The homology of free spectral Lie algebras and machine computation in algebraic topology

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Abstract

In his three seminal papers Thomas Goodwillie constructed the Goodwillie tower, with which one can approximate a homotopy functor on spaces similar to how the Taylor series approximates a smooth function in ordinary calculus. In the case of the identity functor on spaces, Michael Ching showed that the derivatives form an operad in spectra. The algebras for this operad are called spectral Lie algebras. It turns out that the mod 2 homology of a free spectral Lie algebra can be described in terms of the homology of the original spectrum. We will construct a machine computational tool to compute the mod 2 homology of a free spectral Lie algebra as a module over the dual Steenrod algebra.

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1 Introduction

As of late, the area of algebraic topology and homotopy theory has seen a lot of interesting developments with regard to doing machine-based computation. Take for example the Adams spectral sequence, which is one of the main tools to compute the stable homotopy groups of a space, and primarily where we take our space to be an n-dimensional sphere. To determine the entries of the spectral sequence we require a lot of computational work. To do this computation automatically the *Ext*-code was written by Bruner in 1992. Due to this and other improvements, the Adams spectral sequences for the 2-local sphere spectrum is currently known up to dimension 90 as of 2020 [IWX20].

In this thesis, our goal is to extend the list of machine-based tools by making a website which will compute the mod-2 homology of the free spectral Lie algebra of a spectrum. With that the thesis consists of three parts: we explain what it is that we compute, how we have written the computation into machine code and lastly we will give an example of how these computations can be used.
For the first part, we start off in Chapter 3 by taking a look at Goodwillie calculus, where the spectral Lie operad will reveal itself. In Chapter 4 we will dive into operads, which are objects that parametrize operations, and we will see that the spectral Lie operad is indeed an operad. In chapter 5 we will study the homology operations that we get on spectral Lie algebras, with which we can express the homology of the free spectral Lie algebra of a spectrum in terms of the homology of the original spectrum. Next, we will discuss our implementation of our computational tool in Chapter 6 where we will end with a discussion on future work and other ways the code might be generalized.
Lastly, we will return in Chapter 7 to the Goodwillie tower and discuss its interaction with the EHP sequence due to Behrens [Beh12] by making use of our tool.
To clarify, the only original work in this thesis is the machine computational tool as discussed in Chapter 6, in the other chapters we will be discussing the work of others.

2 Symmetric monoidal categories and spectra

To be able to define operads we will first need to construct the kinds of categories in which we are able to have operations. These will be the symmetric monoidal categories, which we can think of categories with a well-behaved product, together with a unit for this product. As we would like to give an operad structure on $\partial_* \operatorname{id}$ we will construct a symmetric monoidal category of spectra. With this, we will then be able to define operads, which we should think of as objects in a category that parametrize operations.

2.1 Symmetric monoidal categories

To be able to define operads we will first need to construct the kinds of categories in which we are able to have operations. These will be the symmetric monoidal categories. This introduction will be based on the discussion in Michael Ching's thesis [Chi05]

Definition 2.1 (symmetric monoidal category)**.** Let $\mathcal{C}$ be a category, then a *monoidal structure* on $\mathcal{C}$ consists of a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ which we will write as a tensor product $(c, d) \longmapsto c \otimes d$, together with a unit object $1 \in \mathrm{Ob}(\mathcal{C})$, that satisfy the following conditions

1. The multiplication functor is associative, in that we have the following natural isomorphism $c_1 \otimes (c_2 \otimes c_3) \xrightarrow{\simeq} (c_1 \otimes c_2) \otimes c_3$.

2. The unit objects serves indeed as a unit in that we have the following two natural isomorphisms $1 \otimes c \xrightarrow{\simeq} c$ and $c \xleftarrow{\simeq} c \otimes 1$.

Besides these conditions we also want them to satisfy two extra coherence conditions. We will call a monoidal category *symmetric*, if there is also a natural isomorphism that makes it symmetric switch: $c_1 \otimes c_2 \xrightarrow{\simeq} c_2 \otimes c_1$ together with four extra coherence conditions so that it behaves properly with the tensor product and the unit.

The most important example in this thesis will be the symmetric monoidal category of pointed topological spaces $(\mathrm{Top}_*, \wedge, S^0)$. This has the smash product

$$\begin{aligned} \wedge\colon \mathrm{Top}_* \times \mathrm{Top}_* &\longrightarrow \mathrm{Top}_* \\ (X,Y) &\longmapsto (X \times Y)/(X \vee Y) \end{aligned}$$

as its tensor product, and the 0-sphere S^0 as its unit. We note that the smash product between based spaces without any extra conditions, is not symmetric monoidal. So to make $(\mathrm{Top}_*, \wedge, S^0)$ into a symmetric monoidal category we will let Top_* be the category of based compactly generated spaces. We also note that this is not the only symmetric monoidal structure on Top_*, as we will also have $(\mathrm{Top}_*, \times, *)$.
One thing that a symmetric monoidal category lets us do, is define a monoid object in it, which will be a generalization standard monoid.

Definition 2.2 (monoid object)**.** Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category. Then a monoid object object in $\mathcal{C}$ is an object $m \in \mathcal{C}$, together with a multiplication map $m \otimes m \xrightarrow{\mu} m$ and a unit map $1 \xrightarrow{\eta} m$ such that they satisfy the multiplication is associative and unital, i.e. in that the following two diagrams commute

$$\begin{tikzcd} m \otimes m \otimes m \arrow[r, "\mu\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & m \otimes m \arrow[d, "\mu"] \\ m \otimes m \arrow[r, "\mu"'] & m \end{tikzcd} \qquad \begin{tikzcd} m \otimes 1 \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[dr, "\simeq"'] & m \otimes m \arrow[d, "\mu"] & 1 \otimes m \arrow[l, "\eta\otimes\mathrm{id}"'] \arrow[dl, "\simeq"] \\ & m & \end{tikzcd}$$

We will call a monoid object *commutative* if the multiplication map is commutative in the sense that we have the following commutative diagram

$$\begin{tikzcd} m \otimes m \arrow[r, "\mathrm{switch}", "\simeq"'] \arrow[dr, "\mu"'] & m \otimes m \arrow[d, "\mu"] \\ & m \end{tikzcd}$$

As an example from this definition, we see that a monoid is just a monoid object in the symmetric monoidal category $(\mathrm{Sets}, \times, *)$, and that a monad is a monoid object in the symmetric monoidal category $(\mathrm{Fun}(\mathcal{C}, \mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}})$.
Another interesting example is that a commutative ring is precisely a commutative monoid object in $(\mathrm{AbGrp}, \otimes, \mathbb{Z})$. So we might wonder if we are able to generalize the notion of a module over a commutative ring to some notion over commutative monoid objects, and indeed we can.

Definition 2.3. Let m be a monoid in a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$, then we call an object n in $\mathcal{C}$ a *(right)-module over* m if we have an action map $\xi\colon m \otimes n \longrightarrow n$ that is distributive and unital in the sense that the following two diagrams commute

$$\begin{tikzcd} m \otimes m \otimes n \arrow[r, "\mathrm{id}\otimes\xi"] \arrow[d, "\mu\otimes\mathrm{id}"'] & m \otimes n \arrow[d, "\xi"] \\ m \otimes n \arrow[r, "\xi"'] & m \end{tikzcd} \qquad \begin{tikzcd} 1 \otimes n \arrow[r, "\eta\otimes n"] \arrow[dr, "\simeq"'] & m \otimes n \arrow[d, "\xi"] \\ & n \end{tikzcd}$$

As we alluded to we see that indeed the module over a commutative ring is a right module over the ring as a commutative monoid object.
We note that in Top_* the tensor map has a right adjoint

$$\mathrm{Hom}_{\mathrm{Top}_*}(X \wedge Y, Z) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Top}_*}(X, \mathrm{Map}(Y, Z))$$

with $\mathrm{Map}\colon \mathrm{Top}_*^{\mathrm{op}} \times \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$ taking two based spaces to the space of based maps between them. In order for this adjunction to hold we will let Top_* denote the category of compactly generated weak Hausdorff spaces with a chosen basepoint.

Definition 2.4 (closed symmetric monoidal category)**.** Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category, then we will call it *closed* if there is a bifunctor called the internal hom functor

$$\mathrm{Map}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$$

such that for every object $X \in \mathcal{C}$, the functor $- \otimes X : \mathcal{C} \to \mathcal{C}$ is right adjoint to the functor $\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{C}$ and this adjunction is natural in the choice of X.

We note that this adjunction can be made internal to the underlying category, so that we can replace the homsets with the internal hom functors.

2.2 The symmetric monoidal category of symmetric spectra

Besides the category $(\mathrm{Top}_*, \wedge, S^0)$ we will also be dealing with a closed symmetric monoidal category of spectra. For this, we will be using a model for spectra that will be different from the usual one, as to be able to construct a symmetric monoidal smash product. This will be the model of symmetric spectra, as is discussed in [Sch12].

Definition 2.5 (symmetric spectrum)**.** We define a *symmetric spectrum* to be a sequence of based spaces $\{X_n\}_{n \geq 0}$, with

1. each space X_n carries a left action by the n-th symmetric group Σ_n that is continuous and base-point preserving. This makes the sequence into a symmetric sequence in spaces.
2. a suspension map $\sigma_n : X_n \wedge S^1 \to X_{n+1}$ such that when taking it repeatedly

$$\sigma_n^m : X_n \wedge S^m \to X_{n+1} \wedge S^{m-1} \to \cdots \to X_{n+m}$$

   is $\Sigma_n \times \Sigma_m$ equivarient. Here Σ_n acts on X_n by it left action while the right action on S^m will be explained later. The action of $\Sigma_m \times \Sigma_n$ comes from the right action of Σ_{m+n} on X_{m+n} under the block inclusion of $\Sigma_m \times \Sigma_n$ into Σ_{m+n}. This will be defined later in 3.

A morphism between symmetric spectra $f : X \longrightarrow Y$ is then given by a sequence of maps $f_n : X(n) \longrightarrow Y(n)$ that are Σ_n-equivariant, and commute with respect to the suspension maps. We will write the category of symmetric spectra as Sp.
The usual examples of spectra also give us a symmetric spectrum, but we have to be careful in the construction of the underlying spaces so as to have the desired left action of the symmetric group.
As an example, we will construct the sphere spectrum. To get a left action on S^n, we will start with the left action on $\mathbb{R}^n$ by Σ_n that is given by permuting the coordinates $\sigma(x_1, \cdots, x_n) = (x_{\sigma(1), \cdots, x_{\sigma}(n)})$. We then take the one-point-compactification, to get S^n together with the left action of Σ_n. With this, we can then construct the sphere spectrum in the usual way, as $\mathbb{S} = \{S^n\}_{n \geq 0}$ with the suspension homomorphism being the suspension isomorphism $S^n \wedge S^1 \xrightarrow{\simeq} S^{n+1}$.
Another useful example will be the Eilenberg-Maclane spectrum HA for an abelian group A. Again the idea is to give an explicit construction of the Eilenberg-Maclane spaces $K(A, n)$, so that we can give them a left action by Σ_n. We then construct HA as the sequence $\{K(A, n)\}_{n \geq 0}$ together with the suspension homomorphism being the adjoint of the equivalence $K(A, n) \xrightarrow{\simeq} \Omega K(A, n)$.

Having constructed the category of symmetric spectra, we are left with giving it a symmetric monoidal structure. We will build this out of the smash product on spaces following the discussion in [Sch12, Sec. I.5]. The construction will be similar to the construction of the tensor product of R-modules for R a commutative ring. We recall that this was constructed in such a way to carry the universal property that for any bilinear map $f : A \times B \to C$ there is a unique linear map $\tilde{f} : A \otimes B \to C$ such that the following diagram commutes.

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[d, "i"'] & C \\ A \otimes B \arrow[ur, dashed, "\exists ! \tilde{f}"'] & \end{tikzcd}$$

where $i : A \times B \to A \otimes B$ is the universal bilinear map.
The construction of the smash product of symmetric spectra is similar. We will first construct our notation of a bimorphism of symmetric spectra. This will be a map $f : (X, Y) \to Z$ where (X, Y) is a pair of symmetric spectra and Z is a symmetric spectrum. This will consist of maps of pointed spaces $f_{m,n} : X_m \wedge X_n \to Z_{m+n}$

that are $\Sigma_m \times \Sigma_n$-equivarient, that behave properly with the structure of the suspension maps of the spectra, in that the following diagram commutes

$$\begin{tikzcd}
X_m \wedge S^1 \wedge Y_n \arrow[r, "\sigma_m^X"] & X_{m+1} \wedge Y_n \arrow[r, "f_{m+1,n}"] & Z_{m+1+n} \arrow[d, "1\times\mathrm{shuffle}"] \\
X_m \wedge Y_n \wedge S^1 \arrow[u, "\mathrm{id}\wedge\mathrm{shuffle}"] \arrow[r, "f_{m,n}"'] \arrow[d, "\mathrm{id}"] & Z_{m+n} \wedge S^1 \arrow[r, "\sigma_{m+n}^Z"] & Z_{m+n+1} \\
X_m \wedge Y_n \wedge S^1 \arrow[r, "\sigma_n^Y"] & X_m \wedge Y_{n+1} \arrow[r, "f_{m,n+1}"] & Z_{m+n+1} \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

Here shuffle corresponds to both the shuffle map and the corresponding permutation. We will write the set of bimorphisms between the pair (X, Y) and Z as $\mathrm{Bimor}((X,Y),Z)$. Then the smash product of spectra $X \wedge Y$ is constructed in such a way as to have a universal bimorphism $i\colon (X,Y) \to X \wedge Y$, in that if there was a bimorphism $f\colon (X,Y) \to Z$, then this would give a unique morphism of spectra $\tilde{f}\colon X \wedge Y \to Z$ so that the bimorphism $\tilde{f} \circ i$ is equal to f. In a sense we have the following commutative diagram

$$\begin{tikzcd}
(X,Y) \arrow[r, "f"] \arrow[d, "i"'] & Z \\
X \wedge Y \arrow[ur, dashed, "\exists! f"'] &
\end{tikzcd}$$

We will now give a direct construction of the smash product spectrum, and show that it has the universal property we want it to have. For two symmetric spectra X and Y their smash product is the symmetric spectrum defined as the following coequalizer

$$(X \wedge Y)_n = \mathrm{Coeq}\Big(\bigvee_{p+1+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_1 \times \Sigma_q} X_p \wedge S^1 \wedge Y_q \rightrightarrows \bigvee_{p+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \Big)$$

where the top map is induced by the composite

$$(\alpha_X)_{(p,1,q)}\colon X_p \wedge S^1 \wedge Y_q \xrightarrow{\sigma_p^X \wedge \mathrm{id}} X_{p+1} \wedge Y_q$$

and the bottom map is induced by the composite

$$(\alpha_Y)_{(p,1,q)}\colon X_p \wedge S^1 \wedge Y_q \xrightarrow{\mathrm{id}\wedge\mathrm{shuffle}} X_p \wedge Y_q \wedge S^1 \xrightarrow{\mathrm{id}\wedge\sigma_q^Y} X_p \wedge Y_{q+1} \xrightarrow{\mathrm{id}\wedge\mathrm{shuffle}} X_p \wedge Y_{1+q}$$

We now have two choices for the suspension map for $(X \wedge Y)$, one induced from the suspension map from Y by $X_m \wedge Y_n \wedge S^1 \xrightarrow{\mathrm{id}\wedge\sigma_n} X_m \wedge Y_{n+1}$, and the other coming from the suspension map from X in the same way. But under the coequalizer these precisely coincide. With that, we see that a map of spectra $(X \wedge Y) \xrightarrow{f} Z$ is indeed the same as a bimorphism $(X,Y) \xrightarrow{f} Z$ because as f is morphism of spectra we get that it commutes with the suspension maps so if we write $f_{m,n}\colon X_m \wedge Y_n \longrightarrow Z$ as the map under the wedge sum in the coequalizer then we get the diagram back which we used to define a bimorphism.
This makes Sp into a symmetric monoidal category with the sphere spectrum $\mathbb{S}$ as its unit [Sch12, Thm. 5.10]. We now also note that it is in fact a closed symmetric monoidal category. There is an internal hom functor, which we will write as $\mathrm{Map}_{\mathrm{Sp}}(X,Y)$ which gives us the adjunction

$$\mathrm{Hom}_{\mathrm{Sp}}(X \wedge Y, Z) \simeq \mathrm{Hom}_{\mathrm{Sp}}(X, \mathrm{Map}_{\mathrm{Sp}}(Y,Z))$$

We will call the commutative monoid objects in $(\mathrm{Sp}, \wedge, \mathbb{S})$ the commutative ring spectra, mirroring the terminology for $(\mathrm{AbGrp}, \otimes, \mathbb{Z})$, and we will denote the category of modules over a commutative ring spectrum E by Mod_E.

3 Goodwillie calculus

Recall from calculus that if we had a smooth function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, then we could approximate it by a polynomial of degree n, given by the McClaurin series

$$f(x) = f(0) + \frac{\partial_1 f(0)}{1!}x + \frac{\partial_2 f(0)}{2!}x + \cdots \frac{\partial_n f(0)}{n!}x^n$$

Note that this consists of i-homogeneous polynomials $\frac{\partial_i f(0)}{i!}x^i$ and that if we were to write the previous polynomial of degree n as $P_n(f)(x)$, and $\frac{\partial_{n+1} f(0)}{(n+1)!}x^{n+1}$ as $D_{n+1}(f)(x)$, that we would get

$$P_{n+1}(f)(x) = P_n(f)(x) + D_{n+1}(f)(x)$$

and when we were to let n go to infinity, then we would retrieve our original function again if f were to be analytic around 0.
In Goodwillie/functor calculus we will basically construct the same thing as a polynomial approximation for a smooth function, but then as a polynomial approximation to a well-behaved functor of spaces. This was originally constructed Goodwillie's three papers. To define these we will first need to define the notions of a homotopy limit and a homotopy colimit.

3.1 Homotopy limits and homotopy colimits

In Algebraic Topology 2 we have encountered the notions of a homotopy fiber, which we could think of as the homotopy correct way of taking the fiber of a map of spaces. As an example, we had that the following homotopy equivalent maps result in different fiber sequences

$$\begin{tikzcd}
* \arrow[d] & \Omega(X) \arrow[d] \\
* \arrow[r, "\simeq"'] \arrow[d, "f"'] & P(X) \arrow[d, "g"] \\
X \arrow[r, "\mathrm{id}_Y"] & X
\end{tikzcd}$$

By constructing the homotopy fiber, we were able to fix this problem

$$\mathrm{hofib}(f) = \big\{(x,\gamma) \in X \times P(X) \mid f(x) = \gamma(1)\big\}$$

We encounter the same problem with taking pushouts. For example, if we had the following two pushouts $D^2 \sqcup_{S^1} *$ and $* \sqcup_{S^1} *$ then even tho these are homotopy equivalent, the pushouts still result in different spaces.

$$\begin{tikzcd}
 & S^1 \arrow[rr] \arrow[dd] & & * \arrow[dd] \\
S^1 \arrow[ur, "\mathrm{id}"] \arrow[rr] \arrow[dd, "i"'] & & * \arrow[ur, dashed, "\simeq"'] \arrow[dd] & \\
 & * \arrow[rr] & & * \\
D^2 \arrow[ur, dashed, "\simeq"'] \arrow[rr] & & S^2 &
\end{tikzcd}$$

This turns out to be a problem for any kind of limit or colimit over a small category. So we would like to construct a homotopy colimit and homotopy limit. For this discussion, we will follow Dugger's primer [Dug08].

Definition 3.1 (homotopy limit)**.** Let $X\colon \mathcal{I} \to \mathrm{Top}$ be a diagram over a small category $\mathcal{I}$. Then we define its homotopy limit as the following equalizer

$$\mathrm{holim}_{\mathcal{I}}(X) = \mathrm{Eq}\Big(\prod_i X(i)^{B(\mathcal{I}\downarrow i)} \rightrightarrows \prod_{\alpha\colon i\to j} X(j)^{B(\mathcal{I}\downarrow i)}\Big)$$

where the top is defined as

$$\theta_{(\alpha\colon i\to j)}\Big(f\colon B(\mathcal{I}\downarrow i) \to X(i)\Big) = X(\alpha)\circ f\colon B(\mathcal{I}\downarrow i) \to X(j)$$

and the bottom map as

$$\psi_{(\alpha\colon i\to j)}\Big(f\colon B(\mathcal{I}\downarrow j) \to X(j)\Big) = f\circ \alpha_*\colon B(\mathcal{I}\downarrow i) \to X(j)$$

with $\alpha_*\colon B(\mathcal{I}\downarrow i) \to B(\mathcal{I}\downarrow j)$ being the induced map by $\alpha\colon i \to j$.

As an example, we will compute that the homotopy pullback of $X \xrightarrow{f} A \xleftarrow{g} Y$ is given by

$$X \times_A^h Y = \Big\{(x,\gamma,y) \in X \times A^I \times Y \mid f(x) = \gamma(0), g(y) = \gamma(1)\Big\}$$

We recall that the pullback category $\mathcal{I}$ is given by $1 \to 0 \leftarrow 2$. So we see that the under categories are given by $\mathcal{I}\downarrow 1 \simeq *$ and $\mathcal{I}\downarrow 2 \simeq *$, both only consisting of the identity map. On the other hand $\mathcal{I}\downarrow 0$ is isomorphic to $\mathcal{I}$, as indicated by the following diagram

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr] & 0 \arrow[d, "\mathrm{id}"'] & 2 \arrow[l] \arrow[dl] \\ & 0 & \end{tikzcd}$$

So when taking the geometric realizations of the nerves, we find that $B(\mathcal{I}\downarrow 1) \simeq B(\mathcal{I}\downarrow 2) \simeq *$ and $B(\mathcal{I}\downarrow 0) \simeq [0,1] \underset{1\simeq 0}{\sqcup} [0,1] \simeq [0,1]$. So we see that

$$\prod_i F(i)^{B(\mathcal{I}\downarrow i)} \simeq X^* \times A^I \times Y^*$$

We now then get for the morphism $1 \to 0$ in $\mathcal{I}$ the following two maps

$$\theta_{1\to 0}(x\colon * \to X) = \big(f(x)\colon * \to A\big) \text{ and } \psi_{1\to 0}(\gamma\colon I \to A) = \big(\gamma(0)\colon * \to A\big)$$

and for the morphism $2 \to 0$ in $\mathcal{I}$ we get

$$\theta_{2\to 0}(y\colon * \to Y) = \big(g(y)\colon * \to A\big) \text{ and } \psi_{2\to 0}(\gamma\colon I \to A) = \big(\gamma(1)\colon * \to A\big)$$

So that we the equalizer becomes precisely as we wanted to show

$$\mathrm{hocolim}_{\mathcal{I}}(F) = X \times_A^h Y$$

Definition 3.2 (homotopy colimit)**.** Let $X\colon \mathcal{I} \to \mathrm{Top}$ be a diagram over a small category $\mathcal{I}$. Then we define its homotopy colimit as the following coequalizer

$$\mathrm{hocolim}_{\mathcal{I}}(X) = \mathrm{Coeq}\Big(\coprod_{\alpha\colon i\to j} X(i) \times B(j\downarrow \mathcal{I}) \rightrightarrows \coprod_i X(i) \times B(i\downarrow \mathcal{I})\Big)$$

where the top map is given by

$$\theta_{(\alpha\colon i\to j)}\colon X(i) \times B(j\downarrow \mathcal{I}) \xrightarrow{\mathrm{id}\times\alpha^*} X(i) \times B(j\downarrow \mathcal{I})$$

and the bottom map is given by

$$\psi_{(\alpha\colon i\to j)}\colon X(i) \times B(j\downarrow \mathcal{I}) \xrightarrow{X(\alpha)\times\mathrm{id}} X(j) \times B(j\downarrow \mathcal{I})$$

As an example, we will show that the homotopy pushout for the diagram $X \xleftarrow{f} A \xrightarrow{g} Y$ is given by

$$X \sqcup_A^h Y = X \underset{f(a)\sim(a,0)}{\cup} A \times I \underset{g(a)\sim(a,1)}{\cup} Y$$

In much the same way as before we find that $B(1 \downarrow \mathcal{I}) \simeq B(2 \downarrow \mathcal{I}) \simeq *$ and that $B(0 \downarrow \mathcal{I}) \simeq I$. It follows that

$$\coprod_i X(i) \times B(i \downarrow \mathcal{I}) \simeq (X \times *) \cup (A \times I) \cup (Y \times I).$$

so we are left with determining the equivalence relations for the equalizer. For the morphism $0 \to 1$ we get

$$\theta_{0\to 1} \colon A \times * \xrightarrow{\mathrm{id} \times 0} A \times I \text{ and } \psi_{0\to 1} \xrightarrow{f \times \mathrm{id}} X \times *$$

and for the morphism $0 \to 2$ we get

$$\theta_{0\to 2} \colon A \times * \xrightarrow{\mathrm{id} \times 1} A \times I \text{ and } \psi_{0\to 1} \xrightarrow{g \times \mathrm{id}} Y \times *$$

So when taking the equalizer this gives us what we were after $\mathrm{hocolim}_{\mathcal{I}}(F) = X \sqcup_A^h Y$.
We will now discuss some notions we have seen before but then constructed as a special kind of homotopy limit or homotopy colimit.

