Universiteit Utrecht

Faculty of Science

An ∞-categorical perspective on spectral sequences

Master Thesis

Antonie de Potter

Mathematical Sciences

First supervisor:

Dr. F.L.M. Meier
Mathematical Institute

Second supervisor:

Dr. G.S.K.S. Heuts
Mathematical Institute

October 11, 2023

Abstract

In this thesis, we describe spectral sequences from the perspective of ∞-categories. We focus on the approach using 'décalage' taken by Hedenlund in her PhD-thesis, that describes multiplicative structures on these spectral sequences. We use these results to show that the Leray-Serre-Atiyah-Hirzebruch spectral sequence admits a multiplicative structure.

Contents

1 Introduction

Spectral sequences were introduced by Jean Leray in 1946. He developed this technique to compute sheaf cohomology. It was realized that the concept of a spectral sequence was something more general. A major next step was Serre's thesis, where he introduced a spectral sequence related to fibrations. This allowed him to compute the mod 2 cohomology of the Eilenberg-Mac Lane spaces. The thesis had great impact on spreading the use of spectral sequences [16].

Given the prevalence of spectral sequence, it makes sense to study them from a categorical perspective. The Grothendieck spectral sequence is an example of this approach for some spectral sequences in algebraic geometry. Ordinary category theory fails to capture all the structure that is present on topological spaces. The concept of ∞-categories seeks to remedy this problem, and has proven useful in studying general structures in algebraic topology. In Higher Algebra [13], Lurie introduces a spectral sequence of a filtration in a stable ∞-category. In her thesis [8], Hedenlund shows that a multiplication on filtrations induces a multiplicative structure on the spectral sequence.

In this thesis, we give an overview of these results. We start by introducing spectral sequences in Section 2, and show how they make up a category. We then give a construction of the Atiyah-Hirzebruch spectral sequence as an example. In Section 3 we give an exposition of selected topics in higher category theory that are needed to understand spectral sequences in this context. The basic theory of ∞-categories is not treated here, we refer to [12] for an introduction. In Section 4 we demonstrate how a filtration or tower of spectra induces a spectral sequence, and investigate multiplicative properties of this procedure. In the last section, Section 5, we use these results to show that the Leray-Serre-Atiyah-Hirzebruch spectral sequence admits a multiplicative structure.

Acknowledgements
I want to first and foremost thank my supervisor Lennart Meier for all his help writing this thesis. His knowledge of the subjects is tremendous, and I was always surprised at the completeness of his answers. I am also thankful for his patience with the delays of my thesis. I also want to thank my family for their patience when I was busy again with work on the thesis. While my work is nowhere close to the quality of that from the famous composer J.S. Bach, I want to end with the same dedication as he did his compositions.

S.D.G

2 Spectral sequences

In this section we introduce the concept of spectral sequences. They are general tools in homological algebra, and can be seen as a generalization of exact sequences. In this section, we study them from a classical perspective, and define the associated notions. Afterwards we give an example: the Atiyah-Hirzebruch spectral sequence.

2.1 An introduction

Spectral sequences come in all shapes and sizes. To be more specific, they can have no grading, single grading or a bigrading. When the spectral sequence is graded, it can be homological and cohomological. Generally speaking, the more grading is present, the more structure is represented by the spectral sequence.

2.1.1 Definitions

We start with the definition of a spectral sequence without grading structure.

Definition 2.1. Let R be a ring. A **spectral sequence** is a collection of triples $(E_r, d_r, \phi_r)_{r\geq 1}$, where for all r

1. E_r is an R-module,
2. $d_r : E_r \to E_r$ is a homomorphism of R-modules and a differential, i.e. $d_r \circ d_r = 0$,
3. $\phi_r : E_{r+1} \to H(E_r, d_r) = \frac{\ker d_r}{\operatorname{im} d_r}$ is an isomorphism or R-modules.

As the isomorphisms ϕ_r are fixed, they are sometimes taken to be an equality in the definition. As we will be working in a categorical context, using isomorphisms makes more sense. It is common to call E_r the rth page of the spectral sequence, and by taking (co)homology we turn the page to E_{r+1}.

The structure of a grading on the spectral sequence allows a spectral sequence to contain more information. In fact, the most common used spectral sequences are the bigraded ones, that we will introduce shortly. Often, we will call all these just spectral sequences and the notation will make it clear what type of spectral sequence is used.

Definition 2.2. A **grading** on a module is a decomposition into a direct sum of abelian subgroups $M = \bigoplus_{i\in I} M_i$, where $I = \mathbb{Z}$ or $\mathbb{N}$. A morphism $f : M \to N$ is said to have degree p if $f(M_i) \subseteq N_{i+p}$ for all $i \in I$. Repeating this process results in a bigrading of the module, and a morphism $f : M \to N$ has bidegree (p, q) if $f(M_{i,j}) \subseteq N_{i+p,j+q}$.

There are three major grading conventions that are used for bigraded spectral sequences. In the literature, the notation often differs to reflect the type of grading present on the spectral sequence.

Definition 2.3. Let R be a ring and let $(E_r, d_r, \phi_r)_{r\geq 1}$ be a spectral sequence. Furthermore, assume that $E_r^{p,q}$ is a bigraded R-module. We say that the spectral sequence is

1. **homologically graded** if d_r has bidegree $(-r, r-1)$,
2. **cohomologically graded** if d_r has bidegree $(r, -r+1)$,
3. **Adams graded** if d_r has bidegree $(r, r-1)$.

Remark 2.4. We can shift between grading conventions using linear transformations on the indices. One common example is to present an Adams spectral sequence $E_r^{s,t}$ with degree $t-s$ on the horizontal axis and degree s on the vertical axis. The (Adams) bidegree of the differentials is then $(-1, r)$.

Remark 2.5. Notice that the difference of homology and cohomology only appears when grading is present. In Section 2.2 we will see that spectral sequences can easily be constructed without looking at grading, and later uncovering the grading structure that is present.

For our purposes, we want to encapsulate this information in an ordinary category. To do this, we need morphisms of spectral sequences.

