UTRECHT UNIVERSITY

GRADUATE SCHOOL OF NATURAL SCIENCES

INSTITUTE FOR THEORETICAL PHYSICS

Giving Love to modified gravity; an examination of neutron star properties in scalar-tensor gravity

Dáire SCULLY, B.Sc.

October 2023

MASTER'S THESIS

UNDER THE SUPERVISION OF

Gastón CRECI KEINBAUM, M.Sc.

AND

Dr. Tanja HINDERER

Abstract

A major goal of astrophysics involves the study of neutron stars to determine their nuclear matter equation of state. The first detection of gravitational waves from a binary pair neutron star merger, GW170817, in 2017 opened a new channel for the way in which the equations of state may be constrained or eliminated. This is namely through the extraction of the binary system's *tidal Love number*, which encodes how the neutron stars statically respond to tidal fields. It is possible, however, that deviations from general relativity in strong gravity regimes could lead to differences in measurements that are consistent with uncertainties in the equation of state. To accurately untangle these differences means that efficient modelling of neutron stars in modified theories of gravity is essential.

In this thesis, we review some recent work in calculating neutron star properties - namely the mass, radius and tidal Love number - in scalar-tensor gravity before attempting to extend this work for a larger class of theories - *generalized scalar-tensor theories*. We will do this through the use of an effective action, which in the so-called *Einstein frame* allows for a decoupling of the scalar field from the gravitational part of the action. We find that the formalism for extracting the Love number is not straightforward to extend, due in part to the generic nature of the theory. We finally focus on a numerical case study which involves solving for the equilibrium configuration of the neutron star, such that we can examine how generalized scalar-tensor theories impact the mass and radius. The case study has a successful preliminary calculation, but results in a degeneracy of the mass-radius curves that also correspond to so-called $f(R)$ gravity. We conclude that further analysis is needed to determine the authenticity of this degeneracy. Nevertheless we have still paved the way for future neutron star calculations in generalized scalar-tensor theories, such that they can be better included in the constraining of the neutron star equation of state.

Contents

Chapter 1

Introduction

Gravity is everywhere. Of the four fundamental forces of nature, it is the most apparent in our daily lives. It holds us to the floor, keeps the moon in orbit around the Earth, the Earth in orbit around the Sun, and the Sun in orbit around the centre of the galaxy - at which lies a supermassive black hole. All of this exists in a universe that is expanding due to gravitational effects. Therefore, if one expects to understand the working structure of the universe, an accurate theory of gravity is crucial.

The first successful theory of gravity emerged in the 17th century. Developed by Sir Isaac Newton, it is appropriately referred to as Newtonian gravity. His theory described gravity as a universal attractive force between all massive objects and began the first mathematical study of gravitational effects, leading to far-reaching consequences on the 17th century understanding of the world. Newton's one simple equation, describing gravity as a force, allowed for an almost complete description of motion within the solar system. The successful predictive capabilities of Newtonian gravity allowed it to remain as the dominant gravitational paradigm for the next three centuries. Newtonian gravity worked exceptionally well in most circumstances but did not come without its own shortcomings - namely the instantaneous transmission of the gravitational field, as well as its inability to account for the perihelion precession of Mercury's orbit.

These shortcomings were overcome in the next gravitational revolution by Einstein in 1915. Einstein's theory of gravity, referred to as General Relativity (GR), radically changed the way gravity was understood - objects no longer experienced a universal force drawing them together, but instead followed the natural path set out for them by the curvature of spacetime. Einstein's theory was vastly unlike its predecessor, yet successfully filled in the gaps left behind by Newtonian theory. GR brought a wealth of new gravitational phenomena with it. Black holes, gravitational waves, gravitational lensing, expanding universe, the big bang, time dilation and parallel universes are only some examples of the rich phenomena unaccounted for by Newtonian theory. Despite the major successes of GR, it again had shortcomings of its own.

Several gravitational related phenomena still remain unexplained or unaccounted for even today. Some examples include dark matter, dark energy, cosmic inflation, and the lack of a fully consistent quantum theory of gravity. Furthermore, the singularities predicted at the centre of black holes strongly motivate that GR is incomplete. Just as Einstein revolutionised the gravitational framework that preceded him, it is expected that another gravitational revolution may produce a theory that supersedes GR. This expectation has incentivized research into modified theories of gravity. These theories extend the framework of GR, but still reduce to Einstein's theory in low-energy regimes. An instructive approach to modifying GR is possible and is the basis for the theories examined in this thesis. The approach is to violate Lovelock's theorem - the theorem that guarantees GR as the unique theory of gravity under certain assumptions. Breaking one of these assumptions opens new possibilities for the nature of gravity.

For this thesis, we are primarily interested in three classes of modified gravity theories, the first two being scalar-tensor theories and f(R) theories. Scalar-tensor theories involve introducing a second dynamical gravitational field, a scalar field, into the action in addition to the metric tensor of GR.

This is motivated by results in string theory, a potential quantum theory of gravity, which predicts a coupling of the Ricci scalar of GR to another independent scalar field. Meanwhile, f(R) theories have a Lagrangian that goes beyond linear order in the Ricci scalar, in contrast to GR. As we will demonstrate later, f(R) theories can be recast as scalar-tensor theories, further incentivizing both classes of theories for study. One may even be so ambitious as to study f(R) theories coupled to scalar-fields, which we refer to as *generalized scalar-tensor* theories and are the third class of modified gravity theories of interest in this work. The study of modified gravity is futile however if we cannot determine which, if any, corresponds to nature. To give credibility to a theory, its predictions must be comparable with observation. Therefore, computing observables is a key step in constraining the correct theory. In the case of GR, these observables must be compatible with what has already been observed, but also diverge in stronger, untested, regimes of gravity.

A key class of observables are those related to tidal effects. Tidal effects apply to bodies of mass with a finite size i.e., a body that is not a point mass. Due to its finite size, a body will not experience a homogeneous gravitational field. An example is best seen in the tides on the Earth, which are induced by gravitational interactions with the Moon. For this reason, such effects are appropriately referred to as tidal effects. Similar phenomena may occur in neutron stars or other astrophysical objects. An inhomogeneous gravitational field can cause a neutron star to deform, as the matter throughout the star is unevenly influenced by the gravitational field. The characteristics of this tidal deformation are accurately encapsulated within a single number, referred to as the body's Love number (named after British scientist Augustus E.H. Love [1]).

Love numbers are extremely interesting as they are encoded in the gravitational waves emitted by a binary system, offering a cleaner channel of information for neutron star observations. Measuring the gravitational waves and extracting the Love number can give insight into the star's internal structure. This is done by choosing a theoretical model (an equation of state) for the star's interior structure and calculating its respective Love number. The theoretically calculated Love number may be compared with the observed one from the gravitational wave, putting constraints on the possible equations of state that can correspond to the interior structure. This has already been done for [2], which was successful in using Love number measurements from the binary neutron star merger GW170817 to rule out particular equations of state at a 90% confidence.

As useful as gravitational waves are for the extraction of a Love number, they are limited in the additional information they can obtain about the system. For example a gravitational wave cannot determine the radius of a neutron star. Another project for measuring properties of neutron stars is the Neutron Star Interior Composition Explorer (NICER) which uses electromagnetic means to measure the mass and radius. NICER has already been successful in measuring the mass and radius of two neutron stars [3, 4], making these physical quantities a valuable part of the union between theory and experiment.

However, to properly probe the neutron star equation of state, one must take into account potential deviations from general relativity. The intention of this thesis is to model neutron stars in modified theories of gravity such that their mass, radius and tidal Love numbers can be computed. Some work has already been done [5] in computing the tidal Love number in $f(R)$ gravity, and several works have been conducted for computing the tidal Love number in scalar-tensor gravity [6, 7, 8]. For this work, we will primarily review the formalism developed in [6], and attempt to generalized it for the case of generalized scalar-tensor theories.

In chapter 2 we start with a review of tidal effects in Newtonian binaries. We give a pedagogical introduction to the Love number in Newtonian gravity and demonstrate how it can be used to encapsulate finite size effects in an effective action for a binary system on the orbital scale. The chapter concludes with the basic recipe for computing the Love number in Newtonian gravity.

In chapter 3, we take the relevant parts of chapter 2, namely the effective action and the recipe for computing the Love number, and put them in a relativistic context.

Chapter 4 then gives an introduction to modified theories of gravity. We motivate why such theories warrant study and explain a systematic way in which general relativity may be modified. We then choose to focus on a particular class of theories, scalar-tensor theories. We conclude with an introduction to the Einstein frame, which allows one to decouple the scalar field from the Ricci scalar through a conformal rescaling of the metric.

With the necessary tools introduced for studying scalar-tensor theories, chapter 5 then reviews how to generalize the effective action from chapter 3 to a scalar-tensor context. We follow closely the approach of [6]. This will allow us to relate Love numbers from the Einstein frame back to the physical frame as well as shed light on how the Einstein frame Love number can be extracted. We conclude the chapter with a review of the results in [6].

After this, we then attempt in chapter 6 a final extension to examine tidal effects in generalized scalar-tensor theories, in particular we attempt to generalize the formalism of [6] to see how the Love number may be extracted in the Einstein frame.

Finally, in chapter 7 we perform a numerical case study for the neutron star background (equilibrium) spacetime in generalized scalar-tensor gravity, which allows for a theoretical extraction of the mass and radius of the neutron star in this modified gravity theory.

Chapter 2

Tidal effects in Newtonian binaries

In this chapter, we will give a brief introduction to the study of tidal effects in Newtonian binaries. We begin with a multipolar expansion of the gravitational potential, analogues to electromagnetism, and define the relevant mass and tidal multipole moments. We will then go on to derive the full action representing finite-size effects in the so-called adiabatic limit, and treating one of the bodies as a point mass. Lastly we introduce the concept of the Love number, which encapsulates the finite size effects of a body, and motivate its relevance for study.

We will primarily follow Chapter 10 of reference [9]. A similar, but more brief, introduction to tidal effects can also be found in [10]. Lastly, an extensive treatment on Newtonian tidal effects can be found in Chapters 1 & 2 of [11].

2.1 Multipoles and definitions

2.1.1 Multipole moments Q^L

The natural starting point for discussing the effects of Newtonian gravity is the Poisson equation

$$\nabla^2 U = -4\pi\rho. \tag{2.1.1}$$

It represents the gravitational potential, $U(t,\boldsymbol{x})$, generated by a continuous mass distribution $\rho(t,\boldsymbol{x})$, where the bold symbol denotes a 3-vector. The solution for the potential $U(t,\boldsymbol{x})$ can be obtained with the use of a Green's function and is given by

$$U(t,\boldsymbol{x}) = \int d^3x' \rho(t,\boldsymbol{x}')\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{2.1.2}$$

It is also possible to verify by substitution that (2.1.2) is indeed a solution of (2.1.1). Outside the mass distribution ρ, one can Taylor expand the potential around some (dynamical) reference point $z(t)$

$$U(t,\boldsymbol{x}) = \int d^3x' \rho(t,\boldsymbol{x}') \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(x'-z)^L \left(\frac{\partial}{\partial x'^L}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\Bigg|_{\boldsymbol{x}'=\boldsymbol{z}}, \tag{2.1.3a}$$

$$= \int d^3x' \rho(t,\boldsymbol{x}') \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!}(x'-z)^L \partial_L \frac{1}{|\boldsymbol{x}-\boldsymbol{z}|}, \tag{2.1.3b}$$

were we have introduced a symmetric notation for tensors, denoted by a capital L, which represents a string of indices. To demonstrate the notation, take for example x^L and ∂_L in the case that $\ell = 3$. One has

$$x^{L=3} \equiv x^i x^j x^k, \quad \partial_{L=3} \equiv \partial_i\partial_j\partial_k. \tag{2.1.4}$$

To go to the second line of (2.1.3), a direct evaluation of ∂_L on $\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}$ shows the equivalence

$$\left(\frac{\partial}{\partial x'^L}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)|_{\boldsymbol{x}'=\boldsymbol{z}} = (-1)^\ell \partial_L \frac{1}{|\boldsymbol{x}-\boldsymbol{z}|}, \tag{2.1.5}$$

so long as $\boldsymbol{x} > \boldsymbol{z}$. From here, one can group together the $\boldsymbol{x}'$ dependent terms in (2.1.9) to define the Newtonian mass multipole moments:

$$\boxed{Q^L = \int d^3x' \rho(t, \boldsymbol{x}')(x' - z)^L}. \tag{2.1.6}$$

For $L = 0$, we have

$$Q^0 = \int d^3x' \rho\,(t, \boldsymbol{x'}) = M, \tag{2.1.7}$$

and merely recover the total mass of the body. Also, if $\boldsymbol{z}$ is taken to be the body's center of mass, the Q^L for $L = 1$ vanish by the definition of the centre of mass

$$Q^i = \int d^3x' \rho\,(t, \boldsymbol{x'})\,(x' - z)^i = 0. \tag{2.1.8}$$

Thus the Newtonian potential can so far be written as:

$$\begin{aligned} U(t, \boldsymbol{x}) =& \int d^3x' \rho(t, \boldsymbol{x}') \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} (x' - z)^L \partial_L \frac{1}{|\boldsymbol{x} - \boldsymbol{z}|}, \\ =& \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} Q^L \partial_L \frac{1}{|\boldsymbol{x} - \boldsymbol{z}|}, \\ =& \frac{M}{|\boldsymbol{x} - \boldsymbol{z}|} + \sum_{\ell=2}^{\infty} \frac{(-1)^\ell}{\ell!} Q^L \partial_L \frac{1}{|\boldsymbol{x} - \boldsymbol{z}|}. \end{aligned} \tag{2.1.9}$$

We can further simplify the Newtonian potential using the relationship

$$\partial_L \frac{1}{|\boldsymbol{x} - \boldsymbol{z}|} = (-1)^\ell (2\ell - 1)!! \frac{n^{\langle L \rangle}}{|\boldsymbol{x} - \boldsymbol{z}|}. \tag{2.1.10}$$

Here, the $n^i \equiv x^i/r$ denote unit vectors, such that $n_i n^i = 1$. The angular brackets in $n^{\langle L \rangle}$ denote that we take only the Symmetric Trace Free (STF) part. As an example, consider $n^{\langle L \rangle}$ for $L = 2$. Similarly, to (2.1.4), we begin by make n^i symmetric in 2 indices, but also remove the trace:

$$n^{\langle ij \rangle} = n^i n^j - \frac{1}{3} \delta^{ij}. \tag{2.1.11}$$

,
One can then see that $n^{\langle ij \rangle}$ is symmetric in its indices ij such that $n^{\langle ij \rangle} = n^{\langle ji \rangle}$, and also trace-free

$$\delta_{ij} n^{\langle ij \rangle} = 1 - \frac{1}{3}(3), \tag{2.1.12}$$

$$= 0. \tag{2.1.13}$$

For a more comprehensive introduction to STF tensors, the reader is referred to [12, 13]. One should recall that from here on, any tensors defined in terms of angular brackets are STF.

With (2.1.10), the gravitational potential of a body with finite size becomes:

$$U(t, \boldsymbol{x}) = \frac{M}{r} + \sum_{\ell=2}^{\infty} \frac{(2\ell - 1)!!}{\ell!} \frac{Q_L n^{\langle L \rangle}}{r^{\ell+1}}. \tag{2.1.14}$$

The tensors Q^L also takes on the properties of an STF tensor. This is a result of its contraction with $n^{\langle L \rangle}$, another STF tensor. The contraction of a tensor with an STF tensor, leaves only the STF part of the original tensor. Thus $Q^L \equiv Q^{\langle L \rangle}$. The meaning of the mass multipole moments should hopefully be more apparent with (2.1.10). They are coefficients in the correction to the gravitational potential, resulting from the body's finite size. An easy consistency check is for the case of a point mass, with density profile $\rho(\boldsymbol{x}) = M\delta(\boldsymbol{x}-\boldsymbol{z})$. A direct calculation for $\ell > 2$ verifies that the Q^L vanish.

One can similarly look at the case of a body with spherical symmetry i.e. $\rho \equiv \rho(r)$. Evaluating Q^L in spherical coordinates shows that the angular integrals vanish, giving the equivalent contribution to $U(t,\boldsymbol{x})$ as that of a point mass. In this way, one can also think of Q^L as coefficients in the correction to the body's potential energy as it is perturbed from the shape of a sphere.

2.1.2 Tidal moments $\mathcal{E}^L$

Consider the case where a body also experiences a potential due to an external source such as an orbiting companion. Denote this potential by $U^{\text{ext}}(t,\boldsymbol{x})$. One can again expand the potential around the body's centre of mass $\boldsymbol{z}$:

$$U^{\text{ext}}(t,\boldsymbol{x}) = U^{\text{ext}}(t,\boldsymbol{z}) + (\boldsymbol{x}-\boldsymbol{z})^i \left[\partial_i U^{\text{ext}}(t,\boldsymbol{x})\right]_{\boldsymbol{x}=\boldsymbol{z}} - \sum_{\ell=2}^{\infty} \frac{1}{\ell!}(\boldsymbol{x}-\boldsymbol{z})^L \mathcal{E}_L. \tag{2.1.15}$$

Here we have defined the body's tidal multipole moments as

$$\boxed{\mathcal{E}_L = -\left(\partial_L U^{\text{ext}}(t,\boldsymbol{x})\right)_{\boldsymbol{x}=\boldsymbol{z}(t)}}. \tag{2.1.16}$$

For example, if we look at the $L = 2$ tidal moment, for an external gravitational potential Φ, we have

$$\mathcal{E}_{ij} = -\partial_i \partial_j \Phi. \tag{2.1.17}$$

When we go on to discuss tidal effects in the context of GR, we will use (2.1.17) as a reference to ensure that we have obtained the correct non-relativistic limit.

We can also combine the expression (2.1.16) for the external potential in terms of $\mathcal{E}_L$ with the potential (2.1.14) due to the body's finite size to get the full potential for the body as

$$U(t,\boldsymbol{x}) = \frac{M}{r} + \sum_{\ell=2}^{\infty} \frac{1}{\ell!}\left[(2\ell-1)!!\frac{Q_L n^{\langle L\rangle}}{r^{\ell+1}} - n^{\langle L\rangle}\mathcal{E}_L\, r^{\ell}\right], \tag{2.1.18}$$

where we have used

$$(\boldsymbol{x}-\boldsymbol{z})^L = n^L r^{\ell}. \tag{2.1.19}$$

2.2 Constructing the action

2.2.1 Lagrangian approach

We wish to construct the relevant action for a binary system, using the mass and tidal moments defined previously. Consider 2 bodies, A and B, in orbit around one another. The Lagrangian for any system is constructed according to:

$$L = T - V, \tag{2.2.1}$$

where T and V respectively are the total kinetic and potential energies of the system. The motion of each body can be decomposed into the motion of its centre of mass plus internal contributions denoted by "int". The relevant kinetic and potential energies for body A are thus:

$$T_A = \frac{1}{2}M_A \dot{\boldsymbol{z}}_A^2 + T_A^{\text{int}}, \quad V_A = \frac{1}{2}\int_A d^3x \rho_A U_B + V_A^{\text{int}}. \tag{2.2.2}$$

The expression for body B is similar with A and B interchanged in (2.2.2). Note the factor of $\frac{1}{2}$ for V_A in (2.2.2). This factor may at first appear unfamiliar in the continuum limit, but we can quickly check that it is to avoid double counting the interactions between the bodies. We can do this by treating each body as if they were composed of discrete point particles.

For example, take body A to have a density profile consisting of several point masses $\rho(\boldsymbol{x}) = \sum_i m_i \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}_i)$, and body B to be a single point mass with potential $U_B(\boldsymbol{x}) = -\frac{GM_B}{|\boldsymbol{x}-\boldsymbol{z}_B|}$. Filling these into (2.2.2) (and ignoring the V_A^{int}), one finds:

$$V_A = -\frac{1}{2}\int_B d^3x \sum_i m_i \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_i)\frac{GM_B}{|\boldsymbol{x}-\boldsymbol{z}_B|}, \tag{2.2.3}$$

$$= -\frac{1}{2}\sum_i \frac{Gm_i M_B}{|\boldsymbol{x}_i - \boldsymbol{z}_B|}, \tag{2.2.4}$$

which is simply the potential energy for a system of point particles interacting with a point mass M_B (up to a factor of $\frac{1}{2}$).

Similarly for V_B, one can fill in $\rho_B(\boldsymbol{x}) = M_B \delta^{(3)}(\boldsymbol{x} - \boldsymbol{z}_B)$ and $U_A(\boldsymbol{x}) = \sum_i \frac{Gm_i}{|\boldsymbol{x}-\boldsymbol{x}_i|}$. This leads to an explicit expression for V_B as:

$$V_B = -\frac{1}{2}\int_A M_B \delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}_B)\sum_i \frac{Gm_i}{|\boldsymbol{x}-\boldsymbol{x}_i|}, \tag{2.2.5}$$

$$= -\frac{1}{2}\sum_i \frac{Gm_i M_B}{|\boldsymbol{z}_B - \boldsymbol{x}_i|}. \tag{2.2.6}$$

Adding V_A and V_B together gives the total interaction energy between the bodies as:

$$V = V_A + V_B, \tag{2.2.7}$$

$$= \sum_i \frac{-Gm_i M_B}{|\boldsymbol{z}_B - \boldsymbol{x}_i|}, \tag{2.2.8}$$

which is exactly the total potential energy expected from a discrete system of interacting point particles. We can thus see that the $\frac{1}{2}$ in (2.2.2) is essentially an overcounting factor.

We can now assemble all terms to construct the total Lagrangian. To simplify the analysis, and to better eludicate the finite size effects that emerge, we will specialize to the case in which body B is a point mass. From (2.2.1) and (2.2.2), the total Lagrangian is therefore

$$L = \frac{1}{2}\left(M_A \dot{\mathbf{z}}_A^2 + M_B \dot{\mathbf{z}}_B^2\right) - \int_A d^3x \rho_A U_B + L_{\text{int}}, \tag{2.2.9}$$

where all internal interactions are contained in L_{int}. One can further expand U_B in a similar way to the multipole expansion in (2.1.15). By filling in the definitions (2.1.6) and (2.1.16) for Q^L and $\mathcal{E}^L$ one finds

$$L = \frac{1}{2}\left(M_A \dot{\mathbf{z}}_A^2 + M_B \dot{\mathbf{z}}_B^2\right) + M_A U_B|_{x=z} - \sum_{\ell=2}^{\infty}\frac{1}{\ell!}Q^L \mathcal{E}_L + L_{\text{int}}, \tag{2.2.10}$$

where the $\ell = 1$ term has vanished by definition of the centre of mass. The problem may further be simplified by working in the barycentric frame. We can define:

$$M = M_A + M_B, \quad \mu = \frac{M_A M_B}{M}, \quad \mathbf{r} = \mathbf{z}_A - \mathbf{z}_B, \tag{2.2.11}$$

as the total mass, reduced mass and relative separation vector respectively. Filling in $U_B = -\frac{M_B}{r}$ and defining $r \equiv |\mathbf{r}|$, we can work out the total Lagrangian (2.2.10) to be

$$L = \frac{1}{2}\mu v^2 - \frac{\mu M}{r} - \sum_{\ell=2}^{\infty}\frac{1}{\ell!}Q^L \mathcal{E}_L + L_{\text{int}}. \tag{2.2.12}$$

Summarising the final Lagrangian, we have

$$L = L_{\rm pp} + L_{\rm int} - \sum_{\ell=2}^{\infty} \frac{1}{\ell!} Q^L \mathcal{E}_L, \tag{2.2.13}$$

where

$$L_{\rm pp} \equiv \frac{1}{2}\mu v^2 - \frac{\mu M}{r}, \tag{2.2.14}$$

encapsulates the point-particle-like dynamics of each body's centre of mass, and $L_{\rm int}$, encoding the internal dynamics, is yet to be specified.

To conclude, we have neatly summarized all dynamics within three concise terms. One term representing the centre of mass motion of each body; one term representing internal dynamics for body A; and one term representing the interaction energy between the induced deformations of body A with the tidal fields of body B. Equation (2.2.13) is so far exact, the only assumption made being body B is a point mass.

2.2.2 Internal dynamics

To complete the Lagrangian in (2.2.13), we must specify the term corresponding to the internal dynamics. We assume that body A is spherically symmetric, and that its multipole moments Q_L are induced by the externally perturbing tidal moments $\mathcal{E}_L$. For a star made of a fluid undergoing perturbations, it has been demonstrated that the tidally induced multipole moments behave in a way resembling a harmonic oscillator [14, 15, 16, 17]. For this case, the internal Lagrangian takes the form:

$$L_{\rm int} = \sum_{\ell=2} \frac{1}{2\ell! \lambda_\ell \omega_{0\ell}^2} \left[\dot{Q}_L \dot{Q}^L - \omega_{0\ell}^2 Q_L Q^L \right], \tag{2.2.15}$$

The above Lagrangian introduces two constants $\omega_{0\ell}$ and λ_ℓ which we will soon address.

As seen in [18], the multi-pole moments Q^L now represent the dynamical degrees of freedom of the theory, and so may be treated as independent fields. Thus we can take the equation of motion for a single Q^L from the total Lagrangian made up by equations (2.2.13) and (2.2.15). Varying the action (2.2.13) with respect to Q^L results in the equation of motion

$$\ddot{Q}_L + \omega_{0\ell}^2 Q_L = -\omega_{0\ell}^2 \lambda_\ell \mathcal{E}_L. \tag{2.2.16}$$

We can see that the mass multipole moments behave as nothing more than a harmonic oscillator (as postulated in the Lagrangian), with an analogues driving force generated by the tidal moments $\mathcal{E}_L$. The constants $\omega_{0\ell}$ simply characterize the frequency of the oscillations and the constants λ_ℓ act analogously to an inverse spring constant. More discussion will be given to the λ_ℓ soon. We will now go on to simplify (2.2.15), by making further assumptions on the nature of the mass multipole moments, and allowing for a more concise representation of the action (2.2.13).

2.2.3 Adiabatic limit

If the internal timescales of the body, $\tau^{\rm int} \sim \omega_{0\ell}^{-1} \sim \sqrt{R^3 M_A}$ are fast relative to the time-scale of tidal variations, $\tau_{\rm orb} \sim \sqrt{r^3/(M_A + M_B)}$, we may treat the multi-pole moments Q^L as essentially time independent i.e. $\dot{Q}_L = 0$.
With this, we see that the EoM (2.2.16) further reduces to:

$$\boxed{Q_L^{\rm adiab} = -\lambda_\ell \mathcal{E}_L}. \tag{2.2.17}$$

The meaning of λ_ℓ now becomes clear. It characterises the strength of the induced mass moments Q_L by the tidal moments $\mathcal{E}_L$, in the adiabatic limit.

We can go one step further to simplify our total Lagrangian. By re-substituting the EoM (2.2.17) back into the Lagrangian (2.2.13), one can in essence "integrate out" the internal degrees of freedom Q^L. Doing so, and integrating the resulting Lagrangian over time, gives the final action:

$$\boxed{S_{\mathrm{adiab}} = S_{\mathrm{pp}} + \int dt \left[\frac{\lambda_\ell}{2\ell!}\mathcal{E}_L\mathcal{E}^L\right]}, \tag{2.2.18}$$

where the summation over ℓ is implicit and $S_{\mathrm{pp}} = \int dt\, L_{\mathrm{pp}}$, with L_{pp} given by (2.2.14).

The multi-pole moments Q^L have been completely eliminated from the action. The importance of λ_ℓ in the study of finite size effects should now be apparent. They completely characterize the internal dynamics on the orbital scale and are appropriately dubbed the *tidal deformability coefficients*. The advantage here is that all finite-size effects in (2.2.18) are now described by only the tidal field $\mathcal{E}_L$ and the tidal deformability coefficients λ_ℓ. Setting either to zero merely reduces the system to a pair of orbiting point particles. We will go on to give some further attention to the λ_ℓ as well as elaborate on their usefulness.

2.3 Love numbers

2.3.1 Relation to λ_ℓ

We devote the remainder of this section to the discussion of the tidal deformability coefficients and further introduce the concept of the *Love number*. Recall from (2.2.17) that the tidal deformability coefficient is given as:

$$\lambda_\ell = -\frac{Q_L^{\mathrm{adiab}}}{\mathcal{E}_L}, \tag{2.3.1}$$

that is, it is the ratio of the induced mass moments to the tidal moments in the adiabatic limit.

A body's tidal deformability coefficient can be explicitly expressed in terms of the body's tidal Love numbers, k_ℓ, and its radius, R as:

$$\lambda_\ell = \frac{2}{(2\ell-1)!!}k_\ell R^{2\ell+1}. \tag{2.3.2}$$

The Love number was first introduced by its namesake, British scientist A.E.H. Love, in [1]. Although the Love number k_ℓ and tidal deformability coefficient λ_ℓ are distinct quantities, they encapsulate the same information about the body. For this reason, the tidal deformability coefficient λ_ℓ is sometimes referred to as the Love number. We will adopt that practise for the remainder of this thesis.

2.3.2 Relation to gravitational Waves

The importance of the Love number λ_ℓ should be clear from the work seen in the previous section. It allows one to completely characterize the effects of the internal body structure on the two-body dynamics in the adiabatic limit.

Furthermore, if the Love number is known, it can further allow one to constrain the equation of state (EoS) of the body. This is done by choosing a particular EoS, computing the Love number and then constraining the EoS based on the actual observed Love number for the body. In order to do this, one obviously needs the Love number of the body itself, which begs the question - how is the Love number for a body experimentally measured? The answer lies with gravitational waves.

First, one can define a dimensionless Love number for a body, A, through:

$$\Lambda_L^A = \frac{\lambda_\ell}{M_A^{2\ell+1}} = \frac{2}{(2\ell-1)!!}k_\ell\left(\frac{M_A}{R}\right)^{-(2\ell+1)}, \tag{2.3.3}$$

where M_A is the mass of the body. Next, we assume that the binary system emits gravitational waves. In the TaylorF2 waveform template [19], the amplitude of the corresponding gravitational waves in frequency space is given by:

$$\tilde{h}(f) = \mathcal{A} f^{-\frac{7}{6}} \exp\left[i(\psi_{\rm pp} + \psi_{\rm tidal})\right], \tag{2.3.4}$$

where f is the GW frequency, $\mathcal{A} \propto \mathcal{M}^{\frac{5}{6}}/D$, where $\mathcal{M} = \mu^{\frac{3}{5}} M^{\frac{2}{5}}$ is the chirp mass, and D is the distance between the GW detector and the binary. $\psi_{\rm pp}$ represents the point mass contribution to the phase and $\psi_{\rm tidal}$ is the contribution from tidal effects. In the adiabatic limit, the contribution to $\psi_{\rm tidal}$ from the mass quadrupole, Q_{ij}, of a non-spinning neutron star binary with equal masses is

$$\psi_{\rm tidal} = \frac{3M\tilde{\Lambda}}{128\mu x^{5/2}}\left[-\frac{39}{2}x^5 - \frac{3115}{64}x^6\right]. \tag{2.3.5}$$

Here, $\tilde{\Lambda} = (\Lambda^A + \Lambda^B)/2$, and $x = (M\Omega)^{2/3}$, with Ω the angular frequency of the orbit. One can see that the Love number of each neutron star is clearly recoverable from the phase of the GW. This indeed demonstrates GWs as a valuable resource for the probing of astrophysical phenomena. We can also see why the Love number is such a quantity of interest, and indeed it shall remain as such for the duration of this thesis.

2.3.3 Computation

The action (2.2.18) is useful for highlighting how the Love numbers λ_ℓ encapsulate finite size effects at the orbital scale in the binary system. However, this is only an effective action, and is not where the actual computation for the Love number is done. Heuristically, the Newtonian Love number is computed by first specifying an EoS for the neutron star. Then, the star is taken to be spherically symmetric in an equilibrium state, and in the absence of perturbations. One then solves the Poisson equation, the continuity equation, and the Euler equation to obtain the background configuration of the star;

$$\nabla^2 U = -4\pi\rho, \quad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0, \quad \frac{\partial v^i}{\partial t} + (\boldsymbol{v}\cdot\nabla)v^i = -\frac{\partial^i p}{\rho} + \partial_i U. \tag{2.3.6}$$

The star is then assumed to be perturbed by the tidal field, $\mathcal{E}_L$, from a companion, resulting in perturbations to the potential, pressure and density

$$p = p_0 + \delta p, \quad \rho = \rho_0 + \delta\rho, \quad U = U_0 + \delta U, \tag{2.3.7}$$

where the background quantities are denoted with a subscript $_0$. The potential perturbation can be expanded as

$$\delta U = H_\ell(r) Y_{\ell m}(\theta, \phi), \tag{2.3.8}$$

and a master equation for $H(r)$ obtained by substituting (2.3.7) back into (2.3.6) and eliminating the other quantities. A more detailed derivation is presented in [9]. The resulting equation is

$$H_\ell'' + \frac{2}{r}H_\ell' - \frac{l(l+1)}{r^2}H_\ell = -4\pi\left(\frac{1}{\rho_0}\frac{dp_0}{d\rho_0}\right)^{-1} H_\ell, \quad r \leq R. \tag{2.3.9}$$

To proceed further, an equation of state must be specified. We assume a cold, barotropic EoS such that the pressure is only a function of the density i.e. $p = p(\rho)$. The master equation (2.3.9) is then integrated numerically up to the surface at $r = R$. For the vacuum region $r > R$, the pressure and density vanish, and (2.3.9) can be solved exactly

$$H_\ell^{\rm ext} = \frac{Q_{\ell m}}{r^{\ell+1}} - \frac{1}{(2l-1)!!}\mathcal{E}_{\ell m} r^\ell. \tag{2.3.10}$$

We wish to extract the Love number λ_ℓ by computing the ratio

$$\lambda_\ell = -\frac{Q_{\ell m}}{\mathcal{E}_{\ell m}}, \tag{2.3.11}$$

which represents (2.3.1), expressed in a spherical harmonic basis [9]. We can do this by first considering the so-called logarithmic derivative

$$y_\ell(r) = \frac{r}{H_\ell^{\text{ext}}} \frac{dH_\ell^{\text{ext}}}{dr}, \tag{2.3.12}$$

and seeing that

$$\lambda_\ell = \frac{R^{2l+1}}{(2l-1)!!} \frac{\ell - y_\ell(R)}{\ell + 1 + y_\ell(R)}, \tag{2.3.13}$$

$$= -\frac{Q_{\ell m}}{\mathcal{E}_{\ell m}}. \tag{2.3.14}$$

Given that the numerical and analytical solutions solutions should match at the surface, we can then evaluate (2.3.14) using the numerical solution, which extracts the Love number.

However, this solution for the Love number pertains to the Newtonian case. In order to account for the full effects of gravity, we wish to discuss Love numbers in a relativistic context. In the next chapter, we will start building our machinery for the analysis of tidal effects in GR, and present how the above computation may be generalized for a relativistic context.

Chapter 3

Tidal effects in General Relativity

In the previous chapter we discussed the formalism for tidal effects in Newtonian gravity. We would now like to generalize this formalism to a relativistic context, to better understand tidal effects in GR. We begin this chapter by generalizing the Newtonian action of the previous chapter to a relativistic one, through the equivalence principle (also called the principle of covariance or "comma to semi-colon" rule). The form of this action will be the basis for the effective field theory (EFT) action that we make use of in later chapters. Afterward, we present the basic recipe, following [20] and [9], for computing the Love number λ_ℓ using perturbation theory in GR.

3.1 Effective action for relativistic binaries

3.1.1 Equivalence principle

Recall the action (2.2.18), which encodes Newtonian tidal effects in the adiabatic limit

$$S_{\text{adiab}} = S_{\text{pp}} + \int dt \left[\frac{\lambda_l}{2l!}\mathcal{E}_L\mathcal{E}^L\right], \tag{3.1.1}$$

where S_{pp} denotes the same point particle action. One can intuitively make this action relativistic by replacing each term with its relativistic counterpart, through the equivalence principle. Recall that the above action described a binary system, consisting of a finite body and a point mass. This system was then reformulated in terms of the motion of a single particle with reduced mass μ. The relativistic action takes a similar form and describes the system as a point particle (plus tidal multipole moments) moving along a worldline $x(\tau)$. The tangent to the wordline is the four-velocity $u^\mu = \frac{dx^\mu}{d\tau}$, and τ parameterizes the evolution of the point particle along x^μ. This is also referred to as the "skeletonized worldline" approach.

