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Masses of moduli in IIB flux compactifications

Master Thesis

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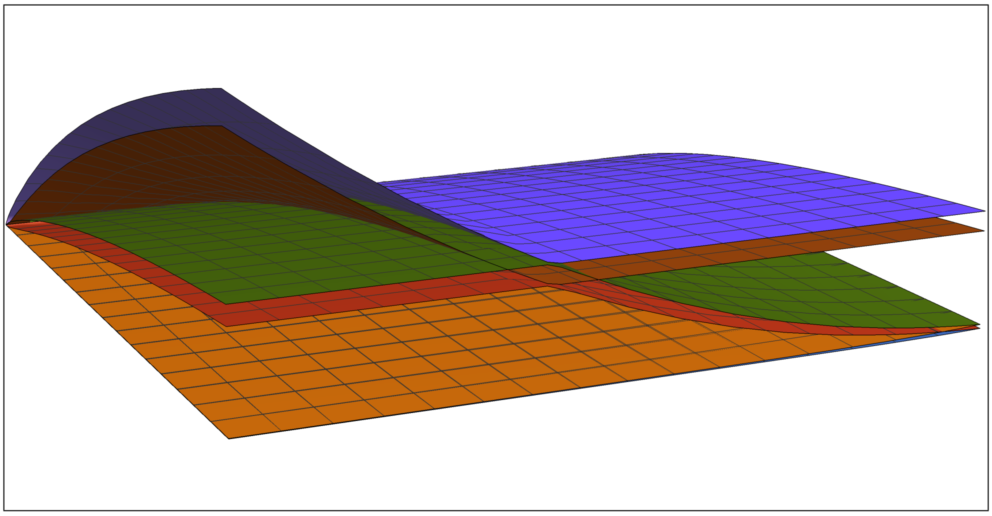
Theoretical Physics



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Abstract

In this thesis we calculate and discuss the mass spectrum of two examples of type IIB flux compactifications. Firstly in the introduction we discuss how string theory leads to higher dimensional spaces, how we can compactify the extra dimensions on an internal manifold creating extra massless modes (moduli) and how fluxes can generate a mass for the moduli. In the last part of the introduction we discuss our setting: four-dimensional $\mathcal{N} = 1$ supergravity.
In chapter 2 and 3 we calculate the masses of the moduli for the case with one complex structure modulus ($h_-^{2,1} = 1$), one complex structure modulus in the large-complex-structure limit and two complex structure moduli ($h_-^{2,1} = 2$). In chapter 4 we discuss certain aspects of the masses we found in chapter 2 and 3. For each case we describe when there can be degeneracies in the masses and we consider the masses in certain limits of moduli space. In chapter 5 we recap and discuss our results.
In this thesis we find the following features for the masses of the moduli in the cases we consider:

- For a general flux all moduli receive a mass and are stabilized when turning on these fluxes. Only for very specific cases the masses of one or a few of the moduli are zero.
- Even having degenerate masses seems to be the exception. For example for all the masses to be equal we need either h^0 or h^i to vanish such that $H = h^i \chi_i + \overline{h}^{\bar{j}} \overline{\chi}_{\bar{j}}$ or $H = h^0 \Omega + \overline{h^0 \Omega}$.
- When going to extremes in the parameters that determine the fluxes the masses approximate degenerate pairs and one pair of masses stays small while the other masses diverge.

We note that restricting the options for compactifications on the basis of the moduli masses in these examples is difficult. Possibly these results combined with an analysis of the stabilisation of the Kähler moduli using KKLT [1] would be able to provide more restrictions.

Contents

Chapter 1

Introduction

In this section we will discuss a brief overview of the basic concepts of string theory. We will mostly follow the book by Ralph Blumenhagen, Dieter Lüst and Stefan Theisen [2] chapters 1, 2, 3, 10, 14 and 17. Note that what is discussed in this thesis is far from a complete review. For a better understanding of these concepts I recommend reading [2].

1.1 String theory, the idea

String theory is a theory of quantum gravity. Therefore it is a candidate for a theory that unifies the standard model with a model for gravity and maybe gives us an understanding of a bigger range of phenomena, for example in regions of space with very strong gravity such as black holes or the beginning of the universe.

The initial premise of string theory is to not consider fundamental particles as a 0-dimensional, point like, object but instead as a one dimensional string. These strings can be either open like a guitar string or closed like a rubber band. The vibrational modes on these strings then provide the degrees of freedom of the theory.

An interesting feature of this model is that, depending on whether the degrees of freedom are bosonic or fermionic, self-consistency of the theory requires the space time that it lives in to be respectively 26- or 10-dimensional. This is not something we observe in the universe we live in. We only experience a 4-dimensional space-time. If string theory is to describe the universe we live in we need to explain why we only experience part of the fundamental dimensions that our universe is build out of.

The way to do this is to compactify the 26- or 10-dimensional manifold that is our space-time. This way it is possible to "roll-up" dimensions so small that movement in these directions is not noticeable compared to movement in the "big" dimensions. A bit like a 2-dimensional sheet of paper that is rolled up so tight that from a bit of a distance it effectively looks like a 1-dimensional line. More on the compactifications in section 1.8.

Once the space-time is compactified we can study the now 4-dimensional effective theory. How exactly the compactification is done has effects on the theory and the different states and their masses that the effective theory contains. By studying these states and masses we can try to narrow down what kinds of compactifications are candidates to describe a string theory that is self consistent and describes the universe we live in.

1.2 From points to strings

Classically we are used to describe the position of particles, using their coordinates X^i, depending on t in the following way:

$$t \rightarrowtail X^i(t). \tag{1.2.1}$$

These are functions depending on one variable giving us the value of the space-coordinates of the particle for each moment in time. That way the position of the particle for each moment in time is given.

Later with the introduction of the theory of relativity it turned out that a more natural way of describing the trajectories of particles is to incorporate time as a (zeroth) coordinate and to take not (necessarily) time t but more naturally the eigentime τ as the parameter for the trajectory of the particle through space-time. This resulted in

$$\tau \rightarrowtail X^\mu(\tau) \tag{1.2.2}$$

as the description of such a trajectory. This better reflects the reparameterization invariance of space-time and therefore more naturally describes phenomena related to gravity. But alongside the theory of relativity also quantum theory was developed into the very successful standard model which describes the fundamental interactions/forces excluding gravity. To have a complete theory describing all fundamental forces including gravity we would need to combine both theories somehow. This turns out to be quite challenging. String theory suggests a solution by changing the way we look at the trajectory of fundamental particles again.

We introduce the idea that the position of a fundamental particle is not point-like and 0-dimensional but rather like a 1-dimensional string. This means that it is not enough to parametrize the trajectory of the particle with a single parameter but we need a second "spacelike" parameter to describe not just where one point of the particle is at every moment in time, but to describe where all parts of this string are at each moment in time. This leads to the description

$$\sigma^\alpha = (\sigma, \tau) \rightarrowtail X^\mu(\sigma, \tau). \tag{1.2.3}$$

We can imagine these strings in two variants. First an open string where $\tau \in \mathbb{R}$ and $\sigma \in [0, l]$ with $l > 0$. This means that σ^α lives on a 2-dimensional sheet that is unbounded in one direction and is bounded in the other. This corresponds to the idea that the string has a finite length and 2 endpoints, like a guitar string.

The other option is a closed string where still $\tau \in \mathbb{R}$ but $\sigma \in S^1$ this corresponds to imposing conditions on $X^\mu(\sigma^\alpha)$ such that

$$X^\mu(\sigma + l, \tau) = X^\mu(\sigma, \tau). \tag{1.2.4}$$

This means that the "ends" of the string always have the same location and therefore that the string is closed and has become more of a loop like a rubber band. In that case σ^α lives on a cylinder that is unbounded in the τ direction but it is compact in the σ direction.

1.3 A dynamical theory

To make a dynamical theory out of these strings we need to write down an action for X^μ. The action we use is the Polyakov action. In this action T is a constant that can be interpreted as a tension of the string but is not relevant for the dynamics of X^μ in this case without interactions. $h_{\alpha\beta}$ is a new field that is introduced and it acts as a metric on the world-sheet Σ where σ^α lives, $h^{\alpha\beta}$ is its inverse and h is the determinant. $\eta_{\mu\nu}$ is the (Minkowski)-metric on the space-time where X^μ lives. This then leads to the action being:

$$S_p = -\frac{T}{2}\int_\Sigma d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{1.3.1}$$

The action can be simplified by introducing light-cone coordinates on Σ, denoting contraction with the light-cone metric by $\cdot$ and fixing the symmetries of the action to arrive at the form

$$S_p = 2T\int_\Sigma d^2\sigma \partial_+ X \cdot \partial_- X, \tag{1.3.2}$$

which is easier to work with.

Varying this action leads to the following solutions for the fields $X^\mu(\sigma^\alpha)$. First of all $X^\mu(\sigma^\alpha)$ splits in two parts: the left $(X^\mu_L(\sigma^+))$ and right $(X^\mu_R(\sigma^-))$ moving part. Making the solution:

$$X^\mu(\sigma^\alpha) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-). \tag{1.3.3}$$

These left $(X^\mu_L(\sigma^+))$ and right $(X^\mu_R(\sigma^-))$ moving parts of the solution should also obey certain boundary conditions. We separately consider the open and the closed string options.

1.3.1 Closed string

The closed string solutions need to satisfy

$$X^\mu(\sigma + l, \tau) = X^\mu(\sigma, \tau). \tag{1.3.4}$$

This leads to a solution that can be written in a oscillator expansion in the following way:

$$\begin{aligned} X^\mu(\sigma^\alpha) =& X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \text{ with} \\ X^\mu_R(\sigma^-) =& \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{l}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\backslash 0}\frac{1}{n}\alpha^\mu_n \mathrm{e}^{-\frac{2\pi}{l}in\sigma^-} \text{ and} \\ X^\mu_L(\sigma^+) =& \frac{1}{2}(x^\mu + c^\mu) + \frac{\pi\alpha'}{l}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\backslash 0}\frac{1}{n}\overline{\alpha}^\mu_n \mathrm{e}^{-\frac{2\pi}{l}in\sigma^+}. \end{aligned} \tag{1.3.5}$$

Here $\sigma^\pm = \tau \pm \sigma$ and c^μ are (in principle complex) parameters. To have $X^\mu \in \mathbb{R}$ we need $x^\mu, p^\mu \in \mathbb{R}$ and also $\alpha^\mu_{-n} = (\alpha^\mu_{-n})^*$ and $\overline{\alpha}^\mu_{-n} = (\overline{\alpha}^\mu_{-n})^*$ where * denotes complex conjugation. p^μ turns out to be the 4-momentum of the string and x^μ the centre of mass.

1.3.2 Open string

The open string solutions need to satisfy either of two options for the boundary conditions. The Dirichlet boundary conditions (D)

$$\delta X^\mu \mid_{\sigma=0,l} = 0 \tag{1.3.6}$$

or the Neumann boundary conditions (N)

$$\partial X^\mu \mid_{\sigma=0,l} = 0. \tag{1.3.7}$$

Both the right and the left end ($\sigma = 0, l$) of the solution can satisfy these conditions independently. This means that we can get both of them to fulfil the same conditions, (NN) or (DD), or they can fulfil different boundary conditions, (ND) or (DN). We list the oscillator expansions for all 4 options:

$$\begin{aligned} X^\mu(\sigma,\tau) =& \; x^\mu + \tfrac{2\pi\alpha'}{l}p^\mu\tau + i\sqrt{2\alpha'}\textstyle\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{1}{n}\alpha^\mu_n \mathrm{e}^{-\frac{\pi}{l}in\tau}\cos\left(\frac{n\pi\sigma}{l}\right) & \text{(NN)}, \\ X^\mu(\sigma,\tau) =& \; x^\mu_0 + \tfrac{1}{l}(x^\mu_1 - x^\mu_0)\sigma + \sqrt{2\alpha'}\textstyle\sum_{n\in\mathbb{Z}\backslash\{0\}}\frac{1}{n}\alpha^\mu_n \mathrm{e}^{-\frac{\pi}{l}in\tau}\sin\left(\frac{n\pi\sigma}{l}\right) & \text{(DD)}, \\ X^\mu(\sigma,\tau) =& \; x^\mu + i\sqrt{2\alpha'}\textstyle\sum_{r\in\mathbb{Z}+\frac{1}{2}}\frac{1}{r}\alpha^\mu_r \mathrm{e}^{-\frac{\pi}{l}ir\tau}\cos\left(\frac{r\pi\sigma}{l}\right) & \text{(ND)}, \\ X^\mu(\sigma,\tau) =& \; x^\mu + \sqrt{2\alpha'}\textstyle\sum_{r\in\mathbb{Z}+\frac{1}{2}}\frac{1}{r}\alpha^\mu_r \mathrm{e}^{-\frac{\pi}{l}ir\tau}\sin\left(\frac{r\pi\sigma}{l}\right) & \text{(DN)}. \end{aligned} \tag{1.3.8}$$

Note that in principle for each dimensional direction a different one of these options can be chosen. So for each μ we can pick the open string to have one of these 4 options.

In the rest of this chapter we will mostly consider closed strings to sketch the idea and again refer for more details on the open string to [2].

1.4 Gauge fixing Lorenz invariance

To make this theory into a theory of quantum gravity we need to quantize it but before we do so it is useful to extract the unphysical degrees of freedom such that we are left with just the physical ones. This will be the easiest way to get the excitation spectrum and the corresponding masses.

To fix the Lorenz invariance we first go to light-cone coordinates such that

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^1\right). \tag{1.4.1}$$

Using these coordinates we can gauge fix such that

$$X^{+}(\tau,\sigma) = \frac{2\pi\alpha'}{l}p^{+}\tau. \tag{1.4.2}$$

This means that all $\alpha_n^+ = 0$ and (due to gauge fixing the translational symmetry) $x^\mu = 0$. Now we remember that originally when we solved the equations of motion of the action (1.3.2) this action was a gauge fixed version of (1.3.1). We got rid of $h_{\alpha\beta}$ this way. But this means that we also still need to impose the gauge fixed equations of motion of $h_{\alpha\beta}$ on the solutions we find. In short when $\dot{X} = \partial_\tau X$ and $X' = \partial_\sigma X$, this means that

$$\left(\dot{X}^\mu \pm X'^\mu\right)^2 = 0. \tag{1.4.3}$$

Therefore we can also fix X^- in therms of the X^i where i runs from 2 to $d-1$ with d the dimension of the target-space where X^μ lives.

All of this results in the remaining dynamical variables being: p^+, p^-, p^i, α_n^i and $\overline{\alpha}_n^i$.

1.5 Going Quantum

To now promote this to a quantum theory we follow the usual procedure of promoting the degrees of freedom to operators and assigning the commutators. This results in the following:

$$\begin{aligned}[p^-,\ p^+] =& -i,\\ [p^i,\ p^j] =& i\delta^{ij},\\ [\alpha_m^i,\ \alpha_n^j] =& [\overline{\alpha}_m^i,\ \overline{\alpha}_n^j] = m\delta_{m+n,0}\delta^{ij}.\end{aligned} \tag{1.5.1}$$

Here $i,j = 2,\ldots,d-1$.

If we now let these operators act on a vacuum state we can build a whole range of states.

Using that $p^\mu p_\mu = -m^2$ we can find the masses of these states depending on the operators α^i_m and $\overline{\alpha}^j_n$ acting on these states.

To give it's expression we need to define the level operator N. So N is the "level" of the state. It is the sum over all the $-n$ of the α^i_n that act on the vacuum state $|0\rangle$ to make up the new state. For example $|0\rangle$ has level 0, $\alpha^i_{-1}\overline{\alpha}^j_{-1}|0\rangle$ has level 1 and both $\alpha^i_{-2}\overline{\alpha}^j_{-2}|0\rangle$ and $\alpha^i_{-1}\alpha^j_{-1}\overline{\alpha}^k_{-2}|0\rangle$ have level 2.
The mass operator turns out to be

$$\alpha' m^2 = 4\left(N - \frac{d-2}{24}\right). \tag{1.5.2}$$

In this way we have found the states and their masses in this theory.

1.6 More dimensions

Now we know what states are available in this theory we can see if they still obey Lorentz symmetry. After all we started with a Lorentz invariant theory but after going to Light cone gauge we went away form this symmetry being apparent by picking out the zeroth and first coordinate to be mixed. This means the Lorentz invariance is not apparent anymore but our theory should still be.

Let's look at the level 1 state, $\alpha^i_{-1}\overline{\alpha}^j_{-1}|0\rangle$. This is a 2 tensor with mass $\alpha' m^2 = 4\left(1 - \frac{d-2}{24}\right)$. If these are to obey Lorentz symmetry we need to be able to write them as a decomposition of irreducible representations of $SO(d-1)$ if it is a massive excitation or of $SO(d-2)$ if it is a massless excitation. We can write $\alpha^i_{-1}\overline{\alpha}^j_{-1}|0\rangle$ as

$$\alpha^i_{-1}\overline{\alpha}^j_{-1}|0\rangle = \left(\alpha^{(i}_{-1}\overline{\alpha}^{j)}_{-1} - \frac{1}{d-2}\delta^{ij}\alpha^i_{-1}\overline{\alpha}^i_{-1}\right)|0\rangle + \alpha^{[i}_{-1}\overline{\alpha}^{j]}_{-1}|0\rangle + \frac{1}{d-2}\delta^{ij}\alpha^i_{-1}\overline{\alpha}^i_{-1}|0\rangle. \tag{1.6.1}$$

As can be found in [2] equation 3.57.
Here the round brackets denote the symmetric part and the square brackets the anti-symmetric part.
This is a $d-2$ dimensional (i,j run from 2 to $d-1$) 2 tensor decomposed into a symmetric traceless part, a anti-symmetric part and a trace. These are the irreducible representations of $SO(d-2)$. Therefore we find that these excitations are massless and $\alpha' m^2 = 4\left(1 - \frac{d-2}{24}\right) \stackrel{!}{=} 0 \Rightarrow d = 26$.

This means that for this theory to be consistent we need the dimension of the space-time, where X^μ lives, to not be the familiar 4 but rather 26 in the case of the bosonic string. For the superstring, which also contains fermionic degrees of freedom but which we will not discuss in detail, it turns out that the dimension space-time needs to take is 10. This is not what we perceive in nature and we will need to discuss how these higher dimensional spaces can be consistent with space-time we live in.

1.7 Mass spectrum

Now we have found the dimension of the target space we can use this to find the mass spectrum of the bosonic closed string excitations. Plugging $d = 26$ into the mass formula we find that

$$\begin{aligned}\alpha' m^2 =& 4\left(N - \frac{26-2}{24}\right) \\ =& 4N - 4.\end{aligned} \tag{1.7.1}$$

The masses we find are:

- $|0\rangle$ with mass $\alpha' m^2 = -4$
- $\alpha^i_{-1}\overline{\alpha}^j_{-1}|0\rangle$ with mass $\alpha' m^2 = 0$,

together with a whole "tower" of (positive) massive states.

Note that there is one state with negative mass. This is the Tachyon. The tachyon is a feature of this specific theory. The supersymmetric theory that we will actually adopt in the end does not contain a tachyon. Then there are the states with zero mass. As we have seen these can be decomposed into a massless spin-2 particle, the graviton, a anti-symmetric tensor field and a massless scalar, the dilaton.

1.8 Compactification of manifolds

Now we have seen that for string theory to be self-consistent it needs a space-time of high dimension (10 or 26). To understand how we still perceive to be living in a 4-dimensional space time we need to consider a high dimensional manifold that is compactified in certain directions. This can be imagined as "rolling up" the space-time to "hide" certain dimensions. Next we will discuss an illustration of this idea.

1.8.1 Paper roll as an intuitive example

We can imagine a sheet stretching infinitely far in both the length and the width of the paper. This represents a 2-dimensional manifold. Anything that is bound to this sheet of paper can move in 2 independent directions. Now imagine deforming this sheet of paper by rolling it up in one direction creating a cylinder like in figure 1.1. This is still a 2-dimensional manifold and anything bound to this paper can travel along the cylinder as well as around it. Now if the sheet gets rolled tighter the diameter of the cylinder becomes smaller and so does the distance that something bound to this surface can travel around the cylinder before ending up at the same location.

Now imagine an ant that lives on this cylinder. The size of the ant is about 1 cm. In principle the ant can move both along the cylinder and around it. But if the sheet is rolled up so tight that the diameter of the cylinder is much smaller than the size of the ant, say 10^{-10}cm, than if the ant moves around the cylinder it doesn't notice any change in position because this change is so much smaller than anything the ant is used to deal with. The ant feels like the only direction it can travel in is along the cylinder. Now by rolling this sheet up tight enough it went from being perceived as the 2-dimensional space it is to being perceived as a one dimensional space by the observers that live in this space.

This is the idea of rolling up dimensions of a space to "hide" them. With this example we can imagine how compactifications of the high dimensional space we live in can make it so that we only perceive 4 of these dimensions.

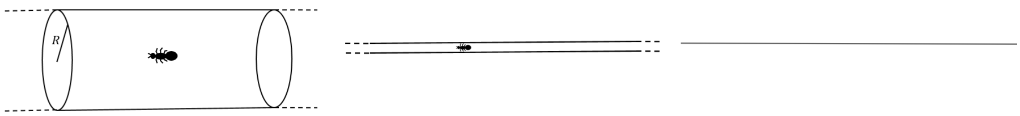

Figure 1.1: Visualisation of a 2-dimensional manifold beïng "rolled up" to effectively be 1-dimensional.

