On the connection between Hodge Theory and Integrable Systems

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Abstract

We investigate the relationship between Hodge theory, a field of mathematics, and integrable systems, a concept in physics. There are several reasons to believe the two notions are related and this suspicion has been strengthened in two recent papers by Grimm and Monnee. There, the authors managed to show that the Weil operator from Hodge theory provides a solution to both the λ-model and the bi-Yang-Baxter model, which are integrable systems. To be precise, they showed that for the bi-Yang-Baxter model the SL(2)-approximation of the Weil operator, coming from the SL(2)-orbit theorem of Hodge theory, provided a solution, whereas the full Weil operator solved the λ-model. In this work we try to build upon their construction to further strengthen the connection between Hodge theory and integrable systems. In particular, we review the relevant mathematical and physical background, as well as the construction of Grimm and Monnee. A concept deeply intertwined with integrable systems, called Poisson-Lie T-duality, seems to play a role in the relation between Hodge theory and integrable systems. We give a self-contained review of the duality and provide one of the necessary steps in the case of SU(2). Moreover, we identify opportunities and difficulties regarding the interplay of Hodge theory, integrable systems and Poisson-Lie T-duality.

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Introduction

THERE have been many instances where fundamental physics and pure mathematics meet. For example, in the theory of quarks introduced by Murray Gell-Mann, the irreducible representations of SU(3) are foundational. Eventually, this was integrated in the formulation of the Standard Model as a non-Abelian gauge theory, in which principal bundles play a key role. Moreover, Einstein his theory of general relativity dictates that gravity is an emergent phenomena due to the curvature of spacetime, a concept from (pseudo)-Riemannian geometry.

An area of theoretical physics where the description of the universe and abstract mathematics are strongly intertwined is *string theory*. One of these connections is studied in this work. Within string theory extended objects called *strings* are studied. The string vibrates and the different harmonics on the string correspond to distinct particles. String theory attempts to solve the holy grail of high energy physics: a consistent unification of gravity and quantum field theory. Over the last fifty years it received much attention, due to the inherent existence of the graviton in the theory. This is the quantum particle associated to gravity, making string theory into *a* theory of quantum gravity. The main question is whether string theory is *the* theory of quantum gravity that describes our universe.

One major obstacle is that a consistent string theory that includes both bosons and fermions requires a ten dimensional spacetime, while we only observe four (macroscopic) dimensions. To proceed, one can curl up the superfluous six dimensions and make them small enough such that the resulting theory effectively lives in a four-dimensional spacetime. This procedure is called *compactification* and it is one of the sources of pure mathematics in string theory. Indeed, the extra six dimensions cannot be arbitrarily curled up. Physical principles force them to constitute a special kind of complex manifold: a *Calabi-Yau manifold*. Interestingly, the effective physics in four dimensions depends on geometry of the chosen Calabi-Yau manifold in the compactification. To be precise, it depends on the isomorphism class of the Calabi-Yau manifold.

This dependence leads to the concept of *moduli space*. Roughly speaking, in the context of string theory, it is the collection of inequivalent Calabi-Yau structures on a complex manifold. The moduli space is itself an interesting geometric space that is extensively studied. From this point of view, the effective four dimensional theory depends on the point in moduli space corresponding to the chosen Calabi-Yau manifold. We say the theory is *moduli dependent*. A special property of Calabi-Yau manifolds is that their cohomology groups admit a particular decomposition. This is known as the *Hodge decomposition*. It turns out, the effective four-dimensional theory also depends on this decomposition, beside the moduli dependence. Moreover, the Hodge decomposition is moduli dependent as well.

Consequently, to understand the effective four-dimensional theory we need to understand the moduli space *and* the dependence of the Hodge decomposition on this space. The latter leads to the concept of a *variation of Hodge structure* first introduced by Griffiths in [Gri68a]. In his work, the properties of the Hodge decomposition and its moduli dependence were axiomatized and shown to be equivalent to a

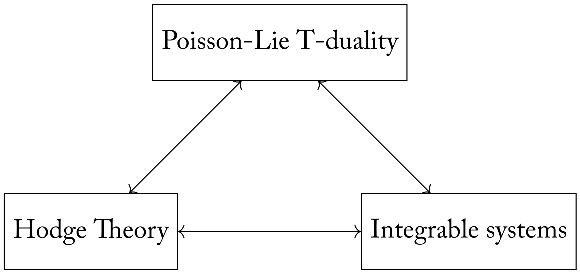

Figure 1: The areas and connections of interest in this work.

mapping

$$\Phi : \mathcal{M} \to \Gamma\backslash D$$

called the *period map*. Here, $\mathcal{M}$ denotes the moduli space and $\Gamma\backslash D$ the collection of all Hodge decompositions, called the *classifying space*[1]. Moreover, the period map satisfies two conditions: *holomorphicity* and *horizontality*. The study of variation of Hodge structures is part of *Hodge theory* and they have found many applications in string theory (see e.g. [GPV18; LLW22]). Especially, two main theorems in Hodge theory have proven to be particularly useful: the *nilpotent orbit theorem* and SL(2)-*orbit theorem*. They give approximations of the period map near the boundary of the moduli space $\mathcal{M}$. However, the theory of variation of Hodge structures and the formulation of the orbit theorems is rather formal. A desire to rephrase this formulation in physical terms leads to the driving motivation of this work. In special cases, the moduli space $\mathcal{M}$ can be viewed as a (singular) Riemann surface. Furthermore, it can be shown in general that the classifying space is a homogeneous space, i.e. a quotient of Lie groups. Hence, the period map can schematically be thought of as a map from a Riemann surface Σ to a Lie group G. Theories that describe the dynamics of such maps are called *non-linear sigma-models*. The underlying question of this work is:

> *Is there a non-linear sigma-model whose equations of motion are precisely holomorphicity and horizontality, i.e. has the period map as a solution?*

However, the landscape of non-linear sigma-models is vast. So, where does one start when attempting to answer this question?

There is a particular class of non-linear sigma-models that seems particularly suited. They have a property called *integrability*, which is studied in the field of *integrable systems*. It has been suggested that Hodge theory is related to integrable systems, see for example [DWS08]. This connection has been straightened in two recent papers [GM22; GM23]. In an attempt to answer the main question above, these papers considered two integrable non-linear sigma-models and constructed Hodge theoretic solutions to them. Interestingly, it was not the period map that constituted the solution, but a different central object in Hodge theory called the *Weil operator*. To be precise, in [GM23] they showed the SL(2)-approximation of the Weil operator, coming from the SL(2)-orbit theorem, produced a solution. Then, a natural question is whether the nilpotent approximation and full Weil operator also produce solutions. It was already noted in [GM23] that the parameters of the integrable model need to be altered to answer this question. Looking for these adjustments is one of the initial objectives of this work.

It was shown in [Kli15; Kli16] that the integrable models considered in [GM22; GM23] are related via a duality called *Poisson-Lie T-duality*. Moreover, all known example of integrable non-linear sigma-models have a symmetry associated to this duality called Poisson-Lie symmetry [DHT19]. This indicates

[1] To be precise, D is called the classifying space and Γ is a symmetry group that needs to be quotiented out to describe a variation of Hodge structure. We elaborate more on this in Chapter 3.

a relation between Poisson-Lie T-duality and integrable systems. Since integrable systems are seemingly related to Hodge theory, there might be a connection between Poisson-Lie T-duality and Hodge theory. Trying to get an inside in this relation is the second objective in this work. Consequently, understanding the web of connections indicated in Figure 1 is the main focus of this thesis.

The outline of the thesis is as follows, in Chapter 1 we start with a general description of string theory and compactifications. To illustrate the qualitative properties of string compactifications, we work out the Kaluza-Klein example in reasonable detail. Furthermore, we argue why the superfluous six dimensions must form a Calabi-Yau manifold. Finally, we discuss the moduli and Hodge decomposition dependence of the effective theory.

Since the moduli space of Calabi-Yau manifolds plays such a central role in string theory, we discuss its properties in Chapter 2. For this, we start with some preliminaries on Kähler geometry and (classical) Hodge theory. Afterwards, we consider deformations of complex manifolds and prove the classical theorem on the obstructions to deformations by Kuranishi [Kur65]. Finally, we prove the Bogomolov-Tian-Todorov [Bog78; Tia87; Tod89] theorem on the unobstructedness of deformations on Calabi-Yau manifolds. We do this without the classical power series argument, but using global methods. We close the chapter with a discussion about the global geometry of the Calabi-Yau moduli space.

In Chapter 3 we discuss the Hodge theory introduced by Griffiths. In particular, we give equivalent definitions of Hodge structures and discuss properties of the classifying space. Furthermore, we describe the concept of a variation of Hodge structure and its relation to the period map. Finally, we give the statements of the celebrated nilpotent and SL(2)-orbit theorems. Along the way we try to clarify the abstract concepts by means of explicit examples.

Subsequently, we discuss integrable systems in Chapter 4. We start with the notion of integrability in classical mechanics. The Lax pair formulation is discussed and its constructive nature in proving integrability is explained. The key role of the r-matrix in this and its properties are described. Afterwards, we introduce the notion of integrability in field theory. We conclude the chapter with a discussion of the non-linear sigma-models used in [GM22; GM23] and comment on their integrability. Moreover, we discuss the Weil operator solutions of both sigma-models found in [GM22; GM23].

We conclude this work with an in-depth exhibition of Poisson-Lie T-duality in Chapter 5. We start by introducing the concept of (Abelian) T-duality and discuss the non-Abelian generalization dictated by Poisson-Lie T-duality. In particular, we give sufficient conditions to construct Poisson-Lie T-dual models and derive Buscher-type rules. Furthermore, we discuss how Poisson-Lie T-duality produces a map between solutions of dual models. Afterwards, the notion of an $\mathcal{E}$-model is introduced, which enhances Poisson-Lie T-duality to a symmetry of a larger non-linear sigma-model. We describe how Poisson-Lie T-dual models can be obtained from $\mathcal{E}$-models and show the models considered in [GM22; GM23] fit in this framework. As an aside, a mathematical description of Poisson-Lie T-duality in terms of Courant algebroids is presented as well. The chapter is concluded by stating the relationship of the two models in [GM22; GM23] with Poisson-Lie T-duality and identifying a road map for the duality. Some comments on performing the duality in the case of SU(2), as well as its possible connection to Hodge theory, are made.

CHAPTER I

String Theory

Since the beginning of modern physics in the seventeenth century, physicists have tried to unravel the fundamental structure of nature. In particular, the study of matter and its interactions through forces are of interest. Throughout the centuries, massive progress has been booked and our understanding ranges from scales of order 10^{-19} m to 10^{27} m. These boundaries were reached by means of two major breakthroughs in theoretical physics during the twentieth century: quantum mechanics and general relativity. The former describes the processes at the microscopic scale, while general relativity talks about gravity and therefore the large-scale structure of the universe. Both theories have separately been extremely successful. Hence, for a more complete understanding of the fundamental laws of nature, a unification of the two theories is necessary. Even though the precise form of this unification is still unknown, it already has been named: *Quantum Gravity*.

However, the nature of the two theories is rather different. Where general relativity is completely deterministic, this is not that obvious for quantum mechanics due to its probabilistic characteristics. Therefore, it is unclear how to unify both theories. Straightforward attempts coming from relativistic quantum mechanics, called *quantum field theory* (QFT), are unsuccessful due to the emergence of uncontrollable infinities. To be more precise, it can be shown that the quantization of gravity coming from QFT is *not renormalizable*. Hence, one should look for novel methods to bring the two theories together. One of those proposals originates from the sixties and will be the one relevant in this work: *String Theory*.

In a nutshell, string theory is based on fundamental one-dimensional objects called *strings* [BBS06]. This is in contrast with the description of QFT, where point-like particles are considered. Initially, string theory was an attempt to describe the strong nuclear force. However, this was abandoned after the development of quantum chromodynamics (QCD), which succeeded in capturing the strong force. Nevertheless, people kept developing the theory and noticed it might be suited for the much more ambitious role: a consistent theory of quantum gravity.

In this chapter we will give an introduction to string theory and state some of the challenges that arise when trying to describe physical reality. For example, the theory predicts the possibility of more dimensions. A way to make the connection with our four-dimensional reality is through a process called *compactification*. We will discuss this notion in reasonable detail in this chapter, as it is the origin of the considerations in this work.

1.1 Bosonic String Theory

We will start with a discussion about the so-called bosonic string. This is the original version of string theory from the 1960s. It is called 'bosonic' as it typically only describes bosons. Note, most of the matter in our universe is made out of fermions. So, we already encounter a drawback of the theory. However, it proves to be a valuable toy model that shares plenty of characteristics with other, more realistic, string theories. For example, the generalization to super strings (see Section 1.2) is rather straightforward. We would like to stress that we will merely give an overview of the topic and refer to [GSW12; BLT12; BBS06; Ton12] for more details. These will also be the references throughout this chapter.

As mentioned before, the fundamental assumption of string theory is that particles are not point-like but correspond to extended one-dimensional objects, called strings. There are two types of strings, namely *open* and *closed*. It may seem a small abstraction, however it has huge consequences. For example, whereas a particle traces out a worldline in spacetime, a string sweeps out a surface (see Figure 1.1). Hence, one can view a string as an embedding

$$X : \Sigma \to M_d,$$

where Σ is the sweeped out surface and M_d a Lorentzian spacetime. The space Σ is referred to as **the worldsheet** and M_d the **target space**. Recall, in special relativity, the dynamics of a particle is given by the action that computes the worldlength of its worldline. In string theory one considers the two dimensional analog: the action computes the area of Σ with respect to pull-back of the metric on M_d.

This action is called the **Nambu-Goto** action and to state it we pick coordinates on both Σ and M_d. On Σ we have coordinates $\sigma^\alpha = (\sigma, \tau)$, where τ is timelike σ spacelike. Depending on the type of string (open or closed), we have either $\sigma \in [0, \pi]$ or $\sigma \in [0, 2\pi)$. Moreover, using the coordinates on M_d, we can view the map X as a collection of d scalars $X^\mu(\tau, \sigma)$ on the worldsheet. Now, the Nambu-Goto action is given by

$$S_{NG} = -T \int_\Sigma d^2\sigma \sqrt{-\det \gamma}.$$

Here, $\gamma = X^* g$ with g the metric on M_d. Furthermore, T is the *tension* of the string, which is often written as $T = \frac{1}{2\pi\alpha'}$ for historical reasons.

However, the Nambu-Goto action is difficult to work with, due to the square root. It turns out, one can get rid of the square root at the cost of an auxiliary metric $h_{\alpha\beta}$ on the worldsheet of signature $(-, +)$

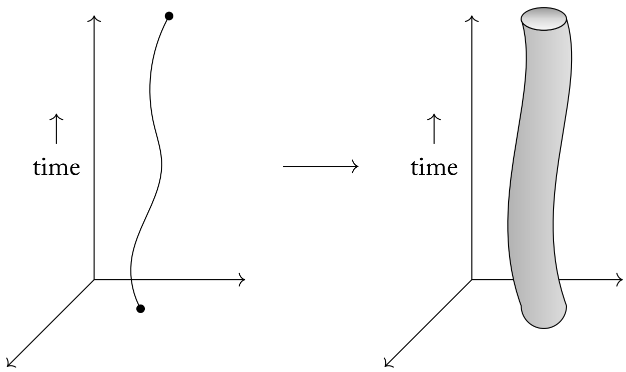

Figure 1.1: Depiction of how the worldline is replaced by a worldsheet for a closed string in string theory.

(cf. [BLT12, Sec. 2.3]). This new, equivalent, action is called the **Polyakov** action and is given by

$$S_P = -\frac{T}{2}\int_\Sigma d^2\sigma\,\sqrt{-h}h^{\alpha\beta}\gamma_{\alpha\beta}, \tag{1.1}$$

where $h = \det h_{\alpha\beta}$. In coordinates, the Polyakov action takes the form

$$S_P = -\frac{T}{2}\int_\Sigma d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}. \tag{1.2}$$

One can now proceed and study the dynamics of this system.

When considering the quantized theory, one encounters the profound implications of our simple assumption. Firstly, the only way to make the theory Lorentz invariant and sensible is by requiring the dimension d of the target space to be *precisely* equal to 26, called the **critical dimension**. Secondly, vibrations on the string can be associated to particles in the target space, labeled by their mass and spin. As the string has an infinite number of harmonics, since it is free to oscillate, there will be an infinite amount of particle states [GSW12, Sec. 1.3]. Moreover, among the first excited states there is a massless spin-2 particle. It was shown by Feynman and Weinberg that *any* theory of interacting spin-2 particles must be equivalent to Einstein gravity [Ton12, Ch. 2]. Hence, string theory inherently incorporates the quantum particle corresponding to gravity, called the **graviton**, making it into a candidate for quantum gravity.

This is very exciting, however there are some caveats. The first of which we already encountered: the theory does not contain fermions. Since most of the matter around us is mostly made out of fermions, a reasonable theory should contain them. Secondly, the ground state of the theory corresponds to a particle with negative mass-squared, called the **tachyon**. Because of this, the theory becomes unstable. One possible way to circumvent both issues is by introducing *supersymmetry*. We will briefly discuss the matter in the next section.

1.2 Superstrings

In an attempt to remedy one of the downfalls of bosonic string theory, the lack of fermions, one can do the most straightforward thing and just add them on the worldsheet. From QFT we know the typical fermionic terms in the Lagrangian and we can extend the action (1.2) accordingly. To be more precise, we introduce additional internal degrees describing fermions $\psi^\mu(\sigma,\tau)$ on the worldsheet, which will be vectors in the target space [BBS06, Ch. 4]. Now, by adding the usual Dirac action for the fermions $\psi^\mu(\sigma,\tau)$ to (1.1), we find in conformal gauge

$$S = -\frac{T}{2}\int_\Sigma d^2\sigma\,\left(\partial^\alpha X^\mu\partial_\alpha X_\mu + \bar\psi^\mu\partial\!\!\!/\psi_\mu\right). \tag{1.3}$$

It turns out that the above action is invariant under infinitesimal transformations that interchange the bosonic and fermionic fields [BBS06, Sec. 4.2]. This is known as **supersymmetry** (SUSY). To be more precise, we have described *worldsheet* supersymmetry and the string corresponding to (1.3) is often referred to as the Ramond-Neveu-Schwarz (RNS) string. There is also the concept of *spacetime* supersymmetry. However, we will not focus on this topic here and refer to [BBS06, Ch. 5] or [GSW12, Ch. 5]. Moreover, as usual, one can construct a (conserved) charge corresponding to a symmetry, which in this case is called the **supercharge** Q. In fact, a theory can have multiple supercharges. The amount of supercharges is denoted by $\mathcal{N}$.

Now, one can proceed as in the bosonic case and one finds the theory is only sensible[1] for a critical dimension of $d = 10$. However, the spectrum of the RNS string still has issues, such as the existence of a

[1] By this we mean no negative-norm states (ghosts) and Lorentz invariant.

tachyon. Hence, one must resort to additional techniques. By projecting the spectrum in a very specific way, the problems are resolved and a supersymmetric ten-dimensional spacetime theory is obtained [BBS06, Sec. 4.6]. This projection is due to Gliozzi, Scherk and Olive and consequently is known as *GSO projection*. Upon investigating the spectrum, after GSO projection, one sees the tachyon is no longer present. Hence, the two main issues of bosonic string theory are solved by considering a supersymmetric string theory in ten dimensions.

It is yet more restrictive, as one cannot write down any arbitrary ten dimensional theory. There are only five consistent superstring theories:

$$\text{Type I}, \quad \text{Type IIA}, \quad \text{Type IIB}, \quad \text{Heterotic } SO(32), \quad \text{Heterotic } E_8 \times E_8$$

The differences lie in the construction, the 'amount' of supersymmetry and the possible existence of open strings in these theories (see [BBS06]). It is not in the scope of this work to quantitatively discuss these differences. However, we point them out to emphasise the restrictive nature of string theory. Moreover, we would like to highlight their similarities: all consistent supersymmetric string theories live in *ten* dimensions. This is strengthened even further by the fact that these, seemingly very different, theories are related to each other via various *dualities*[2] (see [BLT12]). In later sections we will focus on one of the theories, however our findings will typically hold in general.

1.3 Effective Theories

In the previous section we concluded there are five superstring theories. To get a grasp on the behaviour of these theories and relate them to our usual description of particles in QFT, we need a workable framework. One useful observation is that the massive states become very heavy for $\alpha' \to 0$, i.e. for large string tension. Hence, in this limit, which corresponds to the low-energy limit (cf. [BBS06, Ch. 8]), a good approximation of the theory is given by the interactions of just the *massless* excitation. This approximation is called a **low-energy effective theory**. Due to spacetime symmetries (e.g. coordinate invariance) and supersymmetry, the possible effective theories are restricted. Furthermore, by additional requirements, such as anomaly cancellations, the effective ten-dimensional theory is completely fixed, up to higher order corrections (see [BLT12, Ch. 16]). They are given by so-called **supergravity theories**, which are supersymmetric extensions of Einstein gravity. Using the corresponding effective ten-dimensional supergravity action, one can obtain an effective four-dimensional theory through a process of 'compactification'. We will discuss this in Section 1.4. Hence, it is very helpful to obtain those effective actions. In this section we will discuss the effective supergravity theory corresponding to type IIB.

To get there, we first need to discuss the massless spectrum of the type IIB superstring. We will not derive this spectrum here, but refer to reader to [BBS06, Ch. 5]. In the fermionic part of the spectrum there are two left-handed Majorana-Weyl gravitino's (the supersymmetric partner of the graviton) and two right-handed Majorana-Weyl dilatino's. For the bosonic part, there are two sectors: the NS-NS and the R-R sector. The former consists of the metric g, the two form B_2 and the dilaton Φ. In the R-R sector there are three p-form fields, namely C_0, C_2 and C_4. The corresponding field strengths are

$$H_3 = dB_2, \quad F_{p+1} = dC_p.$$

For further purposes, it is convenient to combine the fields as follows

$$\tau := C_0 + ie^{-\Phi}, \quad G_3 := F_3 - \tau H_3, \quad \widetilde{F}_5 := F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3.$$

[2] One of which in particular, or a generalization thereof, is of interest in this work (cf. Chapter 5).

Then, the low-effective action that governs the interactions between the massless states of type IIB is given by (cf. [Den08, Ch. 3])

$$S_{\mathrm{IIB}} = \frac{2\pi}{l_s^8}\int_{\mathcal{M}_{10}} \mathcal{R}\star 1 - \frac{1}{2}\frac{d\tau\wedge\star d\overline{\tau}}{\operatorname{Im}\tau} + \frac{G_3\wedge\star\overline{G}_3}{\operatorname{Im}\tau} + \frac{1}{2}\widetilde{F}_5\wedge\star\widetilde{F}_5 + C_4\wedge H_3\wedge F_3. \tag{1.4}$$

Here, l_s is the string length[3] and $\mathcal{R}$ is the Ricci scalar corresponding to the metric g. Additionally, one has to impose the self-duality constraint $\star\widetilde{F}_5 = \widetilde{F}_5$ by hand, to reproduce the correct equations of motion.

Similar low-effective actions exist for the other types (see [BLT12, Ch. 16]) and they provide a way to do explicit computations. However, there is still a clear hurdle to describe reality with this theory: it is ten-dimensional. To be able to make predictions about nature we need to reduce the theory to a four-dimensional one. There are multiple ways to do this, however the most common approach is to 'curl up' the additional dimensions. This leads to the concept of 'compactification', which is the main topic of the next section. There we will see the effective four-dimensional theory associated to (1.4).

1.4 String Compactifications

As we have seen in Section 1.2, all consistent superstring theories predict spacetime to be ten-dimensional. Yet, all our macroscopic observations seem to indicate that our universe has four macroscopic dimensions. A proposed idea to deal with this issue is to take the additional dimensions and curl them up. Formally, this is known as **string compactification**. The main idea is to assume that the ten-dimensional spacetime is the product of a four-dimensional spacetime and a *'internal' manifold*:

$$\mathcal{M}_{10} = \mathcal{M}_4 \times X_6.$$

Typically, $\mathcal{M}_4$ is assumed to be maximally symmetric, i.e., either Minkowski, de Sitter or anti-de Sitter (see [BBS06, Sec. 9.4]). Furthermore, X_6 inherits a genuine Riemannian metric from $\mathcal{M}_{10}$, as all the negative signatures are absorbed by the metric on $\mathcal{M}_4$. Moreover, the internal manifold is assumed to be 'small' and compact. The terminology 'small' is somewhat ill-defined. Yet, if we assume the compact manifold to be of size l_c, they should be observed at energy scales around $E\sim 1/l_c$. Hence, by making X_6 small enough, the extra dimensions will be invisible to contemporary observations ([BBS06, p. 354]). A very interesting property of string compactifications is that the geometry of the internal manifold effects the physics in the four-dimensional spacetime. To illustrate this phenomenon we first consider a classical example introduced in the 1920's by Kaluza ([Kal21]) and Klein ([Kle26]), in which they obtained general relativity combined with electromagnetism from a higher dimensional gravity theory. Afterwards, in Section 1.4.2 we discuss how certain requirements, such as residual supersymmetry, on $\mathcal{M}_4$ places vast restrictions on the geometry of the internal manifold X_6.

1.4.1 The Kaluza-Klein Example

The concept of compactifying additional dimensions dates back to the 1920's. In 1921 and 1926 Kaluza and Klein, respectively, proposed to start with a five-dimensional gravity theory and compactify on a circle. This results in a gravity theory as well as electromagnetism in the remaining four-dimensional spacetime. Even though their attempt turned out to be an incorrect description of reality, the techniques are still used today in string compactifications. Therefore, we would like to present this example in this section. Before we do this, we consider a simpler example of a free scalar field to highlight some of the important features. Our main reference is [Li22].

[3] It is related to α' by $l_s = 2\pi\sqrt{\alpha'}$.

Consider the five-dimensional manifold $\mathcal{M}_5 = \mathbb{R}^{1,3} \times S^1_r$, where $\mathbb{R}^{1,3}$ is Minkowski space and S^1_r denotes a circle of constant radius $r \geq 0$. Let $x^M = (x^\mu, r\theta)$ denote coordinates on $\mathcal{M}_5$, where $x^\mu \in \mathbb{R}^{1,3}$ and θ denotes the parametrization of the circle. We identify $\theta \sim \theta + 2\pi$. Furthermore, we equip $\mathcal{M}_5$ with the Minkowski metric given, in these coordinates, by $ds_5^2 = \eta_{\mu\nu}dx^\mu dx^\nu + r^2 d\theta^2$. Now, consider a free massless scalar $\hat{\phi}$ with corresponding action

$$\begin{aligned} S[\hat{\phi}] &= -\frac{1}{2}\int_{\mathcal{M}_5} d\hat{\phi}(x^\mu,\theta)\wedge \star d\hat{\phi}(x^\mu,\theta) \\ &= -\frac{1}{2}\int_{\mathcal{M}_5} d^5x\, \eta^{MN}\partial_M\hat{\phi}\partial_N\hat{\phi} \\ &= -\frac{1}{2}\int_{\mathbb{R}^{1,3}\times S^1} d^5x\, \left\{\partial^\mu\hat{\phi}\partial_\mu\hat{\phi} + \frac{1}{r^2}(\partial_\theta\hat{\phi})^2\right\}. \end{aligned} \tag{1.5}$$

Then, the equation of motion of $\hat{\phi}$ is given by

$$\Box\hat{\phi} = 0 \iff \partial^\mu\partial_\mu\hat{\phi} + \frac{1}{r^2}\partial_\theta^2\hat{\phi} = 0. \tag{1.6}$$

Since we identify $\theta \sim \theta + 2\pi$, we have $\hat{\phi}(x^\mu, \theta + 2\pi) = \hat{\phi}(x^\mu, \theta)$. Hence, we can Fourier expand $\hat{\phi}$ as

$$\hat{\phi}(x^\mu,\theta) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\phi_n(x)e^{in\theta}.$$

The scalar field $\hat{\phi}$ being real now translates to $\overline{\phi_n} = \phi_{-n}$. Inserting this expression into the action (1.5) we obtain

$$S[\phi_n] = -\frac{1}{2}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}^{1,3}} d^4x\, \left\{\partial^\mu\phi_n\partial_\mu\overline{\phi_n} + \frac{n^2}{r^2}\overline{\phi_n}\phi_n\right\}. \tag{1.7}$$

Note, by varying the action above we obtain the equation of motion of ϕ_n for every n:

$$\partial^\mu\partial_\mu\phi_n - \frac{n^2}{r^2}\phi_n = 0.$$

Furthermore, these equations and the action (1.7) show that the ϕ_n are scalars in the four-dimensional spacetime. Moreover, the ϕ_n are massive with a mass n/r, for $n > 0$. Hence, one massless scalar in the five-dimensional theory yields a whole tower of massive four-dimensional scalar fields. Elements of this tower are known as *Kaluza-Klein (KK) modes*. Remarkably, the masses depend on the radius r of the circle. This is the first instance of the dependence of the effective four-dimensional physics on the geometry of the internal manifold. Notice, by making the radius r very small, the massive modes ϕ_n for $n > 0$ become undetectable in the low energy effective theory. Yet, the massless mode ϕ_0 will always be present. Such massless modes will appear more often as we will see and they play a key role in connecting string theory with the standard model.

Let us now switch our attention to the five-dimensional gravity compactification. Again, let $\mathcal{M}_5 = \mathbb{R}^{1,3} \times S^1_r$ equipped with the same coordinates as above. Consider the following metric on $\mathcal{M}_5$

$$d\hat{s}^2 = \hat{g}_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu + 2r^2(x)A_\mu(x)dx^\mu d\theta + r^2(x)d\theta^2, \tag{1.8}$$

which is the most general form. Note, we allow r to vary over $\mathbb{R}^{1,3}$. The action we want to consider is the corresponding Einstein-Hilbert action in five dimensions:

$$\hat{S}_{EH} = \frac{1}{2\kappa_5^2}\int_{\mathcal{M}_5} \hat{\mathcal{R}} \star 1,$$

where κ_5 is the five-dimensional Einstein gravitational constant. We would like to proceed as before: integrate over the circle to obtain the lower dimensional theory. For this we need to compute the five-dimensional Ricci scalar $\hat{\mathcal{R}}$. Due to the off-diagonal terms in (1.8), this is a tedious, yet straightforward, computation which is done in [Li22, Sec. 1B]. One finds

$$\hat{\mathcal{R}} = \mathcal{R} - \frac{2}{r}\nabla^\mu\nabla_\mu r - \frac{r^2}{4}F^{\mu\nu}F_{\mu\nu}, \tag{1.9}$$

where $\mathcal{R}$ is the Ricci scalar associated to the four-dimensional metric $g_{\mu\nu}$, ∇_μ is the four-dimensional Levi-Civita connection and $F_{\mu\nu}$ is the usual field strength of A_μ. Note, all the terms in (1.9) are independent of θ by assumption. Furthermore, the second term in (1.9) is a total derivative and thus will not contribute to the action, as $\mathbb{R}^{1,3}$ does not have a boundary. Substituting (1.9) in the action above and integrating over the circle yields

$$\begin{aligned}\frac{1}{2\kappa_5^2}\int_{\mathcal{M}_5} \hat{\mathcal{R}} \star 1 =& \frac{1}{2\kappa_5^2}\int_{\mathbb{R}^{1,3}} (\mathcal{R} - \frac{r^2}{4}F^{\mu\nu}F_{\mu\nu}) \cdot 2\pi r \star 1 \\ =& \frac{2\pi}{2\kappa_5^2}\int_{\mathbb{R}^{1,3}} r\mathcal{R} \star 1 - \frac{r^3}{2}F \wedge \star F.\end{aligned}$$

Up to this point we have been rather sloppy regarding dimensions. Let us first denote the ground state of r by r_0. In a way, r measures the deviation from the ground state radius. Because of this, the field r is often referred to as the *breathing mode*. Then, we can replace $\theta \to r_0\theta$ and $A_\mu \to \kappa_4 A_\mu$, where κ_4 is defined by

$$\frac{1}{2\kappa_4} = \frac{2\pi r_0}{2\kappa_5^2}.$$

In this manner, all fields have the appropriate mass dimension (cf. [Li22, p. 6]) and the four-dimensional action is given by

$$S_4 = \frac{1}{2\kappa_4^2}\int_{\mathbb{R}^{1,3}} r\mathcal{R} \star 1 - \frac{\kappa_4^2 r^3}{2}F \wedge \star F.$$

Note, the first term looks like an Einstein-Hilbert term up to a factor of r. By Weyl rescaling the metric $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = r g_{\mu\nu}$ we can absorb the factor r. However, this affects the Ricci scalar in a particular manner[4], namely

$$\widetilde{\mathcal{R}} = \frac{1}{r}\Big(\mathcal{R} - 6\triangle(\frac{1}{2}\log r) - 6\partial^\mu(\frac{1}{2}\log r)\partial_\mu(\frac{1}{2}\log r)\Big).$$

Here, $\triangle$ denotes the Laplacian, which is a total derivative and thus the second term will drop out of the four-dimensional action as before. Using this we end up with

$$S_4 = \int_{\mathbb{R}^{1,3}} \frac{1}{2\kappa_4^2}\mathcal{R} \star 1 - \frac{3}{2\kappa_4^2 r^2} dr \wedge \star dr - \frac{r^3}{2}F \wedge \star F.$$

[4]For this, see [Lee19, Thm. 7.30]. Note, we applied the formula for dimension four.

Hence, the resulting four-dimensional theory contains an Einstein-Hilbert term, i.e. gravity, a $U(1)$-gauge field and a massless scalar field. Again, notice that the geometry, in the sense of the radius of the circle, effects the physics in the effective theory.

Let us focus on the field r. First notice it depends on the point in $\mathbb{R}^{1,3}$. Moreover, the field r is *dynamical*, meaning the radius is not constant. Combining these two observations tells us that to every point in the four-dimensional spacetime there could be a circle with different radius. In other words, we are not compactifying on “one” circle but on a whole family of circles and the field r keeps track of the data, associating to a point in $\mathbb{R}^{1,3}$ the corresponding radius (see Figure 1.2). The field r is the first example we encounter of a **modulus**: a parameter that captures the geometry of the internal manifold.

In general string compactifications, numerous moduli fields enter the effective four-dimensional theory, parametrizing the ‘size’ and ‘shape’ of the internal manifold, intuitively. At a first glance, this seems very hopeful as it could be used to reproduce the many degrees of freedom of the Standard Model. Yet, there is a problem, which is already apparent in Kaluza-Klein example above. The field r is massless, hence it should be observed in experiments.

For general compactifications the situation is similar (cf. Section 1.4.2). Some moduli fields will be massless and thus should be observed. However, such massless scalars have not been seen in experiments. Moreover, their vacuum expectation values are not determined, meaning they do not predict specific values for physical quantities, e.g. coupling constants. Giving these fields a vacuum expectation value is known as **moduli stabilization** and is an active research area.

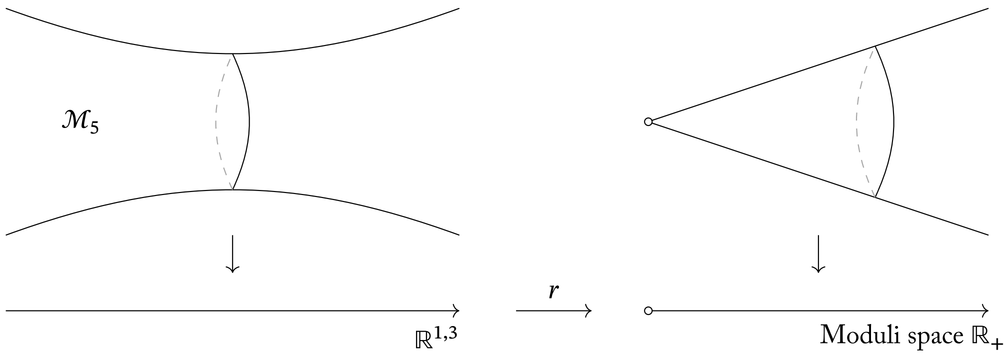

Figure 1.2: On the left we see the five-dimensional spacetime as a collection of circles fibered over $\mathbb{R}^{1,3}$. On the right we see a trivial family of circles fibered above the moduli space, which consists of the radii $r > 0$ in $\mathbb{R}_+$. The moduli field $r(x^\mu)$ maps a point in $\mathbb{R}^{1,3}$ to the corresponding radius in the moduli space of the internal circle. Note, the point $r = 0$ is not in the moduli space as it does not correspond to a regular S^1. In other words, it constitutes the boundary of the moduli space. Such boundary points play an important role in asymptotic Hodge theory. Inspiration from [Li22, Sec. 1.2].

1.4.2 General Compactification

We would like to extend the Kaluza-Klein approach in Section 1.4.1 to the ten-dimensional superstring theories from Section 1.2. Furthermore, by imposing properties on the residual four-dimensional theory we will see how the internal manifold is restricted. For example, we already saw X_6 must be a Riemannian manifold. Eventually, we will focus on type IIB and see moduli fields emerging that parameterize complex structure deformations, which will be extensively studied in Chapter 2. As a reference, we used [BLT12, Sec. 14.3].

Geometry of Internal Manifold

As before, we start we a ten-dimensional spacetime that is a product:

$$\mathcal{M}_{10} = \mathcal{M}_4 \times X_6.$$

Moreover, we impose minimal supersymmetry, i.e. $\mathcal{N} = 1$, in the four-dimensional theory. The reason for this is that supersymmetry is a tool that can solve incompletenesses in the Standard Model. Moreover, one requires *minimal* supersymmetry to have chiral matter. Theories with too much supersymmetry cannot have chiral matter. Since all the matter we observe is chiral, such theories are unrealistic.

Unbroken supersymmetry requires that the vacuum is preserved. This implies[5] that the expectation value of a generic field Φ, under the infinitesimal transformation, must vanish:

$$\langle \delta\Phi \rangle = 0.$$

A general feature from QFT is that the vacuum expectation value of any field other than a scalar must vanish for the theory to be Lorentz invariant. In particular, if one considers a fermionic field Φ_F, we must have $\langle \Phi_F \rangle = 0$. Now, using supersymmetry, we see for a bosonic field Φ_B

$$\langle \delta\Phi_B \rangle \sim \langle \Phi_F \rangle = 0.$$

Thus, the only non-trivial conditions can come from fermionic variations, as those could transform into scalars under supersymmetry ([BBS06, Ch. 9]),

$$\langle \delta\Phi_F \rangle = 0. \tag{1.10}$$

In particular, the above condition must hold for the gravitino ψ_M[6]. One can show it transforms as

$$\delta\psi_M = \nabla_M \epsilon + \chi,$$

with $\langle \chi \rangle = 0$. Here, the spinor ϵ is the infinitesimal parameter associated to the supersymmetry transformation. Furthermore, ∇ denotes the spin connection[7] on $\mathcal{M}_{10}$.

Now, condition (1.10) becomes

$$\overline{\nabla}_M \epsilon := \langle \nabla_M \epsilon \rangle = 0. \tag{1.11}$$

Here, $\overline{\nabla}$ denotes the vacuum expectation value of the spin connection. As $\mathcal{M}_{10}$ is a direct product, we can decompose ϵ as follows

$$\epsilon = \zeta \otimes \eta,$$

where ζ and η denote spinors on $\mathcal{M}_4$ and X_6, respectively. In particular, combining this with (1.11) yields

$$\overline{\nabla}_m \eta = 0. \tag{1.12}$$

Hence, the internal manifold X_6 must have a global covariantly constant spinor. Such spinors are called **parallel spinors** (or Killing spinors) and we have shown that their existence is a necessary condition for a supersymmetric compactification.

[5] We leave out some details here, for a more complete argument we refer to [BLT12, Sec. 14.3].

[6] Here, capital letters are used for spacetime indices on $\mathcal{M}_{10}$.

[7] This is the covariant derivative on spinors induced by the Levi-Civita connection. For more details see [Ham18, Sec. 6.10] or [Joy00, Sec. 3.6.1].

The existence of parallel spinors puts major restrictions on the type of manifolds we are allowed to pick for X_6. For example, it implies that X_6 must be Ricci-flat (cf. [BLT12, Sec. 14.3]). Furthermore, to able to talk about spinors in the first place, X_6 must admit a so-called spin structure (see [Ham18, Sec. 6.9] for details). This puts topological restrictions on X_6. Indeed, the existence of a spin structure is equivalent to X_6 being orientable and having a vanishing second Stiefel-Whitney class [Ham18, Thm. 6.9.7]. Moreover, having a *parallel* spinor produces a differential condition. In particular, it constrains the holonomy group of X_6. For completeness, let us recall the definition of holonomy:

Definition 1.4.1.
Let $E \to M$ be a vector bundle equipped with a connection ∇. Then, the **holonomy group** $\mathrm{Hol}_x(\nabla)$ *of ∇ based at x is defined by*

$$\mathrm{Hol}_x(\nabla) := \{P_\gamma \mid \gamma \textit{ piece-wise smooth loop based at } x\} \subset E_x.$$

Here $P_\gamma : E_x \to E_x$ denotes the parallel transport along γ.

When M is connected, the holonomy groups at different base points are related through conjugation and thus isomorphic [Joy00, Prop. 2.2.3]. Therefore, we omit the subscript and write $\mathrm{Hol}(\nabla)$ for the holonomy group of ∇, in that case. Whenever, (M, g) is a Riemannian manifold, we denote by $\mathrm{Hol}_x(g)$ (or $\mathrm{Hol}(g)$) the holonomy group corresponding to the Levi-Civita connection on M. The holonomy group is important to determine parallel tensors, as can be seen in the following result ([Joy00, Prop. 2.5.2]):

Theorem 1.4.2 (Holonomy principle).
Let M be a connected manifold, $E \to M$ a vector bundle with connection ∇ and $x \in M$. Then, there is a one-to-one correspondence between

i) parallel sections of E

ii) $\mathrm{Hol}_x(\nabla)$*-invariant vectors in E_x*

iii) sections of E, invariant under parallel transport

Now, we can apply this result to spinors on X_6. As said before, X_6 must admit a spin structure. This implies the holonomy $\mathrm{Hol}_x(\overline{\nabla})$ of X_6 is a subgroup of $\mathrm{Spin}(6)$. Here, $\overline{\nabla}$ denotes the spin connection on X_6 from (1.12). In six dimensions we have $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ and spinors have eight components. Moreover, the spin representation decomposes into two irreducible representations of $\mathrm{SU}(4)$ of dimension four [Joy00, Sec. 3.6.1], i.e.

$$\mathbf{8} = \mathbf{4} \oplus \overline{\mathbf{4}}.$$

Here, $\mathbf{4}$ and $\overline{\mathbf{4}}$ denote spinors of opposite chirality. Now, by the holonomy principle, a parallel spinor is equivalent to a $\mathrm{Hol}_x(\overline{\nabla})$-invariant spinor. The largest subgroup of $\mathrm{SU}(4)$ for which a spinor of definite chirality can be invariant is $\mathrm{SU}(3)$ [BBS06, Sec. 9.4]. This is due to the fundamental representation $\mathbf{4}$ of $\mathrm{SU}(4)$ decomposing under $\mathrm{SU}(3)$ as

$$\mathbf{4} = \mathbf{3} \oplus \mathbf{1}.$$

Here, $\mathbf{1}$ denotes a singlet and thus is invariant under $\mathrm{SU}(3)$. Hence, there exists a parallel spinor on X_6 if and only if $\mathrm{Hol}_x(\overline{\nabla}) \subseteq \mathrm{SU}(3)$. In fact, having holonomy contained in $\mathrm{SU}(3)$ already implies the existence of a spin structure (cf. [Joy00, Cor. 3.6.3]). Therefore, we have residual supersymmetry upon compactifying if and only if the holonomy of X_6 is contained in SU(3).

It turns out we can take it one step further. If one assumes the holonomy to be strictly contained in SU(3), i.e. $\mathrm{Hol}_x(\overline{\nabla}) \subseteq \mathrm{SU}(2)$, the resulting four-dimensional theories will *not* have minimal supersymmetry [BLT12, Sec. 14.3]. Therefore, the internal manifold must have holonomy *equal* to SU(3). These are particular complex manifolds[8] known as **Calabi-Yau** manifolds and they play a big role in both geometry as well as string theory. We will discuss the properties of Calabi-Yau manifolds in more detail in Chapter 2.

Effective Four-dimensional Theory

We have seen that compactifying a ten-dimensional supersymmetric theory requires the internal manifold to be a Calabi-Yau threefold. Now, we would like to proceed analogously to Section 1.4.1 and start with an action and integrate out the internal manifold. For exhibition purposes, we restrict ourselves to the effective type IIB theory given by (1.4). For this, we now assume $\mathcal{M}_{10} = \mathcal{M}_4 \times \mathrm{CY}_3$ where CY_3 denotes a Calabi-Yau threefold. We will again see that the geometry of the internal manifold will enter the effective four-dimensional theory. In this section we will only talk about some general features. For more details we refer to [GL04].

Performing the compactification of type IIB results in a four-dimensional $\mathcal{N} = 2$ supergravity theory [BBS06, Sec. 9.7]. Such theories have three types of supermultiplets: the gravity, vector and hypermultiplets. For our purposes, we focus on the gravity and vector multiplet sectors. Now, fixing the amount of vector multiplets in the theory n_V, the bosonic part of the four-dimensional $\mathcal{N} = 2$ supergravity is given by [LV20]

$$S = \int_{\mathcal{M}_4} \frac{1}{2}\mathcal{R} \star 1 - g_{i\bar{j}} dz^i \wedge \star d\bar{z}^{\bar{j}} + \frac{1}{4}\mathrm{Im}\, \mathcal{N}_{IJ} F^I \wedge \star F^J + \frac{1}{4}\mathrm{Re}\, \mathcal{N}_{IJ} F^I \wedge F^J. \tag{1.13}$$

Here, $\star$ denotes the Hodge star on $\mathcal{M}_4$. Furthermore, z^i with $i = 1, \ldots, n_V$ runs over the scalars in the vector multiplet and F^I with $I = 0, \ldots, n_V$ denote the field strengths of the gauge fields A^I in the multiplet, i.e. $F^I = dA^I$. Moreover, the kinetic couplings $g_{i\bar{j}}$, $\mathrm{Im}\, \mathcal{N}_{IJ}$ and $\mathrm{Re}\, \mathcal{N}_{IJ}$ all depend on $z^i, \bar{z}^j$. The reason why two of the couplings are related to a single kinetic matrix $\mathcal{N}_{IJ}$ is important for string theory and is related to 'special geometry' (see e.g. [Cra+97]). However, this goes beyond the scope of this work.

The important thing for us is how the action (1.13) is related to the geometry of the Calabi-Yau threefold. The dependence sits in the scalar fields $z^i, \bar{z}^j$. These fields are closely related to deformations of the complex structure on CY_3. Moreover, they are massless in the effective theory, as can be seen from (1.13). Consequently, they are referred to as **complex structure moduli**. A change of complex structure may result in a different Calabi-Yau manifold. This is reminiscent of the fact we are not compactifying on a single manifold, but on a family as seen in Section 1.4.1. The scalars z^i span a space called the **complex structure moduli space** in which distinct points correspond to non-isomorphic Calabi-Yau manifolds. Hence, understanding the complex structure moduli space of Calabi-Yau manifolds gives insight into the effective four-dimensional physical theory. Describing this space is the topic of Chapter 2.

Besides motivating the study of the complex structure moduli space, some further remarks are in order. Firstly, the number of vector multiplets n_V is directly related to the topology of CY_3, as $n_V = h^{2,1}$. Here $h^{2,1}$ denotes the dimension of the Dolbeault cohomology group $H^{2,1}(\mathrm{CY}_3)$. The appearance of this particular cohomology group is not an coincidence, it is strongly related to the complex structure moduli

[8] We restricted ourselves to the three-dimensional case. In general it would be manifolds with holonomy equal to SU(n). However, we would like to stress that multiple inequivalent definitions are used in the literature. See [Joy00, Sec. 6.1] for examples.

space as we will discuss in the next chapter. Secondly, the kinetic couplings in (1.13) also have to do with the geometric structure of CY_3. They are related to so-called 'periods', which we will encounter later on. For more background, see [GL04].

Finally, since the complex structure moduli are massless, the moduli stabilization problem is also relevant in this case. In fact, this problem is a feature of generic Calabi-Yau string compactifications. One way to produce a vacuum expectation value for the complex structure moduli is through **flux compactification**. This is a procedure in which sources of stress-energy, called *fluxes*, are included in the compactification. In this way, potentials for the moduli are obtained, making them massive. This is a very intricate subject, which we will not discuss in detail here. We refer to [MQ23] for a nice review of the topic. Schematically, the scalar potential coming from flux compactification has the following form

$$V_{\text{flux}} \sim \int_{\mathrm{CY}_3} \frac{G_3 \wedge \star \overline{G}_3}{\operatorname{Im} \tau}. \tag{1.14}$$

Importantly, $\star$ here denotes the Hodge star on the *internal manifold*. Hence, the Hodge theory of CY_3 is important for the four-dimensional physics as well. Moreover, this Hodge structure on CY_3 depends on the complex structure. Thus, understanding how the Hodge structure varies, when deforming the complex structure, is important for string compactifications. This is precisely what we will study in Chapter 3.

CHAPTER 2

Deformation Theory

Typically, in differential geometry people are interested in manifolds with additional (smooth) geometric structures. Examples can be found in Riemannian geometry, symplectic geometry and Poisson geometry. After defining the proper notion of a morphism in the appropriate setting a natural question arises: which objects are isomorphic? Such a question of classification is studied in almost all branches of mathematics.

Deformation theory tries to answer this classification question, at least locally, by starting with a particular geometric structure and trying to perturb it, while retaining the structure. Yet, many of the deformations could, in principle, be isomorphic. Hence, one should take the quotient under the action of isomorphisms. The leads to the idea of a *moduli space*. Schematically, it is thus defined as

$$\mathcal{M} = \mathsf{Struc} \,/\, \mathsf{Iso},$$

where Struc denotes the set of geometric structures of a particular type on a manifold. Moduli spaces themselves are interesting geometric objects. For example, it was shown by Riemann himself that the set of complex structures on a differentiable manifold admits a complex structure. A concept closely related to moduli spaces is that of a *Teichmüller space*. It arises when the above quotient is restricted to the identity component:

$$\mathsf{Teich} = \mathsf{Struc} \,/\, \mathsf{Iso}_0.$$

In a sense, one could view the Teichmüller space as the local moduli space. Consequently, studying the Teichmüller space gives insights into the moduli space.

In this chapter we are interested in deformations of complex manifolds and the geometry of the corresponding Teichmüller/moduli space. Specifically, we prove the classical theorem of Kuranishi on obstructions to complex deformations. To do this, we first discuss some preliminaires on Hodge theory and elliptic operator theory. Futhermore, we study the moduli space of a particular class of complex manifolds, namely Calabi-Yau manifolds. In particular, we prove the Bogomolov-Tian-Todorov theorem that states there are no obstructions to deformations on Calabi-Yau manifolds. We do this without the standard power series argument, but with global methods following [LZ20].

2.1 Mathematical Preliminaries

Motivated by string compactifications, we want to understand the structure of Calabi-Yau manifolds and their deformations. To do this, we need to lay a mathematical foundation, which is done in this section. First we introduce the appropriate notion of a complex manifold that lends itself nicely for deformations. Then, we focus on properties of Kähler manifolds and in particular Calabi-Yau manifolds. One key property of Kähler manifolds is the so-called Hodge decomposition theorem, which is at the heart of Hodge theory. As this is such a fundamental theorem in this work, we will prove it in Section 2.1.4. For this we need techniques from elliptic operator theory, which are introduced as well.

