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On Stochastic Control Theory for Dynamic Carbon Emission Reduction

Master's Thesis

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Abstract

Addressing the reduction of Greenhouse gas emissions in the atmosphere has become crucial, given the significant implications of climate change and global warming. An effective approach to achieve emission reduction is through the implementation of a cap-and-trade system. This system involves a regulator setting a cap on the total emissions allowed and allocating allowances at a predetermined point in time to the participants. The European Union Emissions Trading System (EU ETS), introduced in 2005, serves as an example. However, the EU ETS has shown limitations in effectively compensating for economic shocks, which is necessary to let the system work accurately and to achieve the desired reduction level.

In this thesis, we investigate a novel dynamic policy of allocating allowances, aiming to provide better compensation for economic shocks. The policy is derived through a Stackelberg game, wherein a regulator, the leader, allocates allowances to set of N firms, while minimising a perceived social cost. In response to the regulator, the firms minimise their corresponding costs from abatement and trading. An important challenge is how to realistically model the cumulative Business-As-Usual (BAU) emissions of the firms, allowing for analytically tractable solutions of the policy. Two models are considered for the modelling of the BAU emissions: a Brownian motion, of which the correlation structure is generalised in this thesis, and a Geometric Brownian motion, representing the main contribution of this thesis. The SDEs of these emissions will be controlled by the abatement effort. Within both frameworks, the optimal dynamic allocations are determined through stochastic control theory and variational calculus. The proposed models are both investigated theoretically and numerically.

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Contents

Introduction

In this thesis, we will discuss a dynamic version of a cap-and-trade system, with the aim of exploring and deriving an optimal dynamic allocation. Before delving into this model, we will first discuss what a cap-and-trade system is and why it is useful, supported by its application within the European Union. Afterwards, we will discuss the most important concepts of this thesis and how it is built up.

Nowadays, the impact of climate change on the world is significant, emphasising the urgent need to slow down the global warming. This can be achieved by reducing the Greenhouse gas (GHG) emissions in the atmosphere. These emissions, which stem from human activity, are one of the main contributors of the ongoing global warming phenomenon. Although the GHG emissions consist of multiple gasses, we will predominantly focus on reducing the CO_2 level in the atmosphere, as it is a major contributor to the GHG emissions [LGX19]. Numerous approaches exist for reducing the GHG emission level in the atmosphere. Our primary focus will be on the implementation of a cap-and-trade system, sometimes called an Emission Trading System (ETS).

In essence, a cap-and-trade system involves a regulator, often a government, setting a limit on the total emissions allowed in a predefined system, such as a country or a set of firms. This limit is referred to as the emissions cap. Participants within the system receive a certain number of allowances at a predetermined time, in line with the emission cap. These allowances serve as permits to cover their emissions over a predefined period. If the number of emissions of a participant is higher than the number of allowances at the end of the period, the participant pays a penalty. During the time period, the allowances may be traded on the market of permits, where the market price arises from the market itself. This way, participants that need more allowances can buy them, and others may sell their surplus. The reduction of emissions takes place where it is the cheapest. By lowering the emissions cap step by step, the total emissions in the system can be reduced [GK09].

In the European Union, a cap-and-trade system has been in use since 2005, known as the European Union Emissions Trading System (EU ETS). It covers over 40 percent of the GHG emissions in the EU and is one of the earliest, and largest, emissions trading systems of the world [Wu11, pg.95-100]. The EU ETS has undergone improvements over time and consists of several phases [DT14].

In the second phase, from 2008 to 2012, there was a surplus of allowances in the system [Lai+14]. A possible explanation is the financial crisis the world was in. That is, the demand of emission allowances was reduced, not because participants had invested in cleaner or more effective technologies, but because overall productions were lower. This resulted in a lower market price of permits, with a weaker incentive to lower the emissions as a result. The system seemed unable to respond to changes in the economic circumstances [BVW16]. To solve this problem, the EU has designed the Market Stability Reserve (MSR) mechanism, which has been in operation since 2019. The approach of the MSR is two-sided: when there are too many allowances available, that is, when the number of allowances on the market exceeds a certain threshold, a percentage of allowances is withdrawn from the market. If the number is below another threshold, more allowances are brought onto the market via auctions. This way, the EU ETS is more resistant to economic shocks and keeps the incentive to reduce the emissions as high as possible for the participants [BM16].

Several conclusions can be drawn from this example. Firstly, it highlights that responsive mechanisms to economic shocks are necessary for the accurate functioning of a cap-and-trade system. We have seen that these can be incorporated over time. Additionally, it shows that the allocation of allowances should be modelled as a dynamic process, instead of a static process.

While these systems work well, it is always beneficial to explore new, potentially better, models that can be incorporated along the way. This is where mathematics comes into play, as we would like to predict the impact of such regulations. To achieve this, the system needs to be represented in such a way that it is a mathematically solvable and fairly realistic. This leads us to the main motivation of this thesis: to explore an alternative, potentially more effective way for modelling a cap-and-trade system and addressing economic shocks in the system. This objective is based on an article by René Aïd and Sara Biagini, titled "Optimal dynamic regulation of carbon emissions market" [AB23]. In this article, the goal of the regulator is to lower the emissions by a $100(1-\rho)$ percentage, where $0 < \rho < 1$, in a closed system of N firms. This is achieved by dynamically allocating allowances to a set of N firms. The setting is dynamic in the sense that the allowances are stochastic processes, not deterministic, and allocation is done in continuous time. In response to the regulator, each firm will minimise its costs from emission reduction and trading in the market of permits. The central notion is that in this dynamic setting, there is potential for more effective compensation for economic shocks, which appears to be useful as seen in the example of the MSR.

Several mathematical concepts will be used. As stated, firms would like to minimise their costs from abatement and trading, which will be stochastic quantity. When we deal with a potentially stochastic objective function that needs to be minimised, stochastic control theory comes into play. An extensive overview of this theory is given in the book of Pham [Pha09]. Additionally, the Stackelberg game may be recognised, with a leader, the regulator, and N followers, in this case the set of firms. The equilibrium of the Stackelberg game will be mathematically treated by stochastic control theory and variational calculus, the latter can for example be found in [ET99]. This is what we will focus on, not on the game theoretic side of the Stackelberg game. For the reader interested in the game theory, we refer to [AG99].

The optimal solutions obtained in [AB23] are mostly analytically tractable, which is beneficial. As a consequence, we need to make fairly strong assumptions to achieve this, as there is a trade-off between realistic assumptions and analytically tractable solutions. An important example is the modelling of Business-As-Usual (BAU) emissions, representing emissions when no intervention on pollution occurs. These emissions are inherently random, as they are dependent on the economic shocks. It is crucial to model these realistically, but in a way that we still can achieve analytical solutions. We choose two different approaches for this; modelling by a Brownian motion, which is done in the paper by Aïd and Biagini, and modelling by a Geometric Brownian motion, which is the main contribution of this thesis. In the Brownian framework, we need to make assumptions on the drift and volatility, to make sure the process is non-negative. In the case of the Geometric Brownian motion, this is not necessary anymore, as the process is non-negative by definition. While the structure of the proofs is based on the Brownian motion scenario, the content is entirely new. Modelling emissions by a Geometric Brownian motion can also be found in the literature, for example in [CT12]. Under both frameworks, the optimal dynamic allocation will be solved.

The aim of this thesis is to investigate whether certain assumptions can be relaxed while still obtaining analytical solutions. To achieve this goal, a thorough understanding of the framework and mathematical concepts presented in [AB23] is necessary. Consequently, the first part of this thesis consists of a detailed understanding of the cited paper by Aïd and Biagini, to facilitate the extension in the second part. Building upon the foundational understanding of the main paper, the subsequent part delves into a discussion of the key background elements essential for our study. Note that this overview is far from complete.

We will start with models constructed to reduce emissions. In 1972, a model for an efficient pollution program was constructed by Montgomery [Mon72]. Two years later, Weitzman proposed a partially stochastic and dynamic model for regulating an economic variable to maximise global benefits [Wei74]. This setting is quite general, but is well applicable to the carbon emission market. Although published years ago, they have had a significant influence.

Since the publication of these two seminal papers, extensive research has been done on the modelling of an ETS using stochastic control theory. In a discrete time setting, in the paper of Carmona et al. [CFH09], the first phase of the EU ETS is modelled. Additionally, Seifert et al. [SUW08] consider a model for the EU ETS in continuous time, where the emissions rates are modelled by an SDE involving a Brownian motion. Moreover, Rosemann's PhD thesis in 2023, provides extensive research of the different ways for modelling the emission rates in a continuous time and dynamic setting, extending the framework introduced in the previous article [Ros23]. Lastly, a version of an ETS in continuous time is modelled by Kollenberg and Tascini [KT16], with a focus on accurately modelling the MSR . The bank account framework introduced in their paper will also be adopted in this thesis, facilitating the utilization of allowances at a later time than those initially allocated by the regulator.

This thesis is structured as follows. The first chapter provides a detailed explanation of the setup and stochastic model. For instance, the bank account of allowances is introduced, addressing both the Brownian motion and the Geometric Brownian motion scenarios in the modelling of the BAU emissions. This chapter draws inspiration from [AB23], closely following the proofs. However, to present the complete mathematical story, we have added many details. For example, the space of admissible controls is not covered in [AB23], but is extensively discussed in this thesis. Additionally, the correlation structure of the economic shocks is generalised to allow for direct correlation between all firms. This way, we work with the most general correlation structure possible such that the correlation proposed by Aïd and Biagini is a specific example. Furthermore, an initial value E_0^i is added to the BAU emissions modelled by the Brownian motion, to be able to compare the outcome in the Brownian framework with results of the Geometric Brownian motion in a better way. The models and assumptions introduced in this chapter are necessary to be able to address the optimisation problems in the chapters that follow.

In Chapter 2, we solve the optimal dynamic allocation for the regulator within a Brownian framework using three steps evolving from the Stackelberg game. Multiple insights are required to precisely identify the solutions. Once again, we closely follow the paper [AB23] and reproduce the results introduced there under the general correlation structure. The proofs are carefully modified to fit into our mathematical model. For example, we have added background and proofs on variational calculus to find a sufficient minimiser in the space of admissible controls. This addition can be found in the first nine pages of this chapter. Additionally, the proof of Theorem 2.17 is more precise.

In Chapter 3, we solve the optimal dynamic allocation for the case where the BAU emissions are modelled by a Geometric Brownian motion, which is the main novelty of this thesis. Once more, the steps of the Stackelberg equilibrium are visible, as well as the proofs of the previous chapter. Nevertheless, there are differences between the two chapters, since the bank account of allowances will be different.

In the final chapter, Chapter 4, we compare the optimal dynamic policy and existing policies, such as those from the EU ETS, based on [AB23]. Additionally, we perform numerical comparisons between scenarios involving Brownian motion and Geometric Brownian motion, followed by their interpretations, which represent a novelty as well.

Furthermore, this thesis includes two appendices. In the first appendix, we provide a detailed mathematical background focusing mainly on stochastic calculus and variational calculus. When a mathematical concept is not familiar or clear, the reader is advised to consult this appendix. In some cases, we will explicitly refer to it. Additionally, some proofs that are omitted in the main body of this thesis can be found here. Lastly, an appendix on the optimal allocation in a market with frictions can be found, under the Brownian framework.

In summary, our own contributions consist, first of all, of defining the mathematical framework precisely, and working out the mathematical proofs completely. In terms of content, we have generalised the correlation structure and in the Brownian framework we have added an initial value E_0^i to facilitate comparison to the Geometric Brownian motion setting. Moreover, the situation where the BAU emissions are modelled by a Geometric Brownian motion is introduced, the key contribution of this thesis. A numerical analysis is also performed in this setting. Lastly, the optimal allocation in a market with frictions is elaborated in detail, which, for better flow, is included in the appendix. Overall, this thesis offers a comprehensive overview of the mathematical approach to addressing optimal dynamic allocation.

1 | Model Assumptions

In this chapter, we will first provide a general overview of the subject of this thesis. Second, we will introduce the stochastic model that will be worked with, and relate it to the intuitive idea. Lastly, the stochastic control problems are defined.

Throughout this thesis, we will make use of several mathematical concepts, discussed in Appendix A.1. The first part consists of stochastic processes and stochastic calculus. The second part consists of basic functional analysis and variational calculus. The reader is encouraged to refer to the appendix if certain mathematical definitions or concepts are unfamiliar.

This chapter is based on Section 2 of [AB23], which is the inspiration for this thesis.

1.1 Intuitive idea

First, the intuitive idea will be presented. The overall goal of the regulator is to reduce the emissions by a $100(1-\rho)$ percentage, with $0 < \rho < 1$, in a closed system involving N firms. To achieve this reduction in emissions, we need two steps:

(i) First, the regulator dynamically allocates permits in line with the reduction, while minimising certain social costs. The entire strategy is announced at the beginning of the time period. The allowances that the firms receive, are stochastic processes that are adapted to a later defined filtration. This implies, that the distribution of the allowances is known at time $t = 0$, but the exact realisation is only given at the time the allowances are allocated.

(ii) Given this allocation of the regulator, the firms minimise their own costs from emission reduction and trading. The variables for the emission reduction and trading will also be adapted stochastic processes. The idea of this is that emission reduction and trading depend on unknown, stochastic quantities, and appear in a, to be defined, stochastic state space.

Based on these two steps, a Stackelberg game can be recognised. In this case, there is a leader, the regulator, who announces her policy first. Then, the followers, respond rationally by minimising their costs. To solve such a problem, we need the concept of backward induction:

(i) Given the stochastic process of the allocation, all firms will minimise their cost function from abatement and trading until a market equilibrium occurs.

(ii) Next, given that the firms react rationally, the regulator optimises the social costs function considering the restriction of the reduction with respect to all possible allocations.

For more information on backward induction and Stackelberg games in general, the reader is referred to [CM18]. With this introduction, we are prepared to provide a mathematical definition of this model, covering the steps involved in the Stackelberg game.

1.2 Stochastic model

In this section, the mathematical setup of this thesis will be made clear. We will work with one time period, so that $t \in [0,T]$, and a given probability space $(\Omega, \mathscr{F}, \mathbb{P})$. A filtration on this sigma algebra will be constructed below.

We will write a stochastic process $(X_t)_{t\in[0,T]}$ as (X_t) or X. When referring to the random variable in time $t \in [0,T]$, we will use the notation X_t. The same will hold for a filtration. Furthermore, throughout this thesis, the one-dimensional Lebesgue measure is denoted by λ^1. However, in many instances, whenever it is clear from the context, we will write $\mathrm{d}t$, when we mean $\mathrm{d}\lambda^1$. Here, we make use of the relationship between the Lebesgue integral and the Riemann integral. A variable for a specific firm i will be denoted by X^i. The mean over all N firms will be denoted by $\bar{X} = \frac{1}{N}\sum_{i=1}^N X^i$. The latter will also be done for deterministic functions.

Let $\tilde{B} = (\tilde{B}^0, \tilde{B}^1, \dots, \tilde{B}^N)$ be an $N+1$ dimensional Brownian motion with respect to the natural filtration generated by $\tilde{B}$, denoted by $(\mathscr{F}_t^{\tilde{B}})$. This implies that all individual processes are independent for a given time $t \in [0,T]$. Let $(\mathscr{F}_t)$ be the filtration generated by $\tilde{B}$, augmented with the null sets. That is, $(\mathscr{F}_t)$ is defined as

$$\mathscr{F}_t := \sigma\left(\mathscr{F}_t^{\tilde{B}} \cup \mathscr{N}\right),$$

for all $t \in [0,T]$, where $\mathscr{N}$ is the collection of $\mathbb{P}$-null sets, By [KS91, pg. 91] it holds that $\tilde{B}$ is still a Brownian motion with respect to $(\mathscr{F}_t)$, since adding null sets to the original filtration does not change the distributional properties of B.

Now, for all $t \in [0,T]$ and for all $i \in \{1,2,\dots,N\}$, define the stochastic process

$$W_t^i = \sum_{j=0}^N \kappa_{i,j} \tilde{B}_t^j, \tag{1.1}$$

where $\kappa_{i,j} \in \mathbb{R}$. In fact, the following two properties should hold

$$\sum_{j=0}^N \kappa_{i,j}^2 = 1, \quad -1 \leqslant \sum_{m=0}^N \kappa_{i,m}\kappa_{j,m} \leqslant 1, \tag{1.2}$$

for all possible firms i,j such that $i \neq j$. First, we will show that $W = \left(W^1, W^2, \dots, W^N\right)$ is still a Brownian motion with respect to the filtration $(\mathscr{F}_t)$. For this, we need assumptions (1.2).

Lemma 1.1. *It holds that* $W = (W^1, W^2, \dots, W^N)$ *defined in* (1.1), *under the assumptions of* (1.2), *is a correlated Brownian motion with respect to the filtration* $(\mathscr{F}_t)$. *The correlation for* $i \neq j$ *is given by*

$$Corr\left[W_t^i, W_t^j\right] = \sum_{m=0}^N \kappa_{i,m}\kappa_{j,m}.$$

Proof. First, we will prove that for every i, W^i itself is a Brownian motion. For this, we can use Lévy's theorem in one dimension [KS91, pg. 167]. First, we need that W^i is a martingale with respect to $(\mathscr{F}_t)$. This follows immediately from the fact that $\tilde{B}^j$, for all j, is a one-dimensional Brownian motion. Furthermore, we have $W_0^i = 0$ and continuous paths. Left to prove is that the quadratic variation $\langle W^i \rangle_t = t$ holds, for $t \in [0,T]$. Indeed, it follows by Corollary A.11, Proposition A.12 and assumptions (1.2) that

$$\left\langle W^i \right\rangle_t = \left\langle \sum_{j=0}^N \kappa_{i,j}\tilde{B}^j \right\rangle_t = \sum_{j=0}^N \kappa_{i,j}^2 \langle \tilde{B}^j \rangle_t + \sum_{i=0}^N \sum_{\substack{j=0 \\ j\neq i}}^N \left\langle \tilde{B}^i, \tilde{B}^j \right\rangle_t = \sum_{j=0}^N \kappa_{i,j}^2 t = t,$$

since all separate Brownian motions B^i are independent. By Lévy's theorem, we have for every $i \in \{1, 2, \ldots, n\}$ that W^i is a Brownian motion.

Left to show is that the specific correlation structure holds. Let $i \neq j$. Then we have

$$\begin{aligned}\operatorname{Cov}\left[W_t^i W_t^j\right] &= \mathbb{E}\left[W_t^i W_t^j\right] = \mathbb{E}\left[\left(\sum_{l=0}^{N} \kappa_{i,l}\tilde{B}_t^l\right)\left(\sum_{m=0}^{N} \kappa_{j,m}\tilde{B}_t^m\right)\right] = \sum_{l=0}^{N}\sum_{m=0}^{N} \kappa_{i,l}\kappa_{j,m}\mathbb{E}\left[\tilde{B}_t^l \tilde{B}_t^m\right] \\ &= \sum_{m=0}^{N} \kappa_{i,m}\kappa_{j,m}\mathbb{E}\left[(\tilde{B}_t^m)^2\right] = \sum_{m=0}^{N} \kappa_{i,m}\kappa_{j,m} t.\end{aligned}$$

Here, we used again the independence of the Brownian motions in the vector $\tilde{B}$. Indeed, we then get

$$\operatorname{Corr}\left[W_t^i W_t^j\right] = \frac{\operatorname{Cov}\left[W_t^i W_t^j\right]}{\sqrt{\operatorname{Var}\left[W_t^i\right]}\sqrt{\operatorname{Var}\left[W_t^j\right]}} = \frac{\sum_{m=0}^{N}\kappa_{i,m}\kappa_{j,m} t}{t} = \sum_{m=0}^{N} \kappa_{i,m}\kappa_{j,m}. \tag{1.3}$$

Since correlation factors are always between -1 and 1, we see that the assumptions of (1.2) are necessary. We can conclude that W is a vector of correlated Brownian motions where the correlation coefficient is given as above. □

The Brownian motion W can be interpreted as the random economic shocks in the system. Here, $\tilde{B}^0$ represents the common economic shock, that every firm experiences to some extent. Additionally, $\tilde{B}^i$ is the economic shock that, in principle, only firm i experiences. Due to the correlations between firms, this shock may also impact other firms. Note that this is the most general way to model an N-dimensional correlated Brownian motion based on an $N+1$-dimensional independent Brownian motion. This model extends the model used in Aïd and Biagini in the following way.

Remark 1.1. *In the article [AB23], the Brownian motion W is chosen such that*

$$W_t^i = \sqrt{1-\kappa_i^2}\,\tilde{B}_t^i + \kappa_i \tilde{B}_t^0. \tag{1.4}$$

This is a specific case of the framework presented above, with, for every firm $i > 0$,

$$\kappa_{i,0} = \kappa_i, \quad \kappa_{i,i} = \sqrt{1-\kappa_i^2}, \quad \kappa_{i,j} = 0 \text{ else}. \tag{1.5}$$

Then, the correlation between two firms i, j reduces to

$$Corr\left[W_t^i W_t^j\right] = \sum_{m=0}^{N} \kappa_{i,m}\kappa_{j,m} = \kappa_{i,0}\kappa_{j,0} = \kappa_i\kappa_j.$$

Here, every firm i experiences a common economic shock $\tilde{B}^0$, and a idiosyncratic economic shock $\tilde{B}^i$, which does not impact the other firms.The correlation of the economic shocks between different firms arises solely from the common economic shock $\tilde{B}^0$. This case is a specific example of the framework presented here.

Since we have the sufficient filtration and the corresponding correlated Brownian motion, we are able to define the relation for the BAU cumulative emissions for firm i. First, the following definition is needed.

Definition 1.1 (BAU case). *The Businesses-As-Usual (BAU) case, is the situation in which no intervention with respect to the emissions takes place. In this scenario, all firms may decide their pollution strategy themselves, without any restrictions. The corresponding emissions are called the BAU emissions.*

The emissions in the BAU case need to be modelled, as they depend on the economic shocks, and other random components. This can be done in several ways. In this thesis, two specific methods are worked out in detail. In the first case, modelling by an arithmetic Brownian motion is considered. The second case involves the modelling by a Geometric Brownian motion. In this chapter, both cases will be presented simultaneously. To be able to derive the outcome of the equilibrium the Geometric Brownian motion case, a thorough understanding of the Brownian motion case is crucial. This way, both approaches will be presented in this thesis. The Geometric Brownian motion extends the story of [AB23].

Definition 1.2 (BAU emissions I). *Let (E_t^i) the process of the cumulative emissions of firm i be given by*

$$\begin{aligned} dE_t^i &= \mu_i dt + \sigma_i dW_t^i, \text{ i.e,} \\ E_t^i &= E_0^i + \mu_i t + \sigma_i W_t^i, \end{aligned} \tag{1.6}$$

for $t \in [0,T]$, where $\mu_i, \sigma_i, E_0^i \in \mathbb{R}$ with $\mu_i, \sigma_i > 0, E_0^i \geqslant 0$.

The BAU emissions now indeed depend on the economic shock of the firms. If the firm experiences a positive economic shock, the emissions will increase, aligning with intuition. A possible disadvantage of the model above is that a Brownian motion can become negative, as it is normally distributed. Essentially, this implies that the cumulative emissions could become negative as well, which may be counter intuitive. A negative cumulative emission cannot be easily substantiated. This is particularly true if, in the BAU scenario, firms show no concern for the climate at all. A slight decrease in BAU emissions could be attributed to firms in the BAU case already caring about the climate and taking beneficial actions.

However, from now on, we will assume that the drift is of higher order than the volatility, as done in [AB23]. In this case, the cumulative emissions indeed are positive with probability close to 1. The (arithmetic) Brownian motion case is worked out as a reference for the other scenario.

Ideally, we would have positive BAU emissions without an assumption on the parameters. For this, we will consider a different, non-negative process, to model the BAU emissions. These are denoted by the process (G_t^i), for every firm i.

Definition 1.3 (BAU emissions II). *Let (G_t^i) the process of the cumulative BAU emissions of firm i be modelled by a Geometric Brownian motion [OG19, pg. 19], that is, for every $t \in [0,T]$,*

$$dG_t^i = \mu_i G_t^i dt + \sigma_i G_t^i dW_t^i, \quad G_0^i = E_0^i, \tag{1.7}$$

where $\mu_i, \sigma_i, E_0^i \in \mathbb{R}$ with $\mu_i, \sigma_i, E_0^i > 0$.

Note that in the definition above the initial value G_0^i is chosen to be the same as in the Brownian motion case. This process is non-negative.

Remark 1.2 (Units of variables). *All relevant variables and stochastic processes of this chapter have units. For example, the units of the BAU emissions E^i and G^i will be Gigatons of CO_2. More details on this aspect will be provided in the numerical section, Section 4.3, and in Appendix A.3. In the main part of this thesis, we will assume that the units of the variables correspond with the emissions being measured in Gigaton CO_2.*

The following proposition presents the connection between the strong solution to the GBM scenario and the SDE given in Equation (1.7). For this, the first and second moment of the GBM are used. This is worked out in detail in Proposition A.30 of the appendix.

Proposition 1.2. *Let*

$$G_t^i = E_0^i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right) t + \sigma_i W_t^i\right), \tag{1.8}$$

with $E_0^i > 0$. Then, the SDE of (1.7) *can be obtained from the analytical solution of* (1.8) *. Furthermore, this analytical solution is a strong solution of* (1.7)*.*

Proof. We start with the analytical solution. Let us define for all firms i the function $f^i(t,x)$: $[0,T]\times\mathbb{R}\to\mathbb{R}$ by

$$f^i(t,x) = \exp\left(\left(\mu_i - \frac{1}{2}\sigma^2\right)t + \sigma_i W_t^i\right).$$

This function is twice continuous differentiable and hence we can apply Itô's lemma on $f^i(t,W^i) = G_t^i$. This implies

$$\mathrm{d}G_t^i = \mathrm{d}f^i\left(t,W^i\right) = \left(\mu_i - \frac{1}{2}\sigma_i^2\right)G_t^i\mathrm{d}t + \sigma_i G_t^i\mathrm{d}W_t^i + \frac{1}{2}\sigma_i^2\mathrm{d}t = \mu_i G_t^i\mathrm{d}t + \sigma_i G_t^i\mathrm{d}W_t^i.$$

This is indeed exactly (1.7). Now let us take the SDE and the given probability space $(\Omega,\mathscr{F},\mathbb{P})$. Consider

$$G_t^i = E_0^i\exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i W_t^i\right).$$

In the next part, we are proving that this is a strong solution of the SDE. Since the Brownian motion (W_t^i) is adapted to the filtration $(\mathscr{F}_t)$ and has continuous sample paths, it holds that the process (E_t^i) is adapted and has continuous paths as well. By the above, we know that the SDE can be constructed by Itô's lemma. Now call $b^i(s,G_s^i) = \mu_i G_s^i$ and $\sigma^i(s,G_s^i) = \sigma_i G_s^i$. We need to prove that

$$\int_0^t \left|b^i\left(s,G_s^i\right)\right| + \sigma^i\left(s,G_s^i\right)^2\mathrm{d}s < \infty, \quad \text{a.s,}$$

for all $t\in[0,T]$. It is enough to show that the expected value of the above is finite. Fix $t\in[0,T]$. It holds that

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \left|b^i(s,G_s)\right| + \sigma^i(s,G_s^i)^2\mathrm{d}s\right] &= \mathbb{E}\left[\int_0^t \mu_i G_s^i + \sigma_i^2(G_s^i)^2\mathrm{d}s\right] = \mu_i\,\mathbb{E}\left[\int_0^t G_s^i\mathrm{d}s\right] + \sigma_i^2\,\mathbb{E}\left[\int_0^t \left(G_s^i\right)^2\mathrm{d}s\right] \\
&= \mu_i E_0^i\int_0^t \exp(\mu_i s)\mathrm{d}s + \sigma_i^2\left(E_0^i\right)^2\int_0^t \exp(2\mu_i s + \sigma_i s)\mathrm{d}s \\
&= E_0^i\left(\exp(\mu_i t) - 1\right) + (E_0^i)^2\frac{\sigma_i^2}{2\mu_i + \sigma_i}\left(\exp(2\mu_i t + \sigma_i t) - 1\right) < \infty.
\end{aligned}$$

by Lemma A.30 and Fubini's theorem. By [KS91, pg 285] we can conclude that indeed the analytical solution is a strong solution of (1.7). □

With both BAU emissions defined above, the total emissions in the system at time T can be derived. We have that

$$\mathbb{E}\left[\sum_{i=1}^N E_T^i\right] = \sum_{i=1}^N \mathbb{E}\left[E_0^i + \mu_i T + \sigma_i W_T^i\right] = \sum_{i=1}^N E_0^i + \sum_{i=1}^N \mu_i T = N\bar{E}_0 + N\bar{\mu}T, \tag{1.9}$$

$$\mathbb{E}\left[\sum_{i=1}^N G_T^i\right] = \sum_{i=1}^N \mathbb{E}\left[E_0^i\exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)T + \sigma_i W_T^i\right)\right] = \sum_{i=1}^N E_0^i\exp(\mu_i T), \tag{1.10}$$

by the expectation of a Brownian motion and Proposition A.30. In the above, $\bar{\mu}$ represents the mean drift over the firms, and $\bar{E}_0$ the mean initial emission level. These total BAU emissions need to be reduced by a $100(1-\rho)$ percentage to achieve the desired reduction of the regulator. Therefore, the control variables need to be introduced, which happens in the next subsection. Before we continue, one last remark is made.

Remark 1.3. *Compared to [AB23], in Definition 1.2 the positive initial endowment E_0^i is added. This is done to be able to compare both models of the BAU emissions. That is, when $E_0^i = 0$, we would have by Proposition 1.2 that $G_t^i = 0$ for all $t\in[0,T]$, which is clearly undesirable. We will see that the solutions we obtain are slightly different due to this addition.*

1.2.1 Control variables and spaces

In this part, we will introduce two stochastic control variables. We will often refer to these variables as “the controls". Both controls are stochastic processes, as they depend on the economy, production, and other stochastic quantities. Often, we will denote the dependence on t, but not on $\omega \in \Omega$, unless it is helpful for the interpretation.

First, we introduce the stochastic process α^i, representing the abatement effort for the emissions of firm i. It is a rate per unit of time, measured in Gigatons of CO_2, that expresses how much the BAU emissions change. The higher the abatement effort for a specific firm, the more the firm tries to reduce its emissions. We won't make any assumption about the sign of the abatement effort α_t^i. That is, the abatement effort can be both positive and negative. This may seem counterintuitive at first. One reasoning behind this can be that a negative abatement at some point in time, polluting more than in the BAU scenario, may contribute to a higher, positive abatement in the future. For example, first, a more efficient machine needs to be built, that increases the emissions, to ensure that later the machine can be used to reduce emissions compared to the BAU case.

Second, we will work with the stochastic process β^i, which is the trading rate for firm i, measured in Gigatons of CO_2 per unit of time. We will later see that there is a market of allowances, in which the firms can trade their allowances. The trading rate will identify how much trading costs for a specific firm. The sign of this rate can be both positive and negative. If the rate is positive, it holds that the firm has net bought permits, if it is negative the firm has net sold permits. We will assume throughout that

$$\left|\beta_t^i\right| \leqslant K, \tag{1.11}$$

with $K \in \mathbb{R}$, almost surely, for all $t \in [0, T]$ and firms i. It is reasonable to assume, from the existence of a market, that buying and selling occurs with a bounded amount of permits at every point in time.

Before delving into the mathematics of these variables, we need to define the space to which the controls belong. For this, we first consider a specific L^2 product space. Here we refer the reader to the Appendix to read about the subtle differences between L^2 and $\mathscr{L}^2$. Let $\mathscr{B}([0, T])$ the Borel sigma algebra on $[0, T]$. First, we will look at the properties of a specific product space and product measure $\mathbb{P} \times \lambda^1$, where we recall that λ^1 represents the one-dimensional Lebesgue measure. We consider the following two measure spaces; the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and $([0, T], \mathscr{B}([0, T]), \lambda^1)$. Note that both measure spaces are finite, as $\mathbb{P}(\Omega) = 1$ by the definition of a probability space and $\lambda^1([0, T]) = T$ by the definition of the Lebesgue measure. This implies that both measure spaces are also σ-finite. By Theorem 14.5 of [Sch17], it holds that the measure μ, defined as

$$\mu : \mathscr{F} \otimes \mathscr{B}([0, T]) \to [0, \infty] \qquad \mu(A \times B) := \mathbb{P}(A)\lambda^1(B),$$

where $A \in \mathscr{F}$ and $B \in \mathscr{B}([0, T])$, is a unique σ-finite measure on $(\Omega \times [0, T], \mathscr{F} \otimes \mathscr{B}([0, T])$. Because of this, we can work with the corresponding, unique measure $\mu = \mathbb{P} \times \lambda^1$.

Then, we let the controls α^i, β^i be such that they belong to the space

$$L^2 := L^2\left(\Omega \times [0, T], \mathscr{F} \otimes \mathscr{B}([0, T]), \mathbb{P} \times \lambda^1\right), \tag{1.12}$$

for all $i \in \{1, 2, \ldots, n\}$. In the same way, we can define

$$L^1 := L^1\left(\Omega \times [0, T], \mathscr{F} \otimes \mathscr{B}([0, T]), \mathbb{P} \times \lambda^1\right).$$

A natural question that arises is what it means for a stochastic process to be in the space L^2. We consider this for the abatement effort α, where we ignore the superscript for the firm i. We will first do this from the definition, afterwards in terms of the norm of our L^2 space. By definition, we have that $\alpha \in L^2$, if

$$\int_{\Omega\times[0,T]} |\alpha_t^2| \mathrm{d}\left(\mathbb{P}\times\lambda^1\right) = \int_{\Omega\times[0,T]} \alpha_t^2 \mathrm{d}\left(\mathbb{P}\times\lambda^1\right) < \infty. \tag{1.13}$$

From Fubini's theorem, Theorem A.8 , this implies that

$$\int_{\Omega}\int_{[0,T]} \alpha_t^2 \mathrm{d}\,\lambda^1 \mathrm{d}\,\mathbb{P} = \int_{[0,T]}\int_{\Omega} \alpha_t^2 \mathrm{d}\,\mathbb{P} \mathrm{d}\,\lambda^1 < \infty,$$

By the definition of the Lebesgue integrals and taking integrals with respect to the probability measure $\mathbb{P}$, this means that

$$\int_{\Omega}\int_{[0,T]} \alpha_t^2 \mathrm{d}\,\lambda^1\, \mathrm{d}\mathbb{P} = \mathbb{E}\left[\int_0^T \alpha_t^2 \mathrm{d}t\right] = \int_0^T \mathbb{E}\left[\alpha_t^2\right] \mathrm{d}t = \int_{[0,T]}\int_{\Omega} \alpha_t^2 \mathrm{d}\,\mathbb{P}\, \mathrm{d}\lambda^1 < \infty. \tag{1.14}$$

From this, we can conclude that we can safely interchange the order of integration. Furthermore, again by Fubini's theorem, it holds

$$\int_0^T \alpha_t^2 \mathrm{d}t < \infty \quad \mathbb{P}\text{, almost surely,} \quad \mathbb{E}\left[\alpha_t^2\right] < \infty, \quad \lambda^1 \text{ almost everywhere.} \tag{1.15}$$

Remark 1.4 (Notation). *From now on, we will consistently use the symbol* $\mathrm{d}t$ *in place of* $\mathrm{d}\lambda^1$*. However, when discussing a measure, we will continue to refer to it as* λ^1*.*

In our case, this gives us the following inner product and norm induced by the inner product, which we will denote by $\langle\cdot,\cdot\rangle, \|\cdot\|$, respectively

$$\begin{aligned} \langle X, Y\rangle &:= \mathbb{E}\left[\int_0^T X_t Y_t \mathrm{d}t\right], \\ \|X\| &:= \sqrt{\langle X, X\rangle} = \sqrt{E\left[\int_0^T X_t^2 \mathrm{d}t\right]}. \end{aligned} \tag{1.16}$$

If there is no subscript given for the norm or inner products, we mean the definitions presented above. The next definition defines when two processes are considered equal in L^2.

Definition 1.4. *We identify two processes as equal when* $X = Y$*,* μ *almost everywhere.*

Before we continue, two interesting remarks are made.

Remark 1.5. *Note that when two processes are modifications of each other, by Definition A.5 it holds that* $X_t = Y_t$*,* $\mathbb{P}$ *almost surely (a.s.) for all* $t \in [0,T]$*. This immediately implies that*

$$X = Y,\ \mu = \mathbb{P}\times\lambda^1 \ \textit{a.e.}$$

Hence, a modification is stronger than the equivalence defined in L^2*. When working with equations of stochastic processes, the equations in this thesis will hold* μ *almost everywhere (a.e.), unless stated otherwise. Sometimes, this will be explicitly mentioned.*

Remark 1.6. *Notice that for two processes* X *and* Y *that are equal* μ *a.e, we can construct modifications* $\tilde{X}$ *and* $\tilde{Y}$ *such that* $\tilde{X}_t = \tilde{Y}_t$ $\mathbb{P}$ *a.s, for all* $t \in [0,T]$*. This means that we will often speak about* $X_t = Y_t$ μ *a.e, when we actually mean these modifications.*

By the fact that $\|\cdot\|$ is a norm in the L^2 space, it follows directly for two processes $X, Y \in L^2$,

$$\|X - Y\| = 0 \text{ if and only if } X = Y, \mu \text{ a.e.} \tag{1.17}$$

Unfortunately, requiring only that both controls are in L^2 is not sufficient. The space of admissible controls $\mathscr{A}$, that the controls need to belong to, is defined below.

Definition 1.5 (Space of admissible controls). *The space of admissible controls, denoted by the set $\mathscr{A}$, is given by*

$$\mathscr{A} := \left\{ X : \Omega \times [0.T] \to \mathbb{R} \middle| X \in L^2 \text{ and progressively measurable} \right\}.$$

For the definition of progressive measurability, see Definition A.8. It follows that the controls we are working with will be adapted and measurable, and we have that

$$\left\| \alpha^i \right\|^2 = \mathbb{E}\left[\int_0^T \left(\alpha_t^i \right)^2 \mathrm{d}t \right] < \infty, \quad \left\| \beta^i \right\|^2 = \mathbb{E}\left[\int_0^T \left(\beta_t^i \right)^2 \mathrm{d}t \right] < \infty,$$

where the norm on $\mathscr{A}$ is the same as the norm on L^2. This implies that also in the space $\mathscr{A}$ we can speak of convergence in L^2.

We also define

$$\mathscr{B} := \left\{ X : \Omega \times [0.T] \to \mathbb{R} \middle| X \in L^1 \text{ and progressively measurable} \right\}, \tag{1.18}$$

with corresponding norm for $b \in \mathscr{B}$

$$\|b\|_{\mathscr{B}} = \mathbb{E}\left[\int_0^T |b_t| \mathrm{d}t \right].$$

Since holds that $L^2 \subseteq L^1$, because we work with a finite product measure space, we have that $\mathscr{A} \subseteq \mathscr{B}$. In the next chapters, we will need several properties of the space $\mathscr{A}$, which are proven in the propositions below.

Proposition 1.3. *The space $\mathscr{A}$ in Definition 1.5 is a Hilbert space, with inner product given in* (1.16)*, when we consider two processes $X, Y \in \mathscr{A}$ equivalent when $||X - Y|| = 0$.*

The proof of this lemma is inspired by [Spr11, pg. 34].

Proof. Take a Cauchy sequence $(Y^n)_{n\in\mathbb{N}} \in \mathscr{A}$, thus (Y^n) is a sequence of stochastic processes. We need to prove that this Cauchy sequence has a limit in $\mathscr{A}$. We know that $L^2(\Omega\times[0,T], \mathscr{F} \otimes \mathscr{B}([0,T]), \mathbb{P}\times\lambda^1)$ itself is a Hilbert space, by Lemma A.22 with $\tilde{\Omega} := \Omega \times [0,T]$. Hence, the sequence (Y^n) converges in L^2 to a limit in $L^2(\Omega \times [0,T], \mathscr{F} \otimes \mathscr{B}([0,T]), \mathbb{P} \times \lambda^1)$, this limit is called Y. This means, by Definition A.11, that

$$\lim_{n\to\infty} ||Y^n - Y|| = 0.$$

We do not immediately have that Y is also progressively measurable. However, by the fact that (Y^n) converges to Y in L^2, we have that there exists a sub-sequence (Y^{n^k}) of (Y^n) that converges μ almost everywhere to Y [Sch17, pg. 123]. That is,

$$\mu\left(\lim_{k\to\infty} Y^{n^k} = Y \right) = T.$$

Now call $X_t = \limsup_{k\to\infty} Y_t^{n^k}$. It holds that X is also progressively measurable, by Proposition A.4. Note that it follows immediately that $Y = X$, μ a.e, and thus, by Lemma 1.17, that $||X - Y|| = 0$. We can conclude that X and Y are equivalent processes. Furthermore, we have that X is progressively measurable in $L^2(\Omega \times [0,T], \mathscr{F} \otimes \mathscr{B}([0,T]), \mathbb{P} \times \lambda^1)$ and

$$0 \leqslant ||Y^n - X|| \leqslant ||Y^n - Y|| + ||Y - X|| \to 0.$$

We can conclude that $Y^n \xrightarrow{L^2} X$. From this, it follows that $\mathscr{A}$ is an Hilbert space. $\square$

From the above, it almost immediately follows that $\mathscr{A}$ is a closed subset of L^2. This is written down in the following corollary;

Proposition 1.4. *The space $\mathscr{A}$ is a closed, convex subspace of L^2.*

Proof. The fact that $\mathscr{A}$ is a subspace is trivial. Since progressively measurability is preserved under summations, by Definition A.28, it follows directly that $\mathscr{A}$ is a convex subspace. Left to show is that it is closed. For this, we take a sequence $(Y^n) \subset \mathscr{A}$ such that (Y^n) converges to Y in L^2. We need to prove that $Y \in \mathscr{A}$. Since every converging sequence is a Cauchy sequence, we can use the exact same procedure as in Proposition 1.3. We can conclude that (Y^n) converges to X, where $X \in \mathscr{A}$ and where X is equivalent to Y. This is the desired result. □

Often, we will work with the space

$$\mathscr{A}^2 := \mathscr{A} \times \mathscr{A},$$

as we need that the tuple $(\alpha^i, \beta^i) \in \mathscr{A}^2$ for every firm i. Many equivalent norms can be defined on this product space. However, to ensure that $\mathscr{A}^2$ itself is a Hilbert space, we need that

$$||(X,Y)||_{\mathscr{A}^2} = \sqrt{\|X\|^2 + \|Y\|^2},$$

where $\|\cdot\|$ is the norm in $\mathscr{A}$. If we define the inner product for $(X,Y),(W,Z) \in \mathscr{A}^2$ the inner product by

$$\langle (X,Y),(W,Z)\rangle_{\mathscr{A}^2} = \langle X,W\rangle + \langle Y,Z\rangle,$$

it follows that

$$\|(X,Y)\|_{\mathscr{A}^2} = \sqrt{\|X\|^2 + \|Y\|^2} = \sqrt{\langle X,X\rangle + \langle Y,Y\rangle} = \sqrt{\langle (X,Y),(X,Y)\rangle_{\mathscr{A}^2}}.$$

That is, the norm and inner product now have the relation needed to be a Hilbert space. This follows directly from the properties of an inner product in $\mathscr{A}$.

For notational convenience, we will use the subscript $\mathscr{A}^2$ if we are working with norms or inner products of this product space. Since the norms on L^2 and $\mathscr{A}^2$ coincide, we could also write the subscript $L^2 \times L^2$.

In the next proposition, we prove that $\mathscr{A}^2$ is also a Hilbert space.

Proposition 1.5. *The space $\mathscr{A}^2$ with corresponding norm $||(X,Y)||_{\mathscr{A}^2}$ for $(X,Y) \in \mathscr{A}^2$ defined as*

$$||(X,Y)||_{\mathscr{A}^2} = \sqrt{\|X\|^2 + \|Y\|^2},$$

is a Hilbert space.

Proof. We need to show that $\mathscr{A}^2$ is complete. Let us take a Cauchy sequence of stochastic processes $(X^n, Y^n) \in \mathscr{A}^2$. By definition (X^n) and (Y^n) are both Cauchy sequences with a limit in $\mathscr{A}$, as $\mathscr{A}$ itself is a Hilbert space, by Proposition 1.3. Call these limits $X \in \mathscr{A}$ and $Y \in \mathscr{A}$ respectively. Necessarily it holds that $(X,Y) \in \mathscr{A}^2$. We will prove that (X^n, Y^n) converges to (X,Y) under the chosen norm. Indeed, we have

$$\left\|(X^n, Y^n) - (X,Y)\right\|_{\mathscr{A}^2} = \left\|(X^n - X, Y^n - Y)\right\|_{\mathscr{A}^2} = \sqrt{\|X^n - X\|^2 + \|Y^n - Y\|^2} \to 0,$$

as $n \to \infty$, since both $X^n \xrightarrow{L^2} X$ and $Y^n \xrightarrow{L^2} Y$. □

Before we continue to the next part, we will introduce one more stochastic process $\tilde{A}^i$, which resembles the cumulative allowances given by the regulator to the firm i.

Definition 1.6 (Allowances process). *The stochastic process of the cumulative allowances for firm i, denoted by $(\tilde{A}^i)$, is defined to be a semimartingale, which is square-integrable with respect to the measure $\mathbb{P}$ for all $t \in [0,T]$. The space of the N-dimensional vector of such allowances process $\tilde{A} := (\tilde{A}^1, \dots, \tilde{A}^N)$ is denoted by $\mathscr{S}^N$.*

By Definitions A.10 and A.13, this means that $\mathbb{E}\left[(\tilde{A}^i_t)^2\right] < \infty$ for all $t \in [0,T]$ and that

$$\tilde{A}^i_t = F^i_t + H^i_t, \tag{1.19}$$

where F^i is a process of bounded variation and H^i a martingale, starting in zero, with respect to the filtration $(\mathscr{F}_t)$. The process is defined to be cadlag, and adapted. This implies that the allowances process is right-continuous, but not necessarily continuous. There is again no sign restriction on the allowances process. If for $t \in [0,T]$, $\tilde{A}^i_t > 0$, it means that there are allowances allocated. If $A^i_t < 0$ allowances are withdrawn from the market in some sense. This will become more clear, when the bank account is defined.

By the Martingale Representation Theorem, we can write

$$\tilde{A}^i_t = F^i_t + \sum_{k=0}^{N} \int_0^t \tilde{b}^{k,i}_s \mathrm{d}B^k_s, \tag{1.20}$$

where $\tilde{A}^i_0 = F^i_0$, and $\tilde{b}^{k,i}$ are progressively measurable processes that satisfy

$$\int_0^T \left(\tilde{b}^{k,i}_s\right)^2 \mathrm{d}s < \infty, \quad \text{a.s,}$$

for all relevant k, i. An important observation made in [AB23] is the following.

Remark 1.7. *The process of bounded variation $\tilde{F}^i$ defined in* (1.19) *does not need to be absolutely continuous with respect to the Lebesgue measure. That is, if $dt = 0$, we do not necessarily have that $\tilde{F}^i_t = 0$. Thus, the regulator may allocate permits at discrete moments.*

This remark corresponds with the fact that the process (A^i) is not necessarily continuous. The allowances process can be decomposed further. This is done in the proposition below.

Proposition 1.6. *The process of bounded variation (F^i) can be uniquely decomposed into a sum of a singular part $(\tilde{S}^i)$ and a part absolutely continuous with respect to the Lebesgue measure, as follows*

$$F^i_t = \tilde{S}^i_t + \int_0^t \tilde{a}^i_s ds.$$

Proof. For simplicity, we will leave out the superscript i for each firm. First, fix $\omega \in \Omega$, such that we have a function from $[0,T]$ to $\mathbb{R}$ with $t \to F_t(\omega)$. Since $F(\omega)$ is a function of bounded variation, the integral below is well-defined

$$F_t(\omega) = F_0(\omega) + \int_0^t \mathrm{d}F_s(\omega). \tag{1.21}$$

From now on, we will leave out the dependence on $\omega \in \Omega$. Let μ_F be a signed measure on $([0,T], \mathscr{B}([0,T]))$, defined as

$$\mu_F((s,t]) = F_t - F_s,$$

where $0 \leqslant s < t \leqslant T$. This measure exists and is well-defined by [Doo12, pg. 43-50]. By the same source and [Spr11], it holds that the Lebesgue integral with respect to this measure corresponds with the Stieltjes integral constructed in (1.21). That is, we have

$$\int_0^t \mathrm{d}F_s(\omega) = \int_0^t \mathrm{d}\mu_F.$$

From the Lebesgue Decomposition Theorem [Doo12, pg. 148], it follows that μ_F can be uniquely decomposed into a signed measure which is absolutely continuous with respect to the Lebesgue measure, denoted by $\mu_{F,\text{ac}}$, and a signed measure that is singular with respect to the Lebesgue measure, given by $\mu_{F,\text{sin}}$. Then, it follows that

$$\begin{aligned}\mu_F &= \mu_{F,\text{ac}} + \mu_{F,\text{sin}},\\ F_t(\omega) &= F_0(\omega) + \int_0^t \mathrm{d}\mu_F = F_0(\omega) + \int_0^t \mathrm{d}\mu_{F,\text{ac}} + \int_0^t \mathrm{d}\mu_{F,\text{sin}}.\end{aligned}$$

Because of the fact that μ_F is a finite, signed measure, we can use the Radon-Nikodym Theorem [Doo12, pg. 150] to rewrite $\mu_{F,\text{ac}}$. Let $\tilde{a} = \frac{\mathrm{d}\mu_{F,\text{ac}}}{\mathrm{d}t}$. Then we get,

$$\int_0^t \mathrm{d}\mu_{F,\text{ac}} = \int_0^t \tilde{a}_s \mathrm{d}s.$$

Next, we set

$$\tilde{S}_t := F_0(\omega) + \int_0^t \mathrm{d}\mu_{F,\text{sin}} = F_0(\omega) + \mu_{F,\text{sin}}((0,t]). \tag{1.22}$$

Here, by construction, $\mu_{F,\text{sin}}$ is a measure which is singular with respect to the Lebesgue measure. Furthermore, $F_0(\omega)$ is also singular with respect to the Lebesgue measure, since it can be considered as a product of the Dirac measure in zero and $F_0(\omega)$. It follows that $\tilde{S}_t$ is singular with respect to the Lebesgue measure itself, by Lemma A.9. We can conclude that indeed

$$\tilde{F}_t^i = \tilde{S}_t^i + \int_0^t \tilde{a}_s^i \mathrm{d}s.$$

This is the desired result. □

From the proposition above, it follows that we can write

$$\tilde{A}_t^i = \tilde{S}_t^i + \int_0^t \tilde{a}_s^i \mathrm{d}s + \sum_{k=0}^N \int_0^t \tilde{b}_s^{k,i} \mathrm{d}\tilde{B}_s^k. \tag{1.23}$$

The above formulation of $\tilde{A}^i$ and the expression given in Equation (1.19) will be used interchangeably. Last, we assume that the market price price of permits, denoted by P, also belongs to $\mathscr{A}$. That is, the market price is also in the space of admissible controls, and we assume it is uniformly bounded from below by $C \in \mathbb{R}$, that is,

$$P_t > C, \tag{1.24}$$

a.s, for all $t \in [0, T]$. Note that this constant may be a small, negative number and is not dependent on t. This assumption is induced from the existence of a market of permits. Since all the appropriate variables and spaces are defined, we are ready to continue to the stochastic control problems.

1.3 Stochastic control theory

Now that we have explained the setting of the stochastic model and all corresponding spaces we are working with, we are ready to discuss the stochastic control setting. Before, this, we will first give a general introduction in stochastic control theory.

1.3.1 General

Stochastic optimisation problems can be defined, such that a cost function and the variables with respect to which we optimise, consist of random variables. In general, the following summarises what we need. This subsection is based on [Pha09, ch. 2].

Let (X_t) be a continuous stochastic process with $t \in [0,T]$, where for fixed $t \in [0,T]$ we have that X_t is a random variable defined on the given probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We say that $X_t(\omega)$ represents the state of the system in a world scenario $\omega \in \Omega$. The dynamics of the state can be described by a Stochastic Differential Equation (SDE).

These dynamics are influenced by a control parameter γ, which is often a stochastic process as well. The controlled process will be denoted by X^γ. The controls must satisfy certain assumptions, depending on the situation. The set of all possible controls satisfying the assumptions, is denoted by $\mathscr{A}$, representing the set of admissible controls. To be able to construct a well-posed control problem, we need to make sure that X^γ admits a solution. In this thesis, we will work with a strong solution, such that the controlled SDE is sufficiently solvable [Pha09, pg. 38].