1.8.2 Compactification and the consequences for the theory

To illustrate the effect of compactifications we will discuss a compactification of the 26-dimensional bosonic string. We will compactify one of the dimensions of the target space to a circle. For this we use the coordinate X^{25}. So instead of

$$X^{25} : \mathbb{R} \times S^1 \to \mathbb{R} \tag{1.8.1}$$

now

$$X^{25} : \mathbb{R} \times S^1 \to S^1. \tag{1.8.2}$$

Such that we have the boundary conditions:

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi RL. \tag{1.8.3}$$

Here R is the radius of the circle into which this dimension is compactified and L is called the winding number, this is a whole number that counts how many times the sting is wrapped around this circle.

The mode expansion of X^{25} is now altered such that

$$\begin{aligned} X^{25}(\tau, \sigma) =& X^{25}_R(\tau - \sigma) + X^{25}_L(\tau + \sigma), \\ X^{25}_R(\sigma^-) =& \frac{1}{2}(x^{25} - c) + \frac{\alpha'}{2}\left(\frac{M}{R} - \frac{LR}{\alpha'}\right)\sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}\backslash\{0\}} \frac{1}{n}\alpha^{25}_n e^{-in\sigma^-}, \\ X^{25}_L(\sigma^+) =& \frac{1}{2}(x^{25} + c) + \frac{\alpha'}{2}\left(\frac{M}{R} + \frac{LR}{\alpha'}\right)\sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}\backslash\{0\}} \frac{1}{n}\overline{\alpha}^{25}_n e^{-in\sigma^-}. \end{aligned} \tag{1.8.4}$$

Here M is an operator that takes eigenvalues in $\mathbb{Z}$. This is due to the quantization on a finite space and single-valuedness.

Now if we want to see what the masses are of the states in this theory we use $m^2 = -\sum_{\mu=0}^{24} p^\mu p_\mu$, note we now do not include $\mu = 25$ in the sum because we want to calculate what the mass would be in a theory that is effectively 25-dimensional instead of 26. If we now define N_L and N_R as the left and right level of the state, or the level calculated using the n of the α_n^μ or $\overline{\alpha}_n^\mu$. We can then find:

$$\alpha' m^2 = \alpha' \frac{M^2}{R^2} + \frac{1}{\alpha'} L^2 R^2 + 2(N_L + N_R - 2). \tag{1.8.5}$$

This is a lot like the mass we have seen in (1.7.1). What has changed is that there are 2 extra contributions. One from the internal momentum going around the circle: $\frac{M^2}{R^2}$. The second contribution is due to the energy it takes to wrap the string L times around the circle: L^2R^2.

This means that the spectrum has a few extra states. We still have the tachyon $|0\rangle$ with negative mass -4. Then there still are the now 25-dimensional graviton, anti-symmetric tensor and dilaton $\alpha_{-1}^\mu \overline{\alpha}_{-1}^\nu |0\rangle$. But there are also new states now. First, these are the ones with $M = L = 0$ and mass $m^2 = 0$:

- vector 1: $\alpha_{-1}^{25} \overline{\alpha}_{-1}^\mu |0\rangle$,
- vector 2: $\alpha_{-1}^{\mu} \overline{\alpha}_{-1}^{25} |0\rangle$,
- scalar: $\alpha_{-1}^{25} \overline{\alpha}_{-1}^{25} |0\rangle$.

This scalar has eigenvalues that correspond to the radius R of the circle.

Then we also have new states that have non-trivial M and L. Incorporating the notation $|M, L\rangle$ we can list the following examples:

- vector : $\alpha_{-1}^\mu |1, 1\rangle$ with mass $\alpha' m^2 = \frac{\alpha'}{R^2} + \frac{R^2}{\alpha'} - 2$,
- scalar : $\alpha_{-1}^{25} |1, 1\rangle$ with mass $\alpha' m^2 = \frac{\alpha'}{R^2} + \frac{R^2}{\alpha'} - 2$.

Note that the masses of these states are dependent on the features the compactified manifold and in the case that $R = \sqrt{\alpha'}$ these are massless.

In addition to the extra internal states (Kaluza-Klein modes) there is another feature of the compactification we can notice. The mass formula is equivalent under the transformation $R \to \frac{\alpha'}{R}$. Also the states that we have are symmetric under the interchange $M \leftrightarrow L$. Combining these we find the so called T-duality of this theory:

$$R \to \frac{\alpha'}{R} \text{ and } M \leftrightarrow L. \tag{1.8.6}$$

It is therefore enough to have $0 \leq R \leq \sqrt{\alpha'}$ instead of $0 \leq R < \infty$. This symmetry is an example of different string theories actually being equivalent.

1.9 Calabi-Yau manifolds

The compacitfication of a single dimension on a circle illustrates well what kind of effects compactifications can have on the theory. However this is not a very realistic manifold to consider. A more likely candidate is that the compact manifold is not a circle but a more general compact space. For this we consider Calabi-Yau manifolds.

The definition of a Calabi-Yau manifold is a compact Kähler manifold that is Ricci-flat. Ricci-flatness is a condition on the manifold and it's metric. Via the Einstein equation this also leads to a condition on the field densities of the fields that live on this manifold.

We will now discuss what a Kähler manifold is.

1.9.1 Kähler manifolds

A n (complex-)dimensional Kähler manifold is a complex manifold endowed with a hermitian metric $g_{i\bar{j}}$ such that $g_{i\bar{j}} = \overline{g_{j\bar{i}}}$ and

$$\begin{aligned} J :=& i \sum_{i,j}^{n} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}, \\ dJ =& 0. \end{aligned} \tag{1.9.1}$$

A consequence of this is that the only non-vanishing components of the Riemann tensor are:

$$\begin{aligned} \Gamma^k_{ij} =& g^{k\bar{l}} \partial_i g_{j\bar{l}} \\ \Gamma^{\bar{k}}_{\bar{i}\bar{j}} =& g^{l\bar{k}} \overline{\partial}_{\bar{i}} g_{l\bar{j}} \end{aligned} \tag{1.9.2}$$

So there is no mixing of the i and $\bar{i}$ components.

Also for a Kähler manifold there exists a Kähler potential K such that

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \tag{1.9.3}$$

This K is not unique any $K'(z, \overline{z}) = K(z, \overline{z}) + f(z) + \overline{f}(\overline{z})$, where f is holomorphic and $\overline{f}$ anti-holomorphic, also is a Kähler potential for the same manifold [3].

1.9.2 Hodge numbers

On such a Kähler manifold one can define Hodge numbers, $h^{p,q}$, where $p, q \in \{0, \ldots, n\}$. These numbers are a property of the manifold and contain information about it's structure. They are defined as the dimension of the cohomology groups but we can think of them as a generalization of the Betti numbers, b_p, to complex manifolds [3]. These Betti numbers make precise the notion of the amount of p-dimensional holes in a manifold. For example intuitively it is clear that a annulus is different from a disc because it has a hole in the middle. This would be a 1-dimensional hole and $b_1 = 1$ for a circle but $b_1 = 0$ for a disc. In the same way $b_2 = 0$ for a ball and $b_2 = 1$ for a sphere.

The Hodge numbers of Calabi-Yau manifolds are not all independent. There are the following relations between them:

$$\begin{aligned} h^{p,q} &= h^{q,p} \\ h^{p,q} &= h^{n-p,n-q} \\ h^{n,0} &= h^{0,n} = 1, \\ h^{p,0} &= h^{0,p} = h^{n-p,0} = h^{0,n-p}, \\ h^{p,0} &= 0 \text{ for } 0 < p < n. \end{aligned} \tag{1.9.4}$$

Note that n is the complex dimension of the manifold.

Now because the supersymmetric string theories live in 10 dimensions and we perceive 4 dimensions in our universe we are most interested in compact manifolds with real dimension 6. Therefore we are interested in Calabi-Yau manifolds with complex dimension $n = 3$. These are called Calabi-Yau three-folds. Due to these relations between the Hodge numbers we can arrange them in a shape that reflects their symmetries: the Hodge diamond.

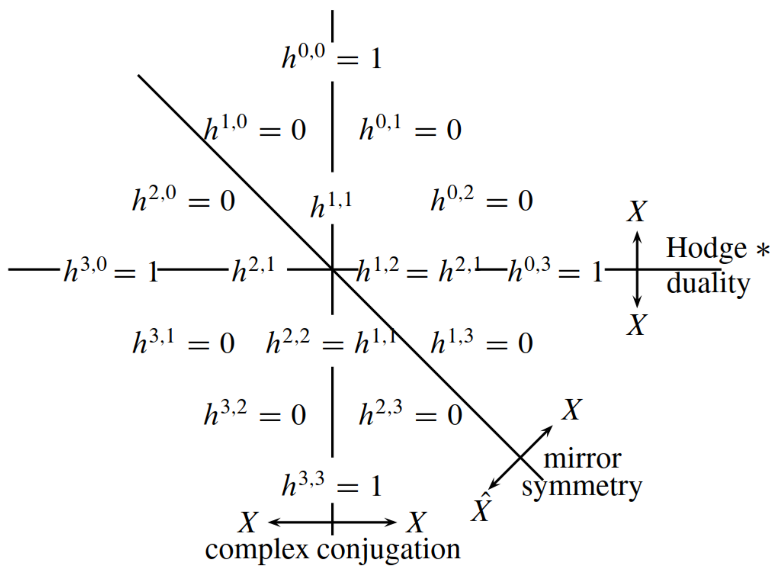

Figure 1.2: Visualisation of the Hodge numbers of the Calabi-Yau three-fold. The symmetries of the Hodge numbers can be show by arranging them in this diamond shape. It becomes apparent that there are only 2 independent Hodge numbers for the Calabi-Yau three-fold. This illustration is taken from [2] page 478.

After getting rid of all the dependent hodge numbers we are only left with the independent $h^{1,1}$ and $h^{2,1}$ for such a Calabi-Yau three-fold. These numbers will be interesting for characterizing the manifold. For example an interesting result for $n = 3$ Calabi-Yau manifolds and their Euler number χ is the following:

$$\chi = 2(h^{1,1} - h^{2,1}). \tag{1.9.5}$$

For us the hodge numbers are especially interesting because they count the "extra" massless fields that the theory has after compactification on the Calabi-Yau threefolds. These are analogues to the internal states $\alpha_{-1}^{25}\overline{\alpha}_{-1}^{25}|0\rangle$ we found in section 1.8.2. These states do not have a counterpart in nature like the other massless states have for example the dilaton. This poses a problem with making a connection to the real world. Therefore it would be useful if it turns out that these do actually have a mass such that it is safe to ignore them in effective theories because the mass is so big, due to the small size of the threefold, that we do not encounter situations where the energy is high enough to produce them. That would explain why we have not observed these particles in nature.

1.10 Supersymmetric IIB

Now we have illustrated how we can get the spectrum of (massless) states of a string theory we can discuss the theory we want to study. Until now we have only considered bosonic states. A similar discussion can be had for a theory with fermionic states. The theory we will discuss is the 10-dimensional type IIB supersymmetric theory. This theory is constructed for both a bosonic and a fermionic part. The theory is called supersymmetric because the action as well as the rest of the theory is symmetric under interchanging the bosons and fermions. This is crucial to get rid of the Tachyon (negative mass state). Therefore this theory is stable in the space-time vacuum. We will only consider the (originally) massless states in this theory to get a low-energy effective theory.

The states we will consider to illustrate how internal massless states can be stabilized by receiving a mass are contained in a two form C_2.

We start with this theory in a 10-dimensional Minkowski space, $\mathbb{R}^{1,9}$. The states in this theory have interactions subject to an action S_{10}. To connect to the 4-dimensional effective theory we compactify 6 of these dimensions on a Calabi-Yau three-fold $\mathcal{X}$ such that $\mathbb{R}^{1,9}$ becomes $\mathbb{R}^{1,3} \times \mathcal{X}$.

1.11 Kähler metric

In this space we consider one therm of the action S_{10} that contains the 2-form $C_2 = c^\alpha \omega_\alpha + \dots$ expanded in a basis where c^α are scalars that live on $\mathbb{R}^{1,3}$ and ω_α is a basis of the cohomology group $H^2(\mathcal{X})$ of the Calabi-Yau three-fold. The $\dots$ indicate terms that we will not discuss here. The term we consider to illustrate how this action results is a action for $\mathbb{R}^{1,3}$ is:

$$\begin{aligned}\tilde{S}_{10} =& \int_{\mathbb{R}^{1,3}\times\mathcal{X}} \mathrm{d}(c^\alpha\omega_\alpha) \wedge * \,\mathrm{d}(c^{\overline{\beta}}\omega_{\overline{\beta}}) \\ =& \int_{\mathbb{R}^{1,3}} \mathrm{d}c^\alpha \wedge * \,\mathrm{d}c^{\overline{\beta}} \int_{\mathcal{X}} \omega_\alpha \wedge *\omega_{\overline{\beta}} \\ =& G_{\alpha\overline{\beta}} \int_{\mathbb{R}^{1,3}} \mathrm{d}c^\alpha \wedge * \,\mathrm{d}c^{\overline{\beta}}.\end{aligned} \tag{1.11.1}$$

Here $M, N = 0, \ldots, h^{1,1} + h_-^{2,1}$ run over the moduli, $\alpha, \beta = 0, \ldots, h^{1,1}$.

This shows we can integrate out the degrees of freedom that live on the internal manifold $\mathcal{X}$ and encode the relevant information in a quantity $G_{\alpha\overline{\beta}}$ which we will call the Kähler metric. It turns out that this $G_{\alpha\overline{\beta}}$ has the nice property, as we have seen in equation (1.9.3), that there exists a real potential K such that

$$G_{\alpha\overline{\beta}} = \partial_\alpha\partial_{\overline{\beta}}K. \tag{1.11.2}$$

This K is the Kähler potential. The information about the manifold $\mathcal{X}$ is encoded in this K when we work in the effective 4-dimensional theory.

1.12 Fluxes

Now we have managed to get an effective theory in 4-dimensions but we still have the massless states on the internal manifold. We call these moduli. For the theory to be well defined perturbatively and to make sure that we do not have many more particles than we expect to have from experiments, we need to stabilize these moduli. We do this by generating a potential and masses for them using fluxes. The construction of flux compactifications was developed in [4],[5] and applied to the IIB theory in [6, 7].

The metric of the manifold that our theory lives on, $\mathbb{R}^{1,3} \times \mathcal{X}$, is a solution to the Einstein equations. One of the solutions to this is a vacuum expectation value, vev, that vanishes for the so called fluxes. This leads to Ricci flat solutions. If instead these fluxes have a non zero vev they carry energy and therefore impact the metric and the shape of the compact manifold via the Einstein equations. The manifold is then deformed away form a completely Ricci-flat manifold and is therefore not exactly a Calabi-Yau manifold anymore. As an example we look at the flux

$$F_3 = \mathrm{d}C_2. \tag{1.12.1}$$

If the vev of this flux $\langle F_3\rangle \neq 0$ then we can redefine C_2 such that we can write the flux as a vev and fluctuations around this. We then get:

$$F_3 = \langle F_3\rangle + \mathrm{d}C_2. \tag{1.12.2}$$

If we plug this into the action we retrieve the dynamical part for $\mathrm{d}C_2$ but we also get a part containing $\langle F_3 \rangle$. This part together with the parts that contain the vev's of the other fluxes are called the scalar potential. These couple to the moduli to generate a potential for them and therefore (potentially) generate masses such that the moduli are stabilized.

This way we arrive at an effective theory in 4-dimensions with stabilized moduli.

1.13 Super-potential

By turning on these fluxes, giving them non trivial vev's, we effectively generate a potential in the 4-dimensional action. This potential is called the scalar-potential V or the F-term potential and needs to be of a specific form. We have the following [6–8]:

$$\begin{aligned} V =& \mathrm{e}^K \left(F_M G^{M\overline{N}} \overline{F}_{\overline{N}} - 3|W|^2 \right) \text{ where} \\ F_M =& \partial_M W + (\partial_M K) W. \end{aligned} \tag{1.13.1}$$

Here $G^{M\overline{N}}$ is the inverse of the Kähler metric $G_{M\overline{N}}$ and W is the super-potential. To match a potential of this form with the potential that comes out of the action we need the super-potential to be [9–11]:

$$W = \int_{\mathcal{X}} \Omega \wedge (F - H\tau). \tag{1.13.2}$$

Here $F = dC_2$ and $H = dB_2$ are the fluxes in this theory and Ω is the holomorphic three-form of this Calabi-Yau three-fold. This three-form is build up out of the moduli in the manner described in (1.14.3).

1.14 Our setting

Now we turn to a more specific discussion that is oriented to finding the masses of the moduli in specific cases. To this end we follow the paper by Plauschinn [9] and recommend reading this for more details.

1.14.1 Potential

We will be interested in calculating the masses of the n scalar fields Φ^M of a 4-dimensional $\mathcal{N} = 1$ supergravity theory in the minimum of the F-term potential V. As we have seen in (1.13.1) this potential is given by:

$$V = \mathrm{e}^K \left(F_P G^{P\overline{Q}} \overline{F}_{\overline{Q}} - 3|W|^2 \right). \tag{1.14.1}$$

Here $P, Q = 0, \ldots, h^{1,1} + h^{2,1}_{-}$ run over the moduli, K is the real Kähler potential, W is the holomorphic super-potential and $G_{M\overline{N}} = \partial_M \partial_{\overline{N}} K$ is the Kähler metric.

Also the F-therms relate to W and K via $F_M = \partial_M W + (\partial_M K)W$.

Setting $\partial_N V = 0$ shows that a minimum is attained when $F_N = 0$. If we consider W in this minimum we will adopt the notation W_0. This will be the minimum we are interested in. An interesting observation we can make is that in the minimum we have [7]:

$$\begin{aligned} 0 =& F_N = \partial_N W_0 + (\partial_N K)W_0 \\ \partial_N W_0 =& -(\partial_N K)W_0 \\ \partial_N F_M =& \partial_N \partial_M W_0 + K_{NM} W_0 + K_M \partial_N W_0 \\ =& \partial_N \partial_M W_0 + K_{NM} W_0 - K_M K_N W_0 \end{aligned} \tag{1.14.2}$$

So we see that ∂F is symmetric in the minimum of the potential V.

We can apply this set up to the case of a type-IIB theory that is compactified on a Calabi-Yau manifold $\mathcal{X}$ that is subject to an orientifold projection. This projection will split the cohomology in to even and odd eigenspaces. This introduces the need for an extra index ($\pm$) for the hodge numbers denoted as a subscript. This would result in the fields Φ^M being the following.

There is one modulus $\tau = c + is$, a set of moduli we call T^A where $A = 1, \dots, h^{1,1}$ and there are $h^{2,1}_-$ complex structure moduli z^i ($i = 1, \dots, h^{2,1}_-$). They can be combined into a holomorphic three-form Ω in the following way:

$$\Omega = X^I \alpha_I - \partial_I \mathcal{F} \beta^I \text{ with } z^i = \frac{X^i}{X^0}, \tag{1.14.3}$$

here $I = 0, \dots, h^{2,1}_-$ and α_I and β^I form a symplectic basis of the three-forms on the internal Calabi-Yau manifold $\mathcal{X}$ such that [12]

$$\int_{\mathcal{X}} \alpha_I \wedge \beta^J = \delta_I^J = -\int_{\mathcal{X}} \beta^J \wedge \alpha_I. \tag{1.14.4}$$

Also the periods $\partial_I \mathcal{F}$ depend on z^i holomorphicaly.

Having defined the fields we turn to the Kähler potential K. It is real and consists of 2 parts. It splits up in a part that depends on τ and the Kähler moduli T^A and a part that depends on z^i [4, 6, 9]:

$$\begin{aligned} K =& K_K + K_{cs}, \\ K_K =& -\log\left[-i(\tau - \overline{\tau})\right] - 2\log\left[\mathcal{V}\right], \\ K_\tau =& -\log\left[-i(\tau - \overline{\tau})\right], \\ K_{cs} =& -\log\left[i \int \Omega \wedge \overline{\Omega}\right]. \end{aligned} \tag{1.14.5}$$

The bar here denotes complex conjugation. The dependence on the Kähler moduli T^A is hidden in the Einstein-frame volume of $\mathcal{X}$: $\mathcal{V}$. K technically also receives corrections due to the slight deformation away from Ricci-flatness but we will ignore these here.

We will take second derivatives of the potential V to find a mass matrix and compute it's eigenvalues to find the masses that the moduli τ and z^i receive from the fluxes. Before we do this we can further simplify the expression for V in the case we are considering. Because K splits in different terms for the dilaton τ, the complex structure moduli z^i and the Kähler moduli T^A as we have seen also $G^{M\overline{N}}$ splits in different blocks corresponding to these groups of moduli [16, (6.3)]. It turns out that [8, 9, 12]

$$\begin{aligned} V =& \mathrm{e}^K \left(F_P G^{P\overline{Q}} \overline{F}_{\overline{Q}} - 3|W|^2 \right) \\ =& \mathrm{e}^K \left(F_\tau G^{\tau\overline{\tau}} \overline{F}_{\overline{\tau}} + F_A G^{A\overline{B}} \overline{F}_{\overline{B}} + F_j G^{j\overline{i}} \overline{F}_{\overline{i}} - 3|W|^2 \right) \\ =& \mathrm{e}^K \left(F_\tau G^{\tau\overline{\tau}} \overline{F}_{\overline{\tau}} + 3|W|^2 + F_j G^{j\overline{i}} \overline{F}_{\overline{i}} - 3|W|^2 \right) \\ =& \mathrm{e}^K F_P G^{P\overline{Q}} \overline{F}_{\overline{Q}}. \end{aligned} \tag{1.14.6}$$

Where now in the last line P and $\overline{Q}$ only run over τ and $i = 1, \ldots, h^{2,1}_-$. In this way we find the potential is only dependent on τ and the z^i. The other moduli will be be stabilized in a different way that we will not discuss.