2.1.1 Complex Geometry

As mentioned above, Calabi-Yau manifolds are complex manifolds. Thus, it makes sense to first discuss some properties of complex manifolds. However, we will be very brief in this section and only discuss the relevant topics. For more background we refer to [Huy05], which is also our main reference.

Firstly, we would like to recall that a complex manifold is a differentiable manifold modelled on $\mathbb{C}^n$ such that the transition functions are holomorphic. Although this is a natural generalization from smooth manifolds, it is not optimal for studying deformations. For, this we need a *complex structure*. To define it, we first need the following concept:

Definition 2.1.1.
An **almost complex structure** *on a smooth manifold X is a bundle morphism*

$$J : TX \to TX, \quad J^2 = -\mathrm{id}.$$

A manifold equipped with an almost complex structure is an **almost complex manifold**.

It is rather simple to see that every complex manifold carries an almost complex structure (cf. [Huy05, Prop. 2.6.2]). However, the converse is not always true. To quantify this, we first need some machinery.

The existence of an almost complex structure J on X, induces a splitting of the complexification of the tangent bundle $T_{\mathbb{C}}X = TX \otimes \mathbb{C}$:

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X, \tag{2.1}$$

where $T^{1,0}X$ and $T^{0,1}X$ denote the $+i$ and $-i$ eigenspace of J, respectively[1]. Hence, we have

$$\overline{T^{1,0}X} = T^{0,1}X. \tag{2.2}$$

In the literature, $T^{1,0}X$ is referred to as the **holomorphic tangent bundle**. These subbundles yield an equivalent definition of an almost complex structure, which is captured in the following proposition

Proposition 2.1.2.
An almost complex structure on a smooth manifold X is completely determined by a subspace $L \subset T_{\mathbb{C}}X$ such that

$$T_{\mathbb{C}}X = L + \overline{L}, \quad L \cap \overline{L} = 0.$$

[1] To be precise, we consider the $\mathbb{C}$-linear extension of J.

Proof. — The proof is elementary linear algebra and is omitted here. ■

One should think about the subbundle L as the $+i$-eigenspace of the almost complex structure. Note, the space $\overline{L}$ completely determines the almost complex structure, as well. Hence, there is an equivalent proposition using $\overline{L}$.

Moreover, the subbundles of $T_{\mathbb{C}}X$ appearing in (2.1) measure whether the almost complex structure arises from complex coordinates on X, meaning it is a complex manifold.

Definition 2.1.3.
We say an almost complex structure J on X is **integrable** *if it comes from complex coordinates. Furthermore, we say an almost complex structure J is* **infinitesimally integrable** *if $T^{0,1}X$ is involutive, i.e.*

$$[T^{0,1}X, T^{0,1}X] \subset T^{0,1}X.$$

Note, because of the relation stated in (2.2) we could equivalently have defined infinitesimal integrability in terms of $T^{1,0}X$.

The motivation for this definition lies in the fact that the distribution $T^{0,1}X$ *is* involutive when X is complex (see [Cav22, Sec. 3.2]). However, involutivity is a infinitesimal condition and it is not clear whether there are other obstructions for an almost complex structure to come from complex coordinates. It turns out integrability is a necessary and sufficient condition, which is the result of the deep theorem:

Theorem 2.1.4 (Newlander-Nirenberg).
Any infinitesimally integrable almost complex structure is integrable.

Consequently, complex manifolds and infinitesimally integrable almost complex manifolds are the same thing. From now on we fix a complex manifold X with its induced integrable almost complex structure J, which we call the *complex structure*.

The decomposition (2.2) induces a decomposition of complex differential forms into so-called (p, q)-forms

$$\Omega^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X), \tag{2.3}$$

where

$$\Omega^{p,q}(X) = \Gamma(\bigwedge^p (T^{1,0}X)^* \otimes_{\mathbb{C}} \bigwedge^q (T^{0,1}X)^*).$$

The bundle on the right is usually denoted by

$$\bigwedge^{p,q} X := \bigwedge^p (T^{1,0}X)^* \otimes_{\mathbb{C}} \bigwedge^q (T^{0,1}X)^*.$$

The exterior derivative d on X behaves nicely with respect to the decomposition (2.3). Indeed, if we define

$$\partial := \pi^{p+1,q} \circ d : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \quad \bar{\partial} := \pi^{p,q+1} \circ d : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$$

we have the following properties

Proposition 2.1.5.
Let X be a complex manifold. Then, we have

i) $d = \partial + \bar{\partial}$

ii) $\partial^2 = 0$ *and* $\bar{\partial}^2 = 0$

iii) $\partial\bar{\partial} + \bar{\partial}\partial = 0$

Proof. — For the first property, we refer to [Huy05, Prop. 2.6.15]. The last two follow directly from $d^2 = 0$ and property *i*):

$$\begin{aligned} 0 &= d^2 \\ &= (\partial + \bar{\partial})(\partial + \bar{\partial}) \\ &= \partial^2 + \bar{\partial}^2 + \partial\bar{\partial} + \bar{\partial}\partial. \end{aligned}$$

Since, ∂^2, $\bar{\partial}^2$ and $\partial\bar{\partial} + \bar{\partial}\partial$ all land in different spaces, they should individually vanish. This proves the claim. ∎

By property *ii*) in the above proposition, it makes sense to define the corresponding cohomology. Typically, people are interested in the $\bar{\partial}$-cohomology.

Definition 2.1.6.
The **Dolbeault cohomology** *of a complex manifold X is defined by*

$$H^{p,q}(X) = \frac{\ker\left(\bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)\right)}{\operatorname{im}\left(\bar{\partial} : \Omega^{p,q-1}(X) \to \Omega^{p,q}(X)\right)}.$$

The dimensions of the Dolbeault cohomology groups $h^{p,q} := \dim H^{p,q}(X)$ are known as **Hodge numbers**.

Now, one could wonder whether the decomposition (2.3) descends to de Rham cohomology, where the (p,q)-forms are replaced by the corresponding classes in Dolbeault cohomology. It turns out this is not true in general. However, for a special kind of (compact) complex manifolds, called *Kähler manifolds*, it is true. The corresponding decomposition of de Rham cohomology is known as a *Hodge structure* and these play a central role in this work.

2.1.2 Kähler Geometry

We already hinted towards special properties of particular manifolds, called Kähler manifolds. In this section we define the notion of a Kähler manifold and work our way to the Hodge decomposition theorem. Finally, we encounter our first example of a Hodge structure and make some remarks about Calabi-Yau manifolds.

Kähler manifolds are right in the middle of the triple intersection between complex, Riemannian and symplectic geometry. Hence, they found their way into many applications in different areas of geometry. To define it, let us fix a complex manifold X with complex structure J.

Definition 2.1.7.
A Riemannian metric g *on* X *is called a* **Hermitian structure** *if the complex structure* J *is an orthogonal transformation with respect to* g*, i.e.* $g(J\cdot, J\cdot) = g(\cdot, \cdot)$*. The triple* (X, J, g) *then is referred to as a* **Hermitian manifold**. *The induced real* $(1, 1)$*-form* $\omega := g(J\cdot, \cdot)$ *is called the* **fundamental form**.

A **Kähler manifold** *is a Hermitian manifold* (X, J, g) *such that the fundamental form is closed, i.e.* $d\omega = 0$.

From this definition it is clear that a Kähler manifold is a complex, symplectic and Riemannian manifold. Moreover, when (X, J, g) is Kähler the fundamental form is referred to as the Kähler form and g is called the Kähler metric. Examples of Kähler manifolds are the complex Euclidean space $\mathbb{C}^n$, Riemann surfaces and complex projective space.

However, the most important class of examples to us is one we already encountered: Calabi-Yau's. At this stage we can refine our previous definition from Section 1.4.2.

Definition 2.1.8.
A **Calabi-Yau** *manfiold is a compact Kähler manifold* (M, J, g) *with* $\mathrm{Hol}(X) := \mathrm{Hol}(g) = \mathrm{SU}(n)$.

Furthermore, originally we defined a Calabi-Yau in terms of the holonomy of the spin connection. However, as $\mathrm{SU}(n)$ is simply connected they yield the same holonomy [Joy00, Sec. 3.6.2]. We would like to emphasize that the Kähler condition in the above definition is redundant. This is due to the fact that $\mathrm{SU}(n) \subset \mathrm{U}(n)$ and the holonomy group being contained in $\mathrm{U}(n)$ implies admission of a Kähler structure (cf. [Joy00, Prop. 4.4.2]). Moreover, the structure can be directly computed from the parallel spinor, as is done in [BBS06, Sec. 9.4].

Once again, we stress that various (in-)equivalent definitions are used in the literature. Our definition has some interesting implications that are very useful for our purposes:

Proposition 2.1.9.
Let X *be a Calabi-Yau* n*-fold. Then,*

i) The canonical bundle $K_X := \bigwedge^{n,0} T^*X$ *of* X *is trivial as a holomorphic line bundle*

ii) X *admits a global non-vanishing holomorphic* $(n, 0)$*-form* Ω

iii) The first Chern class of X *vanishes, i.e.* $c_1(X) = 0$

iv) X *admits a unique Ricci-flat Kähler metric*

Proof. — We will sketch the idea of the proof following [Joy00, Sec. 6.1]. First, note the first two assertions are equivalent. We will focus on the second one. Locally, one defines

$$\Omega = dz_1 \wedge \cdots \wedge dz_n.$$

Note, this defines non-vanishing a $(n, 0)$-form that, which furthermore is holomorphic. Now, the group $\mathrm{SU}(n)$ preserves Ω. Since $\mathrm{Hol}(X) = \mathrm{SU}(n)$, we can extend Ω to a well-defined global form, through parallel transport. In this way, Ω will be non-vanishing and holomorphic everywhere. Hence, we constructed a global holomoprhic non-vanishing $(n, 0)$-form on X. This proves assertions *i*) and *ii*).

For the third assertion, recall that the first Chern class of a manifold is given by the first Chern class of its tangent bundle. Furthermore, the first Chern class of a vector bundle is equal to that of its determinant bundle. Combining this, we find

$$c_1(X) = c_1(TX) = c_1(\det\ TX) = c_1(\bigwedge^{n,0} TX) = c_1(\overline{K_X}) = -c_1(K_X).$$

We already showed K_X to be trivial, meaning $c_1(K_X) = 0$. Hence, we conclude $c_1(X) = 0$.

Finally, the fourth condition is a direct consequence of the famous Yau's theorem, which we do not cover here. For more background, we refer to [Joy00, Ch. 5-6]. ■

Now, we would like to work towards the Hodge decomposition theorem. For this, we introduce two operators that will play an important role, using the structure present on a Kähler manifold[2] (X, J, g). Firstly, the **Lefschetz operator**

$$L : \Omega^k(X) \to \Omega^{k+2}(X), \quad \alpha \mapsto \alpha \wedge \omega.$$

Secondly, using the natural orientation on X together with the metric g, we get an induced **Hodge $\star$-operator**

$$\star : \Omega^k(X) \to \Omega^{2n-k}(X).$$

Here, $2n$ denotes the real dimension of X.

In particular, when X is compact we can define an L^2-metric on the space of complex forms $\Omega^k_{\mathbb{C}}(X)$, using the Hodge star. It is given by (cf. [WG07, Sec. 5.2])

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge \star\overline{\beta}. \tag{2.4}$$

Given this inner product, we can look for formal adjoints of the differential operators d, ∂ and $\bar{\partial}$. If we restrict ourselves to the exterior derivative, by formal adjoint we mean an operator d^* satisfying

$$(d\alpha, \beta)_{L^2} = (\alpha, d^*\beta)_{L^2}.$$

By integration by parts, one rather straightforwardly finds [Huy05, Lem. 3.2.3]

Proposition 2.1.10.
Let X be a compact Kähler manifold. Then,

$$d^* = -\star \circ d \circ \star, \quad \partial^* = -\star \circ \partial \circ \star, \quad \bar{\partial}^* = -\star \circ \bar{\partial} \circ \star.$$

To arrive at the decomposition theorem, we need some machinery from analysis. In particular, we need the theory of elliptic operators. We will introduce this subject and state the relevant results in the next section.

2.1.3 Elliptic Operators

Within differential geometry, one encounters various differential operators such as the exterior derivative and connections. A special kind of differential operators are those that are *elliptic*. They have very nice properties that can be used in geometry, functional analysis and partial differential equations. In this

[2] Actually, one only needs a Hermitian manifold. Yet, we would like to restrict ourselves to Kähler manifolds.

section we will introduce these operators and state the properties of interest. In particular, we will focus on *linear* differential operators. Our exhibition will be quite dense, so for a more detailed description we refer to [WG07, Ch. 4] or [Cav22, Ch. 9].

Before we give the definition, let us introduce some notation. Let X be a compact n-dimensional manifold. Then, a **multi-index** α is a collection $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. Furthermore, if $x = (x_1, \dots, x_n)$ denote local coordinates on X, we define $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ with α a multi-index. Now we can state the definition.

Definition 2.1.11.
Let $E, F \to X$ be vector bundles of rank p, q, respectively. We say a map $D : \Gamma(E) \to \Gamma(F)$ is a **linear differential operator** *if for any choice of local coordinates and local trivializations there exists a linear partial differential operator $\widetilde{D}$, such that for $s = (s_1, \dots, s_p) \in \Gamma(E|_U)$*

$$\widetilde{D}(s)_i = \sum_{\substack{j=1,\\ |\alpha| \le k}}^{p} C_{\alpha,ij}(x) \partial^\alpha s_j.$$

A differential operator is said to be of **order** k if there are no derivatives of order $\geq k+1$ in the local representation. We denote the space of linear differential operators $\Gamma(E) \to \Gamma(F)$ of order k by $\mathrm{Diff}_k(E, F)$.

Example 2.1.12.
Linear differential operators $\Gamma(E) \to \Gamma(F)$ of order zero are tensors. Furthermore, the operators $d, \partial, \bar{\partial}$ are linear differential operators of order 1. ◆

Now, we would like a tool to classify differential operators. Note, the order of a differential operator is defined by the highest degree derivative in its local expression. However, the explicit form of this highest order term is highly dependent on the coordinates. Hence, we want to construct an invariant object. A naive idea would be to consider differential operators of order precisely k, i.e. only derivatives of degree k in the local expression. Unfortunately, these do not exist, as a change in coordinates may introduce lower degree derivatives. This can be nicely seen when we consider the Laplacian $\triangle$ on $\mathbb{R}^2$ and rewrite it in polar coordinates:

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \xrightarrow[\text{coordinates}]{\text{polar}} \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

However, the degree two part on the right-hand-side is precisely how $\partial_x^2 + \partial_y^2$ would transform when viewed as an element in $\mathrm{Sym}^2 T\mathbb{R}^2$. Moreover, the highest degree part should send vectors from E to F. Hence, our invariant object corresponding to a differential operator D of order k is the associated element

$$\sigma(D) \in \mathrm{Sym}^k TX \otimes \mathrm{Hom}(E, F).$$

Or equivalently, by Serre-Swan, as a symmetric map

$$\sigma(D) : \mathrm{Sym}^k T^*X \to \mathrm{Hom}(E, F).$$

As symmetric maps are fully determined by their value on the diagonal, we may view $\sigma(D)$ as a map $T^*X \to \mathrm{Hom}(E, F)$. This leads to the definition (cf. [Cav22, Def. 9.6]).

Definition 2.1.13.
Let D be a differential operator of order k. The **principal symbol** *of D is defined as*

$$\sigma(D) : T^*X \times E \to F, \quad \sigma(D)|_x(\xi, v) := D(f^k s)|_x,$$

where $f \in C^\infty(X)$ is any function such that $f(x) = 0$ and $(df)_x = \xi$ and $s \in \Gamma(E)$ any section such that $s(x) = v$.

It can easily be shown that the above definition is independent of choices. Moreover, the principal symbol will reproduce the highest degree part of D because the only non-vanishing term at x will be the one where all the derivatives act on f, since $f(x) = 0$.

The principal symbol allows us to define a particular class of differential operators.

Definition 2.1.14.
A linear differential operator of order k $D : \Gamma(E) \to \Gamma(F)$ is **elliptic** *if for any $x \in X, \xi \in T_x^*X \setminus \{0\}$, the linear map*

$$\sigma(D)|_x(\xi, \cdot) : E_x \to F_x$$

is an isomorphism.

One could wonder about examples of elliptic operators. For this, we again consider the exterior derivative d. Its principal symbol can be easily computed and is given by $\sigma(d)(\xi, \cdot) = \xi \wedge \cdot$ (see [WG07, p. 117]). However, this is typically not an isomorphism. Similar results hold true for ∂ and $\bar{\partial}$, meaning that none of them are elliptic.

To construct elliptic operators, we consider a family of differential operators. Let $\{E_i\}$ be a collection of vector bundles over X and $\{D_i\}$ a collection of linear differential operators of order k, that fit into the following sequence

$$\cdots \to \Gamma(E_{i-1}) \xrightarrow{D_{i-1}} \Gamma(E_i) \xrightarrow{D_i} \Gamma(E_{i+1}) \to \cdots. \tag{2.5}$$

Associated to this sequence, we obtain for every $x \in X, \xi \in T_x^*X \setminus \{0\}$ a symbol sequence

$$\cdots \to E_{i-1}|_x \xrightarrow{\sigma(D_{i-1})(\xi,\cdot)} E_i|_x \xrightarrow{\sigma(D_i)(\xi,\cdot)} E_{i+1}|_x \to \cdots.$$

Definition 2.1.15.
The sequence (2.5) is called an **elliptic complex** *if it is a complex, i.e. $D_i \circ D_{i-1} = 0$, and the associated symbol complex is exact.*

If (2.5) is an elliptic complex, we can focus at E_i. At $\Gamma(E_i)$ we can define the self-adjoint operator

$$\triangle_D := D_i^* D_i + D_{i-1} D_{i-1}^*, \tag{2.6}$$

where D_i^* denotes the formal adjoint with respect to an L_2-inner product similar to (2.4), denoted by $(\cdot, \cdot)$. These Laplacians turn out to be elliptic [WG07, Sec. 4.5]:

Proposition 2.1.16.
Let (2.5) be an elliptic complex of Hermitian vector bundles. Then, the operator $\triangle_D$ is elliptic.

Proof. — To see $\triangle_D$ is elliptic, we must compute its symbol. By the properties of the symbol, we have

$$\sigma(\triangle_D) = \sigma(D_i^*) \circ \sigma(D_i) + \sigma(D_{i-1}) \circ \sigma(D_{i-1}^*) = \sigma(D_i)^* \circ \sigma(D_i) + \sigma(D_{i-1}) \circ \sigma(D_{i-1})^*.$$

As this should be pointwise an isomorphism, for $\triangle_D$ to be elliptic, the problem reduces to linear algebra. Suppose the following sequence between vector spaces of equal dimension

$$V_{i-1} \xrightarrow{A_{i-1}} V_i \xrightarrow{A_i} V_{i+1}$$

is exact at V_i and let $v \in \ker(A_i^* A_i + A_{i-1}A_{i-1}^*)$. Then,

$$0 = \left((A_i^* A_i + A_{i-1}A_{i-1}^*)v, v\right) = \|A_i v\|^2 + \|A_{i-1}^*\|^2. \tag{2.7}$$

Thus, $A_i v = 0$ and $A_{i-1}^* v = 0$. The former implies $v = A_{i-1}w$ for some $w \in A_{i-1}$, by exactness. Analogous to (2.7), we see $\|A_i w\| = \|v\| = 0$. This implies $v = 0$, proving that $A_i^* A_i + A_{i-1}A_{i-1}^*$ is an isomorphism. ■

By this proposition, we obtain multiple elliptic operators.

Example 2.1.17.

i) Consider the de Rham complex

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to \cdots.$$

We mentioned the symbol is given by $\sigma(d)(\xi, \cdot) = \xi \wedge \cdot$. One can show, given this symbol, that the associated symbol sequence is exact. Hence, the Laplacian $\triangle_d = d^*d + dd^*$ is elliptic.

ii) Consider the Dolbeault complex

$$\Omega^{p,0}(X) \xrightarrow{\bar\partial} \Omega^{p,1}(X) \xrightarrow{\bar\partial} \Omega^{p,2}(X) \to \cdots.$$

Since $\bar\partial$ is the projection onto the anti-holomorphic part, it is straightforward to see the symbol is given by $\sigma(\bar\partial)(\xi, \cdot) = \xi^{0,1} \wedge \cdot$. Again, it turns out the associated symbol sequence is exact, meaning $\triangle_{\bar\partial} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$ is elliptic. Completely analogous, it follows $\triangle_\partial = \partial^*\partial + \partial\partial^*$ is elliptic as well. ◆

Typically, people drop the subscripts on the differential operators in the elliptic complex (2.5) and replace it with a single symbol D, like the example above. From the context it should be clear which operator to apply. We will use this convention as well.

Elliptic operators associated to elliptic complexes are closely related to the cohomology of the complex. Let us study this relation. For this, we consider an elliptic complex (2.5) of Hermitian vector bundles with associated Laplacian (2.6). Let $\mathcal{H}_D := \ker \triangle_D$ denote the space of **harmonic sections**. This space plays a central role.

Proposition 2.1.18.
Let (2.5) be an elliptic complex of Hermitian vector bundles on a compact manifold. Then,

i) *A section* $s \in \Gamma(E_i)$ *is harmonic if and only if* $Ds = 0$ *and* $D^*s = 0$

ii) *The spaces* $\mathcal{H}_D$, $\operatorname{im}(D)$ *and* $\operatorname{im}(D)^*$ *are orthogonal w.r.t.* $(\cdot, \cdot)$

iii) $\ker(D)$ *is orthogonal to* $\operatorname{im}(D^*)$

iv) $\ker(D^*)$ *is orthogonal to* $\operatorname{im}(D)$

Proof. — i) Let $s \in \Gamma(E_i)$ be a section. If $Ds = 0$ and $D^*s = 0$, then

$$\triangle_D s = (D^*D + DD^*)s = 0.$$

Therefore, $s \in \mathcal{H}_D$. Conversely, if $s \in \mathcal{H}_D$, we have

$$0 = (\triangle_D s, s) = ((D^*D + DD^*)s, s) = \|Ds\|^2 + \|D^*s\|^2.$$

Since the right-hand-side is positive, both norms should vanish independently. Consequently, $Ds = 0$ and $D^*s = 0$.

ii) For $s \in \mathcal{H}_D$ and $Dt \in \operatorname{im}(D)$, we see using *i)*

$$(s, Dt) = (D^*s, t) = 0.$$

Analogously, it follows that $\mathcal{H}_{\mathcal{D}} \perp \operatorname{im}(D^*)$. Finally, for $D^*s \in \operatorname{im}(D^*)$ and $Dt \in \operatorname{im}(D)$ we see

$$(D^*s, Dt) = (s, D^2t) = 0.$$

iii) This follows directly from the properties of formal adjoints. Indeed, for $s \in \ker(D)$ and $D^*t \in \operatorname{im}(D^*)$ we have

$$(s, D^*t) = (Ds, t) = 0.$$

iv) Analogous to *iii)*. ■

The main result in elliptic operator theory is the following [WG07, Thm. 4.4.12]

Theorem 2.1.19.
Let $E \to M$ *be an Hermitian vector bundle and* $\triangle : \Gamma(E) \to \Gamma(E)$ *a self-adjoint elliptic operator. Then, there exists a linear mapping*

$$G : \Gamma(E) \to \Gamma(E)$$

such that

i) G *is self-adjoint such that* $\ker G = \ker \triangle$, $\triangle G = G\triangle$ *and*

$$\mathrm{id} = \triangle G + \mathbb{H} = G\triangle + \mathbb{H},$$

where $\mathbb{H}$ *denotes the orthogonal projection onto* $\mathcal{H}_\triangle$.

ii) *There is an orthogonal decomposition* $\Gamma(E) = \mathcal{H}_\triangle \oplus \operatorname{im}(\triangle)$

iii) $\dim \mathcal{H}_{\triangle} < \infty$

Typically, the mapping G is called *Green's operator*. In the case of p-forms, the above theorem was first proved by Hodge. It has some interesting consequences when applied to elliptic complexes (cf. [WG07, Thm. 4.5.2], [Cav22, Cor. 9.18]).

Corollary 2.1.20.
Let

$$\cdots \to \Gamma(E_{i-1}) \xrightarrow{D} \Gamma(E_i) \xrightarrow{D} \Gamma(E_{i+1}) \to \cdots$$

be an ellipic complex of Hermitian bundles. Then,

i) There is an orthogonal decomposition

$$\Gamma(E_i) = \mathcal{H}_{\triangle_D} \oplus \operatorname{im}(D^*) \oplus \operatorname{im}(D)$$

ii) D and D^* commute with G

iii) $\ker(D) = \operatorname{im}(D) \oplus \mathcal{H}_{\triangle_D}$ *and the projection* $\ker(D) \to \mathcal{H}_{\triangle_D}$ *induces an isomorphism*

$$\mathcal{H}^k_{\triangle_D} \cong H^k_D = \frac{\ker(D : \Gamma(E_k) \to \Gamma(E_{k+1}))}{\operatorname{im}(D : \Gamma(E_{k-1}) \to \Gamma(E_k))}.$$

In particular, H^k_D is finite-dimensional.

Proof. — i) Note, by Theorem 2.1.19 *i)* we have

$$s = (\triangle G + \mathbb{H})s = \mathbb{H}s + D^*(DGs) + D(D^*Gs).$$

Hence, we have

$$\Gamma(E_i) = \mathcal{H}_{\triangle_D} + \operatorname{im}(D^*) + \operatorname{im}(D).$$

In Proposition 2.1.18 we already showed these spaces to be orthogonal. This proves the first assertion.

ii) Let us show $DG = GD$, as the other claim is analogous. Note, both D and G vanish on $\mathcal{H}_{\triangle_D}$ by 2.1.18 and 2.1.19, respectively. Hence, it suffices to show the claim on $\mathcal{H}^{\perp}_{\triangle_D} = \operatorname{im}(\triangle_D)$. For $s \in \Gamma(E_i)$, we see by 2.1.19 *i)*

$$\begin{aligned} DG\triangle_D s - GD\triangle_D s &= D(\mathrm{id} - \mathbb{H})s - GDD^*Ds \\ &= Ds - GDD^*Ds \\ &= Ds - G\triangle_D Ds \\ &= Ds - (\mathrm{id} - \mathbb{H})Ds \\ &= Ds - Ds \\ &= 0. \end{aligned}$$

Here, we used $\mathbb{H}D = 0$. Indeed, since $\mathbb{H}$ is an orthogonal projection in a Hilbert space it is self-adjoint, thus

$$(\mathbb{H}Ds, t) = (Ds, \mathbb{H}t) = 0$$

for every $s, t \in \Gamma(E_i)$. Here we applied the orthogonality from 2.1.18 *ii*).

iii) By Proposition 2.1.18 we know $\ker(D)$ is orthogonal to $\operatorname{im}(D^*)$ and thus must lie in $\mathcal{H}_{\triangle_D} \oplus \operatorname{im}(D)$. On the other hand, any element in $\mathcal{H}_{\triangle_D} \oplus \operatorname{im}(D)$ is annihilated by D, as $D^2 = 0$ and by Proposition 2.1.18. Hence, $\ker(D) = \operatorname{im}(D) \oplus \mathcal{H}_{\triangle_D}$. Clearly, the projection $\pi : \ker(D) \to \mathcal{H}_{\triangle_D}$ is surjective and $\ker(\pi) = \operatorname{im}(D)$. Hence in degree k, by the first isomorphism theorem, π descends to a linear isomorphism

$$\tilde{\pi} : \ker(D)/\operatorname{im}(D) = H^k_D \xrightarrow{\sim} \mathcal{H}^k_{\triangle_D}.$$ ■

This concludes our discussion on elliptic operators. In the next section we will apply the above results to the elliptic complexes corresponding to d and $\bar{\partial}$.

2.1.4 Hodge Decomposition Theorem

The machinery developed in the previous section on elliptic operator theory has deep consequences when we apply them to compact Kähler manifolds. This is what we will do in this section. Recall, on a compact Kähler we have three differential operators, namely $d, \partial, \bar{\partial}$. In the previous section we discussed that their corresponding complexes were elliptic, making the Laplacians $\triangle_d, \triangle_\partial$ and $\triangle_{\bar{\partial}}$ elliptic. It turns out it is possible to connect these Laplacians on compact Kähler manifolds. Note, for d and $\bar{\partial}$ the cohomology associated to the elliptic complexes are precisely the de Rham and Dolbeault cohomology, respectively.

The precise connection between the Laplacians is the content of Hodge's theorem

Theorem 2.1.21 (Hodge).
Let (X, J, g) be a compact Kähler manifold. Then,

$$\triangle_d = 2\triangle_\partial = 2\triangle_{\bar{\partial}}.$$

Moreover, $\triangle$ commutes with $\star, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*, L$.

Proof. — The proof relies on numerous identities between the operators at play and is not that insightful, Consequently, we ommit it here and refer to [Huy05, Prop. 3.1.12]. ■

A direct consequence of this result is our sought-after Hodge decomposition theorem

Corollary 2.1.22 (Hodge decomposition).
If (X, J, g) is a compact Kähler manifold, we have

$$H^k(X, \mathbb{C}) = \bigoplus_{k=p+q} H^{p,q}(X), \qquad \overline{H^{p,q}(X)} = H^{q,p}(X).$$

Proof. — By Corollary 2.1.20 it suffices to show the decomposition at the level of harmonic forms:

$$\mathcal{H}^k_{\triangle_d} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\triangle_{\bar{\partial}}}.$$

Let α be d-harmonic, i.e. $\triangle_d \alpha = 0$. By Hodge's theorem we have $\triangle_{\bar{\partial}}\alpha = 0$. By the decomposition of forms (2.3), we can write $\alpha = \alpha^{k,0} + \cdots + \alpha^{0,k}$. Since $\triangle_{\bar{\partial}}$ respects the (p,q)-grading, we see $\triangle_{\bar{\partial}}\alpha = 0$

implies $\triangle_{\bar\partial}\alpha^{k,0} = \cdots = \triangle_{\bar\partial}\alpha^{0,k} = 0$. This proves one inclusion. For the other inclusion, let $\alpha = \alpha^{k,0} + \cdots + \alpha^{0,k}$ be the sum of $\bar\partial$-harmonic (p,q)-forms such that $p+q=k$. In particular, $\triangle_{\bar\partial}\alpha = 0$. Now, again by Hodge's theorem we see $\triangle_d \alpha = 0$, meaing α is d-harmonic.

For the final claim, note Hodge's theorem implies $\triangle_{\bar\partial}$ is a real operator, as $\triangle_d$ is real. Combining this with the fact that conjugation yields an isomorphism $\Omega^{p,q}(X) \to \Omega^{q,p}(X)$ completes the proof. ■

The fact that de Rham cohomology of compact Kähler manifolds admit such a nice decomposition is very useful and is the primary example of a *pure Hodge structure*. We will study those in more detail in Chapter 3. Interestingly, the Hodge decomposition does not depend on the Kähler structure [Huy05, Cor. 3.2.12]. Hence, it is solely connected to the complex structure. Moreover, the decomposition may vary whenever the complex structure is deformed. This is known as *variation of Hodge structure* and we study this in Section 3.3.

There is even more structure at the level of cohomology. To make it apparent, we should not work with the full cohomology group $H^k(X,\mathbb{C})$, as it is too large. This leads to the concept of *primitive cohomology*. To define it, let us fix a compact Kähler manifold (X,J,g) of complex dimension n with Kähler form ω. Furthermore, recall the Lefschetz operator from Section 2.1.2. It descends to cohomology

$$\begin{aligned} L : H^k(X,\mathbb{C}) &\to H^{k+2}(X,\mathbb{C}), \\ [\alpha] &\mapsto [\alpha \wedge \omega]. \end{aligned}$$

Note, this only depends on the Kähler class $[\omega]$. Now, the kth **primitive cohomology group** is defined by

$$H^k_{\mathrm{p}}(X,\mathbb{C}) := \ker(L^{n-k+1} : H^k(X,\mathbb{C}) \to H^{2n-k+2}(X,\mathbb{C})).$$

Since the primitive cohomology groups sit inside $H^k(X,\mathbb{C})$, we get an induced decomposition

$$H^k_{\mathrm{p}}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\mathrm{p}}(X). \tag{2.8}$$

Here, we defined $H^{p,q}_{\mathrm{p}}(X) := H^{p,q}(X) \cap H^k_{\mathrm{p}}(X,\mathbb{C})$. However, the decomposition of primitive cohomology groups comes with additional structure, absent in the original decomposition: it admits a *polarization*. This is a bilinear form on $H^k(X,\mathbb{C})$ given by (cf. [Voi02, Sec. 6.3.2])

$$S([\alpha],[\beta]) = (-1)^{\frac{k(k-1)}{2}} \int_X \omega^{n-k} \wedge \alpha \wedge \beta \tag{2.9}$$

when restricted to $H^k_{\mathrm{p}}(X,\mathbb{C})$ satisfies [WG07, Thm. 5.6.3]

Proposition 2.1.23.
The bilinear form in (2.9) satisfies

i) $S([\alpha],[\beta]) = (-1)^k S([\alpha],[\beta])$

ii) $S(H^{p,q}_{\mathrm{p}}, H^{r,s}_{\mathrm{p}}) = 0$ *for* $(p,q) \neq (s,r)$

iii) $i^{p-q} S(H^{p,q}_{\mathrm{p}}, \overline{H^{p,q}_{\mathrm{p}}}) > 0$

The decomposition (2.8) combined with the bilinear form (2.9) gives $H^p_{\mathrm{p}}(X,\mathbb{C})$ a so-called *polarized Hodge structure*. We will study those structures in more detail in Chapter 3. To make the structures we discussed more concrete, let us consider the 2-torus.

Example 2.1.24 (Torus).
Note, the 2-torus $\mathbb{T}^2$ is a compact Kähler manifold as it is a compact surface. Hence, by the Hodge decomposition

$$H^1(\mathbb{T}^2, \mathbb{C}) = H^{1,0}(\mathbb{T}^2) \oplus H^{0,1}(\mathbb{T}^2), \quad \overline{H^{1,0}(\mathbb{T}^2)} = H^{0,1}(\mathbb{T}^2).$$

We know $H^1(\mathbb{T}^2, \mathbb{C}) \cong \mathbb{C}^2$. Furthermore, we have a unique (up to scalar multiples) non-vanishing holomorphic $(1,0)$-form Ω, that spans $H^{1,0}(\mathbb{T}^2)$. Following [CMP17], let γ^*, δ^* denote the basis of $H^1(\mathbb{T}^2, \mathbb{C})$ Poincaré dual to the homology generators γ, δ indicated in Figure 2.1. Then, we can write

$$\Omega = A\gamma^* + B\delta^*,$$

where

$$A = \int_\gamma \Omega, \quad B = \int_\delta \Omega.$$

The coefficients A, B are known as *periods* and they combine into a so-called *period vector* $\mathbf{\Pi} = (A, B)$.

Since $\mathbb{T}^2$ is a surface, we note that the Lefschetz operator vanishes on $H^1(\mathbb{T}^2, \mathbb{C})$. Consequently, (2.9) for $n = k = 1$ should define a polarization. Note, it satisfies

$$\int_{\mathbb{T}^2} \Omega \wedge \Omega = 0, \quad \int_{\mathbb{T}^2} \overline{\Omega} \wedge \overline{\Omega} = 0, \quad i \int_{\mathbb{T}^2} \Omega \wedge \overline{\Omega} = \mathrm{Im}(B\overline{A}) > 0.$$

The third integral computes the volume of $\mathbb{T}^2$, hence the inequality. Note, these computations show the properties in Proposition 2.1.23 are met. Furthermore, note that the period vector $\mathbf{\Pi}$ determines the polarized Hodge structure completely. This turns out to be a feature that holds in more generality, as we will see in Chapter 3. ♦

To conclude this section, we would like to make some remarks about the Hodge theory on Calabi-Yau manifolds, in particular threefolds. By the existence of the (unique) holomorphic $(n,0)$-form Ω on a Calabi-Yau n-fold, we know $h^{n,0} = h^{0,n} = 1$. Moreover, by conjugation and Hodge-$\star$ duality we find $h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-q,n-p}$. This true for general compact Kähler manifold. However, on a Calabi-Yau we have [Joy00, Prop. 6.2.6]

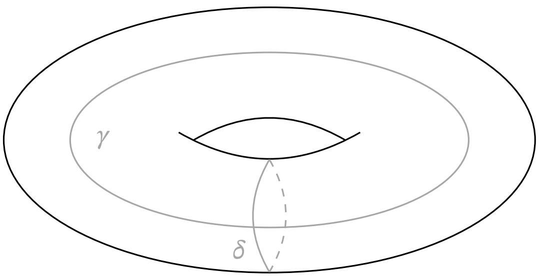

Figure 2.1: Graphical depiction of the 2-torus with its canonical $H_1(\mathbb{T}, \mathbb{Z})$ homology generators γ, δ.

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Figure 2.2: On the left we see the general Hodge diamond for an n-dimensional complex manifold. On the right we see the Hodge diamond for a Calabi-Yau threefold, where we identified $h^{2,1} = h^{1,2}$ and $h^{2,2} = h^{1,1}$.

Proposition 2.1.25.
Let (X, J, g) be a Calabi-Yau n-fold, then $h^{p,0} = 0$ for $0 < p < n$.

Using this, we obtain the *Hodge diamond* for Calabi-Yau threefolds in Figure 2.2. Note, the Kähler form ω is of type $(1, 1)$ and defines a non-trivial class[3] in $H^{1,1}(X)$, therefore $h^{1,1} \geq 1$. We see that for Calabi-Yau threefolds, the only independent Hodge numbers are $h^{1,1}$ and $h^{2,1}$. Moreover, since $h^{3,2} = h^{2,3} = 0$ we see $H^3_{\mathrm{p}}(X, \mathbb{C}) = H^k(X, \mathbb{C})$. Consequently, the middle cohomology of a Calabi-Yau threefold carries a polarized Hodge structure.

2.2 Deformations of Complex Structures

Now, having discussed the preliminaries, we can focus on the deformation theory of general complex manifolds. This is an old idea that dates back to Bernhard Riemann himself, who studied complex structures on Riemann surfaces. In his famous memoir *'Theorie der Abel'schen Functionen'* (Theory of Abelian functions) from 1857, Riemann proved a formula that computed number of independent parameters on which the deformation depended and called them 'moduli' [KS58a; KS58b]. In the years after Riemann, this question received quite a lot of attention. However, the higher-dimensional counter-part, deformations of complex manifolds of dimension at least two was not considered. Max Noether was the first to look at such higher-dimensional deformations in 1888, yet he restricted himself to algebraic surfaces [KS58a]. The general description of deformation of complex structures began in the 1950s with papers by Kodaira, Spencer, Frölich and Nijenhuis [KS57; FN57; KS58a; KS58b]. In [KS58a; KS58b], Kodaira and Spencer give an in-depth discussion of infinitesimal, or first order, deformations of complex structures and multiple stability theorems (including Kähler manifolds[4]) are proven in [KS60]. However, the question remained which first order deformations could be 'integrated' to proper deformations (cf. [KS58b]). The first sufficient conditions were proposed and proven in [KNS58]. Yet, they used a power series argument, which requires a highly analytical proof of convergence. For a complete exhibition of their methods we refer to [Kod86, Ch. 4-7].

In this section we want to discuss the existence of deformations of complex structures and possible obstructions without a power series argument. For this we will follow the methods of Kuranishi [Kur65; Kur62]. We will start by characterizing deformations of almost complex structures in terms of differential

[3]This is a general fact from symplectic geometry. The symplectic form on a compact manifold (without boundary) cannot be exact.

[4]To be specific, they proved that small deformations of Kähler manifolds remain Kähler.

forms with values in a vector bundle. Additionally, we will prove a necessary and sufficient condition on these differential forms such that they correspond to deformations of *complex* structures. Finally, we will discuss the existence of the *Kuranishi family* which was the main result of [Kur65], that ensures that for each compact complex manifold there exists a so-called 'semiuniversal' family of deformations.

2.2.1 Deforming Almost Complex Structures

Our starting point will be a compact smooth manifold X of real dimension $2n$ equipped with a fixed almost complex structure J. We will formulate the meaning of deforming an almost complex structure. This lays the foundation for integrable deformations. Our main references are [Kur65], [Huy05, Ch. 6] and to some extend [Gua04, Ch. 5].

Recall, an almost complex structure on X is completely determined by its $\pm i$-eigenspace (cf. Proposition 2.1.2). Thus, one can construct a new almost complex structure that is 'nearby' by tilting $\overline{T^{0,1}X}$ by a small amount to a subspace $\overline{K} \subset T_{\mathbb{C}}X$ (see Figure 2.3). To be precise, this tilting is parameterized by a bundle map

$$\epsilon : T^{0,1}X \to T^{1,0}X, \quad \eta \mapsto \pi_{1,0} \circ (\pi_{0,1}|_{\overline{K}})^{-1}(\eta)$$

so that $\overline{K} = \operatorname{gr} \epsilon$. Note, the map $\pi_{0,1}|_{\overline{K}}$ does not have an inverse for too big tilts[5]. Furthermore, the zero bundle map corresponds to the original almost complex structure. Hence, we have the following equivalence[6]

$$\left\{\begin{matrix}\text{Almost complex structures} \\ \text{sufficiently close to } J\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Bundle maps } \epsilon : T^{0,1}X \to T^{1,0}X \\ \text{sufficiently close to the zero map}\end{matrix}\right\}. \tag{2.10}$$

Note, by the Serre-Swan theorem we may regard the bundle map ϵ as a section in $\Gamma((T^{0,1}X)^* \otimes T^{1,0}X)$, i.e. $\epsilon \in \Omega^{0,1}(X, T^{1,0}X)$.

Now, consider a continuous family of almost complex structures $J(t)$ such that $J = J(0)$. Similarly to the above, we obtain a continuous splitting $T_{\mathbb{C}}X = T_t^{1,0}X \oplus T_t^{0,1}X$ for every t. By the equivalence in (2.10) above, we can encode the deformation $J(t)$, for small t at least, by maps

$$\epsilon(t) : T^{0,1}X \to T^{1,0}X, \quad \eta \mapsto \pi_{1,0} \circ (\pi_{0,1}|_{T_t^{0,1}X})^{-1}(\eta).$$

Again, so that $T_t^{0,1}X = \operatorname{gr} \epsilon(t) = \{ \epsilon(t)\eta + \eta \mid \eta \in T^{0,1}X \}$.

2.2.2 Integrability Condition and the Maurer-Cartan equation

Our initial interest was deformations of *complex* structures not of almost complex structures. However, we know that every complex structure on X induces an almost complex structure. Recall, such almost complex structures were called integrable. Moreover, by the celebrated Newlander-Nirenberg theorem 2.1.4 we have a necessary and sufficient condition for an almost complex structure to be integrable, namely the involutivity of the (anti-)holomorphic tangent bundle. Combining this result with equivalence (2.10) of the previous section will give us a way to identify deformations corresponding to complex structures.

To do so, we fix a complex structure J on X and consider a continuous family of almost complex structures $J(t)$ such that $J = J(0)$. We are looking for the $J(t)$ that are integrable, as they correspond to

[5] For example, if one crosses $T^{1,0}X$, as the projection is identically zero in that case.

[6] The word 'sufficiently' in the equivalence can be made precise in the appropriate Whitney topology. However, this is not necessary for our purposes.

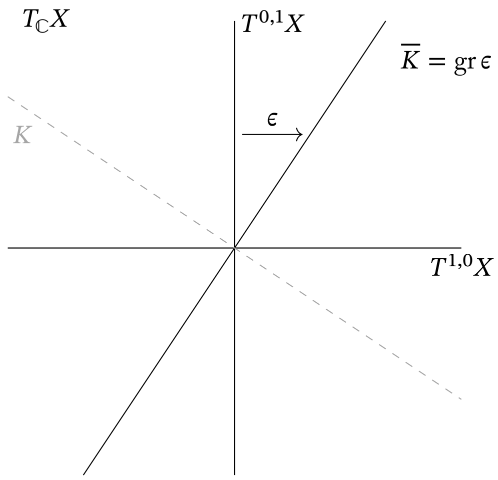

Figure 2.3: Deformation of an almost complex structure by a bundle morphism $\epsilon : T^{0,1}X \to T^{1,0}X$.

complex structures on X. The Newlander-Nirenberg theorem says the almost complex structure $J(t)$ is integrable if and only if the distribution $T_t^{0,1}X$ is involutive, i.e.,

$$[T_t^{0,1}X, T_t^{0,1}X] \subset T_t^{0,1}X.$$

We would like to rewrite this condition in terms of the bundle maps $\epsilon(t)$. To do this, we first note there exists a natural generalization of the $\bar{\partial}$-operator that acts on $\Omega^\bullet(X, T^{1,0}X)$ [Cav22, Ch. 5]. We will denote this operator by $\bar{\partial}$ as well. Furthermore, we need to define a 'Lie bracket' [Huy05, Sec. 6.1]

$$[\cdot,\cdot] : \Omega^{0,p}(X, T^{1,0}X) \times \Omega^{0,q}(X, T^{1,0}X) \to \Omega^{0,p+q}(X, T^{1,0}X)$$

by taking the usual bracket in $T^{1,0}X$ and wedging on the form part. In local coordinates, for $\alpha = \sum_I d\bar{z}_I \otimes X_I$ and $\beta = \sum_J d\bar{z}_J \otimes Y_J$, it is given by

$$[\alpha, \beta] = \sum d\bar{z}_I \wedge d\bar{z}_J \otimes [X_I, Y_J].$$

The bracket is well-defined and has the following properties [Kur65]

$$\begin{gathered}
[\alpha,\beta] = (-1)^{pq+1}[\beta,\alpha] \\
\bar{\partial}[\alpha,\beta] = [\bar{\partial}\alpha,\beta] + (-1)^p[\alpha,\bar{\partial}\beta] \\
(-1)^{pr}[\alpha,[\beta,\gamma]] + (-1)^{pq}[\beta,[\gamma,\alpha]] + (-1)^{qr}[\gamma,[\alpha,\beta]] = 0,
\end{gathered}$$

for $\gamma \in \Omega^{0,r}(X, T^{1,0}X)$.

Using this bracket, we can characterize which deformations stay integrable (cf. [KNS58]).

Proposition 2.2.1.
The almost complex structure $J(t)$ is integrable if and only if the associated bundle morphism $\epsilon(t)$ satisfies the Maurer-Cartan equation

$$\bar{\partial}\epsilon(t) + \frac{1}{2}[\epsilon(t), \epsilon(t)] = 0. \tag{2.11}$$

Proof. — We follow the proof in [Huy05, Lem. 6.1.2][7]. Note, it suffices to prove the statement locally. In local coordinates[8] of X, we can write

$$\epsilon(t) = \sum_{k,l} \epsilon_{kl}(t) d\bar{z}_k \otimes \frac{\partial}{\partial z_l}. \tag{2.12}$$

Suppose, the almost complex structure $J(t)$ is integrable. Then, the distribution $T_t^{0,1}X$ is involutive. Hence,

$$[\frac{\partial}{\partial \bar{z}_i} + \epsilon(t)\left(\frac{\partial}{\partial \bar{z}_i}\right), \frac{\partial}{\partial \bar{z}_j} + \epsilon(t)\left(\frac{\partial}{\partial \bar{z}_j}\right)] \in T_t^{0,1}X. \tag{2.13}$$

Note,

$$[\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}] = 0,$$

as J is integrable. Therefore, (2.13) is equivalent to

$$[\frac{\partial}{\partial \bar{z}_i}, \epsilon(t)\left(\frac{\partial}{\partial \bar{z}_j}\right)] + [\epsilon(t)\left(\frac{\partial}{\partial \bar{z}_i}\right), \frac{\partial}{\partial \bar{z}_j}] + [\epsilon(t)\left(\frac{\partial}{\partial \bar{z}_i}\right), \epsilon(t)\left(\frac{\partial}{\partial \bar{z}_j}\right)] \in T_t^{0,1}X. \tag{2.14}$$

We investigate the above terms separately. Using (2.12), we see the first term in (2.14) becomes[9]

$$\begin{aligned}
[\frac{\partial}{\partial \bar{z}_i}, \epsilon(t)\left(\frac{\partial}{\partial \bar{z}_j}\right)] &= \sum_l [\frac{\partial}{\partial \bar{z}_i}, \epsilon_{jl} \frac{\partial}{\partial z_l}] \\
&= \sum_l \frac{\partial \epsilon_{jl}}{\partial \bar{z}_i} \frac{\partial}{\partial z_l}.
\end{aligned}$$

Combining the first two terms in (2.14) then yields

$$\begin{aligned}
[\frac{\partial}{\partial \bar{z}_i}, \epsilon(t)\left(\frac{\partial}{\partial \bar{z}_j}\right)] + [\epsilon(t)\left(\frac{\partial}{\partial \bar{z}_i}\right), \frac{\partial}{\partial \bar{z}_j}] &= \sum_l \left(\frac{\partial \epsilon_{jl}}{\partial \bar{z}_i} - \frac{\partial \epsilon_{il}}{\partial \bar{z}_j}\right) \frac{\partial}{\partial z_l} \\
&= (\bar{\partial}\epsilon)(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}).
\end{aligned}$$

For the third term in (2.14) we have,

$$[\epsilon(t)\left(\frac{\partial}{\partial \bar{z}_i}\right), \epsilon(t)\left(\frac{\partial}{\partial \bar{z}_j}\right)] = \sum_{l,m} [\epsilon_{il} \frac{\partial}{\partial z_l}, \epsilon_{jm} \frac{\partial}{\partial z_m}]$$

[7] The statement there is missing the factor $\frac{1}{2}$.
[8] Note, these are holomorphic coordinates with respect to the fixed complex structure J.
[9] We suppressed the t-dependence for clarity.

$$= \frac{1}{2}\left(\sum_{l,m}\sum_{p,q} d\bar{z}_l \wedge d\bar{z}_p[\epsilon_{lm}\frac{\partial}{\partial z_m}, \epsilon_{pq}\frac{\partial}{\partial z_q}]\right)(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j})$$
$$= \frac{1}{2}[\epsilon, \epsilon](\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}).$$

We conclude that integrability of $J(t)$ implies

$$\left(\bar{\partial}\epsilon + \frac{1}{2}[\epsilon, \epsilon]\right)(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}) \in T_t^{0,1}X.$$

Note, formally we have $\bar{\partial}\epsilon + \frac{1}{2}[\epsilon, \epsilon] \in \Omega^{0,2}(X, T^{1,0}X)$. Hence,

$$\bar{\partial}\epsilon + \frac{1}{2}[\epsilon, \epsilon] \in \Omega^{0,2}(X, T^{1,0}X \cap T_t^{0,1}X).$$

However, for small t we have $T^{1,0}X \cap T_t^{0,1}X = 0$. Thus, ϵ satisfies the Maurer-Cartan equation.