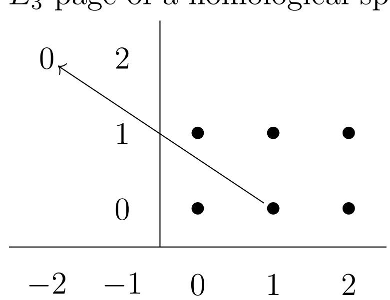
Definition 2.6. A **morphism of spectral sequences** $f : (D, d^D, \psi) \to (E, d^E, \phi)$ is a sequence of morphisms $f_r : D_r \to E_r$ of fixed degree, which satisfies the commutation relations $f_r \circ d_r = d_r \circ f_r$ and $H(f_r) \circ \psi_r = \phi_r \circ f_{r+1}$

The spectral sequences combined with these morphisms give an ordinary category of spectral sequences, which we denote by SSeq.

2.1.2 Convergence

To compute with a spectral sequence, it is useful to know what it converges to. The intuitive picture behind convergence is best understood when looking at a first-quadrant spectral sequence, that is a sequence with $E_r^{p,q} = 0$ if either $p < 0$ or $q < 0$. In homological grading, the differential has bidegree $(-r, r-1)$. Therefore, the incoming differential at $E_r^{p,q}$ starts at 0 if $q < r-1$ and ends at 0 if $p < r$. Then $E_{r+1}^{p,q} = \frac{\ker d_r}{\operatorname{im} d_r} \cong E_r^{p,q}$, so the spectral sequence stabilizes at this point. We denote the collection of these stabilized modules by E^∞.

Example of an E_3-page of a homological spectral sequence



The first-quadrant spectral sequence is only one example where convergence occurs. By different assumptions we can also get a certain stability which allows us to define an infinity page E^∞. For other types of convergence, we refer to [15] or [2]. We only describe one type of convergence which is most suited for our applications, namely in the setting of cohomological spectral sequences.

To describe convergence we first introduce the necessary background on filtrations.

Definition 2.7. Let R be a ring. A **descending filtration** of an R-module M is a sequence $\{F^pM\}_p$

$$M \supseteq \dots F^{-1}M \supseteq F^0M \supseteq F^1M \supseteq \dots \supseteq 0$$

of submodules of M. It is said to be convergent if the union $\bigcup F^pM = M$ and the intersection $\bigcap F^pM = 0$.

Definition 2.8. Let $\{F^pM\}_p$ be a descending filtration on an R-module M. The **associated graded** is defined to be $\mathrm{Gr}^p(M) := F^pM/F^{p+1}M$.

We can now define what it means for a spectral sequence to converge. This consists of two parts. The first is that the spectral sequence should stabilize in some way, this might even be in the limit. The second is that the stabilized spectral sequence is isomorphic to the associated graded of the chosen filtration.

Definition 2.9. [3, Definition 9.21] Let $(E_r^{p,q}, d_r, \phi_r)$ be a cohomological spectral sequence and M^* a graded R-module. Then E_r **converges** to M^* if

1. for each (p,q) there exists an r_0 such that $d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}$ is zero for all $r \geq r_0$; in particular there is an injection $E_{r+1}^{p,q} \hookrightarrow E_r^{p,q}$ for all $r \geq r_0$,
2. there is a convergent filtration of M^*, so that for each n, the limit $E_\infty^{p,q} = \cap_{r \geq r_0} E_r^{p,q}$ is isomorphic to the associated graded $\mathrm{Gr}^p(M^*)$.

We commonly denote this convergence as $E_2^{p,q} \implies M^{p+q}$

Strictly speaking a spectral sequence does not 'converge', but has a 'convergence structure'. This structure is then given by the chosen isomorphisms.

2.1.3 Multiplication

There are multiple examples of spectral sequences being multiplicative. However, the category of spectral sequences as we defined above is not monoidal (we introduce this notion in the next section). The notion of a bilinear map is well-defined.

Definition 2.10. Let $(C_r, d_r^C, \phi_r^C), (D_r, d_r^D, \phi_r^D)$ and (E_r, d_r^E, ϕ_r^E) be spectral sequences. A **bilinear map/pairing of spectral sequences** $\psi : (C_*, D_*) \to E_*$ is a collection of

$$\psi_r : C_r^{p,q} \otimes D_r^{p',q'} \to E_r^{p+p',q+q'}$$

such that

1. $d_r^E \psi_r = \psi_r(d_r^C \otimes 1 + 1 \otimes d_r^D)$,

2. the following diagram commutes

$$\begin{tikzcd}
C_{r+1} \otimes D_{r+1} \arrow[r, "\psi_{r+1}"] \arrow[d, "\phi_r^C \otimes \phi_r^D"'] & E_{r+1} \arrow[dd, "\phi_r^E"] \\
H(C_r) \otimes H(D_r) \arrow[d] & \\
H(C_r \otimes D_r) \arrow[r, "H(\psi_r)"'] & H(E_r)
\end{tikzcd}$$

Here the unnamed map is from the Künneth theorem.

Remark 2.11. In general, the lower left map of the second condition fails to be an isomorphism. This is why the category of spectral sequences is not expected to be monoidal.

2.2 Example: The Atiyah-Hirzebruch spectral sequence

We give a construction of the Atiyah-Hirzebruch spectral sequence as an example. This spectral sequence will be a corollary of the main result of this thesis.

2.2.1 Exact couples

The construction we give uses the concept of exact couples. This is one of the main ways of constructing spectral sequences in general. However, it is difficult to use this method for giving multiplicative structures. Therefore, the general approach we take in the rest of the thesis is not a generalization of the approach using exact couples. Because of the frequent use of exact couples in the literature, we still want to give this example using this approach.

Definition 2.12. An **exact couple** is a pair of R-modules (A, E) with maps

$$\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

such that the triangle is exact.

Note that $jk \circ jk = j(kj)k = 0$, so jk is a differential.

Lemma 2.13. *From an exact couple (A, E) arises a new exact couple (A_2, E_2), called the **derived exact couple**, with*

- $E_2 = H(E, jk)$ *and* $A_2 = \mathrm{im}(i)$
- i_2 *is the restriction of* i
- $k_2([e]) = k(e)$

- *$j_2(a) = j(b)$ for some $b \in A$ with $i(b) = a$*

The composition j_2k_2 is a differential at E_2.

The proof consists of checking that everything is well-defined and that the derived couple is exact, see [15, Proposition 2.7]. We can iterate this process to get a exact couples E_n with differentials d_n. Note that $E_{n+1} = H(E_n, d_n)$, so this sequence of exact couple actually form a spectral sequence.

For later use, we give an explicit description of E_{n+1} in terms of the original exact couple using induction.

