Utrecht University

(No) Categorical Semantics for Classical Protocols

MSC THESIS

HISTORY AND PHILOSOPHY OF SCIENCE

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Abstract

Abramsky and Coecke's categorical quantum mechanics (Cat-QM) is an enticing start to what might one day be a reaxiomatisation of quantum mechanics. Unfortunately, most of its applications have been in the realm of technology, not philosophy. We examine Cat-QM from a philosopher's point of view, and ask what it can do for quantum foundations. We show that applying the methods of Cat-QM to classical mechanics leads to a categorically-formulated quantum-classical distinction, and take some steps in justifying its principles. Since most philosophers of physics are unfamiliar with category theory, we also provide an introduction to 2-category theory, written for this audience.

Acknowledgements

I came into the HPS master looking to be a historian or philosopher of science, with a particular interest in the theory and practice of history and in early modern science. But when I took Guido Bacciagaluppi's space and time course, I encountered a style of scholarship that I had never known existed, and knew I wanted to do too. F.A. Muller's quantum course in the next block further strengthened my conviction, as did Rosalie Iemhoff's logic and computation course, where I had the opportunity to explore quantum logic.

This Thesis is the result of two years in which I have tried to retool myself from a historian to a philosopher of physics. Whether or not the project has been successful, I leave to the reader. Guido and Fred's teaching certainly has been a great success: I don't think there are many places where one can move from the humanities into rigorous mathematical philosophy over the course of one masters programme, or many instructors who could teach such courses.

I would like to thank Guido, for his erudite and patient supervision, even when everything took longer than it should have. I also thank Jaap van Oosten for agreeing to co-supervise at such late notice, and for an enlightening conversation on category theory. Nor could I have written this Thesis without the routine and support of the HPS room, and all its koele kikkers, including, but not limited to: Antonios Papaioannou, Elian Schure, Elise Alkemade, Marien Raat, and Ori Morales Hernández. Finally, I'd like to thank those who proofread and commented on various parts and drafts: Antonios Papaioannou, Juliette Bouillon, Marien Raat, and Mary Cunningham. All the remaining errors are mine, of course.

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Chapter 1

Introduction

The cold beauty of category theory has tempted philosopher-physicists for almost as long as the field exists [4]. To Hans Primas for example, writing in the 1980's, it is obvious that

> the mathematical structure of [Von Neumann-Dirac] quantum mechanics is a category having Hilbert spaces as objects and unitary or antiunitary transformations as morphisms [56, p. 66].

It is equally obvious to him that quantisation could be done by a functor from the category of symplectic manifolds and symplectic transformations into the quantum category, but that nowadays a better approach is possible in terms of the representation theory of of kinematical groups [56, p. 66]. More recently, James Owen Weatherall has translated this newer approach to the language of category theory [64]. Hans Halvorson and Dimitris Tsementzis have recently revived the idea that a category can represent a theory, and that a category of such categories can encode relations between theories [31]. These are broad-strokes applications of category theory to the philosophy of science and the foundations of physics.

Applications of category theory to more concrete physical problems have been less frequent, and frequently less successful. Robert Geroch's textbook on mathematical physics — again, written in the 1980's — uses category theory as a unifying framework, but only as a stage on which to set more conventional mathematics [24]. The book starts with categories, but by the time it has treated sets, vector spaces, topological spaces, and measure spaces, and we reach the chapter on Hilbert spaces [24, pp. 277–284], there is little to no category theory to be found. This is not at all surprising, as a fully categorical treatment of Hilbert spaces did not exist until 2022, when Chris Heunen and Andre Kornell published their axioms for the category of Hilbert spaces [34]. In the 1960's, Bill Lawvere tried to construct a categorical theory of continuum mechanics, and more recently category theory has seen some applications in quantum gravity and string theory [4], but overall its impact on physics remains limited. Developing concrete categorical physics, capable of modelling real-world systems beyond string theory and quantum gravity, seems like a lost cause.

But it is tempting. Categories consist of objects connected by functors, and (operational) physical theories describe systems whose states are connected by processes. Category theory promises to unify widely different mathematical concepts by finding their common structure, and physics tries to unify widely different real-world processes by finding a common mathematical formalism. Hence, we ask:

> **Question 1.1.** What concrete applications of category theory are possible in physics?

Our point of departure is not string theory or quantum gravity, but quantum computation. In 2004, Samson Abramsky and Bob Coecke published "A categorical semantics of quantum protocols" [1], in which they showed that a categorical concept now known as dagger compact closed structure with biproducts is capable of encoding correlations in finitary quantum mechanics. This allows for a categorical description of such quantum protocols as teleportation and entanglement swapping. Further work on graphical calculi for dagger symmetric monoidal categories has led to a diagram language that promises to demystify some of the stranger aspects of quantum theory [15].

This ties in with an older approach to demystifying quantum mechanics, called *reaxiomatisation.* The idea here is that the weirdness of quantum theory is due to its inefficient — or maybe even incorrect — formulation. There might be a quantum formalism in which the phenomena we consider weird now, are actually logical consequences of some easily accepted principles. A good reaxiomatisation consists of a set of such principles from which all of quantum mechanics can be derived, and nothing else. An even better reaxiomatisation accounts for the differences between quantum and classical mechanics by showing which principles must be added, altered, or removed to go from the one theory to the other. Philosopher-physicists who try to reaxiomatise quantum theory therefore ask:

Question 1.2. What are the correct principles of quantum and classical mechanics?

Abramksy and Coecke's *categorical quantum mechanics* (Cat-QM) provides a bridge between the two questions. By formulating their categorical semantics, they have made an Ansatz towards a reaxiomatisation of (a fragment of) quantum theory, and found a way to model certain quantum processes in completely categorical language. To fully answer question 1.2, Cat-QM should also give us a way of doing categorical classical physics. We therefore ask:

Question 1.3. Can Abramsky and Coecke's categorical semantics for quantum protocols be translated to classical physics? If so, how?

In the rest of this introduction, we first discuss some basic concepts of and motivations for category theory, to convince the reader that it is a worthwhile field of study (§ 1.1). Then, in § 1.2, we discuss the basic concepts of quantum mechanics and provide some context for Abramsky and Coecke's Cat-QM. We end with an outline of this Thesis (§ 1.3).

We have tried to make this Thesis as self-contained as possible and, while writing, have kept in mind a reader who is not necessarily familiar with category theory or quantum mechanics, but ideally is familiar with at least one of these fields. No knowledge of physics is necessary, except for some basic Hamiltonian mechanics and Liouville's theorem. This information can be found in any textbook on classical mechanics, for example §§ 1-2, 40-46 of [44]. The only mathematical prerequisites — beyond what is necessary for the physics prerequisites — are some linear and abstract algebra, some set theory and logic, and a smattering of measure theory and functional analysis.

1.1 Category theory

Roughly speaking, category theory is the study of mathematical structure. We can try to describe in abstract terms what that means, but it is more illuminating to begin with some examples. We first provide three examples of natural isomorphisms, one example of a natural transformation that is not an isomorphism, and one non-example. Having done that, we briefly motivate the concepts of *functor* and *category*, and then outline some important lines of categorical enquiry.

Example 1.4 (Eilenberg and Mac Lane [22]). Let $\mathbb{K}$ be a field and let $f : V \to W$ be a linear map between finite-dimensional $\mathbb{K}$-vector spaces. For all such f, V, W there exist double dual spaces V^{**}, W^{**}, a double dual linear map $f^{**} : V^{**} \to W^{**}$, and linear bijections $i_V : V \to V^{**}$, $i_W : W \to W^{**}$ such that for all $v \in V$:

$$i_W(f(v)) = f^{**}(i_V(v)). \tag{1.1}$$

We can represent this equation as a directed graph in which vertices represent mathematical objects and edges represent functions:

$$\begin{array}{ccc} V & \xrightarrow{\quad i_V \quad} & V^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ W & \xrightarrow[\quad i_W \quad]{} & W^{**} \end{array} \tag{1.2}$$

We can see that both sides of the equation correspond to paths over the graph, and that both paths start at the same place and take what is there to the same destination.

Example 1.5 (Mac Lane [47, pp. 17–18]). Let S be a finite set with m elements, and let $\#S = \{1, \dots, m\}$. Let T be another set and f any function $S \to T$. Then there exist bijections $i_S : S \to \#S$, $i_T : T \to \#T$, and a function $\#f : \#S \to \#T$ such that for all $s \in S$:

$$i_T(f(s)) = \#f(i_S(s)). \tag{1.3}$$

Diagrammatically, we represent this as:

$$\begin{array}{ccc} S & \xrightarrow{i_S} & \#S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \#f} \\ T & \xrightarrow[i_T]{} & \#T \end{array} \tag{1.4}$$

Example 1.6. Let G be a group with operation $\bullet$, and G^{op} its opposite. Let H a group with operation $*$, and H^{op} its opposite. Then any group homomorphism $f : G \to H$ is also a homomorphism $G^{\mathrm{op}} \to H^{\mathrm{op}}$, since

$$f(a \bullet^{\mathrm{op}} b) = f(b \bullet a) = f(b) * f(a) = f(a) *^{\mathrm{op}} f(b). \tag{1.5}$$

Again, given group isomorphisms $i : G \to G^{\mathrm{op}}$ and $i_H : H \to H^{\mathrm{op}}$, we have a diagram:

$$\begin{array}{ccc} G & \xrightarrow{i_G} & G^{\mathrm{op}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ H & \xrightarrow[i_H]{} & H^{\mathrm{op}} \end{array} \tag{1.6}$$

These are all examples of *natural isomorphisms:* maps of maps between mathematical structures that preserve certain structure. In the examples above we saw that taking the double dual of finite-dimensional vector spaces preserves — in some sense of the word — linear maps, that an isomorphism of sets can preserve functions between sets, and that mapping a group to its opposite preserves group homomorphism. Such rectangular diagrams appear in every branch of mathematics, and we will see many more of them throughout this thesis. A slightly more general concept is the *natural transformation.* We can find these by letting go of the requirement that the horizontal arrows must be invertible:

Example 1.7. Let $\mathbb{K}$ be a field, and let X, Y be $\mathbb{K}$-vector spaces of finite or infinite dimension. Let $g : V \to W$ be a linear map, let * denote the dual, and let $m_X : X \to X^{**}$, $m_Y : Y \to Y^{**}$ be the canonical injective linear maps. Again, we have a diagram:

$$\begin{array}{ccc} X & \xrightarrow{m_X} & X^{**} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g^{**}} \\ Y & \xrightarrow[m_Y]{} & Y^{**} \end{array} \tag{1.7}$$

Note that the diagram in example 1.4 can be "reversed" along the horizontal axis to form the equally valid diagram

$$\begin{array}{ccc} V^{**} & \xrightarrow{i_V^{\text{inv}}} & V \\ {\scriptstyle f^{**}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ W^{**} & \xrightarrow[i_W^{\text{inv}}]{} & W \end{array} \tag{1.8}$$

but that such a reversal is not generally possible for g, X, Y since a bijection from X^{**} to X does not exist if X is infinite-dimensional.

To better understand what is special about natural isomorphisms and natural transformations, we might also consider a non-example and ask *why* it fails:

Example 1.8 (Mac Lane [47, exc. I.3.4])**.** Let $\mathrm{Z}G$ be the centre of a group G, and let $\mathrm{Z}h$ be the restriction of a group homomorphism $h : G \to H$ to $\mathrm{Z}G$. Let S_n be the symmetric group of degree n. Since S_2 consists of an identity element and one self-inverse element, and S_3 has three self-inverse elements in addition to its identity (along with two non-self-inverse elements), there exist three non-trivial group homomorphisms $S_2 \to S_3$. Let f be one of them.

Since S_2 is Abelian, the canonical isomorphism $h_2 : S_2 \to \mathrm{Z}S_2$ is an identity, and $\mathrm{Z}f = f$. Because S_3 is centreless, $\mathrm{Z}S_3$ is the trivial group and there exists exactly one homomorphism $h_3 : S_3 \to \mathrm{Z}S_3$.

Now we have a problem. The image of $\mathrm{Z}S_2 = S_2$ in $\mathrm{Z}f = f$ is larger than $\mathrm{Z}S_3$, so $\mathrm{Z}f$ is not a homomorphism $\mathrm{Z}S_2 \to \mathrm{Z}S_3$. We therefore cannot draw the kind of rectangular diagram we drew for the previous four examples: the right edge does not exist.

In the first four examples, we had some kind of procedure that turns some kind of mathematical object (vector space, finite set, group) into a related object (double dual space, finite ordinal, opposite group), and turns homomorphisms of the first kind of object into homomorphisms of the second kind of object in a way that preserves the homomorphism relations between objects. In the non-example above, we have a procedure that restricts one type of objects (groups) to another type (Abelian groups), and tries but fails to restrict homomorphisms in the same way.

Category theory was born in the 1940's when Samuel Eilenberg and Saunders Mac Lane [22] realised that natural isomorphisms are an interesting field of study in themselves. In order to study them rigorously, we must be able to tell which procedures allow for natural isomorphisms. This leads to the notion of *functors:* in the above examples, $(-)^{**}$, $\#$, and $(-)^{\mathrm{op}}$ are functors, but Z is not. Before we can discuss functors in any detail, we must be able to talk about their domains and targets, which are *categories,* collections of mathematical objects connected by homomorphisms. *Category theory* classifies and characterises such categories, constructs functors between them, and studies natural transformations and isomorphisms.

We will make all of this more rigorous in due time. For now, we will sketch a few prominent features of the category-theoretical landscape.

One line of enquiry involves studying categories themselves and looking for generalisations of categories. For example, if $\mathbf{C}$ and $\mathbf{D}$ are categories, we can view them as objects and view the functors $F : \mathbf{C} \to \mathbf{D}$ as homomorphisms of these objects. And if we take functors to be objects, we can regard natural transformations as homomorphisms of functors. We will also make all of this more rigorous in due time: the main idea here is that categorical structure can exist at multiple levels, so we can expand our understanding of categories to two-tiered, three-tiered, n-tiered, ∞-tiered categorical structures.[1] Any m-tiered category, it turns out, can be quite easily described as an $m + n$-tiered category satisfying certain conditions, which suggests that the most natural setting in which to study categories is that of ∞-tiered categories. Most of this Thesis takes place in the realm of 2-tiered categories, and we will rarely go higher. See [5] for more on higher category theory.

Another approach is topos theory. This investigates categories called *topoi* which are equipped with some structure that makes them sufficiently similar to the category of sets and functions that they can act as a setting in which to study both geometry and logic. These have some applications in quantum field theory and quantum gravity, and might have further-reaching implications for more branches of physics. See e.g. [19]. We will not use any topos theory in this Thesis.

Adjunctions are a third prominent feature of category theory. We provide two examples, and then discuss their use:

[1] We say "n-tiered category" instead of the more common "n-category" because to use more formal language would imply a higher level of rigour and formality than we have provided so far. Our discussion up to now is informal, and our terminology should reflect some of that.

Example 1.9. First, recall the construction of the *free group*. For S any set, we can construct an alphabet set A_S whose elements are: a symbol $e \notin S$, all the members of S, and for every $s \in S$ a symbol s^{inv}. The free group $\mathrm{F}S$ consists of all the equivalence classes of finite sequences of elements of A_S, where two sequences are equivalent if the one can be transformed into the other by (repeated) insertion or removal of the element e or subsequences ss^{inv}, $s^{\text{inv}}s$, with $s \in S$. The group operation of $\mathrm{F}S$ is sequence concatenation.

Now let F be a functor that maps every set S to the corresponding free group $\mathrm{F}S$ and every function of sets to the corresponding homomorphism of free groups. Let U be the functor that sends any group to its underlying set, and any group homomorphism to the corresponding function between sets. Since S is a generating set of the group $\mathrm{F}S$, any function from S to the underlying set $\mathrm{U}G$ of a group G uniquely defines a homomorphism from $\mathrm{F}S$ to G. Also, every homomorphism $\mathrm{F}S \to G$ defines a unique set-to-set function from S to $\mathrm{U}G$. Hence, we have a bijection $\phi_{S,G}$ from the set $\mathrm{Hom}(\mathrm{F}S, G)$ of group homomorphisms $\mathrm{F}S \to G$ to the set $[S, \mathrm{U}G]$ of functions $S \to \mathrm{U}G$. For $g : \mathrm{F}S \to G$ a group homomorphism, $\phi_{S,G}(g)$ is the restriction $\mathrm{U}g|_S$ of $\mathrm{U}g$ to S.

For any function $f : T \to S$ and group homomorphism $h : G \to H$, we construct the functions:

$$\begin{aligned} &\mathrm{Hom}(\mathrm{F}f, h) : \mathrm{Hom}(\mathrm{F}S, G) \to \mathrm{Hom}(\mathrm{F}T, H) \\ &\mathrm{Hom}(\mathrm{F}f, h)(g) = h \circ g \circ \mathrm{F}f \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} &[f, \mathrm{U}h] : [S, \mathrm{U}G] \to [T, \mathrm{U}H] \\ &[f, \mathrm{U}h](i) = \mathrm{U}h \circ i \circ f \end{aligned} \tag{1.10}$$

where $\circ$ denotes function composition. Then:

$$\begin{aligned} [f, \mathrm{U}h]\,(\phi_{S,G}(g)) &= [f, \mathrm{U}h]\,(\mathrm{U}g|_S) \\ &= \mathrm{U}h \circ \mathrm{U}g|_S \circ f \\ &= \mathrm{U}h \circ \mathrm{U}g \circ f \end{aligned} \tag{1.11}$$

where the last equality holds because S is the target of f, and also:

$$\begin{aligned} \phi_{T,H}\,(\mathrm{Hom}(\mathrm{F}f, h)(g)) &= \phi_{T,H}(h \circ g \circ \mathrm{F}f) \\ &= \mathrm{U}(h \circ g \circ \mathrm{F}f)|_T \\ &= \mathrm{U}h \circ \mathrm{U}g \circ f \end{aligned} \tag{1.12}$$

where the last equality holds because $\mathrm{UF}f|_T$ is the restriction to T of the set-to-set function corresponding to the group homomorphism $\mathrm{F}f$, so $\mathrm{UF}f|_T = f$. This gives us a natural isomorphism:

$$\begin{CD} \mathrm{Hom}(\mathrm{F}S, G) @>{\phi_{S,G}}>> [S, \mathrm{U}G] \\ @V{\mathrm{Hom}(\mathrm{F}f,h)}VV @VV{[f,\mathrm{U}h]}V \\ \mathrm{Hom}(\mathrm{F}T, H) @>>{\phi_{T,H}}> [T, \mathrm{U}H] \end{CD} \tag{1.13}$$

which encodes a relationship between the functors F and U. We call them *adjoint functors*, and we say that F is the *left adjoint* to U and U is the *right adjoint* to F. Such a relationship is called an *adjunction*, and exists wherever we have some notion of a *free object* with an underlying set.

Example 1.10. Let A a poset ordered by $\leq$, and B a poset ordered by $\preccurlyeq$. The order-reversing functions $\mathrm{L} : A \to B$ and $\mathrm{R} : B \to A$ form an *antitone*

Galois connection if for all $a \in A$, $b \in B$:

$$a \leq \mathrm{RL}a \qquad\qquad b \preccurlyeq \mathrm{LR}b. \tag{1.14}$$

An immediate consequence is that $(\mathrm{L}a \succcurlyeq b) \iff (a \leq \mathrm{R}b)$. Now let $a' \leq a$ and $b' \preccurlyeq b$: then $\mathrm{L}a \succcurlyeq b$ implies $\mathrm{L}a' \succcurlyeq b'$, and $a \leq \mathrm{R}b$ implies $a' \leq \mathrm{R}b'$. This lets us draw a diagram of logical implication:

$$\begin{array}{ccc} \mathrm{L}a \succcurlyeq b & \longleftrightarrow & a \leq \mathrm{R}b \\ \downarrow & & \downarrow \\ \mathrm{L}a' \succcurlyeq b' & \longleftrightarrow & a' \leq \mathrm{R}b' \end{array} \tag{1.15}$$

which is very similar to eq. (1.13). The fact that the horizontal edges are biconditionals while the vertical edges need only be implications in one direction corresponds to the fact that the horizontal arrows in our examples of natural isomorphisms have to be isomorphisms, while be the vertical arrows need only be homomorphisms. Here too, we have an adjunction, and here too, we call L and R adjoint functors, with L the left and R the right adjoint.

Adjunctions encode similarities between functors which in turn encode similarities between categories. In example 1.9, the functor U sends every group to the set that is most similar to it, which it does by “forgetting” the group properties. (We call such functors *forgetful*.) The functor F takes a set, and uses it to build a group in the most efficient way it can: by only positing those relations which are required by the group axioms, and extending the set where necessary. The *free-forgetful-adjunction* encodes a duality between these two ways in which two categories are similar.

The fact that L and R in example 1.10 are also adjoint functors tells us that these are dual to each other in a way similar to how F and U are dual to each other. Postulating that $a \leq \mathrm{RL}a$ and $b \preccurlyeq \mathrm{LR}b$ imposes a duality on our functors: the abundance of Galois connections across all of mathematics is evidence of its usefulness.

All of this is hand-wavy: in what sense are L and R functors, and what are the morphisms of posets-considered-as-categories? We will return to these questions later. For now, we have shown that the concepts of category theory — adjunction, natural isomorphism or transformation, functor, category — unify a broad range of mathematical structures. We have also shown that the core concepts — natural isomorphism, adjunction — depend on quite some lower-level concepts — functor, category, morphism — and that we cannot study the former rigorously until we understand the latter.

1.2 Quantum reaxiomatisations

We now turn to the second line of enquiry that underlies this Thesis. We begin by stating the postulates of quantum mechanics in two forms: the pure state formalism, which is probably the most familiar, and the more general density operator formalism. We then show how some strange effects follow, and discuss reaxiomatisation programmes in more detail.

The postulates for pure states are as follows:

Postulates 1.11 (Pure state formalism)**.**

Pure-I to every isolated quantum system $\mathfrak{S}$ there corresponds a complex Hilbert space $\mathcal{H}$, and the state of $\mathfrak{S}$ at any time is a unit vector $|\psi\rangle \in \mathcal{H}$;

Pure-II the time evolution of a quantum system is governed by the Schrödinger equation: $i\hbar\partial_t|\psi\rangle = \hat{\mathrm{H}}|\psi\rangle$, where $\hbar$ is a constant and $\hat{\mathrm{H}}$ is a self-adjoint operator, called the *Hamiltonian,* on $\mathcal{H}$;

Pure-III to every observable M there corresponds a set $\{M_m\}$ of *measurement operators,* where the indices m refer to the possible outcomes, with $\sum_m M_m^\dagger M_m$ equal to the identity operator I (the dagger † denotes the adjoint of an operator, which is not in general a categorical adjoint);

Pure-IV the probability of an M-measurement on a system in state $|\psi\rangle$ yielding the outcome m is $\langle\psi|M_m^\dagger M_m|\psi\rangle$, and the state of the system immediately after the measurement is $\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$;

Pure-V if the system $\mathfrak{S}$ with state $|\psi\rangle \in \mathcal{H}$ is combined with the system $\widetilde{\mathfrak{S}}$ with state $|\widetilde{\psi}\rangle \in \widetilde{\mathcal{H}}$, then the state of the combined system is $|\psi\rangle \otimes |\widetilde{\psi}\rangle \in \mathcal{H} \otimes \widetilde{\mathcal{H}}$.

A consequence of the second postulate is that for every time t there exists a unitary operator $\hat{U}_t : \mathcal{H} \to \mathcal{H}$, so that if our system is in state $|\psi\rangle$ at time 0, it occupies state $\hat{U}_t|\psi\rangle$ at time t. Also, note that we are using Dirac notation: $|\psi\rangle$ ("ket ψ") denotes a vector, and $\langle\psi|$ ("bra ψ") its adjoint.

When dealing with closed systems, we may represent measurements by projection operators. To every observable X there corresponds a self-adjoint operator $\hat{X}$, and the expectation value of X conditional on a state $|\psi\rangle$ is $\langle\psi|\hat{X}|\psi\rangle$. It follows, then, that X is guaranteed to take the value m if $\hat{X}|\psi\rangle = m|\psi\rangle$. The probability of X taking the value m, conditional on $|\psi\rangle$, is $\langle\psi|P_m|\psi\rangle$, where P_m is a projection operator onto the eigensubspace of $\hat{X}$ corresponding to the eigenvalue m.

Every ket is a linear map $\mathbb{C} \to \mathcal{H}$ and every bra is a linear map $\mathcal{H} \to \mathbb{C}$, so a ket-bra $|\psi\rangle\langle\psi|$ is a linear map $\mathcal{H} \to \mathcal{H}$. If $\langle\psi|\psi\rangle = 1$, then $|\psi\rangle\langle\psi|$ is a projection operator onto the subspace $[|\psi\rangle]$ spanned by $|\psi\rangle$, since for all $|\phi\rangle \in \mathcal{H}$:

(i) $(|\psi\rangle\langle\psi|)|\phi\rangle = |\psi\rangle\langle\psi|\phi\rangle \in [|\psi\rangle]$, so the range of $|\psi\rangle\langle\psi|$ is $[|\psi\rangle]$;

(ii) $(|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|) = |\psi\rangle(\langle\psi|\psi\rangle)\langle\psi| = |\psi\rangle\langle\psi|$, so $|\psi\rangle\langle\psi|$ is idempotent.

This gives us a convenient way of representing unit vectors in $\mathcal{H}$ as operators on $\mathcal{H}$, and leads to a more general set of postulates:

Postulates 1.12 (Density operator formalism)**.**

DensOp-I to every isolated quantum system $\mathfrak{S}$ there corresponds a Hilbert space $\mathcal{H}$, and the state of $\mathfrak{S}$ at any time is given by a *density operator* $\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|$ acting on $\mathcal{H}$, where the $|\psi_k\rangle$ are vectors in $\mathcal{H}$, and the p_k are positive real numbers that sum to 1;

DensOp-II if $\hat{U}$ describes the time-evolution of the vectors in $\mathcal{H}$, then the time evolution of the density operator is given by $\rho_t = \hat{U}\rho\hat{U}^\dagger$;

DensOp-III same as Pure-III;

DensOp-IV the probability of an M-measurement yielding the outcome m when $\mathfrak{S}$ is in state ρ is $\mathrm{Tr}\left(M_m^\dagger M_m\rho\right)$, and the state of $\mathfrak{S}$ immediately after the measurement is $\frac{M_m\rho M_m^\dagger}{\mathrm{Tr}\left(M_m^\dagger M_m\rho\right)}$;

DensOp-V if $\mathfrak{S}$ is in state $\rho : \mathcal{H} \to \mathcal{H}$ and $\widetilde{\mathfrak{S}}$ is in state $\widetilde{\rho} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$, then the combined system is in state $\rho \otimes \widetilde{\rho} : \mathcal{H} \otimes \widetilde{\mathcal{H}} \to \mathcal{H} \otimes \widetilde{\mathcal{H}}$.

We call ρ a *pure state* if it can be written as $|\psi\rangle\langle\psi|$ or, equivalently, if $\mathrm{Tr}\left(\rho^2\right) = 1$. If ρ is not a one-dimensional projection operator or, equivalently, if $\mathrm{Tr}\left(\rho^2\right) < 1$, then we call ρ a *mixed state*.

In both sets of postulates, the second contradicts the fourth: is measurement a preferred physical process which interrupts the unitary evolution?

Does this mean that it is physically relevant whether or not we measure a system? And what kind of probabilities are we dealing with here, anyways? To answer these questions is to solve the measurement problem. We will not do that in this Thesis.

For more details on the postulates of quantum mechanics, see §§ 2.2, 2.4 of [54]. Their consequences include the possibility of entanglement and teleportation: we discuss entanglement here, and teleportation later in this Thesis.

Definition 1.13. System $\mathfrak{S}$ with corresponding Hilbert space $\mathcal{H}$ and system $\widetilde{\mathfrak{S}}$ with the corresponding Hilbert space $\widetilde{\mathcal{H}}$ are *entangled* if their combined state $|\Psi\rangle \in \mathcal{H} \otimes \widetilde{\mathcal{H}}$ cannot be written as the tensor product of one vector in $\mathcal{H}$ and one vector in $\widetilde{\mathcal{H}}$.

Definition 1.14. A *qubit* is a system whose corresponding Hilbert space is $\mathbb{C}^2$. A *computational basis* is an orthonormal basis of a qubit.

Example 1.15 (Einstein, Podolsky, and Rosen [23])**.** Let Q_1 and Q_2 be qubits, and equip both with the computational basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. We choose $M_\uparrow = |\uparrow\rangle\langle\uparrow|$ and $M_\downarrow = |\downarrow\rangle\langle\downarrow|$ as our measurement operators.

One physical realisation of this setup is an electron, of which we measure the spin. The state $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$ corresponds to a probability $|\alpha|^2$ of measuring spin up, and a probability $|\beta|^2$ of measuring spin down.

Now we put our two electrons in the EPR-state

$$|\mathsf{EPR}\rangle = \frac{1}{2}\sqrt{2}\,(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle) \tag{1.16}$$

where the first term in each tensor product refers to Q_1, and the second to Q_2. We then move the electrons a great distance apart: we give Q_1 to Alice, and Q_2 to Bob. $|\mathsf{EPR}\rangle$ cannot be written as a tensor product $|\phi\rangle \otimes |\psi\rangle$

with $|\phi\rangle, |\psi\rangle \in \mathbb{C}^2$, so the qubits are entangled and it is impossible for a measurement operator to affect only one of the electrons: any measurement of one is a measurement of both.

When Alice and Bob receive their electrons, there is no matter of fact about the spin of either electron: there is an ontic probability of $\frac{1}{2}$ of spin up, and an ontic probability of $\frac{1}{2}$ of spin down for each. However, when Alice measures the spin of her particle and gets the result spin up, this changes the state to:

$$(|\uparrow\rangle\langle\uparrow| \otimes I)|\mathsf{EPR}\rangle = |\uparrow\rangle \otimes |\downarrow\rangle \tag{1.17}$$

Now Bob's electron is guaranteed to have spin down. If the quantum state truly encodes everything concerning the electron about which there is a matter of fact, then Bob's electron could not be ascribed a spin property until Alice measured hers, and a measurement at one location has instantly affected something arbitrarily far away. If the quantum state — and therefore quantum theory — provides an incomplete description of reality, then what is the complete theory? This is the EPR paradox.

The above setup also lets us illustrate the difference between pure and mixed states. First, we choose a second basis:

$$|+\rangle = \frac{1}{2}\sqrt{2}\,(|\uparrow\rangle + |\downarrow\rangle) \qquad |-\rangle = \frac{1}{2}\sqrt{2}\,(|\uparrow\rangle - |\downarrow\rangle)\,, \tag{1.18}$$

in which both vectors encode a probability $\frac{1}{2}$ of spin up and $\frac{1}{2}$ of spin down. Then we put an electron in the superposition state $|\psi\rangle = \frac{1}{2}\sqrt{2}\,(|+\rangle + |-\rangle)$. Both terms in the sum are superpositions of states; they *interfere,* resulting in an eigenstate of our spin operator:

$$|\psi\rangle = \frac{1}{2}\,(|\uparrow\rangle + |\downarrow\rangle + |\uparrow\rangle - |\downarrow\rangle) = |\uparrow\rangle \tag{1.19}$$

so the electron is guaranteed to have spin up. Now consider the density matrix $\rho = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-|$, built up out of two states which both have a $\frac{1}{2}$ probability of spin up and a $\frac{1}{2}$ probability of spin down. Here the spin down states do not interfere away:

$$\begin{aligned}\rho &= \frac{1}{4}\left[(|\uparrow\rangle + |\downarrow\rangle)(|\uparrow\rangle + |\downarrow\rangle) + (|\uparrow\rangle - |\downarrow\rangle)(|\uparrow\rangle - |\downarrow\rangle)\right] \\ &= \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow|\end{aligned} \tag{1.20}$$

so each measurement outcome still has a probability of $\frac{1}{2}$. We might say that in a superposition $|\psi\rangle = \alpha|\phi_1\rangle + \beta|\phi_2\rangle$ the system is in the state $|\psi\rangle$ with probability 1 and in either of the states $|\phi_1\rangle$ or $|\phi_2\rangle$ with probability 0, while in a mixed state $\rho = |\alpha|^2|\phi_1\rangle\langle\phi_1| + |\beta|^2|\phi_2\rangle\langle\phi_2|$, the system has a probability $|\alpha|^2$ of being in the state $|\phi_1\rangle$ and a probability $|\beta|^2$ of being in the state $|\phi_2\rangle$.

The above formalisms are unintuitive, unanschaulich, and derive paradoxical consequences in unsatisfying ways. Worse yet, they might be incomplete, incompatible with relativity, or even self-contradicting. One way to approach this problem, is to *interpret* quantum mechanics. Interpreting a theory might mean establishing correspondences between entities in the theory and the physical world, to attach a comprehensible picture to the not-yet-comprehended theory. It might also involve adding or removing postulates to make the theory more palatable [42]. But all such interpretations take the original postulates as their starting point, and must therefore inherit at least some of their problems. Might there be a better theory, built on an entirely different foundation, that recovers all the predictions of quantum mechanics without the weirdness?

The postulates of quantum theory were first formalised by John von Neumann in 1932 [52], but already in November 1935, half a year after EPR, he wrote in a letter to Garrett Birkhoff:

> I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more (quoted in [57, p. 493]).

Less than a year later, Birkhoff and Von Neumann published "The logic of quantum mechanics" [12]. They identify experimental propositions about $\mathfrak{S}$ with closed subspaces V, W of $\mathcal{H}$, negation with the orthogonal complement $V^{\perp}$, conjunction of V and W with the intersection $V \cap W$, and disjunction with the closed linear sum $V + W$ or, equivalently, $\left(V^{\perp} \cap W^{\perp}\right)^{\perp}$. We then have a non-distributive orthocomplemented lattice of experimental propositions about $\mathfrak{S}$ corresponding to closed subspaces of $\mathcal{H}$. This gives rise to *quantum logic*, and forms a first step towards a reaxiomatisation of quantum theory.

A reaxiomatisation or reconstruction tries to dissolve the problems of interpretation by building a theory that makes the same predictions, out of different postulates. It takes a set of physical principles that are considered basic, foundational, or intuitive, represents them mathematically and then deduces the formalism of the theory from them [2, 29, 32].

Put differently, a reaxiomatisation differs from an interpretation in that an interpretation takes the current theory as its starting point and looks for physical principles to match it, while a reconstruction takes physical principles, and builds a theory around them. As Rovelli famously puts it:

> quantum mechanics will cease to look puzzling only when we will be able to *derive* the formalism of the theory from a set of simple physical assertions ("postulates," "principles") about the world. Therefore, we should not try to *append* a reasonable

interpretation to the quantum mechanics *formalism,* but rather to *derive* the formalism from a set of experimentally motivated postulates [61, p. 1639].

From the 1960's onwards, a horde of reaxiomatisations has been proposed, often based on Birkhoff and Von Neumann's: see [29] for an overview. Many of these flounder on postulates Pure-V and DensOp-V. Just like the Birkhoff-Von Neumann reaxiomatisation, they can describe single systems, but not the correlations between combined systems [28].

A more modern line of enquiry, started in the 1990's, asks which constraints we must place on information transfer between systems in order to recover quantum mechanics. See [32] for a brief overview. Perhaps the most spectacular result of such programmes is the proof given by Rob Clifton, Jeff Bub, and Hans Halvorson (collectively known as CBH) in 2003, that three information-theoretic conditions — no cloning, no signalling, and no secure bit commitment — guarantee that the observables of a theory are quantum mechanical [13]. In 2004, a seemingly unrelated paper by Samson Abramsky and Bob Coecke showed that if one takes physical systems to be objects of a category, and operations performed on them to be morphisms, then imposing dagger compact closed structure with biproducts on that category allows one to model a good number of quantum processes [1]. The axioms of Cat-QM also give rise to a flow of quantum information [15].

Both approaches can be criticised for assuming too much. CBH prove their theorem within the framework of C*-algebras, which is already quite restrictive [32]. In fact, Halvorson himself would later criticise CBH for this assumption, and wonder whether it is at all possible and desirable to construct a theory from a small number of principles [30]. Similar attacks can be launched against Cat-QM, though Abramsky and Coecke do not seem to have attracted any yet. One reason for this might be that Cat-QM has never been a complete theory: the requirement that the quantum mechanical category be dagger compact closed limits us to quantum mechanics in finite-dimensional Hilbert spaces. But that requirement is quite an assumption. We therefore ask:

Question 1.16. Which physical principles can justify dagger compact closed structure with biproducts?

We have now posed four questions, none of which we are capable of answering in one Thesis. In the next section we narrow these down further, and describe our approach to answering them.

1.3 Outline of the Thesis

This Thesis consists of two parts. In the first, we provide an introduction to the study of 2-categories aimed at philosophers, physicists and those who work in between. In the second, we examine categorical physics in general, and categorical classical mechanics in particular.

Now why should we write *another* treatise on category theory? Surely there are enough of those already. Nevertheless, most philosophers of physics are unfamiliar with the field, and the introductory texts available today are not well suited to this audience.

For example, Saunders Mac Lane's classic *Categories for the Working Mathematician* [47] provides an in-depth discussion of adjunctions and universal properties, but the topics important to us are almost afterthoughts in the later chapters (monoidal, symmetric monoidal, and closed categories), or did not exist at the time it was written (dagger compact closure). Emily Riehl's *Category Theory in Context* takes a similar approach and introduces some key structures early on — monoidal categories, rings, and monoids in the section on diagram chasing [59, § 1.6], for example — but again the topics important to our purposes are relegated to a brief appendix [59, § E.2].

Other commonly cited texts take too slow a pace and provide too little coverage of the material leading up to dagger compact closed structure, or no coverage at all. Examples include *Basic Category Theory* by Tom Leinster [46], and *Category Theory* by Steve Awodey [3].

Coecke, co-inventor of Cat-QM, has written and co-written quite some expository material on the topic (eg. [15, 17]), but we consider much that work unsuitable too since he does his best to avoid discussing the underlying mathematical details.

The texts most suited to our audience and purposes might be Bob Coecke and Éric Paquette's "Categories for the practising physicist" [18], and Chris Heunen and Jamie Vicary's *Categories for Quantum Theory* [35], but these too have problems. The former contains much material we do not need, and its treatment of adjoints is unsatisfying. It provides solid physical motivations, but very little mathematical motivation for the structures it discusses. The latter contains excellent coverage of dagger compact closed categories, but here too, the mathematics is unmotivated. Many definitions are little more than long lists of axioms that lead to very little understanding of the material. To those who are already convinced that category theory is the right approach to quantum theory and want to learn to do the computations, it is a perfect book. To philosophers (whether or not they are "of physics"), it is unsatisfactory.