1.14.2 Masses

Now we have the potential we can from this calculate the mass matrix m^2 by taking the second derivatives and evaluating these in the minimum $F_N = 0$.

First we take the first derivative and find:

$$\begin{aligned} \partial_N V =& \partial_N \left(\mathrm{e}^K F_P G^{P\overline{Q}} \overline{F}_{\overline{Q}} \right) \\ =& \mathrm{e}^K (\partial_N K F_P G^{P\overline{Q}} \overline{F}_{\overline{Q}} + \partial_N F_P G^{P\overline{Q}} \overline{F}_{\overline{Q}} + F_P \partial_N G^{P\overline{Q}} \overline{F}_{\overline{Q}} + F_P G^{P\overline{Q}} \partial_N \overline{F}_{\overline{Q}}). \end{aligned} \tag{1.14.7}$$

To find $m^2_{M\overline{N}}$ we calculate $\partial_M \partial_{\overline{N}} V|_{F=0}$:

$$\begin{aligned} \partial_{\overline{M}} \partial_N V|_{F=0} =& \partial_{\overline{M}} \Big[\mathrm{e}^K (K_N F_P G^{P\overline{Q}} \overline{F}_{\overline{Q}} + \partial_N F_P G^{P\overline{Q}} \overline{F}_{\overline{Q}} \\ & + F_P \partial_N G^{P\overline{Q}} \overline{F}_{\overline{Q}} + F_P G^{P\overline{Q}} \partial_N \overline{F}_{\overline{Q}}) \Big] |_{F=0} \\ =& \partial_{\overline{M}} \left[\mathrm{e}^K (\partial_N F_P G^{P\overline{Q}} \overline{F}_{\overline{Q}} + F_P G^{P\overline{Q}} \partial_N \overline{F}_{\overline{Q}}) \right] |_{F=0} \\ =& \mathrm{e}^K \partial_{\overline{M}} \left[\partial_N F_P G^{P\overline{Q}} \overline{F}_{\overline{Q}} + F_P G^{P\overline{Q}} \partial_N \overline{F}_{\overline{Q}} \right] |_{F=0} \\ =& \mathrm{e}^K \left[\partial_N F_P G^{P\overline{Q}} \partial_{\overline{M}} \overline{F}_{\overline{Q}} + \partial_{\overline{M}} F_P G^{P\overline{Q}} \partial_N \overline{F}_{\overline{Q}} \right] |_{F=0}. \end{aligned} \tag{1.14.8}$$

To get to the second line we found that all therms coming from the first and third therm would still have a F in there so will contribute nothing when evaluating in the minimum. To then get to the third line we used that when the derivative acts on the e^K all therms still contain an F. To get to the fourth line we only kept the therms where the derivative acts on the remaining F's.

Now to go on we calculate

$$\begin{aligned}\partial_{\overline{M}}F_N =&\partial_{\overline{M}}\left[\partial_N W+\partial_N K W\right]\\ =&\partial_{\overline{M}}\partial_N K W\\ =&G_{\overline{M}N}W,\\ \partial_N\overline{F}_{\overline{M}} =&\partial_N\left[\partial_{\overline{M}}\overline{W}+\partial_{\overline{M}}K\overline{W}\right]\\ =&G_{N\overline{M}}\overline{W}.\end{aligned} \tag{1.14.9}$$

Where we used that W is holomorphic.

Now we plug these into (1.14.8) to find:

$$\begin{aligned}m^2_{\overline{M}N} =&e^K\left[\partial_N F_P G^{P\overline{Q}}\partial_{\overline{M}}\overline{F}_{\overline{Q}}+G_{\overline{M}P}WG^{P\overline{Q}}G_{N\overline{Q}}\overline{W}\right]\\ =&e^K\left[\partial_N F_P G^{P\overline{Q}}\partial_{\overline{M}}\overline{F}_{\overline{Q}}+G_{\overline{M}N}|W|^2\right].\end{aligned} \tag{1.14.10}$$

Then using the same principles, dropping therms that will contain F and become 0. We find

$$\begin{aligned}m^2_{MN} =&\partial_M\partial_N V|_{F=0}\\ =&\partial_M\left[e^K(K_N F_P G^{P\overline{Q}}\overline{F}_{\overline{Q}}+\partial_N F_P G^{P\overline{Q}}\overline{F}_{\overline{Q}}+F_P\partial_N G^{P\overline{Q}}\overline{F}_{\overline{Q}}+F_P G^{P\overline{Q}}\partial_N\overline{F}_{\overline{Q}})\right]\\ =&e^K\left[\partial_N F_P G^{P\overline{Q}}\partial_M\overline{F}_{\overline{Q}}+\partial_M F_P G^{P\overline{Q}}\partial_N\overline{F}_{\overline{Q}}\right]\\ =&e^K\left[\partial_N F_P G^{P\overline{Q}}G_{M\overline{Q}}\overline{W}_0+\partial_M F_P G^{P\overline{Q}}G_{N\overline{Q}}\overline{W}_0\right]\\ =&e^K\overline{W}_0\left[\partial_N F_M+\partial_M F_N\right]\\ =&2e^K\overline{W}_0\partial_M F_N.\end{aligned} \tag{1.14.11}$$

Note that to get to the last line we used a symmetry by equation (1.14.2).

Now we can combine these results with their complex conjugates into the mass matrix:

$$\begin{aligned}m^2 =&\begin{bmatrix} m^2_{M\overline{N}} & m^2_{MN}\\ m^2_{\overline{MN}} & m^2_{\overline{M}N}\end{bmatrix}\\ =&e^K\begin{bmatrix}\partial_M F_P G^{P\overline{Q}}\partial_{\overline{N}}\overline{F}_{\overline{Q}}+G_{\overline{N}M}|W_0|^2 & 2\partial_M F_N\overline{W}_0\\ 2\partial_{\overline{M}}\overline{F}_{\overline{N}}W_0 & \partial_N F_P G^{P\overline{Q}}\partial_{\overline{M}}\overline{F}_{\overline{Q}}+G_{\overline{M}N}|W_0|^2\end{bmatrix}\end{aligned} \tag{1.14.12}$$

If we want to be able to compare these masses to the literature we have to make sure that we consider the canonical mass matrix. This is due to different conventions that can be chosen for the re-normalization of the fields. To this end we define Γ using that G is hermitian and positive definite such that we can write:

$$G = \Gamma\Gamma^{\dagger}. \tag{1.14.13}$$

Using this we define Q as

$$Q = \Gamma^{-1}(\partial F)\Gamma^{-T}. \tag{1.14.14}$$

Using these we can rewrite the canonical mass matrix m^2_{can} to be:

$$\begin{aligned} m^2_{\text{can}} =& \begin{bmatrix} \Gamma^{-1} & 0 \\ 0 & \overline{\Gamma^{-1}} \end{bmatrix} m^2 \begin{bmatrix} \left(\Gamma^{\dagger}\right)^{-1} & 0 \\ 0 & \Gamma^{-T} \end{bmatrix} \\ =& \mathrm{e}^K \begin{bmatrix} QQ^{\dagger} + |W_0|^2 & 2Q\overline{W}_0 \\ 2Q^{\dagger}W_0 & Q^{\dagger}Q + |W_0|^2 \end{bmatrix}. \end{aligned} \tag{1.14.15}$$

The matrix $QQ^{\dagger}$ has eigenvalues we denote by σ^2_{α} with $\alpha = 0, \ldots, h^{2,1}_{-}$. The eigenvalues of the mass matrix, m^2_{can}, and therefore the masses that the moduli receive are then:

$$m^2_{\alpha\pm} = \mathrm{e}^K \left(\sigma_{\alpha} \pm |W_0|\right)^2. \tag{1.14.16}$$

This is equation (5.5) in the paper[9].

1.14.3 Q

We want to find the eigenvalues of $QQ^{\dagger}$. To later make contact with the canonical form of the masses it is useful to transform Q into $\mathbf{Q}$.
Define $\mathbf{Q} = G^{-1}\partial F$ evaluated at the minimum. There for $\overline{\mathbf{Q}} = \overline{G^{-1}\partial F} = \overline{\Gamma^{-1\dagger}\Gamma^{-1}\partial F} = \Gamma^{-T}\overline{\Gamma^{-1}\partial F}$. Now we show that $QQ^{\dagger}$ has the same eigenvalues as $\mathbf{Q}\overline{\mathbf{Q}}$

$$\begin{aligned} \det\left(QQ^{\dagger} - \lambda\right) =& \det\left(\Gamma^{-1}(\partial F)\Gamma^{-T}\overline{\Gamma^{-1}}(\partial F)^{\dagger}\Gamma^{-1\dagger} - \lambda\right) \\ =& \det\left(\Gamma^{-1\dagger}\Gamma^{-1}(\partial F)\Gamma^{-T}\overline{\Gamma^{-1}}(\partial F)^{\dagger} - \lambda\right) \\ =& \det\left(G^{-1}(\partial F)\Gamma^{-T}\overline{\Gamma^{-1}}\overline{(\partial F)} - \lambda\right) \\ =& \det\left(\mathbf{Q}\overline{\mathbf{Q}} - \lambda\right). \end{aligned} \tag{1.14.17}$$

So indeed the eigenvalues of $\mathbf{Q}\overline{\mathbf{Q}}$ are the same as those of $QQ^{\dagger}$.
To be able to find an expression for $\mathbf{Q}$ we introduce the following.

D_i are the covariant derivatives with respect to the z^i. Acting with these on Ω gives a basis $\chi_i = D_i\Omega = \partial_i\Omega + (\partial_i K)\Omega$ for the (2,1) forms on $\mathcal{X}$ such that we can express H in this basis in the following way:

$$H = h^0\Omega + h^i\chi_i + \overline{h}^{\bar{j}}\overline{\chi}_{\bar{j}} + \overline{h^0\Omega}. \tag{1.14.18}$$

This defines the components of H in this basis. They can also be expressed as

$$h^{\bar{i}} = -i\mathrm{e}^{K_{cs}}G^{\bar{i}j}\int_{\mathcal{X}}\chi_j \wedge H. \tag{1.14.19}$$

Note that here only the components of $G^{M\overline{N}}$ that are associated with the z^i show up.

We also need the Yukawa-couplings κ_{ijk}. These are expressed in therms of Ω:

$$\kappa_{ijk} = -\int_{\mathcal{X}} \Omega \wedge D_i D_j D_k \Omega \tag{1.14.20}$$

With this expression we can write down $\mathbf{Q}$ in the following way:

$$\begin{aligned}
\mathbf{Q}_{\tau}^{\overline{\tau}} &= 0, \\
\mathbf{Q}_{\tau}^{\overline{i}} &= -i\mathrm{e}^{-K_{cs}} h^{\overline{i}}, \\
\mathbf{Q}_{j}^{\overline{\tau}} &= i(\tau - \overline{\tau})^2 \mathrm{e}^{-K_{cs}} G_{j\overline{i}} h^{\overline{i}}, \\
\mathbf{Q}_{j}^{\overline{i}} &= (\tau - \overline{\tau}) \kappa^{\overline{i}}{}_{jk} h^k.
\end{aligned} \tag{1.14.21}$$

Here i, j, k run from 1 to $h^{2,1}$.
If we now define $R_{\overline{i}\overline{j}m\overline{n}}$ as the Riemann tensor of the complex-structure moduli-space metric, this leads to $\mathbf{Q}\overline{\mathbf{Q}}$ being:

$$\begin{aligned}
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{\tau}}_{\overline{\tau}} &= \mathrm{e}^{-2(K_\tau + K_{cs})} h^i G_{i\overline{j}} h^{\overline{j}}, \\
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{\tau}}_{\overline{j}} &= \mathrm{e}^{-(K_\tau + K_{cs})} (\tau - \overline{\tau})^2 \kappa_{\overline{j}\overline{m}\overline{n}} h^{\overline{m}} h^{\overline{n}}, \\
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{i}}_{\overline{\tau}} &= -\mathrm{e}^{-(K_\tau + K_{cs})} \kappa^{\overline{i}}{}_{mn} h^m h^n, \\
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{i}}_{\overline{j}} &= -\mathrm{e}^{-2K_{cs}} (\tau - \overline{\tau})^2 h^{\overline{i}} G_{\overline{j}k} h^k - (\tau - \overline{\tau})^2 \kappa^{\overline{i}}{}_{lk} h^k \kappa^{l}{}_{\overline{j}\overline{m}} h^{\overline{m}} \\
&= \mathrm{e}^{-2(K_\tau + K_{cs})} \left(-R^{\overline{i}}{}_{\overline{j}m\overline{n}} h^{\overline{n}} h^m + \delta^{\overline{i}}_{\overline{j}} h^m G_{m\overline{n}} h^{\overline{n}} + 2 h^{\overline{i}} h_{\overline{j}} \right).
\end{aligned} \tag{1.14.22}$$

Note that we have now two expressions for $(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{i}}_{\overline{j}}$. One in therms of κ and one in therms of the Riemann tensor. We will later be able to use this to switch between ways to express our quantities.

This concludes the introduction.

Chapter 2

Calculations on $h_-^{2,1} = 1$

In this and consecutive sections we will calculate and discuss the masses which these moduli receive from the fluxes. We will do so by calculating the eigenvalues of $\mathbf{Q}\overline{\mathbf{Q}}$ in 3 cases. In the first case we consider $h_-^{2,1} = 1$ such that there is only one z^i. In the second case we consider a specialization of the first case: $h_-^{2,1} = 1$ and we assume to be in the large-complex-structure limit such that $\mathcal{F}$ has a known form. In the last case we consider $\mathcal{F}$ with arbitrary form and $h_-^{2,1} = 2$ such that there are two z^i.

From equation (4.3.1) we have the expression $m_{\alpha\pm}^2 = \mathrm{e}^K \left(\sigma_\alpha \pm |W_0|\right)^2$. This is an expression of the masses in terms of the eigenvalues of $\mathbf{Q}\overline{\mathbf{Q}}$ and the super-potential W_0. We can Express $\mathbf{Q}\overline{\mathbf{Q}} = aM$. Where a is a scalar and M a matrix. Now note that for an eigenvector x and an eigenvalue λ of M we have

$$\begin{aligned} (M - \lambda I)\, x =& 0 \\ a\,(M - \lambda I)\, x =& 0 \\ (aM - a\lambda I)\, x =& 0. \end{aligned} \tag{2.0.1}$$

So if λ is an eigenvalue of M then $a\lambda = \sigma_\alpha^2$ is an eigenvalue of $aM = \mathbf{Q}\overline{\mathbf{Q}}$.

We will use this to first find an expression for the masses in therms of the eigenvalues λ of M. This procedure will be very similar for all the cases we consider. After this we will find expressions for the λ for each case.

2.1 $h_-^{2,1} = 1$

For this section it is more convenient to use $h = h^1$, $\overline{h} = h^{\overline{1}}$, $G = G_{1\overline{1}}$, $\kappa = \kappa_{111}$, $z = z^i$ and $R = R_{1\overline{1}1\overline{1}} G^{1\overline{1}} G^{1\overline{1}}$ the Ricci scalar.

2.1.1 Masses in therms of λ

We pick $a = \mathrm{e}^{-2(K_\tau + K_{cs})} G |h^1|^2$. We then have $\sigma_\alpha^2 = \mathrm{e}^{-2(K_\tau + K_{cs})} G |h^1|^2 \lambda$ and therefore

$$\sigma_\alpha = \mathrm{e}^{-(K_\tau + K_{cs})} \sqrt{G} |h^1| \sqrt{\lambda} \tag{2.1.1}$$

This we can plug (4.3.1) to find:

$$m^2_{\alpha\pm} = \mathrm{e}^K \left(\mathrm{e}^{-(K_\tau + K_{cs})} \sqrt{G} |h^1| \sqrt{\lambda} \pm |W_0| \right)^2 . \tag{2.1.2}$$

We want to express the masses in more natural quantities. The quantities we will use are a prefactor $16\mathrm{e}^K |W_0|^2$ where $|W|$ should be taken to be in the minimum $F_N = 0$ and we use the flux number N_{flux}. We will combine these into a quantity μ later. The definition of the flux number is the one in [2] equation (17.50) [12, 13]:

$$N_{\text{flux}} = \int F \wedge H \in \mathbb{N}. \tag{2.1.3}$$

Note that this is a whole number due to F and H being integer fluxes [6].

In order to express G and h in therms of these quantities we use from the literature: [2] equation 14.128a and page 517 [6]:

$$\star\Omega = -\, i\Omega, \quad \star\chi = i\chi, \tag{2.1.4}$$

and equation (3.2) in the paper [6, 14]

$$G = -\frac{\int \chi \wedge \overline{\chi}}{\int \Omega \wedge \overline{\Omega}}, \tag{2.1.5}$$

as well as [13]

$$s = \frac{N_{\text{flux}}}{\int H \wedge \star H}. \tag{2.1.6}$$

and last equation (5.6) in [9]

$$W = \int_{\mathcal{X}} \Omega \wedge (F - H\tau), \tag{2.1.7}$$

Recall that s is the complex part of τ the dilaton.
This last expression we can use to find an expression for $|W_0|$. We use the first line from (1.14.2) to find in the minimum of V:

$$\begin{aligned} 0 =& F_\tau = \partial_\tau W + (\partial_\tau K) W \\ 0 =& \int_{\mathcal{X}} \Omega \wedge \left(-H + \frac{iF}{2s} - \frac{i\tau}{2s} H \right) \\ -2si \int_{\mathcal{X}} \Omega \wedge H =& \int_{\mathcal{X}} \Omega \wedge (F - \tau H) \\ W_0 =& -\, 2si \int_{\mathcal{X}} \Omega \wedge H \end{aligned} \tag{2.1.8}$$

Now we can use the expression (1.14.18) we have for H, the expression (1.14.5) for K_{cs} as well as the expressions we just quoted from the literature to calculate

$\int H \wedge \star H$. This leads to

$$\begin{aligned}\int H \wedge \star H =& 2i \int \left(|h^0|^2 \Omega \wedge \overline{\Omega} - |h|^2 \chi \wedge \overline{\chi}\right) \\ =& 2|h^0|^2 \mathrm{e}^{-K_{cs}} - 2i|h|^2 \int \chi \wedge \overline{\chi} \\ =& 2|h^0|^2 \mathrm{e}^{-K_{cs}} + 2i|h|^2 G \int \Omega \wedge \overline{\Omega} \\ =& 2\mathrm{e}^{-K_{cs}} \left(|h^0|^2 + |h|^2 G\right)\end{aligned} \tag{2.1.9}$$

Now we have related $\int H \wedge \star H$ to h^0 and h. Using (2.1.6) we can relate $\int H \wedge \star H$ to N_{flux} and we can relate W to h^0 using (2.1.8):

$$\begin{aligned}W_0 =& -2i \int \Omega \wedge H_3 s \\ =& -\overline{h^0} e^{-(K_\tau + K_{cs})}.\end{aligned} \tag{2.1.10}$$

With these relations we can, starting form (2.1.9), express h in therms of N_{flux} and W:

$$\begin{aligned}|h|^2 G =& \frac{1}{2} \mathrm{e}^{K_{cs}} \int H \wedge \star H - |h^0|^2, \\ =& \mathrm{e}^{K_\tau + K_{cs}} N_{\text{flux}} - \mathrm{e}^{2(K_\tau + K_{cs})} |W_0|^2.\end{aligned} \tag{2.1.11}$$

We plug this into (2.1.2) and find the mass in therms of h^μ:

$$\begin{aligned}m^2_{\alpha\pm} =& \mathrm{e}^K \left(\mathrm{e}^{-(K_\tau + K_{cs})} \sqrt{G} |h^1| \sqrt{\lambda} \pm |W_0|\right)^2, \\ =& \mathrm{e}^{-K_\tau - K_{cs} - 2\log[\mathcal{V}]} \left(\sqrt{h^1 G \overline{h^1}} \sqrt{\lambda} \pm |\overline{h^0}|\right)^2.\end{aligned} \tag{2.1.12}$$

Now we define

$$\mu^2 = \frac{N_{\text{flux}}}{\mathrm{e}^{K_\tau + K_{cs}} |W_0|^2}. \tag{2.1.13}$$

This leads to

$$\begin{aligned}m^2_{\alpha\pm} =& \mathrm{e}^K \left(\mathrm{e}^{-(K_\tau + K_{cs})} \sqrt{\mathrm{e}^{K_\tau + K_{cs}} N_{\text{flux}} - \mathrm{e}^{2(K_\tau + K_{cs})} |W_0|^2} \sqrt{\lambda} \pm |W_0|\right)^2, \\ =& \mathrm{e}^K \left(|W_0| \sqrt{\frac{N_{\text{flux}}}{\mathrm{e}^{K_\tau + K_{cs}} |W_0|^2} - 1} \sqrt{\lambda} \pm |W_0|\right)^2, \\ =& \mathrm{e}^K |W_0|^2 \left(\sqrt{\lambda} \sqrt{\mu^2 - 1} \pm 1\right)^2, \\ =& \frac{N_{\text{flux}}}{\mu^2} \mathrm{e}^{-2\log[\mathcal{V}]} \left(\sqrt{\lambda} \sqrt{\mu^2 - 1} \pm 1\right)^2, \\ =& \frac{N_{\text{flux}}}{\mathcal{V}^2} \left(\sqrt{\lambda} \sqrt{1 - \frac{1}{\mu^2}} \pm \frac{1}{\mu}\right)^2.\end{aligned} \tag{2.1.14}$$

2.2 $h_-^{2,1} = 1$ but not Large-complex-structure limit

$h_-^{2,1} = 1$ Having found an expression for the masses dependent on λ we will in this section calculate λ when $h_-^{2,1} = 1$ or when there is only one complex structure modulus z^1. This means that the i index only takes the value 1 and $\mathbf{Q}\overline{\mathbf{Q}}$ is a 2 by 2 matrix. Applying this to (1.14.22) we find:

$$\begin{aligned}
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{\tau}}_{\overline{\tau}} =& \mathrm{e}^{-2(K_\tau+K_{cs})} h^1 G_{1\overline{1}} h^{\overline{1}}, \\
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{\tau}}_{\overline{1}} =& -4\mathrm{e}^{-(K_\tau+K_{cs})} s^2 \kappa_{\overline{111}} h^{\overline{1}} h^{\overline{1}}, \\
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{1}}_{\overline{\tau}} =& -\mathrm{e}^{-(K_\tau+K_{cs})} \kappa^{\overline{1}}_{\ 11} h^1 h^1, \\
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{1}}_{\overline{1}} =& 4\mathrm{e}^{-2K_{cs}} s^2 h^{\overline{1}} G_{\overline{1}1} h^1 + 4s^2 \kappa^{\overline{1}}_{\ 11} h^1 \kappa^{1}_{\ \overline{11}} h^{\overline{1}} \\
=& \mathrm{e}^{-2(K_\tau+K_{cs})} \left(-R^{\overline{1}}_{\ \overline{1}1\overline{1}} h^{\overline{1}} h^1 + h^1 G_{1\overline{1}} h^{\overline{1}} + 2h^{\overline{1}} h_{\overline{1}} \right).
\end{aligned} \tag{2.2.1}$$

Here we have used that the indices only run over 1 and we have used that $\tau = c+si$.