Lemma 2.14. *Let*

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

be an exact couple. Then

$$E_{n+1} = \frac{k^{-1}(i^n A)}{j(\ker i^n)}.$$

Proof. We give a proof by induction. For the base case $n = 1$, notice that $\ker jk = k^{-1}(\ker j) = k^{-1}(\operatorname{im} i) = k^{-1}(iA)$ and $\operatorname{im} jk = j(\operatorname{im} k) = j(\ker i)$ by exactness. It follows that $E_2 = \frac{k^{-1}(iA)}{j(\ker i)}$.

Now assume that $E_n = \frac{k^{-1}(i^{n-1}A)}{j(\ker i^{n-1})}$. By definition, $E^{n+1} = \frac{\ker j_n k_n}{\operatorname{im} j_n k_n}$. Notice that $A_n = i^{n-1}A$ so $\operatorname{im} i_n = i^n(A)$. Again by exactness $\ker j_n k_n = k_n^{-1}(\operatorname{im} i_n) = k_n^{-1}(i^n A)$, and as $k_n([e]) = k_{n-1}(e)$ it follows by the induction hypothesis that $\ker j_n k_n = \frac{k^{-1}(i^n A)}{j(\ker i^{n-1})} \subseteq E_n$, which is the image of $k^{-1}(i^n A)$ under the quotient map.

By exactness we have

$$\begin{aligned} \operatorname{im}(j_n k_n) &= j_n(\ker i_n) \\ &= \{j_n(c) \mid c \in A_n : i_n(c) = 0\} \\ &= \frac{\{j(a) \mid a \in A : i^n(c) = 0\}}{j(\ker i^{n-1})} \\ &= \frac{j(\ker i^n)}{j(\ker i^{n-1})} \end{aligned}$$

It follows that

$$E_{n+1} = \frac{k^{-1}(i^n A)}{j(\ker i^n)}. \qquad \square$$

2.2.2 Cohomology theories

The Atiyah-Hirzebruch spectral sequence relates (co)homology theories to ordinary homology with local coefficients. There are many types of (co)homology, such as singular (co)homology, K-theories and even cobordism classes of manifolds. These all satisfy certain properties, like homotopy invariance and having exact sequences. This has led to the axiomatization of general (co)homology theories. The set of these axioms that describes these is called the Eilenberg-Steenrod axioms. We present reduced generalized cohomology theories and cohomology with local coefficients in this way.

Definition 2.15. A **reduced generalized cohomology theory** is a collection of functors

$$\left(\tilde{h}^s : \mathrm{Top}_* \to \mathrm{Ab}^{\mathrm{op}}\right)_{s\in\mathbb{Z}}$$

from pointed topological spaces to abelian groups, together with natural isomorphisms $\delta^s : \tilde{h}^s \circ \Sigma \simeq \tilde{h}^{s-1}$ called the **suspension isomorphism**, that satisfy the following properties:

1. (Homotopy invariance) If f_1, f_2 are two pointed homotopic morphisms in Top_*, then $f_1^* = f_2^*$,

2. (Exactness) For a morphism $f : X \to Y$, let C_f be the mapping cone and $i : Y \hookrightarrow C_f$ the inclusion. Then
$$\tilde{h}^s(C_f) \xrightarrow{\tilde{h}^s(i)} \tilde{h}^s(Y) \xrightarrow{\tilde{h}^s(f)} \tilde{h}^s(X)$$
is exact,

3. (Additive) Given a collection of pointed spaces X_i, then the natural maps induced by including each space combine to an isomorphism $\tilde{h}^s(\bigvee_{i\in I} X_i) \cong \prod_{i\in I} \tilde{h}^s(X_i)$.

Furthermore, the cohomology theory is called

4. **Ordinary**, if it has the **dimension** axiom: $\tilde{h}^s(S^0) = 0$.

When we restrict our attention to CW-complexes, these axioms also produce long exact sequences in the following way. Let (X, A) be a relative CW-complex, and $f : A \hookrightarrow X$ be the inclusion, then $C_f \simeq X/A$. Then we get a sequence

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \Sigma X/A$$

where every three terms form a cofiber sequence. Applying the cohomology theory to this sequence for different s we get the following long exact sequence

$$\ldots \longrightarrow \tilde{h}^{s-1}(X/A) \longrightarrow \tilde{h}^{s-1}(X) \longrightarrow \tilde{h}^{s-1}(A) \longrightarrow \tilde{h}^s(X/A) \longrightarrow \tilde{h}^s(X) \longrightarrow \tilde{h}^s(A) \longrightarrow \ldots$$

where the boundary map is induced by the composite of the suspension isomorphism with the map induced by $X/A \to \Sigma A$.

The Brown representability theorem states that every generalized cohomology theory is represented by a spectrum

Theorem 2.16 (Brown)**.** *[6, §4.E] Let $\tilde{h}^*$ be a reduced generalized cohomology theory of pointed CW-complexes. Then there exists an Ω-spectrum h such that $\tilde{h}^s(X) \simeq [X, h_s]$.*

Similarly, every spectrum defines a generalized homology theory by $\tilde{h}_s(X) \simeq \pi_s(\Sigma^\infty X \wedge h)$.

The following lemma is a tool that allows us to use the degree of a map in generalized cohomology.

Lemma 2.17. *[9, Lemma 2.3] Let $f : S^n \to S^n$ be a continuous map for some $n \geq 1$, and let $\tilde{h}$ be a reduced generalized cohomology theory. Then the induced map $f^* : \tilde{h}(S^n) \to \tilde{h}(S^n)$ is multiplication by* $\deg(f)$*.*

We now define cohomology with local coefficients in a similar axiomatic way. In all coming definitions we follow [19]. The general idea is that the coefficient group of the cohomology can vary along the space. We make the notion of varying coefficients precise in the following definition.

Definition 2.18. Let X be a topological space. A **local coefficient system** on X is a functor $\Pi_1(X) \to \mathrm{Ab}$ from the fundamental groupoid of X to the category of abelian groups. Local coefficient systems form a functor category.

To define cohomology with local coefficients axiomatically, we first need a correct category to map out of. In [19] this is defined for pairs of compactly generated spaces (X, A), we take the slightly more general stance with arbitrary topological spaces.