In chapter 2, we therefore provide a bottom-up introduction to the study of 1-categories. We do not discuss adjunctions and universal properties which many of the established texts begin with, but take as direct a route to

2-categories as possible. These are the natural setting in which to study monoidal categories, which are the topic of chapter 3.

In the second part, we discuss categorical physics. Chapter 4 examines the categorical structure of physical theories. We take as our starting point Coecke's observation that the general structure of an operational physical theory is that of a category [15], and then ask:

> **Question 1.17.** Which categorical structure must every operational physical theory have?

We show that every such theory must have symmetric monoidal structure. We then discuss Abramsky and Coecke's categorical semantics [1] in detail: adding three assumptions — that our physical category is a subcategory of the category of complex Hilbert spaces and linear maps, that adjunctions encode correlations, and that biproducts encode indeterministic branching — leads to finitary quantum mechanics. Hence, the correct principles of a significant part of quantum mechanics are these three.

But a good reaxiomatisation should also distinguish quantum from classical mechanics. In chapter 5 we ask:

> **Question 1.18.** How many of these principles apply to classical mechanics as well?

To answer this, we attempt to translate the categorical semantics for quantum protocols to the classical protocols. Koopman-Von Neumann mechanics [41, 53] provides a Hilbert space formulation of classical mechanics, and therefore is the natural setting for such a translation. We show that classical mechanics cannot be done in finite-dimensional Hilbert spaces, and that a category for classical physics therefore cannot have the same kind of adjunctions as the quantum category. And because there are no adjunctions, biproducts do not appear there in the same natural way as in quantum mechanics.

We conclude that neither Hilbert spaces nor tensor product structure are characteristics of quantum theory, but that any categorical quantum theory must allow for adjunctions and biproducts. We discuss some implications that a recent axiomatisation of the category of Hilbert spaces [35] has for these results.

Part I

Mathematical Preliminaries

Chapter 2

Category theory

Many mathematical structures form natural collections connected by structure-preserving maps, such as sets with functions, groups with group homomorphisms, vector spaces with linear maps, and topological spaces with continuous functions. Set theory is too austere to study these collections without significant added baggage: they are the subject of category theory.

Originally intended as a language for algebraic topology, category theory — like set theory — has grown into an independent mathematical field with elegant and surprising results of its own, and quite some foundations potential — though in the opposite direction to set theory. Whereas set theorists axiomatise the two-place predicate $\in$, and thereby break mathematical structures down to their many smallest components, category theorists axiomatise a three-place composition predicate for structure-preserving maps. They generalise and abstract, so as to show that seemingly separate structures are all instances of a far larger single thing ([43] attributes this two-place-three-place distinction to William Lawvere).

The most elegant and rigorous introduction to category theory available today might be Mac Lane's *Categories for the Working Mathematician* [47];

one of the best reference works (and a moderately good introduction) is the nLab [55], a collaborative website written by and for category theorists. Unfortunately, both are nigh-indecipherable to anyone who is not a working mathematician. Other textbook discussions are more accessible, but they often discuss category theory in an unmotivated way (such as the otherwise excellent *Categories for Quantum Theory* by Heunen and Vicary [35]), or hide it behind hundreds of pages of motivating mathematical preliminaries, as a topic among topics (such as George Bergman's *Invitation to General Algebra and Universal Constructions* [11]).

We strive for a balance between the three extremes. In this chapter and the next, we try to be as rigorous as possible while keeping our narrative intuitively motivated and readable to non-mathematicians. This often means delaying generalisations for longer than most texts: for example, we discuss products and coproducts before functors, and only later point out that the former are instances of the latter. This allows us to provide several examples of products, coproducts, and other constructions, so that we have a stock of well-understood examples at hand before tackling natural transformations and similarly abstract material. It also means re-deriving some well-known results from the perspective of the quantum mechanic or the philosopher of physics. While most authors are content to build structures out of loose components that mathematics already had "lying around" and then show that the result fits some aspect of quantum mechanics, we try to justify the structures and generalisations we use, with the mathematical structure of quantum mechanics as our starting point. Our goal, then, is to write a text as readable and motivated for philosophers as Steve Awodey's *Category Theory* [3], but faster-paced, as focused for quantum theory as Heunen and Vicary's [35], but better motivated, and perhaps with a shadow of Mac

Lane's [47] elegance and rigour.

In this chapter, we define some of the most important algebraic structures for category theory. We start with categories themselves (§ 2.1), and then discuss functors (§ 2.4), and finally natural transformations (§ 2.6). We define a number of categories, some of which ($\mathbf{Hilb}_{\mathbb{C}}$, $\mathbf{FdHilb}_{\mathbb{C}}$) are important for categorical approaches to quantum mechanics. Others (**NToset**, $\mathbf{N^2Poset}$) are useful examples of concepts that might otherwise be difficult to illustrate. Along the way, we also discuss other structures including poset and toset categories (§ 2.2); products, coproducts, and lattices (§ 2.3); as well as groups and their representations (§ 2.5). Our presentation of the material is mostly based on Heunen and Vicary [35] and Mac Lane [47].

2.1 Categories

It is possible, in principle, to define categories without referring to sets and classes, types, or any other foundational system of mathematics (e.g. [45]), though few sources actually do this. It is also possible to define categories in a foundations-agnostic way, which many texts do. Because foundations of mathematics are not our main concern, and because we expect our audience is more familiar with set theory than with other foundational systems, we define categories in terms of sets and classes.

Definition 2.1. A *category* $\mathbf{C}$ consist of a class $\mathbf{C}_0$ of *objects*, a class $\mathbf{C}_1$ of *morphisms*, and a partial function $\circ : \mathbf{C}_1 \times \mathbf{C}_1 \to \mathbf{C}_1$ called the *composition rule*, such that:

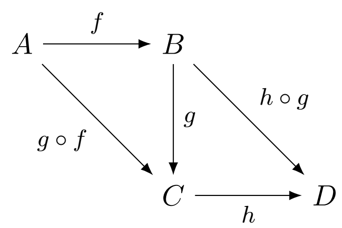
(i) $\mathbf{C}_1$ is partitioned by the ordered pairs of $\mathbf{C}_0$: for all $A, B \in \mathbf{C}_0$, there exists a *hom-set* or *hom-class* $\mathbf{C}_1(A, B) \subseteq \mathbf{C}_1$, and every morphism $f \in \mathbf{C}_1$ belongs to exactly one such hom-class. We write $A \xrightarrow{f} B$ for

$f \in \mathbf{C}_1(A, B)$; in that case we say that A is the domain of f (or: $A = \operatorname{dom} f$) and that B is the codomain of f (or: $B = \operatorname{cod} f$);

(ii) $g \circ f$ is defined if and only if $\operatorname{dom} g = \operatorname{cod} f$;

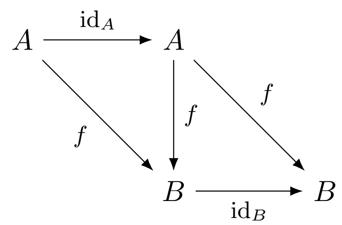
(iii) for all $A, B, C \in \mathbf{C}_0$, the composition rule maps $\mathbf{C}_1(B, C) \times \mathbf{C}_1(A, B)$ to $\mathbf{C}_1(A, C)$;

(iv) the composition rule is associative, i.e. for all $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, and $C \xrightarrow{h} D$, with $A, B, C, D \in \mathbf{C}_0$ the following diagram commutes:



(2.1)

(v) for all $A, B \in \mathbf{C}_0$ there exist *identity morphisms* $A \xrightarrow{\mathrm{id}_A} A$ and $B \xrightarrow{\mathrm{id}_B} B$ such that the following diagram commutes for all $A \xrightarrow{f} B$:



(2.2)

A diagram *commutes* if all paths that follow the arrows and have the same start and end-points are equivalent.

Many mathematical structures naturally form categories, with instances of the structure as objects, and structure-preserving maps as morphisms. Here are some examples:

Example 2.2. In the category **Rel**:

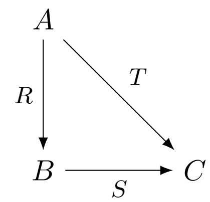
(i) $\mathbf{Rel}_0$ is the class of all sets;

(ii) $\mathbf{Rel}_1(A, B)$ is the class of all binary relations between A and B;

(iii) $\mathrm{id}_A = \{(a, a) \mid a \in A\}$;

(iv) the composition of $A \xrightarrow{R} B$ and $B \xrightarrow{S} C$ is $\{(a, c) \mid \exists b(aRb \wedge bSc)\}$.

The diagram



(2.3)

commutes in **Rel** if for all $(a, c) \in T$ there exists a $b \in B$ such that aRb and bSc.

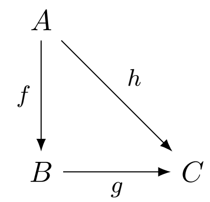
Example 2.3. In the category **Set**:

(i) $\mathbf{Set}_0$ is the class of all sets;

(ii) for all $A, B \in \mathbf{Set}_0$, $\mathbf{Set}_1(A, B)$ is the class of all functions $A \to B$;

(iii) identity morphisms are identity functions.

The diagram



(2.4)

commutes in **Set** if for all $a \in A$ we have $h(a) = g(f(a))$.

Example 2.4. In the category $\mathbf{Vect}_{\mathbb{K}}$:

(i) the objects are all the vector spaces over the field $\mathbb{K}$;

(ii) the morphisms are all the bounded linear transformations between $\mathbb{K}$-vector spaces;

(iii) composition is function composition;

(iv) identity morphisms are identity functions.

$\mathbf{FdVect}_{\mathbb{K}}$ is the restriction of $\mathbf{Vect}_{\mathbb{K}}$ to only the finite-dimensional $\mathbb{K}$-vector spaces. $\mathbf{Hilb}_{\mathbb{K}}$ has all the Hilbert spaces over the field $\mathbb{K}$ as its objects and all the bounded linear transformations between them as its morphisms. $\mathbf{FdHilb}_{\mathbb{K}}$ has all the finite-dimensional $\mathbb{K}$-Hilbert spaces as its objects and all the linear transformations between them as its morphisms The commutation conditions for eq. (2.4) in $\mathbf{Vect}_{\mathbb{K}}$, $\mathbf{FdVect}_{\mathbb{K}}$, $\mathbf{Hilb}_{\mathbb{K}}$, and $\mathbf{FdHilb}_{\mathbb{K}}$ are the same as those in **Set**.

Clearly, there is some kind of inclusion relation between **Set** and **Rel**, and a similar relation between $\mathbf{Hilb}_{\mathbb{K}}$ and $\mathbf{FdHilb}_{\mathbb{K}}$. We formalise this by defining subcategories.

Definition 2.5. A category $\mathbf{C}$ is a *subcategory* of the category $\mathbf{D}$ if:

(i) $\mathbf{C}_0 \subseteq \mathbf{D}_0$,

(ii) for all $A, B \in \mathbf{C}_0$, $\mathbf{C}_1(A, B) \subseteq \mathbf{D}_1(A, B)$,

(iii) $\mathbf{C}$ and $\mathbf{D}$ have the same identities, and the same composition rule.

$\mathbf{C}$ is a *full subcategory* of $\mathbf{D}$ if $\mathbf{C}_1(A, B) = \mathbf{D}_1(A, B)$ for all $A, B \in \mathbf{C}_0$, and $\mathbf{C}$ is a *wide subcategory* of $\mathbf{D}$ if $\mathbf{C}_0 = \mathbf{D}_0$.

Hence, **Set**, $\mathbf{Vect}_{\mathbb{K}}$, $\mathbf{FdVect}_{\mathbb{K}}$, $\mathbf{Hilb}_{\mathbb{K}}$ and $\mathbf{FdHilb}_{\mathbb{K}}$ are all subcategories of **Rel;** the categories $\mathbf{Vect}_{\mathbb{K}}$, $\mathbf{FdVect}_{\mathbb{K}}$, $\mathbf{Hilb}_{\mathbb{K}}$, and $\mathbf{FdHilb}_{\mathbb{K}}$ are subcategories of **Set;** and $\mathbf{FdHilb}_{\mathbb{K}}$ is a full subcategory of both $\mathbf{Hilb}_{\mathbb{K}}$ and $\mathbf{FdVect}_{\mathbb{K}}$.

Recall that according to definition 2.1, the objects and the morphisms in a category **C** both form classes. If we had only spoken of sets, category theory would be vulnerable to the paradoxes of set theory: the category of all categories invites the same problems as the set of all sets. We can make precise the way we dodge these paradoxes, by imposing "size restrictions":

Definition 2.6. A category **C** is *locally small* if every hom-class $\mathbf{C}_1(A, B)$ is a set. It is *small* if $\mathbf{C}_0$ and $\mathbf{C}_1$ are both sets.

Note that smallness implies local smallness, but that the inverse does not hold. For example, the category **Set** is locally small — the class of all functions between two sets is a set — but is is not small since $\mathbf{Set}_0$ is the class of all sets.

The categorical analogue to the class of all sets is the category of all small categories. (It is possible to create a category-like structure whose objects are categories that are not small [47, p.23 f.], but we will not do that.) Locally small categories let us dodge set-theoretical paradoxes in certain circumstances (for example, when working with enriched categories) while preserving some of the flexibility of non-small categories.

Many more familiar mathematical concepts have exact categorical equivalents, such as the following:

Definition 2.7. A morphism f is an *endomorphism* if $\operatorname{dom} f = \operatorname{cod} f$; the morphism $A \xrightarrow{f} B$ is an *isomorphism* if there exists a morphism $B \xrightarrow{f^{\text{inv}}} A$ such that

$$f^{\text{inv}} \circ f = \text{id}_A \quad \text{and} \quad f \circ f^{\text{inv}} = \text{id}_B \tag{2.5}$$

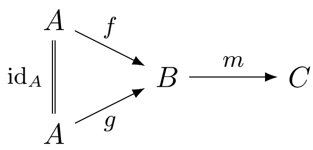
In that case, A and B are said to be *isomorphic objects,* and we write $A \simeq B$. Every object is isomorphic to itself by its identity morphism, and isomorphism is transitive: $A \simeq B$ and $B \simeq C$ imply $A \simeq C$.

In **Set** the isomorphisms are exactly the bijective functions.

Definition 2.8. A morphism $B \xrightarrow{m} C$ is *monic* (or: m is a *monomorphism*) if for each object A and all morphisms $A \xrightarrow{f} B$, $A \xrightarrow{g} B$ we have

$$m \circ f = m \circ g \Rightarrow f = g. \tag{2.6}$$

That is, the commutativity of:



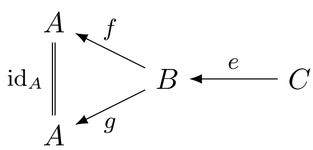
(2.7)

implies $f = g$.

The dual notion to monomorphism is the *epimorphism.* A morphism $C \xrightarrow{e} B$ is *epic* if for all $B \xrightarrow{f} A$, $B \xrightarrow{g} A$ we have

$$f \circ e = g \circ e \Rightarrow f = g. \tag{2.8}$$

That is, the commutativity of



(2.9)

implies $f = g$.

As becomes clear from the above diagrams, mono- and epimorphisms are dual in the sense that each can be defined as the other with the arrows reversed. We will see several more such dualities in the remainder of this Thesis.

In **Set**, the monomorphisms are all and only all the injective functions, and the epimorphisms are all and only all the surjective functions.

Every isomorphism is both monic and epic, but the converse does not hold. By taking a subcategory of **Set**, imposing some extra structure on its objects, and discarding all morphisms that do not respect that structure, we can construct epimorphisms that are not surjective and therefore not iso:

Example 2.9 (nLab, [60, prop. 2.1])**.** In the category **Ring** whose objects are all the rings, and whose morphisms are all the ring homomorphisms, the canonical inclusion $\mathbb{Z} \xrightarrow{i} \mathbb{Q}$ is monic, since it is injective.

Now let $f, g : \mathbb{Q} \to \mathbb{Q}$ be ring homomorphisms. For any $\frac{a}{b} \in \mathbb{Q}$ we then have $f\left(\frac{a}{b}\right) = f(a)\,(f(b))^{-1}$ and $g\left(\frac{a}{b}\right) = g(a)\,(g(b))^{-1}$, so $f = g$ if and only if for all $z \in \mathbb{Z}$: $f(z) = g(z)$. Therefore $f \circ i = g \circ i$ implies $f = g$, so i is epic but not surjective.

A fundamental dogma of category theory is that the most abstract concepts deserve the most attention. Hence, we will not speak of objects when describing something that can be defined in terms of morphisms and — later on — we will always prefer functors over morphisms, and natural transformations over functors.

2.2 Constructions on categories

We now move on to constructions on categories. We have already seen one kind, namely subcategory-of. Here we consider two more.

Definition 2.10. The *product category* of categories $\mathbf{A}$ and $\mathbf{B}$ is the category $\mathbf{A} \times \mathbf{B}$ in which:

(i) the class $(\mathbf{A} \times \mathbf{B})_0$ of objects of $\mathbf{A} \times \mathbf{B}$ is the class of all ordered pairs (A, B) with $A \in \mathbf{A}_0$ and $B \in \mathbf{B}_0$;

(ii) the class $(\mathbf{A} \times \mathbf{B})_1((A, B), (A', B'))$ of morphisms $(A, B) \to (A', B')$ in $\mathbf{A} \times \mathbf{B}$ is the class of all ordered pairs (f, g) with $A \xrightarrow{f} A'$ and $B \xrightarrow{g} B'$;

(iii) composition is pairwise: $(f, g) \circ (f', g') = (f \circ f', g \circ g')$;

(iv) identity morphisms are ordered pairs of identity morphisms: $\mathrm{id}_{(A,B)} = (\mathrm{id}_A, \mathrm{id}_B)$.

Consider the category **NToset** in which:

(i) the objects are the natural numbers;

(ii) for every object n there exists a morphism $n \to n + 1$;

(iii) no two objects have more than one morphism between them;

(iv) all diagrams commute (as long as their morphisms exist). In fact, a morphism in **NToset** is nothing more than an arrow from one object to another.

This category is a totally ordered set under the relation

$$A \leq B \iff \text{there exists a morphism } A \to B, \tag{2.10}$$

since:

(i) by definition 2.1, there exists a morphism $A \to A$ for every object A (reflexivity);

(ii) by definition 2.1, if morphisms $A \to B$ and $B \to C$ exist, then so does a morphism $A \to C$ (transitivity);

(iii) morphisms $A \to B$ and $B \to A$ both exist if and only if $A = B$ (antisymmetry);

(iv) for all objects A and B, there exists a morphism $A \to B$ or a morphism $B \to A$ (totality).

In the product category $\mathbf{N^2Poset} = \mathbf{NToset} \times \mathbf{NToset}$, we define:

$$(A, B) \preceq (A', B') \iff \text{there exists a morphism } (A, B) \to (A', B'). \quad (2.11)$$

$\mathbf{N^2Poset}$ is a partially ordered set under $\preceq$: it still satisfies the axioms of reflexivity, transitivity, and antisymmetry, but it has incomparable objects such as $(2, 1)$ and $(1, 2)$.

Having seen the above examples, we can define poset and toset categories:

Definition 2.11. A category $\mathbf{C}$ is a *poset* if for all $A, B \in \mathbf{C}_0$, $\mathbf{C}_1(A, B) \cup \mathbf{C}_1(B, A)$ is either empty, or a singleton set. $\mathbf{C}$ is a *toset* if for all $A, B \in \mathbf{C}_0$, $\mathbf{C}_1(A, B) \cup \mathbf{C}_1(B, A)$ is a singleton set.

In both cases, reflexivity and transitivity follow from definition 2.1, and antisymmetry follows from the current definition plus the requirement that every object has an identity morphism. In the case of a toset, totality follows from the requirement that for all $A, B \in \mathbf{C}_0$, there must exist either a morphism $A \to B$ or a morphism $B \to A$, since otherwise $\mathbf{C}_1(A, B) \cup \mathbf{C}_1(B, A)$ would be empty.

Our final construction on categories is the opposite category:

Definition 2.12. The *opposite category* $\mathbf{C}^{\text{op}}$ of a category $\mathbf{C}$ has the same objects as $\mathbf{C}$, but has morphisms in the opposite direction. Where $\mathbf{C}$ has a morphism $A \xrightarrow{f} B$, $\mathbf{C}^{\text{op}}$ has $B \xrightarrow{f^{op}} A$.

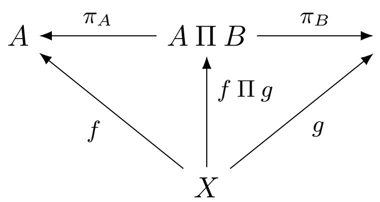
For example, $\mathbf{NToset}^{\text{op}}$ is a total order of the natural numbers under the relation $\geq$.

2.3 Constructions in categories

The constructions we have seen so far — subcategory, product category, opposite category — relate categories to each other. We now examine five ways objects and morphisms within a category may relate to each other. We first discuss products and their dual notion, coproducts. We then discuss initial objects and their dual notion, terminal objects. We end by defining zero objects.

Consider the Cartesian product of sets. For A, B, X arbitrary sets, any two functions $f : X \to A$ and $g : X \to B$ define a canonical function $f \times g : X \to A \times B$, such that the values of $f(x)$ and $g(x)$ are recoverable from $(f \times g)(x)$. The category-theoretical product generalises the Cartesian product on sets, and reveals to us a class of similar constructions, such as the meet of a poset.

Definition 2.13. Let A and B be objects in a category. Their *product* is an object $A \,\Pi\, B$ along with morphisms $A \,\Pi\, B \xrightarrow{\pi_A} A$ and $A \,\Pi\, B \xrightarrow{\pi_B} B$, such that any morphisms $X \xrightarrow{f} A$ and $X \xrightarrow{g} B$ define a unique morphism $X \xrightarrow{f \Pi g} A \,\Pi\, B$ for which the following diagram commutes:



(2.12)

A category *has finite products* if for all objects A and B, it contains an object $A \,\Pi\, B$.

Consider the following examples:

Example 2.14. In the category $\mathbf{FdVect}_{\mathbb{K}}$, let $\{|a_i\rangle\}_i$, $\{|b_j\rangle\}_j$, $\{|x_k\rangle\}_k$ form bases of A, B, X. Also let $f = \sum_{i,k} f_{ik} |a_i\rangle\langle x_k|$ be a morphism $X \xrightarrow{f} A$ and $g = \sum_{j,k} g_{jk} |b_j\rangle\langle x_k|$ a morphism $X \xrightarrow{g} B$. We then define

$$A \,\Pi\, B = A \oplus B \tag{2.13}$$

with $f \,\Pi\, g$, π_A, and π_B given by the following block matrices:

$$f \,\Pi\, g = \begin{bmatrix} f \\ \\ g \end{bmatrix} \tag{2.14}$$

$$\pi_A = \begin{bmatrix} \mathrm{id}_A & \mathbf{0} \end{bmatrix} \tag{2.15}$$

$$\pi_B = \begin{bmatrix} \mathbf{0}' & \mathrm{id}_B \end{bmatrix} \tag{2.16}$$

where $\mathbf{0}$ and $\mathbf{0}'$ are appropriately-sized zero matrices.

The product morphism $f \,\Pi\, g$ maps each $|\psi\rangle \in X$ to $f|\psi\rangle \oplus g|\psi\rangle$, where $f|\psi\rangle \in A$ and $g|\psi\rangle \in B$. The morphisms π_A and π_B send the subspaces $f(X)$ and $g(X)$ of $A \oplus B$ to A and B. It follows then, that

$$\begin{aligned} \pi_A \circ (f \,\Pi\, g) &= \mathrm{id}_A \cdot f + \mathbf{0} \cdot g \\ &= f \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \pi_B \circ (f \,\Pi\, g) &= \mathbf{0}' \cdot f + \mathrm{id}_B \cdot g \\ &= g \end{aligned} \tag{2.18}$$

so eq. (2.12) commutes. Hence, $\mathbf{FdVect}_{\mathbb{K}}$ has finite products given by the direct sum.

Example 2.15. Products in **Rel** are disjoint unions:

$$A \,\Pi\, B = A \sqcup B := \{(a, \star), (b, \bullet) \mid a \in A, b \in B\} \tag{2.19}$$

where $\star$ and $\bullet$ may be chosen freely, as long as $\star \neq \bullet$. For $X \xrightarrow{R} A$ and $X \xrightarrow{S} B$, we define:

$$R \,\Pi\, S = \{(x, (a, \star)), (x, (b, \bullet)) \mid x \in X, xRa, xSb\} \tag{2.20}$$

$$\pi_A = \{((a, \star), a) \mid (a, \star) \in A \sqcup B\} \tag{2.21}$$

$$\pi_B = \{((b, \bullet), b) \mid (b, \bullet) \in A \sqcup B\} \tag{2.22}$$

According to the definition in example 2.2, eq. (2.12) commutes if for all $x \in X$:

$$xRa \iff \exists c(x(R \,\Pi\, S)(c, \star) \wedge (c, \star)\pi_A a) \tag{2.23}$$

and

$$xRb \iff \exists c(x(R \,\Pi\, S)(c, \bullet) \wedge (c, \bullet)\pi_B b) \tag{2.24}$$

To see that the first of these conditions holds, note that if $c = a$, then $(c, \star)\pi_A a$. All that we need to prove, then, is $xRa \iff x(R \,\Pi\, S)(a, \star)$, which follows directly from the definition of $R \,\Pi\, S$. The proof for the second condition is similar, so eq. (2.12) commutes.

Example 2.16. For objects $A = (a, a')$ and $B = (b, b')$ in $\mathbf{N^2Poset}$, let $A \,\Pi\, B = (\min(a, b), \min(a', b'))$. Clearly, $A \,\Pi\, B \preceq A$ and $A \,\Pi\, B \preceq B$. Now let $X = (x, x')$ also be an object in $\mathbf{N^2Poset}$. If $X \preceq A$ then $x \leq a$ and $x' \leq a'$; if $X \preceq B$ then $x \leq b$ and $x' \leq b'$. So if $X \preceq A$ and $X \preceq B$, then

$X \preceq A \,\Pi\, B$, and the following diagram commutes:

$$\begin{array}{ccccc} A & \longleftarrow & A \,\Pi\, B & \longrightarrow & B \\ & \nwarrow & \uparrow & \nearrow & \\ & & X & & \end{array} \tag{2.25}$$

Hence, $A \,\Pi\, B$ is a product in $\mathbf{N^2Poset}$. As $A \,\Pi\, B$ is defined for all objects A and B, $\mathbf{N^2Poset}$ has finite products.

Recall that in a poset, for all A, B, and X, the following properties:

(i) $A \,\Pi\, B \preceq A$;

(ii) $A \,\Pi\, B \preceq B$;

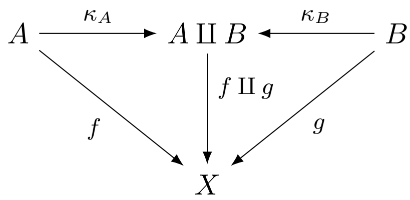
(iii) if $X \preceq A$ and $X \preceq B$ then $X \preceq A \,\Pi\, B$.

are exactly the definition of the meet, and note that they always define a product. Therefore, products in posets are always meets and vice versa.

Now consider the disjoint union of sets. For any sets A and B there exist canonical functions $\kappa_A : A \to A \sqcup B$ and $\kappa_B : B \to A \sqcup B$. Any two functions $f : A \to X$ and $g : B \to X$ then define a unique function $f \sqcup g : A \sqcup B \to X$ such that $(f \sqcup g) \circ \kappa_A = f$ and $(f \sqcup g) \circ \kappa_B = g$. The coproduct generalises this property of $\sqcup$, and reveals connections to other constructions.

Definition 2.17. Let A and B be objects in some category. Their *coproduct* is an object $A \amalg B$ along with morphisms $A \xrightarrow{\kappa_A} A \amalg B$ and $B \xrightarrow{\kappa_B} A \amalg B$ such that any two morphisms $A \xrightarrow{f} X$ and $B \xrightarrow{g} X$ define a unique morphism

$A \amalg B \xrightarrow{f \amalg g} X$ for which the following diagram commutes:



(2.26)

A category *has finite coproducts* if for all objects A and B, it has an object $A \amalg B$.

Here are some examples:

Example 2.18. For any objects A, B, X, and morphisms $A \xrightarrow{f} X$, $B \xrightarrow{g} X$ in $\mathbf{FdVect}_{\mathbb{K}}$, let:

$$A \amalg B = A \oplus B \tag{2.27}$$

$$f \amalg g = \begin{bmatrix} f & g \end{bmatrix} \tag{2.28}$$

$$\kappa_A = \begin{bmatrix} \mathrm{id}_A \\ \mathbf{0} \end{bmatrix} \tag{2.29}$$

$$\kappa_B = \begin{bmatrix} \mathbf{0}' \\ \mathrm{id}_B \end{bmatrix} \tag{2.30}$$

where, again, $\mathbf{0}$ and $\mathbf{0}'$ are appropriately-sized zero matrices. It follows then, that

$$\begin{aligned} (f \amalg g) \circ \kappa_A &= f \cdot \mathrm{id}_A + g \cdot \mathbf{0} \\ &= f \end{aligned} \tag{2.31}$$

and

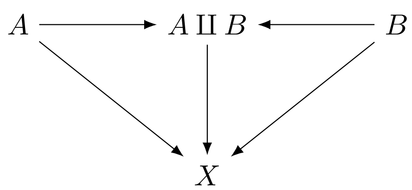
$$
\begin{aligned}
(f \amalg g) \circ \kappa_B &= f \cdot \mathbf{0}' + g \cdot \mathrm{id}_B \\
&= g
\end{aligned}
\tag{2.32}
$$

so eq. (2.26) commutes. Therefore, $\mathbf{FdVect}_{\mathbb{K}}$ has finite coproducts given by the direct sum.

Example 2.19. The disjoint sum is a coproduct in **Rel**; for relations $A \xrightarrow{R} X$ and $B \xrightarrow{S} X$, let $R \amalg S$, κ_A, and κ_B be the converses of the morphisms $R \amalg S$, π_A, and π_B defined in example 2.15. The diagram in definition 2.17 then commutes.

In both $\mathbf{FdVect}_{\mathbb{K}}$ and **Rel**, the products are also coproducts, and vice versa. This is because a product in a category is a coproduct in its opposite, and because $\mathbf{FdVect}_{\mathbb{K}}$ and **Rel** are equal to their opposites. In our next example, this is not the case.

Example 2.20. For any objects $A = (a, a')$ and $B = (b, b')$ in $\mathbf{N^2Poset}$, let $A \amalg B = (\max(a, b), \max(a', b')$. Clearly, $A \preceq A \amalg B$ and $B \preceq A \amalg B$. Now let $X = (x, x')$ also be an object in $\mathbf{N^2Poset}$. If $A \preceq X$ then $a \leq x$ and $a' \leq x'$; if $B \preceq X$ then $b \leq x$ and $b' \leq x'$. So if $A \preceq X$ and $B \preceq X$ then $A \amalg B \preceq X$ and the following diagram commutes:



(2.33)

Hence, $A \amalg B$ is a coproduct in $\mathbf{N^2Poset}$. As $A \amalg B$ is defined for all objects A and B, $\mathbf{N^2Poset}$ has finite coproducts.

Recall that in a poset, for all A, B, and X, the following properties:

(i) $A \preceq A \amalg B$;

(ii) $B \preceq A \amalg B$;

(iii) if $A \preceq X$ and $B \preceq X$ then $A \amalg B \preceq X$.

are exactly the definition of a join. Therefore, coproducts in posets are always joins, and vice versa. This lets us define a lattice in category-theoretical language.

Definition 2.21. A *lattice* is a poset category with finite products and coproducts.

We return to **Set** again. Note that for each set A there exists an empty function $\emptyset \to A$. Also, for each singleton set $\{\star\}$, there exists a unique function $A \to \{\star\}$. The following notions generalise this:

Definition 2.22. An object I is *initial* in a category $\mathbf{C}$ if there exists a unique morphism $I \to A$ for each object A of $\mathbf{C}$. An object T is *terminal* in $\mathbf{C}$ if for each $A \in \mathbf{C}_0$ there exists a unique morphism $A \to T$. An object that is both initial and terminal is a *zero object.*

Here are some examples:

(i) 0 is initial in **NToset** and terminal in $\mathbf{NToset}^{\mathrm{op}}$;

(ii) a *bounded lattice* is a lattice with an initial object and a terminal object;

(iii) the zero vector space is a zero object in $\mathbf{Vect}_{\mathbb{K}}$;

(iv) in **Rel**, the empty set is a zero object;

(v) in the category of all groups, with group homomorphisms as morphisms, all groups of order 1 are zero objects.

Note that a category may have many — even infinitely many — initial, terminal, or zero objects. In **Set**, for example, there are infinitely many singletons, and in the category of groups there are infinitely many trivial groups. Since these are all isomorphic, it does not matter which we use, so we can still speak of *the* initial/terminal/zero object. This use of *the* occurs frequently in category theory.

A property that will turn out useful later on is that in a category with a zero object 0, any objects A and B have at least one morphism $0_{A,B}$ between them, given by the composition of $A \to 0$ and $0 \to B$.

2.4 Functors

The existence of constructions on and in categories suggest the idea of functors, which are mappings between categories. Many constructions on and in categories, as we shall see later on, can be described and studied in a more concise and general way if we work at the level of functors instead of objects and morphisms. We first define several kinds of functors and provide examples. Then we discuss equivalence functors in more detail.

Definition 2.23. A *functor* between categories $\mathbf{C}$ and $\mathbf{D}$ is a mapping $F : \mathbf{C} \to \mathbf{D}$ which:

(i) sends objects to objects;

(ii) sends morphisms to morphisms;

(iii) sends identity morphisms to identity morphisms: $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$;

(iv) preserves commutativity of diagrams: for all $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ in $\mathbf{C}$, $F(g \circ f) = F(g) \circ F(f)$.

Additionally, every functor is either *covariant* or *contravariant*. A covariant functor preserves the directions of morphisms:[1] $\mathbf{C}_1(A, B)$ is mapped to $\mathbf{D}_1(F(A), F(B))$. A contravariant functor reverses their directions: $\mathbf{C}_1(A, B)$ is mapped to $\mathbf{D}_1(F(B), F(A))$. From now on, when we do not specify the variance of a functor, it is covariant.

An *endofunctor* is a functor from a category to itself. A *contravariant endofunctor* is a functor from a category to its opposite. A *bifunctor* is a functor whose domain is a product category.

If a functor $\mathbf{C} \to \mathbf{D}$ maps every object of $\mathbf{C}$ to the same object $D \in \mathbf{D}_0$, and maps every morphism in $\mathbf{C}$ to id_D, we call it a *constant functor*. An endofunctor $F : \mathbf{C} \to \mathbf{C}$ is an *identity* if $F(C) = C$ and $F(f) = f$ for every object C and morphism f of $\mathbf{C}$.

Here are some examples of functors:

Example 2.24. If $\mathbf{C}$ is a category with finite products, then the product is given by a bifunctor $\Pi : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$. If $\mathbf{C}$ has finite coproducts, then we have a bifunctor $\amalg : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$.

Example 2.25. There exists a contravariant endofunctor $F : \mathbf{NToset} \to \mathbf{NToset}^{\mathrm{op}}$, in which every object k of $\mathbf{NToset}$ is mapped to the object $2k$ in $\mathbf{NToset}^{\mathrm{op}}$. This functor maps the toset of natural numbers with $\leq$ to that of even natural numbers with $\geq$.

Example 2.26. There exists a constant functor $G : \mathbf{Rel} \to \mathbf{NToset}$ which maps every object to the number 12, and every morphism to id_{12}.

[1] This is one of only two cases in all of category theory where "co" does not mean something is flipped around. The other instance is "cone", but we will not discuss those.

Example 2.27. Consider the category $\mathbf{FdMat}_{\mathbb{K}}$ in which the objects are the natural numbers, the morphisms $m \to n$ are all the $n \times m$ matrices over the field $\mathbb{K}$, identity objects are identity matrices, and morphism composition is matrix multiplication.

There exists a functor $H : \mathbf{FdMat}_{\mathbb{K}} \to \mathbf{FdVect}_{\mathbb{K}}$ which maps every object n of $\mathbf{FdMat}_{\mathbb{K}}$ to the vector space $\mathbb{K}^n$, and every $n \times m$ matrix $\mathbf{M}$ to the function $\mathbb{K}^m \to \mathbb{K}^n :: |\psi\rangle \mapsto \mathbf{M}|\psi\rangle$.

Intuitively, it is clear that $\mathbf{FdMat}_{\mathbb{K}}$ and $\mathbf{FdVect}_{\mathbb{K}}$ are very similar categories, and that the functor H expresses this similarity. The objects of $\mathbf{FdMat}_{\mathbb{K}}$ may seem to 'carry less information' or 'be simpler' than those of $\mathbf{FdVect}_{\mathbb{K}}$, but the category-theoretician is not very interested in the internal structure of objects: morphisms matter more. Since every $n \times m$ matrix over $\mathbb{K}$ defines a linear map from an m-dimensional to an n-dimensional $\mathbb{K}$-vector space, there seems to be a canonical mapping for morphisms between the two categories.

But the morphisms have to belong to objects: without a canonical object-to-object map, there can be no canonical morphism-to-morphism map. It is not at all clear from the definition of $\mathbf{FdVect}_{\mathbb{K}}$ whether we regard, say, the Banach spaces $\mathbb{K}^n_{\text{Banach}}$ and the inner product spaces $\mathbb{K}^n_{\text{inner}}$ as separate objects. If they are indeed separate objects, to which one should we map the object n of $\mathbf{FdMat}_{\mathbb{K}}$? And what of the vector space of all $m \times n$ $\mathbb{K}$-matrices: is this the same object as $\mathbb{K}^{mn}_{\text{Banach}}$ or $\mathbb{K}^{mn}_{\text{inner}}$, or is it yet another object?

The most category-theoretical solution is not to worry about such fine-grained distinctions between objects. As long as $\mathbb{K}^{mn}_{\text{matrices}}$, $\mathbb{K}^{mn}_{\text{Banach}}$, $\mathbb{K}^{mn}_{\text{inner}}$, and all other mn-dimensional K-vector spaces are at least isomorphic, it does not matter which one we use. The isomorphisms will 'give us access' to all the others.

In this spirit, a functor like H, which expresses some similarity between two categories, is called an equivalence if it provides a bijective mapping of objects and morphisms of the one category, to objects and morphisms of the other, up to some isomorphism of objects. Here is a more formal definition:

Definition 2.28. A functor $K : \mathbf{C} \to \mathbf{D}$ is an *equivalence* if it is:

(i) *full,* i.e. for all $A, B \in \mathbf{C}_0$, the function

$$\begin{aligned} \mathbf{C}_1\,(A, B) &\to \mathbf{D}_1\,(K(A), K(B)) \\ f &\mapsto K(f) \end{aligned} \tag{2.34}$$

is surjective;

(ii) *faithful,* i.e. for all $A, B \in \mathbf{C}_0$, the function

$$\begin{aligned} \mathbf{C}_1\,(A, B) &\to \mathbf{D}_1\,(K(A), K(B)) \\ f &\mapsto K(f) \end{aligned} \tag{2.35}$$

is injective;

(iii) *essentially surjective on objects (eso),* i.e. for each $D \in \mathbf{D}_0$, there exists a $C \in \mathbf{C}_0$ such that $D \simeq K(C)$.

