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The phononic Klein paradox using magneto-elastic coupling

Thesis by
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Abstract

In this Thesis we will investigate how phonons and magnons interact with each other. Using the Klein paradox, where matter-antimatter interactions can cause interesting phenomena in scattering theory. In this thesis we will work out the phononic Klein effect, where phonons will couple with antimagnons with negative energy states. This interaction, facilitated by magneto-elastic coupling from magneto-elastic theory, will cause the same phenomenon as the bosonic Klein effect published a few years ago. An increase of, in our case, phonons in the system, due to the reflection coefficient, $R > 1$. By working out the dispersion relation for the coupled magnons and phonons, we can see how antimagnons can interact with phonons, giving us the Klein effect. We will end this thesis with an expression for the reflection coefficient, by using scattering theory.

Contents

1 Introduction

> ”Spintronics is a rapidly emerging field of science and technology that will most likely have a significant impact on the future of all aspects of electronics as we continue to move into the 21st century” [1]

Spintronics being spin transport electronics, also known as spin electronics, is the study of the intrinsic spin and magnetic moment of the electron. Where in regular electronics, almost all machines contain some form on integrated circuits like computer chips, microchips, or just multiple interconnected electronic components such as transistors, resistors, and capacitors. They all have integrated circuits, where the device relies on the transport of electrons as information carriers. In magnon spintronics we achieve the information transfer in a different manner. In a magnon spintronic device, it is not a electron that is the information carries, and thus is not transported, but a magnon is the information carrier. magnons are the quanta of spin waves. The integrated circuit would be an insulator, where the electrons stay fixed in position. Electrons have a tiny magnetic moment and intrinsic spin in them, making it possible from them to carry over spin waves and thus magnons. [2, 3]

1.1 Relevance

The big difference between spintronics and electronics is the way information is carried over, in what form and in what way. In electronic devices the transport of information is like track runners carrying the information, where the runners represent the electrons. However, in spintronic devices, the transport of information is more like a line of people passing buckets with water, where the people in the line are the electrons, and the buckets are spin-waves. A perfectly simple way of information transport. And it has huge benefit. In electronics, electrons are the information carries and thus transported. Electrons are transported by electric currents, which have a cost. Electric currents induce Joule-heating, which converts electric current into heat. This is a very well known fact about electronics, we just lose a bit of the energy to the resistance of the medium of the wires or integrated circuits. Throughout the history of technology, making a device more energy efficient, by increasing the density of transistors. However, nowadays the size of electronics can not be made any smaller and this thus plateauing [4]. Not only can magnons be used for not only the usual classical information processing in electrons, think of logic gates [5, 6], transistors [7–9] and diodes [10], but also for quantum sciences and technologies, think of single-magnon states, squeezed states and entanglement with other quantum platforms [11–14]. Furthermore, in spintronics there will be no joule-heating, so non of the information carriers will be converted into heat and lost. As in the magnon spintronics way of transport of information, the electrons do not move, and it are the magnons that carry the information. This will mean that there will be zero percent energy loss due to Joule heating, making spintronic devices are a lot more efficient and cost a lot less energy.

Then the question arises, how much energy is loss by electronics to heat. If it is not much, then why would this paragraph be called relevant? It is estimated that the total information and communications technology ecosystem is responsible for 2 - 4 % of global carbon emissions, and is very likely this percentage will increase in the coming decennium. [15–17]

With that in mind, magnonic devices might be interesting to do more research on and try to achieve this room temperature superconductor. We now understand the difference between how electronic devices and magnonic devices transport their information. And the benefits thereof.

But now there remains a big challenge; get control over magnons. For without control over magnons, we can not utilize them for information transport. The big problem with magnons is that they are not really particles, but packets of spin, which just dissipate over time. We need to investigate ways to counteract this magnon dissipation, and find a reliable ways to sustain magnon currents even for long distance transport.

1.2 Magnons explained

First of all, let us dive in magnons, what exactly are they? Magnons are a different way of describing spin waves. To get a better understanding of what this means and what spin waves are, we will look at an example where we have magnons. We have a periodic lattice, filled with electrons, with the spin for all electrons aligned. This is the definition of a ferromagnet, which gets its magnetism through its spin alignment. [18, 19] Where we see in the ground state is achieved when there is full alignment, that is the lowest energy state for the ferromagnet.

When we place this lattice in an external magnetic field, we get Larmor precession of the spin. Creating spin waves [20]

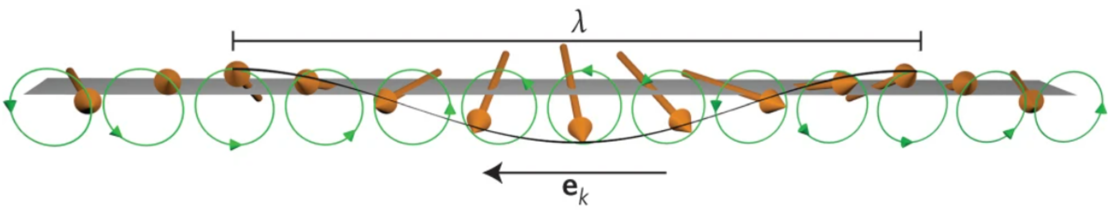

Figure 1: Spin wave moving to the left. The orange arrows indicating the spin direction. Figure is from [21]

In figure 1, the propagation of a spin wave is visible. At the peaks of the waves, the spin of the electron is most deviated from the alignment. This deviation from the alignment can be seen as a certain amount of spin or energy, this packet of energy/spin we know as a magnon. Magnons are the quanta of spin waves, just like photons are to light waves. So, magnons are collective excitations that occur in ordered magnets.

As magnons have been researched, the big challenge that arises in realising magnonic devices is the dissipation of magnons in the form of the amplitude and coherence of magnon currents. This dissipation is detrimental for efficient application of magnons in nanoscale spintronic devices.

In order to be able to use magnons for information carriers in nanoscale spintronic devices, we need to be able to control magnons and there movement, to ensure that our information carried by the magnons doesn't dissipate into thin air. Fortunately, recently it has been proved that it is possible to enhance magnon spin currents using the bosonic Klein effect. Which means we are able to 'call up' or even create more magnons.

Which exactly what we are looking for, counteracting the dissipation. Using the Klein effect has yielded what we were looking for, that's why we will also use it in this thesis. Therefore let us quickly get a grasp of what the Klein effect entails:

1.3 Klein paradox

The physicist Oskar Klein discovered an interesting result in 1929. His discovery was named after him: the Klein paradox. Klein applied the Dirac equation to the familiar problem of electron scattering from a potential barrier. In non-relativistic quantum mechanics, electron tunneling into a barrier is observed, which resulting in exponential damping. However, Klein's result showed that if the potential is at least of the order of the electron mass, $Ve \sim mc^2$ the barrier is nearly transparent. Moreover, as the potential approaches infinity, the reflection diminishes and the electron is always transmitted. [22]

This phenomenon can be explained when we consider the dispersion relation of the set up. We have a region on ground level, and a region wit a high potential. For these regions we can look at the dispersion relations, as depicted in figure 2:

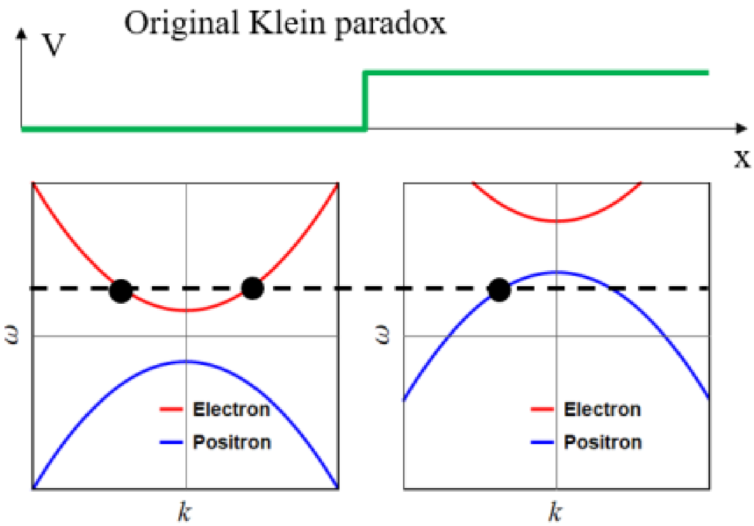

Figure 2: Dispersion relations of electrons and positrons in different regions have overlapping frequencies [3]

In this figure we can see that the dispersion relation of the electron in the ground level region has overlapping frequencies with positrons in the higher potential region. This matter-antimatter interaction, causes the electrons glide over to the higher potential area. [23]

Moreover, the Klein paradox, is actually the fermionic Klein effect, for it involves electrons, which are fermions. And this paradox comes from 1929, which is ancient, but recently the Klein effect has gotten more interest. However, this time not the fermionic Klein effect, but the bosonic Klein effect:

1.4 Bosonic Klein Effect

Recently the Klein effect has been research in the spintronics pictures. From this we get the bosonic Klein effect, and the enhanced magnonic Klein reflections. Here we see an enhanced reflections of magnons being send in from thin film metal to heavy metal. Due to the conductivity of the heavy metal layer there more spin in here, it is a high magnon potential. a kind of external magnetic field can be seen as a high potential region for the magnons.

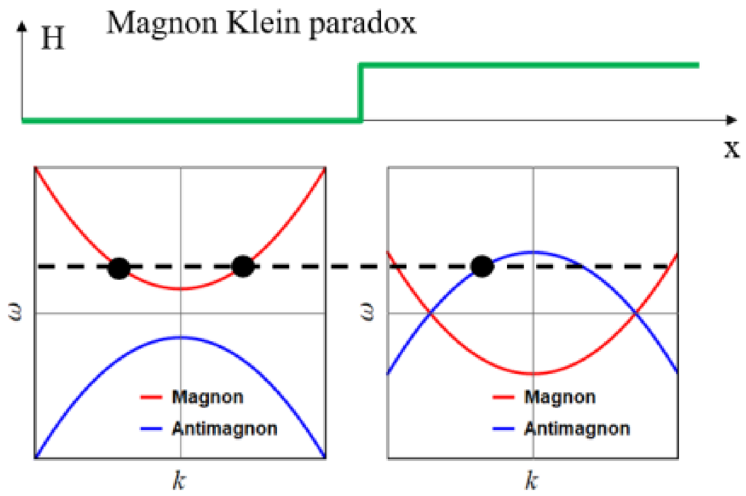

Figure 3: Dispersion relations of magnons and antimagnons in different regions have overlapping frequencies [3]

The enhanced effect can be explained by the magnon being sent in the higher magnon region. This magnon has the opposite direction to the external high potential region. For certain wave numbers, the antimagnons of the higher magnon region share the same frequencies as the incoming magnons. These magnons and antimagnons interact, causing an enhanced magnon current. This is the magnonic Klein effect, with the magnon-antimagnon interaction as the matter-antimatter interaction in a barrier or interface. With the magnon enhancing physics in mind one can create magnon amplifying devices. [24]

1.5 Phononics

Since the bosonic Klein effect yields such interesting results, it might be interesting to further inspect the Klein effect, in more modern field theory physics, such as phononics. Phonics is the field of controlling and understanding phononic properties of materials. Phononics can help to thermally insulate buildings, transform waste heat into electricity, reduce environmental noise and develop earthquake protection. [25] With spintronics and phononics in mind, we will focus on the magneto-elastic coupling and study what happened when we use the Klein effect. For the Klein effect we need to regions with an interface. In one region (a ferromagnet) there will be magneto-elastic coupling, and the other region (an insulator) will be no magnons, only phonons.