1. We can construct the homotopy fiber of a map $f \colon X \to Y$ as a homotopy pullback of the diagram $X \xrightarrow{f} Y \leftarrow *$. Likewise, we can construct the homotopy cofiber of f as the homotopy pushout of the diagram $* \leftarrow X \xrightarrow{f} y$.
2. We can construct the loop space of X as a homotopy pullback over two points $\Omega X \simeq \mathrm{holim}(* \to X \leftarrow *)$ and the suspension of X as the homotopy pushout over two points $\Sigma X \simeq \mathrm{hocolim}(* \leftarrow X \to *)$.
3. Another example that we will encounter often will be the construction of taking the *homotopy orbits*. We recall that the orbit space is given by the quotient X/G with $[x] = [y]$ if there is a $g \in G$ such that $x \cdot g = y$. We can think of this as a colimit in the following way. We first view G as a one object category, so that a G-space is the same as a functor $X \colon G \longrightarrow \mathrm{Top}_*$. Then the orbit space is the colimit of this diagram $\mathrm{colim}_G(X) \simeq X/G$. With this construction, we define the homotopy orbits by replacing a colimit with the homotopy correct version

$$X_{hG} \simeq \mathrm{hocolim}_G(X)$$

3.2 Construction of the Goodwillie tower

Using the new tools of the homotopy limit and the homotopy colimit, we will construct a tower of approximations to a homotopy functor of spaces, that will closely resemble the Taylor approximations to a smooth function. By a homotopy functor, we will mean a functor $F \colon \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$ that preserves homotopy equivalences. So if we were to have two homotopy equivalent spaces $X \simeq Y$, then we would also have that $F(X) \simeq F(Y)$. In this case, the notion of a polynomial of degree n will be a functor that is n-excisive. So to start, we will define what we mean by an n-excisive functor. For this discussion we will be following Kuhns notes [Kuh07].

Definition 3.3. Let $[n]$ be the set of n elements viewed as a category, then its power set $\mathcal{P}(n)$ has the structure of a partially ordered set, where the morphisms are given by the inclusion. An *n-cube* in Top_* is a functor $F \colon \mathcal{P}(n) \to \mathrm{Top}_*$.

For example in the case that $n = 2$ we see that a 2-cube in Top_* is given by a diagram of the following kind

$$\begin{tikzcd}
F(\emptyset) \arrow[r] \arrow[d] \arrow[dr] & F(\{2\}) \arrow[d] \\
F(\{1\}) \arrow[r] & F(\{1,2\})
\end{tikzcd}$$

so we see that we do indeed get an n dimensional cube. This construction makes us able to generalize the notions of a homotopy pullback and a homotopy pushout to cartesian and cocartesian n-cubes

Definition 3.4 (cartesian and cocartesian n-cubes)**.** Suppose we have an n-cube $F\colon \mathcal{P}(n) \to \mathrm{Top}_*$. Then

1. We will call the n-cube *cartesian* if the initial vertex $F(\emptyset)$ is homotopy equivalent to the homotopy limit of the rest of the n-cube, i.e.

$$F(\emptyset) \simeq \mathrm{holim}_{\mathcal{P}(n)-\emptyset}(F(i))$$

2. We will call an n-cube *cocartesian* if the terminal vertex $F(\{1,\cdots,n\})$ is homotopy equivalent to the homotopy colimit of the rest of the n-cube

$$F([n]) \simeq \mathrm{hocolim}_{\mathcal{P}(n)-[n]} F(i)$$

As an example, we see that a cartesian 2-cube is given by a homotopy pullback diagram and that a cocartesian 2-cube is a homotopy pushout diagram. If $n \geq 2$ we will call a n-cube *strongly cocartesian* if it is cocartesian and all the 2d faces of the n-cube are homotopy pushouts. We will now define what we mean by a functor $F\colon \mathrm{Top}_* \to \mathrm{Top}_*$ to be n-excisive.

Definition 3.5 (n-excisive functor)**.** We will call a homotopy functor $F\colon \mathrm{Top}_* \to \mathrm{Top}_*$ an n-excisive functor if it takes every strongly cocartesian $n+1$-cube to a cartesian $n+1$-cube. We will write the subcategory of n-excisive functors as $\mathrm{Exc}_n \subset \mathrm{Fun}(\mathrm{Top}_*, \mathrm{Top}_*)$.

For example, we see that a functor is 1-excisive, precisely when it takes homotopy pushout squares to homotopy pullback squares. This is very closely related to satisfying excision: if we take the homotopy groups after taking the functor, $\pi_*(F(-))\colon \mathrm{Top}_* \to \mathrm{grGrp}$, then this carries a Mayer-Vietoris sequence. So we can think of an n-excisive functor to satisfy a worse version of excision.
Suppose we have a functor $F\colon \mathrm{Top}_* \to \mathrm{Top}_*$, then the Goodwillie tower gives a sequence of n-excisive approximations to F. To construct these n-excisive approximations we will first give a construction that will be n-excisive for a special kind of strongly cartesian $(n+1)$-cubes.

$$\begin{aligned} T_n(F)\colon \mathrm{Top}_* &\longrightarrow \mathrm{Top}_* \\ X &\longmapsto \underset{i\in(\mathcal{P}(n+1)-\emptyset)}{\mathrm{holim}} F\left(\left(\coprod_{|i|} CX)\right)/\sim\right) \end{aligned}$$

where we identify the individual cones at their base. For example, for $n=1$ we get that $T_1(F)(X)$ would be the homotopy limit of the diagram $F(CX) \leftarrow F(\Sigma X) \to F(CX)$. With this we also get a natural transformation $\tau_n(F)\colon F \to T_n(F)$.
The n-excisive approximation to F is then given by taking the homotopy colimit over repeatedly taking approximation T_n:

$$\begin{aligned} P_n(F)\colon \mathrm{Top}_* &\longrightarrow \mathrm{Top}_* \\ X &\longmapsto \mathrm{hocolim}\left(F(X) \xrightarrow{\tau_n(F)} T_n(F)(X) \xrightarrow{\tau_n(T_n(F))} T_k(T_n(F))(X) \longrightarrow \cdots\right) \end{aligned}$$

together with natural transformations $e_n(F)\colon F \to P_n(F)$. These n-excisive approximations are universal in the sense that if there was a natural transformation $F \to G_n$, with $G_n \in \mathrm{Exc}_n$ being a n-excisive functor, we get a unique morphism $P_n(F) \to G_n$ of n-excisive functors making the following diagram commute

$$\begin{tikzcd} F(X) \arrow[d, "\nu_X"'] \arrow[dr, "(e_n(F))_X"] & \\ G_n(X) & P_n(F)(X) \arrow[l, "\exists!"] \end{tikzcd}$$

We now note that when $F\colon \mathrm{Top}_* \to \mathrm{Top}_*$ is a reduced functor, in that $F(*) \simeq *$, we see that $F(C(X)) \simeq *$ and thus we find $T_1(F) = \mathrm{hocolim}(* \to F(\Sigma X) \leftarrow *) \simeq \Omega\Big(F\big(\Sigma(X)\big)\Big)$. With this we can now compute $P_1(F)(X)$ to be

$$P_1(F)(X) \simeq \mathrm{hocolim}\left(F(X) \longrightarrow \Omega(F(\Sigma(X))) \longrightarrow \Omega^2(F(\Sigma^2(X))) \longrightarrow \cdots\right) \simeq \mathrm{hocolim}_{n\to\infty}(\Omega^n(F(\Sigma^n(X))))$$

In the case when we take F to be the identity functor $\mathrm{id}\colon \mathrm{Top}_* \to \mathrm{Top}_*$, we find that its 1-excisive approximation is given by $P_1(\mathrm{id})(X) = \Omega^\infty\Sigma^\infty(X)$, as we have seen before.
We now note that any n-excisive functor is also $n+1$-excisive. So using the universal property of an n-excisive approximation of F, we see that we get commutative diagrams of the form

$$\begin{tikzcd} F(X) \arrow[d, "(e_n(F))_X"'] \arrow[dr, "(e_{n+1}(F))_X"] & \\ P_n(X) & P_{n+1}(F)(X) \arrow[l, "\exists!"] \end{tikzcd}$$

which combine together into the Goodwillie tower of the functor F at X:

$$\begin{tikzcd} F(X) \arrow[d] \arrow[dr] \arrow[drr] & & & \\ P_1(X) & P_2(F)(X) \arrow[l] & P_3(F)(X) \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

The existence of this tower was first introduced by Tom Goodwillie in [Goo03, Thm. 1.8].

3.3 The n-homogeneous layers of the Goodwillie tower

So suppose we have a homotopy functor F, then we get a tower of n-excisive approximations. We would now like to study the layers $D_n(F)$ of this tower, that being the functor defined by taking homotopy fibers of the tower maps

$$\begin{aligned} D_n(F)\colon \mathrm{Top}_* &\longrightarrow \mathrm{Top}_* \\ X &\longmapsto \mathrm{hofib}\left(P_n(F)(X) \longmapsto P_{n-1}(F)(X)\right) \end{aligned}$$

To study these we first note that if we have a fiber sequence of functors $F(X) \longrightarrow G(X) \longrightarrow H(X)$, then this also gives a fiber sequence between the Goodwillie towers, i.e. we get

$$P_n(F)(X) \longrightarrow P_n(G)(X) \longrightarrow P_n(H)(X)$$

As the $(n-i)$-th excessive approximation of $P_n(F)$ is of course $P_{n-i}(F)$ for $i \geq 0$, we see that $P_{n-1}(P_n(F)) \simeq P_{n-1}(P_{n-1}(F)) \simeq P_{n-1}(F)$. From the definition of $D_n(F)$ we now get the following fiber sequence

$$P_{n-1}(D_n(F))(X) \longrightarrow P_{n-1}(P_n(F))(X) \xrightarrow{\simeq} P_{n-1}(P_{n-1}(F))(X)$$

from which we conclude that $D_n(F)$ is a n-homogeneous functor: $P_k(D_n(f)) \simeq *$ for $k < n$. So the tower is build up out of n-homogenous functors.
We are then able to identify these n-homogenous functors in the following way. Suppose we have an n-homogenous functor $F\colon \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$ then it factors as a functor to the category of spectra by taking taking its 0-th space, so there is a n-homogenous homotopy functor $F_{\mathrm{Sp}}\colon \mathrm{Top}_* \longrightarrow \mathrm{Sp}$ such that we have the following homotopy equivalence [Goo03, Thm. 2.1]

$$F(X) \simeq \Omega^\infty(F_{\mathrm{Sp}}(X))$$

In the case for the homotopy fibers of the Goodwillie tower of a homotopy functor F, we will write this as $D_i(F)(X) \simeq \Omega^\infty\mathbb{D}_i(F)(X)$. We can actually do even more. The n-homogeneous functors from spaces to spectra can be described as a special case of a symmetric n-linear functor from $(\mathrm{Top}_*)^{\times n}$ to Sp. These types of functors are defined as follows

Definition 3.6 (symmetric n-linear functor)**.** Let $F\colon (\mathrm{Top}_*)^{\times n} \longrightarrow \mathrm{Sp}$ be a functor, then it is *n-linear* if when we fix $n-1$ spaces X_i, then $F(X_1, \cdots, -, \cdots, X_{n-1})$ is a 1-homogenous functor, and it is *symmetric* if F is invariant under permuting its entries. So for a permutation $\sigma \in \Sigma_n$ we have that $F(X_1, \cdots, X_n) = F(X_{\sigma(1)}, \cdots, X_{\sigma(n)})$.

We now find that an n-homogenous functor $F\colon \mathrm{Top}_* \longrightarrow \mathrm{Sp}$ gives rise to a symmetric n-linear functor $LF\colon (\mathrm{Top}_*)^{\times n} \longrightarrow \mathrm{Sp}$ so that we get that [Goo03, Thm. 3.5]

$$F(X) \simeq LF(X, \cdots, X)_{h\Sigma_n}$$

where we take the homotopy orbits in spectra. With this, we then find that for homotopy functor $F\colon \mathrm{Top}_* \to \mathrm{Top}_*$ the layers of its Goodwillie tower for a space X are given by

$$D_k(F)(X) \simeq \Omega^\infty \mathbb{D}_k(F)(X) \simeq \Omega^\infty \Big(L\mathbb{D}_k(F)(X, \cdots, X)_{h\Sigma_k} \Big)$$

However in the case of Top_* we also have can also write the symmetric k-linear functors as $L\mathbb{D}_k(F)(X, \cdots, X) \simeq L\mathbb{D}_k(F)(S^0, \cdots, S^0) \otimes \Sigma^\infty X^{\otimes k}$. We will call the first part the k-th derivative of F and will write it as $\partial_k(F) = L\mathbb{D}_k(F)(S^0, \cdots, S^0)$. Here we note that this spectrum carries an action of the k-th symmetric group Σ_k by permuting its factors. With this, we can finally write our layers in the same way as we did for the n-homogeneous parts of the Taylor series in the case that F preserves filtered homotopy colimits or when X is a finite CW complex [Kuh07, Cor. 2.5]:

$$D_k(F)(X) \simeq \Omega^\infty \mathbb{D}_k(X) \simeq \Omega^\infty \Big(\big(\partial_k(F) \otimes \Sigma^\infty X^{\otimes n}\big)_{h\Sigma_k} \Big)$$

When we take F to be the identity functor $\mathrm{id}\colon \mathrm{Top}_* \to \mathrm{Top}_*$ this gives us that the layers of the identity are given by

$$\mathbb{D}_n(X) \simeq \big(\partial_n \mathrm{id} \otimes \Sigma^\infty X^{\otimes n}\big)_{h\Sigma_n}$$

Just as for any other homotopy functor, the derivatives of the identity form a symmetric sequence $\partial_* \mathrm{id}$. However in this case it turns out that the derivatives carry more structure: that of an operad. We will now introduce the notion of an operad so that we in the end can give a good description of the mod-2 homology of the layers $\big(\partial_n \mathrm{id} \otimes \Sigma^\infty X^{\otimes n}\big)_{h\Sigma_n}$.

4 Operads and the spectral Lie operad

We recall that the set of homotopy classes of based maps $[S^1, X]_*$ carries a group structure, that being the fundamental group. We note that the underlying space of based maps, that being the loop space $\Omega(X) = \mathrm{Map}_{\mathrm{Top}_*}(S^1, X)$ does still carry a product, by concatenation of loops, but this does not make it into a topological group. If we for example take three loops $\alpha, \beta, \gamma \in \Omega(X)$, then we already see that $(\gamma \cdot \beta) \cdot \alpha$ is not the same as $\gamma \cdot (\beta \cdot \alpha)$ as the reparametrization of the speeds will be different: for the first composition we first go trough α for $t \in [0, \frac{1}{2}]$ and then through $(\gamma \cdot \beta)$ for $t \in [\frac{1}{2}, 1]$, whereas for the second composition we first go trough $(\beta \cdot \alpha)$ for $t \in [0, \frac{1}{2}]$ and then through γ for $t \in [\frac{1}{2}, 1]$.
However, in some sense it does still carry some notion of a group, but then as some kind of group 'up to homotopy'.
One way to make this precise is by constructing a sequence of based spaces that will encode all the different choices of multiplications we have on $\Omega(X)$. When a space carries a product with some extra structure, then this can be encoded by means of an operad. For example, when a space carries the structure of a commutative monoid, then that is equivalent to being an Comm-algebra, where Comm denotes the commutative operad. The notion of an operad first appeared in 1972 by Peter May in [May06]. In this operads are defined for the symmetric monoidal category $(\mathrm{Top}_*, \times, *)$, however as $\partial_* \mathrm{id}$ will be an operad in spectra we would like to define operads for a general category. However not being able to work with points/operations does make the definitions a lot less clear. So we have chosen to follow the discussion as in [May06] and to refer to the corresponding definitions in Michael Mandell's operad book [Man19] for the categorical treatment. Afterwards, we will look at the little-n-cubes operad $\mathscr{C}_n$ for which the n-fold loop spaces are an of $\mathscr{C}(n)$-algebras. The rest of this chapter is dedicated to the discussion about the operad structure of $\partial_* \mathrm{id}$.

4.1 Operads and algebras over an operad

To start we will first give the definition of what we mean by an operad. As mentioned in the introduction, we will be defining this for $(\mathrm{Top}_*, \times, *)$, but writing the conditions in terms of commutative diagrams will give the definition for an arbitrary symmetric monoidal category. When the category in question has a forgetful functor to Sets, then we the definition we will give will work as well.

Definition 4.1 (operad)**.** An operad in a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ is a sequence of objects $\{\mathcal{O}(n)\}_{n\geq 0}$, together with the following data

1. each object $\mathcal{O}(n)$ carries a right action by the n-th symmetric group Σ_n. For a permutation $\sigma \in \Sigma_n$ and an operation $\alpha \in \mathcal{O}(n)$ we will write this action as $\alpha \cdot \sigma$.
2. It carries a unit operation $1 \in \mathcal{O}(1)$ which we can think of the operation that does nothing.
3. It carries a composition map which composes the n-ary operation with the n j_i-ary operations to give a new j-ary operation, where $j = j_1 + \cdots + j_n$

$$\begin{aligned}\Gamma^n_{j_1,\cdots,j_n} \colon \mathcal{O}(n) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_n) &\longrightarrow \mathcal{O}(j)\\ \alpha_n \otimes \alpha_1 \otimes \cdots \otimes \alpha_n &\longmapsto \alpha_n(\alpha_1, \cdots, \alpha_n)\end{aligned}$$

And we want this triple $(\{\mathcal{O}(n)\}_{n\geq 0}, \Gamma, 1)$ to satisfy the following relations

1. We want the composition map to be associative. Suppose we have operations $\alpha \in \mathcal{O}(a)$, $\beta_i \in \mathcal{O}(b_i)$ for $1 \geq i \geq a$ and $\gamma_j \in \mathcal{O}(c_j)$ for $1 \geq j \geq b$, with $b = b_1 + \cdots b_a$ and $c = c_1 + \cdots c_b$. Then this means that

$$\alpha(\beta_1, \cdots, \beta_a)(\gamma_1, \cdots, \gamma_b) = \alpha(\beta_1(\gamma_1, \cdots, \gamma_{b_1}), \cdots, \beta_a(\gamma_{b-b_a+1}, \cdots, \gamma_b)) \in \mathcal{O}(c)$$

2. The unit map is indeed unital, in that for an operation $\alpha_1 \in \mathcal{O}(1)$ we have that $\alpha_1(1) = \alpha_1$ and for an operation $\alpha_n \in \mathcal{O}(n)$ we have that $\alpha_n(1, \cdots, 1) = \alpha_n$.
3. Every $\Gamma^a_{b_1,\cdots,b_a} \colon \mathcal{O}(a) \otimes \mathcal{O}(b_1) \otimes \cdots \otimes \mathcal{O}(b_a) \to \mathcal{O}(b)$ is $\Sigma_{b_1} \times \cdots \times \Sigma_{b_a}$ equivariant, where it acts on $\mathcal{O}(b)$ via the block inclusion. Here the block sum inclusion is defined in the following way. Suppose we have $(\sigma_1, \cdots, \sigma_n) \in \Sigma_{b_1} \times \cdots \times \Sigma_{b_a}$, then this gives permutation which we will also denote by $(\sigma_1, \cdots, \sigma_n)$ by taking the partition of $\{1, \cdots, b\}$ into a block each of size b_i for $1 \leq i \leq a$ and then permuting each i-th block by the σ_i.
   Let $\alpha \in \mathcal{O}(a)$ and $\beta_i \in \mathcal{O}(b_i)$ for $1 \leq i \leq a$ then this gives us that for $(\sigma_1, \cdots, \sigma_n) \in \Sigma_{b_1} \times, \cdots \times \Sigma_{b_a}$ that
$$\alpha(\beta_1 \cdot \sigma_1, \cdots, \beta_a \cdot \sigma_a) = \alpha(\beta_1 \cdots, \beta_a) \cdot (\sigma_1, \cdots, \sigma_a)$$
4. Suppose we have an operation $\alpha \in \mathcal{O}(a)$ and $\beta_i \in \mathcal{O}(b_i)$ for $1 \leq i \leq a$, and a permutation $\sigma \in \Sigma_a$ then we get that
$$(\alpha \cdot \sigma)(\beta_{\sigma(1)}, \cdots, \beta_{\sigma(a)}) = \alpha(\beta_1, \cdots, \beta_a)$$

A morphism of operads $f \colon \mathcal{O}_1 \longrightarrow \mathcal{O}_2$ is then given by a sequence of maps $f_n \colon \mathcal{O}_1(n) \longrightarrow \mathcal{O}_2(n)$ that are Σ_n-equivariant, preserve the unit $f(1_1) = 1_2$ and preserve composition. We will write the category of operads in the symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ as Operads$(\mathcal{C})$. For the definition of an operad for an arbitrary symmetric monoidal category we will refer to [Man19, Def. 2.1] and for a morphism of operads to [Man19, Def. 2.2].
As operads encode operations, we want them to act on objects in our category, yielding the operations on it as governed by the operad. If we have such an action, then we call the object an algebra over the corresponding operad. With this, we can now define what we mean by an algebra over an operad

Definition 4.2 (algebra over an operad)**.** An object X is an $\mathcal{O}$-algebra, precisely if there are action maps which we will think of letting an operation act on its arguments

$$\begin{aligned}\xi_n \colon \mathcal{O}(n) \otimes X^{\otimes n} &\longrightarrow X\\ \alpha \otimes (x_1 \otimes \cdots \otimes x_n) &\longmapsto \alpha(x_1, \cdots, x_n)\end{aligned}$$

such that it satisfies the following conditions:

1. They are equivariant with respect to the action of Σ_n, which acts trivially on the target and the action on the right side is given by the *diagonal action*, or in other words we let $\sigma \in \Sigma_n$ act on $\mathcal{O}(n) \otimes X^{\otimes n}$ by acting on $\mathcal{O}(n)$ by its left action, and on $X^{\otimes n}$ the right action given by permuting the factors along σ^{-1}.
$$(\alpha \cdot \sigma)(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)}) = \alpha(x_1, \cdots, x_n)$$

2. It preserves composition, in that if we have operations $\alpha \in \mathcal{O}(a)$ and $\beta_i \in \mathcal{O}(b_i)$ for $1 \leq i \leq a$ then we have that
$$\big(\alpha(\beta_1, \cdots, \beta_a)\big)(x_1, \cdots, x_b) = \alpha\big(\beta_1(x_1, \cdots, x_{b_1}), \cdots, \beta_a(x_{b-b_a+1}, \cdots, x_b)\big)$$
3. Letting the unit operation act on an object should be the same as the identity map $1(x) = x$.