Definition 2.19. We define $\mathcal{L}^*$ to be the category with objects triples (X, A, G), where (X, A) is a pair of topological spaces, and G is a local coefficient system on X. A morphism $(\phi_1, \phi_2) : (X, A, G) \to (Y, B, H)$ consists firstly of a morphism of pairs $\phi_1 : (X, A) \to (Y, B)$. Taking fundamental groupoids is a functor, hence we get a functor $\Pi_1(\phi_1) : \Pi_1 X \to \Pi_1 Y$. Taking composition, we get a new local system $\phi_1^* H := H \circ \Pi_1(\phi_1) : \Pi_1 X \to \mathrm{Ab}$.

$$\begin{tikzcd}
\Pi_1 X \arrow[r, "G"] \arrow[d, "\Pi_1(\phi_1)"'] \arrow[dr, "\phi_1^* H"] & \mathrm{Ab} \\
\Pi_1 Y \arrow[r, "H"'] & \mathrm{Ab}
\end{tikzcd}$$

Then ϕ_2 is a natural transformation $\phi_1^* H \implies G$.

We note that the order of H and G is switched in the requirements for ϕ_2, and this results in the contravariance of cohomology. The notion of homotopy on $\mathcal{L}^*$ is quite intuitive.

Definition 2.20. Let $i_t : X \to I \times X$ be the map $i_t(x) = (t, x)$ and $p : I \times X \to X$ the projection $p(t, x) = x$. We define $I \times (X, A, G) = (I \times X, I \times A, p^* G)$ and $j_t = (i_t, 1)$. Let $\phi, \psi : (X, A, G) \to (Y, B, H)$ be morphisms in $\mathcal{L}^*$. A homotopy between them is a map $h : I \times (X, A, G) \to (Y, B, H)$ such that $h \circ j_0 = \phi$ and $h \circ j_1 = \psi$.

We can now define the cohomology axiomatically.

Theorem 2.21. *[19, §VI.2] There exists a collection of contravariant functors $H^n : \mathcal{L}^* \to$* Ab*, called cohomology with local coefficients, that satisfies the following properties:*

1. *(Homotopy invariance) If f_1, f_2 are homotopic morphisms in $\mathcal{L}^*$, , then the induced morphisms on cohomology groups are equal $f_1^* = f_2^*$.*

2. *(Exactness) If $(X, A, G) \in \mathcal{L}^*$ and $i : (A, *) \to (X, A)$, $j : (X, *) \to (X, A)$ inclusion maps, then there is a boundary morphism such that*

$$\ldots \longrightarrow H^{q-1}(A; G|_A) \xrightarrow{\delta^{q-1}} H^q(X, A; G) \xrightarrow{H^q(j)} H^q(X; G) \xrightarrow{H^q(i)} H^q(A; G|_A) \longrightarrow \ldots$$

   is exact.

3. *(Excision) Let X, X_1, X_2 be compactly generated spaces such that $X = \mathrm{int}(X_1) \cup \mathrm{int}(X_2)$ and G a local coefficient system in X. The inclusion induces an isomorphism for all q*

$$H^q(X, X_2; G) \cong H^q(X_1, X_1 \cap X_2; G|_{X_1})$$

4. *(Additive) Given a collection of pairs of spaces (X_i, A_i) and G a local coefficient system in $\bigcup_i X_i$, then*

$$H^q(\bigcup_{i\in I} X_i, \bigcup_{i\in I} A_i, G) = \bigoplus_{i\in I} H^q(X_i, A_i, G|_{X_i})$$

5. *(Dimension) For all $q \neq 0$ $H^q(*; G) = 0$ and $H^0(*; G) = G(*)$.*

The construction can be found in [19, §VI.2], together with a similar construction for homology with local coefficients. In particular, they are generalizations of ordinary (co)homology. While they are interesting in their own right and have applications such as in characteristic classes, we will only use this in the Atiyah-Hirzebruch spectral sequence.

2.2.3 The spectral sequence

We are now ready to prove the theorem on the spectral sequence.

Theorem 2.22. *Let X be a finite dimensional CW-complex and let h be an Ω-spectrum. Then there exists a spectral sequence*

$$E_2^{s,t} = H^p(X; \pi_{-q}h) \implies h^{p+q}X$$

We only prove the cohomological version, but a similar statement holds for homology. This proof takes inspiration from [18], [20] and [9].

Proof. As X is a finite dimensional CW-complex, it has a skeletal filtration

$$\emptyset = X^{-1} \subseteq X^0 \subseteq \ldots X^n = X$$

.

These fit in a short exact sequence

$$X^{s-1} \to X^s \to \mathrm{gr}^s X$$

where $\mathrm{gr}^s X = X^s/X^{s-1}$ This in turns gives us a long exact sequence, which we directly grade with Serre grading. That is, for every s there is an exact sequence

$$\cdots \to \tilde{h}^{s+t}(\mathrm{gr}^s X) \xrightarrow{k} \tilde{h}^{s+t}(X^s) \xrightarrow{i^*} \tilde{h}^{s+t}\left(X^{s-1}\right) \xrightarrow{j} \tilde{h}^{s+t+1}(\mathrm{gr}^s X) \to \cdots$$

From the long exact sequence we construct an exact couple with terms $A = \bigoplus_{s,t} \tilde{h}^{s+t}(X^s)$, $E_1 = E = \bigoplus_{s,t} \tilde{h}^{s+t}(\mathrm{gr}^s X)$, and direct sums of all the maps in the sequence. We therefore have a resulting spectral sequence, where we grade the first page $E_1^{s,t} = \tilde{h}^{s+t}(\mathrm{gr}^s X)$.

The differential on each page is $d_r = j_r \circ k_r$. We describe this differential in terms of the original maps, we start with an element $a \in E_r^{t,s}$, which is the domain of $d_r^{s,t}$. Then a has a representative b in $\tilde{h}^{s+t}(\mathrm{gr}^s X)$, to which we apply $k : \tilde{h}^{s+t}(\mathrm{gr}^s X) \to \tilde{h}^{s+t}(X^s)$. Then we write $k(b) = (i^*)^{r-1}(c)$ for some representative c from the image of $(i^*)^{r-1} : \tilde{h}^{s+t}(X^{s+r-1}) \to \tilde{h}^{s+t}(X^s)$ and then apply $j : \tilde{h}^{s+t}(X^{s+r-1}) \to \tilde{h}^{s+t+1}(\mathrm{gr}^{s+r} X)$ to c, and $j(c)$ is a representative for an element in $E_r^{s+r,t+1-r}$, so the differential has bidegree $(r, 1-r)$.