The relativistic analogue of (3.1.1) can be obtained through making the covariant replacements

$$dt \to z\, d\tau, \tag{3.1.2}$$

$$S_{\text{pp}} \to -m\int z\, d\tau = -m\int z\, d\tau\sqrt{-\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}g_{\mu\nu}}. \tag{3.1.3}$$

$$\tag{3.1.4}$$

Here $z = \sqrt{-u^\mu u_\mu}$ is the redshift factor, which ensures the invariance of the integration measure $d\tau$ under reparameterization. The tidal moments also take on a different definition in the relativistic context. For $\ell = 2$, the relativistic tidal moments are given as

$$\mathcal{E}_{\mu\nu} = W_{\mu\alpha\nu\beta}\frac{u^\alpha u^\beta}{z^2}, \tag{3.1.5}$$

where $W_{\mu\alpha\nu\beta}$ is the Weyl tensor, defined as the trace-free part of the Riemann tensor. It is possible to check, by decomposing the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \quad \partial_0 h_{\mu\nu} = 0, \tag{3.1.6}$$

and identifying $h_{00} = -2\Phi$, that (3.1.5) reduces to the Newtonian tidal moment (2.1.17)

$$\mathcal{E}_{ij} = -\partial_i \partial_j \Phi, \tag{3.1.7}$$

for weak, static gravitational fields. Analogously to the Newtonian case, higher order tidal moments are obtained by taking covariant derivatives of the lower order moments

$$\mathcal{E}_L = \frac{1}{(l-2)!} \nabla_{\langle L-2} \mathcal{E}_{L_1, L_2 \rangle}, \tag{3.1.8}$$

where angular brackets denote the STF part. We will often omit the angular brackets when talking about tidal multipole moments, however the reader should be aware that they are implied. As a demonstration of the notation, the $\ell = 4$ tidal moments look like

$$\mathcal{E}_{\alpha\beta\mu\nu} = \frac{1}{2} \nabla_{\langle \alpha} \nabla_{\beta} \mathcal{E}_{\mu\nu \rangle}. \tag{3.1.9}$$

To summarize, we have generalized the Newtonian point particle action to a relativistic one

$$S = -m \int z \, d\tau + \int z \, d\tau \left[\frac{\lambda_l}{2l!} \mathcal{E}_L \mathcal{E}^L \right], \tag{3.1.10}$$

with the new tidal moments $\mathcal{E}_L$ given by (3.1.8) and (3.1.5).

3.1.2 Comments on the effective field theory approach for relativistic binaries

Although the non-minimal coupling technique applied in the previous section was sufficient to obtain the effective action for relativistic binaries (3.1.10), this was only one possible way to do so. We did this by first writing down the Newtonian action and promoting the relevant terms to their relativistic counterparts. We would like to be able to generalize this procedure for relativistic actions which do not have a distinct Newtonian counterpart, namely modified theories of General Relativity.

We will be specifically interested in studying tidal effects in scalar-tensor theories of gravity in subsequent chapters, for which an EFT description of the system will be needed and for which there is no Newtonian analogue to build from. The basic recipe for obtaining the effective action in such theories, is to first write down an action that contains all terms satisfying the relevant symmetries for Newtonian gravity and GR: general coordinate invariance, time reversal invariance and spacetime parity invariance among others. A more detailed discussion can be found in [21]. The action may then be further simplified using equations of motion and field redefinitions, and the form factors may be determined by matching the action onto its low energy counterpart. This is similar to the Wilsonian EFT approach often employed in quantum field theories [22].

For unmodified GR, this process is carried out in [23] and results in the effective action identical to (3.1.10) for the adiabatic limit.

3.2 Computing Love numbers in relativistic binaries

Now that we have discussed the basic effective action for describing the binary system at the orbital scale, we would like to again proceed with a computation of the Love number in GR, similarly to how we did previously in the Newtonian case. We will begin this section by defining the mass and tidal multipole moments in GR, and how they can be extracted from the metric. After this we will show explicitly the recipe for computing the Love number of a perturbed body using perturbation theory. In what follows we will we use geometrical units such that $G = c = 1$.

3.2.1 Defining the multipole moments and the Love number

As we saw in the previous chapter, the multipole moments in Newtonian theory were obtained by expanding the Newtonian potential and identifying the mass moments Q_L with the terms that run as $r^{\ell+1}$ and the tidal moments $\mathcal{E}_L$ as those that run as r^l. We can make use of this approach in order to define Q_L and $\mathcal{E}_L$ in the GR context. The time-time component of the metric is analogous to an effective Newtonian potential U_{eff}

$$g_{tt} = -\left(1 - 2U_{\text{eff}}\right). \tag{3.2.1}$$

For example, in the Schwarzchild spacetime, which is valid for a point mass or in the vacuum region of a spherically symmetric object, one intuitively has $U_{\text{eff}} = M/r$. If we expand U_{eff} sufficiently far away from the source for a *non-spherical object*, we should expect it to run similarly to the Newtonian potential

$$\lim_{r\to\infty} U_{\text{eff}} = \frac{M}{r} + \frac{3n^{\langle ij\rangle}Q_{ij}}{2r^3} + \mathcal{O}(r^{-4}) - \frac{1}{2}n^{\langle ij\rangle}\mathcal{E}_{ij}r^2 + \mathcal{O}(r^3). \tag{3.2.2}$$

Thus, if one can write down the Einstein equations for a non-spherically symmetric spacetime, and solve for the g_{tt} component of the metric, they should be able to expand the metric according to (3.2.2) and extract the Love number through the ratio

$$\lambda_\ell = -\frac{Q_L}{\mathcal{E}_L}. \tag{3.2.3}$$

With the basic outline explained, we now give the detailed explanation of how this computation may be done.

3.2.2 Background configuration and mass-radius curves

The TOV equations

We begin with the background metric for a spherically symmetric non-rotating spacetime

$$ds_0^2 = -e^{\nu(r)}dt^2 + e^{\gamma(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.2.4}$$

where $\nu(r)$ and $\gamma(r)$ are yet to be determined functions. The mass of the star is defined as

$$m(r) \equiv \frac{\left[1 - e^{-\gamma(r)}\right]r}{2}, \tag{3.2.5}$$

which can be equivalently stated as

$$e^{-\gamma(r)} = \left(1 - \frac{2m(r)}{r}\right). \tag{3.2.6}$$

This is in analogy with the rr component of the Schwarzchild metric for which $m(r) = M$. The neutron star matter is also modelled as perfect fluid, through the energy-momentum tensor

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu + pg_{\mu\nu}, \tag{3.2.7}$$

where p and ρ are the neutron star pressure and energy density respectively. The fluid's four-velocity is represented by u^μ. We will work in the rest frame of the fluid, such that the average velocity is 0, for which the four-velocity looks like

$$u^\mu = (u^0, 0, 0, 0). \tag{3.2.8}$$

We can determine the value of u^0 through the normalization condition $u^\mu u_\mu = u^\mu u^\nu g_{\mu\nu} = -1$. Plugging in the metric (3.2.4) tells us that

$$u^0 = e^{-\frac{\nu}{2}}. \tag{3.2.9}$$

Beginning with the Einstein equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{3.2.10}$$

we can also plug in the ansatz metric (3.2.4) as well the form of the energy momentum tensor (3.2.7). Looking at the tt and rr components, we get differential equations for $\gamma(r)$ and $\nu(r)$

$$\frac{1}{r^2}\frac{d}{dr}\left[r(1-e^{-\gamma(r)})\right] = 8\pi\rho, \tag{3.2.11}$$

$$\frac{e^{-\gamma(r)}}{r}\frac{d\nu}{dr} - \frac{1}{r^2}\left(1-e^{-\gamma(r)}\right) = 8\pi p. \tag{3.2.12}$$

We can also use the conservation of the energy-momentum tensor

$$\nabla^\mu T_{\mu\nu} = 0, \tag{3.2.13}$$

to get an additional differential equation for p

$$\frac{dp}{dr} = -\frac{p+\rho}{2}\frac{d\nu}{dr}. \tag{3.2.14}$$

To rewrite these equations in terms of more familiar quantities, we can replace $\gamma(r)$ with $m(r)$ in (3.2.11) and (3.2.12) to yield

$$\frac{dm}{dr} = 4\pi r^2\rho, \tag{3.2.15}$$

$$\frac{d\nu}{dr} = 2\frac{4\pi r^3 p + m}{r(r-2m)}. \tag{3.2.16}$$

We can also simplify the energy conservation equation (3.2.14) by replacing $\frac{d\nu}{dr}$ with equation (3.2.16) to get

$$\frac{dp}{dr} = -\frac{(4\pi r^3 p + m)(\rho + p)}{r(r-2m)}. \tag{3.2.17}$$

To summarize, we have 2 coupled differential equations for the mass $m(r)$ and pressure $p(r)$ of the system

$$\frac{dm}{dr} = 4\pi r^2\rho, \quad \frac{dp}{dr} = -\frac{(4\pi r^3 p + m)(\rho + p)}{r(r-2m)}, \tag{3.2.18}$$

as well as a third differential equation for the metric component $\nu(r)$

$$\frac{d\nu}{dr} = 2\frac{4\pi r^3 p + m}{r(r-2m)}. \tag{3.2.19}$$

Together, these 3 equations comprise the Tolman-Oppenheimer-Volkoff (TOV) equations. To solve them, it is only necessary to integrate equations (3.2.18). The solutions can be substituted into (3.2.19) to obtain $\nu(r)$.

The background spacetime is completely specified by solving these 3 equations. To do so, one must first specify an equation of state $p = p(\rho)$, relating the pressure and density for the neutron star matter. The initial conditions close to the center of the star, $r \to 0$, are given by

$$\rho = \rho_c + \mathcal{O}(r^2), \quad p = p_c + \mathcal{O}(r^2), \quad m = \frac{4\pi}{3}\rho_c r^3 + \mathcal{O}(r^5). \tag{3.2.20}$$

Here, ρ_c is referred to as the *central density*. It is typically chosen to be on the order of $\sim 10^{18}\mathrm{kg/m}^3$, in accordance with observational constraints. We will usually integrate the TOV equations for a variety of central densities. When a central density is chosen, the TOV equations are integrated from the center of the star (which we numerically take to be $r_{min} \sim 10^{-10}\mathrm{km}$). The surface of the star is defined to be the radius for which the pressure vanishes i.e.

$$p(r = R) = 0. \tag{3.2.21}$$

When the pressure (and thus the density) are zero, the vacuum TOV equations are merely

$$\frac{dm}{dr} = 0, \quad \frac{dp}{dr} = 0, \quad \frac{d\nu}{dr} = 2\frac{m}{r(r-2m)}, \tag{3.2.22}$$

implying a constant mass and pressure. The mass of the star may thus be evaluated as

$$m(R) = M. \tag{3.2.23}$$

Similarly the differential equation for ν can be solved as

$$\nu(r) = \ln\left(1 - \frac{2M}{r}\right), \tag{3.2.24}$$

implying that $\nu(r) = \gamma(r)$ in the vacuum.

Mass-radius curves

We can solve the TOV equations numerically in Mathematica. As discussed previously, the radius of the star is determined by the vanishing of the pressure, and the mass can be extracted by evaluating $m(r)$ at the surface. We present a solution of the TOV equations in 3.1for the SLy equation of state, which is popular in the literature. We use a central density of $\rho_c = 10^{17.59}\mathrm{kg/m}^3$.

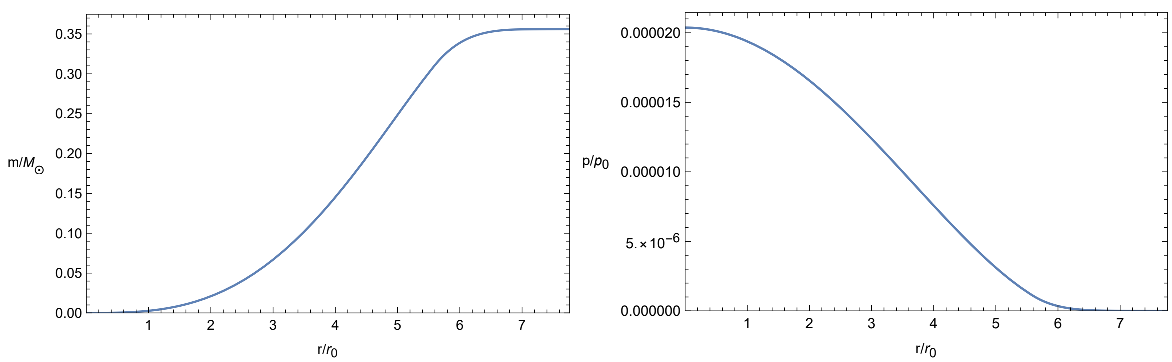

Figure 3.1: Mass and pressure as a function of radius for the SLy equation of state with a central density of $\rho_c = 10^{17.59}\mathrm{kg/m}^3$. The radius of the neutron star can be determined from where the pressure goes to zero. The radius is plotted in geometrical solar units, where $r_0 = GM./c^2 \simeq 1.5\mathrm{km}$. The pressure is plotted in dynes per centimetre squared such that $p_0 = 1\ \mathrm{dyn/cm}^2$.

Furthermore, by varying the central density and successively calculating each M and R, we may plot the mass against the radius for various neutron star equations of state. The resulting diagrams are referred to as *mass-radius* diagrams or *mass-radius* curves. Some simple examples for the SLy, H4 and AP44 and WFF1 equations of state[1] are presented in Figure 3.2. As we can see, the various equations of state allow for slightly different ranges of the masses and radii of the neutron stars.

The maximum mass of such stars appear to be on the order of two solar masses. As the radii are decreased (or the compactness $C = M/R$ is increased), the mass of the neutron star will reach a maximum and then begin to decline. This maximum mass represents the point at which the neutron star configuration becomes unstable and begins to collapse to a black hole (this is the result of a time dependent analysis however, beyond the scope of this thesis). As a result the parts of the curves with radii smaller than the radii corresponding to the maximum masses are rarely of interest.

We will later go on to modify the TOV equations in the presence of a scalar field ϕ and numerically calculate the mass-radius curves. Thus the plots in Figure 3.2 serve as a benchmark for comparing our results in modified gravity to GR.

[1]Credit to existing code from Frank Visser and Gastón Creci Keinbaum.

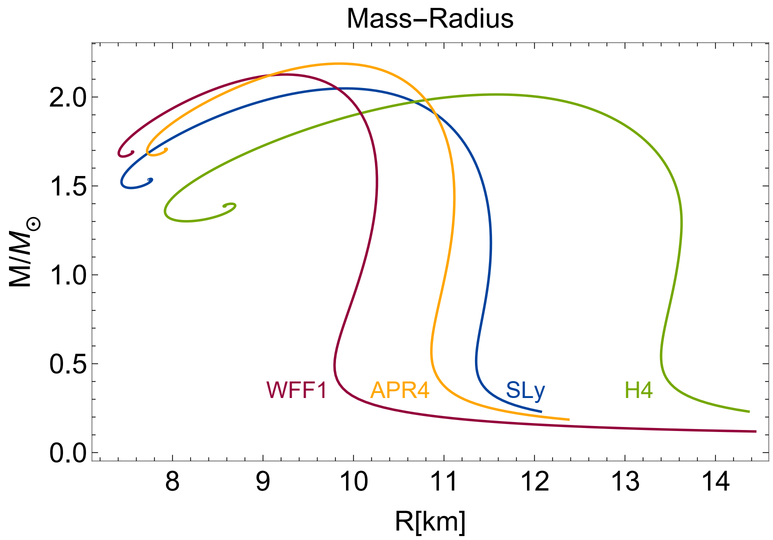

Figure 3.2: Mass radius curves for various popular equations of state in the literature. The central densities start from $10^{17.3}\text{kg/m}^3$ on the right hand side and are slowly increased until the maximum mass configuration is reached. The solutions to the left of the maximum mass then correspond to unstable configurations. The total range of the central densities here is $10^{18.3}\text{kg/m}^3 \leq \rho_c \leq 10^{21.5}\text{kg/m}^3$.

3.2.3 Perturbations and extracting the Love number

To compute the neutron star Love number we continue by considering linear, static perturbations to the equilibrium metric (3.2.4)

$$ds^2 = ds_0^2 + h_{\mu\nu}dx^\mu dx^\nu, \tag{3.2.25}$$

where ds_0^2 denotes the metric (3.2.4). We will usually refer to the unperturbed spacetime ds_0^2 as the *background spacetime*. Working in the Regge-Wheeler gauge [24, 25, 26], $h_{\mu\nu}$ can be decomposed into tensorial spherical harmonics, which are characterized by mode integers (l, m), and a parity π which is either $(-1)^l$ or $(-1)^{l+1}$. Here we consider the even-parity (also known axial or electric-type) perturbations $\pi = (-1)^l$. With this, $h_{\mu\nu}$ takes the form

$$h_{\mu\nu} = \sum_{\ell,m} Y_\ell^m(\theta,\phi) \times \text{Diag}\left[-e^\nu H_0^{\ell m}(r), e^\gamma H_2^{\ell m}(r), r^2 K^{\ell m}(r), r^2\sin^2(\theta)K^{\ell m}(r)\right], \tag{3.2.26}$$

where $Y_\ell^m(\theta,\phi)$ are the spherical harmonics and H_0^ℓ, H_1^ℓ, and K^ℓ are functions of r which characterize the perturbations. Similarly we can perturb the other physical quantities, including the pressure and density as

$$p = p_0 + \delta p Y_\ell^m(\theta,\phi), \tag{3.2.27a}$$

$$\rho = \rho_0 + \frac{d\rho}{dp}\delta p Y_\ell^m(\theta,\phi), \tag{3.2.27b}$$

$$u^\mu = u_0^\mu + \delta u^\mu. \tag{3.2.27c}$$

By re-substituting the now perturbed metric, pressure, density and four-velocity back into the Einstein equations, it is possible to derive equations of motion for the perturbations. This is done by expanding (3.2.10) to first order in the perturbations, and cancelling off the background TOV equations (3.2.18) and (3.2.19), which constitute the "zero order" contributions. This leaves behind differential equations for the linear perturbations. Using the various components of the Einstein equations as well as the conservation of the energy-momentum tensor, it is then possible to obtain the relations

$$H_0^\ell = -H_2^\ell \equiv H^\ell, \quad \frac{\delta p}{\rho + p} = -\frac{1}{2}H^\ell. \tag{3.2.28}$$

The function K^ℓ and its derivative may be eliminated in favour of H^ℓ and its derivatives using the rr and $r\theta$ components of the Einstein equations. Lastly, the tt component leads to the differential equation

$$\begin{aligned}
0 =& \frac{d^2H^\ell}{dr^2} + \left\{\frac{2}{r} + e^\gamma\left[\frac{2M}{r^2} + 4\pi r(p_0 - \rho_0)\right]\right\}\frac{dH^\ell}{dr} \\
&+ \left\{e^\gamma\left[-\frac{l(l+1)}{r^2} + 4\pi(\rho_0 + p_0)\frac{d\rho_0}{dp_0} + 4\pi(5\rho_0 + 9p_0)\right] - \left(\frac{d\nu}{dr}\right)^2\right\} H^\ell, \qquad (3.2.29)
\end{aligned}$$

for the metric perturbation $H^\ell(r)$. This equation represents the relativistic counterpart to the master equation from the Newtonian case (2.3.9). Such as in the Newtonian case, the master equation (3.2.29) is solved numerically inside the star. The initial conditions [20], close to the center of the star $r \to 0$, are

$$H^\ell \propto r^\ell, \quad H'_\ell \propto \ell r^{\ell-1}. \qquad (3.2.30)$$

These ensure the regularity of the solution at the origin. The proportionally constant in (3.2.30) is irrelevant and can be chosen arbitrarily, as it will drop out when extracting the tidal deformability coefficients.

Outside the star, we have $\rho = p = 0$ by definition of the vacuum. This means that (3.2.29) can be further simplified to

$$0 = \frac{d^2H^\ell}{dr^2} + \left[\frac{2}{r} + e^\gamma\frac{2M}{r^2}\right]\frac{dH^\ell}{dr} - \left[e^\gamma\frac{l(l+1)}{r^2} + \left(\frac{d\nu}{dr}\right)^2\right]H^\ell. \qquad (3.2.31)$$

The above equation can be further simplified using the fact that $m(r)$ and $\nu(r)$ are constant in the vacuum. Using the relationship (3.2.5) between $\gamma(r)$ and $m(r)$, and the vacuum form (3.2.24) for $\nu(r)$, the master equation (3.2.31) becomes

$$0 = \frac{d^2H^\ell}{dr^2} + \left[\frac{2(M-r)}{r(2M-r)}\right]\frac{dH^\ell}{dr} + \left[\frac{l(l+1)r(2M-r) - 4M^2}{r^2(2M-r)^2}\right]H^\ell. \qquad (3.2.32)$$

By defining a new variable $x = \frac{r}{M} - 1$, the above equation can be written as an associated Legendre equation. The result is

$$0 = (x^2-1)H''_\ell(x) + 2xH'_\ell(x) - \left(\ell(\ell+1) + \frac{4}{x^2-1}\right)H_\ell(x). \qquad (3.2.33)$$

This equation has an exact solution (conveniently obtainable with Mathematica) that looks like

$$H^\ell = c_l^Q Q_{l2}(x) + c_l^P P_{l2}(x). \qquad (3.2.34)$$

Here $P_{\ell 2}(x)$ and $Q_{\ell 2}(x)$ are the normalized associated Legendre functions of the first and second kind respectively, and c_ℓ^Q and c_ℓ^P are integration constants. The leading order terms of $P_{\ell 2}(x)$ and $Q_{\ell 2}(x)$ in the limit of $r \to \infty$ is

$$P_{\ell 2}\left(\frac{r}{M} - 1\right) \sim r^\ell, \qquad (3.2.35)$$

$$Q_{\ell 2}\left(\frac{r}{M} - 1\right) \sim r^{-(\ell+1)}, \qquad (3.2.36)$$

which ensures that the solution $H^\ell(r)$ asymptotically behaves like the Newtonian potential (3.2.2). The integration constants c_ℓ^Q and c_ℓ^P are then determined by matching the analytical exterior solution (3.2.34) to the numerical interior solution at the surface of the star $r = R$. The Love number λ_ℓ is then extracted by comparing the solution for $H^\ell(r)$ with the tt component of the metric (3.2.1) using the expression (3.2.2) for U_{eff}. As an instructive example we demonstrate how the $\ell = 2$ Love number can be extracted. Combining equations (3.2.1) and (3.2.2), one finds the asymptotic behavior of the metric to be

$$
\begin{aligned}
\lim_{r\to\infty} g_{tt} &= -(1-2U_{\text{eff}}),\\
&= -\left(1-\frac{2M}{r}-\frac{3}{r^3}n^{\langle ij\rangle}Q_{ij}+\mathcal{O}(r^{-4})+r^2 n^{\langle ij\rangle}\mathcal{E}_{ij}+O(r^3)\right).
\end{aligned}
\tag{3.2.37}
$$

Similarly, using the vacuum solutions for the metric function $\nu(r)$ (3.2.24) and perturbation $H^\ell(r)$ (3.2.34) we find

$$
\begin{aligned}
\lim_{r\to\infty} g_{\text{tt}} &= -e^{\nu(r)}\left(1+H^{\ell=2}(r)\right),\\
&= -\left(1-\frac{2M}{r}-c_\ell^Q\frac{8M^3}{5r^3}+\mathcal{O}(r^{-4})-r^2\frac{3c_\ell^P}{M^2}+\mathcal{O}(r)\right).
\end{aligned}
\tag{3.2.38}
$$

Comparing powers of r in both equations (3.2.37) and (3.2.38), we can deduce that the expression for the Love number is

$$
\lambda_2 = -\frac{Q_{ij}}{\mathcal{E}_{ij}}, \tag{3.2.39}
$$

$$
= \frac{8}{45}\frac{c_2^Q}{c_2^P}M^5. \tag{3.2.40}
$$

Furthermore, a generic expression for the Love number λ_ℓ at any ℓ can be written as

$$
\frac{(2\ell-1)!!}{M^{2\ell+1}}\lambda_\ell = -\left.\frac{P'_{\ell 2}(x)-\frac{M}{R}y_\ell P_{\ell 2}(x)}{Q'_{\ell 2}(x)-\frac{M}{R}y_\ell Q_{\ell 2}(x)}\right|_{x=\frac{R}{M}-1}, \tag{3.2.41}
$$

where $y_\ell(r)$ denotes the logarithmic derivative

$$
y_\ell(r) \equiv= \frac{r}{H^\ell}\frac{dH^\ell}{dr}. \tag{3.2.42}
$$

We include some plots for the Love numbers λ_ℓ for $\ell = 2$ in Figure 3.3. Plots for the Love numbers k_ℓ, defined through (2.3.2), and the dimensionless Love number Λ_ℓ (2.3.3) can be found in Figures 3.4 and 3.5 respectively.

We can use the plots from the mass-radius curves in Figure 3.2 to get some insight into the behavior of the Love number. We can see that the H4 equation of state has a larger radius for its masses compared to the other equations of state. This implies a smaller compactness $C = \frac{M}{R}$. Similarly, we can see that H4 in Figure 3.4 has the largest Love number of the other equations of state. We can get some qualitative understanding from this by seeing that as a neutron star becomes more compact, it's Love number decreases. This would be in accordance with our understanding of a black hole, which has the largest possible compactness $C = \frac{1}{2}$ (in geometrical units) and a vanishing Love number $\lambda_\ell = 0$.

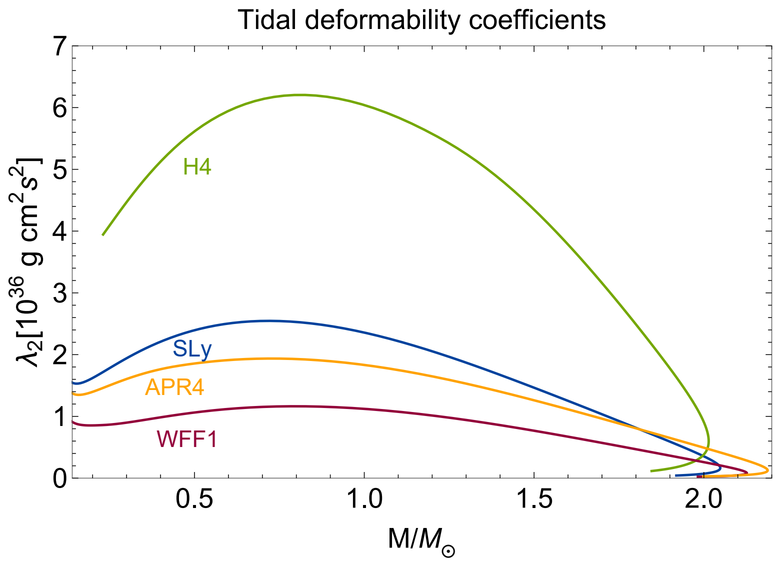

Figure 3.3: Tidal deformability coefficients λ_ℓ ("Love numbers") for for $\ell = 2$. We use the various equations of state shown previously. The central densities range from $10^{17.3}$kg/m$^3 \leq \rho_c \leq 10^{18.5}$kg/m^3. One can compare with the mass-radius curves in Figure 3.2 to see that smaller compactness $C = M/R$ usually results in a larger Love number.

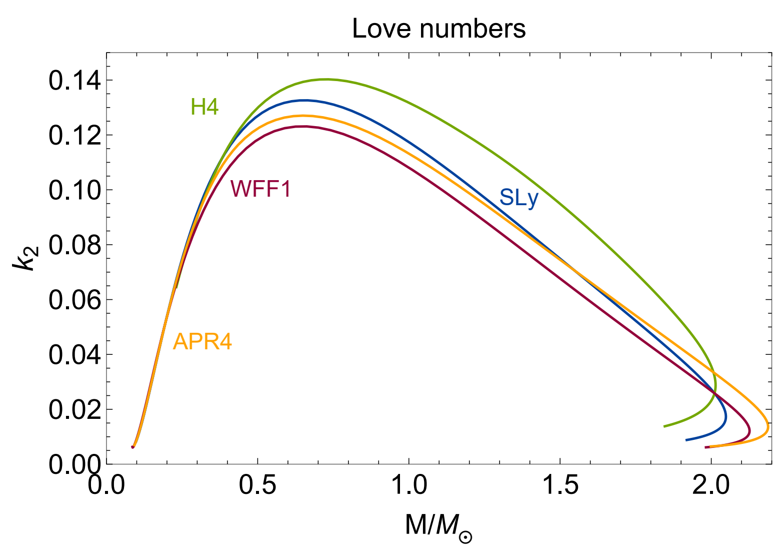

Figure 3.4: Love numbers k_ℓ for $\ell = 2$. The central densities range from $10^{17.3}$kg/m$^3 \leq \rho_c \leq 10^{18.5}$kg/m^3.

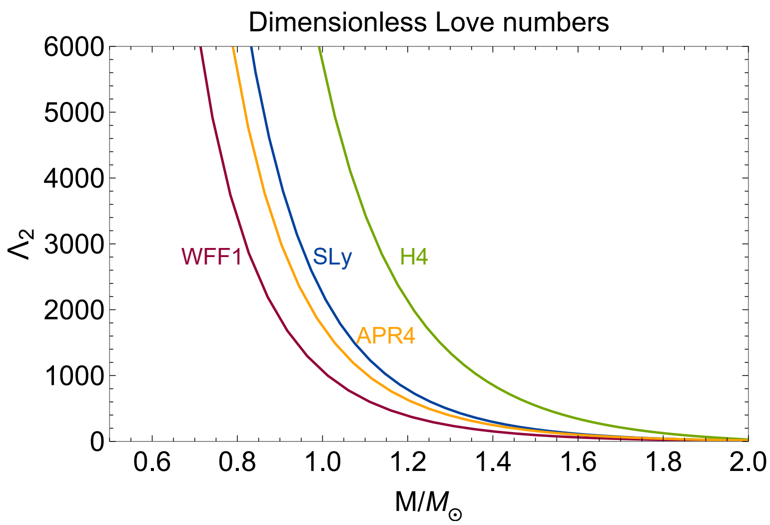

Figure 3.5: The dimensionless Love numbers Λ_ℓ for $\ell = 2$. The central densities range from $10^{17.3}\mathrm{kg/m^3} \leq \rho_c \leq 10^{18.5}\mathrm{kg/m^3}$.

Chapter 4

Scalar-tensor theories of gravity

4.1 Introduction to modified theories

For this chapter we provide a brief introduction to modified theories of general relativity. We start with motivation for why such theories warrant study and discuss a systematic way in which one may modify general relativity - namely by violating Lovelock's theorem. Following this, we will specialize to a particular class of modified gravity theories - scalar-tensor theories. We will further break down scalar-tensor theories into their many sub-classifications. Finally, we will give a brief mathematical introduction to the Jordan and Einstein frames of scalar-tensor theories, and discuss the mathematical tools that one will need to pass between them - conformal transformations.

An introduction to scalar-tensor theories in the context of cosmology can be found in [27, 28]. A more broad overview of modified gravity theories can be found in [29], which also dedicates an entire chapter to the various motivations. Lastly, suitable reviews of the many various modified theories can also be found in [30, 31].

4.1.1 Why do we need modified gravity?

Einstein's theory of General Relativity (GR) is currently the most accepted theory of gravity. It describes gravity as a consequence of the curvature of space and time, which is induced by the presence of matter and energy. It has remained the dominant theory of gravity for over 100 years and has been used to study phenomena across all scales of the macroscopic world, ranging from planet Earth to the entire universe itself. Despite being tested to ultra-high precision within the solar system (and passing), GR has not come without its shortcomings. These problems mainly arise when one studies the universe in cosmological contexts i.e. on large length and time scales, or when one studies gravity in very small regimes i.e. attempts at quantizing gravity.

Despite the success of General Relativity with regard to experimental testing, these tests have been largely confined to what can be conducted within the solar system. GR so far remains largely untested for stronger regimes of gravity. Furthermore, some problems that arise when studying GR at a cosmological scale include the Big Bang singularity and the flatness, horizon, and monopole problems. As well as this, GR, is not seen to incorporate Mach's principle (a search for which lead to the birth of the first modified GR theories). GR in its original formulation also does not account for the rapid present-day expansion of the universe, the source of which has been dubbed "dark energy". Even the simplest dark energy model, the cosmological constant Λ, requires a modification to the Einstein equations. Despite its success in matching to observation, the cosmological constant also comes with its own problem, namely the coincidence problem. As a result, even more complex models have been proposed such as quintessence, which instead incorporates a dynamical Λ in the form of a scalar field. Such a theory is a simple example of a scalar-tensor theory, and demonstrates how such modified theories of gravity see use in mainstream physics. More-over, scalar tensor and $f(R)$ theories also naturally exhibit inflationary properties, making them natural choices for the study of cosmological inflation - where they indeed find frequent application.

On the other extreme scale one also runs into trouble when attempting to quantize gravity. The resulting quantum theories of gravity are only semi-renormalizable i.e. at the first loop level. In order to provide a fully renormalizable theory, an infinite number of higher order counter terms are necessary in the action. Including higher order terms in the in the Ricci scalar, R, into the action results in equations of motion that are at least 4th order in derivatives of the metric $g_{\mu\nu}$ (as opposed to 2nd order in GR), adding vast complexity to the task of solving such equations. The problem with quantizing gravity lies with the fact that, in a quantum theory of gravity, the metric tensor $g_{\mu\nu}$ acts as both the matter field for the graviton as well as a way of representing the curvature of the background space time on which the graviton propagates. This apparent contradiction is at the heart of many of the problems with quantizing gravity and further motivates why a revised theory might be needed at the quantum level. Other attempts at producing a quantum theory of gravity, for example example string theory, should also reduce to the equations of General Relativity in the low-energy limit. This in general does not happen with string theory, and instead a Brans-Dicke [28] like theory, emerges. Thus we are forced to conclude that GR might not be the final word on gravity. Further work needs to be done in order to allow a fully consistent quantum theory of gravity and also explain the unaccounted for phenomena that emerge at cosmological scales.

4.1.2 How to modify gravity?

As we will see, there are many possible ways in which one can construct a modified theory of GR. The most systematic way involves studying *Lovelock's theorem* [32, 33]. Originally presented by David Lovelock, in 1971, it proposes that the Einstein equations are the unique field equations for gravity under certain assumptions. Informally, the theorem can be described as saying that the metric tensor $g_{\mu\nu}$, and the Einstein tensor $G_{\mu\nu}$ are the only rank-2, divergence free, symmetric tensors that can be constructed from the metric in 4 space-time dimensions, using no more than 2 derivatives of the metric tensor.

In order words the Einstein equations are uniquely determined as

$$aG_{\mu\nu} + bg_{\mu\nu} = 0, \tag{4.1.1}$$

under the previous assumptions, and in the absence of matter. Thus, to ensure that we obtain field equations that differ from (4.1.1), one must break one or more of the previously listed assumptions - that is we must violate Lovelock's theorem. Such violations should result in a theory which is distinct to GR. We can provide a systematic way in which one can modify GR, by listing all of the possible ways in which these assumptions can be broken:

1. Include fields other than the metric tensor $g_{\mu\nu}$.
2. Work in greater or fewer than $3 + 1$ spacetime dimensions.
3. Allow for tensors in the field equations that go beyond second-order in the derivatives of the metric.
4. Allow for terms that violate locality.
5. Allow for a violation of energy conservation i.e. $\nabla^{\mu} G_{\mu\nu} \neq 0$

4.2 Scalar-tensor theories

4.2.1 Classification of theories

The primary modified gravity theory of choice for this thesis are *scalar tensor* theories. As suggested, these involve introducing an additional scalar field into the gravitational sector of the action. There are different forms in which the scalar-tensor theory can take, and here we settle some dispute on conventions and discuss in which form we will be studying the scalar-tensor theory.

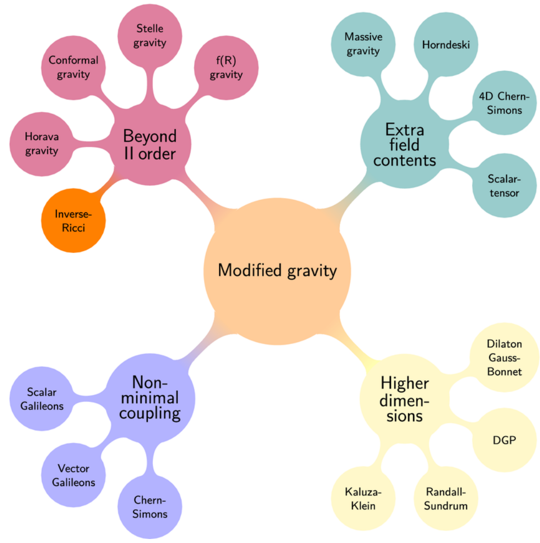

Figure 4.1: Diagram [34] showing the different modifications that one can make to GR and a classification of the different theories that emerge.