Conversely, if $\epsilon(t)$ satisfies the Maurer-Cartan equation, the above computation shows that $T_t^{0,1}X$ is involutive in any local frame. Hence, it holds for all sections of $T_t^{0,1}X$. ∎

If one considers first order (or infinitesimal) deformations, Proposition 2.2.1 says that such deformations must be $\bar{\partial}$-closed, as the bracket yields a non-linear term. This is the first hint towards the infinitesimal description of the moduli space of complex structures on a manifold. We want to emphasise that precisely the non-linear behaviour of the bracket makes the existence of deformations non-trivial.

At this stage we have identified which nearby deformations are integrable. However, these integrable almost complex structures could give rise to isomorphic complex structures. Hence, we would like to identify such equivalent nearby deformations. We follow [Kur65] and say two deformations are equivalent if they are related by a small diffeomorphism. Hence, one should consider the action of $\mathsf{Diff}_0(X)$, the identity component of $\mathsf{Diff}(X)$. So, technically we are describing Teich, which for our purposes is sufficient. Note, for every element in $\mathsf{Diff}_0(X)$, there is a path to the identity. Hence, we should consider deformations coming from one-parameter families $F_t : X \to X$ of diffeomorphisms. It is known that such families are equivalent to flows of vector fields on X. Heuristically, this can be seen by viewing $\mathsf{Diff}(X)$ as an infinite dimensional Lie group with Lie algebra $\mathfrak{X}(X)$. Actually, this can be made precise in the language of Fréchet Lie groups, see e.g. [Ham82, Sec. 4.6]. Consequenlty, it suffices to restrict our discussion to flows of vector fields on X.

Let $V \in \mathfrak{X}(X)$ be a real vector field and denote its flow by F_t. Using the flow we obtain a new complex structure on X

$$\tilde{J} := dF_t \circ J \circ (dF_t)^{-1} \tag{2.15}$$

on X. Moreover, it induces continuous family of complex structures. Our objective is to translate the fact $\tilde{J}$ comes from a diffeomorphism to a condition on the deformation $\epsilon(t)$.

Note, (2.15) implies the anti-holomorphic tangent $T^{0,1}\widetilde{X}$ corresponding to $\tilde{J}$ is given by

$$T^{0,1}\widetilde{X} = dF_t(T^{0,1}X). \tag{2.16}$$

Hence, we should study the map dF_t. Local holomorphic coordinates $\{z_i\}$ on X yield $2n$ real coordinates $(z, \bar{z})$. In these local coordinates, we can write

$$\left.\frac{d}{dt}\right|_{t=0} F_t = V = \sum_i f_i(z, \bar{z})\frac{\partial}{\partial z_i} + \overline{f_i}(z, \bar{z})\frac{\partial}{\partial \bar{z}_i} \tag{2.17}$$

The diffeomorphism F_t yields new local coordinates $(\tilde{z}, \bar{\tilde{z}})$. Locally, this transformation can be described by [Huy05, Ch. 6]

$$\begin{aligned}\tilde{z}_i &= z_i + t f_i(z,\bar{z}) + \mathcal{O}(t^2)\\ \bar{\tilde{z}}_i &= \bar{z}_i + t\overline{f_i}(z,\bar{z}) + \mathcal{O}(t^2).\end{aligned}$$

This can been seen by locally defining $\tilde{z}_i := F_t(z_i)$ and applying Taylor's theorem at $t = 0$ and similarly for $\bar{\tilde{z}}_i$. Moreover, in these coordinates we have

$$\begin{aligned}dF_t =& \sum_i d\tilde{z}_i \otimes \frac{\partial}{\partial z_i} + d\overline{\tilde{z}_i} \otimes \frac{\partial}{\partial \bar{z}_i}\\
=& \sum_{i,j}\left(\frac{\partial \tilde{z}_i}{\partial z_j} dz_j \otimes \frac{\partial}{\partial z_i} + \frac{\partial \tilde{z}_i}{\partial \bar{z}_j} d\bar{z}_j \otimes \frac{\partial}{\partial z_i} + \frac{\partial \bar{\tilde{z}}_i}{\partial z_j} dz_j \otimes \frac{\partial}{\partial \bar{z}_i} + \frac{\partial \bar{\tilde{z}}_i}{\partial \bar{z}_j} d\bar{z}_j \otimes \frac{\partial}{\partial \bar{z}_i}\right)\\
=& \sum_i \left(dz_i \otimes \frac{\partial}{\partial z_i} + d\bar{z}_i \otimes \frac{\partial}{\partial \bar{z}_i}\right) + t\sum_{i,j}\left(\frac{\partial f_i}{\partial z_j} dz_j \otimes \frac{\partial}{\partial z_i} + \frac{\partial f_i}{\partial \bar{z}_j} d\bar{z}_j \otimes \frac{\partial}{\partial z_i}\right.\\
&\left. + \frac{\partial \overline{f_i}}{\partial z_j} dz_j \otimes \frac{\partial}{\partial \bar{z}_i} + \frac{\partial \overline{f_i}}{\partial \bar{z}_j} d\bar{z}_j \otimes \frac{\partial}{\partial \bar{z}_i}\right)\\
=& \mathrm{id} + t\sum_{i,j}\left(\frac{\partial f_i}{\partial z_j} dz_j \otimes \frac{\partial}{\partial z_i} + \frac{\partial f_i}{\partial \bar{z}_j} d\bar{z}_j \otimes \frac{\partial}{\partial z_i} + \frac{\partial \overline{f_i}}{\partial z_j} dz_j \otimes \frac{\partial}{\partial \bar{z}_i} + \frac{\partial \overline{f_i}}{\partial \bar{z}_j} d\bar{z}_j \otimes \frac{\partial}{\partial \bar{z}_i}\right).\end{aligned}$$

When we consider the image of an element $\eta \in T^{0,1}X$ under dF_t (cf. (2.16)) in these local coordinates, we see

$$\begin{aligned}dF_t(\eta) &= \eta + t\sum_{i,j}\left(\underbrace{\frac{\partial f_i}{\partial z_j} dz_j \otimes \frac{\partial}{\partial z_i}}_{=0 \text{ on } T^{0,1}X} + \frac{\partial f_i}{\partial \bar{z}_j} d\bar{z}_j \otimes \frac{\partial}{\partial z_i} + \underbrace{\frac{\partial \overline{f_i}}{\partial z_j} dz_j \otimes \frac{\partial}{\partial \bar{z}_i}}_{=0 \text{ on } T^{0,1}X} + \frac{\partial \overline{f_i}}{\partial \bar{z}_j} d\bar{z}_j \otimes \frac{\partial}{\partial \bar{z}_i}\right)\eta\\
&= \eta + \sum_{i,j}\left(\frac{\partial f_i}{\partial \bar{z}_j} d\bar{z}_j \otimes \frac{\partial}{\partial z_i} + \frac{\partial \overline{f_i}}{\partial \bar{z}_j} d\bar{z}_j \otimes \frac{\partial}{\partial \bar{z}_i}\right)\eta. \end{aligned} \tag{2.18}$$

Note, the third term in (2.18) maps $T^{0,1}X$ onto itself. Hence, from this computation we see $T^{0,1}\widetilde{X}$ is the graph of the bundle morphism

$$T^{0,1}X \to T^{1,0}X, \quad \frac{\partial}{\partial \bar{z}_j} \mapsto \sum_i \frac{\partial f_i}{\partial \bar{z}_j}\frac{\partial}{\partial z_i}. \tag{2.19}$$

Note, using (2.17) we see (2.19) can be written in a coordinate-free way. Hence, we found (cf. [Huy05, Lem. 6.1.4])

Proposition 2.2.2.
First order deformations of J induced by flows F_t of vector fields on X are given by

$$\tilde{\epsilon} := \bar{\partial}\left(\left(\frac{d}{dt}\Big|_{t=0} F_t\right)^{1,0}\right) : T^{0,1}X \to T^{1,0}X.$$

Another way to interpret the above proposition is that, infinitesimally, the difference of two equivalent first order deformations lies in the image of $\bar{\partial}$, i.e. are $\bar{\partial}$-exact. We previously saw that first order deformations had to be $\bar{\partial}$-closed, thus they define classes in cohomology. Consequently, we expect the tangent space to the moduli space to lie in[10] $H^1(X, T^{1,0}X)$.

2.2.3 The Kuranishi Family

We will show the deformation theorem due to Kuranishi [Kur65] that constructs a specific family of deformations for any complex manifold X that is compact. The proof differentiates itself from earlier proofs (e.g. [KS57; Kur62]) as it does not apply power series techniques. The argument relies on analysis and elliptic operator theory. In particular, the operator $\triangle_{\bar{\partial}}$ is central. Consequently, the theory developed in Section 2.1.3 will be important. Our main references are [Kur65] and [Gua04, Sec. 5.2]

Let us state the main theorem

Theorem 2.2.3 (Kuranishi).
Let (X, J) be a complex manifold. There exists an open neighbourhood $U \subset H^1(X, T^{1,0}X)$ containing zero, a smooth family $\widetilde{\mathcal{T}} = \{\epsilon(t) \mid t \in U,\ \epsilon(0) = 0\}$ of almost complex deformations of J and an obstruction map $\Phi : U \to H^2(X, T^{1,0}X)$, such that $\mathcal{T} = \{\epsilon(t) \mid t \in \mathcal{Z} := \Phi^{-1}(0)\}$ are precisely the integrable deformations. The space $\mathcal{T}$ is called the **Kuranishi family**.

Proof. — For the proof we use some theory about Sobolev spaces, which we do not cover. For the relevant material we refer to [WG07, Sec. 4.1].

Since we want to describe deformations modulo isomorphisms, i.e. a quotient, we should propose a 'gauge slice' to parameterize this quotient. To motivate which slice to pick, note the set of integrable deformations sit inside the vector space $\Omega^{0,1}(X, T^{1,0}X)$. Around an integrable deformation ϵ, the set of integrable deformations can be approximated by the vector space $W := \ker \bar{\partial} : \Omega^{0,1}(X, T^{1,0}X) \to \Omega^{0,2}(X, T^{1,0}X)$ (see Figure 2.4). The orbit through ϵ can then be approximated by $\operatorname{im} \bar{\partial}$, by Proposition 2.2.2. As W is a local model of the set of integrable deformations, the vectors in W orthogonal to $\operatorname{im} \bar{\partial}$ will produce the desired slice. Note, $(\operatorname{im} \bar{\partial})^{\perp} = \ker \bar{\partial}^*$ by Proposition 2.1.18. Hence, we expect a small neighbourhood in

$$\mathcal{G} := \{\epsilon \in \Omega^{0,1}(X, T^{1,0}X) \mid \bar{\partial}\epsilon + \frac{1}{2}[\epsilon, \epsilon] = 0,\ \bar{\partial}^*\epsilon = 0\}$$

to be the desired family.

To see this, we apply elliptic operator theory. For, $\epsilon \in \mathcal{G}$ we have

$$\triangle_{\bar{\partial}}\epsilon = (\bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*)\epsilon = \bar{\partial}^*\bar{\partial}\epsilon = -\frac{1}{2}\bar{\partial}^*[\epsilon, \epsilon].$$

[10] Here, $H^1(X, T^{1,0}X)$ denotes the sheaf cohomology of the holomorphic tangent bundle. See [WG07, Ch. 2] for details.

Hence, $\triangle_{\bar\partial}\epsilon + \frac{1}{2}\bar\partial^*[\epsilon,\epsilon] = 0$. Applying Green's operator from Theorem 2.1.19 to this equation yields

$$
\begin{aligned}
0 &= G\triangle_{\bar\partial}\epsilon + \frac{1}{2}G\bar\partial^*[\epsilon,\epsilon] \\
&= \epsilon - \mathbb{H}\epsilon + \frac{1}{2}G\bar\partial^*[\epsilon,\epsilon] \\
&= \epsilon - \mathbb{H}\epsilon + \frac{1}{2}Q[\epsilon,\epsilon].
\end{aligned}
$$

Here, we set $Q := G\bar\partial^*$. Thus, every $\epsilon \in \mathcal{G}$ satisfies $\epsilon + \frac{1}{2}Q[\epsilon,\epsilon] = \mathbb{H}\epsilon$. So, we showed $\mathcal{G}$ is a subset of

$$\mathcal{F} := \{\epsilon \in \Omega^{0,1}(X, T^{1,0}X) \mid \epsilon + \frac{1}{2}Q[\epsilon,\epsilon] \in \mathcal{H}^1\},$$

where $\mathcal{H}^1$ denotes $\triangle_{\bar\partial}$-harmonic forms.

Now, consider the map

$$F : \Omega^{0,1}(X, T^{1,0}X) \to \Omega^{0,1}(X, T^{1,0}X), \quad \epsilon \mapsto \epsilon + \frac{1}{2}Q[\epsilon,\epsilon].$$

The map F extends to a continuous map between Hilbert spaces

$$F : W^k(X,\mathcal{E}) \to W^k(X,\mathcal{E}),$$

for sufficiently large k [Pal68, Thm. 5.4]. Here, $\mathcal{E}$ denotes the bundle $\bigwedge^{0,1} T^*X \otimes T^{1,0}X$ and $W^k(X,\mathcal{E})$ the completion of $\Omega^{0,1}(X, T^{1,0}X)$ with respect to the Sobolev inner product[11] $(\cdot,\cdot)_k$. The norm corresponding to $(\cdot,\cdot)_k$ is denoted by $\|\cdot\|_k$. Now, since Q is linear and $[\cdot,\cdot]$ is bilinear, it follows that F is smooth. Moreover, its differential at zero is the identity as $Q[\epsilon,\epsilon]$ is quadratic in ϵ. Hence, by the inverse function theorem on Banach spaces, F^{-1} is a smooth map that sends a neighbourhood of the origin bijectively to a different neighbourhood of zero. Consequently, for $\delta > 0$ sufficiently small, the open set

$$U := \{t \in \mathcal{H} \mid \|u\|_k < \delta\} \subset W^k(X,\mathcal{E})$$

is diffeomorphically mapped by F^{-1} to

$$\widetilde{\mathcal{F}} := \{\epsilon(t) = F^{-1}(t) \mid t \in U\}.$$

[11] This is a specific inner product that reduces to (2.4) for $k = 0$. It is defined in [Kur65] as well.

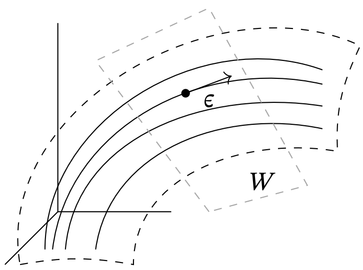

Figure 2.4: Graphical depiction of procedure to find gauge slice. Here the area encapsulated by the dashed line denotes the set of integrable deformations inside the vector space $\Omega^{0,1}(X, T^{1,0}X)$. Furthermore, the arrow denotes a vector $\bar\partial X^{1,0}$.

Since, $\mathcal{H}^1$ is a finite dimensional vector space (see Theorem 2.1.19), U is a smooth manifold. Consequently, $\widetilde{\mathcal{T}}$ the smooth (in fact, holomorphic) family. Left to check is that $\epsilon(t)$ defines an almost complex structure deformation, i.e. are smooth sections of $\mathcal{E}$. To see this, note $\epsilon(t) \in \widetilde{\mathcal{T}}$ satisfies $t = F(\epsilon(t))$. Applying $\triangle_{\bar\partial}$ to both sides yields

$$\begin{aligned}
0 &= \triangle_{\bar\partial}\left(\epsilon(t) + \frac{1}{2}Q[\epsilon(t), \epsilon(t)]\right) \\
&= \triangle_{\bar\partial}\epsilon(t) + \frac{1}{2}\triangle_{\bar\partial}\bar\partial^* G[\epsilon(t), \epsilon(t)] \\
&= \triangle_{\bar\partial}\epsilon(t) + \frac{1}{2}\bar\partial^* \triangle_{\bar\partial} G[\epsilon(t), \epsilon(t)] \\
&= \triangle_{\bar\partial}\epsilon(t) + \frac{1}{2}\bar\partial^*(\mathrm{id} - \mathbb{H})[\epsilon(t), \epsilon(t)] \\
&= \triangle_{\bar\partial}\epsilon(t) + \frac{1}{2}\bar\partial^*[\epsilon(t), \epsilon(t)].
\end{aligned} \tag{2.20}$$

By a standard result about elliptic PDEs [Mor+54], we can conclude solutions $\epsilon(t)$ to (2.20) are smooth. Hence, $\widetilde{\mathcal{T}}$ sits inside $\Omega^{0,1}(X, T^{1,0})$ and thus is the desired family of almost complex deformations. Furthermore, the tangent space of $\widetilde{\mathcal{T}}$ at the origin is equal to the tangent space of U, which is $\mathcal{H}^1 \cong H^1(X, T^{1,0}X)$.

Our next task is to identify the integrable deformations in $\widetilde{\mathcal{T}}$, i.e. elements satisfying the Maurer-Cartan equation (2.11). Let $\epsilon(t) \in \widetilde{\mathcal{T}}$, then by construction $\epsilon(t) + \frac{1}{2}Q[\epsilon(t), \epsilon(t)] = t$. Note, as t is harmonic we have $\bar\partial t = 0$, thus

$$\bar\partial\epsilon(t) = -\frac{1}{2}\bar\partial Q[\epsilon(t), \epsilon(t)]. \tag{2.21}$$

We can rewrite $\mathrm{id} = \triangle_{\bar\partial} G + +\mathbb{H}$ using Corollary 2.1.20 as

$$\mathrm{id} = \bar\partial Q + Q\bar\partial + \mathbb{H}.$$

Combining this with (2.21), we obtain

$$\begin{aligned}
\bar\partial\epsilon(t) + \frac{1}{2}[\epsilon(t), \epsilon(t)] &= -\frac{1}{2}\bar\partial Q[\epsilon(t), \epsilon(t)] + \frac{1}{2}[\epsilon(t), \epsilon(t)] \\
&= -\frac{1}{2}Q\bar\partial[\epsilon(t), \epsilon(t)] - \frac{1}{2}\mathbb{H}[\epsilon(t), \epsilon(t)].
\end{aligned}$$

Note, the images of Q and $\mathbb{H}$ are orthogonal by Corollary 2.1.20. Hence, $\epsilon(t)$ is integrable if and only if $Q\bar\partial[\epsilon(t), \epsilon(t)] = \mathbb{H}[\epsilon(t), \epsilon(t)] = 0$. We claim $\mathbb{H}[\epsilon(t), \epsilon(t)] = 0$ implies $Q\bar\partial[\epsilon(t), \epsilon(t)] = 0$. To see this, note

$$\begin{aligned}
Q\bar\partial[\epsilon(t), \epsilon(t)] &= 2Q[\bar\partial\epsilon(t), \epsilon(t)] \\
&= -Q[\bar\partial Q[\epsilon(t), \epsilon(t)], \epsilon(t)] \\
&= -Q[(\mathrm{id} - Q\bar\partial - \mathbb{H})[\epsilon(t), \epsilon(t)], \epsilon(t)].
\end{aligned}$$

Assuming $\mathbb{H}[\epsilon(t), \epsilon(t)] = 0$ implies

$$Q\bar\partial[\epsilon(t), \epsilon(t)] = Q[Q\bar\partial[\epsilon(t), \epsilon(t)], \epsilon(t)].$$

Let us write $\xi(t) = Q\bar\partial[\epsilon(t), \epsilon(t)]$, then we found

$$\xi(t) = Q[\xi(t), \epsilon(t)].$$

As explained in [Kur65], the map $(\alpha, \beta) \mapsto Q[\alpha, \beta]$ satisfies $\|Q[\alpha, \beta]\|_k < c\|\alpha\|_k \|\beta\|_k$ for sufficiently large k and some $c > 0$. Now, take δ small enough so that $\|\epsilon(t)\|_k < \frac{1}{c}$ holds. In that case, we find

$$\|\xi(t)\|_k < c\|\xi(t)\|_k \|\epsilon(t)\|_k < \|\xi(t)\|_k.$$

Therefore, $\xi(t) = 0$ proving our claim. Hence, $\epsilon(t)$ is integrable if and only if $\mathbb{H}[\epsilon(t), \epsilon(t)] = 0$. Consequently, we identified the space of integrable deformations

$$\mathcal{T} = \{\epsilon(t) \mid t \in \mathcal{Z} = \Phi^{-1}(0)\}$$

with holomorphic obstruction map

$$\Phi : H^1(X, T^{1,0}X) \supset U \to H^2(X, T^{1,0}X), \qquad t \mapsto \mathbb{H}[\epsilon(t), \epsilon(t)].$$

This makes $\mathcal{T}$ an analytic set.

To conclude the proof, we show our initial expectation. We claim $\mathcal{T}$ is a neighbourhood of zero in $\mathcal{G}$. Clearly, $0 \in \mathcal{T}$. For $\epsilon(t) \in \mathcal{T}$ we have $t = F(\epsilon(t)) = \epsilon(t) + \frac{1}{2}Q[\epsilon(t), \epsilon(t)]$. By rearranging and applying $\bar{\partial}^*$, we find

$$\bar{\partial}^* \epsilon(t) = \bar{\partial}^* t - \frac{1}{2}\bar{\partial}^* Q[\epsilon(t), \epsilon(t)] = 0.$$

Thus, $\mathcal{T} \subset \mathcal{G}$. Conversely, for $\epsilon \in \mathcal{G}$ we showed $F(\epsilon) = \epsilon + \frac{1}{2}Q[\epsilon, \epsilon] \in \mathcal{H}^1$. Hence, for ϵ sufficiently close to 0 we have $F(\epsilon) \in U$, i.e. $\epsilon \in \mathcal{T}$. ■

If $\epsilon(t) \in \mathcal{T}$ is a complex structure deformation, let us denote the corresponding complex manifold by X_t.

> **Remark.** In [Kur65] it is also shown that $\mathcal{T}$ defines a holomorphic family, i.e. there is a holomorphic surjective submersion $\pi : \mathcal{X} \to \mathcal{T}$ such that $\mathcal{X}$ is a complex manifold and the fibers correspond to X_t. The notion of holomorphic families will play a role in Chapter 3.

Kuranishi's theorem tells us that the space of complex deformations $\mathcal{T}$ has a smooth structure, if the obstruction map vanishes. If that happens, we say the complex manifold X is **unobstructed**. In that case, one has $\widetilde{\mathcal{T}} = \mathcal{T}$. One trivial example is when $H^2(X, T^{1,0}X) = 0$. However, this was proven before Kuranishi in [KNS58]

> **Corollary 2.2.4.**
> *Let X be a complex manifold. If $H^2(X, T^{1,0}X) = 0$, then X is unobstructed.*

Moreover, it turns out Calabi-Yau manifolds are unobstructed as well. We will prove this in the next section. On the other hand, we would like to stress that the complex structure moduli space, in general, is far from smooth. This can be seen, for example, in [Ver15]. In this work it is shown particular complex manifolds, such as certain complex tori and hyperkähler manifolds, admit so-called *ergodic complex structures*. These are complex structures that have a dense orbit in Teich under the mapping class group $\Gamma := \mathsf{Diff}(X)/\mathsf{Diff}_0(X)$. Consequently, if CS denotes the space of complex structures, the moduli space

$$\mathcal{M} = \mathsf{CS}/\mathsf{Diff} = \mathsf{Teich}/\Gamma$$

will be highly convoluted. In particular, $\mathcal{M}$ will be 'co-Hausdorff', in the sense that *any* two open sets in $\mathcal{M}$ will have non-trivial intersection. In other words, $\mathcal{M}$ is as far from being Hausdorff as it can be.

Finally, one might wonder whether *all* complex deformations of J are contained in $\mathcal{T}$, i.e. whether the family $\mathcal{T}$ is complete. It turns out locally this assertion is true: any sufficiently small deformation ϵ of J is equivalent to at least one element of $\mathcal{T}$ [Gua04, Thm. 5.4]. We say $\mathcal{T}$ is **semiuniversal**[12]. Thus, we have the result

Theorem 2.2.5.
The Kuranishi family $\mathcal{T}$ is semiuniversal.

Proof. — The proof is rather technical and we omit it here. We refer to [Kur65] or [Gua04, Thm. 5.2]. The latter is on deformations of generalized complex structures. However, the argument is completely analogous. Moreover, our statement is a corollary of the theorem for generalized complex structures, as complex structures are naturally generalized complex structures. ■

2.3 The Calabi-Yau Moduli Space

In chapter 1 we argued that the moduli space of Calabi-Yau manifolds is strongly connected to the four-dimensional effective theory. In particular, the complex structure moduli space is of interest to us. It suffices for our purposes to restrict ourselves to Teich and apply Kuranishi's theorem from the previous section. Within this section we study the local structure of the Calabi-Yau moduli space. We will see the local moduli space splits into two parts: complex structure deformations and Kähler deformations. Furthermore, we will prove a classical result due to Tian [Tia87] and Todorov [Tod89] which states that Calabi-Yau manifolds are unobstructed. This result was first announced by Bogomolov in [Bog78] and is therefore known as the Bogomolov-Tian-Todorov theorem. The original arguments by Tian and Todorov both relied on power series arguments. In this work we use global techniques to prove the theorem, following [LZ20].

Let us start with the unobstructedness of Calabi-Yau manifolds. Let (X, J, g) be a Calabi-Yau manifold and let Ω denote its nonvanishing holomorphic $(n, 0)$-form. It induces an isomorphism

$$\bigwedge^{p} T^{1,0}X \to \bigwedge^{n-p,0} X, \quad V_1 \wedge \cdots \wedge V_p \mapsto i_{V_1} \cdots i_{V_p}\Omega$$

by contraction. Therefore,

$$\bigwedge^{p} T^{1,0}X \otimes \bigwedge^{0,q} X \cong \bigwedge^{n-p,0} X \otimes \bigwedge^{0,q} X \cong \bigwedge^{n-p,q} X.$$

Thus, contraction with Ω induces an isomorphism

$$\eta : \Omega^{0,q}(X, \bigwedge^{p} T^{1,0}X) \to \Omega^{n-p,q}(X).$$

Moreover, η induces an isomorphism $H^q(X, \bigwedge^p T^{1,0}X) \cong H^{n-p,q}(X)$. Using this map, we can construct a differential operator

$$\Delta : \Omega^{0,q}(X, \bigwedge^{p} T^{1,0}X) \xrightarrow{\eta} \Omega^{n-p,q}(X) \xrightarrow{\partial} \Omega^{n-p+1,q}(X) \xrightarrow{\eta^{-1}} \Omega^{0,q}(X, \bigwedge^{p-1} T^{1,0}X).$$

[12] Semiuniversal deformations are also called *miniversal*. Moreover, *versal* and *universal* deformations also exist. For a nice review see [Man05].

This map is crucial for the unobstructedness, as it satisfies

Lemma 2.3.1 (Tian-Todorov).
For $\alpha \in \Omega^{0,p}(X, T^{1,0}X)$ *and* $\beta \in \Omega^{0,q}(X, T^{1,0}X)$ *we have*

$$(-1)^p[\alpha, \beta] = \Delta(\alpha \wedge \beta) - \Delta(\alpha) \wedge \beta - (-1)^{p+1}\alpha \wedge \Delta(\beta)$$

Proof. — In the proof they first argue why it is sufficient to check the $p = q = 0$ case. Afterwards, it is a long yet straightforward computation. Hence, we omit it here and refer to [Huy05, Lem. 6.1.9] ∎

At this point, the operator Δ and the Tian-Todorov lemma feel like merely technical tools. However, it turns out they fit in a more general framework, which makes it very apparent which algebraic structures are necessary for the unobstructedness of Calabi-Yau manifolds. The appropriate setting is that of differential Gerstenhaber-Batalin-Vilkovisky (dGBV) algebras and their deformations. However, we will not discuss these structures here. For more background we refer to [Man99].

With the above machinery at hand, we state the main result

Theorem 2.3.2 (Bogomolov-Tian-Todorov).
For a Calabi-Yau manifold, its deformation is unobstructed.

Proof. — Let us assume the above setting. Furthermore, let U denote the open from Kuranishi's theorem. If we use the Kähler metric g, η will be an isomorphism between the Hodge theories on $\Omega^{0,k}(X, T^{1,0}X)$ and $\Omega^{n-1,k}(X)$ [Huy05, Rmk. 6.1.12]. Hence, for $t \in U$

$$\Phi(t) = \mathbb{H}[\epsilon(t), \epsilon(t)] = 0 \iff \mathbb{H}\big(\eta([\epsilon(t), \epsilon(t)])\big) = 0.$$

Then, by the Tian-Todorov lemma

$$\begin{aligned}\eta([\epsilon(t), \epsilon(t)]) &= -\eta\Big(\Delta\big(\epsilon(t) \wedge \epsilon(t)\big) - \Delta\big(\epsilon(t)\big) \wedge \epsilon(t) - \epsilon(t) \wedge \Delta\big(\epsilon(t)\big)\Big) \\ &= -\partial\eta\big(\epsilon(t) \wedge \epsilon(t)\big) + 2\eta\big(\epsilon(t) \wedge \Delta(\epsilon(t))\big) \\ &= -\partial\big(i_{\epsilon(t)}i_{\epsilon(t)}\Omega\big) + 2i_{\epsilon(t)}\eta\big(\Delta(\epsilon(t))\big) \\ &= -\partial\big(i_{\epsilon(t)}i_{\epsilon(t)}\Omega\big) + 2i_{\epsilon(t)}\partial\big(i_{\epsilon(t)}\Omega\big).\end{aligned}$$

Since $\mathbb{H}\partial = 0$, it suffices to show

$$\partial(i_{\epsilon(t)}\Omega) = \partial\Big(\eta(\epsilon(t))\Big) = 0.$$

Recall, we had $\epsilon(t) = t - \frac{1}{2}Q[\epsilon(t), \epsilon(t)]$. Applying η to this equation yields

$$\begin{aligned}\eta\big(\epsilon(t)\big) &= \eta(t) - \eta\left(\frac{1}{2}Q[\epsilon(t), \epsilon(t)]\right) \\ &= \eta(t) - \frac{1}{2}Q\eta([\epsilon(t), \epsilon(t)]) \\ &= \eta(t) - Q\Big(i_{\epsilon(t)}\partial\big(i_{\epsilon(t)}\Omega\big)\Big) + \frac{1}{2}Q\partial\big(i_{\epsilon(t)}i_{\epsilon(t)}\Omega\big).\end{aligned}$$

Here we used that η and Q commute, as they act on different spaces. Define $\Psi := \eta\big(\epsilon(t)\big)$. Then,

$$\Psi = \eta(t) - Q(i_{\epsilon(t)}\partial\Psi) + \frac{1}{2}\partial Q(i_{\epsilon(t)}i_{\epsilon(t)}\Omega).$$

Since t is $\bar{\partial}$-harmonic, $\eta(t)$ is $\bar{\partial}$-harmonic as well, implying that $\eta(t)$ is ∂-harmonic by Hodge's theorem. Therefore, $\partial\eta(t) = 0$ and we see

$$\partial\Psi = -\partial Q(i_{\epsilon(t)}\partial\Psi).$$

Similarly to the proof of Kuranishi's theorem, we have [LZ20, Thm. 3.7]

$$\|\partial\Psi\|_k < c\|\epsilon(t)\|_k\|\partial\Psi\|_k < \|\partial\Psi\|_k$$

for $t \in U$. Hence, $\|\partial\Psi\| = 0$, meaning $\partial\Psi = 0$. This concludes the proof. ■

From this result, we conclude the local complex structure moduli space of a Calabi-Yau manifold to be smooth, in fact complex. This is an important result which we will use in the next chapter.

Now, we would like to discuss the Kähler deformations. However, these are not our main interest. Therefore, the discussion will be less in depth. Firstly, if we fix a Calabi-Yau manifold (X, J, g) we have the following result [Joy00, Thm. 6.2.1]

Theorem 2.3.3.
The Ricci-flat Kähler metrics on X form a smooth family of dimension $h^{1,1}$, isomorphic to the Kähler cone

$$\mathcal{K} := \{[\omega] \mid \omega \text{ is the Kähler class of a Kähler metric on } X\} \subseteq H^{1,1}(X) \cap H^2(X, \mathbb{R}).$$

It is straightforward to show that $\mathcal{K}$ is an open convex cone, hence the name. For details, see [Joy00, Sec. 4.7].

Theorem 2.3.3 gives of the Kähler moduli space for a fixed complex structure J. We wonder what happens when we deform J. By a classical stability result due to Kodaira and Spencer [KS60] we have

Theorem 2.3.4.
There is an open neighbourhood $\mathcal{V}$ of zero in $\mathcal{T}$ such that X_t is Kähler for all $\epsilon(t) \in \mathcal{V}$.

The proof of this theorem is beyond the scope of this work. For the interested reader, we refer to [Voi02, Thm. 9.23]. Moreover, the Hodge numbers are well-behaved with respect to the deformation. If $h_t^{p,q}$ denote the Hodge numbers on X_t, we have [Voi02, Prop. 9.20]

Proposition 2.3.5.
Let X be a compact Kähler manifold. Then, for $\epsilon(t)$ near 0 in $\mathcal{T}$ we have $h_t^{p,q} = h^{p,q}$.

Hence, upon possibly shrinking $\mathcal{V}$, the Hodge numbers are constant. Finally, X_t and X are isomorphic as *smooth* manifolds. Therefore, $c_1(X_t) = c_1(X) = 0$, as the first Chern class is a topological invariant. Now, if we restrict to $\mathcal{V}$, X_t is a compact Kähler manifold with vanishing first Chern class. This is equivalent to $\mathrm{Hol}(X_t) \subseteq \mathrm{SU}(n)$ (cf. [BLT12, Sec. 14.5]). However, holonomy is lower semicontinuous [Mül22], meaning that the holonomy of the deformation can only increase. In other words, $\mathrm{SU}(n) \subseteq \mathrm{Hol}(X_t)$ and thus $\mathrm{Hol}(X_t) = \mathrm{SU}(n)$. Hence, X_t is Calabi-Yau. Combining this with Theorem 2.3.3 and Theorem 2.3.2 we obtain the following result [Joy00, Cor. 6.8.2]

Theorem 2.3.6.
Let (X, J, g) be a Calabi-Yau n-fold. Then, the local moduli space of deformations of the Calabi-Yau structure of X is a smooth manifold of dimension $h^{1,1} + 2h^{n-1,1}$.

Up to now, we restricted ourselves to local deformation theory, i.e. equivalence up to isomorphism connected to identity. However, much is known about the geometry of the genuine moduli space of Calabi-Yau manifolds. To conclude this chapter, we would like to state some of its main properties. Firstly, we need to impose some technical conditions on the Calabi-Yau for the construction to work. We say a Calabi-Yau manifold X is **polarized** if its Kähler class $[\omega]$ is integral. This is a technical condition, irrelevant to our discussion, closely related to the existence of a polarization. See [Voi02, Sec. 7.1.3] for more background. Then, we have the following result [LS04, Thm. 2.20] due to Viehweg [Vie95, Thm. 1.13]

Theorem 2.3.7.
The moduli space $\mathcal{M}$ of polarized Calabi-Yau manifolds is a quasi-projective variety. Moreover, there exists a compact projective variety $\overline{\mathcal{M}}$ such that $\mathcal{M} \subset \overline{\mathcal{M}}$ is a subvariety.

A projective variety is a particular (algebraic) subset of projective space. The precise definition is not of interest to us and we refer to [Har77]. Furthermore, a quasi-projective variety is an open subset of a projective variety. We do not wish to prove the above theorem, as the proof is highly technical. Furthermore, it uses a lot of algebraic geometry, which we did not introduce here, as already displayed. An important remark is that in general a quasi-projective variety may have singularities, i.e. is not smooth. However, by a classical theorem by Hironaka [Hir64], these singularities can be resolved, making $\mathcal{M}$ into a smooth manifold. Moreover, $\overline{\mathcal{M}}$ is called the **compactification** of the moduli space and can be made smooth by the same theorem.

Finally, the existence of $\overline{\mathcal{M}}$ is important to us in the next chapter. There, we are interested in the asymptotic behaviour of Hodge structures near the boundary $\mathcal{M}_{\text{sing}} := \overline{\mathcal{M}} \setminus \mathcal{M}$ of 'singular' points[13]. Consequently, $\mathcal{M}_{\text{sing}}$ is also called the singular locus. The behaviour near the singular locus is of particular interest to string theorists, as many *swampland conjectures* make statements about structures that may appear near the boundary (see e.g. [GPV18; LLW22; GLV20; Hei22; Li22]). In the swampland program, people try to make general statements, i.e. conjectures, about arbitrary theories of quantum gravity. In a nutshell, the aim is to narrow down the amount possibilities for *the* theory of quantum gravity. This motivates the study of singular points in the moduli space. It turns out they admit a useful local description [LS04, Cor. 2.21], namely $\mathcal{M}_{\text{sing}}$ is a divisor of normal crossing. In practice, this means for $x \in \mathcal{M}_{\text{sing}}$ there is a neighbourhood such that we can write $\mathcal{M}$ as

$$(\Delta^*)^l \times \Delta^{m-l},$$

where Δ (Δ^*) denotes the open (punctured) unit disc and m is the complex dimension of $\overline{\mathcal{M}}$. We will exploit this local expression in Section 3.5.

[13] The terminology comes from the fact that often one can assign singular manifolds to these boundary points.

CHAPTER 3

Asymptotic Hodge Theory

In the first chapter we argued the dependence of the four-dimensional effective action (1.13) as well as the flux potential on the complex structure moduli and objects from Hodge theory of the internal manifold, such as the Hodge star. Recall, in the compactification procedure, we consider a family of internal manifolds. Because of this we studied the moduli space in the previous chapter. It turns out the relevant objects in effective theory are strongly related to the way the cohomology of the internal manifold decomposes in the Hodge decomposition theorem. Moreover, we have seen that this decomposition depends on the complex structure, i.e. on the complex structure moduli. Therefore, as the internal manifold deforms, the Hodge decomposition will change, altering the objects appearing in the effective theory. Hence, we want to understand the change in the Hodge decomposition as we deform the internal manifold. This leads to the concept of *variation of Hodge structure*, which is central in this work.

To do this, we formalize the concepts appearing in the decomposition theorem and argue abstractly. In doing so, we unravel interesting differential- and algebraic-geometric structures. In particular, we will see variation of Hodge structures are completely determined by a holomorphic map

$$\Phi : \mathcal{M} \to \Gamma\backslash D$$

called the *period map*. One should think of $\mathcal{M}$ as the complex structure moduli space of a Calabi-Yau manifold. Moreover, $\Gamma\backslash D$ denotes the set of inequivalent Hodge decompositions on the underlying manifold X, called the classifying space. As the period map determines the variation of Hodge structure, the behaviour of the Hodge structure near $\mathcal{M}_{\text{sing}}$ is equivalent to the asymptotic form of Φ. These asymptotic forms are captured by two main theorems: the nilpotent orbit theorem and the SL(2)-orbit theorem. We will state those theorems and discuss their relation.

To already make the connection with the subsequent chapter, if a Calabi-Yau has only one complex structure modulus, $\mathcal{M}$ has complex dimension one. Hence, in a way, it can be seen as a (singular) Riemann surface, i.e. a worldsheet. Furthermore, we will see that D is a homogeneous space. Therefore, we can schematically view the period map as a string embedding into a Lie group, i.e. a non-linear sigma-model. Now, the period map satisfies certain equations, as we will see. Then, the main question is: can we find an action such that the period map solves its equations of motion, i.e. produces variation of Hodge structures. This is the central question in this work, as mentioned before.

3.1 Hodge Structures

At this point we have seen how the de Rham cohomology groups of a compact Kähler manifold decompose into so-called (p,q)-forms. As is typical in mathematics, we now try to isolate the structure at play and generalize it. The abstract object we are interested in in this section is a so-called *Hodge structure*. In this section we will define and develop the theory of Hodge structures. Throughout this section one should constantly compare the results with those from Section 2.1.2. We use [CKS86] and [Sch73] as main references.

As mentioned above, *the* remarkable conclusion of the Hodge decomposition theorem was that the vector space $H^k(X;\mathbb{C})$ splits into subspaces, satisfying a certain property. This is what we want to capture in the definition of a Hodge structure.

Definition 3.1.1.
A **(pure) Hodge structure of weight k** *is a finite dimensional real vector space* $H_\mathbb{R}$ *whose complexification* $H_\mathbb{C} = H_\mathbb{R} \otimes \mathbb{C}$ *carries a decomposition*

$$H := H_\mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

into a finite amount of subspaces $H^{p,q}$*, such that* $H^{p,q} = \overline{H^{q,p}}$*. The dimension of these subspaces* $h^{p,q} := \dim H^{p,q}$ *are called the* **Hodge numbers**.

For clarity, even though p, q and k were non-negative integers in the Hodge decomposition theorem, this is not necessary in the above definition. They are merely labels. Yet, for the purposes of this thesis it suffices to solely consider non-negative integer values for p, q and k, as we will be interested in the middle cohomology of particular complex manifolds.

Definition 3.1.1 is a nice abstraction of the results we encountered before. Yet, it has some downsides which are not apparent at this point. We will comeback to these disadvantages in Section 3.3. Luckily, there is an equivalent definition of a Hodge structure that circumvents these issues. Namely, through the **Hodge filtration**. In this language, a Hodge structure of weight k is a pair $(H_\mathbb{R}, F)$, with $H_\mathbb{R}$ a real vector space as above and F a finite decreasing filtration

$$\cdots \supseteq F^p \supseteq F^{p+1} \supseteq \cdots, \qquad p \in \mathbb{Z}$$

of subspaces of $H = H_\mathbb{C}$, satisfying

$$H = F^p \oplus \overline{F^{k-p+1}} \tag{3.1}$$

for every $p \in \mathbb{Z}$.

Proposition 3.1.2.
There is a one-to-one correspondence between Hodge decompositions of weight k *as in Definition 3.1.1 and Hodge filtrations described above. The correspondence is given by equations (3.2) and (3.3) below.*

Proof. — From a Hodge decomposition one easily constructs a Hodge filtration by setting

$$F^p := \bigoplus_{i \geq p} H^{i,k-i}. \tag{3.2}$$

We then see

$$
\begin{aligned}
F^p \oplus \overline{F^{k-p+1}} &= \bigoplus_{i \geq p} H^{i,k-i} \oplus \bigoplus_{j \geq k-p+1} \overline{H^{j,k-j}} \\
&= \bigoplus_{i \geq p} H^{i,k-i} \oplus \bigoplus_{j \geq k-p+1} H^{k-j,j} \\
&= \left(\cdots \oplus H^{p+1,k-p-1} \oplus H^{p,k-p}\right) \oplus \left(H^{p-1,k-p+1} \oplus H^{p-2,k-p+2} \oplus \dots\right) \\
&= \bigoplus_{p+q=k} H^{p,q} \\
&= H.
\end{aligned}
$$

Hence, the filtration in (3.2) defines a Hodge filtration. On the other hand, given a Hodge filtration, we can recover the Hodge decomposition by setting

$$
H^{p,q} = F^p \cap \overline{F^q}. \tag{3.3}
$$

Then, one readily sees that $\overline{H^{q,p}} = H^{p,q}$ holds. Furthermore[1],

$$
\begin{aligned}
\bigoplus_{p+q=k} H^{p,q} = \bigoplus_{p+q=k} F^p \cap \overline{F^q}& \\
&= (F^k \cap \overline{F^0}) \oplus (F^{k-1} \cap \overline{F^1}) \oplus \cdots \oplus (F^1 \cap \overline{F^{k-1}}) \oplus (F^0 \cap \overline{F^k}) \\
&= F^k \oplus (F^{k-1} \cap \overline{F^1}) \oplus \cdots \oplus (F^1 \cap \overline{F^{k-1}}) \oplus \overline{F^k} \\
&= (F^{k-1} \cap (F^k \oplus \overline{F^1})) \oplus \cdots \oplus ((F^1 \oplus \overline{F^k}) \cap \overline{F^{k-1}}) \\
&= (F^{k-1} \cap H) \oplus (F^{k-2} \cap \overline{F^2}) \oplus \cdots \oplus (F^2 \cap \overline{F^{k-2}}) \oplus (H \cap \overline{F^{k-1}}) \\
&\ \ \vdots \\
&= \begin{cases} F^{\frac{k}{2}} \oplus \overline{F^{\frac{k}{2}+1}}, & k \text{ even} \\ F^{\frac{k+1}{2}} \oplus \overline{F^{\frac{k+1}{2}}}, & k \text{ odd} \end{cases} \\
&= H.
\end{aligned}
$$

In this computation we repeatedly used the identity $F^p \oplus (F^{p-1} \cap \overline{F^{k-p+1}}) = (F^p \oplus \overline{F^{k-p+1}}) \cap F^{p-1}$ and the condition $F^p \oplus \overline{F^{k-p+1}} = H$. We conclude that Hodge structures and Hodge filtrations are in bijection. ■

Because of the equivalence established in the above proposition, we use both formulations interchangeably.

Now that we have generalized some of the findings from before, it makes sense to wonder which objects can translated from Section 2.1.2 to this new language. One object that will be important for us later is the generalization of the Hodge star, the so-called **Weil operator**. Given a Hodge structure of H, it is defined as a linear map $C : H \to H$ such that

$$
Cv = i^{p-q} v, \quad v \in H^{p,q}.
$$

[1] For simplicity, we assume p, q, k to be non-negative integers.

The Weil operator will be a pivotal object in our discussion in Chapter 4. Therefore, we investigate how it interacts with Hodge structures. Firstly, we can define an operator $Q : H \to H$ that measures the Hodge decomposition of H by

$$Qv := (p - k/2)v, \quad \text{for } v \in H^{p,q}.$$

It is called the **charge operator** and it can be used to compute the Weil operator via

$$C = e^{i\pi Q} = (-1)^Q.$$

Notice, it satisfies $\overline{Q} = -Q$, i.e. it is purely imaginary. Secondly, inspiration is found in Section 2.1.2, where we saw that the k-th cohomology group carried a bilinear form that, combined with the Hodge star, defined a positive definite Hermitian form compatible with the Hodge decomposition. This leads to the notion of a polarization:

Definition 3.1.3.
A **polarization** *for a Hodge structure* $(H_{\mathbb{R}}, F)$ *of weight* k *is is a bilinear form* S *defined over* $\mathbb{R}$*, such that*

$$\begin{aligned} &S(v,w) = (-1)^k S(w,v) && v,w \in H \\ &S(F^p, F^{k-p+1}) = 0, && \textit{for every } p \\ &S(Cv, \overline{v}) > 0, && \textit{for } v \in H \setminus \{0\}. \end{aligned}$$

The triple $(H_{\mathbb{R}}, F, S)$ *is called a* **polarized Hodge structure**.

In terms of the Hodge decomposition of H, the second property in the definition above becomes

$$S(H^{p,q}, H^{r,s}) = 0, \quad \text{if } (p,q) \neq (s,r), \tag{3.4}$$

which is called *the first bilinear relation*. Moreover, the third condition in the definition above is often referred to as *the second bilinear relation*.

Note, the primitive cohomology on a compact Kähler manifold equipped with the bilinear form (2.9) is an example of a polarized Hodge structure. This proves existence of such structures. However, given a complex vector space H, one could ask how many inequivalent (polarized) Hodge structures it enjoys. This is the central question of the next section.

3.2 The Classifying Space of Hodge Structures

We have seen that the primitive cohomology groups on compact Kähler produce examples of polarized Hodge structures. However, given such a cohomology group, one might wonder whether there exist multiples Hodge structures. This leads to the concept of the **classifying space**, which was first introduced by Griffiths in [Gri68a]. Initially, it was known under the name of *period domain* and is denoted by D. The idea is that every point in the classifying space corresponds to a (polarized) Hodge structure. In this section will we discuss the geometry and properties of this space. One of the main results, that is particularly interesting for our purposes, is the fact that the classifying space is a homogeneous space. In this section our main references are [Sch73; Voi02; CMP17].

Before we can start our discussion, we need some initial data. To describe the classifying space we fix (see [Sch73, Sec. 3])

- a real vector space $H_{\mathbb{R}}$ with complexification $H := H_{\mathbb{C}}$,

- a weight k and a collection of Hodge numbers $\{h^{p,q}\}_{k=p+q} \subset \mathbb{N}$, such that $h^{p,q} = h^{q,p}$ and $\sum h^{p,q} = \dim H_{\mathbb{R}}$.

Keeping the equivalent description through filtrations in mind (cf. Prop. 3.1.2), we define

$$f^p := \sum_{i \geq p} h^{i,k-i}.$$

Now, let $\hat{\mathcal{F}}$ denote the set of all decreasing filtrations of H such that $\dim F^p = f^p$. Note, elements of $\hat{\mathcal{F}}$ will not necessarily be Hodge filtrations, however they naturally sit inside $\hat{\mathcal{F}}$. Therefore, we take a closer look at the geometry of $\hat{\mathcal{F}}$.

First, note that an element in $\hat{\mathcal{F}}$ corresponds to a collection of subspaces of $H_{\mathbb{C}}$ of particular dimensions, namely f^p. Hence, we can view $\hat{\mathcal{F}}$ as a subset of a product of Grassmannians[2]. To be more concrete,

$$\hat{\mathcal{F}} \subset \mathrm{Grass}(f^0, H) \times \cdots \times \mathrm{Grass}(f^k, H).$$

It is known that the Grassmannian is a compact complex manifold (see [Voi02, Sec. 10.1.1]) and the above embedding defines a complex submanifold of $\mathrm{Grass}(f^0, H) \times \cdots \times \mathrm{Grass}(f^k, H)$ (cf. [Voi02, Sec. 10.1.3]). This suggests that $\hat{\mathcal{F}}$ has the structure of a complex manifold. This turns out to be the case, as $\hat{\mathcal{F}}$ admits a transitive and holomorphic action of $\mathrm{GL}(H)$ [Sch73, Sec. 3]. For completeness, the action is given by rotating all the subspaces in the filtration by $g \in \mathrm{GL}(H)$, i.e.

$$(g \cdot F)^p := g(F^p) \tag{3.5}$$

for $F \in \mathcal{F}$.

Using this description, we can denote the set of Hodge filtrations by

$$\mathcal{F} = \left\{ F \in \hat{\mathcal{F}} \mid H = F^p \oplus \overline{F^{k-p+1}} \ \text{ for each } p \right\} \subset \hat{\mathcal{F}}.$$

Note, the condition (3.1) is open. Hence, $\mathcal{F}$ is an open subset of $\hat{\mathcal{F}}$ and therefore inherits a complex manifold structure. Moreover, by Proposition 3.1.2, we see $\mathcal{F}$ parameterizes Hodge structures on H of weight k with Hodge numbers $h^{p,q}$, i.e. it is the unpolarized classifying space.

Let us now look into polarized Hodge structures. For this, we additionally fix a bilinear form S defined over $\mathbb{R}$ on H, such that $S(v,w) = (-1)^k S(w,v)$. Recall, a polarized Hodge structure is a Hodge structure satisfying two conditions regarding the polarization S (cf. Definition 3.1.3). With this in mind, we define

$$\hat{D} := \left\{ F \in \hat{\mathcal{F}} \mid S(F^p, F^{k-p+1}) = 0 \ \text{ for all } p \right\}.$$

Moreover, in view of Definition 3.1.3, it also makes sense to define

$$D := \left\{ F \in \hat{D} \mid S(Cv, \overline{v}) > 0 \ \text{ for } v \in H \setminus \{0\} \right\}. \tag{3.6}$$

We have suggestively denote this subset of $\hat{D}$ by D, as it precisely is the classifying space for polarized Hodge structures of weight k, with Hodge numbers $\{h^{p,q}\}$. Indeed, for $F \in D$, one readily sees that $\dim F^p + \dim \overline{F^{k-p+1}} = f^p + f^{k-p+1} = \dim H$ and that $F^p \cap F^{k-p+1} = 0$. Hence, every $F \in D$ satisfies condition (3.1), meaning it defines a Hodge structure.