We now calculate the E_2-page. As the X^s form the skeletal filtration of a CW-complex, it holds that $\mathrm{gr}^s X = X^s/X^{s-1} \cong \bigvee_{e \in C_s} S^s$, where C_s indexes all the s-cells of X. Then

$$\begin{aligned} E_1^{s,t} &= \tilde{h}^{s+t}(\mathrm{gr}^s X) \\ &\cong \tilde{h}^{s+t}\Big(\bigvee_{e \in C_s} S^s\Big) \\ &\cong \bigoplus_{e \in C_s} \tilde{h}^{s+t}(S^s) \\ &\cong \bigoplus_{e \in C_s} \pi_{-t}(h_0) \end{aligned}$$

where the last step follows from Brown's representability theorem. By [19, Theorem 4.1*] this is the cellular cochain complex of degree s of X with values in $\pi_{-t}h$. If the differentials of the spectral sequence match the differentials of this complex, then the E_2-page is as claimed, by [19, Theorem 4.4*].

The differentials can be written as a matrix, decomposed on cells in the domain and image, as is common for differentials of cellular cochain complexes. The coefficients of this matrix for cellular cochain complexes are then given by the mapping degrees of the following compositions

$$f_{\alpha,\beta} : \partial D_\alpha^{s+1} \xrightarrow{\phi_\alpha} X^s \xrightarrow{\pi} X^s/X^{s-1} \xrightarrow{g_\beta} D_\beta^s/\partial D_\beta^s \xrightarrow{\cong} \partial D_\beta^{s+1}$$

In words, this means that we get the coefficients that determine the differential in the cellular complex as follows. We first take the attaching map $\phi_\alpha : \partial D_\alpha^{s+1} \to X^s$ of the $s+1$-cell α, then collapse the $(s-1)$-skeleton X_{s-1}, with a quotient map π. Then collapse all (s)-cells except β, thus picking out this single cell, with a quotient map g_β. Lastly we identify the resulting space $D^s/\partial D^s$ with ∂D^{s+1} using a fixed homeomorphism ψ. The degree of this map then gives the coefficient $d_{\alpha\beta}$.

The differential of the spectral sequence is given by $d_1 = j \circ k$. We look at the (α, β)-component of this map, as depicted in the following diagram.

$$\begin{tikzcd}
\tilde{h}^{s+t}(\operatorname{gr}^s X) \arrow[r, "k=\pi^*"] & \tilde{h}^{s+t}(X^s) \arrow[r, "j"] \arrow[d, "\phi^*_\alpha"] & \tilde{h}^{s+t+1}(\operatorname{gr}^{s+1} X) \arrow[d, dashed, "i^*_\alpha"] \\
\tilde{h}^{s+t}(D^s_\beta/\partial D^s_\beta) \arrow[u, "g^*_\beta"] & \tilde{h}^{s+t}(\partial D^{s+1}_\alpha) \arrow[r, "\cong"] & \tilde{h}^{s+t+1}(D^{s+1}_\alpha/\partial D^{s+1}_\alpha)
\end{tikzcd}$$

The lower right isomorphism results from the suspension isomorphism with the isomorphism ψ. Hence it is a connecting homomorphism in a long exact sequence, and by naturality of long exact sequences the lower right square is commutative. Now $i^*_\alpha \circ j \circ k \circ g^*_\beta$ is the (α, β)-component of d_1 and coincides with $f^*_{\alpha,\beta}$. By Lemma 2.17 this map is multiplication by $\deg(f_{\alpha,\beta})$, which proves that the differentials are the same. Hence $E_2^{s,t} = H^s(X, \pi_{-t})$

We now show the convergence of this sequence. Note that by the finiteness assumption on X, the first page $E_1^{s,t} = \tilde{h}^{s+t}(\operatorname{gr}^s X)$ is trivial for $s > n$ and $s < 0$. Hence, the spectral sequence stabilizes on $E_\infty = E_{n+1}$. By Lemma 2.14

$$E_\infty = \frac{k^{-1}(i^n A)}{j(\ker i^n)}.$$

Note that i^n maps out of the strip $0 \leq i \leq n$, so the kernel is everything. Hence, the denominator is $\operatorname{im} j = \ker k$. The numerator is $i^n \tilde{h}^{s+t}(X^{s+n}) = \operatorname{im}(\tilde{h}^{s+t}(X) \to \tilde{h}^{s+t}(X^s))$ as $X^{s+n} = X$. Hence, the ∞-page becomes

$$E_\infty^{s,t} = \frac{k^{-1}(\operatorname{im}(\tilde{h}^{s+t}(X) \to \tilde{h}^{s+t}(X^s)))}{\ker k : (\tilde{h}^{s+t}(X^s/X^{s-1}) \to \tilde{h}^{s+t}(X^s))}$$

There is an exact sequence by the first isomorphism theorem,

$$0 \to \ker k \to k^{-1}(\operatorname{im}(\tilde{h}^{s+t}(X) \to \tilde{h}^{s+t}(X^s))) \xrightarrow{\gamma} \operatorname{im}(\tilde{h}^{s+t}(X) \to \tilde{h}^{s+t}(X^s)) \cap \operatorname{im} k \to 0$$

which shows, combined with the fact that $\operatorname{im} k = \ker i$, that

$$E_\infty^{s,t} \cong \operatorname{im}(\tilde{h}^{s+t}(X) \to \tilde{h}^{s+t}(X^s)) \cap \ker i$$

.

We now have to provide a graded module with a convergent filtration, such that its associated graded is isomorphic to the E_∞-page. We will use the skeletal filtration on the CW-complex to construct this filtration. The inclusions in the skeletal filtration induce a sequence of maps on the cohomology groups

$$\tilde{h}^t(X) = \tilde{h}^t(X^n) \xrightarrow{i^*} \ldots \xrightarrow{i^*} \tilde{h}^t(X^2) \xrightarrow{i^*} \tilde{h}^t(X^1) \xrightarrow{i^*} \tilde{h}^t(X^0) = \tilde{h}^t(X^{-1}) = 0$$

which stabilizes by the finiteness assumption. We define a filtration on $\tilde{h}^t(X)$ by declaring

$$F^s \tilde{h}^t(X) = \ker(i^*)^{n-s} : \tilde{h}^t(X) \to \tilde{h}^t(X^s).$$

This gives us a descending filtration

$$\tilde{h}^t(X) = F^{-1}\tilde{h}^t(X) \supseteq \cdots \supseteq F^{n-1}\tilde{h}^t(X) \supseteq F^n\tilde{h}^t(X) = 0$$

