that transversality gives us exactly this result. One should thus think of transversality as the image of $j^\infty s$ intersecting Σ in a 'clean' way. In this chapter we will also see that being transversal is a *generic property*, i.e. it can be achieved by the slightest perturbation. This is what is known as the Thom-transversality theorem. The classical phrasing of this theorem considers the space $C^\infty(M,N)$. In this chapter we will also see a similar statement for the space $\Gamma(E)$ of a fibre bundle $\pi : E \to M$.

A typical illustration of transversality can be found in figure 4.1. The black line intersects the green one transversely in the point B, but non-transversely in the point A. The intersection in A can be easily avoided by shifting the black line down by an arbitrarily small amount. However, the transverse intersection at B cannot be so easily avoided. It turns out that under certain dimension constraints, being transverse to Σ is equivalent to avoiding Σ altogether. In such cases, not intersecting Σ thus becomes a generic property.

This observation is what makes transversality such a desirable property when we want to 'remove a singularity'. Indeed, we will see that it is the main tool in reducing the problem of homotoping a formal immersion to an actual one to proposition 3.23. It is therefore important to make sure that if we apply some transversality argument in each step of this proces, we do not 'destroy' the holonomicity we previously achieved. In the last section of this chapter, we will therefore discuss some transversality results that will exactly preserve this partial holonomicity.

4.1 Transversality

We wil first introduce the notion of transversality and discuss some of its elementary properties. The following definitions and properties are based on [9, section 2.4, pp.50-59].

Definition 4.1. *Let M and N be smooth manifolds and $f : M \to N$ be a smooth map. Let $\Sigma \subset N$ be a submanifold and $x \in M$. We say that **f intersects Σ transversely at x**, denoted by $f \pitchfork \Sigma$ at x, if either of the following holds*

(i) $f(x) \notin \Sigma$.
(ii) $f(x) \in \Sigma$ *and* $T_{f(x)}N = T_{f(x)}\Sigma + (df)_x(T_xX)$.

Example 4.2. Figure 4.1 illustrates both a transverse and a non-transverse intersection. At the point $A = f(0)$ we see that the image of f is tangent to the submanifold Σ. Therefore, $(df)_0(T_0\mathbb{R})$ does not contain a complementary direction to $T_A\Sigma$, and the transversality condition (ii) is not satisfied. At the point $B = f(3)$, the image of f is not tangent to Σ, thus condition (ii) is satisfied at $x = 3$. Furthermore at any $x \in \mathbb{R}$ with $x \neq 0, 3$, condition (i) is satisfied. Thus we see that in the example below $f \pitchfork \Sigma$ at any $x \neq 0$. △

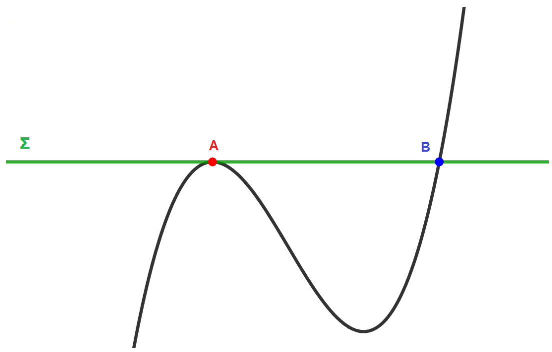


Figure 4.1: Illustration of the image of the map $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (x, (x-1)^3 - 3(x-1) - 2)$ and the submanifold $\Sigma = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$. The point B illustrates a transverse intersection of f with Σ, whereas A illustrates a non-transverse intersection of f and Σ.

Definition 4.3. *Let M and N be smooth manifolds and $f \in C^\infty(M, N)$. Let $\Sigma \subset N$ be a submanifold and $A \subset M$ a subset.*

- *We say that* $\boldsymbol{f \pitchfork \Sigma}$ ***on A*** *if $f \pitchfork \Sigma$ at x for all $x \in A$.*
- *We say that* $\boldsymbol{f \pitchfork \Sigma}$ *if $f \pitchfork \Sigma$ on M.*

It is easy to see that if Σ has enough codimension, being transversal to Σ is equivalent to avoiding Σ altogether.

Lemma 4.4. *Let M and N be manifolds and $\Sigma \subset N$ be a submanifold of codimension q with $q > m := \dim(M)$. Then for $f \in C^\infty(M, N)$, $f \pitchfork \Sigma$ if and only if $f(M) \cap \Sigma = \emptyset$.*

Proof. Note that the if statement is trivial. To see that the only if statement holds follows from a simple dimension count.

Assume that $f \pitchfork \Sigma$. For any $x \in M$, note that $\dim\big((df)_x(T_xM)\big) \leq \dim(T_xM) = m$. Now assume that $f(x) \in \Sigma$. Then $\dim\big(T_{f(x)}\Sigma\big) = n - q < n - m$. Thus it follows that

$$\dim\big((df)_x(T_xM) + T_{f(x)}\Sigma\big) < n$$

which means that condition (ii) cannot be satisfied. This gives a contradiction, hence $f(x) \notin \Sigma$. □

As discussed in the introduction of this chapter, imposing transversality on a smooth map $f : M \to N$ w.r.t. some submanifold $\Sigma \subset N$, means that the inverse image $f^{-1}(\Sigma) \subset M$ is also a submanifold. To see this we will first discuss an alternative definition of transversality.

Remark 4.5. Let N be a manifold and let $\Sigma \subset N$ be a submanifold of codimension q. Note that by the submersion theorem, for any $y \in \Sigma$, we can find an open $V \subset N$ and a submersion $\varphi : V \to \mathbb{R}^q$, such that $V \cap \Sigma = \varphi^{-1}(0)$. The tangent space $T_y\Sigma$ is then given by $\ker\big((d\varphi)_y\big)$. ⋄

Lemma 4.6. *Let M and N be smooth manifolds and $\Sigma \subset N$ a submanifold of codimension q. Let $x \in M$ with $f(x) \in \Sigma$. Let $V \subset N$ be an open around $f(x)$ and $\varphi : V \to \mathbb{R}^q$ as in remark 4.5. Then $f \pitchfork \Sigma$ at x if and only if $\varphi \circ f$ is a submersion at x.*

Proof. Note that by remark 4.5 we get that $f \pitchfork \Sigma$ at x if and only if

$$T_{f(x)}N = \ker\left((d\varphi)_{f(x)}\right) + (df)_x(T_xM)$$

Since φ is a submersion, this is equivalent to saying that $\left((d\varphi)_{f(x)} \circ (df)_x\right)(T_xM) = T_{\varphi\circ f(x)}\mathbb{R}^q$. By the chain rule this means exactly that $\varphi \circ f$ is a submersion at x. □

Corollary 4.7. *Let M and N be smooth manifolds and $\Sigma \subset N$ a submanifold of codimension q. Let $f \in C^\infty(M,N)$ with $f \pitchfork \Sigma$. Then $f^{-1}(\Sigma) \subset M$ is a submanifold of codimension q.*

Remark 4.8. Note that from lemma 4.6, we can also deduce something about the tangent space of $\Sigma(f)$, namely that it is of the form

$$T_x\Sigma(f) := \left((df)_x\right)^{-1}\left(T_{f(x)}\Sigma\right)$$

This observation will play a key role in the construction of the Thom-Boardman singularities in chapter 5. ⋄

The following proposition is proven in for example [9, proposition 4.5]. This makes transversality, under certain conditions, into a so-called *stable* property.

Proposition 4.9. *Let M and N be manifolds and $\Sigma \subset N$ a closed submanifold. Then the set*

$$T_\Sigma := \left\{f \in C^\infty(M,N) \mid f \pitchfork \Sigma\right\} \subset C^\infty_S(M,N)$$

is an open subset.

Note that this property holds only if we endow $C^\infty(M,N)$ with the strong topology. A similar statement for the weak topology would require Σ to be compact. [18, theorem 3.2.1.]

4.2 Thom-transversality theorem

In this section we will state the well-known Thom-transversality theorem. For a proof of this theorem we refer the reader to [9, Theorem 4.9, pp. 54-56] or the original paper by Thom [29]. This theorem is classically stated for the space $C^\infty_S(M,N)$. We will then deduce a similar result for the space $\Gamma_S(E)$, where $\pi : E \to M$ is a fibre bundle.

Theorem 4.10. *Let M and N be manifolds, $k \geq 0$ be finite and $\Sigma \subset J^k(M,N)$ be a submanifold. We define*

$$T_\Sigma = \left\{f \in C^\infty(M,N) \mid j^k f \pitchfork \Sigma\right\}$$

Then T_Σ is a residual subset of $C^\infty_S(M,N)$. If Σ is a closed submanifold, then T_Σ is an open set.

Remark 4.11. Note that we could also translate the Thom-transversality theorem to the pf-manifold setting in the following way. Assume that $\Sigma \subset J^\infty(E)$ is a pf-submanifold of level l. Then we say that $j^\infty f \pitchfork \Sigma$ if and only if $j^l f \pitchfork a_l(\Sigma)$. The set T_Σ could then also be defined as above with $k = \infty$. Given that everything can be translated to the finite setting of $k = l$ the same result applies.

In the rest of this chapter we will only consider the case where k is finite. However, similarly, any of these results translate to the $k = \infty$ setting. That setting has the advantage that it makes some of the terminology later on in this thesis easier. ⋄

Remark 4.12. Note also, that the countable intersection of residual subsets remains residual. Therefore, the Thom-transversality theorem also holds for transversality to a countable collection of submanifolds. We shall specify this for the case of sections in corollary 4.16 ⋄

We can now deduce a Thom-transversality statement for the space $\Gamma(E)$. Note that sections locally are simply maps to the fibre and from lemma 1.16 it follows that as topological spaces, $\Gamma(E)$ is locally also the space of maps. We will use this property to prove the following corollary.

Corollary 4.13. *Let $\pi : E \to M$ be a fibre bundle and $\Sigma \subset J^k(E)$ be a submanifold. Then the set*

$$T_\Sigma := \left\{ s \in \Gamma(E) \mid j^k s \pitchfork \Sigma \right\}$$

is a residual subset of $\Gamma_S(E)$.

Proof. Let $U \subset M$ be a trivializing open of the bundle $\pi : E \to M$. Then we define

$$\begin{aligned} \alpha : C^\infty_S(U, F) &\to \Gamma_S(\pi^{-1}(U)) \\ f &\mapsto s_f \end{aligned}$$

where s_f is defined by $s_f(x) = (x, f(x)) \in U \times F \cong \pi^{-1}(U)$. It is easy to see that this defines a homeomorphism of topological spaces. Furthermore we know from lemma 1.18 that $J^k(U \times F) \cong J^k(U, F)$. Thus it follows that

$$T'_{\Sigma|_U} := \left\{ s \in \Gamma\left(\pi^{-1}(U)\right) \mid j^k s \pitchfork \Sigma \cap \pi^{-1}(U) \right\}$$

is a residual subset of $\Gamma(\pi^{-1}(U))$

Furthermore, note that the restriction map

$$\begin{aligned} |_U : \Gamma_S(E) &\to \Gamma_S(\pi^{-1}(U)) \\ s &\mapsto s|_U \end{aligned}$$

is an open and continuous map. Note that from openness it follows that for any dense subset $D \subset \Gamma(\pi^{-1}(U))$, $|_U^{-1}(D)$ is also dense. In combination with continuity it then follows that

$$T_{\Sigma|_U} = |_U^{-1}\left(T'_{\Sigma_U}\right) = \left\{ s \in \Gamma(E) \mid j^k s \pitchfork \Sigma \text{ on } U \right\}$$

is a residual subset of $\Gamma(E)$.

Note that by the second countability axiom we can cover M with a countable amount of such opens $\{U_i\}_{i \in I}$. It then follows that

$$T_\Sigma = \left\{ s \in \Gamma(E) \mid j^k s \pitchfork \Sigma \right\} = \bigcap_{i \in I} T_{\Sigma|_{U_i}}$$

is the countable intersection of residual sets, which is thus once again residual.

□

We have now seen that transversality to a submanifold of jet space is a dense property for sections. However, the Thom-transversality theorem says that for functions, this is also an open property, given that the submanifold is closed. This is also true for sections.

Corollary 4.14. *Let $\pi : E \to M$ be a fibre bundle and $\Sigma \subset J^k(E)$ a closed submanifold. Then the set $T_\Sigma \subset \Gamma_S(E)$ is open.*

Note that by proposition 4.9 we already know that the set

$$T'_\Sigma := \left\{\zeta \in C^\infty\left(M, J^k(E)\right) \mid \zeta \pitchfork \Sigma\right\} \subset C^\infty_S\left(M, J^k(E)\right)$$

is an open subset. Thus proving the following proposition implies corollary 4.14.

Proposition 4.15. *The inclusion*

$$\begin{aligned} \iota_k : \Gamma_S(E) &\hookrightarrow C^\infty_S\left(M, J^k(E)\right) \\ s &\mapsto j^k s \end{aligned}$$

is continuous.

Proof. Let us consider for any $l \geq 0$ the following inclusion

$$\begin{aligned} \iota_{k,l} : J^{l+k}(E) &\hookrightarrow J^l(M, J^k(E)) \\ j^{l+k}_x s &\mapsto j^l_x\left(j^k s\right) \end{aligned}$$

Writing this out in coordinates gives exactly a coordinate inclusion, thus this is a smooth (and hence continuous) map.

Now consider $\mathcal{O}_U \subset C^\infty_S(M, J^k(E))$ with $U \subset J^l(M, J^k(E))$ open. Then $\iota_k^{-1}(\mathcal{O}_U) = \mathcal{O}_{\iota_{k,l}^{-1}(U)} \subset \Gamma_S(E)$ is also open. □

Note that corollaries 4.13 and 4.14 together give the same results for the space $\Gamma_S(E)$ as the Thom-transversality theorem does in the specific case where E is the product bundle of two manifolds (i.e. $\Gamma_S(E) \cong C^\infty_S(M, N)$).

Furthermore, as discussed in remark 4.12, we alsoget the following corollary.

Corollary 4.16. *Let $\pi : E \to M$ be a fibre bundle and $\{\Sigma_j\}_{j\in J}$ be a countable collection of pf-submanifolds of $J^\infty(E)$. Then the set*

$$T := \left\{s \in \Gamma(E) \mid j^\infty s \pitchfork \Sigma_j \text{ for all } j \in J\right\}$$

is a residual set. Furthermore, if the Σ_j's are closed and J is finite, then T is open.

4.3 Partially holonomic sections

In this section we will study sections s of which some components are holonomic, while others might not be. The problem we will consider is then whether we can still apply some transversality argument, without losing the holonomicity of the components that we knew to be holonomic already. Note that this is exactly the problem considered in question 2 of remark 3.20. It turns out that we can indeed apply such transversality arguments, as is specified in the theorem below.

Theorem 4.17. *Let $s \in \Gamma\left(J^1(M,\mathbb{R}^n)\right)$ be a section. Assume that the first i components of s are holonomic. Furthermore, let J be a countable index collection and for $j \in J$, let $\Sigma_j \subset J^\infty\left(J^1(M,\mathbb{R}^n)\right)$ be a pf-submanifold.*

Then any neighbourhood $\mathcal{N} \subset \Gamma_S\left(J^1(M,\mathbb{R}^n)\right)$ of s contains another element s', such that the following hold:

1. *The first i components of s' are holonomic.*
2. *$j^\infty s' \pitchfork \Sigma_j$ for all $j \in J$.*

Thus this theorem says that if some components of s are holonomic, then we can satisfy any transversality condition arbitrarily close to s while keeping those components holonomic.

To prove this theorem we will construct a certain submersion $\left(J^1\left(J^1(M,\mathbb{R}^n)\right) \to J^1(M,\mathbb{R}^n)\right.$. By lemma 4.6, this will allow us to consider the transversality condition on sections of $\left(J^1\left(J^1(M,\mathbb{R}^n)\right)\right)$ instead. Furthermore, by the specific construction of the submersion, this will preserve holonomicity exactly for those components we are interested in when translating it back to sections of $J^1(M,\mathbb{R}^n)$.

Note that for any trivial neighbourhood $U \subset M$, we get the following trivialisation of $J^1(U,\mathbb{R}^n)$

$$\begin{aligned} J^1\left(U,\mathbb{R}^n\right) &\cong U \times \mathbb{R}^n \times \mathrm{Hom}\left(\mathbb{R}^m,\mathbb{R}^n\right) \\ j^1_x f &\mapsto \left(x, f(x), (df)_x\right) \end{aligned}$$

We will the denote the U-coordinate by x, the $\mathbb{R}^n$ coordinate by y and the $\mathrm{Hom}(\mathbb{R}^m,\mathbb{R}^n)$ coordinate by z.

Considering the bundle $J^1\left(J^1(M,\mathbb{R}^n)\right)$, we can write $s \in \Gamma\left(J^1(M,\mathbb{R}^n)\right)$ as $s = \left(x, y_s, z_s\right)$. If we then restrict to some trivial neighbourhood $U \subset M$, we get

$$\begin{aligned} J^1\left(J^1\left(U,\mathbb{R}^n\right)\right) &\cong U \times \mathbb{R}^n \times \mathrm{Hom}\left(\mathbb{R}^m,\mathbb{R}^n\right) \times \mathrm{Hom}\left(\mathbb{R}^m,\mathbb{R}^n\right) \times \mathrm{Hom}\left(\mathbb{R}^m,\mathbb{R}^{m+n}\right) \\ j^1_x(s) &\mapsto \left(x, y_s(x), z_s(x), d\left(y_s\right)_x, d\left(z_s\right)_x\right) \end{aligned}$$

We will denote these coordinates by x, y, z, y' and z' respectively. Note that the functions z and y' map to the same space. We can use this to define for any $1 \le j \le n$ locally the following projection map

$$\begin{aligned} \mathrm{pr}_i : J^1\left(J^1\left(U,\mathbb{R}^n\right)\right) &\to J^1\left(U,\mathbb{R}^n\right) \\ (x,y,z,y',z') &\mapsto \left(x, y, \left(y'_1,\ldots,y'_i, z_{i+1},\ldots,z_n\right)\right) \end{aligned} \tag{4.1}$$

This map is actually independent of the chosen chart and thus we get a global projection map $\mathrm{pr}_i : J^1\left(J^1(M,\mathbb{R}^n)\right) \to J^1(U,\mathbb{R}^n)$.

Lemma 4.18. *Let M be a manifold and let $1 \le i \le n$. Then the map $pr_i : J^1\left(J^1(M,\mathbb{R}^n)\right) \to J^1(M,\mathbb{R}^n)$ is a submersive bundle morphism over the map $id_M : M \to M$.*

Proof. Note that the local definition over $J^1(U,\mathbb{R}^n)$ is a projection of coordinates, thus pr_j is a submersion. Furthermore, the map does not change the base point x, and is thus indeed a fibre bundle morphism over $\mathrm{id}_M : M \to M$. □

Let E' denote the bundle $J^1\left(J^1(M,\mathbb{R}^n)\right)$ and let E denote the bundle $J^1(M,\mathbb{R}^n)$. First of all, note that by lemma 1.14 (ii) we then get induced smooth maps

$$\begin{aligned}\mathrm{pr}_i^k : J^k(E') &\to J^k(E)\\ j_x^k s &\mapsto j_x^k(\mathrm{pr}_i \circ s)\end{aligned}$$

Writing this out in local coordinates immediately shows that these maps are also submersions for all finite k's.

Combining this fact with lemma 4.6 gives the following corollary.

Corollary 4.19. *Let $\Sigma \subset J^k(E)$ be a submanifold. Then for any section $\sigma \in \Gamma(E')$, the following are equivalent*

(i) $j^k(pr_i \circ \sigma) \pitchfork \Sigma$
(ii) $j^k \sigma \pitchfork (pr_i)^{-1}(\Sigma)$

Recall that we were interested in sections $s \in \Gamma(J^1(M, \mathbb{R}^n))$ of which the first i components were already holonomic. Note that for such sections, the definition of pr_i ensures (and was constructed such that) we get $\mathrm{pr}_i \circ j^1 s = s$. We now have all the ingredients to prove theorem 4.17.

Proof of Theorem 4.17. Let $s \in \Gamma(J^1(M, \mathbb{R}^n))$ be a section whose first i components are holonomic. Furthermore let $\mathcal{U} \subset \Gamma_S(J^1(M, \mathbb{R}^n))$ be an open around s. Then it follows from corollary 2.7 that the set

$$\tilde{\mathcal{U}} := \{\sigma \in \Gamma(J^1(J^1(M, \mathbb{R}^n))) \mid \mathrm{pr}_i \circ s' \in \mathcal{U}\} \subset \Gamma_S(J^1(J^1(M, \mathbb{R}^n)))$$

is open. Furthermore, note that $j^1 s \in \tilde{\mathcal{U}}$, hence it follows that $\tilde{\mathcal{U}}$ is a non-empty open set. Thus by corollary 4.16 it follows that there exists some $\sigma' \in \mathcal{U}$, that satisfies $j^\infty \sigma' \pitchfork \Sigma_j$ for all $j \in J$. Then by definition we get that $s' := \mathrm{pr}_i \circ \sigma' \in \mathcal{U}$ and by corollary 4.19 it follows that $j^\infty s' \pitchfork \Sigma_j$ for all $j \in J$. □

This theorem also gives the following corollary, which will be the form we will actually need in chapter 7. The notation used here was introduced in chapter 3.

Corollary 4.20. *Let $s \in \Gamma(J^1(M, \mathbb{R}^n))$ be a section. Assume that the first $i - 1$ components of s are holonomic. Furthermore, let J be a countable index collection and for $j \in J$, let $\Sigma_j \subset J^\infty(J^1(M, \mathbb{R}^{n-1}))$ be a pf-submanifold.*

Then any neighbourhood $\mathcal{N} \subset \Gamma_S(J^1(M, \mathbb{R}^n))$ of s contains another element s', such that the following hold:

1. *The first $i - 1$ components of s' are holonomic.*
2. $j^\infty \hat{s}'_i \pitchfork \Sigma_j$ *for all $j \in J$.*

Proof. Note that the projection map

$$\begin{aligned}P_i : J^1(M, \mathbb{R}^n) &\to J^1(M, \mathbb{R}^{n-1})\\ j_x^1 f &\mapsto j_x^1 \hat{f}_i\end{aligned}$$

is a submersion and by definition we have $\hat{s}_i = P_i \circ s$ from any $s \in \Gamma(J^1(M, \mathbb{R}^n)$. Furthermore, it is easy to see that the induced maps

$$P_i^* : J^k(J^1(M, \mathbb{R}^n)) \to J^k(J^1(M, \mathbb{R}^{n-1}))$$

as defined in lemma 1.14 are submersions.

Note that from lemma 4.6 it then follows that

$$j^{k_j}\hat{s}_i \pitchfork \left(\Sigma_j\right)_{k_j} \iff j^{k_j} s \pitchfork \left(P_i^*\right)^{-1}\left(\left(\Sigma_j\right)_{k_j}\right)$$

where k_j is the level of Σ_j and $\left(\Sigma_j\right)_{k_j} = a_{k_j}\left(\Sigma_j\right)$. The result then follows from theorem 4.17. □

5 Thom-Boardman singularities

In his 1967 paper [2] Boardman defined pf-submanifolds of infinite jet space, $\Sigma^{\mathscr{J}}_{\mathrm{TB}} \subset J^\infty(M,N)$, for M and N manifolds, that generalized the singularity subsets of a map $f: M \to N$, as described earlier by René Thom. By Thom's definition, the j^{th} singularity of a map f, $\Sigma^j_{\mathrm{TB}}(f)$, consists of the points s in M where the rank of the kernel of $(df)_x$ is exactly j. In his 1956 paper [30], Thom had shown using transversality that for maps f in a dense subset of $C^\infty_S(M,N)$, the singularities $\Sigma^j_{\mathrm{TB}}(f)$ are manifolds again.