Consequently, to understand the geometry of D, one can first study $\hat{D}$. First of all, note that the defining property in (3.6) is an open condition, meaning that D is an open set inside $\hat{D}$. Furthermore, interesting

[2] Even more so, as elements are decreasing filtrations, one can view $\hat{\mathcal{F}}$ as a flag space.

features of the latter were noticed in [Gri68a]. To state these, we need to consider the orthogonal group of the form S, i.e.

$$G_{\mathbb{C}} := \{g \in \mathrm{GL}(H) \mid g^*S = S\}.$$

Note, as elements of $G_{\mathbb{C}}$ preserve the polarization, the first bilinear relation is left invariant by the action of $G_{\mathbb{C}}$. Hence, the group $G_{\mathbb{C}}$ acts naturally on $\hat{D}$ via (3.5). This action allows us to probe the geometry of $\hat{D}$, as can be seen by the following result, first shown by Griffiths in [Gri68a].

Proposition 3.2.1.
The group $G_{\mathbb{C}}$ acts transitively on $\hat{D}$.

The proof of the above proposition relies on elementary, yet tedious, arguments from linear algebra. For our purposes, it is not very enlightening and thus we omit it here. However, it has interesting consequences:

Corollary 3.2.2.
The space $\hat{D}$ is a complex manifold.

Proof. — From Proposition 3.2.1 it follows that $\hat{D}$ is a so-called nonsingular variety[3]. It is a known fact from algebraic geometry that a nonsingular variety can be made into a complex manifold (see e.g. [Ara12, Cor. 2.5.16]). ■

From Corollary 3.2.2 we directly conclude that D inherits the structure of a complex manifold, as it is an open subset of $\hat{D}$. Even more so, a result similar to Proposition 3.2.1 holds true. To see this, we consider the real elements in $G_{\mathbb{C}}$:

$$G_{\mathbb{R}} := \left\{g \in \mathrm{GL}(H_{\mathbb{R}}) \mid g^*S|_{H_{\mathbb{R}}} = S|_{H_{\mathbb{R}}}\right\}.$$

Note, $G_{\mathbb{R}}$ defines a Lie group since it is a closed subgroup of $\mathrm{GL}(H_{\mathbb{R}})$. Furthermore, as $G_{\mathbb{R}}$ consists of real elements, it preserves the third condition in Definition 3.1.3. Hence, $G_{\mathbb{R}}$ defines an action on D. Again, using linear algebra, one can show (cf. [Gri68a]):

Proposition 3.2.3.
The group $G_{\mathbb{R}}$ acts transitively on D.

Propositions 3.2.1 and 3.2.3 show $\hat{D}$ and D are homogeneous spaces, meaning they can be realized as quotients of $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$, respectively. To make this manifest, we fix a reference Hodge filtration $F_0 \in D$. Using this reference structure, we can define the following isotropy groups, corresponding to the action (3.5),

$$B := \{g \in G_{\mathbb{C}} \mid g \cdot F_0 = F_0\}, \quad V = G_{\mathbb{R}} \cap B.$$

Now, we can identify

$$\hat{D} \cong G_{\mathbb{C}}/B, \quad D \cong G_{\mathbb{R}}/V.$$

[3] This is a concept from algebraic geometry, which is not particularly crucial to us. Hence, we omit the details here. For more background, we refer to [Har77].

For later reference, let $\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{R}, \mathfrak{b}, \mathfrak{v}$ denote the Lie algebras of $G_\mathbb{C}, G_\mathbb{R}, B$ and V, respectively. Note, one readily sees that $S(Q\cdot,\cdot) + S(\cdot, Q\cdot) = 0$, meaning $Q \in \mathfrak{g}_\mathbb{C}$. Consequently, as Q is purely imaginary and $C = \exp(\pi i Q)$, we see $C \in G_\mathbb{R}$.

The isotropy V is particularly nice, as it easily follows that [Sch73, Sec. 3]

Proposition 3.2.4.
The isotropy group V is compact.

A direct consequence of this is that any discrete subgroup Γ of $G_\mathbb{R}$ acts properly discontinuous[4] on D. Consequently, the complex structure on D makes the quotient $\Gamma\backslash D := D/\Gamma$ into a complex analytic variety [Sch73]. This does not mean that $\Gamma\backslash D$ is smooth, however its singularities are well-behaved. This is the key difference between D and $\hat{D}$, as the analogous statement for $\hat{D}$ is false. This is one of the main motivations of considering *polarized* Hodge structures.

One could wonder which homogeneous spaces could realize the classifying space of (polarized) Hodge structures. This depends on the weight k and can be deduced through elementary linear algebra considerations. This is done in [CMP17, Ch. 4]. Moreover, if one restricts itself to polarized Hodge structures, a complete classification of D is known (see [CMP17, Prop. 4.4.4])

Proposition 3.2.5.
Let $D = G_\mathbb{R}/V$ be the classifying space of a weight k polarized Hodge structure $(H_\mathbb{R}, F, S)$ with Hodge numbers $\{h^{p,q}\}$ and $\dim H = 2n$. Then,

i) For odd weight $k = 2m+1$, we have

$$G_\mathbb{R} \cong \mathrm{Sp}(2n, \mathbb{R}), \qquad V \cong \prod_{p \leq m} \mathrm{U}(h^{p,q}).$$

In this situation, $D = G_\mathbb{R}/V$ is connected and non-compact

ii) For even weight $k = 2m$, we have

$$G_\mathbb{R} \cong \mathrm{SO}(s,t), \qquad V \cong \prod_{p<m} \mathrm{U}(h^{p,q}) \times \mathrm{SO}(h^{m,m}),$$

where $s = \sum_{p \text{ even}} h^{p,q}$ and $t = \sum_{p \text{ odd}} h^{p,q}$. In this situation, $D = G_\mathbb{R}/V$ is compact and connected if either $s = 0$ or $t = 0$, otherwise it consists of two isomorphic connected components.

Example 3.2.6 (Period domain torus).
Recall, on the 2-torus $\mathbb{T}^2$, the Hodge decomposition is completely determined by the period vector $\mathbf{\Pi} = (A, B)$. This fixed the holomorphic $(1,0)$-form Ω. However, scalar multiples of $\mathbf{\Pi}$ produce the same Hodge decomposition. Hence, we may set $A = 1$. In that case, we know from Example 2.1.24

$$\mathrm{Im}(B) > 0.$$

Moreover, we say that any Hodge structure was automatically polarized. Hence, the classifying

[4]This means every element of D has a neighbourhood U such that $\gamma \cdot U \cap U = \varnothing$ for every $\gamma \in \Gamma \setminus \{e\}$.

space of a weight one polarized Hodge structure on $\mathbb{T}^2$ is given by $D = \mathbb{H}$, the upper half-plane. This agrees with the proposition above, as $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$ and $\mathbb{H} = \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. ◆

3.3 Variation of Hodge Structure

In Section 1.4 we saw that the effective four dimensional theory depends on the moduli parameters of the internal Calabi-Yau threefold. Moreover, the flux potential (1.14) depends heavily on objects coming from the Hodge structure on the middle cohomology of the threefold. However, the decomposition of the middle cohomology is not constant in the complex structure moduli. Therefore, we need to understand how the Hodge structure changes when we deform the internal manifold. This is captured in the notion of a **variation of Hodge structure**. In this section we will define and describe features of such a variation, mostly from an abstract point of view. Our main references are [CKS86; Sch73; Voi02; CMP17].

If we started with the abstract definition of a variation of Hodge structure directly, it would feel rather arbitrary. Therefore, we would like to motivate the definition by first considering the compact Kähler case and extrapolate the properties. Hence, we consider a family of compact Kähler manifolds. Up to now we have been rather informal with the term 'family'. Let us define it properly

Definition 3.3.1.
A **family of complex manifolds** *is a proper holomorphic surjective submersion $\pi : \mathcal{X} \to \mathcal{M}$ between connected complex manifolds.*

Given a family, it follows the fibers $X_t := \pi^{-1}(t)$ are compact complex submanifold of $\mathcal{X}$. If we pick a reference point $t_0 \in \mathcal{M}$ and set $X := X_{t_0}$, we can view $\mathcal{M}$ as the space parameterizing deformations of X. For example, $\mathcal{M}$ could correspond to the complex structure moduli space of a Calabi-Yau threefold. This is justified by the following result[5] [Voi02, Prop. 9.5]

Theorem 3.3.2 (Ehresmann).
Let $\pi : \mathcal{X} \to \mathcal{M}$ be a family of complex manifolds with reference point $t_0 \in \mathcal{M}$. Then, in a neighbourhood U of t_0 there exists a trivialization

$$T = (T_0, \pi) : \mathcal{X} \xrightarrow{\cong} X \times U$$

over $\mathcal{M}$, such that the fibers of T_0 are complex submanifolds of $\mathcal{X}$.

Note, the map T_0 induces a diffeomorphism $X_t \cong X$ for every $t \in U$. However, in general T_0 is not holomorphic as the fibers will typically not be isomorphic as complex manifolds. Yet, we can use this diffeomorphism to transport the complex structure on X_t to X, meaning the family describes a complex structure deformation. Since the fibers of T_0 are complex submanifolds, it follows that the family of complex structures parameterized by $U \subset \mathcal{M}$ varies holomorphically with $t \in U$. We already encountered examples of families in Chapter 2, namely the Kuranishi family of compact manifolds and compact Kähler manifolds. The latter is of particular interested this section. By Theorem 2.3.4, upon shrinking $\mathcal{M}$ we may assume all fibers X_t are compact Kähler manifolds.

Now, fix a family $\pi : \mathcal{X} \to \mathcal{M}$ of compact Kähler manifolds. By the Hodge decomposition theorem, the kth cohomology group of the fibers X_t decompose into (p,q)-forms. However, since $X_t \cong X$

[5] To be precise, Ehresmann's theorem is about smooth manifolds, see [Voi02, Thm. 9.3]. However, we only need the more precise result for complex manifolds.

as smooth manifolds, T_0 induces an isomorphism $H^k(X_t, \mathbb{C}) \cong H^k(X, \mathbb{C})$. Meaning we can transport the decomposition on $H^k(X_t, \mathbb{C})$ to $H^k(X, \mathbb{C})$. Thus, we can view $\mathbf{H}^{p,q} := \{H^{p,q}(X_t, \mathbb{C})\}$ as a family or 'variation' of Hodge structures on the fixed vector space $H^k(X, \mathbb{C})$, parameterized by $t \in \mathcal{M}$. Furthermore, we can assemble the $H^k(X_t, \mathbb{C})$ together into a holomorphic vector bundle [CMP17, Sec. 4.3]

$$\mathbf{H}^k := \{H^k(X_t, \mathbb{C})\}_{t \in \mathcal{M}} \to \mathcal{M}$$

over $\mathcal{M}$. By restricting to cohomology with values in $\mathbb{R}$ and $\mathbb{Z}$, we obtain a subbundle $\mathbf{H}^k_{\mathbb{R}}$ and lattice bundle $\mathbf{H}^k_{\mathbb{Z}}$ in $\mathbf{H}^k$. Furthermore, since the Hodge numbers are locally constant (cf. Proposition 2.3.5), the $\mathbf{H}^{p,q}$ define smooth subbundles of $\mathbf{H}^k$ (again, possibly after shrinking $\mathcal{M}$). Note, intuitively, the geometry of the subbundles $\mathbf{H}^{p,q}$ dictate the variation of the Hodge structure on $H^k(X, \mathbb{C})$. Thus we will study its geometric structure.

Firstly, it turns out the bundles $\mathbf{H}^{p,q}$ do not define *holomorphic* vector bundles, i.e. the fibers to do not vary holomorphically over the base $\mathcal{M}$. However, we can define the filtration

$$\mathbf{F}^p := \bigoplus_{i \geq p} \mathbf{H}^{i,k-i}$$

analogous to (3.2), which will behave more nicely. As the Hodge numbers are locally constant, so are f^p, making $\mathbf{F}^p$ into smooth subbundles. Note, the geometry of $\mathbf{H}^{p,q}$ is encoded in the geometry of the $\mathbf{F}^p$. Hence, we study the latter.

To vector bundle $\mathbf{H}^k$ comes equipped with a flat connection. To see this, note

$$\mathbf{H}^k = \mathbf{H}^k_{\mathbb{Z}} \otimes \mathcal{O}(\mathcal{M}),$$

where $\mathcal{O}(\mathcal{M})$ denotes the sheaf of holomorphic functions[6]. For completeness, this means we can identify $\mathbf{H}^k_{\mathbb{Z}}$ with a local system (cf. [Voi02, Sec. 9.2.1]). Then, the connection

$$\nabla : \Gamma(\mathbf{H}^k) \to \Omega^1(\mathcal{M}, \mathbf{H}^k), \quad \nabla\sigma = \sum_i \sigma_i \otimes d\alpha_i,$$

where $\sigma = \sum_i \alpha_i \sigma_i$ in a local trivialization, is well-defined and flat. This is a generic feature of local systems. This connection is called the **Gauss-Manin connection**.

It is now possible to state one of the fundamental theorems of variations of Hodge structures, which is due to Griffiths [Gri68a]

Theorem 3.3.3 (Transversality).
The subbundles $\mathbf{F}^p \subset \mathbf{H}^k$ are holomorphic subbundles. Furthermore, for each p they satisfy

$$\nabla\Gamma(\mathbf{F}^p) \subset \Omega^1(\mathcal{M}, \mathbf{F}^{p-1}). \tag{3.7}$$

The fact $\mathbf{F}^p$ are holomorphic subbundles as opposed to $\mathbf{H}^{p,q}$, make them the preferred objects to study. This was the property we were hinting at in Section 3.1. Moreover, property (3.7) is referred to as Griffiths transversality and implies that the bundles $\mathbf{F}^p$ can not change arbitrarily.

From the discussion above, we extrapolate the following definition

[6] One should think of this space as the set $\{f : U \to \mathbb{C} \mid U \subset \mathcal{M} \text{ open}, f \text{ holomorphic}\}$.

Definition 3.3.4.
A **variation of Hodge structure** *is a flat real vector bundle* $(\mathbf{H}_{\mathbb{R}}, \nabla)$ *over a connected complex manifold* $\mathcal{M}$ *such that* $\mathbf{H} := \mathbf{H}_{\mathbb{R}} \otimes \mathbb{C}$ *comes together with a filtration by holomorphic subbundles*

$$\cdots \supset \mathbf{F}^p \supset \mathbf{F}^{p-1} \supset \cdots$$

defining a Hodge structure on every fiber such that

$$\nabla\Gamma(\mathbf{F}^p) \subset \Omega^1(\mathcal{M}, \mathbf{F}^{p-1}). \tag{3.8}$$

Note, the rank of $\mathbf{H}$ determines the weight of the Hodge structures on the fibers. Furthermore, in the discussion above there also was a flat lattice $\mathbf{H}_{\mathbb{Z}}$. One could replace the connection by a local system $\mathbf{H}_{\mathbb{Z}}$ in the definition above such that ∇ becomes the Gauss-Manin connection, as is done in [Sch73]. However, for our purposes the above definition is sufficient. Moreover, we will denote a variation of Hodge structure by the data $(\mathcal{M}, \mathbf{H}, \nabla, \mathbf{H}_{\mathbb{R}}, \mathbf{F}^\bullet)$. Finally, condition (3.8) is also referred to as **horizontality** and is frequently denoted by [CMP17, Ch. 4]

$$\frac{\partial \mathbf{F}^p}{\partial z} \subset \mathbf{F}^{p-1}.$$

Finally, one can repeat the discussion above for polarized Hodge structures. Since it is completely analogous, we omit it here. In the end, the relevant structure is captured in the following definition

Definition 3.3.5.
A **variation of polarized Hodge structure** *consists of a variation of Hodge structure* $(\mathcal{M}, \mathbf{H}, \nabla, \mathbf{H}_{\mathbb{R}}, \mathbf{F}^\bullet)$ *together with a bilinear form* $\mathbf{S} \in \Gamma(\mathbf{H}^* \otimes \mathbf{H}^*)$ *defined over* $\mathbb{R}$ *such that*

i) $\mathbf{S}$ *induces a polarization on every fiber* H_t

ii) $\mathbf{S}$ *is flat, meaning*

$$d\mathbf{S}(\sigma, \sigma') = \mathbf{S}(\nabla\sigma, \sigma') + \mathbf{S}(\sigma, \nabla\sigma'),$$

for $\sigma, \sigma' \in \Gamma(\mathbf{H})$.

Example 3.3.6 (Calabi-Yau threefolds).
To illustrate the usefulness of the horizontality condition (3.8), let us consider a family of Calabi-Yau threefolds $Y_3(t)$. Here t denotes the complex structure modulus. Then, the Hodge structure on the middle cohomology is given by

$$F^3(t) \subset F^2(t) \subset F^1(t) \subset F^0(t) = \mathbb{C}^{2(h^{2,1}+1)},$$

where $F^3(t)$ is spanned by the holomorphic $(3,0)$-form $\Omega(t)$ on $Y_3(t)$. Analogously to Example (2.1.24), we pick an integral homology basis $\Gamma_i \in H_3(Y_3, \mathbb{Z})$. Then, we can expand $\Omega(t)$ in the dual basis $\gamma_i \in H^3(Y_3, \mathbb{Z})$

$$\Omega(t) = \Pi^i(t)\gamma_i, \quad \Pi^i(t) = \int_{\Gamma_i} \Omega(t).$$

Now, in the Calabi-Yau threefold case, horizontality yields a *completeness principle* [Gri68b]: the Hodge filtration is completely determined by the period vector $\mathbf{\Pi}(t)$. To be precise, we have

$$F^p(t) = \operatorname{span}\{\partial_{i_1} \cdots \partial_{i_m} \mathbf{\Pi}(t) \mid 0 \le m \le 3-p\}.$$

Consequently, the period vector contains all the information about the variation of Hodge structure for Calabi-Yau threefolds. ◆

3.4 The Period Map

The intuitive idea behind a variation of Hodge structure is to assign a (polarized) Hodge structure on a fixed vector space H to every t in a parameter space $\mathcal{M}$ in a holomorphic fashion. If we restrict ourselves to polarized Hodge structures, this hints to a map $\mathcal{M} \to D$. This map is central in this section and its well-definedness depends on the topology of $\mathcal{M}$.

Indeed, given a variation of polarized Hodge structure $(\mathcal{M}, \mathbf{H}, \nabla, \mathbf{H}_{\mathbb{R}}, \mathbf{F}^\bullet, \mathbf{S})$, one can transfer the data on a typical fiber $H := \mathbf{H}_{t_0}$ to different fiber $\mathbf{H}_t$ via parallel transport. Since ∇ is flat, this only depends on the homotopy class of the path connecting t_0 and t in $\mathcal{M}$. In particular, one can consider loops based at t_0. In general, the transported data along a loop is not equal to the original. This failure is captured by the **monodromy representation**

$$\rho : \pi_1(\mathcal{M}, t_0) \to \mathrm{GL}(H), \quad [\gamma] \to P_\gamma.$$

Therefore, if the topology of $\mathcal{M}$ is non-trivial there might exist a loop γ based at t_0 such that

$$\mathbf{F}^p_{t_0} \neq P_\gamma(\mathbf{F}^p_{t_0}).$$

Thus, to every $t \in \mathcal{M}$ we can assign a polarized Hodge structure *up to monodromy*. To make this precise, let $\Gamma := \rho(\pi_1(\mathcal{M}, t_0))$. Then, we have a well-defined map

$$\Phi : \mathcal{M} \to \Gamma \backslash D,$$

called the **period map**. Here, $\Gamma \backslash D$ denotes the quotient of D under the monodromy action described above. As D already is a quotient, the notation $\Gamma \backslash D$ is introduced for the quotient.

The period map was introduced by Griffiths and its properties were studied in [Gri68b]. We want to state some of those properties. For this, note Γ is discrete and $\Gamma \subset G_{\mathbb{R}}$. The latter follows from the flatness of $\mathbf{S}$. Hence, $\Gamma \backslash D$ is a complex analytic variety (cf. Section 3.2). Practically, this means a notion of holomorphicity exists. Therefore, the following statement is sensible.

Theorem 3.4.1 (Griffiths).
The period map $\Phi : \mathcal{M} \to \Gamma \backslash D$ is holomorphic.

For a proof we refer to [Voi02, Thm. 10.9]. In the literature, the holomorphicity is often written as

$$\frac{\partial \mathbf{F}^p}{\partial \bar{z}} \subset \mathbf{F}^p.$$

Upon locally lifting to the universal covering $\widetilde{\mathcal{M}}$ of $\mathcal{M}$, we obtain a holomorphic map $\widetilde{\Phi} : \widetilde{\mathcal{M}} \to D$ such that

$$\begin{tikzcd} \widetilde{\mathcal{M}} \arrow[r, "\widetilde{\Phi}"] \arrow[d] & D \arrow[d] \\ \mathcal{M} \arrow[r, "\Phi"'] & \Gamma \backslash D \end{tikzcd}$$

commutes. In other words,

$$\widetilde{\Phi}([\gamma]\cdot z) = \rho([\gamma])\cdot\widetilde{\Phi}(z), \quad z\in\widetilde{\mathcal{M}}, [\gamma]\in\pi_1(\mathcal{M}).$$

One can use this lift to translate the horizontality condition (3.8) in terms of Φ. It turns out that the holomorphic tangent bundle $T^{1,0}D$ of D may be viewed as a subbundle of $\bigoplus_p \mathrm{Hom}(\mathbf{F}^p, \mathbf{H}/\mathbf{F}^p)$ [Sch73]. By intersecting $T^{1,0}D$ with $\mathrm{Hom}(\mathbf{F}^p, \mathbf{F}^{p-1}/\mathbf{F}^p)$ one obtains a holomorphic subbundle $T_h^{1,0}D$, called the **horizontal tangent bundle**. The horizontality condition then becomes

$$d\widetilde{\Phi}(T^{1,0}\mathcal{M}) \subset T_h^{1,0}D. \tag{3.9}$$

We say Φ is **locally liftable** and its local lifts satisfy the horizontality condition.

Conversely, given a locally liftable holomorphic map $\mathcal{M}\to\Gamma\backslash D$ who local lifts are horizontal, i.e. satisfy (3.9), the variation of Hodge structure is completely determined. Hence, we found an alternative description of variation of Hodge structures.

3.5 Asymptotic Behaviour

With the concepts developed so far, we are able to describe the behaviour of Hodge structures near the boundary of the moduli space. There, the Hodge decomposition theorem could in principle fail and we are interested in the structures that may arise. By the previous section, we know these structures are captured by the form of the period map Φ near $\mathcal{M}_{\mathrm{sing}}$. Furthermore, by Theorem 2.3.7 and the discussion afterwards, we may assume that $\mathcal{M}$ is Zariski-open in a compact complex manifold $\overline{\mathcal{M}}$ [CKS86]. Now, if the codimension of $\mathcal{M}_{\mathrm{sing}}$ is at least two, it turns out Φ has a holomorphic, locally liftable continuation to $\overline{\mathcal{M}}$ [Sch73]. Hence, we may restrict ourselves to the codimension-one case, in which we can take $\mathcal{M}_{\mathrm{sing}}$ to be a divisor of normal crossing. Finally, in the compact Kähler case one can show the local monodromy to be quasi-unipotent. Hence, we may assume this as well. Before we can state the celebrated orbit theorems, we need to introduce the setup.

We are interested in the local study of the singularities of the period map. Hence, we may view Φ as a map $(\Delta^*)^l\times\Delta^{m-l}\to\Gamma\backslash D$ (cf. Section 2.3). The associated monodromy group Γ is Abelian, as $\pi_1(\Delta^*)$ is. Let $\gamma_1,\dots,\gamma_l$ denote a set of commuting generators. Here $\gamma_j\cdot v$ denotes the clockwise parallel transport of v around the jth puncture. Note, the universal cover of the punctured disc Δ^* is given by the upper-half plane $\mathbb{H}$ via the exponential map $t\mapsto e^{2\pi i t} = z$. In this case the singularity at $z=0$ corresponds to $t\to i\infty$ (see Figure 3.1). Furthermore, encircling the puncture in Δ^* corresponds to $t\mapsto t+1$ in $\mathbb{H}$. Consequently, the lift of Φ is a mapping

$$\widetilde{\Phi} : \mathbb{H}^l\times\Delta^{m-l}\to D,$$

satisfying

$$\widetilde{\Phi}(t_1,\dots,t_j+1,\dots,t_l,w_{l+1},\dots,w_m) = \gamma_j\cdot\widetilde{\Phi}(t_1,\dots,t_l,w_{l+1},\dots w_m).$$

Here, we dropped the representation ρ from the notation. By assumption, the monodromy generators γ_j are quasi-unipotency, meaning there exist positive integers s_j, n_j such that

$$(\gamma_j^{s_j}-1)^{n_j} = 0.$$

By the coordinate transformation $z\to z^k$ on the base, for some k, it follows that one can set $s_j=1$, for all j. Consequently, we may assume the γ_j to be unipotent.

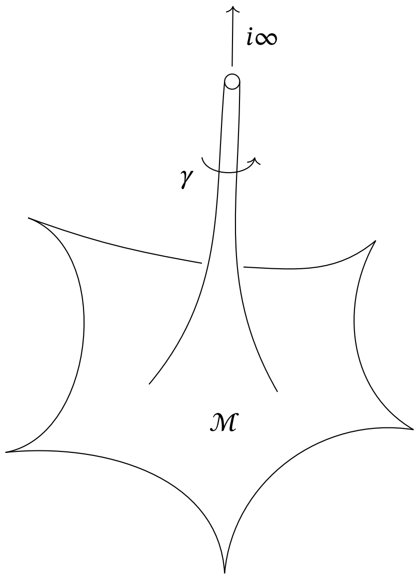

Figure 3.1: An example of a non-trivial monodromy generator γ looping around a point on the boundary of the moduli space $\mathcal{M}$, corresponding to $t \to i\infty$.

Then, we may define

$$N_j := \log(\gamma_j) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} (\gamma_j - 1)^k$$

These transformations are known as **monodromy logarithms** or **log-monodromy** transformations. Since $\Gamma \subset G_{\mathbb{R}}$, we see $N_j \in \mathfrak{g}_{\mathbb{R}}$. Moreover, from their definition we readily see that the N_j are nilpotent.

Using the monodromy logarithms, we can construct a holomorphic map

$$\widetilde{\Psi} : \mathbb{H}^l \times \Delta^{m-l} \to \hat{D}, \quad \widetilde{\Psi}(t, w) = \exp\left(-\sum_{j=1}^{l} t_j N_j \right) \cdot \widetilde{\Phi}(t, w).$$

Some remarks are in order. Firstly, the target has been enlarged. Indeed, in general the element $\sum t_j N_j$ is complex and thus lies in $\mathfrak{g}_{\mathbb{C}}$. Furthermore, from their construction we see $G_{\mathbb{C}} \cdot D = \hat{D}$. Therefore, $\widetilde{\Psi}$ maps into $\hat{D}$. Secondly, even though the definition of $\widetilde{\Psi}$ looks complicated, it is a simpler map regarding monodromy. Note, $\widetilde{\Psi}$ is *invariant* under $t_j \mapsto t_j + 1$. Intuitively, the exponent in $\widetilde{\Psi}$ precisely undoes the monodromy of $\widetilde{\Psi}$. Consequently, $\widetilde{\Psi}$ descends to a holomorphic map

$$\Psi : (\Delta^*)^l \times \Delta^{m-l} \to \hat{D}.$$

Now, the nilpotent orbit theorem states that this map captures the asymptotic behaviour of the period map. Let us focus on the precise statement in the next section.

3.5.1 Nilpotent Orbit Theorem

At this stage we can state the first main theorem in asymptotic Hodge theory due to Schmid [Sch73]: the nilpotent orbit theorem. Let us start with the definition of a nilpotent orbit following [CKS86]. For this, let us write $t_j = x_j + iy_j \in \mathbb{H}^l$. Moreover, any $F \in \hat{D}$ defines a decreasing filtration on $\mathfrak{g}_{\mathbb{C}}$ by

$$F^p_{\mathfrak{g}_{\mathbb{C}}} := \{X \in \mathfrak{g}_{\mathbb{C}} \mid X(F^r) \subset F^{r+p}\}.$$

In particular, the space $F^{-1}_{\mathfrak{g}_{\mathbb{C}}}$ corresponds to elements of $\mathfrak{g}_{\mathbb{C}}$ whose flow are horizontal near the origin. Consequently, they are called **horizontal** at F.

Then we say

Definition 3.5.1.
A **nilpotent orbit** *is a map* $\theta : \mathbb{C}^l \to \hat{D}$ *of the form*

$$\theta(t) = \exp\left(\sum_{j=1}^{l} t_j N_j\right) \cdot F,$$

where

i) $F \in \hat{D}$

ii) $\{N_j\}_{j=1}^{l}$ *is a set of commuting nilpotent elements of* $\mathfrak{g}_{\mathbb{R}}$*, horizontal at* F

iii) There exists $\alpha \in \mathbb{R}$ *such that* $\theta(t) \in D$ *for* $y_j > \alpha$*,* $1 \leq j \leq l$*.*

Note, from horizontality of the nilpotent elements N_j, it follows that θ is a horizontal map. Moreover, θ describes an orbit of F in $\hat{D}$ under elements in $G_{\mathbb{C}}$ corresponding to nilpotent elements of $\mathfrak{g}_{\mathbb{R}}$, hence the name.

Finally, we state the main result

Theorem 3.5.2 (Nilpotent orbit theorem).

i) The map Ψ *extends holomorphically to* Δ^m

ii) Let $F_{\lim}(w) = \Psi(0, w)$ *and*

$$\theta_w(t) = \exp\left(\sum_{j=1}^{l} t_j N_j\right) \cdot F_{\lim}(w),$$

where N_j *are the monodromy logarithms. Then,* $\theta_w(z)$ *is a nilpotent orbit for every* $w \in \Delta^m$

iii) Moreover, for any $G_{\mathbb{R}}$*-invariant distance* d *on* D*, there exist constants* $\alpha, \beta, K \geq 0$ *such that under the restrictions*

$$y_j \geq \alpha, \quad 1 \leq j \leq l$$

we have

$$\begin{aligned} &\theta_w(t) \in D, \\ &d(\widetilde{\Phi}(t, w), \theta_w(t)) \leq K \sum_{j=1}^{l} (y_j)^{\beta} \exp(-2\pi y_j). \end{aligned} \tag{3.10}$$

The above phrasing of the nilpotent orbit theorem can be found in [CKS86]. The original statement in [Sch73] contained a different distance estimate. The one presented here was observed by Deligne and is stronger. As it is necessary in the proof of the multi-variable SL(2)-orbit theorem in [CKS86], we state the nilpotent orbit theorem as above. Furthermore, the proof of the above theorem is highly non-trivial and will not be presented here.

The **limiting filtration** $F_{\mathrm{lim}}(w)$ can be obtained from the nilpotent orbit as follows:

$$F_{\mathrm{lim}}(w) = \lim_{t\to i\infty} \exp\left(-\sum_{j=1}^{l} t_j N_j\right) \cdot \theta_w(t). \tag{3.11}$$

Furthermore, by the distance estimate, the limiting filtration is the asymptotic value of $\widetilde{\Psi}(t,w)$ in $\hat{D}$. Indeed, we have

$$\lim_{t\to i\infty} \widetilde{\Psi}(t,w) = \lim_{t\to i\infty} \exp\left(-\sum_{j=1}^{l} t_j N_j\right) \cdot \widetilde{\Phi}(t,w) = \lim_{t\to i\infty} \exp\left(-\sum_{j=1}^{l} t_j N_j\right) \cdot \theta_w(t) = F_{\mathrm{lim}}(w).$$

In particular, the limiting filtration may not correspond to a Hodge structure. A priori, it is an arbitrary element in $\hat{D}$. It turns out, there is additional structure: the combined data $(F_{\mathrm{lim}}, N_1, \dots N_l)$ of the limiting filtration and monodromy logarithms describes a *mixed Hodge structure*. This is a (non-trivial) consequence of the second main theorem in asymptotic Hodge theory: the SL(2)-orbit theorem. In the next section we will discuss its content.

From the physics point of view, we are interested in a family of Calabi-Yau threefolds. From Example 3.3.6 we know the period vector $\mathbf{\Pi}(t,w)$ completely determines the variation of Hodge structure. Therefore, we expect the nilpotent orbit theorem to yield an approximation for the period vector, as well. Indeed, the period vector $\mathbf{\Pi}(t,w)$ is just the $F^3(t,w)$ part of the period map. Then, sufficiently close to the singular locus, the nilpotent orbit theorem tells us

$$\mathbf{\Pi}(t,w) = \exp\left(\sum_{j=1}^{l} t_j N_j\right) \mathbf{A}(t,w),$$

with $\mathbf{A}$ holomorphic in both t and w. Moreover, $\mathbf{A}$ accounts for the exponential corrections coming from (3.10), as well. Hence, this is not yet an approximation. Now, consider an expansion in t

$$\mathbf{A}(t,w) = \mathbf{a}_0(w) + \sum_{j=1}^{l} \mathbf{a}_j(w) e^{2\pi i t_j} + \sum_{j,k=1}^{l} \mathbf{a}_{jk}(w) e^{2\pi i t_j} e^{2\pi i t_k} + \dots .$$

Then, the nilpotent orbit theorem asserts

$$\mathbf{\Pi}_{\mathrm{nil}}(t,w) = \exp\left(\sum_{j=1}^{l} t_j N_j\right) \mathbf{a}_0(w) + \mathcal{O}(e^{2\pi i t_j}) \tag{3.12}$$

is a good approximation of the period vector $\mathbf{\Pi}(t,w)$ near the singular locus. However, we would like to stress that in practice the exponential corrections in (3.12) are important and *cannot* be neglected. Note, from (3.11) it follows that $\mathbf{a_0}(w) \in F_{\mathrm{lim}}(w)$. It is this approximation, alongside the SL(2)-orbit approximation, that has turned out useful in string theory, in particular in the swampland program. See for instance [GLP19; LLW22; GLV20].

Example 3.5.3 (Nilpotent orbit theorem).
Let us conclude this section by considering an example of the behaviour dictated by the nilpotent orbit theorem near a particular singular point. Namely, we will consider a variation of polarized Hodge structure near a so-called *large complex structure* (LCS) limit of a Calabi-Yau threefold Y_3. For simplicity, we assume $h^{2,1} = 1$, i.e. Y_3 has only one complex structure modulus t. Moreover, it is chosen in such a way that the LCS limit corresponds to $t \to i\infty$.

It is known the period vector takes the form

$$\mathbf{\Pi}(t) = \begin{pmatrix} 1 \\ t \\ \frac{1}{2}t^2 \\ \frac{1}{6}t^3 + i\chi \end{pmatrix} + \mathcal{O}(e^{2\pi i t}) \tag{3.13}$$

near the LSC limit (see [GMH22]). Here, $\chi \in \mathbb{R}$ denotes some constant that is related to the Euler characteristic of Y_3. Now, using the completeness principle from Example 3.3.6, we find (in leading order) the Hodge filtration

$$F(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ \frac{1}{2}t^2 & t & 1 & 0 \\ \frac{1}{6}t^3 + i\chi & \frac{1}{2}t^2 & t & 1 \end{pmatrix}. \tag{3.14}$$

By this notation we mean $F^p(t)$ is the $\mathbb{C}$-linear span of the first $4-p$ columns of the above matrix.

Now, we want to study the nilpotent approximation $\mathbf{\Pi}_{\text{nil}}$ of the period vector. A special property of the LCS limit is that we can write

$$\mathbf{\Pi}_{\text{nil}}(t) = e^{tN}\mathbf{a}_0$$

for some log-monodromy matrix N. In other words, the exponential corrections can be neglected and we only consider the leading term in (3.12). Thus, we have

$$e^{tN}\mathbf{a}_0 = (1 + tN + \frac{1}{2}t^2N^2 + \frac{1}{6}t^3N^3 + \dots)\mathbf{a}_0.$$

Comparing this to the leading term in (3.13), we must have $N^4 = 0$ as there is no t^4 term. Thus, N is nilpotent, as expected. Consequently,

$$\begin{pmatrix} 1 \\ t \\ \frac{1}{2}t^2 \\ \frac{1}{6}t^3 + i\chi \end{pmatrix} = e^{tN}\mathbf{a}_0 = \mathbf{a}_0 + tN\mathbf{a}_0 + \frac{1}{2}t^2N^2\mathbf{a}_0 + \frac{1}{6}t^3N^3\mathbf{a}_0.$$

Hence, as $\mathbf{a}_0$ is independent of t we must have

$$\mathbf{a}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ i\chi \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.15}$$

Finally, we will show the Hodge filtration can be written, in leading order, as

$$F(t) = e^{tN} F_{\text{lim}}.$$

This precisely is the nilpotent orbit approximation. Note,

$$\begin{aligned}
F^3(t) &= \operatorname{span}\left\{e^{tN}\mathbf{a}_0\right\} \\
F^2(t) &= \operatorname{span}\left\{e^{tN}\mathbf{a}_0, e^{tN}N\mathbf{a}_0\right\} \\
F^1(t) &= \operatorname{span}\left\{e^{tN}\mathbf{a}_0, e^{tN}N\mathbf{a}_0, e^{tN}N^2\mathbf{a}_0\right\} \\
F^0(t) &= \operatorname{span}\left\{e^{tN}\mathbf{a}_0, e^{tN}N\mathbf{a}_0, e^{tN}N^2\mathbf{a}_0, e^{tN}N^3\mathbf{a}_0\right\}.
\end{aligned}$$

Hence, from this we see $F(t) = e^{tN}F_{\text{lim}}$ with $F^p_{\text{lim}} = \operatorname{span}\{N^i\mathbf{a}_0 \mid 0 \le i \le 3-p\}$. In the notation used above, this is

$$F_{\text{lim}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\chi & 0 & 0 & 1 \end{pmatrix}.$$

This does *not* define a Hodge filtration as

$$F^2_{\text{lim}} \oplus \overline{F^2_{\text{lim}}} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ i\chi \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i\chi \end{pmatrix} \right\} \neq \mathbb{C}^4 = H^3(Y_3, \mathbb{C}). \tag{3.16}$$

Hence, we have an explicit example where the limiting filtration does not lie in D. We build upon this example in Example 3.5.10. ♦

3.5.2 The SL(2)-orbit Theorem

The nilpotent orbit theorem singles out the leading order of the period map of a variation of Hodge structure near the singular locus. Now, the structure and asymptotic behaviour of nilpotent orbits is the content of the second main theorem: the SL(2)-orbit theorem. The one variable ($l = 1$) version of the theorem was proved by Schmid in [Sch73]. However, its generalization to several variables was only proven over a decade later in [CKS86], reflecting its highly non-trivial nature. In this work we will solely focus on the one-variable case. To state this theorem, we need to introduce some necessary notions. We follow [CKS86].

Filtrations and Gradings

In example 3.5.3 we saw the limiting filtration $F_{\lim}$ did not contain sufficient data to define a Hodge structure. This has to do with the fact that the information contained in the log-monodromy transformations is removed from $F_{\lim}$ (cf. (3.11)). Therefore, we need to use the monodromy logarithms to find the underlying structure. The crucial concept is the *monodromy weight filtration*.

To define it, let H be a finite dimensional complex vector space of complex dimension D and W a finite increasing filtration

$$\cdots \subset W_l \subset W_{l+1} \subset \cdots \subset H.$$

From this, a natural filtration arises on $\mathfrak{gl}(H)$ given by

$$W_r^{\mathfrak{gl}} := \{X \in \mathfrak{gl}(H)\} \mid X(W_l) \subset W_{l+r}\}. \tag{3.17}$$

Elements of $W_r^{\mathfrak{gl}}$ are called r-morphisms. Furthermore, to every filtration W as above we can associated **graded spaces** given by $\mathrm{Gr}_l^W := W_l/W_{l-1}$.

Now, given a nilpotent endomorphism N of H, the **weight filtration** of N is the unique increasing filtration $W = W(N)$ given by [CKS86]

$$W_{-1} := 0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2D-1} \subset W_{2D} = H$$

such that

$$\begin{aligned} N \in W_{-2}^{\mathfrak{gl}}, &\quad \text{i.e.} \quad NW_l \subset W_{l-2}, \\ N^j : \mathrm{Gr}_{D+j}^W \to \mathrm{Gr}_{D-j}^W &\quad \text{is an isomorphism for } j \geq 0. \end{aligned}$$

In the case of Calabi-Yau D-folds[7], it is explicitly given by (cf. [GLP19])

$$W_l = \sum_{j \geq \max(-1, l-D)} \ker N^{j+1} \cap \operatorname{im} N^{j-l+D}. \tag{3.18}$$

Note, by construction we have $NW_l \subset W_{l-2}$. Finally, when N corresponds to log-monodromy transformation corresponding to a variation of Hodge structure on H we call the filtration $W(N)$ the **monodromy weight filtration**. We will refer to this as the **geometric setting**.

Mixed Hodge Structures and Splittings

Let us assume there exist a real vector space $H_{\mathbb{R}}$ such that $H = H_{\mathbb{R}} \otimes \mathbb{C}$. Then, using the concepts introduced above we can define

Definition 3.5.4.
A (real) **mixed Hodge structure** *(MHS) on* $H_{\mathbb{R}}$ *is a pair of filtrations* (W, F) *of* H

$$\begin{aligned} \cdots \subset W_l \subset W_{l+1} \subset \cdots &\quad (\textit{the weight filtration}) \\ \cdots \subset F^p \subset F^{p-1} \subset \cdots &\quad (\textit{the Hodge filtration}) \end{aligned}$$

such that

[7] It probably holds in general, as only uses abstract properties of the objects are used in the proof. However, for our purposes it suffices to restrict ourselves to Calabi-Yau D-folds.

i) *W is defined over $\mathbb{R}$*

ii) *For any l, the filtration $F(\mathrm{Gr}_l^W)$ given by*

$$F^p(\mathrm{Gr}_l^W) := (F^p \cap W_l)/(F^p \cap W_{l-1})$$

is a pure Hodge structure of weight l on Gr_l^W.

An important example of a mixed Hodge structure is due to Schmid (cf. [Sch73, Thm. 6.16]). If we consider a one-parameter nilpotent orbit

$$t \mapsto \exp(tN) \cdot F_{\mathrm{lim}} \tag{3.19}$$

we have

Theorem 3.5.5.
The pair $(W(N), F_{\mathrm{lim}})$ is a mixed Hodge structure.

If the one-parameter nilpotent orbit is the approximation of a variation of Hodge structure coming from geometry, the above statement can be shown geometrically [Ste76]. In general, it follows from the SL(2)-orbit theorem, showing its power.

Note, as $t \mapsto \exp(tN) \cdot F_{\mathrm{lim}}$ is a nilpotent orbit, we have $NF^p_{\mathrm{lim}} \subset F^{p-1}_{\mathrm{lim}}$ by horizontality. Furthermore, we already argued $NW_l \subset W_{l-2}$ for the monodromy weight filtration. Finally, N is a real operator, thus defines an element in

$$W^{\mathfrak{gl}}_{2r} \cap F^r_{\mathfrak{gl}} \cap \mathfrak{gl}(H_{\mathbb{R}}) \tag{3.20}$$

for $r = -1$. Here $F^r_{\mathfrak{gl}}$ is defined analogous to $W^{\mathfrak{gl}}_l$ (cf. (3.17)). In general, elements in (3.20) are called (r, r)-morphisms. Thus, N is a $(-1,-1)$-morphism. Furthermore, if $H_{\mathbb{R}}$ admits a polarization S, we can define

Definition 3.5.6.
A **polarized mixed Hodge structure** *is a mixed Hodge structure (W, F) on $H_{\mathbb{R}}$ and a nilpotent element $N \in \mathfrak{g}_{\mathbb{R}}$ such that*

i) $W = W(N)$

ii) $S(F^p, F^{k-p+1}) = 0$

iii) $NF^p \subset F^{p-1}$

iv) *The Hodge structure on the primitive part P_l : $\ker\left(N^{l-D+1} : \mathrm{Gr}^W_l \to \mathrm{Gr}^W_{2D-l-2}\right)$ is polarized by $S(\cdot, N^l \cdot)$.*

In the geometric setting one can explicitly check that $(W(N), F, N)$ defines a polarized mixed Hodge structure, using Theorem 3.5.5. Therefore, we have the following consequence

Corollary 3.5.7.
If a nilpotent orbit $t \mapsto \exp(tN) \cdot F_{\mathrm{lim}}$ approximates a variation of polarized Hodge structure, the triple $(W(N), F_{\mathrm{lim}}, N)$ defines a polarized mixed Hodge structure.

The definition of a (polarized) mixed Hodge structure stated above is rather convoluted. However, there is a more workable concept which contains the same data. To define it, we need the following concept

Definition 3.5.8.
A **splitting** *of a mixed Hodge structure (W, F) is a bigrading $H = \bigoplus J^{p,q}$ such that*

$$W_l = \bigoplus_{p+q \leq l} J^{p,q}, \quad F^p = \bigoplus_s \bigoplus_{r \geq p} J^{r,s}.$$

When the mixed Hodge structure is polarized, there is an extra compatibility requirement for the splitting with the polarization. This is not crucial for our purposes and for details we refer to [CKS86]. In both cases, there is one particular splitting with very useful properties.

Theorem 3.5.9.
Given a mixed Hodge structure (W, F), the **Deligne splitting** *is the unique splitting given by*

$$I^{p,q} := F^p \cap W_{p+q} \cap \left(\overline{F^q} \cap W_{p+q} + \sum_{j \geq 1} \overline{F^{q-j}} \cap W_{p+q-j-1} \right)$$

satisfying

$$I^{p,q} = \overline{I^{q,p}} \mod \bigoplus_{r<p, s<q} I^{r,s}. \tag{3.21}$$

In particular, mixed Hodge structures are in one-to-one correspondence to bigradings satisfying (3.21).

Note, the graded spaces can be recovered from the Deligne splitting, as

$$\mathrm{Gr}_l^W = \bigoplus_{p+q=l} I^{p,q}.$$

To be precise, the above equality is formally an isomorphism. However, the graded space can be identified with subspaces of H. Under this identification, the above equality holds. Then, condition (3.21) tells us that Deligne splitting gives a Hodge structure on Gr_l^W, up to lower-degree terms. From this, we already see that $I^{p,q} = \overline{I^{q,p}}$ is an interesting property. Whenever the Deligne splitting satisfies this condition, we say it is $\mathbb{R}$-**split**. We will see later on that every splitting can be made $\mathbb{R}$-split. Note, as N is a $(-1,-1)$-morphism in the geometric setting, it follows from the definition that in that case N respects the Deligne splitting: $N I^{p,q} \subset I^{p-1,q-1}$

Analogous to the Hodge diamond, we can set $i^{p,q} := \dim_{\mathbb{C}} I^{p,q}$ and group them together in the **Hodge-Deligne diamond** (see Figure 3.2). Recall, one should think about H as the middle cohomology

$$
\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 h^{3,0} & & \boxed{h^{2,1}} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & i^{3,3} & & & \\
 & & i^{3,2} & & i^{2,3} & & \\
 & i^{3,1} & & i^{2,2} & & i^{1,3} & \\
 i^{3,0} & & i^{2,1} & & i^{1,2} & & i^{0,3} \\
 & i^{2,0} & & i^{1,1} & & i^{0,2} & \\
 & & i^{1,0} & & i^{0,1} & & \\
 & & & i^{0,0} & & &
\end{array}
$$

Figure 3.2: On the right we see a general Hodge-Deligne diamond corresponding to a mixed Hodge structure on a real vector space $H_{\mathbb{R}}$ of dimension three. In the geometric setting this corresponds to a limiting filtration of a variation of Hodge structure on Calabi-Yau threefolds. Moreover, we indicate that the Hodge-Deligne splitting yields a finer decomposition of the middle cohomology of the threefold. In particular, we indicate how $h^{2,1}$ decomposes according to (3.22).

of a Calabi-Yau threefold. Consequently, the Deligne splitting can be viewed as a finer decomposition of this middle cohomology. Moreover, we can decompose the Hodge numbers as follows [KPR19]

$$h^{p,D-p} = \sum_{q=0}^{D} i^{p,q}. \tag{3.22}$$

In the geometric setting, the Hodge-Deligne numbers satisfy

$$
\begin{aligned}
i^{p,q} &= i^{q,p} = i^{n-p,n-q}, && \text{for all } p, q \\
i^{p-1,q-1} &\leq i^{p,q}, && \text{for } p+q \leq D.
\end{aligned}
$$

The first equality follows from (3.21) and the second is a consequence of $N^{p+q-D} : I^{p,q} \to I^{D-p,D-q}$ being an isomorphism. Moreover, it is related to $NI^{p,q} \subset I^{p-1,q-1}$.

Interestingly, in the Calabi-Yau threefolds case, the Hodge-Deligne diamond can be used to classify the singularities occurring at the singular locus (see [GLP19; GRH21]). To get an insight into this, let us focus on Calabi-Yau threefolds. In that case, there is a huge restriction posed by the fact that $h^{3,0} = 1$. Indeed, (3.22) implies $i^{3,d} = 1$ for precisely one $d = 0, 1, 2, 3$, while the others vanish. Consequently, every limiting mixed Hodge structure must be one of those four types. The four different types are denoted by I,II,III and IV corresponding to $d = 0, 1, 2, 3$, respectively. Moreover, due to the symmetries of the Hodge-Deligne, the only independent ones are $i^{2,1}$ and $i^{2,2}$. The LCS limit point from Example 3.5.3 is a type IV singularity and its Hodge-Deligne diamond is depicted in Figure 3.3.

Example 3.5.10 (Mixed Hodge structure).
Let us return to the LCS limit of Calabi-Yau threefolds from Example 3.5.3. There we found the limiting filtration and by Theorem 3.5.5 we know it defines a mixed Hodge structure when combined with the monodromy weight filtration. Here we want to explicitly compute this weight filtration and show it defines a mixed Hodge structure. Furthermore, we will compute the corresponding Hodge-Deligne diamond. In particular, we will see it corresponds to a type IV singularity.