Again, for the definition for an $\mathcal{O}$-algebra for an operad in a general symmetric monoidal category we refer to [Man19, Def. 4.1]. As a first interesting example of an $\mathcal{O}$-algebra, we note that a commutative monoid object is precisely a Comm-algebra, where Comm denotes the commutative operad.
We will now discuss an operad that will play a major role in this thesis, that being the E_n operad. For this discussion we will follow [May06].

Definition 4.3 ($\mathscr{C}_n$ operad)**.** The little-n-cubes operad $\mathscr{C}_n$ will be an operad in $(\mathrm{Top}_*, \times, *)$. Let I^n denote the n-dimensional unit cube. Then $\mathscr{C}_n(k)$ is given by the space of maps $f\colon \coprod_k I^n \to I^n$ such that if we write $f = f_1 \sqcup \cdots \sqcup f_k$, then each f_i is an *affine parallel axis* embedding of I^n into itself so that each of their images are disjoint. Here we mean by an affine parallel axis embedding that the maps $f_i\colon I^n \to I^n$ are of the form $f_i(t_1, \cdots, t_n) = (p_1 + a_1 t_1, \cdots, p_n + a_n t_n)$ with both $p = (p_1, \cdots, p_n)$ and $a = (a_1, \cdots, a_n)$ points in I^n such that $p_i + a_i \leq 1$.
Here the action of the symmetric group Σ^k is free: it is given by permuting the embedding maps, so $f \cdot \sigma = f_{\sigma(1)} \sqcup \cdots \sqcup f_{\sigma(k)}$.
The unit in $\mathscr{C}_n(1)$ is given by the identity embedding $\mathrm{id}\colon I^n \longrightarrow I^n$, and the composition rule is again given by a composition of functions

$$\begin{aligned}\Gamma^k_{l_1, \cdots, l_k}\colon \mathscr{C}_n(k) \times \mathscr{C}_n(l_1) \times \cdots \times \mathscr{C}_n(l_k) &\longrightarrow \mathscr{C}_n(l)\\ (f, (g_1, \cdots, g_k)) &\longmapsto f(g_1, \cdots, g_k) = f_1 \circ g_{1,1} \sqcup \cdots \sqcup f_k \circ g_{k,l_k}\end{aligned}$$

We have an interesting example of a $\mathscr{C}_n$-algebra, that being the n-fold loop space $\Omega^n(X) = \mathrm{Map}((S^n, 1), (X, x_0))$. First, we will identify S^n with $I^n/\partial I^n$, so that we can think of the loop space as relative maps $(I^n, \partial I^n) \longrightarrow (X, x_0)$. The algebra map is then given by

$$\begin{aligned}\mathscr{C}_n(k) \times (\Omega^n(X))^{\times k} &\longrightarrow \Omega^n(X)\\ (f, (g_1, \cdots, g_k)) &\longmapsto \begin{pmatrix} \tilde{f}\colon (I^n, \partial I^n) \longrightarrow (X, x_0) & \\ \qquad\qquad x \longmapsto \begin{cases} g_i(f_i^{-1}(x)) & \text{if } x \in \mathrm{im}(f_i) \\ x_0 & \text{else} \end{cases} \end{pmatrix}\end{aligned}$$

The spaces of the little-n-cubes operad are homotopy equivalent to a configuration space

$$\mathscr{C}_n(k) \simeq \mathrm{Conf}_k(\mathbb{R}^n)$$

where $\mathrm{Conf}_k(\mathbb{R}^n)$ is the space of k-tuples pairwise disjoint points in $\mathbb{R}^n$. As an example we see from this that $\mathscr{C}_n(2) \simeq \mathrm{Conf}_2(\mathbb{R}^n) \simeq S^{n-1}$ on which Σ_2 freely via the antipodal action.

If we think about what kind of structure the little-n-cubes operad imposes on their algebras, then we see that they give a homotopy unital product that is homotopy associative with all higher homotopy coherences. Besides that for $n \geq 1$ we see that the product will be homotopy commutative as well, and this commutativity will be more homotopy coherent for larger n. We will call the operads that encode this kind of structure an E_n operad. So we see that the little-n-cubes operad will be an model for this. When an operad induces a product that is homotopy unital, homotopy associative will all higher coherences and homotopy commutative with all higher coherences, then we will call this operad an E_∞ operad. An operad turns out to be an E_∞ operad if and only if all its underlying spaces are contractible, and when the actions of the symmetry groups Σ_n are free.
One interesting thing of the little cube operads is that we can embed $\mathscr{C}_n(k)$ into the $\mathscr{C}_{n+1}(k)$. For $f \in \mathscr{C}_n(k)$ this map is given by $f \longmapsto f \times \mathrm{id}_I$ where $f \times \mathrm{id}_I$ is given by

$$(f \times \mathrm{id}_I)(t_1, \cdots, t_{n+1}) = (f(t_1, \cdots, t_n), t_{n+1})$$

so we extend the images over the whole cube. So we find that using these embeddings we can build a sequence of operads $\mathscr{C}_1 \to \mathscr{C}_2 \to \cdots$. Taking the colimit with respect to this sequence then gives us a new operad $\mathscr{C}_\infty$ whose underlying spaces acted on freely by Σ and which are also contractible. So we see that $\mathscr{C}_\infty$ is a model for an E_∞ operad, which was what we were after. We note that just like the $\Omega^n(X)$ were algebras for $\mathscr{C}_n$, the infinite loop spaces $\Omega^\infty(X)$ are algebras for $\mathscr{C}_\infty(X)$.

We again have that we can describe the spaces $\mathscr{C}_\infty(k)$ as a configuration space $\mathscr{C}_\infty(k) \simeq \mathrm{Conf}_k(\mathbb{R}^\infty)$, and that for $k = 2$ we find that $\mathscr{C}_\infty(2) \simeq \mathrm{Conf}_2(\mathbb{R}^\infty) \simeq S^\infty$ where Σ_n acts freely by the antipodal map.

We now note that the $\mathscr{C}_n$ and $\mathscr{C}_\infty$ also give operads in spectra. By taking the suspension functor $\mathrm{Top}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp}$ to its underlying spaces, we get a new operad on spectra, which we will write as $\Sigma^\infty\mathscr{C}_n$ and as $\Sigma^\infty\mathscr{C}_\infty$, or just simply as E_n and E_∞.

4.2 The operad structure on $\partial_* id$

In 1995 Johnson gave a construction of the derivatives of the identity functor as a *Spanier-Whitehead dual* of a space Δ_n in [Joh95]

$$\partial_n \,\mathrm{id} \simeq \mathrm{Map}_{\mathrm{Sp}}(\Delta_n, \mathbb{S})$$

Arone and Mahowald gave a reformulation of this result in [AM99] by constructing the Δ_n as a finite complex. We will now give this construction. We start with the set of partitions Part_n of the set $\{1, \cdots, n\}$. This has the structure of a poset by refinement of the partitions. We can then form a complex by taking the geometric realization of its simplicial nerve $|N(\mathrm{Part}_n)_*|$. Our space Δ_n is then given by the quotient complex

$$\Delta_n = |N(\mathrm{Part}_n)_*| / \partial |N(\mathrm{Part}_n)_*|$$

where $\partial|N(\mathrm{Part}_n)_*|$ is the boundary of the complex $|N(\mathrm{Part}_n)_*|$. We will give two examples

1. For $n = 2$ we see that the partition poset Part_2 looks like

$$(12) \longrightarrow (1)(2)$$

so the geometric realization of the nerve will be an interval, and the boundary will be given by the vertices indicated by (12) and (1)(2). From this we see that $\Delta_2 \simeq S^1$. Taking the Spanier-Whitehead dual then shows us that $\partial_2 \,\mathrm{id} \simeq \mathbb{S}^{-1}$.

2. For $n = 3$ we see that the partition pose Part_3 looks as

$$\begin{tikzcd} & (123) \arrow[dl] \arrow[d] \arrow[dr] & \\ (1)(23) \arrow[dr] & (2)(13) \arrow[d] & (3)(12) \arrow[dl] \\ & (1)(2)(3) & \end{tikzcd}$$

so the geometric realization of the nerve will be three triangles glued along a common edge. The boundary is then given by the 1-cells of the complex, except for the 1-cell from the map $(123) \to (1)(2)(3)$. By taking the quotient we get a circle together with three inner disks, which gives us that $\Delta_3 \simeq S^2 \vee S^2$ so in the end we get that $\partial_3 \,\mathrm{id} \simeq \mathbb{S}^{-2} \vee \mathbb{S}^{-2}$.

In general, Johnson deduced that the derivatives of the identity are non-equivariently equivalent to a wedge of spheres

$$\partial_n \,\mathrm{id} \simeq \bigvee_{n-1!} \mathbb{S}^{1-n}$$

4.2.1 The operadic bar construction

The main reference for this part will be [Chi05]. In this section we review the operadic bar construction, which will be used to give the structure of an operad on $\partial_* \,\mathrm{id}$. To define this we will first need to construct the symmetric monoidal category of symmetric sequences.

Definition 4.4. By a symmetric sequence X in Top_* we mean a sequence of spaces $(X(n))_{n\geq 0}$ such that each $X(n)$ has a right action of the symmetric group Σ_n.

The symmetric sequences are the objects of a category $\mathrm{SymSeq}(\mathrm{Top}_*)$. This category has a symmetric monoidal structure where the symmetric monoidal product is given by the composition product of symmetric sequences

Definition 4.5. Let X and Y be two symmetric sequences in Top_*, then the composition product $X \circ Y$ is defined as

$$(X \circ Y)(n) = \bigvee_{\bar{n}=\sqcup_{i\in I}\bar{n}_j} X(|I|) \wedge \bigwedge_{i\in I} Y(|\bar{n}_i|)$$

so we take a wedge over the partitions of $\{1, \cdots, n\}$ where I is the indexing set for the parts a partition.

To get some intuition we note that when $n = 3$, this gives us that

$$(X \circ Y)(n) = \big(X(1) \wedge Y(2)\big) \vee \big(X(2) \wedge Y(1) \wedge Y(1)\big)$$

where the first part comes from the partition (12) and the second from the partition (1)(2). When $n = 3$ we get some extra terms as

$$(X \circ Y)(3) = \big(X(1) \wedge Y(3)\big) \vee \Big(\bigvee_{\substack{(12)(3)\\(1)(23)\\(2)(13)}} X(2) \wedge Y(1) \wedge Y(2)\Big) \vee \big(X(3) \wedge Y(1) \wedge Y(1) \wedge Y(1)\big)$$

where the first part comes from the partition (123) and last part comes from the partition (1)(2)(3).
By taking the product multiple times, we get the iterated composition product. Instead of being indexed by partitions of n this will then be indexed by $k-1$ refinements of partitions of n, for k being the amount of symmetric sequences in the composition product. As an example we see that the refinement $(12)(345) \to (1)(2)(345)$ will give a wedge summand

$$X(2) \wedge \big(Y(2) \wedge Z(1) \wedge Z(1)\big) \wedge \big(Y(1) \wedge Z(3)\big)$$

in the iterated composition product $(X \circ Y \circ Z)(5)$.
The composition product makes the category of symmetric sequences in Top_* into a symmetric monoidal category, and the monoid objects in this are precisely the operads.

Definition 4.6 (simplicial bar construction)**.** Let $\mathcal{O}$ be an operad in Top_*. Then its simplicial bar construction $\mathrm{Bar}(\mathcal{O})_*$ is a simplicial symmetric sequence, given by

$$\mathrm{Bar}(\mathcal{O})_n \simeq \underbrace{\mathcal{O} \circ \cdots \circ \mathcal{O}}_{n \text{ times}}$$

where the face maps come from the composition map $\Gamma\colon \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$, and the degeneracy maps come from the unit map $I \to \mathcal{O}$ where I denotes the unit symmetric sequence.

We now would like to build a new symmetric sequence out of the simplicial bar construction. This is done by taking the geometric realization of a simplicial symmetric sequence. To define this we will first need to define how we take the geometric realization of a simplicial space.

Definition 4.7. Let X_* be a simplicial space, then its geometric realization $|X_*|$ is the new space given by

$$|X_*| = \mathrm{Coeq}\left(\bigsqcup_{[n]\to[k]} X_k \times \Delta^n \rightrightarrows \bigsqcup_n X_n \times \Delta^n\right)$$

where the top map is given by

$$\theta_{\sigma\colon [n]\to[k]}\colon X_k \times \Delta^n \xrightarrow{\sigma^*\times \mathrm{id}} X_n \times \Delta^n$$

where $\sigma^*\colon X_k \to X_n$ is the induced map on X_*, and the bottom map is given by

$$\psi_{\sigma\colon [n]\to[k]}\colon X_k \times \Delta^n \xrightarrow{\mathrm{id}\times\sigma_*} X_k \times \Delta^k$$

where $\sigma_*\colon \Delta^n \to \Delta^k$ is the induced map on the cosimplicial space Δ^*.

The geometric realization of a simplicial symmetric sequence is now defined pointwise, so for a simplicial symmetric sequence X_* it is defined as $|X_*|(n) = |X_*(n)|$.
We now note that the spaces Δ_n which we used to define $\partial_* \operatorname{id}$ can equally be defined as the geometric realization of a simplicial set $(T_n)_*$ by

$$(T_n)_* = N(\mathrm{Part}_n)_*/(N(\mathrm{Part}_n^+)_* \cup N(\mathrm{Part}_n^-)_*)$$

where Part_n^+ is the partition poset with where we exclude the most refined partition $(1)(2)\cdots(n)$ and Part_n^- is the partition poset where we exclude the least refined partition $(12\cdots n)$.
It is now shown by Ching that $\Delta_n = |(T_n)_*|$ can be described as the geometric realization of the simplicial bar construction of the operad Comm in Top_*. This operad is given by $\mathrm{Comm}(n) \simeq S^0$, and has the commutative monoidal spaces as it algebras.

Theorem 4.1 ([Chi05, Lem. 8.6])**.** *The partition poset complex Δ_n is homeomorphic to the simplical bar construction of the operad* Comm*. In fact the underlying simplicial symmetric sequences are isomorphic $(T_n)_* \simeq \mathrm{Bar}(\mathcal{O})_*(n)$.*

Here we see that an n-simplex of $(T_k)_*$ is given by a refining sequence of $n-1$ partitions of $\bar{k}$. And an n-simplex in $\mathrm{Bar}(\mathcal{O})_*(k)$ is a point in the space $(\mathcal{O} \underset{n-1 \text{ times}}{\circ \cdots \circ} \mathcal{O})(k)$ where the wedge sum was also indexed by refining sequences of $n-1$ partitions of $\bar{k}$.
We recall that the face maps of the simplicial nerves of $(T_n)_*$ are given by removing a partition in the sequence, for example

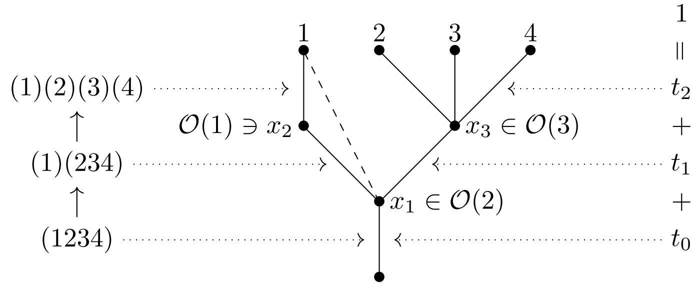
$$d_2\Big((1234) \leq (13)(24) \leq (13)(2)(4) \leq (1)(2)(3)(4)\Big) = (1234) \leq (13)(2)(4) \leq (1)(2)(3)(4)$$

and the degeneracy maps are given by repeating a partition

$$s_2\Big((123) \leq (1)(23) \leq (1)(2)(3)\Big) = (123) \leq (1)(23) \leq (1)(23) \leq (1)(2)(3)$$

Now the face and degeneracy maps on $\mathrm{Bar}(\mathcal{O})_*$ which were induced by the structure map and unit map of $\mathcal{O}$, give the same face and degeneracy maps on the sequence of partitions. This indeed shows that $(T_n)_*$ and $\mathrm{Bar}(\mathcal{O})_*(n)$ are isomorphic.
So with this we have shown that $\partial_n \operatorname{id} \simeq \mathrm{Map}_{\mathrm{Sp}}(|\mathrm{Bar}(\mathrm{Comm})|(n), \mathbb{S})$. So far this only means that $\partial_n \operatorname{id}$ is in some sense related to the commutative operad on Top_*, but we want to use this fact to give $\partial_* \operatorname{id}$ the structure of an operad in spectra.

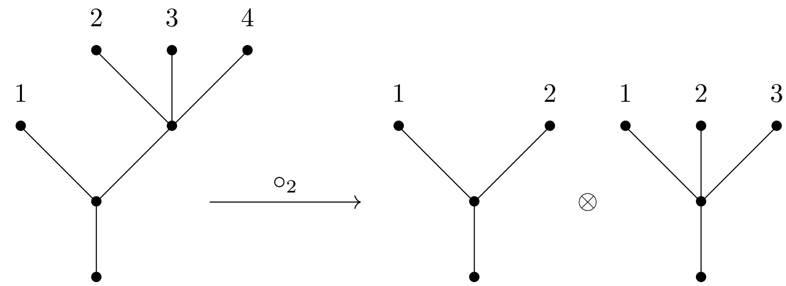
1. The first step is to show that the the geometric realization of the bar construction of a reduced operad spaces carries the structure of a cooperad in spaces. Here we recall that an n-simplex in $|\mathrm{Bar}\,\mathcal{O}_*|(k)$ is given by a point in $(\mathcal{O} \underset{n-1 \text{ times}}{\circ \cdots \circ} \mathcal{O})(k) \times \Delta^n$ (under the equivalence relation of the coequalizer). Here a point in $(\mathcal{O} \underset{n-1 \text{ times}}{\circ \cdots \circ} \mathcal{O})(k)$ is given by a refining sequence of $n-1$ partitions of $\bar{k}$, which indexes the particular wedge summand, and then a smash product of points in the corresponding spaces. The cooperad structure is now given by rethinking the sequences of partitions as a rooted tree with k leaves, where the inner vertices v correspond to the points of the $\mathcal{O}(i_v)$, where i_v are the amount of incoming edges. We recall that Δ^n was defined as the space $\{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1\}$, so that a point in Δ^n corresponds to a metric of our tree.



$$\wr$$

$$(x,t) \in \big(\mathcal{O}(2) \otimes (\mathcal{O}(1) \otimes \mathcal{O}(2))\big) \times \Delta^2$$

The benefit of interpreting the geometric realization of the bar construction as a space of rooted metric trees, is that we can give the space of trees the structure of a cooperad. We will write this cooperad as Tree with Tree(n) being the spaces of rooted trees with n leaves. The cooperad structure is then a map $\mathrm{Tree}(n-1+k_2) \xrightarrow{\circ_2} \mathrm{Tree}(n) \wedge \mathrm{Tree}(k_2)$, which is given by ungrafting a tree on the corresponding edge.



Of course for the space of trees that describe the geometric realization of the simplicial bar construction of a reduced operad there is more going on to define the structure of a cooperad in that we also have to deal with the metric and the labeling of the inner vertices, however the idea as to why this gives a cooperad remains the same.

2. Now that we have shown that the geometric realization of the simplicial bar construction of a reduced operad carries the structure of a cooperad, we will make use of [Chi05, Lem. 6.1] which tells us that the for a cooperad in spaces, taking the Spanier-Whitehead dual of its suspension spectrum gives us an operad in spectra.

As the symmetric sequence $\partial_* \mathrm{id}$ is precisely constructed as the Spanier-Whitehead dual of the geometric realization of the simplicial bar construction of the operad Comm in spaces, we hereby see that it does indeed carry the structure of an operad.
Alternatively, as is shown in [Chi05, Cor. 8.8] we can also think of $\partial_* \mathrm{id}$ as the cobar construction in spectra of the cooperad Comm with $\mathrm{Comm}(n) \simeq \mathbb{S}$ the sphere spectrum. This is shown by

$$\partial_n \mathrm{id} \simeq \mathrm{Map}_{\mathrm{Sp}}(\mathrm{Bar}(S^0), \mathbb{S})(n) \simeq \mathrm{coBar}(\mathrm{Map}_{\mathrm{Sp}}(S^0, \mathbb{S})) \simeq \mathrm{coBar}(\mathbb{S})$$

4.3 The free algebra over an operad

Let $\mathcal{O}$ be an operad and X be an algebra over this operad. Then we have a forgetful functor from $\mathrm{Alg}_{\mathcal{O}}$ to the underlying category, by forgetting the $\mathcal{O}$-algebra structure on X. This then has a right adjoint, the free $\mathcal{O}$-algebra functor $\mathrm{Free}_{\mathcal{O}} \colon \mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ which is on objects given by

$$\mathrm{Free}_{\mathcal{O}}(X) = \coprod_n \mathrm{Sym}^n_{\mathcal{O}}(X) = \coprod_n \Big(\big(\mathcal{O}(n) \otimes X^{\otimes n}\big)_{h\Sigma_n} \Big)$$

where Σ_n acts on $\mathcal{O}(n)$ be its right action, and on $X^{\otimes n}$ by permuting the elements. We will call the functors that are in the coproduct $\mathrm{Sym}^n_{\mathcal{O}} \colon \mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ the extended power constructions.
We will now first show that the free $\mathcal{O}$ algebra of X is indeed an $\mathcal{O}$ algebra, so will need to define the structure maps $\mathcal{O}(n) \xrightarrow{\xi_n} \mathrm{Free}_{\mathcal{O}}(X)^{\otimes n}$. To do this we will first rewrite $\mathrm{Free}_{\mathcal{O}}(X)^{\otimes n}$ as

$$\begin{aligned}
\mathrm{Free}_{\mathcal{O}}(X)^{\otimes n} \simeq \Big(\coprod_{k \geq 0} \big(\mathcal{O}(k) \otimes X^{\otimes n}\big)_{h\Sigma_k} \Big)^{\otimes n} &\simeq \coprod_{k_1 \geq 0} \cdots \coprod_{k_n \geq 0} \Big(\big(\mathcal{O}(k_1) \otimes X^{\otimes k_1}\big)_{h\Sigma_{k_1}} \otimes \cdots \otimes \big(\mathcal{O}(k_n) \otimes X^{\otimes k_n}\big)_{h\Sigma_{k_n}} \Big) \\
&\simeq \coprod_{k \geq 0} \coprod_{\substack{k_1, \cdots, k_n \\ k_1 + \cdots + k_n = k}} \Big(\mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \otimes X^{\otimes k} \Big)_{h(\Sigma_{k_1} \times \cdots \times \Sigma_{k_n})}
\end{aligned}$$

Using this we can now use the composition map from $\mathcal{O}$ to write the structure map under the coproduct as

$$\begin{aligned}
\mathcal{O}(n) \otimes \Big(\mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \otimes X^{\otimes k} \Big)_{h(\Sigma_{k_1} \times \cdots \times \Sigma_{k_n})} &\longrightarrow \Big(\mathcal{O}(k) \otimes X^{\otimes k} \Big)_{h\Sigma_k} \\
\alpha \otimes (\alpha_{k_1} \otimes \cdots \otimes \alpha_{k_n}) \otimes (x_1 \otimes \cdots \otimes x_n) &\longmapsto \alpha(\alpha_{k_1}, \cdots, \alpha_{k_n}) \otimes (x_1 \otimes \cdots \otimes x_n)
\end{aligned}$$

This construction indeed gives us a $\mathcal{O}$-algebra which is free, in the sense that we have a free-forgetful adjunction

$$\mathrm{Hom}_{\mathrm{Alg}(\mathcal{O})}(\mathrm{Free}_{\mathcal{O}}(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

where a map $f\colon X \to Y$ with (Y, ξ) being an $\mathcal{O}$-algebra corresponds to the map of $\mathcal{O}$-algebras $\tilde{f}\colon \mathrm{Free}_{\mathcal{O}}(X) \to Y$ is given by

$$\tilde{f}(\alpha \otimes x_1 \otimes \cdots \otimes x_n) = \alpha(f(x_1), \cdots, f(x_n))$$

where we use the action map of Y. Using this adjunction we also note that a map $\mathrm{Free}_{\mathcal{O}}(X) \longrightarrow X$ corresponds to an $\mathcal{O}$-algebra structure on X.
Looking at the definition of the extended power constructions again, we might notice that we have encountered these before as we found that the layers of the Goodwillie tower were precisely given by

$$\mathbb{D}_n(X) \simeq \left(\partial_n \operatorname{id} \otimes (\Sigma^\infty X)^{\otimes n}\right)_{h\Sigma_n}.$$

So with now that we know that $\partial_* \operatorname{id}$ we see that to understand the mod-2 homology of the layers, we want to study the mod-2 homology of free spectral Lie algebras.