Boardman's approach took this a step further. In his paper he considers iterative singularities, by examining the rank of the map f after restricting it to $\Sigma^j_{\mathrm{TB}}(f)$ and so on. More specifically, for a tuple $\mathscr{J}$, he defined subsets $\Sigma^{\mathscr{J}}_{\mathrm{TB}} \subset J^\infty(M,N)$ and showed that these subsets are pf-submanifolds. Furthermore, he showed that for any tuple $\mathscr{J}$, a given number $j \in \mathbb{N}$ and any map $f: M \to N$ whose jet section $j^\infty f: M \to J^\infty(M,N)$ is transversal to $\Sigma^{\mathscr{J}}_{\mathrm{TB}}$

$$\Sigma^{\mathscr{J},j}_{\mathrm{TB}}(f) := j^\infty f^{-1}\left(\Sigma^{\mathscr{J},j}_{\mathrm{TB}}\right) = \Sigma^j_{\mathrm{TB}}\left(f|_{\Sigma^{\mathscr{J}}_{\mathrm{TB}}(f)}\right) \tag{5.1}$$

Example 5.1. Let us consider what Thom-Boardman singularities[1] exist for the function

$$\begin{aligned} f: \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x_1,x_2) &\mapsto \left(\frac{1}{3}x_1^3 + x_1x_2^2 - x_1, (x_2-1)^2\right) \end{aligned}$$

Calculating its differential gives

$$(df)_x = \begin{pmatrix} x_1^2 + x_2^2 - 1 & 2x_1x_2 \\ 0 & 2(x_2-1) \end{pmatrix}$$

Determining the kernel of this linear map tells us that $\Sigma^2_{\mathrm{TB}}(f) = \{(0,1)\}$ and $\Sigma^1_{\mathrm{TB}}(f)$ is the union of $\mathbb{S}^1$ and the line $x_2 = 1$, but without $\Sigma^2_{\mathrm{TB}}(f)$.

Since $\Sigma^2_{\mathrm{TB}}(f)$ is only a point, there are no further singularities to be considered there. However, $\Sigma^1_{\mathrm{TB}}(f)$ is a 1-dimensional manifold. On the part where $x_2 = 1$, we see that $\ker\big((df)_x\big) = \langle 2\partial_{x_1} - x_1\partial_{x_2}\rangle$. Since the ∂_{x_2}-component does not disappear, it follows that in this part of $\Sigma^1_{\mathrm{TB}}(f)$, there are no further singularities to consider. On the other hand, for the elements $x \in \Sigma^1_{\mathrm{TB}}(f)$ with $x_1^2 + x_2^2 = 1$, we get $\ker\big((df)_x\big) = \langle \partial_{x_1}\rangle$, which is tangent to the singularity $\Sigma^1_{\mathrm{rank}}(f)$ at $(0,-1)$. Hence this point forms the singularity $\Sigma^{1,1}_{\mathrm{TB}}(f)$. These singularities are illustrated in figure 5.1. △

[1] Note that to consider whether these singularities are well-defined we should check that $j^\infty f$ is transversal to the Σ^j's. However, that is highly technical and does not add to the illustrating nature of the example. However, note that all the first order singularities of f, $\left(\Sigma^j(f)\right)$ are manifolds, so we can restrict f and its differential to these singularities.

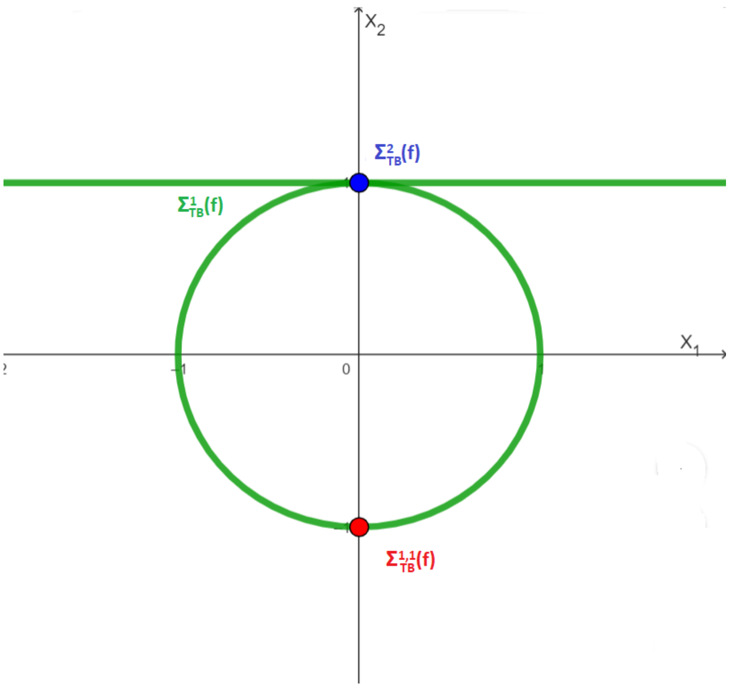


Figure 5.1: Illustration of the Thom-Boardman singularities of the map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, (x_1, x_2) \mapsto \left(\frac{1}{3}x_1^3 + x_1 x_2^2 - x_1, (x_2 - 1)^2\right)$$

Let us discuss why we would be interested in these singularites. Let $\mathscr{I}_k$ denote the tuple consisting of k 1's. Then any formal immersion $j^1 \hat{f}_i \oplus s_i$ that satisfies

$$j^\infty \hat{f}_i \pitchfork \Sigma_{\text{TB}}^{\mathscr{I}_k}$$

for all $k \geq 0$ satisfies the conditions of proposition 3.23. This is because of equation (5.1) and the fact that the $\Sigma_{\text{TB}}^{\mathscr{I}_k} \subset J^\infty\left(M, \mathbb{R}^{n-1}\right)$ are pf-submanifolds of increasing codimension. We will see all these properties in theorem 5.21, which was proven by Boardman [2, theorem 6.1].

These Thom-Boardman singularities thus exactly provide us with the information we need to put the generic condition on $j^1 \hat{f}_i \oplus_i s_i$ that will allow us to apply proposition 3.23 to $j^1 \hat{f}_i \oplus_i s_i$. However, note that this argument only applies to formal immersions for which all but 1 component are holonomic already. In his proof Gromov remarked that a similar argument can be used to make any formal immersion $s \in \Gamma\left(J^1\left(M, \mathbb{R}^n\right)\right)$ generic so that proposition 3.23 may be applied to $\hat{s}_i$.

Thus, in order to give the proof Gromov described, we need to define similar hierarchies to the ones Boardman defined, that consider higher order derivatives of sections in stead of functions. Thus these hierarchies will need to be defined in the tower $J^\bullet\left(J^1(M, \mathbb{R}^{n-1}\right)$. Since the construction of those singularities will be very similar to the Thom-Boardman construction, we first need to understand how Boardman constructed his singularities, which is what we will do in this chapter. We will introduce an alternative definition of the Thom-Boardman singularities, which will make it easier to give a more general definition in chapter 6.

In the first section of this chapter we discuss the intuition behind our alternative characterisation of the Thom-Boardman singularities. In the second and third section we then discuss some technical notions and (slight generalisations of) results that Boardman described in his paper. In section 5.4 we will then use those technical notions and results to show that our characterisation of the Thom-Boardman singularities in fact describe the Thom-Boardman singularities.

5.1 Rank and holonomic tangencies

Let us now shortly discuss a bit of the intuition behind the Thom-Boardman singularities. It is a well-known fact, for example proven in [9, Theorem 2.5.4], that for $j \geq 0$, the sets

$$\Sigma^j_{\text{rank}} := \{ j^1_x f \in J^1(M,N) \mid \dim\big(\ker\big((df)_x\big) = j\big)\} \subset J^1(M,N)$$

are submanifolds. Thus, as we already saw in the introduction to this chapter, for any map $f \in C^\infty(M,N)$ with $j^1 f \pitchfork \Sigma^j_{\text{rank}}$, we indeed get that $\Sigma^j_{\text{rank}}(f) \subset M$ is a manifold. We then want to examine the rank of such a map f when restricted to the singularity $\Sigma^j_{\text{rank}}(f)$. Thus for $x \in \Sigma^j_{\text{rank}}(f)$ we want to examine the rank of the map $(df)_x$ after restricting it to $T_x\Sigma^j_{\text{rank}}(f)$. Note that to determine this rank, it suffices to determine the dimension of the intersection

$$\ker(df)_x \cap T_x\Sigma^j_{\text{rank}}(f) \subset T_x M$$

To do this, observe the following. From remark 4.8 we know that $V \in T_x\Sigma^j_{\text{rank}}(f)$ if and only if $(dj^1 f)_x(V) \in T_{j^1 f(x)}\Sigma^j_{\text{rank}}$. Also, since any jet-section $j^k f$ is an immersion, we know that $\big(dj^1 f\big)_x$ is injective. Thus if we set $K_{f,x} := \ker\big((df)_x\big) \subset T_x M$, it suffices to consider the image of this subspace under $(dj^1 f)_x$ and compare it to the tangent space $T_x\Sigma^j_{\text{rank}}(f)$. Specifically, we need to determine the rank of the following intersection

$$\big(dj^1 f\big)_x \big(K_{f,x}\big) \cap T_{j^1_x f}\Sigma^j_{\text{rank}} \subset T_{j^1_x f} J^1(M,N)$$

From proposition 1.59 it follows that this information is contained in the second jet space, i.e., the following set

$$\Sigma^{j_1,j_2}_{\text{rank}} := \left\{ j^2_x f \in \pi_2^{-1}\left(\Sigma^{j_1}_{\text{rank}}\right) \mid \dim\left(\big(dj^1 f\big)_x \big(K_{f,x}\big) \cap T_{j^1_x f}\Sigma^j_{\text{rank}}\right) = j_2 \right\} \subset J^2(M,N)$$

is a well-defined set.[2] We will see that this set exactly describes the set $\Sigma^{j_1,j_2}_{\text{TB}}$ that Boardman defined. If we can show that this new set $\Sigma^{j_1,j_2}_{\text{rank}}$ is once again a manifold, we can repeat this process to construct $\Sigma^{\mathscr{J}}_{\text{rank}}$ for all lengths of $\mathscr{J}$ as follows.

Definition 5.2. *Let $\Sigma \subset J^\infty(M,N)$ be a pf-submanifold of level k with $a_k(\Sigma) =: \Sigma_k \subset J^k(M,N)$. Then we define the **j-lift of Σ** as*

$$(\Sigma)^j := \left\{ j^\infty_x f \in \Sigma \mid \dim\left(\left(dj^k f\right)_x \big(K_{f,x}\big) \cap T_{j^k_x f}\Sigma_k\right) = j \right\}$$

We would then like to give the following inductive definition.

Definition 5.3. *Let $\mathscr{J}$ be a non-empty sequence of non-negative integers. Then for $j \geq 0$, we define the set*

$$\Sigma^{\mathscr{J},j}_{rank} = \left(\Sigma^{\mathscr{J}}_{rank}\right)^j$$

[2] Here $\pi_2 : J^2(M,N) \to J^1(M,N)$ denotes the projection map.

However, to give this definition we need to show that each $\Sigma^{\mathscr{I}}_{\text{rank}} \subset J^\infty(M,N)$ is a pf-submanifold. This is exactly what Boardman did in his paper [2] for the singularities $\Sigma^{\mathscr{I}}_{\text{TB}}$. The main strategy of Boardman's proof in his paper is to locally write the sets $\Sigma^{\mathscr{I}}_{\text{TB}}$ as the vanishing set of some ideal of functions $I \subset C^\infty(J^k(M,N))$. To complete the argument that these sets are then manifolds, Boardman showed, by using these ideals, that locally the sets $\Sigma^{\mathscr{I}}$'s are the null-set of some submersion into Euclidean space.

The full proof that Boardman gave is beyond the scope of this thesis. Instead, we will use a characterisation of the Thom-Boardman singularities given by Boardman (theorem 5.35) to show that

- $\Sigma^j_{\text{rank}} = \Sigma^j_{\text{TB}}$ for all j (lemma 5.28)
- $\Sigma^{\mathscr{I},j}_{\text{TB}} = \left(\Sigma^{\mathscr{I}}_{\text{TB}}\right)^j$ for all sequences $\mathscr{I}$ and indices j (lemma 5.39)

which implies that definition 5.2 is well-defined, and actually yields the Thom-Boardman singularities. To do this, we will focus on the ideals Boardman constructed to describe his singularities.

5.2 Vanishing ideals

As discussed above, we want to describe the Thom-Boardman singularities as the null-set of some ideal $I \subset C^\infty\left(J^k(M,N)\right)$. Note that all the information about holonomic tangencies as described in the previous section is contained in the infinite Cartan distribution. As we have seen in chapter 1, the infinite Cartan distribution is a construction on the infinite jet bundle $J^\infty(M,N)$ with some very nice and regular properties. To be able to use the Cartan distribution, we will define the higher singularities as subsets of infinite instead of finite jet space. The vanishing ideals will then also consist of functions defined on infinite jet space.

Definition 5.4. *Let $I \subset C^\infty\left(J^k(M,N)\right)$ be an ideal. Then we define **the lift of I to $J^\infty(M,N)$** as the ideal $\tilde{I} \subset C^\infty(J^\infty(M,N))$ generated by functions of the form $\varphi \circ a_k : J^\infty(M,N) \to \mathbb{R}$ with $\varphi \in I$.*

Remark 5.5. If $I \subset C^\infty(U_k)$ is an ideal with $U_k \subset J^k(M,N)$ open, then we define the lift of I to $J^\infty(U)$, $\tilde{I} \subset C^\infty(J^\infty(U))$ by the same condition. Here $U := a_k^{-1}(U_k) \subset J^\infty(M,N)$. ⋄

Since $\Sigma^j_{\text{rank}} \subset J^1(M,\mathbb{R}^n)$ is a submanifold of some codimension q, we can locally write it as the vanishing set of a smooth submersion $\Phi : U \to \mathbb{R}^q$, where U is an open of $J^1(M,\mathbb{R}^n)$. The tangent space $T_{j^1_x f}\Sigma^j_{\text{rank}}$ is then given by the kernel of the map $(d\Phi)_{j^1_x f}$. We can also put this into terms of ideals as follows.

Notation 5.6. *Let $\pi : E \to M$ be a fibre bundle, let $U \subset J^\infty(E)$ be open and $\Phi_1, \dots, \Phi_n \in C^\infty(U)$.*

- *Then we set $\Phi = (\Phi_1, \dots, \Phi_n) : U \to \mathbb{R}^n$.*
- *We also write $I_\Phi = \langle \Phi_1, \dots, \Phi_n \rangle \subset C^\infty(U)$ for the ideal generated by the Φ_i's.*
- *We will write $\text{null}(I)$ for the zero-set of any ideal $I \subset C^\infty(U)$.*

Recall from definition 1.55 the differential of a smooth function between pf-manifolds. Thus, if we equip $\mathbb{R}^n$ with the trivial pf-structure, we can consider the differential of a map $\Phi : U \to \mathbb{R}^n$. We will mostly be interested in how this differential acts on the Cartan distribution.

Notation 5.7. *Let* $\Phi \in C^\infty(U,\mathbb{R}^n)$ *and* $\sigma \in U$*, then we will write*

$$(\mathscr{d}\Phi)_\sigma := (d\Phi)_\sigma|_{(\mathcal{C}_\infty)_\sigma} : (\mathcal{C}_\infty)_\sigma \to \mathbb{R}^n$$

As mentioned in the previous section, we want to construct vanishing ideals for the higher singularities. The construction of these ideals will be done inductively (definition 5.34). The following lemma will be essential in showing that this inductive construction indeed produces vanishing ideals of the higher singularities.

Lemma 5.8. *Let* $U \subset J^\infty(E)$ *be open and* $\Phi_1,\ldots,\Phi_n \subset C^\infty(U)$*. Set* $K = \ker(\mathscr{d}\Phi)$*. Take* $\Psi \in I_\Phi$ *and* $\sigma \in \text{null}(I_\Phi)$*. Then* $K_\sigma \subset \ker(\mathscr{d}\Psi)_\sigma$*.*

Proof. Since $\Psi \in I_\Phi$, we can write

$$\Psi = \sum_{i=1}^n \alpha_i \Phi_i$$

with $\alpha_i \in C^\infty(U)$. Let $\mathcal{V} \in K_\sigma$, then we know that $(d\Phi_i)_\sigma(\mathcal{V}) = 0$ for all i. Also, since $\sigma \in \text{null}(I)$, it follows that $\Phi_i(\sigma) = 0$ for all i. Then since $\mathcal{V}$ acts as a derivation, it follows from the Leibniz rule that

$$(d\Psi)_\sigma(\mathcal{V}) = \sum_{i=1}^n \alpha_i(\sigma)(d\Phi_i)_\sigma(\mathcal{V}) + \Phi_i(\sigma)(d\alpha_i)_\sigma(\mathcal{V}) = 0$$

Since $\mathcal{V} \in K_\sigma \subset \mathcal{C}_\infty|_\sigma$, it then follows that $\mathcal{V} \in \ker(\mathscr{d}\Psi)$. □

Definition 5.9. *Let* $A \subset C^\infty(U)$ *be a subset. Then we define the* $\mathcal{C}_\infty$***-kernel of* A *at* $\sigma \in$ *U*** *to be*

$$\ker_\sigma^{\mathcal{C}_\infty}(A) := \bigcap_{\varphi \in A} \ker((\mathscr{d}\varphi)_\sigma)$$

Remark 5.10. Note that it immediately follows from lemma 5.8 that for $\sigma \in \text{null}(I_\Phi)$, $\ker_\sigma^{\mathcal{C}_\infty}(I_\Phi) := \ker^{\mathcal{C}_\infty}((\mathscr{d}\Phi)_\sigma)$. ◇

Remark 5.11. Actually, with a similar proof as was given for lemma 5.8 we can show that for a generating subset A of the ideal I, it follows that for any $\sigma \in \text{null}(I)$, we get that

$$\ker_\sigma^{\mathcal{C}_\infty}(I) = \ker_\sigma^{\mathcal{C}_\infty}(A)$$

◇

We will finish this section with the following definition which will be used for the construction of the vanishing ideals of the higher singularities.

Notation 5.12. *Let* $\mathcal{S} \subset \mathcal{C}_\infty|_U$ *a smooth subbundle. Then we denote the (local) sections of* $\mathcal{C}_\infty$ *on* $\pi_M(U)$ *with values in* $\mathcal{S}$ *by* $\Gamma_\mathcal{S}$*.*

Definition 5.13. *Let* $A \subset C^\infty(U)$ *be a subset and* $\mathcal{S} \subset \mathcal{C}_\infty|_U$ *a smooth subbundle. We define* ***the ideal*** $\mathbf{\Gamma_\mathcal{S}(A)} \subset C^\infty(U)$ *to be the ideal generated by elements of the form* $X(\varphi)$ *with* $X \in \Gamma_\mathcal{S}$ *and* $\varphi \in A$*.*

Remark 5.14. As mentioned earlier, we will construct the vanishing ideals of the higher singularities inductively. Definition 5.13 will play an important role in that construction. Let us therefore shortly consider the null-sets of such ideals. In particular, note that $j_x^\infty f \in \text{null}(\Gamma_{\mathcal{S}}(A))$ if and only if

$$\mathcal{S}|_{j_x^\infty f} \subset \ker^{\mathcal{C}_\infty}_{j_x^\infty f}(A)$$

◇

5.3 Totally independent functions

Next we will look at so-called *totally independent functions* on some open $U \subset J^\infty(E)$. Being totally independent means that the differential w.r.t. the Cartan distribution, as described in notation 5.7, is of full rank. Boardman defined this property in his paper [2, definition 1.20, p.28]. We will finish this section with a discussion on how totally independent functions are related to the $\mathcal{C}_\infty$-kernel of a set of functions as defined in the previous section.

Definition 5.15. *Let $D = \{\Phi_1, \dots \Phi_r\} \subset C^\infty(U)$ and $\sigma \in U$. We say that D is **totally independent at** σ if $(\mathscr{d}\Phi)_\sigma$ is of rank r. We say that D is **totally independent on U** if it is totally independent at σ for all $\sigma \in U$.*

Example 5.16. Let us consider $J^\infty(\mathbb{R},\mathbb{R})$ and the smooth pf-function of level 1

$$\begin{aligned}\Phi : J^\infty(\mathbb{R},\mathbb{R}) &\to \mathbb{R}\\ j_x^\infty f &\mapsto f'(x)\end{aligned}$$

Note that Φ is exactly the function for which $\Sigma^1_{\text{rank}} \subset J^\infty(\mathbb{R},\mathbb{R}) = \Phi^{-1}(0)$.

Also note that the set $D = \{\Phi\}$ is totally independent at $j_x^\infty f$ if and only if $(d\Phi)_{j_x^\infty f}\left(\tilde{\partial_x}|_{j_x^\infty f}\right) \neq 0$. We can calculate this expression as follows

$$\begin{aligned}(d\Phi)_{j_x^\infty f}\left(\tilde{\partial_x}|_{j_x^\infty f}\right) &= (d\Phi_1)_{j_x^1 f}\left(\left(d j^1 f\right)_x(\partial_x|_x)\right)\\ &= d\left(\Phi_1 \circ j^1 f\right)_x\left(\partial_x|_x\right) = f''(x)\end{aligned}$$

Thus $\{\Phi\}$ is a totally independent function at any jet $j_x^\infty f$ with $f''(x) \neq 0$.

Let us now consider the singularity Σ^1_{rank}. By definition we have

$$\begin{aligned}\Sigma^{1,1}_{\text{rank}} &= \left\{j_x^\infty f \in \Sigma^1_{\text{rank}} \mid \dim\left(\left(d j^1 f\right)_x(T_x\mathbb{R}) \cap T_{j_x^1 f}\Sigma^1_{\text{rank}}\right) = 1\right\}\\ &= \left\{j_x^\infty f \in \Sigma^1_{\text{rank}} \mid \dim\left(\left(d j^1 f\right)_x(T_x\mathbb{R}) \cap \ker(d\Phi_1)_{j_x^1 f}\right) = 1\right\}\end{aligned}$$

Because of the definition of the infinite Cartan distribution and the map $d\Phi$, it then follows that

$$\begin{aligned}\Sigma^{1,1}_{\text{rank}} &= \left\{j_x^\infty f \in \Sigma^1_{\text{rank}} \mid \dim\left(\mathcal{C}_\infty|_{j_x^\infty f} \cap \ker\left((d\Phi)_{j_x^\infty f}\right)\right) = 1\right\}\\ &= \left\{j_x^\infty f \in \Sigma^1_{\text{rank}} \mid \dim\left(\ker(\mathscr{d}\Phi)_{j_x^\infty f}\right) = 1\right\}\end{aligned}$$

thus we can determine $\Sigma^{1,1}_{\text{rank}}$ by determining the rank of the map $\mathscr{d}\Phi$. In particular, $\Sigma^{1,1}_{\text{rank}}$ contains exactly those jets $j_x^\infty f$ of Σ^1_{rank} at which $\{\Phi\}$ is not totally independent. In the next section we will see that we can apply this strategy more generally. △

Total independence of a set $D = \{\Phi_1,\dots,\Phi_r\}$ thus means that the differentials of the functions 'see' r different holonomic directions. Note that such a set can only be totally independent if $r \le m := \dim(M)$. In this thesis we will also make use of the following slightly adjusted definition.