Brans-Dicke theory

The original scalar-tensor theory, as introduced by Brans and Dicke in 1961 [35], involves non-minimally coupling a scalar field ϕ to the Ricci scalar, R. The action takes the form

$$S_{\mathrm{BD}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[\phi R - 2\frac{\omega}{\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] + S_{\mathrm{matter}}\left[\psi_m, g_{\mu\nu}\right]. \tag{4.2.1}$$

Here ω is a parameter of the theory, and in the classic Brans Dicke (BD) theory is taken to be a constant. The ψ_m denote matter fields. The equations of motion in the absence of matter for $g_{\mu\nu}$ and ϕ respectively are

$$\phi G_{\mu\nu} = 2\frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi + \nabla_\nu\nabla_\mu\phi - g_{\mu\nu}\left[\nabla_\alpha\nabla^\alpha\phi + \frac{\omega}{\phi}\nabla_\alpha\phi\nabla^\alpha\phi + \frac{1}{2}V(\phi)\right], \tag{4.2.2}$$

$$\frac{\omega}{\phi}\Box\phi = \frac{\omega}{2\phi^2}\nabla_\alpha\phi\nabla^\alpha\phi\frac{1}{4}V'(\phi) - \frac{R}{4}, \tag{4.2.3}$$

where the coupling between the scalar field and the Ricci scalar has made itself more apparent through the complexity of the equations.

One can alternatively express the BD action (4.2.1) in a canonical form by redefining the scalar field ϕ in terms of a new scalar field φ as

$$\phi = \frac{\varphi}{4\omega}. \tag{4.2.4}$$

The BD action then becomes

$$S_{\mathrm{BD}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[\frac{1}{4\omega}\varphi^2 R - 2\partial_\mu\varphi\partial^\mu\varphi - V\left(\frac{\varphi^2}{4\omega}\right)\right] + S_{\mathrm{matter}}\left[\psi_m, g_{\mu\nu}\right]. \tag{4.2.5}$$

This field redefinition has given a canonical kinetic term for the scalar field, but indeed has done little to simplify the coupling between φ and R. We will soon seek a redefinition that can do both, and will find that the answer lies with a rescaling of the metric tensor - indeed a conformal rescaling.

Scalar-tensor theories - $F(\phi)R$

The action (4.2.1) can be generalized further generalized by taking ω to be a function of the scalar field ϕ, and by making a more arbitrary coupling of the scalar field to R. Such an action looks like

$$S_{\mathrm{ST}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[F(\phi)R - 2\frac{\omega(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] + S_{\text{matter}}\left[\psi_m, g_{\mu\nu}\right], \tag{4.2.6}$$

and is what we will later refer to as *scalar tensor theories* or $F(\phi)R$ theories. The equations of motion for $g_{\mu\nu}$ and ϕ increase in intricacy again

$$\begin{aligned} F(\phi)G_{\mu\nu} =& 2\frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla_\nu\phi + F'(\phi)\nabla_\mu\nabla_\nu\phi + F''(\phi)\nabla_\mu\phi\nabla_\nu\phi \\ &- g_{\mu\nu}\left[\frac{\omega(\phi)}{\phi}\nabla_\alpha\phi\nabla^\alpha\phi + F'(\phi)\Box\phi + F''(\phi)\nabla_\alpha\phi\nabla^\alpha\phi + \frac{1}{2}V(\phi)\right], \end{aligned} \tag{4.2.7}$$

$$\frac{\omega(\phi)}{\phi}\Box\phi = \left[\frac{\omega(\phi)}{2\phi^2} - \frac{\omega'(\phi)}{2\phi}\right]\nabla_\alpha\phi\nabla^\alpha\phi + \frac{1}{4}V'(\phi) - \frac{1}{4}F'(\phi)R. \tag{4.2.8}$$

As we can see, the equations of motion become more coupled for ϕ and $g_{\mu\nu}$. To simplify the action (4.2.6), one might try to canonicalize the scalar field ϕ through a redefinition, similarly to how was done in the Brans Dicke action. However, this will depend on the differential relation

$$\frac{d\varphi}{d\phi} = \sqrt{\frac{\omega(\phi)}{\phi}}, \tag{4.2.9}$$

being integrable, allowing for a substitution of

$$\partial_\mu\varphi = \sqrt{\frac{\omega(\phi)}{\phi}}\partial_\mu\phi, \tag{4.2.10}$$

in the action. This is not always possible for particular choices of $\omega(\phi)$, as we will encounter later in chapter 7.

Generalized scalar-tensor theories - $f(\phi, R)$

Finally, we will generalize the BD action one more time. This will be to include arbitrary couplings of both ϕ *and* R within the action. Thus the action looks like

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[f(\phi, R) - 2\frac{\omega(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi\right] + S_{\text{matter}}\left[\psi_m, g_{\mu\nu}\right], \tag{4.2.11}$$

where $f(\phi, R)$ is an arbitrary, algebraic coupling of ϕ and R - meaning it contains no derivatives of ϕ or R. We will from here refer to such theories as *generalized scalar-tensor theories* or $f(\phi, R)$ theories. Note that the function $f(\phi, R)$ has also absorbed the scalar field potential $V(\phi)$. In order to recover the scalar-tensor action (4.2.6), one can set

$$f(\phi, R) = F(\phi)R - V(\phi). \tag{4.2.12}$$

Alternatively one can set $\phi = 1$ in $f(\phi, R)$ to recover so called $f(R)$ theories

$$f(\phi, R) = f(R), \tag{4.2.13}$$

which are another class of modified gravity theories. As we will see in chapter 6, $f(R)$ theories are equivalent to scalar-tensor theories. The equations of motion for the generalized scalar-tensor action (4.2.11) are given by

$$\begin{aligned} \frac{\partial f}{\partial R}G_{\mu\nu} =& 2\frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla_\nu\phi + \nabla_\mu\nabla_\nu\left(\frac{\partial f}{\partial R}\right) \\ &- g_{\mu\nu}\left[\frac{\omega(\phi)}{\phi}\nabla_\alpha\phi\nabla^\alpha\phi - \frac{1}{2}\left(f - \frac{\partial f}{\partial R}R\right) + \Box\left(\frac{\partial f}{\partial R}\right)\right], \end{aligned} \tag{4.2.14}$$

$$\frac{\omega(\phi)}{\phi}\Box\phi = \left[\frac{\omega(\phi)}{2\phi^2} - \frac{\omega'(\phi)}{2\phi}\right]\nabla_\alpha\phi\nabla^\alpha\phi - \frac{1}{4}\frac{\partial f}{\partial\phi}. \tag{4.2.15}$$

Due to the potential higher order terms of the Ricci scalar in $f(\phi, R)$, the equation (4.2.14) is not necessarily of second order in derivatives of the metric. This can be made more apparent by noting that

$$\nabla_\mu \left(\frac{\partial f}{\partial R}\right) = \left(\frac{\partial^2 f}{\partial R^2}\right) \nabla_\mu R + \left(\frac{\partial^2 f}{\partial R \partial \phi}\right) \nabla_\mu \phi. \tag{4.2.16}$$

Substituting this into (4.2.14) convolutes the equations considerably and leads to terms which are proportional to $\nabla_\mu \nabla_\nu R$, and thus are fourth order in the derivatives of the metric.

As a final note, it is intentional that the generalized coupling between ϕ and R be only algebraic, and contain no derivatives of R. As seen above, introducing higher orders of R into the action, will result in equations of motion that are fourth order in the metric, essentially introducing additional degrees of freedom into the theory. This can be generalized to say that for every 2 derivatives of the metric that are introduced into the action, an additional degree of freedom is present in the theory. What we mean by "degree of freedom" in this context, will become clearer after we have introduced the concept of the Einstein frame, and in chapter 6 where we demonstrate how to transform the above action into this Einstein frame.

4.2.2 Introducing the Einstein frame

As discussed previously, the scalar-tensor equations of motion are made increasingly more intricate by the coupling of the scalar field to the curvature sector of the action. This has immediate consequences on the equations of motion. To later compute the Love number in scalar-tensor gravity, it is important that the equations of motion be in a more convenient form to work with. The solution to this problem is found by employing a commonly used trick in the study of scalar-tensor theories - the Einstein frame. We can do this by taking the scalar-tensor action (4.2.6),

$$S_{\rm ST} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[F(\phi) R - 2\frac{\omega(\phi)}{\phi} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + S_{\rm matter} \left[\psi_m, g_{\mu\nu} \right], \tag{4.2.17}$$

(which is said to be in the physical frame), and rescaling the metric according to

$$g_{\mu\nu} = \Omega^{-2} \tilde{g}_{\mu\nu}, \tag{4.2.18}$$

where $\Omega \equiv \Omega(x)$ is an arbitrary spacetime function. Such a rescaling is a *conformal* rescaling. The consequence of this conformal rescaling, is that the scalar-tensor action (4.2.17) may be neatly written as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - 2\tilde{\partial}_\mu \varphi \tilde{\partial}^\mu \varphi - U(\varphi) \right] + S_{\rm matter} \left[\psi_m, \Omega^{-2}(\varphi) \tilde{g}_{\mu\nu} \right], \tag{4.2.19}$$

where quantities with a *tilde* denote that they depend on the Einstein frame metric $\tilde{g}_{\mu\nu}$ and the scalar field φ is a yet to be specified function of the Jordan frame scalar field ϕ. The action (4.2.19) is said to be in the *Einstein* frame, for the reason that it's curvature sector resembles that of GR i.e. it is linear in $\tilde{R}$ and minimally coupled to the scalar field. There are in principle an infinite number of frames one can transform to, depending on the choice of the conformal factor Ω, however the Jordan and Einstein frame are the ones of most interest and so receive their own names. The vacuum equations of motion for the newly obtained Einstein frame action are

$$\tilde{G}_{\mu\nu} = 2\tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi - \tilde{g}_{\mu\nu} 2\tilde{\nabla}_\alpha \varphi \tilde{\nabla}^\alpha \varphi - \frac{1}{2} \tilde{g}_{\mu\nu} U(\varphi), \tag{4.2.20}$$

$$\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha \varphi = \frac{1}{4} U'(\varphi), \tag{4.2.21}$$

where we emphasize the simplicity of these equations compared to Jordan frame scalar-tensor equations (4.2.7) and (4.2.8). We will go on to derive the Einstein frame action (4.2.19) shortly, but first discuss its most important consequence.

The coupling to matter

While the transformation to the Einstein frame has given us the advantage of decoupling the curvature sector from the scalar field, there is a trade off to this benefit. One can now see from equation (4.2.19), that the scalar field φ has coupled to the matter section of the action through $\Omega(\varphi)$. In this sense, we merely traded the coupling to gravity for a coupling to matter.

This coupling of φ to the matter sector of the action will have consequences for particle motion in the Einstein frame. For example, the energy momentum tensor of the Einstein frame $\tilde{T}_{\mu\nu}$ will have explicit dependence on φ, as will the physical quantities that it depends on. We will shortly see this for the case of a perfect fluid.

An additional interesting consequence of this coupling to matter, is that test particles no longer follow geodesics in the Einstein frame - in other words the weak equivalence principle (WEP) has been violated. In the Einstein frame, the geodesic equation changes from

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0, \tag{4.2.22}$$

to

$$\frac{d\tilde{u}^\mu}{d\tilde{\tau}} + \tilde{\Gamma}^\mu_{\nu\lambda} \tilde{u}^\nu \tilde{u}^\lambda = \frac{\tilde{u}^\nu}{\Omega(\varphi)} \left(\partial_\nu \Omega(\varphi) \tilde{u}^\mu - \tilde{u}_\nu \tilde{\partial}^\mu \Omega(\varphi) \right). \tag{4.2.23}$$

Given that Ω is a function of φ, this now implies a scalar-dependent correction to the geodesic equation. This apparent "fifth" fundamental force, present only in the conformally transformed frame, violates the universality of free fall. The trajectory of a massive body in free-fall is no longer only determined by it's initial position and velocity, but also influenced by the presence of the scalar field.
This scalar-dependent correction to the geodesic equation manifests from the fact that, in the new frame, the conservation of the energy-momentum tensor

$$\nabla^\mu T_{\mu\nu} = 0, \tag{4.2.24}$$

changes [27] as

$$\tilde{\nabla}^\mu \tilde{T}_{\mu\nu} = \frac{\Omega'(\varphi)}{\Omega(\varphi)} \tilde{T} \tilde{\nabla}_\nu \varphi, \tag{4.2.25}$$

where $\tilde{T}$ is the trace of the Einstein frame energy-momentum tensor. This is the trade-off for the neater equations of motion (4.2.21). As we can see from this discussion, the Einstein and Jordan frame are not physically equivalent due to how the scalar field couples to matter. As for which frame is the true physical one, this is outside the scope of this thesis. For the purposes of this work, we merely employ the Einstein frame as a mathematical tool and are not interested in the philosophy of frames.

4.2.3 Performing the conformal rescaling

With the Einstein frame action previously presented, we now like to discuss the basic recipe for transforming from the Jordan to Einstein frame. The results of this subsection will be summarized in (4.2.68), such that they can easily be made reference to in future chapters.

We begin with the conformal rescaling of the metric tensor

$$g_{\mu\nu} = \Omega^{-2} \tilde{g}_{\mu\nu}. \tag{4.2.26}$$

With this scaling, we work in a convention that keeps our coordinates unchanged such that

$$dx^\mu = d\tilde{x}^\mu. \tag{4.2.27}$$

The invariant line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ then changes as

$$ds^2 = \Omega^{-2}(x) d\tilde{s}^2. \tag{4.2.28}$$

This allows us to relate the proper time τ between frames as

$$d\tau = -ds = -\Omega^{-1} d\tilde{s} = \frac{1}{\Omega} d\tilde{\tau}. \tag{4.2.29}$$

We can then deduce that the four-velocity u^μ changes as

$$\begin{aligned} u^\mu &= \frac{dx^\mu}{d\tau}, && (4.2.30) \\ &= \Omega \frac{d\tilde{x}^\mu}{d\tilde{\tau}}, && (4.2.31) \\ &\equiv \tilde{u}^\mu. && (4.2.32) \end{aligned}$$

To obtain the relationship between the inverse metrics, we make the ansatz

$$g^{\mu\nu} = C \tilde{g}^{\mu\nu}. \tag{4.2.33}$$

We can solve for the constant C by using the invariance of the Kronecker delta δ^μ_ν, giving

$$\begin{aligned} \delta^\mu_\nu &= \tilde{\delta}^\mu_\nu, \\ &= \tilde{g}^{\mu\rho} \tilde{g}_{\rho\nu}, \\ &= C\Omega^{-2} g^{\mu\rho} g_{\rho\nu}, \\ &= C\Omega^{-2} \delta^\mu_\nu, \end{aligned}$$

and so we can deduce that $C = \Omega^2$. Next, we would like to find how the metric determinant g changes between frames. Using the determinant formula $\det(c\mathcal{A}) = c^n \det(\mathcal{A})$, for an $n \times n$ matrix $\mathcal{A}$ and constant c, we can see that the metric determinants are related as $\tilde{g} = \Omega^8 g$, which further implies that their square roots change as

$$\sqrt{-\tilde{g}} = \Omega^4 \sqrt{-g}. \tag{4.2.34}$$

Next, we would like to see how the Ricci scalar will change upon conformally rescaling the metric. To do this however, we will first need to transform the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$. This calculation is carried out in Appendix A and results in

$$\Gamma^\mu_{\alpha\beta} = \tilde{\Gamma}^\mu_{\alpha\beta} - \left(\delta^\mu_\alpha f_\beta + \delta^\mu_\beta f_\alpha - \tilde{g}_{\alpha\beta} \tilde{f}^\mu \right), \tag{4.2.35}$$

where $\tilde{\Gamma}^\mu_{\alpha\beta}$ are the Christoffel symbols defined in terms of the Einstein frame metric

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2} \tilde{g}^{\mu\rho} \left(\partial_\alpha \tilde{g}_{\beta\rho} + \partial_\beta \tilde{g}_{\rho\alpha} - \partial_\rho \tilde{g}_{\alpha\beta} \right). \tag{4.2.36}$$

We have also introduced some shorthand notation for $\ln \Omega$ and its derivatives in terms of the variable f:

$$f \equiv \ln \Omega, \quad f_\mu \equiv \partial_\mu f, \quad \tilde{f}^\mu \equiv \tilde{g}^{\mu\nu} \partial_\nu f. \tag{4.2.37}$$

Using the transformed Christoffel symbols (4.2.35), we can now go on to find how the Ricci will change between frames. The calculation is again shown explicitly in Appendix A and results in

$$R = \Omega^2 \left(\tilde{R} + 6 \tilde{\Box} f - 6 \tilde{f}^2 \right), \tag{4.2.38}$$

were we have defined another shorthand notation $\tilde{f} \equiv \tilde{g}^{\mu\nu} f_\mu f_\nu$. The second term in the the transformed Ricci scalar (4.2.38) contains the D'Alembertian in curved spacetime $\tilde{\Box} \equiv \tilde{\nabla}_\mu \tilde{\nabla}_\nu$ acting on f. As f is a scalar, this term is proportional to

$$\frac{1}{\sqrt{-\tilde{g}}} \partial_\mu \left(\sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\nu f \right). \tag{4.2.39}$$

Thus, when it is multiplied by $\sqrt{-\tilde{g}}$ in the action, it will amount to nothing more than a boundary via term Stokes' theorem. For this reason, $\tilde{\Box} f$ can mostly be ignored.

With the transformation rules for the metric, metric determinant, and the Ricci scalar obtained, we are now ready to begin transforming the scalar-tensor action (4.2.17) to the Einstein frame. Applying the previously derived transformation rules to (4.2.17), and omitting the matter action, we find

$$\begin{aligned} S &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[F(\phi)R - 2\frac{\omega(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right], \\ &= \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\Omega^{-4}\left[F(\phi)\Omega^2\left(\tilde{R} + 6\tilde{\Box}f - 6\tilde{f}^2\right) - 2\frac{\omega(\phi)}{\phi}\Omega^2\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \\ &= \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\left[F(\phi)\Omega^{-2}\left(\tilde{R} + 6\tilde{\Box}f - 6\tilde{f}^2\right) - 2\frac{\omega(\phi)}{\phi}\Omega^{-2}\tilde{\partial}_\mu\phi\tilde{\partial}^\mu\phi - \Omega^{-4}V(\phi)\right]. \end{aligned} \tag{4.2.40}$$

We see that there is now an explicit coupling between ϕ and $\tilde{R}$. We can cancel this coupling by freely choosing

$$\Omega^2(\phi) = F(\phi), \tag{4.2.41}$$

such that

$$F(\phi)\Omega^{-2} = 1. \tag{4.2.42}$$

By filling in the choice for Ω, we can further rewrite the Einstein frame action (4.2.40) as

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\left[\left(\tilde{R} - 6\tilde{f}^2\right) - 2\frac{\omega(\phi)}{\phi F(\phi)}\tilde{\partial}_\mu\phi\tilde{\partial}^\mu\phi - \frac{V(\phi)}{F^2(\phi)}\right], \tag{4.2.43}$$

where we have dropped the previously mentioned boundary term that was proportional to $\sqrt{-\tilde{g}}\tilde{\Box}f$. We can further simplify this action by evaluating $\tilde{f}^2$ for our choice of coupling (4.2.41). Using that the derivative of the logarithm can be written as

$$\partial_\mu \ln\Omega = \frac{1}{2\Omega^2}\partial_\mu\Omega^2, \tag{4.2.44}$$

we find that $\tilde{f}^2$ may be written as

$$\tilde{f}^2 = \tilde{g}^{\mu\nu}\partial_\mu\left(\ln\Omega\right)\partial_\nu\left(\ln\Omega\right), \tag{4.2.45}$$

$$= \frac{\tilde{g}^{\mu\nu}}{4\Omega^4}\partial_\mu\Omega^2\partial_\nu\Omega^2. \tag{4.2.46}$$

We can now plug in our choice (4.2.41) for Ω^2 to find that $\tilde{f}$ takes the form

$$\tilde{f}^2 = \frac{1}{4}\left(\frac{F(\phi)}{F'(\phi)}\right)^2\tilde{\partial}_\mu\phi\tilde{\partial}^\mu\phi, \tag{4.2.47}$$

where we have used the chain rule $\partial_\mu F(\phi) = \partial_\mu\phi\frac{\partial F}{\partial\phi}$.

We can now return to our Einstein frame action (4.2.43) and plug in our expression (4.2.47) for $\tilde{f}^2$. The resulting action looks like

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - \left(2\frac{\omega(\phi)}{\phi F(\phi)} + \frac{3}{2}\left(\frac{F'(\phi)}{F(\phi)}\right)^2\right)\tilde{\partial}_\mu\phi\tilde{\partial}^\mu\phi\right]. \tag{4.2.48}$$

We would like to further simplify this action by writing it in terms of a canonicalized scalar field. We can do this by defining a new scalar field φ through the differential relation

$$\frac{d\varphi}{d\phi} = \sqrt{\frac{\Delta(\phi)}{2}}, \tag{4.2.49}$$

where the function $\Delta(\phi)$, is given by

$$\Delta(\phi) = 2\frac{\omega(\phi)}{\phi F(\phi)} + \frac{3}{2}\left(\frac{F'(\phi)}{F(\phi)}\right)^2. \tag{4.2.50}$$

Rewriting the action in terms of the new scalar field φ, and restoring the previously omitted matter action $S_{\text{matter}}\left[\psi_m, g_{\mu\nu}\right]$, we find that the final Einstein frame action takes the form

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - 2\tilde{\partial}_\mu\varphi\tilde{\partial}^\mu\varphi - U(\varphi)\right] + S_{\text{matter}}\left[\psi_m, \Omega^{-2}(\varphi)\tilde{g}_{\mu\nu}\right], \tag{4.2.51}$$

where the potential $U(\varphi)$ is given by

$$U(\varphi) \equiv \frac{V(\phi)}{F^2(\phi)}. \tag{4.2.52}$$

Comments on additional conventions

Initially we have specified our conformal rescaling in the Jordan frame according to (4.2.26). In the literature, it is sometimes common to define the conformal rescaling instead as

$$g_{\mu\nu} = A^2(\varphi)\tilde{g}_{\mu\nu}. \tag{4.2.53}$$

This new conformal factor is a function of the Einstein frame scalar field φ, but is related to our original conformal factor Ω through $A = -1/\Omega$. Specifying a form for $\omega(\phi)$ and $F(\phi)$ defines a scalar-tensor theory in the Jordan frame. Alternatively, one can define a scalar-tensor theory in terms of its Einstein frame counterpart by specifying $A(\varphi)$ along with $F(\phi)$. The two can then be related through (4.2.41) such that

$$A(\varphi) = F^{-1/2}(\phi). \tag{4.2.54}$$

Solving this equation gives φ as a function of ϕ or vice versa. One can then use (4.2.49) to substitute for $\Delta(\phi)$ in (4.2.50), and then solve for $\omega(\phi)$. Thus one can recover the Jordan frame action frame action from one which has been originally specified in the Einstein frame. This will be a useful aid for our case study of generalized scalar-tensor theories in chapter 7.
Furthermore, using equations (4.2.49) and (4.2.54), we can relate $A(\varphi)$, $F(\phi)$ and $\Delta(\phi)$ as

$$\Delta = \left(\frac{F'}{2F\alpha}\right) = \left(\frac{A^2F'}{2\alpha}\right), \tag{4.2.55}$$

where F' denotes a derivative of F with respect to ϕ. The function $\alpha(\varphi)$ is another commonly used convention in the literature, defined as

$$\alpha(\varphi) \equiv -\frac{1}{A}\frac{dA}{d\varphi}. \tag{4.2.56}$$

Energy momentum tensor

So far in our demonstration of Einstein frame transformations, we focused purely on transforming the gravitational part of the the action. It is now time to turn our attention toward the Einstein frame energy-momentum tensor $\tilde{T}_{\mu\nu}$, such that the full Einstein equations can be specified. As discussed in chapter 3, we are interested in modelling neutron stars that have an energy momentum tensor described by a perfect fluid

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}, \tag{4.2.57}$$

$$= (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \tag{4.2.58}$$

We demand that the energy-momentum tensor in the Einstein frame $\tilde{T}_{\mu\nu}$ have a similar form

$$\begin{aligned}\tilde{T}_{\mu\nu} &= \frac{-2}{\sqrt{-\tilde{g}}}\frac{\delta S_{\text{matter}}}{\delta \tilde{g}^{\mu\nu}}, \\ &= (\tilde{\rho}+\tilde{p})\tilde{u}_\mu \tilde{u}_\nu + \tilde{p}\tilde{g}_{\mu\nu},\end{aligned} \tag{4.2.59}$$

where the Einstein frame density $\tilde{\rho}$ and pressure $\tilde{p}$ are distinct from their Jordan frame counterparts ρ and p. When solving the Einstein equations later on, it will be convenient to be able to express them in terms of the Jordan frame quantities ρ and p. Beginning with the previously derived identities for the metric and four-velocity

$$\sqrt{-g} = \Omega^{-4}\sqrt{-\tilde{g}}, \quad \delta g^{\mu\nu} = \Omega^2 \delta \tilde{g}^{\mu\nu}, \quad u_\mu = \Omega^{-1}\tilde{u}_\mu, \tag{4.2.60}$$

we can relate the Jordan and Einstein frame energy-momentum tensors

$$\begin{aligned}T_{\mu\nu} &= \frac{-2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}, \\ &= \frac{-2\Omega^2}{\sqrt{-\tilde{g}}}\frac{\delta S_{\text{matter}}}{\delta \tilde{g}^{\mu\nu}}, \\ &= \Omega^2 \tilde{T}_{\mu\nu}.\end{aligned} \tag{4.2.61}$$

Next we can take the relationship for the energy-momentum tensors between frames (4.2.61) and replace the four-velocity $\tilde{u}_\mu$ with its Jordan frame counterpart $\Omega\ u_\mu$ to get

$$\begin{aligned}T_{\mu\nu} &= \Omega^2 \tilde{T}_{\mu\nu}, \\ &= \Omega^2 \left[(\tilde{\rho}+\tilde{p})\,\tilde{u}_\mu \tilde{u}_\nu + p\tilde{g}_{\mu\nu} \right], \\ &= \Omega^2 \left[(\tilde{\rho}+\tilde{p})\,\Omega^2 u_\mu u_\nu + p\Omega^2 g_{\mu\nu} \right], \\ &= \left(\Omega^4\tilde{\rho} + \Omega^4\tilde{p}\right) u_\mu u_\nu + \left(\Omega^4 \tilde{p}\right) g_{\mu\nu}.\end{aligned} \tag{4.2.62}$$

Comparing the form of the energy momentum tensor (4.2.62) with its original Jordan frame counterpart (4.2.58), we can deduce that the pressure and density change between frames as

$$\rho = \Omega^4 \tilde{\rho}, \quad p = \Omega^4 \tilde{p}. \tag{4.2.63}$$

We can now use these relationships in the Einstein frame energy-momentum tensor (4.2.59) to replace $\tilde{\rho}$ and $\tilde{p}$ with their Jordan frame counterparts. The final form of the Einstein frame energy momentum tensor is thus

$$\tilde{T}_{\mu\nu} = \left(\Omega^{-4}\rho + \Omega^{-4}p\right)\tilde{u}_\mu \tilde{u}_\nu + \Omega^{-4}p\tilde{g}_{\mu\nu}. \tag{4.2.64}$$

The Weyl tensor

In chapter 3, we introduced the effective relativistic action for a binary system. This action depended on the tidal moments $\mathcal{E}_{\mu\nu}$, which again depend on the Weyl tensor given by (3.1.5). We will later on be interested in transforming these tidal terms to the Einstein frame, which will also mean knowing how the Weyl tensor $W_{\mu\alpha\nu\beta}$ transforms. Recall that the Weyl tensor is the trace-free part of the Riemann tensor, which in 4 spacetime dimensions looks like

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \left(g_{\alpha[\mu}R_{\nu]\beta} - g_{\beta[}R_{\nu]\alpha}\right) + \frac{1}{3}Rg_{\alpha[\mu}g_{\nu]\beta}. \tag{4.2.65}$$

While we will not show it here (although one can use the Mathematica package *xPand* for example), the Weyl tensor can be transformed to the Einstein frame using our previously derived transformation rules. The resulting transformation is [36]

$$W_{\mu\alpha\nu\beta} = \Omega^{-2}\tilde{W}_{\mu\alpha\nu\beta}. \tag{4.2.66}$$

We can further raise an index on the Weyl tensor $W_{\mu\alpha\nu\beta}$ using the metric, which implies that

$$\begin{aligned} W^{\alpha}_{\ \beta\mu\nu} &= g^{\alpha\rho} W_{\rho\beta\mu\nu} \\ &= \tilde{g}^{\alpha\rho} \cancel{\Omega^2 \Omega^{-2}} \tilde{W}_{\rho\beta\mu\nu} \\ &= \tilde{W}^{\alpha}_{\ \beta\mu\nu}. \end{aligned} \tag{4.2.67}$$

This can be further generalized to say the Weyl tensor with *any* index raised is invariant under conformal rescalings. We record the result (4.2.67) for completeness, however, it is primarily (4.2.66) that we will use throughout the rest of this thesis.

4.2.4 Tool box

To summarize the work done in the previous section, we collect the most relevant conformal transformations that were derived in this chapter and put them in one place for future reference

$$\boxed{\begin{array}{rlrl} g_{\mu\nu} &= \Omega^{-2} \tilde{g}_{\mu\nu}, & R &= \Omega^2 \left(\tilde{R} + 6\tilde{\Box} f - \dfrac{3}{2} \cdot \dfrac{\tilde{g}^{\mu\nu}}{\Omega^4} \partial_\mu \Omega^2 \partial_\nu \Omega^2 \right), \\ g^{\mu\nu} &= \Omega^2 \tilde{g}^{\mu\nu}, & \Gamma^\mu_{\alpha\beta} &= \tilde{\Gamma}^\mu_{\alpha\beta} - \left(\delta^\mu_\alpha f_\beta + \delta^\mu_\beta f_\alpha - \tilde{g}_{\alpha\beta} \tilde{f}^\mu \right), \\ \sqrt{-g} &= \Omega^{-4} \sqrt{-\tilde{g}}, & W_{\alpha\beta\mu\nu} &= \Omega^{-2} \tilde{W}_{\alpha\beta\mu\nu}, \\ u^\mu &= \Omega \tilde{u}^\mu, & T_{\mu\nu} &= \Omega^{-2} \tilde{T}_{\mu\nu}, \\ \rho &= \Omega^{-4} \tilde{\rho}, & p &= \Omega^{-4} \tilde{p}, \end{array}} \tag{4.2.68}$$

where

$$f \equiv \ln \Omega, \quad f_\mu \equiv \partial_\mu \ln \Omega, \quad \tilde{f}^\mu \equiv \tilde{g}^{\mu\nu} f_\nu. \tag{4.2.69}$$

In this chapter, we have successfully shown how a conformal rescaling of the metric tensor can decouple a scalar field ϕ from the gravitational sector of the action at the expense of coupling it to matter. However, this work was only done for the simplest coupling type $F(\phi)R$, corresponding to the scalar-tensor theory. This trick does not account for the generalized scalar-tensor theory with couplings of the type $f(\phi, R)$. We will later go on in chapter 6 to address this issue, and present a recipe to generalize the transformations in this chapter for the generalized scalar-tensor theory.

Chapter 5

Tidal effects in scalar-tensor theories

So far, we have laid the basic theoretical foundation for calculating Love numbers in GR. We have further introduced the idea of modified theories of gravity, namely the scalar tensor theories. For this chapter, we are interested in merging both, that is to obtain the basic recipe for computing the Love number in scalar-tensor theories - that is those of the coupling type $F(\phi)R$.

We will do this by first schematically deriving the effective action for a binary system in scalar-tensor gravity. Then, through a conformal rescaling of the metric, we will transform this effective action to the Einstein frame, allowing us to identify how the Love number(s) λ_ℓ change between frames. Finally, we will derive and solve equations of motion for the metric $g_{\mu\nu}$ and scalar field φ in the Einstein frame in an asymptotic frame. This will allow us to see how the Love number can be extracted in the Einstein frame such that we can convert it back to the Jordan frame.

This chapter will primarily be a review of the most relevant parts of [6] (although other attempts at computing the Love number in $F(\phi)R$ gravity can be found in [7, 8]). The ultimate outcome here is that we would like to review the most important aspects of computing the Love number in $F(\phi)R$ gravity. The intention is then to investigate if these aspects may be generalized for the generalized scalar-tensor theories $f(\phi, R)$.

5.1 The effective action in the Jordan frame

We begin by readopting the skelontized worldline approach, as seen in chapter 3, that allowed us to write an effective for a binary system at the orbital scale. In this approach, the system is treated as a point particle with (static) multipole moments moving along the centre of mass worldline worldline $x^\mu(\tau)$, where τ parameterizes the particle's trajectory along x^μ. This action schematically looks like

$$S = S_{\mathrm{ST}} + S_{\mathrm{pp}} + S_{\mathrm{Tidal}}. \tag{5.1.1}$$

Here S_{ST} is the scalar tensor action of the full theory

$$S = \int d^4x\sqrt{-g}\left[K_R F(\phi)R - K_\phi \frac{\omega(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi\right], \tag{5.1.2}$$

where we have used arbitrary normalization constants K_R and K_ϕ to account for differences in the literature. We have also set the scalar potential $V(\phi) = 0$.

The term S_{pp} in the full action (5.1.1), is the action for the relativistic point-particle

$$S_{\mathrm{pp}} = -\int d\tau\ z\ \sqrt{-u^\mu u_\mu}\ m(\phi), \tag{5.1.3}$$

where here, the mass $m(\phi)$, is a function of the scalar field due to a scalar-dependent binding energy [37]. The redshift factor z is still given as $z = \sqrt{-u^\mu u_\mu}$.

Lastly, the tidal part S_{Tidal} of the full action (5.1.1) represents the tidal multipole moments that move along the worldline with the point particle. This action schematically looks like

$$S_{\text{Tidal}} = \sum_{\ell} \int d\tau \; z \; \sqrt{-u^\mu u_\mu} \left(\frac{\lambda_\ell^T}{2\ell!} \mathcal{E}_L \mathcal{E}^L + \frac{\lambda_\ell^S}{2\ell!} E_L E^L + \frac{\lambda_\ell^{ST}}{\ell!} \mathcal{E}_L E^L \right). \tag{5.1.4}$$

Here, $\mathcal{E}_L$ are the previously discussed tidal multipole moments from GR (3.1.8). We have also included two additional terms in the tidal action that were not present in previous chapters. These terms depend on a scalar tidal term that we denote E_L which is defined as

$$E_L \equiv -\nabla_{\langle L \rangle} \phi. \tag{5.1.5}$$

These new terms encapsulate effect that a tidal scalar field will have on the body. The $E_L E^L$ term characterizes the static response of the induced scalar multipole in the body due the perturbing tidal scalar field from the companion. This internal response in the body is characterized by a scalar Love number that we denote λ_ℓ^S. Similarly, the third term shows a coupling $\mathcal{E}_L E^L$, which characterizes the tensor/scalar multipole induced in the body due to the perturbing tidal scalar/tensor field. We now denote the previous tensor Love number λ_ℓ from GR as λ_ℓ^T.

The form of the scalar tidal moments E_L are determined through an effective field theory approach similar to what was commented on in chapter 3. The terms containing E_L arise naturally in the action when one considers all possible terms with the allowed symmetries of the theory. For the interested reader, a schematic derivation for the the scalar tidal dipole term $E_{\mu\nu}$ using effective field theory can be found in Appendix A of [38].

With the effective action (5.1.1) describing a binary system in the Jordan frame fully specified, our next step is to transform the action to the Einstein frame and identify how the various Love numbers λ_ℓ^T, λ_ℓ^S, and λ_ℓ^{ST} transform along with it.

5.2 The effective action in the Einstein frame

5.2.1 Transforming the tidal moments

With the effective action describing finite size effects in the Jordan frame (5.1.1) fully obtained, we are almost ready to perform our conformal rescaling of the metric $g_{\mu\nu}$ to transform the action to the Einstein frame. The only terms in the effective action (5.1.1) that we have not examined yet in the Einstein frame are the tidal moments $\mathcal{E}_L$ and E_L.

Tensor tidal terms

To transform these terms, we begin by looking at the $\ell = 2$ tensor tidal term

$$\mathcal{E}_{\mu\nu} = \frac{1}{z^2} W_{\mu\alpha\nu\beta} u^\alpha u^\beta, \tag{5.2.1}$$

where $z = \sqrt{-u^\mu u_\mu}$ is the redshift factor. We can transform each term in $\mathcal{E}_{\mu\nu}$ by applying our previously derived toolbox (4.2.68) of transformations. The result is

$$\begin{aligned} \mathcal{E}_{\mu\nu} &= \frac{1}{z^2} W_{\mu\alpha\nu\beta} u^\alpha u^\beta, \\ &= \frac{1}{\tilde{z}^2} \cancel{\Omega^2} \tilde{W}_{\mu\alpha\nu\beta} \cancel{\Omega^{-2}} \tilde{u}^\alpha \tilde{u}^\beta, \\ &= \tilde{\mathcal{E}}_{\mu\nu}, \end{aligned} \tag{5.2.2}$$

Thus we find that the $\ell = 2$ tensor tidal term is conformally invariant, and equivalent in the Jordan and Einstein frames. We can similarly transform the higher order tidal moments starting with the $\ell = 3$ term

$$\mathcal{E}_{\alpha\mu\nu} = \nabla_\alpha \mathcal{E}_{\mu\nu}. \tag{5.2.3}$$

The transformations quickly become cumbersome for $\ell \geq 3$ however due to the increasing presence of the covariant derivative with each ℓ. An explicit derivation for the $\ell = 3$ tensor tidal term is carried out in Appendix B, where we also present a generic recursive formula for calculating higher order tidal moments. For now, we simply state the result for the $\ell \geq 3$ tensor tidal terms as

$$\mathcal{E}_L = \tilde{\mathcal{E}}_L + ..., \tag{5.2.4}$$

where the "..." denotes higher order powers of $1/r$ in the power counting scheme of the EFT [6] and so can be neglected.