The objective function that needs to be minimised is given by the expected value of the integral of a cost function $g(X_t^\gamma, \gamma_t)$ and a terminal penalty $h(X_T^\gamma)$. The objective functional is of the following form

$$J(X,\gamma) = \mathbb{E}\left[\int_0^T g\left(X_t^\gamma, \gamma_t\right) \mathrm{d}t + h\left(X_T^\gamma\right)\right]. \tag{1.25}$$

The goal is to find $\gamma \in \mathscr{A}$ that satisfies

$$v = \inf_{\gamma \in \mathscr{A}} J\left(X^\gamma, \gamma\right), \tag{1.26}$$

such that v is the smallest possible value of the objective functional $J(X^\gamma, \gamma)$. This will then give us the optimal value of the control variable. Stochastic control theory is a generalisation of deterministic control theory, for example explained in [Kap07]. In this article, a method to solve a stochastic control problem using the Hamilton-Jacobi-Bellman equations is introduced. In our case, we will rely on an alternative method, involving variational calculus, explained in Section A.1.3 in the appendix. An important question is whether there exists a solution, possibly unique, to (1.26). In Chapter 2, we will see that a significant part is devoted to this.

1.3.2 Specific stochastic control problem

In this subsection, we aim to provide a clear interpretation of the variables within our stochastic control problem. Recall that for a firm i, we are working with two control variables; $(\alpha^i, \beta^i) \in \mathscr{A}^2$. Consistent with the theory discussed above, the set $\mathscr{A}^2$ is called the set of admissible controls and (α, β) represent the controls. Recall that the objective of the regulator is to reduce the emissions by a $100(1-\rho)$ percentage, while minimising social costs. The regulator achieves this, by allocating permits, to which firm i responds by trading the permits at a rate β^i in the market and abating their emissions at a rate α^i. Next, we will see how these processes are used to reduce emissions.

The abatement effort rate is a process that is directly used to reduce the emissions. With the cumulative BAU emissions modelled in (1.6) for the Brownian motion case, and (1.7) for the Geometric Brownian motion case, it is reasonable to control those emissions. We will work with the following, controlled version of the cumulative emissions E^i, G^i respectively,

$$E_t^{i,\alpha^i} = E_t^i - \int_0^t \alpha_s^i \mathrm{d}s = E_0^i + \int_0^t \left(\mu_i - \alpha_s^i\right) \mathrm{d}s + \sigma_i W_t^i, \tag{1.27}$$

$$G_t^{i,\alpha^i} = G_t^i - \int_0^t \alpha_s^i \mathrm{d}s = E_0^i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right) t + \sigma_i W_t^i\right) - \int_0^t \alpha_s^i \mathrm{d}s. \tag{1.28}$$

We will often refer to these equations as "the abated emission". That is, the abated emissions are the BAU emissions minus the abatement effort, which is defined as a rate, integrated over time.

It is worth mentioning that in differential form we could write

$$\begin{aligned}
\mathrm{d}E_t^{i,\alpha^i} &= \mathrm{d}E_t^i - \alpha_t^i \mathrm{d}t, \quad E_0^{i,\alpha^i} = E_0, \\
\mathrm{d}G_t^{i,\alpha^i} &= \mathrm{d}G_t^i - \alpha_t^i \mathrm{d}t, \quad G_0^{i,\alpha^i} = E_0,
\end{aligned}$$

where the differentials follow from Definitions 1.2 and 1.3. Given the integrability conditions on α^i and the fact that $\mu_i, \sigma_i \in \mathbb{R}$, there exists a unique strong solution for the SDEs above, which is given by (1.27) and (1.28) respectively [Pha09, pg. 38]. For this, we make implicitly use of Proposition 1.2.

We see that, we will only control the drift of the emissions, not the volatility. One reason for this is that according to [AB23] the firm cannot control its volatility properly, since high correlations between the firms may be present. Another, mathematical reason is that analytical solutions can be found more easily if we only have drift control. One could question whether analytical solutions are obtainable if we would incorporate volatility control. This goes beyond the scope of this thesis.

To achieve the desired reduction of the regulator, we need to compare the total emissions in the system in the BAU case (1.9) and (1.10) with the total abated emissions, in both cases separately. That is, the regulator wishes to get, respectively,

$$\mathbb{E}\left[\sum_{i=1}^N E_T^{i,\alpha^i}\right] = \sum_{i=1}^N \mathbb{E}\left[E_t^i\right] - \sum_{i=1}^N \mathbb{E}\left[\int_0^T \alpha_s^i \mathrm{d}s\right] = \sum_{i=1}^N E_0^i + N\bar{\mu}T - \sum_{i=1}^N \mathbb{E}\left[\int_0^T \alpha_s^i \mathrm{d}s\right] =: \rho N\left(\bar{E}_0 + \bar{\mu}T\right), \tag{1.29}$$

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^N G_T^{i,\alpha^i}\right] &= \sum_{i=1}^N \mathbb{E}\left[G_T^i\right] - \sum_{i=1}^N \mathbb{E}\left[\int_0^T \alpha_s^i \mathrm{d}s\right] = \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right) - \sum_{i=1}^N \mathbb{E}\left[\int_0^T \alpha_s^i \mathrm{d}s\right] \\
&=: \rho \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right).
\end{aligned} \tag{1.30}$$

Then, she indeed exactly achieves a $100(1-\rho)$ percentage reduction of the emissions compared to the cumulative BAU emissions, for $0 < \rho < 1$.

To achieve this emission reduction, the regulator implements a dynamic cap-and-trade system. Here, at every time $t \in [0, T]$, each firm receives a specific number of allowances, to use until time T. They do not need to use them immediately, since at time $t = 0$ the regulator opens a bank account X^i for each firm. Firms can put their allowances on the bank account to use them at some later point in time. In the meantime, firms can buy and sell carbon permits for a specific market price P_t when they need more or have a surplus of allowances. The bank account really counts the number of allowances available, and may be negative if a firm has emitted more than the allowances available. In the equation of the bank account X^i, the trading rate β^i, the allowances process $\tilde{A}^i$ and the abated emissions will come into play. Since it depends on those emissions, a superscript for the specific process will be used. That is, $X^{i,E}$ and $X^{i,G}$ for the Brownian motion and GBM case respectively. We will still use X^i if the underlying process of the emissions does not play a role. Combining all, we get the following dynamics for the bank account of the allowances

$$\mathrm{d}X_t^{i,E} = \beta_t^i \mathrm{d}t + \mathrm{d}\tilde{A}_t^i - \mathrm{d}E_t^{i,\alpha^i}, \qquad X_0^{i,E} = \tilde{A}_0^i - E_0^i, \qquad X_t^{i,E} = \tilde{A}_t^i + \int_0^t \beta_s^i \mathrm{d}s - E_t^{i,\alpha^i}, \tag{1.31}$$

$$\mathrm{d}X_t^{i,G} = \beta_t^i \mathrm{d}t + \mathrm{d}\tilde{A}_t^i - \mathrm{d}G_t^{i,\alpha^i}, \qquad X_0^{i,G} = \tilde{A}_0^i - E_0^i, \qquad X_t^{i,G} = \tilde{A}_t^i + \int_0^t \beta_s^i \mathrm{d}s - G_t^{i,\alpha^i}, \tag{1.32}$$

where the abated emissions are given by (1.27)and (1.28). Here, we set $X_0^i = \tilde{A}_0^i - E_0^i$ for each firm i, in both cases, such that the initial bank account consists of the initial allocation minus the initial emission level. This is a realistic assumption, as at time zero these are the only items in the bank account. The initial value of the bank account is not clearly mentioned in [AB23]. Recall that β_t^i, for a fixed $t \in [0,T]$, is positive if the firm buys net extra allowances to use at the specific time, and negative if it sells them. This is in correspondence with the equations for the bank account above. If β_t^i is positive, there are net permits bought at time t and thus the number of permits available, represented by the bank account of permits, increases. Furthermore, the bank account decreases as the firms are emitting more, and increases if the allowances increase. This is all consistent with reality. Last, we see that a negative allowances process results in a decrease in the bank account. From this, we can conclude that a negative allowances process can be interpreted as a penalty on the bank account.

An important observation is that the bank account can be seen as the state space of Section 1.3.1, the uncertainty in the system is modelled via the bank account. Since the state space is stochastic and depends on $\omega \in \Omega$, the controls need to depend on the specific ω as well. This substantiates ones more why the controls are chosen to be stochastic processes.

Note that we can rewrite the equation of the bank account (1.31) by plugging in (1.27) to

$$\mathrm{d}X_t^{i,E} = \left(\alpha_t^i + \beta_t^i\right)\mathrm{d}t + \mathrm{d}\tilde{A}_t^i - \mu_i \mathrm{d}t - \sigma_i \mathrm{d}W_t^i. \tag{1.33}$$

From this, we can introduce a new, transformed variable, which we will call the net allocation process (over the trend), given by

$$A_t^i = \tilde{A}_t^i - \mu_i t. \tag{1.34}$$

Hence, (1.33) can be rewritten to the final expression of the bank account

$$\mathrm{d}X_t^{i,E} = \left(\alpha_t^i + \beta_t^i\right)\mathrm{d}t + \mathrm{d}A_t^i - \sigma_i \mathrm{d}W_t^i, \tag{1.35}$$

which gives solution at time $t \in [0,T]$,

$$X_t^{i,E} = X_0^i + A_t^i - A_0^i + \int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s + E_0^i - E_0^i - \sigma_i W_t^i = A_t^i + \int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s - \sigma_i W_t^i - E_0^i. \tag{1.36}$$

In the case of the GBM, the bank account can only be rewritten to

$$X_t^{i,G} = \tilde{A}_t^i + \int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s - G_t^i. \tag{1.37}$$

Several properties hold for this bank account. These are stated and proven in the lemmas below.

Proposition 1.7. *The bank accounts* (1.36) *and* (1.32) *are square-integrable w.r.t the measure* $\mathbb{P}$, *introduced in Definition A.10.*

Proof. Let $t \in [0,T]$. We start with $(X_t^{i,E})$. By the linearity of the expectation it follows

$$\begin{aligned}\mathbb{E}\left[\left(X_t^{i,E}\right)^2\right] &= \mathbb{E}\left[(A_t^i)^2\right] + \mathbb{E}\left[\left(\int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s\right)^2\right] - \sigma_i^2 \mathbb{E}\left[(W_t^i)^2\right] + (E_0^i)^2 + 2\mathbb{E}\left[A_t^i \int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s\right] \\ &\quad - 2\sigma_i \mathbb{E}\left[A_t^i W_t^i\right] - 2\sigma_i \mathbb{E}\left[W_t^i \int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s\right] - 2E_0^i \mathbb{E}\left[A_t^i + \int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s - \sigma_i W_t^i\right].\end{aligned} \tag{1.38}$$

First, note that

$$\begin{aligned}\mathbb{E}\left[A_t^i + \int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s - \sigma_i W_t^i\right] &= \mathbb{E}\left[A_t^i + \int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s\right] \\ &\leqslant \sqrt{\mathbb{E}\left[\left(A_t^i\right)^2\right] + 2\mathbb{E}\left[A_t^i \int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s\right] + \mathbb{E}\left[\left(\int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s\right)^2\right]} \\ &\leqslant \sqrt{2\mathbb{E}\left[\left(A_t^i\right)^2\right] + 2\mathbb{E}\left[\left(\int_0^t \alpha_s^i + \beta_s^i \mathrm{d}s\right)^2\right]},\end{aligned}$$

where we made use of the fact that in general $2xy \leqslant x^2 + y^2$ and of the Cauchy-Schwarz inequality in $L^2(\Omega)$. The stochastic processes $\tilde{A}$, and thus the process A as well, and W^i are square-integrable. That, if we can prove that

$$\mathbb{E}\left[\left(\int_0^T \alpha_s^i + \beta_s^i \mathrm{d}s\right)^2\right] < \infty,$$

the desired result follows by the same reasoning above. Indeed, it holds that

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T \alpha_s^i + \beta_s^i \mathrm{d}s\right)^2\right] &\leqslant T^2\,\mathbb{E}\left[\int_0^T \left(\alpha_s^i + \beta_s^i\right)^2 \mathrm{d}s\right] \leqslant T^2\,\mathbb{E}\left[\int_0^T (\alpha_s^i)^2 + (\beta_s^i)^2 + 2\alpha_s^i\beta_s^i \mathrm{d}s\right] \\ &\leqslant T^2\left(\mathbb{E}\left[\int_0^T \left(\alpha_s^i\right)^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^T \left(\beta_s^i\right)^2 \mathrm{d}s\right] + 2\left\|\alpha^i\right\|\left\|\beta^i\right\|\right) < \infty,\end{aligned}$$

where the Cauchy-Schwarz inequality in the space $L^2([0,T])$ is used in the first step, and the same inequality in the given product L^2 space of (1.12) in the last step. The exact same approach can be used for $(X_t^{i,G})$, since G_t^i is square-integrable itself, by Proposition A.30. □

Lemma 1.8. *The bank accounts $(X_t^{i,E})$ and $(X_t^{i,G})$ are adapted to the filtration $(\mathscr{F}_t)$.*

Proof. Let $t \in [0,T]$. Since $A_t^i = \tilde{A}_t^i - \mu_i t$, it follows from Definition A.13 for a semimartingale that A_t^i is adapted. As $(\alpha^i, \beta^i) \in \mathscr{A}^2$, it implies that α^i and β^i are progressively measurable. By Proposition A.6, it follows that the time integral is progressively measurable as well, by Proposition A.6. Since a progressively measurable process is adapted, the time integral $\int_0^T \alpha_s^i + \beta_s^i \mathrm{d}s$ is adapted as well. Furthermore, W^i is adapted, as it is a martingale by construction. Hence, $(X_t^{i,E})$ is adapted by representation (1.36). By the same arguments and representation (1.37) $(X_t^{i,G})$ is also adapted, as a continuous function of an adapted process is again adapted. □

With these bank accounts and properties, the cost minimisation of the firms and the regulator can be explained.

1.3.3 Firms cost minimisation

Here we assume that the market price of permits $P \in \mathscr{A}$ and the net allowances $A \in \mathscr{S}^N$ as described are given exogenously.

The cost function g and terminal penalty h for a specific firm i, in accordance with Section 1.3 and Equation (1.25) are assumed to be given by

$$g\left(X_t^i, \alpha_t^i, \beta_t^i\right) = h_i \alpha_t^i + \frac{1}{2\eta_i}\left(\alpha_t^i\right)^2 + P_t \beta_t^i + \frac{\left(\beta_t^i\right)^2}{2\nu}, \quad h\left(X_T^i\right) = \lambda\left(X_T^i\right)^2. \tag{1.39}$$

Note here that indeed X_t depends on α^i and β^i. The objective functional, corresponding with (2.4), that the firm i needs to minimise is given by

$$\mathscr{J}^i\left(\alpha^i, \beta^i\right) := \mathbb{E}\left[\int_0^T h_i \alpha_t^i + \frac{1}{2\eta_i}(\alpha_t^i)^2 + P_t \beta_t^i + \frac{1}{2\nu}(\beta_t^i)^2 \mathrm{d}t + \lambda\left(X_T^i\right)^2\right]. \tag{1.40}$$

Note that this functional is well-defined, as α, β are square-integrable with respect to the measure $\mu = \mathbb{P} \times \lambda^1$ and (X_t^i) is square-integrable with respect to $\mathbb{P}$, by Proposition 1.7. In the above, some parameters are not yet discussed, so that is what we will do next, term by term. First, of all, we define the abatement costs $c_i(\alpha^i)_t$ where $c_i : \mathscr{A} \to \mathscr{B}$ with

$$c_i\left(\alpha_t^i\right) := c_i\left(\alpha^i\right)_t = h_i \alpha_t^i + \frac{1}{2\eta_i}\left(\alpha_t^i\right)^2, \quad h_i, \eta_i \in \mathbb{R}_{>0}, \tag{1.41}$$

where the space $\mathscr{B}$ is defined in (1.18). Indeed, as it is given that $\alpha \in \mathscr{A}$, we have that $\alpha^2 \in \mathscr{B}$, giving that $c_i(\alpha^i) \in \mathscr{B}$. These are the costs of the abatement, which has a linear part and a quadratic part. The linear part consist of the constant h_i. In the quadratic part, η_i is included, which represents the flexibility of the abatement process. It is often called the adjustment cost parameter. The higher η_i the higher the reversibility of the abatement, and thus the smaller the quadratic cost part of $c_i(\alpha_t^i)$. From the cost of abatement the marginal costs of abatement can be derived, but we should be careful as we are dealing with the derivative with respect to a random variable. In words, the marginal costs of abatement are defined as the extra costs that come from abating one Gigaton of emissions.

The third term, $P_t\beta_t^i\mathrm{d}t$, represents the direct costs of trading in the market of permits. Note that this can be both negative and positive. When $\beta_t^i < 0$, the net effect is that there are emissions sold, so that the firm earns $P_t\beta_t^i\mathrm{d}t$, the market price times the number of permits sold. This means that the costs decrease. When $\beta_t^i > 0$, there are emissions bought, which costs money. Hence, the extra costs are then the market price times the trading rate.

The fourth part of (1.39) resembles the indirect costs of trading. Here, $\nu \in \mathbb{R}_{>0}$ is considered to be the market depth parameter. According to [Kyl85], this is defined as the size of an order flow innovation to require that prices change by a given amount. It is assumed to be constant. The idea behind this is that large transactions can have an influence on the price of the permits, which should be incorporated in the model. By dividing by ν, we exactly obtain the change in permit price if we trade with amount $\beta_t\mathrm{d}t$. Note that it is also a reasonable assumption that $\nu = \infty$, that is, no order flow will change the permit price. In the coming chapters, we will mostly work with this assumption. We refer to this case as optimisation in a market without frictions. The case where $\nu < \infty$ will be referred to the case with market frictions, of which the optimisation is worked out in Appendix B.

From the two parts involving the trading rate, the costs of trading

$$f_i\left(\beta^i\right)_t = P_t\beta_t^i + \frac{(\beta_t^i)^2}{2\nu},$$

are defined. With this, the marginal costs of trading can be derived. This is defined, in line with the marginal costs of abatement, as the extra costs of trading one more unit. More on this can be found in Chapter 2.

Last, the terminal penalty here is $\lambda(X_T^i)^2$, where $\lambda \in \mathbb{R}$ is a positive common penalty coefficient, measured in euros per unit of emission. First of all, without the presence of the penalty functions, given an initial allocation, the firms would just pollute their BAU emissions, since there is no incentive for them to lower their emissions. A penalty is thus really necessary to make sure our stochastic control problem is well-posed. This specific choice of penalty is positive in both cases where the bank account is positive and negative. If the bank account $X_T^{i,E} \geqslant 0$, it holds that

$$\tilde{A}_T^i + \int_0^T \beta_s^i \mathrm{d}s \geqslant E_T^{i,\alpha} - E_0^i,$$

This means that the firm has more cumulative allowances than abated emissions. It is not realistic that the firm needs to pay a penalty then, as there are enough allowances to cover the emissions. It would be the best for the firm if $X_T^i = 0$, then it would not need to pay a penalty. If $X_T^i < 0$, it is reasonable that a penalty needs to be paid, as there are not enough permits to cover the emissions. A penalty of the form

$$\lambda \max\left(-X_T^i, 0\right)^2, \tag{1.42}$$

would correspond with this more realistic fact that we only pay a penalty if $X_T^i \leqslant 0$. However, we are still going to work with the given quadratic penalty function. It appears that this function is chosen to be able to obtain an analytical solution. Here, we have chosen mathematical elegance over reality.

The goal of every firm i is to find optimal controls $(\hat{\alpha}^i, \hat{\beta}^i)$ such that their objective function is minimised, that is,

$$\mathscr{J}^i\left(\hat{\alpha}^i, \hat{\beta}^i\right) = \inf_{(\alpha^i, \beta^i)\in\mathscr{A}^2} \mathbb{E}\left[\int_0^T h_i \alpha_t^i + \frac{\left(\alpha_t^i\right)^2}{2\eta_i} + P_t \beta_t^i + \frac{1}{2\nu}(\beta_t^i)^2 \mathrm{d}t + \lambda\left(X_T^i\right)^2\right]. \tag{1.43}$$

1.3.4 Social costs minimisation by regulator

The goal of the regulator is to design a dynamic allocation scheme $A = (A^1, \ldots, A^N)$ or $\tilde{A} = (\tilde{A}^1, \ldots, \tilde{A}^N)$ to reduce expected emissions, while minimising social costs, given that the firms act rationally. The allowances process is announced at time $t = 0$, where for $t > 0$, the distribution of the random variables is known. At time $t = s$, the realisation $\tilde{A}_s^i(\omega)$ is known, since it is adapted. The policy is assumed to be time-consistent, such that there is no deviation from the plan that is announced. We need to have, to reduce expected emissions given by (1.29), that

$$\mathbb{E}\left[\sum_{i=1}^N E_T^{i,\alpha^i}\right] = \rho N\left(\bar{E}_0 + T\bar{\mu}\right), \quad \mathbb{E}\left[\sum_{i=1}^N G_T^{i,\alpha^i}\right] = \rho \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right), \tag{1.44}$$

respectively in both cases. This results in $100(1-\rho)$ percentage reduction compared to the BAU case.

Remark 1.8. *Note that in the expressions above an equal sign is written. From a climate point of view, the constraint*

$$\mathbb{E}\left[\sum_{i=1}^N E_T^{i,\alpha^i}\right] \leqslant \rho N\left(\bar{E}_0 + T\bar{\mu}\right), \quad \mathbb{E}\left[\sum_{i=1}^N G_T^{i,\alpha^i}\right] \leqslant \rho \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right),$$

would also be sufficient. Then there are only fewer emissions in the controlled case than the regulator asks for. However, from a mathematical point of view, it could be argued that this would lead to the same constraint as written above, as it is always more expensive to emit less. More importantly, the equality appears necessary to be able to derive analytical solutions.

The social costs are defined as the sum over all individual costs of the firm, the total costs in the system. To minimise the social costs, the infimum over the specific allocations $A \in \mathscr{S}^N$ will be taken. The specific problem of the regulator of minimising social costs while reducing the BAU emissions modelled by a Brownian motion is given by

$$\inf_{A\in\mathscr{S}^N} \mathbb{E}\left[\sum_{i=1}^N \int_0^T c_i\left(\alpha_t^i\right) + P_t \beta_t^i + \frac{1}{2\nu}(\beta_t^i)^2 \mathrm{d}t + \lambda\left(X_T^{i,E}\right)^2\right], \quad \mathbb{E}\left[\sum_{i=1}^N E_T^{i,\alpha^i}\right] = \rho N(\bar{E}_0 + T\bar{\mu}). \tag{1.45}$$

In the case of the GBM, we see that (1.37) depends on $\tilde{A}$, hence we will minimise over $\tilde{A} \in \mathscr{S}^N$. Then, the specific problem of the regulator becomes

$$\inf_{\tilde{A}\in\mathscr{S}^N} \mathbb{E}\left[\sum_{i=1}^N \int_0^T c_i\left(\alpha_t^i\right) + P_t \beta_t^i + \frac{1}{2\nu}(\beta_t^i)^2 \mathrm{d}t + \lambda\left(X_T^{i,G}\right)^2\right], \quad \mathbb{E}\left[\sum_{i=1}^N G_T^{i,\alpha^i}\right] = \rho \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right). \tag{1.46}$$

This ends the chapter on the mathematical assumptions. We are ready to continue to the calculation of the optimal dynamic policy for both choices of model of the BAU emissions.

2 | The Brownian Framework

In this chapter, we determine the optimal dynamic policy, when the BAU emissions are modelled as an arithmetic Brownian motion. This is done under the assumption that there is a market without frictions. That is, the regulator desires to solve (1.45), where $\nu \to \infty$. The case of a market with frictions, where the regulator is solving (1.45) for $\nu < \infty$, can be found in Appendix B.

We recall that the derivation of the optimal dynamic policy will proceed through three different steps, corresponding to the two steps of backward induction, leading to the Stackelberg equilibrium. First, given an allocation and a market price of permits, the firm will minimise its corresponding cost, given in (1.40) with $\nu \to \infty$. Afterwards, the corresponding market price of the market equilibrium can be deduced, given an allocation of the regulator. Assuming that the firms are acting rationally, the regulator will then solve for the optimal dynamic allocation. These three steps are crucial to the derivation, forming the foundation for the structure of the sections in this chapter.

This chapter is based on Sections 3, 4 and 5 of [AB23].

2.1 Single firm optimisation

In this section, given an allocation A^i satisfying Definition 1.6 and market price $P \in \mathscr{A}$, the cost minimisation for every firm i will be solved. Before we will continue to this derivation, we need to introduce some variables and functions.

Definition 2.1. *Let*

$$M_t^i := \mathbb{E}\left[A_T^i \middle| \mathscr{F}_t\right], \quad R_t^i := \mathbb{E}\left[A_T^i - A_t^i \middle| \mathscr{F}_t\right] = M_t^i - A_t^i, \tag{2.1}$$

$$f(t) := \frac{2\lambda}{1 + 2\lambda\bar{\eta}(T-t)}, \tag{2.2}$$

where A_t^i is the net allocation at time t, given by (1.34)*. In the last expression, $\bar{\eta}$ is the average over the firms adjustment cost of abatement and λ is the common terminal penalty parameter.*

That is, M_t^i is the conditional expectation of the cumulative net allocation A at time T of firm i. We note that allocations with the same cumulative net value at time T, but different values in the interval $[0, T)$, lead to the same expression for M^i. From the definition of R_t^i, we see that it is the difference between the conditional expectation of the total allowances at time T and the realised allowances at time t. The function $f(t)$ is the same for every firm, since it does not depend on the index i.

First of all, the cost function that a firm will minimise in this case, as given in (1.40), with $X_T^i = X_T^{i,E}$, will be determined.

Proposition 2.1. *The cost functional in the case that $\nu \to \infty$ is given by $\tilde{\mathscr{J}}^{i,E}(\alpha^i, \beta^i) : \mathscr{A}^2 \to \mathbb{R}$*

$$\tilde{\mathscr{J}}^{i,E}\left(\alpha^i, \beta^i\right) := \mathbb{E}\left[\int_0^T c_i(\alpha_t^i) + P_t\beta_t^i dt + \lambda\left(X_T^{i,E}\right)^2\right]. \tag{2.3}$$

Proof. We need to calculate the limit of $\mathscr{J}^{i,E}(\alpha^i, \beta^i)$ of (1.40) as $\nu \to \infty$. Since almost all terms do not depend on ν, this can be written as follows

$$\begin{aligned}
\lim_{\nu\to\infty} \mathscr{J}^{i,E}\left(\alpha^i, \beta^i\right) &= \lim_{\nu\to\infty} \mathbb{E}\left[\int_0^T c_i\left(\alpha_t^i\right) + P_t\beta_t^i + \frac{1}{2\nu}(\beta_t^i)^2 \mathrm{d}t + \lambda\left(X_T^{i,E}\right)^2\right] \\
&= \mathbb{E}\left[\int_0^T c_i\left(\alpha_t^i\right) + P_t\beta_t^i + \lambda\left(X_T^{i,E}\right)^2\right] + \lim_{\nu\to\infty} \frac{1}{2\nu}\mathbb{E}\left[\int_0^T \left(\beta_t^i\right)^2 \mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T c_i\left(\alpha_t^i\right) + P_t\beta_t^i + \lambda\left(X_T^{i,E}\right)^2\right] =: \tilde{\mathscr{J}}^{i,E}\left(\alpha^i, \beta^i\right).
\end{aligned}$$

□

This functional needs to be minimised with respect to $(\alpha^i, \beta^i) \in \mathscr{A}^2$. That is, we look for optimal $(\hat{\alpha}^i, \hat{\beta}^i)$ such that

$$\inf_{(\alpha^i,\beta^i)\in\mathscr{A}^2} \tilde{\mathscr{J}}^{i,E}(\alpha^i, \beta^i) = \tilde{\mathscr{J}}^{i,E}\left(\hat{\alpha}^i, \hat{\beta}^i\right). \tag{2.4}$$

First, we need to know whether this stochastic control problem is solvable. Then, we need to investigate how these solutions can be found. The first goal of this section is to prove the following theorem.

Theorem 2.2. *Let $\tilde{\mathscr{J}}^{i,E}$ be given as in* (2.3). *Then, there exists at least one solution to* (2.4).

To prove this result, we will use Proposition A.29. Because it is of importance, it is also stated below.

Proposition A.28 [ET99, pg. 35] *Let X be a Hilbert space, $A \subset X$ a closed convex subspace of this space and $F : A \to \mathbb{R}$ a functional that is convex, continuous and coercive. Then,*

$$\inf_{x\in A} F(x),$$

has at least one solution. It has a unique solution, if F is strictly convex.

Note that this proposition is in analogy with (a part of) the Extreme Value Theorem on the real line.

In this case, we work with $X = L^2 \times L^2$, which is a Hilbert space, by a similar argument as in Proposition 1.5, since the norms on $\mathscr{A}^2$ and $L^2 \times L^2$ coincide. Additionally, $A = \mathscr{A}^2$, a Hilbert space itself by the same proposition. From this, we can easily prove that $\mathscr{A}^2$ is a closed, convex subspace of X, which is done in the next lemma.

Lemma 2.3. *The space $\mathscr{A}^2$ is a closed, convex subspace of $L^2 \times L^2$.*

Proof. By Proposition 1.4, we have that $\mathscr{A}$ is a closed, convex subspace of L^2. As we defined

$$\mathscr{A}^2 = \mathscr{A} \times \mathscr{A},$$

it follows by basic topology that $\mathscr{A}^2$ is a closed, convex subspace of $L^2 \times L^2$. □

Next to this, we need that $\tilde{\mathscr{J}}^{i,E}$ is a coercive, continuous, convex functional over $\mathscr{A}^2$. Continuity will follow from Fréchet differentiability and Proposition A.27. We start by proving that the functional is coercive. For this, we will use Assumptions (1.11) and (1.24), on β^i and P, respectively.

Proposition 2.4 (Coerciveness of the cost functional). *The functional $\tilde{\mathscr{J}}^{i,E}$ is coercive in the controls (α^i, β^i).*

Proof. From Definition A.27, we need to prove that

$$\left|\tilde{\mathscr{J}}^{i,E}(\alpha^i,\beta^i)\right| \to \infty, \quad \text{if} \quad \left\|\left(\alpha^i,\beta^i\right)\right\|_{\mathscr{A}^2} \to \infty.$$

It is sufficient to show that $\tilde{\mathscr{J}}^{i,E}(\alpha^i,\beta^i) \to \infty$. Let $\|(\alpha^i,\beta^i)\|_{\mathscr{A}^2} \to \infty$. Recall that the norm on $\mathscr{A}^2$ is given by

$$\left\|\left(\alpha^i,\beta^i\right)\right\|_{\mathscr{A}^2} = \sqrt{\|\alpha^i\|^2 + \|\beta^i\|^2}.$$

That is, if $\|(\alpha^i,\beta^i)\|_{\mathscr{A}^2} \to \infty$, we have either $\|\alpha^i\|^2 \to \infty$,$\|\beta^i\|^2 \to \infty$, or both. By assumption (1.11) it follows that

$$\left\|\beta^i\right\|^2 = \mathbb{E}\left[\int_0^T \left(\beta_t^i\right)^2 \mathrm{d}t\right] \leqslant \mathbb{E}\left[\int_0^T K^2 \mathrm{d}t\right] = K^2 T < \infty.$$

Hence, from $\|(\alpha^i,\beta^i)\| \to \infty$, it follows that we should have $\|\alpha^i\|^2 \to \infty$. We can thus conclude that $\|\alpha^i\| \to \infty$ and $\|\beta^i\| < \infty$. Furthermore, by (1.24), we obtain

$$\begin{aligned}\tilde{\mathscr{J}}^{i,E}\left(\alpha^i,\beta^i\right) &= \mathbb{E}\left[\int_0^T h_i\alpha_t^i + \frac{(\alpha_t^i)^2}{2\eta_i} + P_t\beta_t^i \mathrm{d}t + \lambda\left(X_T^{i,E}\right)^2\right] \\ &\geqslant \mathbb{E}\left[\int_0^T \frac{(\alpha_t^i)^2}{2\eta_i} + h_i\alpha_t^i + P_t\beta_t^i\mathrm{d}t\right] \geqslant \mathbb{E}\left[\int_0^T \frac{(\alpha_t^i)^2}{2\eta_i} + h_i\alpha_t^i\mathrm{d}t\right] - KCT,\end{aligned}$$

where $K, C \in \mathbb{R}$ are fixed, by the assumptions on β^i and P.

If it can be argued that the expectation term goes to infinity when $\|\alpha^i\| \to \infty$, the desired result follows. Indeed, by completing the square, it holds that

$$\mathbb{E}\left[\int_0^T \frac{(\alpha_t^i)^2}{2\eta_i} + h_i\alpha_t^i\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T \left(\frac{\alpha_t^i}{\sqrt{2\eta_i}} + \frac{h_i\sqrt{2\eta_i}}{2}\right)^2 \mathrm{d}t\right] - \frac{1}{2}\eta_i h_i^2 T.$$

As $\|\alpha^i\|^2 \to \infty$ implies that $\|\alpha^i + c\|^2 \to \infty$, for $c \in \mathbb{R}$, it follows that

$$\begin{aligned}\tilde{\mathscr{J}}^{i,E}\left(\alpha^i,\beta^i\right) &\geqslant \mathbb{E}\left[\int_0^T \left(\frac{\alpha_t^i}{\sqrt{2\eta_i}} + \frac{h_i\sqrt{2\eta_i}}{2}\right)^2 \mathrm{d}t\right] - \frac{1}{2}\eta_i h_i^2 T - KCT \\ &= \left\|\frac{\alpha^i}{\sqrt{2\eta_i}} + \frac{h_i\sqrt{2\eta_i}}{2}\right\| - \frac{1}{2}\eta_i h_i^2 T - KCT \to \infty,\end{aligned}$$

as $||\alpha^i|| \to \infty$. From this, it can be concluded that

$$\lim_{||(\alpha^i,\beta^i)||_{\mathscr{A}^2} \to \infty} \tilde{\mathscr{J}}^{i,E}(\alpha^i,\beta^i) = \infty.$$

Hence, $\tilde{\mathscr{J}}^{i,E}(\alpha^i,\beta^i)$ is coercive in (α^i,β^i). □

Next, we will show that the cost functional $\tilde{\mathscr{J}}^{i,E}$ is convex in (α^i,β^i). To achieve this, the cost functional will be split up in two parts

$$\tilde{C}^i(\alpha^i,\beta^i) = \mathbb{E}\left[\int_0^T h_i\alpha_t^i + \frac{(\alpha_t^i)^2}{2\eta_i} + P_t\beta_t^i\mathrm{d}t\right], \quad F^{i,E}(\alpha^i,\beta^i) = \mathbb{E}\left[\lambda\left(X_T^{i,E}\right)^2\right], \tag{2.5}$$

such that $\tilde{\mathscr{J}}^{i,E} = \tilde{C}^i + F^{i,E}$. We will show that both, $F^{i,E}$ and $\tilde{C}^i$, are convex, from which we can conclude that the whole function $\tilde{\mathscr{J}}^{i,E}$ is convex.

Proposition 2.5. *The functional* $F^{i,E}(\alpha^i,\beta^i) = \mathbb{E}\left[\lambda\left(X_T^{i,E}\left(\alpha^i,\beta^i\right)\right)^2\right]$ *is convex in* (α^i,β^i).

Proof. First, the term $\left(X_T^{i,E}(\alpha^i,\beta^i)\right)^2$ needs to be written out. In this expression, we are only concerned with the dependence on the controls, the remaining components can be treated as constant in this proof. Hence, we can write

$$X_T^{i,E}(\alpha^i,\beta^i) = K_T^{i,E} + \int_0^T \alpha_s^i + \beta_s^i \mathrm{d}s, \text{ with } K_T^{i,E} = A_T^i - E_0^i - \sigma_i W_T^i.$$

Squaring the above, we can write

$$F^{i,E}(\alpha^i,\beta^i) = \mathbb{E}\left[\lambda\left(X_T^{i,E}\right)^2\right] = \lambda\,\mathbb{E}\left[\left(K_T^{i,E}\right)^2\right] + 2\lambda\,\mathbb{E}\left[K_T^{i,E}\int_0^T \alpha_s^i + \beta_s^i \mathrm{d}s\right] + \lambda\,\mathbb{E}\left[\left(\int_0^T \alpha_s^i + \beta_s^i \mathrm{d}s\right)^2\right].$$

Now let $V, Y \in \mathscr{A}^2$, that is, $V = (V_1, V_2)$ and $Y = (Y_1, Y_2)$. Note that V, Y are time-dependent, but this dependence won't be made explicit. Let $\theta \in [0,1]$. Then

$$\begin{aligned}
F^{i,E}(\theta V + (1-\theta)Y) &= \lambda\,\mathbb{E}\left[\left(K_T^{i,E}\right)^2\right] + 2\lambda\,\mathbb{E}\left[K_T^{i,E}\int_0^T \theta V_1 + (1-\theta)Y_1 + \theta V_2 + (1-\theta)Y_2 \mathrm{d}s\right] \\
&\quad + \lambda\,\mathbb{E}\left[\left(\int_0^T \theta V_1 + (1-\theta)Y_1 + \theta V_2 + (1-\theta)Y_2 \mathrm{d}s\right)^2\right] \\
&= \theta\lambda\,\mathbb{E}\left[\left(K_T^{i,E}\right)^2\right] + (1-\theta)\lambda\,\mathbb{E}\left[\left(K_T^{i,E}\right)^2\right] + \theta 2\lambda\,\mathbb{E}\left[K_T^{i,E}\int_0^T V_1 + V_2 \mathrm{d}s\right] \\
&\quad + (1-\theta)2\lambda\,\mathbb{E}\left[K_T^{i,E}\int_0^T Y_1 + Y_2 \mathrm{d}s\right] + \lambda\,\mathbb{E}\left[\left(\int_0^T \theta V_1 + (1-\theta)Y_1 + \theta V_2 + (1-\theta)Y_2 \mathrm{d}s\right)^2\right].
\end{aligned}$$

Note that all the parts are already in desired form, except the last term involving the square. For this part, the following can be done;

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T \theta V_1 + (1-\theta)Y_1 + \theta V_2 + (1-\theta)Y_2 \mathrm{d}s\right)^2\right] &= \mathbb{E}\left[\left(\theta\int_0^T (V_1+V_2)\mathrm{d}s + (1-\theta)\int_0^T Y_1 + Y_2 \mathrm{d}s\right)^2\right] \\
&= \theta^2\,\mathbb{E}\left[\left(\int_0^T V_1 + V_2 \mathrm{d}s\right)^2\right] \\
&\quad + 2\theta(1-\theta)\,\mathbb{E}\left[\int_0^T (V_1+V_2)\mathrm{d}s \int_0^T (Y_1+Y_2)\mathrm{d}s\right] \\
&\quad + (1-\theta)^2\,\mathbb{E}\left[\left(\int_0^T (Y_1+Y_2)\mathrm{d}s\right)^2\right].
\end{aligned}$$

Let $A = \int_0^T (V_1+V_2)\mathrm{d}s$ and $B = \int_0^T (Y_1+Y_2)\mathrm{d}s$. By Young's inequality applied on A and B, it follows that

$$2\theta(1-\theta)\,\mathbb{E}\left[\int_0^T (V_1+V_2)\mathrm{d}s \int_0^T (Y_1+Y_2)\mathrm{d}s\right] \leqslant \theta(1-\theta)\left(\mathbb{E}\left[\left(\int_0^T (V_1+V_2)\mathrm{d}s\right)^2 + \left(\int_0^T (Y_1+Y_2)\mathrm{d}s\right)^2\right]\right). \tag{2.6}$$

Hence,

$$\mathbb{E}\left[\left(\int_0^T \theta V_1 + (1-\theta)Y_1 + \theta V_2 + (1-\theta)Y_2 \mathrm{d}s\right)^2\right] \leqslant \theta\,\mathbb{E}\left[\left(\int_0^T (V_1+V_2)\mathrm{d}s\right)^2\right] + (1-\theta)\,\mathbb{E}\left[\left(\int_0^T (Y_1+Y_2)\mathrm{d}s\right)^2\right].$$

This means that

$$F^{i,E}(\theta V+(1-\theta)Y)\leqslant \theta\lambda\mathbb{E}\left[\left(K_T^{i,E}\right)^2\right]+(1-\theta)\lambda\mathbb{E}\left[\left(K_T^{i,E}\right)^2\right]+\theta 2\lambda\mathbb{E}\left[K_T^{i,E}\int_0^T V_1+V_2\mathrm{d}s\right]$$
$$+(1-\theta)2\lambda\mathbb{E}\left[K_T^{i,E}\int_0^T Y_1+Y_2\mathrm{d}s\right]+\theta\lambda\mathbb{E}\left[\left(\int_0^T (V_1+V_2)\mathrm{d}s\right)^2\right]$$
$$+(1-\theta)\lambda\mathbb{E}\left[\left(\int_0^T (Y_1+Y_2)\mathrm{d}s\right)^2\right].$$

Indeed, the above exactly shows that

$$F^{i,E}(\theta V+(1-\theta)Y)\leqslant \theta F^{i,E}(V)+(1-\theta)F^{i,E}(Y),$$

for any $\theta\in[0,1]$. By Definition A.29, the functional $F^{i,E}$ is convex. □

Good to note is that the functional $F^{i,E}$ is not strictly convex, as in Equation (2.6) the inequality does not become strict. That is, if $X\neq Y$, then we can still have that $A=B$. Next, the convexity is proven for $\tilde{C}^i$.

Proposition 2.6. *The functional*

$$\tilde{C}^i\left(\alpha^i,\beta^i\right)=\mathbb{E}\left[\int_0^T h_i\alpha_t^i+\frac{(\alpha_t^i)^2}{2\eta_i}+P_t\beta_t^i dt\right],$$

is convex, but not strictly convex in the controls. It follows that $\tilde{\mathscr{J}}^{i,E}$ is also convex, but not strictly convex, in the controls.

Proof. Take $(V,Y)\in\mathscr{A}^2$ such that $V\neq Y$. Suppose, without loss of generality, that $V_1=Y_1$, μ a.e, so that

$$2V_2Y_2<V_2^2+Y_2^2\quad\text{and}\quad 2V_1Y_1\leqslant V_1^2+Y_1^2,\ \text{a.e.}$$

Then,

$$\tilde{C}^i(\theta V+(1-\theta)Y)=\theta\mathbb{E}\left[\int_0^T h_iV_1+P_tV_2\mathrm{d}t\right]+(1-\theta)\mathbb{E}\left[\int_0^T h_iY_1+P_tY_2\mathrm{d}t\right]$$
$$+\mathbb{E}\left[\int_0^T\frac{(\theta V_1+((1-\theta)Y_1)^2}{2\eta_i}\mathrm{d}t\right].$$

The last expression can be rewritten to

$$\mathbb{E}\left[\int_0^T\frac{(\theta V_1+(1-\theta)Y_1)^2}{2\eta_i}\right]=\theta^2\mathbb{E}\left[\int_0^T\frac{V_1^2}{2\eta_i}\mathrm{d}t\right]+2\theta(1-\theta)\mathbb{E}\left[\int_0^T\frac{V_1Y_1}{2\eta_i}\mathrm{d}t\right]+(1-\theta)^2\mathbb{E}\left[\int_0^T\frac{Y_1^2}{2\eta_i}\mathrm{d}t\right]$$
$$\leqslant\theta\mathbb{E}\left[\int_0^T\frac{V_1^2}{2\eta_i}\mathrm{d}t\right]+(1-\theta)\mathbb{E}\left[\int_0^T\frac{Y_1^2}{2\eta_i}\mathrm{d}t\right],$$

where we again used Young's inequality. Hence, it holds for all $\theta\in[0,1]$ that

$$\tilde{C}^i(\theta V+(1-\theta)Y)\leqslant\theta\tilde{C}^i(V)+(1-\theta)\tilde{C}^i(Y).$$

We can conclude that $\tilde{C}^i(\alpha^i,\beta^i)$ is convex. By Proposition 2.5, $F^{i,E}(\alpha^i,\beta^i)$ is convex in the controls. It follows that

$$\begin{aligned}\tilde{\mathscr{J}}^{i,E}(\theta V+(1-\theta)Y)&=\tilde{C}^i(\theta V+(1-\theta)Y)+F^{i,E}(\theta V+(1-\theta)Y)\\&\leqslant\theta\tilde{C}^i(V)+(1-\theta)\tilde{C}^i(Y)+\theta F^{i,E}(V)+(1-\theta)F^{i,E}(Y)\\&=\theta\left(\tilde{C}^i(V)+F^{i,E}(V)\right)+(1-\theta)\left(\tilde{C}^i(Y)+F^{i,E}(Y)\right)\\&=\theta\tilde{\mathscr{J}}^{i,E}(V)+(1-\theta)\tilde{\mathscr{J}}^{i,E}(Y),\end{aligned}$$

for all $\theta \in [0,1]$. We can conclude that $\tilde{\mathscr{J}}^{i,E}(\alpha^i,\beta^i)$ is indeed convex. It is not strictly convex as both parts are only convex themselves. $\square$

The last condition to establish, in order to satisfy the assumptions of Proposition A.29, is the continuity of the cost functional $\tilde{\mathscr{J}}^{i,E}$ with respect to (α^i,β^i). We will use the Fréchet derivative, which is sufficient here. A good candidate for the Fréchet derivative is the Gateaux derivative, which can be calculated. How this derivative is exactly derived and found, via the Gateaux derivative, can be found in Proposition A.31 in the appendix. There, it is calculated for $\nu<\infty$, but the exact same procedure works when $\nu=\infty$.

Here, we will show that the expression found is indeed the Fréchet derivative. The goal is to prove the proposition below. Note the subtle difference between $V,Z\in\mathscr{A}^2$, which implies that both V and Z are two-dimensional vectors, and $(V,Z)\in\mathscr{A}^2$. The latter implies that V and Z are scalar.

Proposition 2.7. *Let $\phi=(V,Z)\in\mathscr{A}^2$. The Fréchet derivative $\delta\tilde{\mathscr{J}}^{i,E}((\alpha^i,\beta^i);\phi)$ of $\tilde{\mathscr{J}}^{i,E}$ is given by*

$$\delta\tilde{\mathscr{J}}^{i,E}\left(\left(\alpha^i,\beta^i\right);\phi\right)=\delta\tilde{C}^i\left((\alpha^i,\beta^i);\phi\right)+\delta F^{i,E}\left((\alpha^i,\beta^i);\phi\right), \tag{2.7}$$

where

$$\delta\tilde{C}^i\left((\alpha^i,\beta^i);\phi\right)=\mathbb{E}\left[\int_0^T V_t\left(h_i+\frac{\alpha_t^i}{\eta_i}\right)+Z_tP_t dt\right],\quad \delta F^{i,E}\left((\alpha^i,\beta^i);\phi\right)=2\lambda\mathbb{E}\left[\int_0^T (V_t+Z_t)X_T^{i,E}dt\right].$$

Before this, we first prove the following lemma, which we will need to prove the proposition.

Lemma 2.8. *The operators of Proposition 2.7 are linear and continuous in $\phi\in\mathscr{A}^2$.*

Proof. We will first show that $\delta\tilde{\mathscr{J}}^{i,E}((\alpha,\beta);(V,Z))$ is linear in $(V,Z)\in\mathscr{A}$, Let $(G,H)\in\mathscr{A}^2$. Then, by the linearity of the expectation, it follows that

$$\delta\tilde{\mathscr{J}}^{i,E}\left((\alpha,\beta);(V,Z)+(G,H)\right)=\delta\tilde{\mathscr{J}}^{i,E}\left(\alpha,\beta);(V,Z)\right)+\delta\tilde{\mathscr{J}}^{i,E}\left(\alpha,\beta);(G,H)\right).$$

We can conclude that this a linear operator in $\phi\in\mathscr{A}^2$. Next, we need to show that it is continuous. By Proposition A.24, it is sufficient to show that the operators are bounded in $\phi\in\mathscr{A}^2$. For this, let us fix $(\alpha^i,\beta^i)\in\mathscr{A}^2$ and omit the index i. Then,

$$\begin{aligned}\left|\delta\tilde{\mathscr{J}}^{i,E}\left((\alpha,\beta);(V,Z)\right)\right|&=\left|\mathbb{E}\left[\int_0^T V_t\left(h+\frac{\alpha_t}{\eta}+2\lambda X_T^E\right)+Z_t\left(P_t+2\lambda X_T^E\right)\mathrm{d}t\right]\right|\\&\leqslant\left|\left\langle V,\left(h+\frac{\alpha}{\eta}+2\lambda X_T^E\right)\right\rangle+\left\langle Z,\left(P+\frac{\beta}{\nu}+2\lambda X_T^E\right)\right\rangle\right|\\&=\left|\left\langle (V,Z),\left(h+\frac{\alpha}{\eta}+2\lambda X_T^E,P+2\lambda X_T^E\right)\right\rangle_{\mathscr{A}^2}\right|\\&\leqslant\|(V,Z)\|_{\mathscr{A}^2}\left\|\left(h+\frac{\alpha}{\eta}+2\lambda X_T^E,P+2\lambda X_T^E\right)\right\|_{\mathscr{A}^2},\end{aligned}$$

where the latter step follows from the Cauchy-Schwarz inequality in $\mathscr{A}^2$. From the definition of a bounded operator, we need to show that the right hand term is finite. We can write

$$\left\|h+\frac{\alpha}{\eta}+2\lambda X_T,P+\frac{\beta}{\nu}+2\lambda X_T\right\|_{\mathscr{A}^2}=\sqrt{\left\|h+\frac{\alpha}{\eta}+2\lambda X_T\right\|^2+\left\|P+\frac{\beta}{\nu}+2\lambda X_T\right\|^2}.$$

Since $\alpha,\beta,P\in\mathscr{A}$ and X_T^E is square-integrable w.r.t $\mathbb{P}$, it follows that these norms are finite, by the inequality of Cauchy-Schwarz. We can conclude that the given operators are indeed linear and bounded, and thus continuous. $\square$

Now, we are ready to prove the Fréchet differentiability.

Proof of Proposition 2.7. Again, the index i will be omitted. We need to prove that (2.7) satisfies Definition A.25. The linearity and boundedness follow directly from Lemma 2.8. We will prove the limit separately for $\tilde{C}$ and F^E, by the triangle inequality and the non-negativity of the absolute value, it follows that this will be sufficient. First, we are proving that

$$\frac{\left|\tilde{C}(\alpha+V,\beta+Z)-\tilde{C}(\alpha,\beta)-\delta\tilde{C}\left((\alpha,\beta);\phi\right)\right|}{\|\phi\|_{\mathscr{A}^2}}\to 0, \tag{2.8}$$

as $\|\phi\|_{\mathscr{A}^2}\to 0$. When we plug in ϕ, we see that almost all terms cancel out

$$\begin{aligned}\tilde{C}(\alpha+V,\beta+Z)-\tilde{C}(\alpha,\beta)-\delta\tilde{C}\left((\alpha,\beta);\phi\right)&=\mathbb{E}\left[\int_0^T h(\alpha_t+V_t)+\frac{(\alpha_t+V_t)^2}{2\eta}+P_t(\beta_t+Z_t)\mathrm{d}t\right]\\&\quad-\mathbb{E}\left[\int_0^T h\alpha_t+\frac{(\alpha_t)^2}{2\eta}+P_t\beta_t\mathrm{d}t\right]-\mathbb{E}\left[\int_0^T V_t\left(h+\frac{\alpha_t}{\eta}\right)+Z_tP_t\mathrm{d}t\right]\\&=\frac{1}{2}\mathbb{E}\left[\int_0^T\frac{V_t^2}{\eta}\mathrm{d}t\right].\end{aligned}$$

This implies

$$\begin{aligned}0\leqslant\frac{\left|\tilde{C}(\alpha+V,\beta+Z)-\tilde{C}(\alpha,\beta)-\delta\tilde{C}\left((\alpha,\beta);\phi\right)\right|}{\|\phi\|_{\mathscr{A}^2}}&=\frac{\left|\frac{1}{2}\mathbb{E}\left[\int_0^T\frac{V_t^2}{\eta}\mathrm{d}t\right]\right|}{\|\phi\|_{\mathscr{A}^2}}\leqslant\frac{1}{2\eta}\frac{\left(||V||^2+||Z||^2\right)}{\|\phi\|_{\mathscr{A}^2}}\\&=\frac{1}{2\eta}\frac{\|\phi\|^2_{\mathscr{A}^2}}{\|\phi\|_{\mathscr{A}^2}}=\frac{1}{2\eta}\|\phi\|_{\mathscr{A}^2}\to 0,\quad\text{when }\|\phi\|_{\mathscr{A}^2}\to 0,\end{aligned}$$

by the definition of the norm and inner product in $\mathscr{A}^2$ and the fact that $\eta>0$. We can conclude that indeed $\delta C((\alpha,\beta);\phi)$ satisfies equation (2.8) and thus, it is the Fréchet derivative of $C(\alpha,\beta)$.