1.6 Phonons explained

The phonon is the physical particle representing mechanical vibration and is responsible for the transmission of everyday sound and heat.

Firstly, a quick refresher, what are phonons? phonons are the physical particle representing mechanical vibration, the lattice vibration waves. Just like magnons to spin waves, and photons to light waves. Note that we can not see a phonon with our eyes, but the effect of one, namely the slight displacement of the lattice. The figure 10 demonstrates what the effect of phonons look like.

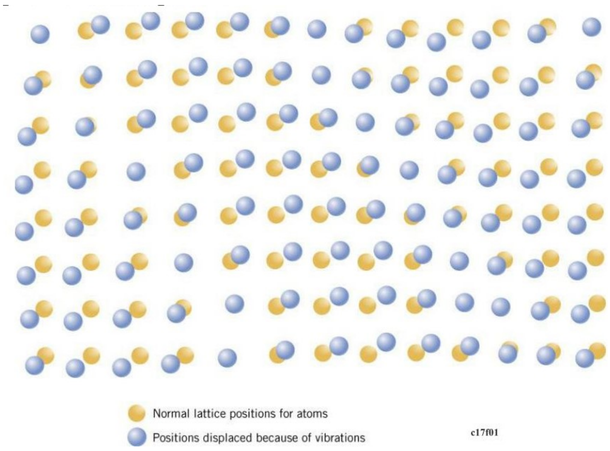

Figure 4: The wave like displacement of the lattice is the phonon

Now knowing what phonons are, one can realise that phonons are everywhere, in every material with a lattice structure. With spintronic devices in mind, the study of how magnons will interact and work with phonons is essential for realising spintronic devices. For phonons can be in almost every medium. In the past, there already have been done a lot of studies about phonons and magnetic material: magneto-elastic theory

1.7 Magneto-elastic theory

Because we want to better understand magnons and how they behave, we should dive into magneto-elastic theory: the theory describing the interaction of phonons, lattice vibrations, and magnons, spin waves. In magneto-elastic theory people have studied the interaction between phonons and magnons, and especially the resonance behavior exhibited when the frequencies and wavelengths, of the phonons and magnons, are equal. This is from a paper from Kittel et al. published in 1958, so it is actually a really old and well known phenomenon in magneto-elastic theory. [26] With this phenomenon in mind, it would be interesting to use the Klein effect for this setting, for which we expect there to also be resonance for equal frequencies. However, in our case with the Klein effect.

1.8 This thesis: the Phononic Klein Effect

With our understanding of the Klein effect, magneto-elastic theory, magnonics and phononics, we will be able to describe an new Klein effect: In this thesis we will investigate the phononic Klein effect. This effect would result in the increase of phonons in the system. Compared to the original Klein paradox, our model will have an interface between an insulator and a ferromagnet. In this interface we will see that matter-antimatter interactions are possible, more specifically, that unstable antimagnons will have overlapping frequencies, causing them to interact. On the left side of the interface we have an insulator, in this region we will only find phonons. On the right side we have a ferromagnet. This region can have both magnons and phonons. Furthermore, by Bogoliubov transforming the equations of motion for coupled magnons and phonons, from magneto-elastic theory. With these Bogoliubov equations we can calculate dispersion relations and work out the reflection coefficient using scattering theory. Ending with an expression for the reflection coefficient of incoming phonons.

2 Set up and general theory

In this section of this thesis, we will talk about what we did and what it is based on. But firstly we will talk about the set up and model I am working in.

2.1 Model and set up

We will work in a simple model: we have a long rod, half made of insulator, half made of ferromagnet, as mentioned in the introduction. Because we are working with the Klein effect, we are interested in the reflected phonons. Figure ?? displays a schematic set-up of the system:

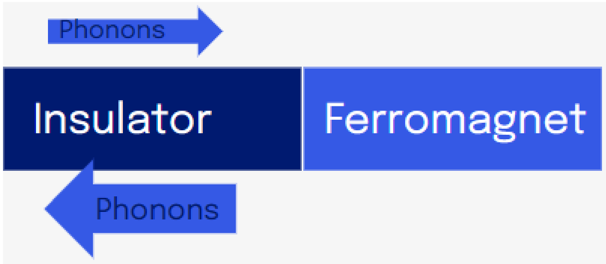

Figure 5: A schematic overview of the set up, the arrows show the expected increase in reflected phonons, compared to the incoming phonons

In our model, we will send in phonons from the left to the right. Because the rod is half insulator and half ferromagnet, we can view this rod as having two regions and one interface. the insulator left and the ferromagnet right.

We expect to see an enhanced phonon output. From magneto-elastic theory we know that the underlying crystal lattice breaks the rotational invariance of the magnetic order. Owing to spin-orbit interaction and dipolar fields, the spins experience elastic deformations in the form of a magneto-elastic coupling (MEC). Vice versa, the lattice is affected by the magnetization in the form of, e.g., magneto-striction. The MEC appears to be the dominant cause for Gilbert damping [27] of the magnetization dynamics of insulators and plays the key role in equilibration of the magnetic system with its surroundings. [28] Since we are trying to better understand magnonic devices, the magneto-elastic equations of motion are essential.

2.2 Equations of motion

In this section we will workout the magneto-elastic coupled equations of mation, derived from the energy density of the system. For this, we closely follow the work of Kamra et al. (2015) on coherent elastic excitation of spin waves. [29]

The free energy density $\mathcal{H}$ of the system can be described by the contributions of the system. in our set up we will have contributions from the Zeeman interaction, anisotropy, exchange interaction, MEC, and elastic energy:

$$\mathcal{H} = \mathcal{H}_Z + \mathcal{H}_{\text{an}} + \mathcal{H}_{\text{ex}} + \mathcal{H}_{\text{MEC}} + \mathcal{H}_{\text{el}}. \tag{1}$$

For small deviations from equilibrium ($M_{x,y} \ll M_z \approx M_s$), the saturation magnetization), Zeeman plus anisotropy energy densities read:

$$\mathcal{H}_Z + \mathcal{H}_{\text{an}} = \frac{\omega_0}{2\gamma M_s}\left(M_x^2 + M_y^2\right). \tag{2}$$

where $\omega_0 = \gamma\mu_0 H$ is the ferromagnetic resonance (FMR) frequency, which comes from magneto elastic theory and $\gamma(> 0)$ is the gyromagnetic ratio.

The exchange energy density can be expressed as:

$$\mathcal{H}_{\text{ex}} = \frac{A}{M_s}\left((\nabla M_x)^2 + (\nabla M_y)^2\right). \tag{3}$$

where A is the material dependent exchange constant. We consider materials with cubic symmetry so that the MEC energy density is parametrized by the MEC constants b_1, b_2 as

$$\mathcal{H}_{\text{MEC}} = \frac{b_1}{M_s^2}\sum_i M_i^2 S_{ii} + \frac{b_2}{M_s^2}\sum_{i\neq j} M_i M_j S_{ij} + \frac{r_0}{3M_s^2}\frac{\partial A}{\partial r}\left((\nabla M_x)^2 + (\nabla M_y)^2\right)\left(\sum_i S_{ii}\right). \tag{4}$$

where the expression is approximated as

$$\mathcal{H}_{\text{MEC}} \approx \frac{2b_2}{M_s}\left(M_x S_{xz} + M_y S_{yz}\right). \tag{5}$$

With $b_{1,2}$ are phenomenological MEC constants that are typically obtained experimentally Furthermore, S_{ij} are the components of the strain tensor given by $S_{ij} = \frac{1}{2}\left(\frac{\partial R_i}{\partial x_j} + \frac{\partial R_j}{\partial x_i}\right)$, r is the distance between the nearest neighbor spins with equilibrium value r_0, and only terms linear in $M_{x,y}$ have been retained. The last term in Eq. 4 represents the exchange mediated MEC due to dependence of the exchange integral on r. We may interpret the MEC as an effective Zeeman field with its x and y components proportional to S_{xz} and S_{yz}.

The elastic energy density for an isotropic solid reads:

$$\mathcal{H}_{\text{el}} = \frac{1}{2}\rho_F(\dot{R}\cdot\dot{R})^2 + \frac{\lambda_F}{2}\left(\sum_i S_{ii}\right)^2 + \mu_F\sum_{ij} S_{ij}^2. \tag{6}$$

where ρ_F, λ_F, and μ_F are the density and Lame's constants of the ferromagnet, respectively. S_{ij} represents the components of the strain tensor, and $\dot{R}$ denotes the time derivative of the position vector.

So in our model we will have the complete energy density is given by:

$$\begin{aligned}\mathcal{H} = &\frac{\omega_0}{2\gamma M_s}\left(M_x^2 + M_y^2\right) + \frac{A}{M_s}\left((\nabla M_x)^2 + (\nabla M_y)^2\right)\\ &+ \frac{2b_2}{M_s}(M_x S_{xz} + M_y S_{yz}) + \frac{1}{2}\rho_F(\dot{R}\cdot\dot{R})^2 + \frac{\lambda_F}{2}\left(\sum_i S_{ii}\right)^2 + \mu_F\sum_{ij} S_{ij}^2.\end{aligned} \tag{7}$$

With this energy density, we can calculate the equations of motions via functional differentiation. With functional derivative we take the derivative of the variable and the gradient of the variable, you make the equation of motion for. Here we will take a Field Theory approach, where we use that the Hamiltonian equations are similar to the equations of motion for fields. The Hamiltonian field equations are given by:

$$\dot{\phi}_i = +\frac{\delta \mathcal{H}}{\delta \pi_i}, \quad \dot{\pi}_i = -\frac{\delta \mathcal{H}}{\delta \phi_i}. \tag{8}$$

where the dots represent partial time derivatives, and the variational derivative with respect to the fields $\delta/\delta\phi_i$ is defined as:

$$\frac{\delta}{\delta \phi_i} = \frac{\partial}{\partial \phi_i} - \nabla \cdot \frac{\partial}{\partial(\nabla \phi_i)}. \tag{9}$$

where in our case ϕ_i will be M_x, M_y, x, y and z, which we will quickly work out. For the magnons, the Hamiltonian equations will be used for M_x & M_y, looking like:

$$\dot{M}_x = +\frac{\delta \mathcal{H}}{\delta M_y}, \quad \dot{M}_y = -\frac{\delta \mathcal{H}}{\delta M_x}. \tag{10}$$

Filling out the functional derivative we get:

$$\dot{M}_x = \frac{\partial \mathcal{H}}{\partial M_y} - \nabla \cdot \frac{\partial \mathcal{H}}{\partial(\nabla M_y)}, \quad \dot{M}_y = \frac{\partial \mathcal{H}}{\partial M_x} - \nabla \cdot \frac{\partial \mathcal{H}}{\partial(\nabla M_x)}. \tag{11}$$

Working this out with with $\mathcal{H}$ described in eq. 7, we will get:

$$\begin{aligned}
\dot{M}_x &= \left(\frac{\omega_0}{\gamma M_s} M_y + \frac{2b_2}{M_s} S_{yz}\right) - \nabla \cdot \frac{2A}{M_s}(\nabla M_y), \\
&= \frac{\omega_0}{\gamma M_s} M_y - \frac{2A}{M_s}\nabla^2 M_y + \frac{2b_2}{M_s} S_{yz}, \\
&= \frac{\omega_0}{\gamma M_s} M_y - \frac{2A}{M_s}\nabla^2 M_y + \frac{2b_2}{M_s}\frac{1}{2}\left(\frac{\partial R_y}{\partial z} + \frac{\partial R_z}{\partial y}\right).
\end{aligned} \tag{12}$$

times γM_s

$$\dot{M}_x = \omega_0 M_y - 2\gamma A \nabla^2 M_y + b_2 \gamma \left(\frac{\partial R_y}{\partial z} + \frac{\partial R_z}{\partial y}\right). \tag{13}$$