Scalar tidal terms

With the transformation rules for the tensor tidal moments $\mathcal{E}_L$ obtained, we can now turn our attention to the scalar tidal moments E_L. Starting with the $\ell = 1$ scalar tidal moment we have that

$$\begin{aligned} E_\mu &= \nabla_\mu \phi, \\ &= \partial_\mu \phi. \end{aligned} \tag{5.2.5}$$

where the covariant derivative ∇_μ reduces to a partial derivative ∂_μ when acting on a scalar. To express the tidal term (5.2.5) in the Einstein frame, we must recall that when passing to the Einstein frame we redefine our scalar field through the differential relation

$$\frac{d\varphi}{d\phi} = \sqrt{\Delta}, \tag{5.2.6}$$

where Δ is a function of the scalar field ϕ. This differential relation can alternatively be expressed as

$$\partial_\mu \phi = \frac{1}{\sqrt{\Delta}} \partial_\mu \varphi. \tag{5.2.7}$$

Putting this relationship back into the $\ell = 1$ scalar tidal moment (5.2.5), we obtain that the Jordan and Einstein frame tidal moments are related as

$$E_\mu = \frac{1}{\sqrt{\Delta}} \tilde{E}_\mu, \tag{5.2.8}$$

where the Einstein frame tidal moment is defined as $\tilde{E}_\mu = \partial_\mu \varphi$. Similarly to the tensor tidal terms, the scalar tidal terms become more cumbersome to transform starting with $\ell = 2$. As a result, the transformations for higher order moments have also been left to Appendix B. For now, we simply state the result as

$$E_L = \frac{1}{\sqrt{\Delta}} \tilde{E}_L + ..., \tag{5.2.9}$$

where the "..." again denote terms that can be neglected due to power counting arguments in the EFT [6].

To summarize, we find that the scalar and tensor tidal terms both transform proportionally to their Einstein frame counterparts + higher order terms, which are suppressed in the EFT

$$\mathcal{E}_L = \tilde{E}_L + ..., \quad E_L = \frac{1}{\sqrt{\Delta}} \tilde{E}_L + ... \tag{5.2.10}$$

For all intents and purposes however, we can thus treat these terms as changing as

$$\mathcal{E}_L \to \tilde{\mathcal{E}}_L, \quad E_L \to \frac{1}{\sqrt{\Delta}} \tilde{E}_L. \tag{5.2.11}$$

With these final transformations, we are now almost ready to express the entire effective action (5.1.1) in the Einstein frame.

Rescaling coordinates

Before proceeding to the final step of transforming our effective Jordan frame action to the Einstein frame, we make a sidenote. From here on, we switch to the convention of expressing the conformal factor $\Omega(\phi)$ instead as $A(\varphi)$, defined through (4.2.53).

Secondly, in order to extract the Love number at spatial infinity, we will later demand that our Einstein frame spacetime asymptotically resembles Minkowski space. We can ensure this condition with a simple rescaling of our Jordan frame coordinates. Recall that the invariant line element is

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu, \\ &= \tilde{g}_{\mu\nu} A^2(\varphi) dx^\mu dx^\nu, \end{aligned} \tag{5.2.12}$$

where we have conformally rescaled the metric according to (4.2.53). In order to demand that

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{5.2.13}$$

at spatial infinity, we must redefine our coordinates according to

$$dx_*^\mu = A(\varphi_\infty) dx^\mu. \tag{5.2.14}$$

This ensures that $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ for the Jordan frame metric at spatial infinity. This redefinition of coordinates will not change our previously established transformation rules to the Einstein frame, with the exception that the covariant derivative now changes with an additional factor dependent on $A(\varphi)$

$$\nabla_L \to A_\infty^\ell \nabla_L^*. \tag{5.2.15}$$

Here $A_\infty \equiv A(\varphi_\infty)$ denotes A with the scalar field φ evaluated at infinity.

5.2.2 Relating the Love numbers between frames

Finally we are ready to transform our effective tidal action (5.1.4)

$$S_{\text{Tidal}} = \sum_\ell \int d\tau \; z \; \sqrt{-u^\mu u_\mu} \left(\frac{\lambda_\ell^T}{2\ell!} \mathcal{E}_L \mathcal{E}^L + \frac{\lambda_\ell^S}{2\ell!} E_L E^L + \frac{\lambda_\ell^{ST}}{\ell!} \mathcal{E}_L E^L \right), \tag{5.2.16}$$

to the Einstein frame. Using our toolbox (4.2.68), the transformation rules for the tidal terms (5.2.11) and the coordinate rescaling (5.2.15), the Jordan frame action (5.2.16) transforms as

$$S_{\text{Tidal}} = \sum_\ell \int d\tilde{\tau} \; \tilde{z} \; \sqrt{-\tilde{u}_\mu \tilde{u}^\mu} \left(\frac{A}{A_\infty} \right)^{1-2\ell} \left(\frac{\lambda_l^T}{2\ell!} \tilde{\mathcal{E}}_L \tilde{\mathcal{E}}^L + \frac{\lambda_\ell^S}{2\ell!\Delta} \tilde{E}_L \tilde{E}^L + \frac{\lambda_\ell^{ST}}{\ell!\sqrt{\Delta}} \tilde{\mathcal{E}}_L \tilde{E}^L \right). \tag{5.2.17}$$

We would now like to be able to compare how the Love numbers change between frames. In the Einstein frame, we will demand that the effective tidal action takes the form

$$S_{\text{Tidal}} = \sum_\ell \int d\tilde{\tau} \; \tilde{z} \; \sqrt{-\tilde{u}_\mu \tilde{u}^\mu} \left(\frac{\tilde{\lambda}_\ell^T}{2\ell!} \tilde{\mathcal{E}}_L \tilde{\mathcal{E}}^L + \frac{\tilde{\lambda}_\ell^S}{2\ell!} \tilde{E}_L \tilde{E}^L + \frac{\tilde{\lambda}_\ell^{ST}}{\ell!} \tilde{\mathcal{E}}_L \tilde{E}^L \right). \tag{5.2.18}$$

We can thus read off how the Love number changes by comparing the prefactors in front of each of the tidal terms in the Jordan frame (5.2.17) and Einstein frame (5.2.18) action. The results are

$$\lambda_\ell^T = (A/A_\infty)^{2\ell-1} \tilde{\lambda}_\ell^T, \tag{5.2.19}$$

$$\lambda_\ell^S = (A/A_\infty)^{2\ell-1} \Delta \tilde{\lambda}_\ell^S, \tag{5.2.20}$$

$$\lambda_\ell^{ST} = (A/A_\infty)^{2\ell-1} \sqrt{\Delta} \tilde{\lambda}_\ell^{ST}, \tag{5.2.21}$$

where

$$\Delta = \left(\frac{F'}{2F\alpha}\right) = \left(\frac{A^2F'}{2\alpha}\right). \tag{5.2.22}$$

Given that the Love numbers are extracted at spatial infinity, we can further simplify the relationships between the Love numbers. Evaluating $A(\varphi) \equiv A_\infty$ at spatial infinity, the relationships (5.3.28) become

$$\lambda_\ell^T = \tilde{\lambda}_\ell^T, \tag{5.2.23a}$$

$$\lambda_\ell^S = \Delta_\infty \tilde{\lambda}_\ell^S, \tag{5.2.23b}$$

$$\lambda_\ell^{ST} = \sqrt{\Delta_\infty}\tilde{\lambda}_\ell^{ST}. \tag{5.2.23c}$$

Conveniently, we find that tensor tidal Love number λ_ℓ^T does not change between frames, whereas the scalar and scalar-tensor Love numbers change according to Δ_∞, due to the redefinition (5.2.6) of the original scalar field ϕ.

To give a sense for the magnitude of Δ_∞, we can evaluate it for the values performed in the case study [6]. For this we have that the coupling function and inverse conformal factor are

$$F(\phi) = \phi, \tag{5.2.24}$$

$$A(\varphi_\infty) = e^{\frac{1}{2}\beta\varphi_\infty^2}. \tag{5.2.25}$$

Note the distinction between the scalar fields ϕ and φ in F and A. The parameter β here measures the strength of the coupling between the scalar field and matter in the Einstein frame. Setting $\beta = -4.5$ and taking the scalar field at infinity to be $\varphi_\infty = 10^{-3}$ as in [6], we find that

$$\Delta \sim 1.2 \times 10^5. \tag{5.2.26}$$

This implies a difference of several orders of magnitude between the values of λ_ℓ^S in each frame, owing to the redefinition of the scalar field ϕ and justifying the need for having a transformation between the two frames. With the relationship between Love numbers obtained between frames, we can now turn our attention to how $\tilde{\lambda}_\ell^S$ may be extracted in the Einstein frame.

5.3 Extracting the Love number in the Einstein frame

5.3.1 The EFT equations of motion

Recall that the total effective action in the Einstein frame looks like

$$S = S_{\text{ST}} + S_{\text{pp}} + S_{\text{Tidal}}, \tag{5.3.1}$$

with

$$S_{\text{ST}} = \int d^4x\sqrt{-\tilde{g}}\left[K_R\tilde{R} - K_\varphi\tilde{\partial}_\mu\varphi\tilde{\partial}^\mu\varphi\right], \tag{5.3.2a}$$

$$S_{\text{pp}} = -\int d\tilde{\tau}\ \tilde{z}\ \sqrt{-\tilde{u}^\mu\tilde{u}_\mu}\tilde{m}(\varphi), \tag{5.3.2b}$$

$$S_{\text{Tidal}} = \sum_l \int d\tilde{\tau}\ \tilde{z}\ \sqrt{-\tilde{u}^\mu\tilde{u}_\mu}\left(\frac{\tilde{\lambda}_l^T}{2l!}\tilde{\mathcal{E}}_L\tilde{\mathcal{E}}^L + \frac{\tilde{\lambda}_\ell^S}{2l!}\tilde{E}_L\tilde{E}^L + \frac{\tilde{\lambda}_l^{ST}}{2l!}\tilde{\mathcal{E}}_L\tilde{E}^L\right). \tag{5.3.2c}$$

When studying how to compute the Love number in chapter 3 for basic GR, we were able to take advantage of the fact that the tt component of the metric in an asymptotically flat frame could be expressed in terms of an effective potential

$$g_{tt} = -\left(1 - 2U_{\text{eff}}\right), \tag{5.3.3}$$

which could then be used to identify which components contain the multipole moments Q_L and the tidal fields $\mathcal{E}_L$

$$\lim_{r\to\infty} U_{\text{eff}} = \frac{M}{r} + \sum_{\ell=2}^{\infty} \frac{1}{\ell!}\left[(2\ell-1)!!\frac{Q_L n^{\langle L\rangle}}{r^{\ell+1}} - n^{\langle L\rangle} E_L\ r^{\ell}\right], \tag{5.3.4}$$

such that the Love number

$$\lambda_\ell = -\frac{Q_L}{\mathcal{E}_L}, \tag{5.3.5}$$

could be extracted.

However, we are now dealing with the computation of a Love number in a scalar-tensor theory, which means our effective action has been modified to contain the presence of the scalar field φ, as well a scalar tidal field E_L with appropriate Love numbers λ_ℓ^S, λ_ℓ^{ST}. We would like to be able to compute the equations of motion of the effective action, such that the effective solution can be obtained. In this sense we will be able to see which coefficients of the solution contain the tidal fields $\mathcal{E}_L$, E_L such that the Love number can be extracted in a similar way to the GR case (5.3.4).

We begin by working in a local aymptotic frame, far away from the system. This means the spacetime is locally Minkowskian such that the metric determinant is 1 i.e. $\sqrt{-\tilde{g}} = 1$. We also work in the rest frame of the body such that the four velocity only has one component $u^0 = 1$ and thus its normalization is $\sqrt{-\tilde{u}^2} = 1$. We would like to start by varying the action with respect to φ, however, we must first address a subtlety

The total effective action (5.3.1) contains 3 terms, however not all have the same integration measure. As an instructive example, consider varying the point particle action S_{pp} with the integration measure $d\tilde{\tau}$ replaced by d^4x. The action in our local asymptotic frame would read

$$S_{\text{pp}} = -\int d^4x\ \tilde{m}(\varphi), \tag{5.3.6}$$

and its contribution to the equations of motion would be

$$\frac{\delta S_{\text{pp}}}{\delta\varphi(z)} = \int d^4x\ \tilde{m}'(\varphi)\delta^{(4)}(x^\mu - z^\mu), \tag{5.3.7}$$

$$= \tilde{m}'(\varphi), \tag{5.3.8}$$

where it is now understood that $\varphi \equiv \varphi(z)$. In actuality, the point-particle action integrates over $\tilde{\tau}$. We can compute its equation of motion by again computing $\frac{\delta S_{\text{pp}}}{\delta\varphi(z)}$, this time obtaining

$$\frac{\delta S_{\text{pp}}}{\delta\varphi(z)} = -\int d\tilde{\tau}\ \tilde{m}'(\varphi)\delta^{(4)}(x^\mu - z^\mu). \tag{5.3.9}$$

We can simplify things by separating the delta function as

$$\delta^{(4)}(x^\mu - z^\mu) = \delta(t - z^i)\delta^{(3)}(x^i - z^i), \tag{5.3.10}$$

and setting $z^\mu = (\tilde{\tau}, 0, 0, 0)$ by invariance of its reparameterization. Performing the integral over $\tilde{\tau}$, the result is

$$\frac{\delta S_{\text{pp}}}{\delta\varphi(z)} = -\tilde{m}'(\varphi)\delta^{(3)}(x - 0), \tag{5.3.11}$$

$$= -\tilde{m}'(\varphi)\delta^{(3)}(x), \tag{5.3.12}$$

$$\equiv -\tilde{m}'(\varphi)\delta(x), \tag{5.3.13}$$

where we have abbreviated $\delta^{(3)}(x) \equiv \delta(x)$. The full equation of motion of for φ can be computed in a similar way, with delta functions arising from each term which contains an integration measure over $\tilde{\tau}$. The resulting equation of motion for φ is then

$$\tilde{\Box}\varphi = -\frac{\tilde{m}'(\varphi)}{2K_\varphi}\delta(x) + \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l!2K_\varphi}\left[\tilde{\lambda}_l^S\tilde{E}^L\partial_L + \tilde{\lambda}_\ell^{ST}\tilde{E}^L\partial_L\right]\delta(x). \tag{5.3.14}$$

Here, the tidal moments $\tilde{E}_L = -\tilde{\nabla}_L\varphi$ are evaluated on the worldine $\tilde{\nabla}_L\varphi \equiv \tilde{\nabla}_L\varphi\Big|_0$, such that they may be treated as constant.

We can now perform a similar procedure for the metric equations of motion. One can take the tt component of the Einstein equations and expand the metric in terms of the Newtonian potential $\tilde{U}_N$ in the same way as (5.3.3). We can then take the Newtonian limit by working to linear order in $\tilde{U}_N$. The resulting equation of motion is

$$\tilde{\Box}\tilde{U}_N = \frac{\tilde{m}(\varphi_\infty)}{4K_R}\delta(x) + \sum_{\ell}^{\infty}\frac{(-1)^{\ell+1}}{\ell!4K_R}\left[\tilde{\lambda}_\ell^T\tilde{\mathcal{E}}^L\partial_L + \tilde{\lambda}_\ell^{ST}\tilde{E}^L\partial_L\right]\delta(x), \tag{5.3.15}$$

where the tidal moments $\tilde{\mathcal{E}}_L$ in their Newtonian limit look like

$$\tilde{\mathcal{E}}_L = -\partial_L\tilde{U}_N. \tag{5.3.16}$$

5.3.2 Solving the equations of motion

With the Einstein frame EFT equations of motion obtained, we would now like to work on solving them. This can be done by passing to Fourier space. An example of the methodology can be found in [39]. We start by transforming the field $\varphi(x)$ to its fourier space counterpart $\tilde{\varphi}(\boldsymbol{k})$

$$\varphi(x) = \int d^3\boldsymbol{k}e^{-i\boldsymbol{k}\boldsymbol{x}}\tilde{\varphi}(\boldsymbol{k}), \quad \tilde{\varphi}(\boldsymbol{k}) = \int\frac{d^3x}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}\varphi(x), \quad \delta(x) = \int d^3\boldsymbol{k}e^{-i\boldsymbol{k}\boldsymbol{x}} \tag{5.3.17}$$

With these definitions, the equation of motion for φ

$$\tilde{\Box}\varphi = -\frac{\tilde{m}'(\varphi)}{2K_\varphi}\delta(x) + \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l!2K_\varphi}\left[\tilde{\lambda}_l^S\tilde{E}^L + \tilde{\lambda}_\ell^{ST}\tilde{E}^L\right]\partial_L\delta(x), \tag{5.3.18}$$

then looks like

$$\begin{aligned}\int d^3\boldsymbol{k}\left[-\boldsymbol{k}^2\tilde{\varphi}(\boldsymbol{k})\right]e^{-i\boldsymbol{k}\boldsymbol{x}} = &-\frac{\tilde{m}'(\varphi_\infty)}{2K_\varphi}\int d^3\boldsymbol{k}\,[1]\,e^{-i\boldsymbol{k}\boldsymbol{x}}\\ &+\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell+1}}{\ell!2K_\varphi}\left(\tilde{\lambda}_l^S\tilde{E}^L + \tilde{\lambda}_\ell^{ST}\tilde{E}^L\right)\int d^3\boldsymbol{k}\left[(-i)^{\ell+1}\boldsymbol{k}^L\right]e^{-i\boldsymbol{k}\boldsymbol{x}}.\end{aligned} \tag{5.3.19}$$

We can isolate the left hand side of this equation by diving by $-\boldsymbol{k}^2$ and performing the remaining integrals to return to real space. The integrals are performed in [39]. The solution for φ is then

$$\varphi = \frac{\tilde{m}'(\varphi_\infty)}{2\Omega_2K_\varphi r} + \sum_{l=1}^{\infty}\frac{(2l-1)!!}{l!2\Omega_2K_\varphi}\frac{(-\tilde{\lambda}_\ell^S\tilde{E}^L - \tilde{\lambda}_\ell^{ST}\tilde{\mathcal{E}}^L)n_L}{r^{l+1}} + \varphi_{\text{hom}}. \tag{5.3.20}$$

Due to the linearity of the equation of motion (5.3.18) in φ, we have also added the solution of the homogeneous equation

$$\tilde{\Box}\varphi_{\text{hom}} = 0. \tag{5.3.21}$$

This is the standard Laplace equation in Euclidean space and the solution can be written as

$$\varphi_{\text{hom}} = -\sum_{\ell=1}^{\infty}\frac{1}{\ell!8\pi K_\varphi}\tilde{E}^L n_L r^\ell. \tag{5.3.22}$$

This completes the solution for φ in the EFT in the local asymptotic frame.

We can now turn our attention to the equation of motion for the Newtonian potential $\tilde{U}_N$

$$\tilde{\Box}\tilde{U}_N = \frac{\tilde{m}(\varphi_\infty)}{4K_R}\delta(x) + \sum_{\ell}^{\infty} \frac{(-1)^{\ell+1}}{\ell! 4K_R}\left[\tilde{\lambda}_\ell^T \tilde{\mathcal{E}}^L + \tilde{\lambda}_\ell^{ST} \tilde{E}^L\right] \partial_L \delta(x). \tag{5.3.23}$$

Fortunately this equation is identical to the scalar φ equation of motion (5.3.18) up to replacing

$$K_\varphi \to 2K_R, \quad \tilde{\lambda}_\ell^S \tilde{E}_L \to \tilde{\lambda}_\ell^T \tilde{\mathcal{E}_L}. \tag{5.3.24}$$

Hence, the solution for $\tilde{U}_N$ can be written immediately as

$$\tilde{U}_N = \frac{\tilde{m}'(\varphi_\infty)}{4\Omega_2 K_R r} + \sum_{l=1}^{\infty} \frac{(2l-1)!!}{l! 4\Omega_2 K_R} \frac{(-\tilde{\lambda}_l^T \tilde{\mathcal{E}}^L - \tilde{\lambda}_\ell^{ST} \tilde{E}^L) n_L}{r^{l+1}} + \tilde{U}_N^{\text{hom}}, \tag{5.3.25}$$

with the homogeneous solution

$$\tilde{U}_N^{\text{hom}} = -\sum_{\ell=1}^{\infty} \frac{1}{\ell! 16\pi K_R} \tilde{\mathcal{E}}^L n_L r^\ell. \tag{5.3.26}$$

With the solutions for the ETF equations of motion in the Einstein frame obtained, we can turn our attention to the extraction of the Love numbers $\tilde{\lambda}_\ell^T$, $\tilde{\lambda}_\ell^S$, $\tilde{\lambda}_\ell^{ST}$.

5.3.3 Extracting the Love numbers

To extract the Love numbers in the Einstein frame requires a similar approach to what was used in chapter 3. In that chapter, we compared an exact solution for the perturbed metric function $H^\ell(r)$ with the the expected Newtonian limit of the tt component of the metric. We take a similar approach in this section. To compute the Love number in the full theory, the metric $\tilde{g}_{\mu\nu}$ and the scalar field φ are both perturbed similarly to how the metric was in chapter 3, this time in terms of $H^\ell(r)$ and a scalar perturbation $\delta\varphi^\ell(r)$. One can then solve the perturbation equations at $r \to \infty$ in a perturbative expansion in r (these equations of motion are later presented in (5.4.9)). The perturbative expansions in H^ℓ and φ^ℓ go as

$$\delta\varphi^\ell(r) = \delta\varphi^{(\ell)} r^\ell \left(1 - \frac{\ell M}{r}\right) + ... + \frac{\delta\varphi^{(-\ell-1)}}{r^{\ell+1}} + \mathcal{O}\left(\frac{1}{r^{\ell+2}}\right), \tag{5.3.27a}$$

$$H^\ell(r) = H^{(\ell)} r^\ell \left(1 - \frac{\ell M}{r}\right) + ... + \frac{H^{(-\ell-1)}}{r^{\ell+1}} + \mathcal{O}\left(\frac{1}{r^{\ell+2}}\right). \tag{5.3.27b}$$

Here the terms $\delta\varphi^{(\ell)}(r)$ and $H^{(\ell)}$ with brackets around ℓ denote coefficients in the solutions for $\delta\varphi^\ell(r)$ and $H^\ell(r)$ respectively. We can then compare these coefficients with those from the solution to the EFT equations (5.3.20) and (5.3.25). Matching the coefficients, we see that the various Love numbers can be extracted from the full theory solutions $\delta\varphi^\ell(r)$ and $H^\ell(r)$ as

$$\tilde{\lambda}_\ell^T = \frac{1}{(2\ell-1)!!} \left.\frac{H^{(-\ell-1)}}{H^{(\ell)}}\right|_{\delta\varphi^{(\ell)}=0}, \tag{5.3.28a}$$

$$\tilde{\lambda}_\ell^S = \frac{1}{(2\ell-1)!!} \left.\frac{\delta\varphi^{(-\ell-1)}}{\varphi^{(\ell)}}\right|_{H^{(\ell)}=0}, \tag{5.3.28b}$$

$$\tilde{\lambda}_\ell^{ST} = \frac{1}{(2\ell-1)!!} \left.\frac{H^{(-\ell-1)}}{2\delta\varphi^{(\ell)}}\right|_{H^{(\ell)}=0} = \frac{1}{(2\ell-1)!!} \left.\frac{2\delta\varphi^{(-\ell-1)}}{H^{(\ell)}}\right|_{\delta\varphi^{(\ell)}=0}. \tag{5.3.28c}$$

Notice that all expressions are evaluated with $\delta\varphi^{(\ell)}$ or $H^{(\ell)} = 0$. This is due to the mingling of the tensor $\tilde{\mathcal{E}}_L$ and scalar $\tilde{E}_L$ tidal fields in the coefficients for the EFT solutions (5.3.20) and (5.3.25). For example, when evaluating the tensor Love number $\tilde{\lambda}_\ell^T$ without $\delta\varphi^{(\ell)}$ we find

$$\frac{1}{(2\ell-1)!!}\frac{H^{(-\ell-1)}}{H^{(\ell)}} = \frac{\tilde{\lambda}_l^T \tilde{\mathcal{E}}^L + \tilde{\lambda}_\ell^{ST} \tilde{E}^L}{\tilde{\mathcal{E}}_L}, \tag{5.3.29}$$

$$= \tilde{\lambda}_\ell^T + \tilde{\lambda}_\ell^{ST} \frac{\tilde{E}_L}{\tilde{\mathcal{E}}_L}. \tag{5.3.30}$$

Thus the necessity of setting $\delta\varphi^{(\ell)} = 0$ really ensures that $\tilde{E}_L = 0$ and we recover $\tilde{\lambda}_\ell^T$. This problem of setting $\delta\varphi^{(\ell)} = 0$ or $H^{(\ell)} = 0$, however is a problem for the numerical aspect of the analysis. It is explained in more detail in [6], where here we merely review the analytical treatment for extracting the Love numbers, in the hope that it can be generalized to generalized scalar-tensor theories in chapter 6.

With the Love numbers now obtained in the Einstein frame, they can be transformed back to their Jordan frame counterparts with the use of the previously derived transformation formulas (5.2.23). For the remainder of this chapter, we conclude by reviewing some of the numerical results that were obtained using the EFT formalism to extract the Love numbers.

5.4 Numerical results

In the previous sections we have given an analytical treatment to scalar-tensor $F(\phi)R$ theories, culminating with a recipe for extracting the Love number in the Einstein frame and transforming it back to the Jordan frame. In this section, we now put this analytical treatment into context by reviewing the numerical results that give the mass-radius curves and Love numbers λ_ℓ in scalar-tensor gravity. This section will take a similar outline to chapter 3 in that we first review the results from the background spacetime i.e. the mass-radius curves, followed by the results from perturbing the spacetime i.e. the Love numbers.

5.4.1 Background equations of motion

We start with the scalar-tensor action in the Einstein frame

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - 2\tilde{\partial}_\mu\varphi\tilde{\partial}^\mu\varphi\right] + S_{\text{matter}}\left[\psi_m, A^2(\varphi)\tilde{g}_{\mu\nu}\right]. \tag{5.4.1}$$

Here, we fix the previous normalization constants as $K_R = 1/16\pi G$ and $K_\varphi = 2K_R$ also use $A^2(\varphi)$ to denote the conformal factor as in (4.2.53). The theory is defined in the Einstein frame by choosing $F(\phi) = \phi$ and $A(\varphi) = e^{\frac{1}{2}\beta\varphi^2}$.

We make the same metric ansatz for a non-rotating, spherically symmetric spacetime, as used in chapter 3. Varying the action (5.4.1) with respect to $\tilde{g}_{\mu\nu}$ and φ, we can derive the scalar-tensor counterpart to the TOV equations. These equations look like

$$\frac{dm}{dr} = 4\pi A^4(\varphi)\rho^2 r^2 + \frac{r^2}{2}\left(1-\frac{2m}{r}\right)\left(\frac{d\varphi}{dr}\right)^2, \tag{5.4.2a}$$

$$\frac{d\nu}{dr} = 2\frac{4\pi A^4(\varphi)r^3 p + m}{r(r-2m)} + r\left(\frac{d\varphi}{dr}\right)^2, \tag{5.4.2b}$$

$$\frac{dp}{dr} = -\frac{\rho+p}{2}\left(\frac{d\nu}{dr} - 2\alpha(\varphi)\frac{d\varphi}{dr}\right), \tag{5.4.2c}$$

where $\alpha(\varphi) = -\frac{1}{A}\frac{dA}{d\varphi}$. The modified TOV equations are then completed by the equation of motion of motion for the scalar field

$$\frac{d^2\varphi}{dr^2} + \left(\frac{1}{2}\frac{d\nu}{dr} - \frac{1}{2}\frac{d\gamma}{dr} + \frac{2}{r}\right)\frac{d\varphi}{dr} = 4\pi\alpha(\varphi)A^4(\varphi)(3p-\rho)e^\gamma. \tag{5.4.3}$$

Here it is more convenient to express the equation in terms of the metric component $\gamma(r)$, related to the mass $m(r)$ by

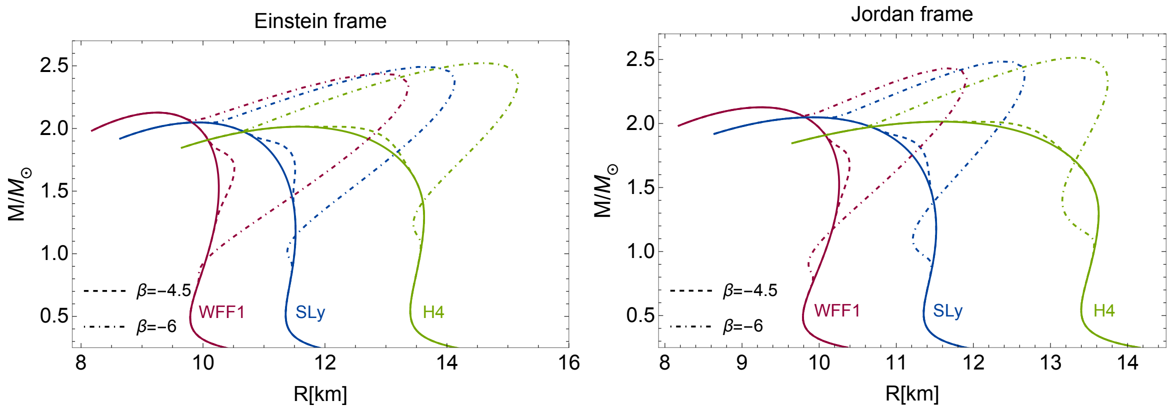

Figure 5.1: Mass-radius curves [6] in the Einstein and Jordan frame for various equations of state. The deviations from GR correspond to “scalarized” [40] neutron star configurations. We can see that lowering the value of β can potentially influence the maximum mass of the neutron star. Setting $\beta = 0$ recovers the GR mass-radius curves in solid. The value of the scalar field at infinity is $\varphi_\infty = 10^{-3}$.

$$\gamma(r) = -\ln\left(1 - \frac{2m(r)}{r}\right). \tag{5.4.4}$$

We can see that setting $\alpha(\varphi) = \varphi = 0$ and $A(\varphi) = 1$ in the modified TOV equations recovers the GR limit.

5.4.2 Numerical solutions to the TOV equations

The modified TOV equations (5.4.2) are then solved similarly to the TOV equations of chapter 3, including using the same initial conditions for the density, pressure and mass close to the star

$$\rho(r \to 0) = \rho_c + \mathcal{O}(r^2), \quad p(r \to 0) = p_c + \mathcal{O}(r^2), \quad m(r \to 0) = \frac{4\pi}{3}\rho_c r^3 + \mathcal{O}(r^5). \tag{5.4.5}$$

However, with the presence of the scalar field, we also require two more initial conditions to complete the numerical integration. These will come from the scalar field $\varphi(r)$, and are given by

$$\varphi(r \to 0) = \varphi_c, \quad \varphi'(r \to 0) = 0, \tag{5.4.6}$$

where $\varphi'(r)$ denotes a derivative of $\varphi(r)$ with respect to r. We will determine the central value of the scalar field φ_c by fixing the value of φ as it approaches infinity instead, i.e.

$$\lim_{r\to\infty} \varphi(r) = \varphi_\infty, \tag{5.4.7}$$

in accordance with observational constraints ($\varphi_\infty = 10^{-3}$ is used in [6]). This means integrating the system using a shooting method to determined the φ_c that gives the desired value of φ_∞. This will need to be done for each value of the central density ρ_c. The resulting mass-radius curves are presented in Figure 5.1. Here we omit some of the numerical details for extracting the mass and converting the diagrams between frames for our own case study in chapter 7. More detail can be found in [6] for the interested reader.

Examining the mass-radius curves in Figure 5.1, we can see that the solutions partially overlap with the GR curves but then display qualitatively different behavior in particular regions. These configurations correspond to so-called “scalarized” neutron star configurations and allow scalar-tensor theories to obey satisfy solar system constraints while still deviating from GR in stronger regimes of gravity.

5.4.3 Perturbation equations

Similarly to chapter 3, we next derive the perturbation equations by perturbing the metric as (3.2.26) and the pressure, density and four-velocity as (3.2.27). In addition, we also include perturbations to the scalar field as

$$\varphi = \varphi_0 + \sum_{\ell,m} \delta\varphi^{\ell m}(r) Y_\ell^m(\theta,\varphi). \tag{5.4.8}$$

Adding these perturbations into the Einstein equations and subtracting the modified background TOV equations, we can write equations of motion for the linear perturbations as

$$\begin{aligned}
&\frac{d^2H^\ell}{dr^2} + \left\{\frac{2}{r} + e^\gamma\left[\frac{1-e^{-\gamma}}{r} + 4\pi A^4(\varphi_0)\left(p_0-\rho_0\right)r\right]\right\}\frac{dH^\ell}{dr} \\
&\quad + \left\{-\frac{l(l+1)}{r^2}e^\gamma + 4\pi A^4(\varphi_0)e^\gamma\left[9p_0 + 5\rho_0 + \frac{\rho_0+p_0}{\tilde{c}_s^2}\right] - \left(\frac{d\nu}{dr}\right)^2\right\}H^\ell \\
&\quad + e^\gamma\left\{-4\left(\frac{d\varphi_0}{dr}\right)r\left[\frac{1-e^{-\gamma}}{r^2} + 8\pi A^4(\varphi_0)p_0 + e^{-\gamma}\left(\frac{d\varphi_0}{dr}\right)^2\right]\right. \\
&\quad \left. - \frac{16\pi}{\sqrt{3}}A^4(\varphi_0)\left[\left(\rho_0 - 3p_0\right) + \left(\rho_0+p_0\right)\frac{1-3\tilde{c}_s^2}{2\tilde{c}_s^2}\right]\right\}\delta\varphi^\ell = 0,
\end{aligned} \tag{5.4.9a}$$

$$\begin{aligned}
&\frac{d^2\delta\varphi^\ell}{dr^2} + \left(\frac{1}{2}\frac{d\nu}{dr} - \frac{1}{2}\frac{d\gamma}{dr} + \frac{2}{r}\right)\frac{d\delta\varphi^\ell}{dr} \\
&\quad - e^\gamma\left\{\frac{l(l+1)}{r^2} + 4e^{-\gamma}\left(\frac{d\varphi_0}{dr}\right)^2 - \frac{8\pi}{3}A^4(\varphi_0)\left[-2\left(\rho_0-3p_0\right) + \left(\rho_0+p_0\right)\frac{1-3\tilde{c}_s^2}{2\tilde{c}_s^2}\right]\right\}\delta\varphi^\ell \\
&\quad + e^\gamma\left\{-e^{-\gamma}\left(\frac{d\nu}{dr}\right)\left(\frac{d\varphi_0}{dr}\right) - \frac{4\pi}{\sqrt{3}}A^4(\varphi_0)\left[\left(\rho_0-3p_0\right) + \left(\rho_0+p_0\right)\frac{1-3\tilde{c}_s^2}{2\tilde{c}_s^2}\right]\right\}H^\ell = 0.
\end{aligned} \tag{5.4.9b}$$

Here $\tilde{c}_s^2 = \partial p_0/\partial\rho_0$ denotes the Jordan frame speed of sound. These equations are solved perturbatively in $1/r$ at $r\to\infty$ to give the series solutions (5.3.27) discussed previously.
Notice the presence of a $\delta\varphi^\ell$ term in the equation for H^ℓ and vice versa. This coupling between perturbation equations is what is accounted for in the effective action through the scalar-tensor coupling term $\lambda_\ell^{ST}\mathcal{E}_L E^L$. Setting $\delta\varphi^\ell = \varphi = 0$ in (5.4.9a) recovers the GR master equation (3.2.29).