Next, we prove that the derivative of F^E fulfils Definition 2.8. After a lengthy computation, as in Equation (1.36), we get

$$\begin{aligned}F^E(\alpha+V,\beta+Z)-F^E(\alpha,\beta)-\delta F^E\left((\alpha,\beta);\phi\right)&=\mathbb{E}\left[\left(\int_0^T(V_t+Z_t)\,\mathrm{d}t\right)^2\right]\\=\mathbb{E}\left[\left(\int_0^T V_t\mathrm{d}t\right)^2\right]+\mathbb{E}\left[\left(\int_0^T Z_t\mathrm{d}t\right)^2\right]&+2\mathbb{E}\left[\int_0^T V_t\mathrm{d}t\int_0^T Z_t\mathrm{d}t\right].\end{aligned}$$

It holds that

$$2\int_0^T V_t\mathrm{d}t\int_0^T Z_t\mathrm{d}t\leqslant\left(\int_0^T V_t\mathrm{d}t\right)^2+\left(\int_0^T Z_t\mathrm{d}t\right)^2.$$

Furthermore, from the Cauchy-Schwarz inequality in $L^2([0,T])$, it follows that,

$$\left(\int_0^T V_t\mathrm{d}t\right)^2\leqslant T\int_0^T V_t^2\mathrm{d}t,$$

by the fact that $(V_t+Z_t)^2\geqslant 0$, for all $t\in[0,T]$. Combining these results in expectation, we obtain

$$F^E(\alpha+V,\beta+Z)-F^E(\alpha,\beta)-\delta F^E\left((\alpha,\beta);\phi\right)\leqslant 2T\mathbb{E}\left[\int_0^T V_t^2\mathrm{d}t\right]+2T\mathbb{E}\left[\int_0^T Z_t^2\mathrm{d}t\right].$$

This gives us that

$$0\leqslant\frac{|F^E(\alpha+V,\beta+Z)-F^E(\alpha,\beta)-\delta F^E((\alpha,\beta);\phi)|}{\|\phi\|_{\mathscr{A}^2}}\leqslant\frac{2T\left(\mathbb{E}\left[\int_0^T V_t^2+Z_t^2\mathrm{d}t\right)\right]}{\|\phi\|_{\mathscr{A}^2}}=2T\|\phi\|_{\mathscr{A}^2}\to 0,$$

when $\|\phi\|_{\mathscr{A}^2}\to 0$. We can conclude that $\delta F^E((\alpha,\beta);\phi)$ is the Fréchet derivative of $F^E(\alpha,\beta)$, and we can conclude that $\tilde{\mathscr{J}}^E$ is Fréchet differentiable. $\square$

Now, we can prove the continuity of the cost functional, which will follow directly from the Fréchet differentiability.

Proposition 2.9. *The cost functional $\tilde{\mathscr{J}}^{i,E}$ is continuous in the controls.*

Proof. This follows directly from Proposition 2.7 and Proposition A.27. □

Next, we are finally able to prove Theorem B.1.

Proof of Theorem B.1 . By Proposition 2.3, it follows that $\mathscr{A}^2$ is a closed, convex Hilbert space of $L^2 \times L^2$. Furthermore, from Proposition B.4, the cost functional $\tilde{\mathscr{J}}^{i,E}$ is continuous. By Proposition 2.4, $\tilde{\mathscr{J}}^{i,E}$ is coercive in the controls and in Propositions 2.5 and 2.6 it is proven that the objective functional is convex. By the aforementioned Proposition A.29, it holds that (2.4) admits at least one solution, which is the desired result. □

Note that there exists at least one solution, but this solution is not necessarily unique, as $\tilde{\mathscr{J}}^{i,E}$ is not strictly convex. The next question is how these solutions can be obtained. For this, we introduce the following proposition, which makes use of the Gateaux gradient, introduced in Appendix A.1.3. Afterwards, a proposition for the first order conditions is given.

Remark 2.1 (Notation). *For a general process Z, we will denote the Gateaux gradient in a specific point in time by ∇Z_t.*

Proposition 2.10. *The two-dimensional Gateaux gradient is given by,*

$$\nabla \mathscr{J}^{i,E}\left(\alpha^i, \beta^i\right) = \left(h_i + \frac{\alpha^i}{\eta} + 2\lambda \mathbb{E}\left[X_T^{i,E}\middle|\mathscr{F}_\cdot\right], P + 2\lambda \mathbb{E}\left[X_T^{i,E}\middle|\mathscr{F}_\cdot\right]\right). \tag{2.9}$$

The gradient in a specific point in time $t \in [0,T]$ will be denoted by $\nabla \mathscr{J}^{i,E}(\alpha^i, \beta^i)_t$, and equals, consequently,

$$\nabla \mathscr{J}^{i,E}\left(\alpha^i, \beta^i\right)_t = \left(h_i + \frac{\alpha_t^i}{\eta} + 2\lambda \mathbb{E}\left[X_T^{i,E}\middle|\mathscr{F}_t\right], P_t + 2\lambda \mathbb{E}\left[X_T^{i,E}\middle|\mathscr{F}_t\right]\right).$$

Proof. Again, the superscript i will be omitted. From the Fréchet derivatives of Proposition 2.7, the gradient of Definition A.26 can be derived, by the fact that the Fréchet derivative determines the Gateaux derivative directly. Indeed, we see that

$$\begin{aligned}
\delta \tilde{C}\left((\alpha,\beta);\phi\right) &= \mathbb{E}\left[\int_0^T V_t\left(h + \frac{\alpha_t}{\eta}\right) + Z_t P_t \mathrm{d}t\right] \\
&= \left\langle V, \left(h + \frac{\alpha}{\eta}\right)\right\rangle + \langle Z, P\rangle \\
&= \left\langle (V,Z), \left(h + \frac{\alpha}{\eta}, P\right)\right\rangle_{\mathscr{A}^2} = \left\langle \phi, \left(h + \frac{\alpha}{\eta}, P\right)\right\rangle_{\mathscr{A}^2},
\end{aligned}$$

and thus the gradient of $\tilde{C}$ is given by $\nabla \tilde{C}(\alpha,\beta) = \left(h + \frac{\alpha}{\eta}, P\right)$. In a specific point $t \in [0,T]$, this gives

$$\nabla \tilde{C}(\alpha,\beta)_t = \left(h + \frac{\alpha_t}{\eta}, P_t\right). \tag{2.10}$$

The gradient itself is a process.

The same procedure will be followed to obtain an expression for the gradient of F^E. Indeed,

$$\begin{aligned}
\delta F^E\left((\alpha,\beta);\phi\right) &= 2\lambda \mathbb{E}\left[\int_0^T X_T^E (V_t + Z_t)\,\mathrm{d}t\right] = \langle 2\lambda X_T^E, V\rangle + \langle 2\lambda X_T^E, Z\rangle \\
&= \langle 2\lambda X_T^E\left(\alpha,\beta\right)(1,1), \phi\rangle_{\mathscr{A}^2}.
\end{aligned}$$

From this, we can conclude that

$$\nabla F(\alpha,\beta) = 2\lambda X_T^E(\alpha,\beta)(1,1).$$

In a specific point $t \in [0,T]$, it still holds that

$$\nabla F(\alpha,\beta)_t = 2\lambda X_T^E(\alpha,\beta)(1,1).$$

This implies that this gradient is not adapted, as the right-hand side is only $\mathscr{F}_T$ measurable. We will rewrite the expression under the expectation sign to make it adapted. Let $Y \in \mathscr{A}$. Then, it holds

$$\begin{aligned} 2\lambda\mathbb{E}\left[\int_0^T X_T^E Y_t \mathrm{d}t\right] &= 2\lambda\mathbb{E}\left[\mathbb{E}\left[\int_0^T X_T^E Y_t \mathrm{d}t \middle| \mathscr{F}_t\right]\right] = 2\lambda\mathbb{E}\left[\int_0^T \mathbb{E}\left[X_T^E Y_t \middle| \mathscr{F}_t\right]\mathrm{d}t\right] \\ &= 2\lambda\mathbb{E}\left[\int_0^T Y_t \mathbb{E}\left[X_T^E \middle| F_t\right]\mathrm{d}t\right]. \end{aligned} \tag{2.11}$$

Here, we used Fubini's theorem for conditional expectations, and the fact that Y is an adapted process, as it is progressively measurable. Note that we can rewrite (2.11) into

$$\begin{aligned} \mathbb{E}\left[\int_0^T \left(2\lambda X_T^E - 2\lambda\mathbb{E}\left[X_T^E \middle| \mathscr{F}_t\right]\right) Y_t \mathrm{d}t\right] = 0, \\ \left\langle 2\lambda X_T^E - 2\lambda\mathbb{E}[X_T|\mathscr{F}_\cdot], Y\right\rangle = 0. \end{aligned}$$

This holds for all $Y \in \mathscr{A}$. By Lemma A.23, it follows that $2\lambda X_T = 2\lambda\mathbb{E}[X_T|\mathscr{F}_\cdot]$, μ a.e, by the equivalence classes defined in $\mathscr{A}$. This can be done for both components of the gradient. We can conclude that, indeed, the gradient considered in the point $t \in [0,T]$, is given by

$$\nabla F^E(\alpha,\beta)_t = \left(2\lambda\mathbb{E}\left[X_T^E(\alpha,\beta)\middle|\mathscr{F}_t\right], 2\lambda\mathbb{E}\left[X_T^E(\alpha,\beta)\middle|\mathscr{F}_t\right]\right).$$

By summing both parts of the gradient, we conclude that we have found the desired Gateaux gradient of the cost functional $\tilde{\mathscr{J}}^{i,E}$.

□

Next, we are ready to identify where the minimiser is located.

Proposition 2.11. *The functional* $\tilde{\mathscr{J}}^{i,E}(\alpha^i,\beta^i)$ *attains its minimum at* $(\hat{\alpha}^i,\hat{\beta}^i)$ *that satisfy*

$$\nabla\tilde{\mathscr{J}}^{i,E}\left(\hat{\alpha}^i,\hat{\beta}^i\right) = 0, \quad \mu \textit{ a.e.}$$

Proof. From Theorem B.1, we know that (1.39) has at least one solution. Now, suppose that $\nabla\tilde{\mathscr{J}}\left(\hat{\alpha}^i,\hat{\beta}^i\right) = 0$. Let $V \in \mathscr{A}$ and define $\hat{U} = (\hat{\alpha}^i,\hat{\beta}^i)$. Then,

$$\tilde{\mathscr{J}}^{i,E}(V) - \tilde{\mathscr{J}}^{i,E}\left(\hat{U}\right) - \left\langle \nabla\tilde{\mathscr{J}}^{i,E}\left(\hat{U}\right), \left(V-\hat{U}\right)\right\rangle_{\mathscr{A}^2} = \tilde{\mathscr{J}}^{i,E}(V) - \tilde{\mathscr{J}}^{i,E}\left(\hat{U}\right) - \left\langle 0, \left(V-\hat{U}\right)\right\rangle \geqslant 0,$$

by [ET99, pg. 24]. By the definition of the inner product,this implies for all $V \in \mathscr{A}^2$,

$$\tilde{\mathscr{J}}^{i,E}(V) \geqslant \tilde{\mathscr{J}}^{i,E}\left(\hat{U}\right).$$

Thus, we can conclude that $\tilde{\mathscr{J}}^{i,E}(\hat{U})$ is a minimum of the functional in question. □

With all this information, we are ready to find the minimiser of the cost functional $\tilde{\mathscr{J}}^{i,E}$. For this, we need the Gateaux gradient and solve the first order conditions. We introduce the following theorem, which is about finding the expressions of the optimal controls of (2.4). It will make use of the previously proven proposition.

Theorem 2.12. *A solution to* (2.4) *exists if and only if the market price of permits P is a martingale. In that case, the optimal abatement effort of firm i is uniquely given by*

$$\hat{\alpha}^i_t = \eta_i(P_t - h_i),$$

The optimal trading rate is not uniquely defined. Instead, define for firm i,

$$\mathbb{E}\left[\int_0^T \beta^i_t dt \middle| \mathscr{F}_t\right] = B^i_t, \quad \int_0^T \beta^i_t dt = \hat{B}^i_T, \tag{2.12}$$

with the process $(\hat{B}^i_t)$ having the following dynamics

$$d\hat{B}^i_t = -\left(\frac{1+2\lambda(T-t)\eta_i}{2\lambda} dP_t + dM^i_t - \sigma_i dW^i_t\right), \tag{2.13}$$

$$\hat{B}^i_0 = -\left(\frac{1+2\lambda\eta_i T}{2\lambda}\hat{P}_0 + M^i_0 - E^i_0 - \eta_i h_i T\right). \tag{2.14}$$

Then, we have that $\beta^i \in \mathscr{A}^2$ is optimal as long as it satisfies (2.12)*. These equations hold μ a.e.*

Proof. Again, we will omit the superscript for firm i in the notation.

The first order conditions of $\tilde{\mathscr{J}}^E$ are as given in Proposition 2.10, and reduce to

$$h + \frac{\alpha_t}{\eta} + 2\lambda\mathbb{E}\left[X^E_T \middle| \mathscr{F}_t\right] = 0, \tag{2.15}$$

$$P_t + 2\lambda\mathbb{E}\left[X^E_T \middle| \mathscr{F}_t\right] = 0, \tag{2.16}$$

as the gradient should be equated to zero. All equalities below are meant in the μ a.e. sense. First starting with (2.1), we see that it results in

$$P_t = -2\lambda\mathbb{E}\left[X^E_T \middle| \mathscr{F}_t\right].$$

Since the right-hand side is a martingale, by Proposition 1.7, this equation only works out if the market price (P_t) itself also is a martingale. When this is not the case this implies that (2.1) won't have a solution and there does not exists an equilibrium. From now on, we will thus work with the assumption that the price process (P_t) is a martingale.

Plugging the result in (2.15), we obtain

$$\hat{\alpha}_t = \eta(P_t - h). \tag{2.17}$$

Since the price process P is given exogenous, we can conclude that this value of (α^i_t) is unique ,as long as the price is determined.

With this, we hope to find some expression for $\hat{\beta}$. Plugging X^E_T in (2.1) with $\hat{\alpha}$, we obtain

$$\begin{aligned} P_t + 2\lambda\mathbb{E}\left[\int_0^T \hat{\alpha}_t + \beta_t \mathrm{d}t + A_T - \sigma W_T - E_0 \middle| \mathscr{F}_t\right] &= P_t + 2\lambda M_t + 2\lambda\mathbb{E}\left[\int_0^T \eta(P_t - h)\mathrm{d}t \middle| \mathscr{F}_t\right] \\ &+ 2\lambda\mathbb{E}\left[\int_0^T \beta_t \mathrm{d}t \middle| \mathscr{F}_t\right] - 2\lambda\sigma W_t - E_0 = 0. \end{aligned} \tag{2.18}$$

We see that the trading rate (β_t) only appears in the integral of the conditional expectation. That implies that we cannot solve directly for this process itself. Instead, we define

$$B_t = \mathbb{E}\left[\int_0^T \beta_t \mathrm{d}t \middle| \mathscr{F}_t\right].$$

First of all, we note that that β is progressively measurable, and the time integral over β is again progressively measurable, by Proposition A.6, and thus adapted. This way, it follows that

$$B_T = \int_0^T \beta_t \mathrm{d}t,$$

and that (B_t) is a martingale and in L^2 itself, as $\beta_t \in \mathscr{A}$. Now, we can rewrite (2.18) to

$$P_t + 2\lambda M_t + 2\lambda \mathbb{E}\left[\int_0^T \eta(P_t - h)\mathrm{d}t \middle| \mathscr{F}_t\right] + 2\lambda B_t - 2\lambda\sigma W_t - E_0 = 0.$$

This can be solved for B_t and gives

$$B_t = \sigma W_t + E_0 - \frac{1}{2\lambda} P_t - M_t - \mathbb{E}\left[\int_0^T \eta(P_t - h)\mathrm{d}t \middle| \mathscr{F}_t\right]. \tag{2.19}$$

We already deduced that the price P needs to be a martingale. This implies that

$$\mathbb{E}\left[\int_0^T \eta(P_s - h)\mathrm{d}s \middle| \mathscr{F}_t\right] = \int_0^t \eta(P_s - h)\mathrm{d}s + \int_t^T \eta\mathbb{E}[P_s|\mathscr{F}_t] - \eta h \mathrm{d}t = \int_0^t \eta(P_s - h)\mathrm{d}s + \eta(P_t - h)(T - t),$$

by a Fubini argument. Furthermore, since $P \in \mathscr{A}$, the time integral is again adapted. Then, (2.19) becomes

$$B_t = \sigma W_t + E_0 - \frac{1}{2\lambda} P_t - M_t - \int_0^t \eta(P_s - h)\mathrm{d}s - \eta(P_t - h)(T - t). \tag{2.20}$$

Note that for given $A, P \in \mathscr{A}$, the process B_t^i is unique. Since we are solving the backward induction of the Stackelberg game, this is indeed the case. A unique solution for β_t^i cannot be obtained, but fortunately a unique solution for B_t^i is available.

Taking $t = 0$ in (2.19), we get

$$\begin{aligned} B_0 &= E_0 - \frac{1}{2\lambda} P_0 - M_0 - \mathbb{E}\left[\int_0^T \eta(P_0 - h)\ dds\right] = E_0 - \frac{1}{2\lambda} P_0 - M_0 - \eta(P_0 - h)T, \\ \hat{B}_0 &= -\left(P_0\left(\frac{1 + 2\lambda\eta T}{2\lambda}\right) + M_0 - E_0 + \eta h T\right). \end{aligned} \tag{2.21}$$

Now we have the initial condition, it makes sense to consider the differential $\mathrm{d}B_t$. For this, we use Lemma A.20. We write

$$\begin{aligned} \mathrm{d}\mathbb{E}\left[\int_0^T \eta(P_t - h)\mathrm{d}t \middle| \mathscr{F}_t\right] &= \eta\mathrm{d}\mathbb{E}\left[\int_0^T P_t \mathrm{d}t \middle| \mathscr{F}_t\right] - \eta h \mathrm{d}\mathbb{E}\left[\int_0^T \mathrm{d}t \middle| \mathscr{F}_t\right] \\ &= \eta(T - t)\mathrm{d}P_t + \mathrm{d}(\eta h(T)) = \eta(T - t)\mathrm{d}P_t. \end{aligned}$$

Taking the differential in our original equation (2.19), we get

$$\begin{aligned} \mathrm{d}\hat{B}_t &= -\frac{1}{2\lambda}\left(\mathrm{d}P_t(1 + 2\lambda\eta(T - t)) + 2\lambda \mathrm{d}M_t - 2\lambda\sigma \mathrm{d}W_t\right) \\ &= -\left(\frac{1 + 2\lambda\eta(T - t)}{2\lambda}\mathrm{d}P_t + \mathrm{d}M_t - \sigma \mathrm{d}W_t\right). \end{aligned} \tag{2.22}$$

The equation above together with the initial condition (2.21), gives the desired dynamics for the martingale $\hat{B}$. The direct formula of (2.20) is equivalent to the differential form given above.

We can conclude that any $\beta^i \in \mathscr{A}$ that satisfies

$$\int_0^T \beta_t^i \mathrm{d}t = \hat{B}_T^i,$$

is optimal. $\square$

In the proof above, first a direct proof of $\hat{B}_t^i$ was given, afterwards the dynamics were deduced. These dynamics are necessary to be able to deduce the optimal market price of permits P in the next section. Although in Theorem B.1 only the existence of a solution is proven, uniqueness of $\hat{\alpha}^i$ and $\hat{B}^i$ can be obtained by arguments involving the Stackelberg game.

In the last part of this section, we will interpret the results obtained.

First of all, it follows from the first order condition , that in the optimum

$$X_T^{i,E} = -\frac{1}{2\lambda} P_T, \tag{2.23}$$

should hold, since $X_T^{i,E}$ is $\mathscr{F}_T$ measurable by Lemma 1.8. We will use this to derive the optimal allocation.

Let f_i' are the marginal trading costs and c_i' the marginal costs of abatement. Furthermore, again from the first order conditions (2.15) and (2.1), we can write

$$c_i'(\alpha^i)_t = h_i + \frac{\alpha_t^i}{\eta_i} = P_t = f_i'(\beta^i)_t.$$

These are derived in the Fréchet sense in Proposition A.6 for $\nu < \infty$. The procedure is the same when $\nu = \infty$. Here, we will just work with these equations. We see here that in the optimal result, the marginal abatement cost are equated to the market price P_t, which represents the marginal trading costs. Equating the market price to the marginal costs is often done in economics, when we want to find an optimum. We can conclude that our result is consistent with economic theory.

Because of the fact that we do not have a market depth parameter, the expression for $\hat{\alpha}$ in (2.17) is relatively easy. The only variable that the optimal abatement effort now depends on is the market price. If the price increases, the abatement effort will increase as well, since it is less attractive to buy more permits on the market, ceteris paribus. However, the expression for the trading rate is quite complicated.

Recall that the direct formula of $\hat{B}_t^i$ is given by

$$\hat{B}_t^i = \sigma W_t + E_0 - \frac{1}{2\lambda} P_t - M_t - \int_0^t \eta(P_s - h)\mathrm{d}s - \eta(P_t - h)(T - t).$$

From the equation above, it can be deduced that $\hat{B}_t$ decreases if the market price of permits decreases, everything else being equal. If it is more expensive to trade, firms will reduce trading. In an exchange, the abatement effort will rise to make sure the desired emission reduction can be obtained. If M_t^i, the net conditional expectation of the cumulative allowances, increases, $\hat{B}_t^i$ will decrease. This has to do with the fact that the instantaneous effort decreases, as there are more allowances expected. Furthermore, a positive economic shock will induce a rise in the integral of the trading rate, as firms can buy more, ceteris paribus. Last, a bigger initial emission level induces an increase in $\hat{B}_t$ as well, since firms need to buy more allowances to get to the same reduction, ceteris paribus.

Note that the optimal abatement effort does not depend on E_0 directly, which may be counter-intuitive. In the next section, we will see that $\hat{P}_t$ depends on this, and so does the abatement effort.

2.2 Market equilibrium

In the previous section, the optimal controls for a single firm were obtained. Here, it is time to achieve the market equilibrium, that is, the market price P for a given, fixed, net allocation A of the regulator. This market equilibrium arises from the trading in the market of permits. We will assume that such an equilibrium exists and the firms trade until an equilibrium sets in. This is needed to be able to deduce the market price in the equilibrium. It is the second step in the solution of the optimal allocation, and the end of the first part of the backward induction to find the Stackelberg equilibrium.

In the market equilibrium, the following market clearing condition is satisfied

$$\sum_{i=1}^{N} \hat{\beta}_t^i = 0, \quad \mathbb{P} \text{ a.s.} \tag{2.24}$$

for all $t \in [0, T]$. This condition can be thought of as the fact that in equilibrium, all allowances sold at some point in time $t \in [0, T]$ should also be bought by another firm at that specific point in time [FT22]. When we would consider $N = 2$ firms, it becomes clearly visible that we indeed need $\beta_1 = -\beta_2$ to make sure the allowances bought are sold by the other firm. The market clearing condition is a generalisation of this fact. Note that this implies, by Remark 1.5 that this condition also holds μ a.e.

Although no uniqueness for the trading rate $\hat{\beta}$ could be obtained, it holds that optimality for the market price $\hat{P}$ can still be deduced. This is done in the following theorem.

Theorem 2.13. *Assume that there is given an exogenous net allocation scheme $A = (A^1, \dots A^N) \in \mathscr{S}^N$. The equilibrium price is then given by, μ a.e,*

$$d\hat{P}_t = -\frac{f(t)}{N} dZ(t), \quad \hat{P}_0 = f(0)(T\bar{H} - \bar{M}_0 + \bar{E}_0). \tag{2.25}$$

Here,

$$Z(t) = \sum_{i=1}^{N} M_t^i - \sigma_i W_t^i, \quad \bar{M}_0 = \frac{1}{N} \sum_{i=1}^{N} M_0^i, \quad \bar{H} = \frac{1}{N} \sum_{i=1}^{N} \eta_i h_i, \quad \bar{E}_0 = \frac{1}{N} \sum_{i=1}^{N} E_0^i. \tag{2.26}$$

Proof. We will use that, in the equilibrium, the market clearing conditions (2.24) hold. If we integrate this over time, we get

$$\sum_{i=1}^{N} \hat{B}_T^i = \sum_{i=1}^{N} \int_0^T \beta_s^i \mathrm{d}s = 0, \quad \mathbb{P} \text{ a.s.}$$

If we now take the conditional expectation on both sides, this should stay zero. As $\mathbb{E}[\hat{B}_T^i | \mathscr{F}_t] = \hat{B}_t^i$ for all firms i, this implies

$$\sum_{i=1}^{N} \hat{B}_t^i = 0, \quad \mathbb{P} \text{ a.s.},$$

for all $t \in [0, T]$. Taking the differential on both sides, the condition

$$\sum_{i=1}^{N} \mathrm{d}\hat{B}_t^i = 0,$$

can be deduced, μ a.e. All the equations from now on hold a.e. First start with $t > 0$. Plugging in the expressions for $\hat{B}_t^i$ of Theorem 2.12, we get

$$\sum_{i=1}^{N} \mathrm{d}\hat{B}_t^i = -\sum_{i=1}^{N} \left(\frac{1 + 2\lambda(T-t)\eta_i}{2\lambda} \mathrm{d}P_t + \mathrm{d}\left(M_t^i - \sigma_i W_t^i\right) \right) = -\left(\frac{N + 2\lambda(T-t)\bar{\eta}N}{2\lambda} \right) \mathrm{d}P_t - \sum_{i=1}^{N} \mathrm{d}\left(M_t^i - \sigma_i W_t^i\right).$$

Equating the above to zero, we obtain

$$\mathrm{d}\hat{P}_t = -\frac{2\lambda}{N\left(1 + 2\lambda(T-t)\bar{\eta}\right)} \sum_{i=1}^{N} \mathrm{d}\left(M_t^i - \sigma_i W_t^i\right) = -\frac{f(t)}{N} \sum_{i=1}^{N} \mathrm{d}\left(M_t^i - \sigma_i W_t^i\right) = -\frac{f(t)}{N} \mathrm{d}Z(t). \tag{2.27}$$

For $t=0$, it follows from (2.21)

$$\sum_{i=1}^{N}\hat{B}_0^i = -\sum_{i=1}^{N}\left(\hat{P}_0\frac{1+2\lambda\eta_i T}{2\lambda}+M_0^i-\eta_i h_i T-E_0^i\right) = -\hat{P}_0\frac{N(1+2\lambda\bar{\eta}T)}{2\lambda}-N\bar{M}_0+T\sum_{i=1}^{N}\eta_i h_i+\sum_{i=1}^{N}E_0^i.$$

Equating to zero and solving for $\hat{P}_0$ results in

$$\hat{P}_0 = -\frac{2\lambda}{N(1+2\lambda\bar{\eta}T)}(N\bar{M}_0-N\bar{E}_0-TN\bar{H}) = f(0)(T\bar{H}-\bar{M}_0+\bar{E}_0). \tag{2.28}$$

Together with (2.27), the equation above gives an expression for the optimal market price. □

The optimal market price depends on the process (Z_t), which is the difference of the conditioned net allowances process and the Brownian motion. In the BAU case, the bank account could be defined, with zero controls in (1.36), as

$$X_T^{i,\mathrm{BAU},E} := A_T^i+\sigma_i W_T^i - E_0^i.$$

We see that then

$$\mathrm{d}Z_t = \sum_{i=1}^{N}\mathrm{d}(M_t^i-\sigma_i W_t^i) = \sum_{i=1}^{N}\mathrm{d}\mathbb{E}\left[A_T^i-\sigma_i W_T^i\middle|\mathscr{F}_t\right] = \sum_{i=1}^{N}\mathrm{d}\mathbb{E}\left[X_T^{i,\mathrm{BAU},E}\middle|\mathscr{F}_t\right].$$

Hence, the market price depends on the summation of the conditional expectations of the bank accounts in the BAU case. If the bank account in this scenario rises, we can see from the equation above that the market price $\hat{P}_t$ decreases, as the instantaneous demand for allowances decreases, ceteris paribus.

The initial market price depends on some parameters, the initial allocation and emission level. First of all, if $\bar{H}$ rises, the costs of abatement for every firm increase, as these depend on h_i and η_i. Then, the market price will increase as more firms will buy allowances on the market, instead of reducing their emissions directly via the abatement effort. Second, if the initial allocation rises, the market prices decreases, as there are fewer allowances needed and the supply of permits will grow. Lastly, we also see that the market price depends on the initial emission level. An increase of the initial emission level induces a decrease in the bank account, increasing the demand of allowances.

The expressions for $\hat{\alpha}$ and $\hat{B}$ stay the same, with $\hat{P}$ substituted in Equations (2.17) and (2.22). Any $\beta\in\mathscr{A}$ that satisfies

$$\hat{B}_T^i = \int_0^T \beta_t^i \mathrm{d}t,$$

is optimal. One could question whether such a trading rate $\beta\in\mathscr{A}$ even exists. This is treated in the next subsection. Before this, one last remark is made.

Remark 2.2. *In this remark, a few comparisons with [AB23] are made. Compared to Proposition 4.2 of [AB23] a factor* $1/N$ *is added in the expression for* $d\hat{P}_t$. *Furthermore, on page 94 of [AB23] it is stated that the relation,*

$$\hat{B}_T^i = \hat{B}_0^i + \int_0^T \hat{\beta}_t^i dt,,$$

holds. However, we think that the following two relations hold separately,

$$(1)\quad \hat{B}_T^i = \int_0^T \beta_t^i dt, \quad \hat{B}_t^i = \mathbb{E}\left[\hat{B}_T^i\middle|\mathscr{F}_t\right],$$

$$(2)\quad \hat{B}_t^i = \hat{B}_0^i + \int_0^t d\hat{B}_s^i,$$

where $d\hat{B}_s^i$ *is given in* (2.22).

2.2.1 Existence of a square-integrable trading rate

From Theorem 2.12, we know that the optimal trading rate $\hat{\beta}^i$ is non-unique, but there exists at least one optimal solution, by Theorem B.1. However, it may be that it satisfies the relation (2.12), but it is not square-integrable and progressively measurable. This is not desirable; therefore, we want to determine whether and when it holds true that $\beta \in \mathscr{A}$, when it fulfils relation (2.12). This subject is covered partially in both Remark 4.3 of the main paper [AB23] and the subsequent article [BZ23].

We start with the remark of [AB23], where the following observation is stated. Suppose that $\hat{B}^i_t = \hat{B}^i_0$ for all $t \in [0,T]$ and a specific firm i. Then, $\hat{B}_t$ is constant and thus square-integrable and progressively measurable. It follows that

$$\beta^i_t = \frac{\hat{B}_0}{T},$$

is optimal, for $t \in [0,T]$. Indeed, then

$$\hat{B}_T = \int_0^T \beta^i_t \mathrm{d}t = \hat{B}_0, \quad \hat{B}_t = \mathbb{E}\left[\int_0^T \beta^i_t \mathrm{d}t \middle| \mathscr{F}_t\right] = \hat{B}_0,$$

which exactly corresponds with the given relation . We will see in the optimal dynamic allocation in Section 2.3 that this corresponds with the optimal $\hat{B}$ determined there. After this, in the article, there are examples are provided in which the trading rate exists and is square-integrable, while in another case, it is not square-integrable. The Martingale Representation Theorem is used, however, it appears that the dimensions of the Brownian motion have not been properly taken into account. Furthermore, the examples are given in a fairly specific case. Therefore, for the remainder of this section, we will rely on [BZ23], since it is more general.

Based on this article, we are in the strongly regular case, as we satisfy the condition

$$\mathbb{E}\left[\int_0^T \left(\beta^i_t\right)^2 \mathrm{d}t\right] < \infty, \tag{2.29}$$

since every $\beta^i \in \mathscr{A}$. This article makes the assumption that all local martingales are continuous, which is stronger than we have in Proposition A.18. In our case, the allowances A are local martingales, but not necessarily continuous. Since the results are interesting, we still show the proposition below, but we should be careful with directly applying it to our case. Recall that we look for the control β^i such that

$$\int_0^T \hat{\beta}^i_t \mathrm{d}t = \hat{B}^i_T, \quad \text{a.s,} \tag{2.30}$$

where $\hat{B}^i_T$ is the given random variable of Theorem 2.12. An important result can be found in the proposition below.

Proposition 2.14. *[BZ23, pg. 3] Let*

$$\hat{\beta}^i_t = \frac{1}{T}\hat{B}^i_0 + \int_0^t \frac{1}{T-s} d\hat{B}^i_s, \tag{2.31}$$

for $t \in [0,T)$. Then, there exists at least one suitable solution to (2.30) *and if only if the given $\hat{\beta}^i \in \mathscr{A}$. Furthermore, the control variable $\hat{\beta}^i$ that satisfies* (2.30) *is uniquely defined, up to a modification, given by* (2.31).

(Sketch of) proof. The proof of the first implication relies on finding a minimum, employing a perturbation argument and an application of Itô's lemma. Additionally, it establishes that every β^i that fulfils (2.31) is a martingale. However, the whole proof is quite lengthy and outside the

scope of this thesis. It can be found in [BZ23, pg. 4]. It even holds that $\hat{\beta}^i$ obtained in the cited article is progressively measurable. Indeed, by Proposition A.15, we have that $(\hat{\beta}^i_t)$ is continuous and adapted, as it holds that $\hat{\beta}^i_t$ is a martingale and we assume that all martingales are continuous. By Proposition A.5, $\hat{\beta}^i_t$ is progressively measurable. Hence, $\hat{\beta}^i \in \mathscr{A}$.

Now assume that $\hat{\beta}^i \in \mathscr{A}$, then, we need to show that it satisfies (2.30). We will use Itô's lemma several times. First, we will apply it to $(T-t)\hat{\beta}^i_t$. Then, it holds for $t \in [0,T)$,

$$\begin{aligned}
&\mathrm{d}(T-t)\hat{\beta}^i_t = -\hat{\beta}^i_t \mathrm{d}t + (T-t)\mathrm{d}\hat{\beta}^i_t = -\hat{\beta}^i_t \mathrm{d}t + (T-t)\mathrm{d}\frac{1}{(T-t)}\mathrm{d}\hat{B}^i_t = -\hat{\beta}^i_t \mathrm{d}t + \mathrm{d}\hat{B}^i_t,\\
&(T-t)\hat{\beta}^i_t = T\hat{\beta}^i_0 + \hat{B}^i_t - \hat{B}^i_0 - \int_0^t \hat{\beta}^i_s \mathrm{d}s = \hat{B}^i_t - \int_0^t \hat{\beta}^i_s \mathrm{d}s.
\end{aligned}$$

by the differential form of (2.31). We can rewrite the above to

$$\frac{1}{(T-t)}\int_0^t \hat{\beta}^i_s \mathrm{d}s = -\hat{\beta}^i_t + \frac{1}{T-t}\hat{B}^i_t. \tag{2.32}$$

We can again apply Itô's lemma on the function on the right-hand side of the equation, giving for $t \in [0,T)$,

$$\frac{1}{T-t}\hat{B}^i_t = \frac{1}{T}\hat{B}^i_0 + \int_0^t \frac{1}{T-s}\mathrm{d}\hat{B}^i_s - \int_0^t \frac{\hat{B}^i_s}{(T-s)^2}\mathrm{d}s = \hat{\beta}^i_t - \int_0^t \frac{\hat{B}^i_s}{(T-s)^2}\mathrm{d}s.$$

Plugging this in (2.32), we can write

$$\int_0^t \hat{\beta}^i_s \mathrm{d}s = \frac{\int_0^t \frac{\hat{B}^i_s}{(T-s)^2}\mathrm{d}s}{\frac{1}{(T-t)}}.$$

We take a limit on both sides, to get

$$\int_0^T \hat{\beta}^i_s \mathrm{d}s = \lim_{t\to T}\int_0^t \hat{\beta}^i_s \mathrm{d}s = \lim_{t\to T}\frac{\int_0^t \frac{\hat{B}^i_s}{(T-s)^2}\mathrm{d}s}{\frac{1}{(T-t)}}. \tag{2.33}$$

By applying L'Hôpital's rule on the stochastic integral and the deterministic denominator, the right-hand side reduces exactly to $\hat{B}^i_T$, from which the desired result is obtained and indeed (2.30) holds, a.s.

Left to prove is that the representation (2.31) is unique. Suppose that there exists another martingale $\beta^{*,i} \in \mathscr{A}$ that fulfils (2.31). Then, by the martingality of $\beta^{*,i}$, we can write for $t \in (0,T]$,

$$\mathbb{E}\left[\int_0^T \hat{\beta}^i \mathrm{d}t \middle| \mathscr{F}_t\right] = \hat{B}^i_t = \mathbb{E}\left[\int_0^T \beta^{*,i}_t \mathrm{d}t \middle| \mathscr{F}_t\right] = \int_0^t \beta^{*,i}_t \mathrm{d}t + \beta^{*,i}_t (T-t),$$

which implies

$$L(t) := \int_0^t \beta^{*,i}_u - \hat{\beta}^i_u \mathrm{d}u + \left(\beta^{*,i}_t - \hat{\beta}^i_t\right)(T-t) = 0,$$

where in the last step we use that the first equation can also be employed for $\hat{\beta}^i$ itself. By taking the differential of $L(t)$ with Itô's lemma, we obtain the desired result for $t \in [0,T)$,

$$\begin{aligned}
&\mathrm{d}L(t) = \left(\beta^{*,i}_t - \hat{\beta}^i_t\right)\mathrm{d}t - \left(\beta^{*,i}_t - \hat{\beta}^i_t\right)\mathrm{d}t + (T-t)\mathrm{d}\left(\beta^{*,i}_t - \hat{\beta}^i_t\right) = (T-t)\mathrm{d}\left(\beta^{*,i}_t - \hat{\beta}^i_t\right) = 0,\\
&\mathrm{d}\left(\beta^{*,i}_t - \hat{\beta}^i_t\right) = 0, \quad \beta^{*,i}_t = \hat{\beta}^i_t + c.
\end{aligned}$$

Since the value for $t = 0$ is the same in both cases, it follows that $c = 0$ and $\hat{\beta}^i_t$ unique. □

In the rest of this thesis, we will assume that, given the process $\hat{B}^i$, $\hat{\beta}^i \in \mathscr{A}$ exists. In some cases, this value can be determined, in other cases, only the value of $\hat{B}^i$ will be available.

2.3 Optimal dynamic policy

In this section, the last step of the Stackelberg game will be handled. That is, the regulator wants to minimise her objective function, with respect to the vector of net allocations $A \in \mathscr{S}^N$, given the optimal controls for every firm and the equilibrium market price of permits. This results in the following social costs minimisation problem, corresponding to (1.45) with $v \to \infty$,

$$\inf_{A\in\mathscr{S}^N} \mathbb{E}\left[\sum_{i=1}^{N}\int_0^T h_i\hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} + \hat{\beta}_t^i\hat{P}_t\mathrm{d}t + \lambda\left(\hat{X}_T^{i,E}\right)^2\right] \text{ s.t. } \mathbb{E}\left[\sum_{i=1}^{N} E_T^{i,\hat{\alpha}^i}\right] = \rho N\left(\bar{E}_0 + T\bar{\mu}\right). \tag{2.34}$$

Since we assume that the market is in an equilibrium, the market clearing condition is satisfied. This means that

$$\mathbb{E}\left[\sum_{i=1}^{N}\int_0^T \hat{\beta}_t^i\hat{P}_t\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T\sum_{i=1}^{N}\hat{\beta}_t^i\hat{P}_t\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T\hat{P}_t\left(\sum_{i=1}^{N}\hat{\beta}_t^i\right)\mathrm{d}t\right] = \mathbb{E}\left[\int_0^T 0\,\mathrm{d}t\right] = 0.$$

With this, the optimisation problem given above reduces to

$$\inf_{A\in\mathscr{S}^N} \mathbb{E}\left[\sum_{i=1}^{N}\int_0^T h_i\hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} + \lambda\left(\hat{X}_T^{i,E}\right)^2\right] \text{ s.t. } \mathbb{E}\left[\sum_{i=1}^{N} E_T^{i,\hat{\alpha}^i}\right] = \rho N\left(\bar{E}_0 + T\bar{\mu}\right). \tag{2.35}$$

This is the objective function we are working with in this chapter. To begin, the constraint on the total emissions in the case of abatement can be reformulated as a constraint on average of the expected value of the cumulative net allocations, denoted by $\bar{M}_0$. For this, recall the definition of (M_t^i) from (2.1). The result is proven in the following proposition.

Proposition 2.15. *The constraint*

$$\mathbb{E}\left[\sum_{i=1}^{N} E_T^{i,\hat{\alpha}^i}\right] = \rho N\left(\bar{E}_0 + T\bar{\mu}\right).$$

induces that

$$\bar{M}_0 = \frac{1}{2\lambda\bar{\eta}}\left(\bar{H} + \frac{(1-\rho)\,\bar{E}_0}{T} + (1+2\lambda\bar{\eta}T)(1-\rho)\bar{\mu} - 2\lambda\bar{\eta}\rho\bar{E}_0\right) =: l(\rho),$$

Proof. Let us start with the given constraint. It holds that

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^{N} E_T^{i,\hat{\alpha}^i}\right] &= \mathbb{E}\left[\sum_{i=1}^{N} E_0^i + \int_0^T\left(\mu_i - \hat{\alpha}_s^i\right)\mathrm{d}s + \sigma_i W_T^i\right] = \sum_{i=1}^{N} E_0^i + \int_0^T \mathbb{E}\left[\mu_i - \hat{\alpha}_s^i\right]\mathrm{d}s \\ &= \sum_{i=1}^{N} E_0^i + \mu_i T - \int_0^T \mathbb{E}\left[\hat{\alpha}_s^i\right]\mathrm{d}s, \end{aligned}\tag{2.36}$$

by a Fubini argument and the fact that a Brownian motion has zero expectation. Using that $\hat{P}$ is a martingale, we obtain

$$\mathbb{E}\left[\hat{\alpha}_s^i\right] = \mathbb{E}\left[\eta_i\left(\hat{P}_s - h_i\right)\right] = \eta_i\left(\mathbb{E}\left[\hat{P}_s\right] - h_i\right) = \eta_i(\hat{P}_0 - h_i).$$

Thus, (2.36) results in

$$\mathbb{E}\left[\sum_{i=1}^{N} E_T^{i,\hat{\alpha}^i}\right] = \sum_{i=1}^{N} E_0^i + \mu_i T - \int_0^T \eta_i\left(\hat{P}_0 - h_i\right)\mathrm{d}s = N\bar{E}_0 + NT\left(\bar{\mu} - \bar{\eta}\hat{P}_0 + \bar{H}\right).$$

When we equate the outcome of the equation above to the constraint in (2.35), the following value for $\hat{P}_0$ can be deduced,

$$\begin{aligned} N\bar{E}_0 + NT(\bar{\mu} - \bar{\eta}\hat{P}_0 + \bar{H}) &= \rho N\left(\bar{E}_0 + T\bar{\mu}\right), \\ \hat{P}_0 &= \frac{1}{\bar{\eta}}\left(\bar{H} + \frac{(1-\rho)\bar{E}_0}{T} + (1-\rho)\bar{\mu}\right). \end{aligned} \tag{2.37}$$

It follows that the constraint on the total emissions results in a constraint on the initial optimal market price. Since the market price is a martingale, it has constant expectation. This means that the initial optimal market price is also the expected value, or average, of the market price. That is,

$$\mathbb{E}\left[\hat{P}_t\right] = \mathbb{E}\left[\hat{P}_0\right] = \hat{P}_0.$$

With the relation (2.37), this implies that the average optimal market price of allowances is influenced by several parameters that are involved in the optimisation problem. It is fixed by the system as long as ρ is known. In Equation (2.28), another expression for $\hat{P}_0$ was deduced. When we equate these two equations and solve for $\bar{M}_0$, we obtain

$$\begin{aligned} f(0)(T\bar{H} - \bar{M}_0 + \bar{E}_0) &= \frac{1}{\bar{\eta}}\left(\bar{H} + \frac{(1-\rho)\bar{E}_0}{T} + (1-\rho)\bar{\mu}\right), \\ \bar{M}_0 &= T\bar{H} + \bar{E}_0 - \frac{1}{\bar{\eta}f(0)}\left(\bar{H} + \frac{(1-\rho)\bar{E}_0}{T} + (1-\rho)\bar{\mu}\right). \end{aligned}$$

By plugging in $f(0)$, this can be simplified to

$$\begin{aligned} \bar{M}_0 &= -\frac{1}{2\lambda\bar{\eta}}\left(-2\lambda\bar{\eta}T\bar{H} - 2\lambda\bar{\eta}\bar{E}_0 + (1+2\lambda\bar{\eta}T)\bar{H} + (1+2\lambda\bar{\eta}T)\left(\frac{(1-\rho)\bar{E}_0}{T} + (1-\rho)\bar{\mu}\right)\right) \\ &= -\frac{1}{2\lambda\bar{\eta}}\left(\bar{H} + \frac{(1-\rho)\bar{E}_0}{T} + (1+2\lambda\bar{\eta}T)(1-\rho)\bar{\mu} - 2\lambda\bar{\eta}\rho\bar{E}_0\right) =: l(\rho). \end{aligned} \tag{2.38}$$

Indeed, the given constraint can be reformulated to a constraint on $\bar{M}_0$. □

Note that all the parameters, including $\bar{\mu}$ and $\bar{E}_0$, are given to be non-negative. The sign of $l(\rho)$ cannot be determined yet. However, when $\bar{E}_0 = 0$, it follows that $\bar{M}_0$ reduces to

$$\bar{M}_0 = -\frac{1}{2\lambda\bar{\eta}}\left(\bar{H} + (1+2\lambda\bar{\eta}T)(1-\rho)\bar{\mu}\right) < 0.$$

Then, $l(\rho) < 0$ holds for every possible value of ρ. Recall that $\bar{M}_0 = \frac{1}{N}\sum_{i=1}^N \mathbb{E}[A_T^i]$, by construction of the variable M. We can conclude that, on average, the cumulative allocation at time T is negative, when $\bar{E}_0 = 0$. As introduced in Chapter 1, negative allowances are indeed possible and can be interpreted as allowances being removed from the market, as a negative allowance can be considered as a penalty on the bank account.

From the above, we see that the condition on the total emissions at time T translates via a condition on the initial, optimal, equilibrium, market price to a condition on $\bar{M}_0$. That is, in (2.35), the constraint can be exchanged by $\bar{M}_0 = l(\rho)$. Not only this is possible, but the complete social costs minimisation problem can be minimised in terms of the martingale process $(M_t^i) = \mathbb{E}[A_T^i | \mathscr{F}_t]$, instead of the semimartingale A^i directly. The following corollary summarises this idea mathematically.

Corollary 2.16. *Let $\mathscr{M}^N$ be the space of N-tuples of square-integrable martingales where M^i belongs to, and denote a tuple of square-integrable martingales in this space by $\vec{M}$. The social cost minimisation problem of* (2.35) *can be rewritten to*

$$\inf_{\vec{M}\in\mathscr{M}^N} \mathbb{E}\left[\sum_{i=1}^N \int_0^T h_i\hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} + \lambda\left(\hat{X}_T^{i,E}\right)^2\right] \text{ s.t. } \bar{M}_0 = l(\rho). \tag{2.39}$$

Proof. The initial condition follows from Proposition 2.15. Since $\hat{\alpha}_t^i = \eta_i(\hat{P}_t - h_i)$, and

$$\hat{P}_t = \hat{P}_0 - \sum_{i=1}^{N}\int_0^T \frac{f(t)}{N}\mathrm{d}M_t^i + \sum_{i=1}^{N}\int_0^T \frac{f(t)\sigma_i}{N}\mathrm{d}W_t^i,$$

we see that $\hat{\alpha}_t^i$ only depends on A^i via M^i. Furthermore, by Equation (2.23), it follows that also $\lambda(\hat{X}_T^{i,E})$ only depends on A^i through M^i. Hence, it holds for all firms i that $\hat{\alpha}_t^i$ and $\hat{X}_T^{i,E}$ only depend on A^i through M^i. The result follows. $\square$

Our social costs problem is now rewritten to an infimum over the conditional allocations. To find the conditional allocations, even more can be said about the tuple of martingales $\vec{M} \in \mathcal{M}^N$. It follows by the Martingale Representation Theorem that every martingale $M^i \in \mathcal{M}$ can be represented as

$$M_t^i = M_0^i + \sum_{j=0}^{N}\int_0^t \gamma_s^{i,j}\mathrm{d}\tilde{B}_s^j, \tag{2.40}$$

where $\gamma^{i,j}$ is a progressively measurable stochastic process, which satisfies

$$\mathbb{E}\left[\int_0^T \left(\gamma_s^{i,j}\right)^2 \mathrm{d}s\right] < \infty,$$

for every i,j. To apply this theorem, we need that M has a cadlag modification, which is the case by Proposition A.3.

Let $\vec{\gamma}^i = \left(\gamma^{i,0},\dots,\gamma^{i,N,}\right)$. It follows by the proposition above, that, to represent martingale $M^i \in \mathcal{M}$, it is enough to identify $(M_0^i, \vec{\gamma}^i)$. This holds for every firm $i \in \{1,\dots,N\}$. Thus, finding the initial conditions M_0^i and vectors $\vec{\gamma}^i$ for every firm in the social costs minimisation problem will give us immediately expressions for $\vec{M} \in \mathcal{M}^N$.

Now, we have gathered enough information to state and prove the main theorem of this chapter.

Theorem 2.17. *The solution to the social costs minimisation problem of the regulator, in rewritten version given in Equation* (2.39)*, is given by*

$$\frac{1}{N}\sum_{i=1}^{N} M_0^i = l(\rho),$$
$$\sum_{i=1}^{N}\left(\gamma_t^{i,j} - \sigma_i\kappa_{i,j}\right) = 0, \quad \textit{for } j = 0,\dots,N, \mu \textit{ a.e.}$$

The optimal martingales $\vec{M}$ are not unique.