Definition 5.17. *Let $\mathcal{S} \subset \mathcal{C}_\infty|_U$ be a smooth subbundle. Then we say that a set of functions $D = \{\Phi_1,\dots,\Phi_r\} \subset C^\infty(U)$ is* ***totally independent w.r.t*** $\mathcal{S}$ ***at*** *σ, if the restricted map*

$$(\mathscr{d}\Phi)|_{\mathcal{S}} : \mathcal{S}_\sigma \to \mathbb{R}^r$$

is of rank r.

Note that in particular, being totally independent is the same as being totally independent w.r.t. $\mathcal{S} = \mathcal{C}_\infty$. Thus any results we prove for totally indpendent functions/sets w.r.t. $\mathcal{S}$, also hold for totally independent functions/sets.

In his paper, Boardman uses a construction of an increasing set of totally independent functions. I.e. he keeps adding functions to some set D, which then needs to stay totally independent (see definition 5.31). We will rephrase this (lemma 5.33) with definition 5.17 using the following lemma.

Lemma 5.18. *Let $D = \{\Phi_1,\dots,\Phi_r\} \subset C^\infty(U)$ be a set of totally independent functions at $\sigma \in U$. Let $D' = \{\Psi_1,\dots\Psi_{r'}\} \subset C^\infty(U)$ be another set of functions, such that $D \cap D' = \emptyset$. We take $\mathcal{S} \subset \mathcal{C}_\infty|_U$ to be the subbundle $\mathcal{S} := \ker(\mathscr{d}\Phi)$. Then the set $D \cup D'$ is totally independent at σ if and only if D' is totally independent with respect to $\mathcal{S}$ at σ.*

Proof. Let $(\Phi,\Psi) : U \to \mathbb{R}^{r+r'}$ be the function of which the components are exactly the Φ_i's and Ψ_i's. Then $D \cup D'$ being totally independent at σ means exactly that the map

$$(\mathscr{d}(\Phi,\Psi))_\sigma : (\mathcal{C}_\infty)_\sigma \to \mathbb{R}^{r+r'}$$

is of rank $r + r'$, i.e. that it has a kernel of dimension $m - r - r'$. Furthermore, note that

$$\ker((\mathscr{d}(\Phi,\Psi))_\sigma) = \ker(\mathscr{d}\Phi)_\sigma \cap \ker(\mathscr{d}\Psi)_\sigma = \mathcal{S}_\sigma \cap \ker(\mathscr{d}\Psi)_\sigma$$

Thus we see that $D \cup D'$ being totally independent at σ is equivalent to saying that $\ker\left((\mathscr{d}\Psi|_{\mathcal{S}})_\sigma\right)$ is of dimension $(m-r)-r'$. Since $\mathcal{S}_\sigma$ is of dimension $(m-r)$ we also get that this is equivalent to

$$(\mathscr{d}\Psi|_{\mathcal{S}})_\sigma : \mathcal{S}_\sigma \to \mathbb{R}^{r'}$$

being surjective. Thus we see that indeed $D \cup D'$ is totally independent at σ if and only if D' is totally independent w.r.t. $\mathcal{S}$ at σ. □

This lemma has the following corollary, which directly links the notions discussed in this section and the previous one.

Corollary 5.19. *Let $A \subset C^\infty(U)$ be a subset with $U \subset J^\infty(E)$ an open around $j^\infty_x s$. Furthermore, let $\mathcal{S} \subset \mathcal{C}_\infty|_U$ be a rank l subbundle. Then there exists a set $D = \{\Psi_1,\dots\Psi_r\} \subset A$ of linearly independent functions w.r.t. $\mathcal{S}$ at $j^\infty_x s$ if and only if $\dim\left(\ker^{\mathcal{C}_\infty}_{j^\infty_x s}(A) \cap \mathcal{S}_{j^\infty_x s}\right) \le l - r$.*

Proof. First we assume that there exists some D as described. Note that in this case $\ker\left((\mathscr{d}\Psi)_{j_x^\infty s}\right) \supset \ker^{\mathcal{C}_\infty}_{j_x^\infty s}(A)$. Since D is linearly independent w.r.t. $\mathcal{S}$, it follows that $\dim\left(\ker\left((\mathscr{d}\Psi)_{j_x^\infty s}\right) \cap \mathcal{S}_\sigma\right) = l - r$, hence the result follows.

Next we assume that $\dim(\ker^{\mathcal{C}_\infty}_{j_x^\infty s}(A) \cap \mathcal{S}_{j_x^\infty s}) \le l - r$. We assume $r \ge 1$, since $r = 0$ is a trivial case. Note that we can then find an r-dimensional subspace $\mathcal{A} \subset \mathcal{S}_{j_x^\infty s}$, such that $\ker_{j_x^\infty s}(A) \cap \mathcal{A} = \{0\}$. Hence it follows that there exists some $\varphi \in A$, such that $\left(\mathscr{d}\varphi\right)_{j_x^\infty s}\Big|_{\mathcal{A}}$ is of rank 1. This means that $\left(\mathscr{d}\varphi\right)_{j_x^\infty s}\Big|_{\mathcal{S}_{j_x^\infty s}}$ is of rank 1. Thus the set $\{\varphi\}$ is linearly independent w.r.t. $\mathcal{S}$ at $j_x^\infty s$. We can use lemma 5.18 to inductively extend this set until it contains r elements. □

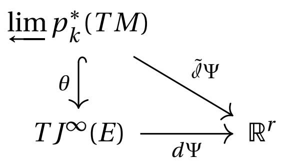
Furthermore, it turns out that in a sense definition 5.17 is a local definition instead of a definition for single jets. Essentially, total independence w.r.t. $\mathcal{S}$ at a given jet $j_x^\infty f$ ensures total indpendence w.r.t. $\mathcal{S}$ on some open neighbourhood of $j_x^\infty f$. The following lemma and proof are based on a remark in the proof of [2, lemma 3.10].

Lemma 5.20. *Let $D = \{\Psi_1, \dots, \Psi_r\} \subset C^\infty(U)$ be a set of totally independent functions w.r.t. some rank l smooth subbundle $\mathcal{S} \subset \mathcal{C}_\infty$ at $j_x^\infty s$. Assume that the Ψ_i's are lifts of some maps $(\Psi_i)_k \in C^\infty(a_k(U))$ and assume that $\mathcal{S}$ is of level $k+1$. Then D is totally independent w.r.t. $\mathcal{S}$ on some open $U' = a_{k+1}^{-1}(U_{k+1})$ where $U_{k+1} \subset J^{k+1}(E)$ is an open around $j_x^{k+1} s$.*

Proof. Since $\mathcal{S}$ is of level $k+1$, we know from lemma 1.70 that there exists some rank l subbundle $\mathcal{S}_{k+1} \subset p_{k+1}^*(TM)$, such that

$$\theta^{-1}(\mathcal{S}) = \left(a'_{k+1}\right)^{-1}(\mathcal{S}_{k+1})$$

where θ denotes the (smooth) map from $\varprojlim p_k^*(TM)$ to $\mathcal{C}_\infty$ as defined in lemma 1.62 and $a'_\bullet$ is the pf-atlas of $\varprojlim p_k^*(TM)$. Then we can study the map $\mathscr{d}\Psi$, by defining a different map $\tilde{\mathscr{d}}\Psi$ with domain $\varprojlim p_k^*(TM)$ as follows



The map $\tilde{\mathscr{d}}\Psi$ is then a smooth pf-map induced by the $k+1$-level map

$$\tilde{\mathscr{d}}_{k+1}\Psi = d\Psi_k \circ \theta_{k+1} : p_{k+1}^*(TM) \to \mathbb{R}^r$$

Furthermore, since both θ_{k+1} and $d\Psi_k$ are smooth maps that are linear in the fibres, it follows that the same holds for $\tilde{\mathscr{d}}_{k+1}\Psi$.

The fact that D is a set of totally independent functions w.r.t. $\mathcal{S}$ at $j_x^\infty s$, then means that the map $\tilde{\mathscr{d}}_{k+1}\Psi$ restricted to the subbundle $\mathcal{S}_{k+1} \subset p_{k+1}^*(TM)$ is of rank r at $j_x^k s$. Since $\mathcal{S}_{k+1}$ is a smooth subbundle, it follows that there is an open $U_{k+1} \subset J^{k+1}(E)$ around $j_x^{k+1} s$, such that $\tilde{\mathscr{d}}_{k+1}\Psi$ restricted to the subbundle $\mathcal{S}_{k+1} \subset p_{k+1}^*(TM)$ is of rank r on U_{k+1}. Thus D is totally independent w.r.t. $\mathcal{S}$ on $a_{k+1}^{-1}(U_{k+1})$. □

5.4 Thom-Boardman singularities

As mentioned at the start of this chapter, in his 1967 paper [2], Boardman proved the following theorem.

Theorem 5.21. *For each sequence* $\mathscr{I} = (j_1, j_2, \dots, j_k)$ *of integers, there exists a pf-submanifold* $\Sigma^I_{TB} \subset J^\infty(M,N)$ *of level k, such that*

(i) For $\mathscr{I} = (j)$ *and* $f \in C^\infty(M,N)$*, we get* $\Sigma^j_{TB}(f) = \Sigma^j_{rank}(f)$ *as defined in section 4.1.*
(ii) for any smooth map $f : M \to N$ *with* $j^\infty f \pitchfork \Sigma^I$*, we get for any integer j*

$$\Sigma^{I,j}_{TB}(f) = \Sigma^j_{TB}\Big(f|_{\Sigma^j_{TB}}(f)\Big)$$

He also gives an invariant definition for these Σ^I's using a so-called *jacobian extension.* However, in this section we will focus on a different definition of these Σ^I's. We will then use some results from Boardman's paper to show that the definition we consider here actually describes the same sets (and thus manifolds) that Boardman described.

5.4.1 Alternative description of Σ^j_{rank}

Note that the characterisation in (i) of theorem 5.21 tells us immediately that $\Sigma^j_{\text{TB}} = \Sigma^j_{\text{rank}}$. In this subsection we will consider a characterisation of Σ^j_{rank}, that is based on Boardman's characterisation of the (higher) singularities.[3] It immediately follows from Boardman's paper that this construction gives $\Sigma^j_{\text{rank}} = \Sigma^j_{\text{TB}}$. However, we will reprove this result, since it gives us some insight into Boardman's construction and characterisation of the singularities.

Recall that we defined Σ^j_{rank} as

$$\Sigma^j_{\text{rank}} := \{ j^1_x f \in J^1(M,N) \mid \dim\big(\ker\big(df\big)_x\big) = j \}$$

Let us now define the following set $A_N \subset C^\infty(J^\infty(M,N))$ that plays an important role in Boardman's paper.

Definition 5.22. *For any open* $U \subset J^\infty(M,N)$*, we define the set* $A_N(U) \subset C^\infty(U)$ *to consist of those functions* $\varphi \in C^\infty(U)$ *for which there exists a function* $\varphi_N \in C^\infty(\pi_N(U))$*, such that the following diagram commutes*

$$\begin{array}{ccc} U & \xrightarrow{\ \varphi\ } & \mathbb{R} \\ {\scriptstyle \pi_N}\big\downarrow & \nearrow_{\varphi_N} & \\ \pi_N(U) & & \end{array}$$

where $\pi_N : J^\infty(M,N) \to N$ *denotes the projection.*

Remark 5.23. Note that any map φ for which there exists a smooth φ_N such that the diagram above commutes is immediately a smooth pf-function of level 0. ⋄

The following lemma and its proof are based on [2, lemma 2.22].

[3] This characterisation will be discussed later in this section, specifically it is given in theorem 5.35.

Lemma 5.24. *Let $\mathcal{V} = \left(dj^\infty f\right)_x(V) \in \mathcal{C}_\infty|_{j^\infty_x f}$ and let $U_N \subset N$ be an open coordinate neighbourhood of x. We denote by $U := \pi_N^{-1}(U_N) \subset J^\infty(M,N)$ the corresponding open around $j^\infty_x f$. Then the following are equivalent*

(i) $\mathcal{V} \in \ker^{\mathcal{C}_\infty}_{j^\infty_x f}(A_N(U))$
(ii) $V \in \ker\left((df)_x\right)$

Proof. Note that $\mathcal{V} \in \ker^{\mathcal{C}_\infty}_{j^\infty_x f}(A_N(U))$ by definition means that $\mathcal{V}(\varphi) = 0$ for all $\varphi \in A_N(U)$.

For $\varphi \in A_N(U)$, we can calculate $\mathcal{V}(\varphi)$ as follows. Let $\varphi_0 \in C^\infty(a_0(U))$, be the map defined as $\varphi_0 = \varphi_N \circ \mathrm{pr}_N$. We use here that $a_0(U) \cong M \times U_N$. Then it follows from the defintion of $\mathcal{V}(\varphi)$ that

$$\begin{aligned}\mathcal{V}(\varphi) &= \mathcal{V}_0\left(\varphi_0\right)\\ &= \left(d\varphi_0\right)_{j^0_x f}\left(\left(dj^0 f\right)_x(V)\right)\end{aligned}$$

By the chain rule we can rewrite this as

$$\begin{aligned}\mathcal{V}(\varphi) &= \left(d\left(\varphi_0 \circ j^0 f\right)\right)_x(V)\\ &= \left(d\left(\varphi_N \circ f\right)\right)_x(V)\end{aligned}$$

Once again applying the chain rule then immediately shows the equivalence between conditions (i) and (ii). □

Thus the kernel of the set $A_N(U)$ at the jet $j^\infty_x f$ tells us exactly what the kernel of $(df)_x$ looks like. If we combine this lemma with proposition 1.60, we get the following corollary.

Corollary 5.25. *The set Σ^j_{rank} as defined at the beginning of this chapter can be described as*

$$\Sigma^j_{rank} = \left\{ j^\infty_x f \in J^\infty(M,N) \mid \dim\left(\ker^{\mathcal{C}_\infty}_{j^\infty_x f}(A_N(U))\right) = j \right\}$$

where U is any open around $j^\infty_x f$.

Proof. Note that from lemma 5.24 and proposition 1.60 the statement follows immediately for any U for which $\pi_N(U)$ is contained in a trivial open of N. Since the $\mathcal{C}_\infty$ kernel of a set is a local property, the general statement follows. □

Remark 5.26. Note that the kernel

$$\ker^{\mathcal{C}_\infty}(A_N(U))\Big|_{\Sigma^j_{\mathrm{rank}}} \subset \mathcal{C}_\infty|_{\Sigma^j_{\mathrm{rank}} \cap U}$$

can locally be extended to a subbundle of $\mathcal{C}_\infty|_U$. To see this, recall the pull-back bundle $p_1^*(TM) \to J^1(M,N)$ as defined in remark 1.61 and let $p_1^*(TM)|_{\Sigma^j_{\mathrm{rank}}}$ denote its restriction to Σ^j_{rank}. Then consider the vector bundle morphism

$$\begin{aligned}H_j : p_1^*(TM)|_{\Sigma^j_{\mathrm{rank}}} &\mapsto TN\\ \left(j^1_x f, V\right) &\mapsto (df)_x(V)\end{aligned}$$

Note that $\ker(H_j)$ is of constant rank j, hence it follows that $\ker\left(H_j\right) \subset p_1^*(TM)|_{\Sigma^j_{\mathrm{rank}}}$ is a subbundle of rank j. Let $j^1_x f \in \Sigma^j_{\mathrm{rank}}$ and let $U_1 \subset J^1(M,N)$ be an open coordinate neighbourhood of $j^1_x f$ in which $\Sigma^j_{\text{rank}} \cong \mathbb{R}^\alpha \times \{0\}$. Then on U_1 we can extend $\ker(H_j)$ to some rank j subbundle $\mathcal{S}^j_1 \subset p^*_1(TM)\big|_U$. This defines the rank j subbundle $\mathcal{S}^j \subset \mathcal{C}_\infty|_U$ on $U := a^{-1}_1(U_1) \subset J^\infty(M,N)$. Note that this subbundle is indeed a local extension of $\ker^{\mathcal{C}_\infty}_{j^\infty_x f}(A_N)\Big|_{\Sigma^j_{\text{rank}}}$. ◇

This means that lemma 5.24 also has the following corollary.

Corollary 5.27. *Let $U_1 \subset J^1(M,N)$ be an open as described in remark 5.26 with $U := a^{-1}_1(U_1)$ and let $\mathcal{S}^j \subset \mathcal{C}_\infty|_U$ be the corresponding smooth subbundle. Furthermore, assume that U is such that the dimension of $\ker^{\mathcal{C}_\infty}(A_N(U))$ is at most j on all of U. Then $\Sigma^j_{rank} \cap U$ can be described as*

$$\Sigma^j_{rank} \cap U = \text{null}\left(\Gamma_{\mathcal{S}^j}(A_N(U))\right)$$

In fact, we can give such a description of the singularities Σ^j_{rank} without specifying $\mathcal{S}_j$ beforehand as follows.

Lemma 5.28. *A jet $j^\infty_x f \in J^\infty(M,N)$ is an element of the singularity Σ^j_{rank} if and only if there exist*

1. *an open U around $j^\infty_x f$.*
2. *a set of functions $C := \{\phi_1, \ldots, \phi_{m-j}\} \subset A_N(U)$ that is totally independent on U.*
3. *a rank j subbundle $\mathcal{S} \subset \mathcal{C}_\infty|_U$.*

such that the following hold

(i) $\Gamma_{\mathcal{S}}(C) = \{0\}$
(ii) $j^\infty_x f \in \text{null}(\Gamma_{\mathcal{S}}(A_N(U)))$

Proof. We will first consider the only if statement. Note by corollary 5.19, lemma 5.20 and remark 5.26 it follows that we can find some U such that $\mathcal{S}^j$ is defined on U and U satisfies the conditions of corollary 5.27. Since $\mathcal{S}_{j^\infty_x f} = \ker^{\mathcal{C}_\infty}_{j^\infty_x f}(A_N(U))$, it follows from corollary 5.19 and lemma 5.20 that there exists a set $C \subset A_N(U)$ consisting of $m-j$ elements, that on some perhaps smaller open $U' \subset U$ is totally independent. Since $\mathcal{S}$ is of rank j, this means exactly that $\Gamma_{\mathcal{S}}(C) = \{0\}$. Condition (ii) is then satisfied by corollary 5.27.

Now assume that U, $\mathcal{S}$ and C exist and satisfy all given conditions for $j^\infty_x f \in \text{null}(\Gamma_S(A_N(U)))$. By the existence of C and corollary 5.19 it follows that $\dim\left(\ker^{\mathcal{C}_\infty}_{j^\infty_x f}(A_N(U))\right) \leq j$. On the other hand, by $j^\infty_x f \in \text{null}(\Gamma_S(A_N(U)))$ it follows that $\dim\left(\ker^{\mathcal{C}_\infty}_{j^\infty_x f}(A_N(U))\right) \geq j$ since $\mathcal{S}$ is of rank j. Thus it follows from corollary 5.25 that $j^\infty_x f \in \Sigma^j_{\text{rank}}$. □

Remark 5.29. Note that in the lemma above, since the set C is totally independent, we get that

$$\dim\left(\ker^{\mathcal{C}_\infty}_{j^\infty_x f}(A_N(U))\right) \leq j$$

By condition (ii) and remark 5.14 it then follows that $\mathcal{S}|_{j^\infty_x f} = \ker^{\mathcal{C}_\infty}_{j^\infty_x f}(A_N(U))$. Also, note that since C is totally independent, we get that

$$\mathcal{S} = \ker^{\mathcal{C}_\infty}(C)$$

◇

Remark 5.30. Note that the existence of the set C in lemma 5.28 means that $(df)_x$ is of rank at least j. Condition (ii) in that lemma then ensures that the rank of $(df)_x$ is not larger than j. ⋄

5.4.2 Higher Thom-Boardman singularities

In his paper [2], Boardman gave a characterisation of the Thom-Boardman singularities that generalises lemma 5.28. He makes use of the following definition.

Definition 5.31. *Let $\mathscr{J} = (j_1, \dots, j_k)$ be a sequence of non-increasing, non-negative integers and $U \subset J^\infty(M,N)$ be an open subset. A* ***special $\mathscr{J}$-flag*** *$\mathcal{F}$ over U is a flag of subbundles*

$$\mathcal{C}_\infty|_U \supset \mathcal{S}_1 \supset \mathcal{S}_2 \supset \cdots \supset \mathcal{S}_k$$

where each $\mathcal{S}_l$ is of dimension j_l, such that there exists a system of subsets $C_l \subset C^\infty(U)$, satisfying the following conditions:

(i) C_l contains exactly $j_{l-1} - j_l$ elements of level $l-1$. Here we use $j_0 = m$.
(ii) The set $C_1 \cup \cdots \cup C_l$ is totally independent on U for all $1 \le l \le k$.
(iii) $\Gamma_{\mathcal{S}_l}(C_l) = \{0\}$ for all $l \ge 1$.
(iv) $C_1 \subset A_N(U)$ and

$$C_l \subset \Gamma_{\mathcal{S}_{l-1}} \dots \Gamma_{\mathcal{S}_1}(A_N(U))$$

for all $1 < l \le k$.

Remark 5.32. Let $\mathscr{J} = (j_1, \dots, j_k)$ be a sequence with special $\mathscr{J}$-flag $\mathscr{F}$. Let $\mathscr{J}'$ denote the shorter sequence $(j_1, \dots, j_{k-1})$. Then we can obtain a special $\mathscr{J}'$-flag by simply forgetting $\mathcal{S}_k$ and C_k. We will denote this special $\mathscr{J}'$-flag by $\mathscr{F}'$. ⋄

Note that by lemma 5.18 we can rephrase condition (ii) as C_l being totally independent on U with respect to $\mathcal{S}_{l-1}$. Also, since C_l is a totally independent set and $\mathcal{S}_l$ is contained in the Cartan distribution, we can rephrase condition (iii) as $\mathcal{S}_l = \ker^{\mathcal{C}_\infty}(C_l) \cap \mathcal{S}_{l-1}$. These observations yield the following alternative characterisation of special flags.

Lemma 5.33. *Let $\mathscr{J}$ be a sequence of non-increasing[4], non-negative integers and $U \subset J^\infty(M,N)$. Furthermore, let $\mathscr{F}'$ be a special $\mathscr{J}'$-flag over some open U' containing U.*

Assume that there exists some $C_k \subset \Gamma_{\mathcal{S}_{k-1}} \dots \Gamma_{\mathcal{S}_1}(A_N(U))$ that contains $j_{k-1} - j_k$ elements and is totally independent with respect to $\mathcal{S}_{n-1}$ on some open U. Then extending $\mathscr{F}'$ by the set C_k and the subbundle $\mathcal{S}_k|_U = \ker^{\mathcal{C}_\infty}(C_k) \cap \mathcal{S}_{k-1}$ yields a special $\mathscr{J}$-flag over U.