Let us start by computing the monodromy Hodge filtration $W = W(N)$. By (3.18) we see

$$W_0 = \sum_{j \geq \max(-1,-3)} \ker N^{j+1} \cap \operatorname{im} N^{j+3}$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & i^{2,2} & & 0 & \\
0 & & i^{2,1} & & i^{2,1} & & 0 \\
 & 0 & & i^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\quad \cong
$$

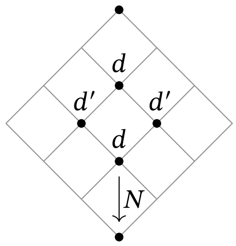

Figure 3.3: The Hodge-Deligne diamond corresponding to a type IV singularity that geometrically corresponds to the LCS limit. Here $d = i^{2,2}$ and $d' = i^{2,1}$. Furthermore, the action of the nilpotent operator N is depicted.

$$
\begin{aligned}
&= \ker N^0 \cap \operatorname{im} N^2 + \ker N \cap \operatorname{im} N^3 + \ker N^2 \cap \operatorname{im} N^4 \\
&= \ker N \cap \operatorname{im} N^3
\end{aligned}
$$

Recall, N was given in (3.15). If we let e_1, e_2, e_3, e_4 denote the standard basis of $\mathbb{C}$, we deduce from this that $W_0 = \mathbb{C}e_4$. Repeating the same computation multiple times yields

$$
\begin{aligned}
&W_1 = \mathbb{C}e_4, && W_2 = \mathbb{C}e_3 \oplus \mathbb{C}e_4, && W_3 = \mathbb{C}e_3 \oplus \mathbb{C}e_4, \\
&W_4 = \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 && W_5 = \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 && W_6 = \mathbb{C}^4.
\end{aligned}
$$

From this, we directly compute the graded spaces

$$
\begin{gathered}
\mathrm{Gr}_0^W = \mathbb{C}e_4, \quad \mathrm{Gr}_1^W = 0, \quad \mathrm{Gr}_2^W = \mathbb{C}e_3, \\
\mathrm{Gr}_3^W = 0, \quad \mathrm{Gr}_4^W = \mathbb{C}e_2, \quad \mathrm{Gr}_5^W = 0, \quad \mathrm{Gr}_6^W = \mathbb{C}e_1.
\end{gathered}
$$

One can check the defining properties are satisfied. For example,

$$
NW_4 = \mathbb{C}Ne_2 \oplus \mathbb{C}Ne^3 \oplus \mathbb{C}Ne^4 = \mathbb{C}Ne^3 \oplus \mathbb{C}Ne^4 = W_2.
$$

Therefore, $NW_4 \subset W_{4-2}$ and $N : W_4 \to W_2$ is an isomorphism.

Now, let us check $F^\bullet_{\lim}(\mathrm{Gr}_2^W)$ defines a pure Hodge structure of weight 2 on Gr_2^W. Using its form (3.16), a straightforward computation shows

$$
F^1_{\lim} \cap W_2 = \mathbb{C}e^3, \quad F^1_{\lim} \cap W_1 = 0, \quad F^2_{\lim} \cap W_2 = 0, \quad F^2_{\lim} \cap W_1 = 0.
$$

Therefore, we obtain the following filtration on Gr_2^W

$$
\begin{array}{ccccc}
F^2_{\lim}(\mathrm{Gr}_2^W) & \subset & F^1_{\lim}(\mathrm{Gr}_2^W) & \subset & F^0_{\lim}(\mathrm{Gr}_2^W). \\
0 & & \mathbb{C}e_3 & & \mathbb{C}e_3
\end{array}
$$

The corresponding Hodge decomposition is then given by

$$
H^{2,0}_{\mathrm{Gr}} \qquad\qquad H^{1,1}_{\mathrm{Gr}} \qquad\qquad H^{0,2}_{\mathrm{Gr}},
$$

$$0 \qquad \mathbb{C}e_3 \qquad 0$$

which clear defines a pure Hodge structure of weight 2 on Gr_2^W. The other filtrations can be checked in a similar fashion, proving that $(W(N), F)$ defines a mixed Hodge structure.

Finally, we discuss the Deligne splitting corresponding to $(W(N), F)$. Using the above data, we see

$$\begin{aligned} I^{3,3} &= F^3_{\lim} \cap W_6 \cap \left(\overline{F^3_{\lim}} \cap W_6 + \overline{F^2_{\lim}} \cap W_4 + \overline{F^1_{\lim}} \cap W_3 + \overline{F^0_{\lim}} \cap W_2 \right) \\ &= F^3_{\lim} \cap \left(\overline{F^3_{\lim}} + \mathbb{C}e_3 + \mathbb{C}e_3 \oplus \mathbb{C}e_4 \right) \\ &= F^3_{\lim} \\ &= \mathbb{C} \begin{pmatrix} 1 \\ 0 \\ 0 \\ i\chi \end{pmatrix}. \end{aligned}$$

From this we conclude $i^{3,3} = 1$ and thus the singularity is type IV, as expected. Furthermore, the Deligne splitting is *not* $\mathbb{R}$-split, as $\overline{I^{3,3}} \neq I^{3,3}$. Analogously, one can compute the other independent Deligne spaces. In the end one finds $I^{2,2} = \mathbb{C}e_2$ and $I^{2,1} = 0$. Hence, the corresponding Deligne diamond is given by:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

We continue this example in 3.5.12. ◆

As mentioned before, a splitting being $\mathbb{R}$-split seems like a desirable property. Luckily, there is a procedure[8] introduced by Deligne that associates an $\mathbb{R}$-split splitting to an arbitrary one. To describe the algorithm, let us fix a mixed Hodge structure (W, F) on $H_{\mathbb{R}}$. Then, we can define the nilpotent Lie algebra

$$\mathfrak{n}^{-1,-1} = \mathfrak{n}^{-1,-1}(W, F) := \left\{ X \in \mathfrak{gl}(H) \mid X(I^{p,q}) \subset \bigoplus_{r \leq p-1, s \leq q-1} I^{r,s} \right\}.$$

Here, $I^{p,q}$ is the Deligne splitting associated to (W, F). Furthermore, we define

$$\mathfrak{n}^{-1,-1}_{\mathbb{R}} := \mathfrak{gl}(H_{\mathbb{R}}) \cap \mathfrak{n}^{-1,-1}.$$

[8] Actually, there are several natural ways to do this. However, the method we discuss works nicely with the Deligne splitting.

Proposition 3.5.11.
Given a mixed Hodge structure (W, F), there is a unique $\delta \in \mathfrak{n}_{\mathbb{R}}^{-1,-1}$ such that $(W, e^{-i\delta} \cdot F)$ is a mixed Hodge structure that is $\mathbb{R}$-split. Moreover, every morphism of (W, F) commutes with δ. In particular, if (W, F, N) is a polarized Hodge structure, then $\delta \in \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{n}_{\mathbb{R}}^{-1,-1}$ and $[\delta, N] = 0$.

In the present form, it may appear that the above statement is not constructive. However, one can make it more explicit, as is done in [Hei22] in the Calabi-Yau threefold case. To get an insight in its structure, let us define a grading operator

$$Yv = (p + q - D)v, \quad v \in I^{p,q}.$$

Note, $Y \in \mathfrak{g}_{\mathbb{C}}$ for a polarized mixed Hodge structure. Moreover, if $I^{p,q}$ is not $\mathbb{R}$-split, Y cannot be a real operator, as $I^{p,q}$ and $I^{q,p}$ lie in the same eigenspace for Y. However, Y and its complex conjugate $\overline{Y}$ are related by some 'rotation'. It is precisely the operator δ that performs the rotation,

$$\overline{Y} = e^{-2i\delta} Y e^{2i\delta}.$$

From this point of view, δ must be the reason for the 'mod' part in (3.21). Concretely, it satisfies

$$\delta(I^{p,q}) \subset \bigoplus_{r<p, s<q} I^{r,s}.$$

Due to this, we can decompose

$$\delta = \sum_{p,q>0} \delta_{-p,-q},$$

where

$$\delta_{-p,-q}(I^{r,s}) \subset I^{r-p,s-q}.$$

The operator δ is crucial for the SL(2)-orbit theorem, for which we now have all the ingredients.

Example 3.5.12 ($\mathbb{R}$-split).
Let us continue the large complex structure limit example (cf. Example 3.5.10). We explicitly compute δ and show $(W, e^{-i\delta} \cdot F_{\text{lim}})$ defines an $\mathbb{R}$-split mixed Hodge structure. Let us write $\widetilde{F}_{\text{lim}} := e^{-i\delta} F_{\text{lim}}$. Previously, we saw $\overline{I^{3,3}} \neq I^{3,3}$. One can verify this is the only Deligne space that fails to be $\mathbb{R}$-split. Recall,

$$I^{3,3} = \mathbb{C}\begin{pmatrix}1\\0\\0\\i\chi\end{pmatrix} = F^3_{\text{lim}}.$$

As the other spaces in the filtration F^3_{lim} are spanned by e_2, e_3, e_4, it seems reasonable to seek for an element $\delta \in \mathfrak{g}_{\mathbb{R}}$ such that

$$e^{-i\delta}\begin{pmatrix}1\\0\\0\\i\chi\end{pmatrix} = e_1, \quad e^{-i\delta} e_2 = e_2, \quad e^{-i\delta} e_3 = e_3, \quad e^{-i\delta} e_4 = e_4.$$

From this, we find

$$e^{-i\delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\chi & 0 & 0 & 1 \end{pmatrix}$$

meaning

$$\delta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \chi & 0 & 0 & 0 \end{pmatrix}.$$

Note, $\delta \in \mathfrak{sp}(4,\mathbb{R})$ and

$$\widetilde{F}_{\text{lim}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, by repeating the arguments in Example (3.5.10) one verifies[a] $(W, \widetilde{F})$ defines a mixed Hodge structure, which is clearly $\mathbb{R}$-split. We use these computations in Example 3.5.16. ◆

[a] It basically amounts to the same computation as in Example 3.5.10 but for $\chi = 0$.

The Statement

To state the SL(2)-orbit theorem, let us fix a polarized mixed Hodge structure (W, F, N) associated to a nilpotent orbit

$$t \mapsto \exp(tN) \cdot F.$$

Then, by Proposition 3.5.11 we obtain an $\mathbb{R}$-split mixed Hodge structure $(W, \widetilde{F})$, where $\widetilde{F} = e^{-i\delta} \cdot F$. As N commutes with δ, we see N is a $(-1,-1)$-morphism of $(W, \widetilde{F})$ as well. Let $\{\widetilde{I}^{p,q}\}$ denote the $\mathbb{R}$-split Deligne splitting associated to $(W, \widetilde{F})$ and consider the grading operator

$$\widetilde{Y}v = (p+q-D)v, \qquad v \in \widetilde{I}^{p,q}.$$

Note, the crucial difference between the grading operator Y from before is that $\widetilde{Y}$ is a *real* transformation, i.e. $\widetilde{Y} \in \mathfrak{g}_{\mathbb{R}}$, because $(W, \widetilde{F})$ is $\mathbb{R}$-split. Using the fact $N(\widetilde{I}^{p,q}) \subset \widetilde{I}^{p-1,q-1}$, we see for $v \in \widetilde{I}^{p,q}$

$$\begin{aligned} [\widetilde{Y}, N]v &= \widetilde{Y}Nv - N\widetilde{Y}v \\ &= (p-1+q-1-D)Nv - N(p+q-D)v \\ &= -2Nv. \end{aligned}$$

In other words, $[\widetilde{Y}, N] = -2N$. There is a natural way to construct a unique element $\widetilde{N}^+ \in \mathfrak{g}_{\mathbb{R}}$ such that $\{\widetilde{N}^+, \widetilde{Y}, N\}$ is an $\mathfrak{sl}_2$-triple [CKS86]. It is completely determined by $\{N, \widetilde{Y}\}$ and the $\mathfrak{sl}_2$-commutation relations. It follows directly from those commutation relations that $\widetilde{N}^+$ is a $(1,1)$-morphism for $(W, \widetilde{F})$. Similar to N, it satisfies $\widetilde{N}^+(\widetilde{I}^{p,q}) \subset \widetilde{I}^{p+1,q+1}$. Note, for a non-$\mathbb{R}$-split mixed Hodge structure, $\widetilde{N}^+$ would in general be complex. Hence, moving to the $\mathbb{R}$-split case is crucial for the existence of a *real* $\mathfrak{sl}_2$-triple. The existence of such an $\mathfrak{sl}_2$-triple is, in its own right, crucial for the SL(2)-orbit theorem, as we will see below.

We started with a nilpotent orbit and had to rotate the Hodge filtration F to make the splitting $\mathbb{R}$-split. Then, one might wonder whether rotated orbit $t \mapsto \exp(tN)\cdot\widetilde{F}$ constitutes a nilpotent orbit. It turns out this is the case and it follows from the existence of an $\mathfrak{sl}_2$-triple [CKS86, Lem. 3.12]

Lemma 3.5.13.
The map $t \mapsto \exp(tN)\cdot\widetilde{F}$ is a nilpotent orbit such that, for $y = \operatorname{im} t > 0$ it holds $\exp(tN)\cdot\widetilde{F} \in D$. Furthermore, at $t = i\infty$ it agrees to first order with the original nilpotent orbit and

$$\exp(iyN)\cdot\widetilde{F} = \exp\Big(-\frac{1}{2}\log(y)\widetilde{Y}\Big)e^{iN}\cdot\widetilde{F}. \tag{3.23}$$

Proof. — We only present the proof of the first assertion and refer to [CKS86] for the rest. The existence of the $\mathfrak{sl}(2)$-triple $\{\widetilde{N}^+, \widetilde{Y}, N\}$ yields a representation

$$\rho : \mathfrak{sl}(2,\mathbb{R}) \to H.$$

Hence, we can decompose H into irreducibles. Note, as $N, \widetilde{Y}, \widetilde{N}^+$ are morphisms of the mixed Hodge structure $(W, \widetilde{F})$ they preserve the splitting $\widetilde{I}^{p,q}$. Consequently, any irreducible subspace of H is the direct sum of $\widetilde{I}^{p,q}$'s. From representation theory we know that any (non-trivial) irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ is isomorphic to tensor products of the standard representation and its dual. As mixed Hodge structures are compatible with those operators, it suffices to check the claim on three the basic types of $\mathbb{R}$-split polarized mixed Hodge structure (see Figure 3.4):

i)

$$H = \mathbb{C} = I^{1,1}, \qquad N = 0,$$

equipped with the standard symmetric bilinear form [CKS86]. This corresponds to the trivial representation $\rho = 0$ and defines a weight one $\mathbb{R}$-split mixed polarized Hodge structure. In fact, as $N = 0$ it defines a genuine polarized Hodge structure. Hence, $\widetilde{F} \in D$ and $t \mapsto \exp(tN)\cdot\widetilde{F} = \widetilde{F}$ trivially defines a nilpotent orbit.

ii)

$$H = \mathbb{C}v_{1,0} \oplus \mathbb{C}v_{0,1}, \qquad I^{p,q} = \mathbb{C}v_{p,q}$$
$$S(v_{1,0}, v_{0,1}) = 1, \qquad N = 0.$$

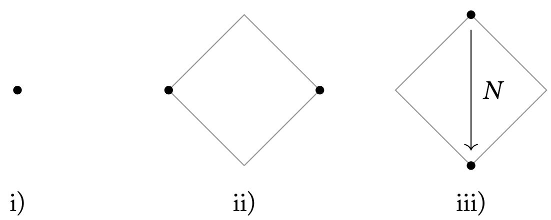

Figure 3.4: The three basic types of $\mathbb{R}$-split polarized mixed Hodge structures we consider in the proof.

This case corresponds to the fundamental representation of $\mathfrak{sl}(2,\mathbb{R})$ and defines to an $\mathbb{R}$-split polarized mixed Hodge structure. Again, as $N = 0$ it defines a genuine polarized Hodge structure and $t \mapsto \exp(tN) \cdot \widetilde{F} = \widetilde{F}$ trivially defines a nilpotent orbit.

iii)

$$H = \mathbb{C}v_{0,0} \oplus \mathbb{C}v_{1,1}, \qquad I^{p,q} = \mathbb{C}v_{p,q}$$
$$S(v_{1,1}, v_{0,0}) = 1, \qquad Nv_{1,1} = v_{0,0}.$$

This case corresponds to the fundamental representation, as well. Note, we have

$$W_0 = \mathbb{C}v_{0,0}, \quad W_1 = W_0, \quad W_2 = H, \quad \widetilde{F}^1 = \mathbb{C}v_{1,1}, \quad \widetilde{F}^0 = H.$$

It is straightforward to see $W = W(N)$ and that $(W, \widetilde{F}, N)$ defines an $\mathbb{R}$-split polarized mixed Hodge structure. Moreover, as $N^2 = 0$, we have

$$\theta^1(t) := \exp(tN) \cdot \widetilde{F}^1 = (1 + tN)\mathbb{C}v_{1,1} = \mathbb{C}v_{1,1} + t\mathbb{C}v_{0,0} = H$$

for $\operatorname{im} t > 0$. Similarly,

$$\theta^0(t) := \exp(tN) \cdot \widetilde{F}^0 = (1 + tN)(\mathbb{C}v_{1,1} + \mathbb{C}v_{0,0}) = \mathbb{C}v_{1,1} + t\mathbb{C}v_{0,0} + \mathbb{C}v_{0,0} = H$$

for any t. Thus, we see $\theta(t)$ defines the trivial polarized Hodge structure of weight one, i.e. $\theta(t) \in D$ for $\operatorname{im} t > 0$. As N is horizontal for $\widetilde{F}$, it is for $\theta(t)$ as well. Hence, $\theta(t)$ defines a nilpotent orbit, completing the proof. ■

A consequence of the lemma above is that for $t = i$, $\exp(tN) \cdot \widetilde{F} = e^{iN} \cdot \widetilde{F}$ defines a Hodge structure. For later purposes we denote this Hodge structure by F_∞ and the corresponding Hodge decomposition by $H_\infty^{p,q}$. In the physics literature, it is referred to as the Hodge structure 'at the boundary'. Yet, this nomenclature is a bit misleading as it does not sit above a point in $\mathcal{M}_{\text{sing}}$.

Furthermore, we already saw that a nilpotent orbit produces a polarized mixed Hodge structure, by the SL(2)-orbit theorem. However, what about the converse? With the above lemma at hand, one can show that the converse it true as well

Corollary 3.5.14.
If (W, F, N) is a polarized Hodge structure, then the map $t \mapsto \exp(tN) \cdot F$ is a nilpotent orbit.

Moreover, as N is part of a $\mathfrak{sl}_2$-triple, the nilpotent orbit $t \mapsto \exp(tN) \cdot \widetilde{F}$ is actually an orbit by an element in $\mathrm{SL}(2,\mathbb{R})$. Consequently, we refer to it as the SL(2)-**orbit**. The content of the SL(2)-orbit theorem is about the relationship between the SL(2)-orbit and the original nilpotent orbit.

If $\operatorname{im} t > \max(0, \alpha)$, then $\exp(tN) \cdot F$ and $\exp(tN) \cdot \widetilde{F}$ both lie in D. As $G_\mathbb{R}$ act transitively on D, there is a $g_t \in G_\mathbb{R}$ such that

$$\exp(tN) \cdot F = g_t \exp(tN) \cdot \widetilde{F}.$$

In general, g_t is far from unique. However, we will construct a natural one. Note, if $t = x + iy$, the operator $\exp(xN)$ is real and thus in $G_\mathbb{R}$. Hence, we can absorb the factors $\exp(\pm xN)$ into g_t. To be precise, the above equation can be written as

$$\exp(iyN) \cdot F = \underbrace{\exp(-xN) g_t \exp(xN)}_{\widetilde{g}(y)} \exp(iyN) \cdot \widetilde{F}$$

$$= \tilde{g}(y)\exp(iyN)\cdot\widetilde{F}.$$

By Lemma 3.5.13, this is equivalent to

$$\begin{aligned}\exp(iyN)\cdot F &= \tilde{g}(y)\exp\Big(-\frac{1}{2}\log(y)\widetilde{Y}\Big)e^{iN}\cdot\widetilde{F}\\ &= \widetilde{h}(y)e^{iN}\cdot\widetilde{F}\\ &= \widetilde{h}(y)\cdot F_\infty \end{aligned}\tag{3.24}$$

with $\widetilde{h}(y) := \tilde{g}(y)\exp\big(-\frac{1}{2}\log(y)\widetilde{Y}\big)$. Note, $F_\infty = e^{iN}\cdot\widetilde{F}$ is just (3.23) evaluated at $y=1$, as mentioned before. Therefore, by Lemma 3.5.13 we have $e^{iN}\cdot\widetilde{F}\in D$. Hence, the connection between the original nilpotent orbit and the SL(2)-orbit is captured by the map $\widetilde{h}$ (or equivalently $\tilde{g}$). To see its properties, let us recall

$$D\cong G_{\mathbb{R}}/V.$$

Without loss of generality, we may assume V is the isotropy group at $e^{iN}\cdot\widetilde{F}$. Furthermore, V is compact (cf. Proposition 3.2.4) and therefore the Killing form of $\mathfrak{g}_{\mathbb{R}}$ is negative-definite on $\mathfrak{v}=\mathrm{Lie}(V)$. Consequently, we have a decomposition

$$\mathfrak{g}_{\mathbb{R}}=\mathfrak{v}\oplus\mathfrak{v}^{\perp}.$$

By left (or right) translating the above decomposition, we obtain a horizontal bundle of the principal bundle

$$V\hookrightarrow G_{\mathbb{R}}\to G_{\mathbb{R}}/V\cong D,$$

which is equivalent to a connection. Therefore, the real analytic curve $y\mapsto\exp(iyN)\cdot F$ in D can be lifted to a *horizontal* real analytic curve $y\mapsto\widetilde{h}(y)$ in $G_{\mathbb{R}}$, meaning it satisfies the following differential equation:

$$\widetilde{h}(y)^{-1}\partial_y\widetilde{h}(y)\in\mathfrak{v}^{\perp}.\tag{3.25}$$

From this, a condition on $\tilde{g}(y)$ is derived. This is the sought-after natural element relating $\exp(iyN)\cdot F$ and $\exp(iyN)\cdot\widetilde{F}$:

Theorem 3.5.15 (SL(2)-**orbit theorem).**
There exists a unique real analytic, $G_{\mathbb{R}}$-valued function $\tilde{g}(y)$ defined for $y>\max(0,\alpha)$, such that

i) $\tilde{g}(y)$ *has a convergent Taylor series around* $y=\infty$,

$$\tilde{g}(y)=\tilde{g}(\infty)(1+\tilde{g}_1y^{-1}+\tilde{g}_2y^{-2}+\dots)$$

ii) $\tilde{g}(\infty)\in\exp\big(L_{\mathbb{R}}^{-1,-1}\cap\ker\operatorname{ad}N\big)$

iii) $(\operatorname{ad}N)^{k+1}\tilde{g}_k=0$

iv) $\exp(iyN)\cdot F=\widetilde{h}(y)e^{iN}\cdot\widetilde{F}$

v) $\widetilde{h}(y)^{-1}\partial_y\widetilde{h}(y)\in\mathfrak{v}$

Furthermore, $\tilde{g}(y)$ also has an inverted series

$$\tilde{g}(y)^{-1} = (1 + \tilde{f}_1 y^{-1} + \tilde{f}_2 y^{-2} + \dots)\tilde{g}(\infty)^{-1}$$

such that

vi) $(\operatorname{ad} N)^{k+1}\tilde{f}_k = 0.$

By the second property, we can write $\tilde{g}(\infty) = e^{\zeta}$ for some $\zeta \in L_R^{-1,-1} \cap \ker \operatorname{ad} N$. Furthermore, the operator ζ is strongly related to δ (see [CKS86, Lem. 6.60]) and can be expressed in terms of the $\delta_{-p,-q}$'s (cf. [GLP19, Appx. B]). Moreover, by restricting to the first term in the Taylor series of $\tilde{g}(y)$ and we construct another filtration

$$\hat{F} = e^{\zeta} \cdot \tilde{F} = e^{\zeta} e^{-i\delta} \cdot F.$$

This filtration is known as the SL(2)-**split**. Again, it is obtained by performing a rotation and $(W, \hat{F}, N)$ defines an $\mathbb{R}$-split polarized mixed Hodge structure. This follows from the fact that ζ is real and commutes with N. The SL(2)-splitting is particularly important in the multi-variable setting, as it produces commuting splittings [CKS86].

The interesting part of the SL(2)-orbit theorem is that it describes a way to get from the SL(2)-orbit to the nilpotent orbit. Moreover, the proof is constructive: [CKS86] provides recursion relations for the the Taylor series coefficients of $\tilde{g}(y)$. However, solving these relations is rather non-trivial, but it has been done in specific cases (see e.g. [GMH22]).

Furthermore, in view of (3.24), objects from the boundary Hodge structure F_∞ can be transported to the nilpotent orbit living in the 'bulk' of moduli space. This is precisely the idea of [Gri21], where a holographic perspective on the moduli space is presented. In particular, the Weil operator is given by

$$C(t) = \tilde{h}(t) C_\infty \tilde{h}(t)^{-1}, \tag{3.26}$$

where $\tilde{h}(t) = e^{xN}\tilde{h}(y)$ and C_∞ the Weil operator of the boundary Hodge structure F_∞. Furthermore, we can write $C_\infty = (-1)^{Q_\infty}$, where Q_∞ is the boundary charge operator. Finally, when looking at first order we have $\tilde{h}(t) = e^{xN} \exp\left(-\frac{1}{2}\log(y)\tilde{Y}\right)$. The corresponding Weil operator

$$C_{\mathrm{SL}(2)}(t) = e^{xN} y^{-\frac{1}{2}\tilde{Y}} C_\infty y^{\frac{1}{2}\tilde{Y}} e^{-xN} \tag{3.27}$$

is what we will call the SL(2)-**orbit approximation** of the Weil operator. Finally, one can explicitly check that the boundary charge operator and the $\mathfrak{sl}(2)$-triple $\{\tilde{N}^+, \tilde{Y}, N\}$ satisfy (cf. [GMH22])

$$[Q_\infty, \tilde{Y}] = i(\tilde{N}^+ + N), \quad [Q_\infty, \tilde{N}^+] = -\frac{i}{2}N^0, \quad [Q_\infty, N] = -\frac{i}{2}N^0. \tag{3.28}$$

Formally, the rotation by e^{ξ} should be incorporated as well. However, this will just result in a rotated $\mathfrak{sl}(2)$-triple, which is qualitatively equivalent. Consequently, we ignore it here.

To summarize: we started with a nilpotent orbit. To this, we were able to associate an $\mathbb{R}$-split polarized mixed Hodge structure, which gave us the SL(2)-orbit. Then, the SL(2)-orbit theorem gave us a way to retrieve the original nilpotent orbit. This procedure is schematically depicted in Figure 3.5.

In [CKS86] the SL(2)-orbit theorem is generalized to several variables. We want to emphasize that this generalization is highly non-trivial, which is indicated by the fact it took thirteen years, after the one-variable case in [Sch73], to prove it. This is due to the fact that multiple moduli can be sent to infinity in different orders. We do not discuss the multi-variable setting here and refer to [CKS86] for more details.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\exp(tN)\cdot F\\ \text{nilpotent orbit}\end{array}} \arrow[r] & \boxed{\begin{array}{c}(W,F,N)\\ \text{pol. MHS}\end{array}} \arrow[d, "\delta"] \\
\boxed{\begin{array}{c}\exp(tN)\cdot \widetilde{F}\\ \mathrm{SL}(2)\text{-orbit}\end{array}} \arrow[u, "\begin{array}{c}\mathrm{SL}(2)\text{-orbit}\\ \text{theorem}\end{array}"] & \boxed{\begin{array}{c}(W,\widetilde{F},N)\\ \mathbb{R}\text{-split}\end{array}} \arrow[l]
\end{tikzcd}$$

Figure 3.5: Schematic depiction of steps leading up to the SL(2)-orbit theorem.

Example 3.5.16 (SL(2)-**orbit theorem).**
In Example 3.5.12 we computed the $\mathbb{R}$-split polarized mixed Hodge structure $(W, \widetilde{F}_{\mathrm{lim}}, N)$. Here we want to build upon this example and go through the steps of the SL(2)-orbit theorem. First, we compute the grading operator $\widetilde{Y}$. For this, let us denote $v^{p,q} \in \tilde{I}^{p,q}$. We saw in Example 3.5.12 that $\tilde{I}^{p,q} = 0$ for $p \neq q$. Furthermore,

$$\widetilde{Y} v^{3,3} = (6-3)v^{3,3} = 3v^{3,3}, \qquad \widetilde{Y} v^{2,2} = (4-3)v^{2,2} = v^{2,2},$$
$$\widetilde{Y} v^{1,1} = (2-3)v^{1,1} = -v^{1,1}, \qquad \widetilde{Y} v^{0,0} = (0-3)v^{0,0} = -3v^{0,0}.$$

As $v^{3,3}, v^{2,2}, v^{1,1}, v^{0,0}$ coincides with the standard basis of $\mathbb{C}^4$, we can write

$$\widetilde{Y} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

The matrix for N was given in Example 3.5.3 and one can verify that $[\widetilde{Y}, N] = -2N$. By assuming a general matrix form for $\widetilde{N}^+$ and imposing the $\mathfrak{sl}_2$-commutation relations, we find

$$\widetilde{N}^+ = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note, $N^i v^{3,3}$, for $0 \leq i \leq 3$, generate $H^3(Y_3, \mathbb{C})$. Hence, $H^3(Y_3, \mathbb{C})$ is an irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$ (cf. proof of Lemma 3.5.13). Finally, we compute the SL(2)-orbit

$$F_{\mathrm{SL}(2)} = e^{tN} \cdot \widetilde{F}_{\mathrm{lim}}.$$

By our matrix definition of the filtration $\widetilde{F}_{\mathrm{lim}}$, the computation above amounts to matrix multiplication:

$$F_{\mathrm{SL}(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ \frac{1}{2}t^2 & t & 1 & 0 \\ \frac{1}{6}t^3 & \frac{1}{2}t^2 & t & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ \frac{1}{2}t^2 & t & 1 & 0 \\ \frac{1}{6}t^3 & \frac{1}{2}t^2 & t & 1 \end{pmatrix}.$$

The original nilpotent orbit is given in (3.14). The difference between the two is a factor $i\chi$ in the bottom left corner. The SL(2)-orbit theorem gives us a way to construct an analytic map $\tilde{g}(y)$ such that

$$e^{iyN} \cdot F_{\lim} = \tilde{g}(y) F_{\mathrm{SL}(2)}.$$

In [GMH22] the algorithm is performed and, adapted to our conventions, it is given by

$$\tilde{g}(y) = \frac{1}{\sqrt{1 - \frac{3\chi}{2y^3}}} \begin{pmatrix} \tilde{g}_{1,1} & 0 & \tilde{g}_{1,3} & 0 \\ 0 & \tilde{g}_{2,2} & 0 & \tilde{g}_{2,4} \\ \tilde{g}_{3,1} & 0 & \tilde{g}_{3,3} & 0 \\ 0 & \tilde{g}_{4,2} & 0 & \tilde{g}_{4,4} \end{pmatrix},$$

where

$$\begin{aligned} \tilde{g}_{1,1} &= \frac{1}{4}\left(1 + 3\beta(y)^{-1}\right), & \tilde{g}_{1,3} &= \frac{3}{2y^2}\left(-1 + \beta(y)^{-1}\right), \\ \tilde{g}_{2,2} &= \frac{3}{4}\left(1 + \frac{1}{3}\beta(y)\right), & \tilde{g}_{2,4} &= \frac{3}{2y^2}(-1 + \beta(y)), \\ \tilde{g}_{3,1} &= -\frac{y^2}{8}\left(1 - \frac{\gamma(y)}{\beta(y)}\right), & \tilde{g}_{3,3} &= \frac{3}{4}\left(1 + \frac{\gamma(y)}{3\beta(y)}\right), \\ \tilde{g}_{4,2} &= -\frac{y^2}{8}(\gamma(y) - \beta(y)), & \tilde{g}_{4,4} &= \frac{1}{4}(\gamma(y) + 3\beta(y)), \end{aligned}$$

and

$$\beta(y) = \sqrt{1 + \frac{3\chi}{y^3}}, \quad \gamma(y) = 1 - \frac{6\chi}{y^3}.$$ ◆

CHAPTER 4

Integrable systems

Hodge theory has proven to be very useful within string theory, as we have mentioned multiples times already. In the previous chapter we saw that the main object of interest is the period map. Yet, its description and properties are quite abstract. However, in the single-complex modulus case, one can view the period map, or its lifting to be precise, as a mapping

$$\Sigma \to G,$$

where Σ is a Riemann surface, i.e a worldsheet, and G a Lie group. In physics, a theory describing such maps is called a *non-linear sigma-model*. The main objective is to find a non-linear sigma-model for which the period map solves the equations of motion.

A first step is made in [GM22] and [GM23], where a connection between two particular non-linear σ-models and objects from Hodge theory has been found. Interestingly, in *both* models, it is not the period map, but the Weil operator that provides a solution. To be precise, they consider a *λ-deformed WZW model* and a *bi-Yang-Baxter model*. These are specific examples of *integrable systems*. This is a manifestation of the general idea that Hodge theory and integrable systems are closely related (see e.g. [DWS08; Fre99; Her03]).

Motivated by this connection, we study integrable systems in this chapter. The field of integrable systems has its origins in classical mechanics. At the time people tried to find exact solutions to Newton's equations of motion. This turned out to be a difficult task, as in about two centuries only a handful instances were found [BBT03]. A systematic approach was needed. This eventually came in the nineteenth century, when Liouville developed a framework in which sufficient conditions were described for the equations of motion to be exactly solvable. The procedure was called *"solvable by quadratures"*. The systems previously found all fell into this category. Nowadays, systems that satisfy Liouville's conditions are dubbed to be *integrable* and will be of interest in this chapter.

Specifically, we will generalize the notion of integrability to field theory. This formalism is used in various fields of physics, such as fluid mechanics (e.g. the Korteweg-De Vries model), condensed matter (the Sine-Gordon model) and also turns out to be relevant for string theory [Dri22]. Furthermore, we will consider the two specific non-linear sigma-models from above, namely deformed WZW models and bi-Yang-Baxter models and discuss their integrability.

4.1 Classical Integrability

Integrability is a powerful tool in solving dynamical systems. A key feature of integrable systems is that they have a lot of symmetry. This, heuristically speaking, gives a way to rewrite the equations of motion, which are typically differential equations, into algebraic equations that are easier to solve, in principle. One should think about inverting algebraic relationships between variables, for example. Note, this can still be highly non-trivial, hence the wording 'in principle'.

For our purposes, we will need to extend the definitions to the field theory setting. Our focus will be on classical integrability and we will mainly follow [Dri22; Hoa22; BBT03; Aru19]. Consequently, we will not touch upon quantum integrability. However, this is a very interesting subject and could be a direction for further research. For an introduction in this topic we refer to [Bom+16; Ret22].

4.1.1 Liouville Integrability

Let us first introduce the concept of integrability in the setting of classical mechanics. Recall, in the Hamiltonian formalism the dynamics are captured in generalized coordinates q^i and their conjugate momenta p_i, which satisfy the Hamilton equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

Mathematically, this data is captured in a symplectic manifold (P, ω) (see e.g. [MS17, Ch. 1]). Here, P can be viewed as the *phase space* of the system. The amount of degrees of freedom is now captured by the half dimension of the manifold P, which coincides with the number of generalized positions $\{q^i\}$. To completely describe a classical dynamical system, one also needs to specify the Hamiltonian H. Thus, the required data is a triple (P, ω, H).

Using the symplectic structure, one can obtain a (non-degenerate) Poisson bracket $\{\cdot,\cdot\}$ by 'inverting' the symplectic form. In local coordinates the bracket is given by

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i}.$$

As usual, the Einstein summation convention is assumed. From this definition, one readily sees

$$\frac{d}{dt} f = \{f, H\},$$

given that the coordinates (q^i, p_i) satisfy the Hamilton equations. Hence, a function $f \in C^\infty(P)$ is constant in time if and only if it Poisson commutes with the Hamiltonian. We say such functions are **conserved quantities** or **conserved charges**.

Now, *Liouville integrability* is defined in terms of the Poisson bracket ([Dri22, Sec. 2.2]):

Definition 4.1.1.
A classical dynamical system (P, ω, H) *is said to be* **Liouville integrable** *if*

i) It has $n = \frac{1}{2}\dim P$ *conserved quantities* $F_i \in C^\infty(P)$.

ii) The conserved quantities are in **involution**, *i.e. they mutually Poisson commute*

$$\{F_i, F_j\} = 0.$$

> *iii) The conserved quantities are* **independent**. *By this we mean that the one forms* $\mathrm{d}F_i$ *are pointwise linearly independent.*

Examples of Liouville integrable classical systems are the (multi-dimensional) harmonic oscillator [Aru19, Ch. 1] and the Kepler problem [Tor16, Sec. 2]. The reason Liouville integrable systems are of interest is because they are exactly solvable. One can find specific canonical coordinates in terms of the conserved quantities. In this new basis, the equations of motion decouple and their solutions are linear in time [Dri22, Sec. 2.2]. Consequently, the equations of motion can be exactly or completely solved by computing and solving finitely many definite integrals and algebraic equations, respectively. This is known as **solvable by quadratures**. Since Liouville integrable systems are, in principle, completely solvable, they are also referred to as *completely integrable systems* [Aru19, Ch. 1].

The statement above (and more) is captured in the Arnold-Liouville theorem:

> **Theorem 4.1.2 (Liouville).**
> *Given a Liouville integrable system* (P, ω, H) *with conserved quantities* $\{F_i\}$*, there exists a canonical transformation* $(q^i, p_i) \to (\Psi^i, F_i)$ *for which the equations of motion can be obtained by "quadrature".*

The proof requires a bit of symplectic geometry and the canonical transformation is difficult to compute in practice. We refer to [BBT03, Ch. 2] for details. In view of Liouville's theorem, we conclude that Liouville integrability is very powerful: a maximal set of independent Poisson commuting conserved quantities guarantees complete solvability of the dynamical system in terms of simple linear solutions.

However, there is an evident complication: one has to find Poisson commuting conserved quantities. Given a dynamical system, this in practice is a rather non-trivial task, as there is no systematic way to compute these quantities [Dri22, Sec. 2.5]. Therefore, it is in general very difficult to determine whether a dynamical system is integrable. Moreover, it is unclear from this formulation of integrability how one would generalize to field theories. Specifically, having as many conserved quantities as degrees of freedom (first requirement in Definition 4.1.1) is not meaningful in a field theory, as such theories typically have an infinite amount of degrees of freedom. To solve these problems we need a different framework: the Lax pair formulation of integrability. We will expand on this in the next subsection.

4.1.2 Lax Pair Formulation

As indicated in the previous section, we need a systematic way to construct conserved charges and a definition of integrability for which the generalization to field theories is apparent. This is done in the **Lax pair formulation**. The approach will consist of two steps: constructing conserved charges and then finding sufficient conditions for them to be in involution. Let us start with the former.

Conserved Charges

To construct conserved charges we will put a constraint on the dynamical system. Let us suppose that the equations of motion of our system can be written as[1]

$$\partial_\tau L = [M, L], \tag{4.1}$$

where L, M are non-singular phase-space valued square matrices [Dri22, Sec. 2.5]. In other words, L and M could be viewed as maps $P \to \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra and P the phase space. Typically, we have

[1]We write τ instead of t for the time derivative, as we want to interpret it as the τ coordinate on the worldsheet later on.

$\mathfrak{g} = \mathfrak{gl}(k)$ for some k, yet we would like to think about it abstractly. The pair (L, M) is called a **Lax pair**. Note, a Lax pair is not unique. One can perform, for instance, a gauge transformation $L \to gLg^{-1}$ and $M \to gMg^{-1} + \partial_\tau g g^{-1}$. A small computation shows that such a transformation leaves (4.1) invariant.

Given a Lax pair, the construction of conserved charges is straightforward. They are encoded in traces of powers of L. Indeed, for every $m \geq 0$

$$\partial_\tau \operatorname{Tr}(L^m) = m \operatorname{Tr}([M, L^m]) = 0.$$

Hence, we have found a tower of conserved charges

$$F_m := \operatorname{Tr}(L^m),$$

which do not necessarily need to be independent [Dri22, Sec. 2.5]. Equivalently, one can show that the eigenvalues of L are conserved. Indeed, write $L = U\Lambda U^{-1}$ with Λ diagonal. Then, by using the gauge symmetry of (4.1), $(\Lambda, \widetilde{M})$ with $\widetilde{M} = U^{-1}MU + \partial_\tau(U^{-1})U$ is a Lax pair. Equation (4.1) then becomes

$$\partial_\tau \Lambda = [\widetilde{M}, \Lambda].$$

However, the right-hand side does not contain diagonal terms as it is skew-symmetric. Therefore, we have $\partial_\tau \Lambda_i = 0$. The next step is to see when these charges are in involution.

Involution of Charges

Now that we found a way to construct conserved quantities we would like to know when the charges are in involution, i.e. Poisson commute. For this, we will construct a Poisson-type bracket on the level of the Lax pair and come to a necessary and sufficient form of this bracket to ensure involution of the charges.

Suppose (L, M) form a Lax pair and L can be diagonalized

$$L = U\Lambda U^{-1}.$$

As seen above, the diagonal elements of Λ are the conserved charges. Furthermore, let E_{ij} be the canonical basis of $n \times n$ matrices, $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ [BBT03, Sec. 2.5]. Then, we can write

$$L = \sum_{ij} L_{ij} E_{ij},$$

where L_{ij} are *functions* on the phase-space. Hence, the Poisson bracket $\{L_{ij}, L_{kl}\}$ makes sense. Now, let

$$L_1 := L \otimes 1 = \sum_{ij} L_{ij} E_{ij} \otimes 1 \in \mathfrak{g} \otimes \mathfrak{g}, \quad L_2 := 1 \otimes L = \sum_{ij} L_{ij} 1 \otimes E_{ij} \in \mathfrak{g} \otimes \mathfrak{g}.$$

If we consider more copies of $\mathfrak{g}$, we write L_n for the embedding of L at position n, e.g $L_3 = 1 \otimes 1 \otimes L \otimes 1 \otimes \cdots$. For a general element $T \in \mathfrak{g} \otimes \mathfrak{g}$, we write

$$T_{12} := T = \sum_{ij,kl} T_{ij,kl} E_{ij} \otimes E_{kl}, \quad T_{21} := \sum_{ij,kl} T_{ij,kl} E_{kl} \otimes E_{ij}$$

Using the definition above we construct a new Poisson-type bracket as follows

$$\{L_1, L_2\} := \sum_{ij,kl} \{L_{ij}, L_{kl}\} E_{ij} \otimes E_{kl}, \tag{4.2}$$

i.e. as the matrix of Poisson brackets of elements of L [BBT03, Sec. 2.5]. The involutivity of the charges is captured in this bracket:

Theorem 4.1.3.
The eigenvalues, i.e. conserved charges, of L *are in involution if and only if there exists a function* $r : P \to \mathfrak{g} \otimes \mathfrak{g}$ *such that the bracket* (4.2) *takes the form*

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \tag{4.3}$$

The object r *is called the* **r-matrix**.

We would like to emphasize that the proof is *constructive* and refer to [BBT03, Sec. 2.5] for the argument. To summarize: given a dynamical system, if one provides a Lax pair and an r-matrix such that equation (4.3) holds, a set of mutually Poisson commuting conserved charges can be constructed. Hence, we succeeded in providing a systematic way to show Liouville integrability. However, finding a Lax pair and an r-matrix is highly non-trivial. Moreover, the classification of integrable systems in the Lax pair formulation, i.e. the setting of Theorem 4.1.3, is still an open problem [Dri22].

However, we can simplify the situation by imposing restrictions on the r-matrix. For instance, note that the bracket (4.2) is skew-symmetric and satisfies the Leibniz identity by construction. Therefore, it defines a Poisson bracket if it satisfies the Jacobi identity. By imposing the Jacobi identity[2] we find [BBT03, Sec. 2.5]

$$[L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] + \text{cyc. perm.} = 0. \tag{4.4}$$

Now, if we assume r to be constant, the last two terms vanish. Hence, a sufficient condition for (4.2) to define a Poisson bracket is

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0. \tag{4.5}$$

When the r-matrix is antisymmetric, i.e. $r_{12} = -r_{21}$, the above equation is referred to as the **classical Yang-Baxter equation** (CYBE). The CYBE and its modifications play an important role in this work. Therefore, we like to understand its structure. We discuss this in the next section.

4.1.3 Algebraic Structure of Classical Yang-Baxter Equation

The current description of the r-matrix is not optimal for our discussion later on. Furthermore, the relevant structures at play are not manifest. In this section we build the appropriate framework for the r-matrices and discuss its relation to integrable systems. We will see certain algebraic structures appearing, which are important in Chapter 5.

For this, let us consider a Lie algebra $\mathfrak{g}$ equipped with a non-degenerate ad-invariant bilinear form $(\cdot, \cdot)$ and a Lax pair L, M that satisfies Theorem 4.1.3. As before, we have $r \in \mathfrak{g} \otimes \mathfrak{g}$. However, using the non-degenerate paring, we could view it as a mapping $R : \mathfrak{g} \to \mathfrak{g}$. Indeed, by pairing we can identify $\mathfrak{g} \cong \mathfrak{g}^*$, meaning $r \in \mathfrak{g} \otimes \mathfrak{g}^*$. As usual, this canonically corresponds to a map $R : \mathfrak{g} \to \mathfrak{g}$. Concretely, if we pick a basis and write $r = r_{12} = \sum_{ij} r^{ij} T_i \otimes T_j$, then

$$RX = \sum_{ij} r^{ij} (T_j, X) T_i.$$

In terms of the map R, the compatibility condition (4.3) becomes

$$\{(L, X), (L, Y)\} = (L, [X, Y]_R), \tag{4.6}$$

[2] For this, we naturally extend to $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

where

$$[X,Y]_R := [RX,Y] + [X,RY].$$

The bracket $[\cdot,\cdot]_R$ is called the R-**bracket**. To see (4.6), take (4.3) and contract with $X \otimes Y$:

$$\begin{aligned}(\{L_1,L_2\}, X\otimes Y) &= ([r_{12},L_1], X\otimes Y) - ([r_{21},L_2], X\otimes Y)\\ \{(L,X),(L,Y)\} &= ([RY,L],X) - ([RX,L],Y)\\ &= -(L,[RY,X]) + (L,[RX,Y])\\ &= (L,[X,Y]_R).\end{aligned}$$

Here, we applied the ad-invariance in the third step. Furthermore, one could write down (4.4) in terms of R. One finds [BBT03, Sec. 4.1]

$$\big(L, \big[X, \{(L,Y), RZ\} - \{(L,Z), RY\} + [RY,RZ] - R[Y,Z]_R\big] + \text{cyc. perm.}\big) = 0$$

As before, if we assume R be constant, it reduces to

$$\big(L, \big[X, [RY,RZ] - R[Y,Z]_R\big] + \text{cyc. perm.}\big) = 0. \tag{4.7}$$

A sufficient condition to fulfill this equation is

$$[RX,RY] - R([RX,Y] + [X,RY]) = -c^2[X,Y]. \tag{4.8}$$

In that way, (4.7) reduces to the Jacobi identity of $[\cdot,\cdot]$. An operator $R : \mathfrak{g} \to \mathfrak{g}$ satisfying (4.8) is called an R-**matrix** or **Yang-Baxter** operator. Equation (4.8) is called the **modified classical Yang-Baxter equation** (mCYBE).

The mCYBE (4.8) is related to the CYBE (4.5). To make this apparent, one can write (4.5) in terms of R [BBT03, Sec. 4.1]:

$$[RX,RY] - R([X,RY] - [R^tX,Y]) = 0,$$

where R^t denotes the transpose of R with respect to the pairing. Now, (4.8) agrees with the above equality when $c = 0$ and $R^t = -R$. The latter translates to $r_{12} = -r_{21}$, which was precisely the condition for the CYBE. Consequently, one can view (4.8) as a generalization of the CYBE (4.5). Note, by a real rescaling of R we may restrict to $c \in \{0,1,i\}$. In the literature, $c = 0$ is known as the homogeneous case. Furthermore, $c = 1$ and $c = i$ are referred to as the *split* and *non-split* inhomogeneous case, respectively. We will mostly be interested in the non-split case.

The existence of an operator $R : \mathfrak{g} \to \mathfrak{g}$ on a Lie algebra satisfying the mCYBE yields an interesting algebraic structure.

Proposition 4.1.4.
If R is a solution of the mCYBE, then the R-bracket $[\cdot,\cdot]_R$ defines a Lie bracket on $\mathfrak{g}$. Furthermore, the operator $R \pm c$ satisfies

$$(R\pm c)([X,Y]_R) = [(R\pm c)X, (R\pm c)Y]. \tag{4.9}$$

Proof. — The skew-symmetry is obvious. The proof of the Jacobi identity is straightforward, yet tedious and not insightful. Thus we omit it. In the end it relies on (4.8) and the Jacobi identity for $[\cdot,\cdot]$. For the final claim, using the mCYBE we see

$$(R\pm c)([X,Y]_R) = R[X,Y]_R \pm c[X,Y]_R$$

$$= [RX, RY] + c^2[X, Y] \pm c[RX, Y] \pm c[X, RY]$$
$$= [(R \pm c)X, (R \pm c)Y]. \qquad \blacksquare$$

Let us denote the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_R)$ by $\mathfrak{g}_R$. This Lie algebra will play an important role in Chapter 5. Property (4.9) suggests the operator $R \pm c$ is a Lie algebra homomorphism between appropriate Lie algebras. This will be discussed in Chapter 5 as well. Furthermore, it is a central object in the classification theorems of solutions to the (modified) CYBE by Belavin-Drinfel'd [BD82] and Semenov-Tian-Shansky [Sem83].

Since the Lie algebra $\mathfrak{g}_R$ will play such an important role, let us end this section by studying its structure. Following [Vic15], we consider both inhomogeneous cases separately. Firstly, we consider the non-split case $c = i$. Let $(\cdot,\cdot)$ be an ad-invariant non-degenerate symmetric pairing on $\mathfrak{g}$. Then, solutions to the mCYBE are related to subalgebras of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$ seen as a real Lie algebra:

Proposition 4.1.5.
The map $R \mapsto \mathfrak{g}_R$ defines a one-to-one correspondence between solutions of the non-split mCYBE on $\mathfrak{g}$ and Lie subalgebras of $\mathfrak{g}_{\mathbb{C}}$ complementary to $\mathfrak{g}$. Moreover, $\mathfrak{g}, \mathfrak{g}_R \subset \mathfrak{g}_{\mathbb{C}}$ are Lagrangian with respect to ad*-invariant non-degenerate pairing on $\mathfrak{g}_{\mathbb{C}}$*

$$\langle Z_1, Z_2\rangle := -i(Z_1, Z_2) + i\overline{(Z_1, Z_2)}, \tag{4.10}$$

where $(\cdot,\cdot)$ is the complex linear extension, if and only if R is skew-symmetric.

Proof. — If R is a solution to the mCYBE, $\mathfrak{g}_R$ is a Lie algebra by Proposition 4.1.4. We claim $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{g}_R$. To see this, note $R - i : \mathfrak{g}_R \to \mathfrak{g}_{\mathbb{C}}$ is an injective Lie algebra homomorphism. Indeed, suppose $(R-i)X = 0$ for $X \in \mathfrak{g}$. Then, by complex conjugation, we have $RX + iX = 0$. Here we used that $\mathfrak{g}$ and R are real. From this we deduce $X = 0$. Furthermore, it is a Lie algebra homomorphism by Proposition 4.1.4. Consequently, we may view $\mathfrak{g}_R$ as a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ by identifying it with the image $(R-i)(\mathfrak{g}_R)$. On the other hand, consider the map

$$\rho : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}, \quad \rho(Z) := \frac{1}{2i}(Z - \overline{Z})$$

Clearly, ρ is surjective and $\ker\rho = \mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$. Furthermore, restricted to $\mathfrak{g}_R$ (seen as the image $(R-i)(\mathfrak{g}_R)$), we have

$$\rho(RX - iX) = \rho(RX) - \rho(iX) = -X.$$

Consequently, ρ is an isomorphism restricted to $\mathfrak{g}_R$. As $\ker\rho = \mathfrak{g}$, we obtain the desired direct sum decomposition $\mathfrak{g}_C = \mathfrak{g} \oplus \mathfrak{g}_R$.