Now we use $h = h^1$, $\overline{h} = h^{\overline{1}}$, $G = G_{1\overline{1}}$, $\kappa = \kappa_{111}$, $z = z^i$ and $R = R_{1\overline{1}1\overline{1}} G^{1\overline{1}} G^{1\overline{1}}$ the Ricci scalar. This results in

$$\begin{aligned}
\mathbf{Q}\overline{\mathbf{Q}} =& \begin{bmatrix} \mathrm{e}^{-2(K_\tau+K_{cs})} hG\overline{h} & -4\mathrm{e}^{-(K_\tau+K_{cs})} s^2 \overline{\kappa}\overline{h}^2 \\ -\mathrm{e}^{-(K_\tau+K_{cs})} G^{-1}\kappa h^2 & \mathrm{e}^{-2(K_\tau+K_{cs})} \left(-RG|h|^2 + 3G|h|^2\right) \end{bmatrix}, \\
=& \mathrm{e}^{-2(K_\tau+K_{cs})} G|h|^2 \begin{bmatrix} 1 & -4\mathrm{e}^{K_\tau+K_{cs}} s^2 G^{-1} \overline{\kappa} \frac{\overline{h}}{h} \\ -\mathrm{e}^{K_\tau+K_{cs}} G^{-2} \kappa \frac{h}{\overline{h}} & 3-R \end{bmatrix}.
\end{aligned} \tag{2.2.2}$$

So when we take out the factor of $a = \mathrm{e}^{-2(K_\tau+K_{cs})} G|h|^2$ we find

$$M = \begin{bmatrix} 1 & -4\mathrm{e}^{K_\tau+K_{cs}} s^2 G^{-1} \overline{\kappa} \frac{\overline{h}}{h} \\ -\mathrm{e}^{K_\tau+K_{cs}} G^{-2} \kappa \frac{h}{\overline{h}} & 3-R \end{bmatrix}. \tag{2.2.3}$$

Now we set out to find the eigenvalues, λ, of M by calculating the 0-points of the characteristic polynomial.

$$\begin{aligned}
0 =& (1-\lambda)(3-R-\lambda) - 4\mathrm{e}^{2(K_\tau+K_{cs})} G^{-3} s^2 |\kappa|^2 \frac{\overline{h}}{h} \frac{h}{\overline{h}}, \\
0 =& \lambda^2 + (R-4)\lambda + 3 - R - \mathrm{e}^{2K_{cs}} G^{-3} |\kappa|^2.
\end{aligned} \tag{2.2.4}$$

We can now use the 4'th and 5'th line of equation (1.14.22) to express κ in therms of R:

$$\begin{aligned}
-\mathrm{e}^{-2K_{cs}} (\tau-\overline{\tau})^2 G|h^1|^2 - (\tau-\overline{\tau})^2 G^{-2} |\kappa|^2 |h^1|^2 =& \mathrm{e}^{-2(K_{cs}+K_\tau)} |h^1|^2 \left(3G - R_{1111} G^{-1}\right) \\
1 + \mathrm{e}^{2K_{cs}} G^{-3} |\kappa|^2 =& -R_{1111} G^{-2} + 3 \\
\mathrm{e}^{2K_{cs}} G^{-3} |\kappa|^2 =& 2 - R, \\
R = 2 - \mathrm{e}^{2K_{cs}} G^{-3} |\kappa|^2.
\end{aligned} \tag{2.2.5}$$

We can now eliminate κ in favour of R. Substituting this into the characteristic polynomial we find (see figure 2.1:

$$\begin{aligned} 0 =& \lambda^2 + (R-4)\lambda + 1, \\ \lambda_\pm =& \frac{(4-R) \pm \sqrt{12 - 8R + R^2}}{2}. \end{aligned} \quad (2.2.6)$$

For a plot of these $\lambda_\pm$ see figure 2.1. We see that these λ only depend on the Ricci

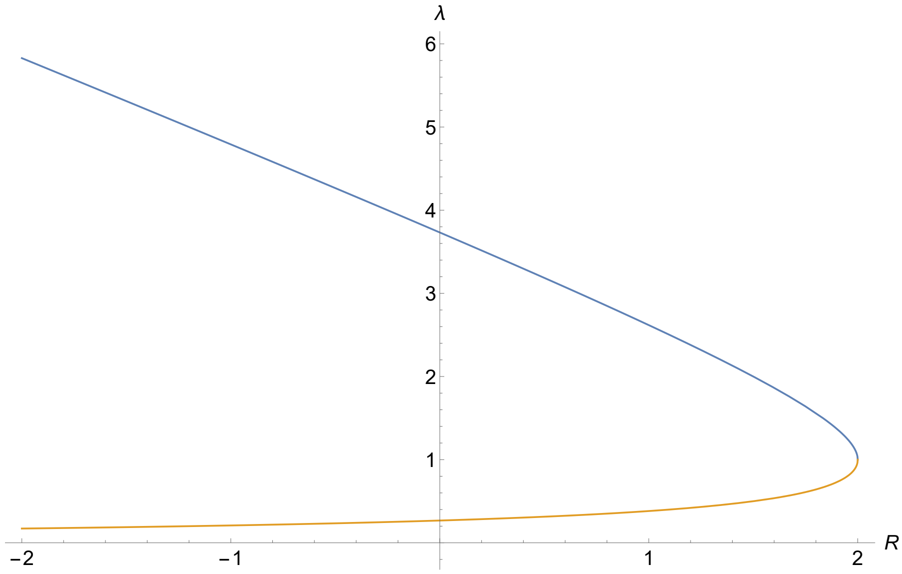

Figure 2.1: This figure shows both λ_- in orange and λ_+ in blue dependent on the Ricci scalar R. These $\lambda_\pm$ are the rescales eigenvalues of $\mathbf{Q}\overline{\mathbf{Q}}$. The case we consider here is $h^{2,1}_{-} = 1$. Note see that for the range $R \leq 2$ we have $\lambda_\pm > 0$ and that for the extremum of $R = 2$ the lambda are degenerate: $\lambda_\pm = 1$.

scalar of the complex-structure moduli-space. Note that R is bound to be at most 2, thus ensuring that $\lambda_\pm \geq 0$ and the masses are real.[15] We found two different λ such that from equation (2.1.14) we find 4 masses. We use notation where $m^2_{\pm\pm}$ represent these masses. The first $\pm$ in the subscript corresponds to whether we take the λ_+ or λ_- option and the second $\pm$ in the subscript corresponds to whether we take the $+$ or $-$ that shows up in equation (2.1.14).

$$m^2_{\pm\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2} \left(\sqrt{\lambda_\pm} \sqrt{1 - \frac{1}{\mu^2}} \pm \frac{1}{\mu} \right)^2. \quad (2.2.7)$$

For a plot of these masses see figure 2.2.

2.3 $h^{2,1}_{-} = 1$ and Large-complex-structure limit

In this section we calculate the masses of the moduli for a special case of the last section. So $h^{2,1}_{-} = 1$ such that there is again only one complex structure modulus z^1

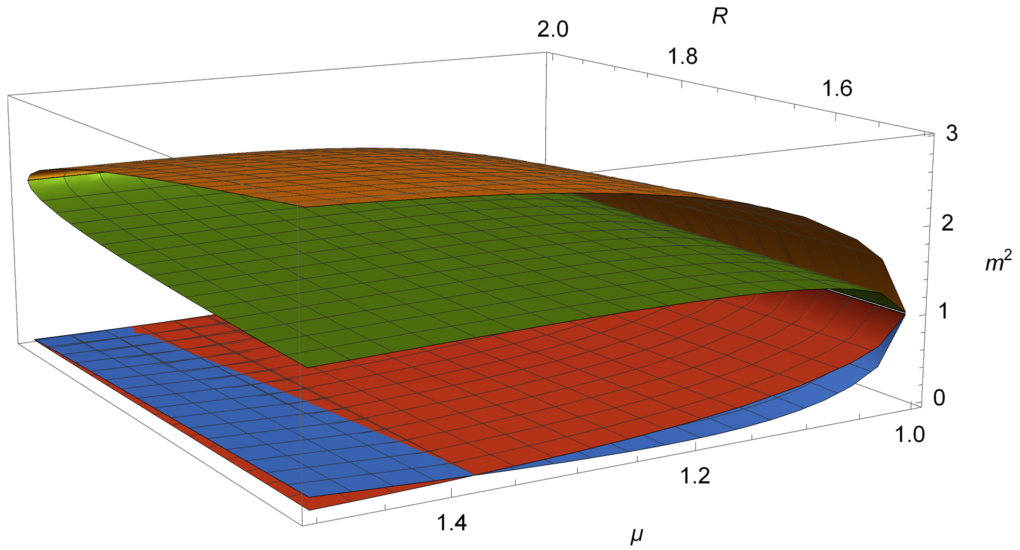

Figure 2.2: This figure shows the $\frac{\mathcal{V}^2}{N_{\text{flux}}} m^2_{\pm\pm}$ dependent on μ and the Ricci scalar R. The case we consider here is $h_-^{2,1} = 1$. Note that all masses are equal when $\mu = 1$.

and again the i index only takes the value 1 as well as $\mathbf{Q}\overline{\mathbf{Q}}$ being a 2 by 2 matrix. The new part is that we assume the large-complex-structure limit such that we have

$$\mathcal{F} = -\frac{1}{3!}\frac{\hat{\kappa}_{111}(X^1)^3}{X^0} \tag{2.3.1}$$

This is taken from [9] equation (4.4) also see [11]. In this expression $\hat{\kappa}_{111}$ is a constant, this constant is related to κ as we will determine in (2.3.15).

We will now determine the value of R in this case such that we can plug this into the expressions we found before. We do this by first preparing some relations that we can then use to calculate a specific value for R in this case.

2.3.1 Calculating $\int_{\mathcal{X}} \Omega \wedge \overline{\Omega}$

From equation (1.14.3) we find that in this setting we have

$$\Omega = X^0\alpha_0 + X^1\alpha_1 - \partial_0\mathcal{F}\beta^0 - \partial_1\mathcal{F}\beta^1. \tag{2.3.2}$$

Now we use

$$z^i = \frac{X^i}{X^0} \Rightarrow X^1 = X^0 z \tag{2.3.3}$$

and (2.3.1) to find

$$\begin{aligned} \mathcal{F} =& -\frac{1}{3!}\frac{\hat{\kappa}_{111}(X^1)^3}{X^0}, \\ \partial_0\mathcal{F} =& \frac{1}{3!}\frac{\hat{\kappa}_{111}(X^1)^3}{(X^0)^2} = \frac{1}{3!}\hat{\kappa}_{111}X^0 z^3, \\ \partial_1\mathcal{F} =& -\frac{1}{2}\frac{\hat{\kappa}_{111}(X^1)^2}{X^0} = -\frac{1}{2}\hat{\kappa}_{111}X^0 z^2. \end{aligned} \tag{2.3.4}$$

This results in:

$$\Omega = X^0 \left(\alpha_0 + z\alpha_1 - \frac{1}{3!}\hat{\kappa}_{111} z^3 \beta^0 + \frac{1}{2}\hat{\kappa}_{111} z^2 \beta^1 \right). \tag{2.3.5}$$

Now to calculate $\int_{\mathcal{X}} \Omega \wedge \overline{\Omega}$ we plug in the relations we just found for Ω and $\overline{\Omega}$ and use the relations in equation (1.14.4). This leads to:

$$\int \Omega \wedge \overline{\Omega} = |X^0|^2 \hat{\kappa}_{111} \left(-\frac{1}{3!}(\overline{z})^3 + \frac{1}{2}(\overline{z})^2 z + \frac{1}{3!}(z)^3 - \frac{1}{2}(z)^2 \overline{z} \right). \tag{2.3.6}$$

Then we split z into its real and imaginary part such that $z = u + vi$. We plug this in and find:

$$\int \Omega \wedge \overline{\Omega} = -|X^0|^2 \hat{\kappa}_{111} \frac{4}{3} i v^3. \tag{2.3.7}$$

2.3.2 K and its z derivative

With the result we found for $\int \Omega \wedge \overline{\Omega}$ in hand we are ready to express K in therms of v and s (using $\tau = c + is$). Recall that from equation (1.14.5) we found that

$$\begin{aligned} K =& -\log\left[-i(\tau - \overline{\tau})\right] - 2\log\left[\mathcal{V}\right] - \log\left[i \int \Omega \wedge \overline{\Omega} \right], \\ =& -\log\left[2s\right] - 2\log\left[\mathcal{V}\right] - \log\left[|X^0|^2 \hat{\kappa}_{111} \frac{4}{3} v^3 \right]. \end{aligned} \tag{2.3.8}$$

Now we have an expression for K we want to also find $\partial_z K$. To this end we first calculate $\partial_z v$:

$$\begin{aligned} z =& u + vi, \\ v =& i\frac{\overline{z} - z}{2}, \\ \partial_z v =& \frac{-i}{2}. \end{aligned} \tag{2.3.9}$$

With this we can calculate $\partial_z K$. Note that only one of it's therms is dependent on z. Therefore

$$\begin{aligned} \partial_z K =& \partial_z \left(-\log\left[2s\right] - 2\log\left[\mathcal{V}\right] - \log\left[|X^0|^2 \hat{\kappa}_{111} \frac{4}{3} v^3 \right] \right), \\ =& -\frac{\partial_z \left[|X^0|^2 \hat{\kappa}_{111} \frac{4}{3} v^3 \right]}{|X^0|^2 \hat{\kappa}_{111} \frac{4}{3} v^3}, \\ =& \frac{3i}{2v}. \end{aligned} \tag{2.3.10}$$

2.3.3 Relating κ and $\hat{\kappa}_{111}$

From (1.14.20) we know that in this case where $h^{2,1}_{-} = 1$ we have

$$\kappa = -\int_{\mathcal{X}} \Omega \wedge D_z D_z D_z \Omega \tag{2.3.11}$$

where $D_z\Omega = \partial_z\Omega + (\partial_z K)\Omega$.

We can expand $D_zD_zD_z\Omega$ using this expression for $D_z\Omega$. Then we want to find expressions for the z derivatives of Ω and K. Using what we found in (2.3.5) we can calculate the z derivatives of Ω:

$$\partial_z\partial_z\partial_z\Omega = -\, X^0\hat{\kappa}_{111}\beta^0. \tag{2.3.12}$$

Then we calculate the z derivatives of K using equation (2.3.10) to find:

$$\partial_z K = \frac{3i}{2v}, \quad \partial_z\partial_z K = -\frac{3}{4v^2}, \quad \partial_z\partial_z\partial_z K = -\frac{3i}{4v^3}. \tag{2.3.13}$$

Next we plug our expressions for the z derivatives into $D_zD_zD_z\Omega$ and we find an expansion of Ω in the α_I β^I basis in therms of v and z:

$$\begin{aligned}\frac{D_zD_zD_z\Omega}{X^0} =&\alpha_0\left[-\frac{15i}{2v^3}\right] - \alpha_1\left[\frac{15i}{2v^3}z + \frac{9}{v^2}\right] - \hat{\kappa}_{111}\beta^0\left[1 + \frac{9i}{2v}z - \frac{9}{2v^2}z^2 - \frac{10i}{8v^3}z^3\right]\\ &+ \hat{\kappa}_{111}\beta^1\left[\frac{9i}{2v} - \frac{9}{v^2}z - \frac{15i}{4v^3}z^2\right].\end{aligned} \tag{2.3.14}$$

Substituting this into (1.14.20) we can use the equations relating the α_I and β^I (1.14.4) as well as the expansion of Ω in this basis (2.3.5) to relate κ and $\hat{\kappa}_{111}$. We find that

$$\kappa = (X^0)^2\hat{\kappa}_{111}. \tag{2.3.15}$$

2.3.4 The Ricci scalar

Now we have this we are almost ready to calculate R in this setting. From equation (1.11.2), (2.3.10) and (2.3.9) we find an expression for G in therms of v:

$$G = \frac{3}{4v^2}. \tag{2.3.16}$$

Also, using equation (1.14.5), (2.3.15) and (2.3.7) we find expressions for the exponential of parts of the Kähler potential in therms of v and s such that:

$$\mathrm{e}^{-K_{cs}} = \frac{\overline{X^0}}{X^0}\kappa\frac{4}{3}v^3 \quad \text{and} \quad \mathrm{e}^{-K_\tau} = 2s. \tag{2.3.17}$$

With what we just found we can give an expression for the Ricci scalar R. From (2.2.5) we find

$$R = 2 - \mathrm{e}^{2K_{cs}}G^{-3}|\kappa|^2. \tag{2.3.18}$$

Now we plug in (2.3.17) and (2.3.16). Then we further simplify using (2.3.15) and find:

$$
\begin{aligned}
R =& 2 - \frac{(X^0)^2}{\overline{X^0}^2}\kappa^{-2}\frac{9}{16}\upsilon^{-6}\frac{64\upsilon^6}{27}|\kappa|^2, \\
=& 2 - \frac{(X^0)^2}{\overline{X^0}^2}\kappa^{-1}\frac{4}{3}\overline{\kappa}, \\
=& 2 - (X^0)^2\kappa^{-1}\frac{4}{3}\hat{\kappa}_{111}, \\
=& 2 - \frac{4}{3}, \\
R =& \frac{2}{3}.
\end{aligned} \tag{2.3.19}
$$

2.3.5 Masses

As we have found the value of R in the large complex structure limit we can plug this into (2.2.6) and find that $\lambda_\pm$ takes the values:

$$
\lambda_+ = 3 \quad \text{and} \quad \lambda_- = \frac{1}{3} \tag{2.3.20}
$$

We can use these results and equation (2.1.14) to calculate the masses in the large complex structure limit:

$$
\begin{aligned}
m^2_{--} =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\left(\frac{1}{\sqrt{3}}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu}\right)^2, \\
m^2_{+-} =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\left(\sqrt{3}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu}\right)^2, \\
m^2_{-+} =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\left(\frac{1}{\sqrt{3}}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu}\right)^2, \\
m^2_{++} =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\left(\sqrt{3}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu}\right)^2.
\end{aligned} \tag{2.3.21}
$$

We have to note that this only holds when $W_0 \neq 0$. If $W_0 = 0$ then, from (2.1.2) we know that

$$
\begin{aligned}
m^2_{--} = m^2_{-+} =& e^K\left(\frac{1}{\sqrt{3}}e^{-(K_\tau+K_{cs})}\sqrt{G}|h|\right)^2 \quad \text{and} \\
m^2_{+-} = m^2_{++} =& e^K\left(\sqrt{3}e^{-(K_\tau+K_{cs})}\sqrt{G}|h|\right)^2.
\end{aligned} \tag{2.3.22}
$$

Then using what we found in (2.1.11) we calculate for $W_0 = 0$ that

$$
\begin{aligned}
m^2_{--} = m^2_{-+} =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\frac{1}{\sqrt{3}} \quad \text{and} \\
m^2_{+-} = m^2_{++} =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\sqrt{3}.
\end{aligned} \tag{2.3.23}
$$

Here $\mathcal{V}$ is the Einstein-frame volume we encountered in (1.14.5). This can be determined during the stabilization of the Kähler moduli.

Chapter 3

Calculations on $h_{-}^{2,1} = 2$

In this section we will again calculate the masses of the moduli. Only this time we set $h_{-}^{2,1} = 2$. This means that we now have an extra modulus, namely the complex modulus τ and both z^1 and z^2. So the index i runs over $1, 2$. Our calculations will follow along similar lines as before. Firstly we will simplify the matrix $\mathbf{Q}\overline{\mathbf{Q}}$ and calculate it's eigenvalues. Secondly we calculate an expression for the masses in therms of the eigenvalues of a matrix M that is a rescaling of $\mathbf{Q}\overline{\mathbf{Q}}$. After that we can discuss under which circumstances there are degenerate masses.

