Utrecht University **Faculty of Science**

Higher order predictors for the LPC curve near Bautin bifurcation in ODEs and DDEs

Bachelor Thesis, 7.5 ECTS

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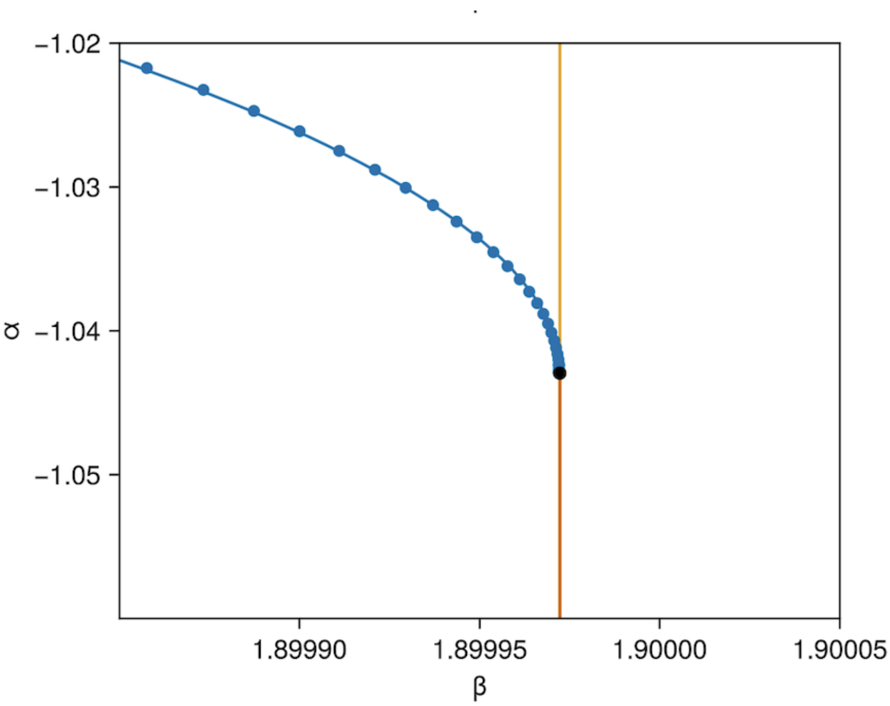

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Abstract

The generalized Hopf (Bautin) bifurcation is a well-studied codimension 2 bifurcation where the system has an equilibrium with a pair of simple purely imaginary eigenvalues and the vanishing first Lyapunov coefficient. This bifurcation can be studied in both ordinary differential equations (ODEs) and delay differential equations (DDEs). Generically, a codimension 1 bifurcation curve of nonhyperbolic limit cycles (LPC curve) emanates from a generalized Hopf point. By performing the parameter-dependent center manifold reduction near the generalized Hopf point, predictors can be derived to initiate the continuation of the LPC curve. In this thesis, we derive higher-order predictors for the LPC curve in ODEs and DDEs for the first time. The new predictors have been implemented, and their effectiveness is demonstrated on several models.

Contents

Chapter 1

Introduction

1.1 The generalized Hopf bifurcation in ODEs and DDEs

Many applications use models that consist of autonomous Ordinary Differential Equations (ODEs)

$$\dot{x}(t) = F(x(t), \alpha), \tag{1.1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$, $\alpha \in \mathbb{R}^p$ and $F : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a smooth mapping.[1] We use dot notation to indicate the derivative with respect to t, denoted as $\dot{x} = \frac{dx}{dt}$. One is often interested in the behaviour of such systems under parameter variations. As the parameters change, the phase portrait might undergo qualitative changes. For instance, equilibria might disappear or change stability. Such an event is called a *bifurcation*. Suppose that system (1.1) has an equilibrium at the origin, i.e. $F(0, 0) = 0$. The type of local bifurcations is determined by the eigenvalues of the linear part $A = D_x F(0, 0)$ of the vector field F. If A has a pair of simple imaginary eigenvalues, the imaginary eigenvalues may cross the imaginary axis under a continuous parameter variation. When this occurs, the system undergoes an *(Andronov-)Hopf bifurcation*. Near a Hopf bifurcation in planar systems depending on one parameter, the system can be transformed into the following *normal form*, through the introduction of a complex variable, and the utilization of smooth invertible coordinate transformations that depend smoothly on the parameters:

$$\dot{w} = \lambda(\alpha)w + c_1(\alpha)w|w|^2 + O(|w|^4), \quad w \in \mathbb{C}, \tag{1.2}$$

where $\lambda(\alpha) = \mu(\alpha) + i\omega(\alpha)$ with $\mu(0) = 0$, $\omega(0) = \omega_0 > 0$. We define $l_1 = \frac{1}{\omega_0}\Re\{c_1(0)\}$ as the *first Lyapunov coefficient* . Under the conditions that $l_1 \neq 0$ and $\mu'(0) \neq 0$, this system is locally topologically equivalent near the origin to the following system in polar coordinates

$$\begin{cases} \dot{\rho} &= \rho(\beta + l_1\rho^2), \\ \dot{\varphi} &= 1, \end{cases} \tag{1.3}$$

where $\beta = \frac{\mu(\alpha)}{\omega(\alpha)}$ is the new unfolding parameter. If $l_1 < 0$, this system has a stable focus at the origin for $\beta \leq 0$ and an unstable focus surrounded by a stable limit cycle for $\beta > 0$. This scenario is known as the *supercritical* Hopf bifurcation. On the other hand, if $l_1 > 0$ the Hopf bifurcation is *subcritical*, and an unstable cycle exists for $\beta < 0$, which vanishes at $\beta = 0$, resulting in an unstable focus at the origin for $\beta \geq 0$. Phase portraits of the system (1.3) for the supercritical case are shown in Figure 1.1.

The condition that $l_1 \neq 0$ in the normal form is also called the *nondegeneracy condition*.

[1] Often, the dependence of the *phase variable* x on *time* t is not explicitly indicated.

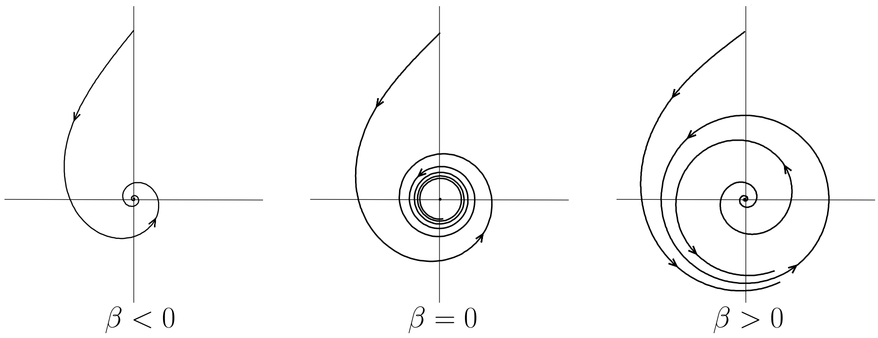

Figure 1.1: Phase portraits near a supercritical ($l_1 < 0$) Andronov-Hopf bifurcation. Figure adapted from [16].

Meanwhile, the condition $\mu'(0) \neq 0$ is referred to as the *transversality condition.* The transversality condition allows the introduction of the new unfolding parameter β. Since we need only one parameter to tune for this bifurcation to occur, i.e. get a pair of simple purely imaginary eigenvalues, it is referred to as a *codimension one* bifurcation. Another well-known codim 1 bifurcation is the fold bifurcation, where two equilibria collide and disappear at the point where a simple real eigenvalue becomes zero.

By allowing the variation of a second parameter, the first Lyapunov coefficient may vanish while the two purely imaginary eigenvalues persist. At such a point a *generalized Hopf bifurcation* (or *Bautin bifurcation*) occurs. This type of bifurcation, which requires two conditions to be satisfied to manifest, is classified as a *codimension two* bifurcation. There are four other well-known local codim 2 bifurcations. For example, a Hopf-Hopf bifurcation occurs when an equilibrium has two pairs of simple purely imaginary eigenvalues. Their normal forms and local behaviour are discussed in detail in [16, Chapter 8]. In this thesis, we will only deal with the generalized Hopf bifurcation.

Near a generalized Hopf bifurcation in planar systems depending on two parameters, the system can be transformed to the following normal form:

$$\dot{w} = \lambda(\alpha)w + c_1(\alpha)w|w|^2 + c_2(\alpha)w|w|^4 + O(|w|^6), \quad w \in \mathbb{C}, \tag{1.4}$$

where we still have $\lambda(\alpha) = \mu(\alpha) + i\omega(\alpha)$ with $\mu(0) = 0$, $\omega(0) = \omega_0 > 0$. If the *second Lyapunov coefficient* $l_2 = \frac{1}{\omega_0}\Re\{c_2(0)\} \neq 0$, this system is generically locally topologically equivalent to the following system in polar coordinates

$$\begin{cases} \dot{\rho} &= \rho(\beta_1 + \beta_2\rho^2 + l_2\rho^4), \\ \dot{\varphi} &= 1, \end{cases} \tag{1.5}$$

Here "generically" means that the map $\alpha \mapsto (\mu(\alpha), l_1(\alpha))$ is regular at $\alpha = 0$, allowing for the introduction of the new unfolding parameters β_1, β_2. From the amplitude equation in (1.5), we see that there always is a trivial equilibrium at the origin. Any non-trivial equilibrium satisfies the equation $\beta_1 + \beta_2\rho^2 + l_2\rho^4 = 0$. Depending on the values of β_1 and β_2 this equation has zero, one or two positive solutions. After a closer inspection, one will find that two codim 1 bifurcation curves emanate from the generalized Hopf point. One curve H along which supercritical/subcritical Hopf-bifurcations occur is the line $\beta_1 = 0$. And a second curve along which two hyperbolic cycles collide and disappear. This curve – along which non-hyperbolic cycles exist – is also referred to as the *LPC curve*, where

LPC stands for limit point of cycles. If $l_2 < 0$, the LPC curve in system (1.5) is given by the half-parabola $\beta_1 = \frac{1}{4l_2}\beta_2^2$ for $\beta_2 > 0$. A sketch of the bifurcation diagram for the case where $l_2 < 0$ is shown in Figure 1.2. In this illustration, we can discern three different regions near the generalized Hopf bifurcation. In region **1** there is only a stable focus at the origin. If we move from region **1** to region **2**, passing the supercritical Hopf bifurcation line H_-, the equilibrium at the origin becomes unstable and a unique stable limit cycle appears. Continuing through the subcritical Hopf bifurcation line H_+ into region **3**, the equilibrium recovers its stability, alongside the emergence of an unstable cycle nested within the pre-existing stable cycle.

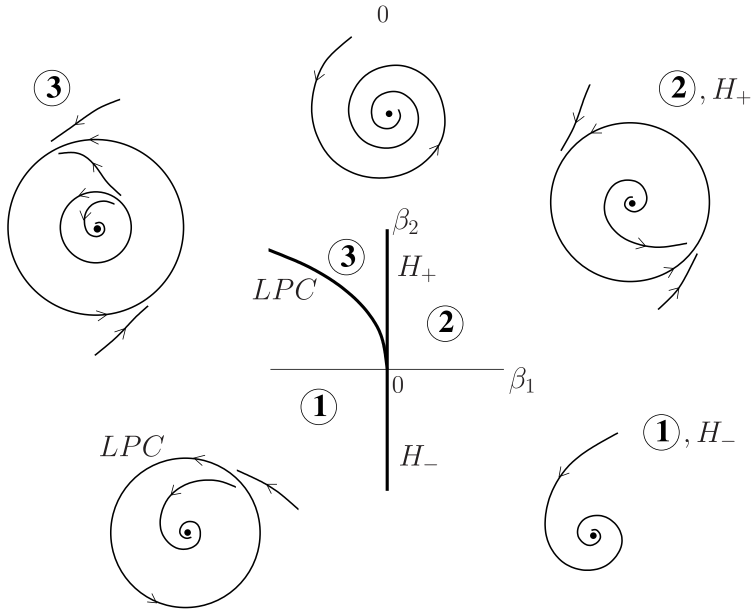

Figure 1.2: Bifurcation diagram near a generalized Hopf bifurcation for the case $l_2 < 0$. Figure adapted from [16].

Up to this point, we have only considered bifurcations in planar systems. However, the same bifurcations can occur in n-dimensional systems. In n-dimensional systems where a local bifurcation arises, a smooth family of parameter-dependent invariant *center manifolds* W^c_α exists for sufficiently small $\|\alpha\|$. All qualitative behaviour near the bifurcation point occurs on this lower dimensional manifold. For the Hopf and the generalized Hopf bifurcations, this manifold will be two-dimensional. If we restrict our system (1.1) to the center manifold, the system can locally be transformed to the normal form (1.4). The existence of an invariant center manifold can be used to derive equations for the normal form coefficients.

Untill now, our focus has been on ODEs. However, another interesting class of differential equations, known as Delay Differential Equations (DDEs) can be encountered in applications. These appear for example in the life sciences [20] and climate physics [12]. Unlike ODEs, which rely only on current state information, DDEs include the values of the phase variables at previous times. This accounts for delays in the system's response, capturing phenomena where the evolution of a system depends not only on its current state but also on its history. A specific case of DDEs that are often encountered in applications is of the following form

$$\dot{x}(t) = F(x(t), x(t-\tau_1), \ldots, x(t-\tau_m), \alpha), \tag{1.6}$$

where $x : \mathbb{R} \to \mathbb{R}^n$, $\alpha \in \mathbb{R}^p$, $F : \mathbb{R}^{n\times(m+1)} \times \mathbb{R}^p \to \mathbb{R}^n$ is a smooth mapping, and $0 < \tau_1 < \ldots < \tau_m =: h$ are constant delays. This is also known as a *discrete DDE*. More generally, a DDE is an equation of the form

$$\dot{x}(t) = F(x_t, \alpha), \quad t \geq 0, \tag{1.7}$$

where $F : X \times \mathbb{R}^p \to \mathbb{R}^n$ for $X = C([-h, 0], \mathbb{R}^n)$. Such equations are referred to as *classical DDEs*. The right hand side depends on the *history* $x_t : [-h, 0] \to \mathbb{R}^n$ defined as

$$x_t(\theta) := x(t + \theta), \qquad \text{for all } \theta \in [-h, 0].$$

We will only consider finite delays $0 < h < \infty$. To guarantee an unique solution to equation (1.7) we need to provide an initial condition

$$x(\theta) = \varphi(\theta), \quad \theta \in [-h, 0].$$

Such systems allow for study of bifurcations, including the generalized Hopf bifurcation (i.e. [13, 14, 24]). In contrast to ODEs, DDEs belong to the class of infinite dimensional dynamical systems, since $\dim X = \infty$. As a result, we will need Functional Analysis to study DDEs. However, the state space $C([-h, 0], \mathbb{R}^n)$ does lead to some technical complications. This can be resolved with the help of perturbation theory for dual semigroups, which has been developed in [5–8] and is also known as *sun-star calculus*. Using this framework, the existence of a finite-dimensional smooth invariant center manifold has been rigorously established for DDEs [3, 9]. On this center manifold, the infinite-dimensional system of DDEs can be reduced to a finite-dimensional system of ODEs. This in turn allows us to "lift" results from the bifurcation theory of ODEs to the theory of DDEs.

1.2 Numerical continuation of the LPC curve

In most cases, the codim 1 bifurcation curves near a generalized Hopf bifurcation cannot be computed analytically. Instead one uses numerical methods for the location of the codim 2 bifurcation and the continuation of the emanating codim 1 bifurcation curves. In this thesis, we will only concern ourselves with the continuation of the LPC curve.

To continue a limit cycle, we first need to know how to compute a periodic solution. We are looking for a solution with $x(T) = x(0)$, where the minimal $T > 0$ satisfying this condition is the period. Generally, the period T of the cycle is unknown. To resolve this, we rescale the time $t = T\tau$ in system (1.1) so that T becomes a parameter. For ODEs, a limit cycle can be found with the following *boundary-value problem (BVP)*

$$\begin{cases} \dot{x}(\tau) - Tf(x(\tau), \alpha) = 0, \quad \tau \in [0, 1], \\ x(1) - x(0) = 0, \\ \displaystyle\int_0^1 \langle x(\tau), \dot{x}^0(\tau) \rangle d\tau = 0, \end{cases} \tag{1.8}$$

where x^0 is a reference periodic solution. Here the first equation arises from system (1.1) after rescaling the time $t = T\tau$ and the second equation defines the periodic boundary conditions. The integral condition is a *phase condition* that ensures the uniqueness of the solution. To solve the resulting system of equations numerically, they first need to be discretized. A method commonly used for discretisation of the BVP is called *orthogonal collocation*. With this method, the solution is approximated by a piecewise polynomial and the approximated solution is required to satisfy the time-scaled system in (1.8) on a number of m collocation points within each subinterval. These collocation points are chosen as the roots of the mth degree Legendre polynomial translated to each interval. Details can be found in for example [16, Section 10.1.4]. Near a point on the LPC curve, two cycles exist that will collide on the curve and then disappear. Consequently, the BVP (1.8) will have two solutions that collide at the critical parameter. This occurs when the linearization of the BVP (1.8) with respect to $(x(\cdot), T)$ has a nontrivial solution. With a bordering technique, the bifurcation point can be detected by including a constraint $G = 0$ to system (1.8) which vanishes precisely when the linearization becomes singular. The function $G = G(x, T, \alpha)$ is determined from a bordered matrix system and more details can be found in [10].

When considering DDEs, having $x(t_0) = x(t_0 + T)$ for some $t_0 \geq 0$ is insufficient to ensure a periodic solution. Instead, periodic solutions of (1.7) must satisfy $x(t) = x(t+T)$ for all $t \in [-h, 0]$. For discrete DDEs, we have the following rescaled system to find a periodic orbit

$$\begin{cases} \dot{x}(s) - Tf(x(s), x(s - \tau_1/T), \ldots, x(s - \tau_m/T), \alpha) = 0, & s \in [0,1], \\ x(\theta + 1) - x(\theta) = 0, & \theta \in [-h/T, 0], \\ \int_0^1 \dot{x}^0(s)(x^0(s) - x(s))ds = 0, & \end{cases} \tag{1.9}$$

where $x^{(0)}$ is a reference periodic solution. This system also needs to be discretized first, which is generally done using orthogonal collocation [11, Section 3.2]. Just as for ODEs, we need to include additional equations to (1.9) to detect LPC points. We again look for a singularity of the linear part of the system (1.9) with respect to the solution x and period T [19]. Similarly to the ODE case, we can add a function to the system which vanishes at the fold point using a bordering technique. For the continuation of the LPC curve for both ODEs and DDEs, we need an initial guess for the periodic solution, the period and the parameters.

Let us now return to the LPC curve emanating from a generalized Hopf bifurcation point. If the approximate location of a generalized Hopf bifurcation point is known, we would like to switch to the LPC curve emanating from this point using only local information that is available at the generalized Hopf point. As mentioned before, to start the continuation, we need an approximation of the LPC curve in the original parameter space, an approximation of the corresponding periodic orbit and the period. Combined, this forms a *predictor*, which can be used to initialize the numerical continuation of the LPC curve starting from the generalized Hopf point. A general method to derive such approximations, using only local information at the codim 2 bifurcation point, has been introduced in [2]. Within the framework of the sun-star calculus, the normalization method developed for ODEs has been extended to DDEs [15]. These methods have already been applied to derive first-order predictors for both ODEs [17] and DDEs [3]. Such predictors, however, do not distinguish the curves H and LPC in the parameter space. In this thesis, we derive a higher-order predictor for both ODEs and DDEs that does not suffer from this drawback.

1.3 Structure

This thesis is structured as follows. We begin by describing the general technique for computing the normal form coefficients on the parameter-dependent center manifold in Chapter 2. We first address this for ODEs, then for DDEs. For DDEs, we provide a summary of relevant results from sun-star calculus. The existing special cases of solutions to the linear operator equations used in [3] are insufficient when computing higher-order coefficients. Therefore, we have derived new, more general solutions. These are presented in Section 2.2.1.

In Chapter 3, we derive higher-order approximations for the parameters of the LPC curve for the normal form. These approximations are then used to obtain a higher-order approximation for the period. We conclude this chapter with a discussion on which coefficients should be included in the parameter and center manifold approximations when extending the higher-order predictor to the general setting of ODEs and DDEs.

Then, in Chapters 4 and 5, the method from Chapter 2 is applied to derive all the coefficients for the higher-order predictor for ODEs and DDEs. All necessary components of the predictors are summarised at the end of these chapters. For the higher-order approximation, we need an expression for the seventh-order critical normal form coefficient. For ODEs, this coefficient was previously derived in [21]. For completeness, we redirived it here and identified a missing term in one of the expressions from [21], although it was included in their calculations. Additionally, the expression for the seventh-order critical normal form coefficient has now been derived for DDEs in this thesis. Finally, the previous method used for deriving parameter-dependent coefficients in [17] and [3] does not work for higher orders. Therefore, we used a slightly different approach, resulting in new equations for the parameter-dependent coefficients.

Finally, the new equations for the computation of all the coefficients in the higher-order predictor have been implemented in the programming language Julia. In Chapter 6, we illustrate the new LPC curve predictors on several models.

Then, there are three appendices. In Appendix A, we present all of the equations collected from the homological equation for ODEs. Meanwhile, Appendix B contains some remaining expressions for coefficients of the center manifold-approximation for both ODEs and DDEs that were not needed in the derivations in Chapters 4 and 5. Finally, Appendix C contains some more background on sun-star and DDEs.

Chapter 2

Parameter-dependent center manifold reduction and normalization

In this chapter, we review the general technique for computing the normal form coefficients on the parameter-dependent center manifold for ODEs and DDEs respectively. For DDEs, this includes a summary of relevant results from sun-star calculus. We also discuss the general methods used to solve the resulting linear systems. This includes additional special cases of representations required for obtaining the higher-order coefficients in DDEs.

2.1 The center manifold reduction and normalization method for ODEs

Consider a system of ODEs depending on two parameters

$$\dot{x} = F(x, \alpha), \tag{2.1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$, $\alpha \in \mathbb{R}^2$ and $F : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$. We will always assume that F is as smooth as necessary, i.e. F is C^k-smooth for some sufficiently large $k \geq 1$. Suppose that $x_0 = 0$ is an equilibrium at $\alpha_0 = 0$, i.e. $F(0, 0) = 0$. We denote the Jacobian matrix at the equilibrium as $A = D_x F(0, 0)$. Let n_c be the number of eigenvalues with zero real part and T_c the corresponding critical eigenspace. For the generalized Hopf bifurcation, we have $n_c = 2$. A procedure to switch to the codim 1 bifurcation curves emanating from a codim 2 bifurcation point was introduced in [2]. We will follow the same approach below.

For each sufficiently small $\|\alpha\|$, system (2.1) has a smooth local n_c-dimensional invariant center manifold W^c_α. At $x = 0$, W^c_0 is tangent to the critical eigenspace T^c of A. Restricted to the center manifold, we can transform the system into a certain normal form using only smooth coordinate and parameter transformations:

$$\dot{w} = G(w, \beta), \quad G : \mathbb{R}^{n_c} \times \mathbb{R}^2 \to \mathbb{R}^{n_c}. \tag{2.2}$$

These normal forms are known for all five codim 2 bifurcations, and details can be found, for example, in [16, Chapter 8]. From the normal form, we can derive an approximation for the codim 1 curves emanating from the codim 2 point. To relate the local behaviour of the normal form on the center manifold to our original system, we need a relation between the original parameters α and the unfolding parameters β:

$$\alpha = K(\beta), \quad K : \mathbb{R}^2 \to \mathbb{R}^2. \tag{2.3}$$

Furthermore, we need a parameterisation of the center manifold depending on the new parameters β:

$$x = H(w, \beta), \quad H : \mathbb{R}^{n_c} \times \mathbb{R}^2 \to \mathbb{R}^n. \tag{2.4}$$

By substituting equations (2.2), (2.3) and (2.4) into equation (2.1), we find the following so-called *homological equation*

$$H_w(w, \beta)G(w, \beta) = f(H(w, \beta), K(\beta)). \tag{2.5}$$

This is essentially a consequence of the invariance of the center manifold. In the following, we will use multi-indices ν and μ to simplify the notation in the expansions of G, H and K. A multi-index μ is defined as $\mu = (\mu_1, \ldots, \mu_n)$ for $\mu_i \in \mathbb{N}_0$ and we have that $w^\mu = w_1^{\mu_1} \ldots w_n^{\mu_2}$, $\mu! = \mu_1!\mu_2! \ldots \mu_n!$ and $|\mu| = \mu_1 + \ldots + \mu_n$. The general form of the normal form expansion is known and is expanded as

$$G(w, \beta) = \sum_{|\nu|+|\mu|\geq 1} \frac{1}{\nu!\mu!} g_{\nu\mu} w^\nu \beta^\mu. \tag{2.6}$$

Meanwhile, H and K are unknown and admit the expansions

$$H(w, \beta) = \sum_{|\nu|+|\mu|\geq 1} \frac{1}{\nu!\mu!} H_{\nu\mu} w^\nu \beta^\mu, \quad K(\beta) = \sum_{|\mu|\geq 1} \frac{1}{\mu!} K_\mu \beta^\mu. \tag{2.7}$$

All of these expansions will be truncated at some sufficiently high order. For the Taylor expansion of F, we define the multilinear forms

$$B(u, v) = \sum_{i,j=1}^{n} \frac{\partial^2 F(x_0, \alpha_0)}{\partial x_i \partial x_j} u_i v_j, \qquad J_1 u = \sum_{i=1}^{2} \frac{\partial F(x_0, \alpha_0)}{\partial \alpha_i} u_i,$$

$$C(u, v, w) = \sum_{i,j,k=1}^{n} \frac{\partial^3 F(x_0, \alpha_0)}{\partial x_i \partial x_j \partial x_k} u_i v_j w_k, \qquad A_1(u, v) = \sum_{i=1}^{n} \sum_{j=1}^{2} \frac{\partial^2 F(x_0, \alpha_0)}{\partial x_i \partial \alpha_j} u_i v_j,$$

$$B_1(u, v, w) = \sum_{i,j=1}^{n} \sum_{k=1}^{2} \frac{\partial^3 F(x_0, \alpha_0)}{\partial x_i \partial x_j \partial \alpha_k} u_i v_j w_k, \qquad \text{etc.}$$

We use the letters A, B, C, D, E, K, and L to denote (in increasing order) the derivatives of F with respect to its first argument evaluated at the critical point. The subscript denotes the order of the derivative of F with respect to the parameters.

If we substitute the expansions (2.6), (2.7) along with the Taylor expansion of F, into the homological equation (2.5), we can collect the coefficients of the $w^\nu \beta^\mu$-terms. This results in a set of equations from which it is possible to recursively solve for the unknown coefficients $g_{\nu\mu}, H_{\nu\mu}$, and K_μ.

2.1.1 Solving the linear systems

If we collect all the coefficients of the $w^\nu \beta^\mu$-terms from the homological equation (2.5), we will find linear systems of equations of the form

$$(\lambda I - A)H_{\nu\mu} = R_{\nu\mu}, \tag{2.8}$$

where λ will be some linear combination of the critical eigenvalues of A. Furthermore, the right-hand side $R_{\nu\mu}$, will depend on the coefficients $g_{\nu'\mu'}, H_{\nu'\mu'}$ with $|\nu'| + |\mu'| \leq |\nu| + |\mu|$, the coefficients $K_{\mu'}$ with $|\mu'| \leq |\mu|$ and on derivatives of F. Two situations need to be considered. Either λ is an eigenvalue, or λ is not an eigenvalue.

If λ is not an eigenvalue, the matrix $\lambda I - A$ will be invertible and system (2.8) has the unique solution

$$H_{\nu\mu} = (\lambda I - A)^{-1} R_{\nu\mu}.$$

If λ is an eigenvalue, we can apply Fredholm's solvability condition. A complex version of the Fredholm solvability condition can be stated as

Lemma 1 (Fredholm Solvability)**.** *Let $L \in \mathbb{C}^{n\times m}$ and $y \in \mathbb{C}^n$. The linear system $Lx = y$ has a solution if and only if for all $p \in \mathbb{C}^n$ satisfying $L^* p = \bar{L}^T p = 0$ we have that $\langle p, y\rangle = \bar{p}^T y = 0$.*

We only have to concern ourselves with the situation where λ is a simple eigenvalue, meaning both the algebraic and geometric multiplicities are equal to one. In that case, there exist, up to scaling, unique eigenvectors $q, p \in \mathbb{C}^n$ such that

$$(\lambda I - A)q = 0, \quad (\bar{\lambda} I - A^T)p = 0, \quad \text{and } \langle p, q\rangle = \bar{p}^T q = 1.$$

The existence of the center manifold implies that the system (2.8) must be solvable. Thus, the Fredholm alternative requires that

$$\langle p, R_{\nu\mu}\rangle = 0. \tag{2.9}$$

When $R_{\nu\mu}$ depends on the unknown normal form coefficient $g_{\nu\mu}$, the solvability condition (2.9) will result in an equation for $g_{\nu\mu}$.

To get to higher-order coefficients, we will also need a solution $H_{\nu\mu}$ to system (2.8) when λ is an eigenvalue. We can obtain the unique solution to equation (2.8) satisfying $\langle p, H_{\nu\mu}\rangle = 0$ by solving the following *bordered system*

$$\begin{pmatrix} \lambda I - A & q \\ \bar{p}^T & 0 \end{pmatrix} \begin{pmatrix} H_{\nu\mu} \\ s \end{pmatrix} = \begin{pmatrix} R_{\nu\mu} \\ 0 \end{pmatrix}, \tag{2.10}$$

with $s \in \mathbb{R}$. The $(n+1) \times (n+1)$ matrix on the left is invertible (see for example [16, Lemma 5.3]). To see that a solution to this bordered system solves equation (2.8) note that this system is equivalent to solving the following equations

$$\begin{cases} (\lambda I - A)H_{\nu\mu} + qs = R_{\nu\mu}, \\ \langle p, H_{\nu\mu}\rangle = 0. \end{cases}$$

Taking the inner product with p on both sides of the first equation yields

$$\langle p, (\lambda I - A)H_{\nu\mu}\rangle + \langle p, q\rangle s = \langle p, R_{\nu\mu}\rangle.$$

From the Fredholm alternative, we had that $\langle p, R_{\nu\mu}\rangle = 0$. Furthermore, we assumed that $\langle p, q\rangle = 1$ and we have that

$$\langle p, (\lambda I - A)H_{\nu\mu}\rangle = \langle (\bar{\lambda} I - A^T)p, H_{\nu\mu}\rangle = 0.$$

Thus, it follows that $s = 0$ and as a result we have indeed that equation (2.8) is satisfied with $\langle p, H_{\nu\mu}\rangle = 0$. We will write $H_{\nu\mu} =: A_\lambda^{INV} R_{\nu\mu}$ for the solution.

2.2 The center manifold reduction and normalization method for DDEs

As presented in [15], the normalization technique for local bifurcations in ODEs can be lifted to the infinite-dimensional setting of DDEs. In [3], this normalization method was further extended to include parameters. We will closely follow the procedure from [3] and summarize some of the results from sun-star calculus that are necessary to apply this technique for the generalized Hopf bifurcation and solve the resulting equations.

We take the nonreflexive Banach space $X := C([-h, 0], \mathbb{R}^n)$ and define for each $t \geq 0$ the history function $x_t : [-h, 0] \to \mathbb{R}^n$ at time t as

$$x_t(\theta) := x(t + \theta), \qquad \text{for all } \theta \in [-h, 0].$$

Consider the classical parameter-dependent DDE

$$\dot{x}(t) = F(x_t, \alpha), \quad t \geq 0, \tag{2.11}$$

where $F : X \times \mathbb{R}^2 \to \mathbb{R}^n$ is a C^k-smooth operator for some $k \geq 1$ and $0 < h < \infty$. Assume that system (2.11) satisfies $F(0, 0) = 0$ and that the trivial equilibrium exhibits a generalized Hopf bifurcation at $\alpha = 0$. As mentioned in the introduction, the existence of a center manifold for DDEs has rigorously been established using the mathematical framework of sun-star calculus. As for ODEs, the proof relies on a variations-of-constants formula describing the solutions. To establish a variations-of-constants formula for DDEs, it turns out that it is convenient to work in the larger space $X^{\odot\star}$, the so-called sun-star dual space of X. Below, we will only present some necessary definitions and results from sun-star calculus that will be needed in the normalisation method. Some background information on sun-star and DDEs is presented in Appendix C, although for a more complete general introduction to sun-star calculus, including proofs, we refer to [9].

There exists a unique matrix-valued function of normalized bounded variation $\zeta : [0, h] \to \mathbb{R}^{n\times n}$ such that the linear part of (2.11) at $\alpha = 0$ can written as

$$D_1F(0, 0)\varphi = \langle \zeta, \varphi \rangle := \int_0^h d\zeta(\theta)\varphi(-\theta). \tag{2.12}$$

The above integral is of Riemann-Stieltjes type. With this notation, we can write the right-hand side of (2.11) in terms of its linear and nonlinear parts

$$F(\varphi, \alpha) = \langle \zeta, \varphi \rangle + D_2F(0, 0)\alpha + G(\varphi, \alpha),$$

where the nonlinear part G is a smooth operator satisfying $G(0, 0) = 0$, $D_1G(0, 0) = 0$, and $D_2G(0, 0) = 0$. We can associate a unique C_0-semigroup[1] T on X with the linear part of (2.11) at $0 \in X$ for the critical parameter value $\alpha = 0$. Its generator A, plays an important role in the stability analysis of the nonlinear DDE (2.11). The generator of a semigroup of operators is defined as the derivative of $T(t)$ at $t = 0$. The generator of the semigroup T corresponding to the linearisation of (2.11) is given by

$$A\varphi = \dot{\varphi}, \qquad \text{with} \qquad D(A) = \{\varphi \in C^1 | \dot{\varphi}(0) = \langle \zeta, \varphi \rangle\}. \tag{2.13}$$

[1] A semigroup is a family $T = \{T(t)\}_{t\geq 0}$ of bounded linear operators with the properties: $T(0) = I$ and $T(s)T(t) = T(s + t)$ for all $t, s \geq 0$. The C_0 indicates the additional property of strong continuity: for all $\varphi \in X$, $\|T(t)\varphi - \varphi\| \to 0$ as $t \downarrow 0$. More details are given in Appendix C.

When we go to the dual space $X^\star$ of X, we lose strong continuity with the adjoint semigroup $T^\star$. Therefore, we consider the maximal subspace of strong continuity $X^\odot$. The space $X^\odot$ has the representation

$$X^\odot = \mathbb{R}^n \times L^1([0,h],\mathbb{R}^n), \tag{2.14}$$

with the duality pairing between $\varphi^\odot = (c,g) \in X^\odot$ and $\varphi \in X$ given by

$$\langle \varphi^\odot, \varphi \rangle = c^T\varphi(0) + \int_0^h g(\theta)\varphi(-\theta)d\theta. \tag{2.15}$$

On the Banach space $X^\odot$ we have a C_0-semigroup $T^\odot$ with generator $A^\odot$. The dual space $X^{\odot\star}$ of $X^\odot$ has the representation

$$X^{\odot\star} = \mathbb{R}^n \times L^\infty([0,h],\mathbb{R}^n). \tag{2.16}$$

On this space, we have the generator

$$A^{\odot\star}(\alpha,\varphi) = (\langle \zeta,\varphi\rangle, \dot{\varphi}), \quad \text{with} \quad D(A^{\odot\star}) = \{(\alpha,\varphi) | \varphi \in \mathrm{Lip}(\alpha)\}, \tag{2.17}$$

where $\mathrm{Lip}(\alpha)$ denotes the subset of $L^\infty([-h,0],\mathbb{C})$ consisting of Lipschitz continuous functions which assume the value α at $\theta = 0$. The duality pairing between $\varphi^{\odot\star} = (a,\psi) \in X^{\odot\star}$ and $\varphi^\odot = (c,g) \in X^\odot$ is given by

$$\langle \varphi^{\odot\star}, \varphi^\odot \rangle = c^T a + \int_0^h g(\theta)\psi(-\theta)d\theta. \tag{2.18}$$

We look again at the maximal subspace of strong continuity $X^{\odot\odot}$. There exists an injection $j : X \to X^{\odot\odot}$ defined by

$$j\varphi = (\varphi(0),\varphi) \in X^{\odot\odot} \text{ for all } \varphi \in X. \tag{2.19}$$

We have that $X^{\odot\odot} = j(X)$. This property is also known as *sun-reflexivity*. We will often move back and forth between the space X and its sun-dual space $X^{\odot\star}$.

Assume that there are $n_0 \geq 1$ eigenvalues of the linearization of (2.11) at $\alpha = 0$ on the imaginary axis, with a corresponding real n_0-dimensional center eigenspace X_0. Then [3, Corollary 20] will imply the existence of a parameter-dependent local center manifold $\mathcal{W}^c_{\mathrm{loc}}(\alpha)$ for (2.11). As in the ODE case, we want to include a relation $\alpha = K(\beta)$ between the original parameters α and some new unfolding parameters β. Let $u : I \to X$ with $u(t) := x_t \in \mathcal{W}^c_{\mathrm{loc}}(\alpha)$ be as in [3, Corollary 20]. Then, u is differentiable on I and satisfies the equation

$$j\dot{u}(t) = A^{\odot\star}ju(t) + (D_2F(0,0)K(\beta))r^{\odot\star} + G(u(t),K(\beta))r^{\odot\star}, \quad \text{for all } t \in I. \tag{2.20}$$

Here $wr^{\odot\star} = (w,0) \in X^{\odot\star}$, for $w \in \mathbb{R}^n$. Now, choose a basis Φ of X_0. With respect to Φ and in terms of the new parameter β, we can consider the locally defined C^k-smooth parameterization $H : \mathbb{R}^{n_c} \times \mathbb{R}^2 \to X$ of the center manifold $\mathcal{W}^c_{\mathrm{loc}}(\alpha)$. Let $z(t)$ be the coordinate with respect to Φ of the projection of $u(t)$ onto the center subspace X_0. Then $z : I \to \mathbb{R}^{n_0}$ satisfies a parameter-dependent ODE where the right-hand side is a C^k-smooth vector field that can be expanded as

$$\dot{z} = \sum_{|\nu|+|\mu|\geq 1} \frac{1}{\nu!\mu!} g_{\nu\mu} z^\nu \beta^\mu. \tag{2.21}$$

We may assume that (2.21) is a smooth normal form in terms of the unfolding parameters β. Furthermore, we have that

$$u(t) = H(z(t), \beta), \quad t \in I. \tag{2.22}$$

If we substitute equation (2.22) into the equation (2.20), we find the following *homological equation*

$$A^{\odot\star} jH(z,\beta) + (J_1 K(\beta)) r^{\odot\star} + G(H(z,\beta), K(\beta)) r^{\odot\star} = jD_z H(z,\beta)\dot{z}, \tag{2.23}$$

where $\dot{z}$ is given by (2.21) and we defined $J_1 := D_2 F(0,0)$. The mappings H and K allow for the expansions

$$H(z,\beta) = \sum_{|\nu|+|\mu|\geq 1} \frac{1}{\nu!\mu!} H_{\nu\mu} z^\nu \beta^\mu, \quad K(\beta) = \sum_{|\mu|\geq 1} \frac{1}{\mu!} K_\mu \beta^\mu. \tag{2.24}$$

Meanwhile, the nonlinear part $G(\varphi, \alpha)$ can be expanded as

$$G(\varphi,\alpha) = \sum_{r+s>1} \frac{1}{r!s!} D_1^r D_2^s F(0,0)(\varphi^{(r)}, \alpha^{(s)}), \tag{2.25}$$

where $\varphi^{(r)} = (\varphi, \ldots, \varphi) \in X^r$, $\alpha^{(s)} = (\alpha, \ldots, \alpha) \in [\mathbb{R}^2]^s$ and $D_1^r D_2^s F(0,0) : X^r \times [\mathbb{R}^2]^s \to \mathbb{R}^n$ is the mixed Fréchet derivative of order $r+s$ evaluated at $(0,0) \in X \times \mathbb{R}^2$. Just as in the ODE-case, we will indicate the multilinear forms in the expansion of F with B, C, D, E, K, L and for the parameter-dependent derivatives B_i, C_i, etc. where the subscript i indicated the number of derivatives with respect to the parameter. Thus,

$$\begin{aligned} &J_1 := D_2F(0,0), && B(u,u) = D_1^2 F(0,0)(u,u), \\ &A_1(u,\alpha) = D_1^1 D_2^1 F(0,0)(u,\alpha), && C(u,u,u) = D_1^3 F(0,0)(u,u,u), \\ &B_1(u,u,\alpha) = D_1^2 D_2^1 F(0,0)(u,u,\alpha), && \textit{etc.} \end{aligned}$$

For the case of discrete DDEs, explicit formulas for the computation of the multilinear forms are presented in [3, Section 5].

Similar to what we did for ODEs, we can now substitute the expansions (2.21), (2.24) and (2.25) into the homological equation (2.23) and collect terms of equal powers $z^\nu \beta^\mu$. Then it is possible to solve recursively for the coefficients $g_{\nu\mu}, H_{\nu\mu}$ and K_μ. Before we discuss how we can solve the resulting equations, we need some facts about the spectrum of the generator A.

The spectrum To determine if bifurcations are present, we need to analyse the spectrum of the generator A of the semigroup corresponding to the linear part of (2.11). For this, we need the *characteristic matrix function* which is defined as

$$\Delta(z) = zI - \int_0^h e^{-z\theta} d\zeta(\theta), \quad \Delta : \mathbb{C} \to \mathbb{C}^{n\times n}, \tag{2.26}$$

and contains all spectral information about A[2]. The integral in the above expression is of Riemann-Stieltjes type and ζ is the same from (2.12). The eigenvalues of A are given by the roots of the *characteristic equation*

$$\det \Delta(\lambda) = 0. \tag{2.27}$$

[2] In general, when we work with operators on infinite-dimensional spaces we divide the spectrum into three parts: the point spectrum, the residual spectrum and the continuous spectrum. The point spectrum consists of all the eigenvalues. Since T will eventually be compact, the spectrum of A consists only of isolated eigenvalues, and the corresponding eigenspaces are finite-dimensional.

We only have to concern ourselves with simple eigenvalues, i.e. eigenvalues for which both the geometric and algebraic multiplicities equal one. If $\lambda \in \mathbb{C}$ is a simple eigenvalue of A, there exist an eigenfunction φ and an adjoint eigenfunction $\varphi^{\odot}$ such that

$$A\varphi = \lambda\varphi, \quad A^{\star}\varphi^{\odot} = \lambda\varphi^{\odot}.$$

Let $q, p \in \mathbb{C}^n$ such that

$$\Delta(\lambda)q = 0, \quad p^T\Delta(\lambda) = 0.$$

Then the corresponding eigenfunctions are given by

$$\varphi(\theta) = e^{\lambda\theta}q, \quad \theta \in [-h, 0] \tag{2.28}$$

and

$$\varphi^{\odot} = \left(p, \theta \mapsto p\int_{\theta}^{h} e^{\lambda(\theta-\tau)}d\zeta(\tau)\right), \quad \theta \in [-h, 0]. \tag{2.29}$$

Furthermore, it is possible to normalize the eigenfunctions to

$$\langle\varphi^{\odot}, \varphi\rangle = p\Delta'(\lambda)q = 1.$$

Here $\Delta'(\lambda)$ is the derivative of $z \mapsto \Delta(z)$ evaluated at $z = \lambda$

$$\Delta'(\lambda) = I + \int_0^h \theta e^{-\lambda\theta}d\zeta(\theta). \tag{2.30}$$

Proofs of the above results on the spectrum of A can be found in [9, Chapter IV]. Note that for $k \geq 2$ we have the higher-order derivatives

$$\Delta^{(k)}(\lambda) = (-1)^{k+1}\int_0^h \theta^k e^{-\lambda\theta}d\zeta(\theta). \tag{2.31}$$

In the following, we will denote the second, third, and fourth derivatives as $\Delta''(\lambda)$, $\Delta'''(\lambda)$, and $\Delta''''(\lambda)$ respectively.

In the special case of the discrete DDE (1.6), the characteristic matrix is given by

$$\Delta(z) = zI - \sum_{j=0}^{m} M_j e^{-z\tau_j}, \quad z \in \mathbb{C},$$

where $M_j := D_{1,j}f(0,0) \in \mathbb{R}^{n\times n}$ is the partial derivative of f with respect to its jth state argument evaluated at the origin [3, Section 6].

Remark. When dealing with the spectrum of A, it is actually necessary to complexify all of the above spaces and the linear operators acting on them. However, as previously remarked by [15, Remark 2.2] this is not a trivial task. Fortunately, this has already been carried out in detail in [9, Section III.7] and therefore will not be discussed here.

2.2.1 Solving the linear operator equations

From the homological equation (2.23) we will find equations of the form

$$(\lambda I - A^{\odot\star})(v_0, v) = (w_0, w), \tag{2.32}$$

for some known $(w_0, w) \in X^{\odot\star}$, $\lambda \in \mathbb{C}$ and an unknown $(v_0, v) \in D(A^{\odot\star})$. We will only have to consider two cases, either λ is a simple eigenvalue or λ is not an eigenvalue.

If λ is not an eigenvalue, then equation (2.32) has a unique solution

$$(v_0, v) = (\lambda I - A^{\odot\star})^{-1}(w_0, w). \tag{2.33}$$

To compute the solutions, we need some representation for the solution (v_0, v). The general representation is given by the following result

Lemma 2 ([15], Lemma 3.3)**.** *Suppose that λ is not an eigenvalue. Then the unique solution of* (2.32) *is given by*

$$v(\theta) = e^{\lambda\theta} v_0 + \int_\theta^0 e^{\lambda(\theta-\sigma)} w(\sigma) d\sigma \quad (\theta \in [-h, 0]), \tag{2.34}$$

with

$$v_0 = \Delta^{-1}(\lambda) \left\{ w_0 + \int_0^h d\zeta(\tau) \int_0^\tau e^{-\lambda\sigma} w(\sigma - \tau) d\sigma \right\}. \tag{2.35}$$

Of course, the above representation is not very nice to work with. Fortunately, we will only encounter equations where the right-hand side of (2.32) has a specific form that allows us to write the above representations in terms of derivatives of the characteristic matrix function $\Delta(z)$. The following Corollary presents some useful cases.

Corollary 1. *Suppose that λ is not an eigenvalue. We have the following special cases*

1. *Suppose that $(w_0, w) = (w_0, 0)$. Then the unique solution $(v_0, v) \in D(A^{\odot\star})$ of* (2.32) *has the representation*

$$v_0 = v(0), \quad v(\theta) = e^{\lambda\theta} \Delta^{-1}(\lambda) w_0.$$

2. *Suppose that $(w_0, w) = \big(0, \theta \mapsto e^{\lambda\theta} \Delta^{-1}(\lambda)\eta\big)$ for some fixed $\eta \in \mathbb{C}^n$. Then the unique solution $(v_0, v) \in D(A^{\odot\star})$ of* (2.32) *has the representation*

$$v_0 = v(0), \quad v(\theta) = \Delta^{-1}(\lambda)[\Delta'(\lambda) - I - \theta\Delta(\lambda)] w(\theta).$$

3. *Let $k \geq 1$. Suppose that $(w_0, w) = \big(0, \theta \mapsto \theta^k e^{\lambda\theta} \Delta^{-1}(\lambda)\eta\big)$ for some fixed $\eta \in \mathbb{C}^n$. Then the unique solution $(v_0, v) \in D(A^{\odot\star})$ of* (2.32) *has the representation*

$$v_0 = v(0), \quad v(\theta) = \frac{1}{k+1} e^{\lambda\theta} \Delta^{-1}(\lambda)[\Delta^{(k+1)}(\lambda) - \theta^{k+1}\Delta(\lambda)]\Delta^{-1}(\lambda)\eta.$$

Proof. These representations follow by applying Lemma 2. The first follows immediately after substitution into equation (2.34) and a derivation of the second case is presented in

[15, Corollary 3.4]. For the third case, we first calculate v_0 using equation (2.35). This yields for $k \geq 1$

$$\begin{aligned}
v_0 &= \Delta^{-1}(\lambda) \int_0^h d\zeta(\tau) \int_0^\tau e^{-\lambda\sigma} e^{\lambda(\sigma-\tau)} (\sigma-\tau)^k d\sigma \Delta^{-1}(\lambda)\eta, \\
&= \Delta^{-1}(\lambda) \int_0^h \int_0^\tau e^{-\lambda\tau} (\sigma-\tau)^k d\sigma d\zeta(\tau) \Delta^{-1}(\lambda)\eta, \\
&= \Delta^{-1}(\lambda) \int_0^h \frac{1}{k+1} (-1)^{k+2} \tau^{k+1} e^{-\lambda\tau} d\zeta(\tau) \Delta^{-1}(\lambda)\eta, \\
&= \frac{1}{k+1} \Delta^{-1}(\lambda) \Delta^{(k+1)}(\lambda) \Delta^{-1}(\lambda)\eta,
\end{aligned}$$

where we used expression (2.31) for the $(k+1)$-th derivative of $\Delta(z)$ at $z = \lambda$. By substituting this into (2.34) we find for $\theta \in [-h, 0]$

$$\begin{aligned}
v(\theta) &= e^{\lambda\theta} v_0 + \int_\theta^0 e^{\lambda(\theta-\sigma)} \sigma^k e^{\lambda\sigma} d\sigma \Delta^{-1}(\lambda)\eta, \\
&= e^{\lambda\theta} \left(v_0 - \frac{1}{k+1} \theta^{k+1} \Delta^{-1}(\lambda)\eta \right), \\
&= \frac{1}{k+1} e^{\lambda\theta} \left(\Delta^{-1}(\lambda)\Delta^{(k+1)}(\lambda)\Delta^{-1}(\lambda)\eta - \theta^{k+1}\Delta^{-1}(\lambda)\eta \right), \\
&= \frac{1}{k+1} e^{\lambda\theta} \Delta^{-1}(\lambda)[\Delta^{(k+1)}(\lambda)\Delta^{-1}(\lambda)\eta - \theta^{k+1}\eta], \\
&= \frac{1}{k+1} e^{\lambda\theta} \Delta^{-1}(\lambda)[\Delta^{(k+1)}(\lambda) - \theta^{k+1}\Delta(\lambda)]\Delta^{-1}(\lambda)\eta.
\end{aligned}$$

□

The special cases from Corollary 1 can be combined to arrive at the following result:

Corollary 2. *Suppose that λ is not an eigenvalue and that the right-hand side of* (2.32) *has the representation*

$$(w_0, w) = \left(w_0, \theta \mapsto e^{\lambda\theta} \Delta^{-1}(\lambda)[\eta + \theta\xi_1 + \theta^2\xi_2] \right),$$

for some fixed $\eta, \xi_1, \xi_2 \in \mathbb{C}^n$. Then the unique solution $(v_0, v) \in D(A^{\odot\star})$ of (2.32) *has the following representation*

$$\begin{aligned}
v(\theta) = e^{\lambda\theta}\Delta^{-1}(\lambda)\Big(w_0 &+ [\Delta'(\lambda) - I - \theta\Delta(\lambda)]\Delta^{-1}(\lambda)\eta \\
&+ \frac{1}{2}[\Delta''(\lambda) - \theta^2\Delta(\lambda)]\Delta^{-1}(\lambda)\xi_1 + \frac{1}{3}[\Delta'''(\lambda) - \theta^3\Delta(\lambda)]\Delta^{-1}(\lambda)\xi_2 \Big),
\end{aligned}$$

and $v_0 = v(0)$.