5.4.4 Numerical solutions to the perturbation equations

The perturbation equations (5.4.9) are then solved numerically inside the neutron star, starting from the origin with similar initial conditions as in GR

$$H^\ell \propto r^\ell, \quad H'_\ell \propto \ell r^{\ell-1}, \tag{5.4.10}$$

$$\delta\varphi^\ell \propto r^\ell, \quad \delta\varphi'_\ell \propto \ell r^{\ell-1}. \tag{5.4.11}$$

The numerical solutions for $H^\ell(r)$ and $\delta\varphi^\ell(r)$ are then matched with the pertubative expansions (5.3.27) to determine the coefficients $H^{(\ell)}$ and $\delta\varphi^{(\ell)}$. The coefficients can then be used in (5.3.28) to extract the various various Love numbers, and be related back to their Jordan frame counterparts through (5.2.23).

The numerical results for the tensor, scalar and scalar-tensor Love numbers can be found in Figures 5.2, 5.3 and 5.4. We see that the tensor Love numbers again show similar behavior to GR but deviate in certain regions corresponding to the same scalarized neutron star configurations as seen in the mass-radius curves. We can see from the plots of the scalar Love numbers $\tilde{\lambda}_\ell^S$ that a larger tensor Love

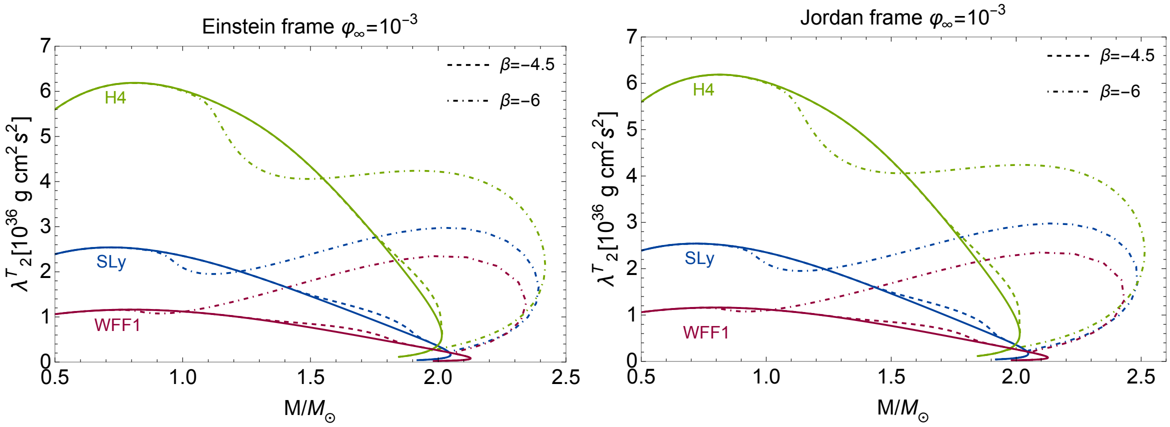

Figure 5.2: Tensor Love numbers in the Jordan and Einstein frame for $\ell = 2$.

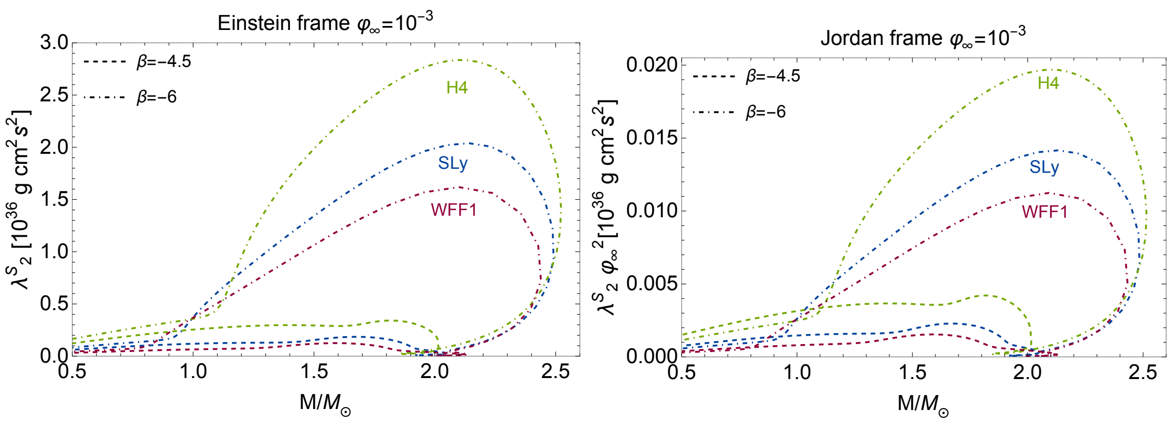

Figure 5.3: Scalar Love numbers in the Jordan and Einstein frame for $\ell = 2$.

number λ_ℓ^T also appears to correlate with a larger scalar Love number, although the order of magnitude is far smaller in the Jordan frame. We can also see that the scalar Love number is essentially zero outside of the scalarized regions, in accordance with there being no scalar field in GR. Interestingly, we see that the scalar-tensor Love numbers λ_ℓ^{ST} are negative.

We omit plots of k_ℓ and Λ_ℓ (as seen in chapter 3) for brevity as they exhibit similar qualitative behavior as the plotted Love numbers λ_ℓ in this section.

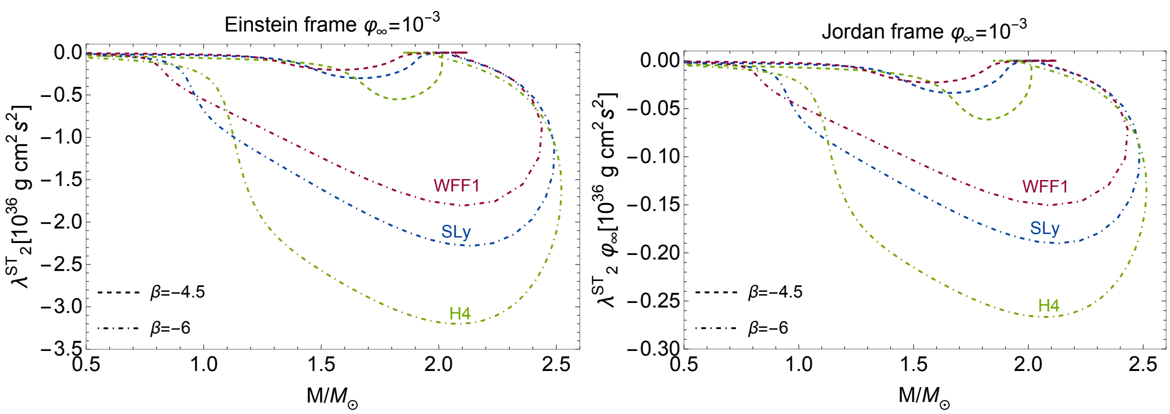

Figure 5.4: Scalar-tensor Love numbers in the Jordan and Einstein frame for $l = 2$.

Chapter 6

Generalized scalar-tensor theories

So far our discussion of scalar-tensor theories has been solely limited to the case of the scalar-tensor $F(\phi)R$ theory

$$S_{\rm ST} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[F(\phi)R - 2\frac{\omega(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] + S_{\rm matter}\left[\psi_m, g_{\mu\nu}\right], \tag{6.0.1}$$

where $F(\phi)$ and $\omega(\phi)$ are arbitrary functions paramaterizing the theory. In chapter 5 we looked at how to describe tidal effects in the scalar-tensor theory and how the Love numbers λ_ℓ may be extracted from it. For this chapter, we would now like to extend the previous discussions of the scalar-tensor action (6.0.1) to the *generalized* scalar-tensor action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[f(\phi, R) - 2\frac{\omega(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi\right] + S_{\rm matter}\left[\psi_m, g_{\mu\nu}\right], \tag{6.0.2}$$

where now $f(\phi, R)$ is an arbitrary, algebraic function of ϕ *and* R (and has also absorbed the potential $V(\phi)$). Specifically, we would like to show how the generalized scalar-tensor action (6.0.2) may be transformed to the Einstein frame by introducing an auxiliary scalar field Φ followed by a conformal rescaling of the metric tensor

$$g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu}. \tag{6.0.3}$$

With the Einstein frame transformations worked out for the general action (6.0.2), we would then like to investigate if the formalism for extracting the Love number in chapter 5 may be extended to the case of $f(\phi, R)$.

We would like to make some comments on the novelty of the work carried out in this chapter. The technique of transforming the generalized scalar-tensor theory (6.0.2) to the Einstein frame is not an original technique. It has been done previously for the arbitrary coupling $f(\phi, R)$ in [41], but *only* at the level of the equations of motion, which were then used to deduce the correct form of the Einstein frame action. The result for the Einstein frame action of $f(\phi, R)$ is also claimed to be in [28] and [29], however their result omits the kinetic term for ϕ and gives little detail on how the transformation may be carried out, suggesting the result may have also been taken from [41].

Thus, to the author's knowledge, the transformation technique for transforming the generalized scalar-tensor theory to the Einstein frame at the *level of the action* is wholly original (or at the very least, the author did not have access to a source showing the correct transformation at the level of the action). The work in the second section of this chapter, which attemps to generalize the work of chapter 5 and [6], is also a wholly original attmept.

6.1 Transforming to the Einstein frame

6.1.1 Introducing the auxiliary field

We begin with the generalized scalar-tensor action (6.0.2) and omit the matter action for brevity

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[f(\phi,R) - 2\frac{\omega(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi\right]. \tag{6.1.1}$$

We can rewrite this action in terms of an auxiliary field Φ by making the replacement

$$f(\phi,R) \to f(\phi,\Phi) + (R-\Phi)f_R(\phi,\Phi), \tag{6.1.2}$$

such that the original action now looks like

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[f(\phi,\Phi) + (R-\Phi)\,f_R(\phi,\Phi) - 2\frac{\omega(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi\right]. \tag{6.1.3}$$

Here we use $f_R(\phi,\Phi)$ to denote a derivative of f with respect to its second argument i.e.

$$f_R(\phi,\Phi) = \left.\frac{\partial}{\partial R}f(\phi,R)\right|_{R=\Phi} = \frac{\partial}{\partial\Phi}f(\phi,\Phi). \tag{6.1.4}$$

Because the new action (6.1.3) is independent of $\partial_\mu\Phi$, we can easily calculate the Φ equation of motion as

$$\begin{aligned}&\frac{\partial}{\partial\Phi}\left[f(\phi,\Phi) + (R-\Phi)f_R(\phi,\Phi)\right] = 0,\\ \implies\ &\cancel{f_R(\phi,\Phi)} - \cancel{f_R(\phi,\Phi)} + (R-\Phi)f_{RR}(\phi,\Phi) = 0.\end{aligned} \tag{6.1.5}$$

Thus we see that Φ has the equation of motion

$$\Phi = R, \tag{6.1.6}$$

provided that $f_{RR}(\phi,\Phi)\neq 0$. We can return to the original generalized scalar-tensor action (6.1.1) by substituting Φ's equation of motion (6.1.6) back into the action (6.1.3), which is permitted at the classical level as both theories produce equivalent equations of motion. In the case that $f_{RR}(\phi,\Phi) = 0$, one can without loss of generality express the coupling function $f(\phi,R)$ as

$$f(\phi,R) = F(\phi)R - V(\phi). \tag{6.1.7}$$

Filling this choice of $f(\phi,R)$ back into the auxiliary field action (6.1.3) we find that

$$\begin{aligned}&f(\phi,\Phi) + (R-\Phi)f_R(\phi,\Phi),\\ =&F(\phi)\Phi - V(\phi) + (R-\Phi)F(\phi),\\ =&F(\phi)R + \cancel{F(\phi)\Phi} - \cancel{F(\phi)\Phi} - V(\phi),\\ =&F(\phi)R - V(\phi).\end{aligned} \tag{6.1.8}$$

Thus the auxilary field drops out for $f_{RR}=0$ and the auxiliary field action (6.1.3) remains equivalent to the original action (6.1.1). For the case that $f_{RR}(\phi,\Phi)\neq 0$, we can tidy up the auxiliary field action as

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[f_R(\phi,\Phi)R - V(\phi,\Phi) - 2\frac{\omega(\phi)}{\phi}\partial_\mu\phi\partial^\mu\phi\right], \tag{6.1.9}$$

where we have defined the potential $V(\phi,\Phi)$ as

$$V(\phi,\Phi) \equiv \Phi f_R(\phi,\Phi) - f(\phi,\Phi). \tag{6.1.10}$$

As we can see, the new action (6.1.12) has been made linear in the Ricci scalar R, which was not necessarily the case depending on the choice of the original coupling function $f(\phi,R)$. Such a linearization requires the introduction of the auxiliary field Φ, as introducing higher order derivatives of the metric (e.g. terms as $\sim R^2$) also introduces an additional scalar degree of freedom into the theory [36]. For example, gravitational waves, which normally contain two polarizations in GR, now come with three polarizations in $f(\phi, R)$ gravity. This additional degree of freedom introduced into the theory must be conserved if the theory is to be made linear in the Ricci scalar, and so the introduction of the auxiliary scalar field Φ is inevitable if one wants to linearize an action in R for which $f_{RR}(\phi, \Phi) \neq 0$.

Before performing the conformal rescaling of the metric to pass to the Einstein frame, we must perform an additional redefinition of the scalar field Φ in terms of a new scalar field χ through

$$\chi = f_R(\phi, \Phi). \tag{6.1.11}$$

The significance of this redefinition will be addressed later, but for now we simply say that it will make for a smoother transformation to the Einstein frame. In terms of the new auxiliary field χ, our action can then be expressed as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[\chi R - V\left(\phi, \Phi(\phi, \chi)\right) - 2\frac{\omega(\phi)}{\phi} \partial_\mu \phi \partial^\mu \phi \right], \tag{6.1.12}$$

where we have inverted the definition (6.1.11) of χ in terms of Φ, and used it to express the potential $V(\phi, \Phi)$ in as a function of ϕ and χ. At this point we will simply denote the potential as V, without reference to its arguments, which will change again through this chapter. One should always recall that the potential is defined with respect to two scalar fields.

6.1.2 Conformal rescaling

With the original scalar-tensor action (6.0.2), now made linear in R, we are ready to perform our conformal rescaling of the metric

$$g_{\mu\nu} = \Omega^{-2} \tilde{g}_{\mu\nu} \tag{6.1.13}$$

to pass to the Einstein frame. We can do this by employing our toolbox of transformations (4.2.68). The initial action in the Einstein frame then looks like

$$S = \int d^4x \sqrt{\tilde{g}} \left[\chi \Omega^{-2} \left(\tilde{R} - \frac{3}{2} \frac{\tilde{g}^{\mu\nu}}{\Omega^4} \partial_\mu \Omega^2 \partial_\nu \Omega^2 \right) - \Omega^{-4} V - 2\frac{\omega(\phi)}{\phi} \Omega^{-2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right], \tag{6.1.14}$$

where we have dropped the $6\tilde{\Box} f$ term in the transformation of R as it soon amounts to a boundary term. We can now kill the coupling between χ and R by choosing the conformal factor Ω as

$$\Omega^2 = \chi, \tag{6.1.15}$$

such that the resulting action simplifies down to

$$S = \frac{1}{16\pi G} \sqrt{-\tilde{g}} \left[\tilde{R} - \frac{3}{2} \frac{1}{\chi^2} \tilde{\partial}_\mu \chi \tilde{\partial}^\nu \chi - \frac{V}{\chi^2} - 2 \frac{1}{\chi} \frac{\omega(\phi)}{\phi} \tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi \right]. \tag{6.1.16}$$

We can further simplify the action to make the χ scalar field canonical. This can be done by defining a new scalar field $\tilde{\chi}$ in terms of χ as

$$\tilde{\chi} = \frac{\sqrt{3}}{2} \ln \chi, \tag{6.1.17}$$

such that

$$2\tilde{\partial}_\mu \tilde{\chi} \tilde{\partial}^\mu \tilde{\chi} = \frac{3}{2} \frac{1}{\chi^2} \tilde{\partial}_\mu \chi \tilde{\partial}^\mu \chi. \tag{6.1.18}$$

With this, the action (6.1.16) becomes

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - 2\tilde{\partial}_\mu \tilde{\chi} \tilde{\partial}^\mu \tilde{\chi} - \Delta^2(\tilde{\chi}) V - 2\Delta(\tilde{\chi}) \Omega(\phi) \tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi \right], \tag{6.1.19}$$

where we have defined

$$\Delta(\tilde{\chi}) \equiv \chi^{-1},$$
$$= e^{-2\tilde{\chi}/\sqrt{3}}, \tag{6.1.20}$$

and abbreviated $\Omega(\phi) \equiv \frac{\omega(\phi)}{\phi}$. This $\Omega(\phi)$ should not be confused with the conformal rescaling factor (6.1.13).

The action (6.1.19) thus represents the final form of the original $f(\phi, R)$ action (6.0.2) in the Einstein frame. We have successfully managed to linearize the action in the Ricci scalar R by rewriting the theory as general relativity plus two coupled scalar fields ϕ and $\tilde{\chi}$. The action (6.1.19) is completely general up to the potential V, which is dependent on the original coupling function through (6.1.10). We will shortly give an explicit example of the potential V may be computed based on the choice of $f(\phi, R)$.

Comment on the field redefinition Φ to χ

We take a moment to emphasize the importance of the field redefinition (6.1.11). Had we not redefined the initial auxiliary field Φ in terms of χ, the conformal factor Ω^2 would instead have been chosen as

$$\Omega^2 = f_R(\phi, \Phi). \tag{6.1.21}$$

This would have lead to a Ricci scalar transformation that looks like

$$R = \Omega^2 \left(\tilde{R}^2 + 6\tilde{\Box} f - \frac{3\tilde{g}^{\mu\nu}}{2\Omega^4} \partial_\mu f_R(\phi, \Phi) \partial_\nu f_R(\phi, \Phi) \right). \tag{6.1.22}$$

While innocent at first, equation (6.1.22) becomes cumbersome to work with, as using the chain rule on $f_R(\phi, \Phi)$ implies that

$$\begin{aligned} \tilde{g}^{\mu\nu} \partial_\mu f_R(\phi, \Phi) \partial_\nu f_R(\phi, \Phi) &= \tilde{g}^{\mu\nu} \left(\partial_\mu \phi \frac{\partial}{\partial \phi} + \partial_\mu \Phi \frac{\partial}{\partial \Phi} \right) f_R(\phi, \Phi) \left(\partial_\nu \phi \frac{\partial}{\partial \phi} + \partial_\nu \Phi \frac{\partial}{\partial \Phi} \right) f_R(\phi, \Phi), \\ &= \tilde{g}^{\mu\nu} f_{\phi R} \partial_\mu \phi \partial_\nu \Phi + 2\tilde{g}^{\mu\nu} f_{RR} f_{\phi R} \partial_\mu \phi \partial_\nu \Phi + \tilde{g}^{\mu\nu} f_{RR} \partial_\mu \Phi \partial_\nu \Phi, \end{aligned} \tag{6.1.23}$$

where we have summed the cross terms by using the symmetry of the metric $\tilde{g}^{\mu\nu}$ under exchange of indices. The above expression (6.1.23) leads to a more complicated action due to the presence of the "cross" kinetic term proportional to $\tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \Phi$. Thus the field redefinition (6.1.11) is necessary if one wants to avoid working with such terms and obtain the simplest possible action.

6.1.3 Special cases

As we have seen, the Einstein frame action posses two coupled scalar fields ϕ and $\tilde{\chi}$, one which is the original Jordan frame field ϕ, and the other, $\tilde{\chi}$, which encodes the additional degrees of freedom from the higher order derivatives of the metric. Removing either of these features from the Jordan frame action leads to an Einstein frame action containing a single scalar field - ϕ or $\tilde{\chi}$. We briefly review these special cases here.

Case $F(\phi)R$

The first subcase, $f(\phi, R) = F(\phi)R - V(\phi)$ we have reviewed extensively in chapters 4 and 5. We repeat it here however as it may not be immediately clear how the action (6.1.19) reduces to the previously seen Einstein frame action (4.2.51). As discussed previously, the auxiliary field Φ drops out in this case. This can be seen by making the replacement

$$\begin{aligned} f(\phi, R) \rightarrow & f(\phi, \Phi) + (R - \Phi) f_R(\phi, \Phi), & (6.1.24) \\ = & R f_R(\phi, \Phi) - [\Phi f_R(\phi, \Phi) - f(\phi, \Phi)], & (6.1.25) \\ = & RF(\phi) - V(\phi), & (6.1.26) \end{aligned}$$

and the action remains independent of the auxiliary field Φ. Thus, when we transform to the Einstein frame, the field redefinition (6.1.11) that introduces χ is no longer necessary and the conformal factor can simply be chosen as

$$\begin{aligned}\Omega^2 &= f_R(\phi, \Phi), \\ &= F(\phi).\end{aligned} \tag{6.1.27}$$

When comparing this choice with the conformal factor (6.1.15) for the general case, this implies that one can simplify the general case by making the replacement

$$\chi \to F(\phi). \tag{6.1.28}$$

Doing so, reduces the generalized scalar-tensor (6.1.16) to

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - \left(2\frac{\omega(\phi)}{\phi F(\phi)} + \frac{3}{2} \left(\frac{F'(\phi)}{F(\phi)} \right)^2 \right) \tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi - \frac{V}{F(\phi)^2} \right], \tag{6.1.29}$$

which is identical to the scalar-tensor action (4.2.48), before we redefined ϕ in terms of φ.

Case $f(R)$

The second special case involves setting the Jordan frame scalar field ϕ to 1 such that the original action looks like

$$S = \int d^4x \sqrt{-g} \left[f(R) \right]. \tag{6.1.30}$$

Such theories are usually referred to as $f(R)$ theories or non-linear theories of gravity due to the fact that the function $f(R)$ can in principle contain arbitrary powers of the Ricci scalar R. We can find this actions counterpart in the Einstein frame by looking at our generalized Einstein frame action (6.1.19) and setting $\phi = 1$. The result is

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - 2\tilde{\partial}_\mu \tilde{\chi} \tilde{\partial}^\mu \tilde{\chi} - \Delta(\tilde{\chi}) V \right], \tag{6.1.31}$$

where $\Delta(\tilde{\chi})$ is still defined by (6.1.20). The field redefinition (6.1.11) is no longer necessary as the conformal factor, Ω, will again only be a function of one field - χ. As a result there are no cross-derivative terms to avoid when transforming R to the Einstein frame, similar to the $F(\phi)R$ case. The conformal factor Ω is then determined by

$$\Omega^2 = f'(\Phi). \tag{6.1.32}$$

The scalar fields Φ and $\tilde{\chi}$ may also be expressed directly in terms of each other as

$$\tilde{\chi} = \frac{\sqrt{3}}{2} \ln \left[f'(\Phi) \right], \tag{6.1.33}$$

and the potential V is

$$U(\tilde{\chi}) = \Phi(\tilde{\chi}) f' \left(\Phi(\tilde{\chi}) \right) - f \left(\Phi(\tilde{\chi}) \right), \tag{6.1.34}$$

where f' denotes a derivative of f with respect to Φ.

6.1.4 Equations of motion

With the Einstein frame action (6.1.19) obtained for the generalized scalar-tensor theory, it is also useful to keep a record of the equations of motion. Varying the action (6.1.19) with respect to $\tilde{g}_{\mu\nu}$, ϕ and $\tilde{\chi}$, one finds

$$\tilde{G}_{\mu\nu} = 2\Delta(\tilde{\chi})\Omega(\phi)\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + 2\tilde{\nabla}_\mu\tilde{\chi}\tilde{\nabla}_\nu\tilde{\chi} - \tilde{g}_{\mu\nu}\left[\Delta(\tilde{\chi})\Omega(\phi)\tilde{\nabla}_\alpha\phi\tilde{\nabla}^\alpha\phi + \tilde{\nabla}_\alpha\tilde{\chi}\tilde{\nabla}^\alpha\tilde{\chi} + \frac{1}{2}\Delta(\tilde{\chi})^2 V\right], \tag{6.1.35a}$$

$$\tilde{\Box}\phi = \frac{2}{\sqrt{3}}\tilde{\nabla}_\alpha\phi\tilde{\nabla}^\alpha\tilde{\chi} - \frac{1}{2}\left(\frac{\Omega'(\phi)}{\Omega(\phi)}\right)\tilde{\nabla}_\alpha\phi\tilde{\nabla}^\alpha\phi + \frac{1}{4}\left(\frac{\Delta(\tilde{\chi})}{\Omega(\phi)}\right)\frac{\partial V}{\partial\phi}, \tag{6.1.35b}$$

$$\tilde{\Box}\tilde{\chi} = -\frac{1}{\sqrt{3}}\Delta(\tilde{\chi})\Omega(\phi)\tilde{\nabla}_\alpha\phi\tilde{\nabla}^\alpha\phi + \frac{1}{4}\Delta^2(\tilde{\chi})\left[\frac{\partial}{\partial\tilde{\chi}} - \frac{4}{\sqrt{3}}\right]V. \tag{6.1.35c}$$

While these equations contain intricacies, they are still preferable to solving fourth order equations for the metric in the Jordan frame. The equations of motion also make it more apparent how the degrees of the freedom of the theory have been conserved with the introduction of the auxiliary field $\tilde{\chi}$. In the Jordan frame, our equations of motion would have been a fourth order differential equation for $g_{\mu\nu}$ and a second order for ϕ. By contrast in the Einstein frame, we can see above that we have a second order differential equation for the metric $\tilde{g}_{\mu\nu}$ as well as two second order differential equations, one for ϕ and $\tilde{\chi}$.

6.1.5 Explicit example

As seen previously, the Einstein frame action (6.1.19) is only defined up until the potential V. In order to completely determin the Einstein frame action, a coupling function $f(\phi, R)$ must be specified in the Jordan frame, and the corresponding potential worked out in the Einstein frame. It is instructive to outline the steps for this procedure with a simple example. After specifying a coupling function $f(\phi, R)$, the basic recipe is as follows

1. Compute the potential $V = \Phi f_R(\phi, \Phi) - f(\phi, \Phi)$.
2. Solve the equation $\chi = f_R(\phi, \Phi)$ for Φ in terms of ϕ and χ.
3. Put this expression for Φ back into V.
4. Replace χ in V with the redefinition $\chi = e^{2\tilde{\chi}/\sqrt{3}}$.

For example, consider one of the simplest coupling functions that contains both ϕ and higher order derivatives of the metric

$$f(\phi, R) = F(\phi)\left(R + \alpha R^2\right), \tag{6.1.36}$$

where setting $\alpha = 0$ and $\phi = 1$ recovers GR.

Following step 1, we compute V

$$V = \Phi f_R(\phi, \Phi) - f(\phi, \Phi), \tag{6.1.37}$$

$$= \Phi F(\phi)(1 + 2\alpha\Phi) - F(\phi)(\Phi + \alpha\Phi^2), \tag{6.1.38}$$

$$= \alpha F(\phi)\Phi^2. \tag{6.1.39}$$

Following step 2, we solve the equation

$$\chi = F(\phi)(1 + 2\alpha\Phi), \tag{6.1.40}$$

for Φ, and obtain

$$\Phi = \frac{1}{2\alpha}\left(\frac{\chi}{F(\phi)} - 1\right). \tag{6.1.41}$$

For step 3, we substitute this expression for Φ back into V to obtain

$$V = \frac{F(\phi)}{4\alpha}\left(\frac{\chi}{F(\phi)} - 1\right)^2, \tag{6.1.42}$$

$$= \frac{1}{4\alpha F(\phi)}\left(\chi - F(\phi)\right)^2. \tag{6.1.43}$$

Finally, for step 4 we replace χ with its expression in terms of $\tilde{\chi}$ to get

$$V = \frac{1}{4\alpha F(\phi)}\left(e^{2\tilde{\chi}/\sqrt{3}} - F(\phi)\right)^2. \tag{6.1.44}$$

Recalling the definition of $\Delta(\tilde{\chi})$ (6.1.20), we can alternatively write the potential as

$$V = \frac{1}{4\alpha F(\phi)}\left(\Delta^{-1} - F(\phi)\right)^2, \tag{6.1.45}$$

which completes the action (6.1.19). While determining V is enough to fully specify the action, recall that V enters the action as

$$\Delta^2(\tilde{\chi})V. \tag{6.1.46}$$

If one finds it more convenient to define a new potential $\tilde{V}$ through

$$\tilde{V} = \Delta^2 V, \tag{6.1.47}$$

this is also possible and is more in convention with what is done in the study of $f(R)$ theories. We find the new potential can be written as

$$\begin{aligned}\tilde{V} &= \frac{1}{4\alpha F(\phi)}\left(1 - \Delta F(\phi)\right)^2, \\ &= \frac{1}{4\alpha F(\phi)}\left(1 - F(\phi)e^{-2\tilde{\chi}/\sqrt{3}}\right)^2.\end{aligned} \tag{6.1.48}$$

Setting $F(\phi) = 1$ gives agreement with the potential found in [5], which studied an $f(R)$ theory of the form $f(R) = R + \alpha R^2$ in the Einstein frame.

6.2 Generalizing the EFT

6.2.1 Transforming to the Einstein frame and relating the Love numbers

For this section, we would like to attempt a generalization of the work done in [6] for extracting the Love numbers in scalar-tensor gravity, which we reviewed in chapter 5. Thus we begin by again writing our effective action in the Jordan frame

$$S = S_{\text{GST}} + S_{\text{pp}} + S_{\text{Tidal}}, \tag{6.2.1}$$

where the generalized scalar-tensor action is given by the $f(\phi, R)$ action (6.1.1) and the point particle S_{pp} and tidal S_{Tidal} actions are still given by (5.3.2b) and (5.3.2c) respectively. We can continue by performing our conformal rescaling of the metric to pass to the Einstein frame. As the point-particle and tidal actions are unchanged from before, their Einstein frame counterparts will be identical to before, the only exception being how the scalar tidal moments E_L will change.

Recall that in the scalar-tensor case, the Jordan frame scalar field ϕ was redefined in terms of a new field φ such that

$$\frac{d\varphi}{d\phi} = \sqrt{\frac{\Delta(\phi)}{2}}, \tag{6.2.2}$$

which lead to the tidal terms E_L changing in the Einstein frame as

$$E_L \to \frac{1}{\sqrt{\Delta}} \tilde{E}_L. \tag{6.2.3}$$

However, we did not perform such a field redefinition in the generalized scalar-tensor case, meaning our action in the Einstein frame action still depends on the original Jordan frame field ϕ. We can thus "generalize" the tidal action to the generalized scalar-tensor case by simply setting $\Delta = 1$.

After this, none of the steps from the case reviewed previously in chapter 5 will change. Therefore to get how the Love numbers are related between frames, we can simply take our result (5.2.23) for the Love numbers and set $\Delta = 1$, giving

$$\lambda_\ell^T = \tilde{\lambda}_\ell^T, \tag{6.2.4a}$$

$$\lambda_\ell^S = \tilde{\lambda}_\ell^S, \tag{6.2.4b}$$

$$\lambda_\ell^{ST} = \tilde{\lambda}_\ell^{ST}. \tag{6.2.4c}$$

We can see from the above equations, that the Love numbers do not change between frames. This is in contrast to the previous results (5.2.23), where there the scalar λ_ℓ^S and scalar-tensor λ_ℓ^{ST} were not necessarily the same in both frames. This change in Love numbers between frames however was owed to the field redefinition of ϕ in terms of φ. Since there is no such redefinition here, we find that there is no consequence to the Love number for changing between frames.

6.2.2 The EFT equations of motion

In order to attempt an extraction of the Love number as was seen previously in chapter 5, we again would like to derive the EFT equations of motion. The procedure for deriving the equations of motion will remain largely unchanged from last time, with the point particle S_{pp} and tidal S_{Tidal} actions giving identical contributions as before. The only part that will change from the equations of motion are those that come from S_{GST}. Fortunately we have presented the equations of motion for this action already and the only thing to do now is combine them with our previously derived equations of motion from chapter 5. We will first rewrite the generalized scalar-tensor action in terms of arbitrary normalization coefficients, such that the Jordan frame action looks like

$$S = \int d^4x \sqrt{-g} \left[K_R f(\phi, R) - K_\phi \frac{\omega(\phi)}{\phi} \partial_\mu \phi \partial^\mu \phi \right]. \tag{6.2.5}$$

Its Einstein frame counterpart is

$$S = \int d^4x \sqrt{-\tilde{g}} \left[K_R \tilde{R} - K_{\tilde{\chi}} \tilde{\partial}_\mu \tilde{\chi} \tilde{\partial}^\mu \tilde{\chi} - K_R \Delta^2(\tilde{\chi}) V - K_\phi \Delta(\tilde{\chi}) \Omega(\phi) \tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi \right], \tag{6.2.6}$$

where the scalar fields $\tilde{\chi}$ and χ are now related through

$$\tilde{\chi} = \sqrt{\frac{3}{2K_{\tilde{\chi}}}} \ln \chi, \tag{6.2.7}$$

such that

$$\begin{aligned} \Delta(\tilde{\chi}) &\equiv \chi^{-1}, \\ &= e^{-\sqrt{2K_{\tilde{\chi}}}\tilde{\chi}/\sqrt{3}}. \end{aligned} \tag{6.2.8}$$

In terms of this general normalization constants, the equations of motion for the generalized scalar-tensor EFT action in the Einstein frame are

$$\tilde{\Box}\tilde{U}_N = \frac{\tilde{m}(\phi_\infty)}{4K_R}\delta(x) + \sum_{\ell}^{\infty} \frac{(-1)^{\ell+1}}{\ell! 4K_R}\left[\tilde{\lambda}_\ell^T \tilde{\mathcal{E}}^L \partial_L + \tilde{\lambda}_\ell^{ST} \tilde{E}^L \partial_L\right]\delta(x), \tag{6.2.9a}$$

$$\begin{aligned}\Delta(\tilde{\chi})\Omega(\phi)\tilde{\Box}\phi =& \sqrt{\frac{2K_{\tilde{\chi}}}{3}}\Delta(\tilde{\chi})\Omega(\phi)\tilde{\nabla}_\alpha\phi\tilde{\nabla}^\alpha\tilde{\chi} - \frac{1}{2}\Delta(\tilde{\chi})\Omega'(\phi)\tilde{\nabla}_\alpha\phi\tilde{\nabla}^\alpha\phi + \frac{K_R}{2K_\phi}\Delta^2(\tilde{\chi})\frac{\partial V}{\partial\phi} \\ &- \frac{\tilde{m}'(\phi)}{2K_\phi}\delta(x) + \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l! 2K_\phi}\left[\tilde{\lambda}_l^S \tilde{E}^L \partial_L + \tilde{\lambda}_\ell^{ST}\tilde{E}^L\partial_L\right]\delta(x),\end{aligned} \tag{6.2.9b}$$

$$\tilde{\Box}\tilde{\chi} = -\frac{1}{\sqrt{6}}\frac{K_\phi}{\sqrt{K_{\tilde{\chi}}}}\Delta(\tilde{\chi})\Omega(\phi)\tilde{\nabla}_\alpha\phi\tilde{\nabla}^\alpha\phi + \frac{K_R}{2K_{\tilde{\chi}}}\Delta^2(\tilde{\chi})\left[\frac{\partial}{\partial\tilde{\chi}} - \frac{2\sqrt{2K_{\tilde{\chi}}}}{\sqrt{3}}\right]V. \tag{6.2.9c}$$

Comparing these equations to what we derived for the scalar-tensor case, we can see that the equation of motion for the Newtonian potential (6.2.9a) is unchanged from the previous chapter. Thus we can similarly use the same solution (5.3.25) for the Newtonian potential.

The same is not necessarily true for the scalar field ϕ. We can see that its equation (6.2.15a) has become considerably more cumbersome than in the $F(\phi)R$ case due to the presence of the additional scalar field $\tilde{\chi}$, which also has its own equation of motion. Our approach previously in chapter 4, was to transform the fields to Fourier space, and attempt to solve for the scalar field and Newtonian potential algebraically, before transforming back to real space. We can see that due to the presence of the generic potential V in the equations of motion, such a solution is not possible. Thus the equations of motion remain too generic to be solved for the generalized scalar-tensor case. We can attempt to see however if filling in an explicit form for V indicates if the equations will be any easier to solve.