Proof. The first required relation for $\bar{M}_0$ follows immediately by the constraint given. To get to the solutions for γ, several mathematical arguments are needed. First, we need to plug in $\hat{\alpha}_t^i$ of (2.17) and (2.23) in the objective function. We obtain

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^{N}\int_0^T h_i\hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i}\mathrm{d}t + \lambda\left(\hat{X}_T^{i,E}\right)^2\right] &= \mathbb{E}\left[\sum_{i=1}^{N}\int_0^T h_i\eta_i\left(\hat{P}_t - h_i\right) + \frac{\left(\eta_i\left(\hat{P}_t - h_i\right)\right)^2}{2\eta_i}\mathrm{d}t + \lambda\left(-\frac{1}{2\lambda}\hat{P}_T\right)^2\right],\\ &= \sum_{i=1}^{N}\mathbb{E}\left[\int_0^T h_i\eta_i\hat{P}_t - h_i^2\eta_i\,\mathrm{d}t\right] + \sum_{i=1}^{N}\mathbb{E}\left[\int_0^T \frac{\eta_i\left(\hat{P}_t\right)^2}{2} - \eta_i h_i\hat{P}_t + \frac{\eta h_i^2}{2}\,\mathrm{d}t\right] + \frac{N}{4\lambda}\mathbb{E}\left[\hat{P}_T^2\right].\end{aligned}$$

This follows from the fact that $(\hat{P}_t)$ is a martingale in the equilibrium, so it has a constant expectation equal to $\hat{P}_0$. When interchanging the integral over time and the expectation, by a Fubini argument, it follows that

$$\sum_{i=1}^{N} \mathbb{E}\left[\int_0^T h_i\eta_i\hat{P}_t - h_i^2\eta_i \mathrm{d}t\right] = T\sum_{i=1}^{N}\left(h_i\eta_i\hat{P}_0 - h_i^2\eta_i\right),$$
$$\sum_{i=1}^{N} \mathbb{E}\left[\int_0^T \frac{\eta_i(\hat{P}_t)^2}{2} - \eta_i h_i\hat{P}_t + \frac{\eta_i h_i^2}{2}\mathrm{d}t\right] = \frac{1}{2}\sum_{i=1}^{N}\eta_i\mathbb{E}\left[\int_0^T \hat{P}_t^2\mathrm{d}t\right] - T\sum_{i=1}^{n}\left(\eta_i h_i\hat{P}_0 - \frac{\eta_i h_i^2}{2}\right).$$

Consequently,

$$\mathbb{E}\left[\sum_{i=1}^{N}\int_0^T h_i\hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} + \lambda\left(\hat{X}_T^{i,E}\right)^2\right] = \frac{1}{2}\sum_{i=1}^{N}\eta_i\mathbb{E}\left[\int_0^T \hat{P}_t^2\mathrm{d}t\right] - \frac{T}{2}\sum_{i=1}^{N}h_i^2\eta_i + \frac{N}{4\lambda}\mathbb{E}[\hat{P}_T^2]. \tag{2.41}$$

Since $\hat{P}$ is a martingale, it follows by Proposition A.18 that it has a continuous modification. If we work on this modification, we can apply Itô's lemma on $\hat{P}_t^2$. Therefore, for $t \in [0,T]$, we have

$$\hat{P}_t^2 = \hat{P}_0^2 + \int_0^T \hat{P}_t\mathrm{d}\hat{P}_t + \langle\hat{P}\rangle_t - \langle\hat{P}\rangle_0.$$

Note that the inner integral is well-defined, because $\hat{P}$ is a progressively measurable, continuous process, if we work on the aforementioned modification. By definition, $\langle\hat{P}\rangle_0 = 0$ as $\hat{P}_0$ is deterministic. Furthermore, $\int_0^T \hat{P}_t\mathrm{d}\hat{P}_t$ is a square-integrable martingale with zero expectation, by Proposition A.15 . This implies

$$\mathbb{E}\left[\hat{P}_t^2\right] = \hat{P}_0^2 + \mathbb{E}\left[\int_0^T \hat{P}_t\mathrm{d}\hat{P}_t\right] + \mathbb{E}\left[\langle\hat{P}\rangle_t\right] = \hat{P}_0^2 + \mathbb{E}\left[\langle\hat{P}\rangle_t\right].$$

This can be used to rewrite (2.41) to

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{N}\eta_i\mathbb{E}\left[\int_0^T \hat{P}_t^2\mathrm{d}t\right] - \frac{T}{2}\sum_{i=1}^{N}h_i^2\eta_i + \frac{N}{4\lambda}\mathbb{E}[\hat{P}_T^2] &= \frac{1}{2}\sum_{i=1}^{N}\eta_i\left(\int_0^T \hat{P}_0^2\mathrm{d}t + \mathbb{E}\left[\int_0^T \langle P\rangle_t\mathrm{d}t\right]\right) - \frac{T}{2}\sum_{i=1}^{N}h_i^2\eta_i \\ &+ \frac{N}{4\lambda}\left(\hat{P}_0^2 + \mathbb{E}[\langle\hat{P}\rangle_T]\right) \\ &= \frac{1}{2}\sum_{i=1}^{N}\eta_i T\hat{P}_0^2 + \sum_{i=1}^{N}\frac{\eta_i}{2}\mathbb{E}\left[\int_0^T \langle P\rangle_t\mathrm{d}t\right] - \frac{T}{2}\sum_{i=1}^{N}h_i^2\eta_i + \frac{N}{4\lambda}\hat{P}_0^2 \\ &+ \frac{N}{4\lambda}\mathbb{E}[\langle P\rangle_T] \\ &= \sum_{i=1}^{N}\left(\frac{T\eta_i}{2}\hat{P}_0^2 - \frac{1}{4\lambda}\hat{P}_0^2 - \frac{\eta_i T}{2}h_i^2\right) + \sum_{i=1}^{N}\frac{\eta_i}{2}\mathbb{E}\left[\int_0^T \langle\hat{P}\rangle_t\mathrm{d}t\right] \\ &+ \frac{N}{4\lambda}\mathbb{E}[\langle\hat{P}\rangle_T].\end{aligned}$$

By (2.28), we see that the initial market price is completely fixed by the constraint on $\bar{M}_0$ and other given, fixed, constants of the model. In the expression on the right all parameters, except for $\langle\hat{P}\rangle$, are given, or fixed by the initial constraint $\bar{M}_0 = l(\rho)$. Thus, the remaining question is what value of $\langle\hat{P}\rangle$, depending on M, makes the equation above as small as possible.

By definition, the quadratic variation process is non-negative. Since $\lambda, \eta_i > 0$ for all firms, the parts that involve the quadratic variation process are also non-negative, by the fact that the time integral of a non-negative integrand is again non-negative. This implies that the social costs above are as small as possible if and only $\langle P\rangle_t = 0$, a.s, for all $t \in [0,T]$.

Since $\hat{P}$ is a martingale itself, according to Proposition A.13, one might attempt to conclude that $\hat{P}_t = 0$, almost surely, for all $t \in [0,T]$. However, caution is required in this scenario, because $\hat{P}_0 \neq 0$. Thus, this proposition does not apply. Otherwise, in the optimum, the price process of permits P would be almost surely zero, which is not what we aim for.

From the fact that we need $\langle \hat{P} \rangle_t = 0$ for all $t \in [0,T]$, we can arrive at a condition regarding the process M. For this, it is enough to identify for every firm $(M_0^i, \vec{\gamma}^i)$. The derived expression for $\hat{P}$ in Equation (2.25) is used. In integral form, the market price $\hat{P}$ is given by

$$\hat{P}_t = \hat{P}_0 - \sum_{i=1}^N \frac{1}{N} \int_0^t f(s) \mathrm{d}\left(M_s^i - \sigma_i W_s^i\right) := \hat{P}_0 - \int_0^t f(s) \mathrm{d}\left(\bar{M}_s - \bar{W}_s\right), \tag{2.42}$$

where

$$\bar{M}_t = \frac{1}{N} \sum_{i=1}^N M_t^i, \quad \bar{W}_t = \frac{1}{N} \sum_{i=1}^N \sigma_i W_t^i.$$

From this, it follows by Itô isometry and the fact that the integrand is deterministic [Shr+04, pg. 149] that

$$P_t \sim \mathcal{N}\left(\hat{P}_0, \int_0^t f(s)^2 \mathrm{d}\langle \bar{M} - \bar{W} \rangle_s\right), \tag{2.43}$$

where $\mathcal{N}$ represents the normal distribution. By the definition of a quadratic variation of a semi-martingale, it follows that

$$\langle \hat{P} \rangle_t = \left\langle \int_0^{\cdot} f(s) \mathrm{d}(\bar{M}_s - \bar{W}_s) \right\rangle_t. \tag{2.44}$$

The expression inside the brackets is a martingale starting at zero. Hence, when we equate $\langle \hat{P} \rangle_t = 0$, we obtain, by Proposition A.13, that

$$\int_0^t f(s) \mathrm{d}(\bar{M}_s - \bar{W}_s) = 0, \tag{2.45}$$

a.s, for all $t \in [0,T]$. However, by properties of the quadratic variation, (2.44) can also be rewritten to

$$0 = \langle \hat{P} \rangle_t = \left\langle \int_0^{\cdot} f(s) \mathrm{d}(\bar{M}_s - \bar{W}_s) \right\rangle_t = \int_0^t f(s)^2 \mathrm{d}\langle \bar{M} - \bar{W} \rangle_s. \tag{2.46}$$

By Itô's isometry, it follows that

$$\mathbb{E}\left[\left(\int_0^t f(s) \mathrm{d}(\bar{M} - \bar{W})_s\right)^2\right] = \mathbb{E}\left[\int_0^t f(s)^2 \mathrm{d}\langle \bar{M} - \bar{W} \rangle_s\right] = 0.$$

Hence, the Equations (2.45) and (2.46) are equivalent. From (2.46), it follows, since $f(s)^2 > 0$, by the fact that $f(s) > 0$, that the following condition should hold to make the integral above equal to zero,

$$\langle \bar{M} - \bar{W} \rangle_t = c,$$

where $c \in \mathbb{R}$, for all $t \in [0,T]$, a.s. Here, we have used that $\bar{M} - \bar{W}$ has a continuous modification. By Definition A.15, it holds that the quadratic variation of this process is also continuous. Since $\langle \bar{M} - \bar{W} \rangle_0 = 0$, it follows directly that $c = 0$.

Note that the process $Y := (\bar{M} - \bar{W})$ is martingale, with initial value $Y_0 = \bar{M}_0 \neq 0$. The same trick as done for $\langle \hat{P} \rangle$ can be done here, using that the process M can be written as in Equation (2.40). Rewriting this at first, we get

$$\bar{M}_t = \frac{1}{N} \sum_{i=1}^N M_t^i = \frac{1}{N} \sum_{i=1}^N M_0^i + \frac{1}{N} \sum_{i=1}^N \int_0^t \sum_{j=0}^N \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j = \bar{M}_0 + \frac{1}{N} \sum_{i=1}^N \int_0^t \sum_{j=0}^N \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j.$$

By properties of the quadratic variation of a semimartingale, we get

$$\langle \bar{M} - \bar{W} \rangle_t = \left\langle \bar{M}_0 + \frac{1}{N} \sum_{i=1}^{N} \int_0^{\cdot} \sum_{j=0}^{N} \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j - \bar{W} \right\rangle_t = \left\langle \frac{1}{N} \sum_{i=1}^{N} \int_0^{\cdot} \sum_{j=0}^{N} \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j - \bar{W} \right\rangle_t = 0.$$

By Proposition A.13, this implies that the condition on the quadratic variation translates to

$$\frac{1}{N} \int_0^t \sum_{i=1}^{N} \sum_{j=0}^{N} \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j - \bar{W}_t = 0,$$

a.s, for all $t \in [0, T]$. This reduces to, a.s,

$$\begin{aligned} 0 = \frac{1}{N} \sum_{i=1}^{N} \int_0^t \sum_{j=0}^{N} \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j - \bar{W}_t &= \frac{1}{N} \sum_{i=1}^{N} \left(\int_0^t \sum_{j=0}^{N} \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j - \sigma_i W_t^i \right) \\ &= \frac{1}{N} \sum_{i=1}^{N} \left(\int_0^t \sum_{j=0}^{N} \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j - \sigma_i \left(\sum_{j=0}^{N} \kappa_{i,j} \tilde{B}_s^j \right) \right). \end{aligned} \tag{2.47}$$

We need to choose $(\vec{\gamma}^1, \dots \vec{\gamma}^N)$ such that $\langle P \rangle_t = 0$ and thus the equality above holds. We get, by writing the Brownian motion as an integral,

$$0 = \sum_{i=1}^{N} \sum_{j=0}^{N} \int_0^t \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j - \sigma_i \left(\sum_{j=0}^{N} \kappa_{i,j} \mathrm{d}\tilde{B}_s^j \right) = \sum_{i=1}^{N} \sum_{j=0}^{N} \int_0^t \left(\gamma_s^{i,j} - \sigma_i \kappa_{i,j} \right) \mathrm{d}\tilde{B}_s^j = \sum_{j=0}^{N} \left(\int_0^t \sum_{i=1}^{N} \left(\gamma_s^{i,j} - \sigma_i \kappa_{i,j} \right) \right) \mathrm{d}\tilde{B}_s^j, \tag{2.48}$$

a.s, for every $t \in [0, T]$. By Itô's isometry, in Proposition A.16, together with Proposition A.12 and the fact that $\tilde{B}$ is an $N+1$-dimensional, independent, Brownian motion, it follows that

$$\begin{aligned} 0 &= \mathbb{E}\left[\left(\sum_{j=0}^{N} \int_0^t \left(\sum_{i=1}^{N} \left(\gamma_s^{i,j} - \sigma_i \kappa_{i,j} \right) \right) \mathrm{d}\tilde{B}_s^j \right)^2 \right] = \mathbb{E}\left[\sum_{j=0}^{N} \int_0^t \left(\sum_{i=1}^{N} \left(\gamma_s^{i,j} - \sigma_i \kappa_{i,j} \right) \right)^2 \mathrm{d}s \right] \\ &\quad + \sum_{j=0}^{N} \sum_{\substack{k=0 \\ k \neq j}}^{N} \mathbb{E}\left[\int_0^t \left(\sum_{i=1}^{N} \gamma_s^{i,j} - \sigma_i \kappa_{i,j} \right) \left(\sum_{i=1}^{N} \gamma_s^{i,k} - \sigma_k \kappa_{i,k} \right) \mathrm{d}\left\langle \tilde{B}^j, \tilde{B}^k \right\rangle_s \right] \\ &= \mathbb{E}\left[\sum_{j=0}^{N} \int_0^t \left(\sum_{i=1}^{N} \left(\gamma_s^{i,j} - \sigma_i \kappa_{i,j} \right) \right)^2 \mathrm{d}s \right]. \end{aligned}$$

From the above and the linearity of the expectation, we recognise a summation of a norm,

$$0 = \mathbb{E}\left[\sum_{j=0}^{N} \int_0^t \left(\sum_{i=1}^{N} \left(\gamma_s^{i,j} - \sigma_i \kappa_{i,j} \right) \right)^2 \mathrm{d}s \right] = \sum_{j=0}^{N} \left\| \sum_{i=1}^{N} \left(\gamma^{i,j} - \sigma_i \kappa_{i,j} \right) \right\|^2.$$

By the non-negativity of the norm, this reduces, according to Equation 1.17 to

$$\sum_{i=1}^{N} \gamma_t^{i,j} - \sigma_i \kappa_{i,j} = 0, \tag{2.49}$$

for all $j = 0, 1, \dots, N$, μ a.e.

To minimise the social costs, we should choose every M_0^i such that $\bar{M}_0 = l(\rho)$ and $\gamma_t^{i,j}$ for all relevant i, j such that the equations of (2.49) is satisfied. Then, the square-integrable martingale $\vec{M}$ is determined. Note that the Equations in (2.49) do not have a unique solution, which implies that the optimal dynamic policy is not unique either.

We are already able to deduce an explicit formulation of the optimal abatement effort in the case of optimal allocation. By the fact that $\langle \bar{M} - \bar{W} \rangle = 0$, we have from (2.43) that $\hat{P}_t$ has zero variance for all $t \in [0, T]$. Hence, $\hat{P}_t$ is constant and equal to $\hat{P}_0$, for all $t \in [0, T]$. As a result,

$$\hat{\alpha}_t = \eta_i(\hat{P}_t - h_i) = \eta_i(\hat{P}_0 - h_i) = \hat{\alpha}_0,$$

for all $t \in (0, T]$. The abatement effort is still unique. This does not hold for the other parameters that we work with. However, the social cost minimum can be quantified in this setting. Call the corresponding costs of Equation (2.39) C^E_{opt}. Then, plugging in all the relations,

$$\begin{aligned}
C^E_{\text{opt}} &= \mathbb{E}\left[\sum_{i=1}^N \int_0^T h_i \hat{\alpha}^i_t + \frac{(\hat{\alpha}^i_t)^2}{2\eta_i} + \lambda \left(\hat{X}^{i,E}_T\right)^2\right] \\
&= \sum_{i=1}^N \left(\frac{T\eta_i}{2}\hat{P}_0^2 + \frac{1}{4\lambda}\hat{P}_0^2 - \frac{\eta_i T}{2} h_i^2\right) + \sum_{i=1}^N \frac{\eta_i}{2}\mathbb{E}\left[\int_0^T \langle \hat{P}\rangle_t \mathrm{d}t\right] + \frac{N}{4\lambda}\mathbb{E}[\langle \hat{P}\rangle_T] \\
&= \sum_{i=1}^N \left(\frac{T\eta_i}{2}\hat{P}_0^2 - \frac{\eta_i T}{2} h_i^2\right) + \frac{N}{4\lambda}\hat{P}_0^2 \\
&= -\frac{T}{2}\sum_{i=1}^N h_i^2 \eta_i + \frac{N}{4\lambda}\left(2\bar{\eta}\lambda T + 1\right)\hat{P}_0^2. \qquad (2.50)
\end{aligned}$$

These are the associated social costs in the optimal dynamic policy. □

The vectors $\vec{\gamma}^i$ are chosen such that they nullify the volatility of the price. This way, the expected value of the price process over time is fixed, by the required emission reduction of the regulator.

Not much yet can be said about the allocation specifically, we only know that Equation (2.49) should be satisfied to get optimal values for the conditional allocations M^i. Since the solutions for the process M^i are non-unique, we will only be able to identify non-unique process A^i as well, since the conditional expectation is identified non-uniquely.

Next, a specific solution of the equations (2.49) will be determined.

Example 2.1 (Standard allocation). *A particular solution of* (2.49) *can be found by setting*

$$\hat{\gamma}^{i,j}_t = \sigma_i \kappa_{i,j},$$

for all relevant i, j and $t \in [0, T]$, a.s. Furthermore, given $\bar{M}_0 = l(\rho)$, we choose $\hat{M}^i_0 = l(\rho)$. Every firm gets the same expected allocation. Based on these values, the expression of the other parameters can be deduced. For completeness, all the parameters are stated here, also those that are already deduced. The equations below hold for all $t \in [0, T]$, a.s.

(i) The price process $\hat{P}$ of permits is constant, by the fact that it has zero volatility, and given by

$$\hat{P}_t = \hat{P}_0 = \frac{f(0)}{\bar{\eta}}\left(T\bar{H} - \bar{M}_0 + \bar{E}_0\right). \qquad (2.51)$$

(ii) As a consequence, the optimal abatement effort is unique and given by

$$\hat{\alpha}^i_t = \hat{\alpha}^i_0 = \eta_i\left(\hat{P}_0 - h_i\right). \qquad (2.52)$$

(iii) With $\hat{\gamma}^{i,j}_t$ and $\hat{M}^i_0$ given above, the solutions of M^i reduce in this case to

$$\begin{aligned}
\hat{M}^i_t = \hat{M}^i_0 + \sum_{j=0}^N \int_0^t \hat{\gamma}^{i,j}_s d\tilde{B}^j_s &= \hat{M}^i_0 + \int_0^t \sum_{j=0}^N \sigma_i \kappa_{i,j} d\tilde{B}^j_s \\
&= \hat{M}^i_0 + \sigma_i \int_0^t d\left(\sum_{j=0}^N \kappa_{i,j}\tilde{B}^j_s\right) = \hat{M}^i_0 + \sigma_i \int_0^t dW^i_s \\
&= \hat{M}^i_0 + \sigma_i W^i_t.
\end{aligned}$$

(iv) *Consequently, the optimal allocation $\vec{A} \in \mathscr{S}^N$ is not unique, as the conditional expectation is not unique. By construction, we need that*

$$\hat{M}_T^i = \mathbb{E}\left[A_T^i \middle| \mathscr{F}_T\right] = A_T^i.$$

This implies that

$$\mathbb{E}\left[\hat{M}_T^i \middle| \mathscr{F}_t\right] = \hat{M}_t^i = \mathbb{E}\left[A_T^i \middle| \mathscr{F}_t\right].$$

Indeed, it holds that $\mathbb{E}[\hat{M}_t^i] = \hat{M}_0^i = l(\rho)$. *It holds that the following martingale A^i is optimal,*

$$\hat{A}_t^i = l(\rho) + \sigma_i W_t^i = \hat{M}_t^i.$$

Then, the shifted allocation $\tilde{A}^i$ is given by

$$\tilde{A}_t^i = \hat{A}_t^i + \mu_i t = l(\rho) + \sigma_i W_t^i + \mu_i t.$$

It is immediate that this corresponds with our formulation of $\hat{M}_t^i$.

(v) *Recall that the trading rate $\hat{\beta}^i$ was already determined to be non-unique. In this example, the trading rate can be determined. It holds that*

$$\begin{aligned}
\hat{B}_t^i &= \sigma_i W_t^i + E_0^i - \frac{1}{2\lambda}\hat{P}_t - \hat{M}_t - \int_0^t \eta_i \left(\hat{P}_s - h\right) ds - \eta_i \left(\hat{P}_t - h_i\right)(T-t) \\
&= \sigma_i W_t^i + E_0^i - \frac{1}{2\lambda}\hat{P}_0 - (\hat{M}_0^i + \sigma_i W_t^i) - \eta_i \left(\hat{P}_0 - h_i\right) t - \eta_i \left(\hat{P}_0 - h_i\right)(T-t) \\
&= E_0^i - \frac{1}{2\lambda}\hat{P}_0 - \hat{M}_0^i - \eta_i \left(\hat{P}_0 - h_i\right) T = -\left(\bar{M}_0 - E_0^i + \frac{1 + 2\lambda\eta_i T}{2\lambda}\hat{P}_0 - \eta_i h_i T\right) = \hat{B}_0^i.
\end{aligned} \tag{2.53}$$

As this is non-random, we see that we can choose

$$\hat{\beta}_t^i = \frac{\hat{B}_0^i}{T},$$

which coincides with the beginning of Subsection 2.2.1.

(vi) *The social costs $C_{opt}^{i,E}$ correspond with those given in the theorem, and equal*

$$C_{opt}^E = -\frac{T}{2}\sum_{i=1}^N h_i^2 \eta_i + \frac{N}{4\lambda}\left(2\bar{\eta}\lambda T + 1\right)\hat{P}_0^2.$$

(vii) *With all the values as above, the bank account $(\hat{X}_t^{i,E})$ reduces to*

$$\begin{aligned}
\hat{X}_t^{i,E} &= \hat{A}_t^i + \int_0^t \hat{\alpha}_s^i + \hat{\beta}_s^i ds - \sigma_i W_t^i - E_0^i \\
&= l(\rho) + \sigma_i W_t^i + \hat{\alpha}_0^i t + \hat{\beta}_0^i t - \sigma_i W_t^i - E_0^i \\
&= l(\rho) + \hat{\alpha}_0^i t + \hat{\beta}_0^i t - E_0^i.
\end{aligned}$$

In this example, the volatility of the price is tackled by allocating volatility of every economic shock, consisting of the common economic shock and the shocks per firm. From Equation (2.52) it appears that the firms will have positive abatement effort, when $\hat{P}_0 > 0$. Since $\bar{M}_0 < 0$, when $\bar{E}_0 = 0$, it follows from (2.51) that indeed $\hat{P}_0 > 0$. In the other cases, this cannot be directly concluded.Furthermore, $\hat{M}_t^i$ and $\tilde{A}_t^i$ are positive for $t \in (0, T]$ as long as the drift of the BAU emissions is of higher order than the volatility, the same reasoning why the BAU emissions are assumed positive. When the initial allocation is negative, we could say that the regulator first distributes a penalty on the bank account, and afterwards allocates true permits for $t \in (0, T]$. If the drift is not of higher order than the volatility, there would be an initial penalty on the bank account, and then also only negative allowances afterwards. This is not realistic. In the GBM case, such an assumption does not need to be made however, which we will see in Chapter 3.

Since the allocations are cumulative, we recognise

$$\tilde{A}_t^i = l(\rho) + E_t^i - E_0^i,$$

the BAU emissions appear in the equation of the allocation. This is exactly the quantity that is distributed by the regulator for time $t \in (0, T]$, if the drift is of higher order than the volatility.

Last, a few comments can be made about the trading rate $\hat{\beta}_t^i$, under the assumption that $E_0^i = \bar{E}_0$, all firms have the same initial emission level. Then, (2.53) can be further simplified to

$$\begin{aligned}
\hat{\beta}_t^i &= -\frac{1}{T}\left(l(\rho) - E_0^i + \frac{1+2\lambda\eta_i T}{2\lambda}\hat{P}_0 - \eta_i h_i T\right) \\
&= -\frac{1}{T}\left(l(\rho) - E_0^i + \frac{1+2\lambda\eta_i T}{2\lambda}\left(\frac{2\lambda}{1+2\lambda\bar{\eta}T}\left(T\bar{H} + \bar{E}_0 - l(\rho)\right)\right) - \eta_i h_i T\right) \\
&= -\frac{1}{T}\left(\frac{2\lambda(\bar{\eta}-\eta_i)T}{1+2\lambda\bar{\eta}T}\left(l(\rho) - \bar{E}_0\right) + T\bar{H}\left(\frac{1+2\lambda\eta_i T}{1+2\lambda\bar{\eta}T} - \frac{h_i\eta_i}{\bar{H}}\right)\right).
\end{aligned}$$

When $\eta_i = \bar{\eta}$, but $h_i \neq \bar{h}$, that is, every firm has identical flexibility parameter, we see that $\hat{\beta}_t^i \geqslant 0$ when

$$\begin{aligned}
-\bar{\eta}\bar{h} + h_i\bar{\eta} &\geqslant 0, \\
h_i &\geqslant \bar{h}.
\end{aligned}$$

The firms that satisfy this condition will buy allowances, since they have a positive trading rate. This is reasonable, as these firms have higher abatement costs than the firms for which $h_i < \bar{h}$. Thus, it is cheaper for them to buy extra allowances. The firms that do not satisfy this condition will sell permits.

When $\eta_i = \bar{\eta}$ and $h_i = \bar{h}$ for all firms, it follows easily that $\hat{\beta}_t^i = 0$ for all firms and all $t \in [0, T]$. Then, the only non-negative control variable is the abatement effort. Since every firm has the same abatement costs and marginal abatement costs, it is not beneficial to trade the permits such that the abatement takes place where it is the cheapest.

This chapter is ended with one last remark.

Remark 2.3. *Equation* (2.49) *is more general than obtained in Section 5 of [AB23], since we work with a more general correlation structure. This structure only appears to be important in the last part of the proof of the previous section, since before that we only look at the correlated Brownian motion* (W). *If we plug in* (1.5) *of the previous chapter, the relevant equations become*

$$\sum_{i=1}^{N} \gamma_t^{i,0} - \sigma_i \kappa_i = 0, \quad \sum_{i=1}^{N} \gamma_t^{i,j} - \sigma_j \sqrt{1-\kappa_j^2} = 0, \tag{2.54}$$

μ a.e, for all $t \in [0, T]$ and $j \in \{0, \ldots, N\}$. This exactly corresponds with the solutions obtained in [AB23]. The example of the standard allocation described before is based on the example in the same source, but extended to the full correlation case. Furthermore, there, the equations are obtained for all $t \in [0, T]$, however, it is not clearly mentioned in what sense. The equations here are obtained $\mu = \mathbb{P} \times \lambda^1$ almost everywhere.

3 | Modelling BAU Emissions with GBM

In this chapter, the optimal allocation in the case where the BAU emissions are modelled as a Geometric Brownian motion, is derived. Furthermore, the solutions of this chapter will be compared theoretically to the solutions obtained in the Brownian framework. A numerical comparison can be found in Section 4.3. The content of this chapter is novel and extends the Brownian framework introduced in [AB23]. The main advantage and motivation of this approach is that we do not need to make any assumptions on the drift and the volatility to get non-negative cumulative BAU emissions, as already discussed in Chapter 1.

There are several similarities in the objective functions in the Brownian framework and this chapter. The main difference will appear from the Geometric Brownian motion in the bank account ($X_t^{i,G}$), instead of the Brownian motion itself. The dependence of α^i and β^i in this bank account does not change. Based on this bank account and the same objective function for the firms, given in (1.43), the optimal dynamic allocation can be found. This is deduced via the two same steps of the Stackelberg game, which consist of the firms minimisation, the market equilibrium, and the optimal dynamic allocation of the regulator. The proof is similar to the proof in the Brownian motion case. Several proofs are omitted, but the main differences are worked out in detail. Before we start, we introduce some notation.

Definition 3.1. *Let $\mathcal{M}^N$ be as in Corollary 2.16 and $\tilde{M} \in \mathcal{M}^N$, be such that $\tilde{M}_t^i = \mathbb{E}[\tilde{A}_T^i | \mathcal{F}_t]$, where $(\tilde{A}_t^i)$ is the cumulative allowances process of Definition 1.6. Additionally, we write*

$$\widetilde{M}_t = \frac{1}{N}\sum_{i=1}^{N} \tilde{M}_t^i.$$

The notation here becomes clear from the dependence on the firm i.

We are ready to start with the firms optimisation and market equilibrium. This is derived in the next section.

3.1 Firms optimisation and market equilibrium

First, the costs minimisation of a single firm will be derived. All solutions will be derived for a specific firm $i \in \{1,\dots,N\}$. This will be done under the assumption that we are in a market without frictions, that is, $\nu \to \infty$. In similar manner as in Proposition 2.1, we can arrive at the following objective functional of firm i, given by $\tilde{\mathcal{J}}^{i,G} : \mathcal{A}^2 \to \mathbb{R}$,

$$\tilde{\mathcal{J}}^{i,G}\left(\alpha^i,\beta^i\right) = \mathbb{E}\left[\int_0^T h_i\alpha_t^i + \frac{(\alpha_t^i)^2}{2\eta_i} + P_t\beta_t^i \mathrm{d}t + \lambda\left(X_T^{i,G}\right)^2\right],$$

where $(\alpha^i,\beta^i) \in \mathcal{A}^2$ and the bank account given by

$$X_T^{i,G} = \tilde{A}_T^i + \int_0^T \alpha_s^i + \beta_s^i \mathrm{d}s - E_0^i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)T + \sigma_i W_T^i\right), \tag{3.1}$$

which corresponds with Equation (1.37).

The goal of every firm i is to find minimise these costs over all possible $(\hat{\alpha}^i, \hat{\beta}^i) \in \mathscr{A}^2$, hence to find

$$\inf_{(\alpha^i,\beta^i)\in\mathscr{A}^2} \tilde{\mathscr{J}}^{i,G}\left(\alpha^i, \beta^i\right) = \tilde{\mathscr{J}}^{i,G}\left(\hat{\alpha}^i, \hat{\beta}^i\right). \tag{3.2}$$

This objective functional has many similarities with $\tilde{\mathscr{J}}^{i,E}$ in the previous chapter. Consequently, we will see that the single firm minimisation will follow the exact same procedure, as in Section 2.1. For the sake of completeness, it is also briefly presented here.

First of all, the properties of the space $\mathscr{A}^2$ do not change. Additionally, the dependence of α^i and β^i in the bank account (3.1) does not change, since the controls are not involved in the exponent. The following proposition follows almost immediately.

Proposition 3.1. *The functional $\tilde{\mathscr{J}}^{i,G}\left(\alpha^i, \beta^i\right)$ is coercive and convex in the controls.*

Proof. The coerciveness follows directly, since

$$\tilde{\mathscr{J}}^{i,G}(\alpha^i, \beta^i) \geqslant \mathbb{E}\left[\int_0^T h_i \alpha_t^i + \frac{(\alpha_t^i)^2}{2\eta_i} + P_t \beta_t^i \mathrm{d}t\right] \to \infty, \quad \text{when } \left\|\left(\alpha^i, \beta^i\right)\right\|_{\mathscr{A}^2} \to \infty,$$

by Proposition 2.4, as $\tilde{\mathscr{J}}^{i,G}$ only differs from $\tilde{\mathscr{J}}^{i,E}$ through the bank account. Note that we can write

$$\tilde{\mathscr{J}}^{i,G}(\alpha^i, \beta^i) = \tilde{C}^i(\alpha^i, \beta^i) + F^{i,G}(\alpha^i, \beta^i), \quad F^{i,G}(\alpha^i, \beta^i) = \mathbb{E}\left[\lambda \left(X_T^{i,G}\right)^2\right],$$

and $\tilde{C}^i$ given in (2.5). By Proposition 2.6 , $\tilde{C}^i(\alpha^i, \beta^i)$ is convex in the controls. Since we can write

$$X_T^{i,G} = K_T^{i,G} + \int_0^T \alpha_s^i + \beta_s^i \mathrm{d}s, \quad K_T^{i,G} = \tilde{A}_T^i + E_0^i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right) T + \sigma_i W_T^i\right) - E_0^i,$$

we follow the same reasoning as in Proposition 2.5 to prove that $F^{i,G}(\alpha^i, \beta^i)$ is convex in the controls as well. This makes $\tilde{\mathscr{J}}^{i.G}(\alpha^i, \beta^i)$ convex in the controls. Again, it is not strictly convex. □

Even more can be said, about the continuity and the Fréchet derivative.

Proposition 3.2. *The functional $\tilde{\mathscr{J}}^{i,G}(\alpha^i, \beta^i)$ is Fréchet differentiable, with the same Fréchet derivative and gradient as $\tilde{\mathscr{J}}^{i,E}(\alpha^i, \beta^i)$, given in Proposition 2.7.*

Proof. Since the dependence of the controls in the bank account is still linear, we can directly conclude that the Fréchet derivative of $F^{i,G}$ is given by,

$$\delta F^{i,G}\left(\left(\alpha^i, \beta^i\right); \phi\right) = 2\lambda \mathbb{E}\left[\int_0^T (V_t + Z_t) X_T^{i,G} \mathrm{d}t\right],$$

where $(V, Z) = \phi \in \mathscr{A}^2$. The functional $\tilde{C}^i$ has not changed at all, and the derivative is given by

$$\tilde{C}^i\left(\left(\alpha^i, \beta^i\right); \phi\right) = \mathbb{E}\left[\int_0^T V_t \left(h_i + \frac{\alpha_t^i}{\eta_i}\right) + Z_t P_t \mathrm{d}t\right].$$

Hence, the Fréchet differential of $\tilde{\mathscr{J}}^{i,G}$ coincides with that of $\tilde{\mathscr{J}}^{i,E}$, noting the slight difference in the bank account. The same holds for the gradient, as the distributional properties of the bank account, such as the adaptedness, do not change. This implies, by Proposition 2.10, that

$$\nabla \tilde{\mathscr{J}}^{i,G}(\alpha^i, \beta^i)_t = \left(h_i + \frac{\alpha_t^i}{\eta} + 2\lambda \mathbb{E}\left[X_T^{i,G}\middle|\mathscr{F}_t\right], P_t + 2\lambda \mathbb{E}\left[X_T^{i,G}\middle|\mathscr{F}_t\right]\right).$$

□

From the Fréchet differentiability, the continuity of $\tilde{\mathscr{J}}^{i,G}(\alpha^i,\beta^i)$ can be deduced, with Proposition A.27. With this, the following proposition can be deduced.

Proposition 3.3. *The stochastic control problem given in Equation* (3.2), *admits at least one solution. The solutions can be found by equating the Gateaux gradient to zero.*

Proof. With the important Proposition A.29, and Propositions 3.1 and 3.2, the optimisation problem admits at least one solution. By Proposition 2.11 of the Brownian framework, the solutions can be found by equating the gradient $\nabla\tilde{\mathscr{J}}^{i,G}(\alpha^i,\beta^i)$ to zero, μ a.e. □

With the last proposition, exact solutions can be obtained. Before delving into that, we will prove a relevant lemma, of which the proof is not difficult.

Lemma 3.4. *It holds that*

$$E_0^i \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right) = E_0^i \exp\left(\mu_i T\right) + E_0^i \sigma_i \int_0^t \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 s + \sigma_i W_s^i\right) dW_s^i.$$

Proof. This is a direct application of the two-dimensional version of Itô's lemma. Indeed, let $f^i : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f^i(t,x) := E_0^i \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i x\right).$$

Since this function twice continuously differentiable in t and x, it follows that

$$\begin{aligned}
\mathrm{d}f^i\left(t,W^i\right) &= -\frac{1}{2}\sigma_i^2 f^i\left(t,W^i\right)\mathrm{d}t + \sigma_i f^i\left(t,W^i\right)\mathrm{d}W_t^i + \frac{1}{2}\sigma_i^2 f\left(t,W^i\right)\mathrm{d}t \\
&= \sigma_i f^i\left(t,W^i\right)\mathrm{d}W_t^i = E_0^i\sigma_i \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right)\mathrm{d}W_t^i.
\end{aligned}$$

We can deduce from this that $f^i(t,W^i)$ is a martingale itself, since an Itô integral is a martingale, by Proposition A.15. Integrating both sides yields the desired relation. □

With this, we are ready to go to the following theorem.

Theorem 3.5. *There exists only a solution to* (3.2) *if and only if the market price* P *is a martingale. Then, the solutions to* (3.2) *are given by*

$$\hat{\alpha}_t^i = \eta_i(P_t - h_i),$$

and any $\beta^i \in \mathscr{A}$ *that satisfies*

$$\mathbb{E}\left[\int_0^T \beta_t^i dt \middle| \mathscr{F}_t\right] = \hat{B}_t^i, \quad \int_0^T \beta_t^i dt = \hat{B}_T^i, \tag{3.3}$$

with the process $(\hat{B}_t^i)$ *having the following dynamics*

$$\begin{aligned}
d\hat{B}_t^i &= -\left(\frac{1+2\lambda\eta_i(T-t)}{2\lambda}dP_t + d\tilde{M}_t^i - E_0^i d\left(\exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right)\right)\right), \\
\hat{B}_0^i &= -\left(\left(\frac{1+2\lambda\eta_i T}{2\lambda}\right)P_0 + \tilde{M}_0^i - E_0^i \exp\left(\mu_i T\right) - \eta_i h_i T\right).
\end{aligned} \tag{3.4}$$

These equations hold a.e.

Proof. All equations in this proof hold in the a.e sense. As indicated in Proposition 3.3, the solutions are found by equating the corresponding Gateaux gradient to zero. Hence the solution can still be found by the first order conditions (2.15) and (2.1), given by

$$h_i + \frac{\alpha_t^i}{\eta_i} + 2\lambda \mathbb{E}\left[X_T^{i,G}\middle|\mathscr{F}_t\right] = 0, \quad P_t + 2\lambda \mathbb{E}\left[X_T^{i,G}\middle|\mathscr{F}_t\right] = 0.$$

Note, first of all, that in the optimum, it holds that

$$P_t = -2\lambda \mathbb{E}\left[X_T^{i,G}\middle|\mathscr{F}_t\right].$$

This equality is only valid when P is a martingale, since the conditional expectation of a square-integrable random variable is a martingale as well. Hence, we will use that the market price is a martingale in what follows. Then, the optimal solution for α^i is given directly, by

$$\hat{\alpha}_t^i = \eta_i(P_t - h_i). \tag{3.5}$$

Since $P \in \mathscr{A}$ is given exogenously, as we are in the first step of the backward induction of the Stackelberg game, the above solution is unique.

The first order conditions cannot be solved directly for β^i. Instead, we solve them for (B_t^i), which satisfies (3.3). For the solution of B_t^i with $t \in [0, T]$, we need to solve the system of equations in the same way as before, with the bank account given in (3.1) and optimal solution of $\hat{\alpha}^i$ of (3.5). That is,

$$\begin{aligned} P_t + 2\lambda \mathbb{E}\left[X_T^{i,G}\middle|\mathscr{F}_t\right] &= P_t + 2\lambda \mathbb{E}\left[\tilde{A}_T^i + \int_0^T \hat{\alpha}_s^i + \beta_s^i \mathrm{d}s - E_0^i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)T + \sigma_i W_T^i\right)\middle|\mathscr{F}_t\right] \\ &= P_t + 2\lambda \mathbb{E}\left[\tilde{A}_T^i\middle|\mathscr{F}_t\right] + 2\lambda \mathbb{E}\left[\int_0^T \eta_i(P_s - h_i)\mathrm{d}s\middle|\mathscr{F}_t\right] + 2\lambda B_t^i \\ &\quad - 2\lambda E_0^i \mathbb{E}\left[\exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)T + \sigma_i W_T^i\right)\middle|\mathscr{F}_t\right]. \end{aligned} \tag{3.6}$$

Since (P_t) is a martingale in the optimum, it holds that

$$\mathbb{E}\left[\int_0^T \eta_i (P_s - h_i)\,\mathrm{d}s\middle|\mathscr{F}_t\right] = \int_0^t \eta_i (P_s - h_i)\,\mathrm{d}s + \eta_i (P_t - h_i)(T - t).$$

Furthermore, by the properties of a Brownian motion and the moment generating function, it holds that

$$\mathbb{E}\left[\exp\left(\sigma_i W_T^i\right)\middle|\mathscr{F}_t\right] = \exp\left(\sigma_i W_t^i\right)\mathbb{E}\left[\exp\left(\sigma_i\left(W_T^i - W_t^i\right)\right)\middle|\mathscr{F}_t\right] = \exp\left(\frac{1}{2}\sigma_i^2(T-t) + \sigma_i W_t^i\right). \tag{3.7}$$

Combining the above and equating (3.6) to zero, it follows

$$\begin{aligned} B_t^i &= -\frac{1}{2\lambda}P_t - \tilde{M}_t^i - \mathbb{E}\left[\int_0^T \eta_i(P_s - h_i)\mathrm{d}s\middle|\mathscr{F}_t\right] + E_0^i \mathbb{E}\left[\exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)T + \sigma_i W_T^i\right)\middle|\mathscr{F}_t\right], \\ &= -\frac{1}{2\lambda}P_t - \tilde{M}_t^i - \int_0^t \eta_i(P_s - h_i)\mathrm{d}s - \eta_i(P_t - h_i)(T-t) + E_0^i \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right). \end{aligned} \tag{3.8}$$

It follows that the process (B_t^i) is a martingale, since the conditional expectation of a sufficiently integrable random variable, and the price process P, are martingales . To find the optimal market price $\hat{P}$, we need the differential form of the process above. We first derive an initial condition for $t = 0$. This gives

$$\hat{B}_0^i = -\frac{1}{2\lambda}P_0 - \tilde{M}_0^i - \eta_i(P_0 - h_i)T + E_0^i \exp\left(\mu_i T\right) = -\left(P_0\left(\frac{1 + 2\lambda\eta_i T}{2\lambda}\right) + \tilde{M}_0^i - E_0^i \exp\left(\mu_i T\right) - \eta_i h_i T\right).$$

For the differential form, for $t \in (0, T]$, it holds by Lemma A.20 that

$$\mathrm{d}\mathbb{E}\left[\int_0^T \eta_i(P_s - h_i)\mathrm{d}s \middle| \mathscr{F}_t\right] = \eta_i(T-t)\mathrm{d}P_t.$$

Hence, from (3.8) it follows

$$\begin{aligned}
\mathrm{d}B_t^i &= -\frac{1}{2\lambda}\mathrm{d}P_t - \mathrm{d}\tilde{M}_t^i - \eta_i(T-t)\mathrm{d}P_t + E_0^i \mathrm{d}\left(\exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right)\right) \\
&= -\left(\frac{1+2\lambda\eta_i(T-t)}{2\lambda}\mathrm{d}P_t + \mathrm{d}\tilde{M}_t^i - E_0^i \mathrm{d}\left(\exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right)\right)\right). \qquad (3.9)
\end{aligned}$$

By Lemma 3.4, the expression for (3.9) can be written as

$$\mathrm{d}\hat{B}_t^i = -\left(\frac{1+2\lambda\eta_i(T-t)}{2\lambda}\mathrm{d}P_t + \mathrm{d}\tilde{M}_t^i - E_0^i\sigma_i \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma W_t^i\right)\mathrm{d}W_t^i\right). \qquad (3.10)$$

With $\hat{B}_0^i$, this gives a well-posed differential form. Both expressions of the differential of $(\hat{B}_t^i)$ are equivalent. Since $P \in \mathscr{A}$ and $\tilde{A}^i$ are given exogenously, the solution for $(\hat{B}_t^i)$ is unique as well. □

Next, the market equilibrium the firms are in after trading can be deduced. We will again assume that such an equilibrium exists. To find the market equilibrium, we make use of the market clearing condition.

Theorem 3.6. *The optimal market price is given by*

$$d\hat{P}_t = -\frac{f(t)}{N}\sum_{i=1}^N d\left(\tilde{M}_t^i - \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right)\right),$$

with

$$\hat{P}_0 = \frac{f(0)}{N}\left(NT\bar{H} - N\widetilde{M}_0 + \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right)\right).$$

Proof. By the market clearing condition, it holds that

$$\sum_{i=1}^N B_0^i = 0, \qquad \sum_{i=1}^N \mathrm{d}B_t^i = 0,$$

a.s, for all $t \in (0, T]$. Let us start with $t = 0$. We achieve

$$\begin{aligned}
\sum_{i=1}^N B_0^i &= -\sum_{i=1}^N \left(P_0\left(\frac{1+2\lambda\eta_i T}{2\lambda}\right) + \tilde{M}_0^i - E_0^i \exp\left(\mu_i T\right) - \eta_i h_i T\right) \\
&= P_0 \frac{N\left(1+2\lambda\bar{\eta}T\right)}{2\lambda} + N\widetilde{M}_0 - \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right) - NT\bar{H} = 0.
\end{aligned}$$

Recall the definition of $f(t)$ of Equation (2.2). Then, solving the above for P_0, we obtain

$$\begin{aligned}
\hat{P}_0 &= \frac{2\lambda}{N(1+2\lambda\bar{\eta}T)}\left(NT\bar{H} - N\widetilde{M}_0 + \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right)\right) \\
&= \frac{f(0)}{N}\left(NT\bar{H} - N\widetilde{M}_0 + \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right)\right). \qquad (3.11)
\end{aligned}$$

For $t > 0$, we get

$$\sum_{i=1}^N \mathrm{d}\hat{B}_t^i = \sum_{i=1}^N -\left(\frac{1+2\lambda\eta_i(T-t)}{2\lambda}\mathrm{d}P_t + \mathrm{d}\tilde{M}_t^i - E_0^i \exp\left(\mu_i T\right)\mathrm{d}\left(\exp\left(-\frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right)\right)\right) = 0.$$

This implies

$$\mathrm{d}\hat{P}_t = -\frac{f(t)}{N}\left(\sum_{i=1}^{N} \mathrm{d}\tilde{M}_t^i - E_0^i \mathrm{d}\left(\exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right)\right)\right).$$

By Equation (3.10), this can also be written as

$$\mathrm{d}\hat{P}_t = -\frac{f(t)}{N}\left(\sum_{i=1}^{N} \mathrm{d}\tilde{M}_t^i - E_0^i \sigma_i \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right)\mathrm{d}W_t^i\right).$$

Both expressions for the market price will be used, as they are equivalent. □

A comparison between the expressions in the Brownian framework and GBM case will now be made. It is noticeable that the formula for the abatement effort $(\hat{\alpha}_t^i)$ still corresponds exactly with the expression in the Brownian motion case, given in (2.17). The modelling of BAU emissions does not impact the abatement effort directly. However, the market price depends on the specific form of BAU emissions, so they influence each other.

The formulas for (B_t^i), for $t \in [0, T]$, look very similar to the Brownian motion case. The differences really stem from the exponent of the Geometric Brownian motion, and the fact that M_0^i has a drift term, by definition. The same can be concluded about the dynamics of the price process. Instead of the volatility times the Brownian part in the Brownian motion framework, we consider here the differential of

$$\exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right).$$

In the expression in the Brownian motion, again, the drift term is incorporated in $\bar{M}_0$. The same holds for $\hat{P}_0$. Last, the extra term in the exponent above comes from the analytical solution of the Geometric Brownian motion.

As in the Brownian scenario, we can define the bank account in the BAU case, without control efforts, as

$$X_T^{i,\mathrm{BAU},G} := \tilde{A}_T^i - E_0^i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right) T + \sigma_i W_T^i\right).$$

Then, we see by a similar argument as in Equation (3.7),

$$\begin{aligned}\sum_{i=1}^{N} \mathrm{d}\tilde{M}_t^i - E_0^i \mathrm{d}\left(\exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right)\right) &= \sum_{i=1}^{N} \mathrm{d}\mathbb{E}\left[\tilde{A}_T^i - E_0^i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right) T + \sigma_i W_T^i\right)\middle|\mathscr{F}_t\right] \\ &= \sum_{i=1}^{N} \mathrm{d}\mathbb{E}\left[X_T^{i,\mathrm{BAU},G}\middle|\mathscr{F}_t\right].\end{aligned}$$

The optimal price process depends on the conditional expectation of the bank account in the BAU case. When this bank account rises, the price decreases. For an influence of the specific variables on $(\hat{B}_t^i)$ and $(\hat{P}_t)$, we refer to the interpretations in the previous chapter.

We can conclude that controls and the market price in the Geometric Brownian motion case look similar to those in the Brownian framework, with some reasonable differences. This substantiates that the derivations in the GBM case are correct.

3.2 Optimal dynamic allocation

In this section, the optimal dynamic allocation of the regulator is derived. This is the last step of the backward induction needed to solve the Stackelberg game. We will see several similarities and differences compared to the proofs in Section 2.3. The structure of the proof will be the same.

Here, we need to find $\tilde{A} = (\tilde{A}^1, \dots \tilde{A}^N) \in \mathscr{S}^N$, such that

$$\inf_{\tilde{A}\in\mathscr{S}^N} \mathbb{E}\left[\sum_{i=1}^N \int_0^T h_i \hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} + \hat{P}_t \hat{\beta}_t^i \mathrm{d}t + \lambda\left(X_T^{i,G}\right)^2\right], \quad \mathbb{E}\left[\sum_{i=1}^N G_T^{i,\hat{\alpha}^i}\right] = \rho \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right).$$

Note the subtle difference with the Brownian framework, where the infimum is taken over the vector $A \in \mathscr{S}^N$, the net allocation, instead of $\tilde{A}$, in which no drift term is involved. By the market clearing condition, the part with the trading rate cancels out. Hence, we can rewrite the above to

$$\inf_{\tilde{A}\in\mathscr{S}^N} \mathbb{E}\left[\sum_{i=1}^N \int_0^T h_i \hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} \mathrm{d}t + \lambda\left(X_T^{i,G}\right)^2\right], \quad \mathbb{E}\left[\sum_{i=1}^N G_T^{i,\hat{\alpha}^i}\right] = \rho \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right). \tag{3.12}$$

First, an expression for the initial market price $\hat{P}_0$ and the average $\widetilde{M}_0$ will be obtained. This is needed to redefine the stochastic control problem from $\tilde{A} \in \mathscr{S}^N$ to $\tilde{M} \in \mathscr{M}^N$.

Proposition 3.7. *From the constraint on the total emissions in the system, an expression for the initial market price $\hat{P}_0$ can be deduced, where*

$$\hat{P}_0 = \frac{1}{TN\bar{\eta}}\left(TN\bar{H} + (1-\rho)\sum_{i=1}^N E_0^i \exp\left(\mu_i T\right)\right).$$

From this equation, a constraint on the average expected allocation $\widetilde{M}_0$ can be deduced, given by

$$\widetilde{M}_0 = \frac{1}{N}\sum_{i=1}^N \tilde{M}_0^i = -\frac{1}{2\lambda\bar{\eta}T}\left(T\bar{H} + \frac{1-\rho(1+2\lambda\bar{\eta}T)}{N}\sum_{i=1}^N E_0^i \exp\left(\mu_i T\right)\right) =: \nu(\rho).$$

Proof. We start with the constraint, given in (3.12). By Proposition A.30, it induces

$$\begin{aligned}
\rho \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right) = \mathbb{E}\left[\sum_{i=1}^N G_T^{i,\hat{\alpha}^i}\right] &= \sum_{i=1}^N E_0^i \mathbb{E}\left[\exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)T + \sigma_i W_T^i\right)\right] - \sum_{i=1}^N \int_0^T \mathbb{E}\left[\hat{\alpha}_s^i\right]\mathrm{d}s \\
&= \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right) - \sum_{i=1}^N \int_0^T \mathbb{E}\left[\eta_i\left(\hat{P}_t - h_i\right)\right]\mathrm{d}t \\
&= \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right) - \hat{P}_0 TN\bar{\eta} + TN\bar{H},
\end{aligned}$$

where we have made use of (3.5), and the fact that the market price $\hat{P}$ is a martingale in the optimum. This can be solved for $\hat{P}_0$ to

$$\hat{P}_0 = \frac{1}{TN\bar{\eta}}\left(TN\bar{H} + (1-\rho)\sum_{i=1}^N E_0^i \exp\left(\mu_i T\right)\right).$$

We see that the average market price of permits $\hat{P}_0$ is again fixed by the parameters in the system and the required level of reduction desired by the regulator. The above, together with the other expression of the initial market price $\hat{P}_0$ of (3.11), can be used to achieve an expression for $\widetilde{M}_0$, as follows

$$\begin{aligned}
\frac{1}{TN\bar{\eta}}\left(TN\bar{H} + (1-\rho)\sum_{i=1}^N E_0^i \exp\left(\mu_i T\right)\right) &= \frac{2\lambda}{N(1+2\lambda\bar{\eta}T)}\left(TN\bar{H} - N\widetilde{M}_0 + \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right)\right), \\
\frac{N\left(1+2\lambda\bar{\eta}T\right)}{N2\lambda\bar{\eta}T}\left(TN\bar{H} + (1-\rho)\sum_{i=1}^N E_0^i \exp\left(\mu_i T\right)\right) - TN\bar{H} - \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right) &= -N\widetilde{M}_0.
\end{aligned}$$

This can be rewritten to

$$\begin{aligned}\widetilde{M}_0 &= -\frac{1}{2\lambda\bar{\eta}T}\left(T\bar{H}+\frac{(1-\rho)}{N}\sum_{i=1}^N E_0^i\exp\left(\mu_i T\right)\right)-T\bar{H}+T\bar{H}-\frac{(1-\rho)}{N}\sum_{i=1}^N E_0^i\exp\left(\mu_i T\right)+\frac{1}{N}\sum_{i=1}^N E_0^i\exp\left(\mu_i T\right)\\ &= -\frac{1}{2\lambda\bar{\eta}T}\left(T\bar{H}+\frac{(1-\rho)}{N}\sum_{i=1}^N E_0^i\exp\left(\mu_i T\right)\right)+\frac{\rho}{N}\sum_{i=1}^N E_0^i\exp\left(\mu_i T\right)\\ &= -\frac{1}{2\lambda\bar{\eta}T}\left(T\bar{H}+\frac{1-\rho(1+2\lambda\bar{\eta}T)}{N}\sum_{i=1}^N E_0^i\exp\left(\mu_i T\right)\right)=:\nu(\rho). \end{aligned} \tag{3.13}$$

Here, $\widetilde{M}_0$ only depends on the desired reduction level of the regulator, since the other terms are constants fixed by the system at this stage of the Stackelberg game. This implies that we can write it as a function of ρ only. □

Based on the proposition above, an important remark is made.