We define a map

$$F^{s-1}\tilde{h}^{s+t}(X) = \ker\left(\tilde{h}^{s+t}(X) \to \tilde{h}^{s+t}\left(X^{s-1}\right)\right) \to \operatorname{im}(\tilde{h}^{s+t}(X) \to \tilde{h}^{s+t}(X^s)) \cap \ker i \cong E_\infty^{s,t}$$

that takes an element of $\tilde{h}^{s+t}(X)$ by application of i^* multiple times, to an element of $\tilde{h}^{s+t}(X^s)$. This map is surjective and its kernel is $F^s\tilde{h}^{s+t}(X)\ker(\tilde{h}^{s+t}(X) \to \tilde{h}^{s+t}(X^s))$. It follows by the first isomorphism theorem that

$$E_\infty^{s,t} \cong \frac{F^{s-1}\tilde{h}^{s+t}(X)}{F^s\tilde{h}^{s+t}(X)}$$

□

Remark 2.23. The Atiyah-Hirzebruch spectral sequence has a generalization to fiber bundles $F \to E \to B$. This spectral sequence also generalizes the well known Serre spectral sequence. The statement is as follows: Let $F \to E \to B$ be a fibration, with B a path-connected finite-dimensional CW-complex. Let h^* be a generalized cohomology theory. There is a spectral sequence

$$E_2^{p,q} = H^p(B; h^q F) \implies h^{p+q}(E)$$

It is this spectral sequence that will be the main application of the theory we develop in the rest of the thesis. In particular, we will prove that is has a multiplicative structure in Section 5.

3 Higher Category Theory

In this chapter, we give an exposition of concepts from higher category theory that are relevant to our goal of describing multiplicative spectral sequences. As such, we first look at how higher category theory tries to capture essential concepts from homotopy category, using stable ∞-categories and t-structures, among others. The main goal is to understand the category of spectra. Afterwards we describe how multiplicative structures are viewed from a categorical perspective.

To give a description of all prerequisites would be unnecessary. For an introduction to ∞-categories, we refer to [12] and [5]. All throughout, we use the word spaces for what is sometimes called ∞-groupoids or anima.

3.1 Stable ∞-categories

In this section we give an exposition of stable ∞-categories and of t-structures on such categories. The guiding example in this theory is the ∞-category of spectra, denoted Sp.

3.1.1 Foundations

There are multiple equivalent ways to define the ∞-category of spectra. Lurie gives four equivalent definitions in [13, §1.4]. We quickly work towards a definition which does not require development of the theory of stable ∞-categories, so that we can use it as an example throughout this section.

Recall that a spectrum is a sequence of pointed spaces $(X_i)_{i\in\mathbb{Z}}$ such that there are continuous maps $\Sigma X_i \to X_{i+1}$. By the loop-suspension adjunction there is a map $X_i \to \Omega X_{i+1}$. If this map is a weak equivalence, then we call X an Omega-spectrum.

We want to generalize this notion to the setting of ∞-categories. We start with determining what a pointed ∞-category is, and how to add a point.

Definition 3.1. [13, Definition 1.1.1.1] Let $\mathcal{C}$ be an ∞-category. A **zero object** is an object of $\mathcal{C}$ that is both initial and terminal. An ∞-category is **pointed** if it admits a zero object.

We can also add a zero object in most cases.

Definition 3.2. [12, p. 7.2.2] Let $\mathcal{C}$ be an ∞-category with a terminal object $*$. A **pointed object** is a morphism $X_+ : * \to X$ in $\mathcal{C}$. The full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by the pointed objects is denoted by $\mathcal{C}_*$. Equivalently we can define this as the under category $\mathcal{C}_* = \mathcal{C}^{*/}$.

The ∞-category of pointed objects has $* \to *$ as zero object, and is therefore pointed. Adding a basepoint is left adjoint to the forgetful functor $+ \dashv - : \mathcal{C} \rightleftarrows \mathcal{C}_*$. Note that the ∞-category $\mathcal{S}$ of spaces has as terminal object, namely the one point space $*$. From this we get the category of pointed spaces $\mathcal{S}_*$.

Definition 3.3. Let $\mathcal{C}$ be a pointed ∞-category.

- A **triangle** in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

  with 0 a zero object in $\mathcal{C}$.

- The triangle is a **fiber sequence** if it is a pullback square, and in this case we call it (or only X) the **fiber** of g.

- Similarly, the triangle is a **cofiber sequence** if it is a pushout square, and in this case we call it (or only Z) the **cofiber** of f.

The information of a triangle also includes a composition $h : X \to Z$ of $g \circ f$ which is null-homotopic. We say that a pointed ∞-category admits (co)fibers if every morphism has a (co)fiber.

In stable homotopy theory fibers and cofibers are the same. We can capture this idea now in ∞-categories.

Definition 3.4. An ∞-category $\mathcal{C}$ is **stable** if

1. $\mathcal{C}$ is pointed.
2. Every morphism of $\mathcal{C}$ admits a fiber and cofiber.
3. Every triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

We will define suspension and loop functors using this terminology. Let $\mathcal{C}$ be a pointed ∞-category, and assume that it admits cofibers. We denote by $\mathcal{C}^\Sigma \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ the full subcategory spanned by pushout diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

which are cofibers of maps $X \to 0$.

Evaluation at the initial vertex induces a trivial fibration $\mathcal{C}^\Sigma \to \mathcal{C}$ [13, pp. 23-24]. A trivial fibration has a section $s : \mathcal{C} \to \mathcal{C}^\Sigma$. Denoting evaluation at the final vertex by $e : \mathcal{C}^\Sigma \to \mathcal{C}$, we then define the suspension functor $\Sigma := e \circ s : \mathcal{C} \to \mathcal{C}$. In the depicted diagram, $\Sigma X = Y$. Dually, if $\mathcal{C}$ is pointed and admits fibers we define $\mathcal{C}^\Omega$ to be spanned by the fibers of $0 \to Y$. We take a section s' of the evaluation at the final vertex and compose this with evaluation at the initial vertex e' to define $\Omega := e' \circ s' : \mathcal{C} \to \mathcal{C}$.