Categories $\mathbf{C}$ and $\mathbf{D}$ are *equivalent* if there exists an equivalence $\mathbf{C} \to \mathbf{D}$.

Consider again the functor $F : \mathbf{NToset} \to \mathbf{NToset}^{\mathrm{op}}$ of example 2.25. For any objects m and n, the hom-sets $\mathbf{NToset}_1\,(m, n)$ and $\mathbf{NToset}^{\mathrm{op}}{}_1\,(2n, 2m)$ either both contain one element (if $m \leq n$), or both are empty (if $m > n$). Therefore, the functions $\mathbf{NToset}_1\,(m, n) \to \mathbf{NToset}^{\mathrm{op}}{}_1\,(2n, 2m)$ are all bijections, so F is full and faithful.

Because morphisms $m \to n$ and $n \to m$ both exist in $\mathbf{NToset}^{\mathrm{op}}$ iff $m = n$, no object in $\mathbf{NToset}^{\mathrm{op}}$ is isomorphic to any object other than itself.

Since, additionally, the object-to-object function of F is not surjective, F is not eso. This means that the categories **NToset** and $\mathbf{NToset}^{\mathrm{op}}$ are not equivalent by F.

Now for the functor $G : \mathbf{Rel} \to \mathbf{NToset}$ of example 2.26. For all objects A and B of **Rel**, G defines a function $\mathbf{Rel}_1(A, B) \to \mathbf{NToset}_1(12, 12)$. Since $\mathbf{NToset}_1(12, 12)$ is a singleton set, all these functions are surjective but they are not all injective, so G is full but not faithful. Since the object-to-object function of G is not surjective on **NToset** and no distinct objects of **NToset** are isomorphic, G is not eso.

As we already hinted, $H : \mathbf{FdMat}_{\mathbb{K}} \to \mathbf{FdVect}_{\mathbb{K}}$ is an equivalence. For each object m of $\mathbf{FdMat}_{\mathbb{K}}$, let $H(m)$ be the inner product space $\mathbb{K}^m$. Since all vector spaces of the same dimension over the same field are isomorphic, every object of $\mathbf{FdVect}_{\mathbb{K}}$ is isomorphic to some object $H(m)$ of $\mathbf{FdVect}_{\mathbb{K}}$, so H is eso.

For all natural numbers m and n, H defines a function

$$H_{m,n} : \mathbf{FdMat}_{\mathbb{K}1}(m, n) \to \mathbf{FdVect}_{\mathbb{K}1}(\mathbb{K}^m, \mathbb{K}^n) \tag{2.36}$$

which sends every $n \times m$ $\mathbb{K}$-matrix $\mathbf{M}$ to the function $|\psi\rangle \mapsto \mathbf{M}|\psi\rangle$. This relation between matrices and linear functions is bijective, so H is both full and faithful. This completes the proof that H is an equivalence.

Remark 2.29. We might expect that the existence of an equivalence functor $\mathbf{C} \to \mathbf{D}$ implies that there also exists an equivalence functor $\mathbf{D} \to \mathbf{C}$. However, this is not immediately obvious from definition 2.28, and the proof given by Mac Lane [47, p. 91 f.] is surprisingly involved. If we had taken a truly categorical approach in this chapter, by introducing universal properties as soon as possible, we could have proven this in a few lines, but that approach has its own problems.

2.5 Groups as categories

The existence of identity morphisms and the associativity of morphism composition are built into the definition of a category. It therefore seems reasonable to define certain algebraic structures by mapping their elements to morphisms. Here we discuss one category-theoretical way to define monoids and groups.

A monoid is usually defined as a set that is closed under some associative binary operation, and has an identity element. Equivalently: a monoid is a group without the requirement that there are inverses. Hence, every group is a monoid. Some examples of monoids that are not groups are the set of natural numbers with the operation of addition, or the rotations of the plane by k radians, where $k = 0, 1, 2, \ldots$, with the operation of composition. Here is a category-theoretical definition:

Definition 2.30. A *monoid* is a (locally) small category with only one object. The identity morphism is the identity element of the monoid, and morphism composition is the monoid operation.

The following examples will help to clarify and motivate the definition:

Example 2.31. Consider the category $\mathbf{RotR^2}$, whose only object is the set $\mathbb{R}^2$. The morphisms of $\mathbf{RotR^2}$ are rotation matrices of the form:

$$R_k = \begin{bmatrix} \cos k & -\sin k \\ \sin k & \cos k \end{bmatrix}, \tag{2.37}$$

where $k = 0, 1, 2, \ldots$ Composition of morphisms is matrix multiplication, which is commutative on these matrices: $R_j \circ R_k = R_{j+k} = R_k \circ R_j$. The identity morphism is

$$R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.38}$$

The set $\mathbf{RotR^2}_1 = \{R_0, R_1, R_2, \ldots\}$ of all morphisms is a monoid: all elements are composable, composition is associative, the set is closed under composition, and there is an identity element. It is easily verified that this applies to all and only all small single-object categories. Hence, the category-theoretic monoid consists of a conventional monoid, *along with an object.*

To see why we have to add an object to the conventional definition, consider the category $\mathbf{N}$ where $\mathbf{N}_1 = \mathbb{N}$, $j \circ k = j + k$, $\mathbf{N}_0 = \{\star\}$, and $\star$ is an unspecified object. Even though the natural numbers do not act on $\star$ in any way, this category fulfils all the requirements of definition 2.1:

(i) $\mathbf{N}_1$ is partitioned into one part by $\mathbf{N}_0$, as $\mathbf{N}_1 = \mathbf{N}_1(\star, \star)$;

(ii) $j \circ k$ is defined for all $j, k \in \mathbf{N}_1(\star, \star)$;

(iii) the composition operation $\circ$ maps $\mathbf{N}_1(\star, \star) \times \mathbf{N}_1(\star, \star)$ to $\mathbf{N}_1(\star, \star)$;

(iv) the operator $\circ$ is associative: $j \circ (k \circ l) = j + (k + l) = (j + k) + l = (j \circ k) \circ l$;

(v) and there is an identity morphism: $0 \circ k = 0 + k = k = k + 0 = k \circ 0$.

Additionally, $\mathbf{N}$ is equivalent to $\mathbf{RotR^2}$ by a functor F, defined as:

$$F(\star) = \mathbb{R}^2 \tag{2.39}$$

$$F(k) = R_k \tag{2.40}$$

We can see then, that $\mathbf{N}$ is an abstract and $\mathbf{RotR^2}$ a concrete monoid, in the same sense that a group can be abstract or concrete. The object of an abstract monoid provides a slot, in which a functor can place an object for a concrete monoid to act upon. In this particular case, the abstract monoid **N** is the natural numbers under addition; $\mathbf{RotR^2}$ is a monoid representation of **N** — a concept completely analogous to that of a group representation.

To move from monoids to groups, we need to add inverses. Since monoid elements are morphisms in a single-object category, and any morphism with an inverse is an isomorphism, we have the following definition of a group:

Definition 2.32. A *group* is a monoid in which all morphisms are isomorphisms.

Here are some examples:

Example 2.33. We can extend the category $\mathbf{RotR^2}$ by adding an inverse morphism

$$R_{-k} = \begin{bmatrix} \cos k & \sin k \\ -\sin k & \cos k \end{bmatrix} \tag{2.41}$$

for each R_k. The result is a group of automorphisms of the plane. It is equivalent to the category **Z**, whose only object is $\star$ and whose morphisms are all the integers, with addition as composition. Just as in **N**, the morphisms of **Z** do not have to act on $\star$ in any way to fulfil the definition of a category.

Example 2.34. Let $\mathbf{C_n}$ denote the category with one object — again, call it $\star$ — and morphisms $c, c^2, c^3, \ldots, c^n$, where $c^n = \mathrm{id}_\star$ and $c^k = \underbrace{c \circ c \circ \cdots \circ c}_{k \text{ copies of } c}$. Every morphism in $\mathbf{C_n}$ can be written as c^k for some k and has c^{n-k} as its inverse, since $c^k \circ c^{n-k} = c^n = c^{n-k} \circ c^k$.

These categories $\mathbf{C_n}$ are exactly the (abstract) finite cyclic groups.

Now let G and H be (conventional) groups, where G has identity element e_G and H has identity e_H. Recall that a group homomorphism from G to H is a function m which:

(i) sends elements of G to elements of H;

(ii) sends identity to identity: $m\,(e_G) = e_H$;

(iii) preserves group structure: for all $f, g \in G$: $m(fg) = m(f)m(g)$.

These are three of the four functor axioms given in definition 2.23; the only thing missing is an object-to-object map.[2] But since a group-as-category has only one object, only one object-to-object map is possible between two groups. Hence, every group homomorphism determines a unique functor between groups, and vice versa. Recall also, that a representation of an abstract group G is a homomorphism from G to the automorphism group of a vector space. This gives us:

Definition 2.35. A *group representation* is a functor from a group to $\mathbf{Vect}_{\mathbb{K}}$.

2.6 Natural transformations

One level higher than the functor, is the natural transformation. Just as a functor is a map between categories, a natural transformation is a map between functors.

Definition 2.36. Given functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$, a *natural transformation* $\tau : F \overset{\bullet}{\to} G$ assigns a morphism τ_A in $\mathbf{D}_1$ to every $A \in \mathbf{C}_0$, such that the following diagram commutes for all $A \xrightarrow{f} B$ in $\mathbf{C}$:

$$\begin{array}{ccc} F(A) & \xrightarrow{\tau_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\tau_B]{} & G(B) \end{array} \tag{2.42}$$

[2] We need not worry about co- or contravariance, because a group is a category with only one object.

We then say that $\tau : F \xrightarrow{\bullet} G$ is *natural in A*. If every τ_A is iso, then τ is a *natural isomorphism* or *natural equivalence* and the functors F and G are *naturally isomorphic* or *naturally equivalent.* Instead of $\tau : F \xrightarrow{\bullet} G$ we then write $\tau : F \cong G$.

Now let F and G be bifunctors $\mathbf{B} \times \mathbf{C} \to \mathbf{D}$. Then τ is *natural in B and C* if the diagrams

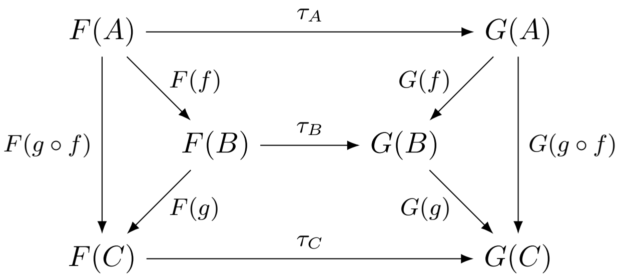
$$\begin{CD} F(B,C) @>{\tau_{B,C}}>> G(B,C) \\ @V{F(f)}VV @VV{G(f)}V \\ F(B',C) @>>{\tau_{B',C}}> G(B',C) \end{CD} \qquad \begin{CD} F(B,C) @>{\tau_{B,C}}>> G(B,C) \\ @V{F(g)}VV @VV{G(g)}V \\ F(B,C') @>>{\tau_{B,C'}}> G(B,C') \end{CD} \tag{2.43}$$

commute for all $(B,C) \xrightarrow{f} (B',C)$ and $(B,C) \xrightarrow{g} (B,C')$. The diagram on the left fixes C and shows that τ is natural in B; the one on the right fixes an object B and shows that τ is natural in C.

Because every morphism $(B,C) \xrightarrow{(f,g)} (B',C')$ in a product category factors as $\left((B,C) \xrightarrow{(f,\mathrm{id}_C)} (B',C)\right) \circ \left((B',C) \xrightarrow{(\mathrm{id}_{B'},g)} (B',C')\right)$, it follows from the functoriality of F and G that naturality in B and C implies naturality in (B,C).

Another, perhaps more intuitive way to phrase the definition of a natural transformation is as follows: $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$ both create an image of $\mathbf{C}_0$ in $\mathbf{D}_0$. When we say that a natural transformation $\tau : F \xrightarrow{\bullet} G$ maps F to G, we mean that it maps the picture drawn by F to the picture drawn by G, while preserving the commutativity of all diagrams. This mapping takes place in the category $\mathbf{D}$, so for every $C \in \mathbf{C}_0$, there must exist a morphism $F(C) \xrightarrow{\tau_C} G(C)$ in $\mathbf{D}$.

Naturality in a given variable means that varying that variable preserves the commutativity of all morphism compositions, as shown in the following diagram, where τ is natural in A:



(2.44)

This commutes because it is composed of three quadrangles ($F(A)-G(A)-G(C)-F(C)$ and the two trapezia), all of which commute by naturality of τ, and two triangles which both commute because F and G are functors.

This definition is best illustrated with some examples:

Example 2.37. Given two $\mathbb{K}$-Hilbert spaces C and D, the set of all bounded linear maps $C \to D$ is itself a $\mathbb{K}$-Hilbert space, so every hom-set in $\mathbf{Hilb}_{\mathbb{K}}$ is also an object of $\mathbf{Hilb}_{\mathbb{K}}$. We can therefore construct a bifunctor $L : \mathbf{Hilb}_{\mathbb{K}}{}^{\mathrm{op}} \times \mathbf{Hilb}_{\mathbb{K}} \to \mathbf{Hilb}_{\mathbb{K}}$ which sends the objects C and D to the Hilbert space $L(C,D) = \mathbf{Hilb}_{\mathbb{K}1}(C,D)$.

For morphisms, L is a map from linear maps between Hilbert spaces to linear maps between Hilbert spaces of linear maps between Hilbert spaces: for all $x \in L(C,D)$, $B \xrightarrow{g} C$, and $D \xrightarrow{h} E$, $L(g,h)$ maps $x \mapsto h \circ x \circ g$. The result is a linear map $B \xrightarrow{g} C \xrightarrow{x} D \xrightarrow{h} E$. If we also have $A \xrightarrow{f} B$ and $E \xrightarrow{i} F$, composition is defined as:

$$(L(f,i) \circ L(g,h))\, x = L(f \circ g, i \circ h)x \tag{2.45}$$

$$= i \circ h \circ x \circ g \circ f. \tag{2.46}$$

This gives a linear function $A \to F$ by a long detour: $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{x} D \xrightarrow{h} E \xrightarrow{i} F$. Note that L is covariant in its second, and contravariant in its first variable: the pair (g, h), consisting of a morphism $B \to C$ and a morphism $D \to E$ is mapped to a morphism $L(C, D) \xrightarrow{L(g,h)} L(B, E)$.

We can also define a bifunctor $\otimes : \mathbf{Hilb}_{\mathbb{K}} \times \mathbf{Hilb}_{\mathbb{K}} \to \mathbf{Hilb}_{\mathbb{K}}$ which maps Hilbert spaces B and C to $B \otimes C$. For every object (B, C) in $\mathbf{Hilb}_{\mathbb{K}} \times \mathbf{Hilb}_{\mathbb{K}}$ there then exists a bounded linear function $\phi_{B,C} : L(B, C) \to B \otimes C$, given by:

$$\sum_{j,k} m_{j,k} |c_k\rangle\langle b_j| \mapsto \sum_{j,k} m_{j,k} |b_j\rangle \otimes |c_k\rangle \tag{2.47}$$

where $\{|b_j\rangle\}_j$ and $\{|c_k\rangle\}_k$ are orthonormal bases of B and C.

Now let $A \xrightarrow{f} B = \sum_{i,j} f_{ij} |b_j\rangle\langle a_i|$ and $C \xrightarrow{g} D = \sum_{k,l} g_{kl} |d_l\rangle\langle c_k|$, where $\{|a_i\rangle\}_i$ and $\{|d_l\rangle\}_l$ are orthonormal bases of A and D. We then have:

$$L(f, g) \sum_{j,k} m_{jk} |c_k\rangle\langle b_j| = \sum_{i,j,k,l} m_{jk} f_{ik} g_{jl} |d_l\rangle\langle a_i| \tag{2.48}$$

$$[f \otimes g] \sum_{i,j} m_{jk} |b_j\rangle \otimes |c_k\rangle = \sum_{i,j,k,l} m_{jk} f_{ik} g_{jl} |a_i\rangle \otimes |d_l\rangle. \tag{2.49}$$

This in turn gives us $\phi_{A,D} \circ L(f, g) = (f \otimes g) \circ \phi_{B,C}$. Since $\phi_{A,D}$, $\phi_{B,C}$, $L(f, g)$, and $f \otimes g$ are all morphisms of $\mathbf{Hilb}_{\mathbb{K}}$, that is equivalent to stating that the following diagram commutes:

$$\begin{array}{ccc} L(B, C) & \xrightarrow{\phi_{B,C}} & B \otimes C \\ {\scriptstyle L(f,g)}\downarrow & & \downarrow{\scriptstyle f \otimes g} \\ L(A, D) & \xrightarrow[\phi_{A,D}]{} & A \otimes D \end{array} \tag{2.50}$$

If we let $f = \mathrm{id}_B$ or $g = \mathrm{id}_C$, then this proves that ϕ is natural in B and C. Because every $\phi_{B,C}$ is a bijection, ϕ is a natural isomorphism $L \cong \otimes$.

Example 2.38 (Group intertwiner)**.** Fix a natural number $n > 0$ and consider the functors $F : \mathbf{C_n} \to \mathbf{FdVect}_{\mathbb{R}}$ and $G : \mathbf{C_n} \to \mathbf{FdVect}_{\mathbb{R}}$, given by:

$$F(\star) = G(\star) = \mathbb{R}^3 \tag{2.51}$$

$$F\left(c^k\right) = \begin{bmatrix} \cos\left(\frac{2k\pi}{n}\right) & -\sin\left(\frac{2k\pi}{n}\right) & 0 \\ \sin\left(\frac{2k\pi}{n}\right) & \cos\left(\frac{2k\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.52}$$

$$G\left(c^k\right) = \begin{bmatrix} \cos\left(\frac{2k\pi}{n}\right) & 0 & -\sin\left(\frac{2k\pi}{n}\right) \\ 0 & 1 & 0 \\ \sin\left(\frac{2k\pi}{n}\right) & 0 & \cos\left(\frac{2k\pi}{n}\right) \end{bmatrix} \tag{2.53}$$

F and G are both group representations, where $F(\mathbf{C_n})$ is the group of rotations of $\mathbb{R}^3$ around the z-axis by increments of $\frac{2\pi}{n}$ radians and $G(\mathbf{C_n})$ is the group of rotations of $\mathbb{R}^3$ around the y-axis by increments of $\frac{2\pi}{n}$ radians.

A natural transformation $\iota : F \xrightarrow{\bullet} G$ must assign to $\star$ a morphism $F(\star) \xrightarrow{\iota_\star} G(\star)$ such that the following commutes for all c^k:

$$\begin{array}{ccc} F(\star) & \xrightarrow{\iota_\star} & G(\star) \\ {\scriptstyle F(c^k)}\downarrow & & \downarrow{\scriptstyle G(c^k)} \\ F(\star) & \xrightarrow[\iota_\star]{} & G(\star) \end{array} \tag{2.54}$$

That is, $\iota_\star$ is a matrix that solves

$$\iota_\star F\left(c^k\right) = G\left(c^k\right)\iota_\star \tag{2.55}$$

for all k. The matrix

$$\iota_\star = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.56}$$

is a solution: it is easily verified that

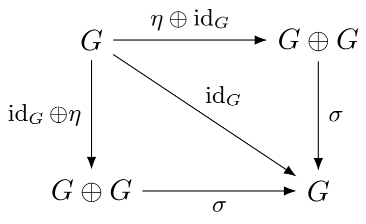
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\left(\frac{2k\pi}{n}\right) & -\sin\left(\frac{2k\pi}{n}\right) & 0 \\ \sin\left(\frac{2k\pi}{n}\right) & \cos\left(\frac{2k\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\left(\frac{2k\pi}{n}\right) & 0 & -\sin\left(\frac{2k\pi}{n}\right) \\ 0 & 1 & 0 \\ \sin\left(\frac{2k\pi}{n}\right) & 0 & \cos\left(\frac{2k\pi}{n}\right) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \quad (2.57)$$

What this tells us is that $F(\mathbf{C_n})$ and $G(\mathbf{C_n})$ are similar, in that the action of $\iota_\star$ — namely, swapping the y and z-axis in $\mathbb{R}^3$ — swaps between the actions of both concrete groups. Since $\iota_\star$ is an invertible matrix, it is an isomorphism on $\mathbb{R}^3$, so ι is not just a natural transformation, but also a natural isomorphism.

Of course, it would have been easier to figure this out by examining the matrices directly instead of looking for a natural transformation. In the next example, we use a natural transformation to define groups — not as categories by themselves, but as objects in categories — and already set up some ideas that will return in the chapter on monoidal categories. The example after that is far shorter, and brings us back to more standard linear algebra.

Example 2.39. Let **Ab** be the category whose objects are all the Abelian groups and whose morphisms are all their homomorphisms. It has finite products, given by the direct sum, and for every object there must exist an associative addition operation, an identity element, and inverses. We first use products and morphisms to provide an identity element, an addition operation, and inverses. Then we postulate a natural transformation to ensure the addition is associative.

First, we require that for each group G in **Ab** there exist a morphism $G \oplus G \xrightarrow{\sigma} G$ such that the following commutes:



(2.58)

We call σ the addition of G: it maps any pair of group elements to the sum of those elements in G. Equation (2.58) tells us that for any element g of G:

$$\sigma(\eta(g), g) = g = \sigma(g, \eta(g)). \tag{2.59}$$

Clearly then, η is a morphism which turns any element of G into the identity element.

By definition, such a morphism must exist. Since **Ab** has a zero object, there must exist homomorphisms $G \xrightarrow{0} 0$ and $0 \xrightarrow{\eta'} G$ for every group G, and since the image of a homomorphism from a group of order 1 to any other group contains only the identity of the target group, we can construct η as

$$\eta = \eta' \circ 0. \tag{2.60}$$

We use a similar tactic to create inverses. Let $G \xrightarrow{i} G$ be a homomorphism that maps each element of G to its inverse. (Note that "element" means "morphism in a category that is an Abelian group", and that such a homomorphism can only exist if G is Abelian.[3]) Then we require that the

[3] A similar construction can be done for the category of all groups, if we take as our morphisms the homomorphisms and the antihomomorphisms.

following diagram commutes for every G:

$$\begin{array}{ccccc} G\oplus G & \xleftarrow{\mathrm{id}_G\oplus\mathrm{id}_G} & G & \xrightarrow{\mathrm{id}_G\oplus\mathrm{id}_G} & G\oplus G \\ \Big\downarrow{\scriptstyle i\oplus\mathrm{id}_G} & & \Big\downarrow{\scriptstyle \eta} & & \Big\downarrow{\scriptstyle \mathrm{id}_G\oplus i} \\ G\oplus G & \xrightarrow[\sigma]{} & G & \xleftarrow[\sigma]{} & G\oplus G \end{array} \tag{2.61}$$

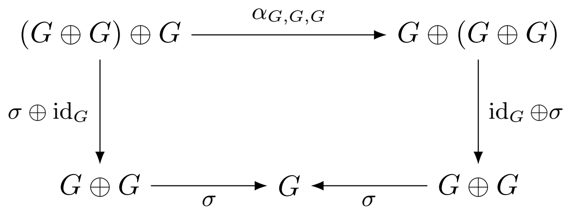
The right-hand rectangle of eq. (2.61) tells us that $\sigma(g, i(g)) = \eta(g)$, i.e. that adding a group element to its inverse on the right is the same as turning that element into the group identity. The rectangle on the left tells us that $\sigma(i(g), g) = \eta(g)$.

So far, we have seen that every object in a category with finite products and a zero object, in which eq. (2.58) and eq. (2.61) commute for every object G, fulfils all the group axioms except associativity. To satisfy this last condition, we need a natural isomorphism. Let $\oplus(\oplus, 1) : (\mathbf{Ab}\times\mathbf{Ab})\times\mathbf{Ab} \to \mathbf{Ab}$ and $\oplus(1, \oplus) : \mathbf{Ab} \times (\mathbf{Ab} \times \mathbf{Ab}) \to \mathbf{Ab}$, where 1 is the identity functor on $\mathbf{Ab}$, be functors that map the three-fold products of the category $\mathbf{Ab}$ to the subcategory of three-fold direct sums of groups, and let there be an *associator natural isomorphism* $\alpha : \oplus(\oplus, 1) \cong \oplus(1, \oplus)$ such that the following commutes for all objects $G_1, G_2, G_3,\ H_1, H_2, H_3$ and morphisms $G_k \xrightarrow{f_k} H_k$:

$$\begin{array}{ccc} (G_1\oplus G_2)\oplus G_3 & \xrightarrow{\alpha_{G_1,G_2,G_3}} & G_1\oplus(G_2\oplus G_3) \\ \Big\downarrow{\scriptstyle (f_1\oplus f_2)\oplus f_3} & & \Big\downarrow{\scriptstyle f_1\oplus(f_2\oplus f_3)} \\ (H_1\oplus H_2)\oplus H_3 & \xrightarrow[\alpha_{H_1,H_2,H_3}]{} & H_1\oplus(H_2\oplus H_3) \end{array} \tag{2.62}$$

The above diagram tells us, that applying group homomorphisms before switching around the parentheses is the same as applying them after moving the parentheses.

In order for G and σ to satisfy the group associativity axiom, we need the following diagram to commute:



(2.63)

This uses the natural isomorphism α to tell us that for any elements f, g, h of G:

$$\sigma(\sigma(f, g), h) = \sigma(f, \sigma(g, h)). \tag{2.64}$$

In general, we call diagrams like eq. (2.62) the *naturality conditions* for natural transformations, and diagrams like eq. (2.63) the *coherence conditions* for categories.

Having seen how functors and natural transformations can work together to define algebraic structures, we now move on to a more concise example:

Example 2.40 (Mac Lane [47, p. 16])**.** Let **CRing** be the category whose objects are all the commutative rings and whose morphisms are all their homomorphisms. Let **Grp** be the category of all groups with all group homomorphisms. Let $\mathrm{GL}_n : \mathbf{CRing} \to \mathbf{Grp}$ be a functor that maps each $K \in \mathbf{CRing}_0$ to the group of all invertible $n \times n$ matrices over K, and each homomorphism $f \in \mathbf{CRing}_1$ to a function that maps the matrix with entries k_{ij} to the matrix with entries $f(k_{ij})$. And let $(-)^\times : \mathbf{CRing} \to \mathbf{Grp}$ be a functor that maps each $K \in \mathbf{CRing}_0$ to its group of units $K^\times$, and each ring homomorphism $K \xrightarrow{f} K'$ to its restriction to $K^\times$.

Since the determinant $\det_K(\mathbf{M})$ of an invertible matrix $\mathbf{M}$ over the ring

K is an element of $K^{\times}$, and since matrix determinants are calculated the same way for all rings, the following diagram commutes for all K:

$$\begin{array}{ccc} \mathrm{GL}_n(K) & \xrightarrow{\det_K} & K^{\times} \\ {\scriptstyle \mathrm{GL}_n(f)}\downarrow & & \downarrow{\scriptstyle f^{\times}} \\ \mathrm{GL}_n(K') & \xrightarrow[\det_{K'}]{} & K'^{\times} \end{array} \tag{2.65}$$

Hence, det is a natural transformation $\mathrm{GL}_n \xrightarrow{\bullet} (-)^{\times}$. Because taking determinants is not an invertible operation, det is not a natural isomorphism.

We now discuss some properties of natural transformations, which will lead up to the concept of a functor category:

Lemma 2.41. Let $\mathbf{C}$ and $\mathbf{D}$ be any categories, let F, G, H be functors $\mathbf{C} \to \mathbf{D}$, and let $\tau : F \xrightarrow{\bullet} G$, $\upsilon : G \xrightarrow{\bullet} H$ be natural transformations. Then τ and υ can be composed to construct a natural transformation $\phi : F \xrightarrow{\bullet} H$.

Proof. The construction is as follows:

$$\begin{array}{ccccc} F(d) & & \xrightarrow{\phi_d = \upsilon_d \circ \tau_d} & & H(d) \\ & \searrow^{\tau_d} & & \nearrow^{\upsilon_d} & \\ & & G(d) & & \\ {\scriptstyle F(f)}\downarrow & & {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle H(f)} \\ & & G(d') & & \\ & \nearrow^{\tau_{d'}} & & \searrow^{\upsilon_{d'}} & \\ F(d') & & \xrightarrow[\phi_{d'} = \upsilon_{d'} \circ \tau_{d'}]{} & & H(d') \end{array} \tag{2.66}$$

The naturality quadrangles for τ and υ, and the two triangles all commute by definition, so the entire diagram commutes, with the rectangle stating the naturality conditions for ϕ. We write $\phi = \upsilon\tau$ for this composition. $\square$

Lemma 2.42. For any functor G, there exists an identity natural isomorphism $1 : G \cong G$:

$$\begin{CD} G(c) @>{1_c = \mathrm{id}_{G(c)}}>> G(c) \\ @V{G(f)}VV @VV{G(f)}V \\ G(c') @>>{1_c = \mathrm{id}_{G(c')}}> G(c') \end{CD} \tag{2.67}$$

For any functors F, H and natural transformations $\tau : F \xrightarrow{\bullet} G$, $\upsilon : G \xrightarrow{\bullet} H$, we then have $1\tau = \tau$ and $\upsilon 1 = \upsilon$.

These properties motivate the following definition:

Definition 2.43. Given categories $\mathbf{C}$ and $\mathbf{D}$, their *functor category* $[\mathbf{C}, \mathbf{D}]$ has all the functors $\mathbf{C} \to \mathbf{D}$ as its objects, and all the natural transformations between such functors as its morphisms. Lemmata 2.41 and 2.42 prove that $[\mathbf{C}, \mathbf{D}]$ is indeed a category.

Here are some examples:

Example 2.44. If $\mathbf{G}$ is a group, then $[\mathbf{G}, \mathbf{Vect}_{\mathbb{K}}]$ has all the representations of $\mathbf{G}$ on $\mathbb{K}$-vector spaces as its objects, and all the intertwiners between representations as its morphisms.

Example 2.45. If $\mathbf{P}$ and $\mathbf{Q}$ are poset categories ordered by $\preceq$, then the objects of $[\mathbf{P}, \mathbf{Q}]$ are the order-preserving maps from $\mathbf{P}$ to $\mathbf{Q}$. Now let F, G be objects of $[\mathbf{P}, \mathbf{Q}]$. In order for there to exist a natural transformation $\tau : F \xrightarrow{\bullet} G$, there must exist morphisms $F(p) \xrightarrow{\tau_p} G(p)$ and $F(p') \xrightarrow{\tau_{p'}} G(p')$ in $\mathbf{Q}$ for all $p, p' \in \mathbf{P}_0$ where $p \preceq p'$, since otherwise the diagram

$$\begin{CD} F(p) @>{\tau_p}>> G(p) \\ @VVV @VVV \\ F(p') @>>{\tau_{p'}}> G(p') \end{CD} \tag{2.68}$$

does not exist, let alone commute. Therefore, $\tau : F \xrightarrow{\bullet} G$ exists if and only if for all p, p':

$$p \preceq p' \Rightarrow F(p) \preceq G(p) \wedge F(p') \preceq G(p'). \tag{2.69}$$

Hence, $[\mathbf{P}, \mathbf{Q}]$ is a partial order of order-preserving maps between posets, where $F \preceq G$ iff there exists a natural transformation $F \xrightarrow{\bullet} G$. If we let $p' = p$ in eq. (2.69), we get:

$$F \preceq G \iff \forall p : F(p) \preceq G(p). \tag{2.70}$$

In particular, any category that can be written as $[\mathbf{P}, \mathbf{P}]$, $[[\mathbf{P}, \mathbf{P}], [\mathbf{P}, \mathbf{P}]]$, $[[[\mathbf{P}, \mathbf{P}], [\mathbf{P}, \mathbf{P}]], [[\mathbf{P}, \mathbf{P}], [\mathbf{P}, \mathbf{P}]]]$, aut cetera, is a poset. It might not even seem far-fetched to combine these various structures into one: a poset, equipped with order-preserving maps, between which there exist order-preserving maps, between which there exist order-preserving maps, et cetera. Since every pair of consecutive rungs in this ladder forms a category, does that mean we can define “higher-order categories”? We answer this question in the next chapter.

Chapter 3

Monoidal categories

Monoidal categories generalise categories of vector spaces equipped with tensor products. This broader structure includes categories of modules with their tensors, functor categories with "horizontal composition", and poset categories with initial or terminal objects and finite coproducts or products. Most important for our purposes is that categorical quantum mechanics takes place inside monoidal categories.

The standard way of introducing monoidal categories is to display one or two pages worth of axioms and commutative diagrams, including the fearsome pentagon equation (examples include Mac Lane [47], and Heunen and Vicary [35]). Others, such as Awodey, make these structures a bit more intuitive by first pointing out the classical monoidal structure of strict monoidal categories [3, p. 79]. But when the time comes to weaken the strict definition down to isomorphism, Awodey too resorts to spelling out the long definition [3, pp. 168–171]. A third approach is that taken by the nLab, which mentions in passing, alongside the usual definition, that a monoidal category is a weak 2-category with only one 0-cell [51]. Our introduction to monoidal categories combines the best of Awodey's and the nLab's pedagogy: first we discuss the strict monoidal structure of a strict 2-category with a single 0-cell, then we generalise to weak 2-categories, after which we justify the pentagon equation in terms of the conventual definition. Finally, we discuss some applications and the graphical calculus.

To do so, we first list several categories with tensor-like bifunctors (§ 3.1), and show in § 3.2 that one of these is equivalent to a strict 2-category with only one 0-cell (independently defined by Samuel Eilenberg and Max Kelly [21], and Jean-Marie Maranda [49]). To describe the other examples in a similar manner, we move on to a more general structure, namely the weak 2-category, as defined by Bénabou [9]. In § 3.3 we define this structure and show how each of our examples is equivalent to a weak 2-category with one 0-cell. That is our first definition of a monoidal category.

Because one of its elements is poorly motivated, we discuss the conventional definition in terms of 1-categories in § 3.4, following the original questions that Bénabou [7] and Mac Lane [48] asked, and the answers they gave. That is our second (but historically the first) definition of a monoidal category.

Equipped with two ways of studying the same structure, we then examine the way in which these two definitions are the same. This leads to the concepts of vertical categorification and oidification. We then have the necessary background to discuss the linear structure of monoidal categories: we start with scalars (§ 3.6), followed by braiding and symmetry (§ 3.7), a more general perspective on adjunctions (§ 3.8), and finally closed and compact closed structure (§ 3.9). This material naturally leads to a graphical calculus for tensor products, which we discuss in § 3.10.

3.1 Some motivating examples

Example 3.1. For any category $\mathbf{C}$, consider the category $[\mathbf{C}, \mathbf{C}]$ of all endofunctors on $\mathbf{C}$. Since the objects of $[\mathbf{C}, \mathbf{C}]$ are functors, they can be composed like functions: let $[G \bullet F]\, x = G\,(F(x))$ for all $F, G \in [\mathbf{C}, \mathbf{C}]_0$, with $x \in \mathbf{C}_0$ or $x \in \mathbf{C}_1$; and for any morphisms $\tau : F \overset{\bullet}{\to} F'$, $\upsilon : G \overset{\bullet}{\to} G'$, let $\upsilon \bullet \tau$ be a natural transformation $G \bullet F \overset{\bullet}{\to} G' \bullet F'$.

$G \bullet F$ is guaranteed to exist in $[\mathbf{C}, \mathbf{C}]$ because it is an endofunctor on $\mathbf{C}$, as is $G' \bullet F'$. Also, note that $\tau : F \overset{\bullet}{\to} F'$ implies $G(\tau) : G \bullet F \overset{\bullet}{\to} G \bullet F'$. Therefore, $\upsilon \bullet \tau$ can be constructed as follows:

$$(\upsilon \bullet \tau)_x = \upsilon_{F'(x)} \circ G(\tau_x), \tag{3.1}$$

so $\upsilon \bullet \tau$ is guaranteed to exist in $[\mathbf{C}, \mathbf{C}]$.

Now let id_C be the identity functor on $\mathbf{C}$. Then for all objects F, F', G, G', H and all morphisms $F \overset{\tau}{\to} F', G \overset{\upsilon}{\to} G'$ of $[\mathbf{C}, \mathbf{C}]$, we have:

$$\mathrm{id}_C \bullet F = F = F \bullet \mathrm{id}_C \tag{3.2}$$

$$[G \bullet F] \xrightarrow{\upsilon \bullet \tau} [G' \bullet F'] \tag{3.3}$$

$$H \bullet [G \bullet F] = [H \bullet G] \bullet F \tag{3.4}$$

Example 3.2. Let $\mathbf{1}$ be the category with a single object $\star$ and a single morphism $\mathrm{id}_\star$, and let $\mathbf{Cat}$ be the category whose objects are all the small categories, and whose morphisms are all the functors between small categories. Let $\mathbf{A} \times \mathbf{B}$ be the product category of $\mathbf{A}$ and $\mathbf{B}$, and let $F \times G = (F, G)$. Then for any $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}', \mathbf{C} \in \mathbf{Cat}_0$, and functors $F : \mathbf{A} \to \mathbf{A}'$, $G : \mathbf{B} \to \mathbf{B}'$, we have:

$$\mathbf{1} \times \mathbf{A} \simeq \mathbf{A} \simeq \mathbf{A} \times \mathbf{1} \tag{3.5}$$

$$(\mathbf{A} \times \mathbf{B}) \xrightarrow{F \times G} (\mathbf{A}' \times \mathbf{B}') \tag{3.6}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \simeq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \tag{3.7}$$

Example 3.3. Let $\otimes$ be the usual tensor product on $\mathbf{Vect}_{\mathbb{K}}$. Then for any objects A, A', B, B', C and morphisms $A \xrightarrow{f} A'$, $B \xrightarrow{g} B'$ of $\mathbf{Vect}_{\mathbb{K}}$, we have:

$$\mathbb{K} \otimes A \simeq A \simeq A \otimes \mathbb{K} \tag{3.8}$$

$$(A \otimes B) \xrightarrow{f \otimes g} (A' \otimes B') \tag{3.9}$$

$$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C \tag{3.10}$$

The common thread between all these examples is the existence of a bifunctor under which the set of *objects* of each category becomes a monoid up to isomorphism. This suggests that we can regard the objects of $[\mathbf{C}, \mathbf{C}]$, $\mathbf{Cat}$, and $\mathbf{Vect}_{\mathbb{K}}$ as morphisms of some categorical monoid, and the morphisms of $[\mathbf{C}, \mathbf{C}]$, $\mathbf{Cat}$, $\mathbf{Vect}_{\mathbb{K}}$ then become higher-order morphisms — i.e. morphisms between morphisms. We now make this idea more precise.