For M_y, we work it out similarly

$$\begin{aligned}
\dot{M}_y &= \left(-\frac{\omega_0}{\gamma M_s} M_x - \frac{2b_2}{M_s} S_{xz}\right) + \nabla \cdot \frac{2A}{M_s}(\nabla M_x), \\
&= -\frac{\omega_0}{\gamma M_s} M_x + \frac{2A}{M_s}\nabla^2 M_x - \frac{2b_2}{M_s} S_{xz}, \\
&= -\frac{\omega_0}{\gamma M_s} M_x + \frac{2A}{M_s}\nabla^2 M_x - \frac{2b_2}{M_s}\frac{1}{2}\left(\frac{\partial R_x}{\partial z} + \frac{\partial R_z}{\partial x}\right), \\
\dot{M}_y &= -\omega_0 M_y + 2\gamma A \nabla^2 M_y - b_2 \gamma \left(\frac{\partial R_x}{\partial z} + \frac{\partial R_z}{\partial x}\right).
\end{aligned} \tag{14}$$

Now having gotten the equations of motion for the magnons, we will now calculate the equations of motion for the phonons, which have three components x, y & z. Again using the Hamiltonian field equations we get

$$\dot{R}_i = +\frac{\delta \mathcal{H}}{\delta P_i}, \quad \dot{P}_i = -\frac{\delta \mathcal{H}}{\delta R_i}. \tag{15}$$

With $i = x, y, z$, and realising how momentum and location is related in our set up.

$$\dot{P}_i = \rho_F \frac{\partial^2 R_i}{\partial t^2}. \tag{16}$$

Where the double time derivative of R is the basis of the equations of motion. Let's first take a look on the R_i dependent part of the energy density:

$$\begin{aligned}\mathcal{H} = &\frac{2b_2}{M_s}(M_x S_{xz} + M_y S_{yz}) + \frac{1}{2}\rho_F(\dot{R}\cdot\dot{R})^2 \\ &+ \frac{\lambda_F}{2}\left(\sum_i S_{ii}\right)^2 + \mu_F \sum_{ij} S_{ij}^2.\end{aligned} \tag{17}$$

No working out the stress tensors we get:

$$\begin{aligned}\mathcal{H} = &\frac{2b_2}{M_s}\left(M_x \frac{1}{2}\left(\frac{\partial R_x}{\partial z} + \frac{\partial R_z}{\partial x}\right) + M_y \frac{1}{2}\left(\frac{\partial R_y}{\partial z} + \frac{\partial R_z}{\partial y}\right)\right) \\ &+ \frac{1}{2}\rho_F(\dot{R}\cdot\dot{R})^2 + \frac{\lambda_F}{2}\left(\sum_i S_{ii}\right)^2 + \mu_F \sum_{ij} S_{ij}^2.\end{aligned} \tag{18}$$

Let's first focus on the second to last term of the energy density:

$$\begin{aligned}\frac{\lambda_F}{2}\left[\sum_i S_{ii}\right]^2 &= \frac{\lambda_F}{2}\left[S_{xx} + S_{yy} + S_{zz}\right]^2, \\ &= \frac{\lambda_F}{2}\left[\frac{1}{2}\left(\frac{\partial R_x}{\partial x} + \frac{\partial R_x}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial R_y}{\partial y} + \frac{\partial R_y}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial R_z}{\partial z} + \frac{\partial R_z}{\partial z}\right)\right]^2, \\ &= \frac{\lambda_F}{2}\left[\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} + \frac{\partial R_z}{\partial z}\right]^2, \\ &= \frac{\lambda_F}{2}\left[\nabla\vec{R}\right]^2.\end{aligned} \tag{19}$$

Hamiltonian is now:

$$\begin{aligned}\mathcal{H} = &\frac{2b_2}{M_s}\left(M_x \frac{1}{2}\left(\frac{\partial R_x}{\partial z} + \frac{\partial R_z}{\partial x}\right) + M_y \frac{1}{2}\left(\frac{\partial R_y}{\partial z} + \frac{\partial R_z}{\partial y}\right)\right) \\ &+ \frac{1}{2}\rho_F(\dot{R}\cdot\dot{R})^2 + \frac{\lambda_F}{2}\left[\nabla\vec{R}\right]^2 + \mu_F \sum_{ij} S_{ij}^2.\end{aligned} \tag{20}$$

After taking the functional derivatives for R_x, we get the following equations of motion:

$$\rho_F \frac{\partial^2}{\partial t^2} R_x = \mu_F \nabla^2 R_x + (\lambda_F + \mu_F)\frac{\partial}{\partial x}\nabla\cdot\vec{R} + \frac{b_2}{M_s}\frac{\partial M_x}{\partial z}. \tag{21}$$

Where we get the R_y and R_z in a similar manner, only then with y and z-derivatives, respectfully.

To summarize, the equation of motions for phonons and magnons in MEC are:

$$\frac{\partial}{\partial t} M_x = \omega_0 M_y - D\nabla^2 M_y + b_2\gamma\left(\frac{\partial R_y}{\partial z} + \frac{\partial R_z}{\partial y}\right). \tag{22}$$

$$\frac{\partial}{\partial t}M_y = -\omega_0 M_x + D\nabla^2 M_x - b_2\gamma\left(\frac{\partial R_x}{\partial z} + \frac{\partial R_z}{\partial x}\right). \tag{23}$$

$$\rho_F\frac{\partial^2}{\partial t^2}R_x = \mu_F\nabla^2 R_x + (\lambda_F+\mu_F)\frac{\partial}{\partial x}\nabla\cdot\vec{R} + \frac{b_2}{M_s}\frac{\partial M_x}{\partial z}. \tag{24}$$

$$\rho_F\frac{\partial^2}{\partial t^2}R_y = \mu_F\nabla^2 R_y + (\lambda_F+\mu_F)\frac{\partial}{\partial y}\nabla\cdot\vec{R} + \frac{b_2}{M_s}\frac{\partial M_y}{\partial z}. \tag{25}$$

$$\rho_F\frac{\partial^2}{\partial t^2}R_z = \mu_F\nabla^2 R_z + (\lambda_F+\mu_F)\frac{\partial}{\partial z}\nabla\cdot\vec{R} + \frac{b_2}{M_s}\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y}\right). \tag{26}$$

3 Magneto-elastic waves

In this section we will work out the equations of motion for the coupled magnons and phonons, starting by getting the dispersion relations.

3.1 Dispersion relation version 1

With constant coefficients, the equations of motion are solved by plane waves in the x direction: $B(x,t) = \mathrm{Re}[b(k,\omega)e^{i(kx-\omega t)}]$. After all the derivatives in the equations of motion have been used, we can write the result as a matrix equation $\mathbf{A}\boldsymbol{\chi} = 0$:

$$\begin{pmatrix} i\omega & \omega_m & 0 \\ -\omega_m & i\omega & -i\beta_2\gamma k \\ \frac{i\beta_2 k}{\rho_F M_s} & 0 & \omega_p - \omega^2 \end{pmatrix}\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{27}$$

where $\omega_m = \omega_m(k) = \omega_0 + Dk^2$ and $\omega_p = \omega_p(k) = k\sqrt{\frac{\mu_F}{\rho_F}}$ are the uncoupled magnonic and phononic dispersion relations. In order to find the dispersion relation of the coupled magnons and phonons, we need to calculate the determinant of eq. 27. equating the determinant to zero and solving it will give us:

$$\omega = \pm\sqrt{\left(\frac{\omega_m^2+\omega_p}{2}\right) \pm \sqrt{\left(\frac{\omega_m^2-\omega_p^2}{2}\right)^2 + \frac{b_2^2k^2\gamma\omega_m}{\rho_F M_s}}}. \tag{28}$$

For us to achieve the Klein effect, we need unstable negative energy states to interact with. We achieve this by setting $\omega_0 = -\omega_0$. Which means that we assume that the magnon we describe has spin in the opposite direction to the rest of the ferromagnet.

3.2 Reformulation as a Bogoliubov-deGennes equation

Now we will rewrite the information of the equations of motion in to a different set up: Bogoliubov deGennes transformation. We choose to transform the equations of motion this way, for we can easily substitute the Bogoliubov deGennes equations into scattering theory, which we will need later on.

For magnons we have:

$$\Psi_m = \frac{M_x + iM_y}{\hbar\gamma\sqrt{M_s}}, \quad \Psi_m^* = \frac{M_x - iM_y}{\hbar\gamma\sqrt{M_s}}. \tag{29}$$

For phonons we have:

$$\Psi_{phonon} = \sqrt{\frac{m\omega}{2\hbar}}\left(R + \frac{i}{m\omega}P\right), \quad \Psi^*_{phonon} = \sqrt{\frac{m\omega}{2\hbar}}\left(R - \frac{i}{m\omega}P\right). \tag{30}$$

From these relations we also now can express M_x, M_y, R_i & P_i in Ψ's: for the M_x & M_y:

$$\Psi_m = \frac{M_x + iM_y}{\hbar\gamma\sqrt{M_s}}, \quad \Psi^*_m = \frac{M_x - iM_y}{\hbar\gamma\sqrt{M_s}}. \tag{31}$$

$$\begin{aligned} \Psi_m &= \frac{M_x + iM_y}{\hbar\gamma\sqrt{M_s}}, & \Psi^*_m &= \frac{M_x - iM_y}{\hbar\gamma\sqrt{M_s}}, \\ iM_y &= \hbar\gamma\sqrt{M_s}\Psi_m - M_x, & iM_y &= M_x - \hbar\gamma\sqrt{M_s}\Psi^*_m. \end{aligned} \tag{32}$$

equating these two expressions we get:

$$\begin{aligned} M_x - \hbar\gamma\sqrt{M_s}\Psi^*_m &= \hbar\gamma\sqrt{M_s}\Psi_m - M_x, \\ 2M_x &= \hbar\gamma\sqrt{M_s}\left(\Psi_m + \Psi^*_m\right), \\ M_x &= \frac{\hbar\gamma\sqrt{M_s}}{2}(\Psi_m + \Psi^*_m). \end{aligned} \tag{33}$$

and then the same but for M_y:

$$\begin{aligned} \Psi_m &= \frac{M_x + iM_y}{\hbar\gamma\sqrt{M_s}}, & \Psi^*_m &= \frac{M_x - iM_y}{\hbar\gamma\sqrt{M_s}}, \\ M_x &= \hbar\gamma\sqrt{M_s}\Psi_m - iM_y, & M_x &= \hbar\gamma\sqrt{M_s}\Psi^*_m + iM_y, \end{aligned} \tag{34}$$

equating these two expressions we get:

$$\begin{aligned} \hbar\gamma\sqrt{M_s}\Psi_m - iM_y &= \hbar\gamma\sqrt{M_s}\Psi^*_m + iM_y, \\ 2(iM_y) &= \hbar\gamma\sqrt{M_s}\left(\Psi_m - \Psi^*_m\right), \\ M_y &= \frac{\hbar\gamma\sqrt{M_s}}{2i}(\Psi_m - \Psi^*_m). \end{aligned} \tag{35}$$