Remark 3.1. *In the Brownian motion case, the expression of $\bar{M}_0$, given by $l(\rho)$ in Equation* (2.38) *can have both signs, but it is negative when $\bar{E}_0 = 0$. In the case of this chapter, we cannot choose $\bar{E}_0 = 0$, as we cannot have negative or zero emissions. That is, $\widetilde{M}_0 = \nu(\rho)$ can have both signs, since the rightmost term can become negative when $\rho(1+2\lambda\bar{\eta}T) > 1$. The sign of $\widetilde{M}_0$ is thus again not yet determined and depends on the specific choice of parameters.*

From the condition on $\widetilde{M}_0$, we can redefine the stochastic control problem of (3.12)of finding the infimum over $\tilde{A} \in \mathscr{S}^N$ to finding the infimum over $\tilde{M} \in \mathscr{M}^N$, where $\tilde{M}^i = E[\tilde{A}_T^i|\mathscr{F}_t]$. This holds, since $\hat{P}$ only depends on $\tilde{A}$ through $\tilde{M}$, and $\hat{\alpha}$ depends on $\hat{P}$. This is equivalent to Corollary 2.16. Then, by an application of the Martingale Representation Theorem with $\tilde{M}$, the problem reduces to finding $(\tilde{M}_0^i, \vec{\gamma})$, for all firms, in correspondence with Equation (2.40). This procedure is not repeated here, as the proofs are exactly applicable.

With this, we are ready to prove the main theorem of this section.

Theorem 3.8. *The stochastic control problem of* (3.12)*, represented as*

$$\inf_{\tilde{M}\in\mathscr{M}^N}\mathbb{E}\left[\sum_{i=1}^N\int_0^T h_i\hat{\alpha}_t^i+\frac{(\hat{\alpha}_t^i)^2}{2\eta_i}dt+\lambda\left(X_T^{i,G}\right)^2\right],\quad \widetilde{M}_0=\nu(\rho),$$

can be solved by

$$\widetilde{M}_0=\nu(\rho),\quad \sum_{i=1}^N\gamma_t^{i,j}-E_0^i\kappa_{i,j}\sigma_i\exp\left(Z_t^i+\sigma_i W_t^i\right)=0,\ \mu\ a.e,\ with\ Z_t^i:=\mu_i T-\frac{1}{2}\sigma_i^2 t,$$

which identifies a non-unique expression of $\tilde{M}\in\mathscr{M}^N$.

Proof. The first part of the proof, until $\widetilde{M}_0$ comes into play, is exactly the same as in Theorem 2.17. It still holds that $\hat{\alpha}_t^i = \eta_i(\hat{P}_t - h_i)$, where $\hat{P}$ is a square - integrable martingale. Hence, we can start with

$$\mathbb{E}\left[\sum_{i=1}^N\int_0^T h_i\hat{\alpha}_t^i+\frac{(\hat{\alpha}_t^i)^2}{2\eta_i}+\lambda(\hat{X}_T^{i,G})^2\right]=\sum_{i=1}^N\left(\frac{T\eta_i}{2}\hat{P}_0^2+\frac{1}{4\lambda}\hat{P}_0^2-\frac{\eta_i T}{2}h_i^2\right)+\sum_{i=1}^N\frac{\eta_i}{2}\mathbb{E}\left[\int_0^T\langle\hat{P}\rangle_t\mathrm{d}t\right]+\frac{N}{4\lambda}\mathbb{E}\left[\langle\hat{P}\rangle_T\right]. \tag{3.14}$$

For the details, we refer to the proof of Theorem 2.17. The minimum of the social costs can be found when $\langle\hat{P}\rangle_t = 0$ for all $t \in [0,T]$ almost surely, where

$$\hat{P}_t=\hat{P}_0-\int_0^t\frac{f(t)}{N}\mathrm{d}\left(\sum_{i=1}^N\tilde{M}_t^i-E_0^i\exp\left(\mu_i T-\frac{1}{2}\sigma_i^2 t+\sigma_i W_t^i\right)\right).$$

We choose this particular expression of $\hat{P}$, to use the same arguments as in the Brownian motion case. Later in this proof, we will rewrite it with Lemma 3.4. With this expression and by the properties of the quadratic variation, the quadratic variation of $\hat{P}$ can be written as

$$\begin{aligned}\langle \hat{P} \rangle_t &= \left\langle \int_0^{\cdot} \frac{f(s)}{N} \mathrm{d}\left(\sum_{i=1}^N \tilde{M}_s^i - E_0^i \exp\left(Z_s^i + \sigma_i W_s^i \right) \right) \right\rangle_t \\ &= \int_0^t \frac{f(s)^2}{N^2} \mathrm{d} \left\langle \left(\sum_{i=1}^N \tilde{M}^i - E_0^i \exp\left(Z^i + \sigma_i W^i \right) \right) \right\rangle_s = 0,\end{aligned}$$

for all $t \in [0, T]$. This reduces, since this is a positive, deterministic integrand with a continuous integrator starting in zero, to

$$\left\langle \sum_{i=1}^N \tilde{M}^i - E_0^i \exp\left(Z^i + \sigma_i W^i \right) \right\rangle_t = 0,$$

a.s, for all $t \in [0, T]$. Here, we have used Proposition A.18 to conclude that the processes in the quadratic variation have a continuous modification.

When we plug the result of Lemma 3.4 in, the quadratic variation above reduces to

$$\begin{aligned}0 = \left\langle \sum_{i=1}^N \tilde{M}^i - E_0^i \exp\left(Z^i + \sigma_i W^i \right) \right\rangle_t &= \left\langle \sum_{i=1}^N \tilde{M}^i - E_0^i \exp\left(\mu_i T \right) + E_0^i \sigma_i \int_0^{\cdot} \exp\left(Z_s^i + \sigma_i W_s^i \right) \mathrm{d}W_s^i \right\rangle_t \\ &= \left\langle \sum_{i=1}^N \tilde{M}^i - E_0^i \sigma_i \int_0^{\cdot} \exp\left(Z_s^i + \sigma_i W_s^i \right) \mathrm{d}W_s^i \right\rangle_t .\end{aligned}$$

By the Martingale Representation theorem, we can write

$$\tilde{M}_t^i = \tilde{M}_0^i + \sum_{i=1}^N \int_0^t \sum_{j=0}^N \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j,$$

where the right part is a martingale starting at zero and $\gamma^{i,j}$ progressively measurable processes. With this, the above reduces to

$$\sum_{i=1}^N \int_0^t \sum_{j=0}^N \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j - E_0^i \sigma_i \int_0^t \exp\left(Z_s^i + \sigma_i W_s^i \right) \mathrm{d}W_s^i = 0,$$

for all $t \in [0, T]$ a.s, by Proposition A.13. By plugging in the expression of (W_t^i), we get

$$\sum_{i=1}^N \int_0^t \sum_{j=0}^N \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j - \sum_{m=0}^N E_0^i \kappa_{i,m} \sigma_i \int_0^t \exp\left(Z_s^i + \sigma_i W_s^i \right) \mathrm{d}\tilde{B}_s^m = 0.$$

When we rearrange and interchange, we get

$$\sum_{j=0}^N \int_0^T \sum_{i=1}^N \left(\gamma_s^{i,j} - E_0^i \kappa_{i,j} \sigma_i \exp\left(Z_s^i + \sigma_i W_s^i \right) \right) \mathrm{d}\tilde{B}_s^j = 0.$$

By Itô's isometry of Proposition A.16 , together with Proposition A.12, it follows that this condition is equivalent to

$$\mathbb{E}\left[\sum_{j=0}^N \int_0^t \left(\sum_{i=1}^N \gamma_s^{i,j} - E_0^i \kappa_{i,j} \sigma_i \exp\left(Z_s^i + \sigma_i W_s^i \right) \right)^2 \mathrm{d}s \right] = \sum_{j=0}^N \left\| \sum_{i=1}^N \gamma^{i,j} - E_0^i \kappa_{i,j} \sigma_i \exp\left(Z_s^i + \sigma_i W_s^i \right) \right\|^2 = 0,$$

following the same reasoning as in Theorem 2.17. This reduces, by Equation (1.17), to

$$\sum_{i=1}^N \gamma_t^{i,j} - E_0^i \kappa_{i,j} \sigma_i \exp\left(Z^i + \sigma_i W^i \right) = 0, \quad \mu \text{ a.e.} \tag{3.15}$$

for all $j \in \{0,\dots,N\}$. This is not uniquely solvable, hence the optimal martingales $\tilde{M} \in \mathscr{M}^N$ are not unique. More can be said.

Again, since the optimal market price has a deterministic integrand, it holds that

$$\hat{P}_t \sim \mathcal{N}\left(\hat{P}_0, \langle \hat{P} \rangle_t\right),$$

where $\mathcal{N}$ represents the normal distribution. Hence, the price process $(\hat{P}_t)$ has zero variance in the optimum. This implies directly that

$$\hat{P}_t = \hat{P}_0, \quad \hat{\alpha}_t^i = \eta_i(\hat{P}_t - h_i) = \eta_i(\hat{P}_0 - h_i).$$

The corresponding social costs C_{opt}^G can be found by plugging in the zero quadratic variation in (3.14), which gives

$$\begin{aligned} C_{\text{opt}}^G := \mathbb{E}\left[\sum_{i=1}^N \int_0^T h_i \hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} \mathrm{d}t + \lambda \left(X_T^{i,G}\right)^2\right] &= \sum_{i=1}^N \left(\frac{T\eta_i}{2}\hat{P}_0^2 - \frac{\eta_i T}{2} h_i^2\right) + \frac{N}{4\lambda}\hat{P}_0^2 \\ &= -\frac{T}{2}\sum_{i=1}^N h_i^2 \eta_i + \frac{N}{4\lambda}\left(2\bar{\eta}\lambda T + 1\right)\hat{P}_0^2. \end{aligned}$$

This ends the proof. □

The three variables $\hat{\alpha}, \hat{P}$ and C_{opt}^G share the same structure as in (2.50) of the Brownian motion case. However, note that $\hat{P}_0$ has a slightly different formula now, so the outcome is different. We will elaborate more on this in the comparison between those situations. To be able to compare this, we need specific outcomes of (3.15). This is done in the following example.

Example 3.1 (Standard allocation). *A particular solution of* (3.15) *can be found by setting*

$$\hat{\gamma}_t^{i,j} = E_0^i \kappa_{i,j} \sigma_i \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right), \tag{3.16}$$

for all relevant i, j and $t \in [0,T]$, a.s. It is in line with Example 2.1. Furthermore, given $\widetilde{M}_0 = \nu(\rho)$, we choose $\tilde{M}_0^i = \nu(\rho)$. Every firm gets the same expected allocation. Based on these values, the expression of the other parameters can be deduced. For completeness, all the parameters are stated here. The equations below hold for all $t \in [0,T]$, a.s.

(i) The market price $(\hat{P}_t)$ is constant and fixed by $\widetilde{M}_0$. As a consequence, the optimal abatement effort $(\hat{\alpha}_t^i)$ is also constant. These are given by

$$\hat{P}_t = \hat{P}_0 = \frac{f(0)}{N}\left(NT\bar{H} - N\widetilde{M}_0 + \sum_{i=1}^N E_0^i \exp\left(\mu_i T\right)\right), \quad \hat{\alpha}_t^i = \eta_i\left(\hat{P}_0 - h_i\right).$$

(ii) The expression for the optimal martingale $\tilde{M}^i$ is as follows

$$\begin{aligned} \tilde{M}_t^i &= \tilde{M}_0^i + \sum_{j=0}^N \int_0^t \hat{\gamma}_s^{i,j} d\tilde{B}_s^j \\ &= \nu(\rho) + \sum_{j=0}^N \int_0^t E_0^i \kappa_{i,j}\sigma_i \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right) d\tilde{B}_s^j \\ &= \nu(\rho) + E_0^i \int_0^t \sigma_i \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right) d\left(\sum_{j=0}^N \kappa_{i,j}\tilde{B}_s^j\right) \\ &= \nu(\rho) + E_0^i \int_0^t \sigma_i \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 s + \sigma_i W_s^i\right) dW_s^i \\ &= \nu(\rho) + E_0^i \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W_t^i\right) - E_0^i \exp\left(\mu_i T\right), \end{aligned}$$

by Lemma 3.4.

(iii) *As a consequence, the allowances process* $(\tilde{A}^i_t)$ *can be represented as*

$$\tilde{A}^i_t = \tilde{M}^i_t,$$

since then, clearly,

$$\tilde{A}^i_t = \tilde{M}^i_t = \mathbb{E}\left[\tilde{A}^i_T \middle| \mathscr{F}_t\right] = \mathbb{E}\left[\tilde{M}^i_T \middle| \mathscr{F}_t\right],$$

since $\tilde{M}^i$ *is a martingale by construction.*

(iv) *The process* $(\hat{B}^i_t)$*, with the expressions above, is given by*

$$\begin{aligned}
\hat{B}^i_t &= -\frac{1}{2\lambda}\hat{P}_t - \tilde{M}^i_t - \int_0^t \eta_i(\hat{P}_s - h_i)ds - \eta_i(\hat{P}_t - h_i)(T-t) + E^i_0 \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W^i_t\right)\\
&= -\frac{1}{2\lambda}\hat{P}_0 - \nu(\rho) - E^i_0 \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W^i_t\right) + \exp\left(\mu_i T\right)\\
&\quad - \eta_i(\hat{P}_0 - h_i)t - \eta_i(\hat{P}_0 - h_i)T + \eta_i(\hat{P}_0 - h_i)t + E^i_0 \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W^i_t\right)\\
&= -\frac{1}{2\lambda}\hat{P}_0 - \nu(\rho) - \eta_i(\hat{P}_0 - h_i)T + E^i_0 \exp \mu_i T\\
&= -\left(\frac{1+2\lambda\bar{\eta}_i T}{2\lambda}\hat{P}_0 + \widetilde{M}_0 - E^i_0 \exp\left(\mu_i T\right) - \eta_i h_i T\right) = \hat{B}^i_0,
\end{aligned}$$

which corresponds with (3.4)*. This way, we can set*

$$\hat{\beta}^i_t = \frac{\hat{B}^i_0}{T},$$

satisfying relation (3.3)*.*

(v) *The social costs* $C^{i,G}_{opt}$ *are given by*

$$C^G_{opt} = -\frac{T}{2}\sum_{i=1}^N h_i^2 \eta_i + \frac{N}{4\lambda}\left(2\bar{\eta}\lambda T + 1\right)\hat{P}_0^2.$$

(vi) *The optimal bank account* $(\hat{X}^{i,G}_t)$ *can now be deduced, and is given by*

$$\begin{aligned}
\hat{X}^{i,G}_t &= \tilde{A}^i_t + \int_0^t \hat{\alpha}^i_s + \hat{\beta}^i_s ds - E^i_0 \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i W^i_t\right)\\
&= \nu(\rho) + E^i_0 \exp\left(\mu_i T - \frac{1}{2}\sigma_i^2 t + \sigma_i W^i_t\right) - E^i_0 \exp\left(\mu_i T\right) + \left(\hat{\alpha}^i_0 + \hat{\beta}^i_0\right)t\\
&\quad - E^i_0 \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i W^i_t\right).
\end{aligned}$$

Note that this cannot be simplified further. At time $t = T$*, it follows that*

$$\hat{X}^{i,G}_T = \nu(\rho) + \left(\hat{\alpha}^i_0 + \hat{\beta}^i_0\right)T - E^i_0 \exp\left(\mu_i T\right).$$

In the last part of this chapter, these results will be interpreted and partially compared theoretically to the results in the Brownian framework of Example 2.1. Note that we are comparing two examples of solutions, and these solutions are not unique. However, the conditions (2.49) and (3.15) are solved in the same way. In the Brownian case, the solution of $\gamma^{i,j}_t$ is constant and fixed. In the case of Geometric Brownian motion, the solution for $\gamma^{i,j}_t$ is random and depends on the Brownian motion W^i.

We remark that a non-random $\gamma^{i,j}$ corresponds with the Malliavin derivative of the Brownian motion, whereas in (3.16), the corresponding Malliavin derivative of the Geometric Brownian motion G^i is obtained [HB15, ch. 1]. This could show that the results for the processes in the examples of both cases are correct and correspond to each other. We won't go into detail about the Malliavin derivative; for further information we refer to the book of Nualart [Nua06].

When we consider Example 3.1 further, it follows that $\hat{\alpha}_t^i > 0$, when $\hat{P}_0 > 0$. The latter can for example be achieved when $\widetilde{M}_0 < 0$ by choosing E_0^i sufficiently small enough. In this situation, we cannot choose $E_0^i = 0$ for a firm, since then by definition of the Geometric Brownian motion the emission path will stay zero for all $t \in (0, T]$. However, there are also other combinations for which this is true. In essence, the price process may become negative here, which would result in a constant, negative abatement effort. This is undesirable.

The sign of $\tilde{M}_t^i$ and $\tilde{A}_t^i$ has not been determined yet, as it depends on the choice of E_0^i, and also on the specific choice of parameters. We say in the Brownian framework, when we choose $E_0 = 0$, we need that $M_t^i = \tilde{M}_t^i - \mu_i t$ is positive. For this, we need an assumption on the drift and the volatility of the Brownian motion to let the stochastic control problem be realistic. Otherwise both the initial allocation and all thereafter would be negative. Here, when we choose E_0 small, this may not be the case anymore. However, this cannot be directly concluded from the expressions itself and should be numerically calculated.

A difference with the previous chapter, is that the BAU emissions do not really appear in the expression for (A_t^i). However, we see many similarities. The differences stem for the fact that we work with an exponent.

The process (B_t^i) and trading rate β^i have the same structure as in the previous chapter. This implies that the same conclusions can be made as before, under the assumption that $E_0^i = \bar{E}_0$ for all firms. That is, when $\eta_i = \bar{\eta}$ but $h_i \neq \bar{h}$, firms have positive trading rate when $h_i \leqslant \bar{h}$, since they have larger abatement costs than firms not satisfying this property. Furthermore, the function of the bank account $\hat{X}_t^{i,G}$ cannot be simplified to an expression similar to the Brownian case. The reason is again the appearance of the exponent, since several parts do not cancel out.

We can conclude that both outcomes look similar and the differences can be substantiated by the exponent, which indicates that the derivations are correct.

4 | Comparisons

In this chapter, the optimal dynamic allocation under the Brownian framework is compared to two other existing policies, as well as the Geometric Brownian motion scenario. The first policy under consideration involves only an initial, static, allocation. This policy has been in use in the early years of the European Union Emissions Trading System (EU ETS). In here, emissions are still modelled by an SDE, which is the only randomness involved, the other variables are deterministic. Additionally, the dynamic policy is compared to the Market Stability Reserve (MSR) mechanism of the EU ETS. To integrate it into the framework that we work with, an alternative, but corresponding, version of the MSR is used. Both cases represent specific instances within the dynamic situation. In the last section of this chapter, the Brownian motion framework of Chapter 2 and the optimal dynamic allocation of Chapter 3 are compared numerically.

The goal in the first two sections is to compare the social costs of these policies with the social costs C_{opt} as discussed in Chapter 2 and deduce when the optimal dynamic policy has lower social costs than the other policies. Then, we could conclude, since the reduction in both frameworks is fixed by the regulator, that the proposed optimal dynamic policy works better, as the costs in the whole system are lower than in the other scenario. This will be calculated in the static case, and partially in the MSR scenario. This chapter is based on Section 6 of [AB23].

The policies are compared under the assumption that $\nu = \infty$ and

$$\eta_i = \eta,$$

for all firms $i = 1, \ldots, N$. From Equation (2.50), it follows that the optimal social costs of the dynamic allocation are then given by

$$C^E_{\text{opt}} = -\frac{T}{2}\sum_{i=1}^{N} h_i^2 \eta_i + \frac{N}{4\lambda}\left(2\bar{\eta}\lambda T + 1\right)\hat{P}_0^2 = -\frac{T}{2}\eta\sum_{i=1}^{N} h_i^2 + \frac{N}{4\lambda}\left(2\eta\lambda T + 1\right)\hat{P}_0^2. \tag{4.1}$$

4.1 Initial, static allocation only

This policy is, in contrast to the optimal dynamic scheme, static, with everything being non-random, except for the emissions. There is only an initial, fixed, endowment given at time $t = 0$ and no further allocations are made thereafter. We assume that $E_0^i = 0$ for all firms. The firms still have the option to trade their permits on the market, if that is needed. Moreover, the BAU and abated emissions are still modelled by a Brownian motion. Let us fix a firm i. Then, the initial bank account, in correspondence with Equation (1.20), becomes

$$X_0^i = \tilde{S}_0^i = \tilde{A}_0^i =: x_0^i \in \mathbb{R}, \quad F_t^i = 0, \quad b_t^{i,j} = 0,$$

for all firms and $t \in [0, T]$. This reflects the assumption that the initial bank account equals the initial allocation. The initial allocation is non-random and given by x_0^i. We recall that the allocations are a cumulative process, so that there is only an initial allocation and no extra given allocation at time $t > 0$. This implies that we need

$$\tilde{A}_t^i = x_0^i, \quad A_t^i = x_0^i - \mu_i t,$$

for all firms and $t \in [0.T]$. Then, it follows that

$$M_0^i = \mathbb{E}\left[A_T^i\right] = x_0^i - \mu_i T, \quad M_t^i = \mathbb{E}\left[A_T^i \middle| \mathscr{F}_t\right] = x_0^i - \mu_i T,$$

where the latter is independent of time $t \in (0, T)$. The objective function of the regulator and the constraint are still the same as in Equation (2.39), with $\eta_i = \eta$. The same holds for the objective function of the firms. That is, we can conclude that $\hat{\alpha}_t^i$ and $\hat{B}_t^i$ of Theorem 2.12 are still optimal. Furthermore, the optimal price process $\hat{P}$ is given by (2.25). Here, this process can be simplified to

$$\begin{aligned}
\hat{P}_t &= \hat{P}_0 - \int_0^T f(s)\mathrm{d}\left(\bar{M}_s - \bar{W}_s\right) = \hat{P}_0 - \int_0^T f(s)\mathrm{d}\left(\bar{x}_0 - \bar{\mu}T - \bar{W}_s\right) \\
&= \hat{P}_0 + \int_0^T f(s)\mathrm{d}\bar{W}_s = \frac{2\lambda}{1+2\lambda\eta T}(T\bar{H} - \bar{M}_0) + \int_0^T \frac{2\lambda}{1+2\lambda\eta(T-s)}\mathrm{d}\bar{W}_s \\
&= \frac{2\lambda}{1+2\lambda\eta T}\left(T\eta\bar{h} - \bar{x}_0 + \bar{\mu}T\right) + \int_0^T \frac{2\lambda}{1+2\lambda\eta(T-s)}\mathrm{d}\bar{W}_s,
\end{aligned}$$

where $\bar{W}$ as in Equation (2.42). To be able to identify the martingales $M \in \mathscr{M}^N$, we only need to set the initial allocation x_0^i under the constraint $\bar{M}_0 = l(\rho)$ of Equation (2.38), with $\bar{E}_0 = 0$. Therefore,

$$\begin{aligned}
\bar{M}_0 &= \bar{x}_0 - \bar{\mu}T, \\
\bar{x}_0 &= \bar{M}_0 + \bar{\mu}T = -\frac{1}{2\lambda\eta}\left(\eta\bar{h} + (1+2\lambda\eta T)(1-\rho)\mu\right) + \bar{\mu}T = -\frac{1}{2\lambda}\left(\bar{h} + \frac{(1-\rho)\bar{\mu}}{\eta}\right) + \rho\bar{\mu}T.
\end{aligned}$$

The sign of the average, initial allocation $\bar{x}_0$ cannot be determined. In the Brownian framework, we have seen that the average, initial allocation is always negative, when $E_0 = 0$. This cannot be concluded here.

The initial price $\hat{P}_0$ is the same as in (2.37) with $\eta_i = \eta$ for all firms, as the process $(\hat{\alpha}_t)$ and $(\hat{P}_t)$ do not change and the main variable is the desired reduction ρ. All the ingredients are there to calculate the social costs in this situation, called C_{stat}. This is done in the theorem below.

Theorem 4.1 (Social costs in static scenario). *The social costs, when there is only an initial, static allocation, are given by*

$$C_{stat} = \frac{N}{4\lambda}\left(1+2\lambda\eta T\right)\hat{P}_0^2 + \frac{N\sigma^2}{2\eta}\log\left(1+2\lambda\eta T\right) - \frac{1}{2}T\eta\sum_{i=1}^N h_i^2. \tag{4.2}$$

Proof. The objective function in the market equilibrium is still equal to (2.41), implying

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^N \int_0^T h_i\hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} + \lambda(\hat{X}_T^i)^2\right] &= \frac{1}{2}\sum_{i=1}^N \eta_i \mathbb{E}\left[\int_0^T \hat{P}_t^2\mathrm{d}t\right] - \frac{T}{2}\sum_{i=1}^N h_i^2\eta_i + \frac{N}{4\lambda}\mathbb{E}\left[\hat{P}_T^2\right] \\
&= \frac{\eta}{2}\sum_{i=1}^N \mathbb{E}\left[\int_0^T \hat{P}_t^2\mathrm{d}t\right] - \frac{T\eta}{2}\sum_{i=1}^N h_i^2 + \frac{N}{4\lambda}\mathbb{E}\left[\hat{P}_T^2\right].
\end{aligned} \tag{4.3}$$

Now, we need to find the expression for the square of $\hat{P}_t$. Let $t \in [0, T]$, then

$$\begin{aligned}
(\hat{P}_t)^2 &= \left(\hat{P}_0 + \int_0^t \frac{2\lambda}{1+2\lambda\eta(T-s)}\mathrm{d}\bar{W}_s\right)^2 \\
&= \hat{P}_0^2 + 2\hat{P}_0\int_0^t \frac{2\lambda}{1+2\lambda\eta(T-s)}\mathrm{d}\bar{W}_s + \left(\int_0^t \frac{2\lambda}{1+2\lambda\eta(T-s)}\mathrm{d}\bar{W}_s\right)^2.
\end{aligned}$$

Since the stochastic integral in the middle is a martingale by Proposition A.15, it has expectation equal to zero here. Furthermore, by Itô's isometry, it holds that

$$\begin{aligned}\mathbb{E}\left[\hat{P}_t^2\right] &= \mathbb{E}\left[\hat{P}_0^2\right] + \mathbb{E}\left[2\hat{P}_0\int_0^t \frac{2\lambda}{1+2\lambda\eta(T-s)}\mathrm{d}\bar{W}_s\right] + \mathbb{E}\left[\left(\int_0^t \frac{2\lambda}{1+2\lambda\eta(T-s)}\mathrm{d}\bar{W}_s\right)^2\right] \\ &= \hat{P}_0^2 + \mathbb{E}\left[\int_0^t \left(\frac{2\lambda}{1+2\lambda\eta(T-s)}\right)^2 \mathrm{d}\langle\bar{W}\rangle_s\right].\end{aligned}$$

With $\bar{W} = \frac{1}{N}\sum_{i=1}^N \sigma_i W^i$, we get

$$\langle\bar{W}\rangle_s = \left\langle \frac{1}{N}\sum_{i=1}^N \sigma_i W^i \right\rangle_s = \frac{1}{N^2}\left\langle \sum_{i=1}^N \sigma_i W^i \right\rangle_s = \frac{1}{N^2}\left\langle \sum_{i=1}^N \sigma_i W^i, \sum_{i=1}^N \sigma_i W^i \right\rangle_s.$$

Since the quadratic variation is bilinear, by Corollary A.11, and W is an N-dimensional correlated Brownian motion with correlation given in (1.3), we have that

$$\begin{aligned}\left\langle\bar{W}\right\rangle_s &= \frac{1}{N^2}\left(\sum_{i=1}^N \sigma_i^2 s + \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \sigma_i\sigma_j \left\langle W^i, W^j\right\rangle_s\right) \\ &= \frac{1}{N^2}\left(\sum_{i=1}^N \sigma_i^2 + \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \sigma_i\sigma_j \sum_{m=0}^N \kappa_{i,m}\kappa_{j,m}\right) s \\ &=: L_N(\sigma)s.\end{aligned}$$

Then, it follows that

$$\begin{aligned}\mathbb{E}\left[\hat{P}_t^2\right] &= \hat{P}_0^2 + \mathbb{E}\left[\int_0^t \left(\frac{2\lambda}{1+2\lambda\eta(T-s)}\right)^2 \mathrm{d}\langle\bar{W}\rangle_s\right] = \hat{P}_0^2 + \mathbb{E}\left[\int_0^t \left(\frac{2\lambda}{1+2\lambda\eta(T-s)}\right)^2 L_N(\sigma)\mathrm{d}s\right] \\ &= \hat{P}_0^2 + L_N(\sigma)\int_0^t \left(\frac{2\lambda}{1+2\lambda\eta(T-s)}\right)^2 \mathrm{d}s = \hat{P}_0^2 + L_N(\sigma)\int_0^t (2\lambda)^2\left(1+2\lambda\eta(T-s)\right)^{-2}\mathrm{d}s \\ &= \hat{P}_0^2 + L_N(\sigma)\frac{2\lambda}{\eta}\left[\frac{1}{1+2\lambda\eta(T-s)}\right]_{s=0}^{s=t} = \hat{P}_0^2 + L_N(\sigma)\frac{2\lambda}{\eta}\left(\frac{1}{1+2\lambda\eta(T-t)} - \frac{1}{1+2\lambda\eta T}\right).\end{aligned}$$

Then, (4.3) reduces to

$$\begin{aligned}C_{\text{stat}} &= \frac{\eta}{2}\sum_{i=1}^N \mathbb{E}\left[\int_0^T \hat{P}_t^2 \mathrm{d}t\right] - \frac{T\eta}{2}\sum_{i=1}^N h_i^2 + \frac{N}{4\lambda}\mathbb{E}[\hat{P}_T^2] \\ &= \frac{\eta}{2}\hat{P}_0^2 NT + \frac{\eta}{2}\sum_{i=1}^N \int_0^T L_N(\sigma)\frac{2\lambda}{\eta}\left(\frac{1}{1+2\lambda\eta(T-t)} - \frac{1}{1+2\lambda\eta T}\right)\mathrm{d}t - \frac{T\eta}{2}\sum_{i=1}^N h_i^2 + \frac{N}{4\lambda}P_0^2 \\ &\quad + \frac{N}{4\lambda}L_N(\sigma)\frac{2\lambda}{\eta}\left(1 - \frac{1}{1+2\lambda\eta T}\right) \\ &= \hat{P}_0^2\left(\frac{NT\eta}{2} + \frac{N}{4\lambda}\right) - \frac{T\eta}{2}\sum_{i=1}^N h_i^2 + L(\sigma)\frac{N}{4\lambda}\frac{2\lambda}{\eta}\left(\frac{2\lambda\eta T}{1+2\lambda\eta T}\right) \\ &\quad + \frac{\eta}{2}\sum_{i=1}^N \int_0^T L(\sigma)\frac{2\lambda}{\eta}\left(\frac{1}{1+2\lambda\eta(T-t)} - \frac{1}{1+2\lambda\eta T}\right)\mathrm{d}t.\end{aligned}$$

Since it holds that

$$\begin{aligned}\int_0^T \left(\frac{1}{1+2\lambda\eta(T-t)} - \frac{1}{1+2\lambda\eta T}\right)\mathrm{d}t &= -\left[\frac{1}{2\lambda\eta}\log(1+2\lambda\eta(T-t))\right]_{t=0}^{t=T} - \frac{T}{1+2\lambda\eta T} \\ &= \frac{1}{2\lambda\eta}\log(1+2\lambda\eta T) - \frac{T}{(1+2\lambda\eta T)},\end{aligned}$$

we can write

$$L_N(\sigma)\frac{N}{4\lambda}\frac{2\lambda}{\eta}\left(\frac{2\lambda\eta T}{1+2\lambda\eta T}\right)=\frac{L_N(\sigma)N\lambda T}{1+2\lambda\eta T},$$

$$\frac{\eta}{2}\sum_{i=1}^{N}\int_0^T L_N(\sigma)\frac{2\lambda}{\eta}\left(\frac{1}{1+2\lambda\eta(T-t)}-\frac{1}{1+2\lambda\eta T}\right)\mathrm{d}t=\frac{1}{2\eta}\sum_{i=1}^{N}L_N(\sigma)\log(1+2\lambda\eta T)-\frac{L_N(\sigma)T\lambda N}{1+2\lambda\eta T}.$$

This results in the following cost function

$$\begin{aligned}
C_{\text{stat}} &= \hat{P}_0^2\left(\frac{NT\eta}{2}+\frac{N}{4\lambda}\right)-\frac{T\eta}{2}\sum_{i=1}^{N}h_i^2+\frac{L_N(\sigma)N\lambda T}{1+2\lambda\eta T}\\
&\quad+\frac{1}{2\eta}\sum_{i=1}^{N}L_N(\sigma)\log(1+2\lambda\eta T)-\frac{L_N(\sigma)T\lambda N}{1+2\lambda\eta T}\\
&=\hat{P}_0^2\left(\frac{NT\eta}{2}+\frac{N}{4\lambda}\right)-\frac{T\eta}{2}\sum_{i=1}^{N}h_i^2+\frac{NL_N(\sigma)}{2\eta}\log\left(1+2\lambda\eta T\right)\\
&=\frac{N}{4\lambda}\left(2\lambda\eta T+1\right)\hat{P}_0^2-\frac{T\eta}{2}\sum_{i=1}^{N}h_i^2+\frac{NL_N(\sigma)}{2\eta}\log\left(1+2\lambda\eta T\right).
\end{aligned}$$

□

It is interesting to compare the social costs in the static scenario with the costs in the dynamic case. This is done in the subsection below.

4.1.1 Comparison static and dynamic

From Theorem 4.1 and Equation (4.1), we recall that

$$\begin{aligned}
C_{\text{stat}} &= \frac{N}{4\lambda}(2\lambda\eta T+1)\hat{P}_0^2-\frac{T\eta}{2}\sum_{i=1}^{N}h_i^2+\frac{NL_N(\sigma)}{2\eta}\log(1+2\lambda\eta T),\\
C_{\text{opt}}^E &= -\frac{T\eta}{2}\sum_{i=1}^{N}h_i^2+\frac{N}{4\lambda}(2\eta\lambda T+1)\hat{P}_0^2.
\end{aligned}$$

We recognise two common terms, as $\hat{P}_0$ is the same in both cases and fixed by the desired reduction level ρ. The difference between the two social costs is given by

$$\begin{aligned}
\Delta_{\text{stat}} := C_{\text{stat}}-C_{\text{opt}}^E &= \frac{N}{4\lambda}(2\lambda\eta T+1)\hat{P}_0^2-\frac{T\eta}{2}\sum_{i=1}^{N}h_i^2+\frac{NL_N(\sigma)}{2\eta}\log(1+2\lambda\eta T)+\frac{T\eta}{2}\sum_{i=1}^{N}h_i^2\\
&\quad-\frac{N}{4\lambda}(2\eta\lambda T+1)\hat{P}_0^2\\
&=\frac{NL_N(\sigma)}{2\eta}\log(1+2\lambda\eta T)\geqslant 0,
\end{aligned}$$

as $\lambda,\eta,T>0$ and $L(\sigma)\geqslant 0$ by the fact that it is part of the quadratic variation of $\bar{W}$. We see that in the static scenario the social costs are always greater or equal than the costs in the dynamic scenario. This is not unexpected, as in the static scenario, there may be less effective compensation for shocks. The difference comes from the uncertainty, since it depends on the variances of the emissions, and the flexibility, as it depends on the flexibility parameter η. Hence, we will check what the effect is when $\eta\to\infty$. This would mean that the system is fully flexible with respect to the abatement decisions and all decisions are completely reversible. Then, since $L(\sigma)$ does not depend on η, we see that

$$\lim_{\eta\to\infty}\Delta_{\text{stat}}=L_N(\sigma)\lim_{\eta\to\infty}\frac{\log(1+2\lambda\eta T)}{2\eta}=0,$$

as $\log(x)$ decreases slower to x than the function x itself, when $x \to \infty$. This implies that the difference in costs vanishes, when the reversibility is high. If all decisions can be reversed regarding the abatement, you do not need the compensation for the shocks anymore to achieve the same social costs.

More about the difference is said in the example below.

Example 4.1 (Quantification of difference). *Assume that $\sigma_i = \sigma$ for all firms and that the correlation structure is as in the scenario of Remark 1.1. Furthermore, suppose that $\kappa_i = \kappa$ for all firms $i \neq j$. The goal is to quantify Δ_{stat}, in the case when $N \to \infty$. Let $\rho := \kappa^2$. Then, $L_N(\sigma)$ becomes*

$$L_N(\sigma) = \frac{1}{N^2}\left(\sum_{i=1}^{N}\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\sigma_i\sigma_j\kappa_i\kappa_j\right) = \frac{\sigma^2}{N} + \frac{\rho\sigma^2}{N^2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} 1 = \frac{\sigma^2}{N} + \frac{\rho\sigma^2(N-1)}{N}.$$

We see that

$$\lim_{N\to\infty} L_N(\sigma) = \lim_{N\to\infty}\frac{\sigma^2}{N} + \frac{\rho\sigma^2(N-1)}{N} = \rho\sigma^2 \lim_{N\to\infty}\frac{N-1}{N} = \rho\sigma^2.$$

The only way when this is equal to zero is when $\rho = 0$, that is, when there is no correlation between the common shocks in the economy. This implies that there is no common shock present in the system. Note that

$$\lim_{N\to\infty}\frac{\Delta_{stat}}{N} = \lim_{N\to\infty}\frac{L_N(\sigma)}{2\eta}\log(1+2\lambda\eta T) = \frac{\log(1+2\lambda\eta T)}{2\eta}\lim_{N\to\infty}L_N(\sigma) = \frac{\rho\sigma^2\log(1+2\lambda\eta T)}{2\eta} > 0,$$

if $\rho,\sigma > 0$. We see when $\rho,\sigma \neq 0$ that $C_{stat} > C_{opt}$ for all possible number of firms N. Note that this is calculated only under the assumption of the correlation structure proposed in [AB23].

4.2 MSR like scenario

In this section, we introduce a scenario inspired by the Market Stability Reserve (MSR) mechanism of the EU ETS. Recall, from the introduction, that the MSR in the EU ETS reacts based on the number of permits available in the market. To connect the MSR with our dynamic case, we will not work with the number of permits, but with the average bank account of the firms. This introduces a slight deviation from the MSR scenario used in practice. The scenario presented here is a deterministic scenario, so the randomness of the allocation process will be removed. However, it differs from the static scenario, since it incorporates reactions based on the level of the bank account. This corresponds with a responsive mechanism based on economic shocks. We will compare this to the Brownian framework.

The situation is mathematically defined as follows. We assume $E_0^i = 0$ and we set an equal initial endowment for the firms, and no randomness, hence, in correspondence with (1.23), we have

$$X_0^i = A_0^i = S_0^i =: x_0, \quad S_t^i = 0, \quad b_t^{i,j} = 0.$$

for all $t \in [0,T]$ and for all firms. Then, the allocation process is assumed to be given by

$$A_t^i = A_0^i + \int_0^t \tilde{a}_s - \mu_i \mathrm{d}s := A_0^i + \int_0^t a_s \mathrm{d}s,$$

where a in this situation is assumed to be

$$a_s = \delta\left(\frac{T-s}{T}x_0 - \bar{X}_s\right),$$

for all firms. The drift term of the emissions is incorporated in the variable a. Note that this way, the allocations are the same for all firms, as they only depend on the average bank account $\tilde{X}$. The idea here, is that the regulator would like to achieve $\mathbb{E}[\bar{X}_T] \approx 0$, as then the system is most efficient and the penalty that need to be paid is low. All the firms start in x_0. The goal will be achieved by a linear path from the initial endowment to the current endowment. Furthermore, a parameter δ is added as a mean-reversion coefficient. This parameter assures that the average bank account goes back to the desired path, when it deviates from it.

Since the allocation is non-random, to achieve the desired reduction level of the regulator, we need

$$\bar{M}_0 = \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left[A_T^i \middle| \mathscr{F}_t\right] = x_0 + \int_0^T \mathbb{E}\left[a_s\right] \mathrm{d}s =: l(\rho), \tag{4.4}$$

which is in line with the condition in Theorem 2.17. In the following proposition, the optimal bank account is derived. Note that the objective functions of the firms and regulator, and the optimal controls of Theorem 2.12 are still valid here, as the current situation is again a simplification of the dynamic situation.

Proposition 4.2 (Optimal average bank account). *The optimal, average bank account is in this situation given by*

$$\bar{X}_t = e^{-\delta t} x_0 + e^{-\delta t}\int_0^t e^{\delta s}\left(\delta \frac{T-s}{T} x_0 + \eta\left(\hat{P}_s - \bar{h}\right)\right) ds - e^{-\delta t}\int_0^t e^{\delta s} d\bar{W}_s, \tag{4.5}$$

from which it follows that

$$x_0 = \frac{\delta T}{1-e^{-\delta T}}\left(l(\rho) + \left(T + \frac{e^{-\delta T}-1}{\delta}\right)\eta\left(\hat{P}_0 - \bar{h}\right)\right).$$

Sketch of proof. The dynamics of the average bank account (1.35), in the market equilibrium, are given by

$$\begin{aligned}
\mathrm{d}\bar{X}_t &= \mathrm{d}\left(\frac{1}{N}\sum_{i=1}^{N} X_t^{i,E}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\alpha}_t^i + \hat{\beta}_t^i\right)\mathrm{d}t + \mathrm{d}\left(\frac{1}{N}\sum_{i=1}^{N} A_t^i\right) - \mathrm{d}\left(\frac{1}{N}\sum_{i=1}^{n}\sigma_i W_t^i\right)\\
&= \frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}_t^i\mathrm{d}t + a_t\mathrm{d}t - \mathrm{d}\bar{W}_t\\
&= \frac{1}{N}\sum_{i=1}^{N}\eta\left(\hat{P}_t - h_i\right)\mathrm{d}t + \delta\left(\frac{T-t}{T}x_0 - \bar{X}_t\right)\mathrm{d}t - \mathrm{d}\bar{W}_t\\
&= \eta\left(\hat{P}_t - \bar{h}\right)\mathrm{d}t + \delta\left(\frac{T-t}{T}x_0 - \bar{X}_t\right)\mathrm{d}t - \mathrm{d}\bar{W}_t,
\end{aligned}$$

with initial condition $\bar{X}_0 = x_0$. By the fact that $\bar{X}_t$ is on both sides of the equation, we see that we are dealing with an SDE. To solve this SDE, we notice that $\bar{X}$ is a semimartingale, with a martingale part and a part of bounded variation. This implies we can apply Itô's lemma to the function $\bar{X}_t e^{\delta t}$. We get

$$\begin{aligned}
\mathrm{d}\left(\bar{X}_t e^{\delta t}\right) &= \bar{X}_t\mathrm{d}\left(e^{\delta t}\right) + e^{\delta t}\mathrm{d}\bar{X}_t = \bar{X}_t\delta e^{\delta t}\mathrm{d}t + e^{\delta t}\mathrm{d}\bar{X}_t\\
&= \bar{X}_t\delta e^{\delta t}\mathrm{d}t + e^{\delta t}\left(\eta\left(\hat{P}_t - \bar{h}\right)\mathrm{d}t + \delta\left(\frac{T-t}{T}x_0 - \bar{X}_t\right)\mathrm{d}t - \mathrm{d}\bar{W}_t\right)\\
&= e^{\delta t}\left(\eta\left(\hat{P}_t - \bar{h}\right)\mathrm{d}t + \delta\left(\frac{T-t}{T}x_0\right)\mathrm{d}t - \mathrm{d}\bar{W}_t\right).
\end{aligned}$$

Integrating this with initial condition x_0, we obtain

$$\bar{X}_t = e^{-\delta t}x_0 + e^{-\delta t}\int_0^t e^{\delta s}\eta\left(\hat{P}_s - \bar{h}\right)\mathrm{d}s + e^{-\delta t}\int_0^t e^{\delta s}\delta\left(\frac{T-s}{T}x_0\right)\mathrm{d}s - e^{-\delta t}\int_0^t e^{\delta s}\mathrm{d}\bar{W}_s.$$

To find x_0, we start with Equation (4.4). Then, it holds that

$$x_0 = l(\rho) - \int_0^T \mathbb{E}[a_t]\mathrm{d}t.$$

The expected values are given by

$$\begin{aligned}
\mathbb{E}[a_t] &= \delta\frac{T-t}{T}x_0 - \delta\,\mathbb{E}[\bar{X}_t],\\
\mathbb{E}[\bar{X}_t] &= e^{-\delta t}x_0 + e^{-\delta t}\mathbb{E}\left[\int_0^t e^{\delta s}\eta\left(\hat{P}_s - \bar{h}\right)\mathrm{d}s\right] + e^{-\delta t}\int_0^t e^{\delta s}\delta\left(\frac{T-s}{T}x_0\right)\mathrm{d}s + e^{-\delta t}\mathbb{E}\left[\int_0^t e^{\delta s}\mathrm{d}\bar{W}_s\right].
\end{aligned}$$

Since $\bar{W}$ is a martingale, the term on the right is equal to zero. To find the desired expression for x_0, we need to use the martingality of $\hat{P}$ and integration by parts. These arguments are quite lengthy and consist of integrating expressions several times. This is not done here. For a part of the details, see [AB23, Appendix A.3].

□

An expression for the optimal price can also be found in [AB23]. However, it relies on an expression for the optimal market price that we could not reproduce [AB23, Eq. (34)], so this part is omitted here as well. To calculate the associated social costs, we require the optimal market price. Consequently, we don't have a specific result that we can compare with the other cases. In the article cited, they rely on numerical results for these costs. We can thus not make any conclusions about the costs in this scenario.

Afterwards, numerical illustrations can be found in that article that show the differences in the three policies. It is found that the optimal allocation results in the least social costs, while the static allocation has the highest cost. The MSR falls in between these scenarios, and can be seen as intermediate policy between the static and dynamic allocation. It has higher costs than the optimal policy, but as an advantage only the observed emissions are needed, and not the observations of economic shocks. Fortunately, Biagini and Aïd can conclude that the optimal dynamic allocation outperforms this version of the MSR, in the situation that $E_0 = 0$ and under the assumption that the correlation structure is as in Remark 1.1.

This ends the section on the MSR. In the final section of this thesis, we will numerically compare the Brownian framework with the Geometric Brownian motion.

4.3 GBM and BM: a numerical approach

In this section, we will validate and visualise the results obtained in the previous chapters through numerical experiments. We will focus on the difference in results between Geometric Brownian motion (GBM) and Brownian motion (BM).

In Section 6.4 of [AB23], parameters are given for the numerical approaches. We will use these parameters in our numerical analysis as well. We need to be careful with the units of these parameters, as we would like to measure our emission processes (E_t^i) and (G_t^i) in Gigatons CO_2, abbreviated by Gton, as this is the sufficient unit of the emissions. An elaboration on this can be found in Appendix A.3. In this section, we will work with the results obtained there. We will primarily write the units for the BM scenario and adjust accordingly for the GBM.

In line with [AB23], the following parameters are chosen for determining the BAU emissions, in case of both the BM and the GBM. We will work with $N = 6$ firms and $\rho = 0.8$, such that the regulator aims for a 20 percent reduction of the BAU emissions. Additionally, we set $T = 5$ years, corresponding with one phase of the EU ETS. Furthermore, we set for every firm i ,

$$E_0^i = 1 \text{ Gton}\,, \quad \mu_i = \frac{1}{3}\frac{\text{Gton}}{\text{year}}, \quad \sigma_i = \frac{0.2}{\sqrt{6}}\frac{\text{Gton}}{\sqrt{\text{year}}},$$

where t is measured in years. With these numbers the drift is of higher order than the volatility, and we fulfill the assumption needed in [AB23]. Last, a specific matrix K needs to be chosen, such that we have

$$W = K\tilde{B},$$

where W is a seven-dimensional correlated Brownian motion, including the common shock, and $\tilde{B}$ the seven-dimensional independent Brownian motion representing the economic shocks, as constructed in (1.1). We choose a correlation matrix K that corresponds with the correlation structure of Aïd and Biagini, and with remark 1.1. We take $\kappa_i = 0.92$.

To be able to simulate the paths of E_t^i and G_t^i, we need to simulate the Brownian motion above. We are going to discretise the interval $[0,5]$ with 3000 steps, such that $\mathrm{d}t \approx 0{,}001667$. Because we have multiple firms, we make use of multidimensional tensors. This way, we can simulate a Brownian motion by sampling a standard normal random variable and multiplying it with $\sqrt{\mathrm{d}t}$ at every time step.

To check whether the simulation of the correlated Brownian motion are correct, we perform a Monte Carlo convergence test. In Figure 4.1, the results of this can be observed, with the number of paths plotted on the x-axis on a log scale, and the absolute error with zero of the mean over the paths depicted on the y-axis in on a log scale. This should converge with a rate of $-\frac{1}{2}$. When we plot a line with slope $-\frac{1}{2}$, we see that on average it seems that it does. Though there are some large fluctuations, we still work with this Brownian motion.

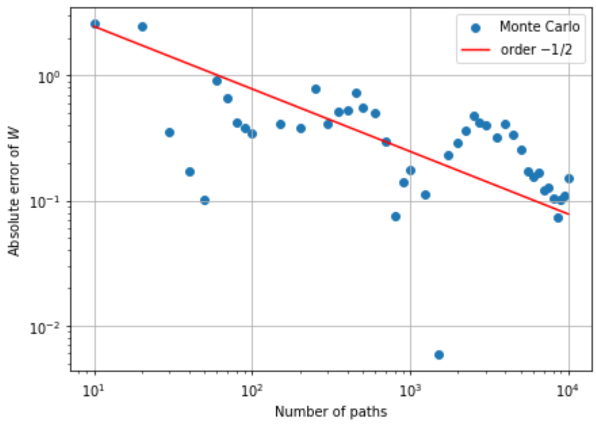

Figure 4.1: Monte Carlo error convergence, from 10 to 10000 paths.

With W and the parameters given above, the plots of (E_t^i) and (G_t^i) for a specific firm i can be obtained, which can be found in Figures 4.2 and 4.3 respectively. Since both emissions are random, we plot the mean and the corresponding standard deviation below and above the mean, over 1000 paths. This way, we can make appropriate conclusions about the dynamics. We will use this method for all the plots from now on.

We see that with the given parameters, the mean path and the standard deviation look similar. Note that the plot, also in the Brownian framework, gives a different outcome compared to

[AB23], as we have chosen $E_0 = 1 > 0$. We indeed see linear and exponential growth, respectively, as expected. Additionally, we observe that, on average, the paths in both cases grow and stay positive. The variance of the GBM is bigger, when time grows. The difference between the GBM and BM can be found in Figure 4.4. The mean grows to $-2,6$ for $t = 5$.

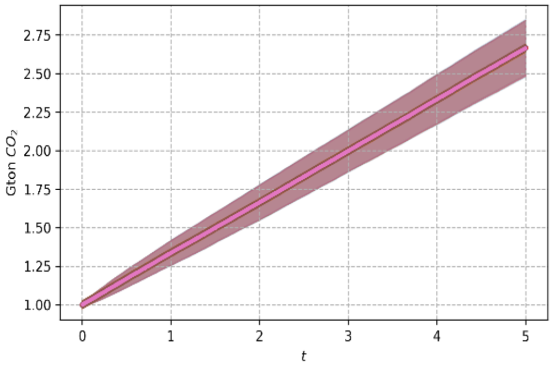

Figure 4.2: BAU emissions (E_t^i), modelled by a Brownian motion.