5 Homology operations on algebras over an operad

One of the reasons that ordinary mod-2 cohomology is such a strong invariant, is that it comes equiped with a lot of structure in the form of cohomology operations. For one we recall that $H^*(-;\mathbb{Z}/2)\colon \mathrm{Top}_* \longrightarrow \mathrm{grAbGrp}$ actually takes a space to a commutative ring where the product is given by the cup product. Here we can think of this as a bilinear operation $H^m(X;\mathbb{Z}/2) \otimes H^n(X;\mathbb{Z}/2) \xrightarrow{\cup} H^{m+n}(X;/2)$, so our goal would be to determine all the cohomology operations. To do this we note that $H^k(-;\mathbb{Z}_2)$ was representable by the Eilenberg Maclane space $K(\mathbb{F}_2, k)$ in that $H^k(X;\mathbb{Z}/2) \simeq [X, K(\mathbb{F}_2, k)]$. Together with the Yoneda lemma we can express the set of unary cohomology operations between two degrees as the cohomology of an Eilenberg-Maclane space

$$\mathrm{Op}^{H\mathbb{F}_2}(k_1, k_2) = \mathrm{Hom}\left(H^{k_1}(-, \mathbb{Z}/2), H^{k_2}(-, \mathbb{Z}/2)\right) \simeq H^{k_2}(K(\mathbb{F}_2, k_1); \mathbb{Z}/2)$$

showing that the left-hand side carries the structure of an abelian group. In Algebriac Topology 2 we found out that these unary operations were generated by the Steenrod squares

$$\mathrm{Sq}^i\colon H^n(X, \mathbb{Z}/2) \longrightarrow H^{n+i}(X, \mathbb{Z}/2).$$

and together with the cup product they generate all of the cohomology operations for cohomology in mod-2 coefficients. These operations turn out to be very useful, one reason of which is that they satisfied the Adem relations

$$\mathrm{Sq}^i \mathrm{Sq}^j = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k} \mathrm{Sq}^k$$

With this we were able to construct a $\mathbb{F}_2$ algebra, being freely generated over monomials of Steenrod squares Sq^I, with the product being the composition of the operations. By taking the quotient with respect to the Adem relations, this gave a new $\mathbb{F}_2$-algebra, the Steenrod algebra $\mathcal{A}^*$. Using the Adem relations we were able to see that a basis for this was given by the admissible monomials, the Sq^I with $I = (i_1, \cdots, i_k)$ and $i_j \geq 2i_{j-1}$.
From this we see that the mod 2 cohomology of a space X carries two structures

1. It carries the structure of a commutative ring given by the cup product.
2. It carries the structure of a module over the Steenrod algebra, given by the action of the Steenrod operations on $H^*(X;\mathbb{Z}/2)$.

These two structures are not independent of each other, the Steenrod operations satisfied the following additional relations

1. For $x \in H^i(X;\mathbb{Z}/2)$ we have that $\mathrm{Sq}^i(x) = x \cup x$.

2. Taking a Steenrod operation of a cup product will follow the Cartan rule

$$\mathrm{Sq}^i(x \cup y) = \sum_{r+s=i} \mathrm{Sq}^r(x) \cup \mathrm{Sq}^s(y)$$

3. Let $x \in H^j(X; \mathbb{Z}/2)$, when $i > j$, then we have that $\mathrm{Sq}^i(x) = 0$.

To be in line with our later terminology, if we have a graded abelian group together with the structure of a commutative ring and a module structure over the Steenrod algebra, such that the above relations are satisfied, then we will call it an *allowable commutative* $\mathcal{A}^*$ *-algebra*.

Another example where we were able to determine all the homology operations were for the case that our space was an iterated loop space $H_*(\Omega^n(X); \mathbb{Z}/2)$. The unary operations were first discovered by Araki and Kudo in [AK56] and later expanded upon by Dyer and Lashof in [DL62] for mod-p homology for $p > 2$. These operations essentially came from $\Omega^n(X)$ carrying the structure of an E_n-algebra.

What we are after is some way to generalize this construction to study the homology operations for a generalized homology theory when studied on the category of spectra that have the structure of an $\mathcal{O}$-algebra. For the construction of these, we will be following the path as is laid out in [Law20].

5.1 Constructing the operations

We recall that by *Brown's representability theorem* that any spectrum E gives rise to both a generalized homology theory $\mathrm{Sp} \xrightarrow{E_*} \mathrm{grAbGrp}$ and a generalized cohomology theory $\mathrm{Sp}^{\mathrm{op}} \xrightarrow{E^*} \mathrm{grAbGrp}$ by

$$\begin{aligned} E_k(X) &= [\mathbb{S}^k, E \otimes X]_{\mathrm{Sp}} \text{ and} \\ E^k(X) &= [\mathbb{S}^{-k}, \mathrm{Map}_{\mathrm{Sp}}(X, E)]_{\mathrm{Sp}} \simeq [X, \Sigma^k E]_{\mathrm{Sp}} \end{aligned}$$

When we take our input spectrum to be a suspension spectrum of a space $\Sigma^\infty X$ then these give a homology/cohomology theory on spaces. For example if we take E to be the Eilenberg Maclane spectrum $H\mathbb{F}_2$ then we find that we recover the usual reduced homology and cohmology theories $(H\mathbb{F}_2)_k(\Sigma^\infty(X)) \simeq H_k(X; \mathbb{Z}/2)$ and $(H\mathbb{F}_2)^k(\Sigma^\infty(X)) \simeq H^k(X; \mathbb{Z}/2)$. One property that makes $H\mathbb{F}_2$ stand out as a homology/cohomology theory is that it is a commutative ring spectrum, so it carries a commutative product $H\mathbb{F}_2 \otimes H\mathbb{F}_2 \to H\mathbb{F}_2$. Suppose we now have a commutative ring spectrum E then we have two interesting examples of modules in Mod_E.

1. For any spectrum X we have that the function spectrum $\mathrm{Map}_{\mathrm{Sp}}(X, E)$ has the structure of an E-module.
2. We note that we have a forgetful functor from the category of E-modules to the category of spectra. This has a left adjoint, which is given by smashing with E:

$$\mathrm{Hom}_{\mathrm{Mod}_E}(E \otimes X, Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Sp}}(X, Y)$$

As we had that $E_k(X) = \pi_k(E \otimes X)$ and $E^k(X) = \pi_{-k}(\mathrm{Map}_{\mathrm{Sp}}(X, E))$ we find that we can generalize our problem statement to finding homotopy operations when restricted to Mod_E. However to say something meaningful about these operations, we want to study them in the case that our input spectrum is both a module over the ring spectrum E as well as having the action of an operad $\mathcal{O}$, so that it becomes $\pi_k \colon h\,\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E) \to \mathrm{AbGrp}$. We now define the group of homotopy operations

$$\mathrm{Op}^E_{\mathcal{O}}(k_1, k_2) = \mathrm{Hom}(\pi_{k_1}, \pi_{k_2}).$$

In the same way as we did for ordinary cohomology, we now want to understand these by trying to represent the functor π_k, and then use Yoneda's lemma to give a more concrete description of the homotopy operations. We first recall that for $\mathrm{Alg}_{\mathcal{O}}$ we had the following free forgetful adjunction

$$\mathrm{Hom}_{\mathrm{Alg}_{\mathcal{O}}}(\mathrm{Free}_{\mathcal{O}}(X), Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Sp}}(X, Y)$$

with $\mathrm{Free}_{\mathcal{O}}(X)$ being the free $\mathcal{O}$ algebra of X. For Mod_E we also had a free-forgetful adjunction given by

$$\mathrm{Hom}_{\mathrm{Mod}_E}(E \otimes X, Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Sp}}(X, Y)$$

where $E \otimes X$ is the free E-module of X. Combining the two adjunctions, we get a free-forgetful adjunction for the category of $\mathcal{O}$-algebras in the category of E-modules, which we will denote by $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)$:

$$\mathrm{Hom}_{\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)}(E \otimes \mathrm{Free}_{\mathcal{O}}(X), Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Sp}}(X, Y)$$

Using the above adjunction, we are now able to represent $\pi_k \colon h\,\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E) \to \mathrm{AbGrp}$ by

$$\pi_k(X) = [\mathbb{S}^k, X]_{\mathrm{Sp}} \simeq [E \otimes \mathrm{Free}_{\mathcal{O}}(\mathbb{S}^k), X]_{\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_E)}$$

so indeed π_k is representable by the object $E \otimes \mathrm{Free}_{\mathcal{O}}(\mathbb{S}^k)$. Using the Yoneda lemma we now find that

$$\mathrm{Op}^E_{\mathcal{O}}(k_1, k_2) = \mathrm{Hom}(\pi_{k_1}, \pi_{k_2}) \simeq \pi_{k_2}(E \otimes \mathrm{Free}_{\mathcal{O}}(\mathbb{S}^{k_1})) \simeq E_{k_2}(\mathrm{Free}_{\mathcal{O}}(\mathbb{S}^{k_1}))$$

So we have now given a description for all the homotopy operations. In the same way, we can also give a description for all the n-ary homotopy operations. These are then given by

$$\mathrm{Op}^E_{\mathcal{O}}(k_1, \cdots, k_n; k) \simeq \pi_k\Big(E \otimes \big(\mathrm{Free}_{\mathcal{O}}(\mathbb{S}^{k_1} \oplus \cdots \oplus \mathbb{S}^{k_n})\big)\Big)$$

However, we are not really interested in all of the unary operations. Recall that the ordinary cohomology operations were generated by some special ones, namely the Steenrod Squares Sq^i. In the terminology we have now, these came from $\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, H\mathbb{F}_2)$ being an E_∞-algebra.
In the same way we want to find some interesting homotopy operations, out of which we can construct all the other homotopy operations. First recall that we constructed the free $\mathcal{O}$-algebra functor as

$$\mathrm{Free}_{\mathcal{O}}(X) = \coprod_k \mathrm{Sym}^k_{\mathcal{O}}(X)$$

With this we see that the abelian group of homotopy operations decompose into homotopy operations with a certain weight

$$\mathrm{Op}^E_{\mathcal{O}}(k_1, k_2)^{\langle k \rangle} = \pi_{k_2}(E \otimes \mathrm{Sym}^k_{\mathcal{O}}(\mathbb{S}^{k_1})) = E_{k_2}\Big(\big(\mathcal{O}(k) \otimes (\mathbb{S}^{k_1})^{\otimes k}\big)_{h\Sigma_k}\Big)$$

The first interesting weight, and for which we get the Dyer-Lashof operations, is in the case that $k = 2$.

$$\mathrm{Op}^E_{\mathcal{O}}(k_1, k_2)^{\langle 2 \rangle} = E_{k_2}\Big(\big(\mathcal{O}(2) \otimes (\mathbb{S}^{k_1})^{\otimes 2}\big)_{h\Sigma_2}\Big)$$

In all the cases we will be studying in this thesis, the weight 2 operations in fact generate all the higher weight unary operations. Spelling out the construction, we see that for a spectrum $X \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Sp})$ we get a weight 2 operation $E_m(X) \xrightarrow{\alpha} E_n(X)$ which corresponds to an element $\tilde{\alpha} \in E_n\Big(\big(\mathcal{O}(2) \otimes (\mathbb{S}^m)^{\otimes 2}\big)_{h\Sigma_2}\Big)$ in the following way: we first note that an element $x \in E_m(X)$ is by definition of the E-homology represented by a map $f_x \colon S^m \to E \otimes X$. Now, by the free-forgetful adjunction of $\mathcal{O}$-algebras in Mod_E, we get that this corresponds to a map $E \otimes \big(\mathcal{O}(2) \otimes (\mathbb{S}^m)^{\otimes 2}\big)_{h\Sigma_2} \to E \otimes X$.

$$\begin{tikzcd}
\mathbb{S}^m \arrow[r, "f_x"] \arrow[d, dotted, "E\otimes\mathrm{Free}_{\mathcal{O}}"'] & E \otimes X \arrow[d, dotted] \\
E \otimes \mathrm{Free}_{\mathcal{O}}(\mathbb{S}^m) \arrow[r] & E \otimes X \\
\mathbb{S}^n \arrow[u, "f_{\tilde{\alpha}}"'] \arrow[ur, "\alpha(x)"'] &
\end{tikzcd}$$

We will now be studying the weight 2 operations when our spectrum is the mod-2 Eilenberg Maclane spectum $H\mathbb{F}_2$. We will do this for E_n and E_∞-algebras and of course also for spectral Lie algebras.

5.2 E_n-algebras

For this discussion about E_n-algberas, we will mostly be using Tyler Lawson's exposition [Law20]. We first recall that $\mathscr{C}_n(2) \simeq S^{n-1}$ together with the antipodal action of Σ_2. Because of this we see that

$$\mathrm{Sym}^2_{\mathscr{C}_n}(\mathbb{S}^m) \simeq \left(\Sigma^\infty S^{n-1} \otimes (\mathbb{S}^m)^{\otimes 2}\right)_{h\Sigma_2} \simeq \Sigma^m \Sigma^\infty \mathbb{R}P^{m+n-1}_m$$

Here $\mathbb{R}P^n_m$ is a stunted projective space, by which we mean a projective space $\mathbb{R}P^n$ where we quotient out the $m-1$-skeleton:

$$\mathbb{R}P^n_m = \mathbb{R}P^n / \mathbb{R}P^{m-1}$$

With this we can now compute that the weight 2 homology operations on the homology of an E_n algebra are indexed by

$$\begin{aligned}\mathrm{Op}^{H\mathbb{F}_2}_{\mathscr{C}_n}(m, m+i)^{\langle 2\rangle} &= (H\mathbb{F}_2)_{m+i}\left(\left(\Sigma^\infty \mathscr{C}_n(2) \otimes (\mathbb{S}^m)^{\otimes 2}\right)\right) = \\ &= (H\mathbb{F}_2)_i\left(\Sigma^\infty \mathbb{R}P^{m+n-1}_m\right)\end{aligned}$$

so the homotopy operations in degree m of weight 2 are given by the E-homology of the space $\mathbb{R}P^{m+n-1}_m$. So we see that we get operations $Q^i \colon (H\mathbb{F}_2)_m(X) \longrightarrow (H\mathbb{F}_2)_{m+i}$ for $m \leq i \leq m+(n-1)$, and by re-indexing by $Q_i(x) = Q^{i+|x|}(x)$ we get operations of the form $Q_i \colon (H\mathbb{F}_2)_k(X) \to (H\mathbb{F}_2)_{2k+i}(X)$ for $0 \leq i \leq n-1$. The relations for composing the operations with eachother are given by Adem relations [Law20, Thm. 5.2]

$$Q_r Q_s = \sum_{t=0} \binom{t-s-1}{2t-r-s} Q_{r+2s-2t} Q_t$$

with which we construct a new $\mathbb{F}_2$-algebra using the same process as before, giving us $\mathcal{R}_n$, *the Dyer-Lashof algebra for the operad* E_n. So we see that the homology of an E_n-algebra carries the structure of an $\mathcal{R}_n$-module, given by the action of the Dyer-Lashof operations.
We now note that there are also homology operations dual to the Steenrod operations. We define the dual Steenrod operations on $(H\mathbb{F}_2)_*(X)$ in the following way. Let $x \in (H\mathbb{F}_2)_m(X)$ then $\mathrm{Sq}_d(x)$ is defined by the following diagram

$$\begin{tikzcd}
\mathbb{S}^m \arrow[r, "x"] \arrow[d, dotted, "\Sigma^{-d}"'] & H\mathbb{F}_2 \otimes X \arrow[d, dotted, "\Sigma^{-d}"] \\
\mathbb{S}^{m-d} \arrow[r] \arrow[dr, "\mathrm{Sq}_d(x)"'] & \Sigma^{-d} H\mathbb{F}_2 \otimes X \arrow[d, "\mathrm{Sq}^d \otimes \mathrm{id}"] \\
 & H\mathbb{F}_2 \otimes X
\end{tikzcd}$$

where we use that $\mathrm{Sq}^d \in [\Sigma^{-d} H\mathbb{F}_2, H\mathbb{F}_2]_{\mathrm{Sp}} \simeq H\mathbb{F}_2^d(H\mathbb{F}_2)$ by the same reasoning as our discussion in the introduction. In the same way as for the Steenrod operations, these make up the dual Steenrod algebra $\mathcal{A}_*$ which acts on the mod-2 homology of any spectrum X. The way the dual Steenrod operations and the Dyer-Lashof operations interact with each other is then given by the Nishida relations [Law20, Thm. 5.18]

$$\mathrm{Sq}_r\, Q_s = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor} \binom{d+s-r}{r-2t} Q_{s-r+2t}, \mathrm{Sq}_t \ \text{ for } s < n-1$$

In the same way, we can also construct the weight 2 binary operations. In this case we get two operations for every two degrees $(-\cdot-) \colon (H\mathbb{F}_2)_r \otimes (H\mathbb{F}_2)_s \to (H\mathbb{F}_2)_{r+s}$ and $[-,-] \colon (H\mathbb{F}_2)_r \otimes (H\mathbb{F}_2)_s \to (H\mathbb{F}_2)_{r+s+(n-1)}$. By composing the binary operations with each other we get the structure of a Poisson algebra [Law20, Thm. 5.2].

Definition 5.1 (Poisson algebra)**.** Let A be a graded $\mathbb{F}_2$-module, then it is a Poisson algebra if it carries a commutative bilinear product

$$- \cdot - \colon A_m \otimes A_n \longrightarrow A_{m+n}$$

together with a Lie bracket

$$[-,-]\colon A_m \otimes A_n \longrightarrow A_{m+n}$$

and that it is antisymmetric, i.e. commutative with $[x,x]=0$, and that it satisfies the Jacobi identity, together with the additional rule that they satisfy the Leibniz identity

$$[x, y\cdot z] = [x,y]\cdot z + y\cdot [x,z]$$

So in the case of an E_n-algebra we will see its mod-2 homology carries the structure of a Poisson algebra, where the Lie bracket is shifted with degree $n-1$, which we will call the Browder bracket. In this case we will call the structure an n-shifted Poisson algebra, which we will write simply as a Pois_n-algebra. This reflects that this can be made into an even stronger statement, in that the mod-2 homology of the $\mathscr{C}_n$-operad is precisely the Poisson operad in symmetric monoidal category $(\mathrm{Mod}_{\mathbb{F}_2}, \otimes, \mathbb{F}_2)$, for which the Pois_n-algebras are its algebras. This is discussed by Dev Sinha in [Sin06].

$$\begin{tikzcd}
E_n(k)\otimes X^{\otimes k} \arrow[r, "\xi_k"] \arrow[d, squiggly, "H\mathbb{F}_2"] & X \arrow[d, squiggly, "H\mathbb{F}_2"] \\
\mathrm{Pois}_n(k)\otimes (H\mathbb{F}_2)_*(X)^{\otimes k} \arrow[r, "(\xi_k)_*"'] & (H\mathbb{F}_2)_*(X)
\end{tikzcd}$$

We will now discuss how this structure interacts with the action of the dual Steenrod algebra on $(H\mathbb{F}_2)_*(X)$. Both for the product and the Browder bracket, these are given by Cartan formulas [Law20, Thm. 5.18]

$$\mathrm{Sq}_i(x\cdot y) = \sum_{r+s=i} \mathrm{Sq}_r(x)\cdot \mathrm{Sq}_s(y) \text{ and } \mathrm{Sq}_i([x,y]) = \sum_{r+s=i} [\mathrm{Sq}_r(x), \mathrm{Sq}_s(y)]$$

The Dyer-Lashof operations and the n-shifted Poisson algebra structure still have some interplay on the homology of a E_n-algebra [Law20, Thm. 5.2].

1. Taking the zeroth Dyer-Lashof operation is the same as squaring: $Q_0(x) = x\cdot x$.
2. Taking a Dyer-Lashof operation of a product follows the Cartan formula, in that $Q_i(x\cdot y) = \sum_{r+s=i} Q_r(x)\cdot Q_s(y)$ for $i<n-1$.
3. Taking the Lie bracket of a Dyer-Lashof operation vanishes, so $[Q_i(x), y] = 0$ for $i<n-1$.

Together with the following additional rules for the top operation Q_{n-1} [Law20, Thm. 5.2].

1. The top operation is not additive like the other operations but gives an extra Browder bracket
$$Q_{n-1}(x+y) = Q_{n-1}(x) + Q_{n-1}(y) + [x,y]$$
2. Taking the top operation on a product will again be given by a Cartan formula, bu this time with an extra Browder bracket
$$Q_{n-1}(x\cdot y) = \sum_{r+s=n-1} Q_r(x)\cdot Q_s(y) + x\cdot [x,y]\cdot y$$
3. Taking the Browder bracket of a top operation will not vanish, but instead give an adjoint identity $[x, Q_{n-1}(y)] = [y,[y,x]]$.

When will call a graded $\mathbb{F}_2$-module that is both a $\mathcal{R}_n$-module and a Pois_n-algebra such that it satisfies the previous relations, an *allowable* $\mathcal{R}_n$-Pois_n*-algebra* . The homology of an E_n-algebra will be an example of this.