Conversely, let $\mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}}$ be subalgebra complementary to $\mathfrak{g}$. Then, the restriction of ρ to $\mathfrak{p}$ yields an isomorphism $\mathfrak{p} \cong \mathfrak{g}$. In particular, for every $X \in \mathfrak{g}$ there is a unique $Y \in \mathfrak{p}$ such that $X = \rho(Y)$. Then, define

$$R_{\mathfrak{p}} : \mathfrak{g} \to \mathfrak{g}, \quad R_{\mathfrak{p}}X := \frac{1}{2}(Y + \overline{Y}).$$

By a straightforward computation, it can be verified (4.8) is satisfied for $c = i$. Moreover, the maps $R \mapsto \mathfrak{g}_R$ and $\mathfrak{p} \mapsto R_{\mathfrak{p}}$ are inverses [Vic15].

Note, $\mathfrak{g}$ is isotropic with respect to the pairing and has dimension $\frac{1}{2}\dim\mathfrak{g}_{\mathbb{C}}$, thus is Lagrangian. For $\mathfrak{g}_R$, we see

$$\begin{aligned}\langle (R-i)X,(R-i)Y\rangle &= -i(RX-iX,RY-iY)+i\overline{(RX-iX,RY-iY)}\\ &= -2\Big((RX,Y)+(X,RY)\Big).\end{aligned}$$

Thus, $\mathfrak{g}_R$ is isotropic if and only if R is skew-symmetric. As $\mathfrak{g}_R$ is complementary to $\mathfrak{g}$ it also has dimension $\frac{1}{2}\dim\mathfrak{g}_{\mathbb{C}}$ and thus is Lagrangian. ∎

A similar reasoning is valid for the split case $c=1$, however then the role of $\mathfrak{g}_{\mathbb{C}}$ is played by $\mathfrak{d}=\mathfrak{g}\oplus\mathfrak{g}$. Let $\mathfrak{g}^{\delta} := \{(X,X)\mid X\in\mathfrak{g}\}$ denote the diagonal subalgebra in $\mathfrak{d}$. Then, we have (cf. [Vic15])

Proposition 4.1.6.
The map $R\mapsto\mathfrak{g}_R$ defines a one-to-one correspondence between solutions of the split mCYBE on $\mathfrak{g}$ and Lie subalgebras of $\mathfrak{d}$ complementary to $\mathfrak{g}$. Moreover, $\mathfrak{g}^{\delta}$ and $\mathfrak{g}_R\subset\mathfrak{d}$ are Lagrangian with respect to the ad*-invariant non-degenerate pairing on $\mathfrak{d}$*

$$\langle (X,Y),(X',Y')\rangle := (X,X')-(Y,Y'),$$

if and only if R is skew-symmetric.

The upshot is that skew-symmetric R-matrices correspond to Lie algebras that admit a decomposition into Lagrangian subalgebras. This data is known as a **Manin triple** and we will discuss them in Chapter 5.

4.2 Integrability in Field Theory

As mentioned before, a direct generalization of Liouville integrability to field theory is ambiguous as the number of degrees of freedom is typically infinite. However, the Lax pair formulation turns out to be more suitable, as we will see in this section. For our purposes, we restrict ourselves to $(1+1)$-dimensional field theories, as those are relevant in string theory contexts. Our main references are [Dri22; BBT03].

Consider a general $(1+1)$-dimensional field theory on a spacetime Σ. In our discussion, we will either have $\Sigma=\mathbb{R}^{1,1}$ or $\Sigma=\mathbb{R}\times S^1$. Furthermore, let us denote the time-direction by τ and the spatial-direction by σ. Similar to the Lax pair formulation, let us assume the dynamics of the theory can be captured into two $\mathfrak{g}$-valued maps[3] $\mathcal{L}_\sigma(\tau,\sigma;\mu)=\mathcal{L}_\sigma(\mu)$ and $\mathcal{L}_\tau(\tau,\sigma;\mu)=\mathcal{L}_\tau(\mu)$, depending on a free **spectral parameter** $\mu\in\mathbb{C}$, such that

$$\partial_\tau\mathcal{L}_\sigma(\mu)-\partial_\sigma\mathcal{L}_\tau(\mu)+[\mathcal{L}_\tau(\mu),\mathcal{L}_\sigma(\mu)]=0,\qquad \forall\mu\in\mathbb{C}. \tag{4.11}$$

Some remarks are in place. The motivation for the above assumption lies in the structure of the Lax equation. It turns out the Lax matrix can be interpreted as an element on a coadjoint orbit in $\mathfrak{g}$ and Lax equation a flow on this orbit (cf. [BBT03, Sec. 3.3]). If one extends this idea to field theory one ends up with the equation above (see [BBT03, Sec. 3.7]). Furthermore, the dependence on the spectral parameter μ is a technicality needed to find the proper infinite set of conserved quantities. To be precise, when incorporating the spectral parameter, $\mathcal{L}_\sigma$ and $\mathcal{L}_\tau$ take values in $\mathfrak{g}\otimes\mathbb{C}$.

[3] Intuitively, one should think about them as matrices, i.e. $\mathfrak{g}=\mathfrak{gl}(n)$.

Requirement (4.11) is called the **zero-curvature condition**. To motivate this, we introduce a one-form on Σ

$$\mathcal{L}(\mu) := \mathcal{L}_\tau(\mu)d\tau + \mathcal{L}_\sigma(\mu)d\sigma,$$

called the **Lax connection**. It can be viewed as a connection on a principal G-bundle over Σ, where G is a Lie group integrating $\mathfrak{g}$, motivating the name. Then, (4.11) means the Lax connection is flat,

$$d\mathcal{L}(\mu) + \mathcal{L}(\mu) \wedge \mathcal{L}(\mu) = 0, \qquad \mu \in \mathbb{C}.$$

Following [Dri22], we will say a field theory is *weakly* (classically) integrable[4] if its equations of motion are captured by a flat Lax connection $\mathcal{L}(\mu)$, for every $\mu \in \mathbb{C}$. If the set of conserved charges are in involution, we say the system is *strongly* (classically) integrable. Analogous to Section 4.1.2, we will find a condition on the Poisson brackets of the Lax matrices that ensures strong integrability.

Conserved charges

Let us focus on constructing an infinite set of conserved charges from a Lax connection. For this, let us introduces an auxiliary field $\Psi(\tau, \sigma; \mu) = \Psi(\mu)$ that satisfies

$$(\partial_\sigma - \mathcal{L}_\sigma(\mu))\Psi(\mu) = 0, \quad (\partial_\tau - \mathcal{L}_\tau(\mu))\Psi(\mu) = 0. \tag{4.12}$$

The field Ψ is completely fixed by the above systems if we require $\Psi(0, 0; \mu) = 1$. Due to its resemblance to the time-dependent Schrödinger equation, Ψ is referred to as the **wave function**. Now, the flatness of the lax connection is equivalent to the compatibility condition

$$\partial_\sigma \partial_\tau \Psi(\mu) = \partial_\tau \partial_\sigma \Psi(\mu)$$

on the wave function.

The system (4.12) can be solved by parallel transporting from the origin to a point (τ, σ) along a path γ in Σ,

$$\Psi(\tau, \sigma; \mu) = P\exp\left(-\int_\gamma \mathcal{L}(\mu)\right). \tag{4.13}$$

Here, $P\exp$ denotes the **path-ordered exponential**, which is defined on a fixed time slice by

$$\begin{aligned} P\exp\left(\int_0^\sigma d\sigma'\, A(\sigma')\right) &= \sum_{n=0}^\infty \frac{1}{n!}\int_0^\sigma \cdots \int_0^\sigma d\sigma_1' \cdots d\sigma_n'\, \overleftarrow{P}\{A(\sigma_1') \cdots A(\sigma_n')\} \\ &= \sum_{n=0}^\infty \int_0^\sigma d\sigma_n' \int_0^{\sigma_n'} d\sigma_{n-1}' \cdots \int_0^{\sigma_2'} d\sigma_1'\, A(\sigma_n') \cdots A(\sigma_1'). \end{aligned}$$

Note, (4.13) only depends the homotopy class of γ, as $\mathcal{L}(\mu)$ is flat.

To obtain an infinite set of conserved charges, we consider a path at a fixed time slice τ. Then, we define the **transport matrix**

$$T(b, a; \mu) := \Psi(\tau, b; \mu)\Psi(\tau, a; \mu)^{-1}$$

[4] We want to mention that this is no universal definition in the literature.

$$= P \exp\left(- \int_a^b d\sigma \, \mathcal{L}_\sigma(\mu) \right).$$

Now, using flatness of the Lax connection, we find (cf. [BBT03, Sec. 3.7])

$$\partial_\tau T(b, a; \mu) = T(b, a; \mu)\mathcal{L}_\tau(\tau, a; \mu) - \mathcal{L}_\tau(\tau, b; \mu)T(b, a; \mu). \tag{4.14}$$

Depending on the type of worldsheet, we have two cases:

i) When $\Sigma = \mathbb{R}^{1,1}$, we impose $\mathcal{L}(\mu) \to 0$ $(\sigma \to \pm\infty)$. Then, any power of the **monodromy matrix** $T(\infty, -\infty; z)$ is conserved by (4.14), i.e.

$$\partial_\tau \big(T(\infty, -\infty; \mu)^n \big) = 0$$

for any $n \in \mathbb{N}$ and $\mu \in \mathbb{C}$.

ii) When $\Sigma = \mathbb{R} \times S^1$ we identify $\sigma \sim \sigma + 2\pi$. Then, the **monodromy matrix** $T(2\pi, 0; \mu)$ satisfies

$$\partial_\tau T(2\pi, 0; \mu) = [T(2\pi, 0; \mu), \mathcal{L}_\tau(\tau, 0; \mu)]$$

by (4.14). Consequently, the trace of any power of $T(2\pi, 0; \mu)$ is conserved, i.e.

$$\partial_\tau \operatorname{Tr} \big(T(2\pi, 0; \mu)^n \big) = 0$$

for any $n \in \mathbb{N}$, $z \in \mathbb{C}$. Note, the monodromy matrix plays the role of the Lax matrix in the field theory context.

Let us for simplicity denote both $T(\infty, -\infty; \mu)$ and $\operatorname{Tr} T(2\pi, 0; \mu)$ by $T(\mu)$, which we call the **transfer matrix**. It will be clear from the context which one is considered. Interestingly, even more conserved charges can constructed upon Taylor expanding $T(\mu)$ around values of μ for which $T(\mu)$ is analytic. For example, if $T(\mu)$ is analytic around $\mu = 0$, we can write

$$T(\mu) = \sum_n Q_n \mu^n.$$

The conservation of the transfer matrix implies the conservation of the coefficients, i.e. $\partial_\tau Q_n = 0$ for every $n \in \mathbb{N}$. Consequently, there are multiple infinite sets of conserved charges that have different properties. We will not focus on these conserved charges in this work and refer to [BBT03; Dri22] for more background.

Charges in Involution

Similar to the classical case, the existence of a tower of conserved charges is not enough for strong integrability. We would like to establish a sufficient condition, similar to Theorem 4.1.3, that ensures strong integrability. For this we consider the spatial Lax component $\mathcal{L}_\sigma(\mu)$ and its Poisson commutation relations. However, in the field theory context the 'if and only if' equivalence is lost and only sufficient conditions can be deduced. Moreover, there are multiple conditions that ensure strong integrability. Here, we will focus on two.

For the first condition, let us assume the Poisson bracket of spatial Lax components satisfies

$$\{\mathcal{L}_{\sigma,1}(\sigma; \mu), \mathcal{L}_{\sigma,2}(\sigma'; \mu')\} = [\mathcal{L}_{\sigma,1}(\sigma; \mu) + \mathcal{L}_{\sigma,2}(\sigma'; \mu'), r_{12}(\mu, \mu')]\delta(\sigma - \sigma') \tag{4.15}$$

at equal times for some r-matrix. Here we use the same notation as in 4.1.2. As only delta functions and not their derivatives appear in the bracket above, we say it is **ultralocal**. To ensure involutivity, we assume r to be non-dynamical, skew-symmetric and satisfies the CYBE

$$[r_{12}(\mu_1,\mu_2), r_{13}(\mu_1,\mu_3)] + [r_{12}(\mu_1,\mu_2), r_{23}(\mu_2,\mu_3)] + [r_{32}(\mu_3,\mu_2), r_{13}(\mu_1,\mu_3)] = 0 \tag{4.16}$$

for all spectral parameters. Using these assumptions, it can be shown the transport matrices satisfy [BBT03, Sec. 3.9]

$$\{T_1(b,a;\mu), T_2(b,a;\mu')\} = [r_{12}(\mu,\mu'), T_1(b,a;\mu)T_2(b,a;\mu)]. \tag{4.17}$$

Equation (4.17) is known as the **Sklyanin exchange relation** and it implies

Proposition 4.2.1.
If the Sklyanin exchange relation holds, we have

$$\{\mathrm{Tr}_1\left(T_1(b,a;\mu)^n\right), \mathrm{Tr}_2\left(T_2(b,a;\mu')^m\right)\} = 0$$

for any $n, m \in \mathbb{Z}$. Consequently, the theory is strongly integrable.

Proof. — Using the relation $\mathrm{Tr}_{12}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$, we have

$$\begin{aligned}
\{\mathrm{Tr}_1\, T_1(b,a;\mu), \mathrm{Tr}_2\, T_2(b,a;\mu)\} &= \mathrm{Tr}_{12}\{T_1(b,a;\mu), T_2(b,a;\mu')\} \\
&= \mathrm{Tr}_{12}[r_{12}(\mu,\mu'), T_1(b,a;\mu)T_2(b,a;\mu)] \\
&= 0.
\end{aligned}$$

Now, by the cyclic property of the trace and the Leibniz rule for the Poisson bracket, this generalizes to higher powers. ■

To above condition proves to be important when trying to quantize the field theory [Skl82]. However, the integrable field theories we are interested in do not possess the Sklyanin exchange relation. This is because the Poisson bracket of their spatial Lax components contain **non-ultralocal** terms, i.e. parts proportional to $\partial_\sigma\delta(\sigma-\sigma')$ (or higher derivatives). Hence, we need to generalize the Sklyanin exchange relation. One possibility that is relevant to us is the r/s **Maillet form** of the Poisson bracket. The Maillet bracket is given by

$$\begin{aligned}
\{\mathcal{L}_{\sigma,1}(\sigma;\mu), \mathcal{L}_{\sigma,2}(\sigma';\mu')\} =& [r_{12}(\mu,\mu'), \mathcal{L}_{\sigma,1}(\sigma;\mu)]\delta(\sigma-\sigma') - \\
& [r_{21}(\mu',\mu), \mathcal{L}_{\sigma,2}(\sigma;\mu')]\delta(\sigma-\sigma') - \\
& s_{12}(\mu',\mu')\partial_\sigma\delta(\sigma-\sigma'),
\end{aligned} \tag{4.18}$$

where $s_{12}(\mu,\mu') = r_{12}(\mu,\mu') + r_{21}(\mu',\mu)$. One can show the CYBE (4.16) implies the Jacobi identity of the Maillet bracket[5]. The non-ultralocal term in the Maillet bracket produces an ambiguity in the Poisson brackets of the transport matrices. This issue can be solved through a regularization procedure due to Maillet in [Mai86]. The result reads,

$$\begin{aligned}
\{T_1(b,a;\mu), T_2(b,a;\mu')\} =& [r(\mu,\mu'), T_1(b,a;\mu)T_2(b,a;\mu')] - \\
& T_2(b,a;\mu')s(\mu,\mu')T_1(b,a;\mu) +
\end{aligned}$$

[5]Actually, there is a more general mixed Yang-Baxter equation that already ensures Jacobi. See [Mai86] for more details.

$$T_1(b,a;\mu)s(\mu,\mu')T_2(b,a;\mu), \tag{4.19}$$

where

$$r(\mu,\mu') := \frac{r_{21}(\mu',\mu) - r_{12}(\mu,\mu')}{2},$$
$$s(\mu,\mu') := \frac{r_{21}(\mu',\mu) + r_{12}(\mu,\mu')}{2}.$$

Hence, by the same reasoning as in Proposition 4.2.1 we obtain

Proposition 4.2.2.
If Poisson bracket of the spatial Lax components is in the r/s Maillet form (4.18), then

$$\{\mathrm{Tr}_1\left(T_1(b,a;\mu)^n\right), \mathrm{Tr}_2\left(T_2(b,a;\mu')^m\right)\} = 0$$

with $n, m \in \mathbb{Z}$. In other words, the theory is strongly integrable.

Proof. — By (4.19) and the cyclic property of the trace we see

$$\begin{aligned}
\{\mathrm{Tr}_1\, T_1(b,a;\mu), \mathrm{Tr}_2\, T_2(b,a;\mu')\} &= \mathrm{Tr}_{12}\{T_1(b,a;\mu), T_2(b,a;\mu')\} \\
&= \mathrm{Tr}_{12}\left(- T_2(b,a;\mu')s(\mu,\mu')T_1(b,a;\mu) + T_1(b,a;\mu)s(\mu,\mu')T_2(b,a;\mu)\right) \\
&= \mathrm{Tr}_{12}\left(- T_2(b,a;\mu')s(\mu,\mu')T_1(b,a;\mu) + T_2(b,a;\mu')s(\mu,\mu')T_1(b,a;\mu)\right) \\
&= 0.
\end{aligned}$$

As before, the general statement follows from the Leibniz rule and the cyclic property. ∎

Note, equations (4.18), (4.19) are generalizations of (4.15), (4.17) that coincide for a skew-symmetric r-matrix[6], i.e. $r_{12}(\mu,\mu') = -r_{21}(\mu',\mu)$. It is the r/s Maillet form that will ensure the strong integrability of the models of interest in this work. We will introduce these models in the next section.

4.3 Non-linear Sigma-models

In this section we will discuss the deformed WZW model and the bi-Yang-Baxter model and discuss their integrability. However, these models are of special type. Namely, they are examples of **non-linear sigma-models**. These are theories that describe the dynamics of mappings between between a (pseudo)-Riemannian manifold and a general target manifold. For us, the target manifold will be a Lie group. The terminology originates from the sixties, where Gell-Mann and Lévy named it after the σ-meson they encountered in [GL60]. As both models fit in this framework, let us first describe non-linear sigma-models.

Two-dimensional non-linear sigma-models are of particular interest to us, as they appear as string worldsheet theories. In Section 1.1 we described strings propagating in flat Minkowski space, where the dynamics was captured in the Polyakov action (cf. (1.2)). A natural generalization is to consider string embeddings into a curved target space[7] M. Moreover, one can couple the string to the other massless

[6] To be precise, they coincide after redefining $r \to -r$.
[7] Mathematically, the target space becomes a general (pseudo)-Riemmanian manifold.

excitations. To be specific, the Kalb-Ramon field B and the dilaton Φ. Following [Ton12, Ch. 7], the resulting string action is in general given by

$$S = \frac{T}{2}\int_\Sigma d^2\sigma\,\sqrt{h}\left(h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X) + \epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X) + \alpha'\Phi(X)\mathcal{R}^{(2)}\right).$$

Here, $\mathcal{R}^{(2)}$ denotes the worldsheet Ricci scalar. This is the type of non-linear sigma-model we will be interested in. However, we do not need the dilaton field in our discussion, so we omit it from now on. Furthermore, when working in the conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$, we can write the action as

$$S = \frac{T}{2}\int d^2\sigma\,\left(G_{\mu\nu}(X) + B_{\mu\nu}(X)\right)\partial_+X^\mu\partial_-X^\nu \tag{4.20}$$

in light-cone coordinates. Here, we impose the skew-symmetry of B by hand. Typically, we define $E := G + B$ and write

$$S = \frac{T}{2}\int d^2\sigma\,E_{\mu\nu}(X)\partial_+X^\mu\partial_-X^\nu.$$

This point of view will be important in Chapter 5. Note, in general $G \in \Gamma(\mathrm{Sym}^2 T^*M)$ and $B \in \Gamma(\wedge^2 T^*M)$. Therefore, $E \in \Gamma(T^*M \otimes T^*M)$.

As mentioned before, we will consider the target space to be a Lie group. It turns out several such non-linear sigma-models are integrable. In the field of integrable systems, people are interested in mapping out the landscape of integrable models. One approach is to start with such a model and deform it, while retaining integrability. This is called an **integrable deformation**. This approach has been very fruitful in discovering new integrable systems, e.g. [Sfe14; DMV13]. There is one worldsheet sigma-model, in particular, whose deformations have received particular interest: the **principal chiral model**. The bi-Yang-Baxter model and deformed WZW model are examples of such deformations. Therefore, we first describe the principal chiral model.

4.3.1 Principal Chiral Model

Let us introduce the principal chiral model (PCM). We will use [Hoa22] as main reference and will follow the notation used in [GM23]. The field content of the PCM consists of a group-valued field

$$g : \Sigma \to G,$$

where G is a simple Lie group with Lie algebra $\mathfrak{g}$. As G is simple there exists a (up to scaling) unique ad-invariant symmetric bilinear form on $\mathfrak{g}$, which we will denote by

$$(\cdot,\cdot) : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}.$$

Furthermore, we introduce the pull-back of Maurer-Cartan form

$$j = g^{-1}dg \in \Omega^1(\Sigma, \mathfrak{g}).$$

Now, the PCM action is given by

$$S_{\mathrm{PCM}}[g] = \frac{k}{4\pi}\int_\Sigma d^2\sigma\,\left(g^{-1}\partial_+g, g^{-1}\partial_-g\right), \tag{4.21}$$

where we work in light-cone coordinates. Furthermore, $g^{-1}\partial_\pm g$ should be understood as the light-cone components of j. In the literature, people write $\mathrm{Tr}(XY) := (X, Y)$, due to the connection of the Killing form with the trace. By naturally extending this to Lie algebra-valued forms, we can rewrite

$$S_{\mathrm{PCM}}[g] = \frac{k}{8\pi} \int_\Sigma \mathrm{Tr}\,(g^{-1}dg \wedge \star g^{-1}dg). \tag{4.22}$$

Note, the principal chiral model enjoys a global $G \times G$ symmetry

$$g \mapsto g_L \cdot g \cdot g_R^{-1}, \quad (g_L, g_R) \in G \times G,$$

by the ad-invariance of the pairing. Furthermore, the variation of the action yields the equation of motion $\partial^\mu j_\mu = 0$, or

$$\partial_+ j_- + \partial_- j_+ = 0$$

in light-cone coordinates (see [Hoa22, Sec. 2.1]).

The principal chiral model is a non-linear sigma-model and can be put into the standard form (4.20) explicitly. This is done in [Hoa22] and the result is the target space G equipped with the bi-invariant metric[8] and vanishing B-field. Furthermore, the principal chiral model is strongly integrable. Firstly, we consider the Lax connection

$$\mathcal{L}_\pm(\mu) = \frac{j_\pm}{1 \mp \mu}.$$

Or equivalently,

$$\mathcal{L}(\mu) = \frac{j + \mu \star j}{1 - \mu^2}. \tag{4.23}$$

A straightforward computation shows the curvature of the Lax connection to be

$$\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] = \frac{1}{1-\mu^2}\big(\partial_+ j_- - \partial_- j_+ + [j_+, j_-] - \mu(\partial_+ j_- + \partial_- j_+)\big).$$

Hence, the flatness of the Lax connection is equivalent to j being flat and conserved. Note, j being conserved is equivalent to $\star j$ being closed, or more generally, $\star j$ being flat when $\mathfrak{g}$ is equipped with the trivial bracket. This is closely related to a hidden symmetry of the PCM: **Poisson-Lie symmetry** [Kli09; Sfe14; DMV15; Šev17b], which turns out to be a feature shared by several well-known integrable deformations. In fact, all known examples of two dimensional sigma-models which are classically integrable have Poisson-Lie symmetry [DHT19]. We discuss Poisson-Lie symmetry in more detail in Chapter 5. Moreover, the specific form of the Lax connection (4.23) generalizes to other deformations as well [Dri22, Sec. 4.5].

As the Maurer-Cartan form is flat and conserved, the Lax connection is flat, meaning the PCM is weakly integrable. To ensure strong integrability, a direct computation of Lax connection Poisson brackets is necessary. This is done in [Dri22, Sec. 4.4], where it is shown the Poisson bracket can be put into r/s Maillet form. This concludes our discussion about the PCM and we will consider a particular integrable deformation in the next section.

[8] To be precise, it is the left- (or right-) translated Killing form on G.

4.3.2 Deformed WZW Model

In this section we will extend the PCM by a topological term to arrive at the famous **Wess-Zumino-Witten (WZW) model**. It was introduced by Witten in [Wit84] and generalizes the ideas of Wess and Zumino to two-dimensional non-linear sigma-models. Interestingly, the WZW model is a conformal field theory when quantized, in contrast to the PCM. Hence, it received a lot of attention. The WZW model enjoys a symmetry that can be gauged, yielding a **gauged WZW model**. These can be deformed, yielding the λ-deformations. These will be relevant for our discussion. We use [GM22; Hoa22; DMS96] as main references.

The WZW Model

Via a conformal transformation, we may regard the worldsheet as the complex plane. Then, by a one-point compactification, we obtain the PCM on the Riemann sphere. Consequently, the field g in the PCM is a map $S^2 \to G$. We extend the PCM action (4.22) by adding the **Wess-Zumino (WZ) term**

$$S_{\mathrm{WZ}}[g] = \frac{k}{12\pi i} \int_B \mathrm{Tr}\,(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg). \tag{4.24}$$

Here B denotes the filled three-dimensional ball whose boundary is S^2. Furthermore, the field g now denotes an extension of the original field to a map $B \to G$. This can always be done as the second homotopy group $\pi_2(G)$ of a Lie group is trivial[9]. However, at this stage it is unclear whether (4.24) is independent of the extension. Suppose we consider two homotopic extensions, i.e. small deformations. Then, one can verify the variation of (4.24) under this transformation is given by [DMS96]

$$\delta S_{\mathrm{WZ}} \sim \int_{S^2} \mathrm{Tr}\,(g^{-1}\delta g\; d(g^{-1}dg)).$$

We assume the two extensions agree on S^2, thus $\delta g|_{S^2} = 0$. Consequently, S_{WZ} is invariant under small deformations.

However, not every two extensions are related via continuous deformations. Note, two topological distinct extensions can be glued together to yield a map

$$\tilde{g} : (B \sqcup B)/\partial B \cong S^3 \to G.$$

Such maps are characterized by the third homotopy group $\pi_3(G)$, which equals $\mathbb{Z}$ for compact simple Lie groups. Therefore, the WZ term is multi-valued, strictly speaking. For the classical theory this is not a problem as the equations of motion can still be obtained through varying the action. However, it has consequences for the quantized theory. Indeed, if g and g' denote two topological in-equivalent extensions, one can show

$$\Delta S_{\mathrm{WZ}} = S_{\mathrm{WZ}}[g'] - S_{\mathrm{WZ}}[g] = 2\pi i k.$$

Consequently, path integrals of the form

$$\langle \mathcal{O} \rangle = \int \mathcal{D}g\; \mathcal{O}[g] e^{-S_{\mathrm{WZ}}[g]}$$

are only well-defined if $k \in \mathbb{Z}$ for *compact* groups. The integer k is called the **level** of the WZ term.

[9] This is not a trivial fact and follows from Morse theory [MSW69, Ch. 24], for example.

Now, the WZW model is given by the following particular combination of (4.22) and (4.24)

$$S_{\mathrm{WZW}}[g] = \frac{k}{8\pi}\int_\Sigma \mathrm{Tr}\,(g^{-1}dg \wedge \star g^{-1}dg) + \frac{k}{12\pi i}\int_B \mathrm{Tr}\,(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg). \tag{4.25}$$

Note, the WZW model has the same global $G \times G$ symmetry as the principal chiral model.

The λ-model

We are ready to introduce the λ-deformation. For this we introduce complex coordinates $t, \bar{t}$ on Σ. The action is given by [GSS20]

$$S_\lambda[g] = S_{\mathrm{WZW}}[g] + \frac{\lambda k}{\pi}\int_\Sigma d^2t\ \mathrm{Tr}\left(g^{-1}\partial g \frac{\mathrm{Ad}_g}{1-\lambda\,\mathrm{Ad}_g} g^{-1}\bar{\partial} g\right). \tag{4.26}$$

By the quotient in the expression above, we mean the inverse operator. This is a different formulation than in the original paper [Sfe14] and the one used in [GM22]. There, they use a gauging procedure, which leads to an action including gauge fields. However, on-shell the two actions agree, as is shown in Appendix C of [GM22]. Since we are interested in solutions of the model, it does not matter which action is used. For our purposes in Chapter 5 we need the action in light-cone coordinates, which is given by

$$S_\lambda[g] = S_{\mathrm{WZW}}[g] + \frac{\lambda k}{\pi}\int_\Sigma d^2\sigma \left(g^{-1}\partial_- g, \frac{\mathrm{Ad}_g}{1-\lambda\,\mathrm{Ad}_g} g^{-1}\partial_+ g\right). \tag{4.27}$$

Integrability

Similar to the PCM, the equations of motion of (4.26) can be written in terms of a connection one-form $\mathbf{A} \in \Omega^1(\Sigma, \mathfrak{g})$. In coordinates, they can be written as

$$\partial\bar{A} + \bar{\partial}A = 0, \qquad \partial\bar{A} = -\frac{1}{1+\lambda}[A, \bar{A}],$$

where

$$\mathbf{A} = A dt + \bar{A} d\bar{t}.$$

Note, the first equation means $\star A$ is closed. This is completely analogous to the PCM. The one-form $\mathbf{A}$ is directly related to the gauging procedure we mentioned above.

Inspired by the Lax connection (4.23), we consider

$$\mathcal{L}(\mu) = \frac{2}{1+\lambda}\frac{\mathbf{A} + \mu \star \mathbf{A}}{1-\mu^2}. \tag{4.28}$$

The prefactor containing λ comes from the fact $\mathbf{A}$ satisfies a re-scaled flatness condition:

$$d\mathbf{A} + \frac{1}{1+\lambda}[\mathbf{A}, \mathbf{A}] = 0.$$

It follows that (4.28) captures the equations of motion and thus defines a Lax connection. This establishes the weak integrability of the λ-model. The strong integrability is established in [GSS20] by proving that the Poisson brackets of spatial Lax connection are in r/s Maillet form.

Hodge-theoretic Solutions

There is a particular class of solutions to the λ-model found in [GM22] corresponding to objects in variation of Hodge structures from Chapter 3. We will describe them here. Their starting point is the charge operator Q. For a general variation of Hodge structure, it depends on the moduli: $Q(t, \bar{t})$. Picking a reference Hodge structure F_{ref}, one can put the coordinate dependence in a map $h : \mathcal{M} \to D$ by

$$Q(t, \bar{t}) = h Q_{\text{ref}} h^{-1},$$

where Q_{ref} is the charge operator on F_{ref}. In this formulation, the map h is just the (local) period map Φ from Chapter 3. Therefore, we identify $\mathcal{M}$ with Σ from now on. Now, the horizontality condition on the period map, translates to a condition on Q:

$$[Q, \partial Q] = -\partial Q, \quad [Q, \bar{\partial} Q] = \bar{\partial} Q, \tag{4.29}$$

as is explained in [GM22]. Actually, this condition on Q is equivalent to the horizontality of h, i.e. the period map.

Suppose a Lie algebra $\mathfrak{g}$ of a Lie group G admits a charge operator[10] $Q \in \mathfrak{g}_{\mathbb{C}}$ such that (4.29) holds. Then, there is a natural group-valued field one can write down, namely

$$g = e^{i\beta Q}.$$

One of the main results of [GM22] is that the above constitutes a solution to the λ-model for $|e^{i\beta}| = 1$ and $|\lambda| = 1$. In particular, for $\beta = \pi$, the map $(-1)^Q$ is a solution. Consequently, any variation of Hodge structure produces a solution to the λ-model, namely the Weil operator, as it carries a canonical charge operator (cf. Chapter 3). In the next section we will discuss a different integrable model, for which the Weil operator is related to a solution as well.

4.3.3 Bi-Yang-Baxter Model

The Action

The second integrable deformation of the principal chiral model we want to discuss is the **bi-Yang-Baxter model**, first introduced by Klimčík in [Kli09]. It is a two parameter deformation of the PCM, extending the so-called Yang-Baxter model. The latter was defined in [Kli02] and to define it one needs the skew-symmetric operator $R : \mathfrak{g} \to \mathfrak{g}$ from Section 4.1.3 satisfying (4.8). Recall, such an operator is called a Yang-Baxter operator, hence the name.

In the notation of Section 4.3.1, the action of the bi-Yang-Baxter model is given by

$$S_{\eta,\zeta}[g] = \int_\Sigma d^2\sigma \left(g^{-1}\partial_+ g, \frac{1}{1 - \eta R - \zeta R^g} g^{-1}\partial_- g \right). \tag{4.30}$$

Here η, ζ are constants parametrizing the deformation and

$$R^g := \text{Ad}_{g^{-1}} \circ R \circ \text{Ad}_g .$$

Note, when $\eta = \zeta = 0$ we recover the principal chiral model. Furthermore, for $\zeta = 0$ we obtain the **Yang-Baxter model**. Since the Yang-Baxter deformation is parametrized by η it is often called the η-model.

[10] By this we mean an element Q such that $\overline{Q} = -Q$ and $\text{ad}(Q)$ has integral spectrum.

Symmetries

In the PCM there was a global $G \times G$ symmetry. For the Yang-Baxter model, i.e $\zeta = 0$, the global $G \times G$ symmetry is broken and one only left G-symmetry remains. Furthermore, for the bi-Yang-Baxter model, both the left *and* right G-symmetry is broken. Interestingly, when $\eta = \zeta$ the symmetry is enhanced and the action is invariant under $g \mapsto g^{-1}$. We refer to this as the **critical line**, following [ST21]. Many interesting physical phenomena happen on the critical line. For example, for $G = \mathrm{SU}(2)$, the bi-Yang-Baxter model on the critical line is equivalent to a Yang-Baxter model on S^3 viewed as coset SO(4)/SO(3) [DMV14; Hoa15]. This is interesting as it is related to deformations of the $\mathrm{AdS}_5 \times S^5$ string, which plays a role in the AdS/CFT correspondence. Furthermore, the Weil operator constitutes a solution to the bi-Yang-Baxter model on the critical line [GM23].

Equations of Motion

To describe the equations of motion of the bi-Yang-Baxter model, it is convenient to define

$$J_\pm := \mp \frac{1}{1 \pm \eta R \pm \zeta R^g} g^{-1} \partial_\pm g.$$

Then, the field equations become [Kli14]

$$\partial_+ J_- - \partial_- J_+ + \eta [J_-, J_+]_R = 0.$$

In other words, J is a flat $\mathfrak{g}_R$-valued one-form on G, up to a rescaling of the R-bracket by η. Similar to the principal chiral model, the fact the equations of motion constitute a flat connection on a different Lie algebra is reminiscent of the hidden Poisson-Lie symmetry of the bi-Yang-Baxter model. We elaborate more on this in Chapter 5.

Integrability

The bi-Yang-Baxter model is integrable. The weak integrability was shown by Klimčík in [Kli14]. He found the following Lax connection

$$\mathcal{L}_\pm(\mu) = \left(\eta(R - i) + \frac{2i\eta \pm (1 - \eta^2 + \zeta^2)}{1 \pm \mu} \right) J_\pm.$$

It is crucial in the argument that R is a skew-symmetric solution to the CYBE. Furthermore, in [Del+16] it was shown the Poisson brackets of the Lax connection can be put into r/s Maillet form, proving its strong integrability.

Drinfel'd-Jimbo Solution

An important remark is that a Yang-Baxter operator R should be fixed before one can speak about solutions of the bi-Yang-Baxter model. Hence, we should specify which Yang-Baxter operator leads to the Weil operator solution. There is a standard construction due to Drinfel'd [Dri85] and Jimbo [Jim85] that produces a solution to the modified CYBE. It is fittingly called the **Drinfel'd-Jimbo solution**. To define it, we consider the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$ and pick a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_\mathbb{C}$. Then, we have the root space decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha,$$

where $\alpha \in \mathfrak{h}^*$ runs over all the roots of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_\alpha$ denotes the corresponding root space[11]. Choosing a base for the roots fixes the notion of positive roots. Then, let $\{H_i, E_{\pm\alpha}\}$ be the Cartan-Weyl basis of $\mathfrak{g}_{\mathbb{C}}$. Here, α exhausts all the positive roots. The Cartan-Weyl basis satisfies

$$[H_i, H_j] = 0, \quad [H_i, E_{\pm\alpha}] = \pm\alpha(H_i)E_{\pm\alpha}.$$

The Drinfel'd-Jimbo solutions is then defined as

$$RH_i = 0, \quad RE_{\pm\alpha} = \mp cE_{\pm\alpha}. \tag{4.31}$$

One can explicitly show it solves (4.8), as is done in [Hoa22, Sec. 4.2], and verify it is skew-symmetric. Note, in general, the Drinfel'd-Jimbo solution defined on $\mathfrak{g}_{\mathbb{C}}$. Therefore, it is not guaranteed that R restricts to an endomorphism of a real form[12] $\mathfrak{g}$. This depends on the choice of real form and constant c.

The SU(2) Case

An interesting case of the bi-Yang-Baxter is when $G = \mathrm{SU}(2)$. Then, there is a particular class of solutions called **unitons**. They were constructed by Uhlenbeck for the SU(n) principal chiral model in [Uhl89], which were extended to the bi-Yang-Baxter model for $n = 2$ in [ST21]. Interestingly, these unitons satisfy $g^2 = -1$ just like the Weil operator of an odd weight Hodge structure. This resemblance was the first hint in [GM23] to expect the Weil operator to be a solution.

As mentioned before, we should first fix a Yang-Baxter operator. For $\mathfrak{su}(2)$, the Drinfel'd-Jimbo solution is essentially the only solution to the modified CYBE. Note, $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ which has the standard Cartan-Weyl basis

$$N^0 := H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad N^+ := E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N^- := E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

satisfying

$$[N^0, N^\pm] = \pm 2N^\pm.$$

Then, (4.31) yields a solution R to the modified CYBE. However, this solution is at the level of $\mathfrak{sl}(2, \mathbb{C})$. To see for which c the Yang-Baxter operator restricts to a real operator on $\mathfrak{su}(2)$, we let $T_j = i\sigma_j$ be the basis of $\mathfrak{su}(2)$. Here σ_j denote the Pauli matrices. Then, a direct computation shows

$$RT_1 = -icT_2, \quad RT_2 = icT_1, \quad RT_3 = 0.$$

Hence, we need $c = i$ for R to be a real endomorphism. With respect to this basis, we may write

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.32}$$

The non-linear sigma-model structure of the SU(2) bi-Yang-Baxter model and a discussion of the uniton solutions is nicely described in [GM23]. The next step is to consider the Weil operator in the simplest case: the torus.

[11] We refer to [Hum72] for the relevant background in Lie theory.
[12] By this we mean a real Lie algebra $\mathfrak{g}$ whose complexification is $\mathfrak{g}_{\mathbb{C}}$.

The Weil Operator

Similar to the λ-model, there is a special role to play for the Weil operator. Below we will show that the Weil operator for the torus produces a solution. Following [GM23], we conveniently write

$$\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}),$$

where τ denotes the usual Teichmüller parameter, which takes values in the upper half-plane[13] $\mathbb{H}$. To write down the Weil operator, we choose two defining periodic coordinates ξ_1, ξ_2 with $\xi_i \sim \xi_i + 1$ on the torus. The normalized metric on the torus is given by

$$ds^2 = \frac{|d\xi_1 + \tau d\xi_2|^2}{\operatorname{im}\tau}.$$

From the metric we deduce the Hodge star

$$\star\, d\xi_1 = \frac{\operatorname{re}\tau}{\operatorname{im}\tau} d\xi_1 + \frac{|\tau|^2}{\operatorname{im}\tau} d\xi_2, \qquad \star\, d\xi_2 = -\frac{1}{\operatorname{im}\tau} d\xi_1 - \frac{\operatorname{re}\tau}{\operatorname{im}\tau} d\xi_2.$$

The Weil C operator is now obtained by viewing the above Hodge star relations as an operation on the middle cohomology of the torus. If we write $\tau = x + iy$, we can represent this action as the following matrix

$$C(x, y) = \frac{1}{y}\begin{pmatrix} x & -1 \\ x^2 + y^2 & -x \end{pmatrix}, \tag{4.33}$$

in the basis $\{[d\xi_1], [d\xi_2]\}$ of $H^1(\mathbb{T}^2, \mathbb{C})$.

One can explicitly check that $C^2 = -1$. In that sense, it resembles a uniton. However, note that C is an element of $\mathrm{SL}(2,\mathbb{R})$ not $\mathrm{SU}(2)$. This is expected as we saw $C \in G_{\mathbb{R}}$ in Section 3.2 and $G_{\mathbb{R}} \cong \mathrm{SL}(2,\mathbb{R})$, by Proposition 3.2.5. To relate the Weil operator to the unitons we apply a two-step procedure that relates $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$. First we relate $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{SU}(1,1)$ using the **Cayley transformation**

$$\mathrm{Ad}_\rho : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SU}(1,1),$$

where

$$\rho = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

The Cayley transformation is an isomorphism of real Lie groups. Note, ρ is an element of $\mathrm{SL}(2,\mathbb{C})$. The second step is to relate $\mathrm{SU}(1,1)$ to $\mathrm{SU}(2)$ via an analytical continuation. Concretely, a general element of $\mathrm{SU}(1,1)$ has the following form

$$\begin{pmatrix} \cosh\theta\, e^{i\phi_1} & \sinh\theta\, e^{i\phi_2} \\ \sinh\theta\, e^{-i\phi_2} & \cosh\theta\, e^{-i\phi_1} \end{pmatrix}. \tag{4.34}$$

Now, using the properties $\cosh(i\theta) = \cos\theta$ and $\sinh(i\theta) = i\sin\theta$, we see $\theta \mapsto i\theta$ maps (4.34) to

$$\begin{pmatrix} \cos\theta\, e^{i\phi_1} & i\sin\theta\, e^{i\phi_2} \\ i\sin\theta\, e^{-i\phi_2} & \cos\theta\, e^{-i\phi_1} \end{pmatrix}.$$

[13] To be precise, to get all the inequivalent tori, i.e. the moduli space, one should quotient by the modular group as is explained in [BLT12, Sec. 6.2].

This is a general element of SU(2). In this way, we constructed a map $\alpha : \mathrm{SU}(1,1) \to \mathrm{SU}(2)$. However, this is not an isomorphism of Lie groups. Moreover, this cannot be expected as SU(2) is compact while SU(1, 1) is not. Post-composing the Cayley transform of C with α yields an element of SU(2) and it was shown in [GM23] that it constitutes a solution to the bi-Yang-Baxter model with R-matrix (4.32).

We would like to emphasise that the described mapping from $\mathrm{SL}(2,\mathbb{R})$ to SU(2) is quite convoluted to do in practice. For example, the Cayley transformation of C is straightforward

$$\mathrm{Ad}_\rho(C) = \frac{1}{2y}\begin{pmatrix} (1+x^2+y^2)i & x^2+y^2-1-2ix \\ x^2+y^2-1+2ix & -(1+x^2+y^2)i \end{pmatrix}.$$

However, finding the corresponding parametrization (4.34) is cumbersome. Luckily, this was done in [GM23] and the result is

$$\phi_1 = \frac{\pi}{2}, \quad \phi_2 = \pi + \frac{i}{2}\log\left(\frac{f}{\bar{f}}\right), \quad \theta = \frac{\pi}{2} + i\arctan\left[\frac{1}{2}\left(|f| + \frac{1}{|f|}\right)\right]. \tag{4.35}$$

Here $f(z)$ is the holomorphic function

$$f(z) = \frac{z-i}{z+i}, \quad z = x+iy,$$

where we identified the Teichmüller parameter τ with the worldsheet coordinate z. As explained in [GM23], (4.35) defines a uniton solution on the *critical line*.

The SL(2)-approximation

A natural question is whether one can generalize the solution of the previous section to arbitrary groups. This is addressed in [GM23] and the answer is affirmative, given the group admits at least one **horizontal $\mathfrak{sl}(2)$-triple**. We will define this notation below. However, the motivation for this can already be seen in the solution (4.33). Notice, one can write

$$C(x,y) = h(x,y)C_\infty h(x,y)^{-1}, \tag{4.36}$$

where

$$h(x,y) = \frac{1}{\sqrt{y}}\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}, \quad C_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Moreover, in terms of the Cartan-Weyl basis we have

$$h(x,y) = e^{xN^-}y^{-\frac{1}{2}N^0}, \quad C_\infty = (-1)^{Q_\infty},$$

where we introduced

$$Q_\infty = \frac{i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This is precisely the SL(2)-orbit approximation (3.27) of the Weil operator. Now, in the case of the torus, the SL(2)-orbit, nilpotent-orbit and the *actual* variation of Hodge structure coincide. Hence, the full Weil operator equals its SL(2)-orbit approximation.

It is this SL(2)-approximation that made the generalization to arbitrary groups possible. Notice, the relevant data was an $\mathfrak{sl}(2)$-triple $\{N^+, N^0, N^-\}$ and a charge operator Q_∞. From Chapter 3 we know $\overline{Q}_\infty = -Q_\infty$ and

$$[Q_\infty, N^0] = i(N^+ + N^-), \quad [Q_\infty, N^\pm] = -\frac{i}{2}N^0, \tag{4.37}$$

whenever they arise from a variation of Hodge structure. Furthermore, it can be explicitly checked that in that case $\mathrm{ad}(Q_\infty)$ has integral eigenvalues. A charge operator combined with an $\mathfrak{sl}(2)$-triple satisfying the above properties is called a **horizontal $\mathfrak{sl}(2)$-triple**. Condition (4.37) precisely is the SL(2)-approximation of (4.29).

It was shown in [GM23] that Lie groups G whose Lie algebra $\mathfrak{g}$ admit a horizontal $\mathfrak{sl}(2)$-triple a solution analogous to (4.36). For the Yang-Baxter operator they again choose the Drinfel'd-Jimbo solution. Then, they proceed to show that the field

$$g(x,y) = h(x,y)(-1)^{Q_\infty} h(x,y)^{-1},$$

with

$$h(x,y) = e^{xN^-} y^{-\frac{1}{2}N^0},$$

provides a solution to the bi-Yang-Baxter model on G. Consequently, for every variation of Hodge structure, the SL(2)-orbit theorem provides a solution to the bi-Yang-Baxter model on $G_\mathbb{R}$. Indeed, the procedure described in Section 3.5.2 yields a horizontal $\mathfrak{sl}(2)$-triple and the SL(2)-orbit theorem gives the map h. In that case, $g(x,y)$ can be interpret as the SL(2)-orbit approximation of the Weil operator.

In view of the SL(2)-orbit theorem one might wonder whether the nilpotent orbit approximation constitutes a solution as well, as the theorem provides a way to go back. However, in [GM23] it was found that this does not hold. At least, for the same Yang-Baxter operator R. Ideally, the Yang-Baxter operator can be modified using objects from Hodge theory so that the nilpotent approximation becomes a solution. However, it is not clear at this stage how to do this, if even possible. As mentioned in [GM23], the connection should be in the map δ from Section 3.5.2. This is because the map δ contains information about the higher order corrections in $\tilde{g}(y)$ from the SL(2)-orbit theorem (cf. Theorem 3.5.15). This can also be seen explicitly in Example 3.5.16. The first guess would be to conjugate R with $\mathrm{Ad}(e^\delta)$. This *does* define a new skew-symmetric Yang-Baxter operator. However, the solutions are conjugated as well, but with $\mathrm{Ad}(e^{-\delta})$. Since the nilpotent and SL(2)-orbit approximation are not related via a conjugation, this method is not successful. It would be interesting to see whether such a connection between R and δ exists.

Up this point, we have discussed two integrable systems for which the Weil operator is a solution. This confirms the belief in a connection between Hodge theory and integrable systems even further. Interestingly, the bi-Yang-Baxter model is related to the (generalized) λ-model via a duality called **Poisson-Lie T-duality** [Kli15; Kli16]. As this duality is closely related to the Poisson-Lie symmetry of the PCM and bi-Yang-Baxter model [Šev17b], this seems a natural place to further investigate the relation between Hodge theory and integrability. This is what we will do in the next chapter.

CHAPTER 5

Poisson-Lie T-duality

Dualities play an important role in physics and string theory, in particular. They connect seemingly different theories and give ways to do computations. In string theory, there is a particular duality discovered in the 1980s called *T-duality*. It produces an equivalence between physical theories. Concretely, in the case of a circle compactification of the bosonic string, T-duality changes the geometry of the internal manifold by sending the radius R to $1/R$, while the spectrum is unchanged. Moreover, it is more than a symmetry, as relates the two distinct type II theories. This can be straightforwardly generalized to toroidal compactifications.

The next step would be to extend the notion of T-duality to non-Abelian groups. The first attempt came, under the fitting name *non-Abelian T-duality*, in 1993 and was due to de la Ossa and Quevedo [dQ93]. However, naming it a 'duality' was a bit optimistic, as the (non-Abelian) isometry group of the dual is always smaller. This is not desirable, when trying to construct a duality. Furthermore, there was no obvious way to get back to the original theory.

A remedy to both issues was introduced in 1995 by Klimčík and Ševera in [KŠ95], under the name *Poisson-Lie T-duality*. They provide sufficient conditions for the existence of a Poisson-Lie T-dual. The group G and the 'dual' group $\widetilde{G}$ should sit in a particular bigger group D, called the *Drinfel'd double*. From the perspective of the double, Poisson-Lie T-duality can be made into a manifest symmetry, using a so-called $\mathcal{E}$*-model*. Moreover, in [KŠ96c] a procedure is described to generate Poisson-Lie T-dual theories, using the double. We describe these various perspectives on Poisson-Lie T-duality in this chapter. In particular, we will relate the Poisson-Lie T-dual of the Yang-Baxter model to the λ-model, following [Kli15]. This approach then directly generalizes to the bi-Yang-Baxter model.

In its purest form, Poisson-Lie T-duality is a statement about equivalent dynamical systems (see e.g. [Šev16]). Hence, a solution to one model can be transfered to a solution of the dual model. However, in the literature, the focus is more on the target space perspective. Therefore, this map between solutions might be obscured in the literature. Our aim is to provide a clear road map in obtaining a solution to the dual model. Moreover, we try to apply this to the Weil operator in the bi-Yang-Baxter model from the previous chapter. Finally, Abelian T-duality can be naturally written in the language of *Courant algebroids* [CG11]. It is expected that the same holds true for Poisson-Lie T-duality, as it is a generalization of Abelian T-duality. This is the case and we discuss the connection with Courant algebroids in this chapter.

5.1 Abelian T-duality

As Poisson-Lie T-duality is a generalization of (Abelian) T-duality, let us first discuss this concept. Abelian T-duality, or just T-duality, is an example of a string duality, which we mentioned in Section 1.2. It relates physical properties of theories with large spacetime radius to quantities in theories of small spacetime radius [AAL95]. For example, in one of the first observations of T-duality [Sat87], the bosonic string is considered with one of the spatial dimensions compactified to a circle with radius R. Then, all physical observable quantities are left unchanged under the transformation $R \to \alpha'/R$, provided the string winding number is exchanged with the so-called Kaluza-Klein excitation number. The latter is related to the momentum of the string, see [BBS06, Ch. 6] for details. Consequently, the circle compactifications with radii R and α'/R are physically indistinguishable.