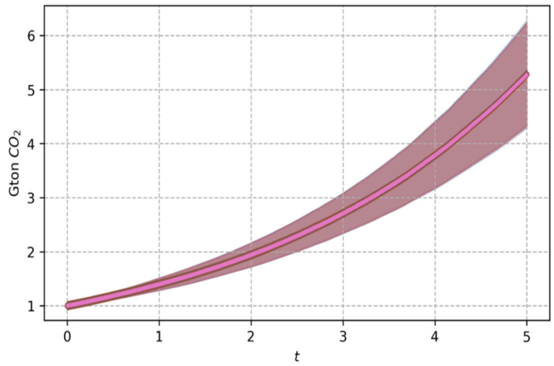

Figure 4.3: BAU emissions (G_t^i), modelled by a Geometric Brownian motion.

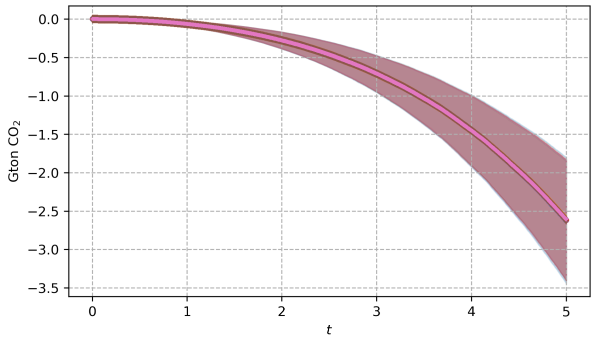

Figure 4.4: The difference between the BAU emissions (E_t^i) and (G_t^i).

Ideally, we would like to compare the outcome of the controls and allowances under the chosen parameters. However, this is not possible yet, as the emission paths are different. For this, we need to choose a sufficient drift and volatility, such that the emission path of the GBM matches the path of the BM in a better way.

This can be achieved by theoretically matching the mean and variance of the GBM with the mean and variance of BM, together with the given numbers in the article by Aïd and Biagini that are calibrated to the Brownian motion. Here, we already plug in $E_0^i = 1$. Then, by Proposition A.30, we need to solve for $\tilde{\mu}_i$ and $\tilde{\sigma}_i$ such that

$$\begin{aligned} \mathbb{E}\left[G_t^i\right] &= \exp\left(\tilde{\mu}_i t\right) = \mathbb{E}\left[E_t^i\right] = 1 + \mu_i t, \\ \mathrm{Var}\left[G_t^i\right] &= \exp\left(2\tilde{\mu}_i t\right)\left(\exp\left(\tilde{\sigma}_i^2 t\right) - 1\right) = \mathrm{Var}\left[E_t^i\right] = \sigma_i^2 t, \end{aligned}$$

where we have used the mean and variance of E_t^i, by 1.6. Solving the system of equations above for $\tilde{\mu}_i$ and $\tilde{\sigma}_i$, we get for $t \in (0, T]$,

$$\tilde{\mu}_i(t) = \frac{\log(1 + \mu_i t)}{t}, \quad \tilde{\sigma}_i(t) = \sqrt{\frac{\log\left(1 + \frac{\sigma_i^2 t}{\exp\left(2\tilde{\mu}_i t\right)}\right)}{t}} = \sqrt{\frac{\log\left(1 + \frac{\sigma_i^2 t}{(1+\mu_i t)^2}\right)}{t}}.$$

Note that the parameters are considered constant in our framework, but are time-dependent here. We solve this problem by plugging in $t = 5 = T$, such that

$$\tilde{\mu}_i = \frac{\log(1+5\mu_i)}{5} \approx 0,1962 \quad \tilde{\sigma}_i = \sqrt{\frac{\log\left(1+\frac{5\sigma_i^2}{\exp(10\tilde{\mu}_i)}\right)}{5}} = \sqrt{\frac{\log\left(1+\frac{5\sigma_i^2}{(1+5\mu_i)^2}\right)}{5}} \approx 0,0306.$$

This way, we make sure that the end points of the expected values, and thus the objective to reduce the emissions, of the regulator, exactly coincide. This implies that

$$\mathbb{E}\left[\sum_{i=1}^{N} E_T^i\right] = NE_0 + \mu_i T = 16 = \exp\left(\tilde{\mu}_i t\right) = E\left[\sum_{i=1}^{N} G_T^i\right],$$
$$\mathbb{E}\left[\sum_{i=1}^{N} E_T^{i,\hat{\alpha}^i}\right] = 0.8 \cdot 16 = 12,8 = \mathbb{E}\left[\sum_{i=1}^{N} G_T^{i,\hat{\alpha}^i}\right]$$

With these new, calibrated, parameters for the GBM, the BAU emissions (G_t^i) and the difference for a single firm are given in Figures 4.5 and 4.6 below. We indeed see that Figures 4.2 and 4.5 look similar and the axis have the same magnitude. The plot of the difference between BM and GBM substantiates this, as the mean only grows to 0,2. We see that both the mean and variance match at time $t = 0$ and $t = 5$, which substantiates that the moment matching is correctly executed.

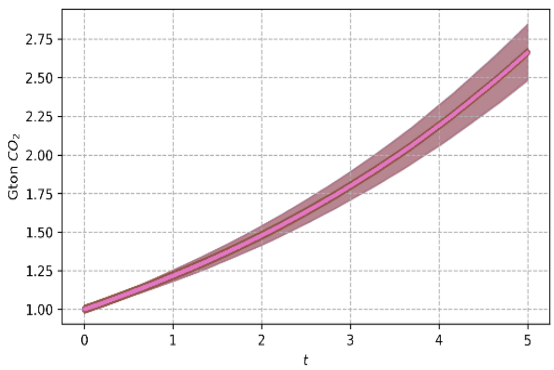

Figure 4.5: The calibrated BAU emissions (G_t^i).

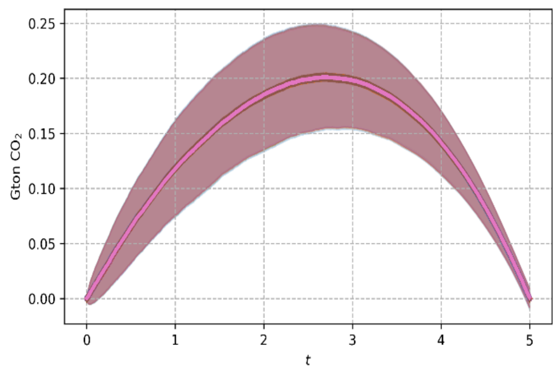

Figure 4.6: The difference between (E_t^i) and the calibrated (G_t^i).

Now we have found a good enough way to calibrate, we can compare the outcome of the variables in the BM and the GBM. First of all, we will look at the total emissions in the system, and the abated emissions, given by

$$\mathbb{E}\left[\sum_{i=1}^{N} E_T^{i,\hat{\alpha}^i}\right] = \mathbb{E}\left[\sum_{i=1}^{N} E_T^i\right] - N\hat{\alpha}_0 T, \quad \mathbb{E}\left[\sum_{i=1}^{N} G_T^{i,\hat{\alpha}^i}\right] = \mathbb{E}\left[\sum_{i=1}^{N} G_T^i\right] - N\hat{\alpha}_0 T, \tag{4.6}$$

since in the optimal dynamic allocation $\hat{\alpha}_t^i$ is constant, and given by the initial value $\hat{\alpha}_0$ for every firm. The abated emission will be a translation of the total emissions. For this, we have

$$\hat{\alpha}_0 = \eta_i\left(\hat{P}_0 - h_i\right), \tag{4.7}$$

where $\hat{P}_0$ is given by (2.28) in the Brownian framework and (3.11) in the GBM case. We will work with the following parameters in the Brownian framework, inspired by [AB23], recalculated to Gton CO_2, for every firm i,

$$\eta_i = 6 \cdot 10^{-10} \frac{(\text{Gton})^2}{€\,\text{year}}, \quad h_i = 2,5 \cdot 10^{10} \frac{€}{\text{year}}, \quad \lambda = 1,25 \cdot 10^{12} \frac{€}{(\text{Gton})^2}.$$

With these, $\hat{P}_0$ and $\hat{\alpha}_0$ can be obtained. With superscripts for the specific BAU emissions, this results in

$$\hat{P}_0^E \approx 2,517 \cdot 10^{10} \frac{€}{\text{Gton}}, \quad \hat{P}_0^G \approx 2,605 \cdot 10^{10} \frac{€}{\text{Gton}}, \quad \hat{\alpha}_0^G = \hat{\alpha}_0^E \approx 0,107 \frac{\text{Gton}}{\text{year}}. \tag{4.8}$$

The market prices are different, but the abatement efforts are still the same. This is interesting, but the difference disappears by (4.7), as η_i is very small. The fact that the abatement efforts coincide is interesting, and substantiates that our results are correct until here. With the abatement effort, we can plot the result of (4.6). The result of the mean and standard deviation, again simulated with 1000 paths, can be found in Figures 4.7 and 4.8 below. Here, the BAU emissions of Figures 4.2 and 4.5 are summed over the 6 firms.

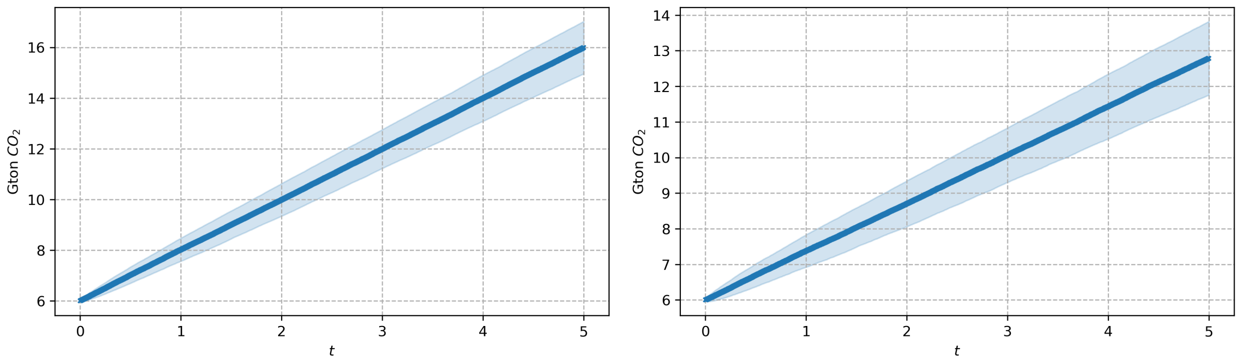

Figure 4.7: The cumulative BAU emissions of 6 firms, modelled by the Brownian motion on the left and the corresponding abated emissions on the right.

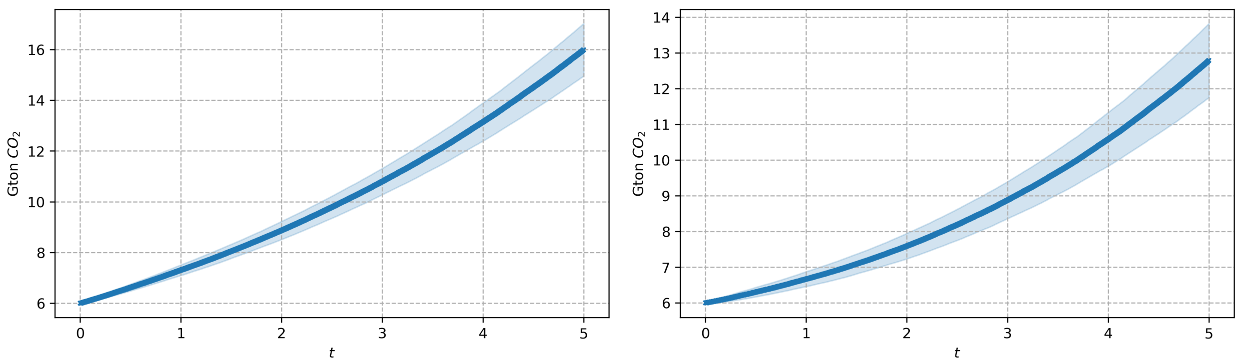

Figure 4.8: The cumulative BAU emissions of 6 firms, modelled by the calibrated Geometric Brownian motion on the left and the corresponding abated emissions on the right.

When we look at the axis of the figures, we see that the abated emissions are only a linear transformation of the total BAU emissions of the firms. Note that the standard deviation of the paths may look bigger in the figures on the right, but the axis is also different there. Hence, these are the same. We also see that both the BM and GBM end in the same point at time $t = 5$, which corresponds with our calibration.

The social costs in both situations can now be calculated, and are given by

$$C_{\text{opt}} = -\frac{6T}{2} h^2 \eta + \frac{6}{4\lambda} (2\lambda\eta T + 1) \hat{P}_0^2.$$

Since $\hat{P}_0^2$ in (4.8) is large, there will be a substantial difference between the social costs in the system in both cases. Indeed, the social costs in both scenarios are given by

$$C^E_{\text{opt}} \approx 8,105 \cdot 10^{10} \text{ euro}, \quad C^G_{\text{opt}} \approx 48,492 \cdot 10^{10} \text{ euro}.$$

These are the costs to reduce the emissions the total emissions in the system from around 16 Gton to 12,8 Gton. We see that the costs in the GBM scenario are around 6 times larger than in the BM scenario. This may be a disadvantage of the GBM and could be an indicator that we should prefer the Brownian framework, as lower costs are beneficial. However, we could argue that the costs have the same order of magnitude.

Next, we will investigate the allowances process $(\tilde{A}^i_t)$, for a specific firm i. Note that in the two examples we are comparing, we have $\tilde{A}^i_t = \tilde{M}^i_t$, such that

$$\begin{aligned}
\tilde{A}^{i,E}_t &= l(\rho) + \mu_i t + \sigma_i W^i_t, \\
\tilde{A}^{i,G}_t &= \nu(\rho) + E^i_0 \exp\left(\tilde{\mu}_i T - \frac{1}{2}\tilde{\sigma}_i{}^2 t + \tilde{\sigma}_i W^i_t\right) - E^i_0 \exp\left(\tilde{\mu}_i T\right),
\end{aligned} \tag{4.9}$$

where the superscripts are given for the BAU emissions and $l(\rho)$ and $\nu(\rho)$ are given in (2.38) and (3.13) and are defined as the average cumulative allocation over all the firms. Since we choose $E^i_0 = 1$, the signs of these constants are not yet determined. With the chosen parameters, they are given by

$$l(\rho) = \frac{1}{6}\sum_{i=1}^{6} \mathbb{E}\left[\tilde{A}^{i,E}_T\right] + \mu_i T \approx 2,123\,, \quad \nu(\rho) = \frac{1}{6}\sum_{i=1}^{t} \mathbb{E}\left[\tilde{A}^{i,G}_T\right] \approx 2,123.$$

On average, the allowances are the same. The processes $(\tilde{A}^i_t)$ are given below.

o Both allowances processes are in general positive. In Figure **??**, we see that the variance is quite small, and increases when time increases. The allowances increase over time as well. Since it is the cumulative allowances process, we see that the regulator allocates true permits at every time step. In Figure **??**, the allowances process is constant over time, which implies only an initial allocation. This is an interesting and unexpected result, as we would expect a similar process to the BM scenario. From a theoretical standpoint, we can see that, since $\tilde{\sigma}_i$ is small and there is no drift term present in $\tilde{A}^{i,G}_t$, the exponent will be almost constant. The result thus makes sense if we look at formula (4.9). In Figure **??**, we see that the mean of the GBM corresponds with the static scenario of Section 4.1, since there is only an initial allocation, and afterwards the allocation is zero. The standard deviation seems fairly large, but note again the differences in the y-axis in the left and right figure. When we look carefully, the standard deviations are of the same order of magnitude.

Clearly, future research is necessary to draw conclusive results about the comparison between the Brownian framework and the modelling by a Geometric Brownian motion. The BAU emissions, while calibrated the GBM to the BM, and the abatement effort are fairly similar. However, there are interesting differences between the market price and the social costs and the allowances processes. We are not yet able to substantiate these differences.

Conclusion

In this thesis we have explored a dynamic policy within a version of a cap-and-trade system, drawing inspiration from the Market Stability Reserve (MSR) mechanism of the European Union. The dynamic policy, allowing for allocation at any point in the time period, is based on allowances modelled as adapted stochastic processes. Building on the work of René Aïd and Sara Biagini [AB23], this thesis extends their method to the case where the BAU emissions are modelled using a Geometric Brownian motion. This eliminates the need to make any assumptions about the drift and volatility of the SDE for the BAU emissions. We have seen that the obtained optimal policy is analytically tractable. Furthermore, a numerical comparison between the Brownian framework and the case of Geometric Brownian motion has been included. We demonstrated that the allowances process and social costs differ, for a specific choice of variable set.

We have described a precise and detailed mathematical model, providing a thorough exposition of the article by Aïd and Biagini and incorporating many relevant mathematical details. In Chapter 1, we have built up the space of admissible controls in detail, as well as the SDEs of the bank account and the BAU emissions. Additionally, in Chapters 2 and 3, several results from variational calculus are used to find the solution to the stochastic control problem. Furthermore, we have generalised the correlation structure between the firms, and added an initial value to the Brownian framework.

In summary, this thesis gives a comprehensive, mathematical framework of the optimal dynamic policy proposed by Aïd and Biagini, and contributes to the existing literature by offering analytical solutions in scenarios where assumptions are relaxed.

Outlook

This section provides an overview of topics that are interesting for future work. It includes ideas that we have discussed, tried, or find interesting to explore in general.

First, as already indicated in Equation (1.42), the penalty function in the objective function of the firms we are currently working with, is not entirely realistic. It would be beneficial to explore the possibility of using another, more realistic, penalty function of which analytical solutions still can be obtained. A suggestion would be

$$\lambda \max\left(-X_T^i, 0\right)^2 = \begin{cases} \lambda \left(X_T^i\right)^2 & \text{when } X_T^i \leqslant 0, \\ 0 & \text{else.} \end{cases}$$

This function aligns with the current penalty function but imposes a penalty only when more emissions are used than allowances available. An important consideration is that the penalty needs to be (Fréchet) differentiable in the controls. A derivative could be found by considering an approximation to smooth out the maximum. However, this aspect is not explored in this thesis, and we are not entirely sure whether it would yield an analytical solution. In general, to stay in the framework presented in Chapter 2, it is crucial that the penalty function is continuous, coercive and convex in the controls. The last property seems to be the most difficult to achieve. The initial choice $\lambda(X_T^i)^2$ satisfies these properties, albeit not being completely realistic.

This thesis focuses on an analytical approach for optimal dynamic regulation. In the final section, we have performed a numerical comparison between the Brownian framework and Geometric Brownian motion. The results regarding the social costs and allowances processes are intriguing and unexplainable, primarily due to time constraints. As a result, we cannot draw conclusions yet regarding the advantages or disadvantages of using a Geometric Brownian motion versus a Brownian motion. Therefore, there is ample room for future research to numerically compare the Brownian framework and Geometric Brownian motion for different sets of variables. For instance, the impact of a small initial value E_0 on $\bar{M}_0$ in both cases could be evaluated, as already indicated in Chapter 3. Additionally, we could explore various correlation matrices for comparison and conducting stress tests on specific variables could reveal advantages or disadvantages of the use of the Geometric Brownian motion.

Another idea involves the incorporation of an exchange option in the model, to allow for an exchange in allowances at terminal time T. In this scenario, a firm that ends up with too few allowances could trade with a firm that has excess of allowances. This option could be purchased at time $t = 0$ for a firm that anticipates having too few or too many allowances. It would facilitate a direct exchange between two firms without interacting with the Stackelberg game. Although we believe it is possible to integrate this into the framework, we have not been able to do so.

Moreover, we can think of other models for the modelling of the BAU emissions. In this thesis, the novel part is the modelling by a Geometric Brownian motion. One could explore the possibility of introducing jumps into the system. An important consideration is that the modelling should be realistic and, ideally, analytical solutions should be attainable.

Additionally, further research could explore the impact of the general correlation structure and assess whether the outcomes regarding the correlations are in line with our expectations. For instance, when considering a few firms that are highly correlated, do their optimal allocations resemble each other? Furthermore, more than one time period could be considered. For this, we should be carefully how the banking of allowances from one time period to the other is modelled. Another idea would be to incorporate the concept of inflation in the system, potentially by adding another SDE. This results in a system of coupled SDEs.

Another consideration would be to precisely connect the MSR-scenario exactly to our framework. In Section 4.2, a version of the MSR scenario is linked to our dynamic framework. However, this representation does not precisely reflect the workings of the MSR. The social costs in this scenario are not obtained explicitly either. As the MSR serves as one of the main motivations for this optimal dynamic policy, a detailed comparison would be beneficial.

The final suggestion involves conducting a numerical analysis of the entire optimal dynamic allocation framework. While in this thesis, as well as in [AB23], the analytical solutions are visualised through some numerical experiments, a comprehensive numerical calculation of the entire framework has not been undertaken. The analytical solutions allow for a comparison between these and the numerical solutions to validate the results . This analysis would involve the three steps of the Stackelberg game. Careful consideration is needed for the programming of the stochastic control problems, with a specific focus on the conditional expectation (M_t^i).

This concludes the overview of some interesting, further possible research. We think that many of these ideas could contribute to the current research, but due to time constraint, we were not able to investigate these thoroughly.

A | Mathematical Insights

In this first appendix, we present several mathematical insights that were omitted in the main part of this thesis. The first section covers the prerequisites and important propositions used in this thesis, presented here for the sake of completeness. The second section includes mathematical proofs that were excluded from the main body. In the third section, we elaborate on the units of the variables used in the models.

A.1 Mathematical background

This section offers a comprehensive presentation of the key mathematical concepts that we use in this thesis. It consists of several subsections that can mostly be read independently of each other.

In this section, we work with a given probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Unless mentioned otherwise, we work with time t such that $0 \leqslant t < \infty$, which will often be denoted by $t \geqslant 0$.

We will write a stochastic process $(X_t)_{t\geqslant 0}$ as (X_t) or X. The same will hold for a filtration. When referring to the random variable in time $t \in [0, T]$, we will use the notation X_t. Furthermore, throughout this thesis, the one-dimensional Lebesgue measure is denoted by λ^1. However, in many instances, whenever it is clear from the context context, we will write $\mathrm{d}t$, when we mean $\mathrm{d}\lambda^1$. Here, we make use of the relationship between the Lebesgue integral and the Riemann integral. More on this can be found in [Sch17, ch. 12].

A.1.1 Stochastic processes and filtrations

The following definitions introduce concepts for filtrations.

Definition A.1 (Right-continuous filtration). *[KS91, pg.4] A filtration $(\mathscr{F}_t)$ is said to be right-continuous if*

$$\mathscr{F}_t = \bigcap_{\varepsilon>0} \mathscr{F}_{t+\varepsilon},$$

for all $t \geqslant 0$.

Definition A.2 (Usual conditions). *[Shr+04, pg. 10] A filtration $(\mathscr{F}_t)$ is said to satisfy the usual conditions if it is right-continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-nullsets.*

The form of filtration used throughout this thesis is specified below.

Definition A.3 (Filtration generated by a Brownian motion). *[KS91, pg. 89] Let W be a d-dimensional Brownian motion. Then, the filtration generated by the Brownian motion W is denoted by $(\mathscr{F}_t^W)$ and defined as*

$$\mathscr{F}_t^W = \sigma(W_s : 0 \leqslant s \leqslant t).$$

With the definition above, we can define the augmentation of a specific sigma-algebra.

Definition A.4 (Augmented Brownian filtration). *[KS91, pg. 89] Let W be a d-dimensional Brownian motion and $(\mathscr{F}_t^W)$ the corresponding filtration generated by W, defined in Definition A.3. Let*

$$\mathcal{N} = \{F \subseteq \Omega : \exists G \in \mathscr{F} \text{ s.t. } F \subseteq G, \mathbb{P}(G) = 0\},$$

that is, $\mathcal{N}$ is the collection of $\mathbb{P}$ null-sets. Then, the augmented Brownian filtration is given by

$$\mathscr{F}_t = \sigma\left(\mathscr{F}_t^W \cup \mathcal{N}\right).$$

Note that in the definition above, we need another "σ"-sign around the union, as the union of two sigma-algebras is not necessarily a sigma-algebra itself. The concepts above can be linked as follows.

Proposition A.1. *The constructed augmented Brownian filtration $(\mathscr{F}_t)$ satisfies the usual conditions.*

Proof. By construction, $(\mathscr{F}_t)$ contains all $\mathbb{P}$ null-sets. By [KS91, pg. 90], this augmented filtration is right-continuous. □

After the relevant information on filtrations is introduced, two concepts for two stochastic processes being equal are presented.

Definition A.5 (Modification). *[Chu13, pg. 28] A process Y is a modification of a process X, if for every $t \geqslant 0$, we have*

$$\mathbb{P}(X_t = Y_t) = 1.$$

That is, for all $t \geqslant 0$, we have

$$X_t = Y_t \quad \mathbb{P} \text{ a.s.}$$

Definition A.6 (Indistinguishable). *[JYC09, pg. 11] Processes X and Y are called indistinguishable if*

$$\mathbb{P}\left(X_t = Y_t, \text{ for all } t \geqslant 0\right) = 1.$$

Note that being indistinguishable implies that almost all sample paths agree. If two processes are indistinguishable, it holds that the processes are also modifications. Although indistinguishability is stronger than a modification, we will see that it is often sufficient enough to work with a modification of a process.

The following proposition is useful, and given without proof.

Proposition A.2. *[CW90, pg. 9] A stochastic process with continuous sample paths is uniquely indistinguishable with a process that has a.s. continuous paths.*

The following definition is helpful to be able to identify when a specific martingale has a modification.

Definition A.7 (Càdlàg). *[KS91, pg. 4] A stochastic process is said to be càdlàg (in French: "continue à droite, limite à gauche") if the process is right-continuous on $[0,\infty)$ and has finite left limits on $(0,\infty)$.*

Often, we will write cadlag, without the accents.

With a couple of the previously mentioned definitions and the constructed filtration satisfying the usual conditions, the following holds.

Proposition A.3. *[Chu13, pg. 30] Let* (X_t) *be a martingale with respect to the filtration* $(\mathscr{F}_t)$ *of Definition A.4. Then,* (X_t) *has a cadlag modification.*

The proof is omitted here and can be found in the reference cited.

Remark A.1. *Note that in [Chu13], it is initially only mentioned that there is a right-continuous modification, not necessarily with left limits. However, from Corollary 1 on page 26 of the same reference, we see that a right-continuous random variable always has left limits. We can conclude that the modification is not only right-continuous, but also a cadlag modification.*

Next, a stronger concept of measurability is defined.

Definition A.8 (Progressively measurable). *[KS91, pg. 4] A* d*-dimensional stochastic process* X *is called progressively measurable with respect to the filtration* $(\mathscr{F}_t)$ *if the mapping* $(s,\omega) \to X_s(\omega)$ *from* $[0,t] \times \Omega$ *to* $\mathbb{R}^d$ *is* $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$ *measurable for all* $t \geqslant 0$.

The latter could also be written as the fact that $X : [0,t] \times \Omega \to \mathbb{R}^d$ is $(\mathscr{B}([0,t]) \otimes \mathscr{F}_t) / \mathscr{B}\left(\mathbb{R}^d\right)$ measurable. It follows immediately that a progressively measurable process is adapted and measurable. More can be said.

Proposition A.4. *The pointwise limit of progressively measurable processes* $(X^n)_{n\in\mathbb{N}}$ *is again progressive. That is, if for all* $n \in \mathbb{N}$, X^n *is progressive, and if the limit* X^n *exists for* (t,ω)*, it holds that*

$$\lim_{n\to\infty} X^n(t,\omega) = X(t,\omega),$$

is also progressive.

Proof. This follows directly from the fact that the pointwise limit of measurable functions is again measurable. □

Proposition A.5. *[KS91, pg. 5] Let* X *be a right-continuous stochastic process. Then,* X *is progressively measurable if it is adapted to the filtration.*

Sketch of proof. The full proof can be found in the reference cited. It relies on writing the process as a limit, since right-continuity is given. This limit appears to be progressively measurable. By the previous proposition, we can conclude that the whole process is progressively measurable. □

The following proposition gives us information about the time integral of a progressively measurable process.

Proposition A.6. *Let* (X_t) *be a progressively measurable process. Then, the process* (Y_t) *given by*

$$Y_t = \int_0^t X_s ds,$$

is progressively measurable as well for all $t \geqslant 0$.

Proof. Let $\{t_1, \dots t_n\}$ be a partition of $[0,t]$. We can write

$$Y_t(\omega) = \lim_{n\to\infty} \sum_{i=1}^n X_{t_i}(\omega)(t_i - t_{i-1}),$$

by the definition of an integral. Since for all i, $t_i \leqslant t$, it holds that $X_{t_i}(\omega)$ is $\mathscr{F}_{t_i}$ measurable. By the properties of a filtration, it follows that Y_t is $\mathscr{F}_t$ measurable. Furthermore, Y is right-continuous by construction, since a Riemann-integral is right-continuous. By Proposition A.5, it follows that (Y_t) is progressively measurable. □

In general, we have the following definition for variables that have a sufficient finite p-th moment.

Definition A.9 ($\mathscr{L}^p$-space). *[Spr12, pg. 39] Let $1 \leqslant p < \infty$ and X a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$. Then, $X \in \mathscr{L}^p(\Omega, \mathscr{F}, \mathbb{P})$, if*

$$\mathbb{E}\left[|X|^p\right] < \infty.$$

Definition A.10 (Square-integrable). *[AB23] A stochastic process (X_t), defined for $t \in [0, T]$, is square-integrable (with respect to the measure $\mathbb{P}$) if*

$$\int_0^T |X_t|^2 \, d\mathbb{P} = \mathbb{E}\left[|X_t|^2\right] < \infty,$$

for all $t \in [0, T]$. It is called square-integrable with respect to the measure $\mathbb{P} \times \lambda^1$ if

$$\int_0^T |X_t|^2 d\left(\mathbb{P} \times t\right) = \mathbb{E}\left[\int_0^T X_t^2 dt\right] < \infty,$$

where we recall that we write dt for the Lebesgue measure $d\lambda^1$.

Often, the measure to which it is square-integrable will be omitted, when it is clear from the context which one we are using.

In the following chapters, we will work with the space $L^2(\Omega, \mathscr{F}, \mathbb{P})$, which will be introduced below. For general $1 \leqslant p < \infty$, there is a subtle difference between $\mathscr{L}^p(\Omega, \mathscr{F}, \mathbb{P})$ and $L^p(\Omega, \mathscr{F}, \mathbb{P})$ [Sch17, pg. 118]. It holds that the norm defined on $\mathscr{L}^p(\Omega, \mathscr{F}, \mathbb{P})$ is defined as

$$||X||_p = \mathbb{E}\left[|X|^p\right]^{1/p}. \tag{A.1}$$

It fulfils almost all the requirements to be a norm. However, it does not hold that $||X||_p = 0$ implies $X = 0$, as this only holds almost surely. To overcome this problem, we can identify an equivalence relation $X \sim Y$ if and only if $\mathbb{P}(X \neq Y) = 0$ for two random variables X, Y. That is, X is equivalent to Y if and only if $X = Y$ almost surely. We then define the quotient space

$$L^p(\Omega, \mathscr{F}, \mathbb{P}) = \mathscr{L}^p(\Omega, \mathscr{F}, \mathbb{P}) \big/ \sim.$$

This way, the space $L^p(\Omega, \mathscr{F}, \mathbb{P})$ has a general norm given by (A.1). To have a properly defined normed vector space, we should work with $L^p(\Omega, \mathscr{F}, \mathbb{P})$. For $p = 2$, the following inner product and norm hold on this space [Sch17, pg. 341]

$$\langle X, Y\rangle_2 := \mathbb{E}[XY],$$
$$||X||_2 = \sqrt{\mathbb{E}\left[X^2\right]} = \sqrt{\langle X, X\rangle_2}.$$

Now that an appropriate variable in a general L^p space is defined, we introduce a notion of convergence and an important inequality.

Definition A.11 (Convergence in L^p). *[Sch17, pg. 120] Let $p \geqslant 1$ and $||\cdot||_p$ the corresponding p-norm. A sequence of random variables $(X^n)_{n\in\mathbb{N}}$ in $L^p(\Omega, \mathscr{F}, \mathbb{P})$ converges in L^p to a process X if*

$$\lim_{n\to\infty} ||X^n - X||_p = 0,$$

We write $X^n \xrightarrow{L^p} X$ then.

Proposition A.7 (Hölder's inequality). *[Sch17, pg. 117] Let $X \in L^p(\Omega, \mathscr{F}, \mathbb{P})$ and $Y \in L^q(\Omega, \mathscr{F}, \mathbb{P})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$||XY||_1 \leqslant ||X||_p ||Y||_q.$$

From Hölder's inequality, the well-known Cauchy-Schwarz inequality can be easily derived, with $p = q = 2$.

The next theorem addresses the interchangeability of double integrals and is commonly referred to as a "Fubini argument".

Theorem A.8 (Fubini's theorem). *[Sch17, pg. 142] Let $(X,\mathscr{B},\mu)$ and $(Y,\mathscr{C},\nu)$ be σ-finite measure spaces and $u : X\times Y \to \mathbb{R}$ be $\mathscr{B}\otimes\mathscr{C}$ measurable. If at least one of the following three integrals is finite,*

$$\int_{X\times Y} |u| d(\mu\times\nu), \int_Y\int_X |u(x,y)|d\mu(x)d\nu(y), \int_X\int_Y |u(x,y)|d\nu(y)d\mu(x),$$

then all are finite, and

$$\int_{X\times Y} |u| d(\mu\times\nu) = \int_Y\int_X |u(x,y)|d\mu(x)d\nu(y) = \int_X\int_Y |u(x,y)|d\nu(y)d\mu(x).$$

Furthermore, then it holds that

$$\int_X |u(x,y)|d\mu(x) < \infty \quad ,\nu \text{ a.e,} \quad \int_Y |u(x,y)|d\nu(y) < \infty \quad ,\mu \text{ a.e.}$$

The first part can be extended to interchanging an integral and a conditional expectations, also referred to as a "Fubini argument for conditional expectations", for example, explained in [Sch17, pg. 354]. The following characterisation of processes is useful for the definition that is mentioned afterwards.

Definition A.12 (Bounded variation). *[JYC09, pg. 12] Let Π be partition of [0,T], given by $\{t_1,\dots t_{n-1},t_n\}$, such that $0 = t_1 \leqslant \cdots \leqslant t_{n-1} \leqslant t_n = T$. The variation of a cadlag process (A_t), where $t \in [0,T]$, over the partition Π is given by*

$$V(A,\Pi) := \sum_{i=1}^{n} \left|A_{t_i}(\omega) - A_{t_{i-1}}(\omega)\right|.$$

The process (A_t) is said to be of bounded variation (over finite intervals) if

$$\sup_{\Pi} V(A,\Pi) < \infty, \quad a.s,$$

where the supremum is taken over all possible partitions of $[0,T]$.

The next definition will involve the concept of a local martingale. In our case, the local martingales in the following definition are always martingales. The reader interested in learning more about local martingales is referred to[KS91].

Definition A.13 (Semimartingale). *[Pha09, pg. 11] A cadlag, adapted, stochastic process X is called a semimartingale if it admits the following decomposition*

$$X = X_0 + A + M,$$

where A is an adapted process of bounded variation with $A_0 = 0$ and M a cadlag local martingale with $M_0 = 0$. A continuous semimartingale is a semimartingale for which A and M are continuous.

At last, we introduce a lemma on the notion of a singular measure, which also contains the definition[Spr12, pg. 61]. The lemma appears to be useful in one of the proofs.

Lemma A.9. *Let (X,Σ,μ) and (X,Σ,ν) be measure spaces, with μ and ν possibly signed measures. The measures μ and ν are called singular, when there exists disjoint $E,F \in \Sigma$ such that for all $A \in \Sigma$ it holds that*

$$\mu(A) = \mu(A\cap E), \quad \nu(A) = \nu(A\cap F).$$

Now let (X,Σ,τ) be another measure space with positive measure τ. If both μ and ν are singular with respect to the same measure τ, it also holds for $\mu+\nu$.

Proof. It is given that there exists disjoint $E_1, F_1 \in \Sigma$ and $E_2, F_2 \in \Sigma$ such that for all $A \in \Sigma$

$$\begin{aligned} &\mu(A) = \mu(A \cap E_1), \quad \tau(A) = \tau(A \cap F_1), \\ &\nu(A) = \nu(A \cap E_2), \quad \tau(A) = \tau(A \cap F_2), \end{aligned} \tag{A.2}$$

Let $E := E_1 \cup E_2$, $F := F_1 \cap F_2$. Then E and F are disjoint. Let $A \in \Sigma$. Then,

$$\tau(A) = \tau(A \cap F_1) = \tau((A \cap F_1) \cap F_2) = \tau(A \cap F),$$

by applying (A.2) twice. Furthermore, it holds that

$$\begin{aligned} \mu(A) &= \mu(A \cap E) + \mu\left(A \cap E^c\right) = \mu(A \cap E) + \mu\left(A \cap E_1^c \cap E_2^c\right) \\ &= \mu(A \cap E) + \mu\left(A \cap E_1^c \cap E_2^c \cap E_1\right) = \mu(A \cap E) + \mu(\emptyset) = \mu(A \cap E), \end{aligned}$$

by the properties of a signed measure. Hence, we can conclude that $\mu(A \cap E) = \mu(A)$. The exact same can be proven for ν. We can conclude that $\mu + \nu$ and τ are singular measures. □

With all this knowledge on stochastic processes, we have sufficient information to proceed to the next section.

A.1.2 Stochastic integration

In this section, the main results to define the quadratic variation and the stochastic integral will be stated. Before we start with the relevant definitions, a space needs to be introduced.

Definition A.14. *By the class $\mathcal{M}_2$ we denote the class of right-continuous square-integrable martingales that start at zero almost surely. That is, $X \in \mathcal{M}_2$, if $X_0 = 0$ a.s, right-continuous and square-integrable with respect to $\mathbb{P}$, as defined in Definition A.10.*

Now, we are ready to start with the first important definition.

Definition A.15 (Quadratic variation). *[KS91, pg. 31] Let $X \in \mathcal{M}_2$. The quadratic variation process of X, denoted by $\langle X \rangle$, is the unique, up to indistinguishability, adapted, natural and increasing process, with $\langle X \rangle_0 = 0$ a.s, and $X^2 - \langle X \rangle$ a martingale. It is continuous, if X is continuous.*

Here, a natural process is a process that satisfies some regularity conditions. From the increasingness, the following corollary can be deduced.

Corollary A.10. *Let $X \in \mathcal{M}_2$. The quadratic variation is a non-negative process. That is, for all $t \geqslant 0$ it holds that*

$$\langle X \rangle_t \geqslant 0,$$

Proof. Since $\langle X \rangle_0 = 0$ and $t \to \langle X \rangle_t$ is non-decreasing, the result follows immediately. □

The definition of quadratic variation can be extended to the following.

Definition A.16 (Quadratic covariation). *[KS91, pg. 31, 35] Let $X, Y \in \mathcal{M}_2$. The quadratic covariation process $\langle X, Y \rangle$ is defined to be the unique, up to indistinguishability, adapted, natural and increasing process such that*

$$XY - \langle X, Y \rangle,$$

is a martingale. It is continuous if X and Y are continuous.

From the above, it follows that $\langle X \rangle = \langle X, X \rangle$. Note that the quadratic covariation and inner product may share the same notation. Often it will be made clear from the context which one is meant. In the following parts, we mean the quadratic covariation. The following corollary will be used without proof, but can be straightforwardly proven with the definitions above.

Corollary A.11 (Bilinear). *The quadratic covaration process is a bilinear and symmetric process. That is, let $X, Y, Z \in \mathscr{M}_2$ and $a, b \in \mathbb{R}$. Then,*

$$\begin{aligned}\langle X, Y\rangle &= \langle Y, X\rangle,\\ \langle aX + bY, Z\rangle &= a\langle X, Z\rangle + b\langle Y, Z\rangle.\end{aligned}$$

This implies that

$$\langle aX + bY, aX + bY\rangle = a^2\langle X\rangle + 2ab\langle X, Y\rangle + b^2\langle Y\rangle.$$

Example A.1 (Quadratic variation of a Brownian motion). *Let W be a Brownian motion. Then, $W \in \mathscr{M}_2$. It is a well-known result that*

$$\langle W\rangle_t = t.$$

This won't be proven here. For more details, see for example, [JYC09, pg. 27] or [Shr+04].

Proposition A.12. *Let W^1, W^2 be two independent Brownian motion with respect to the augmented Brownian filtration $(\mathscr{F}_t)$ generated by these processes. Then,*

$$\left\langle W^1, W^2\right\rangle_t = 0,$$

for all $t \geqslant 0$.

Proof. Clearly, $W^1, W^2 \in \mathscr{M}_2$. By definition A.16, we need to show that W^1W^2 is a martingale. Integrability and adaptedness follow immediately. Left to show is that the martingale property holds. Let $t > s$. Indeed,

$$\mathbb{E}\left[W_t^1W_t^2|\mathscr{F}_s\right] = \mathbb{E}\left[\left(W_t^1 - W_s^1\right)(W_t^2 - W_s^2)|\mathscr{F}_s\right] + \mathbb{E}\left[W_s^1W_t^2|\mathscr{F}_s\right] + \mathbb{E}\left[W_t^1W_s^2|\mathscr{F}_s\right] - \mathbb{E}\left[W_s^1W_s^2|\mathscr{F}_s\right].$$

By well-known properties of the Brownian motion, this reduces to

$$\begin{aligned}\mathbb{E}\left[W_t^1W_t^2|\mathscr{F}_s\right] &= \mathbb{E}\left[\left(W_t^1 - W_s^1\right)(W_t^2 - W_s^2)\right] + W_s^1\mathbb{E}\left[W_t^2|\mathscr{F}_s\right] + W_s^2\mathbb{E}\left[W_t^1|\mathscr{F}_s\right] - W_s^1W_s^2\\ &= \mathbb{E}\left[\left(W_t^1 - W_s^1\right)\right]\mathbb{E}\left[\left(W_t^2 - W_s^2\right)\right] + 2W_s^1W_s^2 - W_s^1W_s^2\\ &= W_s^1W_s^2.\end{aligned}$$

We can conclude that (W^1W^2) is a martingale and thus the result follows. □

Proposition A.13. *Let $X \in \mathscr{M}_2$. Then*

$$X_t = 0,\ \textit{a.s, for all } t \geqslant 0, \quad \textit{if and only if} \quad \langle X\rangle = 0.$$

Proof. Let $X_t = 0$, a.s. By Definition A.15, it holds that $\langle X\rangle$ is the unique process such that $X^2 - \langle X\rangle$ is a martingale. It holds that $X_t^2 = 0$, a.s, hence $X_t^2 = 0$ is a martingale. It suffices to choose $\langle X\rangle = 0$, the desired result.

Now let $\langle X\rangle_t = 0$ for all $t \geqslant 0$. Then it follows that X^2 is a martingale, given the fact that X is a martingale. Now $X_0 = 0$, a.s. (by definition of $\mathscr{M}_2$), we have

$$\mathbb{E}\left[X_t^2\right] = \mathbb{E}\left[X_0^2\right] = 0,$$

by definition of a martingale. It follows that

$$X_t^2 = 0, \quad \text{a.s},$$

for all $t \geqslant 0$. Thus $X_t = 0$, a.s, for all $t \in [0, T]$, by [Sch17, pg. 116]. □

Up to this point, we have focused on the quadratic variation of a sufficient martingale. Next, we will look at the quadratic variation of a semimartingale, where the local martingale part is a martingale starting in zero.

Definition A.17. *[JYC09, pg. 29] Let X and Y be continuous semimartingales given by*

$$X_t = X_0 + A_t + M_t, \quad Y_t = Y_0 + B_t + N_t,$$

where X_0, Y_0 are non-random starting points, A, B processes of bounded variation and M, N square-integrable martingales starting in zero. Then for all $t \geqslant 0$,

$$\langle X \rangle_t = \langle M \rangle_t, \quad \langle X, Y \rangle_t = \langle M, N \rangle_t.$$

that is, the initial value and bounded variation part are not part of the quadratic variation.

Corollary A.14. *Let W a Brownian motion and $Y_t = t$ for all $t \geqslant 0$, then*

$$\langle W, Y \rangle = 0.$$

Proof. Note that Y is a semimartingale with zero martingale part, as it is of bounded variation. Hence, by Definition A.13 it follows that

$$\langle W, Y \rangle = \langle W, 0 \rangle = 0,$$

by the uniqueness of the quadratic covariation. □

Now that the theory on quadratic variations has been introduced, the notion of the stochastic integral will be introduced. Here, only the most important definitions and results are mentioned. This overview is far from complete. Again, the reader who is interested in more information, is referred to [KS91] and [CW90].

Note that integration with respect to the quadratic variation is defined in the Lebesgue-Stieltjes sense, see [KS91, pg. 35]. With this in mind, the following definitions make sense. They consist of several parts.

Definition A.18. *[KS91, pg. 130] Two stochastic processes X, Y are defined to be equivalent if $X = Y$ $\mathbb{P} \times \lambda^1$ almost everywhere. Denote by M a continuous, square-integrable martingale with $M_0 = 0$, a.s, adapted to a filtration $(\mathscr{F}_t)$, satisfying the usual conditions. Let $\mathscr{L}^*(M)$ be the space of equivalence classes of progressively measurable processes with*

$$[X]_{M,T} := \sqrt{\mathbb{E}\left[\int_0^T X_t^2 d\langle M \rangle_t\right]} < \infty,$$

for all $T > 0$. Furthermore, we define

$$[X]_M := \sum_{n=1}^{\infty} 2^{-n} \left(\min(1, [X]_n\right).$$

Definition A.19 (Stochastic integral). *[KS91, pg. 139] Let $X \in \mathscr{L}^*(M)$. Now, the stochastic integral of X with respect to $M \in \mathscr{M}_2$ where M continuous, is the unique, square-integrable martingale $I(X)$, adapted to the same filtration as M, that satisfies*

$$\lim_{n \to \infty} \left\| I(X^n) - I(X) \right\|_M = 0,$$

where $I(X^n)$ is the stochastic integral in terms of a simple process and $||I(X)||_M := [X]_M$. The above needs to hold for every sequence of simple processes $(X^n)_{n \in \mathbb{N}}$ that satisfies

$$\lim_{n \to \infty} \left[X^n - X\right]_M = 0.$$

We write

$$I_t(X) = \int_0^t X_s dM_s,$$

for $t \geqslant 0$.

Sometimes, we will write

$$I_t(X) = \int_0^t X_s \mathrm{d}M_s = (X \cdot M)_t.$$

The following proposition gives us some important properties of the stochastic integral. The proof relies on the extension of the integral in the sense of a simple process, and is omitted here.

Proposition A.15. *[KS91, pg. 139] Let* $M \in \mathscr{M}_2$ *continuous and* $X \in \mathscr{L}^*(M)$*. Then,* $I(X) \in \mathscr{M}_2$ *and continuous as well, where*

$$\langle I(X)\rangle_t = \langle (X \cdot M)\rangle_t = (X^2 \cdot \langle M\rangle)_t = \int_0^t X_s^2 d\langle M\rangle_s.$$

From the fact that the integral is a martingale, it holds that

$$\mathbb{E}\left[\int_0^t X_s \mathrm{d}M_s\right] = 0.$$

Proposition A.16 (Itô's isometry). *[KS91, pg. 144] Let* $M, N \in \mathscr{M}_2$ *and continuous, and let* $I_t = \int_0^t X_s dM_s$ *and* $K_t = \int_0^t Y_s dN_s$*, where the processes* $(X_t), (Y_t)$ *are such that the integrals are well defined. Then, for all* $t \in [0, T]$,

$$\mathbb{E}[I_t K_t] = \mathbb{E}\left[\int_0^t X_s Y_s d\langle M, N\rangle_s\right].$$

Note that the definitions above can easily be extended to $X \in \mathscr{L}^*(M)$, where $M_0 \neq 0$, a.s. Then,

$$\int_0^t X_s \mathrm{d}M_s,$$

is still defined, as $\tilde{M} = M - M_0$ is still a square-integrable martingale that now starts in zero, with $\mathrm{d}M = \mathrm{d}\tilde{M}$. For this martingale, the definitions above apply.

Remark A.2. *Here, we work with a continuous integrator* M*, and a progressively measurable integrand* X*. When we relax the continuity assumption, we need to work with an even stronger notion of measurability, that of a predictable process. This topic is not covered in this thesis. We should be careful that all the integrands we are working with are sufficiently continuous.*

Although we make the remark above, it appears that we often work with a right-continuous martingale, and not with a continuous one. However, the next proposition solves this problem. Before this, we first need an important theorem, of which again the proof is omitted.

Theorem A.17 (Martingale representation theorem). *[KS91, pg. 182] Let* W *be a* d*-dimensional Brownian motion, and let* $(\mathscr{F}_t)$ *be given in Definition A.4. Then for any square-integrable martingale* M *w.r.t this filtration with cadlag paths a.s, there exists a progressively measurable processes* Y^j *such that*

$$\mathbb{E}\left[\int_0^T \left(Y_t^j\right)^2 dt\right] < \infty, \tag{A.3}$$

for any $1 \leqslant j \leqslant d$*, for every* $0 < T < \infty$*, and*

$$M_t = M_0 + \sum_{j=1}^d \int_0^t Y_s^j dW_s^j,$$

for every $t \geqslant 0$*. In particular,* M *is a.s. continuous.*

Note that Equation (A.3) is essential to ensure that the stochastic integral is well-defined, in accordance with Definition A.19.

Proposition A.18. *Let M be a martingale with respect to the augmented Brownian filtration $(\mathscr{F}_t)$ given in Definition A.4. Then, M has a modification that is continuous.*

Proof. First, by Proposition A.3, the martingale has a cadlag modification. When we work on this modification, we can apply Theorem A.17 to conclude that M has a.s. continuous paths. By Proposition A.2 we can say that M is indistinguishable with a continuous process. Hence, we can say that M has a modification that is continuous. □

Recalling Remark A.2, we will use this modification M to ensure continuity, to be able to integrate properly.

We introduce a well-known method to characterise a function of a stochastic process. The proof can be found in the cited reference.

Theorem A.19 (Itô's lemma (in multiple dimensions)). *[KS91, pg. 153] Let $d \geqslant 1$ and $f : [0.\infty] \times \mathbb{R}^d \to \mathbb{R}$ be a twice continuously differentiable function in all variables. Take $t \in [0,\infty)$. Let X_t be a continuous, d-dimensional martingale with initial value X_0. Then, it holds $\mathbb{P}$ a.s.,*

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial}{\partial t} f(s, X_s) ds + \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} f(s, X_s) dX_s^i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s, X_s) d\langle X^i, X^j \rangle_s.$$

Itô's lemma will be mainly used with $d = 1$ or $d = 2$ with $X_t = (t, Y_t)$, for a process Y_t.

We end this section with two lemmas that will be often used throughout this thesis and are partially based on the appendix of [AB23].

Lemma A.20. *Let $(\mathscr{F}_t)$ be the augmented Brownian filtration given in Definition A.4. Let (X_t) be a martingale with respect to the filtration $(\mathscr{F}_t)$, and square-integrable with respect to $\mathbb{P} \times \lambda^1$. Then,it holds that*

$$\tilde{M}_t := \mathbb{E}\left[\int_0^T X_s ds \middle| \mathscr{F}_t\right] = \int_0^t X_s ds + (T - t) X t,$$

with dynamics

$$d\tilde{M}_t = (T - t) dX_t.$$

Furthermore, it holds that $(\tilde{M}_t)$ is also square-integrable with respect to $\mathbb{P} \times \lambda^1$.