We note that we have not yet given one interaction with the dual Steenrod algebra, that being for the top Dyer-Lashof operation Q_{n-1}. This is given by a usual Nishida relation, but together with an ordered Cartan formula of Browder brackets [Law20, Thm. 5.18]

$$\mathrm{Sq}_i\, Q_{n-1} = \sum_{i=0}^{\lfloor \frac{i}{2}\rfloor} \binom{d+n-1-i}{i-2t} Q_{n-1-i+2t}, \mathrm{Sq}_t + \sum_{r+s=i, r<s} [\mathrm{Sq}_r(x), \mathrm{Sq}_s(x)]$$

There is a left adjoint to the forgetful functor that takes an allowable $\mathcal{R}_n$- Pois_n-algebra to its underlying graded $\mathbb{F}_2$-module. We will call this the free $\mathcal{R}_n$- Pois_n-algebra, $\mathrm{Free}_{\mathcal{R}_n\text{-}\mathrm{Pois}_n}$.

$$\mathrm{Hom}_{\mathcal{R}_n\text{-}\mathrm{Pois}_n}(\mathrm{Free}_{\mathcal{R}_n\text{-}\mathrm{Pois}_n}(A), B) \simeq \mathrm{Hom}_{\mathrm{grMod}(\mathbb{F}_2)}(A, B)$$

We will now construct a basis for its underlying graded $\mathbb{F}_2$-module, by constructing bases for its individual structures and then combining them along the relations between them.

1. A basis for the free $\mathcal{R}_n$-algebra of a graded $\mathbb{F}_2$-module, is given by the monomials $Q_I(x)$, with $(i_1, \cdots, i_k)$ such that $i_{j-1} \leq i_j$. So for example if $x \in A_2$ is a basis element, then $Q_0(Q_3(x)) \in \mathrm{Free}_{\mathcal{R}_n}(A)_14$ is a basis element as $0 \leq 3$.
2. We can construct a basis for a free commutative algebra of a graded $\mathbb{F}_2$-module inductively. Let the basis elements of the initial $\mathbb{F}_2$-module A have weight 1. Suppose we have now constructed the basis elements of weight up to n, and ordered them, then $x \cdot y$ is a basis element of weight n, if x and y are both basis elements such that their weights sums to n and $\deg x < \deg y$ in the ordering.
3. The basis of the free shifted Lie algebra is given by the basic products in the basis elements of the initial graded $\mathbb{F}_2$-module. This is already discussed for the spectral Lie algebra case.
4. By the Cartan formula and the top Cartan formula we see that taking an operation of a product can be expressed in terms of products of operations (and a Browder bracket in the top case).
   We also see that taking a Browder bracket of an operation is either trivial or can be expressed of other Browder brackets in the top case.

In the end this gives us that the basis elements are given by products of monomials of Browder brackets, so as a couple examples of basis elements we get $[x_0, [x_0, x_1]] \cdot x_0 \in \mathrm{Free}_{\mathcal{R}_n\text{-}\mathrm{Pois}_n}(A)_{17}$ and $Q_0Q_2x_0 \cdot Q_1x_0 \in \mathrm{Free}_{\mathcal{R}_n\text{-}\mathrm{Pois}_n}(A)_{17}$.
We see that the $\mathcal{R}_n$- Pois_n-algebra structure of the free $\mathcal{R}_n$- Pois_n is regulated by the relations that define the $\mathcal{R}_n$- Pois_n-algebra structure. We will work out what this means in the spectral Lie algebra case.
Recall that we had previously constructed the free E_n-algebra of a spectrum X. With the following theorem we can express its homology in terms of the homology of the original spectrum X.

Theorem 5.1 ([Law20, Thm. 5.5]). *Let X be a spectrum. Then the mod-2 homology of the free E_n-algebra of X is precisely the free allowable $\mathcal{R}_n$- Pois_n-algebra of the mod-2 homology of X*

$$(H\mathbb{F}_2)_*(\mathrm{Free}_{E_n})(X) \simeq \mathrm{Free}_{\mathcal{R}_n\text{-}\mathrm{Pois}_n}\Big((H\mathbb{F}_2)_*(X)\Big)$$

The action of the dual Steenrod algebra is computed using the Nishida and Cartan formulas, in the same way as in the case for $\partial_* \mathrm{id}$.

Now we will study the case for E_∞-algebras in $\mathrm{Mod}_{H\mathbb{F}_2}$. Then by the discussion for the E_n case we see that its mod-2 homology carries a commutative product, together with Dyer-Lashof operations Q^i. Here we will change our indexing by $Q^i(x) = Q_{i-|x|}(x)$ so that the Adem relations are given by [Law20, Thm. 5.8]

$$Q^rQ^s \sum \binom{t-s-1}{2t-r} = Q^{s+r-t}Q^t$$

and the Nishida relations are given by [Law20, Thm. 5.18]

$$\mathrm{Sq}_r\, Q^s = \sum \binom{s-r}{r-2t} Q^{s-r+t}\, \mathrm{Sq}_t$$

The other relations are the same as for the E_n case. We note here that the Browder bracket and top Dyer-Lashof operation vanish as we have taken the colimit $n \to \infty$. This then gives the homology of an E_∞-algebra X the structure of an allowable $\mathcal{R}$- Comm-algebra.

$$\begin{tikzcd}
E^\infty(k) \otimes X^{\otimes k} \arrow[r, "\xi_k"] \arrow[d, squiggly, "H\mathbb{F}_2"] & X \arrow[d, squiggly, "H\mathbb{F}_2"'] \\
\mathrm{Comm}(k) \otimes (H\mathbb{F}_2)_*(X) \arrow[r, "(\xi_k)_*"'] & (H\mathbb{F}_2)_*(X)
\end{tikzcd}$$

And in the same way as for the E_n case, we get that the homology of a free E_n-algebra X is precisely the free allowable $\mathcal{R}$- Comm-algebra over $(H\mathbb{F}_2)_*(X)$.

Theorem 5.2 ([Law20, Thm. 5.5]). *Let X be a spectrum. Then the mod-2 homology of the free E_∞-algebra of X is precisely the free allowable $\mathcal{R}$-Comm-algebra of the mod-2 homology of X*

$$(H\mathbb{F}_2)_*(\mathrm{Free}_{E_\infty})(X) \simeq \mathrm{Free}_{\mathcal{R}\text{-}\mathrm{Comm}}\Big((H\mathbb{F}_2)_*(X)\Big).$$

5.3 Spectral Lie algebras

For the discussion about spectral Lie algebras we will mostly follow Camarenas paper [Cam16]. We recall that $\partial_2\,\mathrm{id} \simeq \mathbb{S}^{-1}$ together with the trivial action by Σ_2. Because of this, we see that

$$\mathrm{Sym}^2_{\partial_*\,\mathrm{id}}(\mathbb{S}^n) \simeq \mathbb{S}^{-1} \otimes (\mathbb{S}^n)^{\otimes 2}_{h\Sigma_2} \simeq \Sigma^{n-1}\Sigma^\infty \mathbb{R}P^\infty_n.$$

Using this we can now deduce that the weight 2 operations on the homology of a spectral Lie algebra are indexed by

$$\begin{aligned}\mathrm{Op}^{H\mathbb{F}_2}_{\partial_*\,\mathrm{id}}(m, m+i)^{\langle 2\rangle} &= (H\mathbb{F}_2)_{m+i}\Big(\big(\partial_2\,\mathrm{id} \otimes (S^m)^{\otimes 2}\big)_{h\Sigma_2}\Big) = \\ &= (H\mathbb{F}_2)_i\big(\Sigma^{-1}(\Sigma^\infty \mathbb{R}P^\infty_m)\big).\end{aligned}$$

so we find that we have operations $\bar{Q}^i \colon (H\mathbb{F}_2)(X) \to (H\mathbb{F}_2)_{m+i-1}(X)$ for $m \leq i < \infty$.

Composing these Dyer-Lashof-like operations are again governed by Adem relations for $j < i \leq 2j$ [Def. 5.6][Cam16]. These Adem relations are the same as in the E_∞-case as by a different construction the Dyer-Lashof-like operations can be built from the usual external Dyer-Lashof operations [**?** , Def. 5.4]camarena2016mod

$$\bar{Q}^i\bar{Q}^j = \sum_{t=0}^{i-j-1} \binom{2j-i+1+2t}{t} \bar{Q}^{2j+1+t}\bar{Q}^{i-j-1-t}$$

So we get a new $\mathbb{F}_2$-algebra, that being the $\mathbb{F}_2$-module freely generated over the monomials of Dyer-Lashof-like operations $\bar{Q}^I$ together with the composition as multiplication. When we take the quotient with respect to these new Adem relations, we get $\bar{\mathcal{R}}$, the *Dyer-Lashof algebra for the operad* $\partial_*(\mathrm{id})$. This makes the homology of a spectral Lie algebra into a $\bar{\mathcal{R}}$-module.
Again the homology of a spectral Lie algebra also carries the structure of a module over the dual Steenrod algebra $\mathcal{A}_*$. Due to their construction, the way the dual Steenrod operations compose with the Dyer-Lashof-like operations is governed by the same Nishida relations as we had for the usual Dyer-Lashof operations [Beh12, Sec. 1.4]

$$\mathrm{Sq}_r\,\bar{Q}^s = \sum_{t=0}^{\lfloor \frac{r}{2}\rfloor} \binom{s-r}{r-2t} \bar{Q}^{s-r+t}\,\mathrm{Sq}_t$$

We can now construct the weight 2 binary operations in the same way as we did for the unary operations, however we will take a different approach this time and construct them by using the $\partial_*\,\mathrm{id}$-algebra structure directly. So let X is a spectral Lie algebra, then we get a bracket on its homology given by $(\Sigma\xi_2)$ where $\xi_2 \colon \partial_2\,\mathrm{id} \otimes X^{\otimes 2} \to X$ is a structure map. Here we use the suspension isomorphism for the last identification.

$$\begin{tikzcd}
\mathbb{S}^{-1} \otimes X^{\otimes 2} \arrow[r, "\xi_2"] \arrow[d, dotted, "\Sigma"'] & X \arrow[d, dotted, "\Sigma"] \\
X \otimes X \arrow[r, "\Sigma\xi_2"] \arrow[d, dotted, "H\mathbb{F}_2"'] & \Sigma X \arrow[d, dotted, "H\mathbb{F}_2"] \\
(H\mathbb{F}_2)_i(X) \otimes (H\mathbb{F}_2)_j(X) \arrow[r, "(\Sigma\xi_2)_*"'] & (H\mathbb{F}_2)_{i+j-1}(X)
\end{tikzcd}$$

It can now be shown that this bracket is indeed a shifted Lie bracket, however the proof of this is rather technical. We refer to the proof of Proposition 5.2 of [Cam16].

Definition 5.2 (shifted Lie algebra)**.** Let A be a graded $\mathbb{F}_2$-module, then it is a shifted Lie ($s\,\mathrm{Lie}$) algebra if it carries a bilinear bracket of degree -1

$$[-,-]\colon A_m \otimes A_n \longrightarrow A_{m+n-1}$$

such that

1. It is commutative, in that for $x \in A_m$ and $y \in A_n$ we have that $[x,y] = [y,x]$.
2. It satisfies the Jacobi identity, in that for $x \in L_m$, $y \in L_n$ and $z \in L_k$ we get $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.

This makes the homology of a spectral Lie algebra into $s\,\mathrm{Lie}$-algebra, justifying its name. We again note that this result can be made stronger, in that the mod-2 homology of $\partial_* \mathrm{id}$ gives the shifted Lie operad in $(\mathrm{Mod}_{\mathbb{F}_2}, \otimes, \mathbb{F}_2)$, whose algebras then carry a shifted Lie bracket. For a proof of this we will refer to [Chi05, Ex. 9.50] which uses $\partial_* \mathrm{id} \simeq \mathrm{coBar}(\mathrm{Comm})$ from before which gives that the homology of $\partial_* \mathrm{id}$ is given by the Koszul dual of the homology of Comm, which is then given by the Lie operad

$$(H\mathbb{F}_2)_k(\partial_n \mathrm{id}) \simeq \begin{cases} \mathrm{Lie}(n) \otimes \mathrm{sgn}_n & \text{for } k = 1-n \\ * & \text{else} \end{cases}$$

where sgn_n is the sign representation of Σ_n.

$$\begin{tikzcd} \partial_n(\mathrm{id}) \otimes X^{\otimes n} \arrow[r, "\xi_n"] \arrow[d, squiggly, "H\mathbb{F}_2"] & X \arrow[d, squiggly, "H\mathbb{F}_2"'] \\ s\,\mathrm{Lie} \otimes (H\mathbb{F}_2)_*(X)^{\otimes n} \arrow[r, "(\xi_n)_*"'] & (H\mathbb{F}_2)_*(X) \end{tikzcd}$$

We would still like to know how this structure interacts with the action of the dual Steenrod operations. These are in fact again given by a Cartan formula [Zha22]

$$\mathrm{Sq}_i([x,y]) = \sum_{r+s=i} [\mathrm{Sq}_r(x), \mathrm{Sq}_s(y)].$$

So the homology of spectral Lie algebras now has two structures: it carries a shifted Lie bracket and it is a module over the Dyer-Lashof algebra for the operad $\partial_* \mathrm{id}$. These two structures are not independent of each other, on the homology of a spectral Lie algebra X, they satisfy the following relations [Cam16, Thm. 6.3].

1. Let $x \in H_i(X)$, then for $j < i$ then we have that $\bar{Q}^j(x) = 0$. (allowable)
2. For $x \in H_i(X)$ we have that $\bar{Q}^i(x) = [x,x]$.
3. Taking the bracket of an operation is trivial in that $[\bar{Q}^i(x), y] = 0$.

We will call a graded $\mathbb{F}_2$-module that is both a $\bar{\mathcal{R}}$-module and a shifted Lie algebra such that the previous relations are satisfied, an *allowable* $\bar{\mathcal{R}}$- sLie-*algebra*. And so we see that the homology of a spectral Lie algebra carries this structure.

We have a left adjoint for the forgetful functor from allowable $\bar{\mathcal{R}}$- sLie algebras to its underlying graded $\mathbb{F}_2$-module, giving us the free allowable $\bar{\mathcal{R}}$- sLie algebra of a graded $\mathbb{F}_2$-module.

$$\mathrm{Hom}_{\bar{\mathcal{R}}\text{- sLie}}(\mathrm{Free}_{\bar{\mathcal{R}}\text{- sLie}}(A), B) \simeq \mathrm{Hom}_{\mathrm{grMod}(\mathbb{F}_2)}(A, B)$$

Here $\mathrm{Free}_{\bar{\mathcal{R}}\text{- sLie}}(A)$ we will first choose a basis for A. Then we freely generate Dyer-Lashof operations and brackets over it and quotient out by the allowable $\bar{\mathcal{R}}$- sLie relations. As an example suppose we would have two basis elements $x \in A_2$ and $y \in A_3$, then an element in the new graded $\mathbb{F}_2$-module would be $\bar{Q}^6\bar{Q}^2(x) \in \mathrm{Free}_{\bar{\mathcal{R}}\text{- sLie}}(A)_8$ and $\bar{Q}^8([y,[x,y]]) \in \mathrm{Free}_{\bar{\mathcal{R}}\text{- sLie}}(A)_{13}$.

In order to be able to work with it better, we would like to construct a basis for $\mathrm{Free}_{\bar{\mathcal{R}}\text{- sLie}}(A)$. We first recall that a basis for $\bar{\mathcal{R}}$ was given by the *completely unadmissible monomials*, i.e. the monomials $\bar{Q}^I$ with $I = (i_1, \cdots, i_n)$ and $i_k \geq 2i_{k-1} + 1$.
As we have the relation that $[x,x] = \bar{Q}^{|x|}(x)$, we would like to find a basis for a free Lie algebra of an $\mathbb{F}_2$-module, such that $[x,x] = 0$ (totally isotropic). It was shown by Hall that a basis of this is given by the

basic products. Here we first start with an ordered basis $\{x_i\}_{i\in I}$ for the initial graded $\mathbb{F}_2$-algebra A, and we will call them the basic products of weight 1. We will now define the basic products inductively. Suppose we already know all the basic products of weight less than n, and have given them an ordering. Then we define the new basic products of weight n, to be the brackets $[\alpha_1, \alpha_2]$ with α_1 and α_2 are basic products where we require the following

1. Their weights should add up to n, i.e. $w(\alpha_1) + w(\alpha_2) = n$.
2. With respect to the ordering of the basic products, α_1 should come before α_2, i.e. $|\alpha_1| < |\alpha_2|$.
3. If α_2 was not of weight 1, we could write it as a bracket of two other basic products $[\beta_1, \beta_2]$. In this case we also require that β_1 either comes before α_1 in the ordering or is the same. In other words that $|\beta_1| \leq |\alpha_1|$.

As an example, if $x_0, x_1 \in A_*$ are basis elements then $\Big[[x_0, x_1], \big[x_1, [x_0, x_1]\big]\Big]$ is a basic product of weight 5. With this we are now able to construct a basis for $\mathrm{Free}_{\bar{\mathcal{R}}\text{-}\,\mathrm{sLie}}(A)$. This is given by the $\bar{Q}^I(\alpha)$ where I is completely undadmissible and α is a basis product. As an example, if we have the following basis elements $x_0 \in A_2$ and $x_1 \in A_3$ in the graded $\mathbb{F}_2$ module A, then we have the basis element $\bar{Q}^{16}\bar{Q}^7[x_1, [x_0, x_1]] \in \mathrm{Free}_{\bar{\mathcal{R}}\text{-}\,\mathrm{sLie}}(A)_{27}$.

We note that $\mathrm{Free}_{\bar{\mathcal{R}}\text{-}\,\mathrm{sLie}}(A)$ has indeed the structure of an allowable $\bar{\mathcal{R}}$-sLie-algebra, governed by the relations that described the $\bar{\mathcal{R}}$-sLie-algebra structure. Suppose we are again in our previous example and we have the element $\bar{Q}^6(x) \in \mathrm{Free}_{\mathrm{sLie}}(A)_7$, then using the Adem relations we see that the action by $\bar{Q}^9$ is given by

$$\bar{Q}^9\big(\bar{Q}^6(x)\big) = \bar{Q}^9\bar{Q}^6(x) = \bar{Q}^{13}Q^2(x) \in \mathrm{Free}_{\bar{\mathcal{R}}\text{-}\,\mathrm{sLie}}(A)_{15}$$

and in the case of the Lie bracket, we get for example for the element $[x, y] \in \mathrm{Free}_{\bar{\mathcal{R}}\text{-}\,\mathrm{sLie}}(A)_4$ and $[x, [x, y]] \in \mathrm{Free}_{\bar{\mathcal{R}}\text{-}\,\mathrm{sLie}}(A)_5$ the following bracket

$$\Big[\big([x, y]\big), \big([x, [x, y]]\big)\Big] = \big[[x, y], [x, [x, y]]\big] \in \mathrm{Free}_{\bar{\mathcal{R}}\text{-}\,\mathrm{sLie}}(A)_8$$

We recall that we had previously constructed the free spectral Lie algebra for a spectrum X, and that (in the case that X comes from a space) these precisely make up the fibers in the Goodwillie tower of the identity of X. We would like to be able to describe the mod-2 homology of a free spectral Lie algebra in terms of the homology of its underlying spectrum, and as it turns out we can.

Theorem 5.3 ([Zha21, Thm. 2.6])**.** *Let X be a spectrum. Then the mod-2 homology of the free spectral Lie algebra of X is precisely the free $\bar{\mathcal{R}}$-*sLie*-algebra of the mod-2 homology of X*

$$(H\mathbb{F}_2)_*(\mathrm{Free}_{\partial_*\,\mathrm{id}})(X) \simeq \mathrm{Free}_{\bar{\mathcal{R}}\text{-}\,\mathrm{sLie}}\Big((H\mathbb{F}_2)_*(X)\Big)$$

The main problem we are left with, is to determine the action of the dual Steenrod algebra on the homology of the free spectral Lie algebra of a spectrum X. Again we see that this is given by us by how the action of the dual Steenrod algebra relates to the $\bar{\mathcal{R}}$-sLie-algebra structure, which was given by the Nishida relations and the Cartan formula.

As an example we will look at the homology of the Moore spectrum $M_2 = S^2/2$ The basis consist of two elements $x_0 \in H_2(M_2)$ and $x_1 \in H_3(M_2)$, together with the action of the dual Steenrod algebra given by a single non-trivial relation $\mathrm{Sq}_1(x_1) = x_0$. We will look at the element $\bar{Q}^8([x_0, x_1])$ and compute the action by Sq_2, we get

$$\begin{aligned}\mathrm{Sq}_2(\bar{Q}^8[x_0, x_1]) = \bar{Q}^6\,\mathrm{Sq}_0([x_0, x_1]) + \bar{Q}^7\,\mathrm{Sq}_1([x_0, x_1]) &= \\ \bar{Q}^6([x_0, x_1]) + \bar{Q}^7\Big([\mathrm{Sq}_1(x_0), \mathrm{Sq}_0(x_1)] + [\mathrm{Sq}_0(x_0), \mathrm{Sq}_1(x_1)]\Big) &= \\ \bar{Q}^7(\bar{Q}^2(x_0)) + \bar{Q}^6([x_0, x_1])&\end{aligned}$$

Where for the first equality, we used the Nishida relations, for the second equality we used the Cartan formula and that $\mathrm{Sq}_0 = \mathrm{id}$ and for the last equality we use the $\mathcal{A}_*$-structure on M_2 together with $[x_0, x_0] = \bar{Q}^2(x_0)$ to write it as a sum of basis elements.

6 Implementation and documentation

We recall the main goal of this thesis, which was to be able to get a useful description of the homology of the free spectral Lie algebra of a spectrum X. We saw that we were in luck, this was precisly the free allowable $\bar{\mathcal{R}}$- sLie-algebra of the homology of the original spectrum. We could determine the action of the dual Steenrod algebra from its action on the homology of the original spectrum X. However, this turned out to be a rather long computation, especially if we were to it not for a single operation, but the whole action on a range of the homology of $\mathrm{Free}_{\partial_* \mathrm{id}}(X)$.
The main goal of this thesis is to be able to determine the structure of the dual Steenrod algebra using machine computation. In the case of the cohomology of a free E_∞-algebra such a program was already built by Robert R. Bruner in 2011 for the MAGMA language [Bru14]. The code for this can be found here. Using this as a starting point, we have written an implementation in both javascript and python for the ∂_* id operad and the E_n operad, together with re-implementation of the E_∞-operad.

6.1 An algorithm for the homology of a free spectral lie algebra

There are currently two versions of the code out there. The first version was written in python, which was then made into a desktop application. As showing the data graphically was hard to accomplish in the desktop application, a small website was already written to accompany for that. However, as the desktop application would only work on Windows, the choice was made to rewrite the code into javascript, so that all of the computation could be done in the website. This website can be found at https://dyer-lashof-machine.netlify.app/, the javascript code can be found at https://github.com/Gerb-24/dyer-lashof-machine and the python code at https://github.com/Gerb-24/Dyer-Lashof.
We will be discussing the javascript code in this thesis, as it will probably be the version, however there are some parts in the code that do not have a use in the computation, but are used later on to be able to give the graphical representation.