The construction can be generalized to toroidal compactifications and even to non-flat conformal backgrounds, as done in [Bus87; RV92]. A key ingredient in the argument was the existence of an isometry of the metric on the target that is a symmetry of the action. In that case, explicit formulas for the metric, B-field and dilaton on an equivalent dual sigma-model can be found. These are known as the **Buscher rules**. See [AAL95] for a good review. Interestingly, the Buscher rules may lead to very different geometries and could even lead to different topologies. Therefore, T-duality relates seemingly distinct theories. However, a necessary condition to construct the dual sigma model is that its isometry group must be Abelian [Kli96]. This is the origin of the nomenclature 'Abelian T-duality'.

The restriction on the isometry group excludes many physically relevant string sigma-models. Therefore, a generalization of the duality to non-Abelian isometries is desirable. As mentioned before, the first attempt came in [dQ93], which built upon the ideas of [RV92]. However, it fundamentally lacks a way to produce the original model from the dual, a wanted property for a duality. This issue is solved by Poisson-Lie T-duality. Its description is the theme of the subsequent section.

5.2 Non-Abelian Generalization

5.2.1 General Features

To discuss Poisson-Lie T-duality, we will adapt the sigma-model point of view, following [KŠ95; Kli96; Kli09]. We consider a two-dimensional sigma-model on a target manifold M equipped with a metric g and a two-form B. We write $E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$. Then, the action is (cf. Section 4.3)

$$S[X] = \int_\Sigma d^2\sigma\, E_{\mu\nu}(X)\partial_+ X^\mu \partial_+ X^\nu.$$

We suppose M admits a free (right) action by a Lie group G. This induces a Lie algebra action $\mathfrak{a} : \mathfrak{g} \to \mathfrak{X}(M)$. Then, in a basis $\{T_A\}$ of $\mathfrak{g}$, the action gives us (left-)invariant vector fields $V_A := \mathfrak{a}(T_A)$ on M. In local coordinates, we can write them as $V_A = V_A^\mu \partial_\mu$. Now, let us consider the variation of the action above with respect to the G-action, parametrized by a worldsheet dependent parameters $\epsilon^A(\tau, \sigma)$

$$\begin{aligned}
\delta S &= S[X + \epsilon^A V_A] - S[X] \\
&= \int_\Sigma d^2\sigma\, E_{\mu\nu}(X + \epsilon^A V_A)\partial_+(X^\mu + \epsilon^A V_A^\mu)\partial_-(X^\nu + \epsilon^A V_A^\nu) - S[X] \\
&= \int_\Sigma d^2\sigma\, \left(E_{\mu\nu}(X) + \epsilon^A \mathcal{L}_{V_A}(E_{\mu\nu})\right)\partial_+(X^\mu + \epsilon^A V_A^\mu)\partial_-(X^\nu + \epsilon^A V_A^\nu) - S[X] \\
&= \int_\Sigma d^2\sigma\, \epsilon^A \mathcal{L}_{V_A}(E_{\mu\nu})\partial_+ X^\mu \partial_- X^\nu + \int_\Sigma d^2\sigma\, E_{\mu\nu} V_A^\mu \partial_- X^\nu \partial_+ \epsilon^A + E_{\mu\nu} V_A^\nu \partial_+ X^\mu \partial_- \epsilon^A
\end{aligned}$$

$$= \int_\Sigma d^2\sigma\, \epsilon^A \mathcal{L}_{V_A}(E_{\mu\nu})\partial_+ X^\mu \partial_- X^\nu + \int_\Sigma J_A \wedge d\epsilon^A,$$

where

$$J_A = E_{\mu\nu}\partial_+ X^\mu V_A^\nu d\sigma^+ - E_{\nu\mu}\partial_- X^\mu V_A^\nu d\sigma^-. \tag{5.1}$$

If the Lie derivatives $\mathcal{L}_{V_A}(E_{\mu\nu})$ vanish, we obtain on-shell closed one-forms J_A. This happens, for example, when G acts by isometries, i.e. is a symmetry. Then, the appearance of the closed one-forms is no surprise: it is Noether's theorem.

Now, suppose they are not closed but satisfy the following zero curvature condition on shell [Kli96]

$$dJ_A = -\frac{1}{2}\tilde{c}_A{}^{BC} J_B \wedge J_C, \tag{5.2}$$

where $\tilde{c}_A^{BC}$ denote the structure constants of some Lie algebra $\tilde{\mathfrak{g}}$. We call $\tilde{\mathfrak{g}}$ the **dual** Lie algebra. Then, we say the sigma-model has G-**Poisson-Lie symmetry with respect to** $\widetilde{G}$, the integration of $\tilde{\mathfrak{g}}$. The conditions (5.2) are part of the field equations and they are exhaustive when the groups action is transitive. This is precisely what happened in the principal chiral model and bi-Yang-Baxter model (cf. Section 4.3).

Using (5.1), we can compute the right-hand-side of (5.2). We find we should require

$$\mathcal{L}_{V_A}(E_{\mu\nu}) = \tilde{c}_A{}^{BC} E_{\mu\alpha}E_{\beta\nu}V_B^\alpha V_C^\beta \tag{5.3}$$

for (5.2) to hold [Kli96]. The property

$$\mathcal{L}_{[V_A,V_B]} = [\mathcal{L}_{V_A}, \mathcal{L}_{V_B}]$$

leads to a condition on the structure constants of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. Indeed, one finds [KŠ95]

$$\tilde{c}_A{}^{BC}c_{EA}{}^D - \tilde{c}_A{}^{BD}c_{EB}{}^C - \tilde{c}_E{}^{BC}c_{AB}{}^D + \tilde{c}_E{}^{BD}c_{AB}{}^C - \tilde{c}_B{}^{DC}c_{EA}{}^B = 0. \tag{5.4}$$

Interestingly, this is precisely the condition for $(\mathfrak{g}, \tilde{\mathfrak{g}})$ to be a **Lie bialgebra**. This is a concept introduced by Drinfel'd in [Dri83]. Formally, it is defined as

Definition 5.2.1.
A **Lie bialgebra** $(\mathfrak{g}, \delta)$ *is a Lie algebra* $\mathfrak{g}$ *equipped with a linear map* $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *such that*

i) $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ *is a Lie bracket on* $\mathfrak{g}^*$

ii) For all $X, Y \in \mathfrak{g}$, *we have*

$$\mathrm{ad}_X^{(2)}\,\delta(Y) - \mathrm{ad}_Y^{(2)}\,\delta(X) - \delta([X,Y]) = 0, \tag{5.5}$$

where $\mathrm{ad}_X^{(2)} = \mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X$.

Condition (5.5) reduces to (5.4) when applied to a basis (see [Dri83; ARH17]). Furthermore, (5.5) is called the **cocycle condition**, as it is related to Lie algebra cohomology (cf. [Koso4]). To give some more context, the dual of a Lie algebra is typically not a Lie algebra, but it is a Poisson manifold[1]. A Lie bialgebra

[1] We do not wish to discuss the details of Poisson geometry and we refer the reader to [CFM21] for an excellent introduction.

is then a Lie algebra for which its dual *is* a Lie algebra, in a compatible manner. Lie bialgebras naturally arise when studying r-matrices, as is discussed in [Kos04].

Any Lie algebra can be integrated to a Lie group. One might wonder what the integration of a Lie bialgebra is. Intuitively, a Lie bialgebra structure on $\mathfrak{g}$ makes it into *both* a Lie algebra and a Poisson manifold, in some compatible fashion by (5.5). This carries over to the integration G, making it a Lie group with a compatible Poisson structure. This is known as a *Poisson-Lie group* and is defined by [LW90]

Definition 5.2.2.
A **Poisson-Lie group** *is a Lie group G equipped with a Poisson structure, such that the multiplication map $\mu : G \times G \to G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure.*

So, for a sigma-model to have Poisson-Lie symmetry, $\mathfrak{g}$ must be a Lie bialgebra by (5.4). Hence, the original group G, being an integration of $\mathfrak{g}$, is a Poisson-Lie group. This is the origin of the name Poisson-Lie T-duality.

Let us get back to the duality. First note, (5.4) is manifestly dual under $c \leftrightarrow \tilde{c}$. Therefore, a dual sigma-model for which the roles of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ is expected [KŠ95]. To see this, let us restrict ourselves to transitive group actions. In that case, the target M can be identified with the Lie group G itself and (5.3) can be solved using a particular Lie bialgebra, namely a *Manin triple*. The general case is covered in [Kli96].

Definition 5.2.3.
A **Manin triple** *is a triple $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$, where $\mathfrak{d}$ is a Lie algebra with a non-degenerate invariant symmetric form $\langle \cdot, \cdot \rangle$ such that $\mathfrak{g}, \tilde{\mathfrak{g}}$ are complementary Lagrangian subalgebras of $\mathfrak{d}$, i.e. $\mathfrak{d} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$, $\dim \mathfrak{g} = \dim \tilde{\mathfrak{g}} = \frac{1}{2} \dim \mathfrak{d}$ and $\langle \mathfrak{g}, \mathfrak{g} \rangle = 0 = \langle \tilde{\mathfrak{g}}, \tilde{\mathfrak{g}} \rangle$. The Lie algebra $\mathfrak{d}$ is called the* **Drinfel'd double**.

The integration D of $\mathfrak{d}$ is also called the Drinfel'd double and contains both groups G and $\widetilde{G}$ as subgroups. As manifolds, one has $D \cong G \times \widetilde{G}$. However, this isomorphism does not respect the group structure, at least globally. Locally, the map $(g, \tilde{g}) \mapsto g\tilde{g}$ is a Lie group isomorphism (cf. [LW90, Thm. 3.12]). Here $g\tilde{g}$ is to be understood as the multiplication in D. We like to emphasise that Abelian T-duality fits in this framework. Indeed, given two sigma models with toroidal targets $\mathbb{T}^k, \widetilde{\mathbb{T}}^k$. Then, the $2k$-torus $\mathbb{T}^{2k}$ forms a Drinfel'd double when its Lie algebra is equipped with the following pairing:

$$\langle T_A, T_B \rangle = \langle \widetilde{T}_A, \widetilde{T}_B \rangle = 0, \quad \langle T_A, \widetilde{T}_B \rangle = \delta_{AB}.$$

Here, $T_A, \widetilde{T}_B$ denote generators of $\mathfrak{t}^k$ and $\tilde{\mathfrak{t}}^k$, respectively. The ad-invariance of the pairing follows from the fact the torus is Abelian.

Note, using the pairing $\langle \cdot, \cdot \rangle$, the subalgebra $\tilde{\mathfrak{g}}$ can be identified with $\mathfrak{g}^*$. In this way, we see that a Manin triple defines a Lie bialgebra. In fact, the converse is also true [Kos04, Sec. 1.6]

Theorem 5.2.4.
There is a one-to-one correspondence between finite dimensional Lie bialgebras and finite dimensional Manin triples.

5.2.2 Constructing Poisson-Lie Symmetric Models

To construct sigma-models with manifest Poisson-Lie symmetry (5.2), we start with a Manin triple $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$, following [Kli96]. Let $2n = \dim \mathfrak{d}$ and consider an n-dimensional linear subspace V_+ of $\mathfrak{d}$, such that $\mathfrak{d} = V_+ \oplus V_-$ and $\langle\cdot,\cdot\rangle$ restricted to V_+ is positive definite. Here $V_- := V_+^\perp$ denotes the orthogonal complement with respect to the pairing. The space V_+ is a so-called *generalized metric* and plays an important role in generalized geometry. We elaborate more on this in Section 5.4. Furthermore, we consider fields $l : \Sigma \to D$ satisfying

$$\langle \partial_\mp l l^{-1}, V_\pm \rangle = 0. \tag{5.6}$$

In other words, the field equations are given by $\partial_\pm l l^{-1} \in V_\pm$. Using this initial data, we will construct a sigma-model with G-Poisson-Lie symmetry with respect to $\widetilde{G}$ and the corresponding dual sigma-model.

As mentioned before, we can decompose l (at least locally) as $l = g\tilde{g}$, where $g : \Sigma \to G$ and $\tilde{g} : \Sigma \to \widetilde{G}$. Inserting this into (5.6) yields

$$\begin{aligned} 0 &= \langle \partial_\mp l l^{-1}, V_\pm \rangle \\ &= \langle \partial_\mp g g^{-1} + \mathrm{Ad}_g(\partial_\mp \tilde{g}\tilde{g}^{-1}), V_\pm \rangle \\ &= \langle g^{-1}\partial_\mp g + \partial_\mp \tilde{g}\tilde{g}^{-1}, \mathrm{Ad}_{g^{-1}} V_\pm \rangle. \end{aligned} \tag{5.7}$$

By the properties of V_+ and $\mathfrak{g}$, it follows that $\mathrm{Ad}_{g^{-1}} V_+ \cap \mathfrak{g} = 0$. The same statement is true for $\tilde{\mathfrak{g}}$. Therefore, we can find a linear map $E(g) : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that

$$\mathrm{Ad}_{g^{-1}} V_+ = \mathrm{gr}\, E(g) = \{X + E(g)X \mid X \in \mathfrak{g}\}.$$

From this we immediately see

$$\mathrm{Ad}_{g^{-1}} V_- = \{X - E(g)^T X \mid X \in \mathfrak{g}\},$$

where $E(g)^T$ denotes the adjoint with respect to the pairing, i.e. the operator such that $(E\cdot,\cdot) = (\cdot, E(g)^T\cdot)$.

To be more concrete, let $\{T_A\}$ be a basis of $\mathfrak{g}$ and $\{\widetilde{T}^A\}$ the dual basis of $\tilde{\mathfrak{g}}$, i.e.

$$\langle T_A, \widetilde{T}^B \rangle = \delta_A^B.$$

Then, we can write

$$E(g)T_A = E_{AB}(g)\widetilde{T}^B$$

where

$$E_{AB}(g) = \langle T_B, E(g)T_A \rangle.$$

Consequently,

$$\begin{aligned} \mathrm{Ad}_{g^{-1}} V_+ &= \mathrm{span}\left\{T_A + E_{AB}(g)\widetilde{T}^B\right\} \\ \mathrm{Ad}_{g^{-1}} V_- &= \mathrm{span}\left\{T_A - E_{BA}(g)\widetilde{T}^B\right\}. \end{aligned} \tag{5.8}$$

Using this and the fact $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are isotropic, (5.7) becomes

$$J_{-,A} := -(\partial_- \tilde{g}\tilde{g}^{-1})_A = E_{AB}(g)(g^{-1}\partial_- g)^B \tag{5.9}$$

$$J_{+,A} := -(\partial_+ \tilde{g}\tilde{g}^{-1})_A = -E_{BA}(g)(g^{-1}\partial_+ g)^B. \tag{5.10}$$

In a basis independent way, we can write $J = d\tilde{g}\tilde{g}^{-1}$ and thus $J \in \Omega^1(\Sigma, \tilde{\mathfrak{g}})$. Consequently, it satisfies the zero curvature condition in $\tilde{\mathfrak{g}}$

$$dJ + \frac{1}{2}[J, J]_{\tilde{\mathfrak{g}}} = 0, \tag{5.11}$$

as it is the right-invariant Maurer-Cartan form. Moreover, we can explicitly write J in terms of $E(g)$,

$$J_+ = -E(g)g^{-1}\partial_+ g, \qquad J_- = E(g)^T g^{-1}\partial_- g. \tag{5.12}$$

Hence, we have constructed a current satisfying (5.2).

In fact, J defines the Poisson-Lie symmetry current of the sigma-model

$$\begin{aligned} S[g] &= \int_\Sigma d^2\sigma\, E_{AB}(g)(g^{-1}\partial_+ g)^A (g^{-1}\partial_- g)^B \qquad (5.13)\\ &= \int_\Sigma d^2\sigma\, (g^{-1}\partial_+ g)^A (g^{-1}\partial_- g)^B \langle T_B, E(g)T_A\rangle \\ &= \int_\Sigma d^2\sigma\, \left\langle g^{-1}\partial_- g, E(g)g^{-1}\partial_+ g\right\rangle, \qquad (5.14) \end{aligned}$$

as (5.11) are its field equations [Kli96]. Note, (5.13) is precisely the standard sigma-model action (4.20), written in left-invariant coordinates on the target Lie group G. Consequently, we succeeded in constructing a sigma-model with G-Poisson-Lie symmetry with respect to $\widetilde{G}$. The required initial data was $(\mathfrak{d}, V_+)$ and the field equations (5.6).

5.2.3 Buscher-type Rules

Note, our discussion above is completely dual in $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. For example, one could have started with the decomposition $l = \widetilde{h}h$. Then, the discussion above can be repeated to obtain the dual sigma-model

$$\begin{aligned} \tilde{S}[\widetilde{h}] &= \int_\Sigma d^2\sigma\, \widetilde{E}^{AB}(\widetilde{h})(\widetilde{h}^{-1}\partial_+\widetilde{h})_A(\widetilde{h}^{-1}\partial_-\widetilde{h})_B \\ &= \int_\Sigma d^2\sigma\, \left\langle \widetilde{h}^{-1}\partial_-\widetilde{h}, \widetilde{E}(\widetilde{h})\widetilde{h}^{-1}\partial_+\widetilde{h}\right\rangle. \end{aligned}$$

Moreover, the background $\widetilde{E}^{AB}(\widetilde{h})$ can be written in terms of $E_{AB}(g)$. To see this, first note at the unit we have

$$\begin{aligned} 0 &= \left\langle T_A + E_{AB}(e)\widetilde{T}^B, \widetilde{T}^C - \widetilde{E}^{DC}(\tilde{e})T_D\right\rangle \\ &= \delta_A^C - E_{AB}(e)\widetilde{E}^{DC}(\tilde{e})\delta_D^B \\ &= \delta_A^C - E_{AB}(e)\widetilde{E}^{BC}(\tilde{e}). \end{aligned}$$

A similar computation can be done where the roles of E and $\widetilde{E}$ are reversed. This results in

$$\widetilde{E}(\tilde{e})E(e) = E(e)\widetilde{E}(\tilde{e}) = 1. \tag{5.15}$$

This is analogous to the $R \to 1/R$ symmetry in Abelian T-duality. In fact, in the Abelian case, (5.15) holds for every g as the Adjoint is trivial (cf. (5.8)). In this way, the Buscher rules are recovered [KŠ96c]. To get a general expression for $E(g)$, we write

$$\mathrm{Ad}_{g^{-1}}\, T_A = a(g)_A^{\ B} T_B, \qquad \mathrm{Ad}_{g^{-1}}\, \widetilde{T}^A = b(g)^{AB} T_B + d(g)^A_{\ B}\widetilde{T}^B.$$

Using this, (5.8) becomes

$$\mathrm{Ad}_{g^{-1}}\, V_+ = \mathrm{span}\left\{\mathrm{Ad}_{g^{-1}}(T_A + E_{AB}(e)\widetilde{T}^B)\right\}$$

$$
\begin{aligned}
&= \operatorname{span}\left\{(a(g)_A^C + E_{AB}(e)b(g)^{BC})T_C + E_{AB}(e)d(g)_C^B\widetilde{T}^C\right\} \\
&= \operatorname{span}\left\{T_C + (a(g)_A^C + E_{AB}(e)b(g)^{BC})^{-1}E_{AB}(e)d(g)_C^B\widetilde{T}^C\right\} \\
&= \operatorname{gr}\left((a(g) + E(e)b(g))^{-1}E(e)d(g)\right).
\end{aligned}
$$

Hence,

$$E(g) = (a(g) + E(e)b(g))^{-1}E(e)d(g).$$

For the dual sigma-model, we obtain an analogous expression

$$\widetilde{E}(\widetilde{h}) = (\tilde{a}(\widetilde{h}) + \widetilde{E}(\tilde{e})\tilde{b}(\widetilde{h}))^{-1}\widetilde{E}(\tilde{e})\tilde{d}(\widetilde{h}).$$

In view of (5.15), we obtain Buscher-type rules for background of the dual sigma-model

$$\widetilde{E}(\widetilde{h}) = (\tilde{a}(\widetilde{h}) + E(e)^{-1}\tilde{b}(\widetilde{h}))^{-1}E(e)^{-1}\tilde{d}(\widetilde{h}). \tag{5.16}$$

5.2.4 Equivalence of Hamiltonian Systems

Up to now, we have realized a sigma-model with Poisson-Lie symmetry in (5.13) and found Buscher-type rules (5.16) to construct the dual sigma-model. Furthermore, the above establishes that solving the sigma-model $S[g]$ (or $\tilde{S}[\widetilde{h}]$) is equivalent to having a lift l that satisfies (5.6). As the latter is independent of the decomposition of l in $G\widetilde{G}$ or $\widetilde{G}G$, we get an equivalence between the sigma-models. In fact, the main result of [KŠ95; KŠ96c] is that the sigma-model $S[g]$ and its dual $\tilde{S}[\widetilde{h}]$ are isomorphic as Hamiltonian systems. In other words, there is a Hamiltonian preserving symplectomorphism between the phase spaces. This is called **Poisson-Lie T-duality**. In particular, the duality implies that any solution to $S[g]$ can be mapped to a solution of $\tilde{S}[\widetilde{h}]$. Let us make this mapping precise.

Let $S[g]$ be a sigma-model that is G-Poisson-Lie symmetric with respect to $\widetilde{G}$ such as (5.13). Then given a solution g, there is an associated flat connection one-form $J \in \Omega^1(\Sigma, \tilde{\mathfrak{g}})$. Then, we can find a field $\tilde{g} : \Sigma \to G$ such that (5.9) holds by

$$\tilde{g} = P\exp\left(-\int_\gamma J\right),$$

which only depends on the homotopy class of γ, by the flatness of J. Moreover, by the periodicity $\tilde{g}(\tau, \sigma + 2\pi) = \tilde{g}(\tau, \sigma)$ we obtain a non-local constraint:

$$P\exp\left(-\int_{a^2} J\right) = \tilde{e}, \tag{5.17}$$

where a denotes the generator of $\pi_1(\Sigma)$. This is called the **unit monodromy constraint** [KŠ96c; KŠ96b]. In other words, we make (5.17) single-valued and can be computed by choosing $\gamma = a$.

The generalized metric V_+ is obtained from the sigma-model via (5.8). By construction, the element $l = g\tilde{g} \in D$ satisfies the field equations (5.6). We also have the decomposition $l = \widetilde{h}h$. By the construction above, $\widetilde{h}$ will be a solution of $\tilde{S}[\widetilde{h}]$, as l satisfies (5.6). Moreover, h will induce the dual flat connection one-form via

$$\tilde{J} = -dhh^{-1}.$$

Thus, the mapping between solutions is given by $g \mapsto \widetilde{h}$.

Even though the discussion above is rather elegant, given a sigma-model, it is hard to see whether it has Poisson-Lie symmetry and what the dual would be. This is due to the fact that the Drinfel'd double D is part of the initial data. However, a priori it is unclear which Drinfel'd double to pick, as it is not unique. Therefore, we would like to have a more systematic way to produce Poisson-Lie T-dual sigma-models. This is done using *$\mathcal{E}$-models* and we will discuss them in the next section. An $\mathcal{E}$-model is a sigma-model on the Drinfel'd double D, whose equations of motion are precisely (5.6). Moreover, it makes Poisson-Lie T-duality into a manifest symmetry. This is closely related to how double field theory makes Abelian T-duality manifest [HZ09; DHT19]. Before we move on, let us end this section with two examples of the construction above.

Example 5.2.5 (Principal Chiral Model).
In Section 4.3.1 we mentioned that the principal chiral model has Poisson-Lie symmetry. Indeed, the Drinfel'd double is given by $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, where we equip the second factor with the trivial Lie bracket. Furthermore, the action has the form (5.14) with $E(g) : \mathfrak{g} \to \tilde{\mathfrak{g}} = \mathfrak{g}$ is the identity (cf. (4.21)). Consequently, the connection one-form $J \in \Omega^1(\Sigma, \mathfrak{g})$ can be computed using (5.12):

$$J = -g^{-1}\partial_+ g + g^{-1}\partial_- g = - \star j.$$

Note, as the second factor of $\mathfrak{d}$ is equipped with the trivial bracket, the flatness of J is just

$$dJ = -d \star j = (\partial_+ j_- + \partial_- j_+) d\sigma^- \wedge d\sigma^+ = 0.$$

Consequently, $\partial_+ j_- + \partial_- j_+ = 0$. This is precisely the equation of motion for the PCM. Now, one can obtain the dual sigma-model using (5.16). This is done in [Kli96]. Actually, this 'semi-Abelian' double, where one of the factors is Abelian is precisely the setting of the original non-Abelian T-duality by de la Ossa and Quevedo [dQ93]. ◆

Example 5.2.6 (Bi-Yang-Baxter Model).
As a second example, we consider the bi-Yang-Baxter model of Section 4.3.3. To define the model we needed a skew-symmetric R-matrix. In particular, for the Weil operator solution, we considered the non-split case. In view of Proposition 4.1.5, the Drinfel'd double is given by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{g}_R$. The action (4.30) of the bi-Yang-Baxter model is of the form (5.14) with

$$E(g)^T = \frac{1}{1 - \eta R - \zeta R^g}.$$

To find $E(g)$, note we can expand

$$\frac{1}{1 - \eta R - \zeta R^g} = \sum_{k=0}^{\infty} (\eta R + \zeta R^g)^k.$$

In first order we have, for $X, Y \in \mathfrak{g}$

$$\begin{aligned}
(X, (\eta R + \zeta R^g) Y) &= (X, \eta R Y) + (X, \zeta R^g Y) \\
&= -(\eta R X, Y) + \zeta (X, \mathrm{Ad}_{g^{-1}} R \, \mathrm{Ad}_g Y) \\
&= -(\eta R X, Y) - (\zeta R^g X, Y) \\
&= ((-\eta R - \zeta R^g) X, Y) .
\end{aligned}$$

This generalizes to higher orders and we obtain

$$E(g) = \frac{1}{1 + \eta R + \zeta R^g}.$$

Then, by (5.12) we find

$$J_\pm = \mp \frac{1}{1 \pm \eta R \pm \zeta R^g} g^{-1} \partial_\pm g.$$

This is precisely the object we defined in Section (4.3.3) to conveniently write the equation of motion. Moreover, this equation of motion is precisely the flatness of J viewed as an element in $\Omega^1(\Sigma, \mathfrak{g}_R)$. In Section 5.5 we will relate the Poisson-Lie T-dual of the bi-Yang-Baxter model to the λ-model.◆

5.3 $\mathcal{E}$-models

In the previous section we discussed sufficient conditions for constructing Poisson-Lie T-dual sigma-models. The crucial ingredients were the Drinfel'd double D, a generalized metric V_+ and the field equations (5.6). Motivated by this, we will construct a sigma-model on the Drinfel'd double that produces the field equations (5.6) called the $\mathcal{E}$-**model**. Furthermore, we discuss how to extract Poisson-Lie T-dual sigma-models from the $\mathcal{E}$-model. Our main references are [Kli15; DHT19].

The $\mathcal{E}$-model was first constructed by Klimčík and Ševera in [KŠ96c] and the necessary components are

- An even dimensional Lie group D, whose Lie algebra $\mathfrak{d}$ admits an ad-invariant non-degenerate pairing $\langle \cdot, \cdot \rangle$ of split signature. The group D will play the role of the Drinfel'd double.
- At least one subgroup $\widetilde{G} \subset D$ such that its Lie algebra $\tilde{\mathfrak{g}} \subset \mathfrak{d}$ is Lagragian with respect to $\langle \cdot, \cdot \rangle$.
- A self-adjoint operator $\mathcal{E} : \mathfrak{d} \to \mathfrak{d}$ such that $\mathcal{E}^2 = \mathrm{id}$ and $\langle \cdot, \mathcal{E} \cdot \rangle$ is positive-definite[2].

As the operator $\mathcal{E}$ is idempotent, i.e. $\mathcal{E}^2 = \mathrm{id}$, it has eigenvalues ± 1. A generalized metric can be obtained from $\mathcal{E}$ by defining V_+ to be the $+1$-eigenspace. Now, to capture the dynamics of the $\mathcal{E}$-model we consider the following Hamiltonian

$$H_\mathcal{E} = \frac{1}{2} \int d\sigma \, \langle j(\sigma), \mathcal{E} j(\sigma) \rangle,$$

where $j(\sigma) = \partial_\sigma l l^{-1}$ for $l : \Sigma \to D$. To completely fix the dynamics of the $\mathcal{E}$-model, we need the Poisson brackets. For this, let $t_A = \{\widetilde{T}^A, T_A\}$ be a basis of $\mathfrak{d}$. Here, $\widetilde{T}^A$ denote the generators of $\tilde{\mathfrak{g}}$ and T_A the remaining generators. Typically, T_A will be the generators of the dual Lie algebra $\mathfrak{g}$. Then, the Poisson brackets of the components $j_A(\sigma) := \langle t_A, j(\sigma) \rangle$ are given by

$$\{j_A(\sigma), j_B(\sigma')\} = F_{AB}{}^C j_C(\sigma) \delta(\sigma - \sigma') + \eta_{AB} \partial_\sigma \delta(\sigma - \sigma'),$$

where $F_{AB}{}^C$ denote the structure constants on $\mathfrak{d}$ and $\eta_{AB} = \langle t_A, t_B \rangle$. Using the Poisson brackets one sees the equations of motion are given by[3] [Kli15]

$$\partial_\tau j(\sigma) = \{H_\mathcal{E}, j(\sigma)\} = \partial_\sigma(\mathcal{E} j(\sigma)) + [\mathcal{E} j(\sigma), j(\sigma)]. \tag{5.18}$$

[2] In [DHT19] this last condition is dropped. By doing this one has to impose $V_+ \cap \mathfrak{g} = 0$ by hand for the corresponding generalized metric V_+.

[3] For this, we assume $\mathcal{E}$ does not depend on the worldsheet coordinates. In [DHT19] the condition is relaxed.

We claim these equations of motion are equivalent to the field equations (5.6). Indeed,

Proposition 5.3.1.
Let $l : \Sigma \to D$. Then, l satisfies (5.6) if and only if $j(\sigma) := \partial_\sigma l l^{-1}$ solves the equations of motion (5.18) of the $\mathcal{E}$-model.

Proof. — On the one hand, suppose l satisfies $\partial_\pm l l^{-1} \in V_\pm$. Here $V_\pm$ denote the ± 1-eigenspaces of $\mathcal{E}$. Then, in light-cone coordinates $\sigma^\pm = \tau \pm \sigma$, we have

$$\begin{aligned} dll^{-1} &= \partial_+ l l^{-1} d\sigma^+ + \partial_- l l^{-1} d\sigma^- \\ &= \underbrace{\left(\partial_+ l l^{-1} - \partial_- l l^{-1}\right)}_{\partial_\sigma l l^{-1}} d\sigma + \underbrace{\left(\partial_+ l l^{-1} + \partial_- l l^{-1}\right)}_{\partial_\tau l l^{-1}} d\tau. \end{aligned} \tag{5.19}$$

Since $\mathcal{E}(\partial_\pm l l^{-1}) = \pm \partial_\pm l l^{-1}$, we see from (5.19)

$$\partial_\tau l l^{-1} = \mathcal{E}(\partial_\sigma l l^{-1}). \tag{5.20}$$

Hence, for $j(\sigma) = \partial_\sigma l l^{-1}$ we have

$$\begin{aligned} \partial_\tau j(\sigma) &= \partial_\tau(\partial_\sigma l l^{-1}) \\ &= \partial_\tau \partial_\sigma l l^{-1} + \partial_\sigma l \partial_\tau (l^{-1}) \\ &= \partial_\tau \partial_\sigma l l^{-1} - \partial_\sigma l l^{-1} \partial_\tau l l^{-1} \\ &= \partial_\sigma \partial_\tau l l^{-1} - \partial_\tau l l^{-1} \partial_\sigma l l^{-1} + [\partial_\tau l l^{-1}, \partial_\sigma l l^{-1}] \\ &= \partial_\sigma(\mathcal{E} j(\sigma)) + [\mathcal{E} j(\sigma), j(\sigma)]. \end{aligned}$$

On the other hand, given a current $j(\sigma) = \partial_\sigma l l^{-1}$ that satisfies (5.18), the above computation shows l satisfies (5.20). Then, using the properties of $\mathcal{E}$, we directly see

$$\begin{aligned} \mathcal{E}\left(\partial_+ l l^{-1}\right) &= \frac{1}{2}\mathcal{E}\left(\partial_\tau l l^{-1} + \partial_\sigma l l^{-1}\right) = \frac{1}{2}\mathcal{E}\left(\mathcal{E}(\partial_\sigma l l^{-1}) + \partial_\sigma l l^{-1}\right) = \frac{1}{2}\left(\partial_\sigma l l^{-1} + \mathcal{E}(\partial_\sigma l l^{-1})\right) = \partial_+ l l^{-1} \\ \mathcal{E}\left(\partial_- l l^{-1}\right) &= \frac{1}{2}\mathcal{E}\left(\partial_\tau l l^{-1} - \partial_\sigma l l^{-1}\right) = \frac{1}{2}\mathcal{E}\left(\mathcal{E}(\partial_\sigma l l^{-1}) - \partial_\sigma l l^{-1}\right) = \frac{1}{2}\left(\partial_\sigma l l^{-1} - \mathcal{E}(\partial_\sigma l l^{-1})\right) = -\partial_- l l^{-1}. \end{aligned}$$

Hence, $\partial_\pm l l^{-1} \in V_\pm$. This completes the proof. ■

The above proposition shows that the $\mathcal{E}$-model precisely captures the initial data needed in Section 5.2. Remarkably, $\mathcal{E}$-models also play an important role in integrability, even though a priori they seem unrelated. For a discussion about this relation, see [DHT19; Kli21].

To make contact with sigma-models, one can perform a Legendre transform to obtain the following action [KŠ96c; DHT19]

$$S[l] = \frac{1}{2}\int_\Sigma d^2\sigma\, \langle \partial_\sigma l l^{-1}, \partial_\tau l l^{-1} \rangle + \frac{1}{12}\int_B \langle dll^{-1}, [dll^{-1}, dll^{-1}] \rangle - \int d\tau\, H_{\mathcal{E}}. \tag{5.21}$$

Note, the first two terms define a WZW model on D[4]. Following [KŠ96b], we split the element l as follows

$$l(\tau, \sigma) = f(\tau, \sigma)\tilde{g}(\tau, \sigma).$$

[4] We are aware the coefficients in (5.21) do not match those stated in (4.25). This is due to the usage of different conventions in [KŠ96c; Kli15]. To keep contact with the relevant literature we will use their conventions from now on.

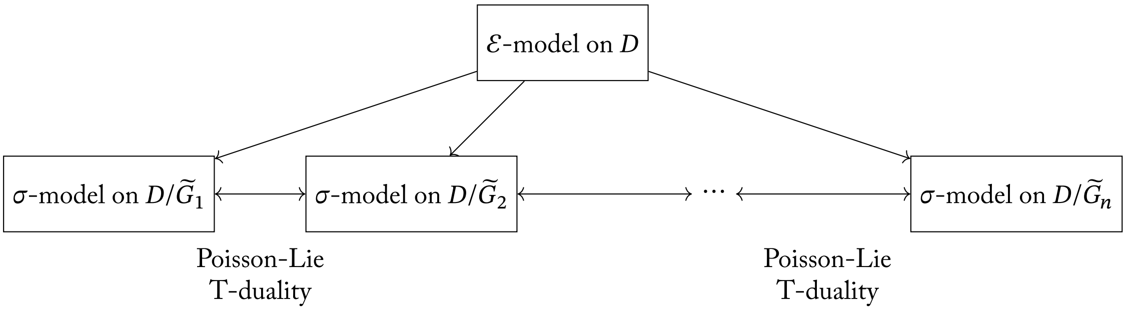

Figure 5.1: Schematic depiction of obtaining Poisson-Lie T-dual sigma-models from an $\mathcal{E}$-model on the double D. Inspired by [DHT19].

Here, $\tilde{g} \in \widetilde{G}$ and $f \in D$ parametrizes elements in the coset $D/\widetilde{G}$. Evaluating (5.21) using the decomposition above and integrating out the $\widetilde{G}$-valued terms[5] we obtain [KŠ96b; Kli15]

$$S_{\mathcal{E}}[f] = S_{\mathrm{WZW},D}[f] - \int_{\Sigma} d^2\sigma\, \langle \mathcal{P}_f(\mathcal{E}) f^{-1}\partial_+ f, f^{-1}\partial_- f\rangle, \tag{5.22}$$

where

$$S_{\mathrm{WZW},D}[f] = \frac{1}{2}\int_{\Sigma} d^2\sigma\, \langle f^{-1}\partial_+ f, f^{-1}\partial_- f\rangle + \frac{1}{12}\int_B \langle f^{-1}df, [f^{-1}df, f^{-1}df]\rangle.$$

Furthermore, $\mathcal{P}_f(\mathcal{E}) : \mathfrak{d} \to \tilde{\mathfrak{g}}$ is the projection operator defined by the relations

$$\operatorname{im} \mathcal{P}_f(\mathcal{E})\mathfrak{d} = \tilde{\mathfrak{g}}, \quad \ker \mathcal{P}_f(\mathcal{E}) = (\mathrm{id} + \mathrm{Ad}_{f^{-1}}\, \mathcal{E}\, \mathrm{Ad}_f)\tilde{\mathfrak{g}}. \tag{5.23}$$

We will use these expressions in Section 5.5 to see that the λ-model and bi-Yang-Baxter model can be obtained from $\mathcal{E}$-models.

Poisson-Lie T-duality from the $\mathcal{E}$-model point of view is the following: given another subgroup $G \subset D$ for which its Lie algebra is Lagrangian, the two resulting sigma-models (5.22) on $D/\widetilde{G}$ and D/G, respectively, are isomorphic as Hamiltonian systems [KŠ96b; KŠ96c]. Schematically, this is depicted in Figure 5.1. In particular, any sigma-model coming from an $\mathcal{E}$-model via (5.22) admits Poisson-Lie T-duals. That is, if D admits two subgroups with Lagrangian Lie algebras. In the special case that $D = G\widetilde{G}$, we recover our discussion in Section 5.2.

Up to this point, we have described two point of views on Poisson-Lie T-duality: the one from Section 5.2 and using $\mathcal{E}$-models. Both perspectives are from the worldsheet point of view. In the next section we discuss a mathematical framework for a target space formulation of Poisson-Lie T-duality in terms of *Courant algebroids*. However, this description is not necessary for Section 5.5 and can therefore be skipped.

5.4 Courant Algebroids and Poisson-Lie T-duality

The main feature of Poisson-Lie T-duality is the fact it is a non-Abelian generalization of Abelian T-duality. It has been noted in [CG11] that Abelian T-duality can be captured in terms of **generalized**

[5] This can be done as those terms will be quadratic in $\partial_\sigma \tilde{g}\tilde{g}^{-1}$ [KŠ96c], by the properties of $\widetilde{G}$.

geometry. In particular, in the language of **Courant algebroids**. Consequently, we expect the same to be true for Poisson-Lie T-duality. The answer turns out to be affirmative, as was first recognized in the famous letters [Šev17a]. Further relationship was established in [Šev15; Šev16; JV18; ŠV20] among others, with several applications in physics. In this section we discuss this relationship and conclude Poisson-Lie T-duality can be interpreted as a *Courant relation*, which is a generalization of morphism, following [Vys20].

5.4.1 Definitions and Examples

Let us start by introducing Courant algebroids. The history behind the development of Courant algebroids is rather fascinating and we refer to [Kos13] for a review. The modern definition of Courant algebroids is due to Roytenberg [Roy99] and Weinstein and Ševera [ŠW02].

Definition 5.4.1.
A **Courant algebroid** *(CA) is a vector bundle $E \to M$ equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the fibers of the bundle, an $\mathbb{R}$-linear bracket $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ on the space of section of E and a vector bundle map (***the anchor***) $\rho : E \to TM$ such that*

i) For any $s, t, u \in \Gamma(E)$,

$$[s, [t, u]] = [[s, t], u] + [t, [s, u]]$$

ii) For any $s, t \in \Gamma(E)$,

$$\rho([s, t]) = [\rho(s), \rho(t)]$$

iii) For any $s, t \in \Gamma(E)$, $f \in C^\infty(M)$,

$$[s, ft] = f[s, t] + \mathcal{L}_{\rho(s)}(f)t$$

iv) For any $s, t, u \in \Gamma(E)$,

$$\rho(s)\langle t, u \rangle = \langle [s, t], u \rangle + \langle t, [s, u] \rangle$$

v) for any $s \in \Gamma(E)$,

$$[s, s] = \mathcal{D}\langle s, s \rangle,$$

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined as $\mathcal{D} = \frac{1}{2}\beta^{-1}\rho^* d$. Here $\beta : E \to E^*$ denotes the isomorphism induced by the pairing $\langle \cdot, \cdot \rangle$.

Alternatively, one can implicitly define the operator $\mathcal{D}$ via

$$\langle \mathcal{D}f, s \rangle = \frac{1}{2}\mathcal{L}_{\rho(s)}(f),$$

for any $s \in \Gamma(E)$. Note, from property $v)$ we see that the bracket $[\cdot, \cdot]$ typically is not skew-symmetric, it satisfies

$$[s, t] + [t, s] = 2\mathcal{D}\langle s, s \rangle.$$

Therefore, it does not define a Lie bracket. Furthermore, let us denote $\rho^t := \beta^{-1}\rho^* : T^*M \to E$, which we call the **transpose** of ρ. Now, since any covector $\xi \in T^*M$ can locally be described as the differential of a function of the form $\langle s, s\rangle$ for some $s \in \Gamma(E)$, we have that

$$0 \to T^*M \xrightarrow{\rho^t} E \xrightarrow{\rho} TM \to 0 \tag{5.24}$$

defines a chain complex, i.e. $\rho \circ \rho^t = 0$ (cf. [Šev15]).

Definition 5.4.2.
We say a Courant algebroid $E \to M$ is

i) **transitive** *if ρ is surjective*

ii) **exact** *if the sequence (5.24) is exact*

A trivial example of Courant algebroids is provided by Lie algebras.

Example 5.4.3 (Quadratic Lie algebras).
A **quadratic Lie algebra** $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, i.e. a Lie algebra equipped with an ad-invariant inner product $\langle\cdot,\cdot\rangle$, is a Courant algebroid over a point. Here, the anchor is the trivial map and thus $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is an example of a transitive CA. ◆

Note, the Drinfel'd double $\mathfrak{d}$ of a Manin triple thus is an example of a Courant algebroid. This is closely tied to its role in Poisson-Lie T-duality, as we will see below. A second example is provided by the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, the central object in generalized geometry.

Example 5.4.4 (Generalized tangent bundle).
Given a manifold M, we equip the generalized tangent bundle $\mathbb{T}M$ with the pairing

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\eta(X) + \xi(Y)) \tag{5.25}$$

and the **Dorfman bracket**

$$[X + \xi, Y + \eta] := [X, Y] + \mathcal{L}_X\eta - i_Y d\xi.$$

Combining this with the anchor $\rho : \mathbb{T}M \to TM$ given by the projection onto the first factor, $(\mathbb{T}M, \rho, \langle\cdot,\cdot\rangle, [\cdot,\cdot])$ defines a Courant algebroid. In fact, one can verify that the sequence (5.24) is exact, making it into an exact CA. In the literature, it is referred to as the **standard CA over** M [Šev15]. ◆

Interestingly, we can obtain a whole class of exact Courant algebroids by twisting the Dorfman bracket. Indeed, for a closed 3-form $H \in \Omega^3(M)$ we define the **twisted Dorfman bracket** as

$$[X + \xi, Y + \eta]_H := [X, Y] + \mathcal{L}_X\eta - i_Y d\xi + i_Y i_X H.$$

Then, $(\mathbb{T}M, \rho, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$ is an exact CA, as well [ŠW02]. In fact, *all* exact CAs are exhausted in this way. This follows from the classification theorem for exact Courant algebroids due to Ševera. For this, we need the notion of B-**field transforms**. Let $B \in \Omega^2(M)$ and view it as a map $B : TM \to T^*M$. From this we obtain an invertible bundle map via exponentiating

$$e^B : \mathbb{T}M \to \mathbb{T}M, \quad X + \xi \mapsto X + \xi + i_X B.$$

It is orthogonal with respect to (5.25) and relates the H-twisted Dorfman bracket to the $H + dB$-twisted Dorfman bracket [BCG07]:

$$[e^B \cdot, e^B \cdot]_H = e^B [\cdot, \cdot]_{H+dB}. \tag{5.26}$$

Theorem 5.4.5.
Exact Courant algebroids E over M, up to B-field transforms, are classified by $H^3(M)$.

Proof. — We give a sketch of the proof, following [BCG07]. Given an exact CA $E \to M$, we can choose a right splitting $\sigma : TM \to E$ of the sequence (5.24) such that $L := \sigma(TM)$ is isotropic. To this splitting we associate a closed curvature 3-form via

$$H(X, Y, Z) := \langle [\sigma(X), \sigma(Y)], \sigma(Z) \rangle.$$

Using the bundle isomorphism $\sigma \oplus \frac{1}{2}\rho^t : \mathbb{T}M \to E$, we may transport the Courant structure of E to $\mathbb{T}M$. A straightforward computation shows the pairing becomes (5.25) and the bracket is the H-twisted Dorfman bracket.

If we choose a different isotropic splitting $\sigma' : TM \to E$, the difference $B := \sigma - \sigma'$ defines an element in $\Omega^2(M)$. Consequently, a different splitting amounts to mapping $X \mapsto X + i_X B$ on TM, i.e. a B-field transform on $\mathbb{T}M$. We know from (5.26) that H is mapped to $H + dB$ under the B-field transform. Consequently, the class $[H] \in H^3(M)$ is independent of isotropic splitting and defines the exact Courant algebroid E completely. ■

Exact Courant algebroids are particularly important for our purposes, as they can be related to non-linear sigma-models [Šev15; Šev16; DMS23]. To see this, we need the notion of a generalized metric [Vys20]:

Definition 5.4.6.
A **generalized metric** *on a Courant algebroid $E \to M$ is a vector bundle morphism $\mathcal{G} : E \to E$ such that $\mathcal{G}^2 = \mathrm{id}$ and $\langle \mathcal{G} \cdot, \cdot \rangle$ is positive definite.*

Equivalently, a generalized metric is given by a maximal positive-definite subbundle $V_+ \subset E$ which corresponds to the $+1$-eigenspace of $\mathcal{G}$. For our purposes, V_+ will be a half-rank subbundle of E. Note, then the notion of a generalized metric coincides with the object considered in Sections 5.2 and 5.3. There, the map $\mathcal{E}$ played the role of $\mathcal{G}$ in the definition above. Furthermore, we saw the geometric data to define Poisson-Lie T-duality between worldsheet theories was a Drinfel'd double $\mathfrak{d}$, i.e. a Courant algebroid, and a generalized metric V_+.

Something similar happens for exact Courant algebroids. Given an exact Courant algebroid $E \to M$ with a generalized metric V_+. By Theorem 5.4.5, we may identify $E \cong \mathbb{T}M$ equipped with the H'-twisted Dorfman bracket, for some closed $H' \in \Omega^3(M)$. Under this identification, the subbundles TM, T^*M are Lagrangian. As the pairing is positive definite on V_+, we have $V_+ \cap TM = V_+ \cap T^*M = 0$. Consequently,

V_+ is the graph of a bundle map $E : TM \to T^*M$. Using the pairing, we can view $E \in \Gamma(T^*M \otimes T^*M)$. Furthermore, we can write $E = g + B$ with g the symmetric and B the anti-symmetric part of E. This is analogous to what we did in Section 4.3. In view of Theorem 5.4.5, we should consider E up to B-field transformations. Therefore, we may assume $E = g$, after performing the B-field transform e^{-B}. We define $H := H - dB$. Consequently, the data (E, V_+) gives us a pair (g, H), consisting of a metric and closed 3-form on M. Using the latter, we can construct a σ-model, describing the dynamics of a field $X : \Sigma \to M$. Here Σ denotes a Riemann surface. The action is given by

$$S[X] = \int_\Sigma g(\partial X, \bar{\partial} X) + \int_B X^*H.$$

Here, B is a three dimensional manifold such that $\partial B = \Sigma$, completely analogous to the WZW model from Section 4.3.2.

5.4.2 Courant Relations

One of the main results of [CG11] is that T-duality can be viewed as an isomorphism of Courant algebroids. To be precise, they produce a vector bundle isomorphism $\varphi : E_1 \to E_2$ between Courant algebroids over a common base, satisfying

$$\langle \varphi(s), \varphi(t) \rangle_2 = \langle s, t \rangle_1, \quad [\varphi(s), \varphi(t)]_2 = \varphi([s,t]_1). \tag{5.27}$$

We wish to produce a similar statement for Poisson-Lie T-duality. Motivated by the above, it is natural to assume the statement will be about morphisms of Courant algebroids.

Defining morphisms between Courant algebrois over a common base, or diffeomorphic bases, is straightforward, as done above. However, defining CA morphisms over arbitrary smooth maps is more subtle. The reason for this is similar to the case of symplectic manifolds. Given a smooth map $\varphi : M_1 \to M_2$ between symplectic manifolds (M_1, ω_1) and (M_2, ω_2), one would say φ is symplectic if $\varphi^*\omega_2 = \omega_1$. This forces φ to be an immersion. Moreover, if we require $\varphi^* : C^\infty(M_2) \to C^\infty(M_1)$ to intertwine the induced Poisson brackets, φ must be a diffeomorphism [Vys20]. Consequently, the notion of morphism in the symplectic category is too restrictive. It is said the symplectic category 'has too few arrows'. Therefore, it was suggested in [Wei82] to weaken the notion of morphism to enlarge the amount of arrow, resulting in the *symplectic 'category'*. Here, an arrow is given by a Lagrangian submanifold $L \subset M_1 \times \overline{M_2}$, where $\overline{M_2}$ denotes the symplectic manifold $(M_2, -\omega_2)$.

As Courant algebroids can be viewed as graded symplectic manifolds [Roy02; Ševo1], we take the same approach. This leads to the concept of **Courant relations**. In this section we give the definition and discuss when two CA relations are composable, following [Vys20]. As mentioned there, the signature of $\langle \cdot, \cdot \rangle$ on a CA E can be general. However, for our purposes, we consider split signature as the main example. Let $(E, \rho, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ be a Courant algebroid.

Definition 5.4.7.
A subset $L \subset E$ is said to be **supported on a submanifold** $S \subset M$, *if it is a subbundle of the restricted vector bundle $E|_S$.*

The pairing $\langle \cdot, \cdot \rangle$ restricts to $E|_S$. Consequently, $L^\perp$ is well-defined and the notion of (co)isotropic exists.

Definition 5.4.8.
Let $L \subset E$ be a subbundle supported on S. We say L is **compatible with the anchor** *if $\rho(L) \subset TS$.*

Finally, we define the compatibility with the third structure of a CA: the bracket. For this we denote

$$\Gamma(E; L) := \{s \in \Gamma(E) \mid s|_S \in \Gamma(L)\},$$

for $L \subset E$ a subbundle supported on S.

Definition 5.4.9.
A subbundle L supported on S is said to be **involutive** *if for $s, t \in \Gamma(E; L)$ we have $[s, t] \in \Gamma(E; L)$.*

Combining the above compatibility's leads to the concept of an *ivolutive structure*.