In [2], Boardman also defines the following ideals.

Definition 5.34. *Let $\mathscr{F}$ be a special $\mathscr{J}$-flag over U with $\mathscr{J} = (j_1, \dots, j_k)$. Then the ideal $I_\mathscr{F} \subset C^\infty(U)$ is defined inductively as follows*

$$I_\mathscr{F} = \begin{cases} \Gamma_{\mathcal{S}_1}(A_N(U)) & \text{if } k = 1 \\ I_{\mathscr{F}'} + \Gamma_{\mathcal{S}_k}(I_{\mathscr{F}'}) & \text{else} \end{cases}$$

The definitions given above are used by Boardman to state and prove the following theorem in his paper [2, Theorem 3.11].

[4]Note that if the sequence of $\mathscr{J}$ is somewhere increasing, then there exists no special $\mathscr{J}$-flag.

Theorem 5.35. *The pf-submanifolds $\Sigma^{\mathcal{J}}_{TB}$ from theorem 5.21 are characterized as follows. A jet $j^\infty_x f \in J^\infty(M,N)$ is contained in $\Sigma^{\mathcal{J}}_{TB}$ if and only if there exists an open $U \subset J^\infty(M,N)$ around $j^\infty_x f$, such that:*

(i) there exists a special $\mathcal{J}$-flag $\mathcal{F}$ over U
(ii) $j^\infty_x f \in \mathrm{null}(I_{\mathcal{F}})$

Remark 5.36. Note that for any special $\mathcal{J}$-flag $\mathcal{F}$, it follows immediately from definition 5.34 that $I_{\mathcal{F}'} \subset I_{\mathcal{F}}$. Therefore, it also follows that $\Sigma^{\mathcal{J}}_{\mathrm{TB}} \subset \Sigma^{\mathcal{J}'}_{\mathrm{TB}}$. ◇

Remark 5.37. Note that lemma 5.28 is the special case of theorem 5.35 for sequences $\mathcal{J}$ of length 1. ◇

Example 5.38. Recall that we saw an example of the Thom-Boardman singularities of the map

$$\begin{aligned} f_0 &: \mathbb{R}^2 \to \mathbb{R}^2 \\ x &\mapsto \left(\frac{1}{3}x_1^3 + x_1x_2^2 - x_1, (x_2-1)^2\right) \end{aligned}$$

in example 5.1. Let us consider the point $(0,-1)$ and what $\mathcal{J}$-flags we get around it from theorem 5.35.

First of all, let us define the function

$$\begin{aligned} \varphi_1 &: J^\infty\left(\mathbb{R}^2,\mathbb{R}^2\right) \to \mathbb{R} \\ j^\infty_x f &\mapsto f_2(x) \end{aligned}$$

where f_2 is the second component of f. I.e., φ_1 maps jets onto their y_2 component. Then note that $\varphi_1 \circ j^\infty f = (x_2-1)^2$, which means that $\tilde{\partial_{x_2}}\big|_{j^\infty_x f_0}(\varphi_1) = 2(x_2-1)$. Thus $\{\varphi_1\}$ is totally independent at $j^\infty_x f_0$ if $x_2 \neq 1$. In particular, it is totally independent at $j^\infty_{(0,-1)} f_0$. Thus, if we set $C_1 = \{\varphi_1\}$ and $\mathcal{S} = \ker^{C^\infty}(C_1)$, together they define a special (1)-flag on some U around $j^\infty_{(0,-1)} f_0$.

Note that around $(0,-1)$, $\mathcal{S}|_{j^\infty_x f_0} = \left(d j^\infty f_0\right)_x\left(\partial_{x_1}\big|_x\right)$. Then from theorem 5.35 we know that around $(0,-1)$, $\Sigma^1_{\mathrm{TB}}(f_0)$ is given by the set of points at which $\partial_{x_1}\left(\varphi \circ j^\infty f_0\right)$ vanishes for all $\varphi \in A_N(U)$. Thus $\Sigma^1_{\mathrm{TB}}(f_0)$ is given by exactly those points x where $\partial_{x_1}|_x \subset \ker(df)_x$, which is indeed what we saw in example 5.1.

Furthermore, around $(0,-1)$, the singularity $\Sigma^{1,1}_{\mathrm{TB}}(f_0)$ is given by those $x \in \Sigma^{1,1}_{\mathrm{TB}}(f_0)$ for which

$$\partial_{x_1}\left(\partial_{x_1}\left(\varphi \circ j^\infty f_0\right)\right)(x) = 0$$

for all $\varphi \in A_N(U)$. If we take φ to be the coordinate functions y_1 and y_2, this gives

$$0 = \begin{cases} \frac{\partial^2 f_1}{\partial x_1^2}(x) = 2x_1 \\ \frac{\partial^2 f_2}{\partial x_1^2}(x) = 0 \end{cases}$$

Thus around $(0,-1)$, the only element that can be contained in $\Sigma^{1,1}_{\mathrm{TB}}(f_0)$ is $(0,-1)$ itself. Note that since any function of $A_N(U)$ only depends on the variables y_1 and y_2, it follows that indeed $(0,-1) \in \Sigma^{1,1}_{\mathrm{TB}}(f_0)$, which is exactly what we saw in example 5.1. △

5.4.3 Inductive construction of higher Thom-Boardman singularities.

The rest of this section will be devoted to showing that that the inductive process of constructing higher singularities described in section 5.1 yields the Thom-Boardman singularities. More specifically, we want to prove the following lemma.

Lemma 5.39. *For any $\mathscr{J} = (j_1, \ldots, j_n)$ with $n \geq 2$, we get*

$$\Sigma_{TB}^{\mathscr{J}} = \left(\Sigma_{TB}^{\mathscr{J}'}\right)^{j_n}$$

where the right-hand side of this equation is defined in definition 5.2.

To prove this lemma, we need to examine these special flags and how one should interpret their meaning. To do so, we first need another result that Boardman deduced in the proof of theorem 5.21 [2, p.47].

Lemma 5.40. *Let $\mathscr{J}$ be a sequence of indices and $\mathscr{F}$ the special $\mathscr{J}$-flag on U around $j_x^\infty f \in \Sigma_{TB}^{\mathscr{J}}$ as described in theorem 5.35. Then on some, perhaps smaller open $U' \subset U$ around $j_x^\infty f$, the ideal $I_{\mathscr{F}}$ is generated by a finite amount of functions $\Phi_1, \ldots, \Phi_\nu \in I_{\mathscr{F}}$ of level k, such that the map*

$$\Phi = (\Phi_1, \ldots, \Phi_\nu) \to \mathbb{R}^{\nu_{\mathscr{J}}}$$

is a submersion.

While we will not show a proof of this lemma, we wíll prove a result in chapter 6 (lemma 6.15) that will also imply this lemma for $j_l = 1$ for all l. For now, however, this lemma gives a description of the tangent spaces of the submanifold $a_k\left(\Sigma_{\text{TB}}^j\right) \subset J^k(M,N)$ and thus also of the tangent space of Σ_{TB}^j. In particular, we get the following result about its intersection with the Cartan distribution.

Corollary 5.41. *Let $j_x^\infty f \in \Sigma_{TB}^{\mathscr{J}}$ and let $\mathscr{F}$ denote a special $\mathscr{J}$-flag over an open neighbourhood U of $j_x^\infty f$ that satisfies the conditions of theorem 5.35. Then*

$$\ker_{j_x^\infty f}^{\mathcal{C}_\infty}(I_{\mathscr{F}}) = \mathcal{C}_\infty|_{j_x^\infty f} \cap T_{j_x^\infty f}\Sigma_{TB}^{\mathscr{J}}$$

We can use this result to deduce how the subbundles of the flags relate to the singularities $\Sigma_{\text{TB}}^{\mathscr{J}}$.

Corollary 5.42. *Let $j_x^\infty f \in \Sigma_{TB}^{\mathscr{J}}$ and let $\mathscr{F}$ denote a special $\mathscr{J}$-flag over an open neighbourhood U of $j_x^\infty f$, such that $j_x^\infty f \in \text{null}(I_{\mathscr{F}})$. Then*

$$(\mathcal{S}_k)_{j_x^\infty f} = T_{j_x^\infty f}\Sigma_{TB}^{\mathscr{J}'} \cap \ker_{j_x^\infty f}^{\mathcal{C}_\infty}(A_N(U))$$

Proof. Note that from condition (iii) of definition 5.31 and the fact that $C = C_1 \cup \cdots \cup C_k$ is a totally independent set, it follows that $\mathcal{S}_k = \ker^{\mathcal{C}_\infty}(C)$. Also note that since $C \setminus C_1 \subset I_{\mathscr{F}'}$, it follows that

$$\ker_{j_x^\infty f}^{\mathcal{C}_\infty}(I_{\mathscr{F}'}) \subset \ker_{j_x^\infty f}^{\mathcal{C}_\infty}(C \setminus C_1)$$

Furthermore, from remark 5.29, it follows that $\ker_{j_x^\infty f}^{\mathcal{C}_\infty}(C_1) = \mathcal{S}_1|_{j_x^\infty f} = \ker_{j_x^\infty f}^{\mathcal{C}_\infty}(A_N(U))$. Thus it follows that

$$\ker_{j_x^\infty f}^{\mathcal{C}_\infty}(I_{\mathscr{F}'}) \cap \ker_{j_x^\infty f}^{\mathcal{C}_\infty}(A_N(U)) \subset \ker_{j_x^\infty f}^{\mathcal{C}_\infty}(C) = \mathcal{S}_k|_{j_x^\infty f}$$

Also, since we know that the ideal $I_{\mathscr{F}} = I'_{\mathscr{F}} + \Gamma_{\mathcal{S}_k}(I_{\mathscr{F}'})$ vanishes at $j_x^\infty f$, it follows that $\mathcal{S}_k|_{j_x^\infty f} \subset \ker^{\mathcal{C}_\infty}_{j_x^\infty f}(I_{\mathscr{F}'})$. Furthermore, from remark 5.29 we know that $\mathcal{S}_k|_{j_x^\infty f} \subset \mathcal{S}_1|_{j_x^\infty f} = \ker^{\mathcal{C}_\infty}_{j_x^\infty f}(A_N(U))$. Hence we get that

$$\mathcal{S}_k|_{j_x^\infty f} = \ker^{\mathcal{C}_\infty}_{j_x^\infty f}(I_{\mathscr{F}'}) \cap \ker^{\mathcal{C}_\infty}_{j_x^\infty f}(A_N(U))$$

The result now follows from corollary 5.41. □

With these results we are now ready to prove lemma 5.39

Proof of lemma 5.39. Note that can we rewrite the definition of $\left(\Sigma^{\mathscr{J}'}_{\mathrm{TB}}\right)^{j_k}$ in terms of the Cartan distribution as follows. From lemma 5.24 it follows that $j_x^\infty f \in \left(\Sigma^{\mathscr{J}'}_{\mathrm{TB}}\right)^{j_k}$ if and only if there exists some small enough open U around $j_x^\infty f$ such that

$$\ker^{\mathcal{C}_\infty}_{j_x^\infty f}(A_N(U)) \cap T_{j_x^\infty f}\Sigma^{\mathscr{J}'}_{\mathrm{TB}} \subset \mathcal{C}_\infty|_{j_x^\infty f}$$

is a j_k-dimensional subspace.

By corollary 5.41 and remark 5.29 we can rephrase this as

$$\mathcal{S}_1|_{j_x^\infty f} \cap \ker^{\mathcal{C}_\infty}_{j_x^\infty f}(I_{\mathscr{F}'}) \subset \mathcal{C}_\infty|_{j_x^\infty f} \tag{5.2}$$

is a j_k-dimensional subspace.

Also, from corollary 5.42 it follows that given that $k \geq 3$, $\ker^{\mathcal{C}_\infty}_{j_x^\infty f}(I_{\mathscr{F}'}) \cap \mathcal{S}_1|_{j_x^\infty f} \subset (\mathcal{S}_{k-1})|_{j_x^\infty f}$, thus it follows that for any $k \geq 2$,

$$\mathcal{S}_1|_{j_x^\infty f} \cap \ker^{\mathcal{C}_\infty}_{j_x^\infty f}(I_{\mathscr{F}'}) \subset (\mathcal{S}_{k-1})|_{j_x^\infty f}$$

and thus

$$\mathcal{S}_1|_{j_x^\infty f} \cap \ker^{\mathcal{C}_\infty}_{j_x^\infty f}(I_{\mathscr{F}'}) = (\mathcal{S}_{k-1})|_{j_x^\infty f} \cap \ker^{\mathcal{C}_\infty}_{j_x^\infty f}(I_{\mathscr{F}'}) \tag{5.3}$$

Let us now examine what it means for a subbundle $\mathcal{S}_k \subset \mathcal{S}_{k-1}$ and set of functions C_k to extend $\mathscr{F}'$ to a special $\mathscr{J}$-flag $\mathscr{F}$ over some open U' around $j_x^\infty f$. From lemma 5.33 it follows that it suffices to construct the set C_k. By lemma 5.18, corollary 5.19 and lemma 5.20 we can deduce that there exist such a $U' \subset U$ and $C_k \subset C^\infty(U')$ if and only if

$$\dim\left(\ker^{\mathcal{C}_\infty}_{j_x^\infty f}\left(\Gamma_{\mathcal{S}_{k-1}} \dots \Gamma_{\mathcal{S}_1}(A_N(U))\right) \cap (\mathcal{S}_{k-1})|_{j_x^\infty f}\right) \leq j_k$$

Note that it follows from the fact that $j_x^\infty f \in \mathrm{null}(I_{\mathscr{F}'})$ that

$$\ker^{\mathcal{C}_\infty}_{j_x^\infty f}\left(\Gamma_{\mathcal{S}_{k-1}} \dots \Gamma_{\mathcal{S}_1}(A_N(U))\right) \cap (\mathcal{S}_{k-1})|_{j_x^\infty f} = \ker^{\mathcal{C}_\infty}_{j_x^\infty f}(I_{\mathscr{F}'}) \cap (\mathcal{S}_{k-1})|_{j_x^\infty f}$$

Thus we see that we can extend $\mathscr{F}'$ to a $\mathscr{J}$-flag $\mathscr{F}$ if and only if $\ker^{\mathcal{C}_\infty}_{j_x^\infty f}(I_{\mathscr{F}'}) \cap (\mathcal{S}_{k-1})|_{j_x^\infty f}$ is a subspace of $\mathcal{C}_\infty|_{j_x^\infty f}$ of dimension $\leq j_k$. The condition $j_x^\infty f \in \mathrm{null}(I_{\mathscr{F}})$ is then equivalent to

$$\mathcal{S}_k|_{j_x^\infty f} \subset \ker^{\mathcal{C}_\infty}_{j_x^\infty f}\left(\Gamma_{\mathcal{S}_{k-1}} \dots \Gamma_{\mathcal{S}_1}(A_N(U))\right) \cap (\mathcal{S}_{k-1})|_{j_x^\infty f} = \ker^{\mathcal{C}_\infty}_{j_x^\infty f}(I_{\mathscr{F}'}) \cap (\mathcal{S}_{k-1})|_{j_x^\infty f}$$

Thus by theorem 5.35 it follows that $j_x^\infty f \in \Sigma_{\mathrm{TB}}^{\mathscr{I}}$ if and only if $\ker_{j_x^\infty f}^{\mathscr{C}_\infty}(I_{\mathscr{F}'}) \cap (\mathscr{S}_{k-1})|_{j_x^\infty f}$ is a j_k-dimensional subspace of $\mathscr{C}_\infty|_{j_x^\infty f}$. By equations (5.2) and (5.3) this means exactly that

$$\Sigma_{\mathrm{TB}}^{\mathscr{I}} = \left(\Sigma_{\mathrm{TB}}^{\mathscr{I}'}\right)^{j_n}$$

□

Note that this means that the higher Thom-Boardman singularities can be described by the dimension of the intersection of the lift of $\ker(df)_x$ to the Cartan distribution with the tangent space of the previous singularity. This description suggests that we can generalise this notition of lifting singularities by considering other subbundles of the Cartan distribution and how they relate to the tangent space of a given pf-submanifold $\Sigma \subset J^\infty(E)$. We will consider this in the next chapter.

6 Hierarchies of singularities

In chapter 7 we will discuss the full removal of singularities proof for the differential relation of immersions into Euclidean space that Gromov described in his book [13]. As discussed in chapter 3, we want to reduce the π_0-surjectivity proof to proposition 3.23. To do so we will need the following lemma.

Lemma 6.1. *Let M be a manifold and let $E \subset J^1(M,\mathbb{R}^n)$ be the open subset*

$$E := \left\{ j^1_x f \in J^1\left(M,\mathbb{R}^{n-1}\right) \mid \dim\left(\ker\left((df)_x\right)\right) \leq 1 \right\}$$

Then E is an open subbundle. Furthermore, there exists a stratification[1] $\bigsqcup_i \Sigma^1_i$ of $\Sigma^1 \subset J^1(M,\mathbb{R}^{n-1})$, such that the set of sections $\mathscr{G} \subset \Gamma(E)$ defined as

$$\mathscr{G} := \left\{ s \in \Gamma(E) \mid \ker(z_s) \text{ is no-where tangent to the stratification of } \Sigma^1 \right\}$$

is a dense and open subset of $\Gamma(E)$.

As discussed at the start of chapter 5, we can do this by constructing hierarchies of singularities similar to the Thom-Boardman singularities. However, instead of considering the restriction of a map $f : M \to N$ to a previous singularity, we now want to consider the restriction of sections $s \in \Gamma\left(J^1(M,N)\right)$. In particular, we want to study the rank of the z-coordinate of s after restricting to the previous singularity. More specifically, we will prove the following theorem.

Theorem 6.2. *Let $\mathscr{I}_k$ denote the sequence consisting of k times the number 1. Then for any $k \geq 1$, there exists a pf-submanifold $\Sigma^{\mathscr{I}_k,\ker(z)} \subset J^\infty\left(J^1(M,N)\right)$ of level $k-1$, such that*

(i) $\Sigma^{1,\ker(z)} = (a_0)^{-1}\left(\Sigma^1_{rank}\right)$, *where* $\Sigma^1_{rank} \subset J^1(M,N)$ *as defined in section 5.1.*

(ii) *for any* $s \in \Gamma\left(J^1(M,N)\right)$ *with* $j^\infty s \pitchfork \Sigma^{\mathscr{I}_k,\ker(z)}$ *we get*

$$\Sigma^{\mathscr{I}_{k+1},\ker(z)}(s) = \left\{ x \in M \mid \ker\left(z_s(x)\right) \subset T_x \Sigma^{\mathscr{I}_k,\ker(z)}(s) \right\}$$

Throughout this chapter we will use the notation $\pi : E \to M$ for a general fibre bundle, with n-dimensional fibre. M and N will also be standard notation for manifolds of dimension m and n respectively. Furthermore $\Sigma \subset E$ will be a submanifold of codimension q. The letter U will be used for an arbitrary open subset of $J^\infty(E)$, unless specified otherwise. $a_\bullet$ will be used for the pf-atlas of $J^\infty(E)$ and $\tilde{a}_\bullet$ for the pf-atlas of $TJ^\infty(E)$.

The concepts and results discussed in this chapter are inspired by the Thom-Boardman singularities and results about them in [2] as discussed in the previous chapter. However, the specific concepts and results discussed in this chapter have not been introduced in literature before and thus form an original contribution of this thesis.

[1] For the definition and properties of stratifications, see appendix A.

6.1 $(j, \mathcal{S})$-lift of a pf-submanifold of $J^\infty(E)$

In this chapter we want to construct a $(\mathscr{J}_k, \mathcal{S})$-lift of a subbundle $\Sigma_0 \subset E$. Here $\mathcal{S} \subset \mathcal{C}_\infty|_\Sigma$ is a smooth subbundle, where we denote $\Sigma := a_0^{-1}(\Sigma_0)$. To do that, we first need to define what a $(j, \mathcal{S})$ lift is for some level k pf-submanifold $W \subset J^\infty(E)$.

In this section we will work with the following data.

- $\Sigma_0 \subset E$ is a subbundle and $\Sigma \subset J^\infty(E)$ is the induced pf-submanifold.
- $\mathcal{S} \subset \mathcal{C}_\infty|_\Sigma$ is a smooth subbundle of rank j_0.
- $W \subset J^\infty(E)$ is a pf-submanifold of some level k that is contained in Σ. In particular, $\mathcal{S}$ is defined on W.

Note that the following definition is very similar to definition 5.2.

Definition 6.3. *Let* $0 \le j \le j_0$ *be an integer. Then we define the* ***$(j, \mathcal{S})$-lift of W*** *as*

$$(W)^{j,\mathcal{S}} = \left\{ j_x^\infty s \in W \mid \dim\left(\mathcal{S}|_{j_x^\infty s} \cap T_{j_x^\infty s} W\right) = j \right\}$$

Example 6.4. Let us consider $\Sigma = \Sigma^1_{\text{rank}} \subset J^1(M, N)$ and define

$$\mathcal{S} = \ker(z) = \left\{ \left(d j^\infty s\right)_x (V) \mid V \in \ker(z_s(x)) \right\}$$

From the discussion in remark 5.26, it follows that this is a level 0 smooth subbundle.[2] For any level k pf-submanifold $W \subset j^\infty(E)$ that is contained in Σ^1, we then get

$$(W)^{j,\mathcal{S}} = \left\{ j_x^\infty s \in W \mid \dim\left(\left(d j^\infty s\right)_x (\ker(z_s(x)) \cap T_{j_x^\infty s} W\right) = j \right\}$$

△

Since we call $(W)^{j,\mathcal{S}}$ a *lift*, one could expect that as W is defined at the level k, its $(j, \mathcal{S})$-lift is defined at the level $k+1$. This is indeed the case if we impose some conditions on $\mathcal{S}$.

Proposition 6.5. *Assume that* $\mathcal{S} \subset \mathcal{C}_\infty|_\Sigma$ *is a smooth subbundle of level* $k+1$. *Then the set* $(W)^{j,\mathcal{S}}$ *is defined at the level* $k+1$.

Proof. Let $s, s' \in \Gamma(E)$ with $j_x^{k+1} s = j_x^{k+1} s'$. Since W is of level k, it follows that $j_x^\infty s \in W$ if and only if $j_x^\infty s' \in W$. Furthermore, since $\mathcal{S}$ is of level $k+1$, it follows from proposition 1.59 that $\tilde{a}_k\left(\mathcal{S}|_{j_x^\infty s}\right) = \tilde{a}_k\left(\mathcal{S}|_{j_x^\infty s'}\right)$. Note that we can rewrite the definition of $(W)^{j,\mathcal{S}}$ as follows

$$(W)^{j,\mathcal{S}} = \left\{ j_x^\infty s \in W \mid \dim\left(\tilde{a}_k\left(\mathcal{S}|_{j_x^\infty s}\right) \cap T_{j_x^k s} a_k(W)\right) = j \right\}$$

thus this is indeed a set defined at the level k. □

While we do not necessarily want to define a completely equivalent notion of the special flags as Boardman did, we do want to focus on the last elements of those flags, i.e. the ones that define the next singularity. Note that in the previous chapter we saw that these special flags defined local vanishing ideals for the singularities. Then in the next step, the $\mathcal{C}_\infty$-kernel of these vanishing ideals could be used to construct the next flag (see the proof of lemma 5.39). We can use this idea to get a similar characterisation of the set $(W)^{j,\mathcal{S}}$.