6.2 Initializing the input data

To be able to compute the mod 2 homology of the free spectral lie algebra of a spectrum X, we will need to be able to give its homology as input, as a module over the dual Steenrod algebra. To do this we will make use of the module definition format that is used by Bruner for the Ext program. In our case we will not be working with the cohomology as a module over the Steenrod algebra however, so our definition will be dual to how it used by Bruner. To define this we will start with an example module, and compare it to how it is written in the module defintion format. Suppose we have a spectrum X such that $H_*(X)$ has three generators, x_2 in degree 5 and x_0 and x_1 in degree 2, together with a

$$\mathrm{Sq}_3(x_2) = x_0 + x_1$$

Then we would write this module in the module definition format as

```
3

2 2 5

2 3 2 0 1
```

So here the first row indicates the amount of generators the module has, the third row tells us in which degrees these generators live, and for the fifth row and onwards we will define the action of the dual steenrod algebra. Here the first index is the the index of the generator in which we have a square, the second indicates the power of the square, the third index tells us the amount of generators in the summand, and afterwards we list the indices of the generators that appear in the summand. So in this case the last line reads as: On x_2 we have a non-trivial action by Sq_3, where $\mathrm{Sq}_3(x_2)$ is the sum of two generators that being x_0 and x_1.
The javascript code now first translates the module definition format into javascript objects that it can work with. In the function `logText` we get the relevant information out of the DOM.

1. First we will need to know the operad over which we will work. Currently this consists of three different operads, that being the spectral Lie operad Lie, E_∞ and E_n. In the case that we will be working over E_n we will also have the value n as part of its inputs.

2. We will require the maximal weight and the maximal dimension for which we want to generate the monomials. These will be stored in the variables `maxWeight` and `maxDim`.

3. The last part will be the input module in the module definition format. This information is then divided into two arrays, `baseDegrees` that stores the degree of the generator at its corresponding index, and `baseOps` which will contain the same information as the bottom section of the module definition format. For the example module from before this array will looks as follows

```
[
    {},
    {},
    {
        2:[0,1],
    },
]
```

6.3 Outline of the computational process

We have now constructed the input data with which we want to do the computation. To do this we will make use of a different function, depending on our choice of operad. These functions are `E_inf_operad`, `Lie_operad` and `E_n_operad`, where the last function has one additional parameter, that being n. As the process in these functions is rather similar, we will in this discussion focus on `Lie_operad`. This function is defined in the file `Lie.js`.
This process will consist of three steps

1. First we will construct the basis elements of $(H\mathbb{F}_2)_*\big(\mathrm{Free}_{\partial_* \mathrm{id}}(X)\big)$. We recall that these were given by completely unadmissible monomials of basic products.
2. When the basis is constructed we will determine the action of the dual Steenrod algebra. We let a dual Steenrod operation act on an element in the basis, and using the Nishida and Cartan rules we factor it all the way through until we reach the basis elements of $(H\mathbb{F}_2)_*(X)$. Lastly, we use then use the action of the dual action of the dual Steenrod algebra on $(H\mathbb{F}_2)_*(X)$ to get the new element in $(H\mathbb{F}_2)_*\big(\mathrm{Free}_{\partial_* \mathrm{id}}(X)\big)$.
3. All that is left is to write this new element as a sum of basis elements of $H_*\big(\mathrm{Free}_{\partial_* \mathrm{id}}(X)\big)$. To do this we will use the relations for an allowable $\bar{\mathcal{R}}$- sLie-algebra structure.

6.3.1 Constructing the basis

We will first construct the basis. As earlier discussed these are given by taking completely unadmissible monomials of basic products with the generating set being the generators of our original module. To construct these we will first generate the basic products with letters in the set of generators in our original module. This is done recursively by the function `productBasisFunc`. Afterwards, we will build up the completely unadmissible monomials, by adding operations while these are still allowed for the corresponding maximal weight and maximal dimension. This is done by `operationBasisFunc` with the following lines.

```
let operationsList;
  if (node instanceof Operation) {
    operationsList = range( 2*node.power + 1, maxDim - node.degree ).map( power  =>
      new Operation( power, node)
    );
  } else {
    operationsList = range( node.degree, maxDim - node.degree ).map( power =>
      new Operation( power, node )
    );
```

```
    }
    newOperations.push(...operationsList);
```

To do this we first construct three new classes `Generator`, `Product` and `Operation`. These will keep track of the corresponding degrees and weights. For example a newly constructed basis element given by $Q^4([x_0, x_1])$ with $|x_0| = 2$ and $|x_1| = 3$ then this corresponds to the following object

```
new Operation(
    4,
    new Product(
        new Generator( 0, 2 ),
        new Generator( 1, 3 ),
    )
)
```

The way we have set this up is to think of the three different classes, as being different nodes that make up a tree (the basis element), where the leaves are given by the `Generator` objects. This is a generalization of Bruners Dyer-Lashof code. For example in the case of the E_∞-operad, the new basis elements are given by products of operations. With this generalization, we can reuse the same structure to be able for this.

6.3.2 The Dual Steenrod action

After all the new basis elements are constructed, we will need to determine the action of the dual Steenrod algebra. This will be done in function `monomialsToData`, which will take our newly generated basis as input. As an example we will look at the computation

$$\mathrm{Sq}_2(\bar{Q}^8[x_0, x_1]) = \bar{Q}^7(\bar{Q}^2(x_0)) + \bar{Q}^6([x_0, x_1])$$

So first we will let Sq_2 act on the monomial $\bar{Q}^8[x_0, x_1]$. This is regulated in the function appropriately called `Steenrod` which has both the power and node as input parameters. As the first part of the monomial is an operation, we will be using the Nishida relations

$$\mathrm{Sq}_2 \bar{Q}^8 = \binom{8-2}{2-0} \bar{Q}^{8-2+0} \mathrm{Sq}_0 + \binom{8-2}{2-2} \bar{Q}^{8-2+1} \mathrm{Sq}_1 = \bar{Q}^6 \mathrm{Sq}_0 + \bar{Q}^7 \mathrm{Sq}_1$$

We see that this takes place in the `Steenrod` function as

```
  if (node instanceof Operation) {
    // nishida relations
    let elt_list = nishida(i, node.power).map(([a, b]) =>
      OperationFunc(a, Steenrod(b, node.next))
    );
    return eltSum(elt_list);
  }
```

For the next part we will need to compute

$$\mathrm{Sq}_1([x_0, x_1]) = [\mathrm{Sq}_1(x_0), \mathrm{Sq}_0(x_1)] + [\mathrm{Sq}_0(x_0), \mathrm{Sq}_1(x_1)]$$

This computation then happens in `Steenrod` at

```
  if (node instanceof Product) {
    // cartan formula
    let elt_list = cartan(i).map(([a, b]) =>
      ProductFunc(Steenrod(a, node.next0), Steenrod(b, node.next1))
    );
    return eltSum(elt_list);
  }
```

Now we have to determine what the squares on the original generators are. In this case we get

$$\mathrm{Sq}_0(x_0) = x_0, \mathrm{Sq}_1(x_0) = 0 \text{ and } \mathrm{Sq}_1(x_1) = x_0$$

We recall that this data was stored in the array `baseOps`, so that in `Steenrod` it is regulated by the following function

```
if (node instanceof Generator) {
  if (i in baseOps[node.index]) {
    let elt = new Element(
      baseOps[node.index][i].map((j) => new Generator(j, baseDegs[j]))
    );
    return elt;
  } else {
    return new Element([]);
  }
}
```

In the end we get the following element

$$\bar{Q}^7([x_0, x_0]) + \bar{Q}^7([0, x_1]) + \bar{Q}^6([x_0, x_1])$$

which is not yet written in terms of basis elements. To do this, we will be working from the bottom to the top again. In the code this is represented in the following way

```
eltSum([
    OperationFunc(
        7,
        eltSum([
            ProductFunc(
                new Element( [new Generator( 0, 2 )] ),
                new Element( [Generator( 0, 2 )] )
            )
        ])
    ),
    OperationFunc(
        7,
        eltSum([
            ProductFunc(
                new Element( [] ),
                new Element( [Generator( 1, 3 )] )
            )
        ])
    ),
    OperationFunc(
        6,
        eltSum([
            new Element([
                Product(
                    Generator( 0, 2 ),
                    Generator( 1, 3 )
                )
            ])
        ])
    )
])
```

This will be regulated by the functions and `OperationFunc` and `ProductFunc`: let x and y both be a sum of basis elements, then the first writes $\bar{Q}^i(x)$ as a sum of basis elements and the second does the same for $[x, y]$. We will define a new class `Element` for working with sums of basis elements. The function `eltSum` takes care of the addition of two elements, which is mod-2.
As an example, for the term $\bar{Q}^7([x_0, x_0])$, we first see that $[x_0, x_0] = \bar{Q}^2(x_0)$. We will now explain where this happens in the code. First, to be precise, we note that we first have to deal with the `Element` objects of which the `Generator` objects are part of.

```
ProductFunc(
    Element( [Generator( 0, 2 )] ),
    Element( [Generator( 0, 2 )] )
)
```

but as the product is bilinear, we just get that this is equal to

```
eltSum([
    ProductFunc(
        Generator( 0, 2 ),
        Generator( 0, 2 )
    )
])
```

where we then check in `ProductFunc` if the product consists of two `Generator` objects, and if they are in fact equal to each other

```
if (
      [Product, Generator].includes(node0.constructor) &&
      [Product, Generator].includes(node1.constructor)
    ) {
      if (node0.isEqual(node1)) {
        let elt = new Element([
          new Operation(node0.degree, node0),
        ]);
        return elt;
      }
```

so that we in the end get the new element `Element([ Operation(2, Generator(0,2))])`, which translates to $[x_0, x_0] = \bar{Q}^2(x_0)$, which is precisely what we wanted.
So we can now look one level higher in our tree structure, at $\bar{Q}^7(\bar{Q}^2(x_0)) = \bar{Q}^7\bar{Q}^2(x_0)$ as $(7, 2)$ is completely unadmissible. This will be handled by `OperationFunc`. Here we first note that this calculation in code will be given by

```
OperationFunc(
    7,
    Element([ Operation(2, Generator(0,2))])
)
```

however as the Dyer-Lashof operations are additive we see that this is the same as

```
Element([
    OperationFunc(
        7,
        Operation(2, Generator(0,2))
    )
])
```

Then we will check in `OperationFunc` if the top node in tree structure has the class of an operation, and if this makes it completely unadmissible

```
    if (node instanceof Operation) {
          if (i > 2 * node.power) {
            let elt = new Element([
              new Operation(i, node),
            ]);
            return elt;
          }
```

This then gives us that the above element is equal to `Element([ Operation(7,Operation(2, Generator(0,2)))])`. To summarize, this process describes the calculation that $\bar{Q}^7([x_0, x_0]) = \bar{Q}^7\bar{Q}^2(x_0)$. In the same way we find that $\bar{Q}^7([0, x_1]) = 0$ and that $\bar{Q}^6([x_0, x_1])$ is already a basis element. We then sum then together, keeping in mind that this is mod-2, to finally get

$$\mathrm{Sq}_2(\bar{Q}^8[x_0, x_1]) = \bar{Q}^7(\bar{Q}^2(x_0)) + \bar{Q}^6([x_0, x_1])$$

In the function `MonomialsToData` we do this computation for every basis element x for every dual Steenrod operation such that its power is smaller than the difference between degree of x and the bottom degree (otherwise the action would necessarily be trivial).

6.3.3 The differences with Bruner's implementation

We would like to highlight the differences between our implementation and that of Bruner's. The main difference is that Bruner's approach relies on the way the basis elements were built up for the E_∞-case. As an example if we were to have the monomial $Q^4(Q^2(x_0)) * Q^4(x_0)$ with x_0 a basis element of degree 2, then this would be represented in Bruner's code as `[[4,2,1],[4,1]]` where we use 1 for x_0 as MAGMA 1-indexes. In our implementation, this monomial would then be represented by

```
Product(
    Operation(
        4,
        Operation(
            2,
            Generator( 0, 2 ),
            )
        )
    ),
    Operation(
        4,
        Generator( 0, 2 ),
    ),
).
```

giving it the structure of a tree, with different kinds of nodes (`Generator` `Operation` and `Product`) that tell us how to interact with them. For one this makes the code a lot more readable. Besides that, to compute the action of the dual Steenrod algebra Bruner's code inherently makes use of the fact that the basis elements are build out of products of operations of generators. If we were to write an implementation for the E_n or ∂_* id we would not be able to reuse our functions and write most from scratch again. With our approach most of the work has already been done, the only thing we need to do is to build in the relations and spell out how we construct the basis elements.

6.4 Generalizations and future work

We have now gone through the whole process of computing the action of the dual Steenrod algebra for the homology of the free spectral Lie algebra of a spectrum X. As we have already mentioned there are also two other versions for the E_∞ and the E_n case. The main question is now if we could set this up in such a way, that we only need to give the relations between the corresponding Dyer-Lashof operations and the algebraic

structure given by the action of the homology of the corresponding operad.
In the python implementation there were two attempts to get closer to this

1. In the file `main.py` we made two functions called `UnaryNode` and `BinaryNode`. Both take a function that computes the degree as input and return a new Operation and Product repsectively. For example for the ∂_* id case we would get the following types of nodes

```
NodeTypes = {
    'Operation':    create_unary_node(
        lambda x,y: x+y-1,                      # degree_func
        lambda x,y: f'Q^{x}({y})'               # output_tex_func
        ),
    'Product' :     create_binary_node(
        lambda x,y: x+y-1,                      # degree_func
        lambda x,y: f'[{x},{y}]'                # output_tex_func
    )
}
```

   However one of the main problems turned out to be the construction of the basis elements.
2. An attempt to bypass this problem can be found in `test.py`. Here the idea was to first freely take products over a set of generators, and then to use the relations to construct quotient classes. While doing this we are taking representatives of the classes to get a basis. Computing the action of the dual Steenrod algebra on a basis element would be a lot easier, as we would have done all the computational work beforehand. The problem with this approach was that constructing quotient classes takes too much memory and computing power to be an effective approach.

So probably the way to go would be to be able to give the relations between the different unary and binary operations, and have a tool that determines a basis from these relations together with a method to decompose any element into a sum basis elements. As an example, suppose we give the structure of a totally isotropic Lie algebra. Then we want it to construct the Hall basis, together with the method that gets any element into the Hall basis again.
We note that this general setup might very well be useful for other mathematical problems. The general problem this might solve is to compute something over a free construction, where the computation is combinatorial in nature.

7 Computing homotopy groups using the Goodwillie tower

So now that we have a tool to compute the homology of free spectral Lie algebras together with the action of the dual Steenrod algebra on it, we will use it to study the Goodwillie tower of the identity for in some sense the easiest spaces we can, that being the spheres. These will then give us spectral sequences from which we can determine the 2-local unstable homotopy groups of spheres. For our discussion we will rely on the work by Mark Behrens in [Beh12], where the Goodwillie spectral sequences of S^n are studied together with its interaction with the EHP spectral sequence. We will discuss how these computations start from mod-2 homology of the free spectral Lie algebra of S^1, and give an outline of the interaction of the GSS with the EHPSS.

7.1 The EHP spectral sequence

It is a theorem by James that there is a splitting of the suspension of the loop space of a suspension into the suspension of a wedge of smash products

$$\Sigma\Omega\Sigma X \xrightarrow{\simeq} \Sigma \bigvee_{k\geq 1} X^{\wedge k}$$

This then gives us a map called the *James-Hopf map* $H_k\colon \Omega\Sigma X \longrightarrow \Omega\Sigma X^{\wedge k}$ which is the adjoint of the suspension of the projection towards the k-th summand in the wedge sum.

$$\Sigma\Omega\Sigma X \xrightarrow{\simeq} \Sigma \bigvee_{k\geq 1} X^{\wedge k} \xrightarrow{\Sigma(\mathrm{pr}_k)} \Sigma X^{\wedge k}$$

For the case that $k = 2$ we will write this as $H\colon \Omega\Sigma X \longrightarrow \Omega\Sigma X^{\wedge 2}$. It is now a theorem by James [Jam55], that after localizing at 2, we get a fiber sequence when taking X to be a sphere

$$S^n \xrightarrow{E} \Omega\Sigma S^n \xrightarrow{H} \Omega\Sigma S^{2n}$$

Here the map $E\colon S^n \longrightarrow \Omega\Sigma S^n$ is the unit of the loop-suspension adjunction, which is also called the suspension map. This fiber sequence then gives us a long exact sequence on homotopy groups

$$\begin{tikzcd}
 & & & \pi_{n-1}(\Omega^2\Sigma S^{2n}) \arrow[dr] & & \\
\cdots \arrow[r] & \pi_n(S^n) \arrow[r, "E_*"] & \pi_n(\Omega\Sigma S^n) \arrow[r, "H_*"] & \pi_n(\Omega\Sigma S^{2n}) \arrow[u, "\simeq"] \arrow[r, "P_*"] & \pi_{n-1}(S^n) \arrow[r] & \cdots
\end{tikzcd}$$

where we note that the connecting map $P\colon \Omega^2\Sigma S^{2n} \longrightarrow S^n$ is closely related to the Whitehead bracket, as on π_{2n-1} it is precisely the map

$$\begin{aligned} \pi_{2n-1}(\Omega^2\Sigma S^{2n}) \simeq \pi_{2n+1}(S^{2n+1}) &\longrightarrow \pi_{2n-1}(S^n) \\ \iota_{2n+1} &\longmapsto [\iota_n, \iota_n] \end{aligned}$$

This long exact EHP sequence was first used by Hirosi Toda in [Tod63] to compute the 2-primary part of the unstable homotopy groups of spheres (as well as the stable homotopy groups) up to the 19 stem, i.e up to $\pi_{n+19}(S^n)$. For that reason this range is now called the Toda range.
We can systematize this approach by constructing a spectral sequence. Here we can first chain the EHP sequences together for different S^n to create a filtration for $\Omega^\infty\Sigma^\infty(S^0)$, the stabilization of S^0.

$$\Omega^1 S^1 \xrightarrow{E} \Omega^2 S^2 \xrightarrow{E} \cdots \longrightarrow \Omega^\infty\Sigma^\infty(S^0)$$

By taking the long exact sequences on homotopy groups for all the fiber sequences, we see that they sew together which gives us the EHP-spectral sequence.

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\pi_{i+n+2}S^{2n+1} \arrow[r, "P"] & \pi_{i+n}S^n \arrow[r, "H"] \arrow[d, "E"] & \pi_{i+n}S^{2n-1} \arrow[r, "P"] & \pi_{(i-1)+(n-1)}S^{n-1} \arrow[r, "H"] \arrow[d] & \pi_{(i-1)+(n-1)}S^{2n-3} \\
\pi_{i+n+3}S^{2n+3} \arrow[r, "P"] & \pi_{i+(n+1)}S^{n+1} \arrow[r, "H"] \arrow[d] & \pi_{i+(n+1)}S^{2n+1} \arrow[r, "P"'] \arrow[urr, dotted, "d_2"] \arrow[rr, dotted, bend right, "d_1"'] & \pi_{(i-1)+n}S^n \arrow[r, "H"] \arrow[d] & \pi_{(i-1)+n}S^{2n-1} \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & \pi_i^s & & \pi_{i-1}^s &
\end{tikzcd}$$

Here we start with the unstable homotopy groups of odd-dimensional spheres, and end up with the stable homotopy groups of spheres.

$$E^1_{i,n} = \pi_{i+n+1}S^{2n+1} \Longrightarrow \pi_i^s$$

As we can read off of the unravelled exact couple, we see that the differentials can be thought of as the composition $d_r = H \circ E^{-(r-1)} \circ P$ so we see that to compute the differentials, we need to understand the P map. As this was closely related to the Whitehead bracket, we see that doing computations in the EHP spectral sequence will be a difficult process.
We still want to note something about what detection means in the EHP spectral sequence. Suppose we have an element $x \in \pi_{t+n}S^{2n-1}$ that survives to the E_∞-page and detects the stable element $\alpha \in \pi_t^s$. This would mean that α is born on S^n, in that we can desuspend α to an unstable element $\tilde{\alpha} \in \pi_{t+n}S^n$ such that $H(\tilde{\alpha}) = x$.

7.2 The Goodwillie spectral sequence and the Atiyah-Hirzebruch spectral sequence

From Goodwillie calculus we get another tool to compute the unstable homotopy groups of spheres, that being the Goodwillie tower of the identity for S^n. Here it was proven by Arone-Mahowald in [AM99, Thm. 3.13] that the fibers $\mathbb{D}_k$ are trivial when k is not a power of 2, so for our tower we will only look at the powers of 2 as only these give interesting homotopical information about the construction of the homotopy groups of S^n.

$$\begin{tikzcd}
 & \pi_{i+1}S^n \arrow[d] & & \pi_i S^n \arrow[d] & \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\pi_{i+1}(\mathbb{D}_{2^k}(S^n)) \arrow[r] & \pi_{i+1}(P_{2^k}(S^n)) \arrow[r] \arrow[d] & \pi_i(\mathbb{D}_{2^{k+1}}(S^n)) \arrow[r] & \pi_i(P_{2^{k+1}}(S^n)) \arrow[r] \arrow[d] & \pi_{i-1}(\mathbb{D}_{2^{k+2}}(S^n)) \\
\pi_{i+1}(\mathbb{D}_{2^{k-1}}(S^n)) \arrow[r] & \pi_{i+1}(P_{2^{k-1}}(S^n)) \arrow[r] \arrow[d] & \pi_i(\mathbb{D}_{2^k}(S^n)) \arrow[r] \arrow[rru, dotted, "d_2"] \arrow[rr, dotted, bend right, "d_1"'] & \pi_i(P_{2^k}(S^n)) \arrow[r] \arrow[d] & \pi_{i-1}(\mathbb{D}_{2^{k+1}}(S^n)) \\
 & \vdots & & \vdots &
\end{tikzcd}$$

This tower then gives us a new spectral sequence, that being the Goodwilllie spectral sequence (GSS) where we start with the homotopy groups of the layers and get the unstable homotopy groups of S^n.

$$E^1_{i,k} = \pi_i \mathbb{D}_{2^k}(S^n) \Longrightarrow \pi_i(S^n)$$

To be able to work with the GSS, we first need to determine the homotopy groups of the fibers $\pi_t(\mathbb{D}_{2^k})$. To do this we will make use of the EHP sequence again. Here we note that the the EHP sequence comes from the sequence of functors given by

$$\mathrm{id} \xrightarrow{E} \Omega\Sigma \xrightarrow{H} \Omega\Sigma\,\mathrm{Sq}$$

where Sq denotes the takes X to $X \wedge X$. When we localize this at 2 and take X to be a sphere, then this results in the EHP sequence. By [Beh12, Lem. 2.1.2] we see that this then also induces a fiber sequence

$$\mathbb{D}_i(\mathrm{id})(S^n) \xrightarrow{E} \mathbb{D}_i(\Omega\Sigma)(S^n) \xrightarrow{H} \mathbb{D}_i(\Omega\Sigma\,\mathrm{Sq})(S^n)$$

By carefully studying these functors, we get by [Beh12, Cor.2.1.4] that for $i = 2^k$, these fiber sequences are actually equivalent to

$$\mathbb{D}_{2^k}(S^n) \xrightarrow{E} \Sigma^{-1}\mathbb{D}_{2^k}(S^{n+1}) \xrightarrow{H} \Sigma^{-1}\mathbb{D}_{2^{k-1}}(S^{2n+1})$$

We now note that when $i = 2^k$ then we can relate the layers of the identity functor on spheres to some other well known class of spectra by

$$\mathbb{D}_{2^k}(S^n) \simeq \Sigma^{n-k}L(k)_n$$

This is was shown by Arone and Dwyer in [AD01]. As we will not need any theory that arose from studying these spectra, and we will only use them to have a more readable indexing of our fiber sequences, we will not go in depth about what these spectra are. We refer to [KMP82] for a discussion on them. Under this 'reindexing' we see that the previous fiber sequence is written as

$$\Sigma^n L(k)_{2n+1} \xrightarrow{P} L(k)_n \xrightarrow{E} L(k)_{n+1}$$

This then gives us a new tower under $L(k)_1$, where we can use Verdier's octahedral axiom to construct a filtration for $L(k)_1$.