Resulting in:

$$\boxed{\begin{aligned} M_x &= \frac{\hbar\gamma\sqrt{M_s}}{2}(\Psi_m + \Psi^*_m), \\ M_y &= \frac{\hbar\gamma\sqrt{M_s}}{2i}(\Psi_m - \Psi^*_m). \end{aligned}} \tag{36}$$

For our phonons we have the momentum and the position we can get an expression for when we rewrite Bogoliubov-deGennes equations for the phonons. We will work in a similar way as above, for further specification of these calculations you can look in the Appendices. The expressions for R and P are:

$$\boxed{\begin{aligned} R &= \frac{1}{2}\sqrt{\frac{2\hbar}{\rho_F\omega}}\left(\Psi_{phonon} + \Psi^*_{phonon}\right), \\ P &= \frac{1}{2i}\sqrt{2\rho_F\omega\hbar}\left(\Psi_{phonon} - \Psi^*_{phonon}\right). \end{aligned}} \tag{37}$$

now we have expressions for $M_x, M_y, R_i \& P_i$, eq. 36 and eq. 80

3.3 Time evolution of Ψ's

We want to get to know the time evolution of the phonon and magnon Bogoliubov-deGennes equations, in the form of a Schrödinger's equation:

$$i\partial t\Psi = H\Psi \quad H\Psi = E\Psi \tag{38}$$

This way we get the energy of the magnons and phonons Working this out for our Ψ_m, Ψ_m^*, Ψ_p&Ψ_p^*, we will find the Hamiltonian matrix,

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi_m &= i\frac{\partial}{\partial t}\left(\frac{M_x + iM_y}{\hbar\gamma\sqrt{M_s}}\right), \\ &= \frac{1}{\hbar\gamma\sqrt{M_s}}\left(i\frac{\partial}{\partial t}M_x - \frac{\partial}{\partial t}M_y\right). \end{aligned} \tag{39}$$

From the equations of motion (22-26), we know the time evolution of the M_x and M_y, which we can fill in:

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi_m = i\frac{1}{\hbar\gamma\sqrt{M_s}}&\left(\omega_0 M_y - D\nabla^2 M_y + b_2\gamma\left(\cancel{\frac{\partial R_y}{\partial z}} + \cancel{\frac{\partial R_z}{\partial y}}\right)\right) \\ - \frac{1}{\hbar\gamma\sqrt{M_s}}&\left(-\omega_0 M_x + D\nabla^2 M_x - b_2\gamma\left(\cancel{\frac{\partial R_x}{\partial z}} + \frac{\partial R_z}{\partial x}\right)\right). \end{aligned} \tag{40}$$

Because we will later only with a plane wave in x direction, all other derivatives, will be zero. So for ease of notion, we will drop the terms that contain a ∂y or ∂z. note that the laplacian operator's only term left is the $\frac{\partial^2}{\partial x^2}$. We are left with:

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi_m &= \frac{1}{\hbar\gamma\sqrt{M_s}}\left[i\left(\omega_0 M_y - D\frac{\partial^2}{\partial x^2}M_y\right) + \omega_0 M_x - D\frac{\partial^2}{\partial x^2}M_x - b_2\gamma\frac{\partial R_z}{\partial x}\right], \\ &= \frac{1}{\hbar\gamma\sqrt{M_s}}\left[i\left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)M_y + \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)M_x + b_2\gamma\frac{\partial}{\partial x}\left[R_z\right]\right], \\ &= \frac{1}{\hbar\gamma\sqrt{M_s}}\left[\left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\left(M_x + iM_y\right) + b_2\gamma\frac{\partial}{\partial x}\left[R_z\right]\right]. \end{aligned} \tag{41}$$

Using eq.29 we know that $\gamma\sqrt{M_s}\Psi_m = M_x + iM_y$ and also that $R = \frac{1}{2}\sqrt{\frac{2\hbar}{\rho_F\omega}}(\Psi_p + \Psi_p^*)$ filling in R expressed in Ψ's from eq.80:

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi_m &= \frac{1}{\hbar\gamma\sqrt{M_s}}\left[\left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\left(\hbar\gamma\sqrt{M_s}\Psi_m\right) + b_2\gamma\frac{\partial}{\partial x}\left[R_z\right]\right], \\ &= \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\Psi_m + \frac{b_2\gamma}{\hbar\gamma\sqrt{M_s}}\frac{\partial}{\partial x}R_z, \\ &= \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\Psi_m + \frac{b_2\gamma}{\hbar\gamma\sqrt{M_s}}\frac{\partial}{\partial x}\left[\frac{1}{2}\sqrt{\frac{2\hbar}{\rho_F\omega}}\left(\Psi_p + \Psi_p^*\right)\right], \\ &= \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\Psi_m + \frac{b_2}{\sqrt{2\rho_F\omega\hbar M_s}}\frac{\partial}{\partial x}\left(\Psi_p + \Psi_p^*\right). \end{aligned} \tag{42}$$

$$\boxed{i\frac{\partial}{\partial t}\Psi_m = \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\Psi_m + \frac{b_2}{\sqrt{2\rho_F\omega\hbar M_s}}\frac{\partial}{\partial x}\left(\Psi_p + \Psi_p^*\right).} \tag{43}$$

Which is the first of time evolution of the Bogoliubov DeGenne equation for magnons and phonons, We will repeat this process for Ψ_m^*, Ψ_p and Ψ_p^*.

$$
\begin{aligned}
i\frac{\partial}{\partial t}\Psi_m^* &= i\frac{\partial}{\partial t}\left(\frac{M_x - iM_y}{\hbar\gamma\sqrt{M_s}}\right), \\
&= \frac{1}{\hbar\gamma\sqrt{M_s}}\left(i\frac{\partial}{\partial t}M_x + \frac{\partial}{\partial t}M_y\right). \qquad (44)
\end{aligned}
$$

From the equations of motion (22-26), we know the time evolution of the M_x and M_y, which we can fill in and do the same procedure as above and we get:

$$
\boxed{i\frac{\partial}{\partial t}\Psi_m^* = \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\Psi_m^* - \frac{b_2}{\sqrt{2\rho_F\omega\hbar M_s}}\frac{\partial}{\partial x}\left(\Psi_p + \Psi_p^*\right).} \qquad (45)
$$

which give us our second time evolution equations for the magnons. Now we will do the same procedure again, now for the Ψ_p:

$$
i\frac{\partial}{\partial t}\Psi_{P_z} = \frac{\partial}{\partial t}\sqrt{\frac{\omega\rho_F}{2\hbar}}\left(iR_x - \frac{1}{\omega\rho_F}P_x\right). \qquad (46)
$$

Which give us:

$$
\boxed{i\frac{\partial}{\partial t}\Psi_{P_z} = \left(\frac{\omega_0}{2} - \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right)\Psi_p + \left(\frac{-\omega_0}{2} - \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right)\Psi_p^* - \frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}\frac{\partial}{\partial x}(\Psi_m + \Psi_m^*).} \qquad (47)
$$

Moreover, and yet again, we will do the same procedure, this time for the Ψ_p^*'s time evolution, giving us the fourth and final time evolution needed:

$$
\boxed{i\frac{\partial}{\partial t}\Psi_{P_z}^* = \left(\frac{\omega_0}{2} + \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right)\Psi_p + \left(\frac{-\omega_0}{2} + \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right)\Psi_p^* + \frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}\frac{\partial}{\partial x}(\Psi_m + \Psi_m^*).} \qquad (48)
$$

In conclusion, our time evolution for the magnon and phonon Bogoliubov-deGennes equations are:

$$
\begin{aligned}
i\frac{\partial}{\partial t}\Psi_m &= \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\Psi_m + \frac{b_2}{\sqrt{2\rho_F\omega\hbar M_s}}\frac{\partial}{\partial x}\left(\Psi_p + \Psi_p^*\right), \\
i\frac{\partial}{\partial t}\Psi_m^* &= \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\Psi_m^* - \frac{b_2}{\sqrt{2\rho_F\omega\hbar M_s}}\frac{\partial}{\partial x}\left(\Psi_p + \Psi_p^*\right), \\
i\frac{\partial}{\partial t}\Psi_{P_z} &= \left(\frac{\omega_0}{2} - \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right)\Psi_p + \left(\frac{-\omega_0}{2} - \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right)\Psi_p^* - \frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}\frac{\partial}{\partial x}(\Psi_m + \Psi_m^*), \qquad (49) \\
i\frac{\partial}{\partial t}\Psi_{P_z}^* &= \left(\frac{\omega_0}{2} + \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right)\Psi_p + \left(\frac{-\omega_0}{2} + \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right)\Psi_p^* + \frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}\frac{\partial}{\partial x}(\Psi_m + \Psi_m^*).
\end{aligned}
$$

Now having found expressions of all the time-evolutions of our Ψ's, we can show it in matrix form:

$$i\partial t \begin{pmatrix} \Psi_m \\ \Psi_m^* \\ \Psi_p \\ \Psi_p^* \end{pmatrix} = \begin{pmatrix} \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right) & 0 & \frac{b_2}{\sqrt{2\rho_F\omega\hbar M_s}}\frac{\partial}{\partial x} & \frac{b_2}{\sqrt{2\rho_F\omega\hbar M_s}}\frac{\partial}{\partial x} \\ 0 & -\left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right) & -\frac{b_2}{\sqrt{2\rho_F\omega\hbar M_s}}\frac{\partial}{\partial x} & -\frac{b_2}{\sqrt{2\rho_F\omega\hbar M_s}}\frac{\partial}{\partial x} \\ \frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}\frac{\partial}{\partial x} & \frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}\frac{\partial}{\partial x} & \left(\frac{\omega_0}{2} - \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right) & \left(-\frac{\omega_0}{2} - \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right) \\ -\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}\frac{\partial}{\partial x} & -\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}\frac{\partial}{\partial x} & \left(\frac{\omega_0}{2} + \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right) & \left(-\frac{\omega_0}{2} + \frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right) \end{pmatrix} \begin{pmatrix} \Psi_m \\ \Psi_m^* \\ \Psi_p \\ \Psi_p^* \end{pmatrix}. \tag{50}$$

As we did before really quickly, but this a little more thorough, we want to get the dispersion relation of the reformulated equations of motion in Bogoliubov-deGennes modes. Our Bogoliubov-deGennes equations can be solved by plane waves in the x direction: $B(x,t) = \mathrm{Re}[u(k,\omega)e^{i(kx-\omega t)}]$. After all the derivatives in the equations of motion have been used, we can write the result as a matrix equation $\mathbf{A}\boldsymbol{\chi} = 0$:

3.4 Dispersion relation and eigenvectors

As we did before with just the equations of motion, we are now again trying to get the dispersion relations for the Bogoliubov-deGennes transformed equations of motion. Because it is just a reformulation of the equations of motion, we should find the same dispersion relation. Which is a good check to see if we are doing everything correctly. Now we want to take the plane wave ansatz and Bogoliubov modes:

$$\begin{pmatrix} \Psi_m \\ \Psi_m^* \\ \Psi_p \\ \Psi_p^* \end{pmatrix} = e^{i(kx-\omega t)} \begin{pmatrix} u_m \\ v_m \\ u_p \\ v_p \end{pmatrix}. \tag{51}$$