Proposition 3.5. *Let $\mathcal{C}$ be a stable ∞-category. Then Σ and Ω are inverse equivalences.*

The suspension and loop functors actually give another characterization of stable ∞-categories

Theorem 3.6. *[5, Theorem 5.12] Let $\mathcal{C}$ be an finitely complete, finitely cocomplete and pointed ∞-category. The following are equivalent:*

1. *The ∞-category $\mathcal{C}$ is stable.*
2. *The functors Σ and Ω are inverse equivalences.*
3. *A square in $\mathcal{C}$ is a pullback square if and only if it is a pushout square.*

Definition 3.7. We denote by $X[n] := \Sigma^n X$ if $n \geq 0$ and $X[n] := \Omega^{-n} X$ if $n \leq 0$. We call this functor the translation functor.

We look at operations under which stability is conserved, which will be needed later in the thesis.

Proposition 3.8. *[13, p. 1.1.3] Let $\mathcal{C}$ be a stable ∞-category.*

- *Let K be a simplicial set. Then $Fun(K, \mathcal{C})$ is stable.*
- *Let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory with a zero object and stable under formation of fibers and cofibers. Then $\mathcal{C}'$ is a stable subcategory.*
- *Let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory which is stable under formation of cofibers and translations. Then $\mathcal{C}'$ is a stable subcategory.*

We will now define the main example of a stable ∞-category, namely the ∞-category of spectra.

Definition 3.9. [13, Remark 1.4.2.25] Let $\mathcal{S}$ be the ∞-category of spaces. The ∞-category of spectra Sp is given by the limit in Cat_∞ of the diagram

$$\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*$$

Now we have an example for the rest of the concepts we introduce in this section. We want to look into ways of creating spectra.

First, we define $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ as the forgetful functor on degree 0. This is an accessible functor that preserves small limits, and hence as a left adjoint, that we denote by $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$.

Example 3.10. Let $* \in \mathcal{S}$ be the final object. The **sphere spectrum** is $\mathbb{S} := \Sigma^\infty_+(*)$. This is an example of a **suspension spectrum** of a space $X \in \mathcal{S}$, namely $\Sigma^\infty_+(X)$.

3.1.2 The structure of the homotopy category

In this subsection we explore the structure of the homotopy category of a stable ∞-category $\mathcal{C}$. It is shown in [13, Theorem 1.1.2.14] that $h\mathcal{C}$ has the structure of a triangulated category. In this subsection, we will explore some implications of the fact that $h\mathcal{C}$ is a triangulated category, but will not spell out every detail.

A triangulated category is additive, has a translation functor and a collection of distinguished triangles satisfying certain axioms. We look at each of these properties in the case of $h\mathcal{C}$.

Definition 3.11. A category is **additive** if it admits finite products and coproducts, has a zero object and for every pair of objects there is an isomorphism $X \sqcup Y \to X \times Y$ given by the identities. Both are often denoted by $X \oplus Y$. Given two morphisms $f, g : X \to Y$, we can define addition as a composite

$$f + g : X \to X \times X \xrightarrow{f,g} Y \times Y \to Y \sqcup Y \to Y.$$

The last requirement for the category to be additive is that for every morphism f, there exists an inverse $-f$.

From the axioms it follows that composition in an additive category is bilinear. The axioms of an additive category can be rephrased to say that it is a finitely complete category enriched over abelian groups.
Let $\mathcal{C}$ be a stable ∞-category. Then $h\mathcal{C}$ is an additive category [13, Lemma 1.1.2.9]. The suspension functor is characterized by natural homotopy equivalences $\mathrm{Map}_{\mathcal{C}}(\Sigma(X), Y) \to \Omega \mathrm{Map}_{\mathcal{C}}(X, Y)$. This implies that $\pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^2(X), Y) \simeq \pi_2 \mathrm{Map}_{\mathcal{C}}(X, Y)$ is abelian. As the suspension functor is an equivalence, we can choose for every $Z \in \mathcal{C}$ an X such that $\Sigma^2(X) \simeq Z$. Hence $\pi_0 \mathrm{Map}_{\mathcal{C}}(Z, Y)$ is abelian. It can also be shown that $h\mathcal{C}$ admits finite coproducts.

The translation functor needed on a triangulated category is given by suspension $X \mapsto X[1]$. The triangles are meant to generalize the notion of fiber, cofiber and short exact sequences. Given a triangle in $\mathcal{C}$, we can extend it to

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0' \arrow[d] \\ 0 \arrow[r] & Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

and in the homotopy category this gives a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$. In this context we sometimes write $\mathrm{cofib}\, f$ for Z. General sequences like this are called **distinguished** or **exact triangles**. We present some of the axioms, that are satisfied in $h\mathcal{C}$ (for all axioms, see [13, Definition 1.1.2.5]):

1. Every morphism $f : X \to Y$ can be extended to a distinguished triangle
2. We can rotate $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ to $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$.
3. Given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

   and extend f, f' to distinguished triangles, then we can form a map $\mathrm{cofib}(f) \to \mathrm{cofib}(f')$ making the whole diagram commute.

4. Given three distinguished triangles extending f, g and $g \circ f$, then there is a distinguished triangle $\mathrm{cofib}\, f \to \mathrm{cofib}\, g \circ f \to \mathrm{cofib}\, g \to \mathrm{cofib}\, f[1]$.

3.1.3 t-structures

On stable ∞-categories, we can axiomatize homotopical properties by t-structures. The idea is that we define subcategories $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ that reflect the degrees in which the homotopy groups are trivial.

Definition 3.12. Let $\mathcal{D}$ be a triangulated category. A **t-structure** consists of a pair of full subcategories $\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0} \subseteq \mathcal{D}$ satisfying the following properties:

1. The subcategories are stable under translations in one direction: $\mathcal{D}_{\geq 0}[1] \subseteq \mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0}[-1] \subseteq \mathcal{D}_{\leq 0}$
2. For $X \in \mathcal{D}_{\geq 0}$ and $Y \in \mathcal{D}_{\leq 0}$ the mapping space $\mathrm{Map}_{\mathcal{D}}(X, Y[-1]) \simeq 0$
3. For any $Y \in \mathcal{D}$, there exists a fiber sequence $X \to Y \to Z$ with $X \in \mathcal{D}_{\geq 0}$ and $Z \in \mathcal{D}_{\leq 0}[-1]$

We write $\mathcal{D}_{\geq n} = \mathcal{D}_{\geq 0}[n]$ and $\mathcal{D}_{\leq n} = \mathcal{D}_{\leq 0}[n]$.