Proof. Note that by construction $(\tilde{M}_t)$ is a martingale, since the square-integrability holds. Using the properties of a martingale and a Fubini argument for conditional expectations gives the following

$$\tilde{M}_t = \mathbb{E}\left[\int_0^T X_s \mathrm{d}s \middle| \mathscr{F}_t\right] = \mathbb{E}\left[\int_0^t X_s \mathrm{d}s + \int_t^T X_s \mathrm{d}s \middle| \mathscr{F}_t\right] = \int_0^t X_s \mathrm{d}s + \int_t^T \mathbb{E}[X_s | \mathscr{F}_t] \mathrm{d}s.$$

Here, in the last step we used that X_s for $s \leqslant t$ is $\mathscr{F}_t$ measurable. Since $\mathbb{E}[X_s | \mathscr{F}_t] = X_t$ for $s > t$, this implies

$$\tilde{M}_t = \int_0^t X_s \mathrm{d}s + X_s (T - t).$$

It follows by Itô's lemma that

$$\mathrm{d}\tilde{M}_t = X_t\mathrm{d}t + \mathrm{d}(X_t(T-t)) = X_t\mathrm{d}t + T\mathrm{d}X_t - \mathrm{d}(tX_t) = X_t\mathrm{d}t + T\mathrm{d}X_t - t\mathrm{d}X_t - X_t\mathrm{d}t = (T-t)\mathrm{d}X_t.$$

Left to prove is that $(\tilde{M}_t)$ is square-integrable with respect to $\mathbb{P}\times\lambda^1$. Rewriting, we get

$$\begin{aligned}\int_0^T\int_\Omega \tilde{M}_s^2\mathrm{d}\mathbb{P}\mathrm{d}s &= \int_0^T\int_\Omega \mathbb{E}\left[\int_0^T X_s\mathrm{d}s\middle|\mathscr{F}_t\right]^2\mathrm{d}\mathbb{P}\mathrm{d}s \leqslant \int_0^T\int_\Omega \mathbb{E}\left[\left(\int_0^T X_s\mathrm{d}s\right)^2\middle|\mathscr{F}_t\right]\mathrm{d}\mathbb{P}\mathrm{d}s\\ &= \int_0^T \mathbb{E}\left[\mathbb{E}\left[\left(\int_0^T X_s\mathrm{d}s\right)^2\middle|\mathscr{F}_t\right]\right]\mathrm{d}s = \int_0^T \mathbb{E}\left[\left(\int_0^T X_s\mathrm{d}s\right)^2\right]\mathrm{d}s = T\mathbb{E}\left[\left(\int_0^T X_s\mathrm{d}s\right)^2\right],\end{aligned}$$

by the conditional Jensen's inequality, the tower property and the fact that $\mathbb{E}\left[\left(\int_0^T X_s\mathrm{d}s\right)^2\right]$ not depends on s anymore. We get, by the Cauchy-Schwarz inequality in $L^2([0,T])$,a.s,

$$\begin{aligned}\left(\int_0^T X_s\mathrm{d}s\right)^2 &\leqslant \int_0^T 1\mathrm{d}s\int_0^T X_s^2\mathrm{d}s = T\int_0^T X_s^2\mathrm{d}s,\\ \mathbb{E}\left[\left(\int_0^T X_s\mathrm{d}s\right)^2\right] &\leqslant T\mathbb{E}\left[\int_0^T X_s^2\mathrm{d}s\right] < \infty,\end{aligned}$$

since inequalities are preserved by taking expectations, and X is a sufficiently square- integrable process. From this, we can conclude that $(\tilde{M}_t)$ is also square-integrable. □

The following corollary has almost the same proof as the lemma above, but is stated here for completeness.

Corollary A.21. *Let (X_t) be a martingale that is square- integrable with respect to $\mathbb{P}\times\lambda^1$. Let the process (N_t), with $t\in[0,T]$, be such that*

$$N_t = \mathbb{E}\left[\int_t^T X_s ds\middle|\mathscr{F}_t\right].$$

Then, (N_t) is also square-integrable with respect to $\mathbb{P}\times\lambda^1$ and can be written as

$$N_t = X_t(T-t).$$

Proof. The part of the proof that N_t is square- integrable is identical to the last part of Lemma A.20. Furthermore, we have again by a Fubini argument that

$$N_t = \mathbb{E}\left[\int_t^T X_s\middle|\mathscr{F}_t\right]\mathrm{d}s = \int_t^T \mathbb{E}[X_s|F_t]\mathrm{d}s = \int_t^T X_t\mathrm{d}s = X_t(T-t).$$

□

A.1.3 Variational calculus

This section explains the most important concepts of variational calculus, needed for this thesis, beginning with the foundational concept of functional analysis.

For the reader seeking more information about vector spaces and Hilbert spaces, the book by Balakrishnan [Bal12] is recommended.

Lemma A.22. *The space $L^2(\Omega,\mathscr{F},\mathbb{P})$ of Section A.1.1 is a Hilbert space.*

The proof of this lemma is omitted here, and can be found in [Sch17, pg.121]. In a Hilbert space, the following lemma applies.

Lemma A.23. *Let H be a Hilbert space with norm $\|\cdot\|_H$ and $X, Z \in H$. If*

$$\langle X - Z, Y\rangle_H = 0,$$

for all $Y \in H$, then we have that $X = Z$.

Proof. As $\langle X - Z, Y\rangle_H = 0$ for all $Y \in H$, we can choose $Y = X - Z$ to get

$$\langle X - Z, Y\rangle_H = \langle X - Z, X - Z\rangle_H = \|X - Z\|_H^2 = 0.$$

This implies that $\|X - Z\| = 0$ and thus $X = Z$, by the definition of a norm. □

In correspondence with Section A.1.1, in $L^2(\Omega, \mathscr{F}, \mathbb{P})$, this would mean that $X_t = Z_t$, $\mathbb{P}$ almost surely, so that X and Z are in the same equivalence class.

Next, several important definitions will be introduced.

Definition A.20 (Operator). *[Lue97, pg. 27] Let X, Y be linear vector spaces. An operator from a domain $D \subset X$ to Y is a rule that associates with every $x \in D$ an element $y \in Y$. We then write*

$$T(x) = y.$$

Definition A.21 (Functional). *[Lue97, pg. 28] Let X be a vector space. A functional is any operator $f : X \to \mathbb{R}$.*

That is, a functional is a function of functions, and a specific case of an operator. From now on, let $T : X \to Y$ be an operator where X is a Hilbert space equipped with norm $||\cdot||_X$ and Y a normed space with norm $||\cdot||_Y$.

Definition A.22 (Linear operator). *The operator T is linear, if for every $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$ holds that*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

Definition A.23 (Bounded operator). *[Lue97, pg. 144] The operator T is bounded if there exists a constant $M \in \mathbb{R}$ such that*

$$||T(x)||_Y \leqslant M||x||_X,$$

for all $x \in X$.

Proposition A.24. *[Lue97, pg.144] A linear operator is called continuous for all $x \in X$ if it is continuous in a single point. It is continuous if and only if it is bounded.*

The proof of this proposition is omitted and can be found in the reference cited.

With these definitions, a definition for a derivative of an operator can be introduced.

Definition A.24 (Gateaux differentiable). *[Lue97, pg. 171] Let $x \in D \subseteq X$, $h \in X$ and $y \in \mathbb{R}$. The operator T is said to be Gateaux differentiable at x if the limit*

$$\delta T(x; h) = \lim_{y \to 0} \frac{T(x + yh) - T(x)}{y},$$

exists. Then, $\delta T(x; h)$ is called the Gateaux differential of T at x with increment h. If it exists for all $h \in X$, the operator T is Gateaux differentiable at x.

For every $x \in X$ the Gateaux differential defines an operator from $\delta T(x; \cdot)$: $X \to Y$ itself. When T itself is a linear operator, it holds that $\delta T(x; h) = T(h)$. In the case of a functional, even more can be said.

Corollary A.25. *[Lue97, pg. 171] When $Y = \mathbb{R}$, T is functional and it holds that the Gateaux differential is given by,*

$$\delta T(x;h) = \frac{d}{dh} T(x + yh)\Big|_{h=0},$$

if it exists.

Now, a stronger concept of differentiability is introduced.

Definition A.25 (Fréchet differentiable). *[Lue97, pg. 172] Let $h \in X$. The operator T is Fréchet differentiable at $x \in X$, if there exists an operator $\delta T(x;h) \in Y$ that is continuous and linear with respect to h such that*

$$\frac{\|T(x+h) - T(x) - \delta T(x;h)\|_Y}{\|h\|_X} \to 0,$$

as $\|h\|_X \to 0$. Then, T is said to be Fréchet differentiable at x, with Fréchet derivative $\delta T(x;h)$. If T is Fréchet differentiable for all $x \in X$, it holds that

$$\delta T(x;h) := A(x)(h),$$

where $A(x)$ is a bounded linear operator from $X \to Y$. We call $A(x)$ the Fréchet derivative T' of T, such that

$$\delta T(x;h) = T'(x)(h).$$

Note that the continuity and linearity are not part of the definition of the Gateaux differential, but are in the case of the Fréchet differential. Another connection between the Gateaux and Fréchet differential is made in the next proposition.

Proposition A.26. *[Lue97, pg. 173] If the Fréchet differential of T exists at x, so does the Gateaux differential at x. In fact, the two functionals coincide.*

Proof. The proof follows easily by writing out the definitions of both differentials. □

This proposition implies that the Fréchet derivative is unique, if it exists, and does not depend on the choice of norm, as long as the norms are equivalent. Since the Fréchet differential does not depend on the norm, we can state a continuity result, of which the proof can be found in the reference cited.

Proposition A.27. *[Lue97, pg. 173] If T is Fréchet differentiable in $x \in X$, then T is continuous at x.*

Note that the concept of the Gateaux differential is often sufficient to achieve the desired results in this thesis. The Fréchet differential will be used to prove continuity of the functionals we will work with. To find the Fréchet differential, we will use the Gateaux differential, and demonstrate that it is linear, continuous and fulfills Definition A.25.

The next definition involves X^*, the dual space of the Hilbert space X. This is included for completeness, and we won't go into details what this space exactly is. The following definition will be helpful in the upcoming chapters.

Definition A.26 (Representation of the Gateaux derivative). *[ET99, pg. 23] Let $Y = \mathbb{R}$, so that T is a functional. The gradient of the Gateaux derivative at $x \in X$, denoted by the Gateaux gradient, is given by a functional $\nabla T(x) \in X^*$ that satisfies*

$$\delta T(x;y) = \langle y, \nabla T(x) \rangle.$$

for every $y \in X$, if it exists.

Proposition A.28. *Since X is a Hilbert space, the Gateaux gradient $\nabla T(x)$ always exists for $x \in X$ if the Gateaux differential $\delta T(x;h)$ exists and is linear and continuous for all $h \in X$.*

Proof. This is a direct consequence of the Riesz Representation Theorem [Bal12, pg. 20]. □

Sometimes the gradient above is called "the Gateaux derivative", as this is defined without an extra $h \in X$. However, we will call it the gradient to keep it clear. The gradient can be seen as a representation of the Gateaux derivative.

With the information introduced above, we can state a proposition for finding an infinum. However, before doing so, we need some more definitions.

Definition A.27 (Coerciveness). *[ET99, pg. 35] The operator T is called coercive if*

$$||T(x)||_Y \to \infty, \quad \textit{if } \|x\|_X \to \infty,$$

for all $x \in X$.

Definition A.28 (Convex subspace). *[ET99, pg. 7] A subspace X of a real vector space V is convex if for every pair $(x, y) \in X$ and every $\theta \in [0,1]$*

$$\theta x + (1-\theta) y \in X.$$

Definition A.29 ((Strictly) Convex functional). *[ET99, pg. 7, 9] Let X be a convex subspace of a real vector space, and F a functional from X to $\mathbb{R}$. F is called convex if for all $(x, y) \in X$ it holds that*

$$F(\theta x + (1-\theta) y) \leqslant \theta F(x) + (1-\theta) F(y),$$

for all $\theta \in [0,1]$.

The functional F is called strictly convex if the above inequality holds strictly, for every $(x, y) \in X$ such that $x \neq y$ and $\theta \in (0,1)$.

We are ready to introduce the aforementioned proposition. The proof can be found in the mentioned reference.

Proposition A.29. *[ET99, pg. 35] Let X be a Hilbert space, $A \subset X$ a closed, convex subspace of this space and $F : A \to \mathbb{R}$ a functional that is convex, continuous and coercive. Then,*

$$\inf_{x \in A} F(x),$$

has at least one solution. It has a unique solution, if F is strictly convex.

This concludes the chapter on the mathematical background.

A.2 Mathematical proofs

In this chapter, some mathematical proofs that have been omitted in the main part of the thesis are written down. These have to do with variational calculus and specific calculations of moments.

First, we will calculate the moment of a Geometric Brownian motion.

Lemma A.30. *Let $t \in [0, T]$ and*

$$G_t^i = E_0^i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i\right) t + \sigma_i W_t^i\right).$$

The first and second moments of G_t^i are given by

$$\mathbb{E}\left[G_t^i\right] = E_0^i \exp(\mu_i t), \quad \mathbb{E}\left[\left(G_t^i\right)^2\right] = \left(E_0^i\right)^2 \exp\left(2\mu_i t + \sigma_i t\right).$$

This implies that $(G_t^i) \in L^2(\Omega \times [0,T], \mathscr{F} \otimes \mathscr{B}([0,T]), \mathbb{P} \times \lambda^1)$.

Proof. Let's start with the expected value. It follows that

$$\begin{aligned}\mathbb{E}\left[G_t^i\right] &= \mathbb{E}\left[E_0^i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i W_t^i\right)\right] = E_0^i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t\right)\mathbb{E}\left[\exp\left(\sigma_i W_t^i\right)\right] \\ &= E_0^i \exp\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t\right)\exp\left(\frac{1}{2}\sigma_i^2 t\right) = E_0^i \exp(\mu_i t),\end{aligned}$$

where in the last step the expected value of the exponent follows from the moment generating function of a Brownian motion, see [OG19, pg. 37]. For the second moment, the same trick can be applied.

$$\begin{aligned}\mathbb{E}\left[\left(G_t^i\right)^2\right] &= \left(E_0^i\right)^2 \exp\left(\left(2\mu_i - \sigma_i^2\right)t\right)\mathbb{E}\left[\exp\left(2\sigma_i W_t^i\right)\right] = \left(E_0^i\right)^2 \exp\left(\left(2\mu_i - \sigma_i^2\right)t\right)\exp\left(\sigma_i^2 t\right) \\ &= \left(E_0^i\right)^2 \exp\left(\left(2\mu_i + \sigma_i^2\right)t\right).\end{aligned}$$

Since $G_t^i > 0$ for all $t \in [0,T]$, it holds by Tonelli's theorem that

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left(G_t^i\right)^2 \mathrm{d}t\right] &= \int_0^T \mathbb{E}\left[\left(G_t^i\right)^2\right]\mathrm{d}t = \int_0^T \left(E_0^i\right)^2 \exp\left(\left(2\mu_i + \sigma_i^2\right)t\right)\mathrm{d}t \\ &= \left(E_0^i\right)^2 \frac{1}{2\mu_i + \sigma_i^2}\left(\exp\left(\left(2\mu_i + \sigma_i^2\right)T\right) - 1\right) < \infty,\end{aligned}$$

as E_0^i is non-random, $\mu_i > 0$ and T fixed. Hence, $(G_t^i) \in L^2(\Omega \times [0,T], \mathscr{F} \otimes \mathscr{B}([0,T]), \mathbb{P} \times \lambda^1)$. □

Now, we will show how we can find the specific Fréchet derivative that we use in Chapter 3. This is done via the Gateaux derivative.

Proposition A.31. *Let $\mathscr{J}^i$ be given as in* (1.40), *and* $(V,Z) = \phi \in \mathscr{A}^2$. *Then the Gateaux derivative of the functional $\mathscr{J}^i$ is given by*

$$\delta\mathscr{J}^i\left(\left(\alpha^i, \beta^i\right); (V,Z)\right) = \mathbb{E}\left[\int_0^T V_t\left(h_i + \frac{\alpha_t^i}{\eta}\right) + V_t\left(P_t + \frac{\beta_t^i}{\nu}\right)dt\right] + \lambda\mathbb{E}\left[\int_0^T X_T^i\left(V_t + Z_t\right)dt\right].$$

Proof. The Gateaux derivative can be derived with Corollary A.25, since we are working with a functional. From the Gateaux derivative, the Gateaux gradient can be obtained.

For this, fix firm i and again split up the cost functional $\mathscr{J}^i$ in two parts, such that we can write

$$\mathscr{J}^i = C^i + F^i.$$

Since the derivative and gradient are linear, we will have that

$$\delta\mathscr{J}^i\left(\left(\alpha^i, \beta^i\right); \phi\right) = \delta C^i\left(\left(\alpha^i, \beta^i\right); \phi\right) + \delta F^i\left(\left(\alpha^i, \beta^i\right); \phi\right), \quad \nabla\mathscr{J}^i = \nabla C^i + \nabla F^i.$$

We will first derive the Gateaux derivative and gradient of the functional C. During the derivations, we will ignore the subscript i for a specific firm.

Let $\phi \in \mathscr{A}^2$ such that $\phi = (V,Z)$ and $\tau \in \mathbb{R}$. Then, the Gateaux derivative with respect to $\phi \in \mathscr{A}^2$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}C\left(\alpha + \tau V_t, \beta + \tau Z_t\right)\Big|_{\tau=0} = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbb{E}\left[\int_0^T h(\alpha_t + \tau V_t) + \frac{(\alpha_t + \tau V_t)^2}{2\eta} + P_t(\beta_t + \tau Z_t) + \frac{(\beta_t + \tau Z_t)^2}{\nu}\mathrm{d}t\right]\Big|_{\tau=0}.$$

Note that $\tau \in \mathbb{R}$ and we are only differentiating deterministic terms, potentially times a random variable. This implies that we interchange the differentiation sign and expected value. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}C\left(\alpha+\tau V_t,\beta+\tau Z_t\right) &= \mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^T h(\alpha_t+\tau V_t)+\frac{(\alpha_t+\tau V_t)^2}{2\eta}+P_t(\beta_t+\tau Z_t)+\frac{(\beta_t+\tau Z_t)^2}{2\nu}\mathrm{d}t\right]\\ &= \mathbb{E}\left[\int_0^T hV_t+\frac{\alpha_t V_t+\frac{1}{2}\tau^2 V_t^2}{\eta}+P_t Z_t+\frac{\beta_t Z_t+\frac{1}{2}\tau^2 Z_t^2}{\nu}\mathrm{d}t\right].\end{aligned}$$

Evaluating in $\tau=0$, it implies

$$\begin{aligned}\delta C\left((\alpha,\beta);\phi\right) &= \mathbb{E}\left[\int_0^T hV_t+\frac{\alpha_t V_t+\frac{1}{2}\tau^2 V_t^2}{\eta}+P_t Z_t+\frac{\beta_t Z_t+\frac{1}{2}\tau^2 Z_t^2}{\nu}\mathrm{d}t\right]\Big|_{\tau=0}\\ &= \mathbb{E}\left[\int_0^T V_t\left(h+\frac{\alpha_t}{\eta}\right)+Z_t\left(P_t+\frac{\beta_t}{\nu}\right)\mathrm{d}t\right].\end{aligned}$$

The above is the Gateaux derivative of C in the direction of ϕ. .

The same procedure will be done to obtain an expression for the Gateaux derivative of F. For this, we will need the chain rule for Gateaux derivatives [Lue97, pg. 176]. Furthermore, this implies, by the same reasoning on interchanging the differentiation and expected value as above, that

$$\begin{aligned}\delta F\left((\alpha,\beta);\phi\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}F(\alpha+V\tau,\beta+Z\tau)\Big|_{\tau=0} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbb{E}\left[\lambda X_T\left(\alpha+\tau V,\beta+\tau Z\right)^2\right]\Big|_{\tau=0}\\ &= \lambda\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}X_T^2(\alpha+\tau V,\beta+\tau Z)\Big|_{\tau=0}\right]\\ &= 2\lambda\mathbb{E}\left[X_T(\alpha,\beta)\left(\delta X((\alpha,\beta);\phi)\right)\right],\end{aligned}\tag{A.4}$$

where $\delta X_T((\alpha,\beta);\phi)$ is the Gateaux derivative of $X_T(\alpha,\beta)$ for $\phi\in\mathscr{A}^2$. Note that here $X_T\left(\alpha+\tau V,\beta+\tau Z\right)^2$ could be written out explicitly. Hence, working out the derivative, we obtain

$$\delta X\left((\alpha,\beta);\phi\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}X_T\left(\alpha+V\tau,\beta+Z\tau\right)\Big|_{\tau=0} = \int_0^T \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\alpha_t+V_t\tau+\beta_t+Z_t\tau\right)\Big|_{\tau=0}\mathrm{d}t = \int_0^T V_t+Z_t\mathrm{d}t.$$

It now holds that (A.4) can be written as

$$\delta F\left((\alpha,\beta);\phi\right) = 2\lambda\mathbb{E}\left[X_T\left(\alpha,\beta\right)\int_0^T V_t+Z_t\mathrm{d}t\right] = 2\lambda\mathbb{E}\left[\int_0^T X_T\left(V_t+Z_t\right)\mathrm{d}t\right].\tag{A.5}$$

This is the Gateaux derivative of F with respect to $\phi\in\mathscr{A}^2$. We conclude that we have indeed found the desired Gateaux derivatives and gradient of the cost functional $\mathscr{J}^i$. □

Proposition A.32. *The abatement costs $c_i(\alpha^i):\mathscr{A}\to\mathscr{B}$ and tradings costs $f_i(\beta^i):\mathscr{A}\to\mathscr{B}$ are given by*

$$c_i\left(\alpha^i\right) = h_i\alpha^i+\frac{(\alpha^i)^2}{2\eta_i},\quad f_i\left(\beta^i\right) = P\beta^i+\frac{(\beta^i)^2}{2\nu}.$$

The marginal abatement costs, $c_i'(\alpha_t^i)$, is an operator from $\mathscr{A}\to\mathscr{B}$, given by, for $x\in\mathscr{A}$,

$$c_i'\left(\alpha^i\right)(x) = x\left(h_i+\frac{\alpha^i}{\eta}\right).\tag{A.6}$$

In like manner, the marginal tradings costs are given by $f_i'(\beta_t^i)$, where for $x\in\mathscr{A}$

$$f_i'\left(\beta^i\right)(x) = x\left(P+\frac{\beta^i}{\nu}\right).$$

Proof. Recall the definition of $\mathscr{B}$ of (1.18). Given that both costs are operators from $\mathscr{A} \to \mathscr{B}$, the derivatives will be taken in the Fréchet and Gateaux sense. This will be written out completely for the marginal abatement costs, the marginal trading costs follow in the exact same manner. We will show that (A.6) fulfils Definition A.25. Linearity in $x \in \mathscr{A}$ follows immediately, continuity follows from the boundedness of the operator by Proposition A.24. Take $x \in \mathscr{A}$ and $\alpha^i \in \mathscr{A}$ arbitrary. Then,

$$\left\| c_i' \left(\alpha^i\right)(x) \right\|_{\mathscr{B}} = \left\| x \left(h_i + \frac{\alpha^i}{\eta_i} \right) \right\|_{\mathscr{B}} \leqslant h_i \|x\|_{\mathscr{B}} + \frac{1}{\eta_i} \left\| \alpha^i x \right\|_{\mathscr{B}}.$$

Since $\left\| x^2 \right\|_{\mathscr{B}} = \|x\|_{\mathscr{A}}$ and by Hölder's inequality, it follows

$$\left\| c_i' \left(\alpha^i\right)(x) \right\|_{\mathscr{B}} \leqslant h_i T^2 \|x\|_{\mathscr{A}} + \frac{1}{\eta_i} \|x\|_{\mathscr{A}} \left\| \alpha^i \right\|_{\mathscr{A}} = \left(h_i T^2 + \frac{1}{\eta_i} \left\| \alpha^i \right\|_{\mathscr{A}} \right) \|x\|_{\mathscr{A}} := M \|x\|_{\mathscr{A}}.$$

We can conclude that this operator is bounded and thus continuous. Left to prove is that it fulfils the uniform limit of the Fréchet derivative. Indeed, by the fact that $\|x\|_{\mathscr{A}}^2 = \left\| x^2 \right\|_{\mathscr{B}}$, it follows

$$\frac{\left\| c_i\left(\alpha^i + x\right) - c_i\left(\alpha^i\right) - c_i'\left(\alpha^i\right)(x) \right\|_{\mathscr{B}}}{\|x\|_{\mathscr{A}}} = \frac{\left\| \frac{x^2}{\eta_i} \right\|_{\mathscr{B}}}{\|x\|_{\mathscr{A}}} = \frac{1}{\eta_i^2} \frac{\|x\|_{\mathscr{A}}{}^2}{\|x\|_{\mathscr{A}}} = \frac{1}{\eta_i^2} \|x\|_{\mathscr{A}} \to 0,$$

if $\|x\|_{\mathscr{A}} \to 0$. We conclude that the marginal costs operator $c_i'(\alpha^i)$ is the Fréchet derivative of $c_i(\alpha^i)$, and in the same manner that that the marginal trading costs operator $f_i'(\beta^i)$ is the Fréchet derivative of $f_i(\beta^i)$. In a specific point $t \in [0, T]$, we will write $c_i'(\alpha^i)_t, f_i'(\beta^i)_t$ respectively. $\square$

A.3 Units of variables

In Section 6.4 of [AB23], the units of the variables in the Brownian framework are given. In this section, we will briefly elaborate on that. After this, we will adapt the units in the Geometric Brownian motion setting, as this is necessary for the well-definedness of the model. In this section, X has unit y will be denoted by $X \sim y$.

First of all, in the Brownian framework, we should be careful, since some parameters in the paper of Aïd and Biagini are given in Gigaton CO_2, while others are given in tons of CO_2. To maintain consistency in this thesis, we converted everything to Gigaton CO_2, which we will abbreviate as Gton. As already indicated, the unit of the BAU emissions E^i is given to be Gton. Both the drift and volatility are defined per year, such that

$$\mu_i \sim \frac{\text{Gton}}{\text{year}}, \quad \sigma_i \sim \frac{\text{Gton}}{\sqrt{\text{year}}}.$$

Although not mentioned in the paper cited, to give E the unit Gton, as per (1.6), we need that the Brownian motion that we work with has a unit of $\sqrt{\text{year}}$, which we can also write as $\sqrt{t}$, where time t in years. We believe this is a reasonable assumption, since

$$\mathbb{E}\left[\left(W_t^i\right)^2\right] = t,$$

by definition of the Brownian motion.

In the Geometric Brownian motion, we consider the strong solution given in (1.8). Here, we should be careful, since we work with an exponent. This means here that we need to make the part in the exponent dimensionless, such that G_t^i has the same unit as E_0^i. That is, we need that the term

$$\left(\left(\mu_i - \frac{1}{2}\sigma_i^2\right) t + \sigma_i W_t^i\right),$$

has no unit. In order to achieve this, we need

$$\mu_i \sim \frac{1}{t}, \quad \sigma_i \sim \frac{1}{\sqrt{t}}, \quad W_t^i \sim \sqrt{\mathrm{t}}. \tag{A.7}$$

The units of the drift and volatility are adjusted such that they fit in this model. This way, (A.7) is dimensionless and G_t^i has a unit of Gton, for every firm i and time $t \in [0, T]$.

B | Market with Frictions

In this appendix, the optimal dynamic allocation in the case of a market with frictions, where the BAU emissions are modelled by a Brownian motion, is solved. Again, the three steps of the Stackelberg game that have been used in Chapters 2 and 3 are present. We will often refer to this chapter, as only the main differences are written out.

This chapter is based on Sections 3 and 4 of [AB23]. Furthermore, it is inspired by Section 5.1.1 of [AB23]. It extends this article, as the derivations are not written out there.

In this chapter, we work with $\nu < \infty$. Furthermore, the structure of the Brownian motion is taken as in [AB23], that is,

$$W_t^i := \kappa_i \tilde{B}_t^0 + \sqrt{1-\kappa_i^2}\tilde{B}_t^i,$$

where $\kappa_i \in \mathbb{R}$. Additionally, we take $E_0^i = 0$ for every firm. These assumptions correspond with Remark 1.1 specifically. With this, we are ready to go to the first step to solve the Stackelberg equilibrium.

Before we start, we define for each firm i,

$$g_i(t) = \frac{2\lambda\eta_i}{1+2\lambda\left(\eta_i+\nu\right)(T-t)},$$

which is the analogue of $f(t)$ in the case of no frictions.

B.1 Single firm optimisation

The first goal of this section is to prove the following proposition

Proposition B.1. *Let $\mathscr{J}^{i,E}$ be given as in* (1.40)*, with $X_T^i = X_T^{i,E}$. Then, the solution couple to*

$$\inf_{(\alpha^i,\beta^i)\in\mathscr{A}^2} \mathscr{J}^{i,E}(\alpha^i,\beta^i),$$

is unique.

First of all, note that

$$\mathscr{J}^{i,E}\left(\alpha^i,\beta^i\right) = \tilde{\mathscr{J}}^{i,E}\left(\alpha^i,\beta^i\right) + L^i\left(\alpha^i,\beta^i\right), \quad \text{with } L^i\left(\alpha^i,\beta^i\right) := \mathbb{E}\left[\int_0^T \frac{(\beta_t^i)^2}{2\nu}\mathrm{d}t\right], \tag{B.1}$$

where $\tilde{\mathscr{J}}^{i,E}$ is given in 2.3 . Next to this representation, the following decomposition will also be used

$$\mathscr{J}^{i,E}\left(\alpha^i,\beta^i\right) = C^i\left(\alpha^i,\beta^i\right) + F^{i,E}\left(\alpha^i,\beta^i\right),$$

where $F^{i,E}$ is given in (2.5) and

$$C^i\left(\alpha^i,\beta^i\right)=\mathbb{E}\left[\int_0^T h_i(\alpha_t^i)+\frac{(\alpha_t^i)^2}{2\eta_i}+P_t\beta_t^i+\frac{(\beta_t^i)^2}{2\nu}\mathrm{d}t\right].$$

We will use both these relations to prove the proposition. Again, Proposition A.29 will be used to achieve this. We need to make sure we satisfy the assumptions of this theorem. The properties of the space $\mathscr{A}^2$ still hold. We need to prove that $\mathscr{J}^{i,E}$ is a coercive, continuous, strictly convex functional over $\mathscr{A}^2$. The proof of the coerciveness is in line with Proposition 2.4 and won't be repeated here. Again, the assumptions on the trading rate β^i in (1.11) and the market price P in Equation (1.24) are used.

Next, we will show that the cost functional $\mathscr{J}^{i,E}$ is strictly convex in (α^i,β^i). From Proposition 2.5, we already know that $F^{i,E}$ is convex. We will show that C^i is strictly convex in the controls, from which we can conclude that the whole function $\mathscr{J}^{i,E}$ is strictly convex.

Proposition B.2. *The cost functional $C^i(\alpha^i,\beta^i)$ is strictly convex in (α^i,β^i). It follows that $\mathscr{J}^{i,E}(\alpha^i,\beta^i)$ is also strictly convex.*

Proof. Let $\theta\in(0,1)$, and $V,Y\in\mathscr{A}^2$ such that $V\neq Y$. This implies that either $V_1\neq Y_1$, $V_2\neq Y_2$, or both. That is, at least one of the two inequalities holds, μ a.e,

$$2V_1Y_1<V_1^2+Y_1^2\quad\text{or}\quad 2V_2Y_2<V_2^2+Y_2^2. \tag{B.2}$$

Using these inequalities, we get

$$\begin{aligned}C^i(\theta V+(1-\theta)Y)=\mathbb{E}&\left[\int_0^T h_i(\theta V_1+(1-\theta)Y_1)+\frac{(\theta V_1+(1-\theta)Y_1)^2}{2\eta_i}+P_t(\theta V_2+(1-\theta)Y_2)\,\mathrm{d}t\right]\\&+\mathbb{E}\left[\frac{(\theta V_2+(1-\theta)Y_2)^2}{2\nu}\mathrm{d}t\right].\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}\left[\int_0^T\frac{(\theta V_1+(1-\theta)Y_1)^2}{2\eta_i}+\frac{(\theta V_2+(1-\theta)Y_2)^2}{2\nu}\mathrm{d}t\right]&=\theta^2\,\mathbb{E}\left[\int_0^T\frac{V_1^2}{2\eta_i}+\frac{V_2^2}{2\nu}\mathrm{d}t\right]\\&+2\theta(1-\theta)\,\mathbb{E}\left[\int_0^T\frac{V_1Y_1}{2\eta_i}+\frac{V_2Y_2}{2\nu}\mathrm{d}t\right]\\&+(1-\theta)^2\,\mathbb{E}\left[\int_0^T\frac{V_1^2}{2\eta_i}+\frac{V_2^2}{2\nu}\mathrm{d}t\right].\end{aligned}$$

By one of the two, or both, inequalities of (B.2), we obtain

$$2\theta(1-\theta)\,\mathbb{E}\left[\int_0^T\frac{V_1Y_1}{2\eta_i}+\frac{V_2Y_2}{2\nu}\mathrm{d}t\right]<\theta\,\mathbb{E}\left[\int_0^T\frac{V_1^2}{2\eta_i}+\frac{V_2^2}{2\nu}\mathrm{d}t\right]+(1-\theta)\,\mathbb{E}\left[\int_0^T\frac{Y_1^2}{2\eta_i}+\frac{Y_2^2}{2\nu}\mathrm{d}t\right].$$

Using the linearity of the integral and the equation above, this gives

$$\begin{aligned}C^i(\theta V+(1-\theta)Y)<\theta\,\mathbb{E}\left[\int_0^T h_iV_1+P_tV_2\mathrm{d}t\right]+\theta\,\mathbb{E}\left[\int_0^T\frac{V_1^2}{2\eta_i}+\frac{V_2^2}{2\nu}\mathrm{d}t\right]+(1-\theta)\,\mathbb{E}\left[\int_0^T h_iY_1+P_tY_2\mathrm{d}t\right]\\+(1-\theta)\,\mathbb{E}\left[\int_0^T\frac{Y_1^2}{\eta_i}+\frac{Y_2^2}{2\nu}\mathrm{d}t\right]=\theta C^i(V)+(1-\theta)C^i(Y).\end{aligned}$$

Since the above holds for all $\theta\in(0,1)$, and $V\neq Y$ is chosen arbitrary, we can conclude that indeed the functional $C^i(\alpha^i,\beta^i)$ is strictly convex in both the controls, by Definition A.29. Since the summation of the strictly convex functional $C^i(\alpha^i,\beta^i)$ and the convex functional $F^{i,E}(\alpha^i,\beta^i)$ is again strictly convex, we can conclude that $\mathscr{J}^{i,E}(\alpha^i,\beta^i)$ is strictly convex in the controls. ☐

The last condition to establish, in order to satisfy the assumptions of Proposition A.29, is the continuity of the cost functional $\mathscr{J}^{i,E}$ with respect to (α^i, β^i). For this, we will use the Fréchet derivative. Since (B.1) holds, by the linearity of the Fréchet derivative, we only need to prove that $L^i(\alpha^i, \beta^i)$ is Fréchet differentiable, since $\tilde{\mathscr{J}}^{i,E}$ is Fréchet differentiable by Proposition 2.7.

How this derivative is exactly derived, via the Gateaux derivative, can be found in Proposition A.31.

Lemma B.3. *The functional $L^i : \mathscr{A}^2 \to \mathbb{R}$ of* (B.1) *is Fréchet differentiable, with following Fréchet derivative and Gateaux gradient respectively, for $\phi = (V, Z) \in \mathscr{A}^2$,*

$$\delta L^i\left(\left(\alpha^i, \beta^i\right); \phi\right) = \mathbb{E}\left[\int_0^T \frac{\beta_t^i Z_t}{\nu} dt\right], \quad \nabla L^i\left(\alpha^i, \beta^i\right) = \left(0, \frac{\beta^i}{\nu}\right).$$

Proof. We will show that $\delta L^i((\alpha^i, \beta^i); \phi)$ satisfies Definition A.25. The linearity and continuity is $\phi \in \mathscr{A}^2$ is not proven here and follows straightforwardly. Furthermore,

$$L^i\left(\left(\alpha^i, \beta^i\right) + \phi\right) - L^i\left(\alpha^i, \beta^i\right) - \delta L^i\left(\left(\alpha^i, \beta^i\right); \phi\right) = \mathbb{E}\left[\int_0^T \frac{Z_t^2}{2\nu} \mathrm{d}t\right].$$

Hence,

$$0 \leqslant \frac{\left|L^i\left((\alpha^i, \beta^i) + \phi\right) - L^i\left(\alpha^i, \beta^i\right) - \delta L^i\left((\alpha^i, \beta^i); \phi\right)\right|}{\|\phi\|_{\mathscr{A}^2}} \leqslant \frac{1}{2\nu} \frac{\left(\mathbb{E}\left[\int_0^T \frac{V_t^2}{2\nu} + \frac{Z_t^2}{2\nu} \mathrm{d}t\right]\right)}{\phi_{\mathscr{A}^2}} = \frac{1}{2\nu} \|\phi\|_{\mathscr{A}} \to 0,$$

when $\|\phi\|_{\mathscr{A}} \to 0$. Hence, the Fréchet derivative has the desired form. Indeed,

$$\delta L^i((\alpha^i, \beta^i); \phi) = \mathbb{E}\left[\int_0^T \frac{\beta_t^i Z_t}{\nu} \mathrm{d}t\right] = \left\langle \left(0, \frac{\beta^i}{\nu}\right), \phi \right\rangle_{\mathscr{A}^2}.$$

Hence, the Gateaux gradient is given by

$$\nabla L^i(\alpha^i, \beta^i) = \left(0, \frac{\beta^i}{\nu}\right).$$

□

Now we are ready to prove the continuity of the cost functional, which will follow directly from the Fréchet differentiability.

Lemma B.4. *The cost functional $\mathscr{J}^{i,E}$ is continuous in the controls.*

Proof. This follows directly from combining Equation (B.1), Proposition B.3 and Proposition 2.7, together with the fact that the Fréchet derivative is linear. By Proposition A.27, $\mathscr{J}^{i,E}$ continuous in the controls. □

Now we are finally ready to prove Theorem B.1.

Proof of Proposition B.1. By Proposition 2.3, it follows that $\mathscr{A}^2$ is a closed, convex Hilbert space of $L^2 \times L^2$. Furthermore, from Lemma B.4, the cost functional $\mathscr{J}^{i,E}$ is continuous. By Lemma 2.4, $\mathscr{J}^{i,E}$ is coercive in the controls and in Propositions 2.5 and B.2 it is proven that $\mathscr{J}^{i,E}$ is a strictly convex functional. By the aforementioned Proposition A.29, it holds that the stochastic control problem admits a unique solution. This is the desired result. □

It won't come as a surprise that this unique solution can be obtained by equating the gradient of $\mathscr{J}^{i,E}$ to zero . The following proposition summarises everything we need.

Proposition B.5. *The two-dimensional Gateaux gradient is given by, for each $t \in [0,T]$,*

$$\nabla \mathscr{J}(\alpha^i, \beta^i) = \left(h_i + \frac{\alpha^i}{\eta} + 2\lambda \mathbb{E}\left[X_T^i \middle| \mathscr{F}_\cdot \right], P + \frac{\beta^i}{\nu} + 2\lambda \mathbb{E}\left[X_T^i \middle| \mathscr{F}_\cdot \right] \right). \tag{B.3}$$

The solution to the optimisation problem is found when we equate this to zero, μ a.e.

Proof. The fact that the gradient is linear in combination with Proposition 2.10, Lemma B.3 and decomposition (B.1) gives the desired result for the gradient. Proposition 2.11 still suffices, where even a strict inequality in the proof will holds, due to the strict convexity. The result follows. □

With all this information, we are finally ready to find the minimiser of the cost functional $\mathscr{J}^{i,E}$.

Theorem B.6. *For a given cumulative allocation scheme A^i and exogenous price $P \in \mathscr{A}$, it holds that*

$$\inf_{(\alpha^i,\beta^i)} \mathscr{J}^{i,E}\left(\alpha^i, \beta^i\right) := \inf_{(\alpha^i,\beta^i)} \mathbb{E}\left[\int_0^T c_i(\alpha_t^i) + P_t \beta_t^i + \frac{1}{2\nu}(\beta_t^i)^2 dt + \lambda \left(X_T^i\right)^2 \right], \tag{B.4}$$

has a unique solution $(\hat{\alpha}^i, \hat{\beta}^i) \in \mathscr{A}^2$. The optimal solution for the abatement effort $\hat{\alpha}^i$ is given by the following SDE

$$\begin{aligned} d\hat{\alpha}_t^i &= -g_i(t)\left(dM_t^i - \sigma_i dW_t^i + d\,\mathbb{E}\left[\int_0^T \nu(h_i - P_s) ds \middle| \mathscr{F}_t \right] \right), \\ \hat{\alpha}_0^i &= -g_i(0)\left(\frac{1}{2\lambda} h_i + M_0^i + \mathbb{E}\left[\int_0^T \nu(h_i - P_t) dt \right] \right). \end{aligned}$$

The optimal solution for the trading rate $\hat{\beta}^i$ is given by

$$\hat{\beta}_t^i = \nu \left(h_i + \frac{\hat{\alpha}_t^i}{\eta_i} - P_t \right).$$

Proof. Throughout this proof, we will consider a specific firm i. This superscript for the corresponding firm will be omitted.

In Proposition B.5, we have seen that the cost functional $\mathscr{J}^{i,E}$ attains its minimum when $\nabla \mathscr{J}^{i,E}(\alpha,\beta) = 0$, μ a.e. Since this is implied by $\nabla \mathscr{J}^{i,E}(\alpha,\beta)_t = 0$ almost surely, for all $t \in [0,T]$, and the solution is unique, we can use this condition to find the optimal result. From now on, all equations in this proof will hold almost surely, unless mentioned otherwise. We can use (B.3) to arrive at the following two first order conditions, which hold almost surely,

$$h + \frac{\alpha_t}{\eta} + 2\lambda \mathbb{E}\left[X_T^E \middle| \mathscr{F}_t \right] = 0, \tag{B.5}$$

$$P_t + \frac{\beta_t}{\nu} + 2\lambda \mathbb{E}\left[X_T^E \middle| \mathscr{F}_t \right] = 0. \tag{B.6}$$

We will rewrite these smartly to arrive at expressions for the optimal values of α_t^i and β_t^i. As can be seen, both first order conditions contain the term $2\lambda \mathbb{E}[X_T^E | \mathscr{F}_t]$. Taking this term to the other side in both expressions, we can equate the given expressions to get

$$\begin{aligned} P_t + \frac{\beta_t}{\nu} &= h + \frac{\alpha_t}{\eta}, \\ \beta_t &= \nu\left(h + \frac{\alpha_t}{\eta} - P_t \right). \end{aligned} \tag{B.7}$$

We see, that if $\hat{\alpha}_t$ has a solution, the solution for $\hat{\beta}$ is given by

$$\hat{\beta}_t = \nu\left(h + \frac{\hat{\alpha}_t}{\eta} - P_t\right). \tag{B.8}$$

This is the desired result for the control variable β.

Next, we need an expression for the optimal control variable $\hat{\alpha}$. Recall from (1.36) that X_T^E is given by

$$X_T^E = \int_0^T \alpha_t + \beta_t \mathrm{d}t + A_T - \sigma W_T,$$

where we used that $A_0 = X_0$. First, we will plug the expression for $\hat{\beta}$ from (B.7) in the expression for X_T^E. We then obtain,

$$X_T^E = A_T + \int_0^T \alpha_t + \nu\left(h + \frac{\alpha_t}{\eta} - P_t\right)\mathrm{d}t - \sigma W_T.$$

We can plug this in the first order condition (B.5), which then gives

$$h + \frac{\alpha_t}{\eta} + 2\lambda\mathbb{E}\left[A_T + \int_0^T \alpha_t + \nu\left(h + \frac{\alpha_t}{\eta} - P_t\right)\mathrm{d}t + \sigma W_T \middle| \mathscr{F}_t\right] = 0. \tag{B.9}$$

This can be rewritten to

$$\begin{aligned}
\alpha_t &= -\eta\left(h + 2\lambda\mathbb{E}\left[A_T + \int_0^T \alpha_t + \nu\left(h + \frac{\alpha_t}{\eta} - P_t\right)\mathrm{d}t - \sigma W_T \middle| \mathscr{F}_t\right]\right) \\
&= -\eta\left(h + 2\lambda\mathbb{E}\left[A_T + \int_0^T \alpha_t + \nu\left(h + \frac{\alpha_t}{\eta} - P_t\right)\mathrm{d}t \middle| \mathscr{F}_t\right]\right) - 2\lambda\sigma W_t,
\end{aligned}$$

as W is a Brownian motion, and thus a martingale. Note that the expression above has α_t on both the left and right hand side, so this is not an immediate solution for the abatement effort. What we can conclude from the expression above, which will also be useful later in this proof, is that the optimal process $(\hat{\alpha}_t)$ will be a martingale with respect to the filtration $(\mathscr{F}_t)$, as the conditional expectation of an integrable process is a martingale. For this to hold, we first need that the optimal process$(\hat{\alpha}_t)$ really exists.. This expression for α_t can be simplified to

$$\begin{aligned}
\alpha_t &= -\eta\left(h + 2\lambda\mathbb{E}\left[A_T + \int_0^T \alpha_t + \nu\left(h + \frac{\alpha_t}{\eta} - P_t\right)\mathrm{d}t \middle| \mathscr{F}_t\right]\right) - 2\lambda\sigma W_t \\
&= -\eta\left(h + 2\lambda M_t + 2\lambda\left(1 + \frac{\nu}{\eta}\right)\mathbb{E}\left[\int_0^T \alpha_t \mathrm{d}t \middle| \mathscr{F}_t\right] + 2\lambda\mathbb{E}\left[\int_0^T \nu(h - P_t)\mathrm{d}t \middle| \mathscr{F}_t\right]\right) - 2\lambda\sigma W_t.
\end{aligned} \tag{B.10}$$

Here we used the linearity of the conditional expectation and $M_t = \mathbb{E}[A_T|\mathscr{F}_t]$ by construction.

This expression and the martingality result can be used to find the optimal value $\hat{\alpha}_t$. First note that from (B.10) we can easily express α_0, by

$$\alpha_0 = -\eta\left(h + 2\lambda M_0 + 2\lambda\left(1 + \frac{\nu}{\eta}\right)\mathbb{E}\left[\int_0^T \alpha_t \mathrm{d}t\right] + 2\lambda\mathbb{E}\left[\int_0^T \nu(h - P_t)\mathrm{d}t\right]\right). \tag{B.11}$$

as $W_0 = 0$, by definition of the Brownian motion and $\mathbb{E}[X|\mathscr{F}_0] = \mathbb{E}[X]$. Let us look at a specific part of the expression above. Since $\alpha \in \mathscr{A}^2$ and a martingale by the reasoning above, we can use Lemma A.20. For this, let

$$\tilde{N}_t = \mathbb{E}\left[\int_0^T \alpha_s \mathrm{d}s \middle| \mathscr{F}_t\right].$$

Now, let us deal with $\tilde{N}_0$. According to the Lemma A.20, this can be expressed as

$$\mathbb{E}\left[\int_0^T \alpha_t \mathrm{d}t\right] = \tilde{N}_0 = T\alpha_0.$$

Plugging this into (B.11), we obtain

$$\alpha_0 = -\eta\left(h + 2\lambda M_0 + 2\lambda\left(1+\frac{\nu}{\eta}\right)T\alpha_0 + 2\lambda\mathbb{E}\left[\int_0^T \nu(h-P_t)\mathrm{d}t\right]\right).$$

Solving this for α_0, we get

$$\begin{aligned}
\alpha_0 + \eta 2\lambda\left(1+\frac{\nu}{\eta}\right)T\alpha_0 &= -\eta\left(h + 2\lambda M_0 + 2\lambda\mathbb{E}\left[\int_0^T \nu(h-P_t)\mathrm{d}t\right]\right)\\
\alpha_0\left(1+2T\lambda(\eta+\nu)\right) &= -\eta\left(h + 2\lambda M_0 + 2\lambda\mathbb{E}\left[\int_0^T \nu(h-P_t)\mathrm{d}t\right]\right)\\
\hat{\alpha}_0 &= -\frac{\eta}{(1+2T\lambda(\eta+\nu)}\left(h + 2\lambda M_0 + 2\lambda\mathbb{E}\left[\int_0^T \nu(h-P_t)\mathrm{d}t\right]\right)\\
&= -g(0)\left(\frac{h}{2\lambda} + M_0 + \mathbb{E}\left[\int_0^T \nu(h-P_t)\mathrm{d}t\right]\right). \qquad \text{(B.12)}
\end{aligned}$$

With the initial optimal value $\hat{\alpha}_0$, it makes sense to derive the dynamics of $\hat{\alpha}_t$. For this, we can make use of (B.10) and take the differential on both sides to obtain

$$\mathrm{d}\alpha_t = -2\lambda\eta\mathrm{d}M_t - 2\lambda\eta\left(1+\frac{\nu}{\eta}\right)\mathrm{d}\mathbb{E}\left[\int_0^T \alpha_t\mathrm{d}t\middle|\mathscr{F}_t\right] - 2\lambda\eta\mathrm{d}\mathbb{E}\left[\int_0^T \nu(h-P_t)\mathrm{d}t\middle|\mathscr{F}_t\right] - 2\lambda\sigma\mathrm{d}W_t. \qquad \text{(B.13)}$$

Again, we can use Lemma A.20 applied to α itself. Then, we get the dynamics of $\tilde{N}_t = \mathbb{E}\left[\int_0^T \alpha_t\mathrm{d}t\middle|\mathscr{F}_t\right]$ as follows

$$\mathrm{d}\tilde{N}_t = \mathrm{d}E\left[\int_0^T \alpha_t\mathrm{d}t\middle|\mathscr{F}_t\right] = (T-t)\mathrm{d}\alpha_t.$$

Plugging this into (B.13) and solving with respect to the dynamics of α_t, we obtain

$$\mathrm{d}\alpha_t = -2\lambda\eta\mathrm{d}M_t - 2\lambda\eta\left(1+\frac{\nu}{\eta}\right)(T-t)\mathrm{d}\alpha_t - 2\lambda\eta\mathrm{d}\mathbb{E}\left[\int_0^T \nu(h-P_t)\mathrm{d}t\middle|\mathscr{F}_t\right] - 2\lambda\sigma\mathrm{d}W_t. \qquad \text{(B.14)}$$

The left and right hand side of this expression depend on α. Moving the terms to one side and dividing by the common factor, we get

$$\begin{aligned}
\mathrm{d}\alpha_t &= -\frac{2\lambda\eta}{1+2\lambda(\eta+\nu)(T-t)}\left(\mathrm{d}M_t + \mathrm{d}\mathbb{E}\left[\int_0^T \nu(h-P_t)\mathrm{d}t\middle|\mathscr{F}_t\right] - \sigma\mathrm{d}W_t\right),\\
\mathrm{d}\hat{\alpha}_t &= -g(t)\left(\mathrm{d}M_t + \mathrm{d}\mathbb{E}\left[\int_0^T \nu(h-P_t)\mathrm{d}t\middle|\mathscr{F}_t\right] - \sigma\mathrm{d}W_t\right). \qquad \text{(B.15)}
\end{aligned}$$

Together with $\hat{\alpha}_0$, we have found a well-defined SDE.

We conclude that the optimal abatement effort is given by

$$\hat{\alpha}_t = \hat{\alpha}_0 - \int_0^T g(t)\mathrm{d}M_t - \nu\int_0^T g(t)\mathrm{d}\mathbb{E}\left[\int_0^T (h-P_t)\mathrm{d}t\middle|\mathscr{F}_t\right] + \sigma\int_0^T g(t)\mathrm{d}W_t, \qquad \text{(B.16)}$$

where $\hat{\alpha}_0$ is given by (B.12). Then, the optimal trading rate is given by

$$\hat{\beta}_t = \nu\left(h + \frac{\hat{\alpha}_t}{\eta} - P_t\right). \qquad \text{(B.17)}$$

This concludes our proof. □

Remark B.1. *Since $\hat{\alpha}$ is a martingale as concluded above, we have that $\hat{\beta}$ is a martingale if and only if the market price of permits P is a martingale. If the market price P would be a martingale, we can use Lemma A.20 to write*

$$dE\left[\int_0^T (h-P_t)dt \middle| \mathscr{F}_t\right] = -d\mathbb{E}\left[\int_0^T P_t dt \middle| \mathscr{F}_t\right] = (T-t)dP_t.$$

A rewritten version of the controls can be found below.