Where the time and place derivative in the matrix will work on the plane wave ansatz giving:

$$+\omega \begin{pmatrix} \Psi_m \\ \Psi_m^* \\ \Psi_p \\ \Psi_p^* \end{pmatrix} = \begin{pmatrix} \left(\omega_0 - Dk^2\right) & 0 & \frac{ib_2k}{\sqrt{2\rho_F\omega\hbar M_s}} & \frac{ib_2k}{\sqrt{2\rho_F\omega\hbar M_s}} \\ 0 & -\left(\omega_0 - Dk^2\right) & -\frac{ib_2k}{\sqrt{2\rho_F\omega\hbar M_s}} & -\frac{ib_2k}{\sqrt{2\rho_F\omega\hbar M_s}} \\ \frac{ikb_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}} & \frac{ikb_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}} & \left(\frac{\omega_0}{2} + \frac{\mu_F k^2}{2\omega_0\rho_F}\right) & \left(-\frac{\omega_0}{2} + \frac{\mu_F k^2}{2\omega_0\rho_F}\right) \\ -\frac{ikb_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}} & -\frac{ikb_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}} & \left(\frac{\omega_0}{2} - \frac{\mu_F k^2}{2\omega_0\rho_F}\right) & \left(-\frac{\omega_0}{2} - \frac{\mu_F k^2}{2\omega_0\rho_F}\right) \end{pmatrix} \begin{pmatrix} u_m \\ v_m \\ u_p \\ v_p \end{pmatrix}. \tag{52}$$

Note that $\omega_m = \omega_0 - Dk^2$ and $\omega_p = k\sqrt{\frac{\mu_F}{\rho_F}}$. Which may simplify the expression a little bit.

$$+\omega\begin{pmatrix}u_m\\v_m\\u_p\\v_p\end{pmatrix}=\begin{pmatrix}\omega_m & 0 & i\frac{b_2}{\sqrt{2\rho_F\omega\hbar M_s}}k & i\frac{ib_2k}{\sqrt{2\rho_F\omega\hbar M_s}}k\\ 0 & -\omega_m & -i\frac{ib_2k}{\sqrt{2\rho_F\omega\hbar M_s}}k & -i\frac{ib_2k}{\sqrt{2\rho_F\omega\hbar M_s}}k\\ i\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}k & i\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}k & \frac{\omega_0}{2}+\frac{\omega_p^2}{2\omega_0} & \frac{\omega_0}{2}+\frac{\omega_p^2}{2\omega_0}\\ -i\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}k & -i\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}k & \frac{\omega_0}{2}-\frac{\omega_p^2}{2\omega_0} & -\frac{\omega_0}{2}-\frac{\omega_p^2}{2\omega_0}\end{pmatrix}\begin{pmatrix}u_m\\v_m\\u_p\\v_p\end{pmatrix}. \tag{53}$$

By calculating the eigenvalues of this matrix, we get the dispersion equation for the magnons and phonons:

$$\det\begin{vmatrix}-\omega+\omega_m & 0 & i\frac{b_2}{\sqrt{2\rho_F\omega_0\hbar M_s}}k & i\frac{b_2}{\sqrt{2\rho_F\omega_0\hbar M_s}}k\\ 0 & -\omega-\omega_m & -i\frac{b_2}{\sqrt{2\rho_F\omega_0\hbar M_s}}k & -i\frac{b_2}{\sqrt{2\rho_F\omega_0\hbar M_s}}k\\ i\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}k & i\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}k & \frac{\omega_0}{2}+\frac{\omega_p^2}{2\omega_0}-\omega & -\frac{\omega_0}{2}+\frac{\omega_p^2}{2\omega_0}\\ -i\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}k & -i\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}k & \frac{\omega_0}{2}-\frac{\omega_p^2}{2\omega_0} & -\frac{\omega_0}{2}-\frac{\omega_p^2}{2\omega_0}-\omega\end{vmatrix}=0. \tag{54}$$

We will be using Wolfram Mathematica, to calculate the determinant, after which we have a quadratic equation equal to zero, which we can easily solve:

$$\omega^4+\frac{b_2^2k^2\gamma\omega_m}{M_s\rho_F}-\omega^2\omega_m^2-\omega^2\omega_p^2+\omega_m^2\omega_p^2=0. \tag{55}$$

Which we can solve for ω and this gives us four dispersion relations:

$$\omega_{++}=\sqrt{\left(\frac{\omega_m^2+\omega_p}{2}\right)+\sqrt{\left(\frac{\omega_m^2-\omega_p^2}{2}\right)^2+\frac{b_2^2k^2\gamma\omega_m}{\rho_F M_s}}}, \tag{56}$$

$$\omega_{+-}=\sqrt{\left(\frac{\omega_m^2+\omega_p}{2}\right)-\sqrt{\left(\frac{\omega_m^2-\omega_p^2}{2}\right)^2+\frac{b_2^2k^2\gamma\omega_m}{\rho_F M_s}}}, \tag{57}$$

$$\omega_{-+}=-\sqrt{\left(\frac{\omega_m^2+\omega_p}{2}\right)+\sqrt{\left(\frac{\omega_m^2-\omega_p^2}{2}\right)^2+\frac{b_2^2k^2\gamma\omega_m}{\rho_F M_s}}}, \tag{58}$$

$$\omega_{--}=-\sqrt{\left(\frac{\omega_m^2+\omega_p}{2}\right)-\sqrt{\left(\frac{\omega_m^2-\omega_p^2}{2}\right)^2+\frac{b_2^2k^2\gamma\omega_m}{\rho_F M_s}}}. \tag{59}$$

Which is still exactly the same as the dispersion relation found earlier for magneto-elastic waves in eq. 28

These $\omega's$ are the eigenvalues of the matrix, and with them we can calculate the eigenvalues and norm of the system. Again using Wolfram Mathematica, we can at least calculate whether norm is negative or positive. We can calculate the norm via:

$$\langle\Psi_i|\,\sigma_z\,|\Psi_i\rangle\,. \tag{60}$$

Where Ψ_i are the eigenvectors. For each eigenvalue, we have eigenvectors, for which we can calculate the norm:

$$\begin{aligned}\omega_{--}&=\omega_{phonon} &&\text{norm}: -AU,\\ \omega_{-+}&=\omega_{phonon} &&\text{norm}: -AU,\\ \omega_{+-}&=\omega_{magnon} &&\text{norm}: +AU,\\ \omega_{++}&=\omega_{magnon} &&\text{norm}: +AU,\end{aligned} \tag{61}$$

The norm is important, for we want to get negative energy states for which we need a negative norm, when we are working with positive frequencies. Note that in eq. 61, we assign the label phonon or magnons to the dispersion relations. This label means that when we are far away form an avoided crossing, this dispersion relation behaves like an uncoupled magnon or phonon, depending on the label. All in all, we now have four dispersion relations describing the energy states for coupled phonons and magnons, with both negative and positive norms, and thus creating negative energy states.

Firstly we will work out and plot the dispersion relation wit stable magnons, and later we will work it our for unstable magnons. Below we plot all four dispersion relations, giving us all the information about the coupled magnons and phonons:

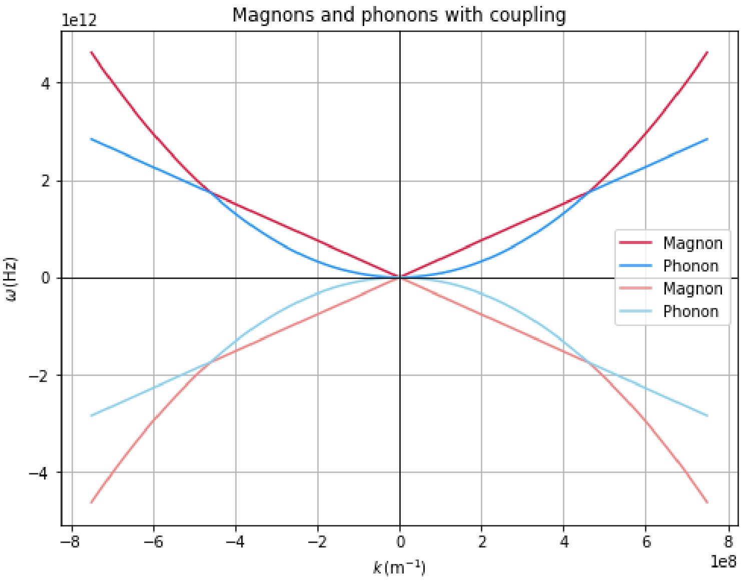

Figure 6: Plot of the dispersion relation for magneto-elastic coupled magnons and phonons

The parameters appropriate for Yttrium Iron Garnet (YIG) are: saturation magnetization $M_s = 1.4 \times 10^5\,\mathrm{A/m}$, anisotropy constant $b_2 = 5.5 \times 10^5\,\mathrm{J/m}^3$, diffusion constant $D = 8.2 \times 10^{-6}\,\mathrm{m}^2/\mathrm{s}$, magnetic field $H = 8 \times 10^4\,\mathrm{A/m}$, gyromagnetic ratio $\gamma = 2.8 \times 10^{10}\,\mathrm{Hz/T}$, density $\rho_F = 5170\,\mathrm{kg/m}^3$, and shear modulus $\mu_F = 74\,\mathrm{GPa}$. Note that our magneto-elastic coupling is not very strong, that's why they are still pretty similar to the independent magnons and phonons. This total picture, we will compare to the case where the phonons and magnons are not coupled:

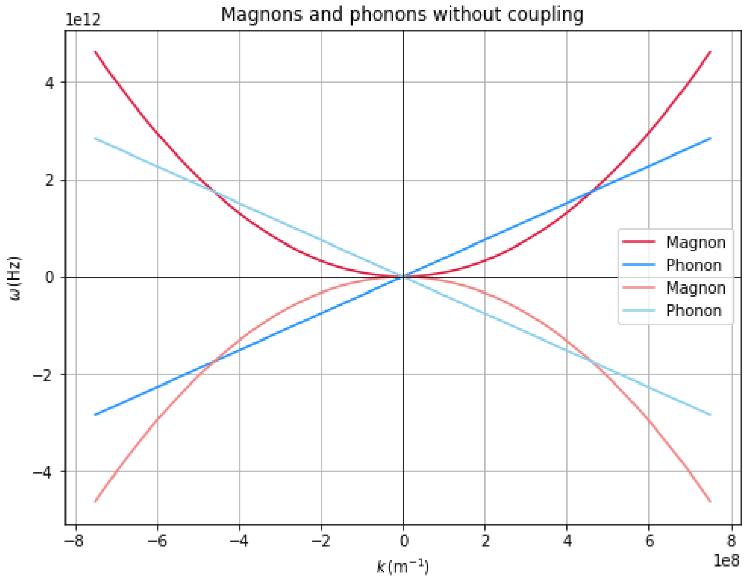

Figure 7: Plot of the dispersion relation for uncoupled magnons and phonons

From a quick comparison, one can see that the shapes stay the same, however instead if the phonons and magnons crossing, the take on each other 'routes', each others dispersion relation. Now focusing on the right positive side, we can zoom in a bit:

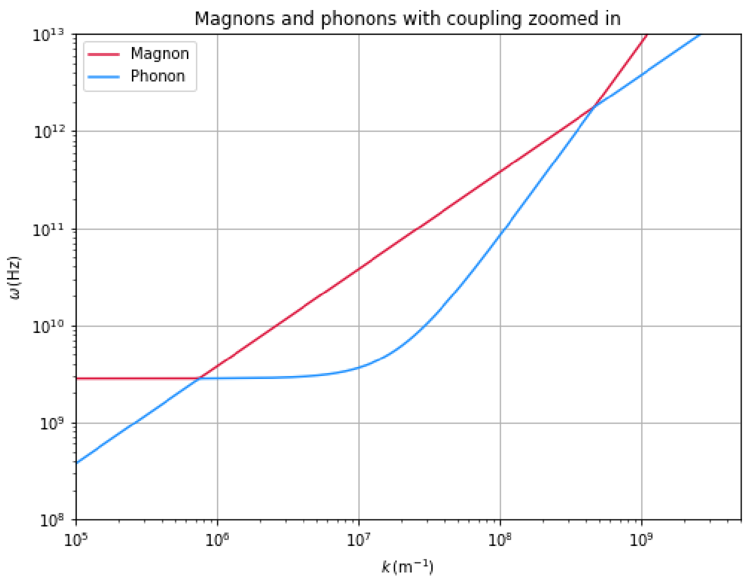

Figure 8: Here one can see how the magneto-elastic coupling correlates the dispersion relation of the magnons and phonons