Proof. Write

$$\begin{aligned}
(w_0, w) = (w_0, 0) &+ \left(0, \theta \mapsto e^{\lambda\theta}\Delta^{-1}(z)\eta\right) + \left(0, \theta \mapsto \theta e^{\lambda\theta}\Delta^{-1}(\lambda)\xi_1\right) \\
&+ \left(0, \theta \mapsto \theta^2 e^{\lambda\theta}\Delta^{-1}(\lambda)\xi_2\right).
\end{aligned}$$

Using the linearity of the inverse operator $(\lambda I - A^{\odot\star})^{-1}$ we can apply the cases from Corollary 1. □

Finally, from this result, we can derive the following special case, which will make our calculations in Section 5.1 easier.

Corollary 3. *Suppose that λ is not an eigenvalue. We have the following special case: Suppose that (w_0, w) has the form*

$$w_0 = w(0), \quad w(\theta) = e^{\lambda\theta}\Delta^{-1}(\lambda)\left(M + [\Delta'(\lambda) - \theta\Delta(\lambda)]\hat{\eta} + [\Delta''(\lambda) - \theta^2\Delta(\lambda)]\hat{\xi}\right),$$

for some fixed $M, \hat{\eta}, \hat{\xi} \in \mathbb{C}^n$. Then the unique solution $(v_0, v) \in D(A^{\odot\star})$ of (2.32) has the representation

$$v(\theta) = e^{\lambda\theta}\Delta^{-1}(\lambda)\left([\Delta'(\lambda) - \theta\Delta(\lambda)]w(0) - \frac{1}{2}[\Delta''(\lambda) - \theta^2\Delta(\lambda)]\hat{\eta} - \frac{1}{3}[\Delta'''(\lambda) - \theta^3\Delta(\lambda)]\hat{\xi}\right).$$

and $v_0 = v(0)$.

Proof. We can rewrite $w(\theta)$ as

$$w(\theta) = e^{\lambda\theta}\Delta^{-1}(\lambda)[M + \Delta'(\lambda)\hat{\eta} + \Delta''(\lambda)\hat{\xi} - \theta\Delta(\lambda)\hat{\eta} - \theta^2\Delta(\lambda)\hat{\xi}].$$

Now we can apply Corollary 2 by taking $\eta = M + \Delta'(\lambda)\hat{\eta} + \Delta''(\lambda)\hat{\xi}$, $\xi_1 = -\Delta(\lambda)\hat{\eta}$ and $\xi_2 = -\Delta(\lambda)\hat{\xi}$. Finally, use that $w_0 = \Delta^{-1}(\lambda)\eta$ to simplify the expression. □

If λ is an eigenvalue, system (2.32) does not have a unique solution. Just as when we solve the matrix equations for the ODE case, there exists a variant of the Fredholm solvability condition for operator equations of the form (2.32).

Lemma 3 ([15], Lemma 3.2)**.** *For arbitrary λ, a solution $(v_0, v) \in D(A^{\odot\star})$ to system (2.32) exists if and only if*

$$\langle (w_0, w), \varphi^{\odot}\rangle = 0, \quad \textit{for all} \quad \varphi^{\odot} \in N(\lambda I - A^{\star}).$$

This condition is also referred to as the Fredholm solvability condition in much of the literature. During our computations in Section 5.1 we will also refer to it as the Fredholm solvability condition. It should be clear from the context which version we use. Similarly to before, we can use a bordered *operator* inverse

$$(\lambda I - A^{\odot\star})^{INV} : R(\lambda I - A^{\odot\star}) \to D(A^{\odot\star})$$

to find a unique solution to system (2.32) satisfying $\langle (v_0, v), \varphi^{\odot}\rangle = 0$ for all (w_0, w) for which (2.32) is consistent, i.e. satisfies the Fredholm solvability condition. A general representation for $(\lambda I - A^{\odot\star})^{INV}$ in the case that λ is a simple eigenvalue is given by the following proposition

Proposition 1 ([15], Proposition 3.6)**.** *Let λ be a simple eigenvalue of A with eigenvector φ and adjoint eigenvector $\varphi^{\odot}$, such that $\langle \varphi^{\odot}, \varphi\rangle = 1$. Suppose that (2.32) is consistent for some given $(w_0, w) \in X^{\odot\star}$. Then the unique solution $(v_0, v) = (\lambda I - A^{\odot\star})^{INV}(w_0, w)$ satisfying $\langle (v_0, v), \varphi^{\odot}\rangle = 0$ is given by*

$$v_0 = \xi + \gamma q, \quad v(\theta) = e^{\lambda\theta}v_0 + \int_\theta^0 e^{\lambda(\theta-\sigma)}w(\sigma)d\sigma \quad (\theta \in [-h, 0]), \tag{2.36}$$

with

$$\xi = \Delta(\lambda)^{INV}\left[w_0 + \int_0^h d\zeta(\tau)\int_0^\tau e^{-\lambda\sigma}w(\sigma - \tau)d\sigma\right], \tag{2.37}$$

and the constant γ is given by

$$\gamma = -p\Delta'(\lambda)\xi - p\int_0^h \int_\tau^h e^{-\lambda s}d\zeta(s)\int_{-\tau}^0 e^{-\lambda\sigma}w(\sigma)d\sigma d\tau. \tag{2.38}$$

The expression for ξ contains the bordered matrix inverse $\Delta(\lambda)^{INV}$. A unique solution $x = \Delta(\lambda)^{INV} y$ satisfying $p^T x = 0$ can be found by solving the following bordered system

$$\begin{pmatrix} \Delta(\lambda) & q \\ p^T & 0 \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix},$$

for $(x, s) \in \mathbb{C}^{n+1}$. We will occasionally encounter the following special case:

Corollary 4 ([15], Corollary 3.7)**.** *Suppose in addition that $(w_0, w) = (\eta, 0) + \kappa(q, \varphi)$ for some $\eta \in \mathbb{C}^n$ and $\kappa \in \mathbb{C}$. Then*

$$v_0 = \xi + \gamma q, \quad v(\theta) = e^{\lambda\theta}(v_0 - \kappa\theta q),$$

with

$$\xi = \Delta^{INV}(\lambda)(\eta + \kappa\Delta'(\lambda)q) \text{ and } \gamma = -p\Delta'(\lambda)\xi + \frac{1}{2}\kappa p\Delta''(\lambda)q.$$

For this case we will use the notation $v = B_\lambda^{INV}(\eta, \kappa)$.

For the derivation of the linear predictor in [3], the special case from Corollary 4 was enough. However, if we want to derive the equations for the higher order coefficients we will find that we need two more special cases for the representation of $(\lambda I - A^{\odot\star})^{INV}$. Before we state these, we derive the following more general result when w is of a polynomial type.

Corollary 5. *Suppose in addition to the assumptions from Proposition 1 that (w_0, w) satisfies*

$$(w_0, w) = \left(w_0, e^{\lambda\theta}(\xi_0 + \theta\xi_1 + \ldots + \theta^m\xi_m)\right),$$

for some $m \in \mathbb{N}$ and constant vectors $\xi_k \in \mathbb{C}^n$, $0 \leq k \leq m$. Then the unique solution $(v_0, v) = (\lambda I - A^{\odot\star})^{INV}(w_0, w)$ satisfying $\langle (v_0, v), \varphi^{\odot} \rangle = 0$ is given by

$$v_0 = \xi + \gamma q, \quad v(\theta) = e^{\lambda\theta}\left(v_0 - \theta\xi_0 - \frac{1}{2}\theta^2\xi_1 - \ldots - \frac{1}{m+1}\theta^{m+1}\xi_m\right),$$

where

$$\xi = \Delta(\lambda)^{INV}\left[w_0 + [\Delta'(\lambda) - I]\xi_0 + \frac{1}{2}\Delta''(\lambda)\xi_1 + \ldots + \frac{1}{m+1}\Delta^{(m+1)}(\lambda)\xi_m\right],$$

and

$$\gamma = -p\Delta'(\lambda)\xi + \frac{1}{2}p\Delta''(\lambda)\xi_0 + \frac{1}{6}p\Delta'''(\lambda)\xi_1 + \ldots + \frac{1}{(m+1)(m+2)}\Delta^{(m+2)}(\lambda)\xi_m.$$

Proof. Write $w(\theta) = e^{\lambda\theta}\sum_{k=0}^m \theta^k\xi_k$ and apply Proposition 1. Filling the expression of $w(\theta)$ into equation (2.36) yields

$$\begin{aligned} v(\theta) &= e^{\lambda\theta}v_0 + \int_\theta^0 e^{\lambda(\theta-\sigma)}e^{\lambda\sigma}\sum_{k=0}^m \sigma^k\xi_k d\sigma, \\ &= e^{\lambda\theta}\left(v_0 + \sum_{k=0}^m \int_\theta^0 \sigma^k d\sigma\xi_k\right), \\ &= e^{\lambda\theta}\left(v_0 - \sum_{k=0}^m \frac{1}{k+1}\theta^{k+1}\xi_k\right). \end{aligned}$$

For the expression of ξ we first evaluate the integral in equation (2.37). This results in

$$\begin{aligned}
Int\xi &= \int_0^h d\zeta(\tau) \int_0^\tau e^{-\lambda\sigma} e^{\lambda(\sigma-\tau)} \sum_{k=0}^m (\sigma-\tau)^k \xi_k d\sigma, \\
&= \int_0^h e^{-\lambda\tau} d\zeta(\tau) \sum_{k=0}^m \left[\frac{1}{k+1} (\sigma-\tau)^{k+1} \right]_{\sigma=0}^{\sigma=\tau} \xi_k, \\
&= \sum_{k=0}^m \frac{(-1)^{k+2}}{k+1} \int_0^h \tau^{k+1} e^{-\lambda\tau} d\zeta(\tau) \xi_k, \\
&= [\Delta'(\lambda) - I]\xi_0 + \sum_{k=1}^m \frac{1}{k+1} \Delta^{(k+1)}(\lambda)\xi_k.
\end{aligned}$$

Here we used expressions (2.30) and (2.31) for the derivatives $\Delta^{(k)}(\lambda)$. Filling this into equation (2.37) will yield the expression for ξ. Finally, evaluating the integral in equation (2.38) results in

$$\begin{aligned}
Int\gamma &= \int_0^h \int_\tau^h e^{-\lambda s} d\zeta(s) \sum_{k=0}^m \int_{-\tau}^0 \sigma^k \xi_k d\sigma d\tau, \\
&= \sum_{k=0}^m \int_0^h \int_\tau^h e^{-\lambda s} d\zeta(s) \frac{(-1)^{k+2}}{k+1} \tau^{k+1} \xi_k d\tau, \\
&= \sum_{k=0}^m \int_0^h e^{-\lambda s} \int_0^s \frac{(-1)^{k+2}}{k+1} \tau^{k+1} d\tau d\zeta(s) \xi_k, \\
&= -\sum_{k=0}^m \frac{(-1)^{k+3}}{(k+1)(k+2)} \int_0^h s^{k+2} e^{\lambda s} d\zeta(s) \xi_k, \\
&= -\sum_{k=0}^m \frac{1}{(k+1)(k+2)} \Delta^{(k+2)}(\lambda)\xi_k.
\end{aligned}$$

Here we used Fubini's theorem to switch the order of integration in the third equality and expression (2.31) in the last step. Filling this expression into equation (2.38) yields the expression for γ. □

From Corollary (7), we have the following two special cases that we will use in our derivations in Section 5.1:

Corollary 6. *Suppose in addition to the assumptions from Proposition 1 that* (w_0, w) *satisfies*

$$w_0 = w(0), \quad w(\theta) = e^{\lambda\theta}(w_0 - \kappa\theta q),$$

for some $\kappa \in \mathbb{C}$. *Then the unique solution* $(v_0, v) = (\lambda I - A^{\odot\star})^{INV}(w_0, w)$ *satisfying* $\langle (v_0, v), \varphi^{\odot} \rangle = 0$ *is given by*

$$v_0 = \tilde{\xi} + \tilde{\gamma} q, \quad v(\theta) = e^{\lambda\theta}\left(v_0 - \theta w_0 + \frac{1}{2}\kappa\theta^2 q \right),$$

where

$$\tilde{\xi} = \Delta(\lambda)^{INV}\left[\Delta'(\lambda) w_0 - \frac{1}{2}\kappa\Delta''(\lambda) q \right],$$

and

$$\tilde{\gamma} = -p\Delta'(\lambda)\tilde{\xi} + \frac{1}{2}p\Delta''(\lambda)w_0 - \frac{1}{6}\kappa p\Delta'''(\lambda)q.$$

For this case we will use the notation $v = \tilde{B}_\lambda^{INV}(w,\kappa)$.

Corollary 7. *Suppose in addition to the assumptions from Proposition 1 that* (w_0, w) *satisfies*

$$w_0 = w(0), \quad w(\theta) = e^{\lambda\theta}(w_0 - \theta\xi + \frac{1}{2}\kappa\theta^2 q),$$

for some constant vector $\xi \in \mathbb{C}^n$ *and some constant* $\kappa \in \mathbb{C}$. *Then the unique solution* $(v_0, v) = (\lambda I - A^{\odot\star})^{INV}(w_0, w)$ *satisfying* $\langle (v_0, v), \varphi^{\odot}\rangle = 0$ *is given by*

$$v_0 = \hat{\xi} + \hat{\gamma}q, \quad v(\theta) = e^{\lambda\theta}\left(v_0 - \theta w_0 + \frac{1}{2}\theta^2\xi - \frac{1}{6}\kappa\theta^3 q\right),$$

where

$$\hat{\xi} = \Delta(\lambda)^{INV}\left[\Delta'(\lambda)w_0 - \frac{1}{2}\Delta''(\lambda)\xi + \frac{1}{6}\kappa\Delta'''(\lambda)q\right],$$

and

$$\hat{\gamma} = -p\Delta'(\lambda)\hat{\xi} + \frac{1}{2}p\Delta''(\lambda)w_0 - \frac{1}{6}p\Delta'''(\lambda)\xi + \frac{1}{24}\kappa p\Delta''''(\lambda)q.$$

For this case we will use the notation $v = \hat{B}_\lambda^{INV}(w,\xi,\kappa)$.

Chapter 3

Higher order LPC curve approximmation for the normal form

In this chapter, we will first derive a parameter approximation of the LPC curve for the normal form. Then, we will use this to derive an approximation for the period of the cycles on the LPC curve. We will conclude the chapter with a detailed discussion on which coefficients need to be included in the center manifold and parameter approximations when extending to general n-dimensional systems of ODEs.

3.1 Approximmation of the LPC curve derived from the normal form

Assume that at $\alpha = 0$ there is an equilibrium at the origin with only one pair of purely imaginary simple eigenvalues

$$\lambda_{1,2} = \pm i\omega_0, \quad \omega_0 > 0.$$

Restricted to the center manifold near a generalized Hopf bifurcation, the system (2.1) can be transformed to the following parameter-dependent normal form

$$\dot{w} = \lambda(\alpha)w + c_1(\alpha)w|w|^2 + c_2(\alpha)w|w|^4 + c_3(\alpha)w|w|^6 + O(|w|^8), \quad w \in \mathbb{C},$$

where $\lambda(0) = i\omega_0$, $d_1 = \Re(c_1(0)) = 0$ and $d_2 = \Re(c_2(0)) \neq 0$. Furthermore, we have the following *transversaility* condition:

$$\text{The map } \alpha \mapsto (\Re(\lambda(\alpha)), \Re(c_1(\alpha))) \text{ is regular at } \alpha = 0. \tag{3.1}$$

If this condition is satisfied, we may introduce new parameters

$$(\beta_1(\alpha), \beta_2(\alpha)) = (\Re(\lambda(\alpha)), \Re(c_1(\alpha))).$$

Then, for $\|\alpha\|$ small enough, the normal form can be expressed in terms of β. This results in the following expression for the normal form

$$\dot{w} = (i\omega_0 + \beta_1 + ib_1(\beta))w + (\beta_2 + ib_2(\beta))w|w|^2 + c_2(\beta)w|w|^4 + c_3(\beta)w|w|^6 + O(|w|^8), \tag{3.2}$$

where b_1 and b_2 are real valued functions with $b_1(0) = 0$ and $b_2(0) = \Im(c_1(0))$. Note that we write $c_i(\beta)$ instead of $c_i(\alpha(\beta))$ for convenience. To get a higher-order approximation of the LPC curve, we first substitute $w = \rho e^{i\psi}$ into (3.2). Taking the real part yields the following amplitude equation

$$\dot{\rho} = \rho(\beta_1 + \beta_2\rho^2 + \Re(c_2(\beta))\rho^4 + \Re(c_3(\beta))\rho^6 + O(\rho^7)). \tag{3.3}$$

3.1.1 LPC curve in the amplitude equation

To derive an approximation for the LPC curve we expand

$$\begin{aligned}\Re(c_2(\beta)) &= d_2 + a_{3210}\beta_1 + a_{3201}\beta_2 + O(\|\beta\|^2),\\ \Re(c_3(\beta)) &= d_3 + O(\|\beta\|).\end{aligned}$$

The LPC curve corresponds to a double equilibrium of the right-hand side of equation (3.3). This occurs when both the right-hand side of equation (3.3) and its first derivative with respect to ρ vanish. This results in the following system of equations:

$$\begin{cases}\beta_1 + \beta_2\rho^2 + \Re(c_2(\beta))\rho^4 + \Re(c_3(\beta))\rho^6 + O(\rho^7) = 0,\\ \beta_2\rho + 2\Re(c_2(\beta))\rho^3 + 3\Re(c_3(\beta))\rho^5 + O(\rho^6) = 0.\end{cases} \tag{3.4}$$

Let us define $P : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$P(\rho, \beta) = \begin{pmatrix}\beta_1 + \beta_2\rho^2 + \Re(c_2(\beta))\rho^4 + \Re(c_3(\beta))\rho^6 + O(\rho^7)\\ \beta_2 + 2\Re(c_2(\beta))\rho^2 + 3\Re(c_3(\beta))\rho^4 + O(\rho^5)\end{pmatrix}. \tag{3.5}$$

Then equation (3.4) is equivalent to $P(\rho, \beta) = 0$. Note that $P(0, 0) = 0$ and $DP_\beta(0, 0)$ is invertible. Thus, by the Implicit Function Theorem, there exists a unique locally smooth function $\beta(\rho) = (\beta_1(\rho), \beta_2(\rho))$ near $\rho = 0$ such that $\beta(0) = 0$ and $P(\rho, \beta(\rho)) = 0$. Consequently, we can expand β_1 and β_2 in ρ as

$$\beta_i = m_{i,1}\rho + m_{i,2}\rho^2 + m_{i,3}\rho^3 + m_{i,4}\rho^4 + m_{i,5}\rho^5 + m_{i,6}\rho^6 + O(\rho^7), \quad i = 1, 2.$$

If we substitute these expansions into the equation $P(\rho, \beta) = 0$, we see that up to ρ^3 we have the following terms

$$\begin{aligned}m_{1,1}\rho + m_{1,2}\rho^2 + (m_{1,3} + m_{1,1})\rho^3 + O(\rho^4) &= 0,\\ m_{2,1}\rho + (m_{2,2} + 2d_2)\rho^2 + (m_{2,3} + 2m_{1,1}a_{3210} + 2m_{2,1}a_{3201})\rho^3 + O(\rho^4) &= 0.\end{aligned}$$

From the ρ-terms it follows that $m_{1,1}, m_{2,1} = 0$ and the ρ^2-terms imply that $m_{1,2} = 0$ and $m_{2,2} = -2d_2$. As a result, the ρ^3 terms yield $m_{1,3}, m_{2,3} = 0$. Collecting the ρ^4-terms yields the equations

$$\rho^4 : \qquad \begin{aligned}d_2 + m_{1,4} - 2d_2 &= 0,\\ m_{2,4} + 3d_3 - 4a_{3201}d_2 &= 0.\end{aligned}$$

Thus, the fourth order coefficients are given by $m_{1,4} = d_2$ and $m_{2,4} = 4a_{3201}d_2 - 3d_3$. From the ρ^5-terms in the first equation of (3.5) we find that $m_{1,5} = 0$. Finally, collecting the ρ^6 terms from the first equation of (3.5) we find that

$$m_{1,6} + m_{2,4} + a_{3201}m_{2,2} + d_3 = 0.$$

Thus, $m_{1,6} = 2(d_3 - a_{3201}d_2)$. We now have the following asymptotic expansions for the beta parameters

$$\begin{aligned}\beta_1 &= d_2\rho^4 + 2(d_3 - a_{3201}d_2)\rho^6 + O(\rho^7),\\ \beta_2 &= -2d_2\rho^2 + (4a_{3201}d_2 - 3d_3)\rho^4 + O(\rho^5).\end{aligned}$$

Let $\rho = \varepsilon > 0$ be small. Then, an approximation of the LPC curve is given by

$$(\rho, \beta_1, \beta_2) = (\varepsilon, d_2\varepsilon^4 + 2(d_3 - a_{3201}d_2)\varepsilon^6 + O(\varepsilon^7), -2d_2\varepsilon^2 + (4a_{3201}d_2 - 3d_3)\varepsilon^4 + O(\varepsilon^5)). \quad (3.6)$$

To see the order of the parameter curve approximation, we derive an expression for β_1 as a function of β_2. For this, we first substitute $\delta = \varepsilon^2$ and write

$$\beta_1(\delta) = d_2\delta^2 + s_1\delta^3 + O(\rho^{\frac{7}{2}}),$$
$$\beta_2(\delta) = -2d_2\delta + s_2\delta^2 + O(\delta^{\frac{5}{2}}),$$

where $s_1 = 2(d_3 - a_{3201}d_2)$ and $s_2 = 4a_{3201}d_2 - 3d_3$. Under the condition that $\beta_2'(0) = -2d_2 \neq 0$ it follows from the Inverse Function Theorem that there exists a unique locally smooth inverse $\delta(\beta_2)$ of $\beta_2(\delta)$. We can expand δ as a function of β_2

$$\delta(\beta_2) = \delta_1\beta_2 + \delta_2\beta_2^2 + O(\beta_2^3).$$

Substituting this into the equation for β_2 yields

$$\beta_2 = -2d_2\delta_1\beta_2 + (s_2\delta_1^2 - 2d_2\delta_2)\beta_2^2 + O(\beta_2^3).$$

From this it follows that $\delta_1 = -\frac{1}{2d_2}$ and $\delta_2 = \frac{s_2\delta_1^2}{2d_2}$ thus $\delta(\beta_2) = -\frac{1}{2d_2}\beta_2 + \frac{s_2}{8d_2^3}\beta_2^2 + O(\beta_2^3)$. Since we set $\delta = \varepsilon^2$, we need that $\delta(\beta_2) \geq 0$. If $d_2 < 0$, this holds for $\beta_2 \geq 0$ small enough. If we substitute this into the expression for β_1 we find

$$\beta_1 = \frac{1}{4d_2}\beta_2^2 - \frac{1}{8d_2^3}(s_2 + s_1)\beta_2^3 + O(\beta_2^{\frac{7}{2}}).$$

Thus, using the approximation (3.6), the LPC parameter curve is approximated up to third order for the normal form (3.2).

Remark For a quadratic approximation in the parameter plane, it is enough to take $\beta_1 = d_2\rho^4$ and $\beta_2 = -2d_2\rho^2$.

3.1.2 Period approximmation

To approximate the period T of the cycle on the LPC curve near the generalized Hopf bifurcation we use that

$$\int_0^T \dot{\psi}dt = 2\pi. \quad (3.7)$$

To obtain the approximation of $\dot{\psi}$, we first take the imaginary part of equation (3.2) after substituting $w = \rho e^{i\psi}$ and partially truncate higher order terms. This results in the equation

$$\dot{\psi} = \omega_0 + b_1(\beta) + b_2(\beta)\rho^2 + \Im(c_2(0))\rho^4 + \Im(c_3(0))\rho^6. \quad (3.8)$$

For $b_1(\beta)$ and $b_2(\beta)$ we make the following expansions

$$b_1(\beta) = b_{1,10}\beta_1 + b_{1,01}\beta_2 + \frac{1}{2}b_{1,20}\beta_1^2 + b_{1,11}\beta_1\beta_2 + \frac{1}{2}b_{1,02}\beta_2^2 + \frac{1}{6}b_{1,30}\beta_1^3 + \frac{1}{2}b_{1,21}\beta_1^2\beta_2$$
$$+ \frac{1}{2}b_{1,12}\beta_1\beta_2^2 + \frac{1}{6}b_{1,03}\beta_2^3 + O(\|\beta\|^4),$$
$$b_2(\beta) = \Im(c_1(0)) + b_{2,10}\beta_1 + b_{2,01}\beta_2 + \frac{1}{2}b_{2,20}\beta_1^2 + b_{2,11}\beta_1\beta_2 + \frac{1}{2}b_{2,02}\beta_2^2 + O(\|\beta\|^3).$$

Since we are only interested in an approximation of the period up to fourth order in ε, the terms in grey would not appear in the period approximation. Their inclusion here is meant to demonstrate this explicitly. Now, substitute the parameter expansions (3.6) of the LPC curve into equation (3.8) and solve the integral (3.7). This results in the following approximation of the period

$$\begin{aligned} T = 2\pi/(\omega_0 + (\Im(c_1(0)) - 2d_2 b_{1,01})\varepsilon^2 + [d_2 b_{1,10} + (4a_{3201}d_2 - 3d_3)b_{1,01} + 2d_2^2 b_{1,02} \\ - 2d_2 b_{2,01} + \Im(c_2(0))]\varepsilon^4 + O(\varepsilon^5)). \end{aligned} \tag{3.9}$$

3.2 Coefficients needed for the center manifold and parameter transformation approximmations

As discussed in [4], it is important to consider the effect of terms that are not present in the normal form used to derive the predictor. The terms that affect the predictor up to the current order of approximation will tell us which coefficients need to be included in the parameter transformation K and center manifold approximation H. We will illustrate this with an example. Consider the system

$$\dot{z} = (\alpha_1 + \xi\alpha_2^2 + i\omega_0)z + \alpha_2 z|z|^2 + (d_2 + a\alpha_2)z|z|^4 + d_3 z|z|^6, \quad z \in \mathbb{C},$$

for some nonzero constants $\xi, \omega_0, d_2, a, d_3 \in \mathbb{R}$. Substituting $z = \rho e^{i\psi}$ and taking the real part results in the following amplitude equation:

$$\dot{\rho} = \rho(\alpha_1 + \xi\alpha_2^2 + \alpha_2\rho^2 + (d_2 + a\alpha_2)\rho^4 + d_3\rho^6)$$

To approximate the LPC curve, we need to solve the following system

$$\begin{cases} \alpha_1 + \xi\alpha_2^2 + \alpha_2\rho^2 + (d_2 + a\alpha_2)\rho^4 + d_3\rho^6 & = 0, \\ \alpha_2 + 2(d_2 + a\alpha_2)\rho^2 + 3d_3\rho^4 & = 0. \end{cases} \tag{3.10}$$

For this, we proceed in the same way as before and expand

$$\alpha_i = m_{i,1}\rho + m_{i,2}\rho^2 + m_{i,3}\rho^3 + m_{i,4}\rho^4 + m_{i,5}\rho^5 + m_{i,6}\rho^6 + O(\rho^7), \quad i = 1, 2.$$

Substituting this into equations (3.10) and collecting terms will result in the following approximation for $\rho = \varepsilon > 0$

$$\begin{aligned} \alpha_1 &= (d_2 - 4d_2^2\xi)\varepsilon^4 + (2d_2 - 2ad_2 + 4d_2(4ad_2 - 3d_3)\xi)\varepsilon^6 + O(\varepsilon^7), \\ \alpha_2 &= -2d_2\varepsilon^2 + (4ad_2 - 3d_3)\varepsilon^4 + O(\varepsilon^5). \end{aligned} \tag{3.11}$$

This yields a different LPC curve approximation than the approximation (3.6) from before. Specifically, in the expansion for α_1, we have two extra terms depending on the constant ξ that have appeared. Therefore, including a term $\xi\alpha_2^2 z$ in the normal form will alter the current predictor. Consequently, this term needs to be transformed away into higher-order terms that do not affect the current order of the predictor. We can achieve this with the following parameter transformation

$$\begin{cases} \alpha_1 = \beta_1 - \xi\beta_2^2, \\ \alpha_2 = \beta_2. \end{cases} \tag{3.12}$$

Then the system becomes

$$\dot{z} = (\beta_1 + i\omega_0)z + \beta_2 z|z|^2 + (d_2 + a\beta_2)z|z|^4 + d_3 z|z|^6, \quad z \in \mathbb{C},$$

The LPC curve approximation for this system, up to the same orders as before, will yield:

$$\beta_1 = d_2\varepsilon^4 + 2(d_2 - ad_2)\varepsilon^6 + O(\varepsilon^7),$$
$$\beta_2 = -2d_2\varepsilon^2 + (4ad_2 - 3d_3)\varepsilon^4 + O(\varepsilon^5),$$

which is the same approximmation as (3.6). Note that if we substitute the approximation for β into the parameter transformation (3.12), we recover our approximation for the original parameters (3.11), as expected.

The key point is that since the additional term $\xi\alpha_2^2 z$ influences the predictor up to the desired order, this term needs to be transformed away into higher order terms. We need to include the quadratic coefficient K_{02} into our parameter approximation to achieve this. Therefore, to find all the coefficients that need to be included in the approximations K and H, we need to see which terms, not included in the normal form (3.2), will affect the predictor up to the desired order. These are then precisely the terms that need to be removed, i.e. transformed into higher order terms, to preserve the accuracy of our predictor up to the desired order.

To derive our parameter approximation of the LPC curve it was enough to consider the truncated normal form

$$\dot{w} = (i\omega_0 + \beta_1 + ib_1(\beta))w + (\beta_2 + ib_2(\beta))w|w|^2 + (c_2(0) + a_{3201}\beta_2)w|w|^4 + d_3 w|w|^6.$$

We can now include terms of the form $g_{nmkl}w^n\bar{w}^m\beta_1^k\beta_2^l$ for $n, m, k, l \in \mathbb{N}$ with $n+m+k+l \geq 1$ to this normal form, and determine if they influence in the predictor approximation. The amplitude equation $\dot{\rho}$ is derived by taking the real part of the normal form after substituting $w = \rho e^{i\psi}$. When you do this, you will notice that terms for which $n-m-1 \neq 0$ will be resonant. These will certainly influence the predictor up to a certain order. Let us write $a_{nmkl}(\psi) = \Re\{g_{nmkl}e^{i\psi(n-m-1)}\}$. Then the amplitude equation will have the following form

$$\dot{\rho} = \beta_1\rho + \beta_2\rho^3 + (d_2 + a_{3201}\beta_2)\rho^5 + d_3\rho^7 + a_{nmkl}(\psi)\rho^{n+m}\beta_1^k\beta_2^l.$$

To find a parameter approximation of the LPC curve, we look for a double zero in the equation

$$\beta_1\rho + \beta_2\rho^3 + (d_2 + a_{3201}\beta_2)\rho^5 + d_3\rho^7 + a_{nmkl}(\psi)\rho^{n+m}\beta_1^k\beta_2^l = 0. \tag{3.13}$$

The goal is to determine the combinations of n, m, k, l for which the resonant term will appear in our expansions of β_1 and β_2, consequently affecting our LPC curve approximation. This will indicate which terms need to be removed in the normal form and thus which coefficients H_{nmkl} need to be included in the center manifold approximation [4]. Similarly, we can see which coefficients K_{kl} need to be included by looking at the nonresonant terms $a_{10kl}\rho\beta_1^k\beta_2^l$ and $a_{21kl}\rho^3\beta_1^k\beta_2^l$. We will first discuss in detail the case when $n + m = 0$, and then provide the coefficients for $n + m \geq 1$.

Case $n + m = 0$: The equations that need to be satisfied for a double zero are in this case given by

$$\beta_1\rho + \beta_2\rho^3 + (d_2 + a_{3201}\beta_2)\rho^5 + d_3\rho^7 + a_{00kl}(\psi)\beta_1^k\beta_2^l = 0,$$
$$\beta_1 + 3\beta_2\rho^2 + 5(d_2 + a_{3201}\beta_2)\rho^4 + 7d_3\rho^6 = 0.$$

As before, we expand

$$\beta_i = m_{i,1}\rho + m_{i,2}\rho^2 + m_{i,3}\rho^3 + m_{i,4}\rho^4 + m_{i,5}\rho^5 + m_{i,6}\rho^6 + O(\rho^7), \quad i = 1, 2.$$

Note that it immediately follows from the second equation that $m_{1,1} = m_{1,2} = 0$. One approach would be to simply substitute the expansions of β_i into the equations and solve for the coefficients up to the desired orders to see where the terms $a_{nmkl}(\psi)$ appear. However, this method would quickly become tedious. Fortunately, we can simplify things a bit by setting $m_{i,1} = m_{i,3} = 0$ for $i = 1, 2$. To see why, let us examine all the ρ-terms in the first equation. This yields

$$a_{0001}(\psi)m_{2,1} + a_{0010}(\psi)m_{1,1} = 0.$$

Since this must hold for all ψ, it follows that $m_{2,1}$ and $m_{1,1}$ have to equal zero. These terms are already zero in our LPC approximation, so this does not present a problem. Now collect all ρ^2-terms in the first equation

$$a_{0001}(\psi)m_{2,2} + a_{0010}(\psi)m_{1,2} + a_{0020}(\psi)m_{1,1}^2 + a_{0002}(\psi)m_{2,1}^2 + a_{0011}(\psi)m_{1,1}m_{2,1} = 0.$$

This can only hold for all ψ if $m_{1,2} = m_{2,1} = m_{1,1} = m_{2,2} = 0$. For $m_{1,2}, m_{2,1}$, and $m_{1,1}$, this is not an issue since they are already zero in the approximation. However, for $m_{2,2}$, it is a problem because this term is not zero in our LPC curve approximation. Thus, if the term $a_{0001}(\psi)$ is present in the normal form, it will influence our predictor. Therefore, this term must be transformed away, implying that the coefficient H_{0001} must be included in the center manifold approximation. Whenever a coefficient like $m_{1,3}$, which is zero in our LPC approximation, appears in front of a term $a_{00kl}(\psi)$, this term can be removed by simply setting $m_{1,3} = 0$ without altering the predictor. There is only a problem if the coefficient in front of $a_{00kl}(\psi)$ depends on coefficients like $m_{2,2}$, which are nonzero in our predictor. Thus, it is enough to substitute

$$\begin{aligned}\beta_1 &= m_{1,4}\rho^4 + m_{1,6}\rho^6,\\ \beta_2 &= m_{2,2}\rho^2 + m_{2,4}\rho^4,\end{aligned}$$

when collecting terms. To arrive at the desired order of the predictor, we need to collect terms up to ρ^7 in the first equation and terms up to ρ^6 in the second equation. Therefore, all terms with $k, j \in \mathbb{N}$ for which $\beta_1^k\beta_2^l$ contains a term ρ^j, $j \leq 7$ need to be removed. All combinations of $k, l \in \mathbb{N}$ for which $\beta_1^k\beta_2^l$ contain such terms have been listed in Table 3.1 below. This would mean that we need the coefficients $H_{0001}, H_{0010}, H_{0002}, H_{0011}$, and H_{0003}.

Table 3.1: The first terms of $\beta_1^k\beta_2^l$ for $k, l \in \mathbb{N}$ which are lower than order ρ^7.

	First term of $\beta_1^k\beta_2^l$
$k = 1, l = 0$	$m_{1,4}\rho^4$
$k = 0, l = 1$	$m_{2,2}\rho^2$
$k = 0, l = 2$	$m_{2,2}^2\rho^4$
$k = 1, l = 1$	$m_{1,4}m_{2,2}\rho^6$
$k = 0, l = 3$	$m_{2,2}^3\rho^6$

Case $n + m \geq 1$: Now consider the following two equations

$$\begin{aligned}\beta_1\rho + \beta_2\rho^3 + (d_2 + a_{3201}\beta_2)\rho^5 + d_3\rho^7 + a_{nmkl}(\psi)\rho^{n+m}\beta_1^k\beta_2^l &= 0,\\ \beta_1 + 3\beta_2\rho^2 + 5(d_2 + a_{3201}\beta_2)\rho^4 + 7d_3\rho^6 + a_{nmkl}(\psi)(n + m - 1)\rho^{n+m-1}\beta_1^k\beta_2^l &= 0.\end{aligned}$$

By the same reasoning as before it is enough to substitute

$$\beta_1 = m_{1,4}\rho^4 + m_{1,6}\rho^6,$$
$$\beta_2 = m_{2,2}\rho^2 + m_{2,4}\rho^4.$$

To derive the parameter approximation of the predictor up to sixth order in ε, we need to collect all terms up to ρ^7 in the first equation and all terms up to ρ^6 in the second equation. If the term $a_{nmkl}(\psi)\rho^{n+m}\beta_1^k\beta_2^l$ contains terms of order ρ^7 or lower, this will influence the approximmation. If $k = l = 0$, then a term $a_{nmkl}(\psi)$ will influence the predictor for all $1 \leq n+m \leq 7$ and thus all of these need to be removed. This implies that we need to include all coefficients H_{nm00} with $1 \leq n+m \leq 7$ in the center manifold approximation.

Now we consider the cases for which k or l are nonzero. For $n+m = 1$, we need to remove all $a_{nmkl}(\psi)\rho\beta_1^k\beta_2^l$ terms for which $\beta_1^k\beta_2^l$ contains terms of ρ^6 or less. This is the case for $(kl) = (10), (01), (02), (11)$ and (03). So we need $H_{nm10}, H_{nm01}, H_{nm02}, H_{nm11}$ and H_{nm03} with $n+m = 1$. From this case, it also follows that the non-resonant terms $a_{nmkl}\rho\beta_1^k\beta_2^k$ with $(kl) = (10), (01), (02), (11)$ and (03) will affect the predictor. Thus they also need to be removed, which can be achieved with the coefficients $K_{10}, K_{01}, K_{02}, K_{11}$, and K_{03} in the parameter approximation.

For $n+m = 2$, we need to remove all $a_{nmkl}(\psi)\rho^2\beta_1^k\beta_2^l$ terms for which $\beta_1^k\beta_2^l$ contains terms of ρ^5 or less. This is the case for $(kl) = (10), (01), (02)$. So we need $H_{nm10}, H_{nm01}, H_{nm02}$ with $n+m = 2$.

For $n+m = 3$, we need to remove the terms for which $\beta_1^k\beta_2^l$ contains terms of ρ^4 or less. This will be the case for $(kl) = (10), (01), (02)$. For $n+m = 4, 5$, only $(kl) = (01)$ will appear and for $n+m \geq 6$ none of the $\rho^{n+m}\beta_1^k\beta_2^l$ terms appear in the approximmation when k or l is nonzero.

To summarise, all coefficients that need to be included in the center manifold approximation are

$$\begin{aligned}
&H_{nm10}, H_{nm01}, H_{nm02}, H_{nm11}, H_{nm03}, && n+m = j, j \in \{0,1\},\\
&H_{nm10}, H_{nm01}, H_{nm02}, && n+m = j, j \in \{2,3\},\\
&H_{nm01}, && n+m = j, j \in \{4,5\},\\
&H_{nm00}, && n+m = j, j \in \{1,2,3,4,5,6,7\}.
\end{aligned}$$

Finally, all coefficients that need to be included in the parameter transformation are $K_{10}, K_{01}, K_{02}, K_{11}$, and K_{03}.

Remark. If one is only interested in a predictor that is second-order in the parameters β, i.e. using the approximation $\beta_1 = d_2\rho^4, \beta_2 = -2d_2\rho^2$, it is enough to collect terms op to order ρ^5 in equation (3.13). Therefore, only terms $a_{nmkl}(\psi)\rho^{n+m}\beta_1^k\beta_2^l$ where $\rho^{n+m}\beta_1^k\beta_2^l$ contains terms of order ρ^5 or lower need to be removed. In that case, the following coefficients are necessary for the center manifold approximation

$$\begin{aligned}
&H_{nm10}, H_{nm01}, H_{nm02}, && n+m = j, j \in \{0,1\},\\
&H_{nm01} && n+m = j, j \in \{2,3\},\\
&H_{nm00}, && n+m = j, j \in \{1,2,3,4,5\}.
\end{aligned}$$

For the parameter transformation, it will be enough to include K_{10}, K_{01} and K_{02}.

Chapter 4

The predictor for ODEs

4.1 Coefficients of the parameter-dependent normal form and the predictor for ODEs

We will now derive the equations to calculate all the coefficients that we need in our predictor for ODEs. We follow the method discussed in Section 2.1. Assume that system (2.1) has an equilibrium at the origin at $\alpha = (0,0) \in \mathbb{R}^2$ with only one pair of purely imaginary simple eigenvalues

$$\lambda_{1,2} = \pm i\omega_0, \quad \omega_0 > 0.$$

Additionally, we assume that all the other eigenvalues have non-zero real parts. This allows us to introduce the complex eigenvectors $p, q \in \mathbb{C}^n$ satisfying

$$Aq = i\omega_0 q, \quad A^T p = -i\omega_0 p, \text{ and } \bar{q}^T q = \bar{p}^T q = 1.$$

Furthermore, we assume that the first Lyapunov coefficient $l_1(0) = 0$ and the second Lyapunov coefficient $l_2(0) \neq 0$. The critical real eigenspace T^c corresponding to $\lambda_{1,2}$ is now two dimensional and we can represent each $y \in T^c$ in terms of the complex coordinate $w = \langle p, y \rangle$ as

$$y = wq + \bar{w}\bar{q}.$$

Then, the homological equation (2.5) has the form

$$D_w H(w, \bar{w}, \beta)\dot{w} + D_{\bar{w}} H(w, \bar{w}, \beta)\dot{\bar{w}} = F(H(w, \bar{w}, \beta), K(\beta)). \tag{4.1}$$

Under the assumption that the transversality condition (3.1) is satisfied, the truncated normal form restricted to the two-dimensional center manifold can be expressed in terms of the unfolding parameters $\beta = (\beta_1, \beta_2)$ as

$$\begin{aligned} \dot{w} = {} & (i\omega_0 + \beta_1 + ib_1(\beta))w + (\beta_2 + ib_2(\beta))w|w|^2 + (c_2(0) + g_{3201}\beta_2)w|w|^4 \\ & + c_3(0)w|w|^6, \end{aligned} \tag{4.2}$$

where it is enough to expand

$$b_1(\beta) = b_{1,10}\beta_1 + b_{1,01}\beta_2 + b_{1,11}\beta_1\beta_2 + \frac{1}{2}b_{1,02}\beta_2^2 + \frac{1}{6}b_{1,03}\beta_2^3, \tag{4.3}$$

$$b_2(\beta) = \Im(c_1(0)) + b_{2,10}\beta_1 + b_{2,01}\beta_2 + b_{2,11}\beta_1\beta_2 + \frac{1}{2}b_{2,02}\beta_2^2 + \frac{1}{6}b_{2,03}\beta_2^3 \tag{4.4}$$

Under the assumption that F is sufficiently smooth, the truncated Taylor expansion of F is taken as

$$\begin{aligned}F(x,\alpha) &= Ax + J_1\alpha + A_1(x,\alpha) + \frac{1}{2}B(x,x) + \frac{1}{2}J_2(\alpha,\alpha) + \frac{1}{6}C(x,x,x) + \frac{1}{2}B_1(x,x,\alpha)\\ &+ \frac{1}{2}A_2(x,\alpha,\alpha) + \frac{1}{6}J_3(\alpha,\alpha,\alpha) + \frac{1}{24}D(x,x,x,x) + \frac{1}{6}C_1(x,x,x,\alpha) + \frac{1}{4}B_2(x,x,\alpha,\alpha)\\ &+ \frac{1}{6}A_3(x,\alpha,\alpha,\alpha) + \frac{1}{120}E(x,x,x,x,x) + \frac{1}{24}D_1(x,x,x,x,\alpha) + \frac{1}{12}C_2(x,x,x,\alpha,\alpha)\\ &+ \frac{1}{12}B_3(x,x,\alpha,\alpha,\alpha) + \frac{1}{720}K(x,x,x,x,x,x) + \frac{1}{120}E_1(x,x,x,x,x,\alpha)\\ &+ \frac{1}{36}C_3(x,x,x,\alpha,\alpha,\alpha) + \frac{1}{5040}L(x,x,x,x,x,x,x).\end{aligned} \tag{4.5}$$

The parameterization of the center manifold is expanded as

$$\begin{aligned}H(w,\bar{w},\beta) = qw + \bar{q}\bar{w} &+ \sum_{n+m=2}^{7}\frac{1}{n!m!}H_{nm00}w^n\bar{w}^m + \sum_{n+m=0}^{5}H_{nm01}\frac{1}{n!m!}w^n\bar{w}^m\beta_2\\ &+ \sum_{n+m=0}^{3}\frac{1}{n!m!}H_{nm10}w^n\bar{w}^m\beta_1 + \sum_{n+m=0}^{3}\frac{1}{2n!m!}H_{nm02}w^n\bar{w}^m\beta_2^2\\ &+ \sum_{n+m=0}^{1}\frac{1}{n!m!}H_{nm11}w^n\bar{w}^m\beta_1\beta_2 + \sum_{n+m=0}^{1}\frac{1}{6n!m!}H_{nm03}w^n\bar{w}^m\beta_2^3\\ &+\frac{1}{6}H_{1103}w\bar{w}\beta_2^3 + \frac{1}{12}H_{2003}w^2\beta_2^3 + \frac{1}{12}H_{2103}w^2\bar{w}\beta_2^3.\end{aligned} \tag{4.6}$$

Note that the image of H lies in $\mathbb{R}^n$ and thus we have that $H_{jk\mu} = \overline{H}_{kj\mu}$. The last three terms, marked in grey, are not needed to approximate the periodic orbit in phase space. However, as will be clear in Section 4.1.2, we do need the expressions for H_{1103}, H_{2003} and H_{2103} to derive the coefficients K_{03}, H_{0003} and H_{1003}. The relation between the parameters is expanded as

$$K(\beta) = K_{10}\beta_1 + K_{01}\beta_2 + \frac{1}{2}K_{02}\beta_2^2 + K_{11}\beta_1\beta_2 + \frac{1}{6}K_{03}\beta_2^3. \tag{4.7}$$

Although the coefficients $b_{1,11}$, $b_{1,03}$ in the expansion of $b_1(\beta)$ and $b_{2,02}, b_{2,11}, b_{2,03}$ in de expansion of $b_2(\beta)$ do not appear in our predictor, we will need their expressions from the homological equation to solve for K_{02}, K_{11} and K_{03}. All the equations collected from the homological equation (4.1) are listed in Appendix A.

In the next two subsections, we will derive all the necessary critical normal form coefficients from (4.2) and the parameter-dependent coefficients from (4.7). Not all coefficients in the center manifold approximation (4.6) are needed in this derivation. Their expressions are presented in Appendix B. An overview of all the coefficients that need to be determined for the higher order predictor is presented in figure 4.1.

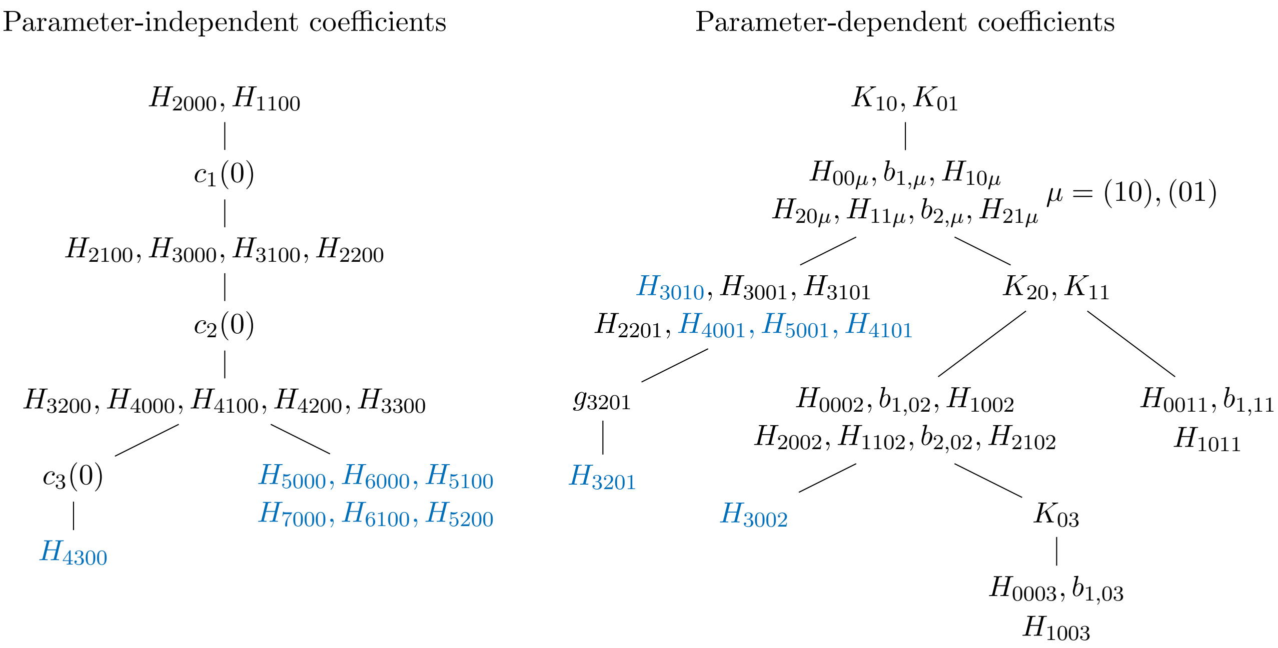

Figure 4.1: Schematic overview of all the coefficients that need to be determined. The coefficients that are marked blue are not needed in the computation of the normal form coefficients but are needed in the approximation of periodic orbit. Their expressions can be found in Appendix B.

Remark. In [17] and [3] a slightly different approach was used to obtain a linear approximation to the LPC curve between the parameters than the one that was taken here. The main difference resides in the fact that they considered the truncated parameter-dependent normal form in terms of the original parameter α:

$$\dot{z} = (i\omega_0 + \gamma_{1,10}\alpha_1 + \gamma_{1,01}\alpha_2)z + (c_1(0) + \gamma_{2,10}\alpha_1 + \gamma_{2,01}\alpha_2)z|z|^2 + c_2(0)z|z|^4.$$

The homological equation will then be simplified to

$$H_z(z, \bar{z}, \alpha)\dot{z} + H_{\bar{z}}(z, \bar{z}, \alpha)\dot{\bar{z}} = F(H(z, \bar{z}, \alpha), \alpha).$$

After deriving the coefficients $\gamma_{1,10}, \gamma_{1,01}, \gamma_{2,10}$ and $\gamma_{2,01}$ from the above phonological equation, one can obtain the following linear relation for the parameters

$$\alpha = \left(\Re\left\{\begin{pmatrix}\gamma_{1,10} & \gamma_{1,01}\\ \gamma_{2,10} & \gamma_{2,01}\end{pmatrix}\right\}\right)^{-1}\beta.$$

Although this method does simplify expressions, one must be careful since the LPC-curve from the normal form is approximated in the parameters β. This should be taken into account when approximating the solution in phase space using the center manifold expansion in α. Furthermore, as remarked by [3], this method only works if one is interested in a linear approximation of α in terms of β.