Proposition B.7. *Let $(\hat{X}_t^{i,E})$ be the corresponding bank account of firm i for the optimal controls $(\hat{\alpha}^i, \hat{\beta}^i)$. Then, the optimal control for $\hat{\alpha}_t{}^i$ can be rewritten to*

$$\hat{\alpha}_t^i = -g_i(t)\left(\frac{h_i}{2\lambda} + \hat{X}_T^{i,E} + R_t^i + \mathbb{E}\left[\int_t^T v(h_i - P_s)\, ds \middle| \mathscr{F}_t\right]\right),$$

where (R_t^i) is defined in (2.1)*. The formula of $\hat{\beta}_t^i$ will stay the same.*

Proof. Again, we omit the index for the firm i. We will use the first order condition (B.5), given in terms of $\hat{X}_T^E$ by

$$h + \frac{\hat{\alpha}_t}{\eta} + 2\lambda\mathbb{E}\left[\hat{X}_T^E \middle| \mathscr{F}_t\right] = 0.$$

Plugging in the expression for $\hat{X}_T^E$ in terms of $\hat{\alpha}, \hat{\beta}$, we get (B.9), but then with the optimal controls. We can add and subtract $2\lambda\hat{X}_t^E$ to that expression to get

$$h + \frac{\alpha_t}{\eta} + 2\lambda\hat{X}_t^E - 2\lambda\hat{X}_t^E + 2\lambda\mathbb{E}\left[A_T + \int_0^T \alpha_t + \nu\left(h + \frac{\alpha_t}{\eta} - P_t\right)\mathrm{d}t + \sigma W_T \middle| \mathscr{F}_t\right] = 0, \tag{B.18}$$

since it holds that

$$\hat{X}_t^E = A_t + \int_0^t \hat{\alpha}_s + \hat{\beta}_s \mathrm{d}s - \sigma W_t.$$

Note that $\hat{X}_t^E$ is $\mathscr{F}_t$ measurable, by Proposition 1.8. Rewriting (B.19), we obtain

$$\begin{aligned} h + \frac{\hat{\alpha}_t}{\eta} + 2\lambda\hat{X}_t^E + 2\lambda\mathbb{E}\left[A_T + \int_0^T \hat{\alpha}_t\left(1+\frac{\nu}{\eta}\right) + \nu(h-P_t)\,\mathrm{d}t - \hat{X}_t^E \middle| \mathscr{F}_t\right] + \sigma W_t = 0, \\ h + \frac{\hat{\alpha}_t}{\eta} + 2\lambda\hat{X}_t^E + 2\lambda\mathbb{E}\left[A_T - A_t + \int_0^T \hat{\alpha}_t\left(1+\frac{\nu}{\eta}\right) + \nu(h-P_t)\,\mathrm{d}t - \int_0^t \hat{\alpha}_t + \hat{\beta}_t \mathrm{d}t \middle| \mathscr{F}_t\right] - \sigma W_t + \sigma W_t = 0, \\ h + \frac{\hat{\alpha}_t}{\eta} + 2\lambda\hat{X}_t^E + 2\lambda\mathbb{E}\left[A_T - A_t + \int_t^T \hat{\alpha}_s\left(1+\frac{\nu}{\eta}\right) + \nu(h-P_s)\,\mathrm{d}s \middle| \mathscr{F}_t\right] = 0. \end{aligned} \tag{B.19}$$

In the above, we used the expression for $\hat{\beta}$. Now, by Corollary A.21, we have that

$$\mathbb{E}\left[\int_t^T \hat{\alpha}_s \mathrm{d}s \middle| \mathscr{F}_t\right] = \hat{\alpha}_t(T-t).$$

Using this and Definition 2.1, we can rewrite (B.19) to

$$h + \frac{\hat{\alpha}_t}{\eta} + 2\lambda\hat{X}_t^E + 2\lambda R_t + 2\lambda\left(1+\frac{\nu}{\eta}\right)(T-t)\hat{\alpha}_t + 2\lambda\nu\mathbb{E}\left[\int_t^T (h-P_s)\,\mathrm{d}s \middle| \mathscr{F}_t\right] = 0. \tag{B.20}$$

Solving (B.20) for $\hat{\alpha}_t$, we obtain

$$\begin{aligned} \hat{\alpha}_t\left(\frac{1}{\eta} + 2\lambda(T-t) + 2\lambda\frac{\nu}{\eta}(T-t)\right) &= -\left(h + 2\lambda\hat{X}_t^E + 2\lambda R_t + 2\lambda\nu\mathbb{E}\left[\int_t^T (h-P_s)\,\mathrm{d}s \middle| \mathscr{F}_t\right]\right), \\ \hat{\alpha}_t\left(\frac{1+2\lambda(\eta+\nu)(T-t)}{\eta}\right) &= -\left(h + 2\lambda\hat{X}_t^E + 2\lambda R_t + 2\lambda\nu\mathbb{E}\left[\int_t^T (h-P_s)\,\mathrm{d}s \middle| \mathscr{F}_t\right]\right), \\ \hat{\alpha}_t &= -g(t)\left(\frac{h}{2\lambda} + \hat{X}_t^E + R_t + \nu\mathbb{E}\left[\int_t^T (h-P_s)\,\mathrm{d}s \middle| \mathscr{F}_t\right]\right). \end{aligned} \tag{B.21}$$

This ends the proof of this proposition. □

Remark B.2. *Note, when P itself would be a martingale, we could again apply Corollary A.21 to write*

$$\nu\mathbb{E}\left[\int_t^T (h-P_s)\,ds\middle|\mathscr{F}_t\right] = \nu h(T-t) - \nu P_t(T-t). \tag{B.22}$$

Recall that marginal costs are defined as the extra costs that arise from an extra unit. The marginal abatement costs and marginal trading rates are given by the operators $\mathscr{A} \to \mathscr{B}$ such that

$$c_i'(\alpha^i) = h_i + \frac{\alpha^i}{\eta_i}, \quad f_i'(\beta^i) = P + \frac{\beta^i}{\nu}.$$

This is proven in Proposition A.32. From the results of our minimisation problem given in (B.5) and (B.6), we see that we could also have written, in the optimum

$$c_i'\left(\hat{\alpha}^i\right) = -2\lambda\mathbb{E}\left[X_T^E|\mathscr{F}_\cdot\right], \quad \hat{\beta}^i = \nu\left(c'\left(\hat{\alpha}^i\right) - P\right).$$

From this, we see that $\hat{\beta}_t^i > 0$, if $c'(\hat{\alpha}_t^i) > P_t$, and $\hat{\beta}_t^i \leqslant 0$ else. This is consistent with what we would expect, if the price of an extra ton of abatement is higher than the market price, there are bought extra permits, as this is cheaper for the firm. If the market price of permits is higher than the marginal abatement costs, it is cheaper to adjust the abatement. Rewriting the latter expression for β_t^i, we obtain

$$c'\left(\hat{\alpha}^i\right)_t = P_t + \frac{\hat{\beta}_t^i}{\nu} = f'\left(\hat{\beta}^i\right)_t.$$

This implies that in the optimum, the marginal costs of trading are equated to the marginal costs of abatement.

As concluded earlier, the optimal abatement effort $\hat{\alpha}^i$ given in (B.16) is a martingale, as it is a stochastic integral. In this expression, the integrand $g_i(t)$ is deterministic and bounded and the integral is taken with respect to three different martingales. The first martingale is (M_t^i), the conditional expectation of A_T^i, the net allowances at time T. If this increases ceteris paribus, we see that the abatement effort decreases. This comes from the fact that the expected allowances increase, hence the instantaneous abatement effort lowers.

The second martingale, $\mathbb{E}\left[\int_0^T (h-P_t)\middle|\mathscr{F}_t\right]$, involves the market price of permits. Note that

$$\mathrm{d}\mathbb{E}\left[\int_0^T (h-P_t)\mathrm{d}t\middle|\mathscr{F}_t\right] = \mathrm{d}hT - \mathrm{d}\mathbb{E}\left[\int_0^T P_t\mathrm{d}t\middle|\mathscr{F}_t\right] = -\mathrm{d}\mathbb{E}\left[\int_0^T P_t\mathrm{d}t\middle|\mathscr{F}_t\right].$$

If the conditional expectation of the aggregate market price goes up, the abatement effort also will rise, again ceteris paribus. This is also reasonable, as an higher aggregate market price means that it is less attractive to buy allowances on the market of permits. To make sure the desired reduction of emissions is achieved, the firm should abate less emissions. The last martingale that is used in (B.16) is the Brownian motion of the economic shocks. If there is a positive economic shock, ceteris paribus, the emissions increase and thus the abatement effort increases as well. This is reasonable, as there are more resources available to lower the emissions.

The optimal trading rate in (B.17) is a function of the optimal abatement effort α. We see that a positive shock in the economy also increases the trading rate $\hat{\beta}$, as then $\hat{\alpha}$ increases. It is less trivial to say what happens if the conditional expectation of the aggregate market price of permits rises, as this variable is also involved in the formula of the trading rate itself.

B.2 Market equilibrium

In this section, the market equilibrium that arises from trading between firms, is obtained. We introduce the following variable, which will be useful in the expression for the optimal market price $\hat{P}$. Let

$$\pi_i(t) := \frac{\frac{g_i(t)}{\eta_i}}{\left(1 - \frac{\nu(T-t)}{N}\sum_{k=1}^{N}\frac{g_k(t)}{\eta_k}\right)}, \tag{B.23}$$

where we recall that $g_i(t) = \frac{2\lambda\eta_i}{1+2\lambda(\eta_i+\nu)(T-t)}$. As $\lambda, \eta_i > 0$, it holds that $g_i(t) > 0$. The same will be proven for $\pi_i(t)$.

Lemma B.8. *The deterministic function $\pi_i(t)$ is positive for all $t \in [0,T]$.*

Proof. It follows

$$\begin{aligned}\frac{\nu(T-t)}{N}\sum_{k=1}^{N}\frac{g_k(t)}{\eta_k} &= \frac{\nu(T-t)}{N}\sum_{k=1}^{N}\frac{2\lambda}{1+2\lambda\eta_k(T-t)+2\lambda\nu(T-t)} \\ &< \frac{\nu(T-t)}{N}\sum_{k=1}^{N}\frac{2\lambda}{1+2\lambda\nu(T-t)} = \frac{2\lambda\nu(T-t)}{1+2\lambda\nu(T-t)} < 1.\end{aligned}$$

This holds as $\lambda, \eta_i > 0$. Since $g_i(t) > 0$, it follows by the argument above that $\pi(t) > 0$ for all $t \in [0,T]$. □

Now, we have all the necessary elements to compute the optimal equilibrium prices $\hat{P}$, which is done in the following main theorem of this section.

Theorem B.9. *Assume that there is given an exogenous, net allocation scheme $A = (A^1, \dots A^N) \in \mathscr{S}^N$. Then, the equilibrium price $\hat{P}$ is given by the SDE,*

$$d\hat{P}_t = -\frac{1}{N}\sum_{i=1}^{N}\pi_i(t)\left(dM_t^i - \sigma_i dW_t^i\right), \quad \hat{P}_0 = \frac{1}{N}\sum_{i=1}^{N}\pi_i(0)\left(\eta_i h_i T - M_0^i\right). \tag{B.24}$$

This can be rewritten to

$$\hat{P}_t = \frac{1}{N}\pi_i(t)\left(\eta_i h_i(T-t) - \left(\hat{X}_T^{i,E} + R_t^i\right)\right). \tag{B.25}$$

Proof. Again, all the equations will hold almost surely, for all $t \in [0,T]$ We start with the market clearing condition in combination with Equation (B.17), which implies

$$\sum_{i=1}^{N}\hat{\beta}_t^i = \sum_{i=1}^{N}\nu\left(h_i + \frac{\hat{\alpha}_t^i}{\eta_i} - P_t\right) = 0,$$

a.s, for all $t \in (0,T]$. We are going to solve this equation for P. As $0 < \nu < \infty$, this reduces to solving

$$\begin{aligned}\sum_{i=1}^{N} h_i + \frac{\hat{\alpha}_t^i}{\eta_i} - P_t = 0, \\ \sum_{i=1}^{N} h_i + \sum_{i=1}^{N}\frac{\hat{\alpha}_t^i}{\eta_i} = NP_t,\end{aligned} \tag{B.26}$$

as the market price is the same for all firms. From this, it immediately follows that $(\hat{P}_t)$ needs to be a martingale, as $(\hat{\alpha}_t^i)$ is given to be one. Taking the differential in (B.26), we get

$$N\mathrm{d}P_t = \sum_{i=1}^{N}\frac{\mathrm{d}\hat{\alpha}_t^i}{\eta_i}, \tag{B.27}$$

as h_i does not depend on t for all firms. Combining Equation (B.15) with (B.27), results in

$$N\mathrm{d}P_t = -\sum_{i=1}^{N}\frac{g_i(t)}{\eta_i}\left(\mathrm{d}M_t^i + \mathrm{d}\mathbb{E}\left[\int_0^T \nu(h-P_t)\mathrm{d}t\middle|\mathscr{F}_t\right] - \sigma_i \mathrm{d}W_t^i\right).$$

Then, Equation (B.27) becomes, with use of Remark B.1,

$$\mathrm{d}P_t = -\frac{1}{N}\sum_{i=1}^{N}\frac{g_i(t)}{\eta_i}\left(\mathrm{d}M_t^i - \nu\mathrm{d}\mathbb{E}\left[\int_0^T P_t\mathrm{d}t\middle|\mathscr{F}_t\right] - \sigma_i \mathrm{d}W_t^i\right). \tag{B.28}$$

It follows immediately that the equilibrium price P is a martingale, by the fact that it is the summation of three stochastic integrals that are all martingales, by Proposition A.15. Note that the stochastic integrals are well-defined, as the integrands are deterministic and thus progressively measurable. The martingality of P is necessary to have, to be able to apply Lemma A.20. From this lemma, we obtain, since $Pin\mathscr{A}$, that

$$\mathrm{d}\mathbb{E}\left[\int_0^T P_t\mathrm{d}t\middle|\mathscr{F}_t\right] = (T-t)\mathrm{d}P_t.$$

Using this in (B.28), we get the following

$$\mathrm{d}P_t = -\frac{1}{N}\sum_{i=1}^{N}\frac{g_i(t)}{\eta_i}\left(\mathrm{d}M_t^i - \nu(T-t)\mathrm{d}P_t - \sigma_i\mathrm{d}W_t^i\right),$$

$$\left(N - \sum_{i=1}^{N}\frac{g_i(t)\nu(T-t)}{\eta_i}\right)\mathrm{d}P_t = -\sum_{i=1}^{N}\frac{g_i(t)}{\eta_i}\left(\mathrm{d}M_t^i - \sigma_i\mathrm{d}W_t^i\right).$$

Solving this for $\mathrm{d}P_t$, we get

$$\begin{aligned}
\mathrm{d}\hat{P}_t &= -\frac{\sum_{i=1}^{N}\frac{g_i(t)}{\eta_i}}{\left(N - \sum_{j=1}^{N}\frac{g_j(t)\nu(T-t)}{\eta_j}\right)}\left(\mathrm{d}M_t^i - \sigma_i\mathrm{d}W_t^i\right)\\
&= -\frac{1}{N}\frac{\sum_{i=1}^{N}\frac{g_i(t)}{\eta_i}}{\left(1 - \frac{1}{N}\sum_{j=1}^{N}\frac{g_j(t)\nu(T-t)}{\eta_j}\right)}\left(\mathrm{d}M_t^i - \sigma_i\mathrm{d}W_t^i\right)\\
&= -\frac{1}{N}\sum_{i=1}^{N}\frac{\frac{g_i(t)}{\eta_i}}{\left(1 - \frac{\nu(T-t)}{N}\sum_{j=1}^{N}\frac{g_j(t)}{\eta_j}\right)}\left(\mathrm{d}M_t^i - \sigma_i\mathrm{d}W_t^i\right) = -\frac{1}{N}\sum_{i=1}^{N}\pi_i(t)\left(\mathrm{d}M_t^i - \sigma_i\mathrm{d}W_t^i\right),
\end{aligned}$$

where $\pi_i(t)$ is defined in Equation (B.23). Now the dynamics for $t > 0$ are obtained, it is time to get the initial condition $\hat{P}_0$. This will follow from $\hat{\beta}_0$ and $\hat{\alpha}_0$ in Equation (B.12). We have

$$NP_0 = \sum_{i=1}^{N}h_i + \sum_{i=1}^{N}\frac{\hat{\alpha}_0^i}{\eta_i} = \sum_{i=1}^{N}h_i - \sum_{i=1}^{N}\frac{g_i(0)}{\eta_i}\left(\frac{h_i}{2\lambda} + M_0 + \mathbb{E}\left[\int_0^T \nu(h_i - P_t)\mathrm{d}t\right]\right).$$

By a simple Fubini argument and the use that P is a martingale, it follows that

$$\mathbb{E}\left[\int_0^T \nu(h_i - P_t)\mathrm{d}t\right] = \nu h_i T - \nu T P_0.$$

This gives

$$NP_0 = \sum_{i=1}^{N}h_i - \sum_{i=1}^{N}\frac{g_i(0)}{\eta_i}\left(\frac{h_i}{2\lambda} + M_0^i + \nu h_i T - \nu T P_0\right),$$

$$\left(N - \sum_{i=1}^{N}\frac{g_i(0)T\nu}{\eta_i}\right)P_0 = \sum_{i=1}^{N}\frac{g_i(0)}{\eta_i}\left(-\frac{h_i}{2\lambda} - M_0^i - \nu h_i T + \frac{\eta_i}{g_i(0)}h_i\right).$$

Since

$$-\frac{h_i}{2\lambda} - \nu h_i T + \frac{\eta_i}{g_i(0)} h_i = -\frac{h_i}{2\lambda} - \nu h_i T + h_i \frac{1+2\lambda(\eta_i+\nu)T}{2\lambda} = h_i \eta_i T,$$

we can write

$$\left(N - \sum_{i=1}^{N} \frac{g_i(0) T \nu}{\eta_i}\right) \hat{P}_0 = \sum_{i=1}^{N} \frac{g_i(0)}{\eta_i} \left(-M_0^i + h_i \eta_i T\right),$$

$$\hat{P}_0 = \frac{\sum_{i=1}^{N} \frac{g_i(0)}{\eta_i}}{\left(N - \sum_{j=1}^{N} \frac{g_j(0) T \nu}{\eta_j}\right)} \left(-M_0^i + h_i \eta_i T\right).$$

From this, we get the final result for the initial condition

$$\hat{P}_0 = \frac{1}{N} \sum_{i=1}^{N} \pi_i(0) \left(h_i \eta_i T - M_0^i\right). \tag{B.29}$$

The equilibrium price $(\hat{P}_t)$ obtained, can be rewritten. For this, the market clearing condition will again be used, but now with another expression for $\hat{\alpha}^i$, which can be found in Proposition B.7, together with Remark B.2. When we plug in $\hat{\alpha}_t^i$ of that proposition in the market clearing condition in (B.26), we get

$$\hat{P}_t = \frac{1}{N} \sum_{i=1}^{n} h_i + \frac{\hat{\alpha}_t^i}{\eta_i} = \frac{1}{N} \sum_{i=1}^{N} h_i - \frac{g_i(t)}{\eta_i} \left(\frac{1}{2\lambda} h_i + \hat{X}_T^{i,E} + R_t^i + \nu(T-t) h_i - \nu \hat{P}_t (T-t)\right),$$

Solving this equation for $\hat{P}$, it follows

$$\left(1 - \frac{1}{N} \sum_{i=1}^{N} \frac{\nu g_i(t)(T-t)}{\eta_i}\right) \hat{P}_t = \frac{1}{N} \sum_{i=1}^{N} h_i - \frac{g_i(t)}{\eta_i} \left(\frac{1}{2\lambda} h_i + \hat{X}_T^{i,E} + R_t^i + \nu(T-t) h_i\right)$$

$$= \frac{1}{N} \sum_{i=1}^{N} h_i \left(1 - \frac{g_i(t)}{\eta_i}\right) \left(\frac{1}{2\lambda} + \nu(T-t)\right) - \frac{g_i(t)}{\eta_i} \left(\hat{X}_T^{i,E} + R_t^i\right).$$

Since it holds that

$$1 - \frac{g_i(t)}{\eta_i} \left(\frac{1}{2\lambda} + \nu(T-t)\right) = 1 - \frac{2\lambda}{1+2\lambda(\eta_i+\nu)(T-t)} \left(\frac{1}{2\lambda} + \nu(T-t)\right)$$

$$= 1 - \frac{1}{1+2\lambda(\eta_i+\nu)(T-t)} - \frac{2\lambda\nu(T-t)}{1+2\lambda(\eta_i+\nu)(T-t)}$$

$$= \frac{2\lambda\eta_i(T-t)}{1+2\lambda(\eta_i+\nu)(T-t)} = g_i(t)(T-t),$$

we can write,

$$\left(1 - \frac{1}{N} \sum_{i=1}^{N} \frac{\nu g_i(t)(T-t)}{\eta_i}\right) \hat{P}_t = \frac{1}{N} \sum_{i=1}^{N} h_i g_i(t)(T-t) - \frac{g_i(t)}{\eta_i} \left(\hat{X}_T^{i,E} + R_t^i\right)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \frac{g_i(t)}{\eta_i} \left(\eta_i h_i (T-t) - \hat{X}_T^{i,E} - R_t^i\right).$$

From this, it follows that

$$\hat{P}_t = \frac{1}{N} \sum_{i=1}^{N} \pi_i(t) \left(\eta_i h_i (T-t) - \hat{X}_T^{i,E} - R_t^i\right).$$

This is all we needed to prove. □

The next corollary follows directly.

Corollary B.10. *The SDEs of the optimal controls $\hat{\alpha}^i$, $\hat{\beta}^i$ are given by*

$$d\hat{\alpha}_t^i = -g_i(t)\left(dM_t^i - \sigma_i dW_t^i - \nu(T-t)d\hat{P}_t\right), \qquad \hat{\alpha}_0^i = -g_i(0)\left(h_i\left(\frac{1}{2\lambda} + \nu T\right) + M_0^i - \nu T\hat{P}_0\right), \quad \text{(B.30)}$$

$$d\hat{\beta}_t^i = \nu d\left(h_i + \frac{\hat{\alpha}_t^i}{\eta_i} - \hat{P}_t\right), \qquad \hat{\beta}_0^i = \nu\left(h_i + \frac{\hat{\alpha}_0^i}{\eta_i} - \hat{P}_0\right). \quad \text{(B.31)}$$

The optimal controls can also be represented as

$$\hat{\alpha}_t^i = g_i(t)\left(\nu(T-t)(\hat{P}_t - h_i) - \left(\frac{h_i}{2\lambda} + \hat{X}_T^{i,E} + R_t^i\right)\right),$$
$$\hat{\beta}_t^i = \nu\left(h_i + \frac{\hat{\alpha}_t^i}{\eta_i} - \hat{P}_t\right).$$

Proof. The proof follows from the fact that $\hat{P}$ is a martingale. The expressions can be immediately seen from the expressions for $\hat{\alpha}^i$, $\hat{\beta}^i$ obtained in Equations (B.21) and (B.8), together with Remark B.1. The alternative representation is obtained from Proposition B.7 together with Remark B.2. □

There are a couple of points to note. Firstly, as $\hat{P}$ is a martingale now, it follows that $\hat{\beta}^i$ is a martingale as well. Additionally, from (B.24) we see that the stochastic process the market price depends on is equal to

$$M_t^i - \sigma_i W_t^i = \mathbb{E}\left[A_T^i - \sigma_i W_T^i \middle| \mathcal{F}_t\right]. \quad \text{(B.32)}$$

We note that $A_T^i - \sigma W_T^i$ can be called the terminal bank account in the BAU scenario, as here no controls are involved. If the bank account in this scenario rises, we can see from the equations that the market price $\hat{P}_t$ decreases, as $\pi_i(t) > 0$ for all $t \in [0,T]$, by Lemma B.8. The decrease of P is realistic and what we would expect, as an increase in bank account decreases the immediate demand for the allowances.

This finalises the section on the equilibrium in the case of market frictions.

B.3 Optimal dynamic policy

This section aims to demonstrate the optimal dynamic allocation in the presence of market frictions, which is based on Section 5.1.1 of [AB23]. In this source, the section regarding this subject is rather short, as they refer to the case without frictions. However, the computations did not seem to be able to fully adapt to account for frictions. Hence, the procedure will follow the same lines as in the case of no frictions, but some steps may significantly differ. We will mainly focus on those differences. Nevertheless, we are able to derive a sufficient solution. The solutions are obtained under the assumption that that all firms have the same flexibility parameter, that is,

$$\eta_i = \eta,$$

for all firms. This implies that

$$\bar{\eta} := \eta, \quad g_i(t) = \frac{2\lambda\eta_i}{1 + 2\lambda(\eta_i + \nu)(T-t)} =: g(t), \quad \pi_i(t) = \frac{\frac{g(t)}{\eta}}{\left(1 - \nu(T-t)\frac{g(t)}{\eta}\right)} =: \pi(t),$$

for all $t \in [0,T]$. The goal of this section is to find

$$\inf_{A\in\mathscr{S}^N} \mathbb{E}\left[\sum_{i=1}^N \int_0^T h_i\hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} + \frac{(\hat{\beta}_t^i)^2}{2\nu}\mathrm{d}t + \lambda\left(\hat{X}_T^{i,E}\right)^2\right] \text{ s.t. } \mathbb{E}\left[\sum_{i=1}^N E_T^{i,\hat{\alpha}^i}\right] = \rho NT\bar{\mu}, \tag{B.33}$$

which corresponds with (1.45) with $E_0^i = 0$. Note that the part $\sum_{i=1}^N P_t\hat{\beta}_t^i$ is not present, by the market clearing condition. The same procedure as in Proposition 2.15 can be followed to prove the following proposition.

Proposition B.11. *From the constraint*

$$\mathbb{E}\left[\sum_{i=1}^N E_T^{i,\hat{\alpha}^i}\right] = \rho NT\bar{\mu},$$

we can get to

$$\bar{M}_0 = -\frac{1}{g(0)(1+T\nu\pi(0))}\left((1-\rho)\bar{\mu} + g(0)\left(\nu T + \frac{1}{2\lambda} - T^2\eta\nu\pi(0)\right)\bar{h}\right) := q(\rho).$$

Proof. The total abated emissions at time T can be expressed by

$$\begin{aligned}
\rho N\bar{\mu}T = \mathbb{E}\left[\sum_{i=1}^N E_T^{i,\hat{\alpha}^i}\right] &= N\bar{\mu}T - \sum_{i=1}^N \int_0^T \mathbb{E}[\hat{\alpha}_s]\mathrm{d}s = N\bar{\mu}T - \sum_{i=1}^N T\hat{\alpha}_0^i \\
&= N\bar{\mu}T + Tg(0)\sum_{i=1}^N \left(\left(h_i\left(\frac{1}{2\lambda}+\nu T\right) + M_0^i - \nu T\hat{P}_0\right)\right),
\end{aligned} \tag{B.34}$$

where we have used the fact that $\hat{\alpha}$ is a martingale and the expression (B.30) for $\hat{\alpha}$.

Together with $\hat{P}_0$ of (B.24), here given by

$$\hat{P}_0 = \frac{1}{N}\pi(0)\left(\eta T\sum_{i=1}^N h_i - \sum_{i=1}^N M_0^i\right),$$

we obtain by (B.34),

$$\begin{aligned}
(\rho-1)N\bar{\mu}T &= Tg(0)\left(\nu T + \frac{1}{2\lambda}\right)\sum_{i=1}^N h_i + Tg(0)\sum_{i=1}^N M_0^i - NT^2\nu g(0)\hat{P}_0 \\
&= Tg(0)\left(\nu T + \frac{1}{2\lambda}\right)\sum_{i=1}^N h_i + Tg(0)\sum_{i=1}^N M_0^i - T^2\nu g(0)\pi(0)\left(\eta T\sum_{i=1}^N h_i - \sum_{i=1}^N M_0^i\right) \\
&= Tg(0)\left(\nu T + \frac{1}{2\lambda}\right)\sum_{i=1}^N h_i + Tg(0)\sum_{i=1}^N M_0^i - T^3\eta\nu g(0)\pi(0)\sum_{i=1}^N h_i + T^2\nu g(0)\pi(0)\sum_{i=1}^N M_0^i.
\end{aligned}$$

As in the previous case, we would like to solve this for $\bar{M}_0$, starting with

$$Tg(0)\sum_{i=1}^N M_0^i + T^2\nu g(0)\pi(0)\sum_{i=1}^N M_0^i = (\rho-1)N\bar{\mu}T - Tg(0)\left(\nu T + \frac{1}{2\lambda}\right)\sum_{i=1}^N h_i + T^3\eta\nu g(0)\pi(0)\sum_{i=1}^N h_i.$$

Dividing both sides by T and solving for $\bar{M}_0$, we obtain

$$\begin{aligned}
\bar{M}_0(Ng(0) + NT\nu g(0)\pi(0)) &= (\rho-1)N\bar{\mu} - g(0)\left(\nu T + \frac{1}{2\lambda}\right)\sum_{i=1}^N h_i + T^2\eta\nu g(0)\pi(0)\sum_{i=1}^N h_i, \\
\bar{M}_0 &= \frac{1}{Ng(0)(1+T\nu\pi(0))}\left((\rho-1)N\bar{\mu} - g(0)\left(\nu T + \frac{1}{2\lambda} - T^2\eta\nu\pi(0)\right)\sum_{i=1}^N h_i\right).
\end{aligned}$$

This can be rewritten to

$$\bar{M}_0 = -\frac{1}{g(0)(1+T\nu\pi(0))}\left((1-\rho)\bar{\mu} + g(0)\left(\nu T + \frac{1}{2\lambda} - T^2\eta\nu\pi(0)\right)\bar{h}\right) =: q(\rho). \tag{B.35}$$

□

In the case of no frictions, this result is deduced by an intermediate step for the expression of the initial market price $\hat{P}_0$. We see here that in the case of frictions, this step is omitted and the relation for $\bar{M}_0$ is obtained directly from the constraint on the total emissions in the system at time T. Ideally, we would again be able to prove that $\bar{M}_0 < 0$, which corresponds to the findings when $\nu = \infty$ and $E_0 = 0$. To obtain the non-negativeness of $q(\rho)$, the following lemma is used.

Lemma B.12. *It holds that*

$$\nu T + \frac{1}{2\lambda} - T^2 \nu\eta\pi(0) > 0. \tag{B.36}$$

Proof. Let's start rewriting $\pi(0)$ in (B.23). The denominator can be rewritten to

$$1 - \nu T \frac{g(0)}{\eta} = 1 - \nu T \frac{2\lambda}{1 + 2\lambda(\eta + \nu)T} = \frac{1 + 2\lambda\eta T}{1 + 2\lambda\eta T + 2\lambda\nu T}.$$

This gives

$$\pi(0) = \frac{g(0)}{\eta}\left(\frac{1 + 2\lambda\eta T}{1 + 2\lambda\eta T + 2\lambda\nu T}\right)^{-1} = \frac{2\lambda}{1 + 2\lambda\eta T}.$$

From this it follows indeed that

$$T^2\nu\eta\pi(0) = \frac{T^2\nu\eta 2\lambda}{1 + 2\lambda\eta T} < \frac{T^2\nu\eta 2\lambda}{2\lambda\eta T} = \nu T < \nu T + \frac{1}{2\lambda}.$$

This is the desired result. □

Since $g(0), \pi(0) > 0$ and $(1-\rho)\bar{\mu} > 0$, and all the other parameters as well, we can conclude with Proposition B.12 that $\bar{M}_0 < 0$ holds, which is the same as in the case without frictions in the scenario where $E_0 = 0$. With condition (B.35), we have an well-posed, rewritten, optimisation problem. From this, the following expression of $\hat{P}_0$ can be obtained in terms of $q(\rho)$

$$\hat{P}_0 = \frac{1}{N}\pi(0)\eta T \sum_{i=1}^{N} h_i - \frac{1}{N}\pi(0)\sum_{i=1}^{N} M_0^i = \pi(0)\eta T\bar{h} - \pi(0)\bar{M}_0 = \pi(0)\eta T\bar{h} - \pi(0)q(\rho).$$

Note that Equation (2.40) holds generally, thus also in this case. Ideally, we would be able to resemble Corollary 2.16, with $\bar{M}_0 = q(\rho)$. This will be done in the following proposition.

Proposition B.13. *The minimisation problem* (B.33) *can be rewritten to*

$$\inf_{\vec{M}\in\mathcal{M}^N} \mathbb{E}\left[\sum_{i=1}^{N}\int_0^T h_i\hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} + \frac{(\hat{\beta}_t^i)^2}{2\nu}dt + \lambda\left(\hat{X}_T^{i,E}\right)^2\right] \text{ s.t. } \bar{M}_0 = q(\rho). \tag{B.37}$$

Proof. The constraint follows from Proposition B.11. From (B.30) we see that $\hat{\alpha}^i$ only depends on A^i through M^i. Since $\hat{\beta}^i$ depends on $\hat{\alpha}$ and $\hat{P}$, with $\hat{P}$ given in (B.24), the same holds for $\hat{\beta}^i$. Furthermore, by (B.5), it follows that

$$X_T^i = \frac{1}{2\lambda}\left(-h_i - \frac{\hat{\alpha}_T^i}{\eta_i}\right), \tag{B.38}$$

a.s. With the above, we can conclude that X_T^i only depends on A^i through M^i. The result follows. □

The next goal would be to rewrite the objective function in terms of $\hat{P}_0$ and $\hat{P}$, such that we can use the same argument as in the cases without market frictions. For this, we first note that from Equation (B.25) we obtain that at time T the market price and bank account are related as follows

$$\hat{P}_T = -\frac{1}{N}\pi(T)\sum_{i=1}^{N}\hat{X}_T^{i,E} = -\frac{1}{N}\frac{g(T)}{\eta}\sum_{i=1}^{N}\hat{X}_T^{i,E} = -2\lambda\frac{1}{N}\sum_{i=1}^{N}\hat{X}_T^{i,E}. \tag{B.39}$$

Unfortunately, this cannot be used directly, as we would like to replace $\sum_{i=1}^{N}(X_T^i)^2$. Instead, we are going to use the relation (B.38) to achieve an optimal result.

Theorem B.14 (Optimal dynamic allocation). *The minimisation problem of the regulator*

$$\inf_{\vec{M}\in\mathcal{M}^N}\mathbb{E}\left[\sum_{i=1}^{N}\int_0^T h_i\hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} + \frac{(\hat{\beta}_t^i)^2}{2\nu}dt + \lambda\left(\hat{X}_T^{i,E}\right)^2\right] \; s.t.\; \bar{M}_0 = q(\rho). \tag{B.40}$$

can be solved for M^i such that the constraint above is met and such that

$$\left\langle \alpha^i\right\rangle_t = 0, \quad \left\langle \alpha^i - P\right\rangle_t = 0, \quad \text{for all firms and a.s, for all } t\in[0,T].$$

The associated costs C_{opt} are then given by

$$C_{opt} = \sum_{i=1}^{N} Th_i\hat{\alpha}_0 + \frac{2h_i\hat{\alpha}_0^i}{4\lambda\eta_i} + \frac{h_i^2}{4\lambda} + \frac{\nu}{2}Th_i^2 + \frac{Th_i\hat{\alpha}_0^i}{\nu\eta_i} - \nu Th_i\hat{P}_0 + \frac{1}{4\lambda(\eta_i)^2}\left(\hat{\alpha}_0^i\right)^2 + \frac{1}{2\eta}\left(\hat{\alpha}_0^i\right)^2$$
$$+\frac{\nu}{2}\left(\frac{\hat{\alpha}_0^i}{\eta_i} - \hat{P}_0\right)^2.$$

Proof. Recall that

$$\hat{X}_T^{i,E} = \frac{1}{2\lambda}\left(-h_i - \frac{\hat{\alpha}_T^i}{\eta_i}\right), \quad \hat{\beta}_t^i = \nu\left(h_i + \frac{\hat{\alpha}_t^i}{\eta_i} - \hat{P}_t\right).$$

We will start rewriting the objective function with the given formulas given. Then,

$$K^i := \mathbb{E}\left[\int_0^T h_i\hat{\alpha}_t^i + \frac{(\hat{\alpha}_t^i)^2}{2\eta_i} + \frac{(\hat{\beta}_t^i)^2}{2\nu}\mathrm{d}t + \lambda(\hat{X}_T^{i,E})^2\right] \tag{B.41}$$
$$= Th_i\hat{\alpha}_0 + \int_0^T\mathbb{E}\left[\frac{(\hat{\alpha}_t^i)^2}{2\eta_i}\right]\mathrm{d}t + \int_0^T\mathbb{E}\left[\frac{(\hat{\beta}_t^i)^2}{2\nu}\right]\mathrm{d}t + \mathbb{E}\left[\lambda\left(\hat{X}_T^{i,E}\right)^2\right]$$
$$= Th_i\hat{\alpha}_0 + \int_0^T\frac{1}{2\eta}\mathbb{E}\left[(\hat{\alpha}_t^i)^2\right]\mathrm{d}t + \int_0^T\frac{1}{2\nu}\mathbb{E}\left[(\hat{\beta}_t^i)^2\right]\mathrm{d}t + \mathbb{E}\left[\frac{1}{4\lambda}\left(h_i^2 + \frac{(\hat{\alpha}_T^i)^2}{(\eta_i)^2} + \frac{2h_i\hat{\alpha}_T^i}{\eta_i}\right)\right]$$
$$= Th_i\hat{\alpha}_0 + \frac{2h_i\hat{\alpha}_0^i}{4\lambda\eta_i} + \frac{h_i^2}{4\lambda} + \int_0^T\frac{1}{2\eta}\mathbb{E}\left[(\hat{\alpha}_t^i)^2\right]\mathrm{d}t + \int_0^T\frac{1}{2\nu}\mathbb{E}\left[(\hat{\beta}_t^i)^2\right]\mathrm{d}t + \frac{1}{4\lambda(\eta_i)^2}\mathbb{E}\left[(\hat{\alpha}_T^i)^2\right],$$

by the fact that $\hat{\alpha}^i$ is a martingale and multiple Fubini arguments. Since

$$\left(\hat{\beta}_t^i\right)^2 = \nu^2\left(h_i^2 + \frac{(\hat{\alpha}_t^i)^2}{\eta_i^2} + \hat{P}_t^2 + \frac{2h_i\hat{\alpha}_t^i}{\eta_i} - 2h_i\hat{P}_t - 2\frac{\hat{\alpha}_t^i\hat{P}_t}{\eta_i}\right),$$

it implies that

$$\frac{1}{2\nu}\mathbb{E}\left[\left(\hat{\beta}_t^i\right)^2\right] = \frac{\nu}{2}h_i^2 + \frac{h_i\hat{\alpha}_0^i}{\nu\eta_i} - \nu h_i\hat{P}_0 + \frac{\nu}{2(\eta_i)^2}\mathbb{E}\left[\left(\hat{\alpha}_t^i\right)^2\right] + \frac{\nu}{2}\mathbb{E}[\hat{P}_t^2] - \frac{\nu}{\eta_i}\mathbb{E}\left[\hat{\alpha}_t^i\hat{P}_t\right].$$

Hence, (B.41) can be rewritten to

$$
\begin{aligned}
K^i = {} & T h_i \hat{\alpha}_0 + \frac{2 h_i \hat{\alpha}_0^i}{4\lambda \eta_i} + \frac{h_i^2}{4\lambda} + \frac{\nu}{2} T h_i^2 + \frac{T h_i \hat{\alpha}_0^i}{\nu \eta_i} - \nu T h_i \hat{P}_0 + \frac{1}{4\lambda (\eta_i)^2} \mathbb{E}\left[(\hat{\alpha}_T^i)^2\right] \\
& + \int_0^T \frac{1}{2\eta} \mathbb{E}\left[(\hat{\alpha}_t^i)^2\right] + \frac{\nu}{2(\eta_i)^2} \mathbb{E}\left[\left(\hat{\alpha}_t^i\right)^2\right] + \frac{\nu}{2} \mathbb{E}[\hat{P}_t^2] - \frac{\nu}{\eta_i} \mathbb{E}\left[\hat{\alpha}_t^i \hat{P}_t\right] \mathrm{d}t.
\end{aligned}
$$

Note that

$$
\frac{\nu}{2(\eta_i)^2} \mathbb{E}\left[\left(\hat{\alpha}_t^i\right)^2\right] + \frac{\nu}{2} \mathbb{E}[\hat{P}_t^2] - \frac{\nu}{\eta_i} \mathbb{E}\left[\hat{\alpha}_t^i \hat{P}_t\right] = \frac{\nu}{2} \mathbb{E}\left[\frac{\left(\hat{\alpha}_t^i\right)^2}{\eta_i^2} - \frac{2}{\eta_i} \hat{\alpha}_t^i \hat{P}_t + \hat{P}_t^2\right] = \frac{\nu}{2} \mathbb{E}\left[\left(\frac{\hat{\alpha}_t^i}{\eta_i} - \hat{P}_t\right)^2\right].
$$

This gives the following expression for K^i,

$$
\begin{aligned}
K^i = {} & T h_i \hat{\alpha}_0 + \frac{2 h_i \hat{\alpha}_0^i}{4\lambda \eta_i} + \frac{h_i^2}{4\lambda} + \frac{\nu}{2} T h_i^2 + \frac{T h_i \hat{\alpha}_0^i}{\nu \eta_i} - \nu T h_i \hat{P}_0 + \frac{1}{4\lambda (\eta_i)^2} \mathbb{E}\left[(\hat{\alpha}_T^i)^2\right] && \text{(B.42)} \\
& + \int_0^T \frac{1}{2\eta} \mathbb{E}\left[\left(\hat{\alpha}_t^i\right)^2\right] + \frac{\nu}{2} \mathbb{E}\left[\left(\frac{\hat{\alpha}_t^i}{\eta_i} - \hat{P}_t\right)^2\right] \mathrm{d}t. && \text{(B.43)}
\end{aligned}
$$

By Itô's lemma, it follows that,

$$
\left(\hat{\alpha}_t^i\right)^2 = \left(\hat{\alpha}_0^i\right)^2 + \int_0^t \hat{\alpha}_s^i \mathrm{d}\hat{\alpha}_s^i + \left\langle \hat{\alpha}^i \right\rangle_t.
$$

By Proposition A.15, we have for $t \in [0, T]$,

$$
\mathbb{E}\left[\left(\hat{\alpha}_t^i\right)^2\right] = \left(\hat{\alpha}_0^i\right)^2 + \mathbb{E}\left[\left\langle \hat{\alpha}^i \right\rangle_t\right].
$$

In the same way, we can derive for $t \in [0, T]$,

$$
\mathbb{E}\left[\left(\frac{\hat{\alpha}_t^i)}{\eta_i} - \hat{P}_t\right)^2\right] = \left(\frac{\hat{\alpha}_0^i}{\eta_i} - \hat{P}_0\right)^2 + \mathbb{E}\left[\left\langle \frac{\hat{\alpha}^i}{\eta_i} - \hat{P} \right\rangle_t\right].
$$

By (B.37) we see that $\hat{P}_0$ can indeed be expressed in terms of $q(\rho)$. If we can do the same for $\hat{\alpha}_0^i$, then we would be able to use the expressions of the quadratic variation to find the minimum value. Indeed,

$$
\begin{aligned}
\sum_{i=1}^N \hat{\alpha}_0^i &= -\sum_{i=1}^N g(0) \left(\frac{1}{2\lambda} h_i + M_0^i + \nu h_i T - \hat{P}_0 T\right) = -g(0) \left(\frac{1}{2\lambda} N\bar{h} + N\bar{M}_0 + \nu T \bar{h} - N\hat{P}_0 T\right) \\
&= -g(0) \left(\frac{1}{2\lambda} N\bar{h} + Nq(\rho) + \nu T\bar{h} - NT\left(\pi(0)\eta T\bar{h} - \pi(0) q(\rho)\right)\right) \\
&= -g(0) \left(\bar{h}\left(\frac{N}{2\lambda} + \nu T - NT^2 \pi(0)\eta\right) + q(\rho) N (1 + T\pi(0))\right).
\end{aligned}
$$

Since all those parameters are given by the constrained, or fixed by the system, this results is sufficient. Hence, K^i can be written as

$$
\begin{aligned}
K^i = {} & T h_i \hat{\alpha}_0 + \frac{2 h_i \hat{\alpha}_0^i}{4\lambda \eta_i} + \frac{h_i^2}{4\lambda} + \frac{\nu}{2} T h_i^2 + \frac{T h_i \hat{\alpha}_0^i}{\nu \eta_i} - \nu T h_i \hat{P}_0 + \frac{1}{4\lambda (\eta_i)^2} \left(\hat{\alpha}_0^i\right)^2 + \frac{1}{2\eta} \left(\hat{\alpha}_0^i\right)^2 \\
& + \frac{\nu}{2} \left(\frac{\hat{\alpha}_0^i}{\eta_i} - \hat{P}_0\right)^2 + \frac{1}{4\lambda (\eta_i)^2} \mathbb{E}[\langle \hat{\alpha}^i \rangle_T] + \int_0^T \frac{1}{2\eta} \mathbb{E}\left[\langle \hat{\alpha}^i \rangle_t\right] + \frac{\nu}{2} \mathbb{E}\left[\left\langle \frac{\hat{\alpha}^i}{\eta_i} - \hat{P} \right\rangle_t\right] \mathrm{d}t.
\end{aligned}
$$

Here, everything is fixed by the system, except for the last three terms. As the quadratic variation is non-negative by assumption, the costs per firm i will be the smallest when

$$
\left\langle \hat{\alpha}^i \right\rangle_t = 0, \quad \left\langle \hat{\alpha}^i - \hat{P} \right\rangle_t = 0, \qquad \text{a.s, for all } t \in [0, T].
$$

This implies that the social costs are as small as possible when the above holds for all firms i. The costs follow by summing K^i over all possible firms. This is indeed what we needed to show. □

In the next proposition a sufficient condition is obtained.

Proposition B.15. *A sufficient condition to achieve the conditions in Theorem B.14 is to have that*

$$\left\langle M^i - \sigma_i W^i \right\rangle_t = 0 \textit{ for all firms a.s, for all } t \in [0,T]$$

Proof. By (2.40), it holds that

$$M_t^i = M_0^i + \int_0^t \sum_{j=0}^N \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j.$$

By definition of the quadratic variation of a semimartingale, this implies that

$$\left\langle M^i - \sigma_i W^i \right\rangle_t = \left\langle M_0^i + \int_0^\cdot \sum_{j=0}^N \gamma_s^{i,j} \mathrm{d}B_s^j - \sigma_i W^i \right\rangle_t = \left\langle \int_0^\cdot \sum_{j=0}^N \gamma_s^{i,j} \mathrm{d}B_s^j - \sigma_i W^i \right\rangle_t$$

Since this is a martingale starting in zero, this implies that

$$\int_0^t \sum_{j=0}^N \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j - \sigma_i W_t^i = 0.$$

Now by the construction of the specific per firm Brownian motion, this gives

$$\int_0^t \sum_{j=0}^N \gamma_s^{i,j} \mathrm{d}\tilde{B}_s^j - \int_0^t \sigma_i \kappa_i \mathrm{d}\tilde{B}_s^0 + \int_0^t \sigma_i \sqrt{1-\kappa_i^2} \mathrm{d}\tilde{B}_s^i = 0.$$

Hence, by the same arguments as in the frictionless case, this system reduces to

$$\gamma_t^{i,0} = \sigma_i \kappa_i, \quad \gamma_t^{i,i} = \sigma_i \sqrt{1-\kappa_i^2}, \quad \gamma_t^{i,j} = 0,$$

for all other j, a.s, for all $t \in [0,T]$. Then,

$$\begin{aligned} M_t^i = M_0^i + \int_0^t \sum_{j=0}^N \gamma_s^{i,j} \mathrm{d}B_s^j &= M_0^i + \int_0^t \sigma_i \left(\kappa_k i \mathrm{d}B_t^0 + \sqrt{1-\kappa_i^2} \mathrm{d}B_t^i \right) = M_0^i + \int_0^t \sigma_i \mathrm{d}W_t^i \\ &= M_0^i + \sigma_i W_t^i. \end{aligned}$$

This immediately implies that

$$\bar{M} - \bar{W} = \bar{M}_0 + \bar{W} - \bar{W} = \bar{M}_0.$$

By properties of the quadratic variation, since $\bar{M}_0$ is a constant fixed by the constraint of the regulator, it holds that

$$\langle \bar{M} - \bar{W} \rangle_t = 0,$$

for all $t \in [0,T]$. Furthermore, by (B.24), it holds that

$$\hat{P}_t \sim \mathcal{N}\left(\hat{P}_0, \int_0^T \pi(t)^2 \mathrm{d} \langle \bar{M} - \bar{W} \rangle_t \right).$$

Since the variance of $\hat{P}$ is zero if the quadratic variation is zero, it implies that

$$\hat{P}_t = \hat{P}_0, \quad \text{for all } t \in [0,T]. \tag{B.44}$$

Hence, $\langle P \rangle_t = 0$, a.s, for all $t \in [0,T]$. By (B.37) the market price is now fixed by the system. With the given assumption and this observation, we are going to prove that we are indeed in the setting of Theorem B.14.

Next, fix a firm i and $t \in [0,T]$. From (B.30), we see that

$$\hat{\alpha}_t^i = \hat{\alpha}_0^i - \int_0^t g(s)\mathrm{d}\left(M_s^i - \sigma_i \mathrm{d}W_s^i\right) + \int_0^t \nu(T-s)g(s)\mathrm{d}\hat{P}_s.$$

When considering quadratic variations, this implies by the bilinearity,

$$\begin{aligned}\langle \alpha^i \rangle &= \int_0^\cdot g(s)^2 \mathrm{d}\left\langle M^i - \sigma_i W^i \right\rangle_s + \int_0^\cdot \nu^2(T-s)^2 g(s)^2 \mathrm{d}\left\langle \hat{P} \right\rangle_s \\ &\quad - 2\int_0^\cdot g(s)\nu(T-s)\mathrm{d}\left\langle M^i - \sigma_i W^i, \hat{P} \right\rangle_s .\end{aligned}$$

Now all three terms are zero because of the assumption on the quadratic variation and (B.44). We can conclude that

$$\left\langle \hat{\alpha}^i - \hat{P} \right\rangle_t = \left\langle \hat{\alpha}^i - \hat{P}_0 \right\rangle_t = \left\langle \hat{\alpha}^i, \hat{\alpha}^i \right\rangle_t = 0,$$

for all firms i, a.s, for all $t \in [0,T]$. □

In the standard example below all optimal control variables and costs are summarised in the sufficient case, with one extra assumption that needs to be made.

Example B.1 (Sufficient, optimal allocation). *We have seen that a particular optimal allocation is found by individually tracking the volatility of the firms. Let $t \in [0,T]$ and fix a firm i. For now, we only know that $\bar{M}_0 = q(\rho)$. Again, we can set $\hat{M}_0^i = q(\rho)$, to obtain the result. Based on this, all other parameters can be deduced. We already know that for every firm i,*

$$\hat{M}_t^i = \hat{M}_0^i + \sigma_i W_t^i,$$

is optimal and that this is not necessarily a unique solution, as it is only sufficient.

(i) The market price of permits is, already suggested in (B.44) *, given by*

$$\hat{P}_t = \hat{P}_0 = \pi(0)\eta T\bar{h} - \pi(0)q(\rho).$$

(ii) The optimal abatement effort $\hat{\alpha}^i$ is also constant, since it has also zero quadratic variation in the optimum. It is given by the initial value

$$\hat{\alpha}_t^i = \hat{\alpha}_0^i = -g(0)\left(h_i\left(\frac{1}{2\lambda} + \nu T\right) + \hat{M}_0^i - \nu T\hat{P}_0\right).$$

(iii) The trading rate $\hat{\beta}$ is constant, since it depends on the optimal abatement effort and market price. Indeed,

$$\hat{\beta}_t^i = \nu\left(h_i + \frac{\hat{\alpha}_t^i}{\eta} - \hat{P}_t\right) = \nu\left(h_i + \frac{\hat{\alpha}_0^i}{\eta} - \hat{P}_0\right) = \hat{\beta}_0^i.$$

(iv) By the same reasoning as in the frictionless case, the optimal allocations $A \in \mathscr{S}^N$ are non-unique. One particular solution is

$$\begin{aligned}A_t^i &= q(\rho) + \sigma_i W_t^i = \hat{M}_t^i, \\ \tilde{A}_t^i &= A_t^i + \mu_i t = q(\rho) + \mu_i t + \sigma_i W_t^i.\end{aligned}$$

Remark B.3. *Compared to Section 5.1.1 of [AB23], we have followed a different procedure. We were unable to deduce a direct relation between the bank account for a firm i and the market equilibrium price. However, we were able to establish this connection for the abatement effort $\hat{\alpha}_T^i$. Although we obtained different optimality conditions at first, Proposition B.15 demonstrates that the conditions stated there are applicable in our case. We cannot prove a necessary condition, but that is not explicitly done in [AB23] either.*

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