$$\begin{tikzcd}
 & L(k)_1 \arrow[ddr] & & & \\
 & \vdots \arrow[u] & & L(k)_1 \arrow[d, "E"] & \Sigma^1 L(k-1)_3 \arrow[l, "P"'] \\
\Sigma^1 L(k-1)_3 & L(k)^1_1 \arrow[l] \arrow[u] \arrow[r] & L(k)_1 \arrow[r] \arrow[ur] \arrow[ddr] & L(k)_2 \arrow[d, "E"] & \\
 & * \arrow[u] \arrow[ur] & & \vdots \arrow[d, "E"] & \\
 & & & * &
\end{tikzcd}$$

So again this filtration induces a spectral sequence in the same way as we described before where we start with the homotopy groups of the $L(k-1)_{2n+1}$ and converge to the homotopy groups of $L(k)_1$. So, as a start we will need to compute the homotopy groups of $L(1)_1$. To do this, we note that the constructed filtration turns out to be precisely the stable CW filtration of the $L(k)_1$. In the case of $L(1)_1$ we in fact have that

$$L(1)_1 \simeq \Sigma^{-1}\mathbb{D}_2(\Sigma^\infty S^1) \simeq \left(\partial_2(\mathrm{id}) \otimes (S^1)^{\otimes 2}\right)_{h\Sigma_2} \simeq \mathbb{S}^{-1} \otimes \Sigma^1\Sigma^\infty \mathbb{R}P^\infty_1 \simeq \Sigma^\infty \mathbb{R}P^\infty$$

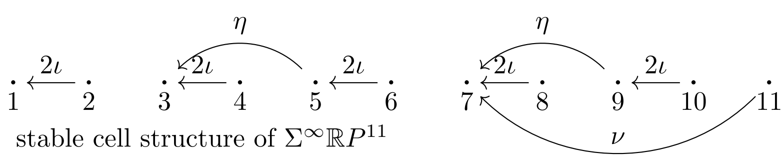
so that the filtration is acually the stable CW filtration for $\mathbb{R}P^\infty$ and the spectral sequence that we are interested in is actually the Atiyah-Hirzebruch spectral sequence for $\Sigma^\infty \mathbb{R}P^\infty$.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
\mathbb{S}^{n-1} \arrow[r] & \mathbb{R}P^{n-1} \arrow[r, "q"] \arrow[d] & \mathbb{S}^{n-1} \\
\mathbb{S}^{n} \arrow[r] & \mathbb{R}P^{n} \arrow[r, "q"] \arrow[d] & \mathbb{S}^{n} \\
 & \vdots \arrow[d] & \\
 & \mathbb{R}P^\infty &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] \\
\pi_i(\mathbb{S}^{n-1}) \arrow[r] & \pi_i(\mathbb{R}P^{n-1}) \arrow[r] \arrow[d] & \pi_i(\mathbb{S}^{n-1}) \simeq \pi_{i-1}(\mathbb{S}^{n-2}) \arrow[r] & \pi_{i-1}(\mathbb{R}P^{n-2}) \arrow[d] \\
\pi_i(\mathbb{S}^{n}) \arrow[r] \arrow[urr, dotted, "d_2"] \arrow[rr, dotted, bend right, "d_1"'] & \pi_i(\mathbb{R}P^{n}) \arrow[r] \arrow[d] & \pi_i(\mathbb{S}^{n}) \simeq \pi_{i-1}(\mathbb{S}^{n-1}) \arrow[r] & \pi_{i-1}(\mathbb{R}P^{n-1}) \arrow[d] \\
 & \vdots \arrow[d] & & \vdots \arrow[d] \\
 & \pi_i(\mathbb{R}P^\infty) & & \pi_{i-1}(\mathbb{R}P^\infty)
\end{tikzcd}$$

So we get a spectral sequence

$$\pi_i(\mathbb{S}^n) \Longrightarrow \pi_i(\mathbb{R}P^\infty)$$

From the unravelled exact couple we see that the differentials come from the attaching map structure of the n-cells. In the following picture we have drawn how far these attaching maps factor through the CW skeleta at the start of the filtration.



stable cell structure of $\Sigma^\infty \mathbb{R}P^{11}$

So for example from this we can read off that $\mathbb{R}P^5$ is build out of $\mathbb{R}P^4$ where the attaching map $\mathbb{S}^4 \longrightarrow \mathbb{R}P^4$ factors through the 3-skeleton so that after quotienting out the 2-skeleton it becomes $\mathbb{S}^4 \xrightarrow{\eta} \mathbb{S}^3$. The stable attaching maps of $\mathbb{R}P^\infty$ is a well-known result that comes from the computation of the vector fields on spheres [Ada62].

7.3 Going transfinite

The goal is now to be able to compute the unstable homotopy groups of spheres, where we assume that we already know about the stable ones by making use of the Goodwillie spectral sequence for the S^n. To do this we will first need to determine the $\pi_*\Sigma^{n-k}(L(k)_n) \simeq \pi_{*-n+k}L(k)_n$.
In order to compute these we will make use of the AHSS for the $L(k)_n$. However, to do this we will need to know the homotopy groups of the $L(k-1)_{2l+1}$ for $l \geq n$. We see that we will need to make use of a sequence of spectral sequences, where we will initially start from the stable homotopy groups of spheres. To have a better view of this process we will discuss it for the towers first, which gives us the following picture

$$\begin{tikzcd}
L(k)_n \arrow[d, "E^{l_k}"'] & \Sigma^{j_k}L(k-1)_{2j_k+1} \arrow[dl, "P"'] \arrow[d, "E^{l_{k-1}}"'] & \Sigma^{j_k+j_{k-1}}L(k-2)_{2j_{k-1}+1} \arrow[dl, "P"'] \arrow[d] & \Sigma^{j_k+\cdots+j_2}L(1)_{2j_2+1} \arrow[dl] \arrow[d] & \Sigma^{j_k+\cdots+j_1}\mathbb{S} \simeq \mathbb{S}^{j_k+\cdots j_1} \arrow[dl, "P"'] \\
L(k)_{j_k} & \Sigma^{j_k}L(k-1)_{j_{k-1}} & \ddots & \Sigma^{j_k+\cdots+j_2}L(1)_{j_1} &
\end{tikzcd}$$

where we have that $l_i = j_{i-1} - (2j_i + 1)$. From this we can read off that $J = (j_1, \cdots, j_k)$ will be a completely unadmissible monomial, together with $e(J) = j_k \geq n$. We note that again these are not the towers that give us the AHSS, for those we also apply Verdier's octahedral axiom. From this sequence of towers, we can then construct our sequence of spectral sequences

$$\bigoplus_{(j_1,\cdots,j_k)CU} \pi_t\mathbb{S}^{j_1+\cdots+j_k} \Longrightarrow \bigoplus_{(j_2,\cdots,j_k)CU} \pi_t(\Sigma^{j_k+\cdots+j_2}L(1)_{2j_2+1} \Longrightarrow \cdots \Longrightarrow \pi_t(L(k)_n)$$

and as we alluded to, we can now feed this information into the GSS for S^n. We will first combine the above sequence of towers together with the Goodwillie tower for S^n.

$$\begin{tikzcd}
S^n \arrow[d, dotted] & & \\
P_{2^k}(S^n) \arrow[d] & \Sigma^{n-k}L(k)_n \arrow[l] \arrow[d, "E^{l_k}"'] & \mathbb{S}^{n-k+j_k+\cdots j_1} \arrow[dl, dash, "\cdots"] \\
P_{2^{k-1}}(S^n) & \Sigma^{n-k}L(k)_{j_k} &
\end{tikzcd}$$

So we see that this give us the following sequence of spectral sequences, where we start with the stable homotopy groups of spheres and end up with the unstable ones.

$$\bigoplus_{k\geq 0} \bigoplus_{(j_1,\cdots,j_k)CU} \pi_t(\mathbb{S}^{n-k+j_1+\cdots+j_n}) \Longrightarrow \cdots \Longrightarrow \bigoplus_{k\geq 0} \pi_t(\Sigma^{n-k}L(k)_n) \Longrightarrow \pi_t(S^n)$$

We now want to feed this into the EHP spectral sequence again. We will again first combine the previous sequence of towers with the EHP filtration

$$\begin{tikzcd}
\Omega^n S^n \arrow[d] & & & \\
\Omega^{n+1}S^{n+1} \arrow[r, "H"'] \arrow[d, dotted] & \Omega^{n+1}S^{2n+1} \arrow[d, dotted] & & \\
\Omega^\infty\Sigma^\infty S^0 & \Omega^{n+1}P_{2^k}(S^{2n+1}) \arrow[d] & \Omega^\infty\Sigma^{n-k}L(k)_{2n+1} \arrow[l] \arrow[d, "E^{l_k}"'] & \Omega^\infty\mathbb{S}^{n-k+j_k+\cdots j_1} \arrow[dl, dash, "\cdots"] \\
& \Omega^{n+1}P_{2^{k-1}}(S^{2n+1}) & \Omega^\infty\Sigma^{n-k}L(k)_{j_k} &
\end{tikzcd}$$

So this then gives us the following sequence of spectral sequence

$$\bigoplus_{n\geq 0}\bigoplus_{k\geq n}\bigoplus_{(j_1,\cdots,j_n)CU} \pi_t\mathbb{S}^{n-k+j_k+\cdots+j_1} \Longrightarrow \cdots \Longrightarrow \bigoplus_{n\geq 0} \pi_{t+n}S^{2n+1} \Longrightarrow \pi_t^s$$

To exploit the full structure of this construction, we would rather want these to not be a sequence of spectral sequences, but a spectral sequence that is indexed by the Grothendieck group of ordinals, that being the natural extension of the the integers.

Definition 7.1 (Grothendieck group of ordinal numbers)**.** We define the Grothendieck group $\mathrm{Groth}(\omega^\nu)$ of ordinal numbers as formal sums

$$a_1 + a_2\omega^1 + \cdots + j_\nu \omega^{\nu-1}$$

with $j_i \in \mathbb{N}$ together with the addition

$$(a_1 + \cdots + a_\nu \omega^{\nu-1}) + (b_1 + \cdots + b_\nu \omega^{\nu-1}) = (a_1 + b_1) + (a_2 + b_2)\omega^1 + \cdots + (a_\nu + b_\nu)\omega^{\nu-1}$$

The benefit of this is that we do not have to solve extension problems after the use of a spectral sequence, we leave this till the end of the process. The construction of these transfinite spectral sequences will be the same as the ordinary ones from before, we will first construct a tower and all the long exact sequences will assemble into a transfinite spectral sequence.
We will first construct the transfinite tower for the iterated Atiyah-Hirzebruch spectral sequences. The underlying idea is that we use Verdier's axiom repeatedly to stitch the sequence of towers together. We will first start with the usual tower under $L(k)_n$, where we write $[L(k)]_{0,\cdots,0,j_k}$ for $L(k)_{j_k}$. The fiber of $L(k)_{j_k} \xrightarrow{E} L(k)_{j_k+1}$ is then given by $\Sigma^{j_k} L(k-1)_{2j_k+1}$, so using the dual tower over it we get maps $\Sigma^{j_k} L(k-1)^{j_{k-1}}_{2j_k+1} \to \Sigma^{j_k} L(k-1)_{2j_k-1} \to L(k)_{j_k}$. Taking the homotopy cofiber of these maps gives us what we want, the same data as the usual tower under $\Sigma^{j_k} L(k-1)_{2j_k+1}$, but then as a tower under $L(k)_{j_k}$. By Verdier's axiom we see that the fibers are the same. We will write these new homotopy cofibers as $[L(k)_{0,\cdots,0,j_{k-1},j_k}]$.

$$\begin{tikzcd}
\Sigma^{j_1+j_2} L(k-2)_{2j_{k-1}+1} \arrow[d] & L(k)_n \arrow[dd] & & \Sigma^{j_1+j_2} L(k-2)_{2j_{k-1}+1} \\
{[L(k)]_{0,\cdots,0,j_{k-1},j_k}} \arrow[dd] & & & \Sigma^{j_k} L(k-1)^{j_{k-1}}_{2j_k+1} \arrow[u] \arrow[dl] \arrow[dll, bend right] \\
& L(k)_{j_k} \arrow[ul] \arrow[dl] & \Sigma^{j_k} L(k-1)_{2j_k-1} \arrow[l] & \\
{[L(k)]_{0,\cdots,0,j_{k-1}+1,j_k}} & & & \Sigma^{j_k} L(k-1)^{j_{k-1}-1}_{2j_k+1} \arrow[uu] \arrow[ul] \arrow[ull, bend left]
\end{tikzcd}$$

We can now do the same trick to construct a new level of this tower. We take the homotopy fiber of the map $[L(k)]_{0,\cdots,0,j_{k-1},j_k}$, and construct the dual tower of it, and then take the homotopy cofiber of the composite to get the next layer of our transfinite tower. We then repeat this process until the fibers of the transfinite tower are given by the $\Sigma^{j_k+\cdots+j_1} L(0)_{2j_1+1} \simeq \mathbb{S}^{j_k+\cdots j_1}$ again. Taking the spectral sequence of this transfinite tower then gives us a transfinite spectral sequence

$$E^1_{t,J} = \pi_t(\mathbb{S}^{j_1+\cdots+j_k}) \overset{\mathrm{TAHSS}(L(k)_n)}{\Longrightarrow} \pi_t(L(k)_n)$$

where the monomials J are completely undamissible with $j_k \geq n$. We will write an element $\alpha \in E^1_{t,J} = \pi_t(\mathbb{S}^{\|J\|})$ as $\alpha[J]$.
We can then combine this with the Goodwillie tower to form the transfinite Goodwillie tower. Here we first recall that the Goodwillie tower is made up of principal fibrations. So we see that the homotopy cofiber of $P_{2^k}(S^n) \to P_{2^{k-1}}(S^n)$ is given by a suspension of its homotopy fiber, i.e. $\Omega^\infty\Sigma^{n-k+1}L(k)$. By taking the transfinite Atiyah-Hirzebruch tower under it, we see that we get maps $P_{2^{k-1}}(S^n) \to \Omega^\infty\Sigma^{n-k+1}L(k) \to \Omega^\infty\Sigma^{n-k+1}[L(k)]_J$. By taking the homotopy fibers of these maps, which we will write as $[S^n]_{J-k\omega^\omega}$ we see

that this gives a tower over $P_{2^{k-1}}(S^n)$.

$$\begin{tikzcd}
\Omega^\infty\Sigma^{n-k+1}\mathbb{S}^{\|J\|} & S^n \arrow[d, dotted] & & \Omega^\infty\Sigma^{n-k+1}\mathbb{S}^{\|J\|} \arrow[d] \\
{[S^n]_{J+1-k\omega^\omega}} \arrow[u] \arrow[dr] & P_{2^k}(S^n) \arrow[d] & & \Omega^\infty\Sigma^{n-k+1}[L(k)]_J \arrow[dd] \\
& P_{2^{k-1}}(S^n) \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & \Omega^\infty\Sigma^{n-k+1}L(k) \arrow[ur] \arrow[dr] & \\
{[S^n]_{J+-k\omega^\omega}} \arrow[uu] \arrow[ur] & & & \Omega^\infty\Sigma^{n-k+1}[L(k)]_{J+1} \\
\Omega^\infty\Sigma^{n-k}\mathbb{S}^{\|J\|} \arrow[u] & & &
\end{tikzcd}$$

Taking the transfinite spectral sequence of this transfinite tower then gives us the transfinite Goodwillie spectral sequence for S^n.

$$E^1_{t,J} = \pi_t(\mathbb{S}^{n-k+j_1+\cdots+j_k}) \overset{\mathrm{TGSS}(S^n)}{\Longrightarrow} \pi_t(S^n)$$

where J can now have an arbitrary length k, but still completely unadmissible with $j_k \geq n$. We will write an element $\alpha \in E^1_{t,J} = \pi_t(\mathbb{S}^{n-k+\|J\|})$ as $\alpha[J]$.

Last but not least we will construct the transfinite EHP filtration. Using the transfinite Goodwillie towers for the cofibers $\Omega^{n+1}S^{2n+1}$ we see that we get a map $\Omega^{n+1}S^{n+1} \xrightarrow{H} \Omega^{n+1}S^{2n+1} \longrightarrow \Omega^{n+1}[S^{2n+1}]_J$. By taking the homotopy fiber of this map we get a filtration of $\Omega^{n+1}S^{2n+1}$ over $\Omega^{n+1}S^{n+1}$ which we will write as $F_{J,n}(QS^0)$.

$$\begin{tikzcd}
\Omega^{n+1}\Sigma^{2n+1-|J|}S^{\|J\|} \simeq QS^{\|J\|+n-|J|} & & & \Omega^{n+1}\Sigma^{2n+1-|J|}S^{\|J\|} \arrow[d] \\
F_{J,n}(QS^0) \arrow[u] \arrow[dr] & \Omega^n S^n \arrow[d] & & \Omega^{n+1}[S^{2n+1}]_J \arrow[dd] \\
& \Omega^{n+1}S^{n+1} \arrow[r] \arrow[d, dotted] \arrow[urr, bend left] \arrow[drr, bend right] & \Omega^{n+1}S^{2n+1} \arrow[ur] \arrow[dr] & \\
F_{J-1,n}(QS^0) \arrow[uu] \arrow[ur] & \Omega^\infty\Sigma^\infty S^0 & & \Omega^{n+1}[S^{2n+1}]_{J+1}
\end{tikzcd}$$

By taking this construction over the whole EHP filtration, we get a transfinite filtration of QS^0. Taking the transfinite spectral sequence of this transfinite filtration then gives a transfinite EHP spectral sequence given by

$$E^1_{t,[J,n]} = \pi_t(QS^{\|J\|+n-|J|}) \overset{\mathrm{TEHPSS}}{\Longrightarrow} \pi^s_t$$

Here we will write an element $\alpha \in E^1_{t,[J,n]}$ as $\alpha[J,n]$ where the n tells us about the sphere in the EHP-filtration it belongs to. For the convergence of these transfinite spectral sequences, we note that we constructed them out of towers whose individual spectral sequences do converge, so they do converge as well. For a more detailed discussion on convergence, we refer to the Appendix of [Beh12]. In order to make use of these we will still need to discuss how to compute their differentials.

7.4 Computing differentials in the AHSS

For $L(1)_1 \simeq \Sigma^\infty\mathbb{R}P^\infty$ we can determine the differentials out of the diagonal, i.e. $\pi_n(\mathbb{S}^n)$, by studying the attaching maps of 7.2. As an example, suppose that we had $\iota \in \pi_4\mathbb{S}^4$, then under composing it with the attaching map, this just gives our attaching map back for the 5-cell. From the attaching map structure we can read off that this factors through the cell structure to $\mathbb{R}P^3$ and after quotienting out $\mathbb{R}P^2$, this gives us

$\eta \in \pi_4(\mathbb{S}^3)$.So this gives us a differential $d_2(1[5]) = \eta[3]$.

$$\begin{tikzcd}
 & \mathbb{S}^3 \arrow[r] & \mathbb{R}P^3 \arrow[r] \arrow[d] & \mathbb{S}^3 \\
\mathbb{S}^4 \arrow[r, "\mathrm{id}"] & \mathbb{S}^4 \arrow[r] \arrow[urr, "\eta"] & \mathbb{R}P^4 \arrow[r] \arrow[d] & \mathbb{S}^4 \\
 & & \mathbb{R}P^5 &
\end{tikzcd}$$

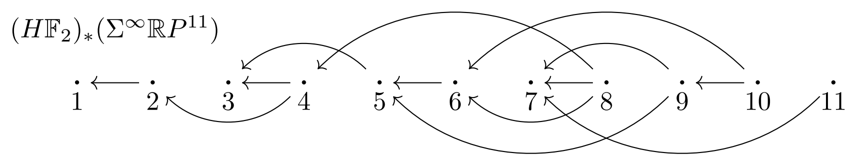
For general differentials in the for the TAHSS of $L(1)_1, L(2)_1$ and $L(3)_1$ we will be using the attaching map structure together with the action of the dual Steenrod algebra over their homology. We recall that we computed that that $\mathbb{D}_{2^n}(S^1)$ was given by the weight 2^k part of $\mathrm{Free}_{\bar{\mathcal{R}}\text{-}\mathrm{sLie}}(H_*(S^1))$, and as $H_*(S^1)$ consists of a single non-trivial element $\iota_1 \in H_1(S^1)$, we get that

$$H_*(L(k)_1) \simeq \mathbb{F}_2\{\sigma^{k-1}\bar{Q}^J(\iota_1) \mid J = (j_1, \cdots, j_k), j_k \geq 1\}$$

where we take the suspension isomorphism σ^{k-1} as $\mathbb{D}_{2^k}(S^1) \simeq \Sigma^{1-k}L(k)_1$. So the action of the dual Steenrod operations on $H_*(L(1))_1$ can be computed from the Nishida relations. We can now make use of the Dyer-Lashof machine to compute these. Our input module would then be given by

$$\begin{array}{cccc}
1 & & & \\
 & & & \\
1 & & & \\
 & & & \\
0 & 0 & 1 & 0
\end{array}$$

where we have added an extra Sq_0 as it cannot parse no operations yet. We have drawn a part of the dual Steenrod algebra in the following picture.



Using this we will give an example computation. We will compute that there is a non-trivial differential $d_2(\eta[4]) = \eta \circ \eta[2] = \eta^2[2]$. From looking at the stable cell structure, we see that for this computation we can safely quotient out the 1-cell, so we will be studying the following diagram

$$\begin{tikzcd}
 & & \mathbb{R}P^2_2 \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{S}^2 \\
\mathbb{S}^4 \arrow[r, "\eta"'] \arrow[urr, dotted, "\eta\circ\eta"] & \mathbb{S}^3 \arrow[r] \arrow[rr, bend right, "2\iota"'] & \mathbb{R}P^3_2 \arrow[r] \arrow[d] & \mathbb{S}^3 \\
 & & \mathbb{R}P^4_2 &
\end{tikzcd}$$

We then see that $\mathbb{R}P^3_2$ build from $\mathbb{R}P^2_2 \simeq \mathbb{S}^2$ by attaching a 3-cell along the constant map, so $\mathbb{R}P^3_2 \simeq \mathbb{S}^3 \vee \mathbb{S}^2$. From this we see that the vertical map $\mathbb{R}P^2_2 \to \mathbb{R}P^3_2$ is the inclusion map from $\mathbb{S}^2$ into $\mathbb{S}^2 \vee \mathbb{S}^3$.
Looking at the action of the dual Steenrod algebra on $\Sigma^\infty \mathbb{R}P^\infty$ between degrees 2 and 4, we see that $\mathbb{R}P^4_2$ is build from $\mathbb{R}P^3_2$ by attaching a 4-cell along the map $\mathbb{S}^3 \xrightarrow{(\eta, 2\iota)} \mathbb{S}^2 \vee \mathbb{S}^3$. As $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\{\eta\}$ we see that $2\iota \circ \eta = 0 \in \pi_4(\mathbb{S}^3)$. So we find that $\mathbb{S}^4 \xrightarrow{(\eta,2\iota)\circ\eta} \mathbb{R}P^3_2 \simeq \mathbb{S}^2 \vee \mathbb{S}^3$ factors through the two skeleton by $\mathbb{S}^4 \xrightarrow{\eta\circ\eta} \mathbb{S}^2$. This proves that there is a non-trivial differential $d_1(\eta[4]) = \eta^2[2]$.