3.2 Strict 2-categories

Consider again the category **Cat**. Its objects are all the small categories, and its morphisms are all the functors between small categories. On top of all this, there are also natural transformations which map morphisms — i.e. functors — to morphisms, and therefore are "higher-order morphisms" or "2-morphisms". We call **Cat**-with-natural-transformations the *strict 2-category* **Cat**. It consists of:

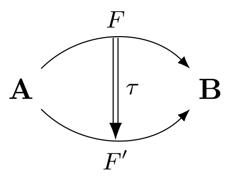
(i) a class $\mathbf{Cat}_0$ of categories;

(ii) a class $\mathbf{Cat}_1$ of functors;

(iii) a class $\mathbf{Cat}_2$ of natural transformations.

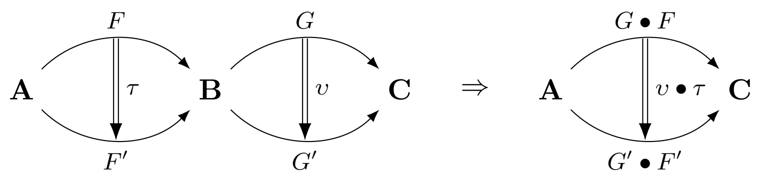
$\mathbf{Cat}_1$ is partitioned by the ordered pairs of $\mathbf{Cat}_0$, and $\mathbf{Cat}_2$ is partitioned by the ordered pairs of $\mathbf{Cat}_1$, so that for all $\mathbf{A}, \mathbf{B} \in \mathbf{Cat}_0$, there exists a category $[\mathbf{A}, \mathbf{B}]$.

In diagrams, we use single arrows for functors, and double arrows for natural transformations:



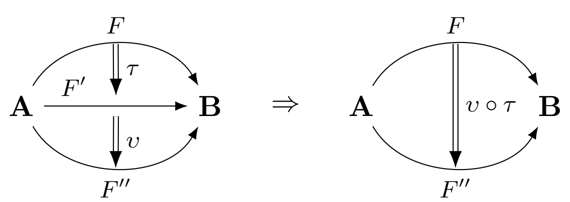
(3.11)

Because **Cat** has categorical structure on multiple levels, natural transformations can be composed in multiple ways. There is "horizontal" composition, like this:



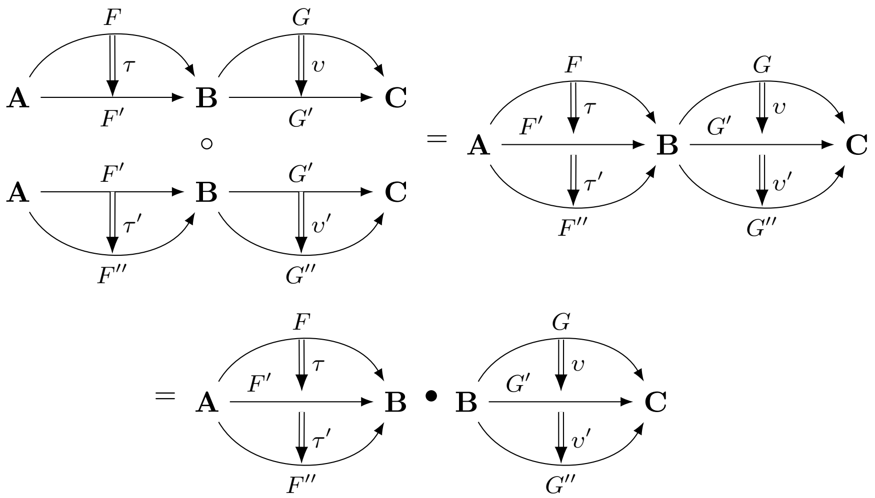
(3.12)

and "vertical" composition, like this:



(3.13)

By pasting curved-morphism diagrams together and then pulling them apart, we get:



(3.14)

which proves that $(\upsilon' \bullet \tau') \circ (\upsilon \bullet \tau) = (\upsilon' \circ \upsilon) \bullet (\tau' \circ \tau)$. Similarly, when $\mathbf{Vect}_{\mathbb{K}}$ is equipped with the tensor product $\otimes$, we have $(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$.

By now, the similarities between examples 3.1 to 3.3 and the strict 2-category **Cat** are clear enough to motivate the following definition:

Definition 3.4 (Eilenberg and Kelly [21], Maranda [49])**.** A *strict 2-category* $\mathbf{C}$ (formerly known as a *hypercategory, category of the second type,* or *2-category*) consists of a class $\mathbf{C}_0$ of *0-cells*, a class $\mathbf{C}_1$ of *1-cells*, and a class $\mathbf{C}_2$ of *2-cells*, such that:

(i) $\mathbf{C}_0$ and $\mathbf{C}_1$ form a category;

(ii) for every pair (A, B) of 0-cells, there exists a *hom-category* $\mathbf{C}(A, B)$, whose objects are all the 1-cells in $\mathbf{C}_1(A, B)$, and whose morphisms are 2-cells in $\mathbf{C}_2(A, B)$;

(iii) every 2-cell is a morphism of exactly one hom-category;

(iv) any pair (τ, υ) of 2-cells with $\operatorname{dom} \upsilon = \operatorname{cod} \tau$ can be *vertically composed;*

(v) for each triple (A, B, C) of 0-cells, there exists a *horizontal composition bifunctor* $\Box : \mathbf{C}(B, C) \times \mathbf{C}(A, B) \to \mathbf{C}(A, C)$.

We write $\tau : f \Rightarrow f' : A \to B$ to indicate that τ is a 2-cell with domain f and codomain f', where f and f' are both 1-cells with domain A and codomain B.

Vertical composition of 2-cells is morphism composition in a hom-category. Given 2-cells $\tau : f \Rightarrow f' : A \to B$ and $\upsilon : f' \Rightarrow f'' : A \to B$ in $\mathbf{C}(A, B)$, their vertical composition is $\upsilon\tau : f \Rightarrow f'' : A \to B$.

Given $\tau : f \Rightarrow f' : A \to B$, $\upsilon : g \Rightarrow g' : B \to C$, and $\phi : h \Rightarrow h' : C \to D$ in a strict 2-category, horizontal composition is associative on 1-cells:

$$h\Box(g\Box f) = (h\Box g)\Box f \tag{3.15}$$

so we can leave out the parentheses and write both sides of the equation as $h\Box g\Box f$. In fact, horizontal composition on 1-cells is simply morphism composition in the category $\mathbf{C}$. The horizontal composition of the 2-cells τ and υ is $\upsilon\Box\tau : gf \Rightarrow g'f' : A \to C$. Here, too, $\Box$ is associative:

$$\phi\Box(\upsilon\Box\tau) = (\phi\Box\upsilon)\Box\tau : h\Box g\Box f \Rightarrow h'\Box g'\Box f' : A \to D \tag{3.16}$$

Remark 3.5. This definition can be extended upwards to make strict 3-categories, 4-categories, and *n-categories*, all the way up to strict *∞-categories*. It also extends downwards: a strict *1-category* is just a category, a *0-category* is a class, and, less obviously, a (-1)-category is a truth value. Alternatively, we could say that every strict n-category is a strict ∞-category in which all the k-cells for $k > n$ are identities [5, pp. 10–13].

A direct consequence of eq. (3.15), is that for any 0-cells A, B, C, D, E and 1-cells $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$, $D \xrightarrow{i} E$ in a strict 2-category, we have:

$$\begin{aligned} i\Box(h\Box(g\Box f)) &= (i\Box h)\Box(g\Box f) = ((i\Box h)\Box g)\Box f \\ &= (i\Box(h\Box g))\Box f = i\Box((h\Box g)\Box f) \end{aligned} \tag{3.17}$$

where $\Box$ is the composition operator for 1-cells, and that for every 0-cell C, there exists a 1-cell id_C such that

$$(g\Box\,\mathrm{id}_C)\,\Box f = g\Box f = g\Box\,(\mathrm{id}_C\,\Box f) \tag{3.18}$$

for all 1-cells f, g with $\operatorname{dom} g = C = \operatorname{cod} f$.

Note also, that if we read the bifunctor $\bullet$ in example 3.1 as a horizontal composition functor on 1-cells and 2-cells, it becomes clear that $[\mathbf{C}, \mathbf{C}]$ is equivalent to a strict 2-category whose only 0-cell is $\mathbf{C}$, whose 1-cells are the functors $\mathbf{C} \to \mathbf{C}$, and whose 2-cells are the natural transformations between the 1-cells.

We have thus defined a 2-category whose horizontal composition has the same effect as the functor $\bullet$. Unfortunately, the rather clean framework of strict 2-categories does not allow enough room to do the same for **Cat** with the bifunctor $\times$, or $\mathbf{Vect}_{\mathbb{K}}$ with $\otimes$. For that, we need a more general structure called the weak 2-category.

3.3 Weak 2-categories

A weak 2-category is identical to a strict 2-category, except that eqs. (3.15) to (3.18) both hold only up to isomorphism. For this to be possible, there must exist a set of iso-2-cells for all composable 1-cells f, g, h, i, by which the pentagon equation:

$$
\begin{array}{ccccc}
 & & i\Box\left(h\Box(g\Box f)\right) & & \\
 & \nearrow & & \nwarrow & \\
(i\Box h)\Box(g\Box f) & & & & i\Box\left((h\Box g)\Box f\right) \\
\Uparrow & & & & \Uparrow \\
 & ((i\Box h)\Box g)\Box f & \Longrightarrow & (i\Box(h\Box g))\Box f &
\end{array}
\tag{3.19}
$$

and the triangle equation:

$$
\begin{array}{ccc}
(g\Box \mathrm{id}_C)\Box f & \Longrightarrow & g\Box(\mathrm{id}_C\Box f) \\
 & \searrow \quad \swarrow & \\
 & g\Box f &
\end{array}
\tag{3.20}
$$

both commute (with $\operatorname{dom} g = C = \operatorname{cod} f$). Unfortunately, this greatly complicates the definition:

Definition 3.6 (Bénabou [9])**.** A *weak 2-category* $\mathbf{C}$ (formerly known as a *bicategory*) consists of a class $\mathbf{C}_0$ of *0-cells*, a class $\mathbf{C}_1$ of *1-cells*, and a class $\mathbf{C}_2$ of *2-cells*, such that:

(i) $\mathbf{C}_0$ and $\mathbf{C}_1$ form a category;

(ii) for every pair (A, B) of 0-cells, there exists a *hom-category* $\mathbf{C}(A, B)$, whose objects are all the 1-cells in $\mathbf{C}_1(A, B)$, and whose morphisms are 2-cells in $\mathbf{C}_2(A, B)$;

(iii) every 2-cell is a morphism of exactly one hom-category;

(iv) any pair (τ, υ) of 2-cells with $\operatorname{dom} \upsilon = \operatorname{cod} \tau$ can be *vertically composed;*

(v) for each triple (A, B, C) of 0-cells, there exists a *horizontal composition bifunctor* $\Box : \mathbf{C}(B, C) \times \mathbf{C}(A, B) \to \mathbf{C}(A, C)$;

(vi) there exists an *associator natural isomorphism* $\alpha : (-\Box-)\Box- \xrightarrow{\bullet} -\Box(-\Box-)$, natural in all three variables;

(vii) for every 0-cell A, there exist a *left unitor natural isomorphism* $\lambda^A : \mathrm{id}_A \Box- \xrightarrow{\bullet} 1$ and a *right unitor natural isomorphism* $\rho^A : -\Box\,\mathrm{id}_A \xrightarrow{\bullet} 1$, where 1 is the identity functor on $\mathbf{C}$.

Vertical composition in a weak 2-category is the same as vertical composition in a strict 2-category, and the notation $\tau : f \Rightarrow f' : A \to B$ has the same meaning in both types of 2-categories.

To every triple $f : A \to B$, $g : B \to C$, $h : C \to D$ of 1-cells, the associator assigns an iso-2-cell $\alpha_{h,g,f} : (h\Box g)\Box f \Rightarrow h\Box(g\Box f)$, so that the following commutes for all 2-cells $\tau : f \Rightarrow f'$, $\upsilon : g \Rightarrow g'$, $\phi : h \Rightarrow h'$:

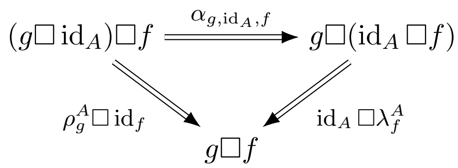
$$\begin{array}{ccc} (h\Box g)\Box f & \xRightarrow{\alpha_{h,g,f}} & h\Box(g\Box f) \\ {\scriptstyle (\phi\Box\upsilon)\Box\tau}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \phi\Box(\upsilon\Box\tau)} \\ (h'\Box g')\Box f' & \xRightarrow[\alpha_{h',g',f'}]{} & h'\Box(g'\Box f') \end{array} \tag{3.21}$$

Additionally, the associator must provide the iso-2-cells for which eq. (3.19) commutes:

$$\begin{array}{ccccc} & & ((i\Box h)\Box g)\Box f & & \\ & {\scriptstyle \alpha_{i,h,g}\Box\,\mathrm{id}_f}\swArrow & & \seArrow{\scriptstyle \alpha_{(i\Box h),g,f}} & \\ (i\Box(h\Box g))\Box f & & & & (i\Box h)\Box(g\Box f) \\ {\scriptstyle \alpha_{i,(h\Box g),f}}\Big\Downarrow & & & & \Big\Downarrow{\scriptstyle \alpha_{i,h,(g\Box f)}} \\ i\Box((h\Box g)\Box f) & & \xRightarrow[\mathrm{id}_i\,\Box\alpha_{h,g,f}]{} & & i\Box(h\Box(g\Box f)) \end{array} \tag{3.22}$$

And finally, the unitors assign iso-2-cells λ^A_g and ρ^A_f to the 1-cells $g\Box\,\mathrm{id}_A$

and $\mathrm{id}_A \Box f$ so that eq. (3.20) commutes:



(3.23)

Corollary 3.7. A strict 2-category is a weak 2-category in which all the α, λ^A, and ρ^A are identities.

Now recall example 3.2, and let the *monoidal category* **Cat** *with tensor product* $\times$ *and unit* **1** be a weak 2-category whose horizontal composition functor is the Cartesian product, whose only 0-cell is an anonymous object o, whose 1-cells are all the small categories, whose 2-cells are all the functors between small categories, and where $\mathrm{id}_o = \mathbf{1}$. Then for all 1-cells $\mathbf{A}, \mathbf{B}, \mathbf{C}$ with objects A, B, C and morphisms a, b, c, the associator assigns to $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ the functor that maps

$$((A, B), C) \mapsto (A, (B, C)) \tag{3.24}$$

$$((a, b), c) \mapsto (a, (b, c)). \tag{3.25}$$

The left unitor assigns to $\mathbf{1} \times \mathbf{A}$ the functor λ^o_A that maps

$$(\star, A) \mapsto A \tag{3.26}$$

$$(\mathrm{id}_\star, a) \mapsto a. \tag{3.27}$$

and the right unitor assigns to $\mathbf{A} \times \mathbf{1}$ the functor ρ^o_A that maps

$$(A, \star) \mapsto A \tag{3.28}$$

$$(a, \mathrm{id}_\star) \mapsto a, \tag{3.29}$$

In general, we define monoidal categories as follows:

Definition 3.8. A *monoidal category with tensor product* $\Box$ *and unit* K is a weak 2-category whose horizontal composition functor is $\Box$, whose only 0-cell is $\star$, and where $\mathrm{id}_\star = K$. Instead of $\lambda_A^\star$ and $\rho_A^\star$, we simply write λ_A and ρ_A.

A *strict monoidal category* is a monoidal category which is also a strict 2-category.

Here are some examples:

Example 3.9. We have already shown that $[\mathbf{C}, \mathbf{C}]$ is equivalent to a strict 2-category with one 0-cell. This makes it a strict monoidal category. Its unit is the identity functor on $\mathbf{C}$.

Example 3.10. $\mathbf{Vect}_{\mathbb{K}}$ with the usual tensor product $\otimes$ is equivalent to a monoidal category with unit $\mathbb{K}$, in which the 1-cells are the $\mathbb{K}$-vector spaces and the 2-cells are the bounded linear maps between them. The associator assigns the bijection

$$\alpha_{A,B,C} : (|a\rangle \otimes |b\rangle) \otimes |c\rangle \mapsto |a\rangle \otimes (|b\rangle \otimes |c\rangle) \tag{3.30}$$

with $|a\rangle \in A$, $|b\rangle \in B$, $|c\rangle \in C$ to each triple (A, B, C) of $\mathbb{K}$-vector spaces, and the unitors assign mappings

$$\lambda_A : 1 \otimes |a\rangle \mapsto |a\rangle \tag{3.31}$$

$$\rho_A : |a\rangle \otimes 1 \mapsto |a\rangle \tag{3.32}$$

to every $\mathbb{K}$-vector space A.

So far, we have defined strict and weak 2-categories, and shown that each of the examples in § 3.1 is equivalent to some type of 2-category with one 0-cell. Unfortunately, our definition of a weak 2-category contains the rather complicated and poorly motivated pentagon equation (3.22). The next section tries to make its presence more palatable.

3.4 Categories with multiplication

Consider some set S with a binary operation $* : S \times S \to S$. If for all $s, t, u \in S : (s * t) * u = s * (t * u)$, then all iterations of the operation $*$ with the same multiplicity, regardless of how we place the parentheses, are equal, and we call $*$ associative. To prove that $*$ is unital, it suffices to show that there exists an element $e \in S$ such that for all $s \in S$: $e * s = s = s * e$. The categorical case is more complicated. For $\mathbf{C}$ a category equipped with a bifunctor $\square : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, associativity of $\square$ is not expressed as an equation, but as a natural isomorphism $\alpha : \square(\square, 1) \cong \square(1, \square)$, where 1 is the identity functor on $\mathbf{C}$. For products of more than three objects, we can "extend" α, as we show below.

A problem then arises: it is sometimes possible for two n-fold products to be isomorphic via two different "extensions" of α, which breaks associativity. We say that $\square$ is *coherently associative* if, for any n-fold iterations F and G of $\square$, there is only one way to "extend" α to form an isomorphism $F \cong G$. As we show below, this fixes associativity.

Unitality is given by two natural isomorphisms, one for the unit on the left, and one for the unit on the right. These can be broken in the same way as the associativity natural isomorphism. $\square$ is a coherently unital operation on $\mathbf{C}$ if such breaking cannot occur.

In the 1960s, various mathematicians working from various directions independently arrived at the question:

Question 3.11. Which limitations must we place on the associativity and unitality transformations in order for $\Box$ to be a coherently associative and unital operation on the objects of $\mathbf{C}$?

In this section, we discuss the answers given by Bénabou [7] and Mac Lane [48]. These lead to a second definition of monoidal categories. While messier than the 2-categorical definition, it provides some more justification for the pentagon equation and provides the necessary background for the coherence theorem, which we will need in order to define the graphical calculus and give monoidal categories their linear structure.

We first define categories with multiplication, and some other auxiliary concepts, and then show how exactly associativity can be broken. We then discuss Mac Lane's proof that the pentagon equation guarantees associativity of all n-fold products, and show how Mac Lane and Bénabou defined what we now call monoidal categories in terms of 1-categories.

Definition 3.12 (Mac Lane [48])**.** $\mathbf{C}$ is a *category with multiplication*[1] $\Box$ if there exists a bifunctor $\Box : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$. Examples include categories with finite products, categories with finite coproducts, and categories with tensor products.

In a category with multiplication, the multiplication can be iterated any number of times:

Definition 3.13. In a category $\mathbf{C}$ with multiplication $\Box$, the identity functor $1 : \mathbf{C} \to \mathbf{C}$ is the *iterate of the functor* $\Box$ *(or:* $\Box$*-iterate) with multiplicity 1.* If F is a $\Box$-iterate with multiplicity m and G is a $\Box$-iterate with multiplicity n, then $\Box(F, G) : \mathbf{C}^{m+n} \to \mathbf{C}$ is a $\Box$*-iterate with multiplicity* $m + n$.

[1] Note that Mac Lane's "categories with multiplication" are not the same as the "catégories avec multiplication" described by Bénabou in [7]. The latter are what we nowadays call monoidal categories.

Now let $\mathbf{C}$ be a category with multiplication $\Box$, and α a natural isomorphism $\Box(\Box, 1) \cong \Box(1, \Box)$. Let K be the unit object of $\Box$ and let λ and ρ be natural isomorphisms $\Box(K, 1) \cong 1$ and $\Box(1, K)$, defined for every $\Box$-iterate A as:

$$\lambda_A : K \Box A \mapsto A \tag{3.33}$$

$$\rho_A : A \Box K \mapsto A \tag{3.34}$$

Also let id be the identity natural isomorphism $1 \cong 1$. We can have α, λ, ρ apply to arbitrarily large $\Box$-iterates by giving them iterates as parameters. If F, G, H are $\Box$-iterates with multiplicities l, m, n, then $\alpha_{F,G,H} : (F \Box G) \Box H \cong F \Box (G \Box H)$ is a natural isomorphism between two iterates of multiplicity $l + m + n$. Similarly, $\lambda_F : K \Box F \cong F$ and $\rho_F : F \Box K \cong F$ both map a $\Box$-iterate of multiplicity $l + 1$ to a $\Box$-iterate of multiplicity l. We can restrict α, λ, ρ to subformulae of iterates by $\Box$-iterating them with id. For $\Box$-iterates E, F, G, H, for example, $\mathrm{id}_E \Box \alpha_{F,G,H}$ is the natural isomorphism $E \Box ((F \Box G) \Box H) \cong E \Box (F \Box (G \Box H))$.

We can now construct a category of iterates of $\Box$ on $\mathbf{C}$:

Definition 3.14. The objects of $\mathbf{It}\,(\mathbf{C}, \Box)$ are all the $\Box$-iterates of all multiplicities. Its morphisms are the following natural transformations (along with all their compositions):

(i) for any iterate A, $\mathbf{It}\,(\mathbf{C}, \Box)$ has λ_A, λ_A^{inv}, ρ_A, ρ_A^{inv}, and an identity natural isomorphism $\mathrm{id}_A : A \xrightarrow{\bullet} A$;

(ii) for any triple A, B, C of iterates, $\mathbf{It}\,(\mathbf{C}, \Box)$ has $\alpha_{A,B,C}$ and $\alpha_{A,B,C}^{\mathrm{inv}}$;

(iii) for any pair $f : A \xrightarrow{\bullet} A'$, $g : B \xrightarrow{\bullet} B'$ of morphisms in $\mathbf{It}\,(\mathbf{C}, \Box)$, there is also a morphism $f \Box g : A \Box B \xrightarrow{\bullet} A' \Box B'$ in $\mathbf{It}\,(\mathbf{C}, \Box)$.

Note that every morphism in $\mathbf{It}\,(\mathbf{C}, \Box)$ is iso.

The morphisms of $\mathbf{It}\,(\mathbf{C}, \Box)$ represent all the ways in which we can eliminate or introduce units, or shift parentheses in iterates of $\Box$. But while the definitions of these isomorphisms closely match the unitality and associativity equations for sets, they are not as powerful. Consider the following example where associativity and unitality are broken in a category with multiplication:

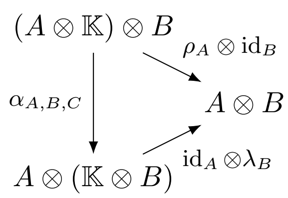
Example 3.15. In the category $\mathbf{Hilb}_{\mathbb{K}}$, let $\otimes$ be the usual tensor product, and $\mathbb{K}$ the unit object. For all vectors $|a\rangle \in A$, $|b\rangle \in B$, $|c\rangle \in C$, we define α, λ, and ρ as follows:

$$\alpha_{A,B,C} : (|a\rangle \otimes |b\rangle) \otimes |c\rangle \mapsto -|a\rangle \otimes (|b\rangle \otimes |c\rangle) \tag{3.35}$$

$$\lambda_A : 1 \otimes |a\rangle \mapsto -|a\rangle \tag{3.36}$$

$$\rho_A : |a\rangle \otimes 1 \mapsto -|a\rangle \tag{3.37}$$

The objects of $\mathbf{It}\,(\mathbf{Hilb}_{\mathbb{K}}, \otimes)$ are all the possible tensors on $\mathbb{K}$-Hilbert spaces; its morphisms are all the ways of shifting parentheses or adding or removing units in $\mathbb{K}$-Hilbert space tensors. Hence every well-formed diagram in $\mathbf{It}\,(\mathbf{Hilb}_{\mathbb{K}}, \otimes)$ should commute, since the order in which we shift around parentheses and add or remove units in a tensor should not matter. But consider the following diagram:



(3.38)

The diagram is well-formed because every morphism is connected to the correct domain and codomain, but it does not commute: the results of both paths from $(A \otimes \mathbb{K}) \otimes B$ to $A \otimes B$ differ by a minus sign. This means that with associativity and unitality defined as above, the order in which we shift parentheses and remove units changes the outcome.

In order to avoid such non-associativity and non-unitality, we need to ensure that any two morphisms in $\mathbf{It}\,(\mathbf{C}, \Box)$ with the same domain and codomain are identical, in which case we call $\mathbf{It}\,(\mathbf{C}, \Box)$ *coherent* and say that $\Box$ is *coherently* unital and associative. Bénabou was the first to observe this [7, ax. 1], albeit in very different terms, but Mac Lane was the first to provide a simple set of constraints on α, λ, and ρ that guarantee such coherence.

To make $\mathbf{It}\,(\mathbf{C}, \Box)$ coherent, we first construct the wide subcategory $\mathbf{it}\,(\mathbf{C}, \Box)$ of $\mathbf{It}\,(\mathbf{C}, \Box)$ as follows:

(i) for each $\Box$-iterate A: id_A is a morphism in $\mathbf{it}\,(\mathbf{C}, \Box)$;

(ii) for all $\Box$-iterates A, A', A'', $\mathbf{it}\,(\mathbf{C}, \Box)$ contains a morphism $\alpha_{A,A',A''}$;

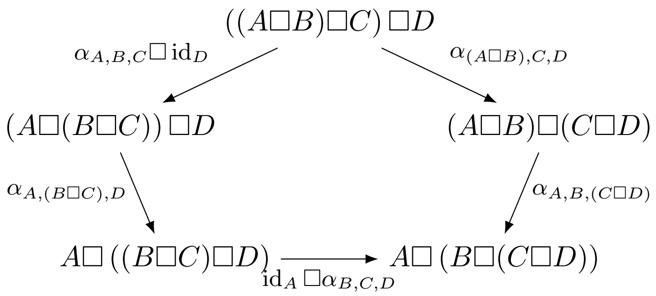
(iii) if $\mathbf{it}\,(\mathbf{C}, \Box)$ contains morphisms $f : A \overset{\bullet}{\to} A'$ and $g : B \overset{\bullet}{\to} B'$, then $\mathbf{it}\,(\mathbf{C}, \Box)$ contains a morphism $f \Box g : A \Box B \overset{\bullet}{\to} A' \Box B'$.

All the morphisms of $\mathbf{it}\,(\mathbf{C}, \Box)$ are iso, and are either identity natural isomorphisms, or natural isomorphisms that shift parentheses around in iterates. We call a morphism $A \xrightarrow{f} A'$ in $\mathbf{it}\,(\mathbf{C}, \Box)$ a *directed path from A to A'* if it does not contain any instances of α^{inv}, that is, if it does not move any parentheses to the left.

Now we can determine which constraints on α (and therefore, mutatis mutandis, on α^{inv}) guarantee that $\mathbf{it}\,(\mathbf{C}, \Box)$ is coherent.

Theorem 3.16 (Mac Lane [48, thm. 3.1])**.** For $\mathbf{C}$ a category with multiplication $\Box$, and associativity given by the natural isomorphism $\alpha : \Box(\Box, 1) \cong$

$\square(1,\square)$, the iterate category $\mathbf{it}\,(\mathbf{C},\square)$ is coherent if and only if for all $\square$-iterates A, B, C, D, the *pentagon equation* commutes:



(3.39)

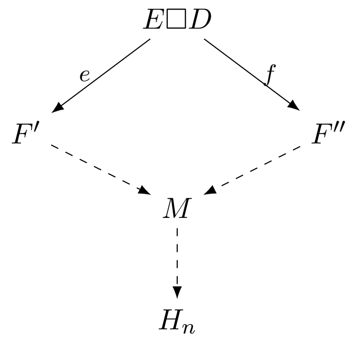
Proof sketch. Let $H_1 = 1$, and for $n \geq 2$, let $H_n = \square(1, H_{n-1})$, which is the $\square$-iterate of multiplicity n that has all the closing parentheses at the end. To prove that $\mathbf{it}\,(\mathbf{C},\square)$ is coherent, all we need to show is that for each $\square$-iterate of multiplicity n, there exists only one directed path $F \xrightarrow{\bullet} H_n$ in $\mathbf{it}\,(\mathbf{C},\square)$. We prove this by rank induction.

For F a $\square$-iterate with multiplicity m and G a $\square$-iterate with multiplicity n, define a rank function:

$$r(F\square G) = r(F) + r(G) + m - 1 \tag{3.40}$$

and let $r(H_1) = 0$. For all n we then have $r(H_n) = 0$.

If $r(F) = 0$, then $F = H_m$, so exists only one directed path $F \xrightarrow{\bullet} H_m$. Now let $F = E\square D$ be a $\square$-iterate of rank r, and let the theorem have been proven for all $\square$-iterates with rank lower than r. Let e and f be two directed paths starting at F, and consider the following diagram:

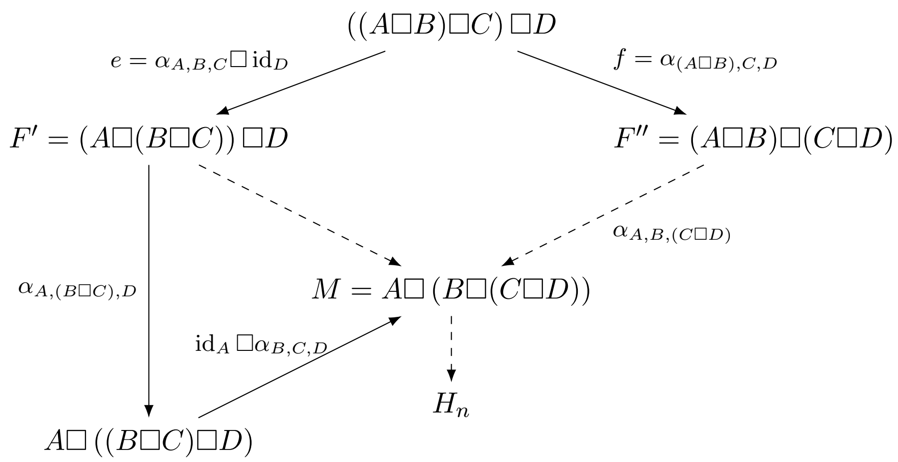


(3.41)

Let all the dashed morphisms be directed paths. The codomain of a directed path must have rank lower than or equal to that of the domain, so if for all F, e, and f there exists an object M such that the diagram commutes, then the theorem holds for iterates of rank r.

If $e = f$, then $F' = M = F''$. If $e \neq f$, there is a small number of possible values of e and f, which Mac Lane exhausts one by one. The only case where the existence of M does not follow trivially is when $E = ((A \square B) \square C)$, $e = \alpha_{A,B,C} \square \mathrm{id}_D$, and $f = \alpha_{(A \square B),C,D}$.

In that case, eq. (3.41) becomes:

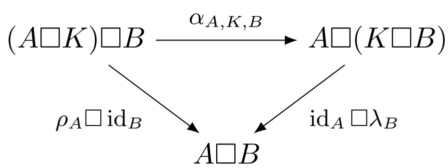


(3.42)

Since this is just a distorted version of the pentagon equation, commutativity of eq. (3.39) implies commutativity of eq. (3.42), and therefore guarantees the existence of M. Commutativity of eq. (3.39) therefore implies coherence of **it** $(\mathbf{C}, \Box)$. Since coherence of **it** $(\mathbf{C}, \Box)$ implies commutativity of eq. (3.39) as a special case, this completes the proof. $\Box$

We can make **It** $(\mathbf{C}, \Box)$ coherent by extending the coherence conditions for **it** $(\mathbf{C}, \Box)$.

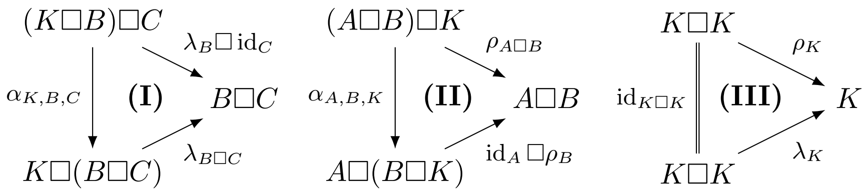
Theorem 3.17 (Mac Lane [48, thm. 5.2], Kelly[2] [39, thm. 3', 6, 7])**.** For $\mathbf{C}$ a category with multiplication $\Box$ and unit object K, with associativity given by α, and unitality by λ and ρ, **It** $(\mathbf{C}, \Box)$ is coherent if and only if eq. (3.39) and the *triangle equation*:



(3.43)

commute for all $\Box$-iterates A, B, C, D

Proof sketch. We have already proven that the coherence of **it** $(\mathbf{C}, \Box)$ follows from eq. (3.39), so all we need to show is that the actions of λ and ρ commute with each other and with the action of α. Again, we need only consider a small number of cases:

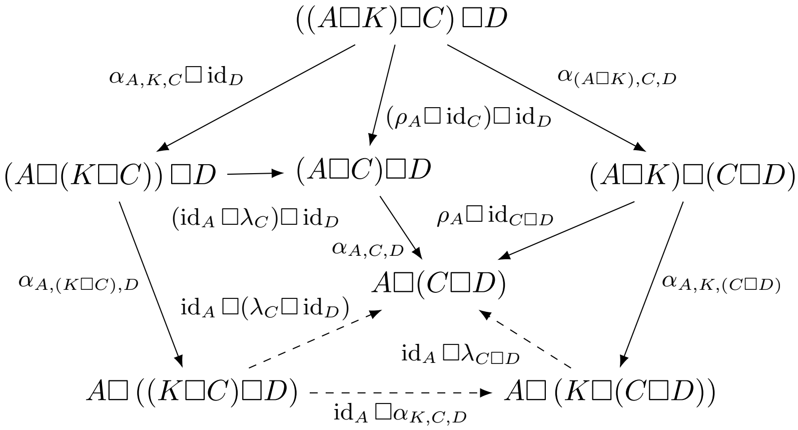


(3.44)

If all these diagrams commute for all $\Box$-iterates A, B, C, then **It** $(\mathbf{C}, \Box)$ is coherent. To prove that (I) commutes, we set $B = K$ in the pentagon equation

[2] Mac Lane's formulation contained some redundant conditions, which Kelly removed.

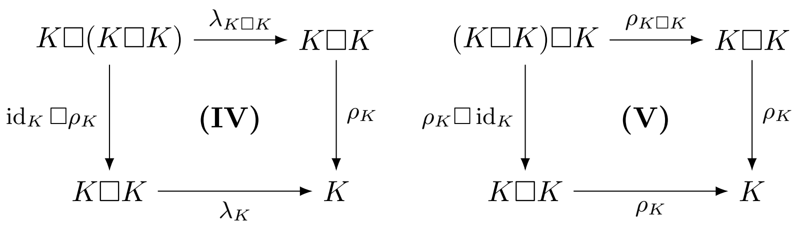
(3.39), and apply α, λ and ρ a few more times [39, p. 400]:



(3.45)

The outside of the diagram commutes as a special case of the pentagon equation; the two quadrangles commute due to the naturality of α; and the top left and bottom right triangles commute due to eq. (3.43). The dashed triangle gives us the commutativity of (I). If we invert all the instances of α in the pentagon, a similar trick proves that (II) commutes.

For case (III), note that the following diagrams commute due to the naturality of λ and ρ:



(3.46)

Due to (V), we have $\rho_{K\square K} = \rho_K\square\,\mathrm{id}_K$, and a similar argument gives us $\lambda_{K\square K} = \mathrm{id}_K\square\lambda_K$. By plugging $A = B = K$ into eq. (3.43), we get $\rho_K\square\,\mathrm{id}_K = (\mathrm{id}_K\square\lambda_K)\circ\alpha_{K,K,K}$, and by setting $A = B = K$ in condition (II), we get $\rho_{K\square K} = (\mathrm{id}_K\;\rho_K)\square\alpha_{K,K,K}$. Since all the morphisms involved are iso, this reduces to $\mathrm{id}_K\square\lambda_K = \mathrm{id}_K\square\rho_K$, and therefore $\lambda_{K\square K} = \mathrm{id}_K\square\rho_K$.

Then (IV) proves that $\lambda_K = \rho_K$, so case (III) commutes.

This proves that commutativity of eqs. (3.39) and (3.43) implies coherence of $\mathbf{It}\,(\mathbf{C}, \Box)$. Because commutativity of eqs. (3.39) and (3.43) is a special case of this coherence, the proof is complete. $\Box$

At the beginning of this section, we asked which conditions were necessary to guarantee coherent associativity and unitality in a category equipped with a bifunctor. The monoidal category was originally defined as an answer to that question:

Definition 3.18. A *monoidal category* is a category with a coherently associative and unital bifunctor, called the *tensor*. Equivalently: a monoidal category is a category $\mathbf{C}$, equipped with a bifunctor $\Box$, an *associator* natural isomorphism α, a unit object K, and *unitors* λ and ρ, such that $\mathbf{It}\,(\mathbf{C}, \Box)$ is coherent. Equivalently: $\mathbf{C}$ is a monoidal category with tensor product $\Box$ and unit K if and only if eqs. (3.39) and (3.43) commute for all $\Box$-iterates on $\mathbf{C}$.

A *strict monoidal category* is a monoidal category in which, for all objects A, B, C, the morphisms $\alpha_{A,B,C}$, λ_A, and ρ_A are identities.

The above definitions are equivalent to definition 3.8 in a way we will make more precise later on.

We sometimes use Bénabou's notation, where $\mathfrak{M} = \langle \mathbf{M}, \Box, K, \alpha, \lambda, \rho \rangle$ stands for "$\mathfrak{M}$ is the monoidal category formed by adding the tensor product $\Box$, the associator α, and the unitors λ, ρ to the category $\mathbf{M}$, and choosing K as the unit object", but not always. We sometimes use this notation to distinguish between a monoidal category and its underlying category, and we never use it for a 2-category with one object.