[2] This is level 0 since we are working in the infinite jet bundle $J^\infty\left(J^1(M, N)\right)$.

Remark 6.6. First of all note that from the submersion theorem, it follows that for any submanifold $a_k(W) \subset J^k(E)$ of codimension q, locally we can find q functions $\Phi_1, \ldots, \Phi_q \in C^\infty(U_k)$, such that

(i) the map $\Phi = (\Phi_1, \ldots, \Phi_q) : U_k \to \mathbb{R}^q$ is a submersion.
(ii) on U_k, the vanishing set of the ideal $I_\Phi \subset C^\infty(U_k)$ is $a_k(W)$.

Let $U = a_k^{-1}(U_k)$ be the corresponding open in $J^\infty(E)$. Also, let $\tilde{I_\Phi}$ denote the ideal of $C^\infty(U)$ generated by lifts of elements of I_Φ to $C^\infty(U)$. Then on U, W is the vanishing set of $\tilde{I_\Phi}$. ⋄

Lemma 6.7. *Let $j_x^\infty s \in W$ and let U be an open neighbourhood of $j_x^\infty s$ as in remark 6.6. Then $j_x^\infty f$ is an element of $(W)^{j,\mathcal{S}}$ if and only if*

$$\dim\left(\ker^{\mathcal{C}_\infty}_{j_x^\infty s}\left(\tilde{I_\Phi}\right) \cap \mathcal{S}|_{j_x^\infty s}\right) = j$$

Proof. By the construction of Φ and lemma 5.8 it follows that

$$T_{j_x^\infty f} W = \ker_{j_x^\infty s}\left(\tilde{I_\Phi}\right) := \left\{\mathcal{V} \in T_{j_x^\infty s} J^\infty(E) \mid \mathcal{V}(\varphi) = 0 \text{ for all } \varphi \in \tilde{I_\Phi}\right\}$$

Since $\mathcal{S}|_{j_x^\infty s}$ is contained in the ∞-Cartan distribution the result follows. □

Corollary 6.8. *Let $j_x^\infty s \in W$ and let U be an open neighbourhood of $j_x^\infty s$ as described in remark 6.6. Then $j_x^\infty f$ is an element of $(W)^{j,\mathcal{S}}$ if and only if there exist*

1. *an open $U' \subset U$ around $j_x^\infty s$ that is defined at the level $k+1$.*
2. *a set $C = \{\varphi_1, \ldots, \varphi_{j_0 - j}\} \subset \tilde{I_\Phi}$ of level $k+1$ functions that are totally independent w.r.t. $\mathcal{S}$ on U'.*

such that for $\mathcal{S}_C := \ker^{\mathcal{C}_\infty}(C) \cap \mathcal{S}$, we get

$$(W)^{j,\mathcal{S}} \cap U' = \text{null}\left(\Gamma_{\mathcal{S}_C}(\tilde{I_\Phi}) + \tilde{I_\Phi}\right) \cap U'$$

The proof of this corollary is essentially the same as the proof of the first half of lemma 5.39.

Proof. Note that by combining lemma 5.18, corollary 5.19 and lemma 5.20 it follows that the existence of U' and C is equivalent to $\dim\left(\ker^{\mathcal{C}_\infty}_{j_x^\infty s}\left(\tilde{I_\Phi}\right) \cap \mathcal{S}|_{j_x^\infty s}\right) \leq j$.[3] In particular, by lemma 6.7 they exist if $j_x^\infty s \in (W)^{j,\mathcal{S}}$.

Since $C \subset \tilde{I_\Phi}$, it follows that $\ker^{\mathcal{C}_\infty}\left(\tilde{I_\Phi}\right) \cap \mathcal{S} \subset \mathcal{S}_C$. Note that $\mathcal{S}_C$ is of rank j. By lemma 6.7, it then follows that for any $j_x^\infty s' \in U'$, $j_x^\infty s' \in (W)^{j,\mathcal{S}}$ if and only if $\mathcal{S}_C|_{j_x^\infty s'} = \ker^{\mathcal{C}_\infty}_{j_x^\infty s'}\left(\tilde{I_\Phi}\right) \cap \mathcal{S}|_{j_x^\infty s'}$. □

Thus, similarly as the Thom-boardman singularities, we can locally define our lifted singularities as the null-sets of ideals. Before we move on to the inductive construction described at the start of this section, we want to examine one more general property of these lifts, or a specific type of lift.

Proposition 6.9. *Let $W \subset J^\infty(E)$ be a closed pf-submanifold and assume that $j = \text{rank}(\mathcal{S})$. Then the lift $(W)^{j,\mathcal{S}} \subset J^\infty(E)$ is also closed. In particular, the set inducing it, $a_{k+1}\left((W)^{j,\mathcal{S}}\right) \subset J^{k+1}(E)$, is closed.*

[3] Note that the functions of C are indeed of level $k+1$, since the condition is of level $k+1$.

Proof. Note that since $j = j_0$, it follows that condition (2) of corollary 6.8 is an empty condition. Thus we can simply take U' to be any U as described in remark 4.5. Thus it follows that we can cover W with such opens U, for which $(W)^{j,\mathcal{S}}$ is then the vanishing set of $\Gamma_{\mathcal{S}}\left(\tilde{I}_\Phi\right)$. This means that for any such U, $(W)^{j,\mathcal{S}} \cap U$ is a closed subset of $W \cap U$. Hence $(W)^{j,\mathcal{S}}$ is a closed subset of W. Since $W \subset J^\infty(E)$ is closed, it follows that $(W)^{j,\mathcal{S}} \subset J^\infty(E)$ is closed. □

Remark 6.10. Note that the condition $j = \mathrm{rank}(\mathcal{S})$ is necessary. Intuitively, one should compare it to the submanifolds $\Sigma^j_{\mathrm{rank}} \subset J^1(M,N)$. If $j \neq 0$, any arbitrarily small open around $j^1_x f \in \Sigma^j_{\mathrm{rank}}$, contains elements of $\Sigma^{j-1}_{\mathrm{rank}}$. Thus if $1 \leq j \leq m$, then Σ^j_{rank} lies in the closure of $\Sigma^{j-1}_{\mathrm{rank}}$. Therefore, the only closed non-empty singularity is Σ^m_{rank} (if $m := \dim(M) \leq \dim(N)$).

Note that if we thus want to consider Σ^1_{rank} as a closed set (which we will in section 6.3), we will need to work in some subbundle of $J^1(M,N)$. ⋄

6.2 $(\mathcal{J}_k, \mathcal{S})$-lift if $\mathrm{rank}(\mathcal{S}) = 1$

In the previous section we saw a general construction of lifiting singularities w.r.t. some smooth subbundle $\mathcal{S} \subset \mathcal{C}_\infty$. We did not make many assumptions on the nature of the singularity. This means that we cannot say much about what these lifted singularities look like. While it can be shown that under some minor conditions on $\mathcal{S}$ the lifted singularities are always at least locally Whitney-A stratifiable, we might very well loose the manifold structure. Also, the co-rank of the higher singularities would very much depend on the original singularity and which Cartan directions are tangent to it.

We will now first consider an example for which the lifted singularity is indeed not a submanifold of jet space.

Example 6.11. Let us consider the following situation. We consider the product bundle $E = \mathbb{R}^3 \to \mathbb{R}^2$, i.e. $\Gamma(E) \cong C^\infty(\mathbb{R}^2, \mathbb{R})$. Then we define

$$\Sigma = \left\{(x_1, x_2, y) \in \mathbb{R}^3 \mid x_2 = y^2\right\} \subset J^0(\mathbb{R}^2, \mathbb{R})$$

Furtermore let $\mathcal{S} = \langle \mathscr{X} \rangle$, where

$$\mathscr{X} = x_1 \tilde{\partial}_{x_1} + (x_1 + 1)\tilde{\partial}_{x_2}$$

Since $\mathscr{X}$ is no-where vanishing, $\mathcal{S}$ is a smooth subbundle (of level 0).

Since Σ is globally the vanishing set of the submersion $\varphi : \mathbb{R}^3 \to \mathbb{R}, (x_1, x_2, y) \mapsto x_1 - y^2$, it follows from the construction in corollary 6.8 that

$$(\Sigma)^{1,\mathcal{S}} = \mathrm{null}\left(\Gamma_{\mathcal{S}}\left(\tilde{I}_\varphi\right) + \tilde{I}_\varphi\right)$$

i.e.

$$(\Sigma)^{1,\mathcal{S}} = \mathscr{X}(\tilde{\varphi})^{-1}(0) \cap \tilde{\varphi}^{-1}(0)$$

Note that these two pf-functions are both defined at level one, with representatives

$$\left(\tilde{\varphi}\right)_1 : J^1\left(\mathbb{R}^2, \mathbb{R}\right) \mapsto \mathbb{R}$$
$$((x_1, x_2), y, (z_1, z_2)) \mapsto x_1 - y^2$$

and

$$\left(\mathscr{X}(\tilde{\varphi})\right)_1 : J^1\left(\mathbb{R}^2,\mathbb{R}\right) \mapsto \mathbb{R}$$
$$((x_1,x_2),y,(z_1,z_2)) \mapsto x_1(1-2yz_1) - 2y(x_1+1)z_2$$

At points satisfying $z_2 = x_1 = y = 0$, the differentials of these two functions together do not form a submersion onto $\mathbb{R}^2$. At such points it this space does not admit a manifold structure, as can be seen in figure △

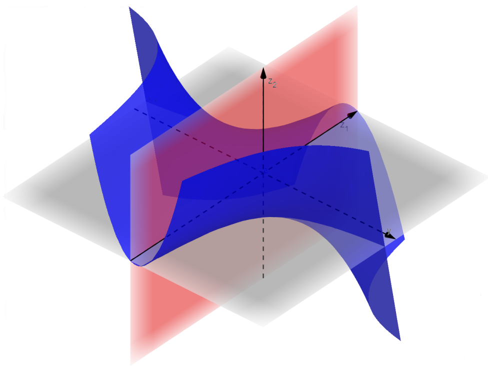


Figure 6.1: The planes $y = 0$ (red) and $y - 2y^2z_1 - 2y^2 - 2z_2 = 0$ (blue) in $\mathbb{R}^3$.

To maintain the manifold structure of such hierarchies of singularities, we would expect to need to have a certain amount of regularity w.r.t. the whole Cartan distribution, or at least the particular bundle $\mathscr{S} \subset \mathscr{C}_\infty$. In this section we will therefore consider subbundles Σ of E. The results we see in this section will apply to a 1-dimensional subbundle $\mathscr{S}$, but it is not unthinkable that there are generalisations to larger subbundles.

The main result of this section is the following theorem. Recall that we denote $\mathscr{I}_k = (\underbrace{1,\ldots,1}_{k \text{ times}})$.

Theorem 6.12. *Let Σ be a subbundle of codimension q of the fibre bundle $\pi : E \to M$. Furthermore, let $\mathscr{S} \subset \mathscr{C}_\infty|_\Sigma$ be a subbundle of rank 1 defined at the level 0. Then for any $k \geq 1$, the singularity $(\Sigma)^{\mathscr{I}_k,\mathscr{S}} \subset J^\infty(E)$ inductively defined as*

$$(\Sigma)^{\mathscr{I}_k,\mathscr{S}} := \left(\Sigma^{\mathscr{I}_{k-1},\mathscr{S}}\right)^{1,\mathscr{S}}$$

is well-defined and a level k pf-submanifold of $J^\infty(E)$ of codimension $q(k+1)$. Furthermore, if Σ is a closed subset, then all $(\Sigma)^{\mathscr{I}_k,\mathscr{S}}$ are closed.

Note that if we can show inductively that the sets $(\Sigma)^{\mathscr{I}_k,\mathscr{S}}$ are indeed pf-manifolds, then the higher singularities are automatically well-defined. Furthermore, combining this with proposition 6.9 also implies the closedness part of the theorem.

We will thus prove this theorem by showing inductively that $a_k\left((\Sigma)^{\mathscr{I}_k,\mathscr{S}}\right)$ is a submanifold of $a_k\left((\Sigma)^{\mathscr{I}_{k-1},\mathscr{S}}\right)$ of codimension q. We want to do this by showing that we can locally write $a_k\left((\Sigma)^{\mathscr{I}_k,\mathscr{S}}\right)$ as the null-set of a submersion $\Psi : a_k\left((\Sigma)^{\mathscr{I}_{k-1},\mathscr{S}}\right) \to \mathbb{R}^q$. Note that by construction of $a_k\left((\Sigma)^{\mathscr{I}_k,\mathscr{S}}\right)$, we can deduce a candidate for this Ψ.

Lemma 6.13. *Let Σ and $\pi : E \to M$ be as above and assume that $U = U_1 \times U_2 \subset E$ is a fibred chart of the bundle such that the last q coordinates of U_2 vanish on Σ. Let $\mathscr{X} \in \Gamma(\mathscr{S}|_U)$ be a non-vanishing section of level 0.*

Set $\tilde{U} := a_0^{-1}(U)$. *Define the map*

$$\tilde{\Psi} : (\Sigma)^{\mathscr{I}_{k-1},\mathcal{S}} \cap \tilde{U} \to \mathbb{R}^q$$

componentwise by $\tilde{\Psi}_i = \underbrace{\mathscr{X}\ldots\mathscr{X}}_{k\ times}(pr_i \circ a_0)$, *where* $pr_i : U \to \mathbb{R}$ *denotes the projection onto the* i*-th coordinate of* U_2.

Then

$$\Sigma^{\mathscr{I}_k,\mathcal{S}} \cap \tilde{U} = \tilde{\Psi}^{-1}(0) \cap \tilde{U}$$

Furthermore, there exists a $\Psi \in C^\infty\left(\pi_{k,0}^{-1}(U)\right)$, *such that* $\tilde{\Psi} = \Psi \circ a_k$.

Proof. First of all, note that while $\mathscr{X}$ is only defined on $\Sigma \cap \tilde{U}$, it follows that since U is trivial and adapted to Σ, $\mathscr{X}$ can be extended to a non-vanishing section on all of $\tilde{U}$.

Let I_0 and I_{k-1} denote the local vanishing ideals on $\tilde{U}$ of Σ and $(\Sigma)^{\mathscr{I}_{k-1},\mathcal{S}}$ respectively.[4] Note that since we are taking the $1 = \text{rank}(\mathcal{S})$ lift w.r.t. $\mathcal{S}$ of $(\Sigma)^{\mathscr{I}_{k-1},\mathcal{S}}$, it follows from corollary 6.8 that on $\tilde{U} \cap \Sigma$ we get that $\underbrace{\Gamma_{\mathcal{S}} \cdots \Gamma_{\mathcal{S}}}_{k \text{ times}}(I_0) + I^{k-1}$ is the induced vanishing ideal of $\Sigma^{\mathscr{I}_k}$, where I^{k-1} is the vanishing ideal of $\Sigma^{\mathscr{I}_{k-1}}$. Thus it follows that

$$\Sigma^{\mathscr{I}_k} = \text{null}\left(\underbrace{\Gamma_{\mathcal{S}} \cdots \Gamma_{\mathcal{S}}}_{k \text{ times}}(I_0)\right) \cap \Sigma^{\mathscr{I}_{k-1}}$$

Note that by the choice of the chart on U, we get that

$$\left.\underbrace{\Gamma_{\mathcal{S}} \cdots \Gamma_{\mathcal{S}}}_{k \text{ times}}(I_0)\right|_{U_i} = \langle \underbrace{\mathscr{X} \cdots \mathscr{X}}_{k \text{ times}}(\text{pr}_1 \circ a_0), \ldots, \underbrace{\mathscr{X} \cdots \mathscr{X}}_{k \text{ times}}(\text{pr}_q \circ a_0)\rangle$$

Thus it follows indeed that $(\Sigma)^{\mathscr{I}_k,\mathcal{S}} = \tilde{\Psi}^{-1}(0)$. Furthermore, it immediately follows from proposition 1.67 that there exists a Ψ as described. □

Hence, the restriction of the map Ψ to $a_k\left((\Sigma)^{\mathscr{I}_{k-1},\mathcal{S}}\right)$ is locally a vanishing map for $a_k\left((\Sigma)^{\mathscr{I}_k,\mathcal{S}}\right)$. To prove theorem 6.12 all that remains to be shown is that this restriction of Ψ is a submersion. To do so we need to consider the tangent space of $a_k\left((\Sigma)^{\mathscr{I}_{k-1},\mathcal{S}}\right)$. Note that one can only consider this tangent space if $a_k\left((\Sigma)^{\mathscr{I}_{k-1},\mathcal{S}}\right)$ is a submanifold. The given argument is thus once again an inductive one and we assume that $a_{k-1}\left((\Sigma)^{\mathscr{I}_{k-1},\mathcal{S}}\right)$ has already been shown to be a submanifold.

Remark 6.14. Note that since $(\Sigma)^{\mathscr{I}_{k-1},\mathcal{S}}$ is defined at the level $k-1$, it follows that the vertical directions of the bundle $J^k(E) \to J^{k-1}(E)$ are tangent to $a_k\left((\Sigma)^{\mathscr{I}_{k-1},\mathcal{S}}\right)$. More specifically, since the 0-vector is contained in any tangent space of $a_{k-1}\left(\Sigma^{\mathscr{I}_{k-1},\mathcal{S}}\right)$, it follows that for any $j_x^\infty s \in \Sigma^{\mathscr{I}_{k-1},\mathcal{S}}$, $\ker\left((d\pi_k)_{j_x^k s}\right) \subset T_{j_x^k s} a_{k-1}\left(\Sigma^{\mathscr{I}_{k-1},\mathcal{S}}\right)$. In section 1.4 we saw how to construct tangent vectors that are in $\ker\left((d\pi_k)_{j_x^k s}\right)$. We will thus use that construction in the proof of the following lemma. ⋄

[4] Note that from corollary 6.8 and the proof of proposition 6.9 it follows that I_{k-1} indeed exists on all of $\tilde{U}$.

Lemma 6.15. *Assume that $\Sigma^{\mathscr{I}_{k-1},\mathcal{S}} \subset J^{\infty}(E)$ is the lift of the submanifold $a_{k-1}\left(\Sigma^{\mathscr{I}_{k-1},\mathcal{S}}\right) \subset J^{k-1}(E)$. Then the restricted map $\Psi : a_k\left(\Sigma^{\mathscr{I}_{k-1},\mathcal{S}}\right) \to \mathbb{R}^q$ as defined in lemma 6.13 is a submersion at any $j^k_{x_0} s \in a_k\left(\Sigma^{\mathscr{I}_k,\mathcal{S}}\right)$.*

Proof. Since we assume that $j^k_{x_0} s \in a_k\left(\Sigma^{\mathscr{I}_k,\mathcal{S}}\right)$, it follows that $\Psi\left(j^k_{x_0} s\right) = 0$.

Note that $X_x := \tilde{a}_0\left(\mathscr{X}_{j^\infty_x f}\right)$ defines a vector field of M around x_0. [5] Furthermore, we know that at x_0, X does not vanish. Hence there exists some open U around x_0 and a submersion $\chi : U \to \mathbb{R}$, such that $X = \left(\frac{\partial}{\partial \chi}\right)$.[6]

Let $U = U_1 \times U_2$ be a trivializing open of E around $s(x_0)$ as described in lemma 6.13, such that (by perhaps further shrinking U) we have that $s|_{U_1}$ maps into U. On U_1, let the section s be given by (id, f), where $f : U_1 \to \mathbb{R}^n \cong U_2$ is some smooth map. Let $I = (-1,1) \subset \mathbb{R}$ be the open interval. For $1 \le i \le q$, we define the following homotopy of sections

$$\begin{aligned} S^i : I \times U_1 &\to U \cong U_1 \times \mathbb{R}^n \\ (t,x) &\mapsto \left(x, (1-t)f + tf^i\right) \end{aligned}$$

where the map f^i is componentwise defined as

$$\left(f^i\right)_j(x) := \begin{cases} f_j & \text{if } j \neq i \\ f_i + \left(\chi(x) - \chi(x_0)\right)^k & \text{if } j = i \end{cases}$$

We will use the notation $s^i_t := S^i(t,-)$. Then it follows that $j^{k-1}_{x_0} s^i_t = j^{k-1}_{x_0} s$ for all $t \in I$. Thus by proposition 1.75 we know that for the induced tangent vector $X_{S^i,x_0} \in T_{j^\infty_{x_0} s} J^\infty(E)$, we get $\tilde{a}_{k-1}(X_{S,x_0}) = 0 \in T_{j^{k-1}_x s} J^{k-1}(E)$. Thus from remark 6.14 it follows that $\tilde{a}_k(X_{S^i,x_0}) \in T_{j^k_x s}\left(a_k\left(\Sigma^{\mathscr{I}_{k-1},\mathcal{S}}\right)\right)$.

Furthermore, note that by definition of X_{S^i,x_0}, we get

$$\tilde{a}_k(X_{S^i,x_0})(\Psi) = \left.\frac{d}{dt}\right|_{t=0}\left(\Psi\left(j^k_{x_0} s^i_t\right)\right)$$

Note that $\Psi_j\left(j^k_{x_0} s^i_t\right)$ is constant in t for $j \neq i$. For the i-th component function, we get

$$\begin{aligned} \Psi_i\left(j^k_x s^i_t\right) &= \underbrace{\mathscr{X} \ldots \mathscr{X}}_{k \text{ times}}(\Psi_i)\left(j^k_{x_0} s^i_t\right) \\ &= \underbrace{X \ldots X}_{k \text{ times}}\left((1-t)f_i + t\left(\chi - \chi(x_0)\right)^k\right)(x_0) \\ &= \underbrace{\frac{\partial}{\partial \chi} \cdots \frac{\partial}{\partial \chi}}_{k \text{ times}}\left((1-t)f_i + t\left(\chi - \chi(x_0)\right)^k\right)(x_0) \end{aligned}$$

Note that since $\Psi\left(j^k_{x_0} s\right) = 0$, it follows that $\underbrace{\frac{\partial}{\partial \chi} \cdots \frac{\partial}{\partial \chi}}_{k \text{ times}}(f_i)(x_0) = 0$, hence we get

$$\begin{aligned} \Psi_i\left(j^k_x s^i_t\right) &= \underbrace{\frac{\partial}{\partial \chi} \cdots \frac{\partial}{\partial \chi}}_{k \text{ times}}\left(t\left(\chi - \chi(x_0)\right)^k\right)(x_0) \\ &= t(k!) \end{aligned}$$

[5] $\mathscr{X}$ is the section of the Cartan distribution as defined in lemma 6.13.
[6] This is a consequence of the flow-box theorem [4, p. 45].

Thus it follows that

$$\tilde{a}_k \left(X_{S^i, x_0} \right) (\Psi) = k! \cdot e_i \in \mathbb{R}^q$$

where e_i denotes the i-th unit vector. Since we can construct this S^i for any $1 \le i \le q$, it follows that $\Psi : a_k \left(\Sigma^{\mathcal{I}_k, \mathcal{S}} \right) \to \mathbb{R}_q$ is indeed a submersion at $j^k_{x_0} s$. □

Corollary 6.16. *Let Σ, $\pi : E \to M$ and $\mathcal{S} \subset \mathcal{C}_\infty$ be as in theorem 6.12. Then for any $k \ge 0$, the singularity $(\Sigma)^{\mathcal{I}_k, \mathcal{S}} \subset J^\infty(E)$ is the lift of the set $a_k \left((\Sigma)^{\mathcal{I}_k, \mathcal{S}} \right) \subset J^k(E)$. Furthermore, this set is a submanifold of codimension $(k+1)q$ of $J^k(E)$.*

Proof. The result follows from inductively applying lemma 6.15. □

Combining this corollary with proposition 6.9 also proves theorem 6.12.