6.2.3 The EFT equations of motion for particular cases

Particular case $f(R)$

The first special case is one that we have discussed previously - the $f(R)$ theories. Recall that to reduce our generalized scalar-tensor theory to the $f(R)$ case we merely need to set $\phi = 1$. Similarly, due to the fact that there is no scalar field in the Jordan frame we can also ignore its associated tidal moments E_L by setting them to zero. The consequence of this is that there is no scalar λ_ℓ^S or scalar-tensor λ_ℓ^{ST} Love numbers in the theory. As a result, our EFT equations of motion (6.2.9) reduce to

$$\tilde{\Box}\tilde{U}_N = \frac{\tilde{m}(\phi_\infty)}{4K_R}\delta(x) + \sum_{\ell}^{\infty}\frac{(-1)^{\ell+1}}{\ell! 4K_R}\tilde{\lambda}_\ell^T\tilde{\mathcal{E}}^L\partial_L\delta(x), \tag{6.2.10a}$$

$$\tilde{\Box}\tilde{\chi} = \frac{K_R}{2K_{\tilde{\chi}}}\Delta^2(\tilde{\chi})\left[\frac{\partial}{\partial\tilde{\chi}} - \frac{2\sqrt{2K_{\tilde{\chi}}}}{\sqrt{3}}\right]V. \tag{6.2.10b}$$

We now only have two uncoupled equations, one for the Newtonian potential $\tilde{U}_N$ and one for the scalar field $\tilde{\chi}$. As we can see from the above, the equation of motion for $\tilde{\chi}$ contains no information about the tidal fields $\tilde{\mathcal{E}}_L$ or the tensor Love number $\tilde{\lambda}_\ell^T$. As it is decoupled from the Newtonian potential $\tilde{U}_N$ it can henceforth be ignored in the extraction of the tensor Love number $\tilde{\lambda}_\ell^T$.

As mentioned previously, we may use our solution (5.3.25) for the Newtonian potential $\tilde{U}_N$. However this is only the solution in the EFT. To obtain the Love number in the full theory requires solving the perturbation equations of motion for the metric perturbation $H^\ell(r)$ and scalar field perturbation $\delta\varphi^\ell$ in the generalized form of (5.4.9). To solve these generalized equations perturbatively, one must again specify a form for the potential V.

We attempted to do so for the simplest $f(R)$ theory $f(R) = R + \alpha R^2$. However we encountered difficulty in first solving the $f(R)$ generalization of the background TOV equations (5.4.2) and (5.4.3) which are required for the solution to the perturbations. The only modification that $f(R)$ theory makes to the background equations is the presence of the potential V. We found (as later discussed in chapter 7) that the potential implied that the field was massive. In flat spacetime, a massive scalar field $\tilde{\chi}$ typically decays exponentially rather than as polynomials. This made constructing a series solution considerably more challenging, and as of the time of writing, we were not successful in obtaining the perturbative solution. As a result, more work must be done here to determine if a perturbative solution for the background quantities is possible.

If such a way for extracting the Love number λ_ℓ^T in the Einstein frame was found, then our equation (6.2.4) would imply that we had also computed the Jordan frame Love number. This is the same as what is claimed in [5], which we have attempted to put on more rigorous grounds through the EFT.

Particular case $f(\phi, R) = \phi(R + \alpha R^2)$

As discussed previously, the EFT equations of motion (6.2.15) remain too generic to be solved without first specifying a coupling function $f(\phi, R)$ such that the potential V can be properly defined. As the two simplest possible cases for study - the scalar tensor case $F(\phi)R$ and the $f(R)$ - have already been examined, the next simplest possible case that we can examine involving both theories is

$$f(\phi, R) = \phi(R + \alpha R^2). \tag{6.2.11}$$

We have already worked out how the potential V looks for this choice of coupling, by taking (6.1.44) and setting $F(\phi) = \phi$ as well as inserting the normalization constant $K_{\tilde{\chi}}$. The result is

$$V = \frac{1}{4\alpha\phi}\left(e^{\tilde{\chi}\sqrt{2K_{\tilde{\chi}}}/\sqrt{3}} - \phi\right)^2. \tag{6.2.12}$$

To substitute this form for V back into the equations of motion (6.2.15), it is necessary to first work out

$$\frac{\partial V}{\partial \phi} = \frac{1}{4\alpha}\left(1 - \frac{1}{\phi^2\Delta^2(\tilde{\chi})}\right), \tag{6.2.13}$$

$$\left[\frac{\partial}{\partial\tilde{\chi}} - \frac{2\sqrt{2K_{\tilde{\chi}}}}{\sqrt{3}}\right] V = \frac{1}{2\alpha}\sqrt{\frac{2K_{\tilde{\chi}}}{3}}\left(\frac{1}{\Delta(\tilde{\chi})} - \phi\right), \tag{6.2.14}$$

where we have expressed the results in terms of $\Delta(\tilde{\chi})$ from (6.2.8). We can then substitute these expressions back into the EFT equations (6.2.9) to obtain

$$\begin{aligned} \Delta(\tilde{\chi})\tilde{\Box}\phi =& \sqrt{\frac{2K_{\tilde{\chi}}}{3}}\Delta(\tilde{\chi})\tilde{\nabla}_\alpha\phi\tilde{\nabla}^\alpha\tilde{\chi} + \frac{K_R}{8\alpha K_\phi}\left(\Delta^2(\tilde{\chi}) - \frac{1}{\phi^2}\right) \\ & - \frac{\tilde{m}'(\phi)}{2K_\phi}\delta(x) + \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l!2K_\phi}\left[\tilde{\lambda}_l^S\tilde{E}^L\partial_L + \tilde{\lambda}_\ell^{ST}\tilde{E}^L\partial_L\right]\delta(x), \end{aligned} \tag{6.2.15a}$$

$$\tilde{\Box}\tilde{\chi} = -\frac{1}{\sqrt{6}}\frac{K_\phi}{\sqrt{K_{\tilde{\chi}}}}\Delta(\tilde{\chi})\tilde{\nabla}_\alpha\phi\tilde{\nabla}^\alpha\phi + \frac{K_R}{4\alpha}\sqrt{\frac{2}{3K_{\tilde{\chi}}}}\Delta(\tilde{\chi})\left(1 - \phi\Delta(\tilde{\chi})\right). \tag{6.2.15b}$$

We have set $\Omega(\phi) = 1$, such that the above equations represent the simplest possible EFT equations in $f(\phi, R)$ gravity. However we can see these equations are not trivial to solve even for the simplest case. One could attempt to transform to Fourier space as was done in chapter 5, however the resulting algebraic equations for ϕ and $\tilde{\chi}$ still remain highly coupled. As well as this, the equations still contain cumbersome exponential terms in the function $\Delta(\tilde{\chi})$, which are still present in some aspect regardless of the form of the potential V. Thus isolating ϕ and $\tilde{\chi}$ in Fourier space is not a straightforward, if even possible, task.

More analysis of the problem is required, as these equations will only become more cumbersome with increasingly complicated choices of the coupling $f(\phi, R)$. One approach to consider in future work is to attempt to linearize the equations in $\tilde{\chi}$. As $\tilde{\chi}$ is a massive scalar field (discussed in the following chapter), it decays exponentially to zero at infinity (a more precise definition of how it behaves at infinity will also be discussed in the superseding chapter). The ϕ scalar field at infinity however is $\sim \mathcal{O}(1)$, as the coupling (6.2.11) should begin to resemble GR far enough away from the system. As a result $\tilde{\chi} \ll \phi$, in our EFT approach meaning an expansion linear in $\tilde{\chi}$ may be a valid approach for solving these equations. However, this remains a task for future work.

We have seen how the formalism for extracting Love numbers in the Einstein frame in $F(\phi)R$ gravity does not straightforwardly extend to generalized scalar-tensor theories $f(\phi, R)$, even for the simplest coupling type. Nevertheless, the transformation to the Einstein frame outlined in this chapter is still a useful result and can be used to tremendously simplify the task of numerically solving the Einstein equations in $f(\phi, R)$ gravity. As a result, we attempt to gain a foothold for future work in $f(\phi, R)$ gravity with a numerical calculation of the mass-radius curves, similarly to how was done in previous chapters.

Chapter 7

Case study - f(R) and f(ϕ,R)

In chapter 6, we attempted to generalize the formalism developed in [6] for studying tidal effects of neutron stars in $F(\phi)R$ to a more general case $f(\phi, R)$. We found that this was challenging, even for the simplest coupling of type $f(\phi, R) = \phi(R + \alpha R^2)$. Despite the lack of a concrete way for extracting the Love number in $f(\phi, R)$ gravity, the transformation to the Einstein frame that we documented still allows one to remove the non-linearities of the Ricci scalar R from the action, leaving equations of motion that are still considerably easier to solve than in the Jordan frame. This provides a good motivation for studying the background spacetime of $f(\phi, R)$ gravity and the resulting mass-radius curves that emerge. As such, obtaining these mass-radius curves shall be the goal of this chapter. To the author's knowledge, no mass-radius curves have been presented for $f(\phi, R)$ gravity before, and so unfortunately there is no literature to fall back on for comparing results.

Before probing into the $f(\phi, R)$ background spacetime, recall first that transforming the $f(\phi, R)$ action to the Einstein frame results in *two* scalar fields, each needing to satisfy their own boundary conditions at infinity (we will soon elaborate on these boundary conditions). To get practise solving such a system numerically, we therefore first put our attention into numerically solving the $f(R)$ background spacetime. This has been done several times, for example in [42, 5, 43, 44, 45]. When working on this section, we initially followed the structure of [5] as a comparison for our equations of motion. We will adopt some similar conventions to that paper. The most notable being that we will denote our Einstein frame scalar field $\tilde{\chi}$ instead as φ. It should be stressed that this φ is not the same as the φ defined by (4.2.49). When referring to the φ scalar field of chapter 5, we will from here on denote it as φ^R. The second convention that we follow from [5] is that we define our conformal rescaling of the metric in terms of the function $A(\varphi)$ such that

$$g_{\mu\nu} = A^2(\varphi)\tilde{g_{\mu\nu}}. \tag{7.0.1}$$

Lastly there is one convention that we differ from [5] in. We instead define the function $\alpha(\varphi)$ as

$$\alpha(\varphi) = -\frac{1}{A}\frac{dA}{d\varphi}, \tag{7.0.2}$$

in accordance with typical convention in the literature. This is in contrast to [5], which defines $\alpha(\varphi)$ without the minus sign. The only consequence here is that our equations of motion will differ from [5] up to a sign flip of $\alpha(\varphi)$.

7.1 Background configuration - f(R)

7.1.1 The action

We begin with the simplest possible $f(R)$ action: one that adds a higher order term of the Ricci scalar R

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R + \alpha R^2\right] + S_{\text{matter}}(g_{\mu\nu}, \psi_m), \tag{7.1.1}$$

where α parameterizes the strength of the quadratic correction to R. Setting $\alpha \to 0$ recovers the GR limit of the theory. As demonstrated in chapter 6, the Einstein frame form of this action looks like

$$S = \frac{1}{16\pi G}\int d^4x \left[\tilde{R} - 2\tilde{g}_{\mu\nu}\partial_\mu\varphi\partial^\mu\varphi - \tilde{V}\right] + S_{\text{matter}}(A^2(\varphi)\tilde{g}_{\mu\nu}, \psi_m). \tag{7.1.2}$$

We can determine the form of the potential $\tilde{V}$ (from here on omitting the tilde) by setting $\phi = 1$ in our previously derived expression (6.1.48) to obtain

$$V(\varphi) = \frac{1}{4\alpha}\left(1 - e^{-\frac{2\varphi}{\sqrt{3}}}\right)^2. \tag{7.1.3}$$

The conformal factor $A(\varphi)$ can be worked out by first determining Ω and then using $A(\varphi) = \Omega^{-1}$. We can determine Ω by using both equations (6.1.32) and (6.1.33) to find

$$\begin{aligned} \Omega^2 &= f'(\Phi), \\ &= e^{\frac{2\varphi}{\sqrt{3}}}, \end{aligned} \tag{7.1.4}$$

which then implies that

$$A(\varphi) = e^{-\frac{\varphi}{\sqrt{3}}}. \tag{7.1.5}$$

Lastly, we recall that φ was originally defined in terms of the auxiliary scalar field Φ. This scalar field obeyed the equation of motion $\Phi = R$, where R is the Jordan frame Ricci scalar. Relating Φ with φ, again through (6.1.32), implies that φ should obey the equation of motion

$$\varphi = \frac{\sqrt{3}}{2}\ln\left(1 + 2\alpha R\right). \tag{7.1.6}$$

This identity will be useful for determining how φ should behave at infinity.

7.1.2 Comments on boundary conditions and the potential

With the $f(R)$ action fully determined in the Einstein frame, we now take a moment to make some comments on the boundary conditions of the field φ and illuminate the role that the potential $V(\varphi)$ plays in the theory. As demonstrated in (7.1.6), the auxiliary field φ obeys an equation of motion that depends on the Ricci scalar. In asymptotic regions of spacetime, far away from any sources of gravity, we demand that

$$\lim_{r\to\infty} R \to 0. \tag{7.1.7}$$

Applying this to (7.1.6), we can thus deduce that

$$\begin{aligned} \lim_{r\to\infty} \varphi &= \frac{\sqrt{3}}{2}\ln(1), \\ &= 0. \end{aligned} \tag{7.1.8}$$

Therefore, the field φ should vanish at spatial infinity. This will be later accounted for with a shooting method when solving for the spacetime numerically.

Secondly, we expect from the action (7.1.1), that taking $\alpha \to 0$ should reproduce the GR limit of the theory. The constant α, however, only appears in the Einstein frame as a reciprocal in the potential $V(\varphi)$. To shed some light on how α should recover the GR limit in the Einstein frame, we can Taylor expand the potential $V(\varphi)$ around $\varphi = 0$ to get

$$V(\varphi) = \frac{\varphi^2}{3\alpha} - \frac{2\varphi^3}{3\sqrt{3}\alpha} + \frac{7\varphi^4}{27\alpha} - \frac{2\varphi^5}{9\sqrt{3}\alpha} + \mathcal{O}(\varphi^6). \tag{7.1.9}$$

Due to the vanishing of $V(\varphi)$ and $V'(\varphi)$ at $\varphi = 0$, the Taylor series begins at order φ^2. By equating the φ^2 term to a mass term

$$2m_\varphi^2\varphi^2 \equiv \frac{\varphi^2}{3\alpha}, \tag{7.1.10}$$

we can deduce that the φ field is massive with an α dependent mass

$$m_\varphi = \frac{1}{\sqrt{6\alpha}}. \tag{7.1.11}$$

Note that the prefactor of 2 in the left-hand side of (7.1.10) is to reconcile with the prefactor of 2 in the kinetic term for φ in the action (7.1.2). We can see from (7.1.11) sending $\alpha \to 0$ should produce a scalar-field with a mass approaching infinity. We can make a qualitative argument about how the mass should influence the behavior of a scalar field by looking at the massive Klein-Gordon equation for a free scalar field φ in flat spacetime

$$\varphi''(r) + \frac{2}{r}\varphi'(r) - m_\varphi^2\varphi(r) = 0. \tag{7.1.12}$$

This equation can be solved straightforwardly with Mathematica. The decaying part of the solution for $\varphi(r)$ behaves as

$$\varphi(r) \sim \frac{e^{-m_\varphi r}}{r}. \tag{7.1.13}$$

We can thus see that sending $m_\varphi \to \infty$ exponentially suppresses the scalar field, essentially killing it. Thus in our Einstein frame $f(R)$ theory, we can similarly expect that taking the limit $\alpha \to 0$ will exponentially send the scalar field to zero, leaving only GR behind.

7.1.3 The equations of motion

Proceeding with the computation, we are now ready to obtain our equations of motion. Recall that in the Einstein frame, we model the background spacetime with the metric for a spherically symmetric, non-rotating spacetime

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\gamma(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{7.1.14}$$

along with an energy-momentum tensor (4.2.64) for a perfect fluid

$$\tilde{T}_{\mu\nu} = A^4(\varphi)\left(\rho + p\right)\tilde{u}_\mu\tilde{u}_\nu + A^4(\varphi)\tilde{g}_{\mu\nu}. \tag{7.1.15}$$

We can then vary the action with respect to $\tilde{g}_{\mu\nu}$ and φ, and use the conservation equation (4.2.25) to obtain

$$\frac{1}{r^2}\frac{d}{dr}\left[r(1-e^{-\gamma})\right] = 8\pi G A^4(\varphi)\rho + e^{-\gamma}\left(\frac{d\varphi}{dr}\right)^2 + \frac{1}{2}V(\varphi), \tag{7.1.16a}$$

$$\frac{1}{r}e^{-\gamma}\frac{d\nu}{dr} - \frac{1}{r^2}(1-e^{-\gamma}) = 8\pi G A^4(\varphi)p + e^{-\gamma}\left(\frac{d\varphi}{dr}\right)^2 - \frac{1}{2}V(\varphi), \tag{7.1.16b}$$

$$\frac{d^2\varphi}{dr^2} + \left(\frac{1}{2}\frac{d\nu}{dr} - \frac{1}{2}\frac{d\gamma}{dr} + \frac{2}{r}\right)\frac{d\varphi}{dr} = -4\pi G\alpha(\varphi)A^4(\varphi)(\rho - 3p)e^{\gamma} + \frac{1}{4}\frac{dV(\varphi)}{d\varphi}e^{\gamma}, \tag{7.1.16c}$$

$$\frac{dp}{dr} = -\frac{\rho + p}{2}\left(\frac{d\nu}{dr} - 2\alpha(\varphi)\frac{d\varphi}{dr}\right). \tag{7.1.16d}$$

Here the first two equations pertain to the tt and rr components of the Einstein equations respectively. The third equation is the equation of motion for the scalar field φ, and the fourth is the energy conservation equation (4.2.25). We express our equations of motion here in terms of the metric function $\gamma(r)$ rather than the mass function $m(r)$, for better comparison with [5]. As previously mentioned, our equations agree up to a sign flip of $\alpha(\varphi)$. One should also send $\gamma \to 2\Lambda$ and $\nu \to 2\Psi$ for an exact comparison, where a different convention for the metric functions has been employed in [5]. We choose however to continue with the convention of [6] as used in previous chapters.

When solving the equations numerically we will reinsert $m(r)$ into (7.1.16) through

$$e^{-\gamma(r)} = 1 - \frac{2m(r)}{r}. \tag{7.1.17}$$

To recover the original TOV equations from chapter 3, one should set $\varphi = 0$, implying that

$$A(\varphi) = 1, \quad V(\varphi) = \alpha(\varphi) = 0. \tag{7.1.18}$$

7.1.4 Discussion of numerical implementation and challenges

Here we discuss some of the subtleties with numerically implementing the shooting method for the scalar field. If one is not interested in numerically reproducing the calculation, they can proceed to the results in subsection 7.1.5.

Initial conditions

With the Einstein frame equations of motion obtained, we can now turn to numerically solving them. Although we have four coupled differential equations (7.1.16), we can reduce this to three by using (7.1.16b) to isolate an expression for $\frac{d\nu}{dr}$, which can be then substituted into (7.1.16d). The result is two first-order differential equations for the mass, $m(r)$ and pressure, $p(r)$, as well as a second-order differential equation for the scalar field, $\varphi(r)$. Overall this constitutes four initial conditions required to solve the system. Similarly to what was seen in chapter 3 and chapter 4, we specify the initial conditions for $m(r)$ and $p(r)$ by first choosing a central density, ρ_c for the neutron star and then using the initial conditions (3.2.20).

We are still left with two initial conditions to specify, which then come from the scalar field. These initial conditions are

$$\varphi(r \to 0) = \varphi_c, \quad \varphi'(r \to 0) = 0. \tag{7.1.19}$$

The second condition ensures the regularity of the solution close to the origin. We are free to choose the initial value, φ_c for the scalar field, and will use it to ensure that the boundary condition (7.1.8) is satisfied. This will mean employing a shooting method in the integrator, which carefully varies the value of φ_c in order to obtain the correct value of φ at infinity.

Divergences for large r

The code used to solve our system of coupled equations is again based on code written by G. Creci and F. Visser for the purposes of [6]. Originally, this code only required an integration of the interior of the neutron star up to the star's surface. This was because an analytical formula for the scalar field at infinity, φ_∞, could be obtained in terms of the neutron star configuration at the surface. This was done by expressing the equations of motion in the so-called "Just" coordinate system (see Appendix B of [6] or reference [46]). We attempted to generalize this formula for the $f(R)$ case, but found this was not possible due to the presence of the potential $V(\varphi)$ in the equations of motion. As a result, the original code has been modified to integrate up to the neutron star surface, and then continue integrating from the surface to a suitable cutoff point, denoted r_{cut}, which we use as a numerical representation of infinity.

We tested the accuracy of this method for the massless $F(\phi)R$ case, and found that $r_{\text{cut}} = 1000$ (in geometrical solar units $G = M_\odot = c = 1$) gave excellent agreement with the analytical formula from the Just coordinate system. We found however that implementing such a cutoff for the $f(R)$ case was more numerically challenging. The $f(R)$ scalar field was found to be extremely sensitive to its central value φ_c. More so, we repeatedly encountered that the scalar field φ would diverge to $\pm\infty$ within a range of two to three times its Compton wavelength

$$\lambda_\varphi = \frac{2\pi}{m_\varphi}, \tag{7.1.20}$$

with m_φ defined the same as in (7.1.10). We found that increasing the numerical precision of the integrator and of the initial guesses for φ_c was not sufficient to suppress these divergences. As a result, we decided to settle on a cutoff of $r \sim \lambda_\varphi$ for the scalar field. We found this to be suitable as pushing the cutoff further, to about $2.5\lambda_\varphi$ lead to an indistinguishable profile for the scalar field, which was already exponentially decaying well before λ_φ. Similar methods were also employed in [43, 5].

The emergence of this diverging solution for the scalar field can be qualitatively explained, by again considering the solution of the massive Klein-Gordon equation (7.1.12) in flat spacetime. The full solution, again obtainable with for example Mathematica, is given by

$$\varphi(r) = c_1 \frac{e^{-m_\varphi r}}{r} + c_2 \frac{e^{m_\varphi r}}{2m_\varphi r}, \tag{7.1.21}$$

where c_1 and c_2 are integration constants. The solution (7.1.21) consists of an exponentially decaying and growing solution. Generalizing to our case, we did not believe it possible, numerically speaking, to completely suppress this exponentially growing part of the solution i.e. set $c_2 = 0$. Thus around $r \sim \lambda_\varphi$, the decaying solution dies out and the growing solution starts to take over, sending the field φ to $\pm\infty$ within a few factors of λ_φ. As a result, we believe that implementing a cut off for r was necessary to obtain a divergent free solution and accurately model the spacetime of the neutron star.

Sensitivity to initial conditions

As discussed previously, the neutron star solution is obtained by first specifying a central density ρ_c, guessing a φ_c, and then integrating from the center of the star outward. When the correct value for φ_c satisfying the boundary conditions is obtained, the central density ρ_c is then slightly incremented and the process starts again. This is continued for as many central densities as one desires. We can denote the collection of these central densities used in the calculation as ρ_c^i, with $\rho_c^{i=1}$ being the first central density used in the calculation.

As well as the likelihood for the scalar field to diverge at large r, we also found that these divergences were extremely sensitive to the central values of the scalar field, denoted φ_c^i. The code initially used the same guess for every φ_c^i. This worked well at first for small increments in ρ_c, but as ρ_c^i was repeatedly incremented, the initial guess for φ_c^i began to deviate too much from the correct value. This resulted in the solution rapidly diverging as soon as new central density ρ_c was selected and the first shooting integration performed.

In order for Mathematica to accurately adjust its initial guesses for the φ_c^i, it requires that the solution does not diverge so that it can extract the value of the field φ at the boundary point ($r = r_{\text{cut}}$). This meant that Mathematica could not continue with the calculation if a single divergence was encountered for any of the φ_c^i. To compensate for this, we fed the code an initial guess for $\varphi_c^{i=1}$ that was extremely close to the correct $\varphi_c^{i=1}$. This initial guess, which we denote $\varphi_c^{i=1,\text{guess}}$ was obtained through trial and error.

Once the correct value for $\varphi_c^{i=1}$ is obtained such that the boundary condition (7.1.8) is satisfied, the code then moves onto $\rho_c^{i=2}$ and begins the shooting method again to determine $\varphi_c^{i=2}$. The code uses $\varphi_c^{i=1}$ as the initial guess $\varphi_c^{i=2,\text{guess}}$, and $\varphi_c^{i=2}$ as the initial guess $\varphi_c^{i=3,\text{guess}}$. Once the first three φ_c^i are determined, the code then uses an extrapolation method to determine a suitably accurate value for $\varphi_c^{i=4}$ and so on.

For example, consider that the correct values for $\varphi_c^{i=1}$, $\varphi_c^{i=2}$, and $\varphi_c^{i=3}$ have been determined (to an acceptable precision) via the shooting method. The initial guess for $\varphi_c^{i=4,\text{guess}}$ is then determined by using $\varphi_c^{i=3}$ plus the difference between $\varphi_c^{i=2}$ and $\varphi_c^{i=3}$, times the ratio of the increment from $\varphi_c^{i=1}$ and $\varphi_c^{i=2}$, to $\varphi_c^{i=2}$ and $\varphi_c^{i=3}$. Mathematically we can express this initial guess for φ_4^c as

	Initial guesses $\varphi_c^{i=1,\text{guess}}$		
α	SLy	H4	APR4
0.5	0.011815	0.021283	0.011094
5	0.0088185	0.014598	0.0082987
50	0.0039040	0.0050307	0.0037299

Table 7.1: Initial guesses for the central value of the scalar field $\varphi_c^{i=1}$, corresponding to $\rho_c^{i=1} = 10^{18.3}\text{kg/m}^3$. Each guess should be sufficient to compute the entire neutron star configurations starting from $\rho_c^{i=1}$.

$$\begin{aligned} \varphi_c^{4,\text{guess}} &= \varphi_c^3 + \left(\varphi_c^3 - \varphi_c^2\right) \times \frac{\varphi_c^3 - \varphi_c^2}{\varphi_c^2 - \varphi_1^c}, \\ &= \varphi_c^3 + \frac{\left(\varphi_c^3 - \varphi_c^2\right)^2}{\varphi_c^2 - \varphi_c^1}. \end{aligned} \tag{7.1.22}$$

The formula can be generalized into a generic guess for any φ_c^i as

$$\varphi_c^{i,\text{guess}} = \varphi_c^{i-1} + \frac{\left(\varphi_c^{i-1} - \varphi_c^{i-2}\right)^2}{\varphi_c^{i-2} - \varphi_c^{i-3}}. \tag{7.1.23}$$

This formula is valid as long as the previous three φ_c^i are known. More sophisticated extrapolation methods are likely possible, but we found this improvised method worked suitably well for any equation of state and value of α in the parameter space, so long as the initial guess $\varphi_c^{i=1,\text{guess}}$ was suitably close to the actual value $\varphi_c^{i=1}$. We include a record of the initial guesses used for the computations in this chapter, found in Table 7.1, for a central density of $\rho_c^{i=1} = 10^{18.3}\text{kg/m}^3$. These initial guesses, combined with our code are the only inputs needed to produce the results of this section. Lastly, we reuse $r_{\text{min}} \sim 10^{-10}$ as the starting radius for the integration as in chapter 3, and used a numerical representation of φ_∞ be $\varphi_\infty = 10^{-10}$. It was found that further decreasing the order of magnitude of φ_∞ made no noticeable difference on results.

7.1.5 Numerical results

Scalar field profiles

With the necessary discussion for numerical implementations out of the way, we can now turn our attention to the results. We numerically solved the modified TOV equations (7.1.16) for three of the tabulated piecewise EoS discussed previously: SLy, H4, and APR4. As in the GR case, the radius of the neutron star R is determined by the point of vanishing pressure i.e. $p(R) = 0$.

It is interesting to discuss the profile of the central values φ_c^i that emerge from the shooting method. We present a plot of the central scalar field values φ_c^i in Figure 7.1 for the SLy equation of state, with three different values of α. For larger central densities, we see that the initial scalar field configuration becomes negative. From equation (7.1.6), we can interpret that a negative scalar field φ corresponds to a negative Jordan frame Ricci scalar i.e.

$$\varphi < 0 \implies R < 0. \tag{7.1.24}$$

Thus the negative central values for the scalar field correspond to negative central values for the Jordan frame Ricci scalar. Furthermore, we can compare the φ_c^i profiles with the profile of the neutron star masses, as seen in Figure 7.2. Interestingly we find that the Jordan frame Ricci scalar at the centre of the star becomes negative very close to maximum mass configuration of the neutron star. Thus the sign of the central Ricci scalar value could potentially highlight a way to determine if a neutron star configuration is unstable, although further analysis and consultation with the literature is needed. Although only shown in Figures 7.1 and 7.2 for the SLy EoS and three values of α, we found similar

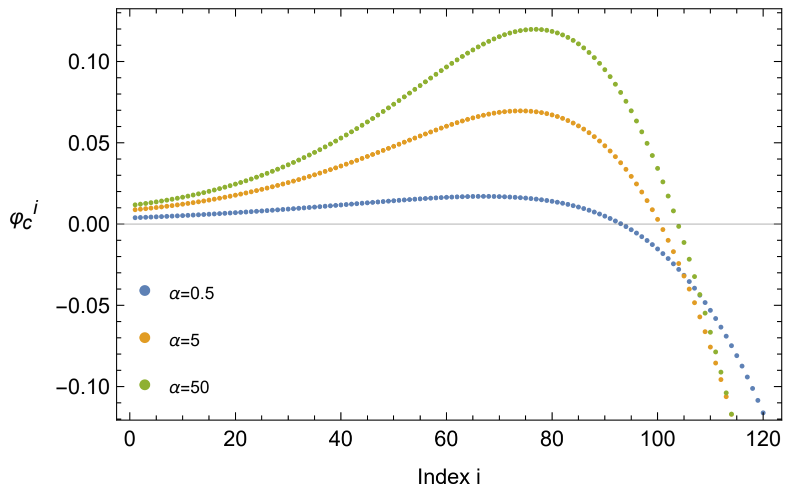

Figure 7.1: Central scalar values φ_c^i for the SLy EoS. These correspond to a central density $10^{17.3}\text{kg/m}^3 \leq \rho_c \leq 10^{18.5}\text{kg/m}^3$, broken into 121 discrete values.

behavior in all of the other studied equations of state, and values of α.

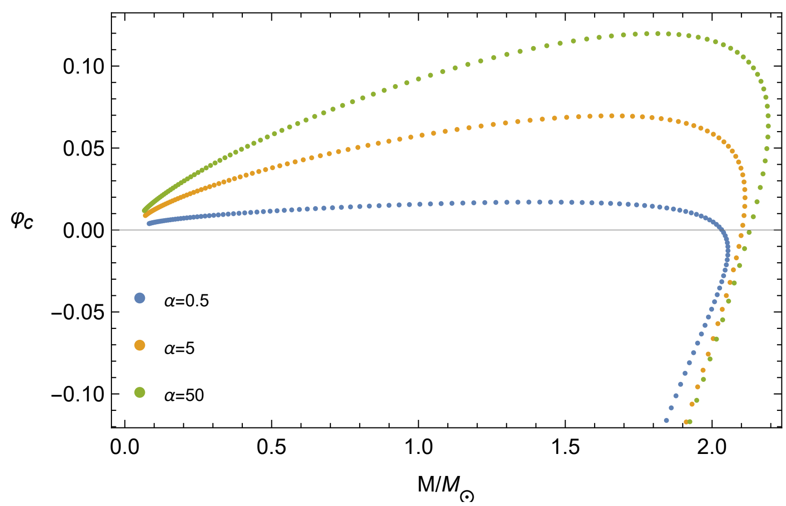

Figure 7.2: The same central values φ_c^i from Figure 7.1 this time plotted as a function of the neutron star mass for the SLy EoS. The central density ranges from $10^{17.3}\text{kg/m}^3 \leq \rho_c \leq 10^{18.5}\text{kg/m}^3$

It is also interesting to note that despite the central value of the scalar field, φ_c, becoming negative for larger central densities ρ_c, this is not necessarily true further away from the star's centre. An example is demonstrated in Figure 7.3, which shows how the scalar field profile changes in response to an increase in ρ_c. The example in Figure 7.3 is for the APR4 equation of state with $\alpha = 5$, but we found this behavior typical for all of the equations of state and values of α that were studied.

Mass-radius curves

Now that we have inspected the qualitative behavior of the background spacetime, in particular, the behavior of the scalar field φ, we can use the results to plot mass-radius curves. Similarly to $F(\phi)R$ gravity however, the mass $m(r)$ no longer becomes constant at the surface. This can be seen by looking at (7.1.16) and setting $\rho = p = 0$ while eliminating $\gamma(r)$ through (7.1.16a). This means the mass $m(r)$ is also integrated in the vacuum region. The mass of the neutron star is determined by evaluating the

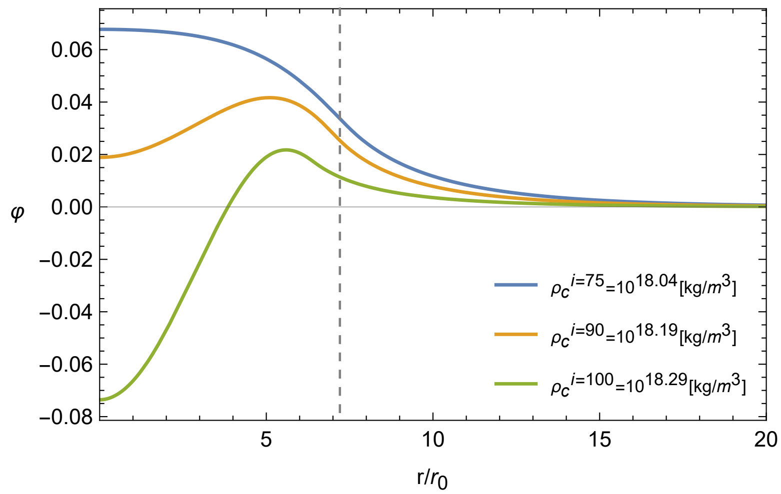

Figure 7.3: Scalar field configurations for the APR4 equation of state with $\alpha = 5$. Here the distance is plotted in geometrical solar units such that $r_0 = \frac{GM_\odot}{c^2} \simeq 1.5\text{km}$. The grey dotted line indicates the average radius of the neutron star for the three different configurations. We can see that the scalar field decays to zero very close to the surface of the star. The cutoff for the integration was $r/r_0 = \lambda_\varphi \simeq 34$.

mass function $m(r)$ sufficiently far away from the star, where the scalar field has essentially decayed to zero. We previously denoted this distance r_{cut}, implying that we define $m(r_{\text{cut}}) = M$.

A plot of the mass radius curves in the Einstein and Jordan frame are compared in Figure 7.4. We relate the radius of the Einstein frame neutron star r_S to the Jordan frame one R_S through $R_S = A\left[\varphi(r_S)\right] r_S$, in accordance with [5]. Interestingly, we found that the formula for relating the masses between frames in [6] led to Jordan frame curves with masses far smaller than the GR curves. This is likely due to the relation of the masses in both frames depending on $\alpha(\varphi)$, which is a constant $\alpha(\varphi) = 1/\sqrt{3}$ in the $f(R)$ case. This is not true for the studied $F(\phi)R$ case, in which $\alpha(\varphi^R)$ has a scalar dependence and is on the order of 10^{-3}, where it would not make any notable difference to the masses. Due to the plots not displaying the qualitatively correct behavior in the Jordan frame, we opted to continue with the formula from [5] until the discrepancy is better understood.

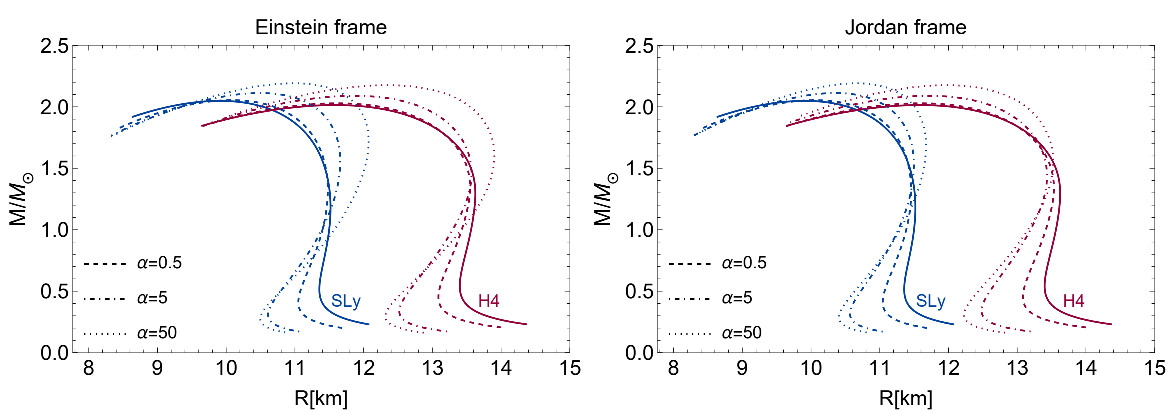

Figure 7.4: Mass-radius curves in the Einstein and Jordan frame for the SLy and H4 equations of state. The solid lines represent the GR mass-radius curves from Figure 3.2. We omit the APR4 mass-radius curves for better visibility, although they demonstrated the same qualitative behavior. The central densities here range from $10^{17.3}\text{kg/m}^3 \leq \rho_c \leq 10^{18.5}\text{kg/m}^3$.