Here we can clearly see the linear line originated from the phonon dispersion, and half a parabola from the magnon dispersion. And upon closer inspection of the plot, we see that

When we zoom in on the place where the magnons and phonons seem to cross, we see that they actually don't cross. In figure 3.4 we see what is called an avoided crossing or Von Neumann–Wigner theorem in quantum chemistry. [30]

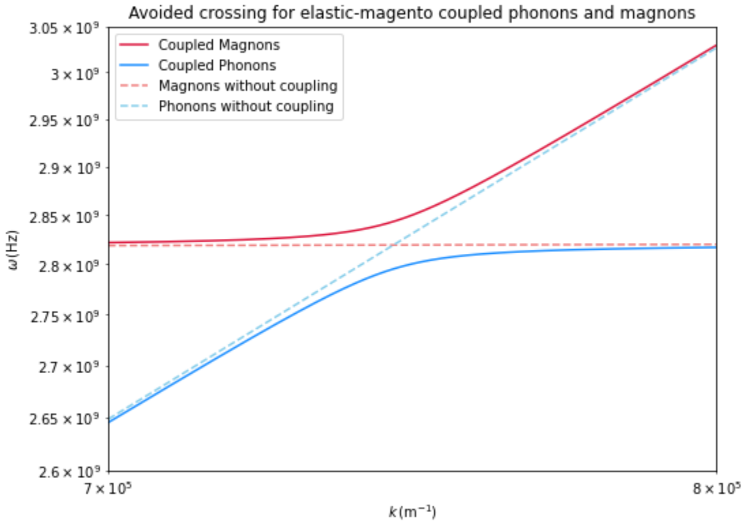

Figure 9: This is zoomed in on the avoided crossing, where compared to without coupling, the dispersion of the magnons and phonons swap for while and return to normal after the second avoided crossing

In the case of no magneto-elastic coupling, the dispersion relation for phonons and magnons do no depend on each other and looks like this:

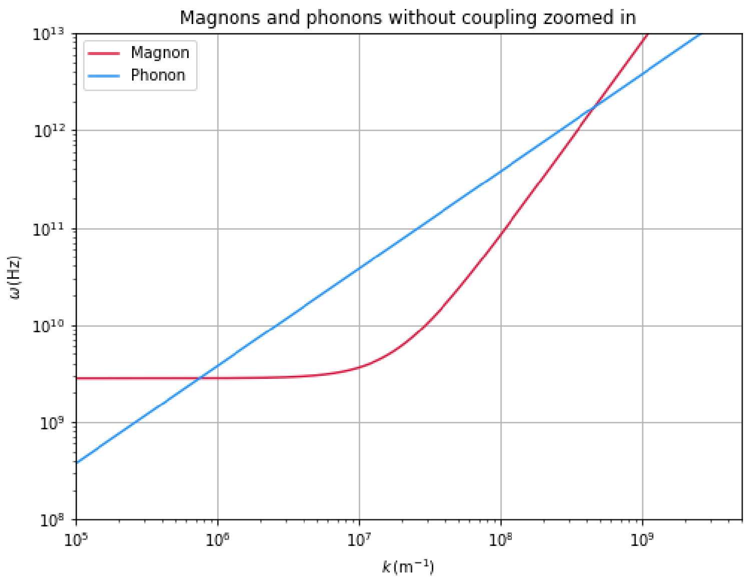

Figure 10: Dispersion relation without coupling

3.5 Opposite magnetic field ω_0

For the Klein effect to be at work, we want to have negative energy- states. The energy of excitations is the product of its frequency and its norm. We do this by, as mentioned earlier, introducing magnons, with opposite sign to the ferromagnet it self. With the spin of the magnons pointing the opposite way of the ferromagnet, one creates a higher energy level, compared to normal, the ground state. But this energy level is unstable, for the spin of the magnon can be pulled back to the ground state. This higher energy level, can be seen as the energy the magnon holds. In a similar way how electrons can be excited, and fall back down releasing a phonon with that energy.

In our case, a magnon with opposite spin sign, means that our ω_0 will be $-\omega_0$. Or in other words, we have $\omega_0 = \gamma * \mu_F * H$, will be $\omega_0 = -\gamma * \mu_F * H$. The magnetization of the magnon described goes against the field. So what will change when we implement this? Well all the ω_0's, and thus the ω_m's, will be affected by this and get a negative sign. However, when then again calculating the dispersion relation, there is no difference, for we worked in the minus sign in the ω_0. Note how the dispersion relation (eq. 28) is not dependent on ω_0, and only on ω_m. So the calculation still works fine and our earlier obtained dispersion relations still hold. But they give different result due to the sign change of ω_0

This dispersion relation describes the relation for magneto-elastic coupling with unstable magnons:

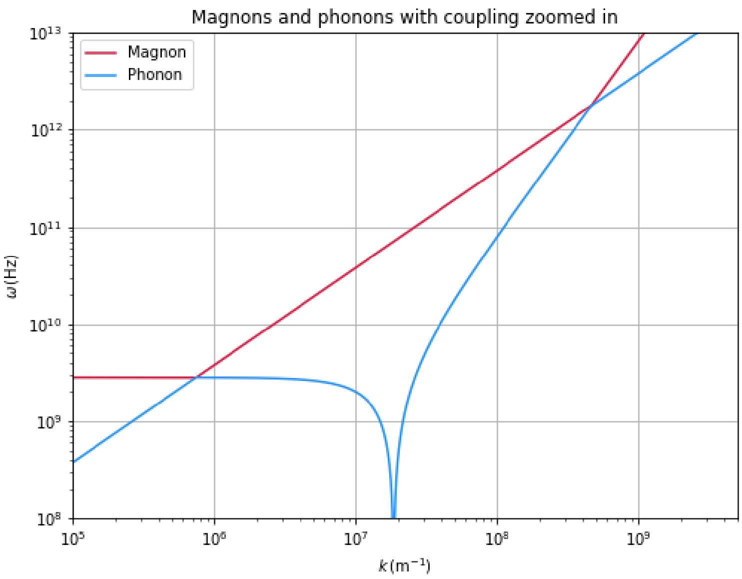

Figure 11: The dispersion relation for phonons and unstable magnons with coupling

Which we can compare to the situation without coupling:

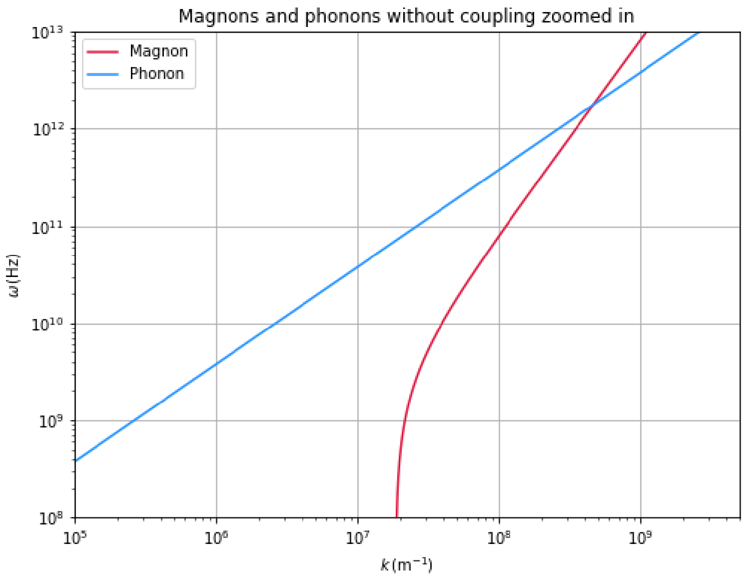

Figure 12: The dispersion relation for phonons and unstable magnons and antimagnons without coupling

We can see when comparing figure 11 and figure 13, that the coupling gives rise to more energy states. Due to the phonons being coupled to the antimagnons, there is another matching frequency that is not there when we look at uncoupled magnons and phonons. In figure 3.5 we see a zoomed in plot of the avoided crossing with unstable antimagnons and phonons

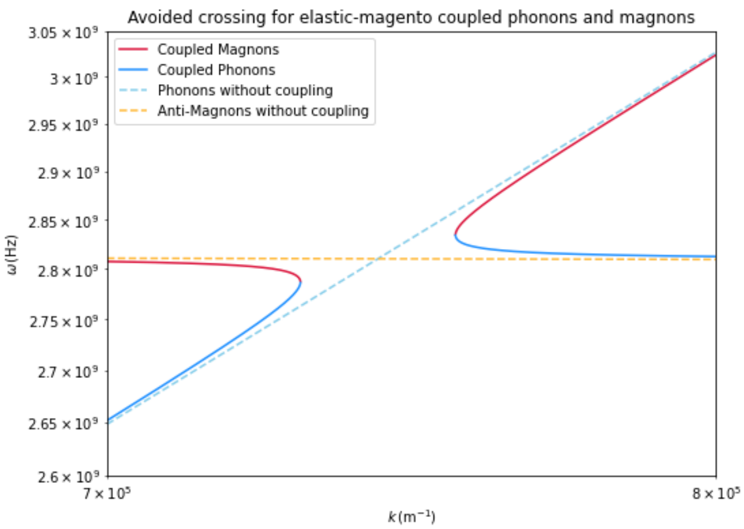

Figure 13: The dispersion relation for phonons and unstable antimagnons with coupling

Notice how the uncoupled phonons and antimagnons are also plotted. These coupled antimagnons, will interact and coupled to the incoming phonons, giving us the matter-antimatter interaction we want for the Klein effect.

4 Scattering Theory

We now want to calculate the reflection coefficient R to see whether we see a similar effect as the bosonic Klein effect. We will do this by using the fact that the waves should be continuous at the interface.

$$1\Psi_i e^{ikx} + R\Psi_r e^{ikx} = T\Psi_t e^{ikx}. \tag{62}$$

To get a better picture of the situation, we can look at figure 4, portraying the dispersion relations for both regions of the model.