4.1.1 Critical normal form coefficients

The critical normal form coefficients up to the fifth order coefficient have already been derived in previous works. For completeness, we include their derivation here following [16, Section 8.7.3]. The expression for the first Lyapunov coefficient $l_1 = \frac{1}{\omega_0}\Re(c_1(0))$ can be derived from the w^2, $w\bar{w}$ and $w^2\bar{w}$ terms in the homological equation. These terms yield the equations

$$H_{2000} = (2i\omega_0 I - A)^{-1}B(q, q), \tag{4.8}$$
$$H_{1100} = -A^{-1}B(q, \bar{q}), \tag{4.9}$$
$$(i\omega_0 I - A)H_{2100} = 2B(q, H_{1100}) + B(\bar{q}, H_{2000}) + C(q, q, \bar{q}) - 2c_1(0)q. \tag{4.10}$$

Since the last equation is singular, we can use the Fredholm solvability condition to find

$$c_1(0) = \frac{1}{2}\bar{p}^T[2B(q, H_{1100}) + B(\bar{q}, H_{2000}) + C(q, q, \bar{q})]. \tag{4.11}$$

The vector H_{2100} with $\langle p, H_{2100}\rangle = 0$ can be found by solving the corresponding bordered matrix system (2.10).

For the second Lyapunov coefficient $l_2 = \frac{1}{\omega_0}\Re(c_2(0))$ we need the w^3, $w^3\bar{w}$, $w^2\bar{w}^2$ and $w^3\bar{w}^2$ terms from the homological equation. The first three terms yield the equations

$$H_{3000} = (3i\omega_0 I - A)^{-1}[3B(q, H_{2000}) + C(q, q, q)], \tag{4.12}$$
$$\begin{aligned}H_{3100} = (2i\omega_0 I - A)^{-1}[&3B(q, H_{2100}) + B(\bar{q}, H_{3000}) + 3B(H_{1100}, H_{2000})\\ &+ 3C(q, q, H_{1100}) + 3C(q, \bar{q}, H_{2000}) + D(q, q, q, \bar{q}) - 6c_1(0)H_{2000}],\end{aligned} \tag{4.13}$$
$$\begin{aligned}H_{2200} = -A^{-1}[&2B(q, \overline{H}_{2100}) + 2B(\bar{q}, H_{2100}) + B(H_{2000}, \overline{H}_{2000})\\ &+ 2B(H_{1100}, H_{1100}) + C(q, q, \overline{H}_{2000}) + 4C(q, \bar{q}, H_{1100})\\ &+ C(\bar{q}, \bar{q}, H_{2000}) + D(q, q, \bar{q}, \bar{q})].\end{aligned} \tag{4.14}$$

Note that we used that $c_1(0) + \bar{c}_1(0) = 2\omega_0 l_1 = 0$ to simplify the expression for H_{2200}. Collecting the $w^3\bar{w}^2$ terms yields the equation

$$
\begin{aligned}
(i\omega_0 I - A)H_{3200} = {} & 3B(q, H_{2200}) + 2B(\bar{q}, H_{3100}) + B(\overline{H}_{2000}, H_{3000}) + 6B(H_{1100}, H_{2100}) \\
& + 3B(\overline{H}_{2100}, H_{2000}) + 3C(q, q, \overline{H}_{2100}) + 6C(q, \bar{q}, H_{2100}) + 3C(q, H_{2000}, \overline{H}_{2000}) \\
& + 6C(q, H_{1100}, H_{1100}) + C(\bar{q}, \bar{q}, H_{3000}) + 6C(\bar{q}, H_{1100}, H_{2000}) + D(q, q, q, \overline{H}_{2000}) \\
& + 6D(q, q, \bar{q}, H_{1100}) + 3D(q, \bar{q}, \bar{q}, H_{2000}) + E(q, q, q, \bar{q}, \bar{q}) \\
& - (12c_2(0)q + 6i\Im(c_1(0))H_{2100}). \qquad (4.15)
\end{aligned}
$$

By applying the Fredholm solvability condition to this equation, we find that

$$
\begin{aligned}
c_2(0) = {} & \frac{1}{12}\bar{p}^T[3B(q, H_{2200}) + 2B(\bar{q}, H_{3100}) + B(\overline{H}_{2000}, H_{3000}) \\
& + 6B(H_{1100}, H_{2100}) + 3B(\overline{H}_{2100}, H_{2000}) + 3C(q, q, \overline{H}_{2100}) \\
& + 6C(q, \bar{q}, H_{2100}) + 3C(q, \overline{H}_{2000}, H_{2000}) + 6C(q, H_{1100}, H_{1100}) \\
& + C(\bar{q}, \bar{q}, H_{3000}) + 6C(\bar{q}, H_{1100}, H_{2000}) + D(q, q, q, \overline{H}_{2000}) \\
& + 6D(q, q, \bar{q}, H_{1100}) + 3D(q, \bar{q}, \bar{q}, H_{2000}) + E(q, q, q, \bar{q}, \bar{q})]. \qquad (4.16)
\end{aligned}
$$

For the higher order coefficients we will also need H_{3200}. The unique solution to equation (4.15) satisfying $\langle p, H_{3200}\rangle = 0$ can again be found by solving the corresponding bordered system (2.10). An expression of the seventh order coefficient $c_3(0)$ has been derived in [21], using the same normalization method. For this we need the coefficients of the terms $w^4, w^4\bar{w}, w^4\bar{w}^2$ and $w^3\bar{w}^3$ from the homological equation. These yield regular systems and the solutions are respectively

$$
\begin{aligned}
H_{4000} = {} & (4i\omega_0 I - A)^{-1}[4B(q, H_{3000}) + 3B(H_{2000}, H_{2000}) \\
& + 6C(q, q, H_{2000}) + D(q, q, q, q)], \qquad (4.17) \\
H_{4100} = {} & (3i\omega_0 I_n - A)^{-1}[4B(q, H_{3100}) + B(\bar{q}, H_{4000}) + 4B(H_{1100}, H_{3000}) \\
& + 6B(H_{2000}, H_{2100}) + 6C(q, q, H_{2100}) + 4C(q, \bar{q}, H_{3000}) \\
& + 12C(q, H_{1100}, H_{2000}) + 3C(\bar{q}, H_{2000}, H_{2000}) + 4D(q, q, q, H_{1100}) \\
& + 6D(q, q, \bar{q}, H_{2000}) + E(q, q, q, q, \bar{q}) - 12c_1(0)H_{3000}], \qquad (4.18)
\end{aligned}
$$

$$
\begin{aligned}
H_{4200} = {} & (2i\omega_0 I - A)^{-1}[4B(q, H_{3200}) + 2B(\bar{q}, H_{4100}) \\
& + B(\overline{H}_{2000}, H_{4000}) + 8B(H_{1100}, H_{3100}) + 4B(\overline{H}_{2100}, H_{3000}) \\
& + 6B(H_{2000}, H_{2200}) + 6B(H_{2100}, H_{2100}) + 6C(q, q, H_{2200}) \\
& + 8C(q, \bar{q}, H_{3100}) + 4C(q, \overline{H}_{2000}, H_{3000}) + 24C(q, H_{1100}, H_{2100}) \\
& + 12C(q, \overline{H}_{2100}, H_{2000}) + C(\bar{q}, \bar{q}, H_{4000}) + 8C(\bar{q}, H_{1100}, H_{3000}) \\
& + 12C(\bar{q}, H_{2000}, H_{2100}) + 3C(\overline{H}_{2000}, H_{2000}, H_{2000}) + 12C(H_{1100}, H_{1100}, H_{2000}) \\
& + 4D(q, q, q, \overline{H}_{2100}) + 12D(q, q, \bar{q}, H_{2100}) + 6D(q, q, \overline{H}_{2000}, H_{2000}) \\
& + 12D(q, q, H_{1100}, H_{1100}) + 4D(q, \bar{q}, \bar{q}, H_{3000}) + 24D(q, \bar{q}, H_{1100}, H_{2000}) \\
& + 3D(\bar{q}, \bar{q}, H_{2000}, H_{2000}) + E(q, q, q, q, \overline{H}_{2000}) + 8E(q, q, q, \bar{q}, H_{1100}) \\
& + 6E(q, q, \bar{q}, \bar{q}, H_{2000}) + K(q, q, q, q, \bar{q}, \bar{q}) - 8(6c_2(0)H_{2000} + 2c_1(0)H_{3100})], \qquad (4.19)
\end{aligned}
$$

$$\begin{aligned}H_{3300} = -A^{-1}[&3B(q, \overline{H}_{3200}) + 3B(\bar{q}, H_{3200}) + 3B(\overline{H}_{2000}, H_{3100}) + B(\overline{H}_{3000}, H_{3000})\\
&+ 9B(H_{1100}, H_{2200}) + 9B(H_{2100}, \overline{H}_{2100}) + 3B(\overline{H}_{3100}, H_{2000})\\
&+ 3C(q, q, \overline{H}_{3100}) + 9C(q, \bar{q}, H_{2200}) + 9C(q, \overline{H}_{2000}, H_{2100})\\
&+ 3C(q, \overline{H}_{3000}, H_{2000}) + 18C(q, H_{1100}, \overline{H}_{2100}) + 3C(\bar{q}, \bar{q}, H_{3100})\\
&+ 3C(\bar{q}, \overline{H}_{2000}, H_{3000}) + 18C(\bar{q}, H_{1100}, H_{2100}) + 9C(\bar{q}, \overline{H}_{2100}, H_{2000})\\
&+ 9C(\overline{H}_{2000}, H_{1100}, H_{2000}) + 6C(H_{1100}, H_{1100}, H_{1100}) + D(q, q, q, \overline{H}_{3000})\\
&+ 9D(q, q, \bar{q}, \overline{H}_{2100}) + 9D(q, q, \overline{H}_{2000}, H_{1100}) + 9D(q, \bar{q}, \bar{q}, H_{2100})\\
&+ 9D(q, \bar{q}, \overline{H}_{2000}, H_{2000}) + 18D(q, \bar{q}, H_{1100}, H_{1100}) + D(\bar{q}, \bar{q}, \bar{q}, H_{3000})\\
&+ 9D(\bar{q}, \bar{q}, H_{1100}, H_{2000}) + 3E(q, q, q, \bar{q}, \overline{H}_{2000}) + 9E(q, q, \bar{q}, \bar{q}, H_{1100})\\
&+ 3E(q, \bar{q}, \bar{q}, \bar{q}, H_{2000}) + K(q, q, q, \bar{q}, \bar{q}, \bar{q}) - 72d_2 H_{1100}].\end{aligned} \tag{4.20}$$

Here we used that $c_1(0) + \bar{c}_1(0) = 0$ and $c_2(0) + \bar{c}_2(0) = \Re(c_2(0)) = d_2$ to simplify the last two expressions. Note that the term $D(\bar{q}, \bar{q}, \bar{q}, H_{3000})$ in the expression for H_{3300} is missing in [21], although it was included in their calculations. Collecting the $w^4\bar{w}^3$ terms results in the equation

$$\begin{aligned}(i\omega_0 I - A)H_{4300} = M_{4300} - (&144c_3(0)q + 72(2c_2(0) + \bar{c}_2(0))H_{2100}\\
&+ 12i\Im\{c_1(0)\}H_{3200}),\end{aligned} \tag{4.21}$$

where we used that $3c_1(0) + 2\bar{c}_1(0) = i\Im\{c_1(0)\}$ and defined

$$\begin{aligned}M_{4300} = &4B(q, H_{3300}) + 3B(\bar{q}, H_{4200}) + 3B(\overline{H}_{2000}, H_{4100}) + B(\overline{H}_{3000}, H_{4000})\\
&+ 12B(H_{1100}, H_{3200}) + 12B(\overline{H}_{2100}, H_{3100}) + 4B(\overline{H}_{3100}, H_{3000})\\
&+ 6B(H_{2000}, \overline{H}_{3200}) + 18B(H_{2100}, H_{2200}) + 6C(q, q, \overline{H}_{3200})\\
&+ 12C(q, \bar{q}, H_{3200}) + 12C(q, \overline{H}_{2000}, H_{3100}) + 4C(q, \overline{H}_{3000}, H_{3000})\\
&+ 36C(q, H_{1100}, H_{2200}) + 36C(q, \overline{H}_{2100}, H_{2100}) + 12C(q, \overline{H}_{3100}, H_{2000})\\
&+ 3C(\bar{q}, \bar{q}, H_{4100}) + 3C(\bar{q}, \overline{H}_{2000}, H_{4000}) + 24C(\bar{q}, H_{1100}, H_{3100})\\
&+ 12C(\bar{q}, \overline{H}_{2100}, H_{3000}) + 18C(\bar{q}, H_{2000}, H_{2200}) + 18C(\bar{q}, H_{2100}, H_{2100})\\
&+ 12C(\overline{H}_{2000}, H_{1100}, H_{3000}) + 18C(\overline{H}_{2000}, H_{2000}, H_{2100})\\
&+ 3C(\overline{H}_{3000}, H_{2000}, H_{2000}) + 36C(H_{1100}, H_{1100}, H_{2100})\\
&+ 36C(H_{1100}, \overline{H}_{2100}, H_{2000}) + 4D(q, q, q, \overline{H}_{3100}) + 18D(q, q, \bar{q}, H_{2200})\\
&+ 18D(q, q, \overline{H}_{2000}, H_{2100}) + 6D(q, q, \overline{H}_{3000}, H_{2000}) + 36D(q, q, H_{1100}, \overline{H}_{2100})\\
&+ 12D(q, \bar{q}, \bar{q}, H_{3100}) + 12D(q, \bar{q}, \overline{H}_{2000}, H_{3000}) + 72D(q, \bar{q}, H_{1100}, H_{2100})\\
&+ 36D(q, \bar{q}, \overline{H}_{2100}, H_{2000}) + 36D(q, \overline{H}_{2000}, H_{1100}, H_{2000})\\
&+ 24D(q, H_{1100}, H_{1100}, H_{1100}) + D(\bar{q}, \bar{q}, \bar{q}, H_{4000}) + 12D(\bar{q}, \bar{q}, H_{1100}, H_{3000})\\
&+ 18D(\bar{q}, \bar{q}, H_{2000}, H_{2100}) + 9D(\bar{q}, \overline{H}_{2000}, H_{2000}, H_{2000})\\
&+ 36D(\bar{q}, H_{1100}, H_{1100}, H_{2000}) + E(q, q, q, q, \overline{H}_{3000}) + 12E(q, q, q, \bar{q}, \overline{H}_{2100})\\
&+ 12E(q, q, q, \overline{H}_{2000}, H_{1100}) + 18E(q, q, \bar{q}, \bar{q}, H_{2100}) + 18E(q, q, \bar{q}, \overline{H}_{2000}, H_{2000})\\
&+ 36E(q, q, \bar{q}, H_{1100}, H_{1100}) + 4E(q, \bar{q}, \bar{q}, \bar{q}, H_{3000}) + 36E(q, \bar{q}, \bar{q}, H_{1100}, H_{2000})\\
&+ 3E(\bar{q}, \bar{q}, \bar{q}, H_{2000}, H_{2000}) + 3K(q, q, q, q, \bar{q}, \overline{H}_{2000}) + 12K(q, q, q, \bar{q}, \bar{q}, H_{1100})\\
&+ 6K(q, q, \bar{q}, \bar{q}, \bar{q}, H_{2000}) + L(q, q, q, q, \bar{q}, \bar{q}, \bar{q}).\end{aligned}$$

To obtain the equation for $c_3(0)$ we apply the Fredholm solvability condition to the singular system (4.21). This yields the equation

$$c_3(0) = \frac{1}{144}\bar{p}^T M_{4300},$$

where we used that $\langle p, H_{2100}\rangle = 0$ and $\langle p, H_{3200}\rangle = 0$. The unique solution to equation (4.21) satisfying $\langle p, H_{4300}\rangle = 0$ can again be found by solving the corresponding bordered sytem (2.10).

4.1.2 Parameter-related coefficients

Linear coefficients K_{10},K_{01} We first determine the coefficients for the linear approximation of the parameter transformation K. Collecting the β_1 and β_2 terms in (4.1) yields for $\mu = (10), (01)$ the systems

$$AH_{00\mu} = -J_1 K_\mu.$$

Let $e_1, e_2 \in \mathbb{R}^2$ be the standard basis vectors. Then we can write

$$K_\mu = \gamma_{1,\mu} e_1 + \gamma_{2,\mu} e_2, \tag{4.22}$$

for some $\gamma_{1,\mu}, \gamma_{2,\mu} \in \mathbb{R}$. Since A is regular, we have

$$H_{00\mu} = -\gamma_{1,\mu} A^{-1} J_1 e_1 - \gamma_{2,\mu} A^{-1} J_1 e_2. \tag{4.23}$$

In the equations that follow we will occasionally use the following notation

$$\delta_\mu^{ij} = \begin{cases} 1, & \text{if } \mu = (ij), \\ 0, & \text{if } \mu = (ji) \end{cases}, \quad \text{for } i, j \in \mathbb{N}. \tag{4.24}$$

The $\beta_1 w$ and $\beta_2 w$ terms yield the systems

$$(i\omega_0 I - A)H_{10\mu} = A_1(q, K_\mu) + B(q, H_{00\mu}) - (\delta_\mu^{10} + ib_{1,\mu})q. \tag{4.25}$$

To reduce the length of the equations, it is convenient to define $\Gamma_i(q) = A_1(q, e_i) + B(q, -A^{-1}J_1 e_i)$. Substituting equations (4.22) and (4.23) into equation (4.25) results in

$$(i\omega_0 I - A)H_{10\mu} = \gamma_{1,\mu}\Gamma_1(q) + \gamma_{2,\mu}\Gamma_2(q) - (\delta_\mu^{10} + ib_{1,\mu})q, \tag{4.26}$$

Applying the Fredholm solvability condition to equation (4.26) results in

$$\delta_\mu^{10} + ib_{1,\mu} = \bar{p}^T[\gamma_{1,\mu}\Gamma_1(q) + \gamma_{2,\mu}\Gamma_2(q)]. \tag{4.27}$$

If we take the real and imaginary parts of the above equation, we find that

$$\delta_\mu^{10} = \gamma_{1,\mu}\Re[\bar{p}^T\Gamma_1(q)] + \gamma_{2,\mu}\Re[\bar{p}^T\Gamma_2(q)], \tag{4.28}$$

and

$$b_{1,\mu} = \gamma_{1,\mu}\Im[\bar{p}^T\Gamma_1(q)] + \gamma_{2,\mu}\Im[\bar{p}^T\Gamma_2(q)]. \tag{4.29}$$

A solution $H_{10\mu}$ of equation (4.26) satisfying $\langle p, H_{10\mu}\rangle = 0$ can be obtained by solving the bordered system

$$\begin{pmatrix} i\omega_0 I - A & q \\ \bar{p}^T & 0 \end{pmatrix}\begin{pmatrix} H_{10\mu} \\ s \end{pmatrix} = \begin{pmatrix} \gamma_{1,\mu}\Gamma_1(q) + \gamma_{2,\mu}\Gamma_2(q) - (\delta_\mu^{10} + ib_{1,\mu})q \\ 0 \end{pmatrix}.$$

However, since the coefficients $b_{1,\mu}$, $\gamma_{1,\mu}$ and $\gamma_{2,\mu}$ are yet unknown, it is not possible to immediately solve this system. Instead, we can derive another equation like (4.28), where the only unknown are $\gamma_{1,\mu}$ and $\gamma_{2,\mu}$. This will then allow us to set up a linear system from which we can solve for $\gamma_{1,\mu}$ and $\gamma_{2,\mu}$. The solution of equation (4.26) can be written as

$$H_{10\mu} = \gamma_{1,\mu} A_{i\omega_0}^{INV} \Gamma_1(q) + \gamma_{2,\mu} A_{i\omega_0}^{INV} \Gamma_2(q) - (\delta_\mu^{10} + ib_{1,\mu}) A_{i\omega_0}^{INV} q, \tag{4.30}$$

where we use the shorthand notation $A_{i\omega_0}^{INV} = (i\omega_0 I_n - A)^{INV}$. The vectors $w = A_{i\omega_0}^{INV} q$ and $v_k = A_{i\omega_0}^{INV} \Gamma_k(q)$ $(k = 1, 2)$ are to be found by solving the following bordered systems

$$\begin{pmatrix} i\omega_0 I - A & q \\ \bar{p}^T & 0 \end{pmatrix} \begin{pmatrix} w \\ s \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} i\omega_0 I - A & q \\ \bar{p}^T & 0 \end{pmatrix} \begin{pmatrix} v_k \\ s_k \end{pmatrix} = \begin{pmatrix} \Gamma_k(q) \\ 0 \end{pmatrix}.$$

From the left bordered system it follows that $A_{i\omega_0}^{INV} q = 0$. To see this, note that we can write the correspoinding bordered system as the following system of equations

$$\begin{aligned} (i\omega_0 I - A)w + qs &= q, \\ \langle p, w \rangle &= 0. \end{aligned}$$

Taking the inner product with p of the first equation yields $s = 1$. Then the first equation becomes $(i\omega_0 I - A)w = 0$, with solution $w = \lambda q$. Filling this into the second equation yields $\langle p, w \rangle = \lambda \langle p, q \rangle = 0$ from which it follows that $\lambda = 0$. Thus, the unique solution to this system is $(w, s) = (0, 1)$. We will use this to simplify the final equations from which we can solve for $\gamma_{1,\mu}$ and $\gamma_{2,\mu}$.

Collecting the $w^2\beta_i$ and $w\bar{w}\beta_i$ terms respectively yield the systems[1]

$$\begin{aligned} (2i\omega_0 I - A)H_{20\mu} &= A_1(H_{2000}, K_\mu) + 2B(q, H_{10\mu}) + B(H_{00\mu}, H_{2000}) \\ &\quad + B_1(q, q, K_\mu) + C(q, q, H_{00\mu}) - 2(\delta_\mu^{10} + ib_{1,\mu})H_{2000}, \\ -AH_{11\mu} &= A_1(H_{1100}, K_\mu) + 2\Re\{B(\bar{q}, H_{10\mu})\} + B(H_{00\mu}, H_{1100}) \\ &\quad + B_1(q, \bar{q}, K_\mu) + C(q, \bar{q}, H_{00\mu}) - 2\delta_\mu^{10} H_{1100}. \end{aligned}$$

Both these systems are regular with solutions

$$\begin{aligned} H_{20\mu} &= A_{2i\omega_0}^{-1}[A_1(H_{2000}, K_\mu) + 2B(q, H_{10\mu}) + B(H_{00\mu}, H_{2000}) \\ &\quad + B_1(q, q, K_\mu) + C(q, q, H_{00\mu}) - 2(\delta_\mu^{10} + ib_{1,\mu})H_{2000}], \end{aligned} \tag{4.31}$$

$$\begin{aligned} H_{11\mu} &= -A^{-1}[A_1(H_{1100}, K_\mu) + 2\Re(B(\bar{q}, H_{10\mu})) + B(H_{00\mu}, H_{1100}) \\ &\quad + B_1(q, \bar{q}, K_\mu) + C(q, \bar{q}, H_{00\mu}) - 2\delta_\mu^{10} H_{1100}], \end{aligned} \tag{4.32}$$

where we write $A_{2i\omega_0}^{-1} = (2i\omega_0 I - A)^{-1}$. We will now substitute equations (4.22), (4.23) and (4.30) into the above expressions. For this it is convenient to define the following functions

$$\begin{aligned} \Lambda_i(u, v, w) &= \Gamma_i(u) + 2B(v, A_{i\omega_0}^{INV} \Gamma_i(w)) + B_1(v, w, e_i) + C(v, w, -A^{-1} J_1 e_i), \\ \Pi_i(u, v, w) &= \Gamma_i(u) + 2\Re\{B\left(v, A_{i\omega_0}^{INV} \Gamma_i(w)\right)\} + B_1(v, w, e_i) + C(v, w, -A^{-1} J_1 e_i). \end{aligned}$$

Then the equations for $H_{20\mu}$ and $H_{11\mu}$ become

$$\begin{aligned} H_{20\mu} &= \gamma_{1,\mu} A_{2i\omega_0}^{-1} \Lambda_1(H_{2000}, q, q) + \gamma_{2,\mu} A_{2i\omega_0}^{-1} \Lambda_2(H_{2000}, q, q) - 2(\delta_\mu^{10} + ib_{1,\mu}) A_{2i\omega_0}^{-1} H_{2000}, \\ H_{11\mu} &= -\gamma_{1,\mu} A^{-1} \Pi_1(H_{1100}, \bar{q}, q) - \gamma_{2,\mu} A^{-1} \Pi_2(H_{1100}, \bar{q}, q) + 2\delta_\mu^{10} A^{-1} H_{1100}. \end{aligned}$$

[1] Since $H_{01\mu} = \overline{H}_{10\mu}$, we can write $B(q, H_{01\mu}) + B(\bar{q}, H_{10\mu}) = 2\Re(B(\bar{q}, H_{10\mu}))$.

The $w^2\bar{w}\beta_i$ terms yield the systems

$$\begin{aligned}(i\omega_0 I_n - A)H_{21\mu} &= A_1(H_{2100}, K_\mu) + 2B(q, H_{11\mu}) + B(\bar{q}, H_{20\mu}) + B(H_{00\mu}, H_{2100}) \\ &+ B(\overline{H}_{10\mu}, H_{2000}) + 2B(H_{10\mu}, H_{1100}) + 2B_1(q, H_{1100}, K_\mu) + B_1(\bar{q}, H_{2000}, K_\mu) \\ &+ C(q, q, \overline{H}_{10\mu}) + 2C(q, \bar{q}.H_{10\mu}) + 2C(q, H_{00\mu}, H_{1100}) + C(\bar{q}, H_{00\mu}, H_{2000}) \\ &+ C_1(q, q, \bar{q}, K_\mu) + D(q, q, \bar{q}, H_{00\mu}) - [2(\delta_\mu^{01} + ib_{2,\mu})q \\ &+ (3\delta_\mu^{10} + ib_{1,\mu})H_{2100} + 2c_1(0)H_{10\mu}]. \end{aligned} \tag{4.33}$$

Applying the Fredholm solvability equation to equation (4.33) results in

$$\begin{aligned}\delta_\mu^{01} + ib_{2,\mu} = \frac{1}{2}\bar{p}^T[&A_1(H_{2100}, K_\mu) + 2B(q, H_{11\mu}) + B(\bar{q}, H_{20\mu}) + B(H_{00\mu}, H_{2100}) \\ &+ B(\overline{H}_{10\mu}, H_{2000}) + 2B(H_{10\mu}, H_{1100}) + 2B_1(q, H_{1100}, K_\mu) + B_1(\bar{q}, H_{2000}, K_\mu) \\ &+ C(q, q, \overline{H}_{10\mu}) + 2C(q, \bar{q}.H_{10\mu}) + 2C(q, H_{00\mu}, H_{1100}) + C(\bar{q}, H_{00\mu}, H_{2000}) \\ &+ C_1(q, q, \bar{q}, K_\mu) + D(q, q, \bar{q}, H_{00\mu})], \end{aligned} \tag{4.34}$$

where we used that $\langle p, H_{2100}\rangle = 0$ and $\langle p, H_{10\mu}\rangle = 0$. If we substitute the expressions for $H_{00\mu}, H_{10\mu}, H_{20\mu}, H_{11\mu}$, K_μ and $b_{1,\mu}$ into equation (4.34), we can finally solve for $\gamma_{1,\mu}$ and $\gamma_{2,\mu}$.

After substitution and some rewriting we arrive at the following system

$$P\begin{pmatrix}\gamma_{1,\mu} \\ \gamma_{2,\mu}\end{pmatrix} = Q_\mu, \tag{4.35}$$

where $P \in \mathbb{R}^{2\times 2}$ is given by

$$P_{1k} = \Re[\bar{p}^T\Gamma_k(q)], \tag{4.36}$$

$$\begin{aligned}P_{2k} = \frac{1}{2}\Re\Big\{\bar{p}^T\Big[&\Gamma_k(H_{2100}) + 2B(q, -A^{-1}\Pi_k(H_{1100}, \bar{q}, q)) + B(\bar{q}, A_{2i\omega_0}^{-1}\Lambda_k(H_{2000}, q, q)) \\ &+ B(H_{2000}, \overline{A_{i\omega_0}^{INV}\Gamma_k(q)}) + 2B(H_{1100}, A_{i\omega_0}^{INV}\Gamma_k(q)) + 2B_1(q, H_{1100}, e_k) + B_1(\bar{q}, H_{2000}, e_k) \\ &+ C(q, q, \overline{A_{i\omega_0}^{INV}\Gamma_k(q)}) + 2C(q, \bar{q}, A_{i\omega_0}^{INV}\Gamma_k(q)) + 2C(q, H_{1100}, -A^{-1}J_1e_k) \\ &+ C(\bar{q}, H_{2000}, -A^{-1}J_1e_k) + C_1(q, q, \bar{q}, e_k) + D(q, q, \bar{q}, -A^{-1}J_1e_k) \\ &+ \Im[\bar{p}^T\Gamma_k(q)]\left(-2iB(\bar{q}, A_{2i\omega_0}^{-1}H_{2000})\right)\Big]\Big\} \end{aligned} \tag{4.37}$$

and $Q_\mu \in \mathbb{R}^2$ with $\mu = (10), (01)$ is given by

$$Q_{1,\mu} = \delta_\mu^{10}, \tag{4.38}$$

$$Q_{2,\mu} = \delta_\mu^{01} + \frac{1}{2}\delta_\mu^{10}\Re\Big\{\bar{p}^T\Big[4B\left(q, -A^{-1}H_{1100}\right) + 2B\left(\bar{q}, A_{2i\omega_0}^{-1}H_{2000}\right)\Big]\Big\}. \tag{4.39}$$

Thus, one first computes the linear coefficients K_{10} and K_{01} by solving the system (4.35). This system should be solvable if the transversality condition holds. Once K_{10}, K_{01} are known, we can determine the coefficients $H_{00\mu}$ from equation (4.23) and $b_{1,\mu}$ from equation (4.29). With these coefficients we can determine $H_{10\mu}$ from equation (4.30), and $H_{20\mu}$ and $H_{11\mu}$ from respectively equations (4.31) and (4.32). The coefficient $b_{2,01}$ is then given by the imaginary part of the right-hand side of equation (4.34). Finally, we can use equation (4.33) to solve for H_{2101} satisfying $\langle p, H_{2101}\rangle = 0$ by solving the corresponding bordered system (2.10).

The coefficient a_{3201} Note that $a_{3201} = \Re(g_{3201})$. To determine this coefficient we also need the coefficients H_{3001}, H_{3101} and H_{2201}. These can be found by collecting the $w^3\beta_2$, $w^3\bar{w}\beta_2$ and $w^2\bar{w}^2\beta_2$ terms from the homological equation (4.1). This results in the following equations

$$\begin{aligned}
H_{3001} = {} & (3i\omega_0 I_n - A)^{-1}[A_1(H_{3000}, K_{01}) + 3B(q, H_{2001}) + B(H_{0001}, H_{3000}) \\
& + 3B(H_{1001}, H_{2000}) + 3B_1(q, H_{2000}, K_{01}) + 3C(q, q, H_{1001}) \\
& + 3C(q, H_{0001}, H_{2000}) + C_1(q, q, q, K_{01}) + D(q, q, q, H_{0001}) - 3ib_{1,01}H_{3000}], \\
H_{3101} = {} & (2i\omega_0 I_n - A)^{-1}[A_1(H_{3100}, K_{01}) + 3B(q, H_{2101}) + B(\bar{q}, H_{3001}) \\
& + B(H_{0001}, H_{3100}) + B(\overline{H}_{1001}, H_{3000}) + 3B(H_{1001}, H_{2100}) \\
& + 3B(H_{1100}, H_{2001}) + 3B(H_{1101}, H_{2000}) + 3B_1(q, H_{2100}, K_{01}) \\
& + B_1(\bar{q}, H_{3000}, K_{01}) + 3B_1(H_{1100}, H_{2000}, K_{01}) + 3C(q, q, H_{1101}) \\
& + 3C(q, \bar{q}, H_{2001}) + 3C(q, H_{0001}, H_{2100}) + 3C(q, \overline{H}_{1001}, H_{2000}) \\
& + 6C(q, H_{1001}, H_{1100}) + C(\bar{q}, H_{0001}, H_{3000}) + 3C(\bar{q}, H_{1001}, H_{2000}) \\
& + 3C(H_{0001}, H_{1100}, H_{2000}) + 3C_1(q, q, H_{1100}, K_{01}) + 3C_1(q, \bar{q}, H_{2000}, K_{01}) \\
& + D(q, q, q, \overline{H}_{1001}) + 3D(q, q, \bar{q}, H_{1001}) + 3D(q, q, H_{0001}, H_{1100}) \\
& + 3D(q, \bar{q}, H_{0001}, H_{2000}) + D_1(q, q, q, \bar{q}, K_{01}) + E(q, q, q, \bar{q}, H_{0001}) \\
& - 6(1 + ib_{2,01})H_{2000} - 6c_1(0)H_{2001} - 2ib_{1,01}H_{3100}],
\end{aligned}$$

$$\begin{aligned}
H_{2201} = {} & -A^{-1}[A_1(H_{2200}, K_{01}) + 2B(q, \overline{H}_{2101}) + 2B(\bar{q}, H_{2101}) + B(H_{0001}, H_{2200}) \\
& + 2B(\overline{H}_{1001}, H_{2100}) + B(\overline{H}_{2000}, H_{2001}) + B(\overline{H}_{2001}, H_{2000}) \\
& + 2B(H_{1001}, \overline{H}_{2100}) + 4B(H_{1100}, H_{1101}) + 2B_1(q, \overline{H}_{2100}, K_{01}) \\
& + B_1(\overline{H}_{2000}, H_{2000}, K_{01}) + 2B_1(H_{1100}, H_{1100}, K_{01}) + 2B_1(\bar{q}, H_{2100}, K_{01}) \\
& + C(q, q, \overline{H}_{2001}) + 4C(q, \bar{q}, H_{1101}) + 2C(q, H_{0001}, \overline{H}_{2100}) \\
& + 4C(q, \overline{H}_{1001}, H_{1100}) + 2C(q, \overline{H}_{2000}, H_{1001}) + C(\bar{q}, \bar{q}, H_{2001}) \\
& + 2C(\bar{q}, H_{0001}, H_{2100}) + 2C(\bar{q}, \overline{H}_{1001}, H_{2000}) + 4C(\bar{q}, H_{1001}, H_{1100}) \\
& + C(H_{0001}, \overline{H}_{2000}, H_{2000}) + 2C(H_{0001}, H_{1100}, H_{1100}) + C_1(q, q, \overline{H}_{2000}, K_{01}) \\
& + 4C_1(q, \bar{q}, H_{1100}, K_{01}) + C_1(\bar{q}, \bar{q}, H_{2000}, K_{01}) + 2D(q, q, \bar{q}, \overline{H}_{1001}) \\
& + D(q, q, H_{0001}, \overline{H}_{2000}) + 2D(q, \bar{q}, \bar{q}, H_{1001}) + 4D(q, \bar{q}, H_{0001}, H_{1100}) \\
& + D(\bar{q}, \bar{q}, H_{0001}, H_{2000}) + D_1(q, q, \bar{q}, \bar{q}, K_{01}) + E(q, q, \bar{q}, \bar{q}, H_{0001}) - 8H_{1100}].
\end{aligned}$$

The coefficient g_{3201} can now be found by applying the Fredholm alternative to the equation that follows from collecting the $w^3\bar{w}^2\beta_2$ terms. This yields the following equation

$$\begin{aligned}
(i\omega_0 I_n - A)H_{3201} = {} & M_{3201} - [12g_{3201}q + 12c_2(0)H_{1001} + (18 + 6ib_{2,01})H_{2100} \\
& + 6i\Im\{c_1(0)\}H_{2101} + ib_{1,01}H_{3200}], \qquad (4.40)
\end{aligned}$$

where

$$
\begin{aligned}
M_{3201} = {} & A_1(H_{3200}, K_{01}) + 3B(q, H_{2201}) + 2B(\bar{q}, H_{3101}) + B(H_{0001}, H_{3200}) \\
& + 2B(\overline{H}_{1001}, H_{3100}) + B(\overline{H}_{2000}, H_{3001}) + B(\overline{H}_{2001}, H_{3000}) \\
& + 3B(H_{1001}, H_{2200}) + 6B(H_{1100}, H_{2101}) + 6B(H_{1101}, H_{2100}) \\
& + 3B(\overline{H}_{2100}, H_{2001}) + 3B(\overline{H}_{2101}, H_{2000}) + 3B_1(q, H_{2200}, K_{01}) \\
& + 2B_1(\bar{q}, H_{3100}, K_{01}) + B_1(\overline{H}_{2000}, H_{3000}, K_{01}) + 6B_1(H_{1100}, H_{2100}, K_{01}) \\
& + 3B_1(\overline{H}_{2100}, H_{2000}, K_{01}) + 3C(q, q, \overline{H}_{2101}) + 6C(q, \bar{q}, H_{2101}) \\
& + 3C(q, H_{0001}, H_{2200}) + 6C(q, \overline{H}_{1001}, H_{2100}) + 3C(q, \overline{H}_{2000}, H_{2001}) \\
& + 3C(q, \overline{H}_{2001}, H_{2000}) + 6C(q, H_{1001}, \overline{H}_{2100}) + 12C(q, H_{1100}, H_{1101}) \\
& + C(\bar{q}, \bar{q}, H_{3001}) + 2C(\bar{q}, H_{0001}, H_{3100}) + 2C(\bar{q}, \overline{H}_{1001}, H_{3000}) \\
& + 6C(\bar{q}, H_{1001}, H_{2100}) + 6C(\bar{q}, H_{1100}, H_{2001}) + 6C(\bar{q}, H_{1101}, H_{2000}) \\
& + C(H_{0001}, \overline{H}_{2000}, H_{3000}) + 6C(H_{0001}, H_{1100}, H_{2100}) + 3C(H_{0001}, \overline{H}_{2100}, H_{2000}) \\
& + 6C(\overline{H}_{1001}, H_{1100}, H_{2000}) + 3C(\overline{H}_{2000}, H_{1001}, H_{2000}) \\
& + 6C(H_{1001}, H_{1100}, H_{1100}) + 3C_1(q, q, \overline{H}_{2100}, K_{01}) + 6C_1(q, \bar{q}, H_{2100}, K_{01}) \\
& + 3C_1(q, \overline{H}_{2000}, H_{2000}, K_{01}) + 6C_1(q, H_{1100}, H_{1100}, K_{01}) + C_1(\bar{q}, \bar{q}, H_{3000}, K_{01}) \\
& + 6C_1(\bar{q}, H_{1100}, H_{2000}, K_{01}) + D(q, q, q, \overline{H}_{2001}) + 6D(q, q, \bar{q}, H_{1101}) \\
& + 3D(q, q, H_{0001}, \overline{H}_{2100}) + 6D(q, q, \overline{H}_{1001}, H_{1100}) + 3D(q, q, \overline{H}_{2000}, H_{1001}) \\
& + 3D(q, \bar{q}, \bar{q}, H_{2001}) + 6D(q, \bar{q}, H_{0001}, H_{2100}) + 6D(q, \bar{q}, \overline{H}_{1001}, H_{2000}) \\
& + 12D(q, \bar{q}, H_{1001}, H_{1100}) + 3D(q, H_{0001}, \overline{H}_{2000}, H_{2000}) \\
& + 6D(q, H_{0001}, H_{1100}, H_{1100}) + D(\bar{q}, \bar{q}, H_{0001}, H_{3000}) + 3D(\bar{q}, \bar{q}, H_{1001}, H_{2000}) \\
& + 6D(\bar{q}, H_{0001}, H_{1100}, H_{2000}) + D_1(q, q, q, \overline{H}_{2000}, K_{01}) + 6D_1(q, q, \bar{q}, H_{1100}, K_{01}) \\
& + 3D_1(q, \bar{q}, \bar{q}, H_{2000}, K_{01}) + 2E(q, q, q, \bar{q}, \overline{H}_{1001}) + E(q, q, q, H_{0001}, \overline{H}_{2000}) \\
& + 3E(q, q, \bar{q}, \bar{q}, H_{1001}) + 6E(q, q, \bar{q}, H_{0001}, H_{1100}) + 3E(q, \bar{q}, \bar{q}, H_{0001}, H_{2000}) \\
& + E_1(q, q, q, \bar{q}, \bar{q}, K_{01}) + K(q, q, q, \bar{q}, \bar{q}, H_{0001}).
\end{aligned}
$$

Applying the Fredholm alternative to (4.40) now yields

$$
g_{3201} = \frac{1}{12}\bar{p}^T M_{3201}. \tag{4.41}
$$

Now the unique solution for H_{3201} satisfying $\langle p, H_{3201} \rangle = 0$ can be solved from equation (4.40) using the corresponding bordered system (2.10).

Quadratic coefficients $K_{02}, K_{11}, b_{1,02}$ The method in which we compute the quadratic coefficients is very similar as what we did for the linear coefficients. We simply have more terms. Collecting the β_2^2 and $\beta_1\beta_2$ terms from (4.1) yields for $\mu = (02), (11)$ the equations

$$AH_{00\mu} = -J_1K_\mu - M_{00\mu}, \tag{4.42}$$

where $M_{00\mu}$ only depends on coefficients that are already known and are given by

$$\begin{aligned} M_{0002} &= 2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01}), \\ M_{0011} &= A_1(H_{0010}, K_{01}) + A_1(H_{0001}, K_{10}) + B(H_{0001}, H_{0010}) + J_2(K_{01}, K_{10}). \end{aligned}$$

Let $e_1, e_2 \in \mathbb{R}^2$ be the standard basis vectors and write

$$K_\mu = \gamma_{1,\mu}e_1 + \gamma_{2,\mu}e_2 \quad \mu = (02), (11), \tag{4.43}$$

where $\gamma_{1,\mu}, \gamma_{1,\mu} \in \mathbb{R}$ are unknown constants that need to be determined. From equation (4.42) it follows that

$$H_{00\mu} = -\gamma_{1,\mu}A^{-1}J_1e_1 - \gamma_{2,\mu}A^{-1}J_1e_2 - A^{-1}M_{00\mu} \tag{4.44}$$

The $w\beta_2^2$ and $w\beta_1\beta_2$ terms yield the equations

$$(i\omega_0 I - A)H_{10\mu} = A_1(q, K_\mu) + B(q, H_{00\mu}) - ib_{1,\mu}q + r_{10\mu}, \tag{4.45}$$

where we define

$$\begin{aligned} r_{1002} &= M_{1002} - 2ib_{1,01}H_{1001}, \\ r_{1011} &= M_{1011} - [(1 + ib_{1,10})H_{1001} + ib_{1,01}H_{1010}], \end{aligned}$$

with multilinear parts given by

$$\begin{aligned} M_{1002} = {} & 2A_1(H_{1001}, K_{01}) + 2B(H_{0001}, H_{1001}) + A_2(q, K_{01}, K_{01}) \\ & + 2B_1(q, H_{0001}, K_{01}) + C(q, H_{0001}, H_{0001}), \\ M_{1011} = {} & A_1(H_{1010}, K_{01}) + A_1(H_{1001}, K_{10}) + B(H_{0001}, H_{1010}) \\ & + B(H_{0010}, H_{1001}) + A_2(q, K_{01}, K_{10}) + B_1(q, H_{0010}, K_{01}) \\ & + B_1(q, H_{0001}, K_{10}) + C(q, H_{0001}, H_{0010}). \end{aligned}$$

Applying the Fredholm alternative to equation (4.45) yields the equations

$$b_{1,\mu}i = \bar{p}^T[A_1(q, K_\mu) + B(q, H_{00\mu}) + M_{10\mu}]. \tag{4.46}$$

Here we used that $\langle p, H_{1010}\rangle = \langle p, H_{1001}\rangle = 0$. Substituting expressione (4.43) and (4.44) into the above equation results in

$$b_{1,\mu}i = \bar{p}^T[\gamma_{1,\mu}\Gamma_1(q) + \gamma_{2,\mu}\Gamma_2(q) + \tilde{M}_{10\mu}], \tag{4.47}$$

where

$$\tilde{M}_{10\mu} = B(q, -A^{-1}M_{00\mu}) + M_{10\mu}.$$

From equation (4.49), it follows that

$$\gamma_{1,\mu}\Re[\bar{p}^T\Gamma_1(q)] + \gamma_{2,\mu}\Re[\bar{p}^T\Gamma_2(q)] = -\Re\{\bar{p}^T\tilde{M}_{10\mu}\}, \tag{4.48}$$

and

$$b_{1,\mu} = \gamma_{1,\mu}\Im[\bar{p}^T\Gamma_1(q)] + \gamma_{2,\mu}\Im[\bar{p}^T\Gamma_2(q)] + \Im\{\bar{p}^T\tilde{M}_{10\mu}\}. \tag{4.49}$$

Furthermore, $H_{10\mu}$ is given by

$$H_{10\mu} = \gamma_{1,\mu}A^{INV}_{i\omega_0}\Gamma_1(q) + \gamma_{2,\mu}A^{INV}_{i\omega_0}\Gamma_2(q) - ib_{1,\mu}A^{INV}_{i\omega_0}q + A^{INV}_{i\omega_0}\tilde{r}_{10\mu}, \tag{4.50}$$

where

$$\begin{aligned}\tilde{r}_{1002} &= \tilde{M}_{1002} - 2b_{1,01}H_{1001},\\ \tilde{r}_{1011} &= \tilde{M}_{1011} - [(1+ib_{1,10})H_{1001} + ib_{1,01}H_{1010}].\end{aligned}$$

Now collect the $w^2\beta^\mu$ terms for $\mu = (02), (11)$. These terms yield the equations

$$\begin{aligned}(2i\omega_0 I - A)H_{20\mu} = {} & A_1(H_{2000}, K_\mu) + 2B(q, H_{10\mu}) + B(H_{00\mu}, H_{2000})\\ & + B_1(q,q,K_\mu) + C(q,q,H_{00\mu}) - 2ib_{1,\mu}H_{2000} + r_{20\mu},\end{aligned} \tag{4.51}$$

where

$$\begin{aligned}r_{2002} &= M_{2002} - 4ib_{1,01}H_{2001},\\ r_{2011} &= M_{2011} - [2(1+ib_{1,10})H_{2001} + 2ib_{1,01}H_{2010}],\end{aligned}$$

with multilinear parts given by

$$\begin{aligned}M_{2002} = {} & 2A_1(H_{2001}, K_{01}) + 2B(H_{0001}, H_{2001}) + 2B(H_{1001}, H_{1001})\\ & + A_2(H_{2000}, K_{01}, K_{01}) + 4B_1(q, H_{1001}, K_{01}) + 2B_1(H_{0001}, H_{2000}, K_{01})\\ & + 4C(q, H_{0001}, H_{1001}) + C(H_{0001}, H_{0001}, H_{2000}) + B_2(q,q,K_{01},K_{01})\\ & + 2C_1(q,q,H_{0001},K_{01}) + D(q,q,H_{0001},H_{0001}),\\ M_{2011} = {} & A_1(H_{2010}, K_{01}) + A_1(H_{2001}, K_{10}) + B(H_{0001}, H_{2010})\\ & + B(H_{0010}, H_{2001}) + 2B(H_{1001}, H_{1010}) + A_2(H_{2000}, K_{01}, K_{10})\\ & + 2B_1(q, H_{1010}, K_{01}) + 2B_1(q, H_{1001}, K_{10}) + B_1(H_{0010}, H_{2000}, K_{01})\\ & + B_1(H_{0001}, H_{2000}, K_{10}) + 2C(q, H_{0001}, H_{1010}) + 2C(q, H_{0010}, H_{1001})\\ & + C(H_{0001}, H_{0010}, H_{2000}) + B_2(q,q,K_{10},K_{01}) + C_1(q,q,H_{0010},K_{01})\\ & + C_1(q,q,H_{0001},K_{10}) + D(q,q,H_{0001},H_{0010}).\end{aligned}$$

Collecting the $w\bar{w}\beta^\mu$ terms for $\mu = (02), (11)$ result in the equations

$$\begin{aligned}-AH_{11\mu} = {} & A_1(H_{1100}, K_\mu) + 2\Re\left(B(\bar{q}, H_{10\mu})\right) + B(H_{00\mu}, H_{1100})\\ & + B_1(q,\bar{q},K_\mu) + C(q,\bar{q},H_{00\mu}) + r_{11\mu},\end{aligned} \tag{4.52}$$

where

$$r_{1102} = M_{1102}, \quad \text{and} \quad r_{1111} = M_{1111} - 2H_{1101},$$

with the multilinear parts given by

$$
\begin{aligned}
M_{1102} &= 2A_1(H_{1101}, K_{01}) + 2B(H_{0001}, H_{1101}) + 2B(\overline{H}_{1001}, H_{1001}) \\
&+ A_2(H_{1100}, K_{01}, K_{01}) + 4\Re\left(B_1(\bar{q}, H_{1001}, K_{01})\right) + 2B_1(H_{0001}, H_{1100}, K_{01}) \\
&+ 4\Re\left(C(\bar{q}, H_{0001}, H_{1001})\right) + C(H_{0001}, H_{0001}, H_{1100}) + B_2(q, \bar{q}, K_{01}, K_{01}) \\
&+ 2C_1(q, \bar{q}, H_{0001}, K_{01}) + D(q, \bar{q}, H_{0001}, H_{0001}), \\
M_{1111} &= A_1(H_{1110}, K_{01}) + A_1(H_{1101}, K_{10}) + B(H_{0001}, H_{1110}) \\
&+ B(H_{0010}, H_{1101}) + 2\Re\left(B(\overline{H}_{1001}, H_{1010})\right) + A_2(H_{1100}, K_{01}, K_{10}) \\
&+ 2\Re\left(B_1(\bar{q}, H_{1010}, K_{01})\right) + 2\Re\left(B_1(\bar{q}, H_{1001}, K_{10})\right) + B_1(H_{0010}, H_{1100}, K_{01}) \\
&+ B_1(H_{0001}, H_{1100}, K_{10}) + 2\Re\left(C(\bar{q}, H_{0001}, H_{1010})\right) + 2\Re\left(C(\bar{q}, H_{0010}, H_{1001})\right) \\
&+ C(H_{0001}, H_{0010}, H_{1100}) + B_2(q, \bar{q}, K_{01}, K_{10}) + C_1(q, \bar{q}, H_{0010}, K_{01}) \\
&+ C_1(q, \bar{q}, H_{0001}, K_{10}) + D(q, \bar{q}, H_{0001}, H_{0010}).
\end{aligned}
$$

From equations (4.51) and (4.52) it follows that

$$
\begin{aligned}
H_{20\mu} &= A_{2i\omega_0}^{-1}[A_1(H_{2000}, K_\mu) + 2B(q, H_{10\mu}) + B(H_{00\mu}, H_{2000}) \\
&\quad + B_1(q, q, K_\mu) + C(q, q, H_{00\mu}) + r_{20\mu}] - 2ib_{1,\mu}A_{2i\omega_0}^{-1}H_{2000}, \qquad (4.53) \\
H_{11\mu} &= -A^{-1}[A_1(H_{1100}, K_\mu) + 2\Re\left(B(\bar{q}, H_{10\mu})\right) + B(H_{00\mu}, H_{1100}) \\
&\quad + B_1(q, \bar{q}, K_\mu) + C(q, \bar{q}, H_{00\mu}) + r_{11\mu}]. \qquad (4.54)
\end{aligned}
$$

Substituting equations (4.43), (4.44) and (4.50) into the expressions for $H_{20\mu}$ and $H_{11\mu}$ yields the equations

$$
\begin{aligned}
H_{20\mu} &= \gamma_{1,\mu}A_{2i\omega_0}^{-1}\Lambda_1(H_{2000}, q, q) + \gamma_{2,\mu}A_{2i\omega_0}^{-1}\Lambda_2(H_{2000}, q, q) \\
&\quad - 2ib_{1,\mu}A_{2i\omega_0}^{-1}H_{2000} + A_{2i\omega_0}^{-1}\tilde{r}_{20\mu}, \qquad (4.55) \\
H_{11\mu} &= -\gamma_{1,\mu}A^{-1}\Pi_1(H_{1100}, \bar{q}, q) - \gamma_{2,\mu}A^{-1}\Pi_2(H_{1100}, \bar{q}, q) \\
&\quad - A^{-1}\tilde{r}_{11\mu}, \qquad (4.56)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{r}_{20\mu} &= r_{20\mu} + 2B\left(q, A_{i\omega_0}^{INV}\tilde{r}_{10\mu}\right) + B(H_{2000}, -A^{-1}r_{00\mu}) + C(q, q, -A^{-1}r_{00\mu}), \\
\tilde{r}_{11\mu} &= r_{11\mu} + 2\Re\left(B\left(\bar{q}, A_{i\omega_0}^{INV}\tilde{r}_{10\mu}\right)\right) + B(H_{1100}, -A^{-1}r_{00\mu}) + C(q, \bar{q}, -A^{-1}r_{00\mu}).
\end{aligned}
$$