Comparing the mass radius curves to the previous case of $F(\phi)R$, we see that the modifications to GR this time have a less subtle effect on the mass-radius curves, as the $F(\phi)R$ only deviated from

GR along certain regions. In contrast, we find in $f(R)$ that there is essentially no overlaps with the GR curves, although it is reassuring to see that the GR limit appears to be recovered as $\alpha \to 0$. The deviations from GR also appear more severe in the Einstein frame, for larger masses. We can see from either frame that a larger α appears to allow for a larger maximum mass of the neutron star. We can conclude that if there are indeed higher order corrections to the Ricci scalar needed in the Einstein-Hilbert action, the effects should have some observable consequences for the allowed masses and radii of a neutron star.

7.2 Background configuration - f(ϕ,R)

Now that we have successfully implemented code to numerically solve the $f(R)$ background spacetime, we have been able to eludicated some aspects that should be approached with caution in the $f(\phi, R)$ case - namely the implementation of the shooting method. This section takes a similar structure to the preceding one as we attempt to generalize the previous work to the background $f(\phi, R)$ spacetime.

7.2.1 The action

We begin as before with the Jordan frame action for the generalized scalar-tensor theory

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[f(\phi, R) - 2\frac{\omega(\phi)}{\phi} \partial_\mu \phi \partial^\mu \phi \right] + S_{\text{matter}} \left(\psi_m, g_{\mu\nu} \right), \tag{7.2.1}$$

where $f(\phi, R) = \phi \left(R + \alpha R \right)^2$ represents the simplest possible coupling of an $f(R)$ theory to a scalar field ϕ. The function $\omega(\phi)$ will be later chosen to coincide with the Jordan frame action of the scalar-tensor theory studied in chapter 5 and in [6]. As demonstrated in chapter 6, the Einstein frame counterpart to this action is

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - 2\tilde{\partial}_\mu \varphi \tilde{\partial}^\mu \varphi - 2\Delta(\varphi)\Omega(\phi)\tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi - V(\phi, \varphi) \right] + S_{\text{matter}} \left(\psi_m, A(\varphi)^2 \tilde{g}_{\mu\nu} \right), \tag{7.2.2}$$

where again φ denotes the auxiliary scalar field that linearizes the Ricci scalar. We also again abbreviate $\Omega(\phi) \equiv \omega(\phi)/\phi$ (separate from the conformal factor Ω, which will not appear again in this chapter). The scalar field φ should not be confused with the φ^R defined in chapter 5. Here the function $\Delta(\varphi)$ is still given by

$$\Delta(\varphi) = e^{-2\varphi/\sqrt{3}}, \tag{7.2.3}$$

and the potential $V(\phi, \varphi)$ has been determined in (6.1.48) as

$$V(\varphi, \phi) = \frac{1}{4\alpha\phi} \left(1 - \phi e^{-2\varphi/\sqrt{3}} \right)^2. \tag{7.2.4}$$

Lastly, the conformal rescaling function $A(\varphi)$ can be determined from equations (6.1.15) and (6.1.17) to give

$$A(\varphi) = e^{-\frac{\varphi}{\sqrt{3}}}. \tag{7.2.5}$$

Although this coupling may at first appear similar to the $f(R)$ case, we can demonstrate that the field φ in $f(\phi, R)$ theory is not equivalent to one studied in $f(R)$. Using equations (6.1.11) and (6.1.17), we can express φ in terms of the Jordan frame Ricci scalar R *and* the scalar field ϕ. The result is

$$\varphi = \frac{\sqrt{3}}{2} \ln \left[\phi \left(1 + 2\alpha R \right) \right]. \tag{7.2.6}$$

Thus, we see that the Einstein frame scalar field φ is not necessarily equivalent to the previous one from $f(R)$ due to its dependence on ϕ. We will now see the implications that this ϕ dependence has on the boundary conditions for φ.

7.2.2 Comments on boundary conditions and potential

We can determine how the scalar field φ should behave at infinity by again sending R to zero in (7.2.6), and by replacing ϕ with its value at infinity, ϕ_∞. Thus we have that

$$\lim_{r\to\infty}\varphi = \frac{\sqrt{3}}{2}\ln(\phi_\infty). \tag{7.2.7}$$

We can see that φ only vanishes at infinity for $\phi_\infty = 1$. We would like to choose the value for ϕ_∞ in our case study that coincides with the Einstein frame scalar field φ^R at infinity as studied in chapter 5 and [6]. Chapter 5 specified the action in the Einstein frame through $A(\varphi) = e^{\frac{1}{2}\beta\varphi_R^2}$ with $F(\phi) = \phi$. We can relate ϕ and φ^R through equation (4.2.54) as

$$\phi = e^{-\beta\varphi_R^2}. \tag{7.2.8}$$

Putting this into (7.2.7), and using $\varphi_\infty^R = 10^{-3}$ as in [6], we find that φ should behave at infinity as

$$\lim_{r\to\infty}\varphi = -\frac{\beta\sqrt{3}}{2}\times 10^{-6}, \tag{7.2.9}$$

which in contrast to $f(R)$, is non-zero. Similarly from (7.2.8) we can write that the field ϕ should behave at infinity as

$$\begin{aligned}\phi_\infty &= e^{-\beta\times 10^{-6}},\\ &= 1-\beta\times 10^{-6} + \mathcal{O}\left(\beta\times 10^{-12}\right).\end{aligned} \tag{7.2.10}$$

We now turn our attention to the form of the potential $V(\phi,\varphi)$. We can Taylor expand (7.2.4) around $\varphi = 0$ to see that it still yields a mass term for the φ field

$$V(\varphi,\phi) = \frac{(\phi-1)^2}{4\phi\alpha} - \frac{(\phi-1)}{\sqrt{3\alpha}}\varphi + \frac{(2\phi-1)}{3\alpha}\varphi^2 + \frac{2(1-4\phi)}{9\sqrt{3}\alpha}\varphi^3 + \mathcal{O}(\varphi^4). \tag{7.2.11}$$

The mass however now contains a ϕ dependence. As the scalar field ϕ is expected to be $\mathcal{O}(1)$ based on its boundary conditions, we can still approximate the mass of the φ term as

$$m_\varphi \sim \frac{1}{\sqrt{6\alpha}}. \tag{7.2.12}$$

We also note the lack of any ϕ^2 terms in the potential, which can be made more explicit by expanding $V(\phi,\varphi)$ as

$$\frac{1}{4\alpha\phi} - \frac{e^{-\frac{2\varphi}{\sqrt{3}}}}{2\alpha} + \frac{e^{-\frac{4\varphi}{\sqrt{3}}}\phi}{4\alpha}. \tag{7.2.13}$$

However we do notice the presence of a term linear in ϕ as well as a term running as $1/\phi^2$. Recall that we expect $\phi \sim \mathcal{O}(1)$ at infinity, such that this $1/\phi^2$ should not contain any numerical surprises as we approach large r.

7.2.3 Determining the form of $\Omega(\phi)$

In order to completely reconcile our Jordan frame action with the one studied in chapter 5, we must determine the form of the function $\Omega(\phi)$. This function was not specified in [6] as the action was first conceived in the Einstein frame. This was done by first specifying $F(\phi)$ and $A(\varphi^R)$, which then fixed the form of $\Omega(\phi)$. We will thus have to work back from the Einstein frame action of chapter 5 to determine $\Omega(\phi)$. This can be done using (4.2.50) and (4.2.55) such that

$$
\begin{aligned}
\frac{d\varphi^R}{d\phi} &= \sqrt{\frac{\Delta(\varphi^R)}{2}}, \\
&= \frac{A(\varphi^R)^2 F(\phi)}{2\alpha(\varphi^R)}, \\
&= \sqrt{\frac{3}{4}\left(\frac{F'(\phi)}{F(\phi)}\right)^2 + \frac{\omega(\phi)}{\phi F(\phi)}}.
\end{aligned}
\tag{7.2.14}
$$

Filling in for F and A, we can then isolate $\Omega(\phi)$ in (7.2.14) to obtain

$$
\omega(\phi) = -\left(\frac{1}{4\beta \ln \phi} + \frac{3}{4}\right), \tag{7.2.15}
$$

such that

$$
\Omega(\phi) = -\left(\frac{1}{4\beta\phi \ln \phi} + \frac{3}{4\phi}\right). \tag{7.2.16}
$$

This now completely specifies our Jordan and thus Einstein frame action. The form of $\Omega(\phi)$ however appears to be highly specific. We thus performed a comparison with the results of chapter 5, by numerically solving for the background spacetime in terms of ϕ, rather than φ^R. This allowed us to verify that we had correctly determined $\Omega(\phi)$. It also came with the additional benefit of verifying the independence of our results on the definition of ϕ, as well as reaffirm our faith in the newly determined boundary condition (7.2.10). For completeness, the action used for this test was

$$
S = \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - 2\Delta(\phi)\tilde{\partial}_\mu\phi\tilde{\partial}^\mu\phi\right] + S_{\text{matter}}\left(\psi_m, A(\phi)^2\tilde{g}_{\mu\nu}\right), \tag{7.2.17}
$$

with $A(\phi) = \frac{1}{\sqrt{\phi}}$. The function $\Delta(\phi)$ is the same as in the first line of (7.2.14). Expressed in terms of ϕ, it works out to be

$$
\Delta(\phi) = -\frac{1}{4\beta\phi^2 \ln \phi}. \tag{7.2.18}
$$

With the specific form of $\Omega(\phi)$ chosen in (7.2.16), it is not possible to canonicalize the field ϕ through a field redefinition. This is due to the fact that redefining ϕ in terms of a canonical field $\tilde{\varphi}$, requires one to be able to integrate the differential equation

$$
\frac{d\tilde{\varphi}}{d\phi} = \sqrt{\Omega(\phi)}. \tag{7.2.19}
$$

This was not possible (at least with standard Mathematical software systems such as Mathematica), and so the ϕ field must be left uncanonicalized for the choice of $\Omega(\phi)$ in (7.2.16). This was our original decision for leaving the field ϕ uncanonicalized with the EFT calculations in chapter 6.

A plot for the mass radius curve generated by the action (7.2.17) can be found in Figure 7.5. We also include a profile of the central scalar field values ϕ_c^i in Figure 7.6 and a profile of several scalar field configurations in Figure 7.7.

It was found that the result for the mass-radius curves gave excellent agreement with the curves in Figure 5.1. We stress however that this is a preliminary result. The test was only conducted for the various equations of state with $\beta = -4.5$. We found that attempting to solve the spacetime for smaller β generated numerical errors in the shooting method. This was possibly due to the presence of the reciprocals containing ϕ in the action, however more analysis is needed. We proceed with a preliminary test of the $f(\phi, R)$ calculation regardless, and keep in mind that we will restrict ourselves to $\beta = -4.5$.

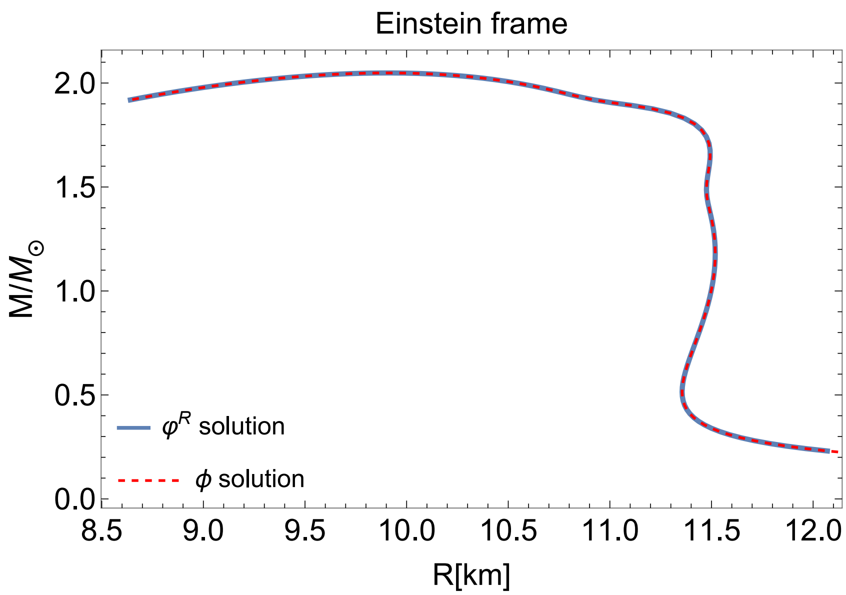

Figure 7.5: Comparison of the mass-radius curves generated by the action (7.2.17) in terms of φ^R and ϕ respectively for the SLy EoS with $\beta = -4.5$

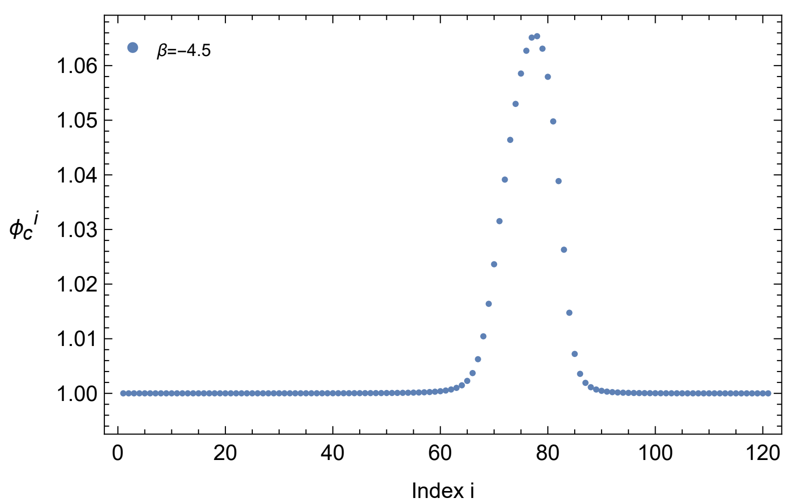

Figure 7.6: Central scalar values ϕ_c^i for the SLy EoS with $\beta = -4.5$. The spike in the central values corresponds to the "scalarized" configuration of the neutron star, where the mass-radius curves deviate from GR. These central values display very different behavior to the ones studied in Figure 7.1 for $f(R)$. The central density is again in the range $10^{17.3}\text{kg/m}^3 \leq \rho_c \leq 10^{18.5}\text{kg/m}^3$, broken into 121 discrete values.

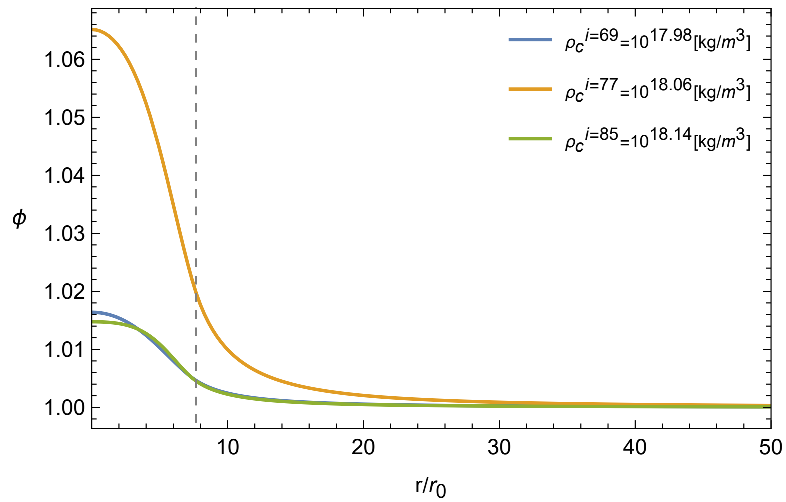

Figure 7.7: Scalar field configurations for the SLy equation of state with $\beta = -4.5$. The distance is again plotted in geometrical solar units and the grey dotted line indicates the average radius of the neutron star for the three configurations. We can see that the scalar field has a stronger presence for the higher central value, corresponding to the spike in Figure 7.6. The cutoff for the integration was $r/r_0 = 1000$.

7.2.4 Equation of motion

With the Einstein frame action fully determined, we are now free to derive the respective equations of motion. We use the same metric ansatz (7.1.14) and energy momentum tensor (7.1.15) as the $f(R)$ case. Given that in the $f(\phi, R)$ Einstein frame there are two scalar fields, we will now have five relevant equations of motion - two from the Einstein equations, two scalar field equations and one energy conservation equation. These equations work out to be

$$\frac{1}{r^2}\frac{d}{dr}\left[r\left(1-e^{-\gamma}\right)\right] = 8\pi G A^4(\varphi)\rho + e^{-\gamma}\left(\frac{d\varphi}{dr}\right)^2 + e^{-\gamma}\Delta(\varphi)\Omega(\phi)\left(\frac{d\phi}{dr}\right)^2 + \frac{1}{2}V(\phi,\varphi), \tag{7.2.20a}$$

$$\frac{e^{-\gamma}}{r}\left(\frac{d\nu}{dr}\right) - \frac{1}{r^2}\left(1-e^{-\gamma}\right) = 8\pi G A^4(\varphi)p + e^{-\gamma}\left(\frac{d\varphi}{dr}\right)^2 + e^{-\gamma}\Delta(\varphi)\Omega(\phi)\left(\frac{d\phi}{dr}\right)^2 - \frac{1}{2}V(\phi,\varphi), \tag{7.2.20b}$$

$$\frac{d^2\varphi}{dr^2} + \left(\frac{1}{2}\frac{d\nu}{dr} - \frac{1}{2}\frac{d\gamma}{dr} + \frac{2}{r}\right)\frac{d\varphi}{dr} = 4\pi G A^4(\varphi)\alpha(\varphi)(\rho - 3p)e^{\gamma} - \frac{1}{\sqrt{3}}\Delta(\varphi)\Omega(\phi)\left(\frac{d\phi}{dr}\right)^2 + \frac{1}{4}e^{\gamma}\frac{\partial V}{\partial\varphi}, \tag{7.2.20c}$$

$$\frac{d^2\phi}{dr^2} + \left(\frac{1}{2}\frac{d\nu}{dr} - \frac{1}{2}\frac{d\gamma}{dr} + \frac{2}{r} - 4\sqrt{3}r\frac{d\varphi}{dr}\right)\frac{d\phi}{dr} = -\frac{1}{2}\left(\frac{\Omega'(\phi)}{\Omega(\phi)}\right)\left(\frac{d\phi}{dr}\right)^2 + \frac{3r}{2\Delta(\varphi)\Omega(\phi)}e^{\gamma}\frac{dV}{d\phi}, \tag{7.2.20d}$$

$$\frac{dp}{dr} = -\frac{1}{2}(p+\rho)\left(\frac{d\nu}{dr} - 2\alpha(\varphi)\frac{d\varphi}{dr}\right). \tag{7.2.20e}$$

While it may appear upon initial inspection that only the φ field couples to p and ρ, one should also recall the presence of ϕ in (7.2.6). This also implies an implicit coupling of ϕ to matter, hopefully allowing for a scalarized neutron star configuration as seen in Figure 7.5. To recover the GR TOV equations from (7.2.20), one should now set $\varphi = 0$ and $\phi = 1$, implying that

$$A(\varphi) = 1, \quad V(\phi,\varphi) = \alpha(\varphi) = 0. \tag{7.2.21}$$

7.2.5 Discussion of numerical implementation and challenges

Again we discuss some of the important aspects of implementing the numerical integration. Readers interested in the results can continue straight to subsection 7.2.6

Initial conditions

As discussed previously for the $f(R)$ case, we begin our numerical integration with the same initial conditions for $m(r)$ and $p(r)$. This depends upon first choosing a central density ρ_c, which we also take in the same range as those used in Figure 7.4. The initial guesses for the central value of the scalar field $\varphi_c^{i=1,\text{guess}}$ are also reused from Table 7.1. Again, a shooting method is employed to determine the correct φ_c^i, such that the scalar field φ obeys the correct boundary condition (7.2.7) at infinity.

We also however require two more initial conditions in our equations due to the second order differential equation for ϕ. We approach this similarly to the previous cases by specifying the initial conditions for ϕ as

$$\phi(r \to 0) = \phi_c, \quad \phi'(r \to 0) = 0. \tag{7.2.22}$$

Similarly to the central values φ_c^i, the central values ϕ_c^i are determined with a shooting method, such that ϕ approaches the correct value (7.2.10) at infinity. This will require a double-shooting method for the integration, to ensure that both boundary conditions are simultaneously satisfied for the scalar fields.

Cutoff for large r

In the $f(R)$, we observed that the scalar field φ would diverge with exponential behavior for large enough r beyond the the Compton wavelength of the field λ_φ. Numerically speaking, we still found this to be true for $f(\phi, R)$. As a result, a cutoff was implemented again for the φ field at $r \simeq \lambda_\varphi$, with λ_φ defined with the same mass from the $f(R)$ case. After this, we set the field φ numerically to a constant for the remainder of the integration, as $\varphi(r) \ll \phi(r)$ and $\varphi(r) \ll m(r)$ in this region. Thus only an integration of $m(r)$ and $\phi(r)$ was necessary beyond this point. We choose again to continue the integration to the same value of spatial infinity $r_\infty \simeq 1000$ as was used in our $F(\phi)R$ test. In this sense we have three regions for the integration, characterized as

$$r_{\text{min}} \leq r \leq R, \quad R < r \leq r_{\text{cut}}, \quad r_{\text{cut}} < r \leq r_\infty. \tag{7.2.23}$$

Here R represents the radius of the star, corresponding to vanishing pressure $p(R) = 0$. We again use a numerical value of $r_{\text{min}} = 10^{-10}$ for the centre of the star.

Next, r_{cut} is the cutoff for the φ field defined as $r_{\text{cut}} = 2\pi\sqrt{6\alpha}$. The field $\varphi(r)$ is then taken to be constant beyond this range i.e. $\varphi(r > r_{\text{cut}}) = \varphi_\infty$.

Lastly, r_∞ is the numerical representation of infinity where the mass is then extracted i.e. $m(r_\infty) = M$.

Sensitivty to initial conditions

During our numerical investigation, we found that the ϕ field was also this time susceptible to divergences - although not as rapidly as the φ field. Suppressing these growing solutions for $\phi(r)$ was challenging for larger r, but again possible with an accurate enough initial value for both $\varphi_c^{i=1\text{guess}}$ and $\phi_c^{i=1,\text{guess}}$.

However, as the parameters α and β were lowered, it was found that the shooting method required considerably more time to obtain accurate values for φ_c^i and ϕ_c^i. This became increasingly so for the neutron star configurations which were closer to the maximum φ_c^i and ϕ_c^i.

7.2.6 Numerical results

We focus on the numerical results for the $f(\phi, R)$ background spacetime. In what follows, all plots are for the SLy equation of state with $\alpha = 50$ and $\beta = -4.5$ unless otherwise stated.

Central values

We present profiles for the central scalar field values ϕ_c^i and φ_c^i in Figure 7.8. Interestingly, we find that the φ_c^i scalar field values overlap perfectly with the φ_c^i values from $f(R)$ and show now visible deviation. This may be a result of the effects of the scalar field ϕ being more suppressed in the $f(\phi, R)$ theory. Comparing Figure 7.8 and Figure 7.6, we can see that the ϕ field in $f(\phi, R)$ has central values that are significantly smaller deviation form 1. Interestingly, we also find that the central ϕ_c^i values are significantly more spread out in $f(\phi, R)$ theory than in $F(\phi)R$, with a slight asymmetry around the maximum ϕ_c configuration.

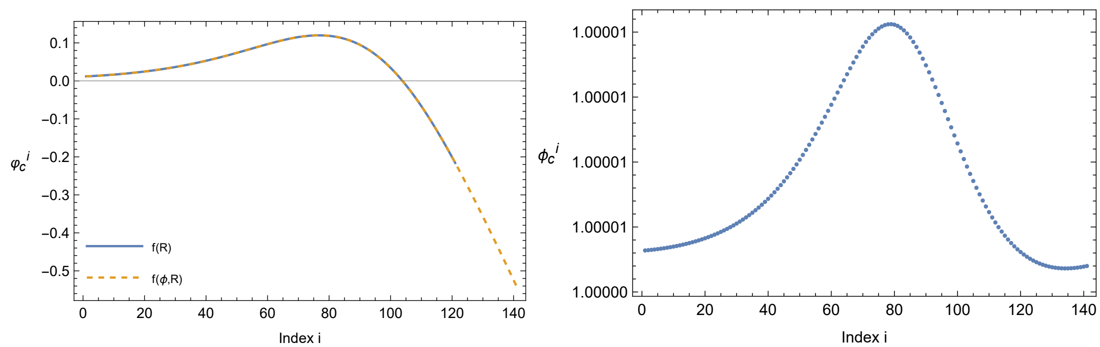

Figure 7.8: Central values φ_c^i and ϕ_c^i for the SLy equation of state in $f(\phi, R)$ theory. We also plot the overlap with the φ_c^i from $f(R)$ theory until $\varphi_c^{i=121}$. Here we take $\alpha = 50$, $\beta = -4.5$. The central densities correspond to $10^{17.3}\text{kg/m}^3 \leq \rho_c \leq 10^{18.7}\text{kg/m}^3$

Scalar field profiles

A plot for some of the ϕ scalar field profiles with the SLy equation of state can be found in Figure 7.9. We omit plots for the $\varphi(r)$ scalar field as they are identical to those of the $f(R)$ case. We can see that the scalar fields ϕ have a much larger drop off range compared to those of the $F(\phi)R$ theory in Figure 7.7, although the $f(\phi, R)$ scalar field is smaller in magnitude.

7.2.7 Mass-radius curves

Now that we have inspected the qualitative behavior of several scalar fields, we can now present our result for the mass-radius curves. The plots can be found in Figure 7.10. As these are only preliminary results, we present only a single equation of state with a single value for α and β. This represents the simplest numerical case study as lowering α and β requires more sensitivity in the initial conditions of the shooting method. Interestingly, we find that the $f(\phi, R)$ solution overlaps perfectly with the $f(R)$ solution for the same value of α, implying the presence of the scalar field ϕ has had no noticeable effect on the mass and radii of the neutron star.

At the time of writing, this is the only neutron star configuration in the parameter space of α and β, as well as the only equation of state, that we have examined for $f(\phi, R)$. As there is no comparison to be made with the literature, a proper analysis to these results is dependent upon the output of further results. One might hope that with a lower value of β and α, the ϕ field might exhibit less suppressed behavior and lead to noticeable differences in the mass-radius curves.

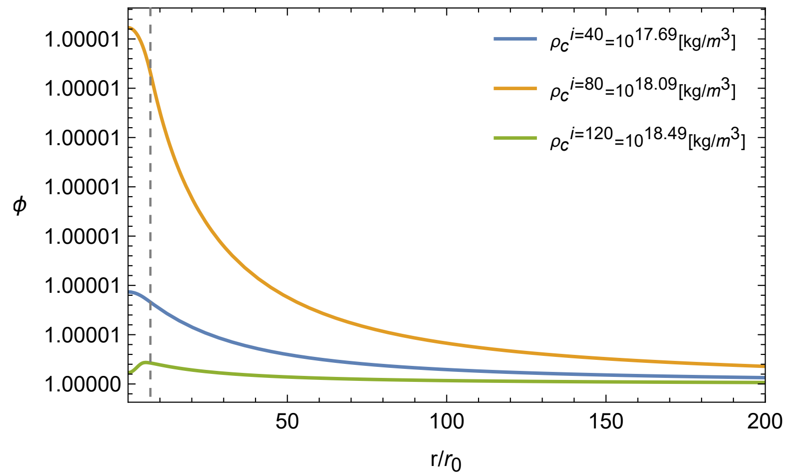

Figure 7.9: Scalar field profiles $\phi(r)$ for different central densities in $f(\phi, R)$ gravity. The magnitude of the field is smaller than those in Figure 7.7 but with a longer drop off range for the scalar field.

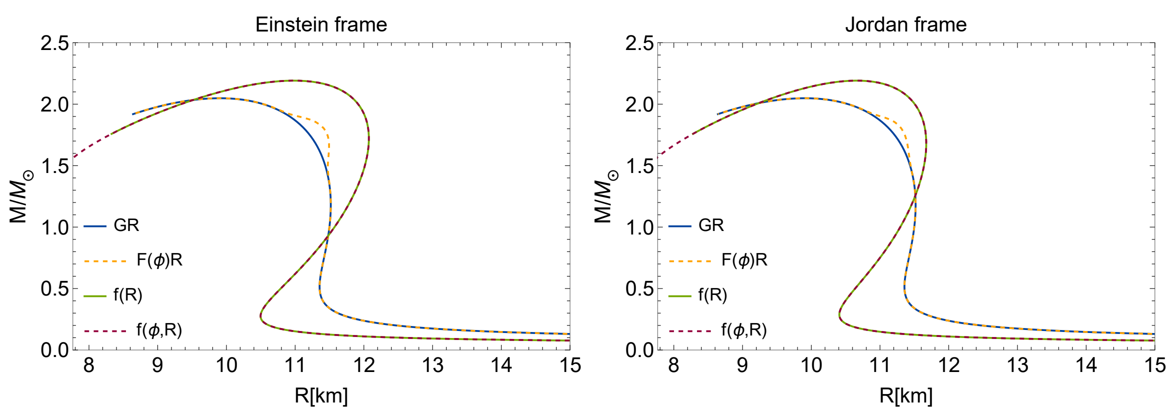

Figure 7.10: Mass-radius curves in the Einstein and Jordan frame for the SLy equation of state. We include GR, $F(\phi)R$, $f(R)$ and $f(\phi, R)$ gravity with $\alpha = 50, \beta = 4.5$ for a wide comparison that the different modifications to gravity can have on the mass-radius curves.

Chapter 8

Discussion, Outlook, Closing statement

8.1 Discussion

We began this thesis with an introduction to the Love number, motivating its relevance and describing how it can encode finite-size effects in a binary system of a binary system. We presented recipes for computing the Love number in both Newtonian gravity and GR. Following this, we then motivated the study of modified gravity theories. We chose to focus on a particular class known as scalar-tensor theories and reviewed the relevant parts of a recent work [6] that proposed a formalism for computing the Love number in $F(\phi)R$ gravity.

The intention of this thesis was to extend that work to a larger class of modified gravity theories that we called *generalized scalar-tensor theories* or $f(\phi, R)$ theories. The attempt at this extension was carried out in chapter 6. Here, we demonstrated a previously poorly documented transformation for the $f(\phi, R)$ theory to the Einstein frame. This allowed us to simultaneously decouple ϕ from R and also linearize R, at the expense of coupling ϕ to matter and introducing a second scalar field respectively.

We were successfully able to relate the Love numbers between frames in the $f(\phi, R)$ theory, and found that the Love numbers were unchanged between frames for the case $f_{RR}(\phi, R) \neq 0$ ($f_{RR}(\phi, R) = 0$ was the unique case addressed in chapter 5). However, we then encountered difficulty when attempting to extract the Love number from the Einstein frame EFT. This was due to the intricacies of the EFT equations of motion, which were ultimately too cumbersome to be solved with the same Fourier transform method as used in chapter 5. We can systematically categorize the differences in these equations compared to the $F(\phi)R$ case in the follow ways.

i The presence of an additional scalar field $\tilde{\chi}$, which is necessary to linearize the Ricci scalar in the Einstein frame.

ii The kinetic term for ϕ in Einstein action could not be decoupled from a $\tilde{\chi}$ dependent prefactor.

iii The Jordan frame coupling $f(\phi, R)$ manifested itself as a non-trivial potential $V(\phi, \tilde{\chi})$ in the Einstein frame, even for the simplest $f(\phi, R)$ coupling.

iv The prefactor $\omega(\phi)/\phi$ in the kinetic term for ϕ was not necessarily removable through a redefinition of the field ϕ.

None of these constraints were present in the $F(\phi)R$ study. Removing these constraints, in particular constraints ii and iii, would have allowed for a significantly better chance at solving the EFT equations. This is because it would have allowed for decoupled equations of motion for ϕ and $\tilde{\chi}$, as well as possibly allowing ϕ to be canonicalized (depending on the choice of $\omega(\phi)$).

Finding that the analytical formalism developed in chapter 5 not easily extended to $f(\phi, R)$, we attempted to at least gain a foothold on potential future work with a numerical case study of the background spacetime. To the best of the author's knowledge, this had not been presented in the literature before.

We began our numerical case study by first solving the $f(R)$ background spacetime, and using it as a toy model to elucidate some of the numerical pitfalls for the $f(\phi, R)$ calculation. With this, we were then successful in obtaining some preliminary results for the $f(\phi, R)$ background spacetime and mass-radius curves.

It was found that the preliminary results for the $f(\phi, R)$ mass-radius curves gave results seemingly degenerate to the $f(R)$ mass-radius curves, with no noticeable contribution appearing to come from the scalar field ϕ. This was despite both scalar fields in the equations of motion qualitatively satisfying the correct boundary conditions. Specifically, it was found that the $\tilde{\chi}$ scalar field (denoted φ in chapter 7) behaved identically to its $f(R)$ counterpart, where as the scalar field ϕ was much smaller in magnitude compared to its $F(\phi)R$ counterpart. It is as of yet undetermined if this degeneracy in the mass-radius curves was the result of an analytical or numerical error, or simply the product of the particular choice of the function $\Omega(\phi)$ and parameters α and β.

8.2 Outlook

With the most relevant work of this thesis discussed, we can now highlight some potential future areas of work and how the work of this thesis may be continued. We can list some of the ideas worth pursuing on the analytical side as

- Review the Einstein frame transformation for $f(\phi, R)$, to ensure that the action we obtained is indeed the simplest possible one. A simpler action might allow for a solving of the EFT equations of motion and thus an extraction of the Love number.
- Revisiting the EFT Einstein frame equations in $f(\phi, R)$ gravity to examine if other techniques beyond Fourier transforming can be used to solve the equations. One might first simplify the problem by attempting to linearize the equations in $\tilde{\chi}$.
- Review other theories that contain Einstein frame transformations such as $f(R, T)$ [47] and $f(R, \Box R)$ [48] gravity, to see if the Love number might be extractable in these theories.
- Reviewing the conditions needed for scalarized neutron stars [40] in $F(\phi)R$ gravity to see if they can be generalized for $f(\phi, R)$ gravity. The intention is that we might potentially constrain α and β for configurations that yield a generalization of scalarized neutron stars to better aid our numerical case study.
- Deriving the equations of motion for the perturbed spacetime in $f(\phi, R)$ gravity, to get a foothold in the event that any future progress on the analytical extraction of the Love number is made.

Similarly, future possible work on the numerical side includes

- Perfecting the shooting method for ϕ in $F(\phi)R$ gravity for $\beta < 4.5$ (as briefly outlined in subsection 7.2.3). This will allow for better insight into the $f(\phi, R)$ shooting method.
- Perfecting the double shooting method for $f(\phi, R)$ to allow for smoother numerical integration and for a larger variety of values in the parameter space.
- Attempting the $f(\phi, R)$ numerical solution with a different choice of $\Omega(\phi)$, ideally taken from the literature to see if the degeneracy from $f(R)$ was $\Omega(\phi)$ specific.
- Using the equations of motion and numerical code to investigate what other astronomical objects might be studied in $f(\phi, R)$ gravity.

8.3 Closing statement

Attempts to better constrain the neutron star equation of state require accurate modelling of neutron stars in modified theories of gravity. In this thesis, we attempted to extend an existing framework for extracting the Love number in scalar-tensor theories to generalized scalar-tensor theories. We found that extracting the Love number in this extension was troublesome, but we were still able to lay some groundwork by accurately documenting the necessary transformation to the Einstein frame. Our Einstein frame transformation still allowed for a modelling of the neutron star equilibrium configuration in generalized scalar-tensor theories. We tested this with a preliminary numerical case study. While there is still work to be done in perfecting the numerical set up, we have demonstrated a proof-of-concept that the background spacetime can in principle be computed numerically. This proof-of-concept calculation opens the door for a larger class of modified gravity theories, the generalized scalar-tensor theories, to be included in future neutron star work, thereby allowing for a more informed constraining of the nuclear matter equation of state and contributing to the solution of one of the biggest open problems in astrophysical research.

Acknowledgements

To Gastón, for being both an awesome supervisor and person, and who taught me the fine balance between optimism, perseverance and creativity when it comes to problem solving. I had a great time with working with you.

To Tanja, who is not only a great physicist, but a great teacher who has demonstrated to me the importance of kindness and empathy that make an inspiring leader. Thank you for all your patience and belief in me.

To Tanja's group, in particular Iris, Ramon, Liam, Niallan, Pablo and Simon, who made for some very lively meetings which introduced me to topics I would otherwise never have heard of. Also double credit to Iris for putting together the meeting slides.

To Lars, who I graciously thank for agreeing to be second examiner under short notice and unusual circumstances.

To my parents, Len and Paula, for being there when I needed them most and for their unconditional love and support in spite of the eternal delays to my studies. Also to my brother, Ciaran, for the occasional "good luck" ;).

To my aunt and uncle, Ger and Sean, who were always invested in my progress and well being, and Sean for his particular words of encouragement during the summer period of the thesis.