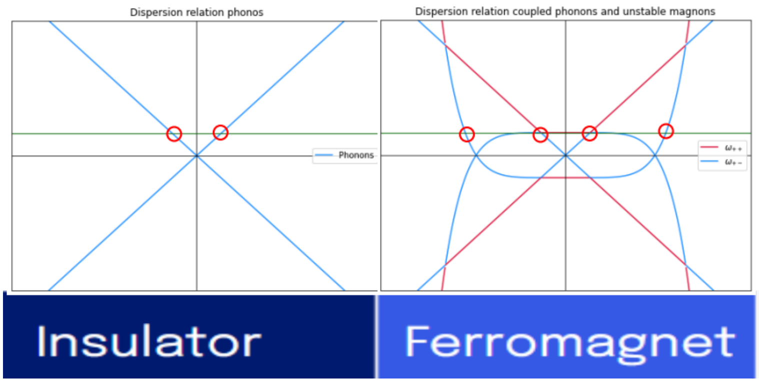

Figure 14: The dispersions in each region shown next to each other, the green line represents a certain frequency which matches with certain wave numbers(red circles)

At x = 0 is the interface between the insulator and the ferromagnet. On the left we have the insulator, which doesn't have any magnons. On the right side, the will be both magnons and phonons. This results in three scattering equations, which are our constrains On both the left and right side we have phonons, and these phonon waves need to be continuous at the inter face. This gives us our first scattering equation:

$$\begin{pmatrix} u_{p,p} \\ v_{p,p} \end{pmatrix} e^{ik_p x} + R \begin{pmatrix} u_{p,p} \\ v_{p,p} \end{pmatrix} e^{-ik_p x} = T_2 \begin{pmatrix} u_{p,2} \\ v_{p,2} \end{pmatrix} e^{ik_2 x} + T_4 \begin{pmatrix} u_{p,4} \\ v_{p,4} \end{pmatrix} e^{ik_4 x}. \tag{63}$$

Where $u_{p,p}, u_{p,2}$ and $u_{p,4}$ are the eigenvectors we can calculate with our matrix, R the reflection coefficient, $T2$ and T_4 the transmission coefficient for the to the right moving coupled magnons and phonons in the ferromagnet. In total there are four possible wave numbers that have the same frequency, as indicated by the red circles in figure 14. Note that only the second and fourth wave number are moving to the right. the first and third wave numbers per frequency on the ferromagnet side are moving to the left, which will not be happening when we send in phonons from the left. Therefore, we only consider the second and fourth frequency match in our scattering equation. T_2 and T_4 are the transmission coefficients for transmitted waves with wave number k_2 and k_4 respectively.

Now at interface $x = 0$, so the exponent will be $e^0 = 1$, gives us our first constraint:

$$(1 + R) \begin{pmatrix} u_{p,p} \\ v_{p,p} \end{pmatrix} = T_2 \begin{pmatrix} u_{p,2} \\ v_{p,2} \end{pmatrix} + T_4 \begin{pmatrix} u_{p,4} \\ v_{p,4} \end{pmatrix}. \tag{64}$$

For our last two constrains, we are going to look at the position derivatives, for these also need to be continuous for the phonons, and needs to be zero for magnons at the interface. Taking the derivative of our scattering equation for phonons, eq. 63, looks like:

$$ik_p \begin{pmatrix} u_{p,p} \\ v_{p,p} \end{pmatrix} e^{ik_p x} - ik_p R \begin{pmatrix} u_{p,p} \\ v_{p,p} \end{pmatrix} e^{-ik_p x} = ik_2 T_2 \begin{pmatrix} u_{p,2} \\ v_{p,2} \end{pmatrix} e^{ik_2 x} + ik_4 T_4 \begin{pmatrix} u_{p,4} \\ v_{p,4} \end{pmatrix} e^{ik_4 x}. \tag{65}$$

Where we still are at $x = 0$ at the interface, giving us our second constraint:

$$k_p(1-R)\begin{pmatrix} u_{p,p} \\ v_{p,p} \end{pmatrix} = k_2 T_2 \begin{pmatrix} u_{p,2} \\ v_{p,2} \end{pmatrix} + k_4 T_4 \begin{pmatrix} u_{p,4} \\ v_{p,4} \end{pmatrix}. \tag{66}$$

Lastly, we get our third and final constraint from the position derivative of the magnons, which looks like:

$$0 = k_2 T_2 \begin{pmatrix} u_{m,2} \\ v_{m,2} \end{pmatrix} + k_4 T_4 \begin{pmatrix} u_{m,4} \\ v_{m,4} \end{pmatrix}. \tag{67}$$

Now with our three equations, eq. 64, eq.66 and eq.67, we can get an expression for the reflection coefficient, R. We start by getting an expression for transmission coefficient $T4$ using eq.67, which gives us:

$$\begin{aligned} k_4 T_4 u_{m,4} &= -k_2 T_2 u_{m,2}, \\ T_4 &= -\frac{k_2 u_{m,2}}{k_4 u_{m,4}} T_2. \end{aligned} \tag{68}$$

Where for ease of notion, we only consider the first element of the magnon eigenvectors, $u_{m,i}$. Substituting this equation in eq.66, we will be able to get an expression for transmission coefficient T_2 expressed in only the eigenvectors, wavenumbers and reflection coefficient R, by using our expression in eq.68

$$\begin{aligned} k_2 T_2 u_{p,2} &= k_p(1-R)u_{p,p} - k_4 T_4 u_{p,4}, \\ k_2 T_2 u_{p,2} &= k_p(1-R)u_{p,p} - k_4(-\frac{k_2 u_{m,2}}{k_4 u_{m,4}} T_2)u_{p,4}, \\ k_2 T_2 u_{p,2} &= k_p(1-R)u_{p,p} + \frac{k_2 u_{m,2}}{u_{m,4}} T_2, \\ T_2(k_2 u_{p,2} - \frac{k_2 u_{m,2}}{u_{m,4}}) &= k_p(1-R), \\ T_2 &= (1-R)k_p \frac{u_{p,p}}{(k_2 u_{p,2} - \frac{k_2 u_{m,2}}{u_{m,4}})}. \end{aligned} \tag{69}$$

4.1 Reflection coefficient

Substituting eq. 68 and eq. 69 into eq.64 will leave us with only eigenvectors, wavenumbers and the reflection coefficient, which we can rewrite for an expression for R, which is:

$$\begin{aligned} (1+R)u_{p,p} &= ((1-R)k_p \frac{u_{p,p}u_{p,2}}{(k_2 u_{p,2} - \frac{k_2 u_{m,2}}{u_{m,4}})}) - \frac{k_2 u_{m,2} u_{p,4}}{k_4 u_{m,4}}((1-R)k_p \frac{u_{p,p}}{(k_2 u_{p,2} - \frac{k_2 u_{m,2}}{u_{m,4}})}), \\ (1+R)u_{p,p} &= ((1-R)k_p \frac{u_{p,p}u_{p,2}}{(k_2 u_{p,2} - \frac{k_2 u_{m,2}}{u_{m,4}})}) - (1-R)\frac{k_p k_2}{k_4}\frac{u_{p,p}u_{m,2}u_{p,4}}{(u_{m,4}(k_2 u_{p,2} - \frac{k_2 u_{m,2}}{u_{m,4}})}), \\ (1+R)u_{p,p} &= ((1-R)\frac{k_p}{k_2}\frac{u_{p,p}u_{p,2}u_{m,4}}{(u_{m,4}u_{p,2} - u_{m,2})}) - (1-R)\frac{k_p}{k_4}\frac{u_{p,p}u_{m,2}u_{p,4}}{(u_{m,4}u_{p,2} - u_{m,2})}, \\ R(u_{p,p} + \frac{k_p}{k_2}\frac{u_{p,p}u_{p,2}u_{m,4}}{u_{m,4}u_{p,2} - u_{m,2}} - \frac{k_p}{k_4}\frac{u_{p,p}u_{m,2}u_{p,4}}{u_{m,4}u_{p,2} - u_{m,2}}) &= (\frac{k_p}{k_2}\frac{u_{p,p}u_{p,2}u_{m,4}}{(u_{m,4}u_{p,2} - u_{m,2})} - \frac{k_p}{k_4}\frac{u_{p,p}u_{m,2}u_{p,4}}{(u_{m,4}u_{p,2} - u_{m,2})} \\ R &= \frac{(\frac{k_p}{k_2}\frac{u_{p,2}u_{m,4}}{(u_{m,4}u_{p,2} - u_{m,2})} - \frac{k_p}{k_4}\frac{u_{m,2}u_{p,4}}{(u_{m,4}u_{p,2} - u_{m,2})} - 1)}{(\frac{k_p}{k_2}\frac{u_{p,2}u_{m,4}}{u_{m,4}u_{p,2} - u_{m,2}} - \frac{k_p}{k_4}\frac{u_{m,2}u_{p,4}}{u_{m,4}u_{p,2} - u_{m,2}} + 1)}. \end{aligned} \tag{70}$$

$$R = \frac{\left(\frac{k_p}{D}\left(\frac{u_{p,2}u_{m,4}}{k_2} - \frac{u_{m,2}u_{p,4}}{k_4}\right) - 1\right)}{\left(\frac{k_p}{D}\left(\frac{u_{p,2}u_{m,4}}{k_2} - \frac{u_{m,2}u_{p,4}}{k_4}\right) + 1\right)}. \tag{71}$$

where $D = u_{m,4}u_{p,2} - u_{m,2}$.

This expression of R is dependent on the frequency, ω. For the wave numbers k_2 and k_4 are the second and fourth option to get that specific frequency from the dispersion relation.

Now we need to find the expressions for k_p, k_2 and k_4. To find the wavenumber k_p is pretty straight forward. Because for phonons, we have the dispersion relation:

$$\omega = \sqrt{\mu_F/\rho_F}k_p. \tag{72}$$

Which gives us:

$$k_p(\omega) = \pm\sqrt{\frac{\rho_F}{\mu_F}}\omega. \tag{73}$$

Where for the reflected wave in the phonon only region, there will be a sign difference from the incoming wave.

For the k_2 and k_4 finding an expression might be a challenge. We already know the dispersion relation for the magneto-elastic waves. And trying to work out the dispersion relation to an expression for the wave numbers, results in a tripolynomial of for k_i. Without this expression, it is difficult to see what we are interested in, namely the value of the reflection coefficient for wave numbers nearing the avoided crossing.

5 Conclusion and Outlook

In this chapter we summarize the derived results in the form of a conclusion and an outlook on future possibilities within this thesis project.

5.1 Conclusion

In this thesis we have verified the equations of motions for phonons and magnons from the free energy of the system. Furthermore, we use Bologiubov-deGennes(BdG) equations to introduce our magnon and phonon wave equations. We use these BdG transforms for it brings a great convenience later on, when working in scattering theory. By using the equations of motion, we were able to calculate the time derivative of our wave equations. By then employing the Bologiubov mode plane wave ansatz, we get not only the dispersion relation, the eigenvalues, but also the eigenvectors and the norms. By plotting the dispersion relations for stable magnons and unstable magnons(opposite spin), we observed how the magneto-elastic coupling causes the magnons and phonons to not cross each other, instead there are avoided crossings preventing the magnons and phonons to overlap. Furthermore, upon inspecting the results for the unstable magnon dispersion with and without coupling, we saw how the magneto-elastic coupling extended the range for the magnon energy states, due to antimagnons also being coupled. This makes for antimagnons and phonons to interact with each other, thus achieving the phononic Klein effect. Moreover, we worked out this set up using scattering theory to get an expression for the reflection coefficient of the incoming phonons.

5.2 Outlook

Firstly, we now have an expression for the reflection coefficient given in eigenvectors and wave numbers, dependent on frequency. Further research could involve getting expressions for the wave numbers and working out whether the reflection coefficient, R 1 for wave numbers nearing the avoided crossing. Moreover, there are also other interesting possibilities for future research, based on assumptions made in this thesis. One of these possiblities could be assuming there is entanglement between excitations on other side of the interface. I would be interesing to see how this would influence our results. Furthermore, another possibility is not lineasrizing the hamiltonian with the BdG equation. But working this out fully and see whether this changes our results.