A useful term introduced by Bénabou [7] is *canonical morphism*. For

$\mathfrak{C} = \langle \mathbf{C}, \Box, K, \alpha, \lambda, \rho \rangle$ a monoidal category, a canonical morphism in $\mathfrak{C}$ is any of the following:

(i) id_A, ρ_A, λ_A for $A \in \mathbf{C}_0$;

(ii) $\alpha_{A,B,C}$ for $A, B, C \in \mathbf{C}_0$;

(iii) f^{inv} for f a canonical morphism;

(iv) $f \circ g$, $f \Box g$ for f, g canonical morphisms.

A useful theorem that follows from that definition is:

Theorem 3.19 (Coherence for monoidal categories)**.** In a monoidal category, any diagram in which all the morphisms are canonical, commutes.

Proof. Immediate from theorem 3.17 and definition 3.18. □

Definition 3.18 suggests several more examples of monoidal categories:

Example 3.20. The monoidal category $\mathfrak{Set}_K = \langle \mathbf{Set}, \times, K, \alpha, \lambda, \rho \rangle$ has the Cartesian product as its tensor, and the singleton set K as its unit.

Example 3.21. Let the category $\mathbf{G}$ be a group, and let some associative bifunctor $-\Box- : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ distribute over morphism composition. That is, let:

$$(h \Box g) \Box f = h \Box (g \Box f) \tag{3.47}$$

$$h \Box (g \circ f) = (h \Box g) \circ (h \Box f) \tag{3.48}$$

$$(h \circ g) \Box f = (h \Box f) \circ (g \Box f) \tag{3.49}$$

for all morphisms f, g, h of $\mathbf{G}$. Since $\mathbf{G}$ has only one object, that object acts as the unit of the operation $\Box$, so $\mathbf{G}$ is a strict monoidal category.

Now let 0 be the identity morphism of the sole object of **G**, and note that:

$$f\Box 0 = f\Box(0 \circ 0) = (f\Box 0) \circ (f\Box 0). \tag{3.50}$$

Composing the leftmost and rightmost side with $(f\Box\,\mathrm{id})^{\mathrm{inv}}$ gives:

$$(f\Box 0) \circ (f\Box 0)^{\mathrm{inv}} = (f\Box 0) \circ (f\Box 0) \circ (f\Box 0)^{\mathrm{inv}} \tag{3.51}$$

$$0 = f\Box 0. \tag{3.52}$$

And by a similar argument, $0\Box f = 0$. If there exists a morphism 1 such that

$$1\Box f = f = f\Box 1 \tag{3.53}$$

for all morphisms f, then we have:

$$(g \circ f)\Box(1 \circ 1) = (g\Box(1 \circ 1)) \circ (f\Box(1 \circ 1)) \tag{3.54}$$

$$= (g\Box 1) \circ (g\Box 1) \circ (f\Box 1) \circ (f\Box 1) \tag{3.55}$$

$$= g \circ g \circ f \circ f \tag{3.56}$$

and

$$(g \circ f)\Box(1 \circ 1) = ((g \circ f)\Box 1) \circ ((g \circ f)\Box 1) \tag{3.57}$$

$$= g \circ f \circ g \circ f \tag{3.58}$$

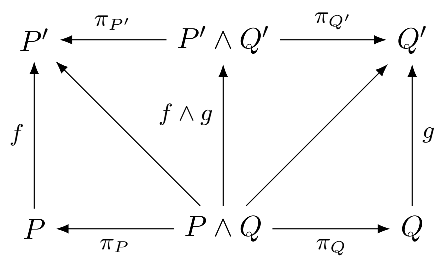
so $g\circ g\circ f\circ f = g\circ f\circ g\circ f$. By composing both sides with g^{inv} and f^{inv}, we get $g \circ f = f \circ g$, so **G** is an Abelian group, and a ring. Morphism composition is addition and $\Box$ is multiplication.

This proves that adding a distributive strict monoidal structure to an Abelian group and choosing a 1 turns that group into a ring (which is not at all surprising: a ring is an additive Abelian group that is also a multiplicative monoid, with multiplication distributing over addition). Note, also, that this $\square$ is exactly the tensor product of Abelian groups.

Example 3.22. Every poset category $\mathbf{P}$ with finite products and a maximal object $\top$ is strict monoidal with the meet $\wedge$ as its tensor product and $\top$ as its unit, since for all $P, Q, R \in \mathbf{P}_0$, we have:

$$(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$$
$$\top \wedge P = P = P \wedge \top$$

The following diagram proves that the existence of morphisms $P \xrightarrow{f} P'$ and $Q \xrightarrow{g} Q'$ implies the existence of a morphism $P \wedge Q \xrightarrow{f \wedge g} P' \wedge Q'$:



(3.59)

The horizontal arrows exist by definition of the meet. Composing f and g, with π_P and π_Q gives the diagonal arrows, and $f \wedge g$ then exists by definition of $\wedge$ (cf. eq. 2.12). This construction guarantees that the morphisms of $\mathbf{P}$ are well-behaved under the tensor product.

Example 3.23. By a similar argument, every poset category with a minimal object $\perp$ is strict monoidal, with the join as its tensor product, and $\perp$ as its unit.

We have now provided and studied two different definitions of monoidal categories, and stated that both are somehow "equivalent". But clearly they are different: one uses 1-categories, and the other uses 2-categories. We now make more precise the way these two definitions are the same.

Recall that a class is a 0-category, and that a group therefore is a 0-category with some added structure. At the same time, a group can be defined as a 1-category with one object and all morphisms iso. In formulating this second definition, we have taken an object defined as an n-category with some "external" structure, and redefined it as an $n+1$ category with some "internal" structure. This second definition is a *vertical categorification* of the first.

In exactly the same way, a (strict) monoidal category can be defined as a (strict) 2-category with one object, but also as a 1-category with some additional structure. The first definition is a vertical categorification of the second.

There is also another kind of categorification: just as we can generalise from a group to a groupoid by using a partial instead of a total operation, we can equivalently generalise a single-object-category-with-all-morphism-iso to a category-with-all-morphisms-iso. This is the *horizontal categorification* or *oidification* of the 0-categorical group. In general, an oidification takes a structure X, defined as an n-category with one object and added structure S, and generalises it to an X-oid, defined as an n-category with additional structure S and any number of objects [36].[3]

In the next section, we make some forays into the categorical theory of C*-algebras by first defining a more general, oidified structure, and then shrinking it down a bit. This directly motivates the categorical study of linear structure.

[3] If this terminology had been applied consistently from the start, then 2-categories would have been called *monoidal monoidoidoids!*

3.5 Enriched categories and C*-algebras

Consider the category $\mathbf{Vect}_{\mathbb{K}}$. For any two $\mathbb{K}$-vector spaces A, B, the hom-set $\mathbf{Vect}_{\mathbb{K}1}(A, B)$ is itself a $\mathbb{K}$-vector space and therefore also an object of $\mathbf{Vect}_{\mathbb{K}}$. Instead of hom-sets, we say that $\mathbf{Vect}_{\mathbb{K}}$ has *hom-objects*. There is, of course, no reason that the hom-objects of some category $\mathbf{C}$ have to be be objects *of* $\mathbf{C}$ *itself:*

Definition 3.24 (Bénabou [8])**.** A category $\mathbf{C}$ is *enriched over the monoidal category* $\mathfrak{M}$ (or: $\mathbf{C}$ *is an* $\mathfrak{M}$*-category*) if every hom-class of $\mathbf{C}$ is an object of the category $\mathfrak{M}$.

The monoidal structure of $\mathfrak{M}$ lets us identify morphism composition in $\mathbf{C}$ with the tensor product of objects in $\mathfrak{M}$.[4] In fact, equipping a category $\mathbf{M}$ with a monoidal structure is equivalent, from $\mathbf{M}$'s point of view, to enriching some category over $\mathfrak{M}$. Enrichment will allow us to define rings and algebras categorically.

Here are some examples:

(i) every locally small category is enriched over $\mathfrak{Set}$, since each of its hom-classes is a set.

(ii) every strict 2-category is equivalent to a 1-category enriched over the monoidal category $\mathfrak{Cat}$.

(iii) let $\mathfrak{Ab}$ be the monoidal category $\langle \mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \lambda, \rho \rangle$, where $\otimes$ is the usual tensor product on Abelian groups. Then a *ringoid* is a small $\mathfrak{Ab}$-category.

(iv) a ring is a one-object $\mathfrak{Ab}$-category: example 3.21 says this same thing

[4] It is sometimes possible to enrich a category over a non-monoidal category, but we will not do this.

from the point of view of the enriching category $\mathfrak{Ab}$ instead of the category being enriched.

(v) an *algebroid* over the field $\mathbb{K}$ is a category enriched over the monoidal category $\mathbf{Vect}_{\mathbb{K}}$. Hence, an algebroid is a ringoid (because $\mathbf{Vect}_{\mathbb{K}}$ is a subcategory of $\mathbf{Ab}$) and all the morphisms in an algebroid can be multiplied by scalars (because the hom-sets are vector spaces). A unital associative *algebra* is an algebroid with only one object: this is exactly equivalent to the usual definition, where an algebra over the field $\mathbb{K}$ is a ring that is also a $\mathbb{K}$-vector space.

Now consider the category $\mathbf{Ban}_{\mathbb{K}}$, which is the full subcategory of $\mathbf{Vect}_{\mathbb{K}}$ with objects restricted to only the $\mathbb{K}$-Banach spaces. $\mathfrak{Ban}_{\mathbb{K}}$ is the corresponding monoidal category with the projective tensor product. This lets us define C*-algebras:

Definition 3.25 (Ghez, Lima, and Roberts [25])**.** A *C*-category* (more precisely: a *C*-algebroid*) is a $\mathfrak{Ban}_{\mathbb{C}}$-category $\mathbf{C^*Cat}$ such that:

(i) there exists an antilinear contravariant endofunctor $(-)^* : \mathbf{C^*Cat} \to \mathbf{C^*Cat}^{\mathrm{op}}$ which is an identity on objects and an involution on morphisms;

(ii) for each morphism f of $\mathbf{C^*Cat}$, $\|f^*f\| = \|f\|^2$;

(iii) for each morphism $A \xrightarrow{f} B$ of $\mathbf{C^*Cat}$, there exists a morphism $A \xrightarrow{g} A$ such that $f^*f = g^*g$.

A *C*-algebra* is a C*-category $\mathbf{C^*Alg}$ with only one object. We can then see that the first condition above guarantees that $\mathbf{C^*Alg}$ is a *-algebra, and the second is the C*-identity. The third condition serves to eliminate some pathological cases, and is entirely trivial when there is only one object. We will not discuss it any further.

We have now seen how to describe C*-algebras in the language of category theory, and how this formulation makes clear the relations between algebras, rings, and vector spaces. However, some of our language was decidedly uncategorical: how should we interpret terms like "antilinear" or $\|f\|^2$ in a categorical way? In the next sections, we answer this by examining the linear and (sometimes) symmetric and closed structure of monoidal categories.

3.6 Linear structure

While it may seem uncategorical to speak of the internal structure of objects, the objects of a monoidal category carry all their structure on the outside, in their morphisms. Consider how every element of a field $\mathbb{K}$ uniquely defines a linear function $\mathbb{K} \to \mathbb{K}$; hence every endomorphism of $\mathbb{K}$ in $\mathbf{Vect}_{\mathbb{K}}$ corresponds to an element of $\mathbb{K}$ and vice versa. Similarly, there exists a bijection between morphisms $\mathbb{K} \to V$ and vectors in V. This motivates the following:

Definition 3.26. In the monoidal category $\langle \mathbf{M}, \Box, I, \alpha, \lambda, \rho \rangle$:

(i) a *scalar* is a morphism $I \to I$;

(ii) a *vector* is a morphism $I \to A$, where A can be any object of $\mathbf{M}$ that is not I.

In $\mathfrak{Set}_K$, where the monoidal unit is a singleton set K, each morphism from K to a non-empty set A selects one element of A. Hence every vector in $\mathfrak{Set}_K$ corresponds to exactly one element of one set, and vice versa. The only scalar in $\mathfrak{Set}_K$ is id_K, which selects the sole element of the singleton set.

In the monoidal category $\mathfrak{Cat}$, the unique functor $\mathbf{1} \to \mathbf{1}$ is the only scalar. Vectors are functors $\mathbf{1} \to \mathbf{A}$, each of which selects from $\mathbf{A}$ a single object along with its identity morphism. Hence every vector in $\mathfrak{Cat}$ corresponds to a smallest-possible subcategory of some small category. In all these cases, a vector selects a smallest possible component of an object.

The remainder of this section is devoted to vectors and scalars in monoidal categories. We first show that the scalars of a monoidal category form a commutative monoid, and that the tensor product on a monoidal category is always bilinear. In the next section, we discuss dual objects, leading to bras and kets, partial trace, and complex conjugates. We can then categorically interpret the non-categorical language we used in our introduction to C*-algebras above.

First, note the following useful lemma:

Lemma 3.27 (Interchange law)**.** Let F be a bifunctor $\mathbf{A} \times \mathbf{B} \to \mathbf{C}$. Then for all morphisms f, f' in $\mathbf{A}$, g, g' in $\mathbf{B}$:

$$F(f, g) \circ F(f', g') = F(f \circ f', g \circ g') \tag{3.60}$$

provided $\mathrm{dom}\, f = \mathrm{cod}\, f'$ and $\mathrm{dom}\, g = \mathrm{cod}\, g'$.

Proof.

$$\begin{aligned} F(f, g) \circ F(f', g') &= F\left((f, g) \circ (f', g')\right) && \text{(by definition 2.23)} \\ &= F(f \circ f', g \circ g') && \text{(by definition 2.10)} \end{aligned}$$

□

In a monoidal category $\mathfrak{C}$ with tensor $\Box$, set $\mathbf{A} = \mathbf{B} = \mathbf{C}$ and $F = \Box$, to obtain

$$(f \Box g) \circ (f' \Box g') = (f \circ f') \Box (g \circ g') \tag{3.61}$$

This monoidal version of the interchange law lets us prove that all scalars in a monoidal category commute:

Proposition 3.28 (Kelly and Laplaza [40, prop. 6.1])**.** Let $\mathfrak{C} = \langle \mathbf{C}, \Box, K, \alpha, \lambda, \rho \rangle$, and let f and g be scalars. Then $f \circ g = g \circ f$.

Proof. The following diagrams commute due to the naturality of λ and ρ:

$$\begin{CD} K \Box K @>{\lambda_K}>> K \\ @V{\mathrm{id}_K \Box f}VV @VV{f}V \\ K \Box K @>>{\lambda_K}> K \end{CD} \qquad \begin{CD} K \Box K @>{\rho_K}>> K \\ @V{g \Box \mathrm{id}_K}VV @VV{g}V \\ K \Box K @>>{\rho_K}> K \end{CD} \tag{3.62}$$

Hence $f = \lambda_K \circ (\mathrm{id}_K \Box f) \circ \lambda_K^{\mathrm{inv}}$ and $g = \rho_K \circ (g \Box \mathrm{id}_K) \circ \rho_K^{\mathrm{inv}}$. Due to the coherence theorem (3.19) $\lambda_K = \rho_K$ and $\lambda_K^{\mathrm{inv}} = \rho_K^{\mathrm{inv}}$, so $g = \lambda_K \circ (g \Box \mathrm{id}_K) \circ \lambda_K^{\mathrm{inv}}$. To finish the proof, we apply the interchange law twice:

$$\begin{aligned} g \circ f &= \lambda_K \circ (g \Box \mathrm{id}_K) \circ (\mathrm{id}_K \Box f) \circ \lambda_K^{\mathrm{inv}} \\ &= \lambda_K \circ (g \Box f) \circ \lambda_K^{\mathrm{inv}} \\ &= \lambda_K \circ (\mathrm{id}_K \Box f) \circ (g \Box \mathrm{id}_K) \circ \lambda_K^{\mathrm{inv}} \\ &= f \circ g \end{aligned} \tag{3.63}$$

$\Box$

In monoidal categories that have only one scalar, such as $\mathfrak{Set}$ and $\mathfrak{Cat}$, this is entirely trivial: the sole scalar cannot but commute with itself. However, proposition 3.28 does place meaningful constraints on monoidal categories with more complex unit objects, as we shall see below. But first, note that proposition 3.28 is an instance of the *Eckmann-Hilton argument*. Let a class S be closed under two unital, associative binary operations $\bullet$ and $\star$, such that $(a \star b) \bullet (c \star d) = (a \bullet c) \star (b \bullet d)$ for all $a, b, c, d \in S$; then Eckmann-Hilton states that $\star$ and $\bullet$ must be the same operation, with the same unit, and that this operation is commutative [20].

Now let K-**Mod** be the category whose objects are all the left K-modules, with K a commutative ring, and whose morphisms are all the homomorphisms between such modules. Let $\mathfrak{M} = \langle K\text{-}\mathbf{Mod}, \otimes, K, \alpha, \lambda, \rho\rangle$, with $\otimes$ the usual tensor product on modules. How should we interpret composition of scalars with other scalars, or with vectors of $\mathfrak{M}$?

Recall that the elements of the ring K are exactly the endomorphisms of the object K, which are homomorphisms $K \to K$. Let $\times$ be the multiplication operation of K, and let $\circ$ denote morphism composition. We can define $\times$ as a bifunctor, so by the interchange law we get:

$$(a \times b) \circ (c \times d) = (a \circ c) \times (b \circ d) \tag{3.64}$$

and then the Eckmann-Hilton argument tells us that composition of an endomorphism k of K with some other morphism (i.e. some element of a K-module) is equal to multiplication by k.

Now consider the more general case where K is a non-commutative ring, and let $\mathfrak{k}$ be the *commutator ideal* of K:

$$\mathfrak{k} = \{ab - ba \mid a, b \in K\} \tag{3.65}$$

If we make K-**Mod** a monoidal category with the usual tensor product, then for all $j, k \in K$, we have $j \circ k = k \circ j$ due to proposition 3.28 (i.e. an

Eckmann-Hilton argument with operations $\circ$ and $\otimes$). The action of K on any module M must then factor through the quotient ring $K/\mathfrak{k}$, so composition of K-endomorphisms, generally speaking, is a multiplication that discards any non-commutativity. Since this is just a roundabout way to define a tensor product on modules of a commutative ring, the proposition essentially tells us that a monoidal category of modules can only be defined over a commutative ring. Similar constraints, of course, apply to all other monoidal categories with multiple scalars.[5]

Another consequence of the interchange law and Eckmann-Hilton is that the bifunctor $\Box$ on a monoidal category is bilinear. For any vectors u, v, and scalar k:

$$\begin{aligned}(u \circ k)\Box v &= (u \circ k)\Box(v \circ \mathrm{id}_K) \\ &= (u\Box v) \circ (k\Box\, \mathrm{id}_K) \\ &= (u\Box v) \circ k \\ &= (u\Box v) \circ (\mathrm{id}_K \Box k) \\ &= u\Box(v \circ k)\end{aligned} \tag{3.66}$$

This is exactly why the monoidal bifunctor is often referred to as a tensor product.

We have thus far shown that the objects and morphisms of a complex monoidal category carry much of the familiar structure of vector spaces, albeit in a far more general way. Now we will discuss categorical generalisations of dual spaces and complex conjugates. First we introduce symmetric and closed monoidal categories, followed by a 2-categorical generalisation of adjunctions. Then we study closed symmetric monoidal categories, and

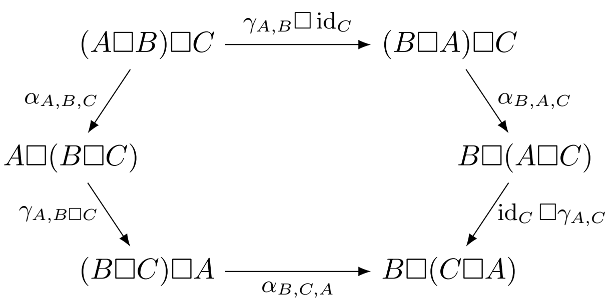
[5]Proposition 3.28 was published in 1980 and, despite its power, was mostly forgotten until Abramsky and Coecke reintroduced it in 2004 as a key component of categorical quantum mechanics [1].

show how some useful properties follow.

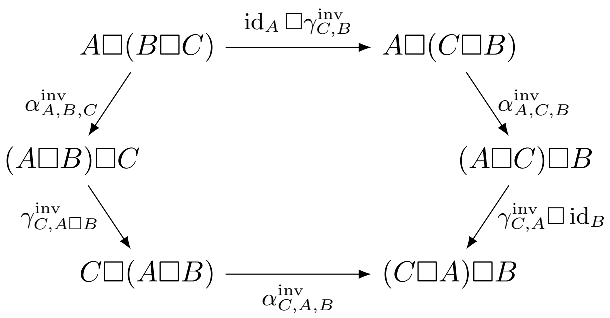
3.7 Braiding and symmetry

Note that in many monoidal categories $A \Box B$ is isomorphic to $B \Box A$ for all objects A, B. If all these isomorphisms together form a natural isomorphism, compatible with the tensor, the category is *braided:*

Definition 3.29 (Joyal and Street [37, def. 2.1, 2.2])**.** $\langle \mathbf{C}, \Box, K, \alpha, \lambda, \rho, \gamma \rangle$ is a *braided monoidal category* if $\langle \mathbf{C}, \Box, K, \alpha, \lambda, \rho \rangle$ is a monoidal category, and the isomorphism $\gamma_{A,B} : A \Box B \overset{\bullet}{\to} B \Box A$ is natural in A and B, and the following diagrams commute:



(3.67)



(3.68)

The natural isomorphism γ is called the *braiding* of the monoidal category.

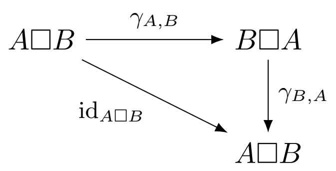
Of the monoidal categories we have seen so far, most were braided though not all. Consider the following not-always-braided category:

Example 3.30. Recall the functor category $[\mathbf{C}, \mathbf{C}]$ of example 3.1, which is strict monoidal with functor composition as the tensor, and the identity functor as the unit. Let C_X be the constant endofunctor onto the object X of $\mathbf{C}$.

Then for all objects A, B, Z of $\mathbf{C}$: $[C_A \bullet C_B]Z = A$ and $[C_B \bullet C_A]Z = B$. Therefore the isomorphism γ_{C_A,C_B} exists if and only if $A \simeq B$, so $[\mathbf{C}, \mathbf{C}]$ is braided if and only if all objects of $\mathbf{C}$ are isomorphic.

The braiding of a monoidal category provides it with some degree of commutativity. If the tensor product is "as commutative as possible", then the category is *symmetric monoidal:*

Definition 3.31 (Mac Lane [48])**.** A braided monoidal category $\langle \mathbf{C}, \Box, K, \alpha, \lambda, \rho, \gamma \rangle$ is a *symmetric monoidal category* if the natural transformation γ is its own inverse:

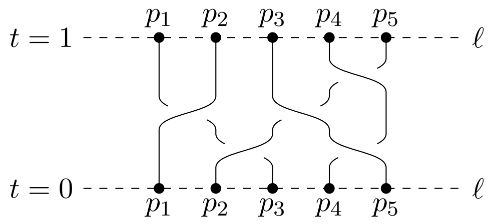


(3.69)

Note that if a braided monoidal category is symmetric, then each of the two diagrams in definition 3.29 implies the other, so we need only specify one.

Examples of symmetric monoidal categories include $\mathfrak{Set}$, where $\gamma_{A,B}$ is the mapping $(a, b) \mapsto (b, a)$ for all $a \in A$, $b \in B$; and **NToset** with the tensor product $m \Box n = \max(m, n)$ and identity 0, where $\gamma_{m,n} = \mathrm{id}_{m \Box n}$. Braided monoidal categories that are not symmetric tend to be a bit more involved. Here is an example by Joyal and Street [37, ex. 2.1]:

Example 3.32. Let P be the Euclidean plane and let C_n be the the space of subsets of P with cardinality n. Now choose n distinct points $p_1, p_2, \ldots, p_n$ along a line ℓ on P, and let $\omega : [0, 1] \to C_n$ be a loop starting at the set $\{p_1, p_2, \ldots, p_n\} \in C_n$. Draw a three-dimensional graph of ω (which will consist of n distinct non-intersecting curves) in $P \times [0,1]$, and then project the graph onto the plane $\ell \times [0,1]$. For $n = 5$ the result may be something like this:

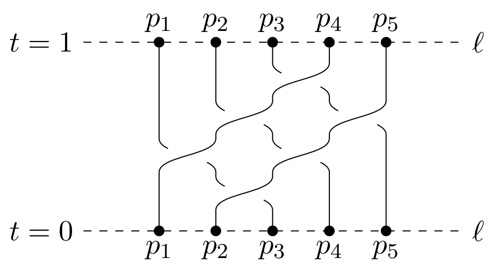


(3.70)

Now let $\pi_1(n)$ be the *fundamental group* of C_n; that is, $\pi_1(n)$ is the group of homotopy equivalence classes of all the loops $[0,1] \to C_n$. The *braid category* $\mathbf{Braid}_P$ has the natural numbers as its objects; morphisms are given by:

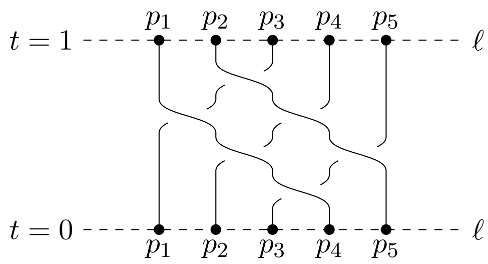
$$\mathbf{Braid}_{P1}(m,n) = \begin{cases} \pi_1(n) & \text{if } m = n \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.71}$$

$\mathbf{Braid}_P$ becomes a strict monoidal category with tensor $\oplus$ and unit object 0, if we take $m \oplus n = m + n$ for objects, and for morphisms ψ, ω, we take $\psi \oplus \omega$ to be the loop formed by placing ψ next to ω (horizontal composition in the most literal sense). To make $\mathbf{Braid}_P$ a braided monoidal category, let $\gamma_{m,n}$ be the loop in $\pi_1(m+n)$ in which the leftmost m strands pass over the rightmost n strands. For example, $\gamma_{2,3}$ is the loop:



(3.72)

and $\gamma_{2,3}^{\text{inv}}$ is:



(3.73)

Note that $\gamma_{2,3}^{\text{inv}}$ is not equal to $\gamma_{3,2}$: in the former, the leftmost three strands pass under the rightmost two, and in the latter they go over. This proves that $\mathbf{Braid}_P$ is a braided monoidal category that is not symmetric.

With braiding and symmetry defined, we now return to a concept we last saw in the introduction. We discuss adjunctions again, but now in a 2-categorical setting.

3.8 Adjunctions again

In the introduction, we described how adjunctions are a central concept of category theory, but only described them in a 1-categorical setting. Given their importance, it seems there should also be higher-categorical notions of adjunction. We begin this chapter with an example of a 1-categorical adjunction in a 2-category, after which we show how the 1-categorical definition can be raised into 2-categories.

Example 3.33 (Tensor-hom adjunction)**.** Let $f : U \to V$ and $g : W \to X$ be morphisms in $\mathbf{FdVect}_{\mathbb{K}}$, let $\otimes$ denote the usual tensor product, and let $- \otimes Z$ denote the functor that sends any object V to $V \otimes Z$, and any morphism f to $f \otimes \text{id}_Z$. Let L be the functor we used in example 2.37.

If we choose bases $\{u\}$, $\{v\}$, $\{w\}$, $\{x\}$, $\{z\}$ of U, V, W, X, Z, and let * denote the dual basis, we get a family of bijections (recall that a vector in

V is a map $\mathbb{K} \to V$, and its dual is a map $V \to \mathbb{K}$):

$$\begin{aligned} &\phi_{V,W} : L\left(V \otimes Z, W\right) \simeq L\left(V, L\left(X, W\right)\right) \\ &\phi_{V,W}\left(w \circ [v^* \otimes z^*]\right) = [w \otimes z^*] \circ v^* \end{aligned} \tag{3.74}$$

Applying $L([v \otimes z] \circ [u^* \otimes z^*], x \circ w^*)$ to $w \circ [v^* \otimes z^*]$ gives

$$x \circ w \circ w^* \circ [v^* \otimes z^*] \circ [v \otimes z] \circ [u^* \otimes z^*] = x \circ [u^* \otimes z^*], \tag{3.75}$$

and applying $L(v \circ u^*, L(1, x \circ w^*))$ to $w \circ [z^* \otimes v^*]$ gives

$$x \circ w^* \circ w \circ [z^* \otimes v^*] \circ v \circ u^* = x \circ [z^* \otimes u^*], \tag{3.76}$$

so by linearity we have for all $V \otimes Z \xrightarrow{l} W$:

$$\phi_{U,X}\left([L(f,g]\,(l)\right) = [L(f, L(\mathrm{id}_Z, g))]\left(\phi_{V,W}(l)\right). \tag{3.77}$$

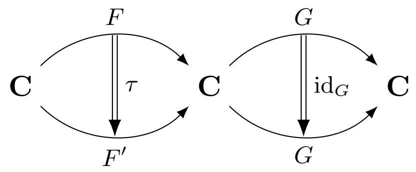
This means we have a natural isomorphism

$$\begin{array}{ccc} L(V \otimes Z, W) & \xrightarrow{\phi_{V,W}} & L(V, L(Z, W)) \\ {\scriptstyle L(f \otimes Z, g)}\downarrow & & \downarrow{\scriptstyle L(f, L(\mathrm{id}_Z, g)} \\ L(U \otimes Z, X) & \xrightarrow[\phi_{U,X}]{} & L(U, L(Z, X)) \end{array} \tag{3.78}$$

which makes $- \otimes Z$ and L adjoint functors.

We will soon generalise our understanding of adjunctions. But first recall how in example 3.1 the natural transformation $\tau : F \xrightarrow{\bullet} F' : \mathbf{C} \to \mathbf{C}$ was composed with the functor $G : \mathbf{C} \to \mathbf{C}$ to form the natural transformation $G \cdot \tau = G(\tau) : G \bullet F \xrightarrow{\bullet} G \bullet F'$, and the natural transformation $\upsilon : G \xrightarrow{\bullet} G'$ combined with the functor F to form the natural transformation $(\upsilon \cdot F)_x =$

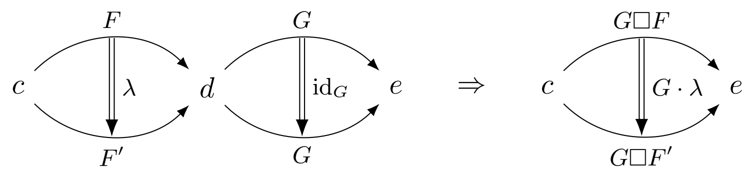
$\upsilon_{F(x)} : F \bullet G \xrightarrow{\bullet} F \bullet G'$. Visually, $G \cdot \tau$ maps the top path to the bottom path in the diagram:



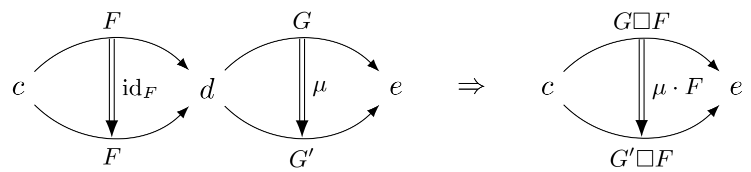
(3.79)

so it is equal to the horizontal composition $\mathrm{id}_G \,\Box \tau$. By a similar argument, $\upsilon \cdot F = \upsilon \Box \, \mathrm{id}_F$.

For any 2-cells $\lambda : F \Rightarrow F' : c \to d$ and $\mu : G \Rightarrow G' : d \to e$ in any weak 2-category, the *whiskerings* $G \cdot \lambda$ and $\mu \cdot F$ are defined as the horizontal compositions:



(3.80)



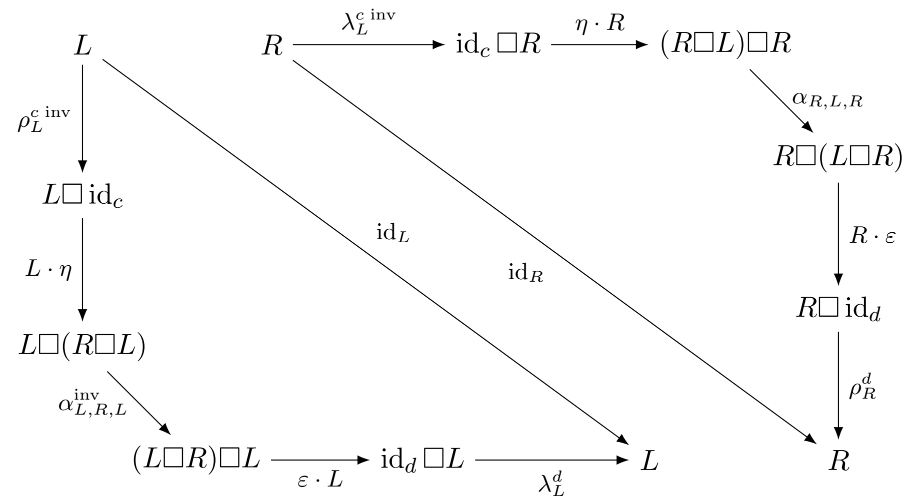
(3.81)

Such composition of 1-cells with 2-cells lets us define adjunctions in 2-categories.

Definition 3.34 (Maranda [49])**.** In a weak[6] 2-category $\mathbf{C}$, an *adjunction* is a 4-tuple $\langle L, R; \eta, \varepsilon \rangle$ consisting of 1-cells $L : c \to d$, $R : d \to c$, and 2-cells

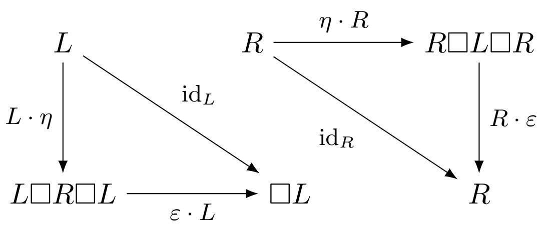
[6]Maranda only discusses strict 2-categories in [49], but the definition he gives is easy to extend to weak 2-categories.

$\eta : \mathrm{id}_c \Rightarrow R\Box L$, $\varepsilon : L\Box R \Rightarrow \mathrm{id}_d$, such that the following diagrams commute:



(3.82)

In a strict 2-category, this collapses down to:



(3.83)

We call L the *left adjoint* of R and R the *right adjoint* of F. We call η the *unit* and ε the *counit* of the adjunction, and we write $L \dashv R$.

Note the similarities of the unit and counit equations to our requirement in example 1.10 that

$$a \leq \mathrm{RL}a \qquad\qquad b \preccurlyeq \mathrm{LR}b. \tag{3.84}$$

We now understand that the posets we used in that example are 1-cells in the strict 2-category **Cat2** of small categories, functors, and natural transformations,[7] and that $a \leq a'$, $b \preccurlyeq b'$ indicate the existence of morphisms $a \to a'$, $b \to b'$. Equation (3.84) then defines natural transformations $\mathrm{id}_A \xrightarrow{\bullet} \mathrm{RL}$ and $\mathrm{L}R \xrightarrow{\bullet} \mathrm{id}_B$. If we set $Z = \mathbb{K}$, we can easily translate example 3.33 to unit-counit language as well:

Example 3.35. In the monoidal category **FdVect**, $\mathbb{K}$ is a 1-cell, as is every vector space. Again, let * denote the dual space. Then under any choice of basis $\{v\}$ for the vector space V, we obtain an adjunction by taking:

$$\eta_V(k) = \sum_j k v_j^* \otimes v_j \tag{3.85}$$

$$\varepsilon_V(v_i \otimes v_j^*) = v_j^* v_i \tag{3.86}$$

and extending by linearity. Hence $V \dashv V^*$. Note that it does not matter which basis we choose, because $\eta_V(k)$ is a multiple of the identity matrix, which is basis-independent.

We end this section with a few useful results.

Theorem 3.36 ([47, thms. IV.1.1, IV.1.2])**.** In **Cat2**, definition 3.34 is equivalent to the informal definition of adjunction that we gave in the introduction.

Proposition 3.37 (Gray [27, prop. I, 6.3])**.** Let $L : c \to d$, $L' : d \to e$, $R' : e \to d$, and $R : d \to c$ be 1-cells in the 2-category **C**. Then $L \dashv R$ and $L' \dashv R'$ together imply $L' \Box L \dashv R \Box R'$.

Proof. Let η, ε be the unit and counit of $L \dashv R$, let $\tilde{\eta}, \tilde{\varepsilon}$ be those of $L' \dashv R'$, and let 1_d be the identity 2-cell on id_d. Because horizontal composition is

[7] Not to be confused with the *monoidal* category **Cat** considered as a weak 2-category with one 0-cell, in which 1-cells are categories, 2-cells are functors, and there are no natural transformations.

bifunctorial, it is subject to the interchange law:

$$(\tilde{\eta}\square 1_d)\circ(1_d\square\varepsilon) = (\tilde{\eta}\square\varepsilon) = (\mathrm{id}_{R'\square L'}\square\varepsilon)\circ(\tilde{\eta}\square\,\mathrm{id}_{L\square R}) \tag{3.87}$$

so the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{id}_d\square L\square R & \xrightarrow{\tilde{\eta}\square\,\mathrm{id}_{L\square R}} & R'\square L'\square L\square R \\
\Big\downarrow{\scriptstyle 1_d\square\varepsilon} & & \Big\downarrow{\scriptstyle \mathrm{id}_{R'\square L'}\square\varepsilon} \\
\mathrm{id}_d\square\,\mathrm{id}_d & \xrightarrow[\tilde{\eta}\square 1_d]{} & R'\square L'\square\,\mathrm{id}_d
\end{array} \tag{3.88}$$

Now assume **C** to be a strict 2-category, and consider the diagram:

$$\begin{array}{ccccc}
R\square R' & \xrightarrow{\eta\cdot(R\square R')} & R\square L\square R\square R' & \xrightarrow{R\cdot\tilde{\eta}\cdot(L\square R)} & R\square R'\square L'\square L\square R\square R' \\
& \searrow{\scriptstyle \mathrm{id}_{R\square R'}} & \Big\downarrow{\scriptstyle R\cdot\varepsilon\cdot R'} & & \Big\downarrow{\scriptstyle (R\square R'\square L')\cdot\varepsilon\cdot R'} \\
& & R\square R' & \xrightarrow[R\cdot\tilde{\eta}\cdot R']{} & R\square R'\square L'\square R' \\
& & & \searrow{\scriptstyle \mathrm{id}_{R\square R'}} & \Big\downarrow{\scriptstyle (R\square R')\cdot\tilde{\varepsilon}} \\
& & & & R\square R'
\end{array} \tag{3.89}$$

The rectangle commutes due to eq. (3.88), and the two small triangles commute due to eq. (3.83), so the entire diagram commutes. Working out a similar diagram starting at $L\square L'$ will prove that $L'\square L \dashv R\square R'$ with unit $(R\cdot\tilde{\eta}\cdot L)\circ\eta$ and counit $\tilde{\varepsilon}\circ(L'\cdot\varepsilon\cdot R')$.