For our last example, we want to compute that

$$d_3(\nu^2[6]) = \langle \eta, 2, \nu^2 \rangle[3] = \epsilon[3]$$

in the TAHSS for $L(1)$. For a discussion on Toda brackets we refer to the Appendix. In order to do this computation we will collapse the lower cells again, as we can see from the attaching structure that these will not play an important role. We want to show that the composite $\mathbb{S}^{11} \xrightarrow{\nu} \mathbb{S}^5 \longrightarrow \mathbb{R}P^5_3$ lifts to a map $\mathbb{S}^{11} \longrightarrow \mathbb{R}P^3_3 \simeq \mathbb{S}^3$, and that this lift is non-trivial. To do this we will first need to determine the attaching maps in this range. Here we will make use of the module structure of $\mathbb{R}P^6_3$ over the dual Steenrod algebra, which we get from truncating the range given in picture 2. From this we can read off that $\mathbb{R}P^6_3$ is constructed from $\mathbb{R}P^3_3$ by first attaching a 4-cell along the map $\mathbb{S}^3 \xrightarrow{2\iota} \mathbb{S}^3$, the 5-cell is then attached to $\mathbb{R}P^4_3$ factors through the 3-skeleton along the map $\mathbb{S}^4 \xrightarrow{\eta} \mathbb{S}^3$, lastly the 6-cell is attached to $\mathbb{R}P^5_3$ to the new 5-cell along the map $\mathbb{S}^5 \xrightarrow{2\iota} \mathbb{S}^5$. This is summarized in the following diagram.

$$\begin{tikzcd}
 & \mathbb{S}^3 \arrow[r, "2\iota"] & \mathbb{R}P^3_3 \arrow[r, "\simeq"] \arrow[d] & \mathbb{S}^3 \\
 & \mathbb{S}^4 \arrow[r] \arrow[ur, "\eta"] & \mathbb{R}P^4_3 \arrow[r] \arrow[d] & \mathbb{S}^4 \\
\mathbb{S}^{11} \arrow[r, "\nu^2"] & \mathbb{S}^5 \arrow[r] \arrow[rr, bend right, "2\iota"'] & \mathbb{R}P^5_3 \arrow[r] \arrow[d] & \mathbb{S}^5 \\
 & & \mathbb{R}P^6_3 &
\end{tikzcd}$$

Here we note that for our computation, the additional 4-cell does not play any role. So to determine our lift we can safely ignore it. So we construct a new spectrum X as the cofiber of $\mathbb{S}^4 \xrightarrow{\eta} \mathbb{S}^3$ and then we attach a 6-cell by the map $\mathbb{S}^5 \xrightarrow{2\iota} \mathbb{S}^5$, to get a spectrum Y. Our goal is now to show that the map $\mathbb{S}^{11} \xrightarrow{\nu^2} \mathbb{S}^5 \longrightarrow X$ indeed lifts to a map $\mathbb{S}^{11} \longrightarrow \mathbb{S}^3$ which we want to be given by the Toda bracket $\langle \eta, 2\iota, \nu^2 \rangle$.

$$\begin{tikzcd}
 & \mathbb{S}^4 \arrow[r, "\eta"] & \mathbb{S}^3 \arrow[d] & \\
\mathbb{S}^{11} \arrow[r, "\nu^2"'] \arrow[urr, dotted, "{\langle \eta, 2\iota, \nu^2 \rangle}"] & \mathbb{S}^5 \arrow[r] \arrow[rr, bend right, "2\iota"'] & X \arrow[r] \arrow[d] & \mathbb{S}^5 \\
 & & Y &
\end{tikzcd}$$

First we will be constructing our lift. Here we note that $[2\iota \circ \nu^2] \in \pi_{11}(\mathbb{S}^5)$ is trivial, so when by taking fibers along the maps $\mathbb{S}^{11} \longrightarrow *$ and $\mathbb{S}^5 \xrightarrow{2\iota} \mathbb{S}^5$, we see that ν^2 induces an associated map on the homotopy fibers which we will denote by $\mathbb{S}^{11} \xrightarrow{\tilde{\nu}^2} \mathbb{S}^4/2$. We now recall that the sequence $\mathbb{S}^3 \to X \to \mathbb{S}^5$ in the construction of X is a fiber sequence, so that we can rewrite the above diagram, given us a new map of homotopy fibers which we will for now denote by $\mathbb{S}^4/2 \xrightarrow{f} \mathbb{S}^3$.

$$\begin{tikzcd}
 & & \mathbb{S}^4 \arrow[d, "\eta"] \\
\mathbb{S}^{11} \arrow[r, dashed, "\tilde{\nu}^2"] \arrow[d] & \mathbb{S}^4/2 \arrow[r, dashed, "f"] \arrow[d] & \mathbb{S}^3 \arrow[d] \\
\mathbb{S}^{11} \arrow[r, "\nu^2"'] \arrow[d] & \mathbb{S}^5 \arrow[r] \arrow[d, "2\iota"'] & X \arrow[d] \\
* \arrow[r] & \mathbb{S}^5 \arrow[r, "\mathrm{id}"'] & \mathbb{S}^5
\end{tikzcd}$$

We will now give a different construction of the Toda bracket, in terms of associated maps on homotopy fibers. So suppose we have the composite $\mathbb{S}^{11} \xrightarrow{2\iota\circ\nu^2} \mathbb{S}^5$ together with a null-homotopy and the composite $\mathbb{S}^5 \xrightarrow{\eta\circ 2\iota} \mathbb{S}^4$ together with a null-homotopy. When we take the associated map of homotopy fibers of the first square we get the same map as before $\tilde{\nu}^2 \colon \mathbb{S}^{11} \to \mathbb{S}^4/2$, and taking the associated map on homotopy fibers for the second square gives us $\tilde{\eta} \colon \mathbb{S}^4/2 \to \mathbb{S}^3$. The composite of these two maps will then be the same as an element in the Toda bracket $\langle \eta, 2\iota, \nu^2 \rangle \subset \pi_{11}(\mathbb{S}^3)$.

$$\begin{tikzcd}
\mathbb{S}^{11} \arrow[r, dashed, "\tilde{\nu^2}"] \arrow[d, "\mathrm{id}"'] & \mathbb{S}^4/2 \arrow[r, dashed, "\tilde{\eta}"] \arrow[d] & \mathbb{S}^3 \arrow[d] \\
\mathbb{S}^{11} \arrow[r, "\nu^2"] \arrow[d] & \mathbb{S}^5 \arrow[r] \arrow[d, "2\iota"'] \arrow[dl, dotted, "h_1"] & * \arrow[d] \\
* \arrow[r] & \mathbb{S}^5 \arrow[r, "\eta"] \arrow[ur, dotted, "h_2"] & \mathbb{S}^4
\end{tikzcd}$$

We want to show that the map $\mathbb{S}^4/2 \xrightarrow{f} \mathbb{S}^3$ is indeed $\tilde{\eta}$. To show this we will look at the definitions of f and $\tilde{\eta}$ as associated maps on homotopy fibers, so that they would fit together into a commutative prism where the new induced map on the homotopy fibers is the identity map on $\mathbb{S}^3$.

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[d] & \mathbb{S}^4 \arrow[l, "\eta"'] \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{S}^4 \arrow[ll, bend left, "\eta"] \arrow[d, "\eta"] \\
\mathbb{S}^3 \arrow[d] & \mathbb{S}^4/2 \arrow[l, dashed, "f"'] \arrow[r, dashed, "\tilde{\eta}"] \arrow[d] & \mathbb{S}^3 \arrow[ll, dashed, bend left] \arrow[d] \\
* \arrow[d] & \mathbb{S}^5 \arrow[l] \arrow[r] \arrow[d, "2\iota"] & X \arrow[ll, bend left] \arrow[d] \\
\mathbb{S}^4 & \mathbb{S}^5 \arrow[l, "\eta"'] \arrow[r, "\mathrm{id}"] & \mathbb{S}^5 \arrow[ll, bend left, "\eta"]
\end{tikzcd}$$

We now see that for $X \to *$ to be the induces map on homotopy cofibers of the top square, we can only have that the map $\mathbb{S}^3 \to \mathbb{S}^3$ is the indentity map.

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[d, "\mathrm{id}"] & \mathbb{S}^4 \arrow[l, "\eta"] \arrow[d, "\eta"] \\
\mathbb{S}^3 \arrow[d] & \mathbb{S}^3 \arrow[l, dashed, "\mathrm{id}"'] \arrow[d] \\
* & X \arrow[l]
\end{tikzcd}$$

This shows that our map f is indeed $\tilde{\eta}$, so our lift is indeed the composite $\mathbb{S}^{11} \xrightarrow{\tilde{\nu}^2} \mathbb{S}^4/2 \xrightarrow{\tilde{\eta}} \mathbb{S}^3$, which we noted before is an element in the Toda bracket $\langle \eta, 2\iota, \nu^2 \rangle \subset \pi_{11}(\mathbb{S}^3)$. In fact we know that this Toda bracket only consists of one element, that being ϵ the generator of $\pi_{11}(\mathbb{S}^3)$. So this shows that we indeed have a non trivial differential $d_3(\nu^2[6]) = \epsilon[3]$.

This way we are able to deduce a lot off differentials in the AHSS for $L(1)_1$ and the same for $L(2)_1$, $L(3)_1$. Although the arguments to determine these become complex very fast, they are still much more tractable then determining the differentials in the EHP spectral sequence. From this we can read off the differentials for $L(k)_n$ by truncating the spectral sequences for $L(k)_1$.

7.4.1 Differentials in the EHPSS and GSS

So now that we are able to compute differentials in the TAHSS, we want to determine differentials in the EHPSS and GSS with them. For this we first recall that the EHP sequence gave fiber sequences over the corresponding Goodwillie towers

$$P_{2^k}(S^n) \xrightarrow{E} \Omega P_{2^k}(S^{n+1}) \xrightarrow{H} \Omega P_{2^{k-1}}(S^{2n+1})$$

by [Beh12, Cor. 2.1.4] and by our construction of the transfinite Goodwillie tower, we find that we get a similar fiber sequences

$$\Omega^2[S^{2n+1}]_J \xrightarrow{P} [S^n]_{[J,n]} \xrightarrow{E} \Omega[S^{n+1}]_{[J,n]}$$

which in turn induce maps of spectral sequence on the transfinite Goodwillie spectral seqeunces.

$$E^1_{t+2,J}(S^{2n+1}) \xrightarrow{P_*} E^1_{t,[J,n]}(S^n) \xrightarrow{E_*} E^1_{t+1,[J,n]}(S^{n+1})$$

where the maps are defined as $P_*(\alpha[J]) = \alpha[J,n]$ and $E_*(\alpha[J]) = \alpha[J]$. These are shown in [Beh12, Thm. 3.4.2] and [Beh12, Thm. 3.4.4]. The idea is now to study what it means to have a non-trivial differential in one of the spectral sequences for the existence of differentials non-trivial differentials in the other two sequences. This idea is covered by the geometric boundary theorem which studies this for a general fiber sequence of transfinite towers. This is discussed in [Beh12, Lem. A.4.1] and we will only make use of Case (5).

Using this we will first look at how to deduce differentials from the EHPSS from differentials in the TGSS which is given in [Beh12, Thm. 5.3.2].

So suppose we have an element $\alpha[J]$ that detects an element $x \in \pi_{t+n+2}S^{2n+1}$ in the TGSS and $\alpha[J,n]$ detects an element $y \in \pi_{t+m+1}(S^{m+1})$ and we have a differential $d^{S^m}(\alpha[J,n]) = \beta[J',m]$ and together with a technical condition, then we will have a differential $d^{m-n}(x) = H(y)$ in the EHPSS.

The idea for this is roughly sketched in the following diagram, where we have highlighted our initial conditions in blue.

$$\begin{tikzcd}
 & \pi_{t+m+1}(S^{m+1}) \arrow[rrr, "H"] \arrow[ddd, "E^{n-m-1}"'] & & & \pi_{t+m+1}(S^{2m+1}) \\
 & & & \beta[J',m] & \beta[J'] \arrow[l, "P_*"] \arrow[u, Rightarrow] \\
 & & \alpha[J,n] \arrow[uul, Rightarrow] \arrow[dd, "E_*^{n-m-1}"] & \alpha[J,n] \arrow[l, "E_*"] \arrow[u, "d^{S^m}"] & \\
\pi_{t+n+2}(S^{2n+1}) \arrow[r, "P"] & \pi_{t+n}(S^n) & & & \\
\alpha[J] \arrow[u, Rightarrow] \arrow[rr, "P_*"] & & \alpha[J,n] \arrow[ul, Rightarrow] & &
\end{tikzcd}$$

Here the top part of the diagram comes from the geometric boundary theorem and we see that this indeed gives us a differential in the EHP spectral sequence. The actual proof of this is rather technical for which we refer to [Beh12, Thm. 5.3.2]. From this result we are able to lift a lot of differentials from the TAHSS's to the EHPSS.

From the fiber sequence of transfinite spectral sequence we can also deduce differentials from the 0-line in the GSS from detection in the TEHPSS. This is [Beh12, Thm. 4.4.1] for which we will discuss its proof. Suppose an element in the stable homotopy groups of spheres $\alpha \in \pi_t^s$ is detected in the EHPSS by an $H(\tilde{\alpha}) \in \pi_{t+n+1}S^{2n+1}$. Then this means that α is born on S^{n+1}. We now note that in the GSS, if an element $\alpha \in \pi_t^s = E^1_{t,0}$ survives till the E_∞-page, then this means that it desuspends to an element $\tilde{\alpha} \in \pi_t(S^n)$. So if a stable element α is born on $\pi_{t+n+1}S^{n+1}$ then this means that in the GSS for S^{n+1} it detects $\tilde{\alpha} \in \pi_{t+n+1}S^{n+1}$, whereas the element α in the GSS for S^n has to be in the source of a non-trivial differential, as it would otherwise survive. By the geometric boundary theorem we thus get the following diagram

$$
\begin{tikzcd}
\alpha & & \\
H(\tilde{\alpha}) \arrow[u, Rightarrow, "\text{EHPSS}"] & & \tilde{\alpha} \arrow[ll, "H"'] \\
\beta[J'] \arrow[u, Rightarrow, "\text{TGSS}(S^{2n+1})"] \arrow[r] & \beta[J',n] & \\
 & \alpha \arrow[u, "d^{S^n}"] \arrow[r] & \alpha \arrow[uu, Rightarrow, "\text{TGSS}(S^{n+1})"'] \\
0 \arrow[uu, dotted, "d^{S^{2n+1}}"] & &
\end{tikzcd}
$$

By the geometric boundary theorem we are able construct an element $\beta[J']$ in the TGSS for S^{2n+1} such that it $d^{S^n}(\alpha) = P_*(\beta[J']) = \beta[J', n]$. We now note that is not the target of any differential longer than the differential from α to $\beta[J', n]$, as this would have its source below the 0-line which is trivial. So $\beta[J']$ does detect $H(\tilde{\alpha})$ in the TGSS of S^{2n+1}, which gives us precisely that $\beta[J', n]$ detects α in the TEHPSS.
So to restate this result, we find that if a stable element $\alpha \in \pi_t^s$ is born on S^{n+1}, then we get a non-trivial differential in the GSS for S^n by $d^{S^n}(\alpha) = \beta[J', n]$ where $\beta[J', n]$ detects α in the TEHPSS.

Ultimately using these methods and many more discussed in [Beh12], a large part of the GSS was computed for S^k for $1 \geq k \geq 6$ and recomputed for the EHPSS in the Toda range. The results of these can be found [Beh12, Sec. 6.5].

8 Appendix

Suppose we have two towers, X_n under X and Y_n under Y, and a sequence of maps $X_n \longrightarrow Y_n$. Then this also induces a map on the fibers of the tower, giving us a map of towers

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & Y_{n-1} \arrow[d] & \\
A_n \arrow[r] \arrow[rrr, bend left] & X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] & B_n \arrow[l] \\
 & X_{n+1} \arrow[r, "f_{n+1}"] \arrow[d] & Y_{n+1} \arrow[d] & \\
 & \vdots & \vdots &
\end{tikzcd}
$$

We will now look at what this means for their corresponding spectral sequences. If we look at the two unravelled couples then we do have the maps on homotopy groups between them, making the diagram commute

$$
\begin{tikzcd}
& \pi_t B_n \arrow[rr] & & \pi_t Y_n \arrow[rr] \arrow[dd] & & \pi_{t-1} B_{n-1} \\
\pi_t A_n \arrow[ur] \arrow[rr] & & \pi_t X_n \arrow[ur] \arrow[rr] \arrow[dd] & & \pi_{t-1} A_{n-1} \arrow[ur] & \\
& \pi_t B_{n+1} \arrow[rr] & & \pi_t Y_{n+1} \arrow[rr] & & \pi_{t-1} B_n \\
\pi_t A_{n+1} \arrow[ur] \arrow[rr] & & \pi_t X_{n+1} \arrow[ur] \arrow[rr] & & \pi_{t-1} A_n \arrow[ur] &
\end{tikzcd}
$$

By tracing this diagram we see that a non-trivial differential in the spectral sequence of X given by $d^r_X(x) = y$, induces a differential in the spectral sequence of Y given by $d^r_Y(f(x)) = f(y)$. Of course this new differential may very well be trivial, as y might be in the kernel of f. And even if this was not the case, then $f(y)$ might have been in the image of a previous shorter differential $d^{r'}_Y(x') = f(y)$, making it trivial on the r-page.

8.1 A special case from Verdier's axiom

We already used several times that if we have a tower, and take its dual tower by using Verdier's axiom, that the resulting spectral sequences from both towers will turn out to be the same. We will now prove this result. Suppose we have a tower under X, together with its dual tower, and we shift by one, then this gives us the following diagram

$$
\begin{tikzcd}
A_\mu \arrow[r, "\mathrm{id}"] & A_\mu & & \\
\Omega X_{\mu+1} \arrow[u] \arrow[r] & X^\mu \arrow[u] \arrow[r] & X \arrow[r] & X_{\mu+1} \\
\Omega X_\mu \arrow[u] \arrow[r] & X^{\mu-1} \arrow[u] \arrow[r] & X \arrow[u, "\mathrm{id}"] \arrow[r] & X_\mu \arrow[u] \\
& & & A_\mu \arrow[u]
\end{tikzcd}
$$

If we now look at how the differentials in the spectral sequences were defined then we get that

$$
\begin{tikzcd}
\pi_{n+1} A_{\mu+1} \arrow[r] \arrow[rr, bend left, "d_1"] \arrow[d, "\mathrm{id}"] & \pi_{n+1} X_{\mu+1} \arrow[r] \arrow[d] & \pi_n(A_\mu) \arrow[d, "\mathrm{id}"] \\
\pi_{n+1} A_{\mu+1} \arrow[r] \arrow[rr, bend right, "d_1"'] & \pi_n X^\mu \arrow[r] & \pi_n(A_\mu)
\end{tikzcd}
$$

So we see that the differentials agree, and we get the same for the higher differentials.

8.2 Toda brackets

Here we recall the construction of the toda bracket. Suppose we have three consecutive maps such that the two compositions are both null-homotopic.

$$
\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d] & X_2 \arrow[r] \arrow[d, "f_2"] \arrow[dl, dotted, "h_1"'] & * \arrow[d] \\
* \arrow[r] & X_3 \arrow[r, "f_3"] \arrow[ur, dotted, "h_2"] & X_4
\end{tikzcd}
$$

Depending on our choice of homotopies we then get a map

$$
\begin{aligned}
\Sigma X_1 &\longrightarrow X_4 \\
[x,t] &\longmapsto \begin{cases} [f_3 \circ (h_1)_t(x), t] & \text{for} t \in [0,1] \\ [(h_2)_t(x) \circ f_1, t] & \text{for} t \in [-1,0] \end{cases}
\end{aligned}
$$

Here we note that for $t = 0$ these two coincide as

$$
f_3 \circ (h_1)_0 = (h_2)_0 \circ f_1 = f_3 \circ f_2 \circ f_1
$$

9 References

[AD01] Greg Z Arone and WG Dwyer. Partition complexes, tits buildings and symmetric products. *Proceedings of the London Mathematical Society*, 82(1):229–256, 2001.

[Ada62] J Frank Adams. Vector fields on spheres. 1962.

[AK56] Shôrô Araki and Tatsuji Kudo. Topology of h -spaces and h-squaring n operations. 1956.

[AM99] Greg Arone and Mark Mahowald. The goodwillie tower of the identity functor and the unstable periodic homotopy of spheres. *Inventiones mathematicae*, 135:743–788, 1999.

[Beh12] Mark Behrens. *The Goodwillie tower and the EHP sequence*, volume 218. American Mathematical Society, 2012.

[Bru14] Robert Bruner. Dyer-lashof, 2014.

[Cam16] Omar Antolín Camarena. The mod 2 homology of free spectral lie algebras. *arXiv preprint arXiv:1611.08771*, 2016.

[Chi05] Michael Ching. Bar constructions for topological operads and the goodwillie derivatives of the identity. *Geometry & Topology*, 9(2):833–934, 2005.

[DL62] Eldon Dyer and Richard K Lashof. Homology of iterated loop spaces. *American Journal of Mathematics*, 84(1):35–88, 1962.

[Dug08] Daniel Dugger. A primer on homotopy colimits. *preprint*, 2008.

[Goo03] Thomas G Goodwillie. Calculus iii: Taylor series. *Geometry & Topology*, 7(2):645–711, 2003.

[IWX20] Daniel C Isaksen, Guozhen Wang, and Zhouli Xu. Stable homotopy groups of spheres. *Proceedings of the National Academy of Sciences*, 117(40):24757–24763, 2020.

[Jam55] Ioan M James. Reduced product spaces. *Annals of Mathematics*, pages 170–197, 1955.

[Joh95] Brenda Johnson. The derivatives of homotopy theory. *Transactions of the American Mathematical Society*, 347(4):1295–1321, 1995.

[KMP82] NJ Kuhn, SA Mitchell, and SB Priddy. The whitehead conjecture and splitting b\left(z/2\right)^k. 1982.

[Kuh07] Nicholas J Kuhn. Goodwillie towers and chromatic homotopy: an overview. *Proceedings of the Nishida Fest (Kinosaki 2003)*, 10:245–279, 2007.

[Law20] Tyler Lawson. E n-spectra and dyer-lashof operations. In *Handbook of homotopy theory*, pages 793–849. Chapman and Hall/CRC, 2020.

[Man19] Michael A Mandell. Operads and operadic algebras in homotopy theory. *arXiv preprint arXiv:1912.13047*, 2019.

[May06] J Peter May. *The geometry of iterated loop spaces*, volume 271. Springer, 2006.

[Sch12] Stefan Schwede. Symmetric spectra. *preprint, available from the author's homepage*, 2012.

[Sin06] Dev Sinha. The homology of the little disks operad. *arXiv preprint math/0610236*, 2006.

[Tod63] Hiroshi Toda. *Composition methods in homotopy groups of spheres*, volume 49. Princeton University Press, 1963.

[Zha21] Adela YiYu Zhang. Quillen homology of spectral lie algebras with application to mod p homology of labeled configuration spaces. *arXiv preprint arXiv:2110.08428*, 2021.

[Zha22] Adela YiYu Zhang. Operations on spectral partition lie algebras and taq cohomology. *arXiv preprint arXiv:2203.15771*, 2022.