3.1 Eigenvalues λ

3.1.1 Simplifying $\mathbf{Q}\overline{\mathbf{Q}}$

To find the σ_α we need the eigenvalues of $\mathbf{Q}\overline{\mathbf{Q}}$. Recall from (1.14.22) that

$$\begin{aligned}
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{\tau}}{}_{\overline{\tau}} =& \mathrm{e}^{-2(K_\tau+K_{cs})} h^i G_{i\overline{j}} h^{\overline{j}}, \\
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{\tau}}{}_{\overline{j}} =& \mathrm{e}^{-(K_\tau+K_{cs})} (\tau-\overline{\tau})^2 \kappa_{\overline{j}\overline{m}\overline{n}} h^{\overline{m}} h^{\overline{n}}, \\
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{i}}{}_{\overline{\tau}} =& -\mathrm{e}^{-(K_\tau+K_{cs})} \kappa^{\overline{i}}{}_{mn} h^m h^n, \\
(\mathbf{Q}\overline{\mathbf{Q}})^{\overline{i}}{}_{\overline{j}} =& -\mathrm{e}^{-2K_{cs}} (\tau-\overline{\tau})^2 h^{\overline{i}} G_{\overline{j}k} h^k - (\tau-\overline{\tau})^2 \kappa^{\overline{i}}{}_{lk} h^k \kappa^{l}{}_{\overline{j}\overline{m}} h^{\overline{m}}.
\end{aligned} \tag{3.1.1}$$

To be able to find it's eigenvalues we will define some other matrix $\hat{Q}$ with the same eigenvalues. To this end we introduce the einbein e_i^a and $\overline{e}_{\overline{j}}^{\overline{b}}$ together with their inverses e_a^i and $\overline{e}_{\overline{b}}^{\overline{j}}$. We define this such that

$$\begin{aligned}
G_{i\overline{j}} =& e_i^a \delta_{a\overline{b}} \overline{e}_{\overline{j}}^{\overline{b}}, \\
e_i^a e_b^i =& \delta_b^a.
\end{aligned} \tag{3.1.2}$$

We can now use this to change all the indices from i, j, k to a, b, c which are raised and lowered by δ_{ab} and δ^{ab} instead of $G_{i\overline{j}}$. This will simplify the expressions we will work with. To this end we define

$$\begin{aligned}
h^a =& e_i^a h^i, \\
\kappa_{abc} =& e_a^i e_b^j e_c^k \kappa_{ijk}.
\end{aligned} \tag{3.1.3}$$

These definitions make it possible to get rid of $G_{i\bar{j}}$ in the matrix and replace it by the more easily to handle $\delta_{a\bar{b}}$. We do this by noting that

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{\overline{a}}_{\overline{i}} \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & e^{\overline{i}}_{\overline{b}} \end{bmatrix} \tag{3.1.4}$$

are each-others inverse such that if we define

$$\hat{Q} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\overline{a}}_{\overline{i}} \end{bmatrix} \mathbf{Q}\overline{\mathbf{Q}} \begin{bmatrix} 1 & 0 \\ 0 & e^{\overline{i}}_{\overline{b}} \end{bmatrix} \tag{3.1.5}$$

it has the same eigenvalues σ^2_α as $\mathbf{Q}\overline{\mathbf{Q}}$.
This leads to $\hat{Q}$ being

$$\begin{aligned} \hat{Q}^{\overline{\tau}}_{\overline{\tau}} =& 4s^2 \mathrm{e}^{-2K_{cs}} h^a \delta_{a\bar{b}} h^{\bar{b}}, \\ \hat{Q}^{\overline{\tau}}_{\overline{a}} =& -8s^3 \mathrm{e}^{-K_{cs}} \kappa_{\overline{a}\bar{b}\bar{c}} h^{\bar{b}} h^{\bar{c}}, \\ \hat{Q}^{\overline{a}}_{\overline{\tau}} =& -2s \mathrm{e}^{-K_{cs}} \delta^{\overline{a}b} \kappa_{bcd} h^c h^d, \\ \hat{Q}^{\overline{a}}_{\bar{b}} =& 4s^2 \mathrm{e}^{-2K_{cs}} h^{\overline{a}} \delta_{\bar{b}c} h^c + 4s^2 \kappa^{\overline{a}}_{\ cd} h^d \kappa^{c}_{\ \overline{f}\bar{b}} h^{\overline{f}}. \end{aligned} \tag{3.1.6}$$

Now we want to define a few things to further simplify this matrix. First we define the vector h and it's norm with respect to G:

$$\begin{aligned} h =& \begin{pmatrix} h^1 \\ h^2 \end{pmatrix} \text{ and} \\ ||h||^2 =& h^a \delta_{a\bar{b}} h^{\bar{b}} = |h^1|^2 + |h^2|^2. \end{aligned} \tag{3.1.7}$$

Note that here all the indices are of the a, b, c type.
With these conventions we can define a normal vector of h, namely $\hat{e}$, such that

$$\hat{e} = \frac{h}{||h||} \text{ and } \overline{\hat{e}} = \frac{\overline{h}}{||h||} \tag{3.1.8}$$

These have the nice property that

$$\hat{e} \cdot \overline{\hat{e}} = \hat{e}^a \delta_{a\bar{b}} \, \overline{\hat{e}}^{\bar{b}} = 1. \tag{3.1.9}$$

Lastly we define the 2 by 2 matrix κ to be

$$[\kappa]^{\overline{a}}_{\ d} = \kappa^{\overline{a}}_{\ cd} \hat{e}^c. \tag{3.1.10}$$

Now adopting matrix notation instead of index notation and denoting that transpose with T, we find an expression for $\hat{Q}$:

$$\hat{Q} = 4s^2 \mathrm{e}^{-2K_{cs}} ||h||^2 \begin{bmatrix} 1 & -2s\mathrm{e}^{K_{cs}} \overline{\hat{e}}^T \overline{\kappa}^T \\ -\frac{\mathrm{e}^{K_{cs}}}{2s} \kappa \hat{e} & \overline{\hat{e}} \hat{e}^T + \mathrm{e}^{2K_{cs}} \kappa \overline{\kappa} \end{bmatrix}. \tag{3.1.11}$$

Then to simplify even further we note from (1.14.20) that κ is symmetric so there exist real numbers κ_1 and κ_2 as well as a unitary matrix U such that

$$U \kappa U^T = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}. \tag{3.1.12}$$

With this we can define the matrix q that has the same eigenvalues as $\hat{Q}$. We set

$$q = \begin{bmatrix} 1 & 0 \\ 0 & -2sU \end{bmatrix} \hat{Q} \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2s}U^\dagger \end{bmatrix}. \tag{3.1.13}$$

Note that the matrices right and left of $\hat{Q}$ are each others inverse such that the eigenvalues of q are still σ_α^2.
This evaluates to

$$q = 4s^2 \mathrm{e}^{-2K_{cs}} ||h||^2 \begin{bmatrix} 1 & \mathrm{e}^{K_{cs}} \overline{\hat{e}}^T U^T \overline{U} \overline{\kappa} \overline{U}^T \\ \mathrm{e}^{K_{cs}} U \kappa U^T \overline{U} \hat{e} & \overline{(\overline{U}\hat{e})} (\overline{U}\hat{e})^T + \mathrm{e}^{2K_{cs}} U \kappa U^T \overline{U} \overline{\kappa} \overline{U}^T \end{bmatrix}. \tag{3.1.14}$$

From this we see that we can define $\tilde{e} = \overline{U}\hat{e}$ that has the same kind of property as $\hat{e}$:

$$\tilde{e} \cdot \overline{\tilde{e}} = \hat{e}^T \overline{U}^T U \overline{\hat{e}} = \hat{e} \cdot \overline{\hat{e}} = 1. \tag{3.1.15}$$

This means we can write the components of $\tilde{e}$ in their polar coordinates in therms of the real r, ϕ_1, ϕ_2 or θ, ϕ_1, ϕ_2 in the following way:

$$\tilde{e} = \begin{pmatrix} r\mathrm{e}^{i\phi_1} \\ \sqrt{1-r^2}\mathrm{e}^{i\phi_2} \end{pmatrix} = \begin{pmatrix} \cos(\theta)\mathrm{e}^{i\phi_1} \\ \sin(\theta)\mathrm{e}^{i\phi_2} \end{pmatrix}. \tag{3.1.16}$$

Using this we find q to be:

$$\begin{aligned} q =& \frac{4s^2}{\mathrm{e}^{2K_{cs}}} ||h||^2 \begin{bmatrix} 1 & \mathrm{e}^{K_{cs}} \kappa_1 r \mathrm{e}^{-i\phi_1} & \mathrm{e}^{K_{cs}} \kappa_2 \sqrt{1-r^2} \mathrm{e}^{-i\phi_2} \\ \mathrm{e}^{K_{cs}} \kappa_1 r \mathrm{e}^{i\phi_1} & \mathrm{e}^{2K_{cs}} \kappa_1^2 + r^2 & r\sqrt{1-r^2} \mathrm{e}^{-i(\phi_1-\phi_2)} \\ \mathrm{e}^{K_{cs}} \kappa_2 \sqrt{1-r^2} \mathrm{e}^{i\phi_2} & r\sqrt{1-r^2} \mathrm{e}^{i(\phi_1-\phi_2)} & \mathrm{e}^{2K_{cs}} \kappa_2^2 + 1 - r^2 \end{bmatrix} \\ =& \frac{4s^2}{\mathrm{e}^{2K_{cs}}} ||h||^2 \begin{bmatrix} 1 & \mathrm{e}^{K_{cs}} \kappa_1 \cos(\theta) \mathrm{e}^{-i\phi_1} & \mathrm{e}^{K_{cs}} \kappa_2 \sin(\theta) \mathrm{e}^{-i\phi_2} \\ \mathrm{e}^{K_{cs}} \kappa_1 \cos(\theta) \mathrm{e}^{i\phi_1} & \mathrm{e}^{2K_{cs}} \kappa_1^2 + \cos^2(\theta) & \cos(\theta)\sin(\theta) \mathrm{e}^{i(\phi_2-\phi_1)} \\ \mathrm{e}^{K_{cs}} \kappa_2 \sin(\theta) \mathrm{e}^{i\phi_2} & \cos(\theta)\sin(\theta) \mathrm{e}^{i(\phi_1-\phi_2)} & \mathrm{e}^{2K_{cs}} \kappa_2^2 + \sin^2(\theta) \end{bmatrix}. \end{aligned} \tag{3.1.17}$$

We can express $q = aM$. Where $a = 4s^2 \mathrm{e}^{-2K_{cs}} ||h||^2$ is a scalar and

$$\begin{aligned} M =& \begin{bmatrix} 1 & \mathrm{e}^{K_{cs}} \kappa_1 r \mathrm{e}^{-i\phi_1} & \mathrm{e}^{K_{cs}} \kappa_2 \sqrt{1-r^2} \mathrm{e}^{-i\phi_2} \\ \mathrm{e}^{K_{cs}} \kappa_1 r \mathrm{e}^{i\phi_1} & \mathrm{e}^{2K_{cs}} \kappa_1^2 + r^2 & r\sqrt{1-r^2} \mathrm{e}^{-i(\phi_1-\phi_2)} \\ \mathrm{e}^{K_{cs}} \kappa_2 \sqrt{1-r^2} \mathrm{e}^{i\phi_2} & r\sqrt{1-r^2} \mathrm{e}^{i(\phi_1-\phi_2)} & \mathrm{e}^{2K_{cs}} \kappa_2^2 + 1 - r^2 \end{bmatrix} \\ =& \begin{bmatrix} 1 & \mathrm{e}^{K_{cs}} \kappa_1 \cos(\theta) \mathrm{e}^{-i\phi_1} & \mathrm{e}^{K_{cs}} \kappa_2 \sin(\theta) \mathrm{e}^{-i\phi_2} \\ \mathrm{e}^{K_{cs}} \kappa_1 \cos(\theta) \mathrm{e}^{i\phi_1} & \mathrm{e}^{2K_{cs}} \kappa_1^2 + \cos^2(\theta) & \cos(\theta)\sin(\theta) \mathrm{e}^{-i(\phi_1-\phi_2)} \\ \mathrm{e}^{K_{cs}} \kappa_2 \sin(\theta) \mathrm{e}^{i\phi_2} & \cos(\theta)\sin(\theta) \mathrm{e}^{i(\phi_1-\phi_2)} & \mathrm{e}^{2K_{cs}} \kappa_2^2 + \sin^2(\theta) \end{bmatrix}. \end{aligned} \tag{3.1.18}$$

a matrix. So if λ_α is an eigenvalue of M then $a\lambda_\alpha = \sigma_\alpha^2$ is an eigenvalue of $aM = q$ and therefore of $\mathbf{Q}\overline{\mathbf{Q}}$.(2.0.1)

3.1.2 Characteristic polynomial

Now we are ready to calculate λ by setting the characteristic polynomial to 0. We find:

$$\begin{aligned}0 =& \lambda^3 - \lambda^2\left[2 + \mathrm{e}^{2K_{cs}}(\kappa_1^2+\kappa_2^2)\right]\\&+\lambda\left[\mathrm{e}^{4K_{cs}}\kappa_1^2\kappa_2^2 + 2\mathrm{e}^{K_{cs}}r^2(\kappa_2^2-\kappa_1^2) + 2\mathrm{e}^{2K_{cs}}\kappa_1^2 + 1\right]\\&-2\mathrm{e}^{2K_{cs}}r^4\kappa_2^2 - 2\mathrm{e}^{2K_{cs}}r^2(1-r^2)\kappa_1\kappa_2\cos(\phi_1-\phi_2) - \kappa_1^2(1-r^2)^2\mathrm{e}^{K_{cs}},\\[1em]0 =& \lambda^3 - \lambda^2\left[2 + \mathrm{e}^{2K_{cs}}(\kappa_1^2+\kappa_2^2)\right]\\&+\lambda\left[\mathrm{e}^{4K_{cs}}\kappa_1^2\kappa_2^2 + 2\mathrm{e}^{K_{cs}}\cos^2(\theta)\kappa_2^2 + 2\mathrm{e}^{2K_{cs}}\kappa_1^2\sin^2(\theta) + 1\right]\\&-2\mathrm{e}^{2K_{cs}}\cos^4(\theta)\kappa_2^2 - 2\mathrm{e}^{2K_{cs}}\cos^2(\theta)\sin^2(\theta)\kappa_1\kappa_2\cos(\phi_1-\phi_2) - \kappa_1^2\sin^4(\theta)\mathrm{e}^{K_{cs}}.\end{aligned} \tag{3.1.19}$$

Now we notice that ϕ_1 and ϕ_2 only appear in the combination $\phi_1 - \phi_2$. Therefore we define $\phi = \phi_1 - \phi_2$ and simplify to:

$$\begin{aligned}0 =& \lambda^3 - \lambda^2\left[2 + \mathrm{e}^{2K_{cs}}(\kappa_1^2+\kappa_2^2)\right]\\&+\lambda\left[\mathrm{e}^{4K_{cs}}\kappa_1^2\kappa_2^2 + 2\mathrm{e}^{K_{cs}}r^2(\kappa_2^2-\kappa_1^2) + 2\mathrm{e}^{2K_{cs}}\kappa_1^2 + 1\right]\\&-2\mathrm{e}^{2K_{cs}}r^4\kappa_2^2 - 2\mathrm{e}^{2K_{cs}}r^2(1-r^2)\kappa_1\kappa_2\cos(\phi) - \kappa_1^2(1-r^2)^2\mathrm{e}^{K_{cs}},\\[1em]0 =& \lambda^3 - \lambda^2\left[2 + \mathrm{e}^{2K_{cs}}(\kappa_1^2+\kappa_2^2)\right]\\&+\lambda\left[\mathrm{e}^{4K_{cs}}\kappa_1^2\kappa_2^2 + 2\mathrm{e}^{K_{cs}}\cos^2(\theta)\kappa_2^2 + 2\mathrm{e}^{2K_{cs}}\kappa_1^2\sin^2(\theta) + 1\right]\\&-2\mathrm{e}^{2K_{cs}}\cos^4(\theta)\kappa_2^2 - 2\mathrm{e}^{2K_{cs}}\cos^2(\theta)\sin^2(\theta)\kappa_1\kappa_2\cos(\phi) - \kappa_1^2\sin^4(\theta)\mathrm{e}^{K_{cs}}.\end{aligned} \tag{3.1.20}$$

From these equations we see that λ depends on 4 real parameters: $\mathrm{e}^{K_{cs}}\kappa_1$, $\mathrm{e}^{K_{cs}}\kappa_2$, r or θ and ϕ.

3.2 Masses in therms of λ

In this section we will express the masses in therms of the λ we just defined and analysed. This will be done along the same lines as in section 2.1.1.

Now we have $a = 4s^2\mathrm{e}^{-2K_{cs}}||h||^2$ and $\sigma_\alpha^2 = 4s^2\mathrm{e}^{-2K_{cs}}||h||^2\lambda$. Therefore we have

$$\sigma_\alpha = 2s\mathrm{e}^{-K_{cs}}||h||\sqrt{\lambda} \tag{3.2.1}$$

This we can plug in to (4.3.1) to find:

$$m_{\alpha\pm}^2 = \mathrm{e}^K\left(2s\mathrm{e}^{-K_{cs}}||h||\sqrt{\lambda} \pm |W_0|\right)^2. \tag{3.2.2}$$

We want to re-express the masses in more natural quantities. In order to express $G_{i\bar{j}}$ and h^i in therms of these quantities we use from the literature: equation (3.2) in the paper [14]

$$G_{i\bar{j}} = -\frac{\int \chi_i \wedge \overline{\chi}_{\bar{j}}}{\int \Omega \wedge \overline{\Omega}}. \tag{3.2.3}$$

Now we can use the expression (1.14.18) we have for H, the expression (1.14.5) for K_{cs}, (2.1.6), (2.1.8), (2.1.4) as well as the expression we just quoted from the literature to calculate $\int H \wedge \star H$. This leads to

$$\begin{aligned}
&\int H \wedge \star H \\
=&2i \int \left(|h^0|^2 \Omega \wedge \overline{\Omega} - |h^1|^2 \chi_1 \wedge \overline{\chi}_1 - |h^2|^2 \chi_2 \wedge \overline{\chi}_2 - \overline{h^1} h^2 \chi_2 \wedge \overline{\chi}_1 - h^1 \overline{h^2} \chi_1 \wedge \overline{\chi}_2 \right) \\
=&2|h^0|^2 \mathrm{e}^{-K_{cs}} + 2i h^i G_{i\bar{j}} h^{\bar{j}} \int \Omega \wedge \overline{\Omega} \\
=&2\mathrm{e}^{-K_{cs}} \left(|h^0|^2 + ||h||^2 \right)
\end{aligned} \tag{3.2.4}$$

Now we have related $\int H \wedge \star H$ to h^0 and $||h||$. With these relations we can, find the analogous relation to (2.1.11):

$$\begin{aligned}
||h||^2 =&\frac{1}{2} \mathrm{e}^{K_{cs}} \int H \wedge \star H - |h^0|^2 \\
=&\mathrm{e}^{K_\tau + K_{cs}} N_{\text{flux}} - \mathrm{e}^{2(K_\tau + K_{cs})} |W_0|^2
\end{aligned} \tag{3.2.5}$$

We plug this in to (3.2.2) and find:

$$\begin{aligned}
m^2_{\alpha\pm} =&\mathrm{e}^K \left(2s\mathrm{e}^{-K_{cs}} ||h|| \sqrt{\lambda_\alpha} \pm |W_0| \right)^2, \\
=&\mathrm{e}^{-K_\tau - K_{cs} - 2\log[\mathcal{V}]} \left(\sqrt{h^i G_{i\bar{j}} h^{\bar{j}}} \sqrt{\lambda_\alpha} \pm |h^0| \right)^2.
\end{aligned} \tag{3.2.6}$$

Using the same definition for μ as before:

$$\mu^2 = \frac{N_{\text{flux}}}{\mathrm{e}^{K_\tau + K_{cs}} |W_0|^2} \tag{3.2.7}$$

leads again to

$$\begin{aligned}
m^2_{\alpha\pm} =&\mathrm{e}^K \left(\mathrm{e}^{-(K_\tau + K_{cs})} \sqrt{\mathrm{e}^{K_\tau + K_{cs}} N_{\text{flux}} - \mathrm{e}^{2(K_\tau + K_{cs})} |W_0|^2} \sqrt{\lambda_\alpha} \pm |W_0| \right)^2, \\
=&\mathrm{e}^K \left(|W_0| \sqrt{\frac{N_{\text{flux}}}{\mathrm{e}^{K_\tau + K_{cs}} |W_0|^2} - 1} \sqrt{\lambda_\alpha} \pm |W_0| \right)^2, \\
=&\mathrm{e}^K |W_0|^2 \left(\sqrt{\lambda_\alpha} \sqrt{\mu^2 - 1} \pm 1 \right)^2, \\
=&\frac{N_{\text{flux}}}{\mu^2} \mathrm{e}^{-2\log[\mathcal{V}]} \left(\sqrt{\lambda_\alpha} \sqrt{\mu^2 - 1} \pm 1 \right)^2, \\
=&\frac{N_{\text{flux}}}{\mathcal{V}^2} \left(\sqrt{\lambda_\alpha} \sqrt{1 - \frac{1}{\mu^2}} \pm \frac{1}{\mu} \right)^2.
\end{aligned} \tag{3.2.8}$$

These masses are shown in figure 3.1.