Remark 6.17. As discussed at the start of this section, it is not unthinkable that theorem 6.12 can be extended to subbundles of other ranks and even $j \ne \mathrm{rank}(\mathcal{S})$ (although we would then obviously lose the closedness part of the theorem). An argument in favor of this is the fact that the construction from section 1.4, which we used to prove that Ψ is a submersion, was introduced by Boardman to prove his statement for all indices. ⋄

Remark 6.18. While we will not need it in the rest of this thesis, the case $j = 0$ (no assumptions on Σ) warrants a further comment. In fact, one should think about this case as the subset where there is no singularity. For example, in the case of the Thom-Boardman singularities, the set $\Sigma^{\mathcal{I},0}_{\mathrm{TB}}$ contains those jets $j^\infty_x f \in \Sigma^{\mathcal{I}}_{\mathrm{TB}}$, where the map $f|_{\Sigma^{\mathcal{I}}_{\mathrm{TB}}(f)}$ is of full rank.

It can be easily shown that for a smooth subbundle $\mathcal{S}$ of level k and a submanifold $\Sigma_k \subset J^\infty(E)$, the condition

$$\dim \left(\mathcal{S}|_{j^\infty_x s} \cap T_{j^\infty_x s} \Sigma_k \right) = 0$$

is an open condition (of level $k+1$). Thus it follows that $(\Sigma_k)^{0, \mathcal{S}} \subset \Sigma_k$ is an open subset of Σ_k of level $k+1$. Hence it is a submanifold of the same codimension as Σ_k. ⋄

6.3 Construction of $\Sigma^{\mathcal{I}_k, \ker(z)}$

So far in this chapter we have mostly focussed on the construction of singularity varieties when starting with a general bundle E and submanifold/subbundle Σ. In this section we will define these singularities for the specific subbundle $\Sigma = \Sigma^1_{\mathrm{rank}} \subset J^1(M,N)$ and use the results from this chapter to prove theorem 6.2. We will also prove a corollary of it, which we will use in chapter 7 to make the necessary genericity assumptions.

First of all, we will be working with the bundle $E = J^1(M,N)$ over N and its infinite jet bundle $J^\infty(E)$. Recall from example 6.4 that over the level 0 subbundle $\Sigma^1_{\mathrm{rank}} \subset J^\infty(E)$ we get the smooth subbundle of rank 1 and level 0

$$\ker(z) := \left\{ \left(d j^\infty s \right)_x (V) \mid V \in \ker\left(z_s(x) \right), s \in \Gamma\left(J^1(M,N) \right) \right\} \subset \mathcal{C}_\infty|_{\Sigma^1_{\mathrm{rank}}}$$

Thus by theorem 6.12, we can now construct the pf-submanifolds $\Sigma^{\mathscr{I}_k,\ker(z)} \subset J^\infty(E)$ as

$$\Sigma^{\mathscr{I}_k,\ker(z)} = \begin{cases} a_0^{-1}\left(\Sigma^1_{\text{rank}}\right) & \text{if } k=1 \\ \left(\Sigma^{\mathscr{I}_{k-1},\ker(z)}\right)^{1,\ker(z)} & \text{if } k>1 \end{cases}$$

Let us now repeat and prove theorem 6.2.

Theorem 6.19. *The pf-submanifolds* $\Sigma^{\mathscr{I}_k,\ker(z)} \subset J^\infty\left(J^1(M,N)\right)$ *satisfy*

(i) $\Sigma^{1,\ker(z)} = (a_0)^{-1}\left(\Sigma^1_{rank}\right)$, *where* $\Sigma^1_{rank} \subset J^1(M,N)$ *as defined in section 5.1.*
(ii) for any $s \in \Gamma\left(J^1(M,N)\right)$ *with* $j^\infty s \pitchfork \Sigma^{\mathscr{I}_k,\ker(z)}$ *we get*

$$\Sigma^{\mathscr{I}_{k+1},\ker(z)}(s) = \left\{x \in M \mid \ker(z_s(x)) \subset T_x\Sigma^{\mathscr{I}_k,\ker(z)}(s)\right\}$$

Proof. Note that condition (i) is trivially satisfied. Futhermore, by the definition of the inductive construction as described in theorem 6.12, it follows that

$$\Sigma^{\mathscr{I}_{k+1},\ker(z)} = \left(\Sigma^{\mathscr{I}_k,\ker(z)}\right)^{1,\mathcal{S}}$$

which means that

$$\Sigma^{\mathscr{I}_{k+1},\ker(z)} = \left\{j^\infty_x s \in \Sigma^{\mathscr{I}_k,\ker(z)} \mid \dim\left(\ker(z)|_{j^\infty_x s} \cap T_{j^\infty_x s}\Sigma^{\mathscr{I}_k,\ker(z)}\right) = 1\right\}$$

Then by the construction of $\ker(z)$ and the definition of the tangent space of a pf-submanifold, this can be rewritten as

$$\Sigma^{\mathscr{I}_{k+1},\ker(z)} = \left\{j^\infty_x s \in \Sigma^{\mathscr{I}_k,\ker(z)} \mid \dim\left(\left(d j^k s\right)_x (\ker(z_s(x))) \cap T_{j^k_x s} a_k\left(\Sigma^{\mathscr{I}_k,\ker(z)}\right)\right) = 1\right\}$$

i.e. this means exactly that for $j^\infty s \pitchfork \Sigma^{\mathscr{I}_k,\ker(z)}$, $j^\infty s$ intersects the singularity $\Sigma^{\mathscr{I}_{k+1},\ker(z)}$ at those points $j^\infty_x s$, for which the (1-dimensional) kernel of $z_s(x)$ is contained in the tangent space of $\Sigma^{\mathscr{I}_k,\ker(z)}(s) \subset M$. This means exactly that condition (ii) holds. □

Remark 6.20. In for example [2, pp.46-47] or [9, Theorem 2.5.4] it is shown that if $\dim(M) \leq \dim(N)$, $\Sigma^1_{\text{rank}} \subset J^1(M,N)$ is a submanifold of codimension $n-m+1$. It thus follows from theorem 6.12 that $\Sigma^{\mathscr{I}_k,\ker(z)} \subset J^\infty(J^1(M,N))$ is a pf-submanifold of codimension $k(n-m+1)$. ⋄

As one might have noticed, these $\Sigma^{\mathscr{I}_k,\ker(z)}$ are not closed in $J^\infty(J^1(M,N))$, since $\Sigma^1_{\text{rank}} \subset J^1(M,N)$ is not closed. We must therefore consider it in a different subbundle $E \subset \Sigma^{\mathscr{I}_k,\ker(z)}$.

Corollary 6.21. *We will now consider* E *to be the subset of* $J^1(M,N)$ *defined as*

$$E := \left\{j^1_x f \in J^1(M,N) \mid \dim\left(\ker\left((df)_x\right)\right) \leq 1\right\}$$

Then E *is an open subbundle of* $J^1(M,N)$. *Furthermore, the singularities* $\Sigma^{\mathscr{I}_k,\ker(z)}$ *are contained in* $J^\infty(E)$ *and are closed pf-submanifolds of codimension* $k(n-m+1)$.

Proof. First of all, note that $E \subset J^1(M,N)$ is indeed an open subbundle. If U and V are trivial neighbourhoods of M and N respectively then we get

$$J^1(U,V) \cap E \cong U \times V \times \left\{A \in \text{Hom}\left(\mathbb{R}^m,\mathbb{R}^n\right) \mid \text{rank}(A) \geq 1\right\}$$

This subset of $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ is open and thus a submanifold. Hence E is indeed an open subbundle.[7]

Also, note that $J^\infty(E) \subset J^\infty(J^1(M,N))$ as follows

$$J^\infty(E) \cong \{j^\infty_x s \in J^\infty(J^1(M,N)) \mid s \in \Gamma(E) \subset \Gamma(J^1(M,N)\} = a_0^{-1}(E) \subset J^\infty\left(J^1(M,N)\right)$$

Since $\Sigma^1_{\mathrm{rank}} \subset E$, it then follows that $\Sigma^{\mathscr{I}_k, \ker(z)} \subset J^\infty(E)$.

Since $\Sigma^1_{\mathrm{rank}} \subset E$ is closed, it also follows from theorem 6.12 that the pf-submanifolds $\Sigma^{\mathscr{I}_k, \ker(z)} \subset J^\infty(E)$ are closed. The codimension follows from remark 6.20 and the fact that $E \subset J^1(M,N)$ is a submanifold with codimension 0. □

Remark 6.22. Note that corollary 6.21 also proves lemma 6.1. ⋄

[7] This argument is inspred by [9, Theorem 2.5.4].

7 The h-principle for immersions

We have now gathered all the results needed to give a removal of singularities proof the Smale-Hirsch theorem for $N = \mathbb{R}^n$ and $\dim(M) < n$. Let us reiterate this theorem.

Theorem 7.1 (Hirsch, 1959, 1961). *Let M be a manifold with $\dim(M) < n$. Then the immersion relation $\mathscr{R}_{\mathrm{Imm},n}$ satisfies the h-principle.*

As discussed in chapter 3, we will first consider the induced map ι_* at the π_0-level. Specifically, the first section of this chapter will be dedicated to proving the following lemma.

Lemma 7.2. *Let M be a manifold and $n > \dim(M)$. Then the inclusion*

$$\iota : \mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right) \hookrightarrow \mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$$

induces a bijection

$$\iota_* : \pi_0\left(\mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right)\right) \to \pi_0\left(\mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)\right)$$

We will see that the surjectivitiy part of this proof follows rather quickly from the results we have seen so far. Before we prove injectivity of this map, we must first consider how to apply transversality to maps of the form

$$F : M \times S \to J^1(M, N)$$

where S is some parameter manifold. We will do this by making use of proposition 1.24. Furthermore we will prove proposition 3.23 in a parametric setting. This will give us the tools needed to prove injectivity of ι_*.

In the second section of this chapter we will consider the higher homotopy groups. To obtain the (full) h-principle, we will then thus have to further consider a parametric setting. These arguments will therefore make use of much of the same tools as required for the injectivity proof.

Throughout this chapter we will encounter many opens of function spaces. Unless specified otherwise, those opens are always assumed to be in the strong topology. Furthermore, we will use $m := \dim(M)$ throughout this chapter. This chapter is inspired by [13, section 2.1.1].

7.1 The h-principle at the π_0-level

Recall from section 3.3 that we want to homope a formal solution of $\mathscr{R}_{\mathrm{Imm},n}$ into an actual solution, by making the components holonomic 1 by 1. To prove injectivity of ι_*, we can make a similar construction, but then homotoping a $[0,1]$-family of formal immersions into a $[0,1]$-family of immersions. Two things to note in that case is that this will require us to already work in a parametric setting (namely on $I=[0,1]$) and that this homotopy will then need to be with respect to endpoints. In the following subsections we will discuss these proofs in detail and thus prove lemma 7.2.

7.1.1 Proof of π_0-surjectivity

We start by proving the following proposition.

Proposition 7.3. *Let M be a manifold and $n > m$. Furthermore, let $s \in \mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$ such that the sections $s_1,\dots,s_{i-1} \in \Gamma\left(J^1(M,\mathbb{R})\right)$ are holonomic. Then there exists some $\tilde{s} \in \mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$ such that*

(i) The sections $\tilde{s}_1,\dots,\tilde{s}_i$ are holonomic
(ii) $\tilde{s} \in \mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$
(iii) There exists a path from s to s' in $\mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$.

Proof. First of all, note that $\mathscr{R}_{\mathrm{Imm},n} \subset J^1(M,\mathbb{R}^n)$ is open. Thus it follows that the set $\mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right) \subset \Gamma\left(J^1(M,\mathbb{R}^n)\right)$ is open in the strong topology. Hence there exists some open $U \subset \mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$ around s, which we assume to be convex. Then by corollary 4.20 it follows that there exists some $s' \in U$ for which $j^\infty \hat{s}'_i \pitchfork \Sigma^{\mathscr{I}_k,\ker(z)}$ for all $1 \le k \le m+1$, of which the first $i-1$ components are holonomic.

It then follows from corollary 4.19 and theorem 6.2 that each $\Sigma^{\mathscr{I}_k,\ker(z)}\left(\hat{s}'_i\right) \subset M$ is a submanifold. What is more, we get a sequence of submanifolds

$$\Sigma^{\mathscr{I}_{m+1},\ker(z)}\left(\hat{s}'_i\right) \subset \Sigma^{\mathscr{I}_m,\ker(z)}\left(\hat{s}'_i\right) \subset \cdots \subset \Sigma^{\mathscr{I}_1,\ker(z)}\left(\hat{s}'_i\right) \subset M$$

Therefore it follows that we get a stratification of $\Sigma^1(\hat{s}_i) = \Sigma^{\mathscr{I}_1,\ker(z)}\left(\hat{s}'_i\right)$ as follows

$$\Sigma^1(\hat{s}_i) = \bigsqcup_{j=1}^{m} \Sigma^{\mathscr{I}_j,\ker(z)}\left(\hat{s}'_i\right) \setminus \Sigma^{\mathscr{I}_{j+1},\ker(z)}\left(\hat{s}'_i\right) =: \bigsqcup_{j=1}^{m} S^j$$

Note that this is indeed a partition, since due to codimension reasons, $j^\infty \hat{s}'_i \pitchfork \Sigma^{\mathscr{I}_{m+1},\ker(z)}$ implies that $\Sigma^{\mathscr{I}_{m+1},\ker(z)}\left(\hat{s}'_i\right) = \emptyset$. Since the consecutive $\Sigma^{\mathscr{I}_j,\ker(z)}\left(\hat{s}'_i\right)$'s are all submanifolds of each other, it follows that this is a stratification of $\Sigma^1(\hat{s}_i)$

Furthermore from theorem 6.2 (ii) it follows that $\hat{s}_i$ then satisfies the conditions of proposition 3.23. Thus there exists some $f_i \in C^\infty(M)$ such that $\tilde{s} := \hat{s}'_i \oplus_i j^1 f_i$ satisfies property (i) (ii) and (iii) from this proposition. Property (i) and (ii) are immediate, while property (iii) follows from proposition 2.10 and the fact that the constructed homotopy in proposition 3.23 is smooth. □

Remark 7.4. In the proof of the lemma above, we used that U around s can be taken to be convex. Note that the image of $j^k s$ is a 1-dimensional embedded submanifold of k-th jet space. Then for any $k \ge 0$ we can construct an arbitrarily small neighbourhood around the image of $j^k s$, which is convex in the fibres. The induced opens of the strong Whitney topology[1] are then also convex. Such an argument will be used more frequently throughout this chapter. ◇

[1] Incidentally, this also holds for induced opens in the weak Whitney topology.

With this proposition we can prove the following lemma.

Lemma 7.5. *Let M be a manifold and $n > m$. Then the inclusion*

$$\iota : \mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right) \hookrightarrow \mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$$

induces a surjection

$$\iota_* : \pi_0\left(\mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right)\right) \to \pi_0\left(\mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)\right)$$

Proof. Let $s \in \mathrm{Sol}_f(\mathscr{R})$. Note that we can inductively apply proposition 7.3 to make the components holonomic step by step to obtain an actual solution $f \in \mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right)$. Property (iii) of 7.3 then implies that there is a path from s to $j^1 f$ in $\mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$. Thus ι_* as described in this lemma is indeed surjective. □

7.1.2 Parametric setting

Before we move on to prove that ι_* from lemma 7.2 is injective, we must first be able to apply some type of transversality arguments to maps of the form

$$F : M \times S \to J^1(M, \mathbb{R}^n)$$

Recall from proposition 1.24 that for any $k \geq 0$, the map

$$\begin{aligned} p_{S,1}^* : J^k\left(J^1\left(M \times S, \mathbb{R}^n\right)\right) &\to J^k\left(J^1(M, \mathbb{R}^n)\right) \\ j^k_{(x,t)} s' &\mapsto j^k_x s'(-,t) \end{aligned}$$

is a submersion. This will be the main tool in making a transversality argument of the form mentioned above.

Lemma 7.6. *Let M be a manifold and S a compact manifold with (perhaps empty) boundary. Let $\mathscr{F} : S \to \Gamma_W\left(J^1(M, \mathbb{R}^n)\right)$ be a continuous map and $\{\Sigma_i\}_{i \in I}$ a countable collection of pf-submanifolds of $J^\infty(M, \mathbb{R}^n)$.*

Then for any open $V \subset \Gamma\left(J^1(M, \mathbb{R}^n)\right)$ (even for the strong topology) containing $\mathscr{F}(S)$, there exists a continuous $\mathscr{F}' : S \to \Gamma_W\left(J^1(M, \mathbb{R}^n)\right)$, such that

- *$\mathscr{F}'(S) \subset V$*
- *the induced map*

  $$F' : M \times S \to J^1(M, \mathbb{R}^n)$$

  is smooth
- *for the map*

  $$\begin{aligned} J^\infty F' : M \times S &\to J^\infty\left(J^1(M, \mathbb{R}^n)\right) \\ (x,t) &\mapsto j^\infty_x F'(-,t) \end{aligned}$$

  we get that $J^\infty F' \pitchfork \Sigma_i$ for all $i \in I$.
- *This F' is homotopic to the map F induced by $\mathscr{F}$, through some homotopy $H : M \times S \times I \to J^1(M, \mathbb{R}^n)$ such that $H(-, t_1, t_2) \in V$ for all $(t_1, t_2 \in S \times I)$.*

Furthermore, for a finite collection $\{A_j\}_{j \in J}$ of closed subsets of S, and opens V_j around $\mathscr{F}(A_j)$ respectively, we can assume that $\mathscr{F}'(A_j) \subset V_j$.

Proof. The idea of this proof is to translate transversality to the Σ_i's to transversality to pf-submanifolds $\left(p^*_{S,1}\right)^{-1}(\Sigma_i)$'s of $J^\infty(M \times S, \mathbb{R}^n)$. First of all it is important to note that these latter are indeed pf-submanifolds by corollary 4.7.

Next let us consider the map

$$F : M \times S \to J^1(M, \mathbb{R}^n)$$

associated to $\mathscr{F}$. By corollary 2.15 we can assume F to be smooth.[2]

We want to construct a section $\mathcal{F} \in \Gamma\left(J^1(M \times S, \mathbb{R}^n)\right)$, such that $p_{1,S} \circ \mathcal{F} = \mathscr{F}$. Note that this is indeed possible, by setting for example

$$\mathcal{F}(x,t) = \left((x,t), \left(y_{\mathscr{F}(-,t)}(x)\right), \left(z_{\mathscr{F}(-,t)}(x) \oplus 0\right)\right)$$

i.e., we simply set the derivatives with respect to S-coordinates to be 0. We now define the following map

$$p^*_S : \Gamma\left(J^1\left(M \times S, \mathbb{R}^n\right)\right) \times S \to \Gamma\left(J^1\left(M, \mathbb{R}^n\right)\right)$$
$$(s,t) \mapsto p^*_t(s)$$

where p^*_t is as defined in proposition 1.24. This is a continuous map, hence $\tilde{V} := \left(p^*_S\right)^{-1}(V)$ is open. We define

$$V' := \left\{s \in \Gamma\left(J^1\left(M \times S, \mathbb{R}^n\right)\right) \mid (s,t) \in \tilde{V} \text{ for all } t \in S\right\}$$

Note that $\mathscr{F} \in V'$. From the fact that S is compact and $\tilde{V}$ open, it then follows that there exists some open U' around $\mathscr{F}$ in V'.

Furthermore, note that $\mathcal{F}(A_j) \subset \left(p^*_{S,1}\right)^{-1}(V_j)$ for all $j \in J$. Thus we can assume $U' \subset V'$ to be such that $U'|_{A_j} \subset \left(p^*_{1,S}\right)^{-1}(V_j)$. Note that by also taking $U' \subset \Gamma\left(J^1(M \times S, \mathbb{R}^n)\right)$ small enough, we can assume it to be convex.

By corollary 4.13 it follows that there exists some $\mathcal{F}' \in U'$, such that

$$j^\infty \mathcal{F}' \pitchfork \Sigma_i$$

for all $i \in I$. By lemma 4.6 and the convexity of U', it then follows that the map

$$F' := p_{1,S} \circ \mathcal{F}' : M \times S \to J^1(M, \mathbb{R}^n)$$

is smooth and the induced map $\mathscr{F}' : S \to \Gamma_W\left(J^1(M, \mathbb{R}^n)\right)$ is continuous and satisfies all the required properties. □

Remark 7.7. As we have seen in the proof of the surjectivity of ι_*, we use an inductive argument on the components. Thus we want to be able to apply the transversality argument described above without losing holonomicity of the components we have already made holonomic. note that we can indeed apply the argument in that way.

[2]While it then might not be the map induced by $\mathscr{F}$ anymore, its image will stay contained in V. Furthermore, restricted to the A_i's, the image will stay in the V_i's respectively.

First of all, note that the smoothing of F can be done component-wise by using the splitting $J^1(M,\mathbb{R}^n) \cong \bigoplus_{i=1}^n J^1(M,\mathbb{R})$. Then by applying corollary 2.13 to the holonomic components and corollary 2.15 to the non-holonomic components gives the desired smooth F.

Also, we could have applied a transversality argument as described in section 4.3. In particular, if for $\mathscr{F}$ it was assumed that the first j components $\mathscr{F}(-,t)$ were holonomic for all $t \in S$, the same could have been asked of $\mathscr{F}'$, by applying theorem 4.17 instead of corollary 4.13. $\diamond$

Next we will consider a result which is the analogue version of proposition 3.23 in a parametric setting.

Proposition 7.8. *Let M be a manifold with $m < n$ and let S be a manifold with (perhaps empty) boundary. Also, take i to be some integer $1 \leq i \leq n$. Furthermore, let*

$$s : M \times S \to J^1\left(M,\mathbb{R}^n\right)$$

be a smooth map such that

(i) $s(-,t) \in \mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$ *for all $t \in S$.*
(ii) $\Sigma_{\mathrm{Imm},n-1}(\hat{s}_i) \subset M \times S$ *is a manifold.*
(iii) $\Sigma_{\mathrm{Imm},n-1}(\hat{s}_i)$ *admits a stratification* $\bigsqcup_{j=1}^k S^k$ *for which the 1-dimensional bundle* $\ker\left(z_{\hat{s}_i}\right) \cap (TM \times \{0\})$ *is no-where tangent to the stratificiation.*

Then there exists some smooth map $f_i : M \times S \to N$ satisfying

1. $\hat{s}_i \oplus_i (p_{S,1} \circ j^1 f_i)$ *is once again a smooth family of formal solutions of* $\mathscr{R}_{\mathrm{Imm},n}$.
2. *s is smoothly homotopic to $\hat{s}_i \oplus_i (p_{S,1} \circ j^1 f_i)$ through families of formal solutions of* $\mathscr{R}_{\mathrm{Imm},n}$.