The $w^2\bar{w}\beta^\mu$ terms yield for $\mu = (02), (11)$ the equations

$$
\begin{aligned}
(i\omega_0 I - A)H_{21\mu} &= A_1(H_{2100}, K_\mu) + 2B(q, H_{11\mu}) + B(\bar{q}, H_{20\mu}) + B(H_{00\mu}, H_{2100}) \\
&+ B(\overline{H}_{10\mu}, H_{2000}) + 2B(H_{10\mu}, H_{1100}) + 2B_1(q, H_{1100}, K_\mu) \\
&+ B_1(\bar{q}, H_{2000}, K_\mu) + C(q, q, \overline{H}_{10\mu}) + 2C(q, \bar{q}, H_{10\mu}) + 2C(q, H_{00\mu}, H_{1100}) \\
&+ C(\bar{q}, H_{00\mu}, H_{2000}) + C_1(q, q, \bar{q}, K_\mu) + D(q, q, \bar{q}, H_{00\mu}) + r_{21\mu} \\
&- (2ib_{2,\mu}q + ib_{1,\mu}H_{2100} + 2c_1(0)H_{10\mu}), \qquad (4.57)
\end{aligned}
$$

where

$$
\begin{aligned}
r_{2102} &= M_{2102} - [4(1 + ib_{2,01})H_{1001} + 2ib_{1,01}H_{2101}], \\
r_{2111} &= M_{2111} - [2ib_{2,10}H_{1001} + 2(1 + ib_{2,01})H_{1010} + (3 + ib_{1,10})\,H_{2101} + ib_{1,01}H_{2110}].
\end{aligned}
$$

with multilinear parts given by

$$\begin{aligned}M_{2102} &= 2A_1(H_{2101}, K_{01}) + 2B(H_{0001}, H_{2101}) + 2B(\overline{H}_{1001}, H_{2001}) + 4B(H_{1001}, H_{1101})\\
&+ A_2(H_{2100}, K_{01}, K_{01}) + 4B_1(q, H_{1101}, K_{01}) + 2B_1(\bar{q}, H_{2001}, K_{01})\\
&+ 2B_1(H_{0001}, H_{2100}, K_{01}) + 2B_1(\overline{H}_{1001}, H_{2000}, K_{01}) + 4B_1(H_{1001}, H_{1100}, K_{01})\\
&+ 4C(q, H_{0001}, H_{1101}) + 4C(q, \overline{H}_{1001}, H_{1001}) + 2C(\bar{q}, H_{0001}, H_{2001})\\
&+ 2C(\bar{q}, H_{1001}, H_{1001}) + C(H_{0001}, H_{0001}, H_{2100}) + 2C(H_{0001}, \overline{H}_{1001}, H_{2000})\\
&+ 4C(H_{0001}, H_{1001}, H_{1100}) + 2B_2(q, H_{1100}, K_{01}, K_{01}) + B_2(\bar{q}, H_{2000}, K_{01}, K_{01})\\
&+ 2C_1(q, q, \overline{H}_{1001}, K_{01}) + 4C_1(q, \bar{q}, H_{1001}, K_{01}) + 4C_1(q, H_{0001}, H_{1100}, K_{01})\\
&+ 2C_1(\bar{q}, H_{0001}, H_{2000}, K_{01}) + 2D(q, q, H_{0001}, \overline{H}_{1001}) + 4D(q, \bar{q}, H_{0001}, H_{1001})\\
&+ 2D(q, H_{0001}, H_{0001}, H_{1100}) + D(\bar{q}, H_{2000}, H_{0001}, H_{0001}) + C_2(q, q, \bar{q}, K_{01}, K_{01})\\
&+ 2D_1(q, q, \bar{q}, H_{0001}, K_{01}) + E(q, q, \bar{q}, H_{0001}, H_{0001}),\end{aligned}$$

and

$$\begin{aligned}M_{2111} &= A_1(H_{2110}, K_{01}) + A_1(H_{2101}, K_{10}) + B(H_{0001}, H_{2110}) + B(H_{0010}, H_{2101})\\
&+ B(\overline{H}_{1001}, H_{2010}) + B(\overline{H}_{1010}, H_{2001}) + 2B(H_{1001}, H_{1110}) + 2B(H_{1010}, H_{1101})\\
&+ A_2(H_{2100}, K_{01}, K_{10}) + 2B_1(q, H_{1110}, K_{01}) + 2B_1(q, H_{1101}, K_{10}) + B_1(\bar{q}, H_{2010}, K_{01})\\
&+ B_1(\bar{q}, H_{2001}, K_{10}) + B_1(H_{0010}, H_{2100}, K_{01}) + B_1(\overline{H}_{1010}, H_{2000}, K_{01}) + 2B_1(H_{1010}, H_{1100}, K_{01})\\
&+ B_1(H_{0001}, H_{2100}, K_{10}) + B_1(\overline{H}_{1001}, H_{2000}, K_{10}) + 2B_1(H_{1001}, H_{1100}, K_{10}) + 2C(q, H_{0001}, H_{1110})\\
&+ 2C(q, H_{0010}, H_{1101}) + 2C(q, \overline{H}_{1001}, H_{1010}) + 2C(q, \overline{H}_{1010}, H_{1001}) + C(\bar{q}, H_{0001}, H_{2010})\\
&+ C(\bar{q}, H_{0010}, H_{2001}) + 2C(\bar{q}, H_{1001}, H_{1010}) + C(H_{0001}, H_{0010}, H_{2100}) + C(H_{0001}, \overline{H}_{1010}, H_{2000})\\
&+ 2C(H_{0001}, H_{1010}, H_{1100}) + C(H_{0010}, \overline{H}_{1001}, H_{2000}) + 2C(H_{0010}, H_{1001}, H_{1100})\\
&+ 2B_2(q, H_{1100}, K_{01}, K_{10}) + B_2(\bar{q}, H_{2000}, K_{01}, K_{10}) + C_1(q, q, \overline{H}_{1010}, K_{01}) + C_1(q, q, \overline{H}_{1001}, K_{10})\\
&+ 2C_1(q, \bar{q}, H_{1010}, K_{01}) + 2C_1(q, \bar{q}, H_{1001}, K_{10}) + 2C_1(q, H_{0010}, H_{1100}, K_{01})\\
&+ 2C_1(q, H_{0001}, H_{1100}, K_{10}) + C_1(\bar{q}, H_{0010}, H_{2000}, K_{01}) + C_1(\bar{q}, H_{0001}, H_{2000}, K_{10})\\
&+ D(q, q, H_{0001}, \overline{H}_{1010}) + D(q, q, H_{0010}, \overline{H}_{1001}) + 2D(q, \bar{q}, H_{0001}, H_{1010})\\
&+ 2D(q, \bar{q}, H_{0010}, H_{1001}) + 2D(q, H_{0001}, H_{0010}, H_{1100}) + D(\bar{q}, H_{0001}, H_{0010}, H_{2000})\\
&+ C_2(q, q, \bar{q}, K_{01}, K_{10}) + D_1(q, q, \bar{q}, H_{0010}, K_{01}) + D_1(q, q, \bar{q}, H_{0001}, K_{10}) + E(q, q, \bar{q}, H_{0001}, H_{0010}).\end{aligned}$$

Applying the Fredholm solvabitlity condition to (4.57) yields the equations

$$\begin{aligned}b_{2,\mu} i = \frac{1}{2}\bar{p}^T[&A_1(H_{2100}, K_\mu) + 2B(q, H_{11\mu}) + B(\bar{q}, H_{20\mu}) + B(H_{00\mu}, H_{2100})\\
&+ B(\overline{H}_{10\mu}, H_{2000}) + 2B(H_{10\mu}, H_{1100}) + 2B_1(q, H_{1100}, K_\mu)\\
&+ B_1(\bar{q}, H_{2000}, K_\mu) + C(q, q, \overline{H}_{10\mu}) + 2C(q, \bar{q}, H_{10\mu}) + 2C(q, H_{00\mu}, H_{1100})\\
&+ C(\bar{q}, H_{00\mu}, H_{2000}) + C_1(q, q, \bar{q}, K_\mu) + D(q, q, \bar{q}, H_{00\mu}) + M_{21\mu}]. \qquad (4.58)\end{aligned}$$

Now substitute the expressions for $H_{00\mu}, H_{10\mu}, H_{20\mu}, H_{11\mu},$ K_μ and $b_{1,\mu}$ into the above expression to solve for $\gamma_{1,\mu}$ and $\gamma_{2,\mu}$.

We find the following system to solve for $\gamma_{1,\mu}$ and $\gamma_{2,\mu}$

$$P\begin{pmatrix}\gamma_{1,\mu}\\ \gamma_{2,\mu}\end{pmatrix} = Q_\mu, \qquad (4.59)$$

where $P \in \mathbb{R}^{2\times 2}$ has the same components (4.36),(4.37) and $Q_\mu \in \mathbb{R}^2$ is given by

$$\begin{aligned}
Q_{\mu,1} &= -\Re\{\bar{p}^T \tilde{M}_{10\mu}\},\\
Q_{\mu,2} &= -\frac{1}{2}\Re\Big\{\bar{p}^T \Big[2B\left(q, -A^{-1}\tilde{r}_{11\mu}\right) + B\left(\bar{q}, A^{-1}_{2i\omega_0}\tilde{r}_{20\mu}\right)\\
&\quad + B\left(H_{2100}, -A^{-1}M_{00\mu}\right) + B\left(H_{2000}, \overline{A^{INV}_{i\omega_0}\tilde{r}_{10\mu}}\right) + 2B\left(H_{1100}, A^{INV}_{i\omega_0}\tilde{r}_{10\mu}\right)\\
&\quad + C\left(q, q, \overline{A^{INV}_{i\omega_0}\tilde{r}_{10\mu}}\right) + 2C\left(q, \bar{q}, A^{INV}_{i\omega_0}\tilde{r}_{10\mu}\right) + 2C\left(q, H_{1100}, -A^{-1}M_{00\mu}\right)\\
&\quad + C\left(\bar{q}, H_{2000}, -A^{-1}M_{00\mu}\right) + D(q, q, \bar{q}, -A^{-1}M_{00\mu}) + M_{21\mu}\\
&\quad + \Im\{\bar{p}^T\tilde{M}_{10\mu}\}\left(-2iB(\bar{q}, A^{-1}_{2i\omega_0}H_{2000})\right)\Big]\Big\}.
\end{aligned}$$

Thus, to obtain the quadratic coefficients K_{02} and K_{11} we solve system (4.59). Once these coefficients are known, one can calculate the coefficients $H_{00\mu}$,$b_{1,02}$ and $H_{10\mu}$ from respectively the equations (4.44),(4.49) and (4.50). Then, the coefficients H_{2002} and H_{1102} are calculated from equations (4.53), (4.54). Finally, the coefficient H_{2102} satisfying $\langle p, H_{2102}\rangle = 0$, can be determined from system (4.57) using the corresponding bordered system (2.10).

The coefficient K_{03} The qubic coefficient K_{03} is determined in the same way as the quadratic coefficients. Essentially, only the terms containing all the known cofficients will be different. Collecting the β_2^3 terms yields the equation

$$AH_{0003} = -J_1K_{03} - M_{0003}, \tag{4.60}$$

with

$$\begin{aligned}
M_{0003} &= 3A_1(H_{0002}, K_{01}) + 3A_1(H_{0001}, K_{02}) + 3B(H_{0001}, H_{0002})\\
&\quad + 3J_2(K_{01}, K_{02}) + 3A_2(H_{0001}, K_{01}, K_{01}) + 3B_1(H_{0001}, H_{0001}, K_{01})\\
&\quad + J_3(K_{01}, K_{01}, K_{01}) + C(H_{0001}, H_{0001}, H_{0001}).
\end{aligned}$$

With $e_1, e_2 \in \mathbb{R}^2$ the standard basis vectors and $\gamma_{1,03}, \gamma_{2,03} \in \mathbb{R}$ we write

$$K_{03} = \gamma_{1,03}e_1 + \gamma_{2,03}e_2.$$

The $w\beta_2^3$ terms yield the equation

$$(i\omega_0 I_n - A)H_{1003} = A_1(q, K_{03}) + B(q, H_{0003}) - ib_{1,03}q + r_{1003}, \tag{4.61}$$

where

$$r_{1003} = M_{1003} - (3ib_{1,02}H_{1001} + 3ib_{1,01}H_{1002}),$$

with multilinear part

$$\begin{aligned}
M_{1003} &= 3A_1(H_{1002}, K_{01}) + 3A_1(H_{1001}, K_{02}) + 3B(H_{0001}, H_{1002})\\
&\quad + 3B(H_{0002}, H_{1001}) + 3A_2(q, K_{01}, K_{02}) + 3A_2(H_{1001}, K_{01}, K_{01})\\
&\quad + 3B_1(q, H_{0002}, K_{01}) + 3B_1(q, H_{0001}, K_{02}) + 6B_1(H_{0001}, H_{1001}, K_{01})\\
&\quad + 3C(q, H_{0001}, H_{0002}) + 3C(H_{0001}, H_{0001}, H_{1001}) + A_3(q, K_{01}, K_{01}, K_{01})\\
&\quad + 3B_2(q, H_{0001}, K_{01}, K_{01}) + 3C_1(q, H_{0001}, H_{0001}, K_{01})\\
&\quad + D(q, H_{0001}, H_{0001}, H_{0001}).
\end{aligned}$$

Collecting the $w^2\beta_2^3$ and $w\bar{w}\beta_2^3$ terms yield the equations

$$
\begin{aligned}
(2i\omega_0 I_n - A)H_{2003} &= A_1(H_{2000}, K_{03}) + 2B(q, H_{1003}) + B(H_{0003}, H_{2000}) \\
&\quad + B_1(q, q, K_{03}) + C(q, q, H_{0003}) - 2ib_{1,03}H_{2000} + r_{2003}, \qquad (4.62)\\
-AH_{1103} &= A_1(H_{1100}, K_{03}) + 2\Re(B(\bar{q}, H_{1003})) + B(H_{0003}, H_{1100}) \\
&\quad + B_1(q, \bar{q}, K_{03}) + C(q, \bar{q}, H_{0003}) + r_{1103}, \qquad (4.63)
\end{aligned}
$$

where

$$
r_{2003} = M_{2003} - (6ib_{1,02}H_{2001} + 6ib_{1,01}H_{2002}), \quad \text{and} \quad r_{1103} = M_{1103},
$$

with multilinear parts given by

$$
\begin{aligned}
M_{2003} &= 3A_1(H_{2002}, K_{01}) + 3A_1(H_{2001}, K_{02}) + 3B(H_{0001}, H_{2002}) + 3B(H_{0002}, H_{2001}) \\
&\quad + 6B(H_{1001}, H_{1002}) + 3A_2(H_{2001}, K_{01}, K_{01}) + 3A_2(H_{2000}, K_{01}, K_{02}) \\
&\quad + 6B_1(q, H_{1002}, K_{01}) + 6B_1(q, H_{1001}, K_{02}) + 6B_1(H_{0001}, H_{2001}, K_{01}) \\
&\quad + 3B_1(H_{0002}, H_{2000}, K_{01}) + 6B_1(H_{1001}, H_{1001}, K_{01}) + 3B_1(H_{0001}, H_{2000}, K_{02}) \\
&\quad + 6C(q, H_{0001}, H_{1002}) + 6C(q, H_{0002}, H_{1001}) + 3C(H_{0001}, H_{0001}, H_{2001}) \\
&\quad + 3C(H_{0001}, H_{0002}, H_{2000}) + 6C(H_{0001}, H_{1001}, H_{1001}) + A_3(H_{2000}, K_{01}, K_{01}, K_{01}) \\
&\quad + 3B_2(q, q, K_{01}, K_{02}) + 6B_2(q, H_{1001}, K_{01}, K_{01}) + 3B_2(H_{0001}, H_{2000}, K_{01}, K_{01}) \\
&\quad + 3C_1(q, q, H_{0002}, K_{01}) + 3C_1(q, q, H_{0001}, K_{02}) + 12C_1(q, H_{0001}, H_{1001}, K_{01}) \\
&\quad + 3C_1(H_{0001}, H_{0001}, H_{2000}, K_{01}) + 3D(q, q, H_{0001}, H_{0002}) + 6D(q, H_{0001}, H_{0001}, H_{1001}) \\
&\quad + D(H_{0001}, H_{0001}, H_{0001}, H_{2000}) + B_3(q, q, K_{01}, K_{01}, K_{01}) + 3C_2(q, q, H_{0001}, K_{01}, K_{01}) \\
&\quad + 3D_1(q, q, H_{0001}, H_{0001}, K_{01}) + E(q, q, H_{0001}, H_{0001}, H_{0001}),
\end{aligned}
$$

and

$$
\begin{aligned}
M_{1103} &= 3A_1(H_{1102}, K_{01}) + 3A_1(H_{1101}, K_{02}) + 3B(H_{0001}, H_{1102}) + 3B(H_{0002}, H_{1101}) \\
&\quad + 3B(\overline{H}_{1001}, H_{1002}) + 3B(\overline{H}_{1002}, H_{1001}) + 3A_2(H_{1101}, K_{01}, K_{01}) \\
&\quad + 3A_2(H_{1100}, K_{01}, K_{02}) + 6\Re(B_1(\bar{q}, H_{1002}, K_{01})) + 6\Re(B_1(\bar{q}, H_{1001}, K_{02})) \\
&\quad + 6B_1(H_{0001}, H_{1101}, K_{01}) + 3B_1(H_{0002}, H_{1100}, K_{01}) + 6B_1(\overline{H}_{1001}, H_{1001}, K_{01}) \\
&\quad + 3B_1(H_{0001}, H_{1100}, K_{02}) + 6\Re(C(\bar{q}, H_{0001}, H_{1002})) + 6\Re(C(\bar{q}, H_{0002}, H_{1001})) \\
&\quad + 3C(H_{0001}, H_{0001}, H_{1101}) + 3C(H_{0001}, H_{0002}, H_{1100}) + 6C(H_{0001}, \overline{H}_{1001}, H_{1001}) \\
&\quad + A_3(H_{1100}, K_{01}, K_{01}, K_{01}) + 3B_2(q, \bar{q}, K_{01}, K_{02}) + 6\Re(B_2(\bar{q}, H_{1001}, K_{01}, K_{01})) \\
&\quad + 3B_2(H_{0001}, H_{1100}, K_{01}, K_{01}) + 3C_1(q, \bar{q}, H_{0002}, K_{01}) + 3C_1(q, \bar{q}, H_{0001}, K_{02}) \\
&\quad + 12\Re(C_1(\bar{q}, H_{0001}, H_{1001}, K_{01})) + 3C_1(H_{0001}, H_{0001}, H_{1100}, K_{01}) \\
&\quad + 3D(q, \bar{q}, H_{0001}, H_{0002}) + 6\Re(D(\bar{q}, H_{0001}, H_{0001}, H_{1001})) + D(H_{0001}, H_{0001}, H_{0001}, H_{1100}) \\
&\quad + B_3(q, \bar{q}, K_{01}, K_{01}, K_{01}) + 3C_2(q, \bar{q}, H_{0001}, K_{01}, K_{01}) + 3D_1(q, \bar{q}, H_{0001}, H_{0001}, K_{01}) \\
&\quad + E(q, \bar{q}, H_{0001}, H_{0001}, H_{0001}).
\end{aligned}
$$

Applying the Fredholm solvability condition to the $w^2\bar{w}\beta_2^3$ terms finally yields the equation

$$
\begin{aligned}
ib_{2,03} &= \frac{1}{2}\bar{p}^T[A_1(H_{2100}, K_{03}) + 2B(q, H_{1103}) + B(\bar{q}, H_{2003}) + B(H_{0003}, H_{2100}) \\
&\quad + B(\overline{H}_{1003}, H_{2000}) + 2B(H_{1003}, H_{1100}) + 2B_1(q, H_{1100}, K_{03}) \\
&\quad + B_1(\bar{q}, H_{2000}, K_{03}) + C(q, q, \overline{H}_{1003}) + 2C(q, \bar{q}, H_{1003}) + 2C(q, H_{0003}, H_{1100}) \\
&\quad + C(\bar{q}, H_{0003}, H_{2000}) + C_1(q, q, \bar{q}, K_{03}) + D(q, q, \bar{q}, H_{0003}) + M_{2103}]. \qquad (4.64)
\end{aligned}
$$

where the expression for r_{2103} can be found on the next page. Equations (4.60),(4.61),(4.62), (4.63) and (4.64) are respectively given by equations (4.42),(4.45),(4.51), (4.52) and (4.58) for $\mu = (03)$. As a result, we can solve for the coefficients $\gamma_{1,03}$ and $\gamma_{2,03}$ by simply using system (4.59) for $\mu = (03)$. Then H_{0003} and H_{1003} can be calculated from respectively the equations (4.60) and (4.61), where the latter is solved using the corresponding bordered system (2.10).

$$
\begin{aligned}
r_{2103} = {} & 3A_1(H_{2102}, K_{01}) + 3A_1(H_{2101}, K_{02}) + 3B(H_{0001}, H_{2102}) + 3B(H_{0002}, H_{2101}) + 3B(\overline{H}_{1001}, H_{2002}) \\
& + 3B(\overline{H}_{1002}, H_{2001}) + 6B(H_{1001}, H_{1102}) + 6B(H_{1002}, H_{1101}) + 3A_2(H_{2101}, K_{01}, K_{01}) \\
& + 3A_2(H_{2100}, K_{01}, K_{02}) + 6B_1(q, H_{1102}, K_{01}) + 6B_1(q, H_{1101}, K_{02}) + 3B_1(\bar{q}, H_{2002}, K_{01}) \\
& + 3B_1(\bar{q}, H_{2001}, K_{02}) + 6B_1(H_{0001}, H_{2101}, K_{01}) + 3B_1(H_{0002}, H_{2100}, K_{01}) + 6B_1(\overline{H}_{1001}, H_{2001}, K_{01}) \\
& + 3B_1(\overline{H}_{1002}, H_{2000}, K_{01}) + 12B_1(H_{1001}, H_{1101}, K_{01}) + 6B_1(H_{1002}, H_{1100}, K_{01}) \\
& + 3B_1(H_{0001}, H_{2100}, K_{02}) + 3B_1(\overline{H}_{1001}, H_{2000}, K_{02}) + 6B_1(H_{1001}, H_{1100}, K_{02}) \\
& + 6C(q, H_{0001}, H_{1102}) + 6C(q, H_{0002}, H_{1101}) + 6C(q, \overline{H}_{1001}, H_{1002}) + 6C(q, \overline{H}_{1002}, H_{1001}) \\
& + 3C(\bar{q}, H_{0001}, H_{2002}) + 3C(\bar{q}, H_{0002}, H_{2001}) + 6C(\bar{q}, H_{1001}, H_{1002}) + 3C(H_{0001}, H_{0001}, H_{2101}) \\
& + 3C(H_{0001}, H_{0002}, H_{2100}) + 6C(H_{0001}, \overline{H}_{1001}, H_{2001}) + 3C(H_{0001}, \overline{H}_{1002}, H_{2000}) \\
& + 12C(H_{0001}, H_{1001}, H_{1101}) + 6C(H_{0001}, H_{1002}, H_{1100}) + 3C(H_{0002}, \overline{H}_{1001}, H_{2000}) \\
& + 6C(H_{0002}, H_{1001}, H_{1100}) + 6C(\overline{H}_{1001}, H_{1001}, H_{1001}) + A_3(H_{2100}, K_{01}, K_{01}, K_{01}) \\
& + 6B_2(q, H_{1101}, K_{01}, K_{01}) + 6B_2(q, H_{1100}, K_{01}, K_{02}) + 3B_2(\bar{q}, H_{2001}, K_{01}, K_{01}) \\
& + 3B_2(\bar{q}, H_{2000}, K_{01}, K_{02}) + 3B_2(H_{0001}, H_{2100}, K_{01}, K_{01}) + 3B_2(\overline{H}_{1001}, H_{2000}, K_{01}, K_{01}) \\
& + 6B_2(H_{1001}, H_{1100}, K_{01}, K_{01}) + 3C_1(q, q, \overline{H}_{1002}, K_{01}) + 3C_1(q, q, \overline{H}_{1001}, K_{02}) \\
& + 6C_1(q, \bar{q}, H_{1002}, K_{01}) + 6C_1(q, \bar{q}, H_{1001}, K_{02}) + 12C_1(q, H_{0001}, H_{1101}, K_{01}) \\
& + 6C_1(q, H_{0002}, H_{1100}, K_{01}) + 12C_1(q, \overline{H}_{1001}, H_{1001}, , K_{01}) + 6C_1(q, H_{0001}, H_{1100}, K_{02}) \\
& + 6C_1(\bar{q}, H_{0001}, H_{2001}, K_{01}) + 3C_1(\bar{q}, H_{0002}, H_{2000}, K_{01}) + 6C_1(\bar{q}, H_{1001}, H_{1001}, K_{01}) \\
& + 3C_1(\bar{q}, H_{0001}, H_{2000}, K_{02}) + 3C_1(H_{0001}, H_{0001}, H_{2100}, K_{01}) + 6C_1(H_{0001}, \overline{H}_{1001}, H_{2000}, K_{01}) \\
& + 12C_1(H_{0001}, H_{1001}, H_{1100}, K_{01}) + 3D(q, q, H_{0001}, \overline{H}_{1002}) + 3D(q, q, H_{0002}, \overline{H}_{1001}) \\
& + 6D(q, \bar{q}, H_{0001}, H_{1002}) + 6D(q, \bar{q}, H_{0002}, H_{1001}) + 6D(q, H_{0001}, H_{0001}, H_{1101}) \\
& + 6D(q, H_{0001}, H_{0002}, H_{1100}) + 12D(q, H_{0001}, \overline{H}_{1001}, H_{1001}) + 3D(\bar{q}, H_{0001}, H_{0001}, H_{2001}) \\
& + 3D(\bar{q}, H_{0001}, H_{0002}, H_{2000}) + 6D(\bar{q}, H_{0001}, H_{1001}, H_{1001}) + D(H_{0001}, H_{0001}, H_{0001}, H_{2100}) \\
& + 3D(H_{0001}, H_{0001}, \overline{H}_{1001}, H_{2000}) + 6D(H_{0001}, H_{0001}, H_{1001}, H_{1100}) \\
& + 2B_3(q, H_{1100}, K_{01}, K_{01}, K_{01}) + B_3(\bar{q}, H_{2000}, K_{01}, K_{01}, K_{01}) + 3C_2(q, q, \bar{q}, K_{01}, K_{02}) \\
& + 3C_2(q, q, \overline{H}_{1001}, K_{01}, K_{01}) + 6C_2(q, \bar{q}, H_{1001}, K_{01}, K_{01}) + 6C_2(q, H_{0001}, H_{1100}, K_{01}, K_{01}) \\
& + 3C_2(\bar{q}, H_{0001}, H_{2000}, K_{01}, K_{01}) + 3D_1(q, q, \bar{q}, H_{0002}, K_{01}) + 3D_1(q, q, \bar{q}, H_{0001}, K_{02}) \\
& + 6D_1(q, q, H_{0001}, \overline{H}_{1001}, K_{01}) + 12D_1(q, \bar{q}, H_{0001}, H_{1001}, K_{01}) + 6D_1(q, H_{0001}, H_{0001}, H_{1100}, K_{01}) \\
& + 3D_1(\bar{q}, H_{0001}, H_{0001}, H_{2000}, K_{01}) + 3E(q, q, \bar{q}, H_{0001}, H_{0002}) + 3E(q, q, H_{0001}, H_{0001}, \overline{H}_{1001}) \\
& + 6E(q, \bar{q}, H_{0001}, H_{0001}, H_{1001}) + 2E(q, H_{0001}, H_{0001}, H_{0001}, H_{1100}) \\
& + E(\bar{q}, H_{0001}, H_{0001}, H_{0001}, H_{2000}) + C_3(q, q, \bar{q}, K_{01}, K_{01}, K_{01}) \\
& + 3E_1(q, q, \bar{q}, H_{0001}, H_{0001}, K_{01}) + K(q, q, \bar{q}, H_{0001}, H_{0001}, H_{0001}).
\end{aligned}
$$

4.2 The higher order LPC predictor for ODEs

We approximate the parameter values on the LPC curve by

$$\beta_1 = d_2\varepsilon^4 + 2(d_3 - a_{3201}d_2)\varepsilon^6, \quad \beta_2 = -2d_2\varepsilon^2 + (4a_{3201}d_2 - 3d_3)\varepsilon^4,$$

for $\varepsilon > 0$. Taking the expansion (4.7) for K, we have the following parameter predictor

$$\alpha = \alpha_0 + K(\beta_1, \beta_2). \tag{4.65}$$

To approximate the solution in phase space we substitute $w = \varepsilon e^{i\psi}$ into the expansion (4.6) of H together with the approximations in the β parameters, where it is not necessary to include the last three terms that are marked grey. Thus, for $\psi \in [0, 2\pi]$ the periodic orbit can be approximated as

$$x = x_0 + H(\varepsilon e^{i\psi}, \varepsilon e^{-i\psi}, \beta_1, \beta_2). \tag{4.66}$$

Finally, the period is approximated by equation (3.9),

$$T = 2\pi/(\omega_0 + (\Im(c_1(0)) - 2d_2b_{1,01})\varepsilon^2 + [d_2b_{1,10} + (4a_{3201}d_2 - 3d_3)b_{1,01} + 2d_2^2b_{1,02} \\ - 2d_2b_{2,01} + \Im(c_2(0))]\varepsilon^4).$$

Chapter 5

The predictor for DDEs

5.1 Coefficients of the parameter-dependent normal form and the predictor in DDEs

We will derive the equations to calculate the coefficients needed in the predictor for DDEs. We follow the procedure as explained in Section 2.2. Assume that system (2.11) has an equilibrium at the origin at $\alpha = (0, 0) \in \mathbb{R}^2$ with only one pair of purely imaginary simple eigenvalues

$$\lambda_{1,2} = \pm i\omega_0, \quad \omega_0 > 0,$$

and no other eigenvalues on the imaginary axis. Furthermore, we assume that the first Lyapunov coefficient $l_1(0) = 0$ and the second Lyapunov coefficient $l_2(0) \neq 0$. Instead of eigenvectors, we now have eigenfunctions φ and $\varphi^{\odot}$ satisfying

$$A\varphi = i\omega_0\varphi, \quad A^{\star}\varphi^{\odot} = i\omega_0\varphi^{\odot}, \quad \langle \varphi^{\odot}, \varphi\rangle = 1,$$

where $\langle \varphi^{\odot}, \varphi\rangle$ is given by the pairing (2.15) not to be confused with the Hermitian inner product. Furthermore, introduce $q, p \in \mathbb{C}^n$ such that

$$\Delta(i\omega_0)q = 0, \quad p^T\Delta(i\omega_0) = 0, \quad p^T\Delta'(i\omega_0)q = 1.$$

With q and p as above, explicit expressions for the eigenfunctions φ and $\varphi^{\odot}$ are respectively given by equations (2.28) and (2.29). Points $y \in X_0$ of the real critical eigenspace can be represented in terms of the complex coordinate $z = \langle \varphi^{\odot}, y\rangle$ as,

$$y = z\varphi + \bar{z}\bar{\varphi}.$$

The homological equation (2.23) becomes

$$\begin{aligned} A^{\odot\star}jH(z,\bar{z},\beta) + J_1K(\beta)r^{\odot\star} + &R(H(z,\bar{z},\beta), K(\beta)) \\ &= j(D_zH(z,\bar{z},\beta)\dot{z} + D_{\bar{z}}H(z,\bar{z},\beta)\dot{\bar{z}}). \end{aligned} \tag{5.1}$$

Restricted to the two-dimensional center manifold, the system can be transformed into the same normalform (4.2) with new unfolding parameters $\beta = (\beta_1, \beta_2)$ if the transversality condition (3.1) holds. Thus, we take the same truncated normal form:

$$\begin{aligned} \dot{z} = &(i\omega_0 + \beta_1 + ib_1(\beta))z + (\beta_2 + ib_2(\beta))z|z|^2 + (c_2(0) + g_{3201}\beta_2)z|z|^4 \\ &+ c_3(0)z|z|^6, \end{aligned} \tag{5.2}$$

where $b_1(\beta)$ and $\beta_2(\beta)$ are expanded as (4.3) and (4.4). The nonlinearity in the homological equation (5.1) is expanded as

$$\begin{aligned}
R(u,\alpha) = [&A_1(u,\alpha) + \frac{1}{2}B(u,u) + \frac{1}{2}J_2(\alpha,\alpha) + \frac{1}{6}C(u,u,u) + \frac{1}{2}B_1(u,u,\alpha) \\
&+ \frac{1}{2}A_2(u,\alpha,\alpha) + \frac{1}{6}J_3(\alpha,\alpha,\alpha) + \frac{1}{24}D(u,u,u,u) + \frac{1}{6}C_1(u,u,u,\alpha) + \frac{1}{4}B_2(u,u,\alpha,\alpha) \\
&+ \frac{1}{6}A_3(u,\alpha,\alpha,\alpha) + \frac{1}{120}E(u,u,u,u,u) + \frac{1}{24}D_1(u,u,u,u,\alpha) + \frac{1}{12}C_2(u,u,u,\alpha,\alpha) \\
&+ \frac{1}{12}B_3(u,u,\alpha,\alpha,\alpha) + \frac{1}{720}K(u,u,u,u,u,u) + \frac{1}{120}E_1(u,u,u,u,u,\alpha) \\
&+ \frac{1}{36}C_3(u,u,u,\alpha,\alpha,\alpha) + \frac{1}{5040}L(u,u,u,u,u,u,u)]r^{\odot\star}.
\end{aligned}$$

Finally, H is expanded as

$$\begin{aligned}
H(z,\bar{z},\beta) = z\varphi + \bar{z}\bar{\varphi} &+ \sum_{n+m=2}^{7} \frac{1}{n!m!} H_{nm00} z^n \bar{z}^m + \sum_{n+m=0}^{5} H_{nm01} \frac{1}{n!m!} z^n \bar{z}^m \beta_2 \\
&+ \sum_{n+m=0}^{3} \frac{1}{n!m!} H_{nm10} z^n \bar{z}^m \beta_1 + \sum_{n+m=0}^{3} \frac{1}{2n!m!} H_{nm02} z^n \bar{z}^m \beta_2^2 \\
&+ \sum_{n+m=0}^{1} \frac{1}{n!m!} H_{nm11} z^n \bar{z}^m \beta_1\beta_2 + \sum_{n+m=0}^{1} \frac{1}{6n!m!} H_{nm03} z^n \bar{z}^m \beta_2^3 \\
&+ \frac{1}{6} H_{1103} z\bar{z}\beta_2^3 + \frac{1}{12} H_{2003} z^2\beta_2^3 + \frac{1}{12} H_{2103} z^2\bar{z}\beta_2^3. \qquad (5.3)
\end{aligned}$$

In contrast to the ODE case, H is now a mapping into $X = C([-h,0],\mathbb{R}^n)$. Since X is real, we still have the property that $H_{ijkl} = \overline{H}_{jikl}$. Finally, the relation between the parameters K is the same as in the ODE case, given by (4.7). The equations collected from the homological equation (5.1) will essentially have the same form as those we collected in the ODE case. For the multilinear forms one only needs to change all q to φ and include an $r^{\odot\star}$ after all the multilinear forms. The remaining vectors H_{ijkl} and q outside the multilinear forms become jH_{ijkl} and $j\varphi$ respectively. However, the solutions to the equations will look quite different, as we are now solving linear operator equations acting on elements in $X^{\odot\star}$.

In the next two subsections we will derive all the critical normal coefficients and parameter-dependent coefficients that appear in the parameter approximation of the LPC curve. All the remaining coefficients needed for the limit cycle approximation on the center manifold are presented in Appendix B. An overview of all the coefficients is presented in Figure 4.1.

5.1.1 Critical normal form coefficients

For the computation of the critical normal form coefficients up to $c_2(0)$ we follow [15]. Collecting the $z^2, z\bar{z}$ and z^3 terms from the homological equation (5.1) results in the following linear systems

$$\begin{aligned}
(2i\omega_0 I - A^{\odot\star})jH_{2000} &= B(\varphi,\varphi)r^{\odot\star}, \\
-A^{\odot\star}jH_{1100} &= B(\varphi,\bar{\varphi})r^{\odot\star}, \\
(3i\omega_0 I - A^{\odot\star})jH_{3000} &= [3B(\varphi,H_{2000}) + C(\varphi,\varphi,\varphi)]r^{\odot\star}.
\end{aligned}$$

All three equations are regular with a right-hand side of the form $(w_0, w) = (w_0, 0)$. Thus, applying Corollary 1.1 results in the following solutions

$$H_{2000}(\theta) = e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)B(\varphi,\varphi), \tag{5.4}$$
$$H_{1100}(\theta) = \Delta^{-1}(0)B(\varphi,\bar\varphi), \tag{5.5}$$
$$H_{3000}(\theta) = e^{3i\omega_0\theta}\Delta^{-1}(3i\omega_0)[3B(\varphi,H_{2000}) + C(\varphi,\varphi,\varphi)]. \tag{5.6}$$

Collecting the $z^2\bar z$ terms will result in the following singular system

$$(i\omega_0 I - A^{\odot\star})jH_{2100} = [2B(\varphi,H_{1100}) + B(\bar\varphi,H_{2000}) + C(\varphi,\varphi,\bar\varphi)]r^{\odot\star} - 2c_1(0)j\varphi.$$

Applying the Fredhom solvability condition to the above system results in

$$c_1(0) = \frac{1}{2}\langle[2B(\varphi,H_{1100}) + B(\bar\varphi,H_{2000}) + C(\varphi,\varphi,\bar\varphi)]r^{\odot\star}, \varphi^{\odot}\rangle.$$

Here we used that $\langle j\varphi,\varphi^{\odot}\rangle = \langle\varphi^{\odot},\varphi\rangle = 1$ which follows from the pairings (2.15) and (2.18). We can evaluate the above pairing using equation (2.18) and the expression (2.29) for $\varphi^{\odot}$. This results in the following formula

$$c_1(0) = \frac{1}{2}p^T[2B(\varphi,H_{1100}) + B(\bar\varphi,H_{2000}) + C(\varphi,\varphi,\bar\varphi)].$$

Finally, to obtain the expression for $H_{2100}(\theta)$ we apply Corollary 4 to obtain the unique solution

$$H_{2100}(\theta) = B^{INV}_{i\omega_0}(2B(\varphi,H_{1100}) + B(\bar\varphi,H_{2000}) + C(\varphi,\varphi,\bar\varphi), -2c_1(0))(\theta)$$

satisfying $\langle\varphi^{\odot},H_{2100}\rangle = 0$. Similarly to the first three systems we find for H_{2200} the equation

$$\begin{aligned}H_{2200}(\theta) = \Delta^{-1}(0)[&2B(\varphi,\overline{H}_{2100}) + 2B(\bar\varphi,H_{2100}) + B(H_{2000},\overline{H}_{2000})\\ &+ 2B(H_{1100},H_{1100}) + C(\varphi,\varphi,\overline{H}_{2000}) + 4C(\varphi,\bar\varphi,H_{1100})\\ &+ C(\bar\varphi,\bar\varphi,H_{2000}) + D(\varphi,\varphi,\bar\varphi,\bar\varphi)].\end{aligned} \tag{5.7}$$

To solve for H_{3100}, a bit more caution is required. The $z^3\bar z$ terms yield the linear system

$$\begin{aligned}(2i\omega_0 I - A^{\odot\star})jH_{3100} = [&3B(\varphi,H_{2100}) + B(\bar\varphi,H_{3000}) + 3B(H_{1100},H_{2000}) + 3C(\varphi,\varphi,H_{1100})\\ &+ 3C(\varphi,\bar\varphi,H_{2000}) + D(\varphi,\varphi,\varphi,\bar\varphi)]r^{\odot\star} - 6c_1(0)jH_{2000}.\end{aligned}$$

Notice that the right hand side is of the form (w_0, w) with $w_0 = [\dots] - 6c_1(0)H_{2000}(0)$ and

$$w(\theta) = -6c_1(0)H_{2000}(\theta) = e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)(-6c_1(0)B(\varphi,\varphi)).$$

Thus, we can apply Corollary 2 with $\eta = -6c_1(0)B(\varphi,\varphi)$ and $\xi_1 = \xi_2 = 0$, which yields

$$\begin{aligned}H_{3100}(\theta) = e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)[&3B(\varphi,H_{2100}) + B(\bar\varphi,H_{3000}) + 3B(H_{1100},H_{2000})\\ &+ 3C(\varphi,\varphi,H_{1100}) + 3C(\varphi,\bar\varphi,H_{2000}) + D(\varphi,\varphi,\varphi,\bar\varphi)]\\ &- 6c_1(0)\Delta^{-1}(2i\omega_0)[\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta).\end{aligned} \tag{5.8}$$

Collecting the $z^3\bar z^2$ terms results in the equation

$$\begin{aligned}(i\omega_0 I - A^{\odot\star})jH_{3200} = [&3B(\varphi,H_{2200}) + 2B(\bar\varphi,H_{3100}) + B(\overline{H}_{2000},H_{3000}) + 6B(H_{1100},H_{2100})\\ &+ 3B(\overline{H}_{2100},H_{2000}) + 3C(\varphi,\varphi,\overline{H}_{2100}) + 6C(\varphi,\bar\varphi,H_{2100}) + 3C(\varphi,H_{2000},\overline{H}_{2000})\\ &+ 6C(\varphi,H_{1100},H_{1100}) + C(\bar\varphi,\bar\varphi,H_{3000}) + 6C(\bar\varphi,H_{1100},H_{2000}) + D(\varphi,\varphi,\varphi,\overline{H}_{2000})\\ &+ 6D(\varphi,\varphi,\bar\varphi,H_{1100}) + 3D(\varphi,\bar\varphi,\bar\varphi,H_{2000}) + E(\varphi,\varphi,\varphi,\bar\varphi,\bar\varphi)]r^{\odot\star}\\ &- (12c_2(0)j\varphi + 6i\Im(c_1(0))jH_{2100}).\end{aligned} \tag{5.9}$$

Applying the Fredholm solvability condition gives

$$\begin{aligned}c_2(0) = \frac{1}{12}p^T[&3B(\varphi, H_{2200}) + 2B(\bar{\varphi}, H_{3100}) + B(\overline{H}_{2000}, H_{3000}) + 6B(H_{1100}, H_{2100}) \\ &+ 3B(\overline{H}_{2100}, H_{2000}) + 3C(\varphi, \varphi, \overline{H}_{2100}) + 6C(\varphi, \bar{\varphi}, H_{2100}) + 3C(\varphi, H_{2000}, \overline{H}_{2000}) \\ &+ 6C(\varphi, H_{1100}, H_{1100}) + C(\bar{\varphi}, \bar{\varphi}, H_{3000}) + 6C(\bar{\varphi}, H_{1100}, H_{2000}) + D(\varphi, \varphi, \varphi, \overline{H}_{2000}) \\ &+ 6D(\varphi, \varphi, \bar{\varphi}, H_{1100}) + 3D(\varphi, \bar{\varphi}, \bar{\varphi}, H_{2000}) + E(\varphi, \varphi, \varphi, \bar{\varphi}, \bar{\varphi})].\end{aligned}$$

We will now proceed by deriving the coefficients that are needed for the computation of the seventh-order coefficient $c_3(0)$. To solve for H_{3200} let us write equation (5.9) as

$$(i\omega_0 I - A^{\odot\star})jH_{3200} = M_{3200}r^{\odot\star} - 12c_2(0)j\varphi - 6i\Im(c_1(0))jH_{2100},$$

where we write M_{3200} for the term that contains all the multilinear forms. We can now use the linearity of the bordered inverse to apply Corollaries 4 and 6 to find the solution

$$H_{3200}(\theta) = B_{i\omega_0}^{INV}(M_{3200}, -12c_2(0))(\theta) - 6i\Im(c_1(0))\tilde{B}_{i\omega_0}^{INV}(H_{2100}, -2c_1(0))(\theta),$$

satisfying $\langle\varphi^{\odot}, H_{3200}\rangle = 0$. Solving for the expressions of the coefficients H_{4000}, H_{4100} and H_{3300} is similar to how we solved for H_{3100} with the application of Corollary 2. This results in the following expressions

$$\begin{aligned}H_{4000}(\theta) &= e^{4i\omega_0\theta}\Delta^{-1}(4i\omega_0)[4B(\varphi, H_{3000}) + 3B(H_{2000}, H_{2000}) \\ &\quad + 6C(\varphi, \varphi, H_{2000}) + D(\varphi, \varphi, \varphi, \varphi)], \qquad (5.10)\\ H_{4100}(\theta) &= e^{3i\omega_0\theta}\Delta^{-1}(3i\omega_0)[4B(\varphi, H_{3100}) + B(\bar{\varphi}, H_{4000}) + 4B(H_{1100}, H_{3000}) \\ &\quad + 6B(H_{2000}, H_{2100}) + 6C(\varphi, \varphi, H_{2100}) + 4C(\varphi, \bar{\varphi}, H_{3000}) \\ &\quad + 12C(\varphi, H_{1100}, H_{2000}) + 3C(\bar{\varphi}, H_{2000}, H_{2000}) + 4D(\varphi, \varphi, \varphi, H_{1100}) \\ &\quad + 6D(\varphi, \varphi, \bar{\varphi}, H_{2000}) + E(\varphi, \varphi, \varphi, \varphi, \bar{\varphi})] \\ &\quad - 12c_1(0)\Delta^{-1}(3i\omega_0)[\Delta'(3i\omega_0) - \theta\Delta(3i\omega_0)]H_{3000}(\theta), \qquad (5.11)\end{aligned}$$

and

$$\begin{aligned}H_{3300}(\theta) = \Delta^{-1}(0)[&3B(\varphi, \overline{H}_{3200}) + 3B(\bar{\varphi}, H_{3200}) + 3B(\overline{H}_{2000}, H_{3100}) + B(\overline{H}_{3000}, H_{3000}) \\ &+ 9B(H_{1100}, H_{2200}) + 9B(H_{2100}, \overline{H}_{2100}) + 3B(\overline{H}_{3100}, H_{2000}) \\ &+ 3C(\varphi, \varphi, \overline{H}_{3100}) + 9C(\varphi, \bar{\varphi}, H_{2200}) + 9C(\varphi, \overline{H}_{2000}, H_{2100}) \\ &+ 3C(\varphi, \overline{H}_{3000}, H_{2000}) + 18C(\varphi, H_{1100}, \overline{H}_{2100}) + 3C(\bar{\varphi}, \bar{\varphi}, H_{3100}) \\ &+ 3C(\bar{\varphi}, \overline{H}_{2000}, H_{3000}) + 18C(\bar{\varphi}, H_{1100}, H_{2100}) + 9C(\bar{\varphi}, \overline{H}_{2100}, H_{2000}) \\ &+ 9C(\overline{H}_{2000}, H_{1100}, H_{2000}) + 6C(H_{1100}, H_{1100}, H_{1100}) + D(\varphi, \varphi, \varphi, \overline{H}_{3000}) \\ &+ 9D(\varphi, \varphi, \bar{\varphi}, \overline{H}_{2100}) + 9D(\varphi, \varphi, \overline{H}_{2000}, H_{1100}) + 9D(\varphi, \bar{\varphi}, \bar{\varphi}, H_{2100}) \\ &+ 9D(\varphi, \bar{\varphi}, \overline{H}_{2000}, H_{2000}) + 18D(\varphi, \bar{\varphi}, H_{1100}, H_{1100}) + D(\bar{\varphi}, \bar{\varphi}, \bar{\varphi}, H_{3000}) \\ &+ 9D(\bar{\varphi}, \bar{\varphi}, H_{1100}, H_{2000}) + 3E(\varphi, \varphi, \varphi, \bar{\varphi}, \overline{H}_{2000}) + 9E(\varphi, \varphi, \bar{\varphi}, \bar{\varphi}, H_{1100}) \\ &+ 3E(\varphi, \bar{\varphi}, \bar{\varphi}, \bar{\varphi}, H_{2000}) + K(\varphi, \varphi, \varphi, \bar{\varphi}, \bar{\varphi}, \bar{\varphi})] \\ &- 72d_2\Delta^{-1}(0)[\Delta'(0) - \theta\Delta(0)]H_{1100}(\theta). \qquad (5.12)\end{aligned}$$

For H_{4200} we need to be more careful. The equation that has to be solved is of the following form

$$(2i\omega_0 I - A^{\odot\star})jH_{4200} = M_{4200}r^{\odot\star} - 48c_2(0)jH_{2000} - 16c_1(0)jH_{3100}, \qquad (5.13)$$

where we use M_{4200} to denote the mulitilinear forms. Using the linearity of the inverse operator, we can apply Corollary 2 for the first two terms. Then to determine the inverse of the last term, we apply Corollary 3 with $M = M_{3100}$, $\hat{\eta} = -6c_1(0)H_{2000}(0)$ and $\hat{\xi} = 0$. This results in the following expression

$$\begin{aligned} H_{4200}(\theta) = e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)[&4B(\varphi, H_{3200}) + 2B(\bar{\varphi}, H_{4100}) \\ &+ B(\overline{H}_{2000}, H_{4000}) + 8B(H_{1100}, H_{3100}) + 4B(\overline{H}_{2100}, H_{3000}) \\ &+ 6B(H_{2000}, H_{2200}) + 6B(H_{2100}, H_{2100}) + 6C(\varphi, \varphi, H_{2200}) \\ &+ 8C(\varphi, \bar{\varphi}, H_{3100}) + 4C(\varphi, \overline{H}_{2000}, H_{3000}) + 24C(\varphi, H_{1100}, H_{2100}) \\ &+ 12C(\varphi, \overline{H}_{2100}, H_{2000}) + C(\bar{\varphi}, \bar{\varphi}, H_{4000}) + 8C(\bar{\varphi}, H_{1100}, H_{3000}) \\ &+ 12C(\bar{\varphi}, H_{2000}, H_{2100}) + 3C(\overline{H}_{2000}, H_{2000}, H_{2000}) + 12C(H_{1100}, H_{1100}, H_{2000}) \\ &+ 4D(\varphi, \varphi, \varphi, \overline{H}_{2100}) + 12D(\varphi, \varphi, \bar{\varphi}, H_{2100}) + 6D(\varphi, \varphi, \overline{H}_{2000}, H_{2000}) \\ &+ 12D(\varphi, \varphi, H_{1100}, H_{1100}) + 4D(\varphi, \bar{\varphi}, \bar{\varphi}, H_{3000}) + 24D(\varphi, \bar{\varphi}, H_{1100}, H_{2000}) \\ &+ 3D(\bar{\varphi}, \bar{\varphi}, H_{2000}, H_{2000}) + E(\varphi, \varphi, \varphi, \varphi, \overline{H}_{2000}) + 8E(\varphi, \varphi, \varphi, \bar{\varphi}, H_{1100}) \\ &+ 6E(\varphi, \varphi, \bar{\varphi}, \bar{\varphi}, H_{2000}) + K(\varphi, \varphi, \varphi, \varphi, \bar{\varphi}, \bar{\varphi})] \\ &- 48c_2(0)\Delta^{-1}(2i\omega_0)[\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta) \\ &- 16c_1(0)e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{3100}(0) \\ &+ 3c_1(0)[\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)]H_{2000}(0)\Big). \end{aligned} \tag{5.14}$$

The third Lyapunov coefficient is derived from the system

$$\begin{aligned} (i\omega_0 I - A^{\odot\star})jH_{4300} = M_{4300}r^{\odot\star} - (&144c_3(0)j\varphi + 72(2c_2(0) + \overline{c_2}(0))jH_{2100} \\ &+ 12i\Im\{c_1(0)\}jH_{3200}), \end{aligned} \tag{5.15}$$

where M_{4300} is given by the same expression as in (4.21) but with all vectors q changed to functions φ. Applying the Fredholm solvability condition to equation (5.15) will then result in

$$c_3(0) = \frac{1}{144}p^T M_{4300},$$

where we used that $\langle\varphi^{\odot}, H_{2100}\rangle = 0$ and $\langle\varphi^{\odot}, H_{3200}\rangle = 0$.