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6 Appendix

6.1 Expression for R and P

In this appendis we will further work out the expression for R and P. for P_i:

$$\begin{aligned}
\frac{i}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}P &= \Psi_{phonon} - \sqrt{\frac{m\omega}{2\hbar}}R, \\
\frac{i}{\sqrt{2m\omega\hbar}}P &= \Psi_{phonon} - \sqrt{\frac{m\omega}{2\hbar}}R, \\
P &= \frac{\sqrt{2m\omega\hbar}}{i}\left(\Psi_{phonon} - \sqrt{\frac{m\omega}{2\hbar}}R\right), \\
P &= \frac{\sqrt{2m\omega\hbar}}{i}\Psi_{phonon} - \frac{m\omega}{i}R. \qquad (74)
\end{aligned}$$

And for the conjugate:

$$\begin{aligned}
-\frac{i}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}P &= \Psi^*_{phonon} - \sqrt{\frac{m\omega}{2\hbar}}R, \\
-\frac{i}{\sqrt{2m\omega\hbar}}P &= \Psi^*_{phonon} - \sqrt{\frac{m\omega}{2\hbar}}R, \\
P &= -\frac{\sqrt{2m\omega\hbar}}{i}\left(\Psi^8_{phonon} - \sqrt{\frac{m\omega}{2\hbar}}R\right), \\
P &= -\frac{\sqrt{2m\omega\hbar}}{i}\Psi^*_{phonon} + \frac{m\omega}{i}R. \qquad (75)
\end{aligned}$$

Now we can equate the two expressions of P, to get an expression for R only dependent on Ψ's:

$$\begin{aligned}
\frac{m\omega}{i}R - \frac{\sqrt{2m\omega\hbar}}{i}\Psi^*_{phonon} &= \frac{\sqrt{2m\omega\hbar}}{i}\Psi_{phonon} - \frac{m\omega}{i}R, \\
\frac{2m\omega}{i}R &= \frac{\sqrt{2m\omega\hbar}}{i}\left(\Psi_{phonon} + \Psi^*_{phonon}\right), \\
R &= \frac{i}{2m\omega}\frac{\sqrt{2m\omega\hbar}}{i}\left(\Psi_{phonon} + \Psi^*_{phonon}\right), \\
R &= \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(\Psi_{phonon} + \Psi^*_{phonon}\right). \qquad (76)
\end{aligned}$$

To find the expression for P, only dependent on Ψ's we do the same thing, but then the opposite, we get expressions for R consisting out of P and Ψ.

$$\sqrt{\frac{m\omega}{2\hbar}}R = \Psi_{phonon} - \frac{i}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}P,$$
$$\sqrt{\frac{m\omega}{2\hbar}}R = \Psi_{phonon} - \frac{i}{\sqrt{2m\omega\hbar}}P,$$
$$R = \sqrt{\frac{2\hbar}{m\omega}}\left(\Psi_{phonon} - \frac{i}{\sqrt{2m\omega\hbar}}P\right),$$
$$R = \sqrt{\frac{2\hbar}{m\omega}}\Psi_{phonon} - \frac{i}{m\omega}P. \tag{77}$$

$$\sqrt{\frac{\rho_F\omega}{2\hbar}}R = \Psi^*_{phonon} + \frac{i}{\rho_F\omega}\sqrt{\frac{\rho_F\omega}{2\hbar}}P,$$
$$\sqrt{\frac{\rho_F\omega}{2\hbar}}R = \Psi^*_{phonon} + \frac{i}{\sqrt{2\rho_F\omega\hbar}}P,$$
$$R = \sqrt{\frac{2\hbar}{\rho_F\omega}}\left(\Psi^*_{phonon} + \frac{i}{\sqrt{2\rho_F\omega\hbar}}P\right),$$
$$R = \sqrt{\frac{2\hbar}{\rho_F\omega}}\Psi^*_{phonon} + \frac{i}{\rho_F\omega}P. \tag{78}$$

Now equating these expressions give us:

$$\sqrt{\frac{2\hbar}{\rho_F\omega}}\Psi^*_{phonon} + \frac{i}{\rho_F\omega}P = \sqrt{\frac{2\hbar}{\rho_F\omega}}\Psi_{phonon} - \frac{i}{v\omega}P,$$
$$\sqrt{\frac{2\hbar}{\rho_F\omega}}\left(\Psi^*_{phonon} - \Psi_{phonon}\right) = -\frac{2i}{\rho_F\omega}P,$$
$$P = -\frac{\rho_F\omega}{2i}\sqrt{\frac{2\hbar}{\rho_F\omega}}\left(\Psi^*_{phonon} - \Psi_{phonon}\right),$$
$$P = -\frac{1}{2i}\sqrt{2\rho_F\omega\hbar}\left(\Psi^*_{phonon} - \Psi_{phonon}\right),$$
$$P = \frac{1}{2i}\sqrt{2\rho_F\omega\hbar}\left(\Psi_{phonon} - \Psi^*_{phonon}\right). \tag{79}$$

Resulting in:

$$\boxed{\begin{aligned} R &= \frac{1}{2}\sqrt{\frac{2\hbar}{\rho_F\omega}}\left(\Psi_{phonon} + \Psi^*_{phonon}\right), \\ P &= \frac{1}{2i}\sqrt{2\rho_F\omega\hbar}\left(\Psi_{phonon} - \Psi^*_{phonon}\right). \end{aligned}} \tag{80}$$

6.2 Time evolution Ψ_m^*

In this appendix we will show in detail work we workout the time evolution for the conjugate BdG equations of magnons:

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi_m^* = i\frac{1}{\hbar\gamma\sqrt{M_s}}\left(\omega_0 M_y - D\nabla^2 M_y + b_2\gamma\left(\cancel{\frac{\partial R_y}{\partial z}} + \cancel{\frac{\partial R_z}{\partial y}}\right)\right) \\ + \frac{1}{\hbar\gamma\sqrt{M_s}}\left(-\omega_0 M_x + D\nabla^2 M_x - b_2\gamma\left(\cancel{\frac{\partial R_x}{\partial z}} + \frac{\partial R_z}{\partial x}\right)\right). \end{aligned} \tag{81}$$

After crossing out the non x derivatives, we can rewrite the equation a bit:

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi_m^* &= \frac{1}{\hbar\gamma\sqrt{M_s}}\left[i\left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)M_y + \left(-\omega_0 + D\frac{\partial^2}{\partial x^2}\right)M_x - b_2\gamma\frac{\partial R_z}{\partial x}\right], \\ &= \frac{1}{\hbar\gamma\sqrt{M_s}}\left[i\left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)M_y - \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)M_x - b_2\gamma\frac{\partial}{\partial x}\left[R_z\right]\right], \\ &= \frac{1}{\hbar\gamma\sqrt{M_s}}\left[\left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\left(M_x - iM_y\right) - b_2\gamma\frac{\partial}{\partial x}\left[R_z\right]\right]. \end{aligned} \tag{82}$$

filling in M_x and M_y using eq.29 and filling in R expressed in Ψ's from eq.80:

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi_m^* &= \frac{1}{\hbar\gamma\sqrt{M_s}}\left[\left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\left(\hbar\gamma\sqrt{M_s}\Psi_m^*\right) - b_2\gamma\frac{\partial}{\partial x}\left[R_z\right]\right] \\ &= \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\Psi_m - \frac{b_2\gamma}{\hbar\gamma\sqrt{M_s}}\frac{\partial}{\partial x}R_z. \\ &= \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\Psi_m^* - \frac{b_2\gamma}{\hbar\gamma\sqrt{M_s}}\frac{\partial}{\partial x}\left[\frac{1}{2}\sqrt{\frac{2\hbar}{\rho_F\omega}}\left(\Psi_p + \Psi_p^*\right)\right]. \\ &= \left(\omega_0 - D\frac{\partial^2}{\partial x^2}\right)\Psi_m^* - \frac{b_2}{\sqrt{2\rho_F\omega\hbar M_s}}\frac{\partial}{\partial x}\left(\Psi_p + \Psi_p^*\right). \end{aligned} \tag{83}$$

Which is the end result.

6.3 Time evolution Ψ_p

In this appendix we will work out the time evolution for the BdG equation for phonons. We start with the definition for our BdG phonon equation with a time derivative. Again start rewriting the equations a bit and cross out any non x derivative:

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi_{P_x} &= \frac{\partial}{\partial t}\sqrt{\frac{\omega\rho_F}{2\hbar}}\left(iR_x - \frac{1}{\omega\rho_F}P_x\right) \\ &= \sqrt{\frac{\omega\rho_F}{2\hbar}}\left(\frac{\partial}{\partial t}iR_x - \frac{1}{\omega\rho_F}\frac{\partial}{\partial t}P_x\right) \\ &= \sqrt{\frac{\omega\rho_F}{2\hbar}}\left(\frac{iP_x}{m} - \frac{1}{\omega\rho_F}\frac{\partial}{\partial t}P_x\right) = \sqrt{\frac{\omega\rho_F}{2\hbar}}\left(\frac{iP_x}{m} - \frac{1}{\omega\rho_F}(\rho_F\frac{\partial^2}{\partial t^2}R_x)\right) \\ &= \sqrt{\frac{\omega\rho_F}{2\hbar}}\left(\frac{iP_x}{m} - \frac{1}{\omega\rho_F}(\mu_F\nabla^2 R_x + (\lambda_F + \mu_F)\cancel{\frac{\partial}{\partial z}}\nabla\cdot\vec{R} + \frac{b_2}{M_s}(\frac{\partial M_x}{\partial x} + \cancel{\frac{\partial M_y}{\partial y}})\right) \\ &= \sqrt{\frac{\rho_F\omega}{2\hbar}}\left(\frac{iP_x}{m} - \frac{\mu_F}{\omega\rho_F}\frac{\partial^2}{\partial x^2}R_x + -\frac{1}{\omega\rho_F}\frac{b_2}{M_s}\frac{\partial M_x}{\partial x}\right) \end{aligned} \tag{84}$$

Now we fill in P_z, R_z and M_x, so the expression only contains other Ψ's

$$
\begin{aligned}
&= \sqrt{\frac{\rho_F\omega_0}{2\hbar}}\frac{i}{\rho_F}\left(\frac{1}{2i}\sqrt{2\rho_F\omega_0\hbar}\left(\Psi_p-\Psi_p^*\right)\right)-\sqrt{\frac{\rho_F\omega}{2\hbar}}\frac{\mu_F}{\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\sqrt{\frac{2\hbar}{\rho_F\omega_0}}(\Psi_{px}+\Psi_{px}^*)\right) \\
&-\frac{1}{\omega\rho_F}\sqrt{\frac{\rho_F\omega_0}{2\hbar}}\frac{b_2}{M_s}\frac{\partial}{\partial x}\left(\frac{\hbar\gamma\sqrt{M_s}}{2}(\Psi_m+\Psi_m^*)\right) \\
&= \frac{\omega_0}{2}\left(\Psi_p-\Psi_p^*\right)-\frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\left((\Psi_p+\Psi_p^*)\right)-\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}\frac{\partial}{\partial x}(\Psi_m+\Psi_m^*) \\
&= \frac{\omega_0}{2}\left(\Psi_p-\Psi_p^*\right)-\frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\left((\Psi_p+\Psi_p^*)\right)-\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}\frac{\partial}{\partial x}(\Psi_m+\Psi_m^*) \\
&= \left(\frac{\omega_0}{2}-\frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right)\Psi_p+\left(\frac{-\omega_0}{2}-\frac{\mu_F}{2\omega_0\rho_F}\frac{\partial^2}{\partial x^2}\right)\Psi_p^*)-\frac{b_2\gamma\sqrt{\hbar}}{2\sqrt{2\rho_F\omega_0 M_s}}\frac{\partial}{\partial x}(\Psi_m+\Psi_m^*)
\end{aligned} \tag{85}
$$

Which is exactly what we are looking for.