By the map $p_{S,1}$ we mean the submersive bundle morphism $J^1(M \times S,\mathbb{R}^n) \to J^1(M,\mathbb{R}^n)$ as defined in lemma 1.11.

Proof. Note that similarly as in proposition 3.23 the bundle $\ker\left(z_{\hat{s}_i}\right) \cap (TM \times \{0\})$ is a 1-dimensional trivial bundle over $\Sigma_{\mathrm{Imm},n-1}(\hat{s}_i)$, where the trivialisation is given by

$$z_{\hat{s}_i} : \ker\left(z_{\hat{s}_i}\right) \cap (TM \times \{0\}) \cong \Sigma_{\mathrm{Imm},n-1}(\hat{s}_i) \times \mathbb{R}$$

Let X be the vector field over $\Sigma_{\mathrm{Imm},n-1}(\hat{s}_i)$ that gets sent to 1 at all $x \in \Sigma_{\mathrm{Imm},n-1}(\hat{s}_i)$. Note that similarly as in proposition 3.23 it suffices to construct a function $f_i \in C^\infty(M,S)$ that satisfies $X_x(f_i) > 0$ for all $x \in \Sigma_{\mathrm{Imm},n-1}(\hat{s}_i)$. This f_i can be constructed inductively on the strata in the same way as was done in the proof of proposition 3.23. $\square$

Through lemma 7.6 we can now argue with a generic position of the maps. However, we want that generic position to imply the conditions we impose on the section in the above lemma. This is ensured by the following corollary.

Corollary 7.9. *Let M, S, s and i be as described in proposition 7.8, such that s satisfies property (i) as described. (Thus we explicitly do not assume property (ii) and (iii)). Let r be the dimension of S. If*

$$J^\infty \hat{s}_i \pitchfork \Sigma^{\mathscr{I}_k,\ker(z)}$$

for all $1 \leq k \leq m + r + 1$, then there exists some smooth map $f_i : M \times S \to N$ satisfying:

1. $\hat{s}_i \oplus_i (p_{S,1} \circ j^1 f_i)$ *is once again a smooth family of formal solutions of* $\mathscr{R}_{\mathrm{Imm},n}$.
2. s *is smoothly homotopic to* $\hat{s}_i \oplus_i (p_{S,1} \circ j^1 f_i)$ *through families of formal solutions of* $\mathscr{R}_{\mathrm{Imm},n}$.

Proof. Note that $\Sigma^{\mathscr{I}_1,\ker(z)} = \Sigma^1 \subset J^1(M,N)$. Furthermore, since $s(-,t) \in \mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$ for all $t \in S$, it follows that $\Sigma_{\mathrm{Imm},n-1}(\hat{s}_i) = \Sigma^1(\hat{s}_i)$. Thus it follows that s satisfies property (ii) of proposition 7.8.

Furthermore, from the definition of the singularities $\Sigma^{\mathscr{I}_k,\ker(z)}$, we know that

$$\Sigma^{\mathscr{I}_k,\ker(z)} = \left\{ j^\infty_x s' \in J^\infty\left(J^1\left(M,\mathbb{R}^{n-1}\right)\right) \mid \dim\left(\left(d j^\infty s'\right)_x (\ker(z_{s'}(x)) \cap T_{j^\infty_x s'}\Sigma^{\mathscr{I}_{k-1},\ker(z)}\right) = 1\right\}$$

Let 0_t denote the 0 vector in T_tS for $t \in S$. Then for any $V \in T_xM$, it follows that

$$\left(dJ^k s\right)_{x,t}(V,0_t) = \left(d j^k s(-,t)\right)_x(V)$$

In particular, if $(x,t) \in \Sigma^{\mathscr{I}_{k-1},\ker(z)}(\hat{s}_i) \setminus \Sigma^{\mathscr{I}_k,\ker(z)}(\hat{s}_i)$, it follows that the 1-dimensional bundle $\ker\left(z_{\hat{s}_i}\right) \cap (TM \times \{0\})$ is not tangent to $\Sigma^{\mathscr{I}_{k-1},\ker(z)}(\hat{s}_i)$ at (x,t). Furthermore, due to the codimension of $\Sigma^{\mathscr{I}_{m+r+1},\ker(z)}$, it follows that $\Sigma^{\mathscr{I}_{m+r+1},\ker(z)}\left(J^\infty \hat{s}'_i\right) = \emptyset$. Thus the following stratification of $\Sigma_{\mathrm{Imm},n-1}(\hat{s}_i)$

$$\Sigma_{\mathrm{Imm},n-1}(\hat{s}_i) = \bigsqcup_{j=1}^{m+r} \Sigma^{\mathscr{I}_j,\ker(z)}\left(J^\infty \hat{s}'_i\right) \setminus \Sigma^{\mathscr{I}_{j+1},\ker(z)}\left(J^\infty \hat{s}'_i\right) =: \bigsqcup_{j=1}^{m+r} S^j$$

satisfies condition (iii) in proposition 7.8. □

7.1.3 Proof of π_0-injectivity

We will now combine the results from the previous subsection with the strategy of the proof of lemma 7.5, to prove injectivity at π_0-level. We will thus consider the results of the previous subsection in the specific parametric setting of $I = [0,1]$

Proposition 7.10. *Let* M *be a manifold and* $n > m$. *Furthermore, let* $g, h \in \mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right)$, *such that there exists a smooth homotopy*

$$H : M \times [0,1] \to J^1\left(M,\mathbb{R}^n\right)$$

with $H(-,0) = j^1 g$, $H(-,1) = j^1 h$ *and* $H(-,t) \in \mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)$ *for all* $t \in [0,1]$. *Furthermore assume that for all* $t \in [0,1]$, *the first* $i-1$ *components of* $H(-,t)$ *are holonomic. Then there exists an* $H' : M \times I \to N$, *such that*

- *(i)* H' *satisfies all of the properties described above for* H
- *(ii) for all* $t \in [0,1]$, *the* i*-th component of* $H'(-,t)$ *is also holonomic.*
- *(iii)* H *and* H' *are themselves homotopic to each other in* $\mathscr{R}_{\mathrm{Imm},n}$ *through sections of* $J^1(M,\mathbb{R}^n)$.

Note in particular that H' needs to be a smooth homotopy from $j^1 g$ to $j^1 h$ once again.

Remark 7.11. Property (iii) of this proposition will not be needed to prove injectivity at the π_0-level. However it will be needed when we prove that ι_* defines an isomorphism on the higher homotopy groups. ⋄

Remark 7.12. Note that since $\mathscr{R}_{\mathrm{Imm},n}$ is open, it follows from corollary 2.15 that for any $g, h \in \mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right)$ with $[g] = [h] \in \pi_0\left(\mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}\right)\right)$ there exists an H as described in the proposition above (with $i = 1$). ⋄

Proof. First of all note that since g and h are solutions of $\mathscr{R}_{\text{Imm},n}$, it follows that there exist opens U_g and U_h around $j^1 g$ and $j^1 h$ that are contained in $\mathscr{R}_{\text{Imm},n}$. Here we use that $\mathscr{R}_{\text{Imm},n} \subset J^1(M,\mathbb{R}^n)$ is open in the strong topology. Furthermore, we can assume these U_g and U_h to be convex.

Let $T \subset \Gamma\left(J^1\left(M,\mathbb{R}^{n-1}\right)\right)$ be the following set

$$T := \left\{ s \in \Gamma\left(J^1\left(M,\mathbb{R}^{n-1}\right)\right) \mid j^k s \pitchfork \Sigma^{\mathscr{I}_k,\ker(z)} \text{ for all } 0 \le k \le m+1 \right\}$$

Let $P_i : J^1(M,\mathbb{R}^n) \to J^1\left(M,\mathbb{R}^{n-1}\right)$ be the map as defined in corollary 4.20. By that corollary it then follows that there exist respectively $s_0 \in U_g$ and $s_1 \in U_h$ that are holonomic and such that $P_i \circ s \in T$ for $l = 0,1$. Furthermore, take $T' := P_i^{-1}(T)$. Note that by corollary 4.14, T is open and hence so is T'. Since $U_g \cap T'$ and $U_h \cap T'$ are open, it follows that there exist V_0 and V_1 respectively around s_0 and s_1 that are contained in $U_g \cap T'$ and $U_h \cap T'$ respectively. Furthermore, we can once again assume these V_0 and V_1 to be convex.

Since U_g and U_h are convex, it follows that there are paths from s_0 to $j^1 g$ and $j^1 h$ to s_1 in U_g and U_h respectively. Thus it follows that there exists some homotopy $\tilde{H} : M \times [0,1] \to J^1(M,\mathbb{R}^n)$ from s_0 to s_1 such that the image of $\tilde{H}$ is a subset of $\mathscr{R}_{\text{Imm},n}$.

Let us now define the set $\tilde{T} \subset C^\infty\left(M \times [0,1],\mathbb{R}^{n-1}\right)$ as

$$\tilde{T} := \left\{ \alpha \in C^\infty\left(M \times [0,1],\mathbb{R}^{n-1}\right) \mid J^k \alpha \pitchfork \Sigma^{\mathscr{I}_k,\ker(z)} \text{ for all } k \ge 0 \right\}$$

and $\tilde{T}' := P_i^{-1}\left(\tilde{T}\right)$.

Since $M \times \{0\}$ and $M \times \{1\}$ are closed subsets of $M \times [0,1]$, it follows from remark 7.7 that there exists some $\tilde{H}' : M \times [0,1] \to J^1(M,\mathbb{R}^n)$, that satisfies

1. the image of $\tilde{H}'$ is contained in $\mathscr{R}_{\text{Imm},n}$
2. $\tilde{H}$ and $\tilde{H}'$ are homotopic to each other in $\mathscr{R}_{\text{Imm},n}$.
3. $\tilde{H}'(-,l) \in V_l$ for $l = 0,1$.
4. $\tilde{H}' \in \tilde{T}'$.
5. the first $i-1$ components of $\tilde{H}'(-,t)$ for all $t \in [0,1]$ are holonomic.

Note that property 1 and 2 are achieved by taking V as described in lemma 7.6 to be the open $\mathscr{R}_{\text{Imm},n}$. Property 3 can be achieved by appropriate A_i's and V_i's as described in lemma 7.6.

Note that since V_0 and V_1 are convex, we can find (smooth) homotopies H_l from s_l to $\tilde{H}'(-,l)$ contained in V_l for $l = 0,1$. Since the images of these smooth homotopies are contained in T', it follows that $J^k H_l \pitchfork \Sigma^{\mathscr{I}_k,\ker(z)}$ for all $k \ge 0$.[3]

We can then thus construct (yet another) homotopy s from s_0 to s_1 by concatenating in consecuctive order H_0, $\tilde{H}'$ and H_1. Note that after a reparametrisation around the concatenation points, we can assume this homotopy to be smooth. Furthermore, this reparametrisation does not affect the transversality to the singularities $\Sigma^{\mathscr{I}_k,\ker(z)}$, since $\tilde{H}'$ is at both endpoints contained in V_0 or V_1 for some time.

Thus it follows that s satisfies the conditions of corollary 7.9. Let $f_i : M \times S \to N$ be the map constructed by this proposition. Then it follows that s is homotopic to $s' := \hat{s}_i \oplus_i \left(p_{[0,1],1} \circ j^1 f_i\right)$.

[3]Note that for $k > m+1$, the statement follows from the fact that any element of T' is disjoint from $\Sigma^{\mathscr{I}_{m+1},\ker(z)}$ and thus also from $\Sigma^{\mathscr{I}_k,\ker(z)}$.

While s' may not a priori start and end at s_0 and s_1 respectively, we can make it so. In particular for $l = 0, 1$ there exists homotopies H'_l from s_l to $s'(-, l)$, which by construction of s' take their values in $\mathscr{R}_{\text{Imm},n}$. Furthermore, this construction ensures that for all $t \in I$, the first i components of $H_l(-, t)$ are holonomic.

Note that since we chose U_g and U_h to be convex, we can now construct a homotopy H' from $j^1 g$ to $j^1 h$ by

- First going from $j^1 g$ to s_0 in U_g (which is possible by the convexity of U_g)
- Then following the homotopy H'_0 from s_0 to $s'(-, 0)$
- Then following s' from $s'(-, 0)$ to $s'(-, 1)$.
- Then following the inverse homotopy of H'_1 from $s'(-, 1)$ to s_1
- Finally going from s_1 to $j^1 h$ in U_h (which is possible by the convexity of U_h)

Then each part of this homotopy is constructed such that the first i components of $H'(-, t)$ are holonomic on that part of H'. Furthermore, by construction, any part of the homotopy is contained in $\mathscr{R}_{\text{Imm},n}$. Finally, note that H' is a piecewise smooth homotopy, thus after reparametrisation, we can assume it to be smooth. Thus H' satisfies the properties (i) and (ii) of the proposition.

Also note that we chose $\tilde{H}$ and $\tilde{H}'$ homotopic to each other in $\mathscr{R}_{\text{Imm},n}$ and that s is homotopic to $\tilde{H}'$ by linear interpolation through elements of $\mathscr{R}_{\text{Imm},n}$. Thus property (iii) is also satisfied. □

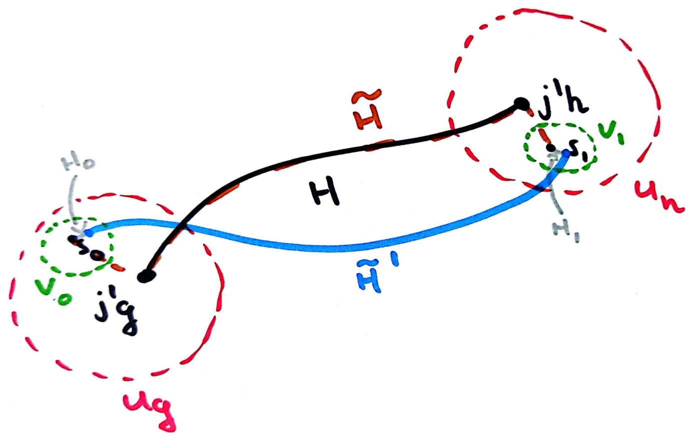


Figure 7.1: Schematic representation of some of the constructed objects in the proof of proposition 7.10.

Lemma 7.13. *Let M be a manifold and $n > m$. Then the inclusion*

$$\iota : \text{Sol}\left(\mathscr{R}_{\text{Imm},n}\right) \hookrightarrow \text{Sol}_f\left(\mathscr{R}_{\text{Imm},n}\right)$$

induces an injection

$$\iota_* : \pi_0\left(\text{Sol}\left(\mathscr{R}_{\text{Imm},n}\right)\right) \to \pi_0\left(\text{Sol}_f\left(\mathscr{R}_{\text{Imm},n}\right)\right)$$

Proof. Assume that $g, h \in \text{Sol}\left(\mathscr{R}_{\text{Imm},n}\right)$, such that $\left[j^1 g\right] = \left[j^1 h\right] \in \pi_0\left(\text{Sol}_f\left(\mathscr{R}_{\text{Imm},n}\right)\right)$. Then there exists some smooth homotopy $H : M \times [0,1] \to J^1(M, \mathbb{R}^n)$ from $j^1 g$ to $j^1 h$, such that $H(-, t)$ is a formal solution of $\mathscr{R}_{\text{Imm},n}$ for all $t \in [0,1]$. Then by inductively applying proposition 7.10 to the components of H it follows that there exists a homotopy H' from $j^1 g$ to $j^1 h$ such that $H'(-,t)$ is a solution of $\mathscr{R}_{\mathrm{Imm},n}$ for all $t \in S$. Thus it follows that $[g] = [h] \in \mathrm{Sol}(\mathscr{R}_{\mathrm{Imm},n})$. □

Together with lemma 7.5, this lemma proves lemma 7.2.

7.2 The h-principle for higher homotopy groups

Note that to prove the Smale-Hirsch theorem as stated at the beginning of this chapter, theorem 7.1, we need to show not only that the induced map on path-components is an ismorophism, but also on higher homotopy groups. Thus, to prove theorem 7.1, we still need to prove the lemma below. Just like the π_0-injectivity part, this lemma makes use of a lot of the same techniques as the proof of the π_0-surjectivity part. However, in stead of working in the $[0,1]$-parametric setting, we now need to work in the $\mathbb{S}^r$-parametric setting. The proof of this lemma is less detailed than we have seen so far. Many of the arguments used are similar to what we have seen in the previous section, which we will also refer back to.

Lemma 7.14. *Let M be a manifold and $n > m$. Then the inclusion*

$$\iota : \mathrm{Sol}(\mathscr{R}_{\mathrm{Imm},n}) \hookrightarrow \mathrm{Sol}_f(\mathscr{R}_{\mathrm{Imm},n})$$

induces a bijection

$$\iota_* : \pi_r(\mathrm{Sol}(\mathscr{R}_{\mathrm{Imm},n})) \to \pi_r(\mathrm{Sol}_f(\mathscr{R}_{\mathrm{Imm},n}))$$

for all $r \geq 0$

Sketch of proof. We will first prove that this map is surjective. Let

$$\mathscr{F} : (\mathbb{S}^r, 1) \to \mathrm{Sol}_f(\mathscr{R}_{\mathrm{Imm},n}, j^1 f)$$

be a pointed continuous map. Furthermore let $F : M \times \mathbb{S}^r \to J^1(M,N)$ be the induced map. Then we must construct a homotopy w.r.t. the subset $M \times \{1\}$ from F to some $F' : M \times \mathbb{S}^r$ such that $F'(-,t)$ is a solution of $\mathscr{R}_{\mathrm{Imm},n}$ for all $t \in \mathbb{S}^r$.

Note that by combining lemma 7.6 and corollary 7.9 we can construct a map

$$\tilde{F} : M \times \mathbb{S}^r \to J^1(M,N)$$

such that

- F and $\tilde{F}$ are homotopic to each other in $\mathscr{R}_{\mathrm{Imm},n}$ by some homotopy $H : M \times \mathbb{S}^r \times I \to J^1(M,N)$.
- $H(-,t_1,t_2) \in \mathrm{Sol}_f(\mathscr{R}_{\mathrm{Imm},n})$ for all $(t_1,t_2) \in \mathbb{S}^r \times I$
- $\tilde{F}(-,t)$ is a solution of $\mathscr{R}_{\mathrm{Imm},n}$ for all $t \in \mathbb{S}^r$.

This is a similar inductive construction on the components of $\tilde{F}$ as described in the proof of lemma 7.5, but now using the parametric results we saw in section 7.1.2.

However, this homotopy $\tilde{F}$ might not be pointed. To fix this, note that $F(-,1)$ and $\tilde{F}(-,1)$ both are solutions of $\mathscr{R}_{\mathrm{Imm},n}$ that are furthermore homotopic to each other in $\mathscr{R}_{\mathrm{Imm},n}$ through the restriction $H|_{M\times\{1\}\times[0,1]}$. Then by proposition 7.10 it follows that $H|_{M\times\{1\}\times[0,1]}$ is homotopic to some $h' : M \times [0,1] \to J^1(M,N)$ where $h'(-,t)$ is an actual solution of $\mathscr{R}_{\mathrm{Imm},n}$ for any $t \in [0,1]$. Furthermore, this homotopy can be taken to be with respect to the endpoints of $[0,1]$.

Note that we can now deform both $\tilde{F}$ to F' and H to some H' as depicted in figure 7.2, such that

- $F' : M \times \mathbb{S}^r \to J^1(M,N)$ is still an $\mathbb{S}^r$-family of solutions of $\mathscr{R}_{\mathrm{Imm},n}$.
- H' is a pointed homotopy from F to F' contained in $\mathscr{R}_{\mathrm{Imm},n}$.

Thus ι_* is indeed surjective.

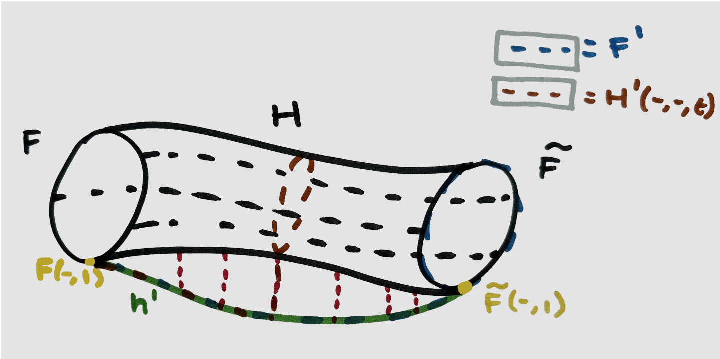


Figure 7.2: Schematic representation of the construction of F' and H' in the proof that ι_* is surjective.

Proving that ι_* is injective on the higher homotopy groups can be done by following similar steps as given in the prove of injectivity at the π_0-level.

Assume we start with some pointed maps $\mathscr{F}, \mathscr{F}' : (\mathbb{S}^r, 1) \to \left(\mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right), f\right)$ such that $\mathscr{F}$ and $\mathscr{F}'$ are in the same homotopy class in $\pi_n\left(\mathrm{Sol}_f\left(\mathscr{R}_{\mathrm{Imm},n}, f\right)\right)$. Then by a similar argument as we have seen in the proofs for π_0, we can assume that $\mathscr{F}$ and $\mathscr{F}'$ induce smooth maps

$$F, F' : M \times \mathbb{S}^r \to J^1(M,N)$$

which are homotopic to each other through a smooth homotopy of families of formal solutions $H : M \times \mathbb{S}^r \times I \to J^1(M,N)$. We let $1 \leq i \leq m$ be some integer and assume that for all $(t_1, t_2) \in \mathbb{S}^r \times I$ the first $i-1$ components of $H(-, t_1, t_2)$ are holonomic. We can then apply the following steps inductively on i (i.e. first applying them for $i = 1$, then $i = 2$ etc.). In all the perturbations we will do we will implicitly assume that the first $i-1$ components stay holonomic (using the same arguments we have seen so far).

- Similarly as we saw in the proof of proposition 7.10, we can define a set $T_i \subset \Gamma\left(J^1(M,\mathbb{R}^n)\right)$ by

$$T_i := \left\{ s \in \Gamma\left(J^1\left(M,\mathbb{R}^n\right)\right) \mid j^k \hat{s}_i \pitchfork \Sigma^{\mathscr{I}_k, \ker(z)} \text{ for all } 0 \leq k \leq m+1 \right\}$$

  This set is open and dense in $\Gamma\left(J^1\left(M,\mathbb{R}^n\right)\right)$.
- We can then perturb f to some g such that
  (i) $j^1 g \in \mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right) \cap T_j$.
  (ii) f and g are homotopic to each other through some homotopy $l : M \times I \to \mathbb{R}^n$, such that $l(-,t) \in \mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right)$ for all $t \in I$.

  Thus by concatenating H with l and the reverse homotopy of l, we can assume that $H(-,1,t) = j^1 g$ for all $t \in \mathbb{S}^r$.
- We can also define the set $T_{r,i} \subset C^\infty(M \times S, \mathbb{R}^n)$ by

$$T_{r,i} := \left\{ R \in C^\infty\left(M \times S, \mathbb{R}^n\right) \mid J^k \hat{R}_i \pitchfork \Sigma^{\mathscr{I}_k, \ker(z)} \text{ for all } 0 \leq k \leq mr+1 \right\}$$

- Then by lemma 7.6, we can then also perturb F and F' close by to G and G' respectively, such that
  (i) $G(-,t), G'(-,t) \in \mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right)$ for all $t \in \mathbb{S}^r$.
  (ii) $G, G' \in T_{r,i}$.
  (iii) F and F' are homotopic to G and G' respectively, through homotopies L and L' respectively, such that $L(-,t_1,t_2), L'(-,t_1,t_2) \in \mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right)$ for all $(t_1,t_2) \in \mathbb{S}^r \times I$.