If **C** is not strict, then we can construct a larger version of eq. (3.89) that includes instances of α, λ, and ρ. $\square$

Above, we have proved that all the scalars of a monoidal category commute. But instead of discussing more linear structure, we digressed on braiding, symmetry and adjunctions. All of this was to set the stage for closed categories which have many useful linear properties, such as dual objects. Those are the topic of the next section.

3.9 Closed categories

Recall how in example 2.37, the functor $L : \mathbf{Hilb}_{\mathbb{K}}{}^{\mathrm{op}} \times \mathbf{Hilb}_{\mathbb{K}} \to \mathbf{Hilb}_{\mathbb{K}}$ mapped any two objects A and B of $\mathbf{Hilb}_{\mathbb{K}}$ to the hom-object of bounded linear maps $A \to B$, and how, in the category **Cat**, $[\mathbf{P}, \mathbf{Q}]$ denotes the category of functors $\mathbf{P} \to \mathbf{Q}$ (definition 2.43). We can easily generalise this idea.

In a category $\mathbf{C}$, let the *internal hom-functor* $[\ ,\ ] : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$ map every pair A, B of objects of $\mathbf{C}$ to the hom-object $[A, B]$ of $\mathbf{C}$, which is as similar to the set $\mathbf{C}_1(A, B)$ as is possible in $\mathbf{C}$. Let Hom denote the functor that sends any objects A, B of $\mathbf{C}$ to the set $\mathbf{C}_1(A, B)$, considered as an object of **Set**. In that case, there will exist a functor $V : \mathbf{C} \to \mathbf{Set}$ such that the following diagram commutes in **Cat**:[8]

$$\begin{array}{ccc} \mathbf{C}^{\mathrm{op}} \times \mathbf{C} & \xrightarrow{[\ ,\ ]} & \mathbf{C} \\ & \underset{\mathrm{Hom}}{\searrow} & \downarrow V \\ & & \mathbf{Set} \end{array} \tag{3.90}$$

A category $\mathbf{C}$ *has internal hom* if there exist an internal hom-functor $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$.

A motivating example by Eilenberg and Kelly [21, pp. 421–422] is the category $\mathbf{B}_{\mathbb{K}}$ whose objects are all the Banach spaces over the field $\mathbb{K}$, and whose morphisms are all the linear maps $f : A \to B$ with norm $\|f\| \leq 1$. The

[8] Stricly speaking, this would limit hom-functors to small categories. While there are ways around this limitation, we will not work them out here: we will simply pretend that all locally small categories are small.

hom-functor maps every pair A, B of Banach spaces to the Banach space $[A, B]$ of *all* linear maps $A \to B$, and the functor V sens this hom-space to the operator ball $\mathbf{B}_1(A, B)$, considered as a set. This hom-functor has a unit object, as every $\mathbb{K}$-Banach space A is isomorphic to $[\mathbb{K}, A]$.

Now consider the functor $L^A = [A, -] : \mathbf{B}_\mathbb{K} \to \mathbf{B}_\mathbb{K}$. When applied to objects, its meaning is clear: $L^A(B) = [A, B]$, but what happens when we apply it to a morphism? Due to eq. (3.90), we should have $V\left(L^A(f)\right) = \mathrm{Hom}\left(\mathrm{id}_A, f\right)$ for all $B \xrightarrow{f} C$, so $L^A(f)$ should be a morphism from $[A, B]$ to $[A, C]$. For all objects B and C, then, L^A lets us define a mapping from $[B, C]$ to $[[A, B], [A, C]]$. All these maps together form a natural transformation

$$L^A_{B,C} : [B, C] \xrightarrow{\bullet} [[A, B], [A, C]] \tag{3.91}$$

that is natural in B and C.

We have now seen the most important data that define the *closed category:*

Definition 3.38 (Eilenberg and Kelly [21, pp. 428–429])**.** A *closed category* is a locally small category $\mathbf{C}$ along with the following:

(i) a functor $V : \mathbf{C} \to \mathbf{Set}$,

(ii) a hom-functor $[\ ,\ ] : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$,

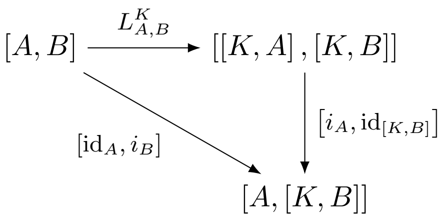
(iii) a unit object K,

(iv) a natural isomorphism $i : - \xrightarrow{\bullet} [K, -]$,

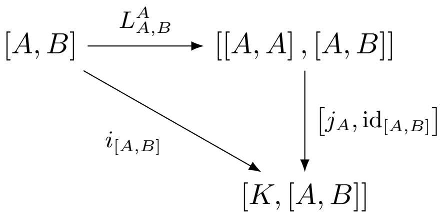
(v) for each object A, a morphism $K \xrightarrow{j_A} [A, A]$,

(vi) for each object A, a natural transformation $L^A : [-, -] \xrightarrow{\bullet} [[A, -], [A, -]]$, such that eq. (3.90) commutes, and several other coherence conditions hold:

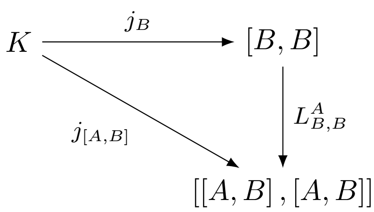
First, the compatibility of [,], i, j, and L, as expressed by the commutativity of the following equations:



(3.92)

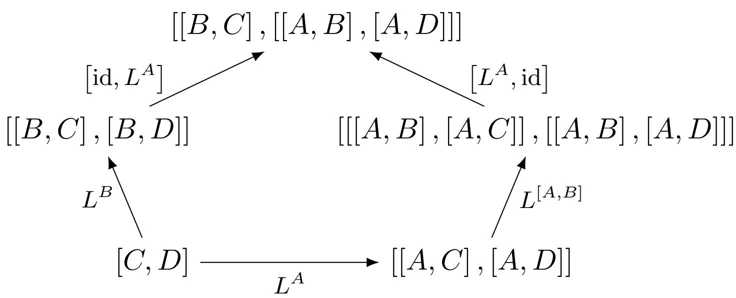


(3.93)



(3.94)

Second, the *pentagon equation for closed categories* must commute:



(3.95)

And finally, for every object A:

$$j_A = i_{[A,A]}\left(\mathrm{id}_A\right). \tag{3.96}$$

This definition is rather unintuitive, and we will only rarely use it explicitly. Before continuing, we show that $\mathbf{B}_{\mathbb{K}}$ is indeed a closed category:

Example 3.39. The category $\mathbf{B}_{\mathbb{K}}$ becomes a closed category if, as stated above, we let $[A, B]$ be the Banach space of linear maps $A \to B$; and we let V be the functor that maps every Banach space B to the unit ball $VB \subset B$ centered around the origin, considered as a set, and maps every morphism $f : A \to B$ to the restriction $Vf = f|_{VA} : VA \to VB$. The unit object K of $\mathbf{B}_{\mathbb{K}}$ considered as a closed category is $\mathbb{K}$, so i_A is the canonical mapping $A \to [\mathbb{K}, A]$ for each A. We have already defined L^A.

For each object A, the canonical isomorphism $i_{[A,A]} : [A, A] \mapsto [\mathbb{K}, [A, A]]$ sends id_A to the $\mathbb{K}$-linear map $f : 1 \mapsto \mathrm{id}_A$, so eq. (3.96) holds if j_A is the restriction $f_{V\mathbb{K}}$ of $1 \mapsto \mathrm{id}_A$ to the ball $V\mathbb{K} = \{k \in \mathbb{K} \mid |k| \leq 1\}$.

That eqs. (3.92) to (3.94) commute is easy to verify by chasing a test function along the two paths of each diagram. For eq. (3.95), we need an additional lemma, for which we interrupt this example.

Lemma 3.40. Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors between locally small categories, and let $F_{A,B}$ denote the function that sends each morphism $A \xrightarrow{f} B$ in $\mathbf{C}$ to $F(A) \xrightarrow{F(f)} F(B)$ in $\mathbf{D}$. Let Hom be the external hom-functor on $\mathbf{D}$. Then τ is a natural transformation $F \overset{\bullet}{\to} G$ if and only if the following diagram commutes for all objects A, B of $\mathbf{C}$:

$$\begin{array}{ccc}
\mathrm{Hom}\,(A, B) & \xrightarrow{\quad F_{A,B} \quad} & \mathrm{Hom}\,(F(A), F(B)) \\
{\scriptstyle G_{A,B}} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Hom}\,(\mathrm{id}_{G(A)}, \tau_B)} \\
\mathrm{Hom}\,(G(A), G(B)) & \xrightarrow[\mathrm{Hom}\,(\tau_A, \mathrm{id}_{G(B)})]{} & \mathrm{Hom}\,(F(A), G(B))
\end{array} \tag{3.97}$$

Proof. Chase a morphism $A \xrightarrow{f} B$ through eq. (3.97):

$$
\begin{array}{ccc}
f & \longmapsto & F(f) \\
\big\downarrow & & \big\downarrow \\
G(f) & \longmapsto & \boxed{G(f) \circ \tau_A = \tau_B \circ F(f)}
\end{array}
\tag{3.98}
$$

The bottom right corner states exactly the definition of natural transformation as given in eq. (2.42). ☐

We can now finish our demonstration that $\mathbf{B}_{\mathbb{K}}$ is a closed category:

Example 3.39 (continued). Due to the above lemma, eq. (3.95) says that, if V is faithful, $L^A_{B,-}$ is a natural transformation $[B, -] \xrightarrow{\bullet} [[A, B], [A, -]]$. Because V is indeed faithful on $\mathbf{B}_{\mathbb{K}}$, we need only prove that the following diagram commutes for all $C \xrightarrow{f} D$:

$$
\begin{array}{ccc}
[B, C] & \xrightarrow{L^A_{B,C}} & [[A, B], [A, C]] \\
{\scriptstyle [\mathrm{id}_B, f]}\big\downarrow & & \big\downarrow{\scriptstyle [\mathrm{id}_{[A,B]}, [\mathrm{id}_A, f]]} \\
[B, D] & \xrightarrow[L^A_{B,D}]{} & [[A, B], [A, D]]
\end{array}
\tag{3.99}
$$

Since this is merely the statement that L^A is natural in its second argument, eq. (3.95) commutes and $\mathbf{B}_{\mathbb{K}}$ is a closed category.

We have now seen that definition 3.38 is quite cumbersome to work with. Fortunately, a shortcut is sometimes available:

Proposition 3.41 (Eilenberg and Kelly [21, prop. II.4.1])**.** Let $\mathbf{C}$ be a category with multiplication $\Box$ and internal hom-functor $[\ ,\ ]$. Let K be an object of $\mathbf{C}$, and let $p_{A,B,C}$ be an isomorphism $[A \Box B, C] \simeq [A, [B, C]]$, natural in A, B, and C. Then the following statements are equivalent:

(i) **C** is a monoidal category with tensor product $\Box$ and unit K;

(ii) **C** is a closed category with unit object K.

This motivates the following definition, formed by applying V to p:

Definition 3.42. A locally small category **C** is a *closed monoidal category* with unit K if it is both a monoidal category with unit K and a closed category with unit K, and the tensor product is left adjoint to the internal hom.

Some examples of closed monoidal categories are **Set** with the Cartesian product, and K-**Mod** with the tensor product of modules. Note that not every monoidal category that is closed is a closed monoidal category: $\mathbf{Vect}_{\mathbb{K}}$ is a closed category, and we can form a monoidal category by choosing the direct sum as our tensor product (and the empty vector space as the unit object), but then the isomorphism $p_{A,B,C}$ does not exist for all objects A, B, C, and the tensor and internal hom do not have the same unit. Therefore $\langle \mathbf{Vect}_{\mathbb{K}}, \oplus, \emptyset, \alpha, \lambda, \rho \rangle$ is a closed *and* monoidal category, but not a *closed monoidal* category.

This relationship between closed and monoidal structure is often said to point out the "correct" tensor product: the most correct tensor is the one that is adjoint to the internal hom. That is why, out of all the possible products, coproducts, and other bifunctors, we almost always use $\otimes$ on modules, and $\times$ on **Set**.

A *closed symmetric monoidal category* is a symmetric monoidal category that is also a closed monoidal category. One common example is $\mathbf{FdHilb}_{\mathbb{K}}$, which also has some other interesting properties:

Example 3.43. Let $\mathbf{FdHilb}_{\mathbb{K},\otimes}$ denote the monoidal category formed by equipping $\mathbf{FdHilb}_{\mathbb{K}}$ with the usual tensor product, and for every object $\mathcal{H}$

of $\mathbf{FdHilb}_{\mathbb{K},\otimes}$, let $[\mathcal{H},\mathbb{K}] = \mathcal{H}^*$, where * is the Hermitean adjoint. Note that the usual definition of * from functional analysis makes it a contravariant endofunctor on $\mathbf{FdHilb}_{\mathbb{K},\otimes}$. Then for all finite-dimensional $\mathbb{K}$-Hilbert spaces A, B, we have $[A,B] = B \otimes A^*$, and there exists an isomorphism

$$p_{A,B,C} : [A \otimes B, C] \cong [A, [B,C]] \tag{3.100}$$

natural in A, B, and C, so by proposition 3.41, $\mathbf{FdHilb}_{\mathbb{K},\otimes}$ is a closed monoidal category.

For any fixed $\mathbb{K}$-Hilbert space $\mathcal{H}$, we can then formulate a tensor-hom adjunction in $\mathbf{FdHilb}_{\mathbb{K},\otimes}$:

$$\phi_{A,B} : \mathrm{Hom}\,(A \otimes \mathcal{H}, B) \cong \mathrm{Hom}\,(A, B \otimes \mathcal{H}^*) \tag{3.101}$$

where the functor $L(A) = A \otimes \mathcal{H}$ is left adjoint to $R(B) = B \otimes \mathcal{H}^* = [\mathcal{H}, C]$. The unit of this adjunction is

$$\eta_A^{\mathcal{H}} = \sum_{i,j} \left(\langle a_i| \otimes \langle h_j| \right) \otimes \left(|a_i\rangle \otimes |h_j\rangle \right) \tag{3.102}$$

with $\{|a_i\rangle\}$ and $\{|h_k\rangle\}$ arbitrary orthonormal bases of A and $\mathcal{H}$. The counit is the *evaluation map:*

$$\mathrm{eval}\,|\chi\rangle \otimes \langle\psi| = \langle\psi|\chi\rangle \tag{3.103}$$

These have the useful property that for any morphism $\mathcal{H} \otimes A \xrightarrow{f} \mathcal{H} \otimes B$:

$$\mathrm{Tr}_{\mathcal{H}}\,(f) = \rho_B \circ \big((\varepsilon_{\mathbb{K}}^{\mathcal{H}} \circ \gamma_{\mathcal{H}^*,\mathcal{H}}) \otimes \mathrm{id}_B \big) \circ \alpha_{\mathcal{H},\mathcal{H}^*,B} \circ \mathrm{id}_{\mathcal{H}} \otimes f \circ \alpha^{\mathrm{inv}}_{\mathcal{H}^*,\mathcal{H},A} \circ \big(\eta_{\mathbb{K}}^{\mathcal{H}} \otimes \mathrm{id}_A \big) \circ \lambda_A^{\mathrm{inv}} \tag{3.104}$$

as is easily, though laboriously, verified by computation.

We can use whiskerings to define trace in a nicer way:

Definition 3.44. The *trace* of a 2-cell $A \Box B \xrightarrow{f} C \Box B$ in a compact closed category is the 2-cell:

$$\mathrm{Tr}_B(f) = \lambda \circ ((\varepsilon \circ \gamma) \cdot C) \circ \alpha \circ (B^* \cdot f) \circ \alpha^{\mathrm{inv}} \circ (\eta \cdot A) \circ \lambda^{\mathrm{inv}} \tag{3.105}$$

where unnecessary subscripts are omitted, and η, ε are the unit and counit of the adjunction $B \dashv B^*$.

Our previous discussion of braiding, symmetry, and adjunctions, combined with closedness, has given us quite a toolbox with which we can examine the linear structure of (certain subtypes of) monoidal categories. We have discussed scalars, dual objects, and tensors in detail; along the way we picked up traces and the relationship between tensors and duals. All of this culminates in the notion of a compact closed category:

Theorem 3.45 (Kelly [38])**.** The following statements are equivalent:

(i) $\mathbf{C}$ is a closed monoidal category, where for all 1-cells A, B, C: $[A, B \Box C] \simeq B \Box [A, C]$;

(ii) $\mathbf{C}$ is a symmetric monoidal category where every 1-cell A has a left adjoint A^*.

In that case, $A^* = [A, K]$, $A^{**} = A$, and $(-)^*$ is a contravariant endofunctor on $\mathbf{C}$. We then call $\mathbf{C}$ a *compact closed category*, and A^* the *dual object* of A.

Now recall the uncategorical language we used earlier: norms, multiplication, and complex conjugates. We have now seen that scalars are endomorphisms of the monoidal unit, so a norm is simply a functor that sends every $A \xrightarrow{f} B$ to some $K \xrightarrow{\|f\|} K$, and multiplication is composition of scalars.

In a compact closed category, applying the *-functor to a scalar gives the complex conjugate. For vectors, the dual gives us bras and kets.

Definition 3.46. In a compact closed category with unit object K, let $K \xrightarrow{|a\rangle} A$ be a vector. We then call $|a\rangle$ a *ket*, and we call $\langle a| \equiv |a\rangle^*$ a *bra*.

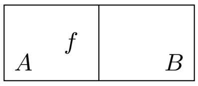
We are now fully equipped to speak in categorical terms about vector spaces, C*-algebras, and many of their constructions. Unfortunately, our diagrams have grown ever larger, and our equations ever longer. We end this chapter by introducing a graphical language that tames such oversized and over-complicated diagrams as eq. (3.89).

3.10 The graphical calculus

In this section, we introduce a second graphical language which (at the cost of some minor details) greatly simplifies our diagrams: so much so, that almost all diagrams that we draw from now on are in this second format. It is also a useful computational tool: theorems 3.48 and 3.50, below, tell us that a statement is derivable from the axioms for a specific type of category if and only if it derivable by the graphical manipulation of diagrams, following rules given by the dialect of the graphical language specific to that type of category. This visual manipulation is often far easier than the diagram chasing and manual equation solving we have been doing so far.

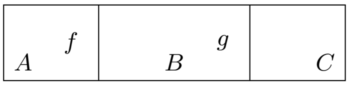
The graphical calculus is a generalisation of Penrose's tensor calculus (see [62] and the references therein for an historical overview). Whereas Penrose used lines to represent vector spaces and boxes for functions, we use lines to represent 1-cells, and boxes (or circles, triangles, ...) for 2-cells. Regions of the plane, separated by 1-cells, represent 0-cells. We read horizontal composition right-to-left, and vertical composition bottom-to-top. So instead of writing

$B \xrightarrow{f} A$, we draw:



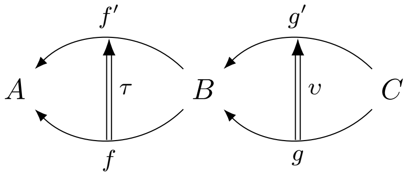
(3.106)

If we also have $C \xrightarrow{g} B$, then we draw $C \xrightarrow{f \Box g} A$ as:



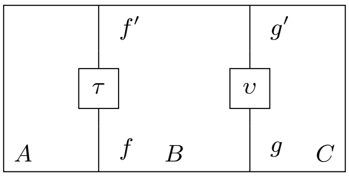
(3.107)

and the diagram



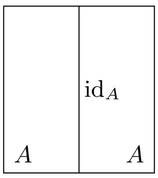
(3.108)

becomes:

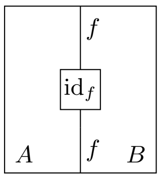


(3.109)

We mentioned above that the graphical calculus loses some detail. In particular, we do not draw any identity 1-cells and 2-cells. So instead of



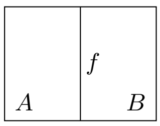
and



(3.110)

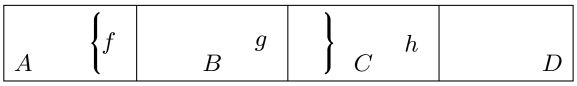
we simply draw:

and



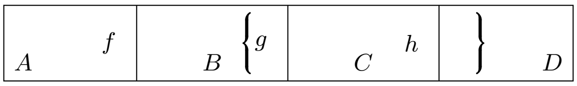
(3.111)

Another lost detail is associativity: the graphical calculus makes no distinction between



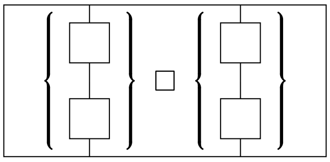
(3.112)

and

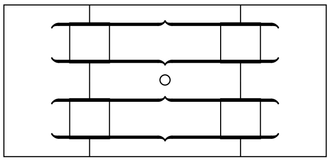


(3.113)

Nor does it distinguish



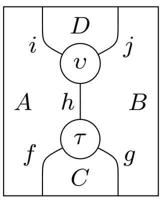
from



(3.114)

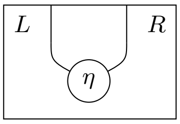
Note that the above is a graphical statement of the interchange law; and note that we've stopped labelling our 0-cells. Almost all the 2-categories we discuss in this thesis are monoidal, so there usually is only one 0-cell anyways.

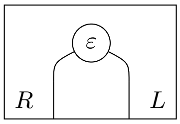
Also note that 2-cells may join and split 1-cells. For example, let $\tau : f\Box g \Rightarrow h : B \to A$ and $\upsilon : h \Rightarrow i\Box j : B \to A$ be 2-cells, and let $f : C \to A$, $g : B \to C$, $h : B \to A$, $i : D \to A$, $j : B \to D$ be 1-cells. We can capture all this information in one simple picture:



(3.115)

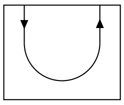
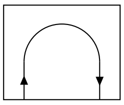
To see the power of this calculus let L be left adjoint to R. We draw the unit η and counit ε as:



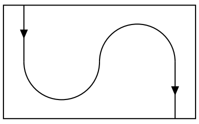
and


(3.116)

It is often nicer to write L as a downward-pointing arrow, and R as an upward-pointing arrow. We can then reduce the above diagrams to:


and

(3.117)

so the monstrous diagrams of eq. (3.82) become: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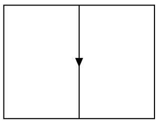
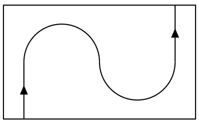

(3.118)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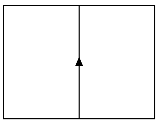
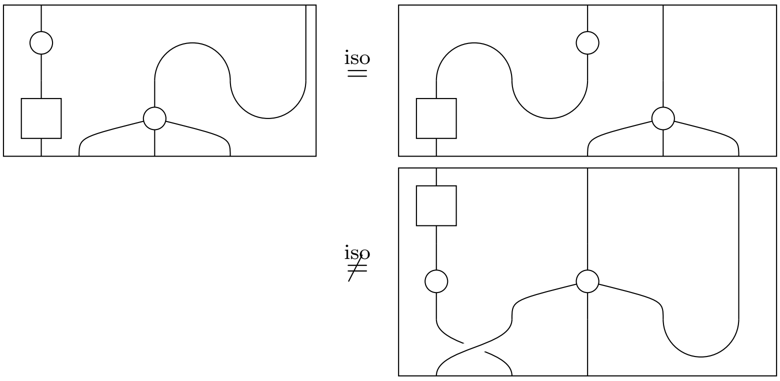

(3.119)

These *yanking equations* are instances of a more general rule:

Definition 3.47. In the graphical calculus, diagrams Γ and Δ are *equivalent by planar isotopy* (or $\Gamma \stackrel{\text{iso}}{=} \Delta$) if each can be continuously deformed into the other within its bounding rectangle, without introducing (or removing) any 1-cell crossings, without sliding any 2-cell over another, and without changing the domain or codomain of the diagram beyond associative and unital isomorphism.

For example:

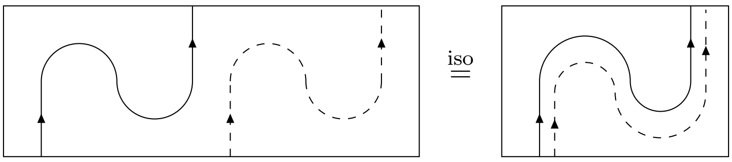


(3.120)

A diagram Γ with input strands $A_1, \ldots, A_m$ and outputs strands $B_1, \ldots, B_n$ is equivalently a 2-cell $\square_{i=1}^{m} A_i \xrightarrow{\gamma} \square_{j=1}^{n} B_j$. This allows for:

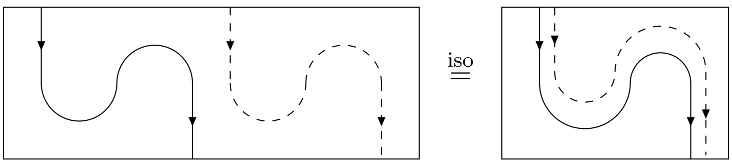
Theorem 3.48. Let Γ, Δ be diagrams in a monoidal category, and let γ, δ be the corresponding 2-cells. Then $\gamma \simeq \delta$ follows from the axioms for monoidal categories if and only if $\Gamma \overset{\text{iso}}{=} \Delta$.

This lets us simplify several complex diagrams and proofs. Proposition 3.37 now reduces to the equations:



(3.121)

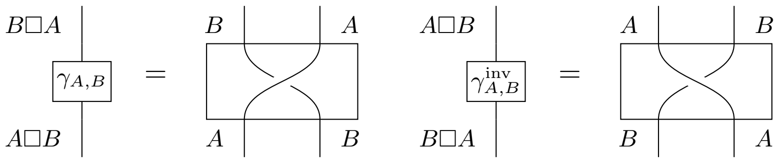
and



(3.122)

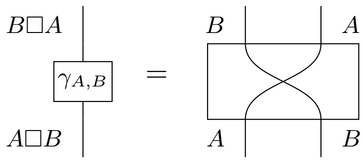
both of which obviously hold.

If our monoidal category is also braided, then the braid isomorphisms are the boxes:



(3.123)

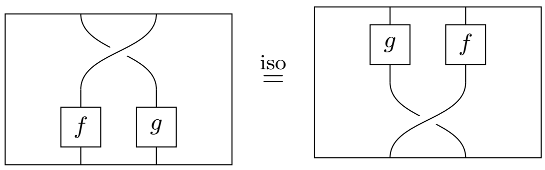
and if the category is symmetric monoidal, we may simply draw:



(3.124)

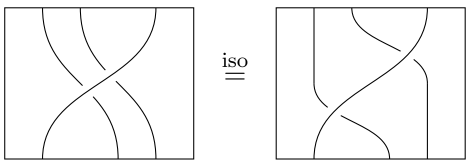
The statement that $\gamma_{A,B}$ is natural in A and B then reads:

$\forall f, g :$

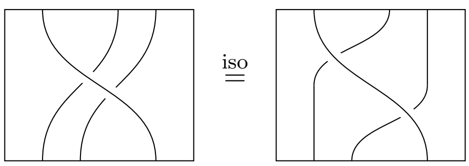


(3.125)

and the braided hexagon eqs. (3.67) and (3.68) become:



(3.126)



(3.127)

Because we can now cross and uncross our 1-cells, and slide 2-cells over these crossings, we have effectively added a third dimension to our graphical calculus: instead of by rectangles, our diagrams are now bounded by rectangular cuboids. Theorem 3.48 then has an obvious three-dimensional extension:

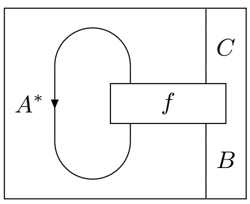
Definition 3.49. In the graphical calculus, diagrams Γ and Δ are *equivalent by spatial isotopy* (or $\Gamma \overset{\text{sp. iso}}{=} \Delta$) if each can be continuously deformed into the other within its bounding cuboid, without any 1-cells or 2-cells touching each other, and without changing the domain or codomain of the diagram beyond braided, associative, and unital isomorphism.

Theorem 3.50. Let Γ, Δ be diagrams in a braided monoidal category, and let γ, δ be the corresponding 2-cells. Then $\gamma \simeq \delta$ follows from the axioms for braided monoidal categories if and only if $\Gamma \overset{\text{sp. iso}}{=} \Delta$.

Note that in a monoidal category, eqs. (3.123) to (3.127) are definitions of 2-cells which must obey theorem 3.48, whereas in a braided monoidal category they are, respectively, an abbreviation, an abbreviation, a statement, a consequence, and a consequence of spatial isotopy.

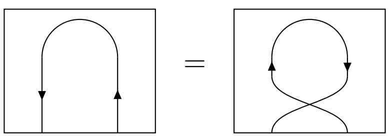
A further advantage of the graphical calculus (besides its simplification of many braid equations) is that it simplifies definition 3.44, of trace in a compact closed category:

Definition 3.51. The *partial trace* over A of a 2-cell $A \square B \xrightarrow{f} A \square C$ in a compact closed category is the 2-cell:

$\mathrm{Tr}_A(f) \quad =$ 

(3.128)

where


. (3.129)

This concludes our mathematical preliminaries. We now have all the background necessary to discuss categorical quantum and classical mechanics.

Part II

Categorical Physics

Chapter 4

The Categorical Structure of Physical Theories

The motivating intuition for our definition of categories at the beginning of chapter 2 was mathematical structures (objects) connected by structure-preserving maps (morphisms). As we studied more and more categories, the structure of definition 2.1 turned out to be more general than that. In monoids, groups, and posets, the morphisms do not necessarily operate on the object or objects in any way, and in $\mathbf{Braid}_P$ all the morphisms are endo, so they cannot connect objects. In this chapter, without changing our mathematical definitions, we further expand our intuitive picture of what categories are, and categorically define physical theories.

We first discuss some pre-theoretical intuitions of what an operational physical theory ought to be able to do, and show that these correspond to the structure of a symmetric monoidal category (§ 4.1). Then we show in § 4.2 how this structure, with the additional assumptions that we are working in $\mathbf{FdHilb}_{\mathbb{C}}$ and that adjunctions encode correlations, lets us describe the quantum teleportation protocol and prove its correctness, and suggests wider applications of categorical physics. Unfortunately, these applications are far less straightforward than we might initially expect; § 4.3 briefly discusses the problems with categorical classical physics.

4.1 Pre-theoretical intuitions

Most of the structures we discussed in chapter 2 were *concrete categories*. That is, categories in the sense of our motivating intuition: **Set**, $\mathbf{Vect}_{\mathbb{K}}$, and **Rel** all consist of mathematical objects — sets, perhaps with some extra structure — each one connected, with specific structure preserved, to all the others by morphisms. In addition, we defined some structures — such as categories, posets, categories with finite products, or monoidal categories — of which the concrete categories are models. These are *abstract categorical structures*, which were the main focus of chapter 3.

But there is no reason to restrict our understanding of categories to those containing mathematical structures: in fact, Bob Coecke and Éric Paquette extend this dichotomy with "*real-world categories*", whose objects are systems that exist in the "real world", and whose morphisms are transformations of real-world systems [18]: one of Coecke's favourite examples is the cooking category where objects are foods (potato, cooked potato, spiced potato, mashed potato, ..., carrot, cooked carrot, ...), and morphisms are preparations (peeling, cooking, spicing, mashing, ...). Since our topic is quanta and their mechanics (not cooking and menus), our first example is the category **PhysOp** of physical systems (both quantum and classical) and laboratory operations (also defined in [18]).

So what kind of category is **PhysOp**? It seems reasonable to ask that every operational physical theory have the following properties, all of which **PhysOp** would have too:

OpTh-I We do not have to take the entire universe into account for every calculation we make: instead, we can get reasonable approximations by considering smaller systems in isolation.

OpTh-II If we can perform an operation f in the laboratory, which turns system A into system B, and an operation g that turns system B into system C, then we can also perform a third operation — call it $g \circ f$ — that transforms A into C.

OpTh-III We can choose to do nothing to a system.

OpTh-IV Any two systems A, B can be composed to form a system $A \square B$.

OpTh-V There exists an empty system.

OpTh-VI If we can perform the operations $A \xrightarrow{f} A'$ and $B \xrightarrow{g} B'$ separately, we can also perform them in parallel: $A \square B \xrightarrow{f \square g} A' \square B'$.

OpTh-VII $A \square B$ and $f \square g$ are, for every conceivable purpose, just as good as $B \square A$ and $g \square f$, though they are generally not the same.

OpTh-VIII Switching the order of the terms in $A \square B$ or $f \square g$ around twice, is the same as doing nothing.

OpTh-I to OpTh-III tell us that physical systems must be the objects, and that operations must be the morphisms of **PhysOp**. OpTh-IV to OpTh-VI imply that **PhysOp** has a monoidal structure, OpTh-VII that it is braided, and OpTh-VIII that it is symmetric.

OpTh-VII is particularly important, as it states that we can keep track of individual systems even when they are composed. For example, if A is a beaker of sodium hydroxide, and B an oscilloscope, then OpTh-VII tells us that

$$\mathsf{NaOH_{aq}} \,\square\, \mathsf{oscilloscope} \xrightarrow{\mathsf{add\ HCl_{aq}} \,\square\, \mathsf{plug\ in}} \mathsf{NaCl_{aq}} \,\square\, \mathsf{oscilloscope} \tag{4.1}$$

is not the same process as

$$\mathsf{NaOH_{aq}} \,\square\, \mathsf{oscilloscope} \xrightarrow{\mathsf{plug\ in} \,\square\, \mathsf{add\ HCl_{aq}}} \mathsf{blown\ fuse} \,\square\, \mathsf{broken\ oscilloscope} \tag{4.2}$$

along with similar statements for other permutations of A, B, f, g.

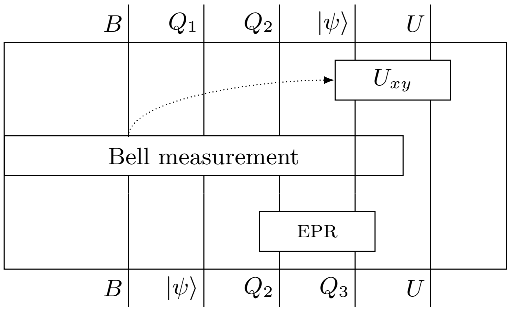
Note that $\square$ is not the usual tensor product or direct sum; nor are A and B vector spaces of any kind. The systems under consideration are, at this stage of reasoning, nothing but physical objects, and $A \square B$ means nothing else than consider both A and B. Also note that we consider $\mathsf{NaOH_{aq}}$ and $\mathsf{NaCl_{aq}}$ two different systems, while oscilloscope and plug in (oscilloscope) are both states of the same system. We could of course have taken a more detailed view, where plug in is not an endomorphism of the oscilloscope system, or a far more general view, where $\mathsf{NaOH_{aq}}$, $\mathsf{NaCl_{aq}}$, oscilloscope, and broken oscilloscope are all states of the system $\mathsf{neutron}^p \,\square\, \mathsf{proton}^q \,\square\, \mathsf{electron}^r$, for very large values of p, q, r. The exact demarcation of systems is context dependent.

4.2 First applications to quantum mechanics

The above description of categorial physics might seem both too abstract to be usable, and too obvious to be of any interest. In this section and the next, we address those criticisms. We first discuss quantum teleportation, the study of which motivated much of the early research in categorical quantum mechanics. This setting lets us flesh out the quantum-mechanical part of

PhysOp, and shows that our initial set of pre-theoretical assumptions needs two additions in order to encode all of non-relativistic quantum mechanics. Then we briefly touch on the difficulties involved in categorical classical physics.

Let Q_1, Q_2, Q_3 be qubits, and let B be a device that performs a Bell measurement, let U be a device that performs a unitary operation on a qubit, and let  be a procedure that puts two qubits in an EPR state. Then the following diagram depicts quantum teleportation [10] in **PhysOp**:

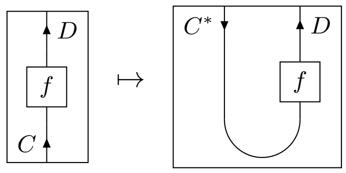


(4.3)

provided the unitary operation U_{xy} performed by U depends on the result of the measurement performed by B in the correct way. The Bell measurement and the unitary correction will perfectly transfer the mystery state of Q_1 to Q_3, while only exchanging two bits of classical information.

Categorical quantum mechanics began when Bob Coecke and Samson Abramsky sought to remove B and U from this diagram, and make the effect of this 2-cell — teleportation — more evident from its structure [1, 15, 16]. The three key insights here are that compact closed structure lets us represent entangled state preparations and measurements, that the left and right adjoint arrows point in the direction of information flow, and that biproducts allow for branching to occur in a single diagram. We discuss each of these, in as categorical a language as possible.

First, note that for all 1-cells C, D in a compact closed category with identity 1-cell K, there exists a bijection $[C, D] \simeq [K, C^* \Box D]$, given by "bending the wire" [15]:



(4.4)

We write the right-hand side of this equation as $\ulcorner f \urcorner$, which we call the *name* of f.

If we take **PhysOp** to be compact closed, identify its objects with finite-dimensional Hilbert spaces, and choose $\otimes$ as the tensor, then every name has the form

$$k \mapsto k \sum_{i,j} f_{i,j} \langle c_i | \otimes | d_j \rangle \tag{4.5}$$

with $\{\langle c_i|\}_i$, $\{|d_j\rangle\}_j$ orthonormal bases of C, D. Therefore, the right-hand side of eq. (4.4) encodes a correlation between C and D.