5.1.2 Parameter-related coefficients

Linear coefficients K_{10},K_{01} To find the equations from which we can solve for K_{10}, K_{01}, we follow the same steps as in the ODE case. Collecting the β_1 and β_2 terms in (5.1) yields for $\mu = (10), (01)$ the systems

$$-A^{\odot\star}jH_{00\mu} = J_1K_\mu r^{\odot\star}. \tag{5.16}$$

Write

$$K_\mu = \gamma_{1,\mu}e_1 + \gamma_{2,\mu}e_2, \tag{5.17}$$

for unknown $\gamma_{1,\mu}, \gamma_{2,\mu} \in \mathbb{R}$ and $e_1, e_2 \in \mathbb{R}^2$ the standard basis vectors. Substituting equation (5.17) into (5.16) and applying Corollary 1.1 results in

$$H_{00\mu}(\theta) = \gamma_{1,\mu}\Delta^{-1}(0)J_1e_1 + \gamma_{2,\mu}\Delta(0)^{-1}J_1e_2. \tag{5.18}$$

The $\beta_1 z$ and $\beta_2 z$ terms yield the systems

$$(i\omega_0 I - A^{\odot\star})jH_{10\mu} = [A_1(\varphi, K_\mu) + B(\varphi, H_{00\mu})]r^{\odot\star} - (\delta_\mu^{10} + ib_{1,\mu})j\varphi, \tag{5.19}$$

where δ_μ^{10} is defined as in (4.24). Define $\Gamma_i(\varphi) = A_1(\varphi, e_i) + B(\varphi, \Delta^{-1}(0)J_1 e_i)$. Substituting equations (5.17) and (5.18) into equation (5.19) then yields

$$(i\omega_0 I - A^{\odot\star})jH_{10\mu} = \gamma_{1,\mu}\Gamma_1(\varphi)r^{\odot\star} + \gamma_{2,\mu}\Gamma_2(\varphi)r^{\odot\star} - (\delta_\mu^{10} + ib_{1,\mu})j\varphi. \tag{5.20}$$

Applying the Fredholm solvability condition to equation (5.20) results in

$$\delta_\mu^{10} + ib_{1,\mu} = p^T[\gamma_{1,\mu}\Gamma_1(\varphi) + \gamma_{2,\mu}\Gamma_2(\varphi)]. \tag{5.21}$$

Taking the real and imaginary parts yield

$$\delta_\mu^{10} = \gamma_{1,\mu}\Re[p^T\Gamma_1(\varphi)] + \gamma_{2,\mu}\Re[p^T\Gamma_2(\varphi)], \tag{5.22}$$

and

$$b_{1,\mu} = \gamma_{1,\mu}\Im[p^T\Gamma_1(\varphi)] + \gamma_{2,\mu}\Im[p^T\Gamma_2(\varphi)]. \tag{5.23}$$

A solution $H_{10\mu}$ of equation (5.20) satisfying $\langle \varphi^\odot, H_{10\mu}\rangle = 0$ can be obtained with Corollary 4. Using the linearity of the bordered inverse we can write the solution as

$$H_{10\mu}(\theta) = \gamma_{1,\mu}B_{i\omega_0}^{INV}(\Gamma_1(\varphi), 0) + \gamma_{2,\mu}B_{i\omega_0}^{INV}(\Gamma_2(\varphi), 0) - (\delta_\mu^{10} + ib_{1,\mu})B_{i\omega_0}^{INV}(0, 1). \tag{5.24}$$

Collecting the $z^2\beta_i$ and $z\bar{z}\beta_i$ terms respectively yield the systems

$$\begin{aligned}
(2i\omega_0 I - A^{\odot\star})jH_{20\mu} &= [A_1(H_{2000}, K_\mu) + 2B(\varphi, H_{10\mu}) + B(H_{00\mu}, H_{2000}) \\
&\quad + B_1(\varphi, \varphi, K_\mu) + C(\varphi, \varphi, H_{00\mu})]r^{\odot\star} - 2(\delta_\mu^{10} + ib_{1,\mu})jH_{2000}, \\
-A^{\odot\star}jH_{11\mu} &= [A_1(H_{1100}, K_\mu) + 2\Re\{B(\bar{\varphi}, H_{10\mu})\} + B(H_{00\mu}, H_{1100}) \\
&\quad + B_1(\varphi, \bar{\varphi}, K_\mu) + C(\varphi, \bar{\varphi}, H_{00\mu})]r^{\odot\star} - 2\delta_\mu^{10}jH_{1100}.
\end{aligned}$$

Both these systems are regular and their solutions follow from Corollary 2. The resulting equations are

$$\begin{aligned}
H_{20\mu}(\theta) &= e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)[A_1(H_{2000}, K_\mu) + 2B(\varphi, H_{10\mu}) + B(H_{00\mu}, H_{2000}) \\
&\quad + B_1(\varphi, \varphi, K_\mu) + C(\varphi, \varphi, H_{00\mu})] \\
&\quad - 2(\delta_\mu^{10} + ib_{1,\mu})\Delta^{-1}(2i\omega_0)[\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta), &(5.25)\\
H_{11\mu}(\theta) &= \Delta^{-1}(0)[A_1(H_{1100}, K_\mu) + 2\Re(B(\bar{\varphi}, H_{10\mu})) + B(H_{00\mu}, H_{1100}) \\
&\quad + B_1(\varphi, \bar{\varphi}, K_\mu) + C(\varphi, \bar{\varphi}, H_{00\mu})] - 2\delta_\mu^{10}\Delta^{-1}(0)[\Delta'(0) - \theta\Delta(0)]H_{1100}(\theta), &(5.26)
\end{aligned}$$

Substituting equations (5.17), (5.18) and (5.24) into these expressions yields

$$\begin{aligned}
H_{20\mu}(\theta) &= \gamma_{1,\mu}e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Lambda_1(H_{2000}, \varphi, \varphi) + \gamma_{2,\mu}e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Lambda_2(H_{2000}, \varphi, \varphi) \\
&\quad - 2(\delta_\mu^{10} + ib_{1,\mu})\Delta^{-1}(2i\omega_0)\left([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta) + e^{2i\omega_0\theta}B\left(\varphi, B_{i\omega_0}^{INV}(0,1)\right)\right), \\
H_{11\mu}(\theta) &= \gamma_{1,\mu}\Delta^{-1}(0)\Pi_1(H_{1100}, \bar{\varphi}, \varphi) + \gamma_{2,\mu}\Delta^{-1}(0)\Pi_2(H_{1100}, \bar{\varphi}, \varphi) \\
&\quad - 2\delta_\mu^{10}\Delta^{-1}(0)\left([\Delta'(0) - \theta\Delta(0)]H_{1100}(\theta) + \Re\left\{B\left(\bar{\varphi}, B_{i\omega_0}^{INV}(0,1)\right)\right\}\right) \\
&\quad - 2b_{1,\mu}\Delta^{-1}(0)\Re\left\{iB\left(\bar{\varphi}, B_{i\omega_0}^{INV}(0,1)\right)\right\},
\end{aligned}$$

where we defined

$$\begin{aligned}
\Lambda_i(u, v, w) &= \Gamma_i(u) + 2B\left(v, B_{i\omega_0}^{INV}(\Gamma_i(w), 0)\right) + B_1(v, w, e_i) + C(v, w, \Delta^{-1}(0)J_1e_i), \\
\Pi_i(u, v, w) &= \Gamma_i(u) + 2\Re\{B\left(v, B_{i\omega_0}^{INV}(\Gamma_i(w), 0)\right)\} + B_1(v, w, e_i) + C(v, w, \Delta^{-1}(0)J_1e_i).
\end{aligned}$$

The $z^2\bar{z}\beta_i$ terms yield the systems

$$\begin{aligned}(i\omega_0 I - A^{\odot\star})jH_{21\mu} &= [A_1(H_{2100}, K_\mu) + 2B(\varphi, H_{11\mu}) + B(\bar{\varphi}, H_{20\mu}) + B(H_{00\mu}, H_{2100})\\ &+ B(\overline{H}_{10\mu}, H_{2000}) + 2B(H_{10\mu}, H_{1100}) + 2B_1(\varphi, H_{1100}, K_\mu) + B_1(\bar{\varphi}, H_{2000}, K_\mu)\\ &+ C(\varphi,\varphi,\overline{H}_{10\mu}) + 2C(\varphi,\bar{\varphi},H_{10\mu}) + 2C(\varphi, H_{00\mu}, H_{1100}) + C(\bar{\varphi}, H_{00\mu}, H_{2000})\\ &+ C_1(\varphi,\varphi,\bar{\varphi},K_\mu) + D(\varphi,\varphi,\bar{\varphi},H_{00\mu})]r^{\odot\star}\\ &- [2(\delta_\mu^{01} + ib_{2,\mu})j\varphi + (3\delta_\mu^{10} + ib_{1,\mu})jH_{2100} + 2c_1(0)jH_{10\mu}]. \end{aligned} \tag{5.27}$$

Applying the Fredholm Alternative to equation (5.27) results in the equation

$$\begin{aligned}\delta_\mu^{01} + ib_{2,\mu} &= \frac{1}{2}p^T[A_1(H_{2100}, K_\mu) + 2B(\varphi, H_{11\mu}) + B(\bar{\varphi}, H_{20\mu}) + B(H_{00\mu}, H_{2100})\\ &+ B(\overline{H}_{10\mu}, H_{2000}) + 2B(H_{10\mu}, H_{1100}) + 2B_1(\varphi, H_{1100}, K_\mu) + B_1(\bar{\varphi}, H_{2000}, K_\mu)\\ &+ C(\varphi,\varphi,\overline{H}_{10\mu}) + 2C(\varphi,\bar{\varphi}.H_{10\mu}) + 2C(\varphi, H_{00\mu}, H_{1100}) + C(\bar{\varphi}, H_{00\mu}, H_{2000})\\ &+ C_1(\varphi,\varphi,\bar{\varphi},K_\mu) + D(\varphi,\varphi,\bar{\varphi},H_{00\mu})],\end{aligned} \tag{5.28}$$

where we used that $\langle\varphi^\odot, H_{2100}\rangle = 0$ and $\langle\varphi^\odot, H_{10\mu}\rangle = 0$. If we substitute the expressions for $H_{00\mu}, H_{10\mu}, H_{20\mu}, H_{11\mu}$, K_μ and $b_{1,\mu}$ into equation (5.28), we can solve for $\gamma_{1,\mu}$ and $\gamma_{2,\mu}$.

After substitution and some rewriting we arrive at the following system

$$P\begin{pmatrix}\gamma_{1,\mu}\\ \gamma_{2,\mu}\end{pmatrix} = Q_\mu, \tag{5.29}$$

where $P \in \mathbb{R}^{2\times 2}$ has for $k = 1,2$ the compenents

$$P_{1k} = \Re[p^T\Gamma_k(\varphi)], \tag{5.30}$$

$$\begin{aligned}P_{2k} &= \frac{1}{2}\Re\Big\{p^T\Big[\Gamma_k(H_{2100}) + 2B\left(\varphi, \Delta^{-1}(0)\Pi_k(H_{1100},\bar{\varphi},\varphi)\right) + B\left(\bar{\varphi}, e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Lambda_k(H_{2000},\varphi,\varphi)\right)\\ &+ B\left(H_{2000}, \overline{B_{i\omega_0}^{INV}(\Gamma_k(\varphi),0)}\right) + 2B\left(H_{1100}, B_{i\omega_0}^{INV}(\Gamma_k(\varphi),0)\right) + 2B_1(\varphi, H_{1100}, e_k) + B_1(\bar{\varphi}, H_{2000}, e_k)\\ &+ C\left(\varphi,\varphi,\overline{B_{i\omega_0}^{INV}(\Gamma_k(\varphi),0)}\right) + 2C\left(\varphi,\bar{\varphi},B_{i\omega_0}^{INV}(\Gamma_k(\varphi),0)\right) + 2C\left(\varphi, H_{1100}, \Delta^{-1}(0)J_1e_k\right)\\ &+ C(\bar{\varphi}, H_{2000}, \Delta^{-1}(0)J_1e_k) + C_1(\varphi,\varphi,\bar{\varphi},e_k) + D(\varphi,\varphi,\bar{\varphi},\Delta^{-1}(0)J_1e_k)\\ &+ \Im[p^T\Gamma_k(\varphi)]\Big(-2iB\left(\bar{\varphi}, \Delta^{-1}(2i\omega_0)[\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta)\right)\\ &- 2iB\left(\bar{\varphi}, e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)B\left(\varphi, B_{i\omega_0}^{INV}(0,1)\right)\right) - 4B\left(\varphi, \Delta^{-1}(0)\Re[iB\left(\bar{\varphi}, B_{i\omega_0}^{INV}(0,1)\right)]\right)\\ &+iB\left(H_{2000}, \overline{B_{i\omega_0}^{INV}(0,1)}\right) - 2iB\left(H_{1100}, B_{i\omega_0}^{INV}(0,1)\right) + iC\left(\varphi,\varphi,\overline{B_{i\omega_0}^{INV}(0,1)}\right)\\ &- 2iC\left(\varphi,\bar{\varphi},B_{i\omega_0}^{INV}(0,1)\right)\Big)\Big]\Big\},\end{aligned} \tag{5.31}$$

and $Q_\mu \in \mathbb{R}^2$ with $\mu = (10), (01)$ is given by

$$Q_{1,\mu} = \delta_\mu^{10}, \tag{5.32}$$

$$\begin{aligned}Q_{2,\mu} &= \delta_\mu^{01} + \frac{1}{2}\delta_\mu^{10}\Re\Big\{p^T\Big[4B\left(\varphi, \Delta^{-1}(0)[\Delta'(0) - \theta\Delta(0)]H_{1100}(\theta)\right)\\ &+ 2B\left(\bar{\varphi}, \Delta^{-1}(2i\omega_0)[\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta)\right)\\ &+ 4B\left(\varphi, \Delta^{-1}(0)\Re\left\{B\left(\bar{\varphi}, B_{i\omega_0}^{INV}(0,1)\right)\right\}\right) + 2B\left(\bar{\varphi}, e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)B\left(\varphi, B_{i\omega_0}^{INV}(0,1)\right)\right)\\ &+ B\left(H_{2000}, \overline{B_{i\omega_0}^{INV}(0,1)}\right) + 2B\left(H_{1100}, B_{i\omega_0}^{INV}(0,1)\right) + C\left(\varphi,\varphi,\overline{B_{i\omega_0}^{INV}(0,1)}\right)\\ &+ 2C\left(\varphi,\bar{\varphi},B_{i\omega_0}^{INV}(0,1)\right)\Big]\Big\}.\end{aligned} \tag{5.33}$$

Compared to equations (4.37) and (4.39) for ODEs we see that there are quite some similarities. If we consider all the terms that do not contain $B_{i\omega_0}^{INV}(0,1)$, we see that the same multilinear forms are present. Where we had matrix inverses A^{-1} and $A_{2i\omega_0}^{-1}$ for the ODE system, these are now replaced by functions of θ containing the inverse of the characteristic matrix $\Delta^{-1}(\lambda)$. Then, we also have some extra multilinear forms containing the bordered inverse $B_{i\omega_0}^{INV}(0,1)$. Compared to the ODE equations, this replaces the term $A_{i\omega_0}^{INV}q$ which we showed to equal zero. Consequently, all the multilinear forms containing $A_{i\omega_0}^{INV}q$ vanish. Finally, notice that instead of $\bar{p}^T$ we have p^T everywhere. This is because we now work with the pairing (2.18), which is different from the complex inner product.

To summarise, we first compute the linear coefficients K_{10} and K_{01} by solving system (5.29). Then we can determine the coefficients $H_{00\mu}$ from equation (5.18),$b_{1,\mu}$ from equation (5.23) and $H_{10\mu}$ from equation (5.24). Once we have these, the coefficients $H_{20\mu}$ and $H_{11\mu}$ can respectively be calculated from equations (5.25) and (5.26). The coefficient $b_{2,01}$ is determined from the imaginary part of equation (5.28). Finally, we will also need H_{2101} and H_{2110} for what follows. The solution to equation (5.27) can be found by separately applying Corollaries 4 and 6. Denote the part containing all the multilinear forms in equation (5.27) by $M_{21\mu}$. This results in the equation

$$\begin{aligned}H_{21\mu}(\theta) = {}& B_{i\omega_0}^{INV}(M_{21\mu}, -2(\delta_\mu^{01} + ib_{2,\mu}))(\theta) - (3\delta_\mu^{10} + ib_{1,\mu})\tilde{B}_{i\omega_0}^{INV}(H_{2100}, -2c_1(0))(\theta)\\ &- 2c_1(0)\tilde{B}_{i\omega_0}^{INV}(H_{10\mu}, -(\delta_\mu^{10} + ib_{1,\mu}))(\theta), \quad \mu = (01), (10).\end{aligned}$$

The coefficient a_{3201} As in the ODE case, to determine the coefficient a_{3201} we first need the coefficients H_{3001}, H_{3101} and H_{2201}. These can be found by collecting the $z^3\beta_2$, $z^3\bar{z}\beta_2$ and $z^2\bar{z}^2\beta_2$ terms from the homological equation (5.1). For convenience, we will write M_{ijkl} for the term containing all the multilinear forms as these are the same as in the corresponding equations for ODEs after changing q to φ. The equations for H_{3001} and H_{2201} follow from Corollary 2. This results in the expressions

$$\begin{aligned}H_{3001}(\theta) &= e^{3i\omega_0\theta}\Delta^{-1}(3i\omega_0)M_{3001} - 3ib_{1,01}\Delta^{-1}(3i\omega_0)[\Delta'(3i\omega_0) - \theta\Delta(3i\omega_0)]H_{3000}(\theta),\\ H_{2201}(\theta) &= \Delta^{-1}(0)M_{2201} - 8\Delta^{-1}(0)[\Delta'(0) - \theta\Delta(0)]H_{1100}(\theta).\end{aligned}$$

For H_{3101}, we need to be more careful. The $z^3\bar{z}\beta_2$ terms yield the equation

$$(2i\omega_0 I - A^{\odot\star})jH_{3101} = M_{3101}r^{\odot\star} - 6(1 + ib_{2,01})jH_{2000} - 6c_1(0)jH_{2001} - 2ib_{1,01}jH_{3100}.$$

To solve for H_{3101}, we can separately apply Corollary 2 for the part $M_{3101}r^{\odot\star} - 6(1 + ib_{2,01})jH_{2000}$ and Corollary 3 for the last two terms. This yields the following solution

$$\begin{aligned}H_{3101}(\theta) = {}& e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)M_{3101} - 6(1 + ib_{2,01})\Delta^{-1}(2i\omega_0)[\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta)\\ &- 6c_1(0)e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2001}(0)\\ &+ ib_{1,01}[\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)]H_{2000}(0)\Big)\\ &- 2ib_{1,01}e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{3100}(0)\\ &+ 3c_1(0)[\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)]H_{2000}(0)\Big).\end{aligned} \tag{5.34}$$

Finally, collecting the $z^3\bar{z}^2\beta_2$ terms results in the following equation

$$\begin{aligned}(i\omega_0 I - A^{\odot\star})jH_{3201} = M_{3201}r^{\odot\star} - [&12g_{3201}j\varphi + 12c_2(0)jH_{1001} + (18 + 6ib_{2,01})jH_{2100}\\ &+ 6i\Im\{c_1(0)\}jH_{2101} + ib_{1,01}jH_{3200}].\end{aligned} \tag{5.35}$$

The coefficient a_{3201} can now be found by applying the Fredholm solvability condition to equation (5.35) and then taking the real part. Thus

$$a_{3201} = \frac{1}{12}\Re\left\{p^T M_{3201}\right\}. \tag{5.36}$$

Quadratic coefficients $K_{02}, K_{11}, b_{1,02}$ In the ODE case, we put all of the terms that contained known coefficients into a rest term r_{ijkl}. Since the multilinear forms in r_{ijkl} will be the same for DDEs but with q changed to φ, we will indicate those with M_{ijkl}. Collecting the β_2^2 and $\beta_1\beta_2$ terms from (5.1) yields for $\mu = (02), (11)$ the equations

$$-A^{\odot\star} jH_{00\mu} = [J_1 K_\mu + M_{00\mu}]r^{\odot\star}. \tag{5.37}$$

Let $e_1, e_2 \in \mathbb{R}^2$ be the standard basis vectors and write

$$K_\mu = \gamma_{1,\mu}e_1 + \gamma_{2,\mu}e_2, \tag{5.38}$$

where $\gamma_{1,\mu}, \gamma_{1,\mu} \in \mathbb{R}$ are unknown constants that need to be determined. From equation (5.37) it follows after applying Corollary 1.1 that

$$H_{00\mu}(\theta) = \gamma_{1,\mu}\Delta^{-1}(0)J_1 e_1 + \gamma_{2,\mu}\Delta^{-1}(0)J_1 e_2 + \Delta^{-1}(0)M_{00\mu}. \tag{5.39}$$

The $z\beta_2^2$ and $z\beta_1\beta_2$ terms yield for $\mu = (02), (11)$ the equations

$$(i\omega_0 I - A^{\odot\star})jH_{10\mu} = [A_1(\varphi, K_\mu) + B(\varphi, H_{00\mu})]r^{\odot\star} - ib_{1,\mu}j\varphi + r_{10\mu}, \tag{5.40}$$

where

$$\begin{aligned} r_{1002} &= M_{1002}r^{\odot\star} - 2ib_{1,01}jH_{1001}, \\ r_{1011} &= M_{1011}r^{\odot\star} - (1 + ib_{1,10})jH_{1001} - ib_{1,01}jH_{1010}. \end{aligned}$$

Applying the Fredholm alternative to equations (5.40) yields the equations

$$ib_{1,\mu} = p^T[A_1(\varphi, K_\mu) + B(\varphi, H_{00\mu}) + M_{10\mu}]. \tag{5.41}$$

Here we used that $\langle\varphi^\odot, H_{1010}\rangle = \langle\varphi^\odot, H_{1001}\rangle = 0$. Substituting equations (5.38) and (5.39) into equation (5.41) results in

$$ib_{1,\mu} = p^T[\gamma_{1,\mu}\Gamma_1(\varphi) + \gamma_{2,\mu}\Gamma_2(\varphi) + \tilde{M}_{10\mu}], \tag{5.42}$$

where

$$\tilde{M}_{10\mu} = B(\varphi, \Delta^{-1}(0)M_{00\mu}) + M_{10\mu}.$$

From equation (5.44), it follows that

$$\gamma_{1,\mu}\Re[p^T\Gamma_1(\varphi)] + \gamma_{2,\mu}\Re[p^T\Gamma_2(\varphi)] = -\Re\{p^T\tilde{M}_{10\mu}\}, \tag{5.43}$$

and

$$b_{1,\mu} = \gamma_{1,\mu}\Im[p^T\Gamma_1(\varphi)] + \gamma_{2,\mu}\Im[p^T\Gamma_2(\varphi)] + \Im\{p^T\tilde{M}_{10\mu}\}. \tag{5.44}$$

Furthermore, applying corollaries 4 and 6 to equations (5.40) yields for $\mu = (02), (11)$ the solutions

$$\begin{aligned} H_{10\mu}(\theta) &= \gamma_{1,\mu}B_{i\omega_0}^{INV}(\Gamma_1(\varphi), 0)(\theta) + \gamma_{2,\mu}B_{i\omega_0}^{INV}(\Gamma_2(\varphi), 0)(\theta) \\ &\quad - ib_{1,\mu}B_{i\omega_0}^{INV}(0, 1)(\theta) + B_{10\mu}(\theta), \end{aligned} \tag{5.45}$$

where

$$B_{1002}(\theta) = B^{INV}_{i\omega_0}(\tilde{M}_{1002}, 0)(\theta) - 2ib_{1,01}\tilde{B}^{INV}_{i\omega_0}(H_{1001}, -ib_{1,01})(\theta),$$
$$B_{1011}(\theta) = B^{INV}_{i\omega_0}(\tilde{M}_{1011}, 0)(\theta) - (1 + ib_{1,10})\tilde{B}^{INV}_{i\omega_0}(H_{1001}, -ib_{1,01})(\theta) - ib_{1,01}\tilde{B}^{INV}_{i\omega_0}(H_{1010}, -(1 + ib_{1,10}))(\theta).$$

Here Corollary 6 was used for the terms with jH_{1001}, jH_{1010}. Now collect the $z^2\beta^\mu$ terms for $\mu = (02), (11)$. These terms yield the equations

$$(2i\omega_0 I - A^{\odot\star})jH_{20\mu} = [A_1(H_{2000}, K_\mu) + 2B(\varphi, H_{10\mu}) + B(H_{00\mu}, H_{2000}) + B_1(\varphi, \varphi, K_\mu) + C(\varphi, \varphi, H_{00\mu})]r^{\odot\star} - 2ib_{1,\mu}jH_{2000} + r_{20\mu}, \quad (5.46)$$

where

$$r_{2002} = M_{2002}r^{\odot\star} - 4ib_{1,01}jH_{2001},$$
$$r_{2011} = M_{2011}r^{\odot\star} - 2(1 + ib_{1,10})jH_{2001} - 2ib_{1,01}jH_{2010}.$$

Using Corollaries 1 and 3, we find that the solutions to equations (5.46) are

$$\begin{aligned} H_{2002}(\theta) = {} & e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)[A_1(H_{2000}, K_{02}) + 2B(\varphi, H_{1002}) + B(H_{0002}, H_{2000}) \\ & + B_1(\varphi, \varphi, K_{02}) + C(\varphi, \varphi, H_{0002}) + M_{2002}] \\ & - 2ib_{1,02}\Delta^{-1}(2i\omega_0)[\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta) \\ & - 4ib_{1,01}e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2001}(0) \\ & + ib_{1,01}[\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)]H_{2000}(0)\Big), \end{aligned} \quad (5.47)$$

$$\begin{aligned} H_{2011}(\theta) = {} & e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)[A_1(H_{2000}, K_{11}) + 2B(\varphi, H_{1011}) + B(H_{0011}, H_{2000}) \\ & + B_1(\varphi, \varphi, K_{11}) + C(\varphi, \varphi, H_{0011}) + M_{2011}] \\ & - 2ib_{1,11}\Delta^{-1}(2i\omega_0)[\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta) \\ & - 2(1 + ib_{1,10})e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2001}(0) \\ & + ib_{1,01}[\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)]H_{2000}(0)\Big) \\ & - 2ib_{1,01}e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2010}(0) \\ & + (1 + ib_{1,10})[\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)]H_{2000}(0)\Big). \end{aligned} \quad (5.48)$$

Substituting equations (5.38), (5.39) and (5.45) into the above expressions result in

$$\begin{aligned} H_{20\mu}(\theta) = {} & \gamma_{1,\mu}e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Lambda_1(H_{2000}, \varphi, \varphi) + \gamma_{2,\mu}e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Lambda_2(H_{2000}, \varphi, \varphi) \\ & - 2ib_{1,\mu}\Delta^{-1}(2i\omega_0)\left([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta) + e^{2i\omega_0\theta}B\left(\varphi, B^{INV}_{i\omega_0}(0, 1)\right)\right) \\ & + e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\tilde{r}_{20\mu}(\theta), \end{aligned} \quad (5.49)$$

where we have for $\mu = (02), (11)$:

$$\begin{aligned}
\tilde{r}_{2002}(\theta) &= M_{2002} + 2B(\varphi, B_{1002}) + B(H_{2000}, \Delta^{-1}(0)M_{0002}) + C(\varphi, \varphi, \Delta^{-1}(0)M_{0002}) \\
&\quad - 4ib_{1,01}\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2001}(0) \\
&\quad + ib_{1,01}[\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)]H_{2000}(0)\Big), \\
\tilde{r}_{2011}(\theta) &= M_{2011} + 2B(\varphi, B_{1011}) + B(H_{2000}, \Delta^{-1}(0)M_{0011}) + C(\varphi, \bar{\varphi}, \Delta^{-1}(0)M_{0011}) \\
&\quad - 2(1 + ib_{1,10})\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2001}(0) \\
&\quad + ib_{1,01}[\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)]H_{2000}(0)\Big) \\
&\quad - 2ib_{1,01}\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2010}(0) \\
&\quad + (1 + ib_{1,10})[\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)]H_{2000}(0)\Big).
\end{aligned}$$

Similarly, collecting the $z\bar{z}\beta^\mu$ terms for $\mu = (02), (11)$ yield the equations

$$\begin{aligned}
-A^{\odot\star}jH_{11\mu} &= [A_1(H_{1100}, K_\mu) + 2\Re\left(B(\bar{\varphi}, H_{10\mu})\right) + B(H_{00\mu}, H_{1100}) \\
&\quad + B_1(\varphi, \bar{\varphi}, K_\mu) + C(\varphi, \bar{\varphi}, H_{00\mu})]r^{\odot\star} + r_{11\mu},
\end{aligned} \tag{5.50}$$

where

$$r_{1102} = M_{1102}r^{\odot\star} \quad \text{and} \quad r_{1111} = M_{1111}r^{\odot\star} - 2jH_{1101}.$$

Using Corollary 2, we find that the solutions to equations (5.50) are

$$\begin{aligned}
H_{1102}(\theta) &= \Delta^{-1}(0)[A_1(H_{1100}, K_{02}) + 2\Re\left(B(\bar{\varphi}, H_{1002})\right) + B(H_{0002}, H_{1100}) \\
&\quad + B_1(\varphi, \bar{\varphi}, K_{02}) + C(\varphi, \bar{\varphi}, H_{0002}) + M_{1102}],
\end{aligned} \tag{5.51}$$

$$\begin{aligned}
H_{1111}(\theta) &= \Delta^{-1}(0)[A_1(H_{1100}, K_{11}) + 2\Re\left(B(\bar{\varphi}, H_{1011})\right) + B(H_{0011}, H_{1100}) \\
&\quad + B_1(\varphi, \bar{\varphi}, K_{11}) + C(\varphi, \bar{\varphi}, H_{0011}) + M_{1111}] \\
&\quad - 2\Delta^{-1}(0)[\Delta'(0) - \theta\Delta(0)]H_{1101}(\theta).
\end{aligned} \tag{5.52}$$

Substituting equations (5.38), (5.39) and (5.45) into the above expressions result in

$$\begin{aligned}
H_{11\mu}(\theta) &= \gamma_{1,\mu}\Delta^{-1}(0)\Pi_1(H_{1100}, \bar{\varphi}, \varphi) + \gamma_{2,\mu}\Delta^{-1}(0)\Pi_1(H_{1100}, \bar{\varphi}, \varphi) \\
&\quad - 2b_{1,\mu}\Delta^{-1}(0)\Re\left\{iB\big((\bar{\varphi}, B^{INV}_{i\omega_0}(0,1)\big)\right\} + \Delta^{-1}(0)\tilde{r}_{11\mu}(\theta),
\end{aligned} \tag{5.53}$$

where $\tilde{r}_{11\mu}$ is given for $\mu = (02), (11)$ by

$$\begin{aligned}
\tilde{r}_{1102}(\theta) &= M_{1102} + 2\Re\{B(\bar{\varphi}, B_{1002})\} + B(H_{1100}, \Delta^{-1}(0)M_{0002}) + C(\varphi, \bar{\varphi}, \Delta^{-1}(0)M_{0002}), \\
\tilde{r}_{1111}(\theta) &= M_{1111} + 2\Re\{B(\bar{\varphi}, B_{1011})\} + B(H_{1100}, \Delta^{-1}(0)M_{0011}) + C(\varphi, \bar{\varphi}, \Delta^{-1}(0)M_{0011}) \\
&\quad - 2[\Delta'(0) - \theta\Delta(0)]H_{1101}(\theta).
\end{aligned}$$

The $z^2\bar{z}\beta^\mu$ terms yield for $\mu = (02), (11)$ the equations

$$\begin{aligned}
(i\omega_0 I - A^{\odot\star})jH_{21\mu} &= [A_1(H_{2100}, K_\mu) + 2B(\varphi, H_{11\mu}) + B(\bar{\varphi}, H_{20\mu}) + B(H_{00\mu}, H_{2100}) \\
&\quad + B(\overline{H}_{10\mu}, H_{2000}) + 2B(H_{10\mu}, H_{1100}) + 2B_1(\varphi, H_{1100}, K_\mu) \\
&\quad + B_1(\bar{\varphi}, H_{2000}, K_\mu) + C(\varphi, \varphi, \overline{H}_{10\mu}) + 2C(\varphi, \bar{\varphi}, H_{10\mu}) + 2C(\varphi, H_{00\mu}, H_{1100}) \\
&\quad + C(\bar{\varphi}, H_{00\mu}, H_{2000}) + C_1(\varphi, \varphi, \bar{\varphi}, K_\mu) + D(\varphi, \varphi, \bar{\varphi}, H_{00\mu}) + r_{21\mu}]r^{\odot\star} \\
&\quad - (2ib_{2,\mu}j\varphi + ib_{1,\mu}jH_{2100} + 2c_1(0)jH_{10\mu}),
\end{aligned} \tag{5.54}$$

where

$$r_{2102} = M_{2102}r^{\odot\star} - [4(1+ib_{2,01})jH_{1001} + 2ib_{1,01}jH_{2101}],$$
$$r_{2111} = M_{2111}r^{\odot\star} - [2ib_{2,10}jH_{1001} + 2(1+ib_{2,01})jH_{1010} + (3+ib_{1,10})\,jH_{2101} + ib_{1,01}jH_{2110}].$$

Applying the Fredholm solvabitlity condition to the equation (5.54) results for $\mu = (02),(11)$ in the equations

$$\begin{aligned} b_{2,\mu}i = \frac{1}{2}p^T[&A_1(H_{2100},K_\mu) + 2B(\varphi,H_{11\mu}) + B(\bar{\varphi},H_{20\mu}) + B(H_{00\mu},H_{2100}) \\ &+ B(\overline{H}_{10\mu},H_{2000}) + 2B(H_{10\mu},H_{1100}) + 2B_1(\varphi,H_{1100},K_\mu) \\ &+ B_1(\bar{\varphi},H_{2000},K_\mu) + C(\varphi,\varphi,\overline{H}_{10\mu}) + 2C(\varphi,\bar{\varphi},H_{10\mu}) + 2C(\varphi,H_{00\mu},H_{1100}) \\ &+ C(\bar{\varphi},H_{00\mu},H_{2000}) + C_1(\varphi,\varphi,\bar{\varphi},K_\mu) + D(\varphi,\varphi,\bar{\varphi},H_{00\mu}) + M_{21\mu}], \end{aligned} \tag{5.55}$$

Finally, substitute equations (5.38), (5.39), (5.45), (5.49) and (5.53) into the above expression and solve for the coefficients $\gamma_{1,\mu}$,$\gamma_{2,\mu}$. We arrive at the following system

$$P\begin{pmatrix}\gamma_{1,\mu}\\ \gamma_{2,\mu}\end{pmatrix} = Q_\mu, \tag{5.56}$$

where the components of $P \in \mathbb{R}^{2\times 2}$ are given by equations (5.30) and (5.31) for $\mu = (02),(11)$. Meanwhile, the components of $Q_\mu \in \mathbb{R}^2$ are

$$\begin{aligned} Q_{1,\mu} &= -\Re\{p^T\tilde{M}_{10\mu}\}, \\ Q_{2,\mu} &= -\frac{1}{2}\Re\Big\{p^T\Big[2B\left(\varphi,\Delta^{-1}(0)\tilde{r}_{11\mu}\right) + B\left(\bar{\varphi},e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\tilde{r}_{20\mu}\right) \\ &\quad + B\left(H_{2100},\Delta^{-1}(0)M_{00\mu}\right) + B\left(H_{2000},\overline{B_{10\mu}}\right) + 2B\left(H_{1100},B_{10\mu}\right) \\ &\quad + C\left(\varphi,\varphi,\overline{B_{10\mu}}\right) + 2C\left(\varphi,\bar{\varphi},B_{10\mu}\right) + 2C\left(\varphi,H_{1100},\Delta^{-1}(0)M_{00\mu}\right) \\ &\quad + C\left(\bar{\varphi},H_{2000},\Delta^{-1}(0)M_{00\mu}\right) + D\left(\varphi,\varphi,\bar{\varphi},\Delta^{-1}(0)M_{00\mu}\right) + M_{21\mu} \\ &\quad + \Im\{p^T\tilde{M}_{10\mu}\}\Big(-4B\left(\varphi,\Delta^{-1}(0)\Re[iB\left(\bar{\varphi},B^{INV}_{i\omega_0}(0,1)\right)]\right) \\ &\quad - 2iB\left(\bar{\varphi},\Delta^{-1}(2i\omega_0)\left([\Delta'(2i\omega_0)-\theta\Delta(2i\omega_0)]H_{2000}(\theta) + e^{2i\omega_0\theta}B\left(\varphi,B^{INV}_{i\omega_0}(0,1)\right)\right)\right) \\ &\quad + iB\left(H_{2000},\overline{B^{INV}_{i\omega_0}(0,1)}\right) - 2iB\left(H_{1100},B^{INV}_{i\omega_0}(0,1)\right) + iC\left(\varphi,\varphi,\overline{B^{INV}_{i\omega_0}(0,1)}\right) \\ &\quad - 2iC\left(\varphi,\bar{\varphi},B^{INV}_{i\omega_0}(0,1)\right)\Big)\Big]\Big\}. \end{aligned}$$

Thus, to obtain the quadratic coefficients K_{02} and K_{11} we solve system (5.56). Once these coefficients are known, one can calculate the coefficients $H_{00\mu}$,$b_{1,\mu}$ and $H_{10\mu}$ from respectively the equations (5.39),(5.44) and (5.45). Then, the coefficients H_{2002} and H_{1102} are calculated from respectively the equations (5.47) and (5.51). Finally, the coefficient H_{2102}, needs to be determined from system (5.54). This can be achieved by a piecewise application of Corollaries 4, 6 and 7.

$$\begin{aligned} H_{2102}(\theta) = B^{INV}_{i\omega_0}(&A_1(H_{2100}, K_\mu) + 2B(\varphi, H_{11\mu}) + B(\bar{\varphi}, H_{20\mu}) + B(H_{00\mu}, H_{2100}) \\ &+ B(\overline{H}_{10\mu}, H_{2000}) + 2B(H_{10\mu}, H_{1100}) + 2B_1(\varphi, H_{1100}, K_\mu) \\ &+ B_1(\bar{\varphi}, H_{2000}, K_\mu) + C(\varphi, \varphi, \overline{H}_{10\mu}) + 2C(\varphi, \bar{\varphi}, H_{10\mu}) + 2C(\varphi, H_{00\mu}, H_{1100}) \\ &+ C(\bar{\varphi}, H_{00\mu}, H_{2000}) + C_1(\varphi, \varphi, \bar{\varphi}, K_\mu) + D(\varphi, \varphi, \bar{\varphi}, H_{00\mu}) + M_{2102}, -2ib_{2,02})(\theta) \\ &- ib_{1,02}\tilde{B}^{INV}_{i\omega_0}(H_{2100}, -2c_1(0))(\theta) - 4(1 + ib_{2,01})\tilde{B}^{INV}_{i\omega_0}(H_{1001}, -ib_{1,01})(\theta) \\ &- 2c_1(0)\hat{B}^{INV}_{i\omega_0}(H_{1002}, -[ib_{1,02}q + 2ib_{1,01}H_{1001}(0)], -2b^2_{1,01}) \\ &- 2ib_{1,01}\hat{B}^{INV}_{i\omega_0}(H_{2101}, -[2(1 + ib_{2,01})q + ib_{1,01}H_{2100}(0) + 2c_1(0)H_{1001}(0)], 4ic_1(0)b_{1,01})(\theta). \end{aligned}$$

The coefficient K_{03} Collecting the β_2^3 terms yields the equation

$$-A^{\odot\star}jH_{0003} = [J_1K_{03} + M_{0003}]r^{\odot\star}. \tag{5.57}$$

With $e_1, e_2 \in \mathbb{R}^2$ the standard basis vectors and $\gamma_{1,03}, \gamma_{2,03} \in \mathbb{R}$ we write

$$K_{03} = \gamma_{1,03}e_1 + \gamma_{2,03}e_2. \tag{5.58}$$

The solution to equation (5.57) follows from Corollary 1.1 and is given by

$$H_{0003}(\theta) = \gamma_{1,03}\Delta^{-1}(0)J_1e_1 + \gamma_{2,03}\Delta^{-1}(0)J_1e_2 + \Delta^{-1}(0)M_{0003}. \tag{5.59}$$

The $z\beta_2^3$ terms yield the equation

$$(i\omega_0 I - A^{\odot\star})jH_{1003} = [A_1(\varphi, K_{03}) + B(\varphi, H_{0003})]r^{\odot\star} - ib_{1,03}j\varphi + r_{1003}, \tag{5.60}$$

where

$$r_{1003} = M_{1003}r^{\odot\star} - 3ib_{1,02}jH_{1001} - 3ib_{1,01}jH_{1002}.$$

Applying the Fredholm solvability condition to equation (5.60), after substituting equations (5.58) and (5.59), yields the following expression:

$$ib_{1,03} = \gamma_{1,03}p^T\Gamma_1(\varphi) + \gamma_{2,03}p^T\Gamma_2(\varphi) + p^T\tilde{M}_{1003}, \tag{5.61}$$

where

$$\tilde{M}_{1003} = B(\varphi, \Delta^{-1}(0)M_{0003}) + M_{1003}.$$

To solve for H_{1003} from equation (5.60), we apply Corollaries 4, 6 and 7. The result is:

$$\begin{aligned} H_{1003}(\theta) = \gamma_{1,03}&B^{INV}_{i\omega_0}(\Gamma_1(\varphi), 0)(\theta) + \gamma_{2,03}B^{INV}_{i\omega_0}(\Gamma_2(\varphi), 0)(\theta) \\ &- ib_{1,03}B^{INV}_{i\omega_0}(0, 1)(\theta) + B_{1003}(\theta), \end{aligned} \tag{5.62}$$

where

$$\begin{aligned} B_{1003}(\theta) = B^{INV}_{i\omega_0}&(\tilde{M}_{1003}, 0)(\theta) - 3ib_{1,02}\tilde{B}^{INV}_{i\omega_0}(H_{1001}, -ib_{1,01})(\theta) \\ &- 3ib_{1,01}\hat{B}^{INV}_{i\omega_0}(H_{1002}, -[ib_{1,02}q + 2ib_{1,01}H_{1001}(0)], -2b^2_{1,01})(\theta). \end{aligned}$$

Collecting the $z^2\beta_2^3$ and $z\bar{z}\beta_2^3$ terms yields the equations:

$$\begin{aligned}(2i\omega_0 I - A^{\odot\star})jH_{2003} &= [A_1(H_{2000}, K_{03}) + 2B(\varphi, H_{1003}) + B(H_{0003}, H_{2000}) \\ &\quad + B_1(\varphi,\varphi,K_{03}) + C(\varphi,\varphi,H_{0003})]r^{\odot\star} - 2ib_{1,03}jH_{2000} + r_{2003}, \end{aligned} \tag{5.63}$$

$$\begin{aligned}-A^{\odot\star}jH_{1103} &= [A_1(H_{1100}, K_{03}) + 2\Re(B(\bar{\varphi}, H_{1003})) + B(H_{0003}, H_{1100}) \\ &\quad + B_1(\varphi,\bar{\varphi},K_{03}) + C(\varphi,\bar{\varphi},H_{0003})]r^{\odot\star} + r_{1103},\end{aligned} \tag{5.64}$$

where

$$r_{2003} = M_{2003}r^{\odot\star} - (6ib_{1,02}jH_{2001} + 6ib_{1,01}jH_{2002}) \quad \text{and} \quad r_{1103} = M_{1103}r^{\odot\star}.$$

To solve equation (5.63), we apply Corollary 1.1 for the terms containing the multilinear forms and Corollary 3 to the remaining terms. This yields the following solution:

$$\begin{aligned}H_{2003}(\theta) &= e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)[A_1(H_{2000}, K_{03}) + 2B(\varphi, H_{1003}) + B(H_{0003}, H_{2000}) \qquad (5.65)\\ &\quad + B_1(\varphi,\varphi,K_{03}) + C(\varphi,\varphi,H_{0003}) + M_{2003}] \\ &\quad - 2ib_{1,03}\Delta^{-1}(2i\omega_0)[\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta) \\ &\quad - 6ib_{1,02}e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2001}(0) \\ &\quad + ib_{1,01}[\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)]H_{2000}(0)\Big) \\ &\quad - 6ib_{1,01}e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2002}(0) \\ &\quad + [\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)](ib_{1,02}H_{2000}(0) + 2ib_{1,01}H_{2001}(0)) \\ &\quad - \frac{4}{3}b_{1,01}^2[\Delta'''(2i\omega_0) - \theta^3\Delta(2i\omega_0)]H_{2000}(0)\Big),\end{aligned} \tag{5.66}$$

Since the right-hand side of equation (5.64) is of the form $(w_0, 0)$, the solution for H_{1103} is obtained by simply applying Corollary 1.1. This results in the equation:

$$\begin{aligned}H_{1103}(\theta) &= \Delta^{-1}(0)[A_1(H_{1100}, K_{03}) + 2\Re(B(\bar{\varphi}, H_{1003})) + B(H_{0003}, H_{1100}) \\ &\quad + B_1(\varphi,\bar{\varphi},K_{03}) + C(\varphi,\bar{\varphi},H_{0003}) + M_{1103}].\end{aligned} \tag{5.67}$$

Substituting equations (5.58), (5.59), and (5.62) into equations (5.68) and (5.69) result in:

$$\begin{aligned}H_{2003}(\theta) &= \gamma_{1,03}e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Lambda_1(H_{2000},\varphi,\varphi) + \gamma_{2,03}e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\Lambda_2(H_{2000},\varphi,\varphi) \\ &\quad - 2ib_{1,03}\Delta^{-1}(2i\omega_0)\left([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2000}(\theta) + e^{2i\omega_0\theta}B\left(\varphi, B_{i\omega_0}^{INV}(0,1)\right)\right) \\ &\quad + e^{2i\omega_0\theta}\Delta^{-1}(2i\omega_0)\tilde{r}_{2003},\end{aligned} \tag{5.68}$$

$$\begin{aligned}H_{1103}(\theta) &= \gamma_{1,03}\Delta^{-1}(0)\Pi_1(H_{1100},\bar{\varphi},\varphi) + \gamma_{2,03}\Delta^{-1}(0)\Pi_1(H_{1100},\bar{\varphi},\varphi) \\ &\quad - 2b_{1,03}\Delta^{-1}(0)\Re\left\{iB((\bar{\varphi}, B_{i\omega_0}^{INV}(0,1))\right\} + \Delta^{-1}(0)\tilde{r}_{1103},\end{aligned} \tag{5.69}$$

where we have

$$\begin{aligned}\tilde{r}_{2003} &= M_{2003} + 2B(\varphi, B_{1003}) + B(H_{2000}, \Delta^{-1}(0)M_{0003}) + C(\varphi,\bar{\varphi},\Delta^{-1}(0)M_{0003}) \\ &\quad - 6ib_{1,02}\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2001}(0) \\ &\quad + ib_{1,01}[\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)]H_{2000}(0)\Big) \\ &\quad - 6ib_{1,01}\Big([\Delta'(2i\omega_0) - \theta\Delta(2i\omega_0)]H_{2002}(0) \\ &\quad + [\Delta''(2i\omega_0) - \theta^2\Delta(2i\omega_0)](ib_{1,02}H_{2000}(0) + 2ib_{1,01}H_{2001}(0)) \\ &\quad - \frac{4}{3}b_{1,01}^2[\Delta'''(2i\omega_0) - \theta^3\Delta(2i\omega_0)]H_{2000}(0)\Big), \\ \tilde{r}_{1103} &= M_{1103} + 2\Re\{B(\bar{\varphi}, B_{1003})\} + B(H_{1100}, \Delta^{-1}(0)M_{0003}) + C(\varphi,\bar{\varphi},\Delta^{-1}(0)M_{0003}).\end{aligned}$$

Applying the Fredholm solvability condition to the $z^2\bar{z}\beta_2^3$ terms yields the equation:

$$\begin{aligned}
ib_{2,03} = \frac{1}{2}p^T[&A_1(H_{2100}, K_{03}) + 2B(\varphi, H_{1103}) + B(\bar{\varphi}, H_{2003}) + B(H_{0003}, H_{2100}) \\
&+ B(\overline{H}_{1003}, H_{2000}) + 2B(H_{1003}, H_{1100}) + 2B_1(\varphi, H_{1100}, K_{03}) \\
&+ B_1(\bar{\varphi}, H_{2000}, K_{03}) + C(\varphi, \varphi, \overline{H}_{1003}) + 2C(\varphi, \bar{\varphi}, H_{1003}) + 2C(\varphi, H_{0003}, H_{1100}) \\
&+ C(\bar{\varphi}, H_{0003}, H_{2000}) + C_1(\varphi, \varphi, \bar{\varphi}, K_{03}) + D(\varphi, \varphi, \bar{\varphi}, H_{0003}) + M_{2103}]. \qquad (5.70)
\end{aligned}$$

Observe that equations (5.59), (5.62), (5.68), (5.69), and (5.70) correspond to equations (5.39),(5.45),(5.49), (5.53), and (5.55) respectively, for $\mu = (03)$. Consequently, we can solve for the coefficients $\gamma_{1,03}$ and $\gamma_{2,03}$ by utilizing system (5.56) for $\mu = (03)$.

5.2 The higher order LPC predictor for DDEs

The parameter approximations for the LPC curve for DDEs remain the same as for ODEs. Thus, we have the β parameter approximations

$$\beta_1 = d_2\varepsilon^4 + 2(d_3 - a_{3201}d_2)\varepsilon^6, \quad \beta_2 = -2d_2\varepsilon^2 + (4a_{3201}d_2 - 3d_3)\varepsilon^4,$$

and then

$$\alpha = \alpha_0 + K(\beta_1, \beta_2),$$

where the expression for K is given by (4.7) and the period is given by (3.9). Similarly, the periodic orbit is approximmated for $\psi \in [0, 2\pi]$ as

$$x = x_0 + H(\varepsilon e^{i\psi}, \varepsilon e^{-i\psi}, \beta_1, \beta_2). \qquad (5.71)$$

However, now this defines a function in $C([-h, 0], \mathbb{R}^n)$. Since, we know that the solution that we are looking for is periodic, we have that $x(\theta) = x(\theta + T)$ for all $\theta \in [-h, 0]$. Thus, it is enough to ensure that $x(0) = x(T)$ when solving the system for the localisation of the periodic cycle. Therefore, we take $H_{ijkl} = H_{ijkl}(0)$ for the coefficients in the approximation of H when we approximate the cycle.