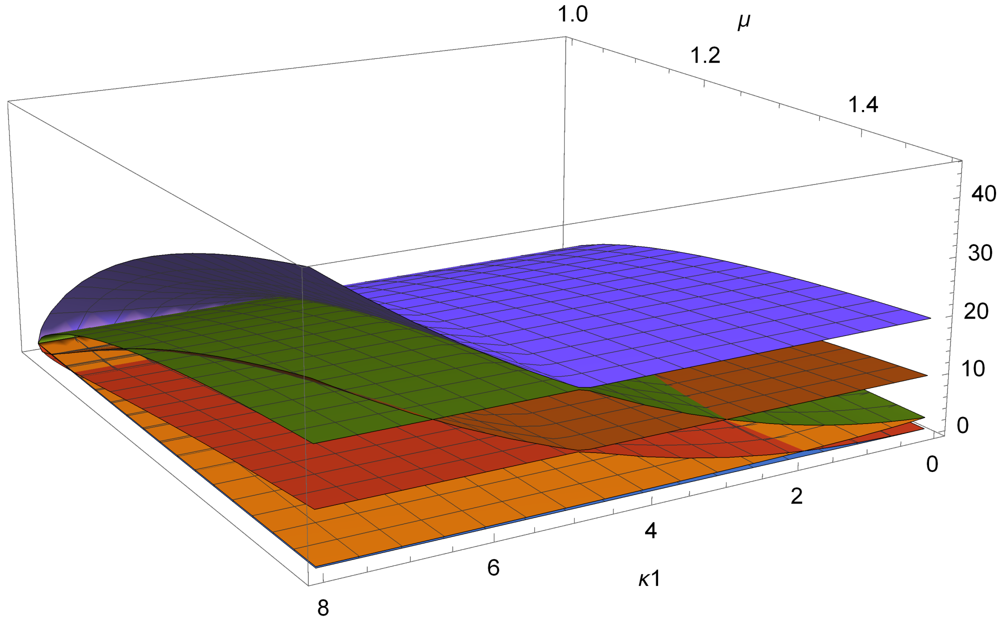

Figure 3.1: This figure shows the six $\frac{\mathcal{V}^2}{N_{\text{flux}}} m^2_{\alpha\pm}$ dependent on μ and $e^{K_{cs}}\kappa_1$ for $e^{K_{cs}}\kappa_2 = 5$ and $\theta = \phi = 0$. The case we consider here is $h^{2,1}_{-} = 2$. Note that all masses are equal when $\mu = 1$ and the masses seem to come in pairs that are close together.

Chapter 4

Masses analysis

In this section we will discuss some of the features of the masses that stand out. To this end we first list the cases when there are degenerate masses. Then we will use this to find interesting features focusing on cases where all masses are equal and limits of the moduli space.

Before we start recall the definition of μ (2.1.13). Note that we can use equation (3.2.4) together with (2.1.6) to write μ in two different ways:

$$\mu^2 = \frac{N_{\text{flux}}}{\mathrm{e}^{K_\tau + K_{cs}} |W_0|^2} = 1 + \frac{||h||^2}{|h^0|^2} \tag{4.0.1}$$

With this in hand we will start analysing the masses we found.

4.1 $h_-^{2,1} = 1$

In the case of only one complex structure modulus z^1 we found the following values for $\lambda_\pm$ (2.2.6):

$$\lambda_\pm = \frac{(4-R) \pm \sqrt{12 - 8R + R^2}}{2}. \tag{4.1.1}$$

Also see figure 2.1.

The 4 masses we found then are (2.2.7):

$$m^2_{\pm\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2} \left(\sqrt{\lambda_\pm} \sqrt{1 - \frac{1}{\mu^2}} \pm \frac{1}{\mu} \right)^2. \tag{4.1.2}$$

Also see figure 2.2.

Now we can analyse when (some of) these masses are equal.

4.1.1 Equal masses 1

Firstly we consider the case $|W_0| \neq 0$:

- if $\mu^2 = 1$ then all masses are equal $m^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}$.
- if $R = 2$ then $\lambda_+ = \lambda_- = 1$ and $m^2_{+\pm} = m^2_{-\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}\left(\sqrt{\mu^2-1} \pm 1\right)^2$.
- if $\sqrt{\lambda_+} + \sqrt{\lambda_-} = \frac{2}{\sqrt{\mu^2-1}} = \frac{2|h^0|}{||h||}$ then $m^2_{--} = m^2_{+-}$.
- if $\sqrt{\lambda_+} - \sqrt{\lambda_-} = \frac{2}{\sqrt{\mu^2-1}} = \frac{2|h^0|}{||h||}$ then $m^2_{+-} = m^2_{-+}$.

Now we can also consider the masses when $|W_0| = 0$. In this case we have to go back to (4.3.1). When we use (2.1.1), (2.1.11) we then find:

$$\begin{aligned} m^2_{\pm\pm} =& \mathrm{e}^K \sigma^2_\pm, \\ =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\lambda_\pm. \end{aligned} \tag{4.1.3}$$

So the masses $m^2_{\pm+} = m^2_{\pm-}$ are equal anyway and also:

- if $R = 2$ then $\lambda_+ = \lambda_- = 1$ and $m^2_{\pm\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2}$.

These are all the options for masses to be equal.

Next we will further analyse some interesting cases.

4.1.2 All 4 masses equal

All 4 masses can only be equal if one of two things are the case. Either $R = 2$ and $|W_0| = 0$, this case we will consider below, or $\mu^2 = 1$, this case we will discuss now.

1: $\mu^2 = 1$

If $\mu^2 = 1$ we can see from (2.2.7) that the eigenvalues $\lambda_\pm$ become irrelevant and all masses take the value $m^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}$. This case becomes interesting when we consider (4.0.1). If $\mu^2 = 1$ we see from this that $h^1 = 0$. This means that the H-flux

$$H = h^0\Omega + \overline{h^0\Omega}. \tag{4.1.4}$$

Thus H can then only be an element of the space span by Ω and $\overline{\Omega}$.

2: $R = 2$ **and** $|W_0| = 0$

The other option for all masses to be equal is to have $R = 2$ and $|W_0| = 0$. In this case $m^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}$. When we then consider (2.1.10) we see that this means that $h^0 = 0$. This makes the H-flux becomes

$$H = h^1\chi_1 + \overline{h}^{\overline{1}}\overline{\chi}_{\overline{1}}. \tag{4.1.5}$$

Thus H can then only be an element of the space span by χ_1 and $\overline{\chi_{\bar{1}}}$.

From these cases we see that all 4 masses are equal if and only if the H-flux is purely an element of the span of just Ω and $\overline{\Omega}$ or just χ_1 and $\overline{\chi_{\bar{1}}}$. Equivalently all 4 masses are equal if and only if $h^0 = 0$ or $h^1 = 0$.

4.1.3 Extremes of R

We also want to see what happens if R takes extreme values.

1: $R = 2$

One option is that it attains it's maximum: $R = 2$. In this case we have seen that $\lambda_+ = \lambda_- = 1$. This also means that the 4 masses split into two pairs of equal masses:

$$
\begin{aligned}
m^2_{+\pm} = m^2_{-\pm} =& \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}\left(\sqrt{\mu^2 - 1} \pm 1\right)^2, \\
=& \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}\left(\frac{|h^1|}{|h^0|} \pm 1\right)^2, \\
=& \frac{\mathrm{e}^{K_\tau + K_{cs}}|W_0|^2}{\mathcal{V}^2}\left(\frac{|h^1|}{|h^0|} \pm 1\right)^2, \\
=& \frac{\mathrm{e}^{-K_\tau - K_{cs}}}{\mathcal{V}^2}\left(|h^1| \pm |h^0|\right)^2.
\end{aligned}
\tag{4.1.6}
$$

Note that in this case some of the masses can vanish namely $m^2_{+-} = m^2_{--} = 0$ if $|h^0| = |h^1|$. In this case these moduli would not be stabilized.

2: Limit R **to** $-\infty$

The other extreme for R is the limit to $-\infty$. From (2.2.6) we find that for R very big and negative we have:

$$
\begin{aligned}
\lambda_+ \approx& -R + 4 \text{ and} \\
\lambda_- \approx& \frac{-1}{R}.
\end{aligned}
\tag{4.1.7}
$$

This means that for R very big and negative the masses split in two pairs the first pair (see figure 4.1) :

$$
m^2_{+\pm} \approx -R\left(1 - \frac{1}{\mu^2}\right)\frac{N_{\text{flux}}}{\mathcal{V}^2} \tag{4.1.8}
$$

and the second pair (see figure 4.2) :

$$
m^2_{-\pm} \approx \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}. \tag{4.1.9}
$$

Note that two of these masses grow very big in this limit while the other two are approximately independent of R.

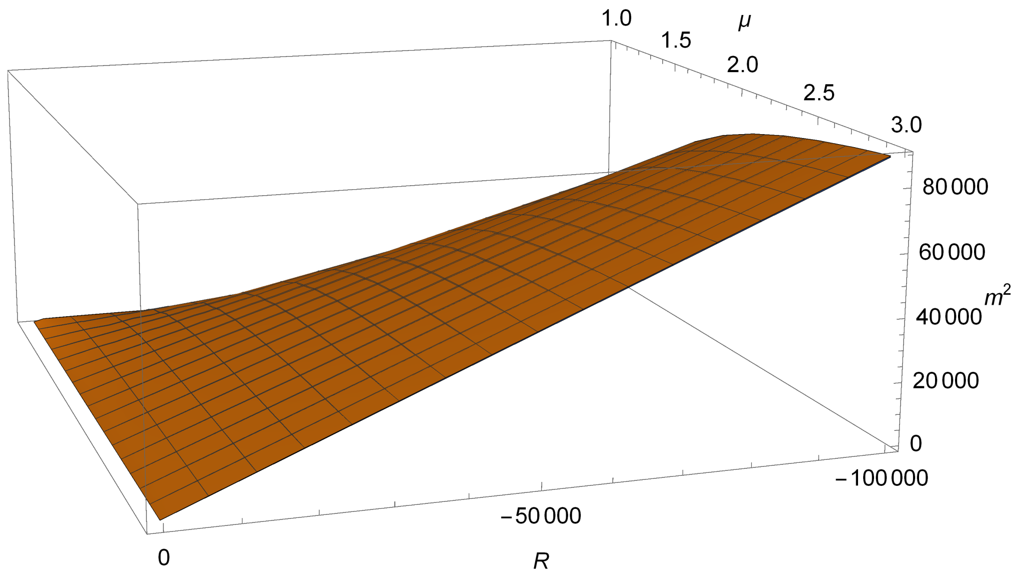

Figure 4.1: This figure shows the 2 masses $m^2_{+\pm}$ dependent on μ and R for large negative values of R. The case we consider here is $h^{2,1}_{-} = 1$. Note that both masses are (almost) equal and linear in R.

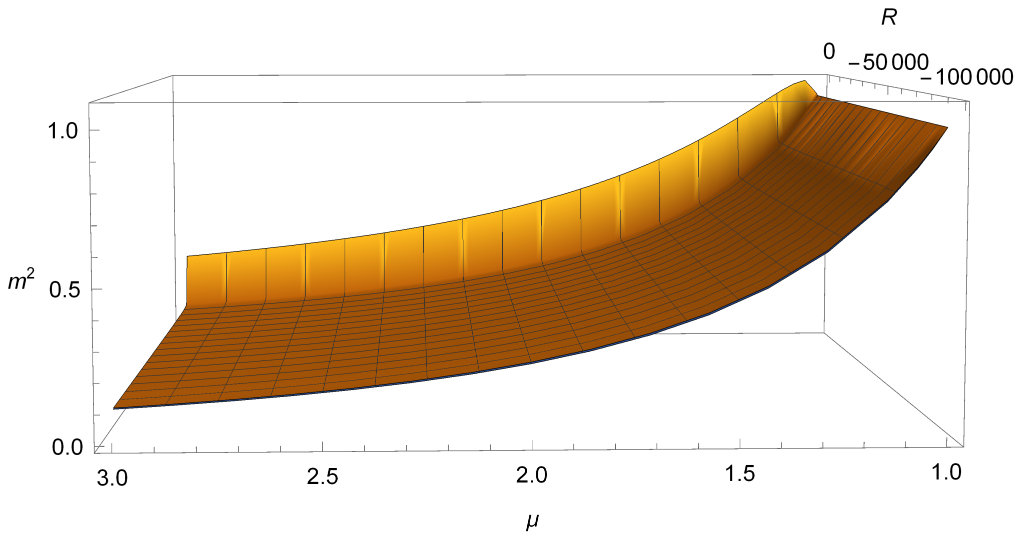

Figure 4.2: This figure shows the 2 masses $m^2_{-\pm}$ dependent on μ and R for large negative values of R. The case we consider here is $h^{2,1}_{-} = 1$. Note that both masses are (almost) equal and independent of R.

4.2 $h^{2,1}_{-} = 1$ in the large complex structure limit

We discuss this special case separately.
From (2.3.21) we know that the 4 masses are:

$$
\begin{aligned}
m_{--}^2 =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\left(\frac{1}{\sqrt{3}}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu}\right)^2, \\
m_{+-}^2 =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\left(\sqrt{3}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu}\right)^2, \\
m_{-+}^2 =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\left(\frac{1}{\sqrt{3}}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu}\right)^2, \\
m_{++}^2 =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\left(\sqrt{3}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu}\right)^2.
\end{aligned}
\tag{4.2.1}
$$

Except when $|W_0| = 0$ in that case we have (2.3.23):

$$
\begin{aligned}
m_{--}^2 = m_{-+}^2 =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\frac{1}{\sqrt{3}} \text{ and} \\
m_{+-}^2 = m_{++}^2 =& \frac{N_{\text{flux}}}{\mathcal{V}^2}\sqrt{3}.
\end{aligned}
\tag{4.2.2}
$$

4.2.1 Equal masses 2

We can use the expressions we found for the masses to analyse when these masses can be equal. First we consider the case $|W_0| \neq 0$:

- if $\mu^2 = 1$ then all masses are equal $m^2 = \mathrm{e}^K |W_0|^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}$.
- if $\mu^2 = \frac{7}{4}$ then $m_{--}^2 = m_{+-}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}\frac{1}{7}$, $m_{-+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}\frac{9}{7}$ and $m_{++}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}\frac{25}{7}$.
- if $\mu^2 = 4$ then $m_{+-}^2 = m_{-+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}$, $m_{--}^2 = 0$ and $m_{++}^2 = \frac{4N_{\text{flux}}}{\mathcal{V}^2}$.

Note that these are the same options as in 4.1.1 except for the option where $R = 2$. That option clearly does not apply here since $R = \frac{2}{3}$.

For the case where $W_0 = 0$ there are no extra options for masses to be equal other that the paring that always occurs in this case: $m_{\pm-}^2 = m_{\pm+}^2$.

Next we will further analyse some interesting cases.

All 4 masses equal

Because $R \neq 2$ the only way all 4 masses can be equal in this case is if $\mu^2 = 1$. This is the same as in 4.1.2. This means $h^1 = 0$.

This concludes the discussion of the masses for the case with only one complex structure modulus ($h_-^{2,1} = 1$). We now proceed to an analysis of the masses in the next case.

4.3 $h_-^{2,1} = 2$

In this case with two complex structure moduli z^i we found for $\lambda_\pm$ a polynomial equation of third degree (3.1.20):

$$\begin{aligned}
0 =& \lambda^3 - \lambda^2 \left[2 + e^{2K_{cs}}(\kappa_1^2 + \kappa_2^2)\right] \\
&+ \lambda \left[e^{4K_{cs}}\kappa_1^2\kappa_2^2 + 2e^{K_{cs}}r^2(\kappa_2^2 - \kappa_1^2) + 2e^{2K_{cs}}\kappa_1^2 + 1\right] \\
&- 2e^{2K_{cs}}r^4\kappa_2^2 - 2e^{2K_{cs}}r^2(1-r^2)\kappa_1\kappa_2\cos(\phi) - \kappa_1^2(1-r^2)^2 e^{K_{cs}}, \\
0 =& \lambda^3 - \lambda^2 \left[2 + e^{2K_{cs}}(\kappa_1^2 + \kappa_2^2)\right] \\
&+ \lambda \left[e^{4K_{cs}}\kappa_1^2\kappa_2^2 + 2e^{K_{cs}}\cos^2(\theta)\kappa_2^2 + 2e^{2K_{cs}}\kappa_1^2\sin^2(\theta) + 1\right] \\
&- 2e^{2K_{cs}}\cos^4(\theta)\kappa_2^2 - 2e^{2K_{cs}}\cos^2(\theta)\sin^2(\theta)\kappa_1\kappa_2\cos(\phi) - \kappa_1^2\sin^4(\theta)e^{K_{cs}}.
\end{aligned} \tag{4.3.1}$$

Note that λ depends on 4 real parameters: $e^{K_{cs}}\kappa_1$, $e^{K_{cs}}\kappa_2$, r or θ and ϕ. We also see that the λ are unchanged when we simultaneously switch $\kappa_1 \leftrightarrow \kappa_2$ and transform $\theta \to -\theta + \frac{\pi}{2}$.

With the 6 masses being (3.2.8):

$$m_{\alpha\pm}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}\left(\sqrt{\lambda_\alpha}\sqrt{1 - \frac{1}{\mu^2}} \pm \frac{1}{\mu}\right)^2. \tag{4.3.2}$$

4.3.1 Equal masses 3

Next, with these expressions for the masses, we will analyse when combinations of these 6 masses are the same. The analysis will be more extensive than before due to the higher number of different masses. In this section we will list the cases where different masses have the same value. Each case is characterized with numbers in brackets. In the brackets we denote the degree of the degeneracy for each distinct value of the masses.

First we exclude two special cases.

Case 1: $W_0 = 0$

If $|W_0| = 0$ we see from that when we use (2.1.1), (2.1.11) we then find:

$$m_{\alpha\pm}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}\lambda_\alpha. \tag{4.3.3}$$

Note for all $\alpha \in \{1, 2, 3\}$ we have $m_{\alpha+}^2 = m_{\alpha-}^2$.
This means there are as many distinct masses as there are distinct λ_α and for each α there is a 2-fold degeneracy. So the options are

- (6) $m_{\alpha\pm}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}\lambda_\alpha$ and $\lambda_1 = \lambda_2 = \lambda_3$ and $|W_0| = 0$,
- (4,2) $m_{\alpha\pm}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}\lambda_\alpha$ and $\lambda_1 = \lambda_2 \neq \lambda_3$ and $|W_0| = 0$,
- (2,2,2) $m_{\alpha\pm}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}\lambda_\alpha$ and $\lambda_\alpha \neq \lambda_\beta$ for $\alpha \neq \beta$ and $|W_0| = 0$.

Case 2: $\mu^2 = 1$

If $\mu^2 = 1$ (but $|W_0| \neq 0$) then by (3.2.8) we have:

$$m^2_{\alpha\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2}\left(\sqrt{\lambda}\sqrt{1-\frac{1}{\mu^2}} \pm \frac{1}{\mu}\right)^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2}. \tag{4.3.4}$$

So there is a 6-fold degeneracy. This happens only when $\mu^2 = \frac{N_{\text{flux}}}{e^{K_\tau + K_{cs}}|W_0|^2} = 1$. So when we replace N_{flux} using (3.2.5) we see that in this case

$$\begin{aligned} N_{\text{flux}} =& e^{K_\tau + K_{cs}}|W_0|^2 \\ e^{K_\tau + K_{cs}}|W_0|^2 + e^{-K_\tau - K_{cs}}||h||^2 =& e^{K_\tau + K_{cs}}|W_0|^2 \\ e^{-K_\tau - K_{cs}}||h||^2 =& 0 \\ ||h||^2 =& 0. \end{aligned} \tag{4.3.5}$$

So we see that this is only possible if $h^1 = h^2 = 0$. This gives one option:

- (6) $m^2_{n\pm} = e^K|W|^2$ and $\mu^2 = 1 \Leftrightarrow h = 0$.

Preparations

Now we look at the other cases so we assume $|W_0| \neq 0$ and $\mu^2 \neq 1$. Because of the amount of options we consider cases that are the same up to permutations of λ_α.

First we have a look at the different ways 2 masses can be equal. There are 4 options:

- $m^2_{\alpha+} = m^2_{\alpha-}$,
- $m^2_{\alpha+} = m^2_{\beta+}$,
- $m^2_{\alpha-} = m^2_{\beta-}$,
- $m^2_{\alpha-} = m^2_{\beta+}$.

Where $\alpha \neq \beta$.
We will consider these cases separately.