In particular, let C and D be qubits, let $\{|c_i\rangle\}_i$, $\{|d_j\rangle\}_j$ both be computational bases, and let f be one of the following matrices:

$$\beta_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \beta_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\beta_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \beta_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

If $f = \beta_4$, for example, then $\ulcorner f \urcorner$ is the mapping

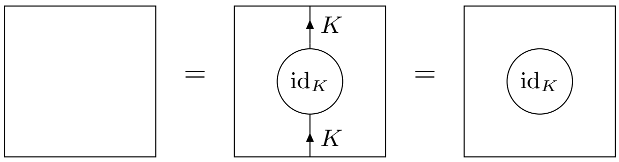
$$k \mapsto \frac{k}{\sqrt{2}} \sum_{i,j} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_{i,j} \langle c_i | \otimes | d_j \rangle = \frac{k}{\sqrt{2}} \left(|c_0 d_1\rangle - |c_1 d_0\rangle \right), \tag{4.6}$$

where $|c_i d_j\rangle \equiv \langle c_i| \otimes |d_j\rangle$. And if we write the Bell states as

$$|b_1\rangle = \frac{1}{\sqrt{2}}(|c_0 d_0\rangle + |c_1 d_1\rangle) \qquad |b_2\rangle = \frac{1}{\sqrt{2}}(|c_0 d_1\rangle + |c_1 d_0\rangle)$$
$$|b_3\rangle = \frac{1}{\sqrt{2}}(|c_0 d_0\rangle - |c_1 d_1\rangle) \qquad |b_4\rangle = \frac{1}{\sqrt{2}}(|c_0 d_1\rangle - |c_1 d_0\rangle)$$

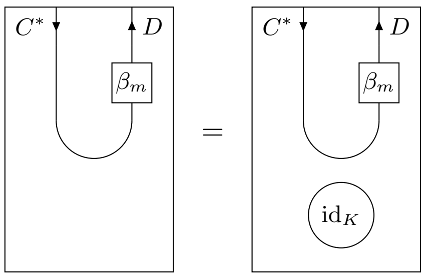
then setting $f = \beta_i$ and bending the wire generates the mapping $\ulcorner f \urcorner = \ulcorner \beta_i \urcorner : k \mapsto \frac{k}{\sqrt{2}}|b_i\rangle$.

Now recall that in the graphical calculus, we don't have to draw identity 1-cells or 2-cells, but it is not forbidden to do so. And because K is the identity 1-cell of our compact closed category:



(4.7)

that is, the empty system is equivalent to id_K. Since $\mathrm{id}_K = 1$ in the category of finite-dimensional Hilbert spaces and linear maps, we then have:



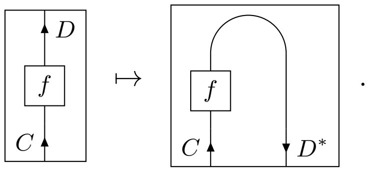
(4.8)

where the right hand side is a mapping

$$\text{empty system} \mapsto 1 \mapsto |b_m\rangle. \tag{4.9}$$

Hence, a name is a procedure that maps the empty system to an entangled state — i.e. an entangled state preparation.

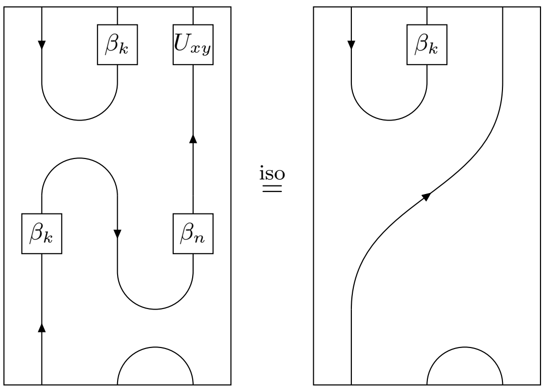
Instead of bending the lower wire up in eq. (4.4), we could have also bent the upper wire down, giving a bijection



(4.10)

The right-hand side of this equation is the *coname* $\llcorner f \lrcorner$ of f. Intuitively, this would represent the opposite of an entangled state preparation — a measurement of an entangled pair, yielding a specific outcome — and this is indeed so (but more on that later).

Theorem 3.3 of [16] then tells us that the teleportation protocol of eq. (4.3) is equal to the left-hand side of:



(4.11)

The isotopic equivalence proves the correctness of the protocol. The right-hand side of the diagram depicts a 2-cell that takes three input strands at the bottom and produces three output strands at the top. The value at the leftmost input strand is sent to the rightmost output strand, the values of the middle and right inputs are discarded, and the values of the left and middle outputs form an entangled pair in the Bell state $|b_k\rangle$. Note that the equivalence holds only when $U_{xy} \circ \beta_n \circ \beta_k = \text{id}$ (but more on that later).

On the right hand side, there appears to be some kind of flow. This cannot be the movement of a physical system, as our notation does not encode that; rather, the arrow is often said to point in the direction of information flow. This is the second key insight underlying categorical quantum mechanics.

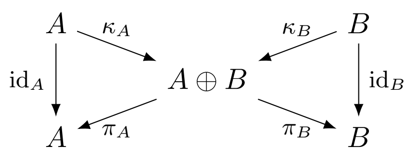
This protocol might at first seem retrocausal, as the arrow in the middle of the left-hand side points backwards in time, but that is not the case. First, we should note that once the Bell measurement is performed, the arrow cannot go forwards in time. The pre-measurement states of Q_1 and Q_2 provide no information on the post-measurement states, so the arrow along Q_1 must either stop or change direction. Also note that if we know the state of Q_1 before the measurement, and we know the measurement outcome — that is, the state at the top of the counit — then we know the state Q_2 had before the measurement. Hence, we can follow the information flow along Q_1, over the top of the counit, and down into Q_2. And if we know the state of Q_2 after the entangled state preparation, and we know how the entangled state was prepared — that is, we know what β_n is — then we know the state in Q_3 after the entangled state preparation. The protocol is not retrocausal, because backwards information flow does not affect the past from the future; it merely allows an observer in the future to calculate past states. (Since all the operations encountered along the way are unitary, this also works in the opposite direction.)

Above, we promised to discuss measurements and the isotopic equivalence in eq. (4.11) in more detail. We now introduce biproducts, and use them to tie up these loose strands.

Recall definition 2.13, which states that the product of two objects A, B in a category $\mathbf{C}$ is an object C of $\mathbf{C}$, along with morphisms $C \xrightarrow{\pi_A} A$ and

$C \xrightarrow{\pi_B} B$, satisfying certain conditions. We then call $\langle C, \pi_A, \pi_B \rangle$ a *product tuple*. Similarly, definition 2.17 lets us characterise the coproduct C of A and B in terms of a *coproduct tuple* $\langle C, A \xrightarrow{\kappa_A} C, B \xrightarrow{\kappa_B} C \rangle$. The biproduct $A \oplus B$ of A and B arises out of $\langle A \Pi B, \pi_A, \pi_B \rangle$ and $\langle A \amalg B, \kappa_A, \kappa_B \rangle$ as follows:

Definition 4.1. For A, B objects in a category $\mathbf{C}$ with a zero object, the *biproduct* of A and B is an object $A \oplus B$ of $\mathbf{C}$, along with morphisms κ_A, κ_B, π_A, π_B such that $\langle A \oplus B, \pi_A, \pi_B \rangle$ is a product tuple, $\langle A \oplus B, \kappa_A, \kappa_B \rangle$ is a coproduct tuple, the following diagram commutes:



(4.12)

and both $\pi_A \circ \kappa_B$ and $\pi_B \circ \kappa_A$ are zero morphisms.

A category *has finite biproducts* if $A \oplus B$ exists for all objects A and B (i.e. zero matrices, empty relations: anything that factors through the zero object).

Examples of categories with finite biproducts are $\mathbf{Vect}_{\mathbb{K}}$, with biproducts given by the direct sum, and $\mathbf{Rel}$, with biproducts given by the disjoint union (cf. examples 2.14, 2.15, 2.18, and 2.19). In both cases the product is also a coproduct, so $A \Pi B = A \amalg B$, and in both cases the projections $A \Pi B \xrightarrow{\pi_A} A$, $A \Pi B \xrightarrow{\pi_B} B$ are the left inverses of the injections $A \xrightarrow{\kappa_A} A \amalg B$, $B \xrightarrow{\kappa_B} A \amalg B$.

Now let $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_n\}$ be finite sets of objects in some category with finite biproducts. Then any morphism $\bigoplus_{j=1}^m A_j \xrightarrow{f} \bigoplus_{i=1}^n B_i$ is uniquely characterised by how every component of its domain is mapped to each component of its codomain — that is, by all the morphisms $f_{i,j} =$

$\pi_{B_i} \circ f \circ \kappa_{A_j}$ — and we may write:

$$f = \begin{bmatrix} f_{1,1} & \cdots & f_{1,m} \\ \vdots & \ddots & \vdots \\ f_{n,1} & \cdots & f_{n,m} \end{bmatrix}. \tag{4.13}$$

This matrix representation of morphisms lets us reconstruct the usual rules for matrix manipulation. First, for morphisms $A_1 \xrightarrow{f} B_1$, $A_2 \xrightarrow{g} B_2$, we define:

$$f \oplus g = \begin{bmatrix} f & 0_{A_2,B_1} \\ 0_{A_1,B_2} & g \end{bmatrix} : A_1 \oplus A_2 \mapsto fA_1 \oplus gA_2. \tag{4.14}$$

Then, we define addition of $A \xrightarrow{f} B$ and $A \xrightarrow{g} B$ as:

$$f + g = (\mathrm{id}_B \amalg \mathrm{id}_B) \circ (f \oplus g) \circ (\mathrm{id}_A \amalg \mathrm{id}_A). \tag{4.15}$$

For example, in $\mathbf{Vect}_{\mathbb{K}}$, $f + g$ is the mapping:

$$|a\rangle \xmapsto{\begin{bmatrix} \mathrm{id}_A \\ \mathrm{id}_A \end{bmatrix}} |a\rangle \oplus |a\rangle \xmapsto{\begin{bmatrix} f & \mathbf{0} \\ \mathbf{0} & g \end{bmatrix}} f|a\rangle \oplus g|a\rangle \xmapsto{\begin{bmatrix} \mathrm{id}_B & \mathrm{id}_B \end{bmatrix}} f|a\rangle + g|a\rangle \tag{4.16}$$

where the last $+$ denotes ordinary addition. In $\mathbf{Rel}$, $R + S$ is the relation:

$$A \mapsto \{(a, \star), (a, \bullet) \mid a \in A\} \mapsto \left\{(b, \star), (b', \bullet) \;\middle|\; aRb, aSb'\right\} \mapsto \left\{b, b' \;\middle|\; aRb, aSb'\right\}, \tag{4.17}$$

so $R + S = R \cup S$.

This addition operation is associative, commutative, and has the zero morphism $0_{A,B}$ as its unit, so every category with finite biproducts is enriched over the category **CMon** of commutative monoids. (In $\mathfrak{Ab}$-enriched categories with finite biproducts, all the usual rules of matrix algebra apply to morphisms-considered-as-matrices: these are the *linear categories*.) For our purposes, it is enough that for any morphisms $\bigoplus_k A_k \xrightarrow{f} \bigoplus_j B_j$, $\bigoplus_k A_k \xrightarrow{g} \bigoplus_j B_j$, and $\bigoplus_j B_j \xrightarrow{h} \bigoplus_i C_i$ the following familiar rules hold:

$$(f+g)_{j,k} = f_{j,k} + g_{j,k} \tag{4.18}$$

$$(h \circ f)_{i,k} = \sum_j h_{i,j} \circ f_{j,k}. \tag{4.19}$$

Note that $\mathbf{Vect}_{\mathbb{K}}$ is $\mathfrak{Ab}$-enriched — and therefore linear — so in that category, this matrix algebra is simply the block matrix algebra; in the limit where all the A_k, B_j, C_i are $\mathbb{K}$, we have reconstructed the usual matrix calculus.

In the generalised matrix calculus, it is easy to show that $\Box$ distributes over $\oplus$. For any finite biproduct $\bigoplus_{k=1}^n A_k$ and object B, there exists an isomorphism

$$\upsilon_B = \begin{bmatrix} \pi_{A_1} \Box \mathrm{id}_B \\ \vdots \\ \pi_{A_n} \Box \mathrm{id}_B \end{bmatrix} : \left(\bigoplus_{k=1}^n A_k \right) \Box B \to \bigoplus_{k=1}^n (A_k \Box B). \tag{4.20}$$

Many more useful constructs in $\mathbf{FdHilb}_{\mathbb{C}}$, such as unitarity and self-adjointness, can be defined in terms of the adjoint. All we need to categorify this, is a contravariant endofunctor $(-)^\dagger$ that is an identity on objects and an involution on morphisms, chosen such that the associators and unitors are unitary, such for all morphisms f, g, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, and such that for all objects A: $\gamma_{A,A^*} \circ \varepsilon_A^\dagger = \eta_A$. If such a †-functor exists, we call our category *dagger compact closed*.

Biproducts and the adjoint in dagger compact closed categories immediately lead to bases and spectral decompositions [1]. If we define

$$n \cdot A \equiv \bigoplus_{k=1}^{n} A \tag{4.21}$$

then a *basis* of an object X in a category with tensor unit K is a unitary isomorphism $n \cdot K \xrightarrow{\mathsf{base}_X} X$. For example, in $\mathbf{FdVect}_{\mathbb{K}}$, a basis of the n-dimensional vector space X is a (non-unique) bijection that sends every $\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in \mathbb{K} \oplus \ldots \oplus \mathbb{K}$ to a vector in X.

Spectral decompositions take some more effort:

Definition 4.2 ([1, §7.4])**.** A *spectral decomposition* of an object A is a unitary isomorphism

$$A \xrightarrow{U} \bigoplus_{k} A_k \tag{4.22}$$

along with morphisms

$$q_k = U^\dagger \circ \kappa_{A_k} : A_k \to A \tag{4.23}$$

$$p_k = \pi_{A_k} \circ U : A \to A_k \tag{4.24}$$

and *projectors:*

$$P_k = q_k \circ p_k : A \to A, \tag{4.25}$$

where κ_{A_k} and π_{A_k} are the biproduct injections and projections.

The above is a generalisation of the familiar projection operators onto disjoint subspaces of a Hilbert space: the P_k are all self-adjoint, idempotent, and orthogonal, and $\sum_k P_k = \mathrm{id}_A$.

A spectral decomposition is *non-degenerate* if it sends A to $n \cdot K$.

Projectors and biproducts then let us discuss branching:

Definition 4.3 ([1, §8])**.** Given a spectral decomposition $A \xrightarrow{U} \bigoplus_{k=1}^{n} A_k$

with projectors P_k, a *measurement* is a morphism

$$\langle P_k \rangle_{k=1}^{n} \equiv \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} : A \to n \cdot A. \tag{4.26}$$

We call the projectors *measurement branches.* If U is non-degenerate, we can also describe the measurement in terms of the *observation branches* $p_k : A \to K$:

$$\langle p_k \rangle_{k=1}^{n} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}. \tag{4.27}$$

Now we return to eq. (4.11). The Bell measurement has four possible outcomes corresponding to the conames $\llcorner \beta_m \lrcorner$. That is, the value of m indicates the measurement outcome, which in turn determines U_{xy}. In terms of definition 4.3, the measurement stage of the teleportation protocol can then be written as a matrix of observation branches:

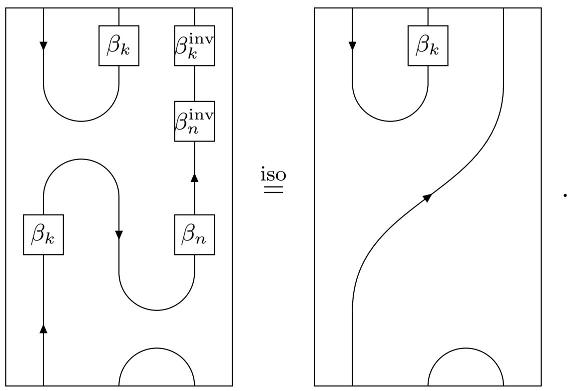
$$\langle \mathsf{Bell} \rangle = \langle \llcorner \beta_k \lrcorner \rangle_{k=1}^{4} = \begin{bmatrix} \llcorner \beta_1 \lrcorner \\ \llcorner \beta_2 \lrcorner \\ \llcorner \beta_3 \lrcorner \\ \llcorner \beta_4 \lrcorner \end{bmatrix}. \tag{4.28}$$

The distributivity of $\square$ over $\oplus$ lets us model classical communication. Given a biproduct $4 \cdot K$ of observation branches, the morphism υ_{Q_3} maps the tensor product $(4 \cdot K) \square Q_3$ — four branches of one measurement, tensored with one system in one branch — to $4 \cdot (K \square Q_3)$ — four branches, each containing a measurement outcome and a qubit.

The teleportation protocol then reads [1, thm. 9.2, simplified — removed some normalising constants and other unnecessary information]:

$$
\begin{array}{rcl}
 & 4 \cdot Q_3 & \\
\bigoplus_{k=1}^{4} \left(\beta_k^{\text{inv}} \circ \beta_n^{\text{inv}}\right) & \Big\uparrow & \text{unitary correction} \\
 & 4 \cdot Q_3 & \\
(4 \cdot \lambda_{Q_3}) \circ \upsilon_{Q_3} & \Big\uparrow & \text{classical communication} \\
 & (4 \cdot K) \Box Q_3 & \\
\langle \beta_k \rangle_{k=1}^{4} & \Big\uparrow & \text{teleportation observation} \\
 & (Q_1 \Box Q_2^*) \Box Q_3 & \\
\alpha^{\text{inv}}_{Q_1, Q_2^*, Q_3} & \Big\uparrow & \text{spatial delocation} \\
 & Q_1 \Box (Q_2^* \Box Q_3) & \\
(\text{id}_{Q_1} \Box \ulcorner \beta_n \urcorner) \circ \rho^{\text{inv}}_{Q_1} & \Big\uparrow & \text{produce EPR pair} \\
 & Q_1 &
\end{array}
\tag{4.29}
$$

For an experimenter in the k-th branch, this protocol looks as follows:



(4.30)

The box U_{xy} in eq. (4.11) has now been expanded to the two boxes β_n^{inv} and β_k^{inv}: the first is there because U_{xy} depends on which entangled state was prepared; the second because U_{xy} depends on the two bits gained from the Bell measurement. The fact that k ranges from 1 to 4 reflects that two classical bits are exchanged.

Equation (4.30) does not show the α, λ, and ρ of eq. (4.29), because the graphical calculus does not show associativity and unitality isomorphisms. It does not show any biproducts, because it only displays one branch. The isotopic equivalence holds because $\beta_k^{\text{inv}} \circ \beta_n^{\text{inv}} \circ \beta_n \circ \beta_k = \text{id}$.

So far, we have shown how to categorically describe certain quantum-mechanical phenomena, but in so doing, we moved out of **PhysOp** and into $\mathbf{FdHilb}_{\mathbb{C}}$. We also had to add two more postulates to our list of pre-theoretical intuitions:

OpTh-IX Adjunctions encode correlations.

OpTh-X There exist finite biproducts.

The eight pre-theoretical postulates, and the two post-theoretical, allow us to formulate teleportation-like protocols and physics-like statements in a compact closed category with biproducts, but leave us with a problem: OpTh-IX and OpTh-X are not at all intuitive or pre-theoretical. Another problem is classical mechanics, as we shall see now.

4.3 The problem of classical mechanics

Common wisdom holds that if quantum mechanics takes place in the category of Hilbert spaces, then classical mechanics should take place in the category of symplectic spaces (eg. [56, p. 66]). This approach comes with two major problems. The first is that until recently no one has actually managed to categorically describe classical mechanics in this way. The symplectic category is ill-behaved [65], and the only categorification of classical mechanics we know of that uses it is due to Baez, Weisbart, and Yassine, published as recently as 2021 [6]. Before 2021, any discussion of classical mechanics in or as the category of symplectic spaces was ill-founded. The second problem is that useful comparisons of quantum and classical mechanics get harder to make as the relevant categories get more different.

Ideally, we would be able to construct a category that models at least the first eight of our pre-theoretical intuitions, so that we may find out which of our post-theoretical additions — OpTh-IX and OpTh-X – are specific to quantum mechanics. Furthermore, since compact closed structure is more general than quantum mechanical structure, constructing categorical classical mechanics might help us identify more categorical structures common to all physical theories.

In the next chapter, we will construct the classical subcategory of **PhysOp**, and try to translate as much of Cat-QM as possible into that category. Our guiding empirical principle there, is that classical observables depend only on the positions and momenta of the particles involved.

Chapter 5

Categorical Classical Mechanics

Koopman-Von Neumann mechanics (KvN) is our bridge from categorical physics to classical mechanics. Developed in the 1930's by Bernard Koopman and John von Neumann [41, 53], it expresses the Liouvillean of classical statistical mechanics as an operator on a Hilbert space of functions from a phase space to the complex numbers. The modulus squared of an appropriately chosen function will be a probability distribution on the phase space, whose time-evolution by the Liouvillean operator is exactly as predicted by classical statistical mechanics.

Our presentation of KvN is somewhat unorthodox. In § 5.1 we discuss Koopman's lemma in the usual way, but where most authors would introduce commutation relations between position and momentum operators, and then derive more physics from that, we take a more general view of observables as nothing more than real-valued functions on the phase space, from which self-adjoint operators arise on the Hilbert space. For a more conventional overview of modern-day KvN, see §2 of [50]. Our unusual treatment of observables triggers a superselection mechanism which Ennio Gozzi and Danilo Mauro noticed and did their best to avoid [26], but which we are quite happy to exploit: we accept a few inelegancies in our classical theory, and in return we get a nice categorical quantum-classical barrier. In classical mechanics, we learn, there can be no pure superpositions.

That is half of our barrier. The other half is the width or narrowness of the quantum or classical subcategory of **PhysOp**. In § 5.2 we discuss the coarse-graining of KvN observables and prove the main theorem of this chapter: that the Hilbert space which one uses for Koopman-Von Neumann mechanics must be infinite-dimensional. We discuss uncorrelated KvN-systems in § 5.3, from which it follows immediately that the KvN subcategory of **PhysOp** is symmetric monoidal.

The main motivation of categorical quantum mechanics was the ability of compact closed structure to encode quantum correlations. In § 5.4 we conjecture, and provide some reasons to believe, that there is no classical analogue to Abramsky and Coecke's categorical description of quantum correlations.

5.1 An overview of Koopman-Von Neumann classical mechanics

Let $\mathfrak{S}$ be a dynamical system with phase space Γ, generalised coordinates q_k, conjugate momenta p_k, and Hamiltonian H. For some energy level E, we define

$$\Omega = \{\omega \in \Gamma \mid \mathrm{H}(\omega) = E\} . \tag{5.1}$$

For every $t \in \mathbb{R}$, let there exist an automorphism S_t of Ω that sends every ω to the phase space point that $\mathfrak{S}$ would occupy at time t if it occupied the point ω at time 0. The S_t compose as $[S_{t_1} S_{t_2}]\,\omega = S_{t_1+t_2}\omega$, so:

$$S_0\omega = \omega \tag{5.2}$$

$$\left(S_1 S_{t_2}\right) S_{t_3} = S_{t_1}\left(S_2 S_{t_3}\right) \tag{5.3}$$

$$S_t S_{-t} = S_0 \tag{5.4}$$

Now recall that in any phase space Γ with general point γ, the integral

$$\int \mathrm{d}\gamma \tag{5.5}$$

is invariant under canonical transformations, so $\int \mathrm{d}\omega$ is invariant under S_t. Also let ϕ be any Lebesgue measurable function $\Omega \to \mathbb{C}$ for which the integrals

$$\int_\Omega |\phi|\,\mathrm{d}\omega \tag{5.6}$$

$$\int_\Omega |\phi|^2\,\mathrm{d}\omega \tag{5.7}$$

are finite. The equivalence classes (with two functions equivalent if they differ only on a region with zero measure) of all such functions together form a Hilbert space $\mathcal{H}$ when equipped with the inner product

$$\langle\phi|\psi\rangle := \int_\Omega \phi^*\psi\,\mathrm{d}\omega, \tag{5.8}$$

where * denotes the complex conjugate. For every S_t, let the operator $U_t : \mathcal{H} \to \mathcal{H}$ be given by

$$[U_t\phi]\,\omega = \phi\left(S_t\omega\right), \tag{5.9}$$

and recall the following useful result:

Lemma 5.1 (Koopman's lemma [41])**.** The operators U_t form a continuous one-parameter unitary group.

Due to Stone's theorem [58, thm. VIII.8], there must exist a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$, such that

$$U_t \phi = e^{itD} \phi. \tag{5.10}$$

We may then write $U_t\phi$ as a time-dependent function $\phi(t, \omega)$. Its derivative is

$$\begin{aligned} \frac{\partial \phi}{\partial t}(t, \omega) &= \left[\frac{\mathrm{d}U_t}{\mathrm{d}t} \right] \phi(\omega) \\ &= [iDU_t]\, \phi(\omega) \\ &= iD\phi(t, \omega) \end{aligned} \tag{5.11}$$

We now have at our disposal a Hilbert space of functions which attach to every point on Ω a value which moves across Ω along the time-evolution paths of $\mathfrak{S}$, and we know that there exists a self-adjoint operator that provides the time-derivative of each function in $\mathcal{H}$. One possible interpretation of the values $\phi(\omega)$ is that for every ϕ such that $\langle \phi | \phi \rangle = 1$ and every suitable region $\Delta \subseteq \Omega$, the integral $\int_\Delta |[U_t \phi](\omega)|^2 \, \mathrm{d}\omega$ represents the epistemic probability of $\mathfrak{S}$ occupying the phase space region Δ at time t.

In classical statistical mechanics, every physically realisable probability distribution $\rho(t, \omega)$ on Ω is a solution to the Liouville equation:

$$\frac{\partial \rho}{\partial t} = \sum_k \left(\frac{\partial \mathrm{H}}{\partial q_k} \frac{\partial \rho}{\partial p_k} - \frac{\partial \mathrm{H}}{\partial p_k} \frac{\partial \rho}{\partial q_k} \right), \tag{5.12}$$

and if ϕ is a solution, then so is $\phi^* \phi$. Hence, we choose $D = -\mathrm{L}$ as the self-adjoint generator of U_t, where L is the Liouvillean operator:

$$\mathrm{L} = i \sum_k \left(\frac{\partial \mathrm{H}}{\partial q_k} \frac{\partial}{\partial p_k} - \frac{\partial \mathrm{H}}{\partial p_k} \frac{\partial}{\partial q_k} \right), \tag{5.13}$$

so that:

$$\begin{aligned} i \frac{\partial \phi}{\partial t} &= -D\phi \\ &= i \sum_k \left(\frac{\partial \mathrm{H}}{\partial q_k} \frac{\partial}{\partial p_k} - \frac{\partial \mathrm{H}}{\partial p_k} \frac{\partial}{\partial q_k} \right) \phi. \\ &= \mathrm{L}\phi \end{aligned} \tag{5.14}$$

This is a classical version of the Schrödinger equation.

We could also have worked the other way around. First we construct a Hilbert space and then we build a unitary group. Choosing our U_t such that the group is generated by $-\mathrm{L}$ guarantees that any everywhere-differentiable unit vector ϕ will evolve in time as a solution to the Liouville equation, and therefore that the S_t represent the correct time-evolution of $\mathfrak{S}$.

Definition 5.2. If the probability distribution on Ω is given by $\rho(\omega) = \phi^*(\omega)\phi(\omega)$, we call $|\phi\rangle$ a KvN-*state* of $\mathfrak{S}$. If $\mathfrak{S}$ occupies the phase space point ω, we call ω the *physical state* of $\mathfrak{S}$.

Having defined the states and dynamics of $\mathfrak{S}$ in terms of $\mathcal{H}$, we can extract expectation values of observables from our probability distributions:

Definition 5.3. An *observable* of $\mathfrak{S}$ is any function $\Omega \to \mathbb{R}$. The expectation value of an observable X conditional on a KvN-state $|\phi\rangle$ is

$$\langle X \,:\, \phi \rangle = \int_\Omega X(\omega) |\phi(\omega)|^2 \, \mathrm{d}\omega. \tag{5.15}$$

Every such observable gives rise to a self-adjoint operator $\hat{X} : \mathcal{H} \to \mathcal{H}$, defined as:

$$\left[\hat{X}\phi\right]\omega = X(\omega)\phi(\omega). \tag{5.16}$$

These operators form a real Abelian algebra on $\mathcal{H}$.

Now we seem to have a problem. Gozzi and Mauro [26] point out that in such a setup, every operator commutes with every other operator and therefore a superselection mechanism sets in. Given two eigenstates of some operator $\hat{Z}$

$$\begin{aligned} \hat{Z}|\phi_1\rangle &= z_1|\phi_1\rangle \\ \hat{Z}|\phi_2\rangle &= z_2|\phi_2\rangle \end{aligned} \tag{5.17}$$

and any observable X, we have $\langle\phi_1|\hat{X}|\phi_2\rangle = 0$ because the supports of the functions $|\phi_1\rangle$ and $|\phi_2\rangle$ are disjoint: $|\phi_1\rangle$ is supported on some subset of $\{\omega \in \Omega \mid Z(\omega) = z_1\}$, and $|\phi_2\rangle$ on some subset of $\{\omega \in \Omega \mid Z(\omega) = z_2\}$. Hence if the time-evolution operator of our system is to be an observable — as is the case in quantum mechanics — no movement is possible, as no observable can change in value. Fortunately, the Liouvillean depends not only on q_k and p_k but also on $\partial_{q_k}\phi$ and $\partial_{p_k}\phi$ and therefore is not an observable. Unfortunately, because our systems are now capable of time-evolving from one superselection sector into another, there is no a priori reason to believe any conservation laws still hold.

Another rather unpleasant consequence of our setup is that $\mathcal{H}$ might not be separable, as it becomes a direct sum (or a direct integral [26]) $\bigoplus_i \mathcal{H}_i$ of perhaps uncountably many superselection sectors.

All of this is unacceptable to Gozzi and Mauro [26]. They show that there is interesting physical information to be extracted from the Liouvillean, so it is desirable for it to be an observable. Also, they point out that there is no point in using a Hilbert space of complex-valued functions if all the observables are to be real-valued functions. Introducing additional observables $\hat{\lambda}_{q_k} = -i\partial_{q_k}$, $\hat{\lambda}_{p_k} = -i\partial_{p_k}$ makes the algebra of observables non-Abelian (since $[\hat{q}_j, \hat{\lambda}_{q_k}] = [\hat{p}_j, \hat{\lambda}_{q_k}] = i\delta^j_k$), prevents the superselection from setting in, and makes L an observable.

We choose instead to accept those inelegancies that are resolved by introducing the $\hat{\lambda}$'s in exchange for a nicer categorical structure, as we will demonstrate below. First, note that yet another consequence of this decision, as pointed out in [26], is that all superpositions of eigenstates corresponding to different eigenvalues are indistinguishable from mixed states.

Let $|\phi\rangle = \alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle$ be a unit vector, where $|\phi_1\rangle$, $|\phi_2\rangle$ are unit eigenstates of $\hat{X}$:

$$\begin{aligned} \hat{X}|\phi_1\rangle &= x_1|\phi_1\rangle \\ \hat{X}|\phi_2\rangle &= x_2|\phi_2\rangle \end{aligned} \tag{5.18}$$

Then $\langle\phi_1|\phi_2\rangle = \langle\phi_2|\phi_1\rangle = 0$, so the expected value of an X-measurement conditional on $|\phi\rangle$ is:

$$\begin{aligned} \langle X \,:\, \phi\rangle &= \langle\phi|\hat{X}|\phi\rangle \\ &= x_1|\alpha_1|^2\langle\phi_1|\phi_1\rangle + x_2|\alpha_2|^2\langle\phi_2|\phi_2\rangle \\ &= x_1|\alpha_1|^2 \operatorname{Tr}(|\phi_1\rangle\langle\phi_1|) + x_2|\alpha_2|^2 \operatorname{Tr}(|\phi_2\rangle\langle\phi_2|)\,. \end{aligned} \tag{5.19}$$

From now on we will write all KvN-states as density operators. In the next section, we explore and prove some consequences of this formulation of KvN, including that classical mechanics is not possible in finite-dimensional Hilbert spaces.

5.2 Coarse-graining of KvN observables

In categorical quantum mechanics, quantum correlations and protocols depend on the compact closed structure of $\mathbf{FdHilb}_{\mathbb{C}}$: the right adjoint of the Hilbert space A is its dual space A^*. This adjunction is well-behaved only if $A^{**} = A$, so Abramsky and Coecke's categorical semantics for quantum protocols only works because finitary quantum mechanics is possible. In this section, we prove that finitary KvN is not possible, which suggests that a difference between the quantum and classical subcategories of **PhysOp** is their width: the quantum subcategory must, and the classical subcategory cannot be wide.

First, we define coarse-graining of observables as a possible route to finite-dimensional classical mechanics. From the functional composition principle (FUNC), we derive the operator relation principle (OPREL), and then we show that every mapping of coarse-grained observables to a finite-dimensional Hilbert space violates OPREL. Throughout this section we assume that we can define a flat probability density function on Ω: to assume otherwise is to assume finitary KvN is impossible, since otherwise we could define states on infinitely many disjoint regions of Ω. These would be pairwise orthogonal, and we would therefore need an infinite-dimensional Hilbert space. Hence, we need only consider those Ω for which $\mu(\Omega) < \infty$, where μ is the Lebesgue measure on Ω. We emphasise that this is not the same as the Lebesgue measure on the total phase space Γ.

Definition 5.4. Let $X : \Omega \to I \subseteq \mathbb{R}$ be an observable, let $I_1, I_2, \ldots, I_n$ be a partition of I, and for each I_k let $\Delta_k = \{\omega \in \Omega \mid X(\omega) \in I_k\}$. For any such partition, we call the observable

$$\overline{X}(\omega) = \begin{cases} 1 & \text{if } \omega \in \Delta_1 \\ & \vdots \\ n & \text{if } \omega \in \Delta_n \end{cases} \tag{5.20}$$

an *n-coarse-graining of* X. If I is a finite set, X is already (isomorphic to) a coarse-graining of itself. We write $\Pr\left(\overline{X} = k\right)$ for the probability of ω lying in Δ_k.

By analogy with finitary QM, we might expect that coarse-graining all the observables of $\mathfrak{S}$ lets us downgrade $\mathcal{H}$ to $\mathbb{C}^d$, for some finite d, without loss of information at the coarse-grained level. Below, we prove that this is impossible.

First, we should define "loss of information at the coarse-grained level". If all we know about $\mathfrak{S}$ is the probabilities $\Pr\left(\overline{X} = k\right)$ for some n-coarse-graining of an observable X, then we can represent our state of knowledge as an element of a Hilbert space $\mathbb{C}^d$, with $n \leq d$. We represent $\overline{X}$ as a self-adjoint operator $\hat{X}$ with eigensubspaces $\mathrm{eig}\,(x_k)$, state the Born rule: a density operator $\rho : \mathbb{C}^d \to \mathbb{C}^d$ corresponds to the probabilities

$$\Pr\left(\overline{X} = k\right) = \mathrm{Tr}\left(P_k^X \rho\right), \tag{5.21}$$

where P_k^X is a projection onto $\mathrm{eig}\,(x_k)$.

Now let $\overline{Y}$ be an m-coarse-graining of the observable Y (with corresponding phase space regions $\Delta'_1, \Delta'_2, \ldots, \Delta'_m$). Since our knowledge of $\mathfrak{S}$ only assigns probabilities to the Δ_k's as a whole, we know nothing of how the probabilities are distributed inside the Δ_k's, so we assume a flat distribution. The same applies to the Δ'_k's, so the probability $\Pr\left(\overline{Y} = j \,:\, \overline{X} = k\right)$ of $\overline{Y} = j$ conditional on $\overline{X} = k$ is proportional to the measure of $\Delta'_j \cap \Delta_k$. This gives us a coarse-grained classical version of the functional composition principle:

Lemma 5.5 (Functional composition (FUNC))**.** For all $\overline{X}$, $\overline{Y}$ as above:

$$\begin{aligned}\Pr\left(\overline{Y} = j\right) &= \sum_{k=1}^{n} \Pr\left(\overline{Y} = j \,:\, \overline{X} = k\right) \Pr\left(\overline{X} = k\right) \\ &= \sum_{k=1}^{n} \frac{\mu\left(\Delta'_j \cap \Delta_k\right)}{\mu\left(\Delta_k\right)} \Pr\left(\overline{X} = k\right),\end{aligned} \tag{5.22}$$

where μ is the Lebesgue measure on Ω.

FUNC guarantees that every coarse-graining of every observable provides us with epistemic probabilities for all the values of all the coarse-grainings of all other observables. If a downgrading of the Hilbert space from infinite to finite dimensions does not respect FUNC, then it loses information at the coarse-grained level.

If $\hat{Y}$ is the self-adjoint operator on $\mathbb{C}^d$ corresponding to $\overline{Y}$, and the density operator $\hat{x}_k$ is a trace-one eigenstate of $\overline{X}$ corresponding to the value k, then we have an operatorial version of FUNC:

Lemma 5.6 (Operator relation principle (OPREL))**.** For every k in the spectrum of $\overline{X}$, there exist trace-one eigenstates $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_m$ of $\hat{Y}$ corresponding to the eigenvalues $1, 2, \ldots, m$, such that each $\hat{x}_k$ can be written as:

$$\hat{x}_k = \sum_{j=1}^{m} p_j \hat{y}_j \tag{5.23}$$

with the real constants p_j defined as:

$$p_j = \frac{\mu\left(\Delta'_j \cap \Delta_k\right)}{\mu\left(\Delta_k\right)}. \tag{5.24}$$

If a map from the coarse-grained observables of $\mathfrak{S}$ to the self-adjoint operators on $\mathbb{C}^d$ violates OPREL, it also violates FUNC and therefore breaks the relations between the $\Pr\left(\overline{X} = k\right)$ and $\Pr\left(\overline{Y} = j\right)$, so knowledge of the probabilities of $\overline{X}$ no longer provides information about the probabilities of $\overline{Y}$. Such a map is incompatible with the laws of physics.

We can now prove the main theorem of this section:

Theorem 5.7. No map from the set of all n-coarse-grained observables of $\mathfrak{S}$ to the self-adjoint operators on any finite-dimensional Hilbert space respects OPREL.

Proof. Let $\overline{X}_1$ be an n-coarse-graining of the observable X, and let $\hat{X}_1 : \mathbb{C}^d \to \mathbb{C}^d$ be the corresponding self-adjoint operator for some $d \geq n$. Let our full knowledge of $\mathfrak{S}$ be given by the density matrix $\rho = p_0\hat{x}_0 + p_1\hat{x}_1$, where $p_0 + p_1 = 1$, and $\hat{x}_0$, $\hat{x}_1$ are eigenstates of $\hat{X}_1$ corresponding to $\omega \in \Delta_0$ and $\omega \in \Delta_1$.

Let $\overline{X}_2$ be another n-coarse-graining of X, with eigenstates $\hat{x}_2$, $\hat{x}_{0,1}$ corresponding to $\omega \in \Delta_2$ and $\omega \in \Delta_{0,1} = (\Delta_0 \cup \Delta_1) \setminus \Delta_2$, where $\Delta_2 \subset \Delta_1$ and $\mu\left(\Delta_2\right) < \mu\left(\Delta_1\right)$.