Chapter 6

Examples

In this chapter, we will demonstrate the performance of the higher-order predictor and compare it to the first-order predictor. We will first present two examples of ODE systems and conclude with one example of a DDE system.

6.1 ODE Examples

We have implemented all of the equations for the higher-order predictor in the programming language Julia. For the numerical computation of the generalized Hopf points and the continuation of the LPC curves, we used the existing Julia package `Bifurcationkit.jl`[23]. We compare the higher-order predictor from Section 4.2 to the first-order predictor from [17].

6.1.1 Bazykin and Khibnik prey-predator model

As a first example, we consider a version of a prey-predator system by Bazykin and Khibnik [1]. The model consists of the following two equations

$$\begin{cases} \dot{x} &= \dfrac{x^2(1-x)}{n+x} - xy, \\ \dot{y} &= -y(m-x), \end{cases} \tag{6.1}$$

where $x, y \geq 0$ and $0 < m < 1$. One can show analytically that an Andronov-Hopf bifurcation occurs along the curve $n = m^2/(1-2m)$. The first Lyapunov coefficient along this curve is positive for $0 < m < \frac{1}{4}$ and negative for $\frac{1}{4} < m < \frac{1}{2}$, while it vanishes at $(m, n) = (\frac{1}{4}, \frac{1}{8})$. Thus, there is a generalized Hopf bifurcation in this system. The bifurcation diagram near the generalized Hopf bifurcation is shown in Figure 6.1.

At the generalized Hopf bifurcation we have the eigenvalues $\lambda_{1,2} = \pm i\omega_0$, with $\omega_0 = \sqrt{2}/4$. We take the following eigenvectors

$$q = \begin{pmatrix} \frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{6}i \end{pmatrix}, \quad p = \begin{pmatrix} \frac{1}{2}\sqrt{3} \\ -\frac{1}{4}\sqrt{6}i \end{pmatrix},$$

satisfying $\bar{q}^T q = \bar{p}^T q = 1$. Using these eigenvectors, the second Lyapunov coefficient has the exact value[1] $l_2 = -\frac{1024}{729}\sqrt{2}$. In our implementation in Julia, this value is numerically approximated by $l_2 = -1.986494770740791$. Figure 6.2 shows a close-up of the bifurcation diagram, including the first-order and the higher-order predictors in parameter space.

[1]This value was calculated in MATLAB

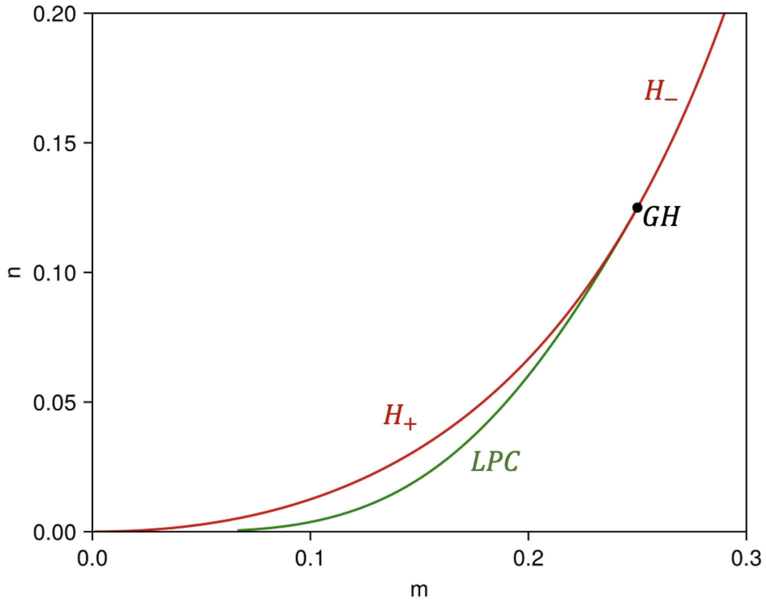

Figure 6.1: The bifurcation diagram near the generalized Hopf bifurcation (GH) in system (6.1). The curve of Hopf bifurcations ($n = m^2/(1-2m)$) is coloured red. The Hopf bifurcation is subcritical on the branch marked by H_+ and supercritical on the branch marked by H_-. The numerically continued LPC curve is plotted in green.

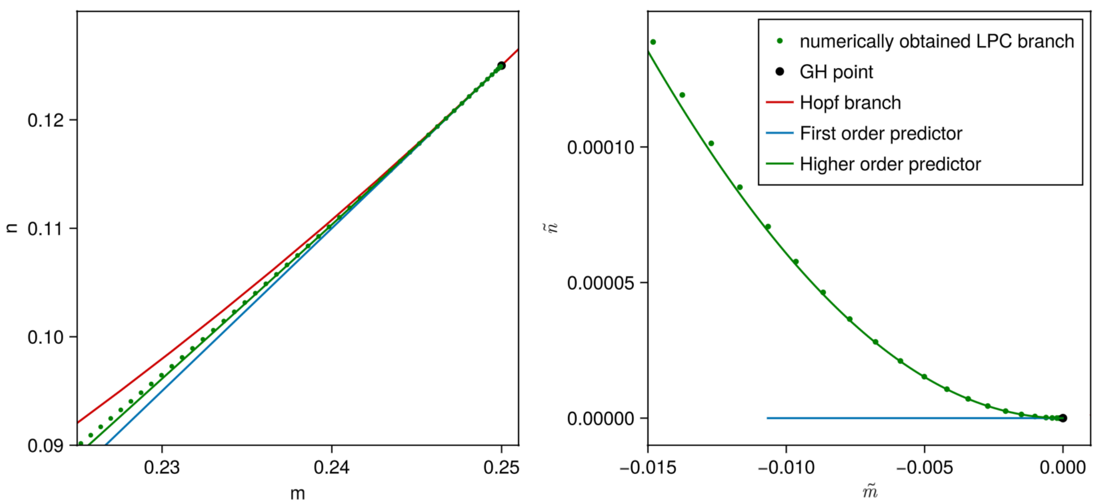

Figure 6.2: On the left, the bifurcation diagram near the generalized Hopf bifurcation in the system (6.1) is shown together with the first-order and the higher-order LPC predictors. The figure on the right shows the LPC curve and the predictors after a translation to the origin and a rotation.

6.1.2 The extended Lorenz-84 model

Our second example is an extended version of the Lorenz-84, which approximates the dynamics of an atmospheric flow model [22]. In this model, X represents the strength of the jet stream, while Y and Z model the sine and cosine coefficients of the baroclinic wave. In [18], the model was extended to include the variable U to study the effect of external parameters, such as temperature, on the jet stream and baroclinic waves. The same model was used as an example in [17]. The extended model consists of four equations:

$$\begin{cases} \dot{X} &= -Y^2 - Z^2 - \alpha X + \alpha F - \gamma U^2, \\ \dot{Y} &= XY - \beta XZ - Y + G, \\ \dot{Z} &= \beta XY + XZ - Z, \\ \dot{U} &= -\delta U + \gamma UX + T. \end{cases} \tag{6.2}$$

We take F and T as the bifurcation parameters and fix the parameters

$$\alpha = 0.25, \quad \beta = 1, \quad G = 0.25, \quad \delta = 1.04, \quad \gamma = 0.987.$$

Then, the system has a generalized Hopf bifurcation at $(F, T) \approx (2.3763, 0.05019)$. At the generalized Hopf bifurcation, we have the purely imaginary eigenvalues $\lambda = \pm i\omega_0$, with $\omega_0 = 0.690367$ and we take the eigenvectors such that $\bar{q}^T q = \bar{p}^T q = 1$. The second Lyapunov coefficient is computed as $l_2 = 0.22567$.

Figure 6.3 shows the first-order and the higher-order predictors in parameter space next to the numerically continued LPC curve.

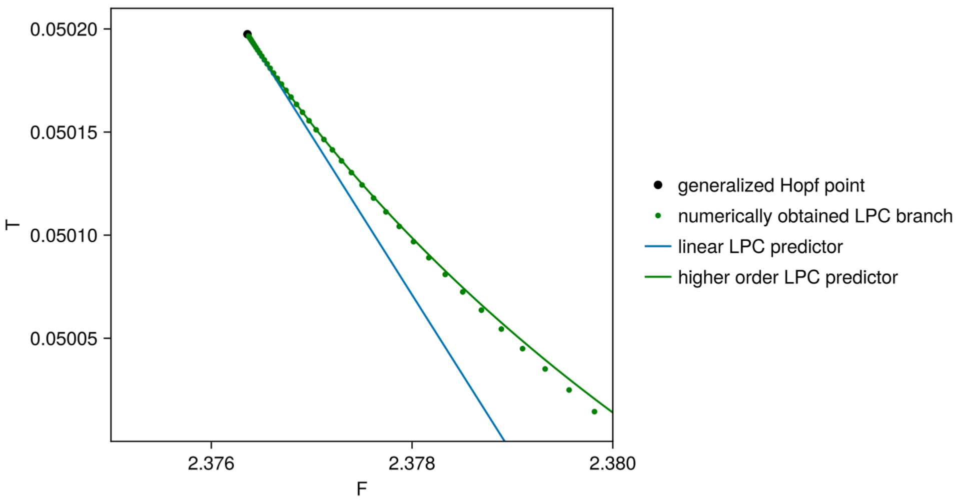

Figure 6.3: The numerically computed LPC curve emanating from the generalized Hopf point in the system (6.2) together with the first-order and higher-order predictors.

6.2 DDE example: Coupled FHN neural system with delay

The numerical computation of the generalized Hopf points and the continuation of the LPC curves has been implemented in Julia by Maikel Bosschaert. The calculation of the higher-order coefficients for the higher-order predictor has been added to this code, which will be released in a future publication. In the following example, we compare the higher order predictor from Section 5.2 to the first order predictor from [3].

We consider the coupled FitzHugh-Nagumo model from [24] which was also used to test the predictor in [3]. This system consists of the following set of equations

$$\begin{cases} \dot{u}_1(t) = -\frac{1}{3}u_1^3(t) + (c+\alpha)u_1^2(t) + du_1(t) - u_2(t) + 2\beta f(u_1(t-\tau)), \\ \dot{u}_2(t) = \varepsilon(u_1(t) - bu_2(t)). \end{cases} \tag{6.3}$$

In this model, α and β measure the synaptic strength in self-connection and neighbourhood interaction, respectively, and $\tau > 0$ represents the time delay in signal transmission. The function f is a sufficiently smooth sigmoidal amplification function. The parameters b and ε are assumed to satisfy $0 < b < 1$ and $0 < \varepsilon \ll 1$.

As in [24], we take β and α as the bifurcation parameters and fix the parameters

$$b = 0.9, \quad \varepsilon = 0.08, \quad c = 2.0528, \quad d = -3.2135, \quad \tau = 1.7722$$

Furthermore, we use $f(u) = \tanh(u)$ for the sigmoid amplification function. There is a generalized Hopf point at $(\beta, \alpha) = (1.9, -1.0429)$. The left and right vectors null vectors of the characteristic matrix function evaluated at the critical eigenvalues are taken such that $\bar{q}^T q = 1$ and $p^T \Delta'(i\omega_0)q = 1$. The second Lyapunov coefficient is negative and has the value $l_2 = -15.6733$. In Figure 6.4, the bifurcation diagram is shown near the generalized Hopf point including the first-order and higher-order predictor curves.

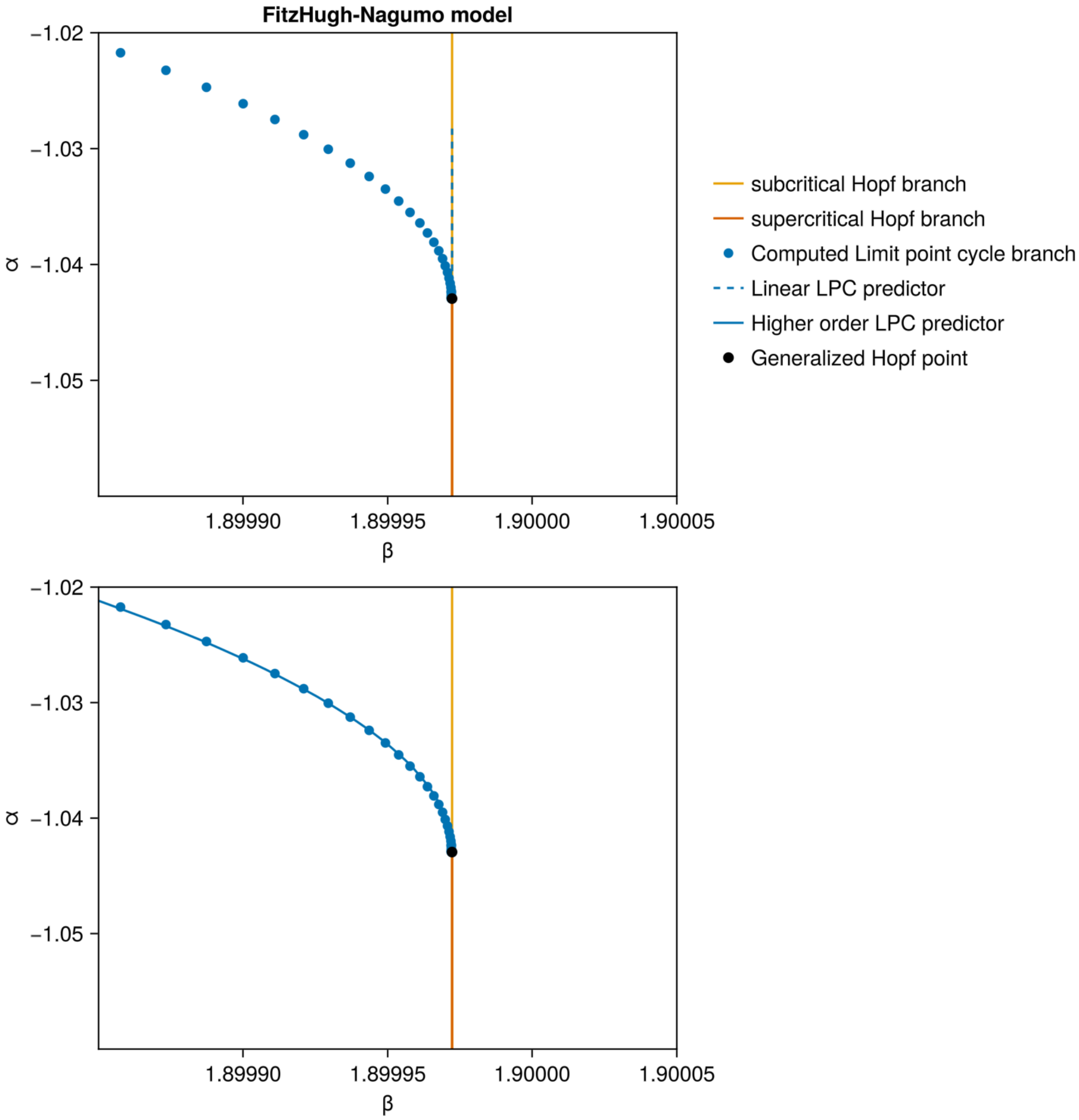

Figure 6.4: The bifurcation diagram near the generalized Hopf bifurcation in system (6.3) together with first-order and higher-order predictors.

Chapter 7

Final remarks

In this thesis, we applied parameter-dependent center manifold reduction to the generalized Hopf bifurcation to derive new higher-order predictors for the LPC curve. We did this for both ODEs and DDEs. In our examples, we have seen a good improvement of the predictor in parameter space. While this marks the end of this Bachelor thesis, there is more to explore. One such thing is creating convergence plots of the full higher-order predictor to compare with the first-order predictor. The inclusion of higher-order terms is expected to result in faster convergence to the LPC curve. Additionally, it would be interesting to look at the approximated periodic orbits with their correction and compare the period approximations to the corrected period.

Finally, as part of this thesis, the equations have only been implemented as a separate code in Julia. The plan is to make these files public in a future publication. The next step would be to include the higher-order predictor in some existing software, like MatCont in MATLAB for ODEs or DDE-BIFTOOL for DDEs. Then it will be easily accessible to anyone interested.

Appendix A

Terms collected from the homological equation for ODEs

In this appendix, all the equations collected from the homological equation for the generalized Hopf bifurcation in ODEs (4.1) are presented. The collected equations are obtained with the help of Mathematica. We use the truncated normal form expressed in terms of β

$$\dot{w} = \lambda(\beta)w + c_1(\beta)w|w|^2 + c_2(\beta)w|w|^4 + c_3(0)w|w|^6, \quad w \in \mathbb{C},\ \beta \in \mathbb{R}^2,$$

where we expand

$$\begin{aligned}
\lambda(\beta) &= i\omega_0 + g_{1010}\beta_1 + g_{1001}\beta_2 + \frac{1}{2}g_{1020}\beta_1^2 + g_{1011}\beta_1\beta_2 + \frac{1}{2}g_{1002}\beta_2^2 + \frac{1}{6}g_{1003},\\
c_1(\beta) &= c_1(0) + g_{2110}\beta_1 + g_{2101}\beta_2 + \frac{1}{2}g_{2120}\beta_1^2 + g_{2111}\beta_1\beta_2 + \frac{1}{2}g_{2102}\beta_2^2 + +\frac{1}{6}g_{2103},\\
c_2(\beta) &= c_2(0) + g_{3210}\beta_1 + g_{3201}\beta_2.
\end{aligned}$$

Compared to (4.2) we have that for $\mu = (10), (01)$

$$\begin{aligned}
g_{10\mu} &= \delta_\mu^{10} + b_{1,\mu}i,\\
g_{21\mu} &= \delta_\mu^{01} + b_{2,\mu}i,
\end{aligned}$$

For $\mu = (20), (02), (11)$ we have $g_{10\mu} = b_{1,\mu}i$ and $g_{21\mu} = b_{2,\mu}i$. The vectorfield F is expanded as (4.5). Note that $H_{ij\mu} = \overline{H}_{ji\mu}$.

A.1 Linear terms

Collecting linear terms of (4.1):

$$\begin{aligned}
w:&\quad Aq = iw_0q\\
\bar{w}:&\quad A\bar{q} = -iw_0\bar{q}\\
\beta_1:&\ AH_{0010} = -J_1K_{10}\\
\beta_2:&\ AH_{0001} = -J_1K_{01}
\end{aligned}$$

A.2 Quadratic terms

Collecting quadratic terms of (4.1):

$$
\begin{aligned}
w^2: \quad (A-2i\omega_0 I_n)H_{2000} &= -B(q,q)\\
w\bar{w}: \quad AH_{1100} &= -B(q,\bar{q})\\
w\beta_1: \quad (A-i\omega_0 I_n)H_{1010} &= g_{1010}q - A_1(q,K_{10}) - B(q,H_{0010})\\
w\beta_2: \quad (A-i\omega_0 I_n)H_{1001} &= g_{1001}q - A_1(q,K_{01}) - B(q,H_{0001})\\
\beta_1^2: \quad AH_{0020} &= -J_1K_{20} - 2A_1(H_{0010},K_{10}) - B(H_{0010},H_{0010}) - J_2(K_{10},K_{10})\\
\beta_2^2: \quad AH_{0002} &= -J_1K_{02} - 2A_1(H_{0001},K_{01}) - B(H_{0001},H_{0001}) - J_2(K_{01},K_{01})\\
\beta_1\beta_2: \quad AH_{0011} &= -J_1K_{11} - A_1(H_{0010},K_{01}) - A_1(H_{0001},K_{10}) - B(H_{0001},H_{0010})\\
&\quad - J_2(K_{01},K_{10})
\end{aligned}
$$

A.3 Qubic terms

Collecting qubic terms of (4.1):

$$
\begin{aligned}
w^3: \quad (A-3i\omega_0 I_n)H_{3000} &= -3B(q,H_{2000}) - C(q,q,q)\\
w^2\bar{w}: \quad (A-i\omega_0 I_n)H_{2100} &= 2c_1(0)q - [2B(q,H_{1100}) + B(\bar{q},H_{2000}) + C(q,q,\bar{q})]\\
\beta_1 w^2: \quad (A-2i\omega_0 I_n)H_{2010} &= 2g_{1010}H_{2000} - [A_1(H_{2000},K_{10}) + 2B(q,H_{1010}) + B(H_{0010},H_{2000})\\
&\quad + B_1(q,q,K_{10}) + C(q,q,H_{0010})]\\
\beta_2 w^2: \quad (A-2i\omega_0 I_n)H_{2001} &= 2g_{1001}H_{2000} - [A_1(H_{2000},K_{01}) + 2B(q,H_{1001}) + B(H_{0001},H_{2000})\\
&\quad + B_1(q,q,K_{01}) + C(q,q,H_{0001})]\\
w\bar{w}\beta_1: \quad AH_{1110} &= (g_{1010}+\bar{g}_{1010})H_{1100} - [A_1(H_{1100},K_{10}) + B(q,H_{0110}) + B(\bar{q},H_{1010})\\
&\quad + B(H_{0010},H_{1100}) + B_1(q,\bar{q},K_{10}) + C(q,\bar{q},H_{0010})]\\
w\bar{w}\beta_2: \quad AH_{1101} &= (g_{1001}+\bar{g}_{1001})H_{1100} - [A_1(H_{1100},K_{01}) + B(q,H_{0101}) + B(\bar{q},H_{1001})\\
&\quad + B(H_{0001},H_{1100}) + B_1(q,\bar{q},K_{01}) + C(q,\bar{q},H_{0001})]\\
w\beta_1^2: \quad (A-i\omega_0 I_n)H_{1020} &= g_{1020}q + 2g_{1010}H_{1010} - [A_1(q,K_{20}) + 2A_1(H_{1010},K_{10}) + B(q,H_{0020})\\
&\quad + 2B(H_{0010},H_{1010}) + A_2(q,K_{10},K_{10}) + 2B_1(q,H_{0010},K_{10})\\
&\quad + C(q,H_{0010},H_{0010})]\\
w\beta_2^2: \quad (A-i\omega_0 I_n)H_{1002} &= g_{1002}q + 2g_{1001}H_{1001} - [A_1(q,K_{02}) + 2A_1(H_{1001},K_{01}) + B(q,H_{0002})\\
&\quad + 2B(H_{0001},H_{1001}) + A_2(q,K_{01},K_{01}) + 2B_1(q,H_{0001},K_{01})\\
&\quad + C(q,H_{0001},H_{0001})]\\
w\beta_1\beta_2: \quad (A-i\omega_0 I_n)H_{1011} &= g_{1011}q + g_{1010}H_{1001} + g_{1001}H_{1010} - [A_1(q,K_{11}) + A_1(H_{1010},K_{01})\\
&\quad + A_1(H_{1001},K_{10}) + B(q,H_{0011}) + B(H_{0001},H_{1010}) + B(H_{0010},H_{1001})\\
&\quad + A_2(q,K_{01},K_{10}) + B_1(q,H_{0010},K_{01}) + B_1(q,H_{0001},K_{10})\\
&\quad + C(q,H_{0001},H_{0010})\\
\beta_2^3: \quad -AH_{0003} &= J_1K_{03} + 3A_1(H_{0002},K_{01}) + 3A_1(H_{0001},K_{02})\\
&\quad + 3B(H_{0001},H_{0002}) + 3J_2(K_{01},K_{02}) + 3A_2(H_{0001},K_{01},K_{01})\\
&\quad + 3B_1(H_{0001},H_{0001},K_{01}) + J_3(K_{01},K_{01},K_{01}) + C(H_{0001},H_{0001},H_{0001})
\end{aligned}
$$

A.4 Quartic terms

Collecting quartic terms of (4.1):

$$
\begin{aligned}
w^4: \quad (4i\omega_0 I_n - A)H_{4000} &= 4B(q, H_{3000}) + 3B(H_{2000}, H_{2000}) + 6C(q, q, , H_{2000}) + D(q, q, q, q) \\
w^3\bar{w}: \quad (2i\omega_0 I_n - A)H_{3100} &= -6c_1(0)H_{2000} + 3B(q, H_{2100}) + B(\bar{q}, H_{3000}) + 3B(H_{1100}, H_{2000}) \\
&+ 3C(q, q, H_{1100}) + 3C(q, \bar{q}, H_{2000}) + D(q, q, q, \bar{q}) \\
w^2\bar{w}^2: \quad -AH_{2200} &= -4(c_1(0) + \bar{c}_1(0))H_{1100} + 2B(q, H_{1200}) + 2B(\bar{q}, H_{2100}) \\
&+ B(H_{0200}, H_{2000}) + 2B(H_{1100}, H_{1100}) + C(q, q, H_{0200}) + 4C(q, \bar{q}, H_{1100}) \\
&+ C(\bar{q}, \bar{q}, H_{2000}) + D(q, q, \bar{q}, \bar{q}) \\
w^3\beta_1: \quad (A - 3i\omega_0 I_n)H_{3010} &= 3g_{1010}H_{3000} - [A_1(H_{3000}, K_{10}) + 3B(q, H_{2010}) + B(H_{0010}, H_{3000}) \\
&+ 3B(H_{1010}, H_{2000}) + 3B_1(q, H_{2000}, K_{10}) + 3C(q, q, H_{1010}) \\
&+ 3C(q, H_{0010}, H_{2000}) + C_1(q, q, q, K_{10}) + D(q, q, q, H_{0010})] \\
w^3\beta_2: \quad (A - 3i\omega_0 I_n)H_{3001} &= 3g_{1001}H_{3000} - [A_1(H_{3000}, K_{01}) + 3B(q, H_{2001}) + B(H_{0001}, H_{3000}) \\
&+ 3B(H_{1001}, H_{2000}) + 3B_1(q, H_{2000}, K_{01}) + 3C(q, q, H_{1001}) \\
&+ 3C(q, H_{0001}, H_{2000}) + C_1(q, q, q, K_{01}) + D(q, q, q, H_{0001})] \\
w^2\bar{w}\beta_1: \quad (A - i\omega_0)H_{2110} &= 2g_{2110}q + (2g_{1010} + \bar{g}_{1010})H_{2100} + 2c_1(0)H_{1010} - [A_1(H_{2100}, K_{10}) \\
&+ 2B(q, H_{1110}) + B(\bar{q}, H_{2010}) + B(H_{0010}, H_{2100}) + B(H_{0110}, H_{2000}) \\
&+ 2B(H_{1010}, H_{1100}) + 2B_1(q, H_{1100}, K_{10}) + B_1(\bar{q}, H_{2000}, K_{10}) \\
&+ C(q, q, H_{0110}) + 2C(q, \bar{q}.H_{1010}) + 2C(q, H_{0010}, H_{1100}) \\
&+ C(\bar{q}, H_{0010}, H_{2000}) + C_1(q, q, \bar{q}, K_{10}) + D(q, q, \bar{q}, H_{0010})] \\
w^2\bar{w}\beta_2: \quad (A - i\omega_0)H_{2101} &= 2g_{2101}q + (2g_{1001} + \bar{g}_{1001})H_{2100} + 2c_1(0)H_{1001} - [A_1(H_{2100}, K_{01}) \\
&+ 2B(q, H_{1101}) + B(\bar{q}, H_{2001}) + B(H_{0001}, H_{2100}) + B(H_{0101}, H_{2000}) \\
&+ 2B(H_{1001}, H_{1100}) + 2B_1(q, H_{1100}, K_{01}) + B_1(\bar{q}, H_{2000}, K_{01}) \\
&+ C(q, q, H_{0101}) + 2C(q, \bar{q}.H_{1001}) + 2C(q, H_{0001}, H_{1100}) \\
&+ C(\bar{q}, H_{0001}, H_{2000}) + C_1(q, q, \bar{q}, K_{01}) + D(q, q, \bar{q}, H_{0001})] \\
w^2\beta_1^2: \quad (A - 2i\omega_0 I_n)H_{2020} &= 2g_{1020}H_{2000} + 4g_{1010}H_{2010} - [2A_1(H_{2010}, K_{10}) + A_1(H_{2000}, K_{20}) \\
&+ 2B(q, H_{1020}) + 2B(H_{0010}, H_{2010}) + B(H_{0020}, H_{2000}) \\
&+ 2B(H_{1010}, H_{1010}) + A_2(H_{2000}, K_{10}, K_{10}) + B_1(q, q, K_{20}) \\
&+ 4B_1(q, H_{1010}, K_{10}) + 2B_1(H_{0010}, H_{2000}, K_{10}) + C(q, q, H_{0020}) \\
&+ 4C(q, H_{0010}, H_{1010}) + C(H_{0010}, H_{0010}, H_{2000}) + B_2(q, q, K_{10}, K_{10}) \\
&+ 2C_1(q, q, H_{0010}, K_{10}) + D(q, q, H_{0010}, H_{0010})] \\
w^2\beta_2^2: \quad (A - 2i\omega_0 I_n)H_{2002} &= 2g_{1002}H_{2000} + 4g_{1001}H_{2001} - [2A_1(H_{2001}, K_{01}) + A_1(H_{2000}, K_{02}) \\
&+ 2B(q, H_{1002}) + 2B(H_{0001}, H_{2001}) + B(H_{0002}, H_{2000}) \\
&+ 2B(H_{1001}, H_{1001}) + A_2(H_{2000}, K_{01}, K_{01}) + B_1(q, q, K_{02}) \\
&+ 4B_1(q, H_{1001}, K_{01}) + 2B_1(H_{0001}, H_{2000}, K_{01}) + C(q, q, H_{0002}) \\
&+ 4C(q, H_{0001}, H_{1001}) + C(H_{0001}, H_{0001}, H_{2000}) + B_2(q, q, K_{01}, K_{01}) \\
&+ 2C_1(q, q, H_{0001}, K_{01}) + D(q, q, H_{0001}, H_{0001})]
\end{aligned}
$$

$$\begin{aligned}
w^2\beta_1\beta_2: \quad (A-2i\omega_0 I_n)H_{2011} &= 2g_{1011}H_{2000}+2g_{1010}H_{2001}+2g_{1001}H_{2010}-[A_1(H_{2010},K_{01})\\
&+A_1(H_{2001},K_{10})+A_1(H_{2000},K_{11})+2B(q,H_{1011})+B(H_{0001},H_{2010})\\
&+B(H_{0010},H_{2001})+B(H_{0011},H_{2000})+2B(H_{1001},H_{1010})\\
&+A_2(H_{2000},K_{01},K_{10})+B_1(q,q,K_{11})+2B_1(q,H_{1010},K_{01})\\
&+2B_1(q,H_{1001},K_{10})+B_1(H_{0010},H_{2000},K_{01})+B_1(H_{0001},H_{2000},K_{10})\\
&+C(q,q,H_{0011})+2C(q,H_{0001},H_{1010})+2C(q,H_{0010},H_{1001})\\
&+C(H_{0001},H_{0010},H_{2000})+B_2(q,q,K_{01},K_{10})+C_1(q,q,H_{0010},K_{01})\\
&+C_1(q,q,H_{0001},K_{10})+D(q,q,H_{0001},H_{0010})]\\
w\bar{w}\beta_1^2: \quad AH_{1120} &= (g_{1020}+\bar{g}_{1020})H_{1100}+2(g_{1010}+\bar{g}_{1010})H_{1110}-[2A_1(H_{1110},K_{10})\\
&+A_1(H_{1100},K_{20})+B(q,H_{0120})+B(\bar{q},H_{1020})+2B(H_{0010},H_{1110})\\
&+B(H_{0020},H_{1100})+2B(H_{0110},H_{1010})+A_2(H_{1100},K_{10},K_{10})\\
&+B_1(q,\bar{q},K_{20})+2B_1(q,H_{0110},K_{10})+2B_1(\bar{q},H_{1010},K_{10})\\
&+2B_1(H_{0010},H_{1100},K_{10})+C(q,\bar{q},H_{0020})+2C(q,H_{0010},H_{0110})\\
&+2C(\bar{q},H_{0010},H_{1010})+C(H_{0010},H_{0010},H_{1100})+B_2(q,\bar{q},K_{10},K_{10})\\
&+2C_1(q,\bar{q},H_{0010},K_{10})+D(q,\bar{q},H_{0010},H_{0010})]\\
w\bar{w}\beta_2^2: \quad AH_{1102} &= (g_{1002}+\bar{g}_{1002})H_{1100}+2(g_{1001}+\bar{g}_{1001})H_{1101}-[2A_1(H_{1101},K_{01})\\
&+A_1(H_{1100},K_{02})+B(q,H_{0102})+B(\bar{q},H_{1002})+2B(H_{0001},H_{1101})\\
&+B(H_{0002},H_{1100})+2B(H_{0101},H_{1001})+A_2(H_{1100},K_{01},K_{01})\\
&+B_1(q,\bar{q},K_{02})+2B_1(q,H_{0101},K_{01})+2B_1(\bar{q},H_{1001},K_{01})\\
&+2B_1(H_{0001},H_{1100},K_{01})+C(q,\bar{q},H_{0002})+2C(q,H_{0001},H_{0101})\\
&+2C(\bar{q},H_{0001},H_{1001})+C(H_{0001},H_{0001},H_{1100})+B_2(q,\bar{q},K_{01},K_{01})\\
&+2C_1(q,\bar{q},H_{0001},K_{01})+D(q,\bar{q},H_{0001},H_{0001})]\\
w\bar{w}\beta_1\beta_2: \quad AH_{1111} &= (g_{1011}+\bar{g}_{1011})H_{1100}+(g_{1010}+\bar{g}_{1010})H_{1101}+(g_{1001}+\bar{g}_{1001})H_{1110}\\
&-[A_1(H_{1110},K_{01})+A_1(H_{1101},K_{10})+A_1(H_{1100},K_{11})+B(q,H_{0111})\\
&+B(\bar{q},H_{1011})+B(H_{0001},H_{1110})+B(H_{0010},H_{1101})+B(H_{0011},H_{1100})\\
&+B(H_{0101},H_{1010})+B(H_{0110},H_{1001})+A_2(H_{1100},K_{01},K_{10})\\
&+B_1(q,\bar{q},K_{11})+B_1(q,H_{0110},K_{01})+B_1(q,H_{0101},K_{10})\\
&+B_1(\bar{q},H_{1010},K_{01})+B_1(\bar{q},H_{1001},K_{10})+B_1(H_{0010},H_{1100},K_{01})\\
&+B_1(H_{0001},H_{1100},K_{10})+C(q,\bar{q},H_{0011})+C(q,H_{0001},H_{0110})\\
&+C(q,H_{0010},H_{0101})+C(\bar{q},H_{0001},H_{1010})+C(\bar{q},H_{0010},H_{1001})\\
&+C(H_{0001},H_{0010},H_{1100})+B_2(q,\bar{q},K_{01},K_{10})+C_1(q,\bar{q},H_{0010},K_{01})\\
&+C_1(q,\bar{q},H_{0001},K_{10})+D(q,\bar{q},H_{0001},H_{0010})]
\end{aligned}$$

$$
\begin{aligned}
w\beta_2^3: \quad (i\omega_0 I_n - A)H_{1003} = {} & A_1(q, K_{03}) + 3A_1(H_{1002}, K_{01}) \\
& + 3A_1(H_{1001}, K_{02}) + B(q, H_{0003}) + 3B(H_{0001}, H_{1002}) \\
& + 3B(H_{0002}, H_{1001}) + 3A_2(q, K_{01}, K_{02}) + 3A_2(H_{1001}, K_{01}, K_{01}) \\
& + 3B_1(q, H_{0002}, K_{01}) + 3B_1(q, H_{0001}, K_{02}) + 6B_1(H_{0001}, H_{1001}, K_{01}) \\
& + 3C(q, H_{0001}, H_{0002}) + 3C(H_{0001}, H_{0001}, H_{1001}) \\
& + A_3(q, K_{01}, K_{01}, K_{01}) + 3B_2(q, H_{0001}, K_{01}, K_{01}) \\
& + 3C_1(q, H_{0001}, H_{0001}, K_{01}) + D(q, H_{0001}, H_{0001}, H_{0001}) \\
& - (g_{1003}q + 3g_{1002}H_{1001} + 3g_{1001}H_{1002})
\end{aligned}
$$

A.5 Quintic terms

Collecting quintic terms of (4.1):

$$
\begin{aligned}
w^5: \quad (5i\omega_0 I_n - A)H_{5000} &= 5B(q, H_{4000}) + 10B(H_{2000}, H_{3000}) + 10C(q, q, H_{3000}) \\
&+ 15C(q, H_{2000}, H_{2000}) + 10D(q, q, q, H_{2000}) + E(q, q, q, q, q) \\
w^3\bar{w}^2: \quad (i\omega_0 I_n - A)H_{3200} &= -12c_2(0)q - 6(2c_1(0) + \bar{c_1}(0))H_{2100} + 3B(q, H_{2200}) + 2B(\bar{q}, H_{3100}) \\
&+ B(H_{0200}, H_{3000}) + 6B(H_{1100}, H_{2100}) + 3B(H_{1200}, H_{2000}) + 3C(q, q, H_{1200}) \\
&+ 6C(q, \bar{q}, H_{2100}) + 3C(q, H_{0200}, H_{2000}) + 6C(q, H_{1100}, H_{1100}) \\
&+ C(\bar{q}, \bar{q}, H_{3000}) + 6C(\bar{q}, H_{1100}, H_{2000}) + D(q, q, q, H_{0200}) \\
&+ 6D(q, q, \bar{q}, H_{1100}) + 3D(q, \bar{q}, \bar{q}, H_{2000}) + E(q, q, q, \bar{q}, \bar{q}) \\
w^4\bar{w}: \quad (A - 3i\omega_0 I_n)H_{4100} &= 12c_1(0)H_{3000} - [4B(q, H_{3100}) + B(\bar{q}, H_{4000}) + 4B(H_{1100}, H_{3000}) \\
&+ 6B(H_{2000}, H_{2100}) + 6C(q, q, H_{2100}) + 4C(q, \bar{q}, H_{3000}) \\
&+ 12C(q, H_{1100}, H_{2000}) + 3C(\bar{q}, H_{2000}, H_{2000}) + 4D(q, q, q, H_{1100}) \\
&+ 6D(q, q, \bar{q}, H_{2000}) + E(q, q, q, q, \bar{q})] \\
w^3\bar{w}\beta_2: \quad (A - 2i\omega_0 I_n)H_{3101} &= 6g_{2101}H_{2000} + 6c_1(0)H_{2001} + (3g_{1001} + \bar{g}_{1001})H_{3100} \\
&- [A_1(H_{3100}, K_{01}) + 3B(q, H_{2101}) + B(\bar{q}, H_{3001}) + B(H_{0001}, H_{3100}) \\
&+ B(H_{0101}, H_{3000}) + 3B(H_{1001}, H_{2100}) + 3B(H_{1100}, H_{2001}) \\
&+ 3B(H_{1101}, H_{2000}) + 3B_1(q, H_{2100}, K_{01}) + B_1(\bar{q}, H_{3000}, K_{01}) \\
&+ 3B_1(H_{1100}, H_{2000}, K_{01}) + 3C(q, q, H_{1101}) + 3C(q, \bar{q}, H_{2001}) \\
&+ 3C(q, H_{0001}, H_{2100}) + 3C(q, H_{0101}, H_{2000}) + 6C(q, H_{1001}, H_{1100}) \\
&+ C(\bar{q}, H_{0001}, H_{3000}) + 3C(\bar{q}, H_{1001}, H_{2000}) + 3C(H_{0001}, H_{1100}, H_{2000}) \\
&+ 3C_1(q, q, H_{1100}, K_{01}) + 3C_1(q, \bar{q}, H_{2000}, K_{01}) + D(q, q, q, H_{0101}) \\
&+ 3D(q, q, \bar{q}, H_{1001}) + 3D(q, q, H_{0001}, H_{1100}) + 3D(q, \bar{q}, H_{0001}, H_{2000}) \\
&+ D_1(q, q, q, \bar{q}, K_{01}) + E(q, q, q, \bar{q}, H_{0001})] \\
w^2\bar{w}\beta_1^2: \quad (A - i\omega_0 I_n)H_{2120} &= 2g_{2120}q + 4g_{2110}H_{1010} + 2c_1(0)H_{1020} + (2g_{1020} + \bar{g}_{1020})H_{2100} \\
&+ (4g_{1010} + 2\bar{g}_{1010})H_{2110} - [2A_1(H_{2110}, K_{10}) + A_1(H_{2100}, K_{20}) \\
&+ 2B(q, H_{1120}) + B(\bar{q}, H_{2020}) + 2B(H_{0010}, H_{2110}) + B(H_{0020}, H_{2100}) \\
&+ 2B(H_{0110}, H_{2010}) + B(H_{0120}, H_{2000}) + 4B(H_{1010}, H_{1110}) \\
&+ 2B(H_{1020}, H_{1100}) + A_2(H_{2100}, K_{10}, K_{10}) + 4B_1(q, H_{1110}, K_{10}) \\
&+ 2B_1(q, H_{1100}, K_{20}) + 2B_1(\bar{q}, H_{2010}, K_{10}) + B_1(\bar{q}, H_{2000}, K_{20}) \\
&+ 2B_1(H_{0010}, H_{2100}, K_{10}) + 2B_1(H_{0110}, H_{2000}, K_{10}) \\
&+ 4B_1(H_{1010}, H_{1100}, K_{10}) + C(q, q, H_{0120}) + 2C(q, \bar{q}, H_{1020}) \\
&+ 4C(q, H_{0010}, H_{1110}) + 2C(q, H_{0020}, H_{1100}) + 4C(q, H_{0110}, H_{1010}) \\
&+ 2C(\bar{q}, H_{0010}, H_{2010}) + C(\bar{q}, H_{0020}, H_{2000}) + 2C(\bar{q}, H_{1010}, H_{1010}) \\
&+ C(H_{0010}, H_{0010}, H_{2100}) + 2C(H_{0010}, H_{0110}, H_{2000}) \\
&+ 4C(H_{0010}, H_{1010}, H_{1100}) + 2B_2(q, H_{1100}, K_{10}, K_{10}) + B_2(\bar{q}, H_{2000}, K_{10}, K_{10}) \\
&+ C_1(q, q, \bar{q}, K_{20}) + 2C_1(q, q, H_{0110}, K_{10}) + 4C_1(q, \bar{q}, H_{1010}, K_{10}) \\
&+ 4C_1(q, K_{10}, H_{0010}, H_{1100}) + 2C_1(\bar{q}, K_{10}, H_{0010}, H_{2000}) \\
&+ D(q, q, \bar{q}, H_{0020}) + 2D(q, q, H_{0010}, H_{0110}) + 4D(q, \bar{q}, H_{0010}, H_{1010}) \\
&+ 2D(q, H_{0010}, H_{0010}, H_{1100}) + D(\bar{q}, H_{2000}, H_{0010}, H_{0010}) \\
&+ C_2(q, q, \bar{q}, K_{10}, K_{10}) + 2D_1(q, q, \bar{q}, K_{10}, H_{0010}) + E(q, q, \bar{q}, H_{0010}, H_{0010})]
\end{aligned}
$$

$$
\begin{aligned}
w^2\bar{w}^2\beta_2: \quad -AH_{2201} &= A_1(H_{2200},K_{01}) + 2B(q,H_{1201}) + 2B(\bar{q},H_{2101}) + B(H_{0001},H_{2200}) \\
&+ 2B(H_{0101},H_{2100}) + B(H_{0200},H_{2001}) + B(H_{0201},H_{2000}) \\
&+ 2B(H_{1001},H_{1200}) + 4B(H_{1100},H_{1101}) + 2B_1(q,H_{1200},K_{01}) \\
&+ B_1(H_{0200},H_{2000},K_{01}) + 2B_1(H_{1100},H_{1100},K_{01}) + 2B_1(\bar{q},H_{2100},K_{01}) \\
&+ C(q,q,H_{0201}) + 4C(q,\bar{q},H_{1101}) + 2C(q,H_{0001},H_{1200}) \\
&+ 4C(q,H_{0101},H_{1100}) + 2C(q,H_{0200},H_{1001}) + C(\bar{q},\bar{q},H_{2001}) \\
&+ 2C(\bar{q},H_{0001},H_{2100}) + 2C(\bar{q},H_{0101},H_{2000}) + 4C(\bar{q},H_{1001},H_{1100}) \\
&+ C(H_{0001},H_{0200},H_{2000}) + 2C(H_{0001},H_{1100},H_{1100}) + C_1(q,q,H_{0200},K_{01}) \\
&+ 4C_1(q,\bar{q},H_{1100},K_{01}) + C_1(\bar{q},\bar{q},H_{2000},K_{01}) + 2D(q,q,\bar{q},H_{0101}) \\
&+ D(q,q,H_{0001},H_{0200}) + 2D(q,\bar{q},\bar{q},H_{1001}) + 4D(q,\bar{q},H_{0001},H_{1100}) \\
&+ D(\bar{q},\bar{q},H_{0001},H_{2000}) + D_1(q,q,\bar{q},\bar{q},K_{01}) + E(q,q,\bar{q},\bar{q},H_{0001}) \\
&- (4(\overline{c_1}(0) + c_1(0))H_{1101} + 4(\bar{g}_{2101} + g_{2101})H_{1100} + 2(\bar{g}_{1001} + g_{1001})H_{2200}) \\
w^4\beta_2: \quad (4i\omega_0 I_n - A)H_{4001} &= A_1(H_{4000},K_{01}) + 4B(q,H_{3001}) + B(H_{0001},H_{4000}) \\
&+ 4B(H_{1001},H_{3000}) + 6B(H_{2000},H_{2001}) + 4B_1(q,H_{3000},K_{01}) \\
&+ 3B_1(H_{2000},H_{2000},K_{01}) + 6C(q,q,H_{2001}) + 4C(q,H_{0001},H_{3000}) \\
&+ 12C(q,H_{1001},H_{2000}) + 3C(H_{0001},H_{2000},H_{2000}) \\
&+ 6C_1(q,q,H_{2000},K_{01}) + 4D(q,q,q,H_{1001}) + 6D(q,q,H_{0001},H_{2000}) \\
&+ D_1(q,q,q,q,K_{01}) + E(q,q,q,q,H_{0001}) - 4g_{1001}H_{4000}
\end{aligned}
$$

$$
\begin{aligned}
w^2\bar{w}\beta_2^2: \quad (A - i\omega_0 I_n)H_{2102} &= 2g_{2102}q + 4g_{2101}H_{1001} + 2c_1(0)H_{1002} + (2g_{1002} + \bar{g}_{1002})H_{2100} \\
&+ (4g_{1001} + 2\bar{g}_{1001})H_{2101} - [2A_1(H_{2101},K_{01}) + A_1(H_{2100},K_{02}) \\
&+ 2B(q,H_{1102}) + B(\bar{q},H_{2002}) + 2B(H_{0001},H_{2101}) + B(H_{0002},H_{2100}) \\
&+ 2B(H_{0101},H_{2001}) + B(H_{0102},H_{2000}) + 4B(H_{1001},H_{1101}) \\
&+ 2B(H_{1002},H_{1100}) + A_2(K_{01},K_{01},H_{2100}) + 4B_1(q,H_{1101},K_{01}) \\
&+ 2B_1(q,H_{1100},K_{02}) + 2B_1(\bar{q},H_{2001},K_{01}) + B_1(\bar{q},H_{2000},K_{02}) \\
&+ 2B_1(H_{0001},H_{2100},K_{01}) + 2B_1(H_{0101},H_{2000},K_{01}) \\
&+ 4B_1(H_{1001},H_{1100},K_{01}) + C(q,q,H_{0102}) + 2C(q,\bar{q},H_{1002}) \\
&+ 4C(q,H_{0001},H_{1101}) + 2C(q,H_{0002},H_{1100}) + 4C(q,H_{0101},H_{1001}) \\
&+ 2C(\bar{q},H_{0001},H_{2001}) + C(\bar{q},H_{0002},H_{2000}) + 2C(\bar{q},H_{1001},H_{1001}) \\
&+ C(H_{0001},H_{0001},H_{2100}) + 2C(H_{0001},H_{0101},H_{2000}) \\
&+ 4C(H_{0001},H_{1001},H_{1100}) + 2B_2(q,H_{1100},K_{01},H_{01}) + B_2(\bar{q},H_{2000},K_{01},K_{01}) \\
&+ C_1(q,q,\bar{q},K_{02}) + 2C_1(q,q,H_{0101},K_{01}) + 4C_1(q,\bar{q},H_{1001},K_{01}) \\
&+ 4C_1(q,H_{0001},H_{1100},K_{01}) + 2C_1(\bar{q},H_{0001},H_{2000},K_{01}) \\
&+ D(q,q,\bar{q},H_{0002}) + 2D(q,q,H_{0001},H_{0101}) + 4D(q,\bar{q},H_{0001},H_{1001}) \\
&+ 2D(q,H_{0001},H_{0001},H_{1100}) + D(\bar{q},H_{2000},H_{0001},H_{0001}) \\
&+ C_2(q,q,\bar{q},K_{01},K_{01}) + 2D_1(q,q,\bar{q},H_{0001},K_{01}) + E(q,q,\bar{q},H_{0001},H_{0001})]
\end{aligned}
$$

$$
\begin{aligned}
w^3\beta_2^2: \quad (3i\omega_0 I_n - A)H_{3002} = {} & 2A_1(H_{3001}, K_{01}) + A_1(H_{3000}, K_{02}) + 3B(q, H_{2002}) \\
& + 2B(H_{0001}, H_{3001}) + B(H_{0002}, H_{3000}) + 6B(H_{1001}, H_{2001}) \\
& + 3B(H_{1002}, H_{2000}) + A_2(H_{3000}, K_{01}, K_{01}) + 6B_1(q, H_{2001}, K_{01}) \\
& + 3B_1(q, H_{2000}, K_{02}) + 2B_1(H_{0001}, H_{3000}, K_{01}) + 6B_1(H_{1001}, H_{2000}, K_{01}) \\
& + 3C(q, q, H_{1002}) + 6C(q, H_{0001}, H_{2001}) + 3C(q, H_{0002}, H_{2000}) \\
& + 6C(q, H_{1001}, H_{1001}) + C(H_{0001}, H_{0001}, H_{3000}) \\
& + 6C(H_{0001}, H_{1001}, H_{2000}) + 3B_2(q, H_{2000}, K_{01}, K_{01}) + C_1(q, q, q, K_{02}) \\
& + 6C_1(q, q, H_{1001}, K_{01}) + 6C_1(q, H_{0001}, H_{2000}, K_{01}) + D(q, q, q, H_{0002}) \\
& + 6D(q, q, H_{0001}, H_{1001}) + 3D(q, H_{0001}, H_{0001}, H_{2000}) \\
& + C_2(q, q, q, K_{01}, K_{01}) + 2D_1(q, q, q, H_{0001}, K_{01}) \\
& + E(q, q, q, H_{0001}, H_{0001}) - [3g_{1002}H_{3000} + 6g_{1001}H_{3001}]
\end{aligned}
$$