Definition 5.4.10.
We say a subbundle L supported on S is an **involutive structure** *if*

i) L is involutive

ii) L is isotropic

iii) $L^\perp$ is compatible with the anchor.

If L is maximally isotropic, we call L a **Dirac structure supported on** S.

In the case L is maximally isotropic, the third condition can be replaced with L being compatible with the anchor [Vys20, Rmk. 2.19]. One might wonder why we do not restrict ourselves to *maximally* isotropic subbundles. This is because their composition might fail to be maximally isotropic [Vys20, Ex. 4.33].

Let $E_1 \to M_1$ and $E_2 \to M_2$ be Courant algebroids. By $\overline{E}_2$ we denote the Courant algebroid $(E_2, \rho_2, -\langle\cdot,\cdot\rangle_2, [\cdot,\cdot]_2)$. Furthermore, the product $E_1 \times \overline{E}_2$ comes with a canonical Courant algebroid structure given by

$$\begin{aligned} \rho(s_1, s_2) &:= \rho_1(s_1) + \rho_2(s_2), \\ \langle\langle (s_1, s_2), (t_1, t_2) \rangle\rangle &:= \langle s_1, t_1 \rangle_1 - \langle s_2, t_2 \rangle_2, \\ [(s_1, s_2), (t_1, t_2)] &:= ([s_1, t_1], [s_2, t_2]). \end{aligned}$$

Using this, we can define Courant algebroid relations.

Definition 5.4.11.
A **Courant algebroid relation from** E_1 **to** E_2 *is an involutive structure $R \subset E_1 \times \overline{E}_2$ supported on a submanifold $S \subset M_1 \times M_2$. We denote it by $R : E_1 \dashrightarrow E_2$.*

If $S = \operatorname{gr}\varphi$, for $\varphi : M_1 \to M_2$ smooth, we say R is a **Courant algebroid morphism from** E_1 **to** E_2 **over** φ. *We denote this by $R : E_1 \rightarrowtail E_2$.*

A trivial example of a CA relation, is the diagonal $\Delta(E) \subset E \times \overline{E}$. An important example for Poisson-Lie T-duality is the *transpose* of a CA relation.

Lemma 5.4.12.
Given a Courant algebroid relation $R : E_1 \dashrightarrow E_2$ supported on $S \subset M_1 \times M_2$, the **transpose relation** *defined by*

$$R^T := \{(s_2, s_1) \in E_2 \times \overline{E}_1 \mid (s_1, s_2) \in R\}$$

is a Courant algebroid relation $R^T : E_2 \dashrightarrow E_1$ supported on S^T, where S^T is defined analogous to R^T. Moreover, the transpose of a Courant algebroid morphism over φ is a Courant algebroid morphism, if and only if φ is a diffeomorphism.

The proof is a straightforward computation and we omit it here.

A natural question is whether Courant algebroid relations compose. After all, their purpose was to enlarge the set of arrows. The answer is *sometimes*. A transversality condition must be satisfied for the relations to be composable [Vys20, Prop. 3.14]. This is the main result of [Vys20]. As a set, the composition of two CA relations $R : E_1 \dashrightarrow E_2$ and $R' : E_2 \dashrightarrow E_3$ supported on S and S', respectively, is defined by

$$R' \circ R = \{(e_1, e_3) \in E_1 \times \overline{E_3} \mid \exists e_2 : (e_1, e_2) \in R,\ (e_2, e_3) \in R'\}. \tag{5.28}$$

In general, $R' \circ R$ fail to be a smooth subbundle of $E_1 \times \overline{E_3}$. By assuming topological conditions on $S \times S'$ and $S' \circ S$, defined analogous to (5.28), it can be assured $R' \circ R$ is a smooth subbundle of $E_1 \times \overline{E_3}$ supported on $S' \circ S$ (cf. [Vys20, Prop. 3.15]). In that case, we say R **and** R' **compose cleanly**. It can be checked $R' \circ R$ defines an involutive structure, when R, R' are CA relations [Vys20, Thm. 3.18]. Hence, we have the following result

Theorem 5.4.13.
Let $R : E_1 \dashrightarrow E_2$ and $R' : E_2 \dashrightarrow E_3$ be Courant algebroid relations that compose cleanly. Then, the composition is a Courant algebroid relation $R' \circ R : E_1 \dashrightarrow E_3$ supported on $S' \circ S$.

5.4.3 Poisson-Lie T-duality as Courant Relation

At this point we are in the position to describe Poisson-Lie T-duality as a Courant algebroid relation. Moreover, we will see Poisson-Lie T-duality is a special kind of CA relation, namely a *generalized isometry*. To do this, we need a reference Courant algebroid, that plays the role of the Drinfel'd double, from which the dual Courant algebroids can be obtained. They are obtained via the reduction of Courant algebroids procedure from [BCG07]. We will restrict ourselves to exact Courant algebroids in this section, however some of the statements hold in more generality. For more details we refer to [Vys20, Ch. 4], which is also our main reference.

Let $\varpi : P \to B$ be a principal D-bundle, for some connected Lie group D.

Definition 5.4.14.
*Let $E \to P$ be an exact Courant algebroid. We say E is a D-***equivariant Courant algebroid** *if there is a linear map $\mathfrak{R} : \mathfrak{d} \to \Gamma(E)$ satisfying*

i) $\rho \circ \mathfrak{R} = \mathfrak{a}$, *where $\mathfrak{a} : \mathfrak{d} \to \mathfrak{X}(P)$ denotes the infinitesimal action corresponding to the right action of D on P*

ii) $\mathfrak{R}([X, Y]_{\mathfrak{d}}) = [\mathfrak{R}(X), \mathfrak{R}(Y)]$

$$\begin{tikzcd}
 & & E \arrow[d] \arrow[ddll, dashed] \arrow[ddrr, dashed] & & \\
 & & P \arrow[ddll] \arrow[ddrr] & & \\
E'_1 \arrow[d] \arrow[rrrr, dashed] \arrow[ddrr, dashed] & & R_{G,\widetilde{G}} & & E'_2 \arrow[d] \arrow[ddll, dashed] \\
M_1 \arrow[ddrr] & & & & M_2 \arrow[ddll] \\
 & & E' \arrow[d] & & \\
 & & B & &
\end{tikzcd}$$

Figure 5.2: Schematic depiction of Poisson-Lie T-duality seen as a Courant algebroid relation $R_{G,\widetilde{G}}$. The two upper dashed arrows correspond to reduction of Courant algebroids, which defines a CA morphism (cf. [Vys20, Prop. 4.21]).

iii) The induced Lie algebra action on E defined by $X \cdot s := [\mathfrak{R}(X), s]$ *integrates to a Lie group action* $G \circlearrowleft E$*, making E into a G-equivariant bundle.*

Given a D-equivariant CA E over P, we may view $\mathfrak{R}$ as a map $P \times \mathfrak{d} \to E$. Note, this map in injective by the first condition. It yields two G-invariant subbundles of E, namely $K := \mathfrak{R}(P \times \mathfrak{d})$ and its orthogonal complement $K^\perp$. It was shown in [BCG07] that the vector bundle

$$E' := \frac{K^\perp/D}{(K \cap K^\perp)/D} \tag{5.29}$$

inherits a Courant algebroid structure from E, making it into a CA over $P/D = B$. It is called the **reduced Courant algebroid**. Furthermore, E' is exact when K is isotropic.

For Poisson-Lie T-duality, we now assume E is a D-equivariant CA over P that admits a G-equivariant isotropic splitting $\sigma : TP \to E$. Furthermore, we assume the pairing on $\mathfrak{d}$ defined by

$$\langle X, Y \rangle_{\mathfrak{d}} = \langle \mathfrak{R}(X), \mathfrak{R}(Y) \rangle$$

to be ad-invariant, non-degenerate and of split-signature (cf. [Vys20, Ex. 4.31]). This makes $\mathfrak{d}$ into a quadratic Lie algebra. Note, the ad-invariance already follows from the properties of an equivariant CA. In that case we have $K \cap K^\perp = 0$, which simplifies E'. Moreover, let $G \subset D$ be a subgroup such that its Lie algebra $\mathfrak{g} \subset \mathfrak{d}$ is a Lagrangian subalgebra. By restricting the map $\mathfrak{R}$ to $\mathfrak{g}$, to obtain a map $\mathfrak{R}_1 : P \times \mathfrak{g} \to E$, we see that $(E, \mathfrak{R}_1)$ defines a G-equivariant CA. Consequently, we can consider the G-reduced Courant algebroid E'_1 over $M_1 := P/G$. Note, $K_1 := \mathfrak{R}_1(P \times \mathfrak{g})$ is isotropic as $\mathfrak{g}$ is Lagragian. Thus, E'_1 is an exact Courant algebroid. If $\varphi_1 : M_1 \to B$ denotes the canonical map, it is shown in [Vys20, Sec. 4.4] that there is bundle morphism $\Psi_G : E'_1 \to E'$, such that $R(G) := \mathrm{gr}(\Psi_G)$ is a Courant algebroid morphism from E'_1 to E' over φ_1.

Given a second subgroup $\widetilde{G} \subset D$ for which the Lie algebra $\mathfrak{g} \subset \mathfrak{d}$ is Lagrangian, the procedure can be repeated, resulting in a Courant algebroid morphism $R(\widetilde{G}) = \mathrm{gr}(\Psi_{\widetilde{G}}) : E'_2 \rightarrowtail E'$ over φ_2. Here, E'_2 denotes the $\widetilde{G}$-reduced exact CA over $M_2 := P/\widetilde{G}$. Furthermore, in [Vys20, Ex. 4.31] it is shown that $R(G)$ and $R(\widetilde{G})^T$ compose cleanly. Hence, the composition $R_{G,\widetilde{G}} := R(\widetilde{G}) \circ R(G)$ defines a Courant algebroid relation $R_{G,\widetilde{G}} : E'_1 \dashrightarrow E'_2$ supported on $\mathrm{gr}(\varphi_2)^T \circ \mathrm{gr}(\varphi_1)$, by Theorem 5.4.13. This relation is Poisson-Lie T-duality. Note, $\mathrm{gr}(\varphi_2)^T \circ \mathrm{gr}(\varphi_1)$ is precisely the fiber product $M_1 \times_B M_2$, or the *correspondence space* of [CG11]. The above discussion is schematically depicted in Figure 5.2.

Note, our setting of Section 5.2 fits in this framework. Given a Drinfel'd double D, its generalized tangent bundle $\mathbb{T}D$ is an exact D-equivariant Courant algebroid under right-multiplication. When we identify $\mathbb{T}D \cong D \times \mathfrak{d} \oplus \mathfrak{d}^*$, we have $\mathfrak{R} = \mathrm{id}$. Then, the reduction by D yields $E' = \mathfrak{d}$, seen as CA over a point. Furthermore, when restricting to the action of G, we find $K_1 = D \times \mathfrak{g}$. Note, $K_1^\perp = D \times \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^*$, as $\mathfrak{g}$ is isotropic. From this we see $K_1 \cap K_1^\perp = 0$. Consequently, $E_1' = D/G \times \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^* \cong \widetilde{G} \times \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}^* \cong \mathbb{T}\widetilde{G}$ as a CA over $\widetilde{G}$. Similarly, reducing with respect to $\widetilde{G}$, we find $E_2' = \mathbb{T}G$. The Courant algebroids $\mathbb{T}G$ and $\mathbb{T}\widetilde{G}$ are precisely the one associated to the Poisson-Lie T-dual sigma-models from Section 5.2.

By our discussion at the end of Section 5.4.1, to make the connection to sigma-models we need to incorporate generalized metrics. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be generalized metrics on Courant algebroids E_1 and E_2. Then, $\mathcal{G} := \mathcal{G}_1 \times \mathcal{G}_2$ defines a generalized metric on $E_1 \times E_2$, as can be easily verified. The CA relations that work nicely with generalized metrics are called *generalized isometries*.

Definition 5.4.15.
Let $R : E_1 \dashrightarrow E_2$ be a Courant algebroid relation and $\mathcal{G}$ as above. We call R a **generalized isometry with respect to $\mathcal{G}_1$ and $\mathcal{G}_2$** *if $\mathcal{G}(R) = R$.*

The Poisson-Lie T-duality as described above can be viewed as a generalized isometry. To see this, let $\mathcal{G}'$ be any generalized metric on E'. Then, there are unique generalized metrics $\mathcal{G}_1, \mathcal{G}_2$ on E_1' and E_2', respectively, making $R(G)$ and $R(\widetilde{G})$ into a generalized isometries [Vys20, Ex. 5.8]. From this, it is not difficult to see that $R(\widetilde{G})^T$ and the composition of generalized isometries define generalized isometries, with respect to the appropriate generalized metrics, as well [Vys20, Prop. 5.7]. Consequently, the Poisson-Lie T-duality Courant algebroid relation $R_{G,\widetilde{G}} : E_1' \dashrightarrow E_2'$ is a generalized isometry with respect to $\mathcal{G}_1$ and $\mathcal{G}_2$.

5.5 Explicit $\mathcal{E}$-models and Poisson-Lie T-duality

After our digression about Poisson-Lie T-duality in the context of generalized geometry, let us shift our focus again to the sigma-models introduced in Section 4.3. In [Vic15; SST15; HT15] a possible relation between λ-models and (bi)-Yang-Baxter models via Poisson-Lie T-duality was first observed. Vicedo [Vic15] focussed on non-compact targets, while [SST15; HT15] restricted themselves to the compact SU(2) case. In this section, we want to describe the relationship for a general compact target space G, following [Kli15; Kli16]. As the discussion for the bi-Yang-Baxter model is an extension of the Yang-Baxter case, we first focus on the latter. We will show that both λ-models and Yang-Baxter models can be obtained from $\mathcal{E}$-models, proving their Poisson-Lie T-dualizability. Furthermore, we will relate the Poisson-Lie T-dual of the Yang-Baxter model to the λ-model via an 'analytical continuation' [Kli15]. Finally, we generalize this to the bi-Yang-Baxter model, as is done in [Kli16].

Consider a simple compact Lie group G with Lie algebra $\mathfrak{g}$. Then, let $\mathfrak{d}_\epsilon$ be a one-parameter family of Lie algebras defined as a vector space as

$$\mathfrak{d}_\epsilon = \mathfrak{g} \oplus \mathfrak{g}.$$

The bracket $[\cdot,\cdot]_\epsilon$ on $\mathfrak{d}_\epsilon$ is given by

$$[(X_1, X_2), (Y_1, Y_2)]_\epsilon := ([X_1, Y_1] + \epsilon[X_2, Y_2], [X_1, Y_2] + [X_2, Y_1]),$$

where $[\cdot,\cdot]$ denotes the bracket on $\mathfrak{g}$. Finally, we equip $\mathfrak{d}_\epsilon$ with the ad-invariant non-degenerate pairing

$$\langle (X_1, X_2), (Y_1, Y_2) \rangle_\epsilon := (X_1, Y_2) + (X_2, Y_1), \tag{5.30}$$

where $(\cdot,\cdot)$ is minus the Killing form on $\mathfrak{g}$, so that (X,X) is positive definite. Note, $\mathfrak{g}$ can be viewed as a subalgebra of $\mathfrak{d}_\epsilon$ using the identification $\mathfrak{g} \cong \mathfrak{g} \oplus 0$. Then, from (5.30) we see $\mathfrak{g}$ is a Lagrangian subalgebra for every ϵ. Consequently, every $\mathfrak{d}_\epsilon$ defines a Drinfel'd double of $\mathfrak{g}$. To obtain the corresponding $\mathcal{E}$-model, we define

$$\mathcal{E} : \mathfrak{d}_\epsilon \to \mathfrak{d}_\epsilon, \quad \mathcal{E}(X,Y) := (Y,X).$$

Note, $\mathcal{E}$ is clear self-adjoint and idempotent. Furthermore, $\langle \cdot, \mathcal{E}\cdot\rangle_\epsilon$ is positive definite, as $\mathfrak{g}$ is a compact Lie algebra. Hence, we obtain a family of $\mathcal{E}$-models on D_ϵ, the integration of $\mathfrak{d}_\epsilon$ (cf. Section 5.3). Note, the generalized metric, i.e. $+1$-eigenspace of $\mathcal{E}$, is given by the diagonal subgroup $V_+ = \mathfrak{g}^\delta = \{(X,X) \mid X \in \mathfrak{g}\}$.

5.5.1 The Lambda-model

The first result of [Kli15] is that the λ-model can be obtained from the above $\mathcal{E}$-model for $\epsilon > 0$. We want to reproduce this result as it is an explicit example of the procedure in Section 5.3. Furthermore, we provide more in-depth computations than the original work and make the connection with Proposition 4.1.6. The precise statement is

Proposition 5.5.1.
The $\mathcal{E}$-model described above, for $\epsilon > 0$, can be identified with the λ-model (4.27) on G with

$$\lambda = \frac{1-\epsilon^{1/2}}{1+\epsilon^{1/2}}. \tag{5.31}$$

Proof. — First we identify $\mathfrak{d}_\epsilon$ with a different Drinfel'd double. Consider the double $\mathfrak{g}\oplus\mathfrak{g}$ from Proposition 4.1.6, i.e. with pairing

$$\langle (X_1,X_2),(Y_1,Y_2)\rangle = (X_1,Y_1) - (X_2,Y_2).$$

Then, the map

$$\Phi_\epsilon : \mathfrak{d}_\epsilon \to \mathfrak{g}\oplus\mathfrak{g}, \quad (X,Y) \mapsto (X+\epsilon^{1/2}Y, X-\epsilon^{1/2}Y)$$

defines a Lie algebra isomorphism. Furthermore, up to a factor, it preserves the pairing[6]

$$\langle \Phi_\epsilon(X_1,X_2), \Phi_\epsilon(Y_1,Y_2)\rangle = 2\epsilon^{1/2}\langle (X_1,X_2),(Y_1,Y_2)\rangle_\epsilon.$$

Consequently, we can transport the data to the double $\mathfrak{g}\oplus\mathfrak{g}$. Firstly, the Lagrangian subalgebra $\mathfrak{g}\oplus 0$ is taken to $\Phi_\epsilon(\mathfrak{g}\oplus 0) = \mathfrak{g}^\delta$. Furthermore, corresponding generalized metric is given by

$$W_+ := \Phi_\epsilon(V_+) = \{X + \epsilon^{1/2}X \mid X \in \mathfrak{g}\}.$$

As expected, it can be verified that W_+ is the $+1$-eigenspace of the transported operator

$$\mathcal{E}_\lambda : \mathfrak{g}\oplus\mathfrak{g} \to \mathfrak{g}\oplus\mathfrak{g}, \quad (X,Y) \mapsto \frac{1}{2}(\epsilon^{1/2}+\epsilon^{-1/2})(X,-Y) + \frac{1}{2}(\epsilon^{1/2}-\epsilon^{-1/2})(Y,-X)$$

defined by $\mathcal{E}_\lambda \circ \Phi_\epsilon = \Phi_\epsilon \circ \mathcal{E}$.

Following our discussion from Section 5.3, the even dimensional Lie group D corresponding to the double $\mathfrak{g}\oplus\mathfrak{g}$ is clearly given by $G \times G$. We saw that the diagonal subalgebra $\mathfrak{g}^\delta$ plays the role of the

[6] In view of Section 5.4, by rescaling Φ_ϵ appropriately, it defines a classical Courant algebroid isomorphism.

Lagrangian subalgebra and its integration is given by the diagonal subgroup $G^\delta \subset G \times G$. To compute the action (5.22) on the target D/G^δ, we need two components: a parametrization f of the coset space D/G^δ and the operator $\mathcal{P}_f(\mathcal{E}_\lambda)$. For the former, note two elements $(g_1, g_2), (h_1, h_2) \in D$ define the same class in D/G^δ if there is $\xi \in G$ such that $(g_1, g_2) = (\xi h_1, \xi h_2)$. Consequently, every class in D/G^δ has a representative of the form $f = (g, e)$. This is the parametrization we choose and from this we also see $D/G^\delta \cong G$.

For the linear projector $\mathcal{P}_{(g,e)}(\mathcal{E}_\lambda)$ we can write $\mathcal{P}_{(g,e)}(\mathcal{E}_\lambda) = (\mathcal{P}_1, \mathcal{P}_2)$, where $\mathcal{P}_i : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$. Its first defining property states (cf. (5.23))

$$\operatorname{im} \mathcal{P}_{(g,e)}(\mathcal{E}_\lambda)(\mathfrak{g} \oplus \mathfrak{g}) = \mathfrak{g}^\delta$$

and thus $\mathcal{P}_1 = \mathcal{P}_2$. In general, we can write $\mathcal{P}_1(X, Y) = AX + BY$, for $A, B : \mathfrak{g} \to \mathfrak{g}$ linear. The second defining property tells us

$$\begin{aligned} 0 =& \mathcal{P}_1 \circ \left(\mathrm{id} + \mathrm{Ad}_{(g^{-1},e)} \mathcal{E}_\lambda \mathrm{Ad}_{(g,e)}\right)(X, X) \\ =& \Big(A + \frac{1}{2}(\epsilon^{1/2} + \epsilon^{-1/2})A + \frac{1}{2}(\epsilon^{1/2} - \epsilon^{-1/2})A \circ \mathrm{Ad}_{g^{-1}} + \\ & B - \frac{1}{2}(\epsilon^{1/2} + \epsilon^{-1/2})B - \frac{1}{2}(\epsilon^{1/2} - \epsilon^{-1/2})B \circ \mathrm{Ad}_g\Big)X. \end{aligned}$$

As this holds for any $X \in \mathfrak{g}$, we find

$$A + B + \frac{1}{2}(\epsilon^{1/2} + \epsilon^{-1/2})(A - B) + \frac{1}{2}(\epsilon^{1/2} - \epsilon^{-1/2})(A \circ \mathrm{Ad}_{g^{-1}} - B \circ \mathrm{Ad}_g) = 0. \tag{5.32}$$

Using (5.31), we can rewrite (5.32) as

$$\frac{2\left(\lambda B(\lambda - \mathrm{Ad}_g) + A(\lambda \mathrm{Ad}_{g^{-1}} - 1)\right)}{\lambda^2 - 1} = 0. \tag{5.33}$$

Hence, the numerator must vanish. Furthermore, as $\mathcal{P}_1$ is a projector, it satisfies $\mathcal{P}_1^2 = \mathcal{P}_1$. This implies $A + B = \mathrm{id}$, if we assume A and B to be invertible. Combining this with (5.33) yields

$$A = \frac{\lambda \mathrm{Ad}_g}{\lambda \mathrm{Ad}_g - 1} = \frac{\lambda}{\lambda - \mathrm{Ad}_{g^{-1}}}, \qquad B = \frac{1}{1 - \lambda \mathrm{Ad}_g}.$$

Note, $f^{-1}\partial_\pm f = (g^{-1}\partial_\pm g, 0)$. Then, the first term in (5.22) becomes

$$\begin{aligned} S_{\mathrm{WZW}, G\times G}[f] &= \frac{1}{2}\int_\Sigma d^2\sigma \,\langle (g^{-1}\partial_+ g, 0), (g^{-1}\partial_- g, 0)\rangle + \frac{1}{12}\int_B \langle (g^{-1}dg, 0), [(g^{-1}dg, 0), (g^{-1}dg, 0)]\rangle \\ &= \frac{1}{2}\int_\Sigma (g^{-1}\partial_+ g, g^{-1}\partial_- g) + \frac{1}{12}\int_B (g^{-1}dg, [g^{-1}dg, g^{-1}dg]) \\ &= S_{\mathrm{WZW}}[g]. \end{aligned}$$

Furthermore,

$$\begin{aligned} I &= -\int_\Sigma d^2\sigma \,\langle \mathcal{P}_f(\mathcal{E}_\lambda) f^{-1}\partial_+ f, f^{-1}\partial_- f\rangle \\ &= -\int_\Sigma d^2\sigma \,\langle (\frac{\lambda \mathrm{Ad}_g}{\lambda \mathrm{Ad}_g - 1} g^{-1}\partial_+ g, \frac{\lambda \mathrm{Ad}_g}{\lambda \mathrm{Ad}_g - 1} g^{-1}\partial_+ g), (g^{-1}\partial_- g, 0)\rangle \\ &= \int_\Sigma d^2\sigma \left(\frac{\lambda \mathrm{Ad}_g}{1 - \lambda \mathrm{Ad}_g} g^{-1}\partial_+ g, g^{-1}\partial_- g\right). \end{aligned}$$

Hence, (5.22) precisely becomes the action (4.27) of the λ-model, up to an overall factor of k/π. ■

5.5.2 The Yang-Baxter Model

Now, one could wonder what happens for $\epsilon < 0$. It turns out, in that case the $\mathcal{E}$-model can be identified with the Yang-Baxter model, as was first shown in [Kli02]. However, a different double is considered: the complexification $G_{\mathbb{C}}$. A key property of the complexification of a compact group is that it admits a specific decomposition:

Theorem 5.5.2 (Iwasawa decomposition).
Let $G_{\mathbb{C}}$ be the complexification of a compact Lie group G. Then, its Lie algebra decomposes as

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Here, $\mathfrak{a}$ is abelian, $\mathfrak{n}$ nilpotent and $\mathfrak{a} \oplus \mathfrak{n}$ solvable Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Furthermore, if A and N denote integrations of $\mathfrak{a}$ and $\mathfrak{n}$, respectively, the multiplication map $G \times A \times N \to G_{\mathbb{C}}, (g, a, n) \mapsto gan$ is a diffeomorphism.

The statement of the theorem is more general, as is discussed in [Kna13, Sec. 6.4]. In particular, if $G_{\mathbb{C}}$ is embedded in $\mathrm{GL}(n, \mathbb{C})$, then G is contained in $U(n)$, N consists of upper triangular matrices with ones on the diagonal and A is the group diagonal matrices with positive real entries [Bum13, Ch. 26].

Proposition 5.5.3.
For $\epsilon < 0$, the above $\mathcal{E}$-model can be identified with the Yang-Baxter model for $\eta = \sqrt{|\epsilon|}$.

Proof. — As hinted before, we consider $\mathfrak{g}_{\mathbb{C}}$. On the complexification, we have the following ad-invariant non-degenerate pairing (cf. Proposition 4.1.5)

$$\langle Z_1, Z_2 \rangle := -i(Z_1, Z_2) + i\overline{(Z_1, Z_2)}.$$

Here, $(\cdot, \cdot)$ is $\mathbb{C}$-linearly extended to $\mathfrak{g}_{\mathbb{C}}$. Consider the map

$$\Psi_\epsilon : \mathfrak{d}_\epsilon \to \mathfrak{g}_{\mathbb{C}}, \quad (X, Y) \mapsto X + i\eta Y.$$

Here we defined $\eta := \sqrt{|\epsilon|}$. By a straightforward computation, one can show Ψ_ϵ is a Lie algebra isomorphism satisfying

$$\langle \Psi_\epsilon(X_1, X_2), \Psi_\epsilon(Y_1, Y_2) \rangle = 2\eta \langle (X_1, X_2), (Y_1, Y_2) \rangle_\epsilon.$$

Under this mapping, the Lagrangian subalgebra $\mathfrak{g} \oplus 0$ is mapped to the real elements of $\mathfrak{g}_{\mathbb{C}}$, i.e. $\mathfrak{g} \subset \mathfrak{g} \oplus i\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$. Furthermore, the generalized metric is taken to

$$W_+ = \Psi_\epsilon(V_+) = \{X + i\eta X \mid X \in \mathfrak{g}\}.$$

It corresponds to the $+1$-eigenspace of the operator $\mathcal{E}_\eta = \Psi_\epsilon \circ \mathcal{E} \circ \Psi_\epsilon^{-1}$, which is explicitly given by[7]

$$\mathcal{E}_\eta(Z) = \frac{i}{2}(\eta - \eta^{-1})Z + \frac{i}{2}(\eta + \eta^{-1})\overline{Z}. \tag{5.34}$$

The corresponding Drinfel'd double is clearly given by $G_{\mathbb{C}}$. Using the Iwasawa decomposition $G_{\mathbb{C}} \cong GAN$, we obtain a sigma-model with target G, by reducing the $\mathcal{E}$-model on $\widetilde{G} = AN$. Its Lie algebra

[7] In [Kli15] there is a minus sign in front of the second term. We suspect this is a typo.

$\tilde{\mathfrak{g}} = \mathfrak{a} \oplus \mathfrak{n}$ is Lagrangian. To see this, let us assume $G_{\mathbb{C}}$ is embedded[8] in $\mathrm{GL}(n, \mathbb{C})$. Then, $\mathfrak{a}$ consists of *real* diagonal matrices and $\mathfrak{n}$ is given by strictly upper triangular matrices. Furthermore, $(\cdot, \cdot)$ is proportional to taking the trace of the matrix product. From this it follows that $\tilde{\mathfrak{g}}$ is Lagrangian. Moreover, the coset space $G_{\mathbb{C}}/\widetilde{G}$ can be identified with G. So, we may choose the parametrization $f = g$ for $g \in G$.

Finally, we need to the projector $\mathcal{P}_g(\mathcal{E}_\eta)$. For this, we note that $\mathcal{E}_\eta$ commutes with Ad_g. Consequently, $\mathcal{P}_g(\mathcal{E}_\eta)$ is independent of g. Furthermore, $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}, \tilde{\mathfrak{g}})$ forms a Manin triple. Hence, by Proposition 4.1.5 there exists a non-split skew-symmetric R-matrix such that $\mathfrak{g}_R$ is identified with $\tilde{\mathfrak{g}}$ via the map $R - i$. It turns out, the R-matrix is precisely the Drinfel'd-Jimbo solution from Section 4.3.3, as is discussed in [Kli09]. Using the fact that every element of $\tilde{\mathfrak{g}}$ is of the form $(R - i)X$ for some[9] $X \in \mathfrak{g}$, we find[10]

$$\mathcal{P}_g(\mathcal{E}_\eta)Z = \frac{1}{2}\frac{R-i}{1+\eta R}\Big((i+\eta)Z - (i-\eta)\overline{Z}\Big).$$

It can be verified this defines a projection operator satisfying the defining properties (5.23).

Finally, we compute the action (5.22). For this, note the $S_{\mathrm{WZW},G_{\mathbb{C}}}$ term vanishes, since $\mathfrak{g}$ is isotropic. Hence, we see

$$\begin{aligned} S_{\mathcal{E}_\eta}[f] &= -\int_\Sigma d^2\sigma \, \langle \mathcal{P}_g(\mathcal{E}_\eta) g^{-1}\partial_+ g, g^{-1}\partial_- g\rangle \\ &= -\eta \int_\Sigma d^2\sigma \, \langle \frac{R-i}{1+\eta R} g^{-1}\partial_+ g, g^{-1}\partial_- g\rangle \\ &= 2\eta \int_\Sigma d^2\sigma \, \left(g^{-1}\partial_+ g, \frac{1}{1-\eta R} g^{-1}\partial_- g\right). \end{aligned} \tag{5.35}$$

This is precisely the action S_η of the Yang-Baxter model (cf. (4.30) for $\zeta = 0$). ■

Let us make some comments about the proof above. Firstly, the R-matrix appearing in the final action (5.35) is not an arbitrary skew-symmetric solution to the non-split mCYBE, it is the Drinfel'd-Jimbo solution. The reasoning generalizes to R-matrices for which $G_{\mathbb{C}} = GG_R$ holds globally. However, the relationship between the λ-model and the Poisson-Lie T-dual of the Yang-Baxter model relies on the Iwasawa decomposition, i.e. the Drinfel'd-Jimbo solution. This is interesting as the Drinfel'd-Jimbo solution was also necessary to the Weil operator solution in Section 4.3.3. Consequently, if Poisson-Lie symmetry is related to the Hodge theoretic solutions, our initial idea of altering the R-matrix (cf. Section 4.3.3) might not be the most natural operation.

5.5.3 Poisson-Lie T-dual of (bi)-Yang-Baxter Model

From the proof of Proposition 5.5.3 we can find a Poisson-Lie T-dual of the Yang-Baxter model. We consider the same $\mathcal{E}$-model, however we choose $\widetilde{G} = G$. Consequently, we obtain a sigma-model with target $G_{\mathbb{C}}/G \cong AN$. Therefore, we can parametrize the coset $G_{\mathbb{C}}/G$ by $f = b \in AN$, as before. Moreover, we have $S_{\mathrm{WZW},G_{\mathbb{C}}}(b) = 0$, since AN is isotropic. Then, analogous to (5.35) we find the dual action has the following form [Kli15]

$$\tilde{S}_\eta[b] = \frac{1}{2}\int_\Sigma d^2\sigma \, \langle \partial_+ b b^{-1}, \widetilde{O}^{-1}(b)\partial_- b b^{-1}\rangle.$$

[8] It is true in general, but for our purposes this setting suffices.
[9] To be precise, for some $X \in \mathfrak{g}_R$. However, recall the underlying vector space of $\mathfrak{g}_R$ is just $\mathfrak{g}$.
[10] Again, there is a sign difference with the expression in [Kli15]. This is the same sign difference as before.

An explicit expression for $\widetilde{O}(b)$ can be found in [Kli02, Eq. (40)]. Yet, it dependence on b is rather complicated. Luckily, there is a different parametrization of the coset $G_{\mathbb{C}}/G$ that is more workable. Using this parametrization, we will show $\tilde{S}_\eta$ can be related to the λ-model. The parametrization is based on the **polar decomposition**: any element of $G_{\mathbb{C}}$ can uniquely written as the product of a positive definite Hermitian element and a unitary one. For this, we assume $G_{\mathbb{C}}$ to be embedded in $\mathrm{GL}(n,\mathbb{C})$. In particular, there is a diffeomorphism between AN and the vector space P of positive definite Hermitian elements given by

$$\Upsilon : AN \to P, \quad b \mapsto \sqrt{bb^\dagger}.$$

Note, $b^\dagger$ makes sense, as we have embedded the group in $\mathrm{GL}(n,\mathbb{C})$. Then, the main result is [Kli15]

Theorem 5.5.4.
The Poisson-Lie T-dual of the Yang-Baxter model and the λ-model are related via

$$\tilde{S}_\eta[b] = -iS_\lambda[bb^\dagger], \quad \lambda = \frac{1-i\eta}{1+i\eta}. \tag{5.36}$$

Proof. — We consider the same operator $\mathcal{E}_\eta$ as in the proof of Proposition 5.5.3 and use $f = \Upsilon(b)$ as the parametrization of the coset $G_{\mathbb{C}}/G$. The corresponding projector is stated in [Kli15]:

$$\begin{aligned}\mathcal{P}_{\Upsilon(b)}(\mathcal{E}_\eta)Z &= \left(\eta - i + (\eta+i)\operatorname{Ad}_{bb^\dagger}\right)^{-1}\left((\eta+i)\operatorname{Ad}_{bb^\dagger} Z - (\eta-i)Z^\dagger\right)\\ &= \left(\lambda \operatorname{Ad}_{bb^\dagger} - 1\right)^{-1}\left(\lambda \operatorname{Ad}_{bb^\dagger} Z + Z^\dagger\right),\end{aligned}$$

where we used the definition of λ in (5.36). Using this, the corresponding action (5.22) becomes

$$\begin{aligned}S_{\mathcal{E}_\eta}[\Upsilon(b)] &= -2iS_{\mathrm{WZW}}[\Upsilon(b)] +\\ &\quad 2i\int_\Sigma d^2\sigma\left(\frac{i+(i+\eta)\operatorname{Ad}_{\Upsilon(b)}}{(i+\eta)\operatorname{Ad}_{\Upsilon(b)} - (i-\eta)\operatorname{Ad}_{\Upsilon(b)^{-1}}}\Upsilon(b)^{-1}\partial_+\Upsilon(b), \Upsilon(b)^{-1}\partial_-\Upsilon(b)\right).\end{aligned}$$

Here, $S_{\mathrm{WZW}}(\Upsilon(b))$ is with respect to $(\cdot,\cdot)$, not $\langle\cdot,\cdot\rangle$. By applying the Polyakov-Wiegmann formula [PW83]

$$S_{\mathrm{WZW}}[bb^\dagger] = S_{\mathrm{WZW}}[\Upsilon(b)^2] = 2S_{\mathrm{WZW}}[\Upsilon(b)] + \int_\Sigma d^2\sigma\left(\operatorname{Ad}_{\Upsilon(b)}\Upsilon(b)^{-1}\partial_+\Upsilon(b), \Upsilon(b)^{-1}\partial_-\Upsilon(g)\right)$$

and the identity

$$(bb^\dagger)^{-1}\partial_\pm(bb^\dagger) = \operatorname{Ad}_{\Upsilon(b)^{-1}}\left(\Upsilon(b)^{-1}\partial_\pm\Upsilon(b)\right) + \Upsilon(b)^{-1}\partial_\pm\Upsilon(b),$$

we can rewrite $S_{\mathcal{E}_\eta}$ as

$$\begin{aligned}S_{\mathcal{E}_\eta}[\Upsilon(b)] &= -iS_{\mathrm{WZW}}[b] - i\int_\Sigma d^2\sigma\left(\frac{\lambda\operatorname{Ad}_{bb^\dagger}}{1-\lambda\operatorname{Ad}_{bb^\dagger}}(bb^\dagger)^{-1}\partial_+(bb^\dagger), (bb^\dagger)^{-1}\partial_-(bb^\dagger)\right)\\ &= -iS_\lambda[bb^\dagger].\end{aligned}$$

Finally, note that $\tilde{S}_\eta[b] = S_{\mathcal{E}_\eta}[\Upsilon(b)]$, as Υ is just a change of coordinates. That proves the statement. ■

The statement of Theorem 5.5.4 is often phrased as: the λ-model is obtained from the Poisson-Lie T-dual of the Yang-Baxter model via *analytical continuation* [SST15; Kli15; Kli16]. Indeed, elements $g \in G$ and $bb^\dagger \in P$ can be written as

$$g = hth^{-1}, \quad b = hah^{-1},$$

for $h \in G, t \in T$ and $a \in A$. Here, T denotes a maximal torus in G and A is the one from the Iwasawa decomposition. Their Lie algebras are related via $\mathfrak{a} = i\mathfrak{t}$. Consequently, replacing g by $bb^\dagger$ in the λ-model (4.27) can be viewed as an analytical continuation.

Interestingly, the value for λ for which Theorem 5.5.4 holds satisfies $|\lambda| = 1$. Recall, this was precisely the condition we needed in Section 4.3.2 to obtain Hodge theoretic solutions to the λ-model. Furthermore, as mentioned before for $G = \mathrm{SU}(2)$, the bi-Yang-Baxter model on the critical line can be viewed as a Yang-Baxter model on S^3 viewed as the coset space $\mathrm{SO}(4)/\mathrm{SO}(3)$. Then, Theorem 5.5.4 suggests there is a way to map the Hodge theoretic solutions of the latter to solutions of the λ-model. It would be interesting to compare these to the full Weil operator solution in the λ-model. This is one way of using Poisson-Lie T-duality to study the connection between Hodge theory and integrable systems.

Furthermore, [Kli16] shows the bi-Yang-Baxter on a simple compact target G can be obtained from an $\mathcal{E}$-model, as well. Moreover, it is a straightforward generalization of the $\mathcal{E}$-model described in Proposition 5.5.3. Again, the Drinfel'd double is $G_{\mathbb{C}}$, while the operator (5.34) is deformed as follows

$$\mathcal{E}_{\eta,\zeta} Z = -Z + \frac{1 + i\eta + \zeta R}{2i\eta}\Big((1 + i\eta - \zeta R)Z + (1 - i\eta - \zeta R)\overline{Z}\Big).$$

Here, R denotes the non-split Drinfel'd-Jimbo solution, as before. In [Kli16], the steps of Proposition 5.5.3 are repeated and the bi-Yang-Baxter action (4.30) is obtained. Furthermore, the reasoning of Theorem 5.5.4 can be extended, relating the Poisson-Lie T-dual of the bi-Yang-Baxter model $\tilde{S}_{\eta,\zeta}[b]$ to the so-called generalized λ-model, after an analytical continuation. The generalized λ-model was introduced in [SST15] and is given by

$$S_{\text{gen. }\lambda}[g] = S_{\mathrm{WZW}}[g] + \int_\Sigma d^2\sigma \left(\left(\frac{1+\alpha+\rho R}{1-\alpha+\rho R} - \mathrm{Ad}_g\right)^{-1} \mathrm{Ad}_g(g^{-1}\partial_+ g), g^{-1}\partial_- g\right).$$

Here we used the conventions of [Kli16]. Furthermore, in [SST15] the generalized λ-model was shown to be weakly integrable and for $\rho = 0$, we obtain the λ-model when identifying

$$\lambda = \frac{1-\alpha}{1+\alpha}.$$

Analogous to Theorem 5.5.4, the precise statement of the relationship is

$$\tilde{S}_{\eta,\zeta}[b] = -iS_{\text{gen. }\lambda}[bb^\dagger],$$

with $\alpha = i\eta$ and $\rho = \zeta$. This produces a second option to map the Weil operator solution to other integrable models. At this point it is unknown whether the generalized λ-model admits Hodge theoretic solutions. This would be an interesting starting point for future work.

5.5.4 Road map for Poisson-Lie T-duality

We saw that performing the duality followed by an analytical continuation is a promising strategy to investigate the connection between Hodge theory and integrability. However, the procedure might be obscured at this stage. Therefore, we present a road map clearly indicating the required steps:

1. Consider a Drinfel'd double $D = G\widetilde{G}$ and a sigma-model $S[g]$ with G-Poisson-Lie symmetry with respect to $\widetilde{G}$.
2. Find a solution g of $S[g]$.
3. Compute the corresponding Poisson-Lie current J, using e.g. (5.12).
4. Evaluate the on-shell current, i.e. the one corresponding to the solution g.
5. Find the dual element $\tilde{g}$, using $J = -d\tilde{g}\tilde{g}$ or
$$\tilde{g} = P\exp\left(-\int_\gamma J\right).$$
Here J denotes the *on-shell* current from the previous step.
6. Find the 'flipping' diffeomorphism $\varphi : D \to D$ that maps $G\widetilde{G}$ to $\widetilde{G}G$.
7. Compute $l = g\tilde{g}$ and find the other decomposition $l = \widetilde{h}h \in \widetilde{G}G$ using φ, where g and $\tilde{g}$ denote the solution and dual element from step 2 and 5, respectively. Then, the field $\widetilde{h} : \Sigma \to \widetilde{G}$ is a solution to the dual sigma-model.
8. In the case of the (bi-)Yang-Baxter model and $D = G_{\mathbb{C}} = GAN$, by the Iwasawa decomposition, denote $b = \widetilde{h}$ and compute $bb^\dagger$. In view of Theorem 5.5.4 and the discussion afterwards, $bb^\dagger$ is a solution of the (generalized) λ-model.

For a general Drinfel'd double D, the above recipe is hard to perform explicitly, especially steps 5 and 6. Furthermore, Theorem 5.5.4 can only be applied for *compact* targets D, while the classifying space for a weight three variation of Hodge structure is non-compact (cf. Proposition 3.2.5).

However, in Section 4.3.3 we saw that, for the torus, the classifying space $\mathrm{SL}(2,\mathbb{C})$ could be mapped to $\mathrm{SU}(2)$ via the Cayley transform and an analytical continuation. Therefore, the $\mathrm{SU}(2)$ bi-Yang-Baxter model should first be examined. In that case, a Drinfel'd double is $\mathrm{SL}(2,\mathbb{C})$ with $G = \mathrm{SU}(2)$ and $\widetilde{G} = AN$, by the Iwasawa decomposition. In Example 5.2.6, we argued that the bi-Yang-Baxter model has $\mathrm{SU}(2)$-Poisson-Lie symmetry with respect to AN. This concludes step 1. In step 2, we pick the $\mathrm{SL}(2)$-orbit approximation $C_{\mathrm{SL}(2)}$ of the Weil operator. The Poisson-Lie current of step 3 was computed in Example 5.2.6. Then on-shell current, corresponding to the Weil operator, is computed in [GM23]. Yet, its explicit form in the $\mathrm{SU}(2)$ case still has to be extracted. Step 5 is the hardest and still open step for the $\mathrm{SU}(2)$ bi-Yang-Baxter model. For $\mathrm{SU}(2)$, steps 6 and 7 can be done for the Iwasawa decomposition. Indeed, then $G_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}) = \mathrm{SU}(2)AN$, where we use the parametrization

$$\mathrm{SU}(2) = \left\{\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \middle|\, \alpha,\beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1\right\},\quad A = \left\{\begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \middle|\, r \in \mathbb{R}\right\},\quad N = \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \middle|\, r \in \mathbb{R}\right\}.$$

Then, given a general element of $\mathrm{SL}(2,\mathbb{C})$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the elements

$$\widetilde{h} = \begin{pmatrix} \tilde{r} & 0 \\ 0 & \frac{1}{\tilde{r}} \end{pmatrix}\begin{pmatrix} 1 & \tilde{z} \\ 0 & 1 \end{pmatrix} \in AN, \quad h = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ -\overline{\tilde{\beta}} & \overline{\tilde{\alpha}} \end{pmatrix} \in \mathrm{SU}(2)$$

such that $M = \widetilde{h}h$ are obtained by setting

$$\tilde{r} = \frac{1}{\sqrt{|c|^2 + |d|^2}}, \quad \widetilde{\alpha} = \tilde{r}\overline{d}, \quad \widetilde{\beta} = -\tilde{r}\overline{c}, \quad \tilde{z} = \frac{a - \tilde{r}^2\overline{d}}{\tilde{r}^2 c}. \tag{5.37}$$

Now, by first applying the multiplication $(g, a, n) \mapsto gan$ and then using the relations (5.37), we find the element $\widetilde{h} \in AN$ from step 7 is defined by

$$\begin{aligned} \tilde{r} &= \frac{1}{\sqrt{r^2|\beta|^2 + r^2|z|^2|\beta|^2 + \frac{1}{r^2}|\alpha|^2 - \alpha\overline{\beta}z - \overline{\alpha}\beta\overline{z}}} \\ \tilde{z} &= -\frac{\alpha}{\overline{\beta}\tilde{r}^2} + \frac{\alpha - r^2\beta\overline{z}}{r\overline{\beta}}. \end{aligned} \tag{5.38}$$

These expression are quite involved, yet they might simplify in the case of the Weil operator. Further research is needed to verify this. Finally, having done all the previous steps, step 8 is straightforward.

To summarize, to apply the above road map to $C_{\mathrm{SL}(2)}$ for the SU(2) bi-Yang-Baxter model, the explicit form of the associated Poisson-Lie current should first be extracted from [GM23]. This should be manageable. Afterwards, the path-ordered exponential must be solved. It is expected this is subtle. If one manages to do so, the rest of the steps can be performed using (5.38) and the resulting solution can be studied. We leave these open problems for future work.

Discussion and Outlook

In this work we studied the connection between Hodge theory and integrable systems. In particular, we focused on the recently discovered Hodge theoretic solutions to the λ-model and the bi-Yang-Baxter model of [GM22; GM23]. We started with an overview of string theory and compactifications, highlighting the dependence on moduli and the Hodge structure. Afterwards, we discussed elliptic operator theory to prove the Hodge decomposition theorem, which provided the first example of a Hodge structure. Furthermore, we discussed deformations of complex manifolds and proved Kuranishi's deformation theorem (cf. Theorem 2.2.3). In particular, for Calabi-Yau manifolds we proved the classical Bogomolov-Tian-Todorov theorem without the typical power series argument and discussed its global geometry. In Chapter 3 we developed the theory of Hodge structures and its variations. The nilpotent and SL(2)-orbit theorem were of particular interest. Subsequently, we introduced the notion of integrability in both classical mechanics as well as field theory. We highlighted the importance of the r-matrix and Poisson-Lie symmetry. Finally, we reviewed Poisson-Lie T-duality from multiple perspectives and discussed how (generalized) λ-models and (bi)-Yang-Baxter models are related via this duality. In particular, we obtained explicit workable formulas to compute the Poisson-Lie current (cf. (5.12)). Furthermore, we identified the 'flipping' diffeomorphism in the case of SU(2). Moreover, throughout this work we have seen multiple similarities between Hodge theory, integrable systems and Poisson-Lie T-duality and opportunities to relate them.

Firstly, an interesting observation is that the relationship between the Poisson-Lie T-dual of the (bi)-Yang-Baxter model and the (generalized) λ-model relies on the choice of the non-split Drinfel'd-Jimbo solution as Yang-Baxter operator (cf. Theorem 5.5.4). Recall, this is precisely the R-matrix used in the bi-Yang-Baxter model to obtain the SL(2)-approximation of the Weil operator as a solution. This strengthens the idea that Poisson-Lie T-duality might play a role in the fact Hodge theoretic solutions can be found in these models. Furthermore, from this point of view, altering the R-matrix to obtain a nilpotent Weil operator solution might not be the most natural way to proceed. Moreover, the required deformation of the R-matrix must be non-trivial, as we established that a simple conjugation does not suffice. At this stage, it is unclear how objects from Hodge theory can be naturally used to achieve this deformation. In view of Propositions 4.1.5 and 4.1.6, one should look for a Lie bialgebra structure on $\mathfrak{g}_{\mathbb{R}}$ (cf. Section 3.2). Note, the dimension of a Lie bialgebra is even. Consequently, finding a bialgebra structure on $\mathfrak{g}_{\mathbb{R}}$ seems difficult, if possible at all, as its dimension depends on the weight of the Hodge structure.

Secondly, the resemblance of horizontality of the period map and field equations (5.6) on the Drinfel'd double. In particular, (3.25) is very reminiscent of (5.6). The reason behind this might be simple: they both describe lifting problems. The map $\widetilde{h}$ in (3.25) is a horizontal lift of $\tilde{g}$, while l is a lift of a solution g to the double. It is unclear at this point whether this is a coincidence or *the* reason the Hodge theoretic solutions are obtained. Interestingly, $\widetilde{h}$ is basically (the SL(2)-approximation of) the period map. Hence, it might be horizontality of the period map that makes the Weil operator into a solution. This seems even more plausible in view of (3.27). To check this, one should extract field equations for $\widetilde{h}$ from the equations of motion of $C_{\mathrm{SL}(2)}$. We leave this for further research.

The most natural next step would be to solve the path-ordered exponential and do step 5 of the road map in Section 5.5.3, in the case of SU(2). For this, the on-shell Poisson-Lie current can be extracted from [GM23]. Afterwards, the duality and analytical continuation can, in principle, be performed to $C_{\mathrm{SL}(2)}$. This would yield a solution to the generalized λ-model. It would be interesting to see whether another Weil operator or other Hodge theoretic objects can be recognized. Furthermore, it would be natural to look for Hodge theoretic solutions to the generalized λ-model.

A different approach would be to identify the SU(2) bi-Yang-Baxter model on the critical line with the Yang-Baxter model on SO(4)/SO(3) (cf. Section 4.3.3) and apply the above road map. This can then be compared to the Weil operator solutions of the λ-model from [GM22]. However, the fact the Yang-Baxter model is defined on a coset might be a problem regarding Poisson-Lie T-duality. For this, *dressing cosets* might be needed [KŠ96a; Kli22; Kli19].

Finally, the Hitchin system might be an interesting integrable system to study (see [BBT03, Sec. 7.11]). This is because it is closely related to so-called Higgs bundles, which in turn are closely related to Hodge theory [Bec19; DWS08; Pea00; Sim91]. Perhaps, this is the integrable system that solves our main question and reproduces variation of Hodge structures as solutions. This is for further research to decide.

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