$m^2_{\alpha+} = m^2_{\alpha-}$

First of all we note that $\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}} \geq 0$ so $|\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}+1| \geq |\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}-1|$ and $m^2_{\alpha+} \geq m^2_{\alpha-}$. Now we check when the equality holds:

$$\begin{aligned}
m^2_{\alpha+} =& m^2_{\alpha-} \\
\left(\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu}\right)^2 =& \left(\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu}\right)^2 \\
\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu} =& \pm\left(\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu}\right) \\
\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu} =& -\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu} \\
\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}} =& -\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}} \\
\sqrt{\lambda_\alpha} =& 0.
\end{aligned} \tag{4.3.6}$$

To get from the third line to the fourth we picked only one option because the other option only gives $-\frac{1}{\mu} = \frac{1}{\mu}$.
So in this case we have

$$m^2_{\alpha\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}. \tag{4.3.7}$$

$m^2_{\alpha+} = m^2_{\beta+}$

In this case we have

$$\begin{aligned}
m^2_{\alpha+} =& m^2_{\beta+} \\
\left(\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu}\right)^2 =& \left(\sqrt{\lambda_\beta}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu}\right)^2 \\
\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu} =& \pm\left(\sqrt{\lambda_\beta}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu}\right) \\
\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu} =& \sqrt{\lambda_\beta}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu} \\
\sqrt{\lambda_\alpha} =& \sqrt{\lambda_\beta}.
\end{aligned} \tag{4.3.8}$$

To get from the third line to the fourth we picked only one option because the other option equates a positive and a negative number.
In this case we have

$$m^2_{\alpha\pm} = m^2_{\beta\pm}. \tag{4.3.9}$$

So then there are two pairs of equal masses.

$m^2_{\alpha-} = m^2_{\beta-}$

In this case we have

$$\begin{aligned}
m^2_{\alpha-} =& m^2_{\beta-} \\
\left(\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu}\right)^2 =& \left(\sqrt{\lambda_\beta}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu}\right)^2 \\
\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu} =& \pm\left(\sqrt{\lambda_\beta}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu}\right) \\
\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu} =& -\sqrt{\lambda_\beta}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu} \\
(\sqrt{\lambda_\alpha}+\sqrt{\lambda_\beta})\sqrt{1-\frac{1}{\mu^2}} =& \frac{2}{\mu} \\
\sqrt{\lambda_\alpha}+\sqrt{\lambda_\beta} =& \frac{2}{\sqrt{\mu^2-1}} = \frac{2|h^1|}{||h||}.
\end{aligned} \tag{4.3.10}$$

To get from the third line to the fourth we picked only one option because the other option is the same as we have seen in the last part ($\lambda_\alpha = \lambda_\beta$).
So in this case we have $\lambda_\alpha = \lambda_\beta$ and $m^2_{\alpha\pm} = m^2_{\beta\pm}$ or $\sqrt{\lambda_\alpha}+\sqrt{\lambda_\beta} = \frac{2}{\sqrt{\mu^2-1}}$ and $m^2_{\alpha-} = m^2_{\beta-} < m^2_{\alpha/\beta+}$.

$m^2_{\alpha-} = m^2_{\beta+}$

In this case we have

$$\begin{aligned}
m^2_{\alpha-} =& m^2_{\beta+} \\
\left(\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu}\right)^2 =& \left(\sqrt{\lambda_\beta}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu}\right)^2 \\
\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu} =& \pm\left(\sqrt{\lambda_\beta}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu}\right) \\
\sqrt{\lambda_\alpha}\sqrt{1-\frac{1}{\mu^2}}-\frac{1}{\mu} =& \sqrt{\lambda_\beta}\sqrt{1-\frac{1}{\mu^2}}+\frac{1}{\mu} \\
(\sqrt{\lambda_\alpha}-\sqrt{\lambda_\beta})\sqrt{1-\frac{1}{\mu^2}} =& \frac{2}{\mu} \\
\sqrt{\lambda_\alpha}-\sqrt{\lambda_\beta} =& \frac{2}{\sqrt{\mu^2-1}} = \frac{2|h^1|}{||h||}
\end{aligned} \tag{4.3.11}$$

To get from the third line to the fourth we picked only one option because the other option only gives $-\frac{1}{\mu} = \frac{1}{\mu}$.
So in this case we have $\sqrt{\lambda_\alpha}-\sqrt{\lambda_\beta} = \frac{2}{\sqrt{\mu^2-1}}$ and $m^2_{\beta-} < m^2_{\beta+} = m^2_{\alpha-} < m^2_{\alpha+}$.

Recap

We found that masses can be equal in the following cases:

- $m^2_{\alpha+} = m^2_{\beta-} \Leftrightarrow \lambda_\alpha = 0$ and $m^2_{\alpha\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$
- $m^2_{\alpha+} = m^2_{\beta+} \Leftrightarrow \lambda_\alpha = \lambda_\beta$ and $m^2_{\alpha\pm} = m^2_{\beta\pm}$
- $m^2_{\alpha-} = m^2_{\beta-} \Leftrightarrow \lambda_\alpha = \lambda_\beta$ and $m^2_{\alpha\pm} = m^2_{\beta\pm}$ or $\sqrt{\lambda_\alpha} + \sqrt{\lambda_\beta} = \frac{2}{1\sqrt{\mu^2-1}} = \frac{2|h^1|}{||h||}$ and $m^2_{\alpha-} = m^2_{\beta-} < m^2_{\alpha/\beta+}$
- $m^2_{\alpha-} = m^2_{\beta+} \Leftrightarrow \sqrt{\lambda_\alpha} - \sqrt{\lambda_\beta} = \frac{2}{\sqrt{\mu^2-1}} = \frac{2|h^1|}{||h||}$ and $m^2_{\beta-} < m^2_{\beta+} = m^2_{\alpha-} < m^2_{\alpha+}$

Now we can describe all the possible combinations. To avoid redundancy's we do not consider perturbations of $\lambda_1, \lambda_2, \lambda_3$.

6 masses the same

If we want all 6 masses to be the same than $m^2_{\alpha+} = m^2_{\alpha-}$ for all α so we have $\lambda_\alpha = 0$ and $m^2_{\alpha\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ for all α. Inspection of the characteristic polynomial shows that this is not possible since the coefficient of the λ^2 therm in the characteristic polynomial can not be 0 (3.1.20). Therefore the only option for all the masses to be equal is:

- (6) $m^2_{\alpha\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2}$ and $\mu^2 = 1 \Leftrightarrow ||h|| = 0$.

5 masses the same

If we want exactly 5 masses to be the same then we need for 2 α's that $m^2_{\alpha+} = m^2_{\alpha-}$ but not all three. Since $\lambda_\alpha \geq 0$ and $m^2_{\alpha+} = m^2_{\alpha-} \Rightarrow \lambda_\alpha = 0$ we have $0 = \lambda_1 = \lambda_2 < \lambda_3$. But this is not possible because the coefficient of the λ therm in the characteristic polynomial can not be 0 (3.1.20).
We conclude that a 5 fold degeneracy is not possible.

4 masses the same

If we want 4 masses to be the same but not 5 then we need for at least 1 α that $m^2_{\alpha+} = m^2_{\alpha-}$ but not all three.

First we look at the case where $m^2_{1/2\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$.
This means we have 4 equal masses so $m^2_{3\pm} \neq m^2_{1/2\pm}$ so we have $\lambda_{1/2} = 0$. This is again not an option for the same reason as before.
Now we look at the case where just $m^2_{1\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$.
$\lambda_1 = 0$ but $\lambda_{2/3} \neq 0$. In this case we still need that one of $m^2_{2\pm}$ and one of the $m^2_{3\pm}$ is the same as $m^2_{1\pm}$. This can not be either of the $m^2_{2/3+}$ because this would lead to all $\lambda_\alpha = 0$ and thus a 6 fold degeneracy. So we need $m^2_{1\pm} = m^2_{2/3-}$, $m^2_{2/3+} = \frac{9N_{\text{flux}}}{\mathcal{V}^2\mu^2}$, $\lambda_1 = 0$ and $\lambda_{2/3} = \frac{4}{\mu^2-1}$.
We have:

- (4,2) $m^2_{1\pm} = m^2_{2/3-} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m^2_{2/3+} = 9\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $\lambda_1 = 0$ and $\lambda_{2/3} = \frac{4}{\mu^2-1}$.

3 masses the same

If we want 3 masses to be the same but not 4 than we have for at most 1 α that $m^2_{\alpha+} = m^2_{\alpha-}$.

First we look at the case where $m^2_{1\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$.
This means we have 2 equal masses so we need $m^2_{2-} = m^2_{1\pm}$ (if $m^2_{2+} = m^2_{1\pm}$ then also $m^2_{2\pm} = m^2_{1\pm}$). So $\lambda_1 = 0$, $\lambda_2 = \frac{4}{\mu^2-1}$ and $m^2_{2+} = 9\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$. Then there are 2 options $m^2_{3-} = m^2_{2+}$ or $m^2_{3-} \neq m^2_{2+}$.
If $m^2_{3-} = m^2_{2+} = 9\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ then $\lambda_3 = 9\lambda_2 = \frac{9}{\mu^2-1}$ and $m^2_{3+} = 16\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ but if $m^2_{3-} \neq m^2_{2+}$ then $0 < \lambda_3 \neq \frac{9}{\mu^2-1}$

Now we look at the case where for none of the α we have $m^2_{\alpha-} = m^2_{\alpha+}$.
So $\lambda_\alpha \neq 0$. To still have 3 the same masses we then need that all these same masses have a different α, so for each α one of the $m^2_{\alpha\pm}$ is in the group. There are 4 options: $m^2_{1+} = m^2_{2+} = m^2_{3+}$, $m^2_{1+} = m^2_{2+} = m^2_{3-}$, $m^2_{1+} = m^2_{2-} = m^2_{3-}$ and $m^2_{1-} = m^2_{2-} = m^2_{3-}$.
If $m^2_{1+} = m^2_{2+} = m^2_{3+}$ then $\lambda_1 = \lambda_2 = \lambda_3$ so $m^2_{1-} = m^2_{2-} = m^2_{3-}$.
If $m^2_{1+} = m^2_{2+} = m^2_{3-}$ then $\lambda_1 = \lambda_2$ and $\sqrt{\lambda_3} - \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$. So $m^2_{1-} = m^2_{2-} < m^2_{1+} = m^2_{2+} = m^2_{3-} < m^2_{3+}$.
If $m^2_{1+} = m^2_{2-} = m^2_{3-}$ we have $\sqrt{\lambda_2}-\sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}}$ so $\sqrt{\lambda_2} > \frac{2}{\sqrt{\mu^2-1}}$ so $\sqrt{\lambda_3}+\sqrt{\lambda_2} \neq \frac{2}{\sqrt{\mu^2-1}}$ because otherwise $\sqrt{\lambda_3} < 0$. So we are left with just the option that $\lambda_2 = \lambda_3$ and $m^2_{2+} = m^2_{3+}$.
Finally if $m^2_{1-} = m^2_{2-} = m^2_{3-}$ either $\lambda_1 = \lambda_2 = \lambda_3$ but we have already considered this case or the second option is $\sqrt{\lambda_1} + \sqrt{\lambda_2} = \sqrt{\lambda_2} + \sqrt{\lambda_3} = \sqrt{\lambda_1} + \sqrt{\lambda_3} = \frac{2}{\sqrt{\mu^2-1}}$ but this also means that $\lambda_1 = \lambda_2 = \lambda_3$. The third and final option is $\lambda_1 = \lambda_2$ and $\sqrt{\lambda_{1/2}} + \sqrt{\lambda_3} = \frac{2}{\sqrt{\mu^2-1}}$. This means $m^2_{1+} = m^2_{2+}$.
So we have

- (3,2,1) $m^2_{1\pm} = m^2_{2-} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m^2_{2+} = m^2_{3-} = 9\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m^2_{3+} = 16\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $\lambda_1 = 0$ and $\lambda_2 = \frac{4}{\mu^2-1}$ and $\lambda_3 = \frac{9}{\mu^2-1}$.
- (3,1,1,1) $m^2_{1\pm} = m^2_{2-} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m^2_{2+} = 9\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2} \neq m^2_{3-} < m^2_{3+} \neq 16\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $\lambda_1 = 0$ and $\lambda_2 = \frac{4}{\mu^2-1}$ and $0 < \lambda_3 \neq \frac{9}{\mu^2-1}$.
- (3,3) $m^2_{1+} = m^2_{2+} = m^2_{3+} \neq m^2_{1-} = m^2_{2-} = m^2_{3-}$ and $0 < \lambda_1 = \lambda_2 = \lambda_3$.
- (3,2,1) $m^2_{1-} = m^2_{2-} < m^2_{1+} = m^2_{2+} = m^2_{3-} < m^2_{3+}$, $0 < \lambda_1 = \lambda_2 < \lambda_3$ and $\sqrt{\lambda_3} - \sqrt{\lambda_{1/2}} = \frac{2}{\sqrt{\mu^2-1}}$.
- (3,2,1) $m^2_{1-} < m^2_{1+} = m^2_{2-} = m^2_{3-} < m^2_{2+} = m^2_{3+}$, $0 < \lambda_1 < \lambda_2 = \lambda_3$ and $\sqrt{\lambda_{2/3}} - \sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}}$.
- (3,2,1) $m^2_{1-} = m^2_{2-} = m^2_{3-} < m^2_{1+} = m^2_{2+}$, $m^2_{\alpha-} < m^2_{3+}$, $0 < \lambda_1 = \lambda_2$, $0 < \lambda_3$ and $\sqrt{\lambda_{1/2}} + \sqrt{\lambda_3} = \frac{2}{\sqrt{\mu^2-1}}$.

2 masses the same

First we look at the option where $m_{1-}^2 = m_{1+}^2 =$ then $\lambda_1 = 0$ and there are a few options. Note none of the other α can have $m_{\alpha-}^2 = m_{\alpha+}^2 =$ because then more then 2 masses will be the same.
First, $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $\lambda_2 = \lambda_3 \neq 0$ then $m_{2\pm}^2 = m_{3\pm}^2$.
Second, $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m_{2-}^2 = m_{3-}^2$ then $\sqrt{\lambda_2} + \sqrt{\lambda_3} = \frac{2}{\sqrt{\mu^2-1}}$ but $0 \neq \lambda_2 \neq \lambda_3 \neq 0$.
Third, $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m_{2-}^2 < m_{2+}^2 = m_{3-}^2 < m_{3+}^2$ then $\sqrt{\lambda_3} - \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$ but $\lambda_{2/3} \neq 0$.
Fourth, $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and none of the $m_{2/3\pm}^2$ are the same, such that $0 \neq \lambda_2 \neq \lambda_3 \neq 0$ and $\sqrt{\lambda_3} \pm \sqrt{\lambda_2} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$.

Now we can consider the cases where for all α we have $m_{\alpha+}^2 \neq m_{\alpha-}^2$. So $\lambda_\alpha \neq 0$.
First, $m_{1\pm}^2 = m_{2\pm}^2$ so $\lambda_1 = \lambda_2 \neq 0$ then neither of the $m_{3\pm}^2$ can be the same as any other so $\lambda_{1/2} \neq \lambda_3 \neq 0$ and $\sqrt{\lambda_3} \pm \sqrt{\lambda_{1/2}} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$.
Second, $m_{1-}^2 = m_{2-}^2$ then $\sqrt{\lambda_1} + \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$ now the m_{3+}^2 can not be the same as any of the others (this would also make all $m_{\alpha-}^2$ the same) but the m_{3-}^2 can be so we have $m_{1+}^2 = m_{3-}^2$ then $\sqrt{\lambda_3} - \sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}}$ or
Third, $m_{1-}^2 = m_{2-}^2$ then $\sqrt{\lambda_1} + \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$ and none of the $m_{3\pm}^2$ are the same as any others.

Than we have one more category left where $m_{1-}^2 < m_{1+}^2 = m_{2-}^2 < m_{2+}^2$ so $\sqrt{\lambda_2} - \sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}}$. In this case $m_{1/2+}^2 \neq m_{3+}^2 \neq m_{2-}^2$ and $m_{1+}^2 = m_{2-}^2 \neq m_{3-}^2$.
First, just $m_{1-}^2 < m_{1+}^2 = m_{2-}^2 < m_{2+}^2$ this gives $\sqrt{\lambda_3} \pm \sqrt{\lambda_2} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$.
Second, $m_{1-}^2 < m_{1+}^2 = m_{2-}^2 < m_{2+}^2 = m_{3-}^2 < m_{3+}^2$ this gives $\sqrt{\lambda_3} - \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$.
All other options are the same as a previous option through permutation of the λ_α.

So we have

- (2,2,2) $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m_{2\pm}^2 = m_{3\pm}^2$ and $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 \neq 0$.
- (2,2,1,1) $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m_{2-}^2 = m_{3-}^2 < m_{2/3+}^2$ and $\lambda_1 = 0$ and $\sqrt{\lambda_2} + \sqrt{\lambda_3} = \frac{2}{\sqrt{\mu^2-1}}$ but $0 \neq \lambda_2 \neq \lambda_3 \neq 0$.
- (2,2,1,1) $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m_{2-}^2 < m_{2+}^2 = m_{3-}^2 < m_{3+}^2$ and $\lambda_1 = 0$ and $\sqrt{\lambda_3} - \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$ but $\lambda_{2/3} \neq 0$.
- (2,1,1,1,1) $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m_{2-}^2 < m_{2+}^2$ and $m_{3-}^2 < m_{3+}^2$ and $m_{3-}^2 \neq m_{2\pm}^2 \neq m_{3+}^2$ and $\lambda_1 = 0$ and $\sqrt{\lambda_3} \pm \sqrt{\lambda_2} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$ but $\lambda_{2/3} \neq 0$.

- (2,2,1,1) $m_{1\pm}^2 = m_{2\pm}^2$ and $m_{1/2\pm}^2 \neq m_{3-}^2 < m_{3+}^2 \neq m_{1/2\pm}^2$ and $\lambda_1 = \lambda_2 \neq 0$ and $\lambda_{1/2} \neq \lambda_3 \neq 0$ and $\sqrt{\lambda_3} \pm \sqrt{\lambda_{1/2}} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$.
- (2,2,1,1) $m_{1-}^2 = m_{2-}^2 < m_{1/2+}^2$ and $m_{1+}^2 \neq m_{2+}^2$ and $m_{1+}^2 = m_{3-}^2 < m_{3+}^2$ and $m_{2+}^2 \neq m_{3+}^2$ and $\lambda_\alpha > 0$ and $\sqrt{\lambda_1}+\sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$ and $\lambda_3 \neq \lambda_1 \neq \lambda_2 \neq \lambda_3$ and $\sqrt{\lambda_3} - \sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}}$.
- (2,1,1,1,1) $m_{1-}^2 = m_{2-}^2 < m_{1/2+}^2$ and $m_{3\pm}^2 \neq m_{1+}^2 \neq m_{2+}^2 \neq m_{3\pm}^2$ and $m_{3-}^2 < m_{3+}^2$ and $\lambda_\alpha > 0$ and $\sqrt{\lambda_1} + \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$ and $\lambda_3 \neq \lambda_1 \neq \lambda_2 \neq \lambda_3$ and $\sqrt{\lambda_3} \pm \sqrt{\lambda_{1/2}} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$.
- (2,1,1,1,1) $m_{1-}^2 < m_{1+}^2 = m_{2-}^2 < m_{2+}^2$ and $m_{1/2\pm}^2 \neq m_{3-}^2 < m_{3+}^2 \neq m_{1/2\pm}^2$ and $\lambda_\alpha \neq 0$ and $\sqrt{\lambda_2} - \sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}}$ and $\sqrt{\lambda_3} \pm \sqrt{\lambda_2} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$.
- (2,2,1,1) $0 < m_{1-}^2 < m_{1+}^2 = m_{2-}^2 < m_{2+}^2 = m_{3-}^2 < m_{3+}^2$ and $\lambda_\alpha \neq 0$ and $\sqrt{\lambda_2} - \sqrt{\lambda_1} = \sqrt{\lambda_3} - \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$.

All masses unique

If none of the masses can be the same then $\lambda_\alpha \neq 0$ for all α. Also $\lambda_\alpha \neq \lambda_\beta$ and $\sqrt{\lambda_\alpha} + \sqrt{\lambda_\beta} \neq \frac{2}{\sqrt{\mu^2-1}}$ for $\beta \neq \alpha$. As well as $\sqrt{\lambda_\alpha} - \sqrt{\lambda_\beta} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$ for all α, β.

So we have

- (1,1,1,1,1,1) $m_{\alpha-}^2 < m_{\alpha+}^2$, $\lambda_\alpha \neq 0$ and $\sqrt{\lambda_\alpha} - \sqrt{\lambda_\beta} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$ for all α, β. As well as $\lambda_\alpha \neq \lambda_\beta$ and $\sqrt{\lambda_\alpha} + \sqrt{\lambda_\beta} \neq \frac{2}{\sqrt{\mu^2-1}}$ for $\alpha \neq \beta$.

Recap

So the full list of options is

- (6) $m_{\alpha\pm}^2 = \mathrm{e}^{-K}|h|^2\lambda_\alpha$ and $\lambda_1 = \lambda_2 = \lambda_3$ and $|W_0| = 0$.
- (6) $m_{\alpha\pm}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $\mu^2 = 1 \Leftrightarrow h = 0$.
- (4,2) $m_{\alpha\pm}^2 = \mathrm{e}^{-K}|h|^2\lambda_\alpha$ and $\lambda_1 = \lambda_2 \neq \lambda_3$ and $|W_0| = 0$.
- (4,2) $m_{1\pm}^2 = m_{2/3-}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m_{2/3+}^2 = 9\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $\lambda_1 = 0$ and $\lambda_{2/3} = \frac{4}{\mu^2-1}$.
- (3,2,1) $m_{1\pm}^2 = m_{2-}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m_{2+}^2 = m_{3-}^2 = 9\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m_{3+}^2 = 16\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $\lambda_1 = 0$ and $\lambda_2 = \frac{4}{\mu^2-1}$ and $\lambda_3 = \frac{9}{\mu^2-1}$.
- (3,1,1,1) $m_{1\pm}^2 = m_{2-}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $m_{2+}^2 = 9\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2} \neq m_{3-}^2 < m_{3+}^2 \neq 16\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$ and $\lambda_1 = 0$ and $\lambda_2 = \frac{4}{\mu^2-1}$ and $0 < \lambda_3 \neq \frac{9}{\mu^2-1}$.
- (3,3) $m_{1-}^2 = m_{2-}^2 = m_{3-}^2 < m_{1+}^2 = m_{2+}^2 = m_{3+}^2$ and $0 < \lambda_1 = \lambda_2 = \lambda_3$.