Since every eigenstate of $\overline{X}_2$ corresponding to outcome 2 is also an eigenstate of $\overline{X}_1$ with outcome 1, we have $\operatorname{eig}\left(x_2\right) \subseteq \operatorname{eig}\left(x_1\right)$ where $\operatorname{eig}\left(x_k\right)$ denotes the eigensubspace of $\hat{X}_k$ corresponding to $\omega \in \Delta_k$. And by definition of Δ_2, OPREL implies that there exist non-zero real constants $p'_{0,1}, p'_2$ such that for some $\hat{x}_1$:

$$\hat{x}_1 = p'_{0,1}\hat{x}_{0,1} + p'_2\hat{x}_2. \tag{5.25}$$

Hence $\text{eig}(x_1) \not\subseteq \text{eig}(x_2)$, so $\dim(\text{eig}(x_1)) > \dim(\text{eig}(x_2))$ where dim denotes the vector space dimension.

We can then choose n-coarse-grainings $\overline{X}_3, \overline{X}_4, \ldots, \overline{X}_{d+1}$ and regions $\Delta_2 \supset \Delta_3 \supset \Delta_4 \supset \ldots \supset \Delta_{d+1}$, and repeat this entire process, showing that $\dim(\text{eig}(x_1)) > \dim(\text{eig}(x_2)) > \dim(\text{eig}(x_3)) > \dim(\text{eig}(x_4)) > \ldots > \dim(\text{eig}(x_{d+1}))$, from which it follows that $\dim(\text{eig}(x_1)) > \dim(\mathbb{C}^d)$. Contradiction: there is no finite d such that the n-coarse-grainings of any observable X can be represented as self-adjoint operators on the Hilbert space $\mathbb{C}^d$ in a way that respects OPREL. □

The fact that the above theorem applies to maps whose domain is the set of *all* n-coarse-grained observables is not an oversight. Due to FUNC, every n-coarse-grained observable provides information about every other n-coarse-grained observable, so if we can send one n-coarse-grained observable to one self-adjoint operator, there is a canonical extension of that map to all n-coarse-grained observables. We also acknowledge that even considering only n-coarse-grained observables for some fixed n, or ignoring the algebra or vector space structure of coarse-grained observables, is already at odds with FUNC: a simpler proof might be possible if we include more of the structure of observables. However, a proof that a small set of observables is already too complex to be embedded in any finite-dimensional Hilbert space strikes us as more powerful than a proof that a larger algebra is unembeddable.

The above proof might suggest that we can choose uncountably many coarse grained observables (for example, for every real number v, we might choose as a 2-coarse-graining of the velocity of some particle the question "Is the velocity greater than v?") for which there exists an uncountable set of eigensubspaces $\{\text{eig}(x_r)\}_{r\in\mathbb{R}}$ such that $r < s$ implies $\dim(\text{eig}(x_r)) > \dim(\text{eig}(x_s))$. It turns out that no such choice of coarse-grainings exists.

This in turn suggests that by limiting our observables to only those which are coarse-grained, we can reduce our unseparable Hilbert spaces to spaces that *are* separable, which is sometimes useful. We explore that further in § 5.4; for now all we will do is prove the theorem.

Let R be an arbitrary subset of $\mathbb{R}$ and let $\left\{\overline{X}_r\right\}_{r\in R}$ be a set of coarse-grained observables. Let $\{\hat{x}_r\}_{r\in R}$ be eigenstates of the $\overline{X}_r$, and $\{\text{eig}\,(x_r)\}_{r\in R}$ eigensubspaces, corresponding to $\omega \in \Delta_r$. Again, we make sure to choose our coarse-grainings such that for all $s > t$: $\Delta_s \subset \Delta_t$ and $\mu\,(\Delta_s) < \mu\,(\Delta_t)$.

Theorem 5.8. If $r < s$ implies $\dim\left(\text{eig}\,(x_r)\right) > \dim\left(\text{eig}\,(x_s)\right)$ for all $r, s \in R$, then R is countable.

Proof. For all $r \in R$, we define:

$$\Delta_r^* = \{\omega \in \Delta_r \mid \forall s > r : \omega \notin \Delta_s\}\,. \tag{5.26}$$

Since all the Δ_r^* are pairwise disjoint subsets of Ω, we have:

$$\sum_{r\in R} \mu\,(\Delta_r^*) \leq \mu(\Omega), \tag{5.27}$$

and because $\mu(\Omega)$ is finite, this implies that only countably many Δ_r^* have non-zero measure. This does not contradict our hypothesis that $s > t$ implies $\mu\,(\Delta_s) < \mu\,(\Delta_t)$, because

$$\mu\,(\Delta_t) - \mu\,(\Delta_s) = \mu\left(\bigcup_{t<u<s} \Delta^* u\right). \tag{5.28}$$

The right hand side may still be greater than zero, even if for all $t < u < s$: $\mu\,(\Delta_u^*) = 0$, since measures need only be countably additive.

However, OPREL implies that there exist eigenstates $\hat{x}_r^*$ and $\hat{x}_*$, corresponding to $\omega \in \Delta_r^*$ and $\omega \in \Delta_r \setminus \Delta_r^*$, such that for every eigenstate $\hat{x}_r$

corresponding to $\omega \in \Delta_r$:

$$\begin{aligned}\hat{x}_r &= \frac{\mu\left(\Delta_r^* \cap \Delta_r\right)}{\mu(\Delta_r)}\hat{x}_r^* + \frac{\mu\left(\left(\Delta_r \setminus \Delta_r^*\right) \cap \Delta_r\right)}{\mu(\Delta_r)}\hat{x}_* \\ &= \frac{\mu\left(\Delta_r^*\right)}{\mu(\Delta_r)}\hat{x}_r^* + \frac{\mu\left(\Delta_r \setminus \Delta_r^*\right)}{\mu(\Delta_r)}\hat{x}_*,\end{aligned} \tag{5.29}$$

so if $\mu\left(\Delta_r^*\right) = 0$, there exists an eigenstate $\hat{x}_s = \hat{x}_*$ for some $s > r$ such that $\hat{x}_r = \hat{x}_s$. Since every eigenstate of $\overline{X}_s = s$ is also an eigenstate of $\overline{X}_r = r$, $\mu\left(\Delta_r^*\right) = 0$ implies that for some $s > r$: $\operatorname{eig}(x_r)$ is isomorphic to $\operatorname{eig}(x_s)$.

There can only be countably many r such that $\mu\left(\Delta_r^*\right) > 0$, so if we are to have $\dim\left(\operatorname{eig}(x_r)\right) > \dim\left(\operatorname{eig}(x_s)\right)$ for all $r < s$, then R must be countable. □

This theorem is useful, because it implies the following:

Corollary 5.9. There exist a separable complex Hilbert space $\mathcal{H}$ such that all the coarse-grained observables of any system $\mathfrak{S}$ can be represented as self-adjoint operators on $\mathcal{H}$ without violating FUNC. Furthermore, there exists a basis $\{|i_n\rangle\}_{n\in\mathbb{N}}$ such that every $|i_n\rangle$ is an eigenstate of every coarse-grained observable. We call this a *finegrained basis* for $\mathfrak{S}$.

5.3 Combined systems

As a prelude to correlated KvN-systems, we now examine combined systems in general. We begin by discussing categories of completely positive maps, and then show that the KvN-subcategory of **PhysOp** is such a category. It follows immediately that all of our pre-theoretical intuitions about **PhysOp** hold in the KvN-subcategory.

Definition 5.10 (Selinger [63])**.** A *dagger* on a category **C** is a contravariant endofunctor $(-)^\dagger : \mathbf{C} \to \mathbf{C}$ that is an identity on objects and an involution on morphisms. A *dagger category* is a category on which a dagger is defined.

A morphism f is a *dagger monomorphism* if $f^\dagger \circ f = \mathrm{id}$, and f is a *dagger epimorphism* if $f \circ f^\dagger = \mathrm{id}$. If a morphism is both dagger monic and dagger epic, it has a left inverse and a right inverse and therefore is iso; such a morphism is called a *dagger isomorphism* or a *unitary morphism*. If $f = f^\dagger$, then f is *self-adjoint*.

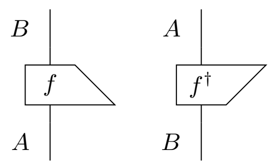
Example 5.11. Let $\mathbb{K}$ be a field on which a conjugation operation † and an absolute value $|\ |$ are defined such that for each $k \in \mathbb{K}$: $|k| \cdot |k| = k^\dagger \cdot k$. Examples include $\mathbb{C}$ with the complex conjugate and $\mathbb{R}$ with the identity. Then an identity-on-objects functor $\mathbf{Vect}_\mathbb{K} \to \mathbf{Vect}_\mathbb{K}$ that sends every linear map to its †-conjugate transpose (and as a special case: every scalar to its †-conjugate) is a dagger. This is a generalisation of the dagger compact closed categories we used in § 4.2.

Definition 5.12 (Selinger [63])**.** A *dagger symmetric monoidal category* is a symmetric monoidal category that is also a dagger category, such that for all morphisms $A \xrightarrow{f} B$, $C \xrightarrow{g} D$:

$$(f \Box g)^\dagger = f^\dagger \Box g^\dagger \tag{5.30}$$

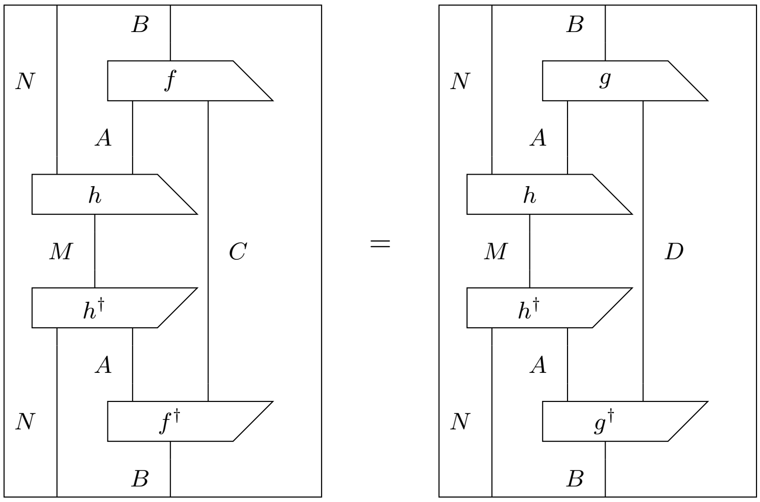
and for all objects A, B, C, the associator $\alpha_{A,B,C}$, the unitors λ_A, ρ_A, and the braiding $\gamma_{A,B}$ are unitary.

In dagger symmetric monoidal categories, we use asymmetric boxes to show the action of the dagger:



(5.31)

We can now define an equivalence relation on the morphisms of any dagger symmetric monoidal category [14]: $\left(A\Box C \xrightarrow{f} B\right) \simeq \left(A\Box D \xrightarrow{g} B\right)$ if and only if for all $M \xrightarrow{h} (N\Box A)$:



(5.32)

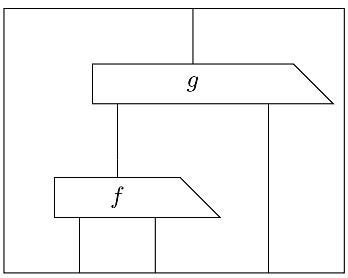
We write $[f]$ for the equivalence class containing f. For any dagger symmetric monoidal category $\mathbf{C}$, we can now construct a mixed-state category $\mathbf{Mix}(\mathbf{C})$:

Definition 5.13 (Coecke [14, def. 1.2])**.** For $\mathbf{C}$ a dagger symmetric monoidal category, $\mathbf{Mix}(\mathbf{C})$ is the category in which:

(i) the objects and identities are those of $\mathbf{C}$;

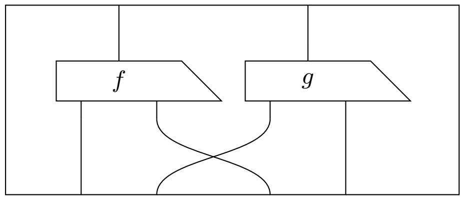
(ii) the morphisms are the equivalence classes $[f]$ of morphism of $\mathbf{C}$;

(iii) $[g] \circ [f]$ is the equivalence class that contains



(5.33)

(iv) $[f]\Box[g]$ is the equivalence class that contains



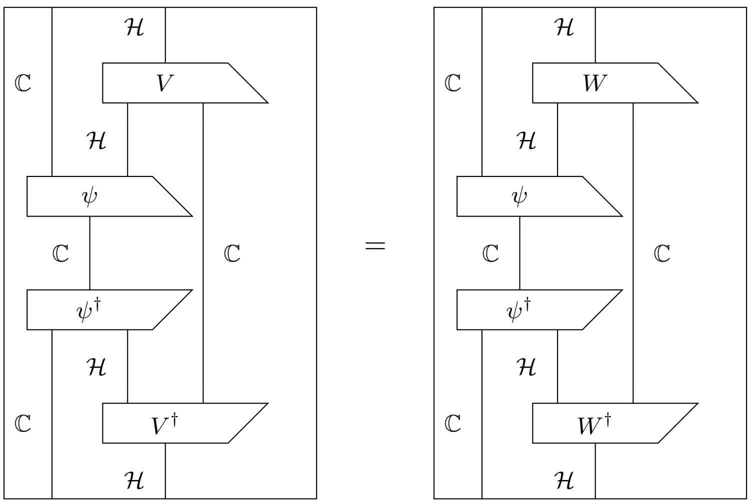
(5.34)

Any such category **Mix** (**C**) is symmetric monoidal, as is easily verified.

Now let $|\phi\rangle$ be a pure state. We recall that $|\phi\rangle$ is a map from $\mathbb{C}$ to $\mathcal{H}$, and that every $\langle\phi| = |\phi\rangle^\dagger$ is a map $\mathcal{H} \to \mathbb{C}$. We also recall that a time evolution operator U_t acts on a density operator ρ as $\rho_t = U_t \rho {U_t}^\dagger$. It follows, then, that two unitary maps $V, W : \mathcal{H} \to \mathcal{H}$ have experimentally indistinguishable actions on pure states if the effect of V on any $|\phi\rangle\langle\phi|$ is the same as that of W. That is, if for all $|\psi\rangle$:

$$V|\phi\rangle\langle\phi|V^\dagger = W|\phi\rangle\langle\phi|W^\dagger. \tag{5.35}$$

Diagrammatically, this becomes:



(5.36)

For the more general case, let $\rho = \sum_k p_k |\phi_k\rangle\langle\phi_k|$ with the $|\phi_k\rangle$ pure unit eigenstates of some operator, corresponding to distinct eigenvalues so that $\langle \phi_j | \phi_k \rangle = \delta^j_k$. Then:

$$\begin{aligned} \rho &= \sum_k p_k |\phi_k\rangle\langle\phi_k| \\ &= \sum_{j,k} \sqrt{p_j}\sqrt{p_k} |\phi_j\rangle\langle\phi_j|\phi_k\rangle\langle\phi_k|, \end{aligned} \tag{5.37}$$

so for every such ρ there exists a self-adjoint operator $\sqrt{\rho} = \sum_k \sqrt{p_k}|\phi_k\rangle\langle\phi_k|$ such that $\sqrt{\rho}\sqrt{\rho}^{\dagger} = \rho$. Again, the unitary maps V and W are indistinguishable if $V\sqrt{\rho}\sqrt{\rho}^{\dagger}V^{\dagger} = W\sqrt{\rho}\sqrt{\rho}^{\dagger}W^{\dagger}$ for all ρ.

Two mixed states ρ and ρ' are experimentally indistinguishable if they provide the same results under any measurement. Since the probability of any outcome x_m of an X-measurement when our system is in a pure state $|\phi\rangle$ is $\int_{\Delta_m} |\phi(\omega)|^2 \, \mathrm{d}\omega = \langle\phi|\chi_m|\phi\rangle$ (where $\Delta_m = \{\omega \in \Omega \mid X(\omega) = x_m\}$ and $\chi_m(\omega)$ equals one for $\omega \in \Delta_m$, zero otherwise), we have:

$$\begin{aligned} \Pr\left(X = x_m \,:\, \rho\right) &= \sum_k p_k \operatorname{Tr}\left(\chi_m|\phi_k\rangle\langle\phi_k|\right) \\ &= \sum_k p_k \langle\phi_k|\chi_m|\phi_k\rangle \\ &= \left(\sum_k p_k\langle\phi_k|\right)\chi_m\left(\sum_k |\phi_k\rangle\right), \end{aligned} \tag{5.38}$$

Now we note that $\chi_m = \chi_m^{\dagger} = \chi_m^{\dagger}\chi_m$ and that all the p_k are positive reals. It then follows that if for all χ_m:

$$\left(\sum_k \sqrt[4]{p_k}\langle\phi_k|\right)\chi_m\left(\sum_k |\phi_k\rangle\sqrt[4]{p_k}\right) = \left(\sum_k \sqrt[4]{p'_k}\langle\phi'_k|\right)\chi_m\left(\sum_k \sqrt[4]{p'_k}|\phi'_k\rangle\right), \tag{5.39}$$

then the pure state $\mathring{\rho} = \sum_k |\phi_k\rangle\sqrt[4]{p_k}$ is experimentally indistinguishable from the similarly defined $\mathring{\rho}'$. Furthermore, $\sqrt{\rho} = \mathring{\rho}^{\dagger}\mathring{\rho}$, so ρ and ρ' are experimentally indistinguishable. Because this holds regardless of our choice of operator that we place between $\mathring{\rho}$ and $\mathring{\rho}'$, or ρ and ρ', we also have $\mathring{\rho} \simeq \mathring{\rho}'$ and $\rho \simeq \rho'$.

From here it follows that any states, time-evolutions, or — by a similar proof — observables are experimentally indistinguishable if and only if they belong to the same equivalence class in $\mathbf{Mix}\,(\mathbf{Hilb}_{\mathbb{C}})$. Hence, up to experimental indistinguishability, the KvN-category is the subcategory of $\mathbf{Mix}\,(\mathbf{Hilb}_{\mathbb{C}})$ whose objects are all the infinite-dimensional complex Hilbert spaces, along with $\mathbb{C}$. Any pair of indistinguishable operators must have the same norm, and the hom-sets of $\mathbf{Mix}\,(\mathbf{Hilb}_{\mathbb{C}})$ inherit their completeness in the operator norm from $\mathbf{Hilb}_{\mathbb{C}}$, so the hom-sets of $\mathbf{Mix}\,(\mathbf{Hilb}_{\mathbb{C}})$ are Banach spaces. This, along with the dagger, is enough to make $\mathbf{Mix}\,(\mathbf{Hilb}_{\mathbb{C}})$ a C*-algebroid.

5.4 Correlated systems

We now have almost all the tools necessary to make plausible the following:

Conjecture 5.14. There is no categorical semantics of classical protocols.

We begin by discussing monoidal equivalence, and make more precise what it means for KvN-systems to be correlated. Then we discuss Heunen and Vicary's Ansatz for describing classical correlations, and show why it cannot work. At the end of this chapter, we are left with one dead end and zero leads towards a categorical semantics for classical protocols.

First, recall that every functor $F : \mathbf{C} \to \mathbf{D}$ respects morphism composition: $F(f \circ g) = F(f) \circ F(g)$. A monoidal functor $F : \mathfrak{C} \to \mathfrak{D}$ is one that, in addition, respects the tensor product: there exists a natural transformation $F(U)\square F(V) \xrightarrow{\bullet} F(U \otimes V)$, satisfying some coherence conditions. The following definition spells these out. If it is overly verbose, that is to make it unambiguously clear which of the two tensor products is being used where.

Definition 5.15 (Bénabou [7, def. 3])**.** Let $\mathfrak{C} = \langle \mathbf{C}, \otimes, i, \alpha, \lambda, \rho \rangle$ and $\mathfrak{D} = \langle \mathbf{D}, \Box, I, \mathrm{A}, \Lambda, \mathrm{P} \rangle$ be monoidal categories. Then a *monoidal functor* (originally: *morphisme de catégories avec multiplication*) $F_{\beta,\mu} : \mathfrak{C} \to \mathfrak{D}$ is an ordinary functor $F : \mathbf{C} \to \mathbf{D}$ along with a morphism $I \xrightarrow{\beta} F(i)$ and a natural transformation $\mu : F(-)\Box F(-) \overset{\bullet}{\to} F(- \otimes -)$, natural in both variables.

$F_{\beta,\mu}$ must respect the associativity (up to isomorphism) of the tensor products. That is, for all objects u, v, w of $\mathfrak{C}$, the *hexagon equation* must commute:

$$\begin{array}{ccc}
(F(u)\Box F(v))\Box F(w) & \xrightarrow{\mathrm{A}_{F(u),F(v),F(w)}} & F(u)\Box(F(v)\Box F(w)) \\
\downarrow{\scriptstyle \mu_{u,v}\Box\, \mathrm{id}_{F(w)}} & & \downarrow{\scriptstyle \mathrm{id}_{F(u)}\,\Box \mu_{F(v),F(w)}} \\
F(u \otimes v)\Box F(w) & & F(u)\Box F(v \otimes w) \\
\downarrow{\scriptstyle \mu_{u\otimes v,w}} & & \downarrow{\scriptstyle \mu_{u,v\otimes w}} \\
F((u \otimes v) \otimes w) & \xrightarrow[F(\alpha_{u,v,w})]{} & F(u \otimes (v \otimes w))
\end{array} \tag{5.40}$$

Note that because α and A are natural equivalences, the horizontal arrows can also point in the other direction. The above diagram tells us that $(F(u)\Box F(v))\Box F(w)$ and $F(u)\Box(F(v)\Box F(w))$ can both be mapped to $F((u \otimes v) \otimes w)$ and $F(u \otimes (v \otimes w))$: both inputs are isomorphic, as are both outputs, so where we place the parentheses does not matter.

$F_{\beta,\mu}$ must also respect the unitality (up to isomorphism) of the tensor products. That is, for any object u of $\mathfrak{C}$, the following must commute:

$$\begin{array}{ccc} I\Box F(u) & \xrightarrow{\beta\Box\,\mathrm{id}_{F(u)}} & F(i)\Box F(u) \\ {\scriptstyle \Lambda_{F(u)}}\downarrow & & \downarrow{\scriptstyle \mu_{i,u}} \\ F(u) & \xleftarrow[F(\lambda_u)]{} & F(i\otimes u) \end{array} \qquad \begin{array}{ccc} F(u)\Box I & \xrightarrow{\mathrm{id}_{F(u)}\Box\beta} & F(u)\Box F(i) \\ {\scriptstyle \mathrm{P}_{F(u)}}\downarrow & & \downarrow{\scriptstyle \mu_{u,i}} \\ F(u) & \xleftarrow[F(\rho_u)]{} & F(u\otimes i) \end{array} \tag{5.41}$$

In both rectangles, the bottom horizontal and left vertical arrows may point in any direction. The diagrams tell us that while a monoidal functor need not map i to I, the object $F(i)$ of $\mathfrak{D}$ should still act like a unit when tensored with the F-image of any object of $\mathfrak{C}$.

Example 5.16. Let $\mathbf{P}$ and $\mathbf{Q}$ be toset categories ordered by $\leq$. The objects of $\mathbf{Q}$ are all the natural numbers, including 0, and the objects of $\mathbf{P}$ can be any non-empty subset of the strictly positive natural numbers, with smallest element p. We can turn both into the monoidal categories $\mathfrak{P}$ and $\mathfrak{Q}$ by choosing the tensor product $m\Box n = \max(m,n)$. Then the unit of $\mathfrak{P}$ is p, and the unit of $\mathfrak{Q}$ is 0.

Now let G be the inclusion functor $\mathbf{P} \hookrightarrow \mathbf{Q}$. To form the monoidal functor $G_{\beta,\mu} : \mathfrak{P} \hookrightarrow \mathfrak{Q}$, let β be the unique morphism $0 \to p$ and let $\mu_{m,n}$ be the morphism $\mathrm{id}_{\max(m,n)}$ in $\mathfrak{Q}$, for all objects m, n of $\mathfrak{P}$.

The hexagon equation holds trivially, because G is an identity on objects and $\mathfrak{P}$ and $\mathfrak{Q}$ both have the same tensor product. It is easily verified that the two diagrams in eq. (5.41) commute for $\mathfrak{P}$ and $\mathfrak{Q}$.

Note that while $G(p)$ is not the unit of $\mathfrak{Q}$, for any object n of $\mathfrak{P}$ we have $\max(G(p), G(n)) = G(n) = \max(G(n), G(p))$, so $G(p)$ behaves like a unit whenever it is tensored with the G-image of an object of $\mathfrak{P}$.

Example 5.17. Recall the category $\mathbf{FdMat}_{\mathbb{K}}$ of example 2.27. It becomes a monoidal category if we take integer multiplication as the tensor product on objects, and the Kronecker product as the tensor product on morphisms.

The equivalence functor $H : \mathbf{FdMat}_{\mathbb{K}} \to \mathbf{FdVect}_{\mathbb{K}}$ becomes the monoidal equivalence $H_{\beta,\mu}$ if we let $\beta = \mathrm{id}_{\mathbb{K}}$ and $\mu_{m,n} : \mathbb{K}^m \otimes \mathbb{K}^n \cong \mathbb{K}^{mn}$. We can form a monoidal equivalence $H'_{\beta',\mu'} : \mathbf{FdVect}_{\mathbb{K}} \to \mathbf{FdMat}_{\mathbb{K}}$ as follows:

$$H'(U) = \dim V \tag{5.42}$$

$$H'(f) = f \tag{5.43}$$

$$\beta' = \mathrm{id}_1 \tag{5.44}$$

$$\mu_{U,V} = \mathrm{id}_{H'(V)\cdot H'(W)} \tag{5.45}$$

for any objects V, W and morphism f of $\mathbf{FdVect}_{\mathbb{K}}$, where dim denotes the vector space dimension.

Note that, even though $H'_{\beta',\mu'}$ replaces vector spaces with natural numbers, nothing is lost: all the vectors, scalars, and linear maps of $\mathbf{FdVect}_{\mathbb{K}}$ still exist in $H'_{\beta',\mu'}(\mathbf{FdVect}_{\mathbb{K}})$. This confirms our dogma from § 2.1, that morphisms are more important than objects.

On to correlated KvN-systems. For all practical purposes, our knowledge of any real-world classical system will be limited to a finite set of values of coarse-grained observables, say $\{\overline{Y}_k = y_k\}_{k\in K}$ for some index set K, where all the $\overline{Y}_k$ are coarse-grained observables. Note that the $\overline{Y}_k$ need not all be coarse-grainings of the same observable: they might be coarse-grainings of the positions of various particles, for example. If every $\overline{Y}_k$ is an n_k-coarse-graining of some observable, then there exists some observable Z which has an n-coarse-graining $\overline{Z}$ (with $n \leq \prod_{k\in K} n_k$) whose eigenstates correspond to all the possible sets $\{\overline{Y}_k = y_k\}_{k\in K}$ for some fixed K and $\{\overline{Y}_k\}_{k\in K}$. For all practical purposes, we can therefore define correlations of KvN-systems as follows:

Definition 5.18. Systems $\mathfrak{S}$ and $\widetilde{\mathfrak{S}}$ are *correlated* if there exist an m-coarse-grained observable $\overline{W}$ of $\mathfrak{S}$ and an n-coarse-grained observable $\overline{U}$ of $\widetilde{\mathfrak{S}}$ (we call these the *correlating observables*), along with real constants $p_k^j \in [0,1]$ for all $1 \leq j \leq m$, $1 \leq k \leq n$, such that

$$\Pr\left(\overline{U} = k \,:\, \overline{W} = j\right) = p_k^j, \tag{5.46}$$

and for all j:

$$\sum_{k=1}^{n} p_k^j = 1. \tag{5.47}$$

Also, there must exist i, j, k for which $p_k^i \neq p_k^j$.

The first part of the definition tells us that every outcome of a $\overline{W}$-measurement defines a probability distribution over $\overline{U}$, and the second tells us that there must exist at least one pair of $\overline{W}$-outcomes which induce different distributions over $\overline{U}$: otherwise there would be no correlation.

Any correlation between KvN-systems defines a stochastic matrix with entries p_k^j, and therefore a morphism in $\mathbf{FdMat}_{\mathbb{C}}$. Due to the monoidal equivalences between $\mathbf{Mix}\left(\mathbf{FdHilb}_{\mathbb{C}}\right)$ and $\mathbf{FdMat}_{\mathbb{C}}$ on the one hand [14, thm. 2.3, 35, example 7.29], and between $\mathbf{FdMat}_{\mathbb{C}}$ and $\mathbf{FdHilb}_{\mathbb{C}}$ on the other, to every diagram that commutes in any one of these categories there corresponds a commuting diagram of the same shape in each of the others.[1] Since $\mathbf{FdHilb}_{\mathbb{C}}$ has compact closed structure, $\mathbf{FdMat}_{\mathbb{C}}$ and $\mathbf{Mix}\left(\mathbf{FdMat}_{\mathbb{C}}\right)$ must have it too. Heunen and Vicary see a possible path to a categorical semantics for classical protocols here [35, pp. 241–242].

Theorem 5.7 prohibit this entire line of reasoning. As we have shown, no element of a finite-dimensional Hilbert space can fully characterise a KvN-

[1] This hand-wavy statement can be made rigorous, but we will not do that: the intuitive picture is clear enough in this case.

state, so any protocol formulated in terms of correlated KvN-systems using finite-dimensional stochastic matrices will hold only up measurement of the correlating observables. More formally, theorem 5.8 and corollary 5.9 imply that any KvN-state of a system $\mathfrak{S}$ can be characterised up to indistinguishability under coarse-grained measurement by an equivalence class $[\rho]$ of density matrices $\rho = \sum_{n\in\mathbb{N}} p_n |i_n\rangle\langle i_n|$, where $\{|i_n\rangle\}_{n\in\mathbb{N}}$ is a finegrained basis for $\mathfrak{S}$. Given any finite set $Y = \{\overline{Y}_k = y_k\}_{k\in K}$ of measurement outcomes, there are infinitely many equivalence classes that are compatible with Y and can be experimentally distinguished by observables outside Y. This stands in stark contrasts with quantum protocols. If we teleport the state $|\psi\rangle \in \mathbb{C}^2$ of qubit 1 to qubit 3, then no measurement operator on $\mathbb{C}^2$ can distinguish the pre-teleportation state of qubit 1 from the post-teleportation state of qubit 3.

In this chapter we have shown that the KvN-subcategory **KvNOp** of **PhysOp** is the C*-algebroid whose objects are the all the infinite-dimensional complex Hilbert spaces, along with the Hilbert space $\mathbb{C}$. For any objects A, B of **KvNOp**, the hom-set $\mathbf{KvNOp}_1(A, B)$ consists of all the $\simeq$-equivalence classes of morphisms in $\mathbf{Hilb}_{\mathbb{C}1}(A, B)$. Since all the objects of **KvNOp** are also objects of $\mathbf{Hilb}_{\mathbb{C}}$ and all the morphisms are linear, **KvNOp** is a subcategory of $\mathbf{Hilb}_{\mathbb{C}}$. Quantum mechanics can be done in terms of pure states as well as in terms of density matrices, in finite-dimensional as well as infinite-dimensional Hilbert spaces, so **QuantOp** can be identified with the entire category $\mathbf{Hilb}_{\mathbb{C}}$.

We had to treat all KvN-states as mixed because of our choice to define the classical observables as all and only all functions $\Omega \to \mathbb{R}$. Had we used the more conventional $\hat{\lambda}$'s instead, then we could have used pure states instead. This would have made our description of KvN simpler, but we would have to accept that the value of a classical observable may depend not only on the coordinates and momenta of our system, but also on our state of knowledge. That contradicts our postulate that all classical observables have values independent of our state of knowledge. We would also have found a less sharp quantum-classical divide: instead of a split along finite-infinite-dimensional and pure-mixed-state lines, we would only have had finite-infinite.

Our formulations of FUNC and OPREL, and our proofs of theorems 5.7 and 5.8 do not depend on whether or not there are $\hat{\lambda}$'s, so no formulation of KvN should allow for finite-dimensional Hilbert spaces. Since all categorical descriptions of correlated systems — and therefore of classical or quantum protocols — depend on compact closed structure, we conjecture, with high confidence, that there is no categorical semantics for classical protocols.

Chapter 6

Conclusion

Over the past few decades, quantum reaxiomatisations have undergone an operational turn. As quantum technology advanced and physics research shifted from describing quantum systems in aggregate to manipulating individual systems, the strange properties of the quantum world have changed from liabilities to resources, and foundations research has moved away from the Birkhoff-Von Neumann style of reaxiomatising towards more information-theoretic and computational approaches. Consequently, the barrier between quantum philosophy and quantum engineering is breaking down. We might draw an analogy to the study of (non-quantum) logic and computation: a good deal of work that a hundred years ago would be labelled as mathematics or the theory of electrical engineering is now being done in philosophy departments.

With the publication of their categorical semantics, Abramsky and Coecke placed themselves on top of this barrier, ready to move in either direction, into engineering or into philosophy. Unfortunately, categorical quantum mechanics has mainly fallen to the engineering side and, however permeable the barrier may be, has mostly stayed there. In this Thesis, we have examined Cat-QM from a philosopher's point of view, and tried to turn it into a good — or perhaps even: better — reaxiomatisation.

Categorical quantum mechanics takes for granted that finite-dimensional quantum mechanics takes place in, or that the mathematical structure of the theory is that of, a dagger compact closed category with biproducts: the empirical validity of the protocols it describes is thought to validate this assumption. But the empirical validity of protocols that cannot be understood without significant physical and mathematical baggage is hardly the kind of principle that should underlie a reaxiomatisation. If Abramsky and Coecke are as serious about reaxiomatising quantum mechanics — or at least its finitary fragment — as they claim to be [1, p. 2], then they need a set of physical principles, and they need to be able to account for differences between quantum and classical protocols.

In the Mathematical Preliminaries to this Thesis, we have explained how compact closed structure arises from symmetric monoidal structure combined, in a compatible way, with an adjunction (theorem 3.45), and in § 4.1 we deduced from eight reasonable assumptions (OpTh-I to OpTh-VIII) that every operational physical theory, when represented as a category, must be symmetric monoidal. In § 4.2 we used Abramsky and Coecke's categorical quantum mechanics to show that by adding three assumptions — that our category is $\mathbf{FdHilb}_{\mathbb{C}}$, that adjunctions encode correlations, and that biproducts encode indeterministic branching — we end up with finitary quantum mechanics in a dagger compact closed category with biproducts. A satisfying reaxiomatisation should account for these three additional assumptions in terms of physical principles, and identify which of them are specific to quantum mechanics.

In chapter 5 we developed categorical classical mechanics along these same lines. We showed that classical mechanics can be done in Hilbert spaces, but only if they are infinite-dimensional (theorem 5.7). This follows from the assumption that observables in classical mechanics are all and only all the real-valued functions of phase space, and that the state of a classical system is a probability distribution over phase space. This contrasts with quantum mechanics, where observables may depend on the state of the system without reference to an underlying phase space, as was the case in example 1.15. It follows then, that there can be no compact closed structure in classical mechanics; nor does the biproduct arise in that context.

We now have a series of implications:

$$\textsf{physical theory} \Rightarrow \textsf{closed symmetric cat.} \tag{6.1}$$

$$\textsf{physical theory} \Rightarrow \textsf{Hilbert spaces} \tag{6.2}$$

$$\textsf{dagger compact closed cat. with biprod.} \Rightarrow \textsf{quantum mechanics} \tag{6.3}$$

$$\textsf{observables } X : \Omega \to \mathbb{R} \Rightarrow \textsf{no compact closed cat.} \tag{6.4}$$

$$\textsf{observables } X : \Omega \to \mathbb{R} \Rightarrow \textsf{no biprod.} \tag{6.5}$$

$$\textsf{observables } X : \Omega \to \mathbb{R} \Rightarrow \textsf{classical methanics} \tag{6.6}$$

from which we can draw one definitive and one tentative conclusion. First, we conclude that, contrary to oft-held wisdom (c.f. [28]), Hilbert spaces and tensor product structure are necessary for, but not characteristic of, quantum theory. Second, we conjecture that due to the absence of compact closed structure in categorical classical mechanics, it is not possible to formulate a categorical semantics for classical protocols (conjecture 5.14). We also note that implications 6.1 and 6.4 to 6.6 are all built on solid physical or metaphysical principles, while 6.2 and 6.3 are in need of further justification.

Further research should try to justify from first principles the use of Hilbert spaces in physics: not just for quantum, but also in classical mechanics. A good starting point might be Heunen and Vicary's recent axiomatisation of the category of Hilbert spaces. In addition to dagger symmetric monoidal structure, their axioms are [34]:

(i) the monoidal unit is a simple monoidal separator: there are exactly two equivalence classes of monomorphisms into $\mathbb{K}$, where $\left(A \xrightarrow{f} \mathbb{K}\right) \simeq \left(B \xrightarrow{g} \mathbb{K}\right)$ iff there exists an isomorphism $A \xrightarrow{i} B$ such that $g \circ i = f$. Also, for all $\mathbb{K} \xrightarrow{h} A$, $\mathbb{K} \xrightarrow{i} B$, any morphisms $f, g : A \otimes B \to C$ are equal iff $f \circ (h \otimes i) = g \circ (h \otimes i)$.

(ii) there is a zero object, and there are finite biproducts; every biproduct injection is a dagger monomorphism.

(iii) for any pair of morphisms $f, g : A \to B$, there is a dagger equaliser: that is, a dagger monomorphism $E \xrightarrow{e} A$ such that for any morphism $X \xrightarrow{m} A$: if $f \circ m = g \circ m$, then m factors through e.

(iv) every dagger monomorphism is the kernel[1] of some morphism.

(v) the wide subcategory whose morphisms are all and only all the dagger monomorphisms has directed colimits. To explain this axiom goes beyond the scope of this Thesis.

The second part of the first axiom could be phrased in terms of state evolutions being indistinguishable if they act the same way on any combined system, but this raises the question why such a requirement must be formulated for combined systems in the first place. Heunen has conjectured that the second axiom might relate to measurement or superselection [33], which

[1]The kernel of a morphism f is the equaliser of f and the zero morphism. In $\mathbf{Vect}_{\mathbb{K}}$ and all its subcategories, this categorical definition is equivalent to the usual definition from linear algebra.

matches the use of biproducts in quantum mechanics. However, biproducts play no meaningful role in categorical classical mechanics, even though every operator superselects in that theory. The dagger also remains to be justified. Heunen has conjectured that it relates to the conservation of information [33], and its role in the formulation of time-reversed quantum mechanics does seem to support this.

Finally, we should note that nothing we have written (beyond the introduction, that is) fully determines quantum mechanics. Nor does our categorical description of the KvN category fully determine Koopman-Von Neumann mechanics. We have provided necessary conditions to characterise quantum and classical mechanics, but have no yet been able to formulate sufficient principles.

Notation

Index of notation and selected terminology. Numbers refer to the pages where the entries are defined.

Categories

Symbols and notation

Terminology

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