$$
\begin{aligned}
w^2\bar{w}\beta_1\beta_2: \quad (A - i\omega_0 I_n)H_{2111} = {} & 2g_{2111}q + 2g_{2110}H_{1001} + 2g_{2101}H_{1010} + 2c_1(0)H_{1011} \\
& + (2g_{1011} + \bar{g}_{1011})\, H_{2100} + (2g_{1010} + \bar{g}_{1010})\, H_{2101} + (2g_{1001} + \bar{g}_{1001})\, H_{2110} \\
& - [A_1(H_{2110}, K_{01}) + A_1(H_{2101}, K_{10}) + A_1(H_{2100}, K_{11}) + 2B(q, H_{1111}) \\
& + B(\bar{q}, H_{2011}) + B(H_{0001}, H_{2110}) + B(H_{0010}, H_{2101}) + B(H_{0011}, H_{2100}) \\
& + B(H_{0101}, H_{2010}) + B(H_{0110}, H_{2001}) + B(H_{0111}, H_{2000}) + 2B(H_{1001}, H_{1110}) \\
& + 2B(H_{1010}, H_{1101}) + 2B(H_{1011}, H_{1100}) + A_2(H_{2100}, K_{01}, K_{10}) \\
& + 2B_1(q, H_{1110}, K_{01}) + 2B_1(q, H_{1101}, K_{10}) + 2B_1(q, H_{1100}, K_{11}) \\
& + B_1(\bar{q}, H_{2010}, K_{01}) + B_1(\bar{q}, H_{2001}, K_{10}) + B_1(\bar{q}, H_{2000}, K_{11}) \\
& + B_1(H_{0010}, H_{2100}, K_{01}) + B_1(H_{0110}, H_{2000}, K_{01}) + 2B_1(H_{1010}, H_{1100}, K_{01}) \\
& + B_1(H_{0001}, H_{2100}, K_{10}) + B_1(H_{0101}, H_{2000}, K_{10}) + 2B_1(H_{1001}, H_{1100}, K_{10}) \\
& + C(q, q, H_{0111}) + 2C(q, \bar{q}, H_{1011}) + 2C(q, H_{0001}, H_{1110}) \\
& + 2C(q, H_{0010}, H_{1101}) + 2C(q, H_{0011}, H_{1100}) + 2C(q, H_{0101}, H_{1010}) \\
& + 2C(q, H_{0110}, H_{1001}) + C(\bar{q}, H_{0001}, H_{2010}) + C(\bar{q}, H_{0010}, H_{2001}) \\
& + C(\bar{q}, H_{0011}, H_{2000}) + 2C(\bar{q}, H_{1001}, H_{1010}) + C(H_{0001}, H^{0010}, H_{2100}) \\
& + C(H_{0001}, H_{0110}, H_{2000}) + 2C(H_{0001}, H_{1010}, H_{1100}) + C(H_{0010}, H_{0101}, H_{2000}) \\
& + 2C(H_{0010}, H_{1001}, H_{1100}) + 2B_2(q, H_{1100}, K_{01}, K_{10}) + B_2(\bar{q}, H_{2000}, K_{01}, K_{10}) \\
& + C_1(q, q, \bar{q}, K_{11}) + C_1(q, q, H_{0110}, K_{01}) + C_1(q, q, H_{0101}, K_{10}) \\
& + 2C_1(q, \bar{q}, H_{1010}, K_{01}) + 2C_1(q, \bar{q}, H_{1001}, K_{10}) + 2C_1(q, H_{0010}, H_{1100}, K_{01}) \\
& + 2C_1(q, H_{0001}, H_{1100}, K_{10}) + C_1(\bar{q}, H_{0010}, H_{2000}, K_{01}) \\
& + C_1(\bar{q}, H_{0001}, H_{2000}, K_{10}) + D(q, q, \bar{q}, H_{0011}) + D(q, q, H_{0001}, H_{0110}) \\
& + D(q, q, H_{0010}, H_{0101}) + 2D(q, \bar{q}, H_{0001}, H_{1010}) + 2D(q, \bar{q}, H_{0010}, H_{1001}) \\
& + 2D(q, H_{0001}, H_{0010}, H_{1100}) + D(\bar{q}, H_{0001}, H_{0010}, H_{2000}) \\
& + C_2(q, q, \bar{q}, K_{01}, K_{10}) + D_1(q, q, \bar{q}, H_{0010}, K_{01}) + D_1(q, q, \bar{q}, H_{0001}, K_{10}) \\
& + E(q, q, \bar{q}, H_{0001}, H_{0010})]
\end{aligned}
$$

$$
\begin{aligned}
w^2\beta_2^3: \quad (2i\omega_0 I_n - A)H_{2003} &= 3A_1(H_{2002}, K_{01}) + 3A_1(H_{2001}, K_{02}) \\
&+ A_1(H_{2000}, K_{03}) + 2B(q, H_{1003}) + 3B(H_{0001}, H_{2002}) \\
&+ 3B(H_{0002}, H_{2001}) + B(H_{0003}, H_{2000}) + 6B(H_{1001}, H_{1002}) \\
&+ 3A_2(H_{2001}, K_{01}, K_{01}) + 3A_2(H_{2000}, K_{01}, K_{02}) + B_1(q, q, K_{03}) \\
&+ 6B_1(q, H_{1002}, K_{01}) + 6B_1(q, H_{1001}, K_{02}) + 6B_1(H_{0001}, H_{2001}, K_{01}) \\
&+ 3B_1(H_{0002}, H_{2000}, K_{01}) + 6B_1(H_{1001}, H_{1001}, K_{01}) \\
&+ 3B_1(H_{0001}, H_{2000}, K_{02}) + C(q, q, H_{0003}) + 6C(q, H_{0001}, H_{1002}) \\
&+ 6C(q, H_{0002}, H_{1001}) + 3C(H_{0001}, H_{0001}, H_{2001}) \\
&+ 3C(H_{0001}, H_{0002}, H_{2000}) + 6C(H_{0001}, H_{1001}, H_{1001}) \\
&+ A_3(H_{2000}, K_{01}, K_{01}, K_{01}) + 3B_2(q, q, K_{01}, K_{02}) + 6B_2(q, H_{1001}, K_{01}, K_{01}) \\
&+ 3B_2(H_{0001}, H_{2000}, K_{01}, K_{01}) + 3C_1(q, q, H_{0002}, K_{01}) \\
&+ 3C_1(q, q, H_{0001}, K_{02}) + 12C_1(q, H_{0001}, H_{1001}, K_{01}) \\
&+ 3C_1(H_{0001}, H_{0001}, H_{2000}, K_{01}) + 3D(q, q, H_{0001}, H_{0002}) \\
&+ 6D(q, H_{0001}, H_{0001}, H_{1001}) + D(H_{0001}, H_{0001}, H_{0001}, H_{2000}) \\
&+ B_3(q, q, K_{01}, K_{01}, K_{01}) + 3C_2(q, q, H_{0001}, K_{01}, K_{01}) \\
&+ 3D_1(q, q, K_{01}, H_{0001}, H_{0001}) + E(q, q, H_{0001}, H_{0001}, H_{0001}) \\
&- (2g_{1003}H_{2000} + 6g_{1002}H_{2001} + 6g_{1001}H_{2002}) \\
w\bar{w}\beta_2^3: \quad -AH_{1103} &= 3A_1(H_{1102}, K_{01}) + 3A_1(H_{1101}, K_{02}) + A_1(H_{1100}, K_{03}) \\
&+ B(q, H_{0103}) + B(\bar{q}, H_{1003}) + 3B(H_{0001}, H_{1102}) + 3B(H_{0002}, H_{1101}) \\
&+ B(H_{0003}, H_{1100}) + 3B(H_{0101}, H_{1002}) + 3B(H_{0102}, H_{1001}) \\
&+ 3A_2(H_{1101}, K_{01}, K_{01}) + 3A_2(H_{1100}, K_{01}, K_{02}) + B_1(q, \bar{q}, K_{03}) \\
&+ 3B_1(q, H_{0102}, K_{01}) + 3B_1(q, H_{0101}, K_{02}) + 3B_1(\bar{q}, H_{1002}, K_{01}) \\
&+ 3B_1(\bar{q}, H_{1001}, K_{02}) + 6B_1(H_{0001}, H_{1101}, K_{01}) + 3B_1(H_{0002}, H_{1100}, K_{01}) \\
&+ 6B_1(H_{0101}, H_{1001}, K_{01}) + 3B_1(H_{0001}, H_{1100}, K_{02}) + C(q, \bar{q}, H_{0003}) \\
&+ 3C(q, H_{0001}, H_{0102}) + 3C(q, H_{0002}, H_{0101}) + 3C(\bar{q}, H_{0001}, H_{1002}) \\
&+ 3C(\bar{q}, H_{0002}, H_{1001}) + 3C(H_{0001}, H_{0001}, H_{1101}) \\
&+ 3C(H_{0001}, H_{0002}, H_{1100}) + 6C(H_{0001}, H_{0101}, H_{1001}) \\
&+ A_3(H_{1100}, K_{01}, K_{01}, K_{01}) + 3B_2(q, \bar{q}, K_{01}, K_{02}) + 3B_2(q, H_{0101}, K_{01}, K_{01}) \\
&+ 3B_2(\bar{q}, H_{1001}, K_{01}, K_{01}) + 3B_2(H_{0001}, H_{1100}, K_{01}, K_{01}) \\
&+ 3C_1(q, \bar{q}, H_{0002}, K_{01}) + 3C_1(q, \bar{q}, H_{0001}, K_{02}) \\
&+ 6C_1(q, H_{0001}, H_{0101}, K_{01}) + 6C_1(\bar{q}, H_{0001}, H_{1001}, K_{01}) \\
&+ 3C_1(H_{0001}, H_{0001}, H_{1100}, K_{01}) + 3D(q, \bar{q}, H_{0001}, H_{0002}) \\
&+ 3D(q, H_{0001}, H_{0001}, H_{0101}) + 3D(\bar{q}, H_{0001}, H_{0001}, H_{1001}) \\
&+ D(H_{0001}, H_{0001}, H_{0001}, H_{1100}) + B_3(q, \bar{q}, K_{01}, K_{01}, K_{01}) \\
&+ 3C_2(q, \bar{q}, H_{0001}, K_{01}, K_{01}) + 3D_1(q, \bar{q}, H_{0001}, H_{0001}, K_{01}) \\
&+ E(q, \bar{q}, H_{0001}, H_{0001}, H_{0001}) \\
&- (g_{1003} + \bar{g}_{1003})H_{1100} + 3(g_{1002} + \bar{g}_{1002})H_{1101} + 3(g_{1001} + \bar{g}_{1001})H_{1102})
\end{aligned}
$$

A.6 Sixth order terms

Collecting sixth order terms of (4.1):

$$
\begin{aligned}
w^6: \quad (6i\omega_0 I_n - A)H_{6000} &= 6B(q,H_{5000}) + 15B(H_{2000},H_{4000}) + 10B(H_{3000},H_{3000}) \\
&+ 15C(q,q,H_{4000}) + 60C(q,H_{2000},H_{3000}) + 15C(H_{2000},H_{2000},H_{2000}) \\
&+ 20D(q,q,q,H_{3000}) + 45D(q,q,H_{2000}.H_{2000}) + 15E(q,q,q,q,H_{2000}) \\
&+ K(q,q,q,q,q,q) \\
w^5\bar{w}: \quad (4i\omega_0 I_n - A)H_{5100} &= 5B(q,H_{4100}) + B(\bar{q},H_{5000}) + 5B(H_{1100},H_{4000}) + 10B(H_{2000},H_{3100}) \\
&+ 10B(H_{2100},H_{3000}) + 10C(q,q,H_{3100}) + 5C(q,\bar{q},H_{4000}) \\
&+ 20C(q,H_{1100},H_{3000}) + 30C(q,H_{2000},H_{2100}) + 10C(\bar{q},H_{2000},H_{3000}) \\
&+ 15C(H_{1100},H_{2000},H_{2000}) + 10D(q,q,q,H_{2100}) + 10D(q,q,\bar{q},H_{3000}) \\
&+ 30D(q,q,H_{1100},H_{2000}) + 15D(q,\bar{q},H_{2000},H_{2000}) + 5E(q,q,q,q,H_{1100}) \\
&+ 10E(q,q,q,\bar{q},H_{2000}) + K(q,q,q,q,q,\bar{q}) - 20c_1(0)H_{4000} \\
w^4\bar{w}^2: \quad (A - 2i\omega_0 I_n)H_{4200} &= 48c_2(0)H_{2000} + 8(3c_1(0) + \overline{c_1}(0))H_{3100} - [4B(q,H_{3200}) + 2B(\bar{q},H_{4100}) \\
&+ B(H_{0200},H_{4000}) + 8B(H_{1100},H_{3100}) + 4B(H_{1200},H_{3000}) \\
&+ 6B(H_{2000},H_{2200}) + 6B(H_{2100},H_{2100}) + 6C(q,q,H_{2200}) \\
&+ 8C(q,\bar{q},H_{3100}) + 4C(q,H_{0200},H_{3000}) + 24C(q,H_{1100},H_{2100}) \\
&+ 12C(q,H_{1200},H_{2000}) + C(\bar{q},\bar{q},H_{4000}) + 8C(\bar{q},H_{1100},H_{3000}) \\
&+ 12C(\bar{q},H_{2000},H_{2100}) + 3C(H_{0200},H_{2000},H_{2000}) + 12C(H_{1100},H_{1100},H_{2000}) \\
&+ 4D(q,q,q,H_{1200}) + 12D(q,q,\bar{q},H_{2100}) + 6D(q,q,H_{0200},H_{2000}) \\
&+ 12D(q,q,H_{1100},H_{1100}) + 4D(q,\bar{q},\bar{q},H_{3000}) + 24D(q,\bar{q},H_{1100},H_{2000}) \\
&+ 3D(\bar{q},\bar{q},H_{2000},H_{2000}) + E(q,q,q,q,H_{0200}) + 8E(q,q,q,\bar{q},H_{1100}) \\
&+ 6E(q,q,\bar{q},\bar{q},H_{2000}) + K(q,q,q,q,\bar{q},\bar{q})] \\
w^3\bar{w}^3: \quad AH_{3300} &= 36(c_2(0) + \overline{c_2}(0))H_{1100} + 18(c_1(0) + \overline{c_1}(0))H_{2200} \\
&- [3B(q,H_{2300}) + 3B(\bar{q},H_{3200}) + 3B(H_{0200},H_{3100}) + B(H_{0300},H_{3000}) \\
&+ 9B(H_{1100},H_{2200}) + 9B(H_{1200},H_{2100}) + 3B(H_{1300},H_{2000}) \\
&+ 3C(q,q,H_{1300}) + 9C(q,\bar{q},H_{2200}) + 9C(q,H_{0200},H_{2100}) \\
&+ 3C(q,H_{0300},H_{2000}) + 18C(q,H_{1100},H_{1200}) + 3C(\bar{q},\bar{q},H_{3100}) \\
&+ 3C(\bar{q},H_{0200},H_{3000}) + 18C(\bar{q},H_{1100},H_{2100}) + 9C(\bar{q},H_{1200},H_{2000}) \\
&+ 9C(H_{0200},H_{1100},H_{2000}) + 6C(H_{1100},H_{1100},H_{1100}) + D(q,q,q,H_{0300}) \\
&+ 9D(q,q,\bar{q},H_{1200}) + 9D(q,q,H_{0200},H_{1100}) + 9D(q,\bar{q},\bar{q},H_{2100}) \\
&+ 9D(q,\bar{q},H_{0200},H_{2000}) + 18D(q,\bar{q},H_{1100},H_{1100}) + D(\bar{q},\bar{q},\bar{q},H_{3000}) \\
&+ 9D(\bar{q},\bar{q},H_{1100},H_{2000}) + 3E(q,q,q,\bar{q},H_{0200}) + 9E(q,q,\bar{q},\bar{q},H_{1100}) \\
&+ 3E(q,\bar{q},\bar{q},\bar{q},H_{2000}) + K(q,q,q,\bar{q},\bar{q},\bar{q})]
\end{aligned}
$$

$$
\begin{aligned}
w^3\bar{w}^2\beta_2: \quad (A - i\omega_0 I_n)H_{3201} &= 12g_{3201}q + 12c_2(0)H_{1001} + (12g_{2101} + 6\bar{g}_{2101})H_{2100} \\
&+ (12c_1(0) + 6\overline{c_1}(0))H_{2101} + (3g_{1001} + 2\bar{g}_{1001})H_{3200} \\
&- [A_1(H_{3200}, K_{01}) + 3B(q, H_{2201}) + 2B(\bar{q}, H_{3101}) + B(H_{0001}, H_{3200}) \\
&+ 2B(H_{0101}, H_{3100}) + B(H_{0200}, H_{3001}) + B(H_{0201}, H_{3000}) \\
&+ 3B(H_{1001}, H_{2200}) + 6B(H_{1100}, H_{2101}) + 6B(H_{1101}, H_{2100}) \\
&+ 3B(H_{1200}, H_{2001}) + 3B(H_{1201}, H_{2000}) + 3B_1(q, H_{2200}, K_{01}) \\
&+ 2B_1(\bar{q}, H_{3100}, K_{01}) + B_1(H_{0200}, H_{3000}, K_{01}) + 6B_1(H_{1100}, H_{2100}, K_{01}) \\
&+ 3B_1(H_{1200}, H_{2000}, K_{01}) + 3C(q, q, H_{1201}) + 6C(q, \bar{q}, H_{2101}) \\
&+ 3C(q, H_{0001}, H_{2200}) + 6C(q, H_{0101}, H_{2100}) + 3C(q, H_{0200}, H_{2001}) \\
&+ 3C(q, H_{0201}, H_{2000}) + 6C(q, H_{1001}, H_{1200}) + 12C(q, H_{1100}, H_{1101}) \\
&+ C(\bar{q}, \bar{q}, H_{3001}) + 2C(\bar{q}, H_{0001}, H_{3100}) + 2C(\bar{q}, H_{0101}, H_{3000}) \\
&+ 6C(\bar{q}, H_{1001}, H_{2100}) + 6C(\bar{q}, H_{1100}, H_{2001}) + 6C(\bar{q}, H_{1101}, H_{2000}) \\
&+ C(H_{0001}, H_{0200}, H_{3000}) + 6C(H_{0001}, H_{1100}, H_{2100}) + 3C(H_{0001}, H_{1200}, H_{2000}) \\
&+ 6C(H_{0101}, H_{1100}, H_{2000}) + 3C(H_{0200}, H_{1001}, H_{2000}) \\
&+ 6C(H_{1001}, H_{1100}, H_{1100}) + 3C_1(q, q, H_{1200}, K_{01}) + 6C_1(q, \bar{q}, H_{2100}, K_{01}) \\
&+ 3C_1(q, H_{0200}, H_{2000}, K_{01}) + 6C_1(q, H_{1100}, H_{1100}, K_{01}) + C_1(\bar{q}, \bar{q}, H_{3000}, K_{01}) \\
&+ 6C_1(\bar{q}, H_{1100}, H_{2000}, K_{01}) + D(q, q, q, H_{0201}) + 6D(q, q, \bar{q}, H_{1101}) \\
&+ 3D(q, q, H_{0001}, H_{1200}) + 6D(q, q, H_{0101}, H_{1100}) + 3D(q, q, H_{0200}, H_{1001}) \\
&+ 3D(q, \bar{q}, \bar{q}, H_{2001}) + 6D(q, \bar{q}, H_{0001}, H_{2100}) + 6D(q, \bar{q}, H_{0101}, H_{2000}) \\
&+ 12D(q, \bar{q}, H_{1001}, H_{1100}) + 3D(q, H_{0001}, H_{0200}, H_{2000}) \\
&+ 6D(q, H_{0001}, H_{1100}, H_{1100}) + D(\bar{q}, \bar{q}, H_{0001}, H_{3000}) + 3D(\bar{q}, \bar{q}, H_{1001}, H_{2000}) \\
&+ 6D(\bar{q}, H_{0001}, H_{1100}, H_{2000}) + D_1(q, q, q, H_{0200}, K_{01}) + 6D_1(q, q, \bar{q}, H_{1100}, K_{01}) \\
&+ 3D_1(q, \bar{q}, \bar{q}, H_{2000}, K_{01}) + 2E(q, q, q, \bar{q}, H_{0101}) + E(q, q, q, H_{0001}, H_{0200}) \\
&+ 3E(q, q, \bar{q}, \bar{q}, H_{1001}) + 6E(q, q, \bar{q}, H_{0001}, H_{1100}) + 3E(q, \bar{q}, \bar{q}, H_{0001}, H_{2000}) \\
&+ E_1(q, q, q, \bar{q}, \bar{q}, K_{01}) + K(q, q, q, \bar{q}, \bar{q}, H_{0001})]
\end{aligned}
$$

$$
\begin{aligned}
w^5\beta_2: \quad (5i\omega_0 I_n - A)H_{5001} &= A_1(H_{5000}, K_{01}) + 5B(q, H_{4001}) + B(H_{0001}, H_{5000}) \\
&+ 5B(H_{1001}, H_{4000}) + 10B(H_{2000}, H_{3001}) + 10B(H_{2001}, H_{3000}) \\
&+ 5B_1(q, H_{4000}, K_{01}) + 10B_1(H_{2000}, H_{3000}, K_{01}) + 10C(q, q, H_{3001}) \\
&+ 5C(q, H_{0001}, H_{4000}) + 20C(q, H_{1001}, H_{3000}) + 30C(q, H_{2000}, H_{2001}) \\
&+ 10C(H_{0001}, H_{2000}, H_{3000}) + 15C(H_{1001}, H_{2000}, H_{2000}) \\
&+ 10C_1(q, q, H_{3000}, K_{01}) + 15C_1(q, H_{2000}, H_{2000}, K_{01}) \\
&+ 10D(q, q, q, H_{2001}) + 10D(q, q, H_{0001}, H_{3000}) + 30D(q, q, H_{1001}, H_{2000}) \\
&+ 15D(q, H_{0001}, H_{2000}, H_{2000}) + 10D_1(q, q, q, H_{2000}, K_{01}) \\
&+ 5E(q, q, q, q, H_{1001}) + 10E(q, q, q, H_{0001}, H_{2000}) \\
&+ E_1(q, q, q, q, q, K_{01}) + K(q, q, q, q, q, H_{0001}) - 5g_{1001}H_{5000}
\end{aligned}
$$

$$
\begin{aligned}
w^4\bar{w}\beta_2: \quad (3i\omega_0 I_n - A)H_{4101} &= A_1(H_{4100}, K_{01}) + 4B(q, H_{3101}) + B(\bar{q}, H_{4001}) \\
&+ B(H_{0001}, H_{4100}) + B(H_{0101}, H_{4000}) + 4B(H_{1001}, H_{3100}) \\
&+ 4B(H_{1100}, H_{3001}) + 4B(H_{1101}, H_{3000}) + 6B(H_{2000}, H_{2101}) \\
&+ 6B(H_{2001}, H_{2100}) + 4B_1(q, H_{3100}, K_{01}) + B_1(\bar{q}, H_{4000}, K_{01}) \\
&+ 4B_1(H_{1100}, H_{3000}, K_{01}) + 6B_1(H_{2000}, H_{2100}, K_{01}) + 6C(q, q, H_{2101}) \\
&+ 4C(q, \bar{q}, H_{3001}) + 4C(q, H_{0001}, H_{3100}) + 4C(q, H_{0101}, H_{3000}) \\
&+ 12C(q, H_{1001}, H_{2100}) + 12C(q, H_{1100}, H_{2001}) + 12C(q, H_{1101}, H_{2000}) \\
&+ C(\bar{q}, H_{0001}, H_{4000}) + 4C(\bar{q}, H_{1001}, H_{3000}) + 6C(\bar{q}, H_{2000}, H_{2001}) \\
&+ 4C(H_{0001}, H_{1100}, H_{3000}) + 6C(H_{0001}, H_{2000}, H_{2100}) \\
&+ 3C(H_{0101}, H_{2000}, H_{2000}) + 12C(H_{1001}, H_{1100}, H_{2000}) \\
&+ 6C_1(q, q, H_{2100}, K_{01}) + 4C_1(q, \bar{q}, H_{3000}, K_{01}) \\
&+ 12C_1(q, H_{1100}, H_{2000}, K_{01}) + 3C_1(\bar{q}, H_{2000}, H_{2000}, K_{01}) \\
&+ 4D(q, q, q, H_{1101}) + 6D(q, q, \bar{q}, H_{2001}) + 6D(q, q, H_{0001}, H_{2100}) \\
&+ 6D(q, q, H_{0101}, H_{2000}) + 12D(q, q, H_{1001}, H_{1100}) \\
&+ 4D(q, \bar{q}, H_{0001}, H_{3000}) + 12D(q, \bar{q}, H_{1001}, H_{2000}) \\
&+ 12D(q, H_{0001}, H_{1100}, H_{2000}) + 3D(\bar{q}, H_{0001}, H_{2000}, H_{2000}) \\
&+ 4D_1(q, q, q, H_{1100}, K_{01}) + 6D_1(q, q, \bar{q}, H_{2000}, K_{01}) \\
&+ E(q, q, q, q, H_{0101}) + 4E(q, q, q, \bar{q}, H_{1001}) + 4E(q, q, q, H_{0001}, H_{1100}) \\
&+ 6E(q, q, \bar{q}, H_{0001}, H_{2000}) + E_1(q, q, q, q, \bar{q}, K_{01}) + K(q, q, q, q, \bar{q}, H_{0001}) \\
&- [12g_{2101}H_{3000} + 12c_1(0)H_{3001} + (4g_{1001} + \bar{g}_{1001})H_{4100}]
\end{aligned}
$$

$$
\begin{aligned}
w^2\bar{w}\beta_2^3: \quad (i\omega_0 I_n - A)H_{2103} &= 3A_1(H_{2102}, K_{01}) + 3A_1(H_{2101}, K_{02}) + A_1(H_{2100}, K_{03}) + 2B(q, H_{1103}) \\
&+B(\bar{q}, H_{2003}) + 3B(H_{0001}, H_{2102}) + 3B(H_{0002}, H_{2101}) + B(H_{0003}, H_{2100}) \\
&+3B(H_{0101}, H_{2002}) + 3B(H_{0102}, H_{2001}) + B(H_{0103}, H_{2000}) + 6B(H_{1001}, H_{1102}) \\
&+6B(H_{1002}, H_{1101}) + 2B(H_{1003}, H_{1100}) + 3A_2(H_{2101}, K_{01}, K_{01}) \\
&+3A_2(H_{2100}, K_{01}, K_{02}) + 6B_1(q, H_{1102}, K_{01}) + 6B_1(q, H_{1101}, K_{02}) \\
&+2B_1(q, H_{1100}, K_{03}) + 3B_1(\bar{q}, H_{2002}, K_{01}) + 3B_1(\bar{q}, H_{2001}, K_{02}) \\
&+B_1(\bar{q}, H_{2000}, K_{03}) + 6B_1(H_{0001}, H_{2101}, K_{01}) + 3B_1(H_{0002}, H_{2100}, K_{01}) \\
&+6B_1(H_{0101}, H_{2001}, K_{01}) + 3B_1(H_{0102}, H_{2000}, K_{01}) + 12B_1(H_{1001}, H_{1101}, K_{01}) \\
&+6B_1(H_{1002}, H_{1100}, K_{01}) + 3B_1(H_{0001}, H_{2100}, K_{02}) + 3B_1(H_{0101}, H_{2000}, K_{02}) \\
&+6B_1(H_{1001}, H_{1100}, K_{02}) + C(q, q, H_{0103}) + 2C(q, \bar{q}, H_{1003}) \\
&+6C(q, H_{0001}, H_{1102}) + 6C(q, H_{0002}, H_{1101}) + 2C(q, H_{0003}, H_{1100}) \\
&+6C(q, H_{0101}, H_{1002}) + 6C(q, H_{0102}, H_{1001}) + 3C(\bar{q}, H_{0001}, H_{2002}) \\
&+3C(\bar{q}, H_{0002}, H_{2001}) + C(\bar{q}, H_{0003}, H_{2000}) + 6C(\bar{q}, H_{1001}, H_{1002}) \\
&+3C(H_{0001}, H_{0001}, H_{2101}) + 3C(H_{0001}, H_{0002}, H_{2100}) + 6C(H_{0001}, H_{0101}, H_{2001}) \\
&+3C(H_{0001}, H_{0102}, H_{2000}) + 12C(H_{0001}, H_{1001}, H_{1101}) \\
&+6C(H_{0001}, H_{1002}, H_{1100}) + 3C(H_{0002}, H_{0101}, H_{2000}) + 6C(H_{0002}, H_{1001}, H_{1100}) \\
&+6C(H_{0101}, H_{1001}, H_{1001}) + A_3(H_{2100}, K_{01}, K_{01}, K_{01}) \\
&+6B_2(q, H_{1101}, K_{01}, K_{01}) + 6B_2(q, H_{1100}, K_{01}, K_{02}) \\
&+3B_2(\bar{q}, H_{2001}, K_{01}, K_{01}) + 3B_2(\bar{q}, H_{2000}, K_{01}, K_{02}) \\
&+3B_2(H_{0001}, H_{2100}, K_{01}, K_{01}) + 3B_2(H_{0101}, H_{2000}, K_{01}, K_{01}) \\
&+6B_2(H_{1001}, H_{1100}, K_{01}, K_{01}) + C_1(q, q, \bar{q}, K_{03}) + 3C_1(q, q, H_{0102}, K_{01}) \\
&+3C_1(q, q, H_{0101}, K_{02}) + 6C_1(q, \bar{q}, H_{1002}, K_{01}) + 6C_1(q, \bar{q}, H_{1001}, K_{02}) \\
&+12C_1(q, H_{0001}, H_{1101}, K_{01}) + 6C_1(q, H_{0002}, H_{1100}, K_{01}) \\
&+12C_1(q, H_{0101}, H_{1001}, , K_{01}) + 6C_1(q, H_{0001}, H_{1100}, K_{02}) \\
&+6C_1(\bar{q}, H_{0001}, H_{2001}, K_{01}) + 3C_1(\bar{q}, H_{0002}, H_{2000}, K_{01}) \\
&+6C_1(\bar{q}, H_{1001}, H_{1001}, K_{01}) + 3C_1(\bar{q}, H_{0001}, H_{2000}, K_{02}) \\
&+3C_1(H_{0001}, H_{0001}, H_{2100}, K_{01}) + 6C_1(H_{0001}, H_{0101}, H_{2000}, K_{01}) \\
&+12C_1(H_{0001}, H_{1001}, H_{1100}, K_{01}) + D(q, q, \bar{q}, H_{0003}) + 3D(q, q, H_{0001}, H_{0102}) \\
&+3D(q, q, H_{0002}, H_{0101}) + 6D(q, \bar{q}, H_{0001}, H_{1002}) + 6D(q, \bar{q}, H_{0002}, H_{1001}) \\
&+6D(q, H_{0001}, H_{0001}, H_{1101}) + 6D(q, H_{0001}, H_{0002}, H_{1100}) + 12D(q, H_{0001}, H_{0101}, H_{1001}) \\
&+3D(\bar{q}, H_{0001}, H_{0001}, H_{2001}) + 3D(\bar{q}, H_{0001}, H_{0002}, H_{2000}) \\
&+6D(\bar{q}, H_{0001}, H_{1001}, H_{1001}) + D(H_{0001}, H_{0001}, H_{0001}, H_{2100}) \\
&+3D(H_{0001}, H_{0001}, H_{0101}, H_{2000}) + 6D(H_{0001}, H_{0001}, H_{1001}, H_{1100}) \\
&+2B_3(q, H_{1100}, K_{01}, K_{01}, K_{01}) + B_3(\bar{q}, H_{2000}, K_{01}, K_{01}, K_{01}) \\
&+3C_2(q, q, \bar{q}, K_{01}, K_{02}) + 3C_2(q, q, H_{0101}, K_{01}, K_{01}) \\
&+6C_2(q, \bar{q}, H_{1001}, K_{01}, K_{01}) + 6C_2(q, H_{0001}, H_{1100}, K_{01}, K_{01}) \\
&+3C_2(\bar{q}, H_{0001}, H_{2000}, K_{01}, K_{01}) + 3D_1(q, q, \bar{q}, H_{0002}, K_{01}) \\
&+3D_1(q, q, \bar{q}, H_{0001}, K_{02}) + 6D_1(q, q, H_{0001}, H_{0101}, K_{01}) \\
&+12D_1(q, \bar{q}, H_{0001}, H_{1001}, K_{01}) + 6D_1(q, H_{0001}, H_{0001}, H_{1100}, K_{01}) \\
&+3D_1(\bar{q}, H_{0001}, H_{0001}, H_{2000}, K_{01}) + 3E(q, q, \bar{q}, H_{0001}, H_{0002}) \\
&+3E(q, q, H_{0001}, H_{0001}, H_{0101}) + 6E(q, \bar{q}, H_{0001}, H_{0001}, H_{1001}) \\
&+2E(q, H_{0001}, H_{0001}, H_{0001}, H_{1100}) + E(\bar{q}, H_{0001}, H_{0001}, H_{0001}, H_{2000}) \\
&+C_3(q, q, \bar{q}, K_{01}, K_{01}, K_{01}) + 3E_1(q, q, \bar{q}, H_{0001}, H_{0001}, K_{01}) + K(q, q, \bar{q}, H_{0001}, H_{0001}, H_{0001}) \\
&-(2g_{2103}q + 6g_{2102}H_{1001} + 6g_{2101}H_{1002} + 2c_1(0)H_{1003} + (2g_{1003} + \overline{g}_{1003})H_{2100} \\
&+(6g_{1002} + 3\overline{g}_{1002})H_{2101} + (6g_{1001} + 3\overline{g}_{1001})H_{2102})
\end{aligned}
$$

A.7 Seventh order terms

Collecting seventh order terms of (4.1):

$$\begin{aligned}
w^7: \quad (7i\omega_0 I_n - A)H_{7000} &= 7B(q, H_{6000}) + 21B(H_{2000}, H_{5000}) + 35B(H_{3000}, H_{4000}) \\
&+ 21C(q, q, H_{5000}) + 105C(q, H_{2000}, H_{4000}) + 70C(q, H_{3000}, H_{3000}) \\
&+ 105C(H_{2000}, H_{2000}, H_{3000}) + 35D(q, q, q, H_{4000}) \\
&+ 210D(q, q, H_{2000}, H_{3000}) + 105D(q, H_{2000}, H_{2000}, H_{2000}) \\
&+ 35E(q, q, q, q, H_{3000}) + 105E(q, q, q, H_{2000}, H_{2000}) \\
&+ 21K(q, q, q, q, q, H_{2000}) + L(q, q, q, q, q, q, q)
\end{aligned}$$

$$\begin{aligned}
w^6\bar{w}: \quad (5i\omega_0 I_n - A)H_{6100} &= 6B(q, H_{5100}) + B(\bar{q}, H_{6000}) + 6B(H_{1100}, H_{5000}) \\
&+ 15B(H_{2000}, H_{4100}) + 15B(H_{2100}, H_{4000}) + 20B(H_{3000}, H_{3100}) \\
&+ 15C(q, q, H_{4100}) + 6C(q, \bar{q}, H_{5000}) + 30C(q, H_{1100}, H_{4000}) \\
&+ 60C(q, H_{2000}, H_{3100}) + 60C(q, H_{2100}, H_{3000}) + 15C(\bar{q}, H_{2000}, H_{4000}) \\
&+ 10C(\bar{q}, H_{3000}, H_{3000}) + 60C(H_{1100}, H_{2000}, H_{3000}) \\
&+ 45C(H_{2000}, H_{2000}, H_{2100}) + 20D(q, q, q, H_{3100}) + 15D(q, q, \bar{q}, H_{4000}) \\
&+ 60D(q, q, H_{1100}, H_{3000}) + 90D(q, q, H_{2000}, H_{2100}) \\
&+ 60D(q, \bar{q}, H_{2000}, H_{3000}) + 90D(q, H_{1100}, H_{2000}, H_{2000}) \\
&+ 15D(\bar{q}, H_{2000}, H_{2000}, H_{2000}) + 15E(q, q, q, q, H_{2100}) \\
&+ 20E(q, q, q, \bar{q}, H_{3000}) + 60E(q, q, q, H_{1100}, H_{2000}) \\
&+ 45E(q, q, \bar{q}, H_{2000}, H_{2000}) + 6K(q, q, q, q, q, H_{1100}) \\
&+ 15K(q, q, q, q, \bar{q}, H_{2000}) + L(q, q, q, q, q, q, \bar{q}) - 30c_1(0)H_{5000}
\end{aligned}$$

$$\begin{aligned}
w^5\bar{w}^2: \quad (3i\omega_0 I_n - A)H_{5200} &= 5B(q, H_{4200}) + 2B(\bar{q}, H_{5100}) \\
&+ B(H_{0200}, H_{5000}) + 10B(H_{1100}, H_{4100}) + 5B(H_{1200}, H_{4000}) \\
&+ 10B(H_{2000}, H_{3200}) + 20B(H_{2100}, H_{3100}) + 10B(H_{2200}, H_{3000}) \\
&+ 10C(q, q, H_{3200}) + 10C(q, \bar{q}, H_{4100}) + 5C(q, H_{0200}, H_{4000}) \\
&+ 40C(q, H_{1100}, H_{3100}) + 20C(q, H_{1200}, H_{3000}) \\
&+ 30C(q, H_{2000}, H_{2200}) + 30C(q, H_{2100}, H_{2100}) + C(\bar{q}, \bar{q}, H_{5000}) \\
&+ 10C(\bar{q}, H_{1100}, H_{4000}) + 20C(\bar{q}, H_{2000}, H_{3100}) + 20C(\bar{q}, H_{2100}, H_{3000}) \\
&+ 10C(H_{0200}, H_{2000}, H_{3000}) + 20C(H_{1100}, H_{1100}, H_{3000}) \\
&+ 60C(H_{1100}, H_{2000}, H_{2100}) + 15C(H_{1200}, H_{2000}, H_{2000}) \\
&+ 10D(q, q, q, H_{2200}) + 20D(q, q, \bar{q}, H_{3100}) + 10D(q, q, H_{0200}, H_{3000}) \\
&+ 60D(q, q, H_{1100}, H_{2100}) + 30D(q, q, H_{1200}, H_{2000}) \\
&+ 5D(q, \bar{q}, \bar{q}, H_{4000}) + 40D(q, \bar{q}, H_{1100}, H_{3000}) + 60D(q, \bar{q}, H_{2000}, H_{2100}) \\
&+ 15D(q, H_{0200}, H_{2000}, H_{2000}) + 60D(q, H_{1100}, H_{1100}, H_{2000}) \\
&+ 10D(\bar{q}, \bar{q}, H_{2000}, H_{3000}) + 30D(\bar{q}, H_{1100}, H_{2000}, H_{2000}) \\
&+ 5E(q, q, q, q, H_{1200}) + 20E(q, q, q, \bar{q}, H_{2100}) + 10E(q, q, q, H_{0200}, H_{2000}) \\
&+ 20E(q, q, q, H_{1100}, H_{1100}) + 10E(q, q, \bar{q}, \bar{q}, H_{3000}) \\
&+ 60E(q, q, \bar{q}, H_{1100}, H_{2000}) + 15E(q, \bar{q}, \bar{q}, H_{2000}, H_{2000}) \\
&+ K(q, q, q, q, q, H_{0200}) + 10K(q, q, q, q, \bar{q}, H_{1100}) + 10K(q, q, q, \bar{q}, \bar{q}, H_{2000}) \\
&+ L(q, q, q, q, q, \bar{q}, \bar{q}) - [120c_2(0)H_{3000} + (40c_1(0) + 10\overline{c_1}(0))H_{4100}]
\end{aligned}$$

$$
\begin{aligned}
w^4\bar{w}^3: \quad (A - i\omega_0 I_n)H_{4300} &= 144c_3(0)q + 72(2c_2(0) + \overline{c_2}(0))H_{2100} + 12(3c_1(0) + 2\overline{c_1}(0))H_{3200} \\
&- [4B(q, H_{3300}) + 3B(\bar{q}, H_{4200}) + 3B(H_{0200}, H_{4100}) + B(H_{0300}, H_{4000}) \\
&+ 12B(H_{1100}, H_{3200}) + 12B(H_{1200}, H_{3100}) + 4B(H_{1300}, H_{3000}) \\
&+ 6B(H_{2000}, H_{2300}) + 18B(H_{2100}, H_{2200}) + 6C(q, q, H_{2300}) \\
&+ 12C(q, \bar{q}, H_{3200}) + 12C(q, H_{0200}, H_{3100}) + 4C(q, H_{0300}, H_{3000}) \\
&+ 36C(q, H_{1100}, H_{2200}) + 36C(q, H_{1200}, H_{2100}) + 12C(q, H_{1300}, H_{2000}) \\
&+ 3C(\bar{q}, \bar{q}, H_{4100}) + 3C(\bar{q}, H_{0200}, H_{4000}) + 24C(\bar{q}, H_{1100}, H_{3100}) \\
&+ 12C(\bar{q}, H_{1200}, H_{3000}) + 18C(\bar{q}, H_{2000}, H_{2200}) + 18C(\bar{q}, H_{2100}, H_{2100}) \\
&+ 12C(H_{0200}, H_{1100}, H_{3000}) + 18C(H_{0200}, H_{2000}, H_{2100}) \\
&+ 3C(H_{0300}, H_{2000}, H_{2000}) + 36C(H_{1100}, H_{1100}, H_{2100}) \\
&+ 36C(H_{1100}, H_{1200}, H_{2000}) + 4D(q, q, q, H_{1300}) + 18D(q, q, \bar{q}, H_{2200}) \\
&+ 18D(q, q, H_{0200}, H_{2100}) + 6D(q, q, H_{0300}, H_{2000}) + 36D(q, q, H_{1100}, H_{1200}) \\
&+ 12D(q, \bar{q}, \bar{q}, H_{3100}) + 12D(q, \bar{q}, H_{0200}, H_{3000}) + 72D(q, \bar{q}, H_{1100}, H_{2100}) \\
&+ 36D(q, \bar{q}, H_{1200}, H_{2000}) + 36D(q, H_{0200}, H_{1100}, H_{2000}) \\
&+ 24D(q, H_{1100}, H_{1100}, H_{1100}) + D(\bar{q}, \bar{q}, \bar{q}, H_{4000}) + 12D(\bar{q}, \bar{q}, H_{1100}, H_{3000}) \\
&+ 18D(\bar{q}, \bar{q}, H_{2000}, H_{2100}) + 9D(\bar{q}, H_{0200}, H_{2000}, H_{2000}) \\
&+ 36D(\bar{q}, H_{1100}, H_{1100}, H_{2000}) + E(q, q, q, q, H_{0300}) + 12E(q, q, q, \bar{q}, H_{1200}) \\
&+ 12E(q, q, q, H_{0200}, H_{1100}) + 18E(q, q, \bar{q}, \bar{q}, H_{2100}) + 18E(q, q, \bar{q}, H_{0200}, H_{2000}) \\
&+ 36E(q, q, \bar{q}, H_{1100}, H_{1100}) + 4E(q, \bar{q}, \bar{q}, \bar{q}, H_{3000}) + 36E(q, \bar{q}, \bar{q}, H_{1100}, H_{2000}) \\
&+ 3E(\bar{q}, \bar{q}, \bar{q}, H_{2000}, H_{2000}) + 3K(q, q, q, q, \bar{q}, H_{0200}) + 12K(q, q, q, \bar{q}, \bar{q}, H_{1100}) \\
&+ 6K(q, q, \bar{q}, \bar{q}, \bar{q}, H_{2000}) + L(q, q, q, q, \bar{q}, \bar{q}, \bar{q})]
\end{aligned}
$$

Appendix B

Remaining coefficients for the center manifold approximmation

In this appendix, we present the coefficients of the center manifold approximation H that were not needed in the computation of the normal form coefficients, i.e. the coefficients marked blue in Figure 4.1. They need to be included in predidic orbit approximation as discussed in Section 3.2.

B.1 Coefficients for ODEs

B.1.1 Parameter-independent coefficients

For our center manifold approximation, we also need the following parameter-independent coefficients whose expressions can be found by respectively collecting the w^5,w^6,$w^5\bar{w}$, w^7, $w^6\bar{w}$,$w^5\bar{w}^2$ terms. The resulting equations are

$$\begin{aligned}
H_{5000} &= (5i\omega_0 I - A)^{-1}[5B(q, H_{4000}) + 10B(H_{2000}, H_{3000}) + 10C(q, q, H_{3000}) \\
&\quad + 15C(q, H_{2000}, H_{2000}) + 10D(q, q, q, H_{2000}) + E(q, q, q, q, q)], \\
H_{6000} &= (6i\omega_0 I_n - A)^{-1}[6B(q, H_{5000}) + 15B(H_{2000}, H_{4000}) + 10B(H_{3000}, H_{3000}) \\
&\quad + 15C(q, q, H_{4000}) + 60C(q, H_{2000}, H_{3000}) + 15C(H_{2000}, H_{2000}, H_{2000}) \\
&\quad + 20D(q, q, q, H_{3000}) + 45D(q, q, H_{2000}.H_{2000}) + 15E(q, q, q, q, H_{2000}) \\
&\quad + K(q, q, q, q, q, q)],
\end{aligned}$$

$$\begin{aligned}
H_{5100} &= (4i\omega_0 I - A)^{-1}[5B(q, H_{4100}) + B(\bar{q}, H_{5000}) + 5B(H_{1100}, H_{4000}) \\
&\quad + 10B(H_{2000}, H_{3100}) + 10B(H_{2100}, H_{3000}) + 10C(q, q, H_{3100}) + 5C(q, \bar{q}, H_{4000}) \\
&\quad + 20C(q, H_{1100}, H_{3000}) + 30C(q, H_{2000}, H_{2100}) + 10C(\bar{q}, H_{2000}, H_{3000}) \\
&\quad + 15C(H_{1100}, H_{2000}, H_{2000}) + 10D(q, q, q, H_{2100}) + 10D(q, q, \bar{q}, H_{3000}) \\
&\quad + 30D(q, q, H_{1100}, H_{2000}) + 15D(q, \bar{q}, H_{2000}, H_{2000}) + 5E(q, q, q, q, H_{1100}) \\
&\quad + 10E(q, q, q, \bar{q}, H_{2000}) + K(q, q, q, q, q, \bar{q}) - 20c_1(0)H_{4000}],
\end{aligned}$$

$$
\begin{aligned}
H_{7000} = (7i\omega_0 I - A)^{-1}[&7B(q, H_{6000}) + 21B(H_{2000}, H_{5000}) + 35B(H_{3000}, H_{4000}) \\
&+ 21C(q, q, H_{5000}) + 105C(q, H_{2000}, H_{4000}) + 70C(q, H_{3000}, H_{3000}) \\
&+ 105C(H_{2000}, H_{2000}, H_{3000}) + 35D(q, q, q, H_{4000}) \\
&+ 210D(q, q, H_{2000}, H_{3000}) + 105D(q, H_{2000}, H_{2000}, H_{2000}) \\
&+ 35E(q, q, q, q, H_{3000}) + 105E(q, q, q, H_{2000}, H_{2000}) \\
&+ 21K(q, q, q, q, q, H_{2000}) + L(q, q, q, q, q, q, q)],
\end{aligned}
$$

$$
\begin{aligned}
H_{6100} = (5i\omega_0 I - A)^{-1}[&6B(q, H_{5100}) + B(\bar{q}, H_{6000}) + 6B(H_{1100}, H_{5000}) \\
&+ 15B(H_{2000}, H_{4100}) + 15B(H_{2100}, H_{4000}) + 20B(H_{3000}, H_{3100}) \\
&+ 15C(q, q, H_{4100}) + 6C(q, \bar{q}, H_{5000}) + 30C(q, H_{1100}, H_{4000}) \\
&+ 60C(q, H_{2000}, H_{3100}) + 60C(q, H_{2100}, H_{3000}) + 15C(\bar{q}, H_{2000}, H_{4000}) \\
&+ 10C(\bar{q}, H_{3000}, H_{3000}) + 60C(H_{1100}, H_{2000}, H_{3000}) \\
&+ 45C(H_{2000}, H_{2000}, H_{2100}) + 20D(q, q, q, H_{3100}) + 15D(q, q, \bar{q}, H_{4000}) \\
&+ 60D(q, q, H_{1100}, H_{3000}) + 90D(q, q, H_{2000}, H_{2100}) \\
&+ 60D(q, \bar{q}, H_{2000}, H_{3000}) + 90D(q, H_{1100}, H_{2000}, H_{2000}) \\
&+ 15D(\bar{q}, H_{2000}, H_{2000}, H_{2000}) + 15E(q, q, q, q, H_{2100}) \\
&+ 20E(q, q, q, \bar{q}, H_{3000}) + 60E(q, q, q, H_{1100}, H_{2000}) \\
&+ 45E(q, q, \bar{q}, H_{2000}, H_{2000}) + 6K(q, q, q, q, q, H_{1100}) \\
&+ 15K(q, q, q, q, \bar{q}, H_{2000}) + L(q, q, q, q, q, q, \bar{q}) - 30c_1(0)H_{5000}],
\end{aligned}
$$

$$
\begin{aligned}
H_{5200} = (3i\omega_0 I - A)^{-1}[&5B(q, H_{4200}) + 2B(\bar{q}, H_{5100}) \\
&+ B(\overline{H}_{2000}, H_{5000}) + 10B(H_{1100}, H_{4100}) + 5B(\overline{H}_{2100}, H_{4000}) \\
&+ 10B(H_{2000}, H_{3200}) + 20B(H_{2100}, H_{3100}) + 10B(H_{2200}, H_{3000}) \\
&+ 10C(q, q, H_{3200}) + 10C(q, \bar{q}, H_{4100}) + 5C(q, \overline{H}_{2000}, H_{4000}) \\
&+ 40C(q, H_{1100}, H_{3100}) + 20C(q, \overline{H}_{2100}, H_{3000}) \\
&+ 30C(q, H_{2000}, H_{2200}) + 30C(q, H_{2100}, H_{2100}) + C(\bar{q}, \bar{q}, H_{5000}) \\
&+ 10C(\bar{q}, H_{1100}, H_{4000}) + 20C(\bar{q}, H_{2000}, H_{3100}) + 20C(\bar{q}, H_{2100}, H_{3000}) \\
&+ 10C(\overline{H}_{2000}, H_{2000}, H_{3000}) + 20C(H_{1100}, H_{1100}, H_{3000}) \\
&+ 60C(H_{1100}, H_{2000}, H_{2100}) + 15C(\overline{H}_{2100}, H_{2000}, H_{2000}) \\
&+ 10D(q, q, q, H_{2200}) + 20D(q, q, \bar{q}, H_{3100}) + 10D(q, q, \overline{H}_{2000}, H_{3000}) \\
&+ 60D(q, q, H_{1100}, H_{2100}) + 30D(q, q, \overline{H}_{2100}, H_{2000}) \\
&+ 5D(q, \bar{q}, \bar{q}, H_{4000}) + 40D(q, \bar{q}, H_{1100}, H_{3000}) + 60D(q, \bar{q}, H_{2000}, H_{2100}) \\
&+ 15D(q, \overline{H}_{2000}, H_{2000}, H_{2000}) + 60D(q, H_{1100}, H_{1100}, H_{2000}) \\
&+ 10D(\bar{q}, \bar{q}, H_{2000}, H_{3000}) + 30D(\bar{q}, H_{1100}, H_{2000}, H_{2000}) \\
&+ 5E(q, q, q, q, \overline{H}_{2100}) + 20E(q, q, q, \bar{q}, H_{2100}) + 10E(q, q, q, \overline{H}_{2000}, H_{2000}) \\
&+ 20E(q, q, q, H_{1100}, H_{1100}) + 10E(q, q, \bar{q}, \bar{q}, H_{3000}) \\
&+ 60E(q, q, \bar{q}, H_{1100}, H_{2000}) + 15E(q, \bar{q}, \bar{q}, H_{2000}, H_{2000}) \\
&+ K(q, q, q, q, q, \overline{H}_{2000}) + 10K(q, q, q, q, \bar{q}, H_{1100}) + 10K(q, q, q, \bar{q}, \bar{q}, H_{2000}) \\
&+ L(q, q, q, q, q, \bar{q}, \bar{q}) - (120c_2(0)H_{3000} + (40c_1(0) + 10\overline{c_1}(0))H_{4100})].
\end{aligned}
$$