- (3,2,1) $m_{1-}^2 = m_{2-}^2 < m_{1+}^2 = m_{2+}^2 = m_{3-}^2 < m_{3+}^2$, $0 < \lambda_1 = \lambda_2 < \lambda_3$ and $\sqrt{\lambda_3} - \sqrt{\lambda_{1/2}} = \frac{2}{\sqrt{\mu^2-1}}$.
- (3,2,1) $m_{1-}^2 < m_{1+}^2 = m_{2-}^2 = m_{3-}^2 < m_{2+}^2 = m_{3+}^2$, $0 < \lambda_1 < \lambda_2 = \lambda_3$ and $\sqrt{\lambda_{2/3}} - \sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}}$.
- (3,2,1) $m_{1-}^2 = m_{2-}^2 = m_{3-}^2 < m_{1+}^2 = m_{2+}^2$, $m_{\alpha-}^2 < m_{3+}^2$, $0 < \lambda_1 = \lambda_2$, $0 < \lambda_3$ and $\sqrt{\lambda_{1/2}} + \sqrt{\lambda_3} = \frac{2}{\sqrt{\mu^2-1}}$.
- (2,2,2) $m_{\alpha\pm}^2 = \mathrm{e}^{-K}|h|^2\lambda_\alpha$ and $\lambda_\alpha \neq \lambda_\beta$ for $m \neq n$ and $|W_0| = 0$.
- (2,2,2) $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$, $m_{2\pm}^2 = m_{3\pm}^2$, $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 \neq 0$.
- (2,2,1,1) $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$, $m_{2-}^2 = m_{3-}^2 < m_{2/3+}^2$, $\lambda_1 = 0$ and $\sqrt{\lambda_2} + \sqrt{\lambda_3} = \frac{2}{\sqrt{\mu^2-1}}$ but $0 \neq \lambda_2 \neq \lambda_3 \neq 0$.
- (2,2,1,1) $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$, $m_{2-}^2 < m_{2+}^2 = m_{3-}^2 < m_{3+}^2$, $\lambda_1 = 0$, $\sqrt{\lambda_3} - \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$ but $\lambda_{2/3} \neq 0$.
- (2,1,1,1,1) $m_{1-}^2 = m_{1+}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$, $m_{2-}^2 < m_{2+}^2$, $m_{3-}^2 < m_{3+}^2$, $m_{3-}^2 \neq m_{2\pm}^2 \neq m_{3+}^2$, $\lambda_1 = 0$, $\sqrt{\lambda_3} \pm \sqrt{\lambda_2} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$ but $\lambda_{2/3} \neq 0$.
- (2,2,1,1) $m_{1\pm}^2 = m_{2\pm}^2$, $m_{1/2\pm}^2 \neq m_{3-}^2 < m_{3+}^2 \neq m_{1/2\pm}^2$, $\lambda_1 = \lambda_2 \neq 0$, $\lambda_{1/2} \neq \lambda_3 \neq 0$ and $\sqrt{\lambda_3} \pm \sqrt{\lambda_{1/2}} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$.
- (2,2,1,1) $m_{1-}^2 = m_{2-}^2 < m_{1/2+}^2$, $m_{1+}^2 \neq m_{2+}^2$, $m_{1+}^2 = m_{3-}^2 < m_{3+}^2$, $m_{2+}^2 \neq m_{3+}^2$, $\lambda_\alpha > 0$, $\sqrt{\lambda_1} + \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$, $\lambda_3 \neq \lambda_1 \neq \lambda_2 \neq \lambda_3$ and $\sqrt{\lambda_3} - \sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}}$.
- (2,1,1,1,1) $m_{1-}^2 = m_{2-}^2 < m_{1/2+}^2$, $m_{3\pm}^2 \neq m_{1+}^2 \neq m_{2+}^2 \neq m_{3\pm}^2$, $m_{3-}^2 < m_{3+}^2$, $\lambda_\alpha > 0$, $\sqrt{\lambda_1}+\sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$, $\lambda_3 \neq \lambda_1 \neq \lambda_2 \neq \lambda_3$ and $\sqrt{\lambda_3}\pm\sqrt{\lambda_{1/2}} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$.
- (2,1,1,1,1) $m_{1-}^2 < m_{1+}^2 = m_{2-}^2 < m_{2+}^2$, $m_{1/2\pm}^2 \neq m_{3-}^2 < m_{3+}^2 \neq m_{1/2\pm}^2$, $\lambda_\alpha \neq 0$, $\sqrt{\lambda_2} - \sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}}$ and $\sqrt{\lambda_3} \pm \sqrt{\lambda_2} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$.
- (2,2,1,1) $0 < m_{1-}^2 < m_{1+}^2 = m_{2-}^2 < m_{2+}^2 = m_{3-}^2 < m_{3+}^2$, $\lambda_\alpha \neq 0$ and $\sqrt{\lambda_2} - \sqrt{\lambda_1} = \sqrt{\lambda_3} - \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$.
- (1,1,1,1,1,1) $m_{\alpha-}^2 < m_{\alpha+}^2$, $\lambda_\alpha \neq 0$ and $\sqrt{\lambda_\alpha} - \sqrt{\lambda_\beta} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$ for all α, β. As well as $\lambda_\alpha \neq \lambda_\beta$ and $\sqrt{\lambda_\alpha} + \sqrt{\lambda_\beta} \neq \frac{2}{\sqrt{\mu^2-1}}$ for $\alpha \neq \beta$.

Note that in all cases except the second case we need $\mu^2 > 1$. Otherwise we will fall back into (6) $m_{\alpha\pm}^2 = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$.

We conclude this with a table of these results.

Code	Masses ($m^2_{\alpha\pm}$)	Eigenvalues (λ_α) and other criteria
(6)	$m^2_{\alpha\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2}\lambda_\alpha$	$\lambda_1 = \lambda_2 = \lambda_3,\ \lvert W_0\rvert = 0$
(6)	$m^2_{\alpha\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$	$\mu^2 = 1 \Leftrightarrow h = 0$
(4,2)	$m^2_{\alpha\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2}\lambda_\alpha$	$\lambda_1 = \lambda_2 \neq \lambda_3,\ \lvert W_0\rvert = 0$
(4,2)	$m^2_{1\pm} = m^2_{2/3-} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2},\ m^2_{2/3+} = 9\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$	$\lambda_1 = 0,\ \lambda_{2/3} = \frac{4}{\mu^2-1}$
(3,2,1)	$m^2_{1\pm} = m^2_{2-} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2},\ m^2_{2+} = m^2_{3-} = 9\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2},\ m^2_{3+} = 16\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$	$\lambda_1 = 0,\ \lambda_2 = \frac{4}{\mu^2-1},\ \lambda_3 = \frac{9}{\mu^2-1}$
(3,1,1,1)	$m^2_{1\pm} = m^2_{2-} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2},\ m^2_{2+} = 9\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2} \neq m^2_{3-} < m^2_{3+} \neq 16\frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2}$	$\lambda_1 = 0,\ \lambda_2 = \frac{4}{\mu^2-1},\ 0 < \lambda_3 \neq \frac{9}{\mu^2-1}$
(3,3)	$m^2_{1-} = m^2_{2-} = m^2_{3-} < m^2_{1+} = m^2_{2+} = m^2_{3+}$	$0 < \lambda_1 = \lambda_2 = \lambda_3$
(3,2,1)	$m^2_{1-} = m^2_{2-} < m^2_{1+} = m^2_{2+} = m^2_{3-} < m^2_{3+}$	$0 < \lambda_1 = \lambda_2 < \lambda_3,\ \sqrt{\lambda_3} - \sqrt{\lambda_{1/2}} = \frac{2}{\sqrt{\mu^2-1}}$
(3,2,1)	$m^2_{1-} < m^2_{1+} = m^2_{2-} = m^2_{3-} < m^2_{2+} = m^2_{3+}$	$0 < \lambda_1 < \lambda_2 = \lambda_3,\ \sqrt{\lambda_{2/3}} - \sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}}$
(3,2,1)	$m^2_{1-} = m^2_{2-} = m^2_{3-} < m^2_{1+} = m^2_{2+},\ m^2_{\alpha-} < m^2_{3+}$	$0 < \lambda_1 = \lambda_2,\ 0 < \lambda_3,\ \sqrt{\lambda_{1/2}} + \sqrt{\lambda_3} = \frac{2}{\sqrt{\mu^2-1}}$
(2,2,2)	$m^2_{\alpha\pm} = \frac{N_{\text{flux}}}{\mathcal{V}^2}\lambda_\alpha$	$\lambda_\alpha \neq \lambda_\beta$ for $m \neq n$, $\lvert W_0\rvert = 0$
(2,2,2)	$m^2_{1-} = m^2_{1+} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2},\ m^2_{2\pm} = m^2_{3\pm}$	$\lambda_1 = 0,\ \lambda_2 = \lambda_3 \neq 0$
(2,2,1,1)	$m^2_{1-} = m^2_{1+} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2},\ m^2_{2-} = m^2_{3-} < m^2_{2/3+}$	$\lambda_1 = 0,\ \sqrt{\lambda_2} + \sqrt{\lambda_3} = \frac{2}{\sqrt{\mu^2-1}},\ 0 \neq \lambda_2 \neq \lambda_3 \neq 0$
(2,2,1,1)	$m^2_{1-} = m^2_{1+} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2},\ m^2_{2-} < m^2_{2+} = m^2_{3-} < m^2_{3+}$	$\lambda_1 = 0,\ \sqrt{\lambda_3} - \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}},\ \lambda_{2/3} \neq 0$
(2,1,1,1,1)	$m^2_{1-} = m^2_{1+} = \frac{N_{\text{flux}}}{\mathcal{V}^2\mu^2},\ m^2_{2-} < m^2_{2+},\ m^2_{3-} < m^2_{3+},\ m^2_{3-} \neq m^2_{2\pm} \neq m^2_{3+}$	$\lambda_1 = 0,\ \sqrt{\lambda_3} \pm \sqrt{\lambda_2} \neq \pm\frac{2}{\sqrt{\mu^2-1}},\ \lambda_{2/3} \neq 0$
(2,2,1,1)	$m^2_{1\pm} = m^2_{2\pm},\ m^2_{1/2\pm} \neq m^2_{3-} < m^2_{3+} \neq m^2_{1/2\pm}$	$\lambda_1 = \lambda_2 \neq 0,\ \lambda_{1/2} \neq \lambda_3 \neq 0,\ \sqrt{\lambda_3} \pm \sqrt{\lambda_{1/2}} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$
(2,2,1,1)	$m^2_{1-} = m^2_{2-} < m^2_{1/2+},\ m^2_{1+} \neq m^2_{2+},\ m^2_{1+} = m^2_{3-} < m^2_{3+},\ m^2_{2+} \neq m^2_{3+}$	$\lambda_\alpha > 0,\ \sqrt{\lambda_1} + \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}},\ \lambda_3 \neq \lambda_1 \neq \lambda_2 \neq \lambda_3,\ \sqrt{\lambda_3} - \sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}}$
(2,1,1,1,1)	$m^2_{1-} = m^2_{2-} < m^2_{1/2+},\ m^2_{3\pm} \neq m^2_{1+} \neq m^2_{2+} \neq m^2_{3\pm},\ m^2_{3-} < m^2_{3+}$	$\lambda_\alpha > 0,\ \sqrt{\lambda_1} + \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}},\ \lambda_3 \neq \lambda_1 \neq \lambda_2 \neq \lambda_3,\ \sqrt{\lambda_3} \pm \sqrt{\lambda_{1/2}} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$
(2,1,1,1,1)	$m^2_{1-} < m^2_{1+} = m^2_{2-} < m^2_{2+},\ m^2_{1/2\pm} \neq m^2_{3-} < m^2_{3+} \neq m^2_{1/2\pm}$	$\lambda_\alpha \neq 0,\ \sqrt{\lambda_2} - \sqrt{\lambda_1} = \frac{2}{\sqrt{\mu^2-1}},\ \sqrt{\lambda_3} \pm \sqrt{\lambda_2} \neq \pm\frac{2}{\sqrt{\mu^2-1}}$
(2,2,1,1)	$0 < m^2_{1-} < m^2_{1+} = m^2_{2-} < m^2_{2+} = m^2_{3-} < m^2_{3+}$	$\lambda_\alpha \neq 0,\ \sqrt{\lambda_2} - \sqrt{\lambda_1} = \sqrt{\lambda_3} - \sqrt{\lambda_2} = \frac{2}{\sqrt{\mu^2-1}}$
(1,1,1,1,1,1)	$m^2_{\alpha-} < m^2_{\alpha+},\ m^2_{\alpha\pm} \neq m^2_{\beta\pm}$ for $\alpha \neq \beta$	$\lambda_\alpha \neq 0,\ \sqrt{\lambda_\alpha} - \sqrt{\lambda_\beta} \neq \pm\frac{2}{\sqrt{\mu^2-1}},\ \lambda_\alpha \neq \lambda_\beta,\ \sqrt{\lambda_\alpha} + \sqrt{\lambda_\beta} \neq \frac{2}{\sqrt{\mu^2-1}}$ for $\alpha \neq \beta$.

Table 4.1: This table gives an overview of the ways in which the masses of the moduli can be equal when $h^{2,1}_- = 2$ depending on the values of μ, $|W_0|$ and the rescaled eigenvalues of $\mathbf{Q}\overline{\mathbf{Q}}$: λ_α. Each case is characterized with numbers in brackets. In the bracts we denote the degree of the degeneracy for each distinct value of the masses.

4.3.2 Limit κ to ∞

Interesting to note is that from an extension of figure 3.1 we can get an idea of the structure of the masses for large κ_1 and κ_2. This extension is shown in figure 4.3. We see that the masses approximately split in pairs and two pairs diverge while one pair stays approximately constant.

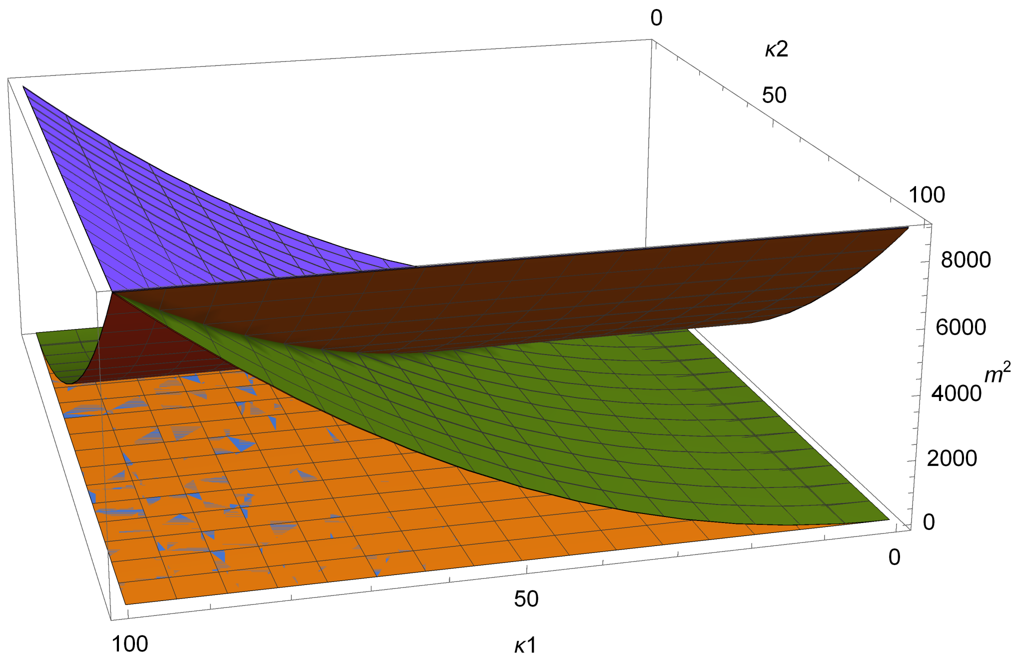

Figure 4.3: This figure shows the six $\frac{\mathcal{V}^2}{N_{\text{flux}}} m^2_{\alpha\pm}$ dependent on $e^{K_{cs}}\kappa_1$ and $e^{K_{cs}}\kappa_2$ for $\mu = 3$ and $\theta = \phi = 0$. The case we consider here is $h^{2,1}_{-} = 2$. Note that for big κ_1 and κ_2 the masses approximately split in pairs and two pairs diverge while one pair stays approximately constant.

Chapter 5

Discussion

In this thesis we have analysed the masses of stabilized moduli generated by the compactification of a 4-dimensional $\mathcal{N} = 1$ supergravity theory in the minimum of the F-term potential. For other work on this see for example [11].

In the introduction 1 we discussed that string theory is a theory of quantum gravity. This is achieved by considering particles to be one-dimensional instead of zero-dimensional. For this to be self consistent we need the space-time that these particles live in to be 26- or 10-dimensional. If we then want to make contact with the very successful standard model that describes particle physics as we understand it today we need to compactify these "extra" dimensions on an internal manifold.

In section 1.8.2 we see that such compactifications generally lead to more massless modes in the theory. This creates a discrepancy with conventional particle physics and therefore constitutes a problem. Massive modes can be ignored in effective theories when we consider energy scales that are not able to produce these modes but extra massless modes should be taken into account.

This can potentially be solved by stabilizing these extra massless modes that arise due to compactification on a internal manifold. This stabilisation can be done by introducing fluxes. These fluxes change the internal manifold slightly via a version of the Einstein-equation such that it is not strictly flat anymore. This makes exact calculations very difficult but it does give masses to these moduli that can then be ignored in effective theories and contact can be made with the particle physics that we know.

In section 1.14 we introduce the setting we will work with. We consider a type IIB-flux compactification with F_3- and H_3-fluxes. We then assume that the moduli can be stabilized, explicitly stabilizing them is difficult due to the fluxes taking the internal manifold away from it being a Kähler manifold, and set out to calculate the masses of these moduli in this case.

This is done in [9] by calculating the eigenvalues of the mass matrix, hessian of the scalar potential, in the minimum of the scalar potential. That leads to an expression (4.3.1) for the masses of the moduli dependent on the eigenvalue σ_α of the matrix $\mathbf{Q}\overline{\mathbf{Q}}$ (1.14.22). These masses are:

$$m^2_{\alpha\pm} = \mathrm{e}^K \left(\sigma_\alpha \pm |W_0|\right)^2 . \tag{5.0.1}$$

With these results we set out in chapter 2 and 3 to find explicit expressions for these masses. As finding the eigenvalues of $\mathbf{Q}\overline{\mathbf{Q}}$ is challenging we instead consider some special cases: one complex structure modulus (2.2 $h^{2,1}_- = 1$), one complex structure modulus in the large complex structure limit (2.3 $h^{2,1}_- = 1$ and $\mathcal{F}$ known) and two comlpex structure moduli (3 $h^{2,1}_- = 2$).

In these cases we are able to find the masses of the moduli 2.2, (2.2.7), (2.3.21), 3.1, (3.2.8). We find that in the case $h^{2,1}_- = 1$ the masses depend on the flux number N_{flux}, the Ricci scalar R, the Einstein-frame volume $\mathcal{V}$ and the quantity μ which we defined in (2.1.13) which also depends on $|W_0|$. Later we find that μ can be expressed depending on the ratio of h^0 and h^i (4.0.1). If we also impose the large-complex structure limit we find that $R = \frac{2}{3}$ and we lose the R dependence of the masses.

If $h^{2,1}_- = 2$ we found that the masses depend on the flux number N_{flux}, the eigenvalue λ, the Einstein-frame volume $\mathcal{V}$ and the quantity μ. In turn we also found and equation for λ that depends on 4 real parameters describing the fluxes that are introduced.

With these expressions for the masses we started analysing them in chapter 4. First we listed all the options for how different combinations of the masses can be equal and we pick out interesting examples of these combinations to see when this actually occurs.

After this analysis we learned that:

- For a general flux all moduli receive a mass and are stabilized when turning on these fluxes. Only for very precise cases the masses of one or a few of the moduli are zero.
- Even having degenerate masses seems to be the exception. For example for all the masses to be equal we need either h^0 or h^i to vanish such that $H = h^i\chi_i + \overline{h}^{\bar{j}}\overline{\chi}_{\bar{j}}$ or $H = h^0\Omega + \overline{h^0\Omega}$.
- When going to extremes in the parameters that determine the fluxes ($R \to -\infty$ or κ_1 and $\kappa_2 \to \infty$) the masses approximate degenerate pairs and one pair of masses stays small while the other masses diverge.

After the findings in this thesis there are some subsequent steps that could be taken to hopefully find interesting results:

- The features of the masses that we describe in this thesis are only studied for explicit examples. It would be interesting to find out if these persist over more general frameworks. For example with an arbitrary number of complex structure moduli.
- There are still some moduli left of which we have not considered the masses here. These are the Kähler moduli. These can be stabilized using KKLT [1, 16]. This gives bounds on W_0. This can then in turn be combined with the findings in this thesis to further restrict the possible masses.

In these ways we will hopefully be able to further restrict the possible compactifications that help string theory touch on measurements in particle physics.

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