  Since we assumed $j^1 g$ to be an element of T_i, we can assume that $G(-,1) = G'(-,1) = j^1 g$.
- Then we get a new pointed homotopy from G to G', $\tilde{H}$, such that $\tilde{H}(-,1,t) = j^1 g$ for all $t \in \mathbb{S}^r$. With a similar argument as used in the proof of proposition 7.10 we can now perturb this $\tilde{H}$ slightly to an H', such that
  (i) $J^\infty \hat{H}'_i \pitchfork \Sigma^{\mathscr{I}_k, \ker(z)}$ for all $k \geq 0$.
  (ii) $H'(-,-,0) = G$, $H'(-,-,1) = G'$ and $H'(-,1,t) = j^1 g$ for all $t \in I$.
- This H' then satisfies the conditions of corollary 7.9 and thus we can homotope it (through families of formal solutions of $\mathscr{R}_{\mathrm{Imm},n}$) to some $\tilde{H}'$, for which the i-th component function of $\tilde{H}'(-,t_1,t_2)$ is holonomic for all $(t_1,t_2) \in \mathbb{S}^r \times I$. This H' and $\tilde{H}'$ only differ in the i-th component and the homotopy is achieved through linear interpolation. Since H' was a pointed homotopy at $j^1 g$, we can, after concatenation, assume that $\tilde{H}'$ is a based homotopy at $j^1 g$. Furthermore, since $H'(-,t,l)$ was already holonomic for all $(t,l) \in \mathbb{S}^r \times \{0,1\}$, it follows that we can assume $\tilde{H}'$ to be a homotopy from G to G'.[4]
- Due to the construction of G, G' and g, we can, after further reparametrisation, make $\tilde{H}'$ into a homotopy from F to F' that is pointed at $j^1 f$.

Hence $\mathscr{F}$ and $\mathscr{F}'$ are in the same class of $\pi_r\left(\mathrm{Sol}\left(\mathscr{R}_{\mathrm{Imm},n}\right)\right)$. From this we can conclude that ι_* is indeed injective for all $r \geq 0$. □

This concludes the proof of the Smale-Hirsch theorem with target manifold $N = \mathbb{R}^n$ and $n > \dim(M)$.

[4] This last step is due to another concatenation which uses that linear interpolation between holonomic jets stays holonomic.

Outlook

In this outlook we will consider the conclusions of this thesis and discuss some insights regarding possible future research beyond this thesis.

Conclusion of this thesis

The main objective of this thesis was to get a better understanding of the method of *removal of singularities*, as introduced by Gromov in [13]. We used the h-principle for immersions to investigate this method. It turns out that the main objects needed to apply this argument to immersions are the hierarchies of singularities $\Sigma^{\mathscr{I}_k,\ker(z)}$ as defined in section 6.3. The main result of this thesis is theorem 6.12, which we shall restate here.

Theorem 6.12. *Let Σ be a subbundle of codimension q of the fibre bundle $\pi : E \to M$. Furthermore, let $\mathcal{S} \subset \mathcal{C}_\infty|_\Sigma$ be a subbundle of rank 1 defined at the level 0. Then for any $k \geq 1$, the singularity $(\Sigma)^{\mathscr{I}_k,\mathcal{S}} \subset J^\infty(E)$ inductively defined as*

$$(\Sigma)^{\mathscr{I}_k,\mathcal{S}} := \left(\Sigma^{\mathscr{I}_{k-1},\mathcal{S}}\right)^{1,\mathcal{S}}$$

is well-defined and a level k pf-submanifold of $J^\infty(E)$ of codimension $q(k+1)$. Furthermore, if Σ is a closed subset, then all $(\Sigma)^{\mathscr{I}_k,\mathcal{S}}$ are closed.

This theorem is applicable to the singularities $\Sigma^{\mathscr{I}_k,\ker(z)}$ and therefore tells us that these higher singularities have all the properties we need them to have. The construction of these singularities and the above theorem was based on the Thom-Boardman singularities as described in [2]. In some way the theorem above is more general than what Boardman described, since it considers general bundles $\pi : E \to M$ and subbundles $\Sigma \subset E$. On the other hand, the theorem above is more limited than what Boardman did in his paper, since we only covered the sequences of indices containing 1's and subbundles $\mathcal{S} \subset \mathcal{C}_\infty$ of rank 1. This remark leads us to the first point of future research.

Relaxing the conditions of theorem 6.12

As stated above, in theorem 6.12 we imposed strict conditions on the sequence of indices and the rank of the subbundle $\mathcal{S}$. However, the definition of the $(\iota, \mathcal{S})$-singularity as defined in definition 6.3 allows for a more general definition, i.e. without such restrictions on $\mathscr{I}$ or $\mathcal{S}$, of the repeated lift $\Sigma^{\mathscr{I},\mathcal{S}}$, as long as the singularities stay pf-manifolds after every lift.

One should note here that from the definition of the lift of a singularity (definition 6.3) it easily follows that

$$(\Sigma)^{\iota,\mathcal{S}} = \emptyset$$

if $\iota > \mathrm{rank}(\mathcal{S})$. Thus in that case, theorem 6.12 clearly holds. However, it is not really of interest, since this describes a property of jets that simply does not exist. We will therefore disregard that case in the following discussion.

An argument in favor of a version of theorem 6.12 with less strict restrictions on $\mathcal{J}$ or $\mathcal{S}$ being true, is that the techniques we employed to prove theorem 6.12 are very similar to the techniques Boardman used in [2]. There he proved that the singularities $\Sigma^{\mathcal{J}}$ are manifolds for all $\mathcal{J}$. Note that defining the singularities $\Sigma^{\mathcal{J}}_{\mathrm{TB}}$ through $(\iota,\mathcal{S})$-lifts as defined in chapter 6 would require relaxation on both the indices allowed and the rank of $\mathcal{S}$.

Furthermore, keeping the assumption that $\Sigma \subset E$ is a subbundle means that Σ is 'regular' with respect to the Cartan Distribution. The fact that $\mathcal{S}$ is still assumed to be of constant rank would presumably mean that Σ is then also 'regular' with respect to $\mathcal{S}$. Therefore, the counting of generators described by Boardman [2, section 5], can reasonably be expected to work in this case too.

A point of note here is the closedness part of the statement of theorem 6.12, i.e. the statement that if Σ is closed, then its hierarchies will consist of closed manifolds. Namely, as we already discussed in remark 6.10, $\iota = \mathrm{rank}(\mathcal{S})$ is a neccessary condition for proposition 6.9. Therefore, it is reasonable to assume that unless the sequence of indices is of the form $\mathcal{J} = (\mathrm{rank}(\mathcal{S}),\ldots,\mathrm{rank}(\mathcal{S}))$, there will appear non-closed manifolds in the hierarchy.

Globalisation argument for the Smale-Hirsch theorem

Something else to remark is that in this thesis we only considered the Smale-Hirsch theorem for immersions into Euclidean space. That result can actually be used to prove the result for general target manifolds, using some 'globalisation argument'. This was already denoted by Gromov in [13, p.53]. To provide full details of that argument, one could use similar techniques as we have seen in the injectivity proofs in chapter 7. The homotopies needed in that case could then be constructed locally (i.e. such that the image is contained in a local chart of the target). With similar arguments as described in chapter 7, these local homotopies could then be constructed with respect to the pieces that were 'already good'. Completing this argument would show that removal of singularities can be used to prove h-principles in a more general setting than having Euclidean space as a target manifold.

Other (local) h-principles

As discussed, the motivation for the construction of the singularity hierarchies in chapter 6 is their application to Gromov's h-principle. More specifically, to the strategy of *removal of singularities* to prove this h-principle. In this thesis, our motivating example is the h-principle of immersions. One could wonder if there are more cases of such an h-principle that can be proven by making use of such hierarchies (and removal of singularities).

The first other example to consider is that of functions that are transverse to a given integrable distribution in the target manifold. For the specific definition of this h-principle and differential relation, we refer the reader to [7, 8.2.2, p. 65]. When the subbundle $\tau \subset TN$ of rank r is integrable, and $r < n - m$, this differential relation can be reduced to the immersion relation in $J^1(M,\mathbb{R}^{n-r})$. Thus the techniques described in this thesis can be applied to prove this, although it is also a direct consequence of the immersion h-principle.

Another local h-principle to consider is that of submersions from $M \to N$, where M is an open manifold [13, section 2.1.2]. In that case, the problem can once again be approached componentwise for $N = \mathbb{R}^n$. However, in this case, to make a formal submersion F into an actual one, it does not suffice to only consider the singularity of $\hat{F}_i$ (notation from chapter 7). This is because submersivity onto $\mathbb{R}^{n-1}$, does not guarantee the same for $\mathbb{R}^n$. Therefore, instead of making the formal submersion componentwise holonomic on all of M, we can only make it holonomic around a submanifold $M_0 \subset M$ of positive codimension. Furthermore, if F is a formal submersion, then $\hat{F}_i$ can have a kernel of rank larger than 1. Therefore, the hierarchies of singularities needed in this case would have to be constructed with the more general version of theorem 6.12 as discussed earlier in this outlook. Also, this means that the vector field along which the constructed component function increases needs to be somehow chosen in $\ker(\hat{F}_i)$. This vector field needs to satisfy certain properties as described in [13, p. 53]. Obtaining these would require a further stratification of the strata of M_0. We would expect those higher singularities to then be constructed in the infinite jet bundle of vector fields $J^\infty(TM)$. This was also remarked by Gromov [13, p.53].

As discussed by Gromov, the removal of singularities technique can thus not be used to show that the h-principle holds for submersion if the domain is closed. This makes sense, given that this h-principle does not hold for compact domains [7, p.64]. Hence, this h-principle is also covered under Gromov's theorem for open diff(M)-invariant relations on open manifolds (theorem 3.15). However, showing that the removal of singularities argument is also applicable to submersions would indicate that it can be useful in a more general setting than just immersions.

Non-local h-principles

Certain non-local properties of functions can also be stated as a differential relation of something called *multijet space*. Multijet space is essentially a space that contains the differential information of one map at multiple points.[5] An example of such a non-local h-principle is A. Haefliger's theorem about embeddings (theorem 0.1).

As discussed in the introduction, the strategy for Szücs's proof of Haefliger's theorem was communicated to Szücs by Gromov and Eliashberg and employs the technique of removal of singularities, by turning the formal data into an actual solution componentwise. Note that for a compact manifold M, a smooth embedding $f : M \to N$ is characterized by being a smooth immersion without any double points [22, prop. 4.2.2, p. 87]. The no double point condition is a non-local property of the function. Szücs tackles (part of) the proof in his paper, by making sure that he can avoid any 'problematic' double points (i.e. double points with some additional bad structure). This characterisation of problematic singularities is done by defining a local singularity property in addition to the non-local double point condition [28, definition 4.1]. In a way these problematic singularities can be seen as a singularity of a level higher then just the double point condition, similar to the lifting of singularities we have seen in this thesis.

This approach of removing singularities by avoiding (increasingly) problematic singularities can also be found in the thesis of T.G. Goodwillie[10]. In his thesis Goodwillie uses multiple types of hierarchies of singularities to prove his result about so-called *concordance embeddings* [10, Theorem D, p.12]. While this author does not claim to understand the full scope of Goodwillie's thesis, we would like to zoom in on a certain part, namely the beforementioned hierarchies used by Goodwillie. Specifically, if we consider the higher singularities he creates

[5] For further information on this topic we refer the reader to [9, Chapter 2].

with what he calls *Operation C*, there are at least similarities to the singularities we considered in the current thesis. However, Goodwillie's approach is of a much more algebraic nature. Instead of manifolds, Goodwillie considers closed Zariski sets, which are thus algebraic varieties. In order to endow these algebraic varieties with a stratification (using a Whitney-A stratification type argument), Goodwillie works over the complex numbers [10, p.151].

A future point of study arising from this thesis could thus also be to compare the notions of this thesis to Goodwillie's approach and investigate whether there are other non-local differential relations that can be studied with these methods.

A Stratifications

The idea of removal of singularities, is to use a Thom-transversality-type argument to show that any non-solution can be made into a solution by a small perturbation, close to the singularity set of the non-solution. Doing so gives us a partition of the singularity locus into submanifolds. This endows the locus with a so-called *stratification.* In this appendix we recall the notion of a stratification and discuss some of its basic properties. We will also briefly discuss Whitney-A stratifications. While these do not appear throughout this thesis, they could certainly be interesting when further studying hierarchies in infinite jet space.

A.1 Stratifications and transversality

Definition A.1. *Let Y be a manifold and let $S \subset Y$ be a subset. A* ***stratification*** *of S is a partition of S into finitely many embedded submanifolds $S^i \subset Y$, such that for any $i = 0, \dots k$ the* ***boundary of*** *S^i, which is defined as follows*

$$\partial S^i := (S \cap \overline{(S^i)}) \setminus S^i$$

is contained in the union $\bigcup_{j>i} S^j \subset S$. We say that the dimension of (the stratification of) S is the dimension of the manifold S^0.

Remark A.2. Note that for a stratification of a set $S \subset Y$, we get that

- $S^0 = S \cap U$, where U is some open subset of Y, i.e. S^0 is an open subset of S endowed with the subspace topology.
- S^i is an open subset of $S \setminus (S^0 \cup \dots \cup S^{i-1})$ endowed with the subspace topology for all i.

Furthermore, note that S^0 is the manifold of the highest dimension, whereas S^k is the manifold of the lowest dimension. ⋄

Example A.3. Any n-dimensional CW-complex X has a natural n-dimensional stratification, given by $S^i = X^{n-i} \setminus X^{n-i-1}$. Here X^j denotes the j-skeleton of X.

Note that we can view stratifications as a generalisation of CW-complexes, where the building blocks are general manifolds instead of just disks. △

Example A.4. Another example of a stratification is that of the first jet bundle by the first-level singularity sets as defined by Thom in [30]. Let $\Sigma^j_{\text{rank}} \subset J^1(M,N)$ with $0 \leq j \leq \dim(M)$ denote the submanifolds as defined in section 5.1. Then clearly these submanifolds define a partition of $J^1(M,N)$. Furthermore, from a linear algebra argument, it follows that any stratum is contained in the closure of the lower strata. Thus this partition defines a stratification of $J^1(M,N)$. △

Since stratifications are built up out of manifolds, we can consider transversality with respect to these structures.

Definition A.5. *Let M and N be manifolds and $f \in C^\infty(M,N)$. Furthermore, let $S \subset N$ be a stratified subset of N, with stratification $\{S^0,\dots,S^k\}$. Then we say that $f \pitchfork S$ if $f \pitchfork S^i$ for all $0 \le i \le k$.*

Being transversal to a manifold W means that the inverse image $f^{-1}(W)$ is also a submanifold of the same codimension of W (corollary 4.7). We get a similar result for stratifications.

Lemma A.6. *Let $\Sigma \subset N$ be a stratified subset and $f \in C^\infty(M,N)$ with $f \pitchfork \Sigma$. Then the partition $f^{-1}\left(\Sigma^i\right)$ defines a stratification, where each stratum $f^{-1}\left(\Sigma^i\right)$ has the same codimension as Σ^i.*

Proof. Note that by using corollary 4.7, the only thing that remains to be proven is that the boundary of the higher strata is contained in the closure of the lower strata. This follows immediately from the fact that f is continuous. □

Before we move on to a specific type of stratifications, we want to introduce 1 last notion that is used throughout this thesis.

Definition A.7. *Let $\mathscr{A} \subset T_xM$ be a linear subspace and let S be a stratified subset of M with $x \in S^i$ for some i. Then we say that $\mathscr{A}$ is **tangent to the stratification** if $\mathscr{A} \subset T_xS^i$.*

A.2 Whitney-A stratifications

The following concepts are not essential for the contents of this thesis. However, if one were to consider other hierarchies than just those that stay manifolds, allowing Whitney-A stratifications would be the first step.

Definition A.8. *Let $\Sigma \subset N$ be a stratified subset. We say that the stratification satisfies the Whitney-A condition if for all $x_l \in S_i$, with some $j < i$ and*

$$\lim_{l\to\infty} x_l = x_\infty \in \Sigma_j$$

and the tangent spaces $T_{x_l}\Sigma_i$ converge to some space $T \subset T_{x_\infty}N$, then T contains $T_{x_\infty}\Sigma_j$.

The following statement was proven by Whitney in 1965 in [32] and [33].

Lemma A.9. *Let $\Sigma \subset \mathbb{R}^n$ be an analytic variety (i.e. the null-set of some analytic functions). Then Σ admits a Whitney-A stratification.*

Note that being a Whitney-A stratification is a local property. This lemma thus has the following corollary.

Corollary A.10. *Let N be a smooth manifold and let $\Sigma \subset N$ be a subset. Assume that we can cover N with charts $\{(U_i,\chi_i)\}_{i\in I}$, such that $\chi_i(\Sigma \cap U_i) \subset \mathbb{R}^n$ is an analytic variety. Then Σ is the union of a countable amount of Whitney stratifiable sets.*

The above corollary can also be rephrased using the following definition.

Definition A.11. *Let $I \subset C^\infty(N)$ be an ideal. We say that I is **locally finitely and analytically***

generated *if around every* $x \in N$*, there exists an open* $U \subset N$ *with* (U, χ) *a chart of* N*, such that on* U $I = \langle \Phi_1, \ldots, \Phi_q \rangle$*. Furthermore, these* Φ_i*'s w.r.t. the chart* χ *are analytical functions.*

Note that the zero-set of such an ideal then satisfies the conditions of corollary A.10.

The upshot of Whitney-A stratifications, as opposed to any stratification, is the following theorem, which was proven by E.A. Feldman [8, Proposition 3.6].

Theorem A.12. *Let* M *and* N *be manifolds and let* $\Sigma \subset N$ *be a closed Whitney-A stratified set. Then the set*

$$T_\Sigma := \left\{ f \in C^\infty(M, N) \mid f \pitchfork \Sigma \right\}$$

is a dense and open subset of $C^\infty_S(M, N)$.

While the dense (or residual) part holds for any stratification as a straightforward consequence of corollary 4.13, the opennes part does not hold for general stratifications.

Bibliography

[1] L. Accornero and M. Crainic. "Haefliger's differentiable cohomology". In: *arXiv: Differential Geometry* (2020).

[2] John. M. Boardman. "Singularties of differentiable maps". In: *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 33 (1967), pp. 21–57.

[3] Ralph Cohen. "The Immersion Conjecture for Differentiable Manifolds". In: *Annals of Mathematics* 122 (1985), pp. 237–328.

[4] M. Crainic, R. L. Fernandes, and I. Mărcuţ. "Lectures on Poisson Geometry". In: *Graduate Studies in Mathematics* (2021).

[5] J. Dieudonné. *Foundations of Modern Analysis*. New york: Academic Press Inc., 1960.

[6] Y. Èliašberg and M. Gromov. "Removal of singularities of smooth mappings". In: *Mathematics of The Ussr-izvestiya* 5 (1971), pp. 615–639.

[7] Y. Èliašberg and N. M. Mishachev. *Introduction to the h-Principle*. Vol. 48. Graduate studies in Mathematics. American Mathematical Society, 2002.

[8] E. A. Feldman. "The geometry of immersions. I". In: *Bulletin of the American Mathematical Society* 70 (1963), pp. 600–607.

[9] M. Golubitsky and V.r W. Guillemin. *Stable mappings and their singularities*. New York: Springer-Verlag, 1973.

[10] T. G. Goodwillie. "A Multiple Disjunction Lemma for Smooth Concordance Embeddings". In: 1990.

[11] M. Gromov. "Stable mappings of foliations into manifolds". In: *Mathematics of The Ussr-izvestiya* 3 (1969), pp. 671–694.

[12] M. Gromov. "Convex integration of differential relations". In: *Mathematics of The Ussr-izvestiya* 7 (1973), pp. 329–343.

[13] M. Gromov. *Partial Differential Relations*. Berlin: Springer-Verlag, 1986.

[14] B. Güneysu and M. J. Pflaum. "The Profinite Dimensional Manifold Structure of Formal Solution Spaces of Formally Integrable PDEs". In: *Symmetry, Integrability and Geometry: Methods and Applications* (2017).

[15] A. Haefliger. "Plongements différentiables dans le domaine stable". In: *Commentarii Mathematici Helvetici* 37 (1962), pp. 155–176.

[16] M. W. Hirsch. "Immersions of manifolds". In: *Transactions of the American Mathematical Society* 93 (1959), pp. 242–276.

[17] M. W. Hirsch. "On Imbedding Differentiable Manifolds in Euclidean Space". In: *Annals of Mathematics* 75 (May 1961), pp. 566–571.

[18] M. W. Hirsch. *Differential Topology*. Vol. 33. Graduate Texts in Mathematics. 1976.

[19] D. H. Husemoller. *Fibre Bundles*. 3rd ed. Springer-Verlag, New York Inc., 1994.

[20] J. Krasil'shchik and A. Verbovetsky. "Geometry of jet spaces and integrable systems". In: *Journal of Geometry and Physics* 61.9 (2011). The Interface between Integrability and Quantization, pp. 1633–1674.

[21] D. Krupka. *Introduction To Global Variational Geometry*. Atlantic Press, 2015.

[22] J.M. Lee. *Introduction to Smooth Manifolds*. 2nd ed. Graduate Texts in Mathematics. Springer, 2013. ISBN: 9781441999825.

[23] P. W. Michor. *Manifolds of differentiable mappings*. Kent: Shiva Publishing Limited, 1980.

[24] J. F. Nash. "C^1 Isometric Imbeddings". In: *Annals of Mathematics* 60 (1954), p. 383.

[25] E. Schechter. *Handbook of Analysis and Its Foundations*. San Diego: Academic Press, 1996.

[26] S. Smale. "A classification of immersions of the two-sphere". In: *Transactions of the American Mathematical Society* 90 (1959), pp. 281–290.

[27] S. Smale. "The Classification of Immersions of Spheres in Euclidean Spaces". In: *Annals of Mathematics* 69 (1959), p. 327.

[28] A. Szücs". "The Gromov-Eliashberg proof of Haefliger's theorem". In: *Studia Scientiarum Mathematicarum Hungarica* 17 (1982), pp. 303–318.

[29] R. Thom. "Quelques propriétés globales des variétés différentiables". In: *Commentarii Mathematici Helvetici* 28 (1954), pp. 17–86.

[30] R. Thom. "Les singularites des applications differentiables". In: *Annales de l'Institut Fourier* 6 (1956), pp. 43–87.

[31] H. Whitney. "On regular closed curves in the plane". In: *Compositio Mathematica* 4 (1937), pp. 276–284.

[32] H. Whitney. "Local Properties of Analytic Varieties". In: *Differential and Combinatorial Topology*. Ed. by S.S. Cairns. Princeton University Press, 1965, pp. 205–244.

[33] H. Whitney. "Tangents to an Analytic Variety". In: *Annals of Mathematics* 81.3 (1965), pp. 496–549.