To Ryan, who always had time for a walk and talk, and whose endless passion and awe for mathematics has more than once helped me rekindle mine.

And lastly to Gabi, my biggest fan. Who always believed in me even when I didn't necessarily believe in myself. It's no exaggeration to say that I would not be the person - and by extension the physicist - that I am today without you. Therefore this achievement is as much yours as it is mine. Cuando todo termine.

Bibliography

[1] Augustus Edward Hough Love. The yielding of the earth to disturbing forces. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 82(551):73–88, 1909.

[2] B. P. Abbott et al. Properties of the binary neutron star merger GW170817. *Physical Review X*, 9(1), jan 2019.

[3] M. C. Miller et al. PSR J0030+0451 mass and radius from NICER data and implications for the properties of neutron star matter. *The Astrophysical Journal*, 887(1):L24, dec 2019.

[4] M. C. Miller et al. The radius of PSR J0740+6620 from NICER and XMM-Newton Data. *The Astrophysical Journal Letters*, 918(2):L28, sep 2021.

[5] Stoytcho S. Yazadjiev, Daniela D. Doneva, and Kostas D. Kokkotas. Tidal love numbers of neutron stars in f(r) gravity. *The European Physical Journal C*, 78(10), oct 2018.

[6] Gastón Creci, Tanja Hinderer, and Jan Steinhoff. Tidal properties of neutron stars in scalar-tensor theories of gravity, 2023.

[7] Stephanie M. Brown. Tidal deformability of neutron stars in scalar-tensor theories of gravity, 2023.

[8] Paolo Pani and Emanuele Berti. Slowly rotating neutron stars in scalar-tensor theories. *Physical Review D*, 90(2), jul 2014.

[9] L. Rezzolla, P. Pizzochero, D.I. Jones, N. Rea, and I. Vidaña. *The Physics and Astrophysics of Neutron Stars*. Astrophysics and Space Science Library. Springer International Publishing, 2019.

[10] Tim Dietrich, Tanja Hinderer, and Anuradha Samajdar. Interpreting Binary Neutron Star Mergers: Describing the Binary Neutron Star Dynamics, Modelling Gravitational Waveforms, and Analyzing Detections. *Gen. Rel. Grav.*, 53(3):27, 2021.

[11] Eric Poisson and Clifford M. Will. *Gravity: Newtonian, Post-Newtonian, Relativistic*. Cambridge University Press, 2014.

[12] Kip S. Thorne. Multipole expansions of gravitational radiation. *Rev. Mod. Phys.*, 52:299–339, Apr 1980.

[13] Torsten Hartmann, Michael H. Soffel, and Thomas Kioustelidis. On the Use of STF-Tensors in Celestial Mechanics. *Celestial Mechanics and Dynamical Astronomy*, 60(1):139–159, September 1994.

[14] Dong Lai. Resonant oscillations and tidal heating in coalescing binary neutron stars. *Monthly Notices of the Royal Astronomical Society*, 270(3):611–629, oct 1994.

[15] Kostas D. Kokkotas and Gerhard Schäfer. Tidal and tidal-resonant effects in coalescing binaries. *Monthly Notices of the Royal Astronomical Society*, 275(2):301–308, jul 1995.

[16] Éanna É. Flanagan and Tanja Hinderer. Constraining neutron-star tidal love numbers with gravitational-wave detectors. *Phys. Rev. D*, 77:021502, Jan 2008.

[17] Y. Rathore, A. E. Broderick, and R. Blandford. A variational formalism for tidal excitation: non-rotating, homentropic stars. *Monthly Notices of the Royal Astronomical Society*, 339(1):25–32, feb 2003.

[18] Sayan Chakrabarti, Té rence Delsate, and Jan Steinhoff. Effective action and linear response of compact objects in newtonian gravity. *Physical Review D*, 88(8), oct 2013.

[19] Curt Cutler and É anna E. Flanagan. Gravitational waves from merging compact binaries: How accurately can one extract the binary's parameters from the inspiral waveform? *Physical Review D*, 49(6):2658–2697, mar 1994.

[20] Tanja Hinderer. Tidal love numbers of neutron stars. *The Astrophysical Journal*, 677(2):1216–1220, apr 2008.

[21] Pawan Kumar Gupta, Jan Steinhoff, and Tanja Hinderer. Relativistic effective action of dynamical gravitomagnetic tides for slowly rotating neutron stars. *Physical Review Research*, 3(1), feb 2021.

[22] H. Georgi. Effective field theory. *Ann. Rev. Nucl. Part. Sci.*, 43:209–252, 1993.

[23] Walter D. Goldberger and Ira Z. Rothstein. Effective field theory of gravity for extended objects. *Physical Review D*, 73(10), may 2006.

[24] Tullio Regge and John A. Wheeler. Stability of a schwarzschild singularity. *Phys. Rev.*, 108:1063–1069, Nov 1957.

[25] Kip S. Thorne and Alfonso Campolattaro. Non-Radial Pulsation of General-Relativistic Stellar Models. I. Analytic Analysis for L >= 2. Astrophysical Journal, vol. 149, p.591, September 1967.

[26] S. Detweiler and L. Lindblom. On the nonradial pulsations of general relativistic stellar models. *Apj*, 292:12–15, May 1985.

[27] Yasunori Fujii and Kei-ichi Maeda. *The Scalar-Tensor Theory of Gravitation*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2003.

[28] Valerio Faraoni. *Cosmology in Scalar-Tensor Gravity*. 01 2004.

[29] Valerio Faraoni and Salvatore Capozziello. *Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics*. Springer, Dordrecht, 2011.

[30] Albert Petrov. *Introduction to Modified Gravity*. Springer International Publishing, 2020.

[31] Timothy Clifton, Pedro G. Ferreira, Antonio Padilla, and Constantinos Skordis. Modified gravity and cosmology. *Physics Reports*, 513(1-3):1–189, mar 2012.

[32] D. Lovelock. The Einstein tensor and its generalizations. *J. Math. Phys.*, 12:498–501, 1971.

[33] D. Lovelock. The four-dimensionality of space and the einstein tensor. *J. Math. Phys.*, 13:874–876, 1972.

[34] S. Shankaranarayanan and Joseph P. Johnson. Modified theories of gravity: Why, how and what? *General Relativity and Gravitation*, 54(5), may 2022.

[35] C. Brans and R. H. Dicke. Mach's principle and a relativistic theory of gravitation. *Phys. Rev.*, 124:925–935, Nov 1961.

[36] Sean Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Benjamin Cummings, 2003.

[37] D M Eardley. Observable effects of a scalar gravitational field in a binary pulsar. *Astrophys. J., Lett., v. 196, no. 2, pp. L59-L62*, 3 1975.

[38] Thibault Damour and Gilles Esposito-Farè se. Gravitational-wave versus binary-pulsar tests of strong-field gravity. *Physical Review D*, 58(4), jul 1998.

[39] Panagiotis Charalambous, Sergei Dubovsky, and Mikhail M. Ivanov. On the vanishing of love numbers for kerr black holes. *Journal of High Energy Physics*, 2021(5), may 2021.

[40] Thibault Damour and Gilles Esposito-Farèse. Nonperturbative strong-field effects in tensor-scalar theories of gravitation. *Phys. Rev. Lett.*, 70:2220–2223, Apr 1993.

[41] Kei-ichi Maeda. Towards the einstein-hilbert action via conformal transformation. *Phys. Rev. D*, 39:3159–3162, May 1989.

[42] Fulvio Sbisà, Oliver F. Piattella, and Sergio E. Jorás. Pressure effects in the weak-field limit of $f(r) = r + \alpha R^2$ gravity. *Phys. Rev. D*, 99:104046, May 2019.

[43] Fulvio Sbisà, Pedro Baqui, Tays Miranda, Sergio Jorás, and Oliver Piattella. Neutron star masses in r 2 -gravity. *Physics of the Dark Universe*, 27, 11 2019.

[44] Plamen P. Fiziev. Compact static stars in minimal dilatonic gravity. *Modern Physics Letters A*, 32(27):1750141, aug 2017.

[45] P. Fiziev and K. Marinov. Modeling of non-rotating neutron stars in minimal dilatonic gravity. *Astrophysics and Space Science*, 362(1), dec 2016.

[46] Kurt Just. Notizen: The motion of mercury according to the theory of thiry and lichnerowicz. *Zeitschrift für Naturforschung A*, 14(8):751–751, 1959.

[47] Tiberiu Harko, Francisco S. N. Lobo, Shin'ichi Nojiri, and Sergei D. Odintsov. $f(r,t)$ gravity. *Phys. Rev. D*, 84:024020, Jul 2011.

[48] S Gottlober, H J Schmidt, and A A Starobinsky. Sixth-order gravity and conformal transformations. *Classical and Quantum Gravity*, 7(5):893, may 1990.

Appendix A

Conformal rescaling of the Ricci scalar

In this appendix we present the steps for how the Ricci scalar transforms in arbitrary spacetime dimensions under a conformal rescaling of the metric tensor

$$g_{\mu\nu} = \Omega^{-2}(x)\tilde{g}_{\mu\nu}. \tag{A.0.1}$$

A.1 Christoffel symbols

Equation (A.0.1) implies a partial derivative of the metric transforms as

$$\begin{aligned}\partial_\alpha(\Omega^{-2}\tilde{g}_{\mu\nu}) &= -\frac{2}{\Omega^2}\frac{\partial_\alpha\Omega}{\Omega}\tilde{g}_{\mu\nu} + \frac{1}{\Omega^2}\partial_\alpha\tilde{g}_{\mu\nu},\\ &\equiv -\frac{2}{\Omega^2}f_\alpha\tilde{g}_{\mu\nu} + \frac{1}{\Omega^2}\partial_\alpha\tilde{g}_{\mu\nu}.\end{aligned} \tag{A.1.1}$$

where we have defined:

$$f_\alpha \equiv \partial_\alpha(\ln\Omega) = \frac{\partial_\alpha\Omega}{\Omega}.$$

The untransformed Christoffel symbols are given by:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\rho}\left(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}\right).$$

Applying equations (A.0.1) and (A.1.1) on the Christoffel symbols we find:

$$\begin{aligned}\Gamma^\mu_{\alpha\beta} &= \frac{1}{2}\frac{\cancel{\Omega^2}}{\cancel{\Omega^2}}\tilde{g}^{\mu\rho}\left(\partial_\alpha\tilde{g}_{\beta\rho} + \partial_\beta\tilde{g}_{\rho\alpha} - \partial_\rho\tilde{g}_{\alpha\beta}\right) - \frac{2}{2}\frac{\cancel{\Omega^2}}{\cancel{\Omega^2}}\tilde{g}^{\mu\rho}\left(f_\alpha\tilde{g}_{\beta\rho} + f_\beta\tilde{g}_{\rho\alpha} - f_\rho\tilde{g}_{\alpha\beta}\right),\\ &= \tilde{\Gamma}^\mu_{\alpha\beta} - \left(\delta^\mu_\alpha f_\beta + \delta^\mu_\beta f_\alpha - \tilde{g}_{\alpha\beta}\tilde{f}^\mu\right),\\ &\equiv \tilde{\Gamma}^\mu_{\alpha\beta} - \delta\tilde{\Gamma}^\mu_{\alpha\beta},\end{aligned}$$

where we define:

$$\tilde{f}^\mu \equiv \tilde{g}^{\mu\rho}f_\rho,$$

and

$$\delta\tilde{\Gamma}^\mu_{\alpha\beta} = \delta^\mu_\alpha f_\beta + \delta^\mu_\beta f_\alpha - \tilde{g}_{\alpha\beta}\tilde{f}^\mu.$$

A.2 Ricci scalar

A.2.1 Preliminaries

Some useful identities and notation are presented here.

With some algebra, one should be able to verify the following

$$\tilde{g}^{\lambda\rho}\left(\partial_\mu \tilde{g}_{\lambda\rho}\right) = 2\tilde{\Gamma}^\lambda_{\mu\lambda}, \tag{A.2.1}$$

$$\delta\tilde{\Gamma}^\lambda_{\mu\lambda} = n f_\mu, \tag{A.2.2}$$

$$\tilde{g}^{\mu\nu}\delta\tilde{\Gamma}^\rho_{\mu\nu} = -(n-2)\tilde{f}^\rho. \tag{A.2.3}$$

where n denotes the number of space-time dimensions, which we for now keep arbitrary.
The following identities require a little more effort, but can also be verified

$$\tilde{\Gamma}^\lambda_{\mu\rho}\tilde{g}^{\mu\nu}\left(\delta\tilde{\Gamma}^\rho_{\nu\lambda}\right) = \tilde{g}^{\mu\nu}\tilde{\Gamma}^\lambda_{\mu\nu} f_\lambda \tag{A.2.4}$$

$$\left(\delta\tilde{\Gamma}^\lambda_{\nu\rho}\right)\left(\delta\tilde{\Gamma}^\rho_{\lambda\mu}\right) = -(n-2)\tilde{f}^2 \tag{A.2.5}$$

where we have defined $\tilde{f}^2 \equiv f_\mu \tilde{f}^\mu = \tilde{g}^{\mu\nu} f_\mu f_\nu$.
Lastly we will make some final definitions:

$$f_{\mu\nu} \equiv \partial_\mu f_\nu = \partial_\mu \partial_\nu f,$$
$$\tilde{f} \equiv \tilde{g}^{\mu\nu} f_{\mu\nu},$$

where $f_{\mu\nu}$ is symmetric under the exchange of μ and ν.

A.2.2 Workings

The Ricci scalar R is defined as

$$R = g^{\mu\nu} R^\lambda_{\mu\lambda\nu},$$

where $R^\lambda_{\mu\sigma\nu}$ denotes the Riemann tensor. The Riemann tensor above can be expanded as:

$$R^\lambda_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\lambda\nu}. \tag{A.2.6}$$

We will systematically expand each Γ above in terms of $\tilde{\Gamma}$ and $\delta\tilde{\Gamma}$ and inevitably take contractions with $g^{\mu\nu} \equiv \Omega^2 \tilde{g}^{\mu\nu}$.

From here, we will temporarily set $\Omega = 1$ and reinsert it later.

Note that we are only interested in simplifying terms containing $\delta\tilde{\Gamma}$ as all of the $\tilde{\Gamma}$ terms can be immediately grouped together to form $\tilde{R}$.
We can compute

$$-\partial_\lambda\left(\delta\tilde{\Gamma}^\lambda_{\mu\nu}\right),$$

and raise with $\tilde{g}^{\mu\nu}$ to obtain

$$\boxed{n\partial_\lambda \tilde{f}^\lambda - 2\tilde{f} + 2\tilde{\Gamma}^\mu_{\mu\lambda}\tilde{f}^\lambda},$$

where at some point we have used equation (A.2.1).
The second term in equation (A.2.6) is easily shown to transform as

$$\boxed{n\tilde{f}},$$

after using equation (A.2.2) and contracting with $\tilde{g}^{\mu\nu}$.

The third term in (A.2.6) is found to vary as

$$\boxed{(n-2)\tilde{\Gamma}^\lambda_{\lambda\rho}\tilde{f}^\rho - n\left(\tilde{g}^{\mu\nu}\tilde{\Gamma}^\rho_{\mu\nu}\right) f_\rho - n(n-2)\tilde{f}^2},$$

after contracting with $\tilde{g}^{\mu\nu}$ and using (A.2.2) and (A.2.3).

Lastly, the fourth term varies as

$$2\tilde{g}^{\mu\nu}\tilde{\Gamma}^{\lambda}_{\mu\rho}\left(\delta\tilde{\Gamma}^{\rho}_{\nu\lambda}\right) - \tilde{g}^{\mu\nu}\left(\delta\tilde{\Gamma}^{\lambda}_{\nu\rho}\right)\left(\delta\tilde{\Gamma}^{\rho}_{\lambda\mu}\right),$$

where we have explicitly inserted $\tilde{g}^{\mu\nu}$. Simplifying with the use of (A.2.4)) and (A.2.5) we find

$$\boxed{2\tilde{g}^{\mu\nu}\tilde{\Gamma}^{\lambda}_{\mu\nu}f_{\lambda} + (n-2)\tilde{f}^2}.$$

Combining the relevant terms in the boxed equations, we obtain

$$n\left(\partial_\mu + \tilde{\Gamma}^{\lambda}_{\lambda\mu}\right)\tilde{f}^{\mu} + (n-2)\tilde{g}^{\mu\nu}\left(f_{\mu\nu} - \tilde{\Gamma}^{\lambda}_{\mu\nu}f_{\lambda}\right) - (n-1)(n-2)\tilde{f}^2. \tag{A.2.7}$$

The last trick is to notice that $\left(\partial_\mu + \tilde{\Gamma}^{\lambda}_{\lambda\mu}\right)\tilde{f}^{\mu}$ and $\tilde{g}^{\mu\nu}\left(f_{\mu\nu} - \tilde{\Gamma}^{\lambda}_{\mu\nu}f_{\lambda}\right)$ are equivalent expressions for the D'Alembertian of f, i.e.

$$\begin{aligned}\tilde{\Box}f &\equiv \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu f,\\ &= \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu f_\nu,\\ &= \tilde{g}^{\mu\nu}\left(f_{\mu\nu} - \tilde{\Gamma}^{\lambda}_{\mu\nu}f_{\lambda}\right).\end{aligned}$$

Equivalently

$$\begin{aligned}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu f_\nu &= \tilde{\nabla}_\mu\tilde{f}^{\mu},\\ &= \left(\partial_\mu + \tilde{\Gamma}^{\lambda}_{\lambda\mu}\right)\tilde{f}^{\mu},\end{aligned}$$

where we have commuted the metric tensor through the covariant derivative on the first line and contracted with f_ν.

Thus equation (A.2.7) further reduces to:

$$2(n-1)\tilde{\Box}f - (n-1)(n-2)\tilde{f}^2.$$

Assembling everything and reinserting Ω we find:

$$R = \Omega^2\left[\tilde{R} + 2(n-1)\tilde{\Box}f - (n-1)(n-2)\tilde{f}^2\right].$$

This further reduces to

$$\boxed{R = \Omega^2\left(\tilde{R} + 6\tilde{\Box}f - 6\tilde{f}^2\right)} \tag{A.2.8}$$

in $n = 4$ spacetime dimensions, which completes the Einstein transformation for the Ricci scalar R.

Appendix B

Transforming tensor and scalar tidal multipoles to the Einstein frame

B.1 Tensor tidal terms

We begin by repeating the $l = 2$ tensor tidal term given by

$$\mathcal{E}_{\mu\nu} = \frac{1}{z^2} W_{\mu\alpha\nu\beta} u^\alpha u^\beta, \tag{B.1.1}$$

where $z = \sqrt{-u^\mu u_\mu}$. Using our toolbox (4.2.68), we can easily show that this term transforms as

$$\begin{aligned} \mathcal{E}_{\mu\nu} &= \frac{1}{\tilde{z}^2} \cancel{\Omega^2} \tilde{W}_{\mu\alpha\nu\beta} \cancel{\Omega^{-2}} \tilde{u}^\alpha \tilde{u}^\beta & \text{(B.1.2)} \\ &= \tilde{\mathcal{E}}_{\mu\nu}. & \text{(B.1.3)} \end{aligned}$$

Thus the $l = 2$ term is invariant under the conformal rescaling. The transformation of the $l = 3$ tidal term happens as

$$\begin{aligned} \mathcal{E}_{\alpha\mu\nu} &= \nabla_\alpha \mathcal{E}_{\mu\nu} \\ &= \partial_\alpha \mathcal{E}_{\mu\nu} - \Gamma^\lambda_{\alpha\mu} \mathcal{E}_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu} \mathcal{E}_{\mu\lambda} \\ &= \partial_\alpha \tilde{\mathcal{E}}_{\mu\nu} - \tilde{\Gamma}^\lambda_{\alpha\mu} \tilde{\mathcal{E}}_{\lambda\nu} - \tilde{\Gamma}^\lambda_{\alpha\nu} \tilde{\mathcal{E}}_{\mu\lambda} \\ &\quad + \left(\delta^\lambda_\alpha f_\mu + \delta^\lambda_\mu f_\alpha - \tilde{g}_{\alpha\mu} \tilde{f}^\lambda \right) \tilde{\mathcal{E}}_{\lambda\nu} \\ &\quad + \left(\delta^\lambda_\alpha f_\nu + \delta^\lambda_\nu f_\alpha - \tilde{g}_{\alpha\nu} \tilde{f}^\lambda \right) \tilde{\mathcal{E}}_{\mu\lambda}, \end{aligned}$$

where we have used again used the toolbox (4.2.68) to transform $\Gamma^\lambda_{\mu\nu}$ and $\mathcal{E}_{\mu\nu}$. Also recall that $f_\mu = \partial_\mu \ln \Omega$. Tidying up the above we find

$$\mathcal{E}_{\alpha\mu\nu} = \tilde{\nabla}_\alpha \tilde{\mathcal{E}}_{\mu\nu} + 2 f_\alpha \tilde{\mathcal{E}}_{\mu\nu} + f_\mu \tilde{\mathcal{E}}_{\alpha\nu} + f_\nu \tilde{\mathcal{E}}_{\mu\alpha} - \tilde{f}^\lambda \left(\tilde{g}_{\alpha\mu} \tilde{\mathcal{E}}_{\lambda\nu} + \tilde{g}_{\alpha\nu} \tilde{\mathcal{E}}_{\mu\lambda} \right). \tag{B.1.4}$$

Higher order tidal moments can be calculated in a similar fashion and it is indeed preferable to do this recursively. An $\mathcal{E}_L$ tidal term can be more easily calculated, assuming that the previous tidal moment, $\mathcal{E}_{L-1}$ is known. This is by noting that $\mathcal{E}_L$ can be written as

$$\mathcal{E}_L = \nabla_\alpha \mathcal{E}_{L-1}. \tag{B.1.5}$$

By substituting in the previously calculated expression for $\mathcal{E}_{L-1}$, only the covariant derivative need be transformed. A generic formula for this is also possible by noting that the $\mathcal{E}_L$ term will have $L - 1$ appearances of the Christoffel symbol Γ when ∇_α is expanded. Thus (B.1.4) can be generalized to higher order moments as

$$
\begin{aligned}
\mathcal{E}_L &= \nabla_\alpha \mathcal{E}_{L-1}, \\
&= \tilde{\nabla}_\alpha \mathcal{E}_{L-1} + (l-1) f_\alpha \mathcal{E}_{L-1} + \sum_\alpha f_1 \mathcal{E}_{\alpha L-2} - \tilde{f}^\lambda \sum_\lambda \tilde{g}_{\alpha 1} \mathcal{E}_{\lambda L-2},
\end{aligned} \tag{B.1.6}
$$

where the sigmas $\sum$ denote a sum over the permutations of the indexed variable on E_{L-1}. For example, the $l = 4$ tidal term looks like

$$
\begin{aligned}
\mathcal{E}_{\alpha\beta\mu\nu} =& \nabla_\alpha \mathcal{E}_{\beta\mu\nu} \\
=& \tilde{\nabla}_\alpha \mathcal{E}_{\beta\mu\nu} + 3 f_\alpha \mathcal{E}_{\beta\mu\nu} + \{ f_\beta \mathcal{E}_{\alpha\mu\nu} + f_\mu \mathcal{E}_{\beta\alpha\nu} + f_\nu \mathcal{E}_{\beta\mu\alpha} \} \\
& - \tilde{f}^\lambda \{ \tilde{g}_{\alpha\beta} \mathcal{E}_{\lambda\mu\nu} + \tilde{g}_{\alpha\mu} \mathcal{E}_{\beta\lambda\nu} + \tilde{g}_{\alpha\nu} \mathcal{E}_{\beta\mu\lambda} \}
\end{aligned} \tag{B.1.7}
$$

After this, the transformed expression for $\mathcal{E}_{L-1}$ may be substituted in to get the full transformed expression for $\mathcal{E}_L$.

B.1.1 A note on Mathematica implementations

To verify that the tidal terms were correctly transformed by hand, the xPert package, part of a larger suite of Mathematica packages known as xAct, was used. The package xPert comes with natural implementation of conformal transformations. Directly transforming tidal terms explicitly defined in terms of the Weyl tensor quickly became computationally expensive however, and indeed transforming the $l = 4$ tidal term took on the order of several hours. This was due to how xPert transforms the Weyl tensor, expressing it in terms of contractions of the Riemann tensor and manually transforming each term. This computational cost becomes larger for each covariant derivative needed to be transformed. There are 2 ways to speed up these transformations.

(i) The recursive formula (B.1.6). One does not even truly need this formula but we include it for completeness. One only need to tell Mathematica to transform the covariant derivative in (B.1.5) and then substitute in the previously calculated transformation for $\mathcal{E}_{L-1}$.

(ii) Manually defining a tensor that transforms like the Weyl tensor and then leaving the code absent of the actual Weyl tensor defined in terms of the Riemann tensor. This means defining a conformally invariant symmetric rank 2 tensor $\mathcal{E}_{\mu\nu}$ and telling Mathematica to transform covariant derivatives of this tensor rather than covariant derivatives of $W_{\mu\alpha\nu\beta} u^\alpha u^\beta$.

This can speed up the transformation of tidal terms from hours to seconds and save valuable time, although it requires some more manual input from the user. For reproducibility, we record the transformed $l = 4$ tensor tidal term as

$$
\mathcal{E}_{\beta\alpha\mu\nu} = \tag{B.1.8}
$$

$$
\begin{aligned}
&\tilde{\mathcal{E}}_{\beta\alpha\mu\nu} + 3\tilde{\mathcal{E}}_{\beta\mu\nu} f_\alpha + 3\tilde{\mathcal{E}}_{\alpha\mu\nu} f_\beta + \tilde{\mathcal{E}}_{\alpha\beta\nu} f_\mu + \tilde{\mathcal{E}}_{\beta\alpha\nu} f_\mu + \tilde{\mathcal{E}}_{\alpha\beta\mu} f_\nu + \tilde{\mathcal{E}}_{\beta\alpha\mu} f_v - \tilde{\mathcal{E}}_{\gamma\mu\nu} \tilde{f}^\gamma \tilde{g}_{\alpha\beta} - \tilde{\mathcal{E}}_{\beta\nu\gamma} \tilde{f}^\gamma \tilde{g}_{\alpha\mu} - \tilde{\mathcal{E}}_{\beta\mu\gamma} \tilde{f}^\gamma \tilde{g}_{\alpha\nu} - \\
&\tilde{\mathcal{E}}_{\alpha\nu\gamma} \tilde{f}^\gamma \tilde{g}_{\beta\mu} - \tilde{\mathcal{E}}_{\alpha\mu\gamma} \tilde{f}^\gamma \tilde{g}_{\beta\nu} + 2 f_\mu f_\nu \tilde{\mathcal{E}}_{\alpha\beta} - \tilde{f}^\gamma f_v \tilde{g}_{\beta\mu} \tilde{\mathcal{E}}_{\alpha\gamma} - \tilde{f}^\gamma f_\mu \tilde{g}_{\beta\nu} \tilde{\mathcal{E}}_{\alpha\gamma} + 4 f_\beta f_v \tilde{\mathcal{E}}_{\alpha\mu} - f_\gamma \tilde{f}^\gamma \tilde{g}_{\beta\nu} \tilde{\mathcal{E}}_{\alpha\mu} + 4 f_\beta f_\mu \tilde{\mathcal{E}}_{\alpha\nu} - \\
&f_\gamma \tilde{f}^\gamma \tilde{g}_{\beta\mu} \tilde{\mathcal{E}}_{\alpha\nu} - \tilde{f}^\gamma f_\nu \tilde{g}_{\alpha\mu} \tilde{\mathcal{E}}_{\beta\gamma} - \tilde{f}^\gamma f_\mu \tilde{g}_{\alpha\nu} \tilde{\mathcal{E}}_{\beta\gamma} + 3 f_\alpha f_\nu \tilde{\mathcal{E}}_{\beta\mu} + 3 f_\alpha f_\mu \tilde{\mathcal{E}}_{\beta\nu} + \tilde{f}^\gamma \tilde{f}^\zeta \tilde{g}_{\alpha\nu} \tilde{g}_{\beta\mu} \tilde{\mathcal{E}}_{\gamma\zeta} + \tilde{f}^\gamma \tilde{f}^\zeta \tilde{g}_{\alpha\mu} \tilde{g}_{\beta\nu} \tilde{\mathcal{E}}_{\gamma\zeta} - \\
&\tilde{f}^\gamma f_\nu \tilde{g}_{\alpha\beta} \tilde{\mathcal{E}}_{\mu\gamma} - 3 f_\beta \tilde{f}^\gamma \tilde{g}_{\alpha\nu} \tilde{\mathcal{E}}_{\mu\gamma} - 2 f_\alpha \tilde{f}^\gamma \tilde{g}_{\beta\nu} \tilde{\mathcal{E}}_{\mu\gamma} + 8 f_\alpha f_\beta \tilde{\mathcal{E}}_{\mu\nu} - 2 f_\gamma \tilde{f}^\gamma \tilde{g}_{\alpha\beta} \tilde{\mathcal{E}}_{\mu\nu} - \tilde{f}^\gamma f_\mu \tilde{g}_{\alpha\beta} \tilde{\mathcal{E}}_{\nu\gamma} - 3 f_\beta \tilde{f}^\gamma \tilde{g}_{\alpha\mu} \tilde{\mathcal{E}}_{\nu\gamma} - \\
&2 f_\alpha \tilde{f}^\gamma \tilde{g}_{\beta\mu} \tilde{\mathcal{E}}_{v\gamma} + 2\tilde{\mathcal{E}}_{\mu\nu} \nabla_\beta f_\alpha - \tilde{g}_{\alpha\nu} \tilde{\mathcal{E}}_{\mu\gamma} \nabla_\beta \tilde{f}^\gamma - \tilde{g}_{\alpha\mu} \tilde{\mathcal{E}}_{\nu\gamma} \nabla_\beta \tilde{f}^\gamma + \tilde{\mathcal{E}}_{\alpha\nu} \nabla_\beta f_\mu + \tilde{\mathcal{E}}_{\alpha\mu} \nabla_\beta f_\nu
\end{aligned}
$$

B.2 Scalar tidal terms

With the tensor tidal terms transformed, we can now move on to doing the same for the scalar tidal terms

$$E_L = -\nabla_L \phi. \tag{B.2.1}$$

To transform (B.2.1) to the Einstein frame, we must recall that there is a field redefinition given by

$$\frac{d\varphi}{d\phi} = \sqrt{\frac{\Delta(\phi)}{2}} \tag{B.2.2}$$

as well as the conformal rescaling of $g_{\mu\nu}$.

For completeness we repeat the transformation of the $l = 1$ scalar tidal term, which transforms straightforwardly. This is because the single covariant derivative will reduce to a partial derivative when acting on ϕ and so is equivalent to its Einstein frame counterpart. The field redefinition (B.2.2) then gives

$$\nabla_\alpha \phi = \frac{1}{\sqrt{\Delta}} \tilde{\nabla}_\alpha \varphi, \tag{B.2.3}$$

Next, the $l = 2$ tidal term is given as

$$\nabla_\mu \nabla_\nu \phi = \partial_\mu \partial_\nu \phi - \Gamma^\lambda_{\mu\nu} \partial_\lambda \phi. \tag{B.2.4}$$

Upon redefining ϕ and conformally rescaling the metric, the first part will change as

$$\partial_\mu \partial_\nu \phi = \frac{1}{\sqrt{\Delta}} \left[\partial_\mu \partial_\nu \varphi - \frac{\Delta'}{2\Delta^{3/2}} \partial_\mu \varphi \partial_\nu \varphi \right], \tag{B.2.5}$$

where we have used that

$$\partial_\mu \left(\frac{1}{\sqrt{\Delta}} \right) = -\frac{\Delta'}{2\Delta^2} \partial_\mu \varphi. \tag{B.2.6}$$

Here Δ' denotes a derivative of Δ with respect to ϕ. The second part of (B.2.4) will change as

$$\Gamma^\lambda_{\mu\nu} \partial_\lambda \phi = \frac{1}{\sqrt{\Delta}} \left[\tilde{\Gamma}^\lambda_{\mu\nu} - \left(2\delta^\lambda_{(\mu} f_{\nu)} - \tilde{g}_{\mu\nu} \tilde{f}^\lambda \right) \right] \partial_\lambda \varphi. \tag{B.2.7}$$

Putting everything together we find that the $l = 2$ tidal term transforms as

$$\nabla_\mu \nabla_\nu \phi = \frac{1}{\sqrt{\Delta}} \left[\tilde{\nabla}_\mu \tilde{\nabla}_\nu \varphi + \left(2\delta^\lambda_{(\mu} f_{\nu)} - \tilde{g}_{\mu\nu} \tilde{f}^\lambda \right) \partial_\lambda \varphi - \frac{\Delta'}{2\Delta^{3/2}} \partial_\mu \varphi \partial_\nu \varphi \right]. \tag{B.2.8}$$

We can employ the same recursive formula for the scalar tidal terms as used for the tensorial tidal terms (B.1.6). This is because the scalar tidal terms are similarily obtained by taking covariant derivatives of lower order moments ((B.1.6) also does not use the symmetry of $\mathcal{E}_{\mu\nu}$ under exchange of indices to simplify the expression). Thus the scalar tidal terms can be computed recursively, similarly to the tensor tidal terms. No speed up is necessary for transforming scalar tidal terms through xAct however due to the absence of the Weyl tensor in these terms. The tranformations are performed on the order of a few seconds, and so (B.1.6) is not necessary for implementing scalar transformation terms in Mathematica. As we will see shortly however, the recursive formula (B.1.6) does fufill another purpose.

Dropping higher order terms

One can see from the recursive formula (B.1.6)

$$\mathcal{E}_L = \tilde{\nabla}_\alpha \mathcal{E}_{L-1} + (l-1) f_\alpha \mathcal{E}_{L-1} + \sum_\alpha f_1 \mathcal{E}_{\alpha L-2} - \tilde{f}^\lambda \sum_\lambda \tilde{g}_{\alpha 1} \mathcal{E}_{\lambda L-2}, \tag{B.2.9}$$

that the tidal terms do not transform neatly from the Jordan frame to the Einstein. They generally transform as

$$\mathcal{E}_L = \tilde{\mathcal{E}}_L + \text{additional terms}, \tag{B.2.10}$$

when one recursively removes all the Jordan frame $\mathcal{E}_{L-1}$ terms and subsequent lower order moments etc.. The purpose of this subsection is to argue why these additional terms may be neglected. We start by looking at $f_\mu = \partial_\mu \ln \Omega$ in (B.2.9). Using the definition

$$\alpha(\varphi) = \frac{1}{\Omega} \frac{d\Omega}{d\varphi}, \tag{B.2.11}$$

f_μ may be rewritten as

$$\begin{aligned} f_\mu &= \partial_\mu \ln \Omega, \\ &= \frac{1}{\Omega} \partial_\mu \Omega, \\ &= \frac{1}{\Omega} \frac{d\Omega}{d\varphi} \partial_\mu \varphi, \\ &= \alpha(\varphi) \partial_\mu \varphi, \\ &= \alpha(\varphi) \tilde{E}_\mu. \end{aligned} \tag{B.2.12}$$

Thus (B.2.9) can be rewritten in terms of Jordan frame tensor and Einstein frame scalar moments as

$$\mathcal{E}_L = \tilde{\nabla}_\alpha \mathcal{E}_{L-1} + \alpha (l-1) \tilde{E}_\alpha \mathcal{E}_{L-1} + \alpha \sum_\alpha \tilde{E}_1 \mathcal{E}_{\alpha L-2} - \alpha \tilde{g}^{\lambda\rho} \tilde{E}_\rho \sum_\lambda \tilde{g}_{\alpha 1} \mathcal{E}_{\lambda L-2}. \tag{B.2.13}$$

We see in the second, third and fourth terms above, the presence of couplings between $\tilde{E}$ and $\tilde{\mathcal{E}}$. These terms represent higher order powers of $1/r$ which are suppressed in the EFT and so can be neglected, leaving $\tilde{\nabla}_\alpha \mathcal{E}_{L-1}$ as the only relevant term. Furthermore, if the recursive formula is then used to replace $\mathcal{E}_{L-1}$, this will result in a

$$\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \mathcal{E}_{L-2}, \tag{B.2.14}$$

plus again additional higher order terms which can be neglected. Repeatedly applying the recursive formula eventually yields

$$\mathcal{E}_L = \tilde{\mathcal{E}}_L + \text{additional terms}, \tag{B.2.15}$$

where the additional terms of higher order can be dropped. We can thus effectively treat $\mathcal{E}_L$ as transforming like

$$\mathcal{E}_L \to \tilde{\mathcal{E}}_L. \tag{B.2.16}$$

Because the recursive formula (B.2.9) also holds for the scalar tidal terms, we can similarly find that the scalar tidal terms will effectively just change as

$$E_L \to \frac{1}{\sqrt{\Delta}} \tilde{E}_L, \tag{B.2.17}$$

.