B.1.2 Parameter-dependent coefficients

We also need the following parameter-dependent coefficients

$$\begin{aligned}H_{3010} = (3i\omega_0 I_n - A)^{-1}[&A_1(H_{3000}, K_{10}) + 3B(q, H_{2010}) + B(H_{0010}, H_{3000}) \\ &+ 3B(H_{1010}, H_{2000}) + 3B_1(q, H_{2000}, K_{10}) + 3C(q, q, H_{1010}) + 3C(q, H_{0010}, H_{2000}) \\ &+ C_1(q, q, q, K_{10}) + D(q, q, q, H_{0010}) - 3(1 + ib_{1,10})H_{3000}],\end{aligned}$$

$$\begin{aligned}H_{4001} = (4i\omega_0 I_n - A)^{-1}[&A_1(H_{4000}, K_{01}) + 4B(q, H_{3001}) + B(H_{0001}, H_{4000}) \\ &+ 4B(H_{1001}, H_{3000}) + 6B(H_{2000}, H_{2001}) + 4B_1(q, H_{3000}, K_{01}) \\ &+ 3B_1(H_{2000}, H_{2000}, K_{01}) + 6C(q, q, H_{2001}) + 4C(q, H_{0001}, H_{3000}) \\ &+ 12C(q, H_{1001}, H_{2000}) + 3C(H_{0001}, H_{2000}, H_{2000}) \\ &+ 6C_1(q, q, H_{2000}, K_{01}) + 4D(q, q, q, H_{1001}) + 6D(q, q, H_{0001}, H_{2000}) \\ &+ D_1(q, q, q, q, K_{01}) + E(q, q, q, q, H_{0001}) - 4ib_{1,01}H_{4000}],\end{aligned}$$

$$\begin{aligned}H_{5001} = (5i\omega_0 I_n - A)^{-1}[&A_1(H_{5000}, K_{01}) + 5B(q, H_{4001}) + B(H_{0001}, H_{5000}) \\ &+ 5B(H_{1001}, H_{4000}) + 10B(H_{2000}, H_{3001}) + 10B(H_{2001}, H_{3000}) \\ &+ 5B_1(q, H_{4000}, K_{01}) + 10B_1(H_{2000}, H_{3000}, K_{01}) + 10C(q, q, H_{3001}) \\ &+ 5C(q, H_{0001}, H_{4000}) + 20C(q, H_{1001}, H_{3000}) + 30C(q, H_{2000}, H_{2001}) \\ &+ 10C(H_{0001}, H_{2000}, H_{3000}) + 15C(H_{1001}, H_{2000}, H_{2000}) \\ &+ 10C_1(q, q, H_{3000}, K_{01}) + 15C_1(q, H_{2000}, H_{2000}, K_{01}) \\ &+ 10D(q, q, q, H_{2001}) + 10D(q, q, H_{0001}, H_{3000}) + 30D(q, q, H_{1001}, H_{2000}) \\ &+ 15D(q, H_{0001}, H_{2000}, H_{2000}) + 10D_1(q, q, q, H_{2000}, K_{01}) \\ &+ 5E(q, q, q, q, H_{1001}) + 10E(q, q, q, H_{0001}, H_{2000}) \\ &+ E_1(q, q, q, q, q, K_{01}) + K(q, q, q, q, q, H_{0001}) - 5ib_{1,01}H_{5000}],\end{aligned}$$

$$
\begin{aligned}
H_{4101} = (3i\omega_0 I_n - A)^{-1}[&A_1(H_{4100}, K_{01}) + 4B(q, H_{3101}) + B(\bar{q}, H_{4001}) \\
&+ B(H_{0001}, H_{4100}) + B(\overline{H}_{1001}, H_{4000}) + 4B(H_{1001}, H_{3100}) \\
&+ 4B(H_{1100}, H_{3001}) + 4B(H_{1101}, H_{3000}) + 6B(H_{2000}, H_{2101}) \\
&+ 6B(H_{2001}, H_{2100}) + 4B_1(q, H_{3100}, K_{01}) + B_1(\bar{q}, H_{4000}, K_{01}) \\
&+ 4B_1(H_{1100}, H_{3000}, K_{01}) + 6B_1(H_{2000}, H_{2100}, K_{01}) + 6C(q, q, H_{2101}) \\
&+ 4C(q, \bar{q}, H_{3001}) + 4C(q, H_{0001}, H_{3100}) + 4C(q, \overline{H}_{1001}, H_{3000}) \\
&+ 12C(q, H_{1001}, H_{2100}) + 12C(q, H_{1100}, H_{2001}) + 12C(q, H_{1101}, H_{2000}) \\
&+ C(\bar{q}, H_{0001}, H_{4000}) + 4C(\bar{q}, H_{1001}, H_{3000}) + 6C(\bar{q}, H_{2000}, H_{2001}) \\
&+ 4C(H_{0001}, H_{1100}, H_{3000}) + 6C(H_{0001}, H_{2000}, H_{2100}) \\
&+ 3C(\overline{H}_{1001}, H_{2000}, H_{2000}) + 12C(H_{1001}, H_{1100}, H_{2000}) \\
&+ 6C_1(q, q, H_{2100}, K_{01}) + 4C_1(q, \bar{q}, H_{3000}, K_{01}) \\
&+ 12C_1(q, H_{1100}, H_{2000}, K_{01}) + 3C_1(\bar{q}, H_{2000}, H_{2000}, K_{01}) \\
&+ 4D(q, q, q, H_{1101}) + 6D(q, q, \bar{q}, H_{2001}) + 6D(q, q, H_{0001}, H_{2100}) \\
&+ 6D(q, q, \overline{H}_{1001}, H_{2000}) + 12D(q, q, H_{1001}, H_{1100}) \\
&+ 4D(q, \bar{q}, H_{0001}, H_{3000}) + 12D(q, \bar{q}, H_{1001}, H_{2000}) \\
&+ 12D(q, H_{0001}, H_{1100}, H_{2000}) + 3D(\bar{q}, H_{0001}, H_{2000}, H_{2000}) \\
&+ 4D_1(q, q, q, H_{1100}, K_{01}) + 6D_1(q, q, \bar{q}, H_{2000}, K_{01}) \\
&+ E(q, q, q, q, \overline{H}_{1001}) + 4E(q, q, q, \bar{q}, H_{1001}) + 4E(q, q, q, H_{0001}, H_{1100}) \\
&+ 6E(q, q, \bar{q}, H_{0001}, H_{2000}) + E_1(q, q, q, q, \bar{q}, K_{01}) + K(q, q, q, q, \bar{q}, H_{0001}) \\
&- (12(1 + ib_{2,01})H_{3000} + 12c_1(0)H_{3001} + 3ib_{1,01}H_{4100})].
\end{aligned}
$$

Finally, we need the coefficient H_{3002} which can be derived by collecting the $w^3\beta_2^2$ terms. This yields the equation

$$
\begin{aligned}
H_{3002} = (3i\omega_0 I - A)^{-1}[&2A_1(H_{3001}, K_{01}) + A_1(H_{3000}, K_{02}) + 3B(q, H_{2002}) \\
&+ 2B(H_{0001}, H_{3001}) + B(H_{0002}, H_{3000}) + 6B(H_{1001}, H_{2001}) \\
&+ 3B(H_{1002}, H_{2000}) + A_2(H_{3000}, K_{01}, K_{01}) + 6B_1(q, H_{2001}, K_{01}) \\
&+ 3B_1(q, H_{2000}, K_{02}) + 2B_1(H_{0001}, H_{3000}, K_{01}) + 6B_1(H_{1001}, H_{2000}, K_{01}) \\
&+ 3C(q, q, H_{1002}) + 6C(q, H_{0001}, H_{2001}) + 3C(q, H_{0002}, H_{2000}) \\
&+ 6C(q, H_{1001}, H_{1001}) + C(H_{0001}, H_{0001}, H_{3000}) \\
&+ 6C(H_{0001}, H_{1001}, H_{2000}) + 3B_2(q, H_{2000}, K_{01}, K_{01}) + C_1(q, q, q, K_{02}) \\
&+ 6C_1(q, q, H_{1001}, K_{01}) + 6C_1(q, H_{0001}, H_{2000}, K_{01}) + D(q, q, q, H_{0002}) \\
&+ 6D(q, q, H_{0001}, H_{1001}) + 3D(q, H_{0001}, H_{0001}, H_{2000}) \\
&+ C_2(q, q, q, K_{01}, K_{01}) + 2D_1(q, q, q, H_{0001}, K_{01}) \\
&+ E(q, q, q, H_{0001}, H_{0001}) - (3ib_{1,02}H_{3000} + 6ib_{1,01}H_{3001})].
\end{aligned}
$$

B.2 Coefficients for DDEs

B.2.1 Parameter-independent coefficients

For our center manifold approximation, we also need the following parameter-independent coefficients

$$\begin{aligned}
H_{5000}(\theta) = {} & e^{5i\omega_0\theta}\Delta^{-1}(5i\omega_0)[5B(\varphi, H_{4000}) + 10B(H_{2000}, H_{3000}) + 10C(\varphi, \varphi, H_{3000}) \\
& + 15C(\varphi, H_{2000}, H_{2000}) + 10D(\varphi, \varphi, \varphi, H_{2000}) + E(\varphi, \varphi, \varphi, \varphi, \varphi)], \\
H_{6000}(\theta) = {} & e^{6i\omega_0\theta}\Delta^{-1}(6i\omega_0)[6B(\varphi, H_{5000}) + 15B(H_{2000}, H_{4000}) + 10B(H_{3000}, H_{3000}) \\
& + 15C(\varphi, \varphi, H_{4000}) + 60C(\varphi, H_{2000}, H_{3000}) + 15C(H_{2000}, H_{2000}, H_{2000}) \\
& + 20D(\varphi, \varphi, \varphi, H_{3000}) + 45D(\varphi, \varphi, H_{2000}.H_{2000}) + 15E(\varphi, \varphi, \varphi, \varphi, H_{2000}) \\
& + K(\varphi, \varphi, \varphi, \varphi, \varphi, \varphi)],
\end{aligned}$$

$$\begin{aligned}
H_{5100}(\theta) = {} & e^{4i\omega_0\theta}\Delta^{-1}(4i\omega_0)[5B(\varphi, H_{4100}) + B(\bar{\varphi}, H_{5000}) + 5B(H_{1100}, H_{4000}) \\
& + 10B(H_{2000}, H_{3100}) + 10B(H_{2100}, H_{3000}) + 10C(\varphi, \varphi, H_{3100}) + 5C(\varphi, \bar{\varphi}, H_{4000}) \\
& + 20C(\varphi, H_{1100}, H_{3000}) + 30C(\varphi, H_{2000}, H_{2100}) + 10C(\bar{\varphi}, H_{2000}, H_{3000}) \\
& + 15C(H_{1100}, H_{2000}, H_{2000}) + 10D(\varphi, \varphi, \varphi, H_{2100}) + 10D(\varphi, \varphi, \bar{\varphi}, H_{3000}) \\
& + 30D(\varphi, \varphi, H_{1100}, H_{2000}) + 15D(\varphi, \bar{\varphi}, H_{2000}, H_{2000}) + 5E(\varphi, \varphi, \varphi, \varphi, H_{1100}) \\
& + 10E(\varphi, \varphi, \varphi, \bar{\varphi}, H_{2000}) + K(\varphi, \varphi, \varphi, \varphi, \varphi, \bar{\varphi})] \\
& - 20c_1(0)\Delta^{-1}(4i\omega_0)[\Delta'(4i\omega_0) - \theta\Delta(4i\omega_0)]H_{4000}(\theta),
\end{aligned}$$

$$\begin{aligned}
H_{7000}(\theta) = {} & e^{7i\omega_0\theta}\Delta^{-1}(7i\omega_0)[7B(\varphi, H_{6000}) + 21B(H_{2000}, H_{5000}) + 35B(H_{3000}, H_{4000}) \\
& + 21C(\varphi, \varphi, H_{5000}) + 105C(\varphi, H_{2000}, H_{4000}) + 70C(\varphi, H_{3000}, H_{3000}) \\
& + 105C(H_{2000}, H_{2000}, H_{3000}) + 35D(\varphi, \varphi, \varphi, H_{4000}) \\
& + 210D(\varphi, \varphi, H_{2000}, H_{3000}) + 105D(\varphi, H_{2000}, H_{2000}, H_{2000}) \\
& + 35E(\varphi, \varphi, \varphi, \varphi, H_{3000}) + 105E(\varphi, \varphi, \varphi, H_{2000}, H_{2000}) \\
& + 21K(\varphi, \varphi, \varphi, \varphi, \varphi, H_{2000}) + L(\varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \varphi)],
\end{aligned}$$

$$\begin{aligned}
H_{6100}(\theta) = {} & e^{5i\omega_0\theta}\Delta^{-1}(5i\omega_0)[6B(q, H_{5100}) + B(\bar{\varphi}, H_{6000}) + 6B(H_{1100}, H_{5000}) \\
& + 15B(H_{2000}, H_{4100}) + 15B(H_{2100}, H_{4000}) + 20B(H_{3000}, H_{3100}) \\
& + 15C(\varphi, \varphi, H_{4100}) + 6C(\varphi, \bar{\varphi}, H_{5000}) + 30C(\varphi, H_{1100}, H_{4000}) \\
& + 60C(\varphi, H_{2000}, H_{3100}) + 60C(\varphi, H_{2100}, H_{3000}) + 15C(\bar{\varphi}, H_{2000}, H_{4000}) \\
& + 10C(\bar{\varphi}, H_{3000}, H_{3000}) + 60C(H_{1100}, H_{2000}, H_{3000}) \\
& + 45C(H_{2000}, H_{2000}, H_{2100}) + 20D(\varphi, \varphi, \varphi, H_{3100}) + 15D(\varphi, \varphi, \bar{\varphi}, H_{4000}) \\
& + 60D(\varphi, \varphi, H_{1100}, H_{3000}) + 90D(\varphi, \varphi, H_{2000}, H_{2100}) \\
& + 60D(\varphi, \bar{\varphi}, H_{2000}, H_{3000}) + 90D(\varphi, H_{1100}, H_{2000}, H_{2000}) \\
& + 15D(\bar{\varphi}, H_{2000}, H_{2000}, H_{2000}) + 15E(\varphi, \varphi, \varphi, \varphi, H_{2100}) \\
& + 20E(\varphi, \varphi, \varphi, \bar{\varphi}, H_{3000}) + 60E(\varphi, \varphi, \varphi, H_{1100}, H_{2000}) \\
& + 45E(\varphi, \varphi, \bar{\varphi}, H_{2000}, H_{2000}) + 6K(\varphi, \varphi, \varphi, \varphi, \varphi, H_{1100}) \\
& + 15K(\varphi, \varphi, \varphi, \varphi, \bar{\varphi}, H_{2000}) + L(\varphi, \varphi, \varphi, \varphi, \varphi, \varphi, \bar{\varphi})] \\
& - 30c_1(0)\Delta^{-1}(5i\omega_0)[\Delta'(5i\omega_0) - \theta\Delta(5i\omega_0)]H_{5000}(\theta),
\end{aligned}$$

$$
\begin{aligned}
H_{5200}(\theta) = {} & e^{3i\omega_0\theta}\Delta^{-1}(3i\omega_0)[5B(\varphi, H_{4200}) + 2B(\bar{\varphi}, H_{5100}) \\
& + B(\overline{H}_{2000}, H_{5000}) + 10B(H_{1100}, H_{4100}) + 5B(\overline{H}_{2100}, H_{4000}) \\
& + 10B(H_{2000}, H_{3200}) + 20B(H_{2100}, H_{3100}) + 10B(H_{2200}, H_{3000}) \\
& + 10C(\varphi, \varphi, H_{3200}) + 10C(\varphi, \bar{\varphi}, H_{4100}) + 5C(\varphi, \overline{H}_{2000}, H_{4000}) \\
& + 40C(\varphi, H_{1100}, H_{3100}) + 20C(\varphi, \overline{H}_{2100}, H_{3000}) \\
& + 30C(\varphi, H_{2000}, H_{2200}) + 30C(\varphi, H_{2100}, H_{2100}) + C(\bar{\varphi}, \bar{\varphi}, H_{5000}) \\
& + 10C(\bar{\varphi}, H_{1100}, H_{4000}) + 20C(\bar{\varphi}, H_{2000}, H_{3100}) + 20C(\bar{\varphi}, H_{2100}, H_{3000}) \\
& + 10C(\overline{H}_{2000}, H_{2000}, H_{3000}) + 20C(H_{1100}, H_{1100}, H_{3000}) \\
& + 60C(H_{1100}, H_{2000}, H_{2100}) + 15C(\overline{H}_{2100}, H_{2000}, H_{2000}) \\
& + 10D(\varphi, \varphi, \varphi, H_{2200}) + 20D(\varphi, \varphi, \bar{\varphi}, H_{3100}) + 10D(\varphi, \varphi, \overline{H}_{2000}, H_{3000}) \\
& + 60D(\varphi, \varphi, H_{1100}, H_{2100}) + 30D(\varphi, \varphi, \overline{H}_{2100}, H_{2000}) \\
& + 5D(\varphi, \bar{\varphi}, \bar{\varphi}, H_{4000}) + 40D(\varphi, \bar{\varphi}, H_{1100}, H_{3000}) + 60D(\varphi, \bar{\varphi}, H_{2000}, H_{2100}) \\
& + 15D(\varphi, \overline{H}_{2000}, H_{2000}, H_{2000}) + 60D(\varphi, H_{1100}, H_{1100}, H_{2000}) \\
& + 10D(\bar{\varphi}, \bar{\varphi}, H_{2000}, H_{3000}) + 30D(\bar{\varphi}, H_{1100}, H_{2000}, H_{2000}) \\
& + 5E(\varphi, \varphi, \varphi, \varphi, \overline{H}_{2100}) + 20E(\varphi, \varphi, \varphi, \bar{\varphi}, H_{2100}) + 10E(\varphi, \varphi, \varphi, \overline{H}_{2000}, H_{2000}) \\
& + 20E(\varphi, \varphi, \varphi, H_{1100}, H_{1100}) + 10E(\varphi, \varphi, \bar{\varphi}, \bar{\varphi}, H_{3000}) \\
& + 60E(\varphi, \varphi, \bar{\varphi}, H_{1100}, H_{2000}) + 15E(\varphi, \bar{\varphi}, \bar{\varphi}, H_{2000}, H_{2000}) \\
& + K(\varphi, \varphi, \varphi, \varphi, \varphi, \overline{H}_{2000}) + 10K(\varphi, \varphi, \varphi, \varphi, \bar{\varphi}, H_{1100}) + 10K(\varphi, \varphi, \varphi, \bar{\varphi}, \bar{\varphi}, H_{2000}) \\
& + L(\varphi, \varphi, \varphi, \varphi, \varphi, \bar{\varphi}, \bar{\varphi})] \\
& - 120c_2(0)\Delta^{-1}(3i\omega_0)[\Delta'(3i\omega_0) - \theta\Delta(3i\omega_0)]H_{3000}(\theta) \\
& - (40c_1(0) + 10\overline{c_1}(0))e^{3i\omega_0\theta}\Delta^{-1}(3i\omega_0)\Big([\Delta'(3i\omega_0) - \theta\Delta(3i\omega_0)]H_{4100}(0) \\
& + 6c_1(0)[\Delta''(3i\omega_0) - \theta^2\Delta(3i\omega_0)]H_{3000}(0)\Big),
\end{aligned}
$$

$$
\begin{aligned}
H_{4300}(\theta) = {} & B^{INV}_{i\omega_0}(M_{4300}, -144c_3(0))(\theta) - 72(2c_2(0) + \overline{c_2}(0))\tilde{B}^{INV}_{i\omega_0}(H_{2100}, -2c_1(0))(\theta) \\
& - 12i\Im\{c_1(0)\}\hat{B}^{INV}_{i\omega_0}(H_{3200}, -[12c_2(0)q + 6i\Im(c_1(0))\, H_{2100}(0)], 12i\Im(c_1(0))\, c_1(0))(\theta).
\end{aligned}
$$

B.2.2 Parameter-dependent coefficients

For our center manifold approximation, we also need the following parameter-dependent coefficients

$$
\begin{aligned}
H_{3010}(\theta) = {} & e^{3i\omega_0\theta}\Delta^{-1}(3i\omega_0)[A_1(H_{3000}, K_{10}) + 3B(\varphi, H_{2010}) + B(H_{0010}, H_{3000}) \\
& + 3B(H_{1010}, H_{2000}) + 3B_1(\varphi, H_{2000}, K_{10}) + 3C(\varphi, \varphi, H_{1010}) + 3C(\varphi, H_{0010}, H_{2000}) \\
& + C_1(\varphi, \varphi, \varphi, K_{10}) + D(\varphi, \varphi, \varphi, H_{0010})] \\
& - 3(1 + ib_{1,10})\Delta^{-1}(3i\omega_0)[\Delta'(3i\omega_0) - \theta\Delta(3i\omega_0)]H_{3000}(\theta)],
\end{aligned}
$$

$$
\begin{aligned}
H_{4001}(\theta) = {} & e^{4i\omega_0\theta}\Delta^{-1}(4i\omega_0)[A_1(H_{4000},K_{01}) + 4B(\varphi,H_{3001}) + B(H_{0001},H_{4000}) \\
& + 4B(H_{1001},H_{3000}) + 6B(H_{2000},H_{2001}) + 4B_1(\varphi,H_{3000},K_{01}) \\
& + 3B_1(H_{2000},H_{2000},K_{01}) + 6C(\varphi,\varphi,H_{2001}) + 4C(\varphi,H_{0001},H_{3000}) \\
& + 12C(\varphi,H_{1001},H_{2000}) + 3C(H_{0001},H_{2000},H_{2000}) \\
& + 6C_1(\varphi,\varphi,H_{2000},K_{01}) + 4D(\varphi,\varphi,\varphi,H_{1001}) + 6D(\varphi,\varphi,H_{0001},H_{2000}) \\
& + D_1(\varphi,\varphi,\varphi,\varphi,K_{01}) + E(\varphi,\varphi,\varphi,\varphi,H_{0001})] \\
& - 4ib_{1,01}\Delta^{-1}(4i\omega_0)[\Delta'(4i\omega_0) - \theta\Delta(4i\omega_0)]H_{4000}(\theta),
\end{aligned}
$$

$$
\begin{aligned}
H_{5001}(\theta) = {} & e^{5i\omega_0\theta}\Delta^{-1}(5i\omega_0)[A_1(H_{5000},K_{01}) + 5B(\varphi,H_{4001}) + B(H_{0001},H_{5000}) \\
& + 5B(H_{1001},H_{4000}) + 10B(H_{2000},H_{3001}) + 10B(H_{2001},H_{3000}) \\
& + 5B_1(\varphi,H_{4000},K_{01}) + 10B_1(H_{2000},H_{3000},K_{01}) + 10C(\varphi,\varphi,H_{3001}) \\
& + 5C(\varphi,H_{0001},H_{4000}) + 20C(\varphi,H_{1001},H_{3000}) + 30C(\varphi,H_{2000},H_{2001}) \\
& + 10C(H_{0001},H_{2000},H_{3000}) + 15C(H_{1001},H_{2000},H_{2000}) \\
& + 10C_1(\varphi,\varphi,H_{3000},K_{01}) + 15C_1(\varphi,H_{2000},H_{2000},K_{01}) \\
& + 10D(\varphi,\varphi,\varphi,H_{2001}) + 10D(\varphi,\varphi,H_{0001},H_{3000}) + 30D(\varphi,\varphi,H_{1001},H_{2000}) \\
& + 15D(\varphi,H_{0001},H_{2000},H_{2000}) + 10D_1(\varphi,\varphi,\varphi,H_{2000},K_{01}) \\
& + 5E(\varphi,\varphi,\varphi,\varphi,H_{1001}) + 10E(\varphi,\varphi,\varphi,H_{0001},H_{2000}) \\
& + E_1(\varphi,\varphi,\varphi,\varphi,\varphi,K_{01}) + K(\varphi,\varphi,\varphi,\varphi,\varphi,H_{0001})] \\
& - 5ib_{1,01}\Delta^{-1}(5i\omega_0)[\Delta'(5i\omega_0) - \theta\Delta(5i\omega_0)]H_{5000}(\theta),
\end{aligned}
$$

$$
\begin{aligned}
H_{4101}(\theta) = {} & e^{3i\omega_0\theta}\Delta^{-1}(3i\omega_0)[A_1(H_{4100}, K_{01}) + 4B(\varphi, H_{3101}) + B(\bar{\varphi}, H_{4001}) \\
& + B(H_{0001}, H_{4100}) + B(\overline{H}_{1001}, H_{4000}) + 4B(H_{1001}, H_{3100}) \\
& + 4B(H_{1100}, H_{3001}) + 4B(H_{1101}, H_{3000}) + 6B(H_{2000}, H_{2101}) \\
& + 6B(H_{2001}, H_{2100}) + 4B_1(\varphi, H_{3100}, K_{01}) + B_1(\bar{\varphi}, H_{4000}, K_{01}) \\
& + 4B_1(H_{1100}, H_{3000}, K_{01}) + 6B_1(H_{2000}, H_{2100}, K_{01}) + 6C(\varphi, \varphi, H_{2101}) \\
& + 4C(\varphi, \bar{\varphi}, H_{3001}) + 4C(\varphi, H_{0001}, H_{3100}) + 4C(\varphi, \overline{H}_{1001}, H_{3000}) \\
& + 12C(\varphi, H_{1001}, H_{2100}) + 12C(\varphi, H_{1100}, H_{2001}) + 12C(\varphi, H_{1101}, H_{2000}) \\
& + C(\bar{\varphi}, H_{0001}, H_{4000}) + 4C(\bar{\varphi}, H_{1001}, H_{3000}) + 6C(\bar{\varphi}, H_{2000}, H_{2001}) \\
& + 4C(H_{0001}, H_{1100}, H_{3000}) + 6C(H_{0001}, H_{2000}, H_{2100}) \\
& + 3C(\overline{H}_{1001}, H_{2000}, H_{2000}) + 12C(H_{1001}, H_{1100}, H_{2000}) \\
& + 6C_1(\varphi, \varphi, H_{2100}, K_{01}) + 4C_1(\varphi, \bar{\varphi}, H_{3000}, K_{01}) \\
& + 12C_1(\varphi, H_{1100}, H_{2000}, K_{01}) + 3C_1(\bar{\varphi}, H_{2000}, H_{2000}, K_{01}) \\
& + 4D(\varphi, \varphi, \varphi, H_{1101}) + 6D(\varphi, \varphi, \bar{\varphi}, H_{2001}) + 6D(\varphi, \varphi, H_{0001}, H_{2100}) \\
& + 6D(\varphi, \varphi, \overline{H}_{1001}, H_{2000}) + 12D(\varphi, \varphi, H_{1001}, H_{1100}) \\
& + 4D(\varphi, \bar{\varphi}, H_{0001}, H_{3000}) + 12D(\varphi, \bar{\varphi}, H_{1001}, H_{2000}) \\
& + 12D(\varphi, H_{0001}, H_{1100}, H_{2000}) + 3D(\bar{\varphi}, H_{0001}, H_{2000}, H_{2000}) \\
& + 4D_1(\varphi, \varphi, \varphi, H_{1100}, K_{01}) + 6D_1(\varphi, \varphi, \bar{\varphi}, H_{2000}, K_{01}) \\
& + E(\varphi, \varphi, \varphi, \varphi, \overline{H}_{1001}) + 4E(\varphi, \varphi, \varphi, \bar{\varphi}, H_{1001}) + 4E(\varphi, \varphi, \varphi, H_{0001}, H_{1100}) \\
& + 6E(\varphi, \varphi, \bar{\varphi}, H_{0001}, H_{2000}) + E_1(\varphi, \varphi, \varphi, \varphi, \bar{\varphi}, K_{01}) + K(\varphi, \varphi, \varphi, \varphi, \bar{\varphi}, H_{0001})] \\
& - 12(1 + ib_{2,01})\Delta^{-1}(3i\omega_0)[\Delta'(3i\omega_0) - \theta\Delta(3i\omega_0)]H_{3000}(\theta) \\
& - 12c_1(0)e^{3i\omega_0\theta}\Delta^{-1}(3i\omega_0)\Big([\Delta'(3i\omega_0) - \theta\Delta(3i\omega_0)]H_{3001}(0) \\
& + \frac{3}{2}ib_{1,01}[\Delta''(3i\omega_0) - \theta^2\Delta(3i\omega_0)]H_{3000}(0)\Big) \\
& - 3ib_{1,01}e^{3i\omega_0\theta}\Delta^{-1}(3i\omega_0)\Big([\Delta'(3i\omega_0) - \theta\Delta(3i\omega_0)]H_{4100}(0) \\
& + 6c_1(0)[\Delta''(3i\omega_0) - \theta^2\Delta(3i\omega_0)]H_{3000}(0)\Big).
\end{aligned}
$$

$$
\begin{aligned}
H_{3201}(\theta) = {} & B_{i\omega_0}^{INV}(M_{3201}, -12g_{3201})(\theta) - 12c_2(0)\tilde{B}_{i\omega_0}^{INV}(H_{1001}, -ib_{1,01})(\theta) \\
& - (18 + 6ib_{2,01})\tilde{B}_{i\omega_0}^{INV}(H_{2100}, -2c_1(0))(\theta) \\
& - 6i\Im\{c_1(0)\}\hat{B}_{i\omega_0}^{INV}(H_{2101}, -[2(1 + ib_{2,01})q + ib_{1,01}H_{2100}(0) + 2c_1(0)H_{1001}(0)], 4ic_1(0)b_{1,01})(\theta) \\
& - ib_{1,01}\hat{B}_{i\omega_0}^{INV}(H_{3200}, -[12c_2(0)q + 6i\Im\left(c_1(0)\right)H_{2100}(0)], 12i\Im\left(c_1(0)\right)c_1(0))(\theta),
\end{aligned}
$$

$$
\begin{aligned}
H_{3002}(\theta) = {} & e^{3i\omega_0\theta}\Delta^{-1}(3i\omega_0)[2A_1(H_{3001}, K_{01}) + A_1(H_{3000}, K_{02}) + 3B(\varphi, H_{2002}) \\
& + 2B(H_{0001}, H_{3001}) + B(H_{0002}, H_{3000}) + 6B(H_{1001}, H_{2001}) \\
& + 3B(H_{1002}, H_{2000}) + A_2(H_{3000}, K_{01}, K_{01}) + 6B_1(\varphi, H_{2001}, K_{01}) \\
& + 3B_1(\varphi, H_{2000}, K_{02}) + 2B_1(H_{0001}, H_{3000}, K_{01}) + 6B_1(H_{1001}, H_{2000}, K_{01}) \\
& + 3C(\varphi, \varphi, H_{1002}) + 6C(\varphi, H_{0001}, H_{2001}) + 3C(\varphi, H_{0002}, H_{2000}) \\
& + 6C(\varphi, H_{1001}, H_{1001}) + C(H_{0001}, H_{0001}, H_{3000}) \\
& + 6C(H_{0001}, H_{1001}, H_{2000}) + 3B_2(\varphi, H_{2000}, K_{01}, K_{01}) + C_1(\varphi, \varphi, \varphi, K_{02}) \\
& + 6C_1(\varphi, \varphi, H_{1001}, K_{01}) + 6C_1(\varphi, H_{0001}, H_{2000}, K_{01}) + D(\varphi, \varphi, \varphi, H_{0002}) \\
& + 6D(\varphi, \varphi, H_{0001}, H_{1001}) + 3D(\varphi, H_{0001}, H_{0001}, H_{2000}) \\
& + C_2(\varphi, \varphi, \varphi, K_{01}, K_{01}) + 2D_1(\varphi, \varphi, \varphi, H_{0001}, K_{01}) \\
& + E(\varphi, \varphi, \varphi, H_{0001}, H_{0001})] \\
& - 3ib_{1,02}\Delta^{-1}(3i\omega_0)[\Delta'(3i\omega_0) - \theta\Delta(3i\omega_0)]H_{3000}(\theta) \\
& - 6ib_{1,01}e^{3i\omega_0\theta}\Delta^{-1}(3i\omega_0)\Big([\Delta'(3i\omega_0) - \theta\Delta(3i\omega_0)]H_{3001}(0) \\
& + \frac{3}{2}ib_{1,01}[\Delta''(3i\omega_0) - \theta^2\Delta(3i\omega_0)]H_{3000}(0)\Big).
\end{aligned}
$$

Appendix C

DDEs and sun-star calculus

A common way of solving DDEs is by the so-called *method of steps.* This will be illustrated in the following example.

Example. Consider the following simple DDE

$$\dot{x}(t) = \alpha x(t-h), \quad t \geq 0, \tag{C.1}$$

for some $\alpha \in \mathbb{R}$ with the initial condition $x(t) = 1$ for $t \in [-h, 0]$. For $t \in [0, h]$ we have that $x(t-h) = 1$ and thus system (C.1) becomes

$$\dot{x} = \alpha, \quad t \in [0, h].$$

Integrating this equation results in the solution

$$x(t) = x(0) + \int_0^h \alpha ds = 1 + \alpha t, \quad t \in [0, h].$$

Using this solution we can proceed to find the solution on the interval $[h, 2h]$. Namely, the equation becomes

$$\dot{x}(t) = \alpha(1 + \alpha(t-h)), \quad t \in [h, 2h].$$

Integrating this equation yields

$$\begin{aligned} x(t) &= x(h) + \alpha \int_h^t (1 + \alpha(s-h))ds, \\ &= 1 + \alpha h + \alpha \left((t-h)(1-\alpha h) + \frac{1}{2}\alpha(t^2 - h^2) \right), \\ &= 1 + \alpha h + \alpha(t-h)(1 + \frac{1}{2}\alpha(t-h)), \quad t \in [h, 2h]. \end{aligned}$$

This second step can also be performed by first translating the solution on the interval $[0, h]$ to the interval $[-h, 0]$ and then integrating the new equation. Proceeding this way one can find a solution for all $t \geq 0$.

Thus, there are two main steps in solving such an equation. First, we *extend* the solution by solving the DDE on the interval $[0, h]$. Then we can *translate* this solution back to the interval $[-h, 0]$ and repeat the process. Motivated by this, a natural state space is the Banach space $X = C([-h, 0], \mathbb{R})$ of continuous functions endowed with the usual supremum norm. To deal with the infinite-dimensional state space X, we need the functional analytic framework of sun-star calculus. We will provide a short introduction of sun-star calculus in the context of DDEs in the next sections following [9].

C.1 The shift semigroup

A good starting point turns out to be the trivial DDE

$$\dot{x}(t) = 0, \quad t \geq 0, \tag{C.2}$$

with some initial condition $\varphi \in X$

$$x(\theta) = \varphi(\theta), \quad \theta \in [-h, 0]. \tag{C.3}$$

Here we take $X = C([-h, 0], \mathbb{C})$ as our state space endowed with the supremum norm. The extension rules for other systems can be considered perturbations of this simple case. The solution to (C.2) is given by

$$x(t) = \begin{cases} \varphi(t), & t \in [-h, 0], \\ \varphi(0), & t \geq 0. \end{cases} \tag{C.4}$$

Based on this solution, we define for each $t \geq 0$ the *shift semigroup*

$$(T_0(t)\varphi)(\theta) = \begin{cases} \varphi(t+\theta), & t+\theta \in [-h, 0], \\ \varphi(0), & t+\theta \geq 0. \end{cases} \tag{C.5}$$

This defines a bounded linear operator $T_0(t) : X \to X$ mapping the initial state φ at time zero to the state x_t at time t. The family $\{T_0(t)\}_{t\geq 0}$ of operators satisfies the following three properties:

1. $T(0) = I$,
2. $T(t)T(s) = T(t+s), \quad t, s \geq 0$,
3. for any $\varphi \in X$, $\|T(t)\varphi - \varphi\| \to 0$ as $t \downarrow 0$.

Such a family bounded linear operators defined on a Banach space X is called a *strongly continuous semigroup of operators* or a *$\mathcal{C}_0$-semigroup*. The first two properties make it a semigroup, while the final property ensures strong continuity. A more rigorous introduction to $\mathcal{C}_0$-semigroup including proofs of some general results can be found in [9, Appendix II]. The *infinitesimal generator* A of a semigroup of operators $\{T(t)\}_{t\geq 0}$ is defined as the derivative at $t = 0$, i.e.

$$A\varphi = \lim_{t\downarrow 0} \frac{1}{t}(T(t)\varphi - \varphi), \tag{C.6}$$

where the domain $\mathcal{D}(A)$ is defined as the set of all $\varphi \in X$ for which the above limit does exist. The operator A is a linear operator on its domain and is generally unbounded. However, it is closed and its domain $\mathcal{D}(A)$ is dense in X. The infinitesimal generator for the shift semigroup (C.5) can be found explicitly and is given by

$$\mathcal{D}(A_0) = \{\varphi \in X | \dot{\varphi} \in C([-h, 0], \mathbb{C}), \dot{\varphi}(0) = 0\}, \quad A_0\varphi = \dot{\varphi}.$$

One problem that arises here is that the extension rule is incorporated in the domain of A_0 through the condition $\dot{\varphi}(0) = 0$. Changing this rule will change the domain of the generator. This will result in certain technical complications when we study perturbations of the trivial equation which is resolved with the help of the sun-star calculus. Before we proceed with sun-star, it is useful to introduce functions of normalized bounded variation.

C.2 Functions of normalized bounded variation

Recall from the functional analysis that a *dual space* of a Banach space X is defined as the space of continuous linear operators on X with values in $\mathbb{R}$ or $\mathbb{C}$. We will denote this by $X^\star$. For the case where $X = C([-h, 0], \mathbb{R})$ it is possible to identify the dual space $X^\star$ with the so-called space of functions of normalized bounded variations. For this, we first need to define functions of bounded variation. Let $f : [a, b] \to \mathbb{R}$. We define the *total variation* of f over the interval $[a, b]$ as

$$V(f) = \sup_{P(a,b)} \sum_{j=1}^{N} |f(\sigma_j) - f(\sigma_{j-1})|,$$

where $P(a, b)$ is a partition $a = \sigma_0 < \sigma_1 < \cdots < \sigma_N = b$ of $[a, b]$. We say that f is of *bounded variation* or $f \in \text{BV}$ if $V(f) < \infty$. We can now define the space of functions of normalized bounded variations as

$$\text{NBV} := \{f \in \text{BV} \,|f(a) = 0 \text{ and } f \text{ is continuous from the right on the interval } (a, b)\}.$$

A vector-valued function $f : [a, b] \to \mathbb{R}^n$ is of normalized bounded variation if and only if every component is of normalized bounded variation. Now consider again the space $X = C([-h, 0], \mathbb{R})$. Then it is convenient to consider functions of NBV on the interval $[0, h]$ and extend the domain to all of $\mathbb{R}$ by setting $f(\theta) = 0$ for $\theta \leq 0$ and $f(\theta) = f(h)$ for $\theta \geq h$. With this convention, we have the following pairing between $f \in \text{NBV}$ and $\varphi \in X$.

$$\langle f, \varphi \rangle = \int_0^h df(\theta)\varphi(-\theta), \tag{C.7}$$

where the integral is a Riemann-Stieltjes integral. More details on the Riemann-Stieltjes integral and its properties can be found in [9, Appendix I].

C.3 Sun-star calculus for the shift semi-group

This section is based on [9, Chapter II]. We will only state some of the main results.

The pairing between $x^\star \in X^\star$ and $x \in X$ will be denoted as $x^\star(x) = \langle x^\star, x \rangle$. We will now consider the family of adjoint semigroup operators $T^\star := \{T^\star(t)\}_{t \geq 0}$ on $X^\star$. For a bounded linear operator $T : X \to X$, the *adjoint operator* $T^\star : X^\star \to X^\star$ is defined by the property that

$$\langle x^\star, Tx \rangle = \langle T^\star x^\star, x \rangle, \qquad \text{for every } x \in X, x^\star \in X^\star.$$

As shown in [9][Section II.4], the adjoint operator of the shift semigroup (C.5) can be found explicitly and is given by

$$(T_0^\star(t)\varphi) = \varphi(t + \theta), \quad \theta > 0. \tag{C.8}$$

The adjoint semigroup is not strongly continuous. For a densely defined unbounded operator A, we define the *adjoint operator* $A^\star : D(A^\star) \to X^\star$ by

$$D(A^\star) = \{x^\star \in X^\star | \text{ There exists } y^\star \in X^\star \text{ such that } \langle x^\star, Ax \rangle = \langle y^\star, x \rangle, \text{ for all } x \in D(A)\},$$

and

$$A^\star x^\star = y^\star.$$

For the infinitesimal generator A defined as (C.6), $A^\star$ only generates the adjoint semigroup $T^\star$ in the *weak***-sense*. This means that

$$\lim_{t\downarrow 0} \frac{1}{t}\langle T^\star(t)x^\star - x^\star, x\rangle = \langle A^\star x^\star, x\rangle, \qquad \text{for all } x \in X \text{ if and only if } x^\star \in D(A^\star).$$

We can consider the restriction of $T^\star$ to the maximal subspace of strong continuity

$$X^\odot := \left\{ x^\star \in X^\star \,\middle|\, \lim_{t\downarrow 0} \frac{1}{t}(T(t)\varphi - \varphi) = 0 \right\}.$$

It turns out that $X^\odot$ is precisely given by the norm closure of $D(A^\star)$, i.e.

$$X^\odot = \overline{D(A^\star)}.$$

The restriction of $T^\star$ to $X^\odot$, denoted by $T^\odot := \{T^\odot(t)\}_{t\geq 0}$ is a C_0-semigroup. The infinitesimal generator $A^\odot$ of $T^\odot$ is the restriction of $A^\star$ to $X^\odot$,

$$D(A^\odot) = \{x^\odot \in D(A^\star) | A^\star x^\odot \in X^\odot\}, \quad A^\odot x^\odot = A^\star x^\odot.$$

For the shift semigroup, we can explicitly determine the sun-dual space $X^\odot$. For the case of the shift semigroup, we have

$$\begin{aligned} X^\odot = \{f \in NBV | f(t) = c + \int_0^t g(\theta)d\theta \text{ for } t > 0, \text{ where } c \in \mathbb{C} \text{ and} \\ g \in L^1 \text{ such that } g(\theta) = 0, \text{ for (almost all) } \theta \geq h\}. \end{aligned}$$

We see that for the shift semigroup, the elements of the space $X^\odot$ are completely specified by the pair $(c, g) \in \mathbb{C} \times L^1([0, h], \mathbb{C})$. Using these coordinates we have from (C.8)

$$T_0^\odot(t)(c, g) = \left(c + \int_0^t g(\sigma)d\sigma, g(t + \cdot)\right) \tag{C.9}$$

Furthermore, the infinitesimal generator $A_0^\odot$ is given by

$$A_0^\odot(c, g) = (g(0), \dot{g}),$$

Let AC denote the space of absolutely continuous functions. Then the domain of $A_0^\odot$ is given by

$$D(A_0^\odot) = \{(c, g) | c \in \mathbb{C} \text{ and } g \in \text{AC with } g(\theta) = 0 \text{ for } \theta \geq h\}.$$

We now have a C_0-semigroup $T_0^\odot$ defined on a Banach space $X^\odot$ with a generator $A_0^\odot$. Repeating the same procedure, we can construct the dual space $X^{\odot\star}$ with the adjoint semigroup $T_0^{\odot\star}$. Finally, we obtain the C_0-semigroup $T_0^{\odot\odot}$ by restricting $T_0^{\odot\star}$ to $X^{\odot\odot} = \overline{D(A_0^{\odot\star})}$. We can use the paring between X and $X^{\odot\star}$ to define an embedding $j : X \to X^{\odot\star}$ as

$$\langle jx, x^\odot\rangle := \langle x^\odot, x\rangle.$$

In the special case where $j(X) = X^{\odot\odot}$, we say that X is $\odot$-*reflexive* with respect to T.

For the shift semigroup we can represent $X^{\odot\star}$ by $\mathbb{C} \times L^\infty([-h, 0], \mathbb{C})$. The pairing between $(\alpha, \varphi) \in X^{\odot\star}$ and $(c, g) \in X^\odot$ is given by

$$\langle(\alpha, \varphi), (c, g)\rangle = \alpha c + \int_0^h \varphi(-\theta)g(\theta)d\theta.$$

From equation (C.9) we can determine the action of $T_0^{\odot\star}$ in terms of the above paring. This yields

$$T_0^{\odot\star}(t)(\alpha,\varphi) = (\alpha, \varphi_t^\alpha),$$

where

$$\varphi_t^\alpha(\theta) = \begin{cases} \varphi(t+\theta), & t+\theta \leq 0, \\ \alpha, & t+\theta > 0. \end{cases}$$

Let $\mathrm{Lip}(\alpha)$ denote the subset of $L^\infty([-h,0],\mathbb{C})$ consisting of Lipschitz continuous functions which assume the value α at $\theta = 0$. Then, the infinitesimal generator $A_0^{\odot\star}$ has domain

$$D(A_0^{\odot\star}) = \{(\alpha,\varphi) | \varphi \in \mathrm{Lip}(\alpha)\},$$

and is given by

$$A_0^{\odot\star}(\alpha,\varphi) = (0, \dot{\varphi}). \tag{C.10}$$

To arrive at $X^{\odot\odot}$ we take the closure of $D(A_0^{\odot\star})$ which yields

$$X^{\odot\odot} = \overline{D(A_0^{\odot\star})} = \{(\alpha,\varphi) | \varphi \in C(\alpha)\},$$

where $\varphi \in C(\alpha)$ are the continuous functions in $L^\infty([-h,0],\mathbb{C})$ satisfying $\varphi(0) = \alpha$. Thus, we see that we can identify eacht $\varphi \in X$ with the pair $(\varphi(0),\varphi) \in X^{\odot\odot}$. This allows us to define the embedding

$$j\varphi = (\varphi(0), \varphi)$$

mapping X onto $X^{\odot\odot}$, i.e. $j(X) = X^{\odot\odot}$. Thus $X = C([-h,0],\mathbb{C})$ is $\odot$-reflexive with respect to the shift semigroup T_0.

C.3.1 Linear DDEs

We can consider linear DDEs as bounded perturbations from the trivial DDE (C.2) following [9, Chapter III]. Let $L : X \to \mathbb{C}^n$ be a continuous linear operator and consider a linear system of DDEs

$$\begin{cases} \dot{x}(t) = Lx_t, & t \geq 0, \\ x(\theta) = \varphi(\theta) & \theta \in [-h, 0] \end{cases} \tag{C.11}$$

A corollary of Riesz representation theorem [9, Theorem 1.1] states that there exists a unique function of normalized bounded variation $\zeta \in NBV$, $\zeta : [0,h] \to \mathbb{C}^n$ such that

$$Lx_t = \langle \zeta, x_t \rangle.$$

Furthermore, define the operator $B : X \to X^{\odot\star}$ as

$$B\varphi = (\langle \zeta, \varphi \rangle, 0).$$

Then we can write the linear DDE as an equation in $X^{\odot\star}$ as

$$j\frac{d}{dt}x_t = A_0^{\odot\star} j x_t + Bx_t, \tag{C.12}$$

where $A_0^{\odot\star}$ is given by (C.10). There exists a unique C_0-semigroup T corresponding to the linear DDE (C.11) which is related to the shift-semigroup (C.5) by the following linear integral equation

$$T(t)x = T_0(t)x + j^{-1}\int_0^t T_0^{\odot\star} BT(\tau)x d\tau, \quad t \geq 0, x \in X. \tag{C.13}$$

It turns out that X is again sun-reflexive with respect to T. Furthermore, the spaces $X^{\odot}$ and $X^{\odot\odot}$ remain the same for T as for T_0. The domain of $A^{\odot\star}$ is not affected by the linear perturbation B. Only the action changes, i.e.

$$A^{\odot\star} = A_0^{\odot\star} + Bj^{-1}, \quad \text{with} \quad D(A^{\odot\star}) = D(A_0^{\odot\star}).$$

For the generator A of T we have

$$D(A) = \{x \in X : jx \in D(A_0^{\odot\star}) \text{ and } A_0^{\odot\star}jx + Bx \in X^{\odot\odot}\}, \tag{C.14}$$

with

$$Ax = j^{-1}(A_0^{\odot\star}jx + Bx). \tag{C.15}$$

Returning to our linear DDE (C.11), we can determine explicit expressions for the generators $A^{\odot\star}$ and A. The generator $A^{\odot\star}$ is given by

$$A^{\odot\star}(\alpha, \varphi) = (\langle \zeta, \varphi \rangle, \dot{\varphi}), \quad \text{with} \quad D(A^{\odot\star}) = \{(\alpha, \varphi) | \varphi \in \text{Lip}(\alpha)\}. \tag{C.16}$$

Meanwhile, the generator A is given by

$$A\varphi = \dot{\varphi}, \quad \text{with} \quad D(A) = \{\varphi \in C^1 | \dot{\varphi}(0) = \langle \zeta, \varphi \rangle\}. \tag{C.17}$$

Observe the change in the domain of the generator A compared to the domain of A_0. Finally, we can relate solutions of the linear DDE (C.11) to the semigroup T defined by (C.13). Namely, if $x(\cdot, \varphi)$ is a solution to equation (C.11), then

$$x_t(\cdot, \varphi) = T(t)\varphi.$$

C.3.2 The variation of constants formula

As we did for linear DDEs, we can associate solutions to a general DDE with solutions to an abstract integral equation by considering nonlinear perturbations of the linear DDE (C.11). This will lead to the so-called variations-of-constants formula for DDEs.

Consider the following perturbed DDE

$$\begin{cases} \dot{x}(t) &= \langle \zeta, x_t \rangle + G(x_t), \quad t \geq 0, \\ x(\theta) &= \varphi(\theta), \quad \theta \in [-h, 0]. \end{cases} \tag{C.18}$$

where $G : X \to \mathbb{R}^n$ is assumed to be sufficiently smooth such that

$$G(0) = 0, \quad DG(0) = 0.$$

Let e_i be the standard basis vectors in $\mathbb{R}^n$ for $i = 1, \ldots, n$. It is convenient to introduce the notation $r_i^{\odot\star} := (e_i, 0) \in X^{\odot\star}$ and

$$wr^{\odot\star} := \sum_{i=1}^{n} w_i r_i^{\odot\star}, \quad w \in \mathbb{R}^n.$$

With this notation we can write $wr^{\odot\star} = (w, 0) \in X^{\odot\star}$. Let $T = \{T(t)\}_{t\geq 0}$ be the C_0-semigroup corresponding to the linear part of (C.18) and define $R : X \to X^{\odot\star}$ as

$$R(\varphi) = G(\varphi)r^{\odot\star}.$$

Let $u(t) = x_t$ and let T be the C_0-semigroup corresponding to the linear part of (C.18) with generator A. Then as was done for linear DDEs, it is tempting to write (C.18) as the abstract equation

$$j\dot{u}(t) = A^{\odot\star} ju(t) + R(u(t)). \tag{C.19}$$

Then, formal integration of (C.19) results in the following version of the variations-of-constants equation

$$u(t) = T(t)\varphi + j^{-1}\int_0^t T^{\odot\star}(t-\tau)R(u(\tau))d\tau, \quad t \geq 0. \tag{C.20}$$

Solutions to (C.20) are continuous functions $u : [0, t_+) \to X$. A one-to-one correspondence exists between solutions to the integral equation (C.20) and solutions to the DDE (C.18) [9, Proposition 6.1]. In particular, if $u(t)$ is a solution to (C.18), then the function $x : [-h, t_\varphi) \to \mathbb{R}^n$ defined as

$$x_0 := \varphi, \quad \text{and} \quad x(t) = u(t)(0), \quad t \in [0, t_\varphi)$$

is the unique solution of (C.18).

As for ODEs, the variations-of-constants equation plays an important role in the study of solutions and their stability of DDEs. In particular, it is used to prove the existence of an invariant local invariant center manifold in [9, Chapter IX]. On this center manifold the solutions $u : I \to X$ actually satisfy the abstract differential equation (C.19). A parameter-dependent version of the variations-of-constants equation results in the parameter-dependent ODE (2.20) on the center manifold, see [3, Corollary 20].

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