Remark 3.13. As the name suggests, a t-structure is not a property but a structure. Hence, it is possible to define t-structures that suit the situation.

Most of the ∞-categories we are interested in are stable. As the homotopy category of a stable ∞-category is triangulated, it is possible to define t-structures on them.

Definition 3.14. Let $\mathcal{C}$ be a stable ∞-category. Then a **t-structure** on $\mathcal{C}$ is a t-structure on the homotopy category h$\mathcal{C}$, and we denote by $\mathcal{C}_{\leq n}, \mathcal{C}_{\geq n}$ the full subcategories of $\mathcal{C}$ spanned on the objects of $(h\mathcal{C})_{\leq n}, (h\mathcal{C})_{\geq n}$.

The inclusions $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ and $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$ admit a left adjoint $\tau_{\leq n}$ and a right adjoint $\tau_{\geq n}$ respectively (see [13, p. 1.2.1.6]). These are called the truncation functors, and sometimes $\tau_{\geq n}$ is called a cover functor. Because $\mathcal{C}_{\leq n} \simeq \mathcal{C}_{\leq 0}[n]$, it follows that $\tau_{\leq n} X \simeq \tau_{\leq 0}(X[-n])[n]$, and similarly for the cover functors.

Remark 3.15. [13, Remark 1.2.1.8] Let $\mathcal{C}$ be a stable ∞-category and X an object. Then for each n there is a fiber sequence $\tau_{\geq n} \to X \to \tau_{\leq n-1} X$.

It can be shown that for all n, m there is an equivalence $\theta : \tau_{\leq m} \circ \tau_{\geq n} \to \tau_{\geq n} \circ \tau_{\leq m}$. These functors are core in the definition of homotopy groups.

Definition 3.16. Let $\mathcal{C}$ be a stable ∞-category. The intersection $\mathcal{C}^{\heartsuit} := \mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq 0}$ is called the **heart** or **core** of the t-structure.

We define the n-th **homotopy group functor** $\pi_n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ to be $\pi_n := \tau_{\geq 0} \circ \tau_{\leq 0} \circ [-n]$.

Note that we can recover the classical result $\pi_{n-1}(X[-1]) = \tau_{\geq 0} \circ \tau_{\leq 0} \circ [-n+1-1] = \pi_n(X)$ almost by definition.

Let $X, Y \in \mathcal{C}^{\heartsuit}$. We can identify $\mathrm{Hom}_{h\mathcal{C}}(X[n], Y) \simeq \pi_n \mathrm{Map}_{\mathcal{C}}(X, Y)$. By the second axiom for t-structures, this vanishes for $n > 0$, and hence $\mathcal{C}^{\heartsuit}$ is equivalent to its homotopy category $h\mathcal{C}^{\heartsuit}$, and we will often refer to $h\mathcal{C}^{\heartsuit}$ by $\mathcal{C}^{\heartsuit}$. Moreover, the homotopy category $h\mathcal{C}^{\heartsuit}$ is an abelian category [13, Remark 1.2.1.12].

Example 3.17. We demonstrate these concepts by looking at the ∞-category of spectra as an example. The t-structure may be defined by stating that $\mathrm{Sp}_{\leq -1}$ is spanned by the objects for which $\Omega^\infty(X)$ is contractible [13, Proposition 1.4.3.6].

Alternatively, the t-structure can be characterized by the full subcategory $\mathrm{Sp}_{\geq 0}$, generated under extensions and colimits by the essential image of the functor Σ^∞_+ [13, Remark 1.4.3.5].

The notation of a t-structure hints to localization to non-negative or non-positive degrees. In Sp this mirrors the notion of n-connected spaces. Precisely, the following equations hold [13, Proposition 1.4.3.6]:

$$\mathrm{Sp}_{\geq 0} = \{X \in \mathrm{Sp} \mid \pi_n X \simeq 0 \quad \forall n < 0\}$$
$$\mathrm{Sp}_{\leq 0} = \{X \in \mathrm{Sp} \mid \pi_n X \simeq 0 \quad \forall n > 0\}$$

The heart of the t-structure is equivalent to the category of abelian groups $\mathrm{Sp}^\heartsuit \simeq N(\mathrm{Ab})$[13, Proposition 1.4.3.6].

Example 3.18. Recall that Eilenberg-Mac Lane spectra associate to an abelian group a certain spectrum that represents (co)homology. We can also construct this for the ∞-category of spectra. Using the results from this section, we can construct a functor

$$H : \mathrm{Ab} \xrightarrow{N} N(\mathrm{Ab}) \simeq \mathrm{Sp}^\heartsuit \hookrightarrow \mathrm{Sp},$$

that we call the **Eilenberg-Mac Lane functor**. Reversing this construction, we see that $\pi_0 HA = A$.

3.2 Monoidal structures

One of the main goals of this section is to understand the symmetric monoidal structure on the ∞-category of spectra Sp given by the smash product. We first give some intuition by giving the classical definitions of all the concepts we are going to introduce. Afterwards, we define symmetric monoidal ∞-categories, strong and lax symmetric monoidal functors and algebra objects. We focus our definitions only on the symmetric monoidal case instead of the more general ordinary monoidal case, as the concepts are very similar. At the end of this thesis, we want to prove that certain functors are symmetric monoidal.

3.2.1 Symmetric monoidal (∞-)categories

To give some intuition for monoidal structures, we first give the definition in the classical case. The definitions reflect that of a monoid $(M, \cdot, 1)$, that is a set M with a multiplication $\cdot : M \times M \to M$ that is associative and a unit $1 \in M$. We call the monoid a commutative monoid in the case that the multiplication is commutative, and commonly write $(M, +, 0)$.

In category theory it is unnatural to require strict equality, so we introduce natural isomorphisms in that place.

Definition 3.19. Let $\mathcal{C}$ be a category. A **monoidal structure** on $\mathcal{C}$ is a tuple $(\otimes, I, \alpha, \lambda, \rho)$ where

- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, called the **tensor product**,
- I is an object of $\mathcal{C}$, called the **unit**,
- α is a natural isomorphism in three variables, $\alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$, expressing associativity,
- λ, ρ are natural isomorphisms, $\lambda_A : I \otimes A \cong A$ and $\rho_A : A \otimes I \cong A$ expressing left and right unitarity respectively.

In addition there are two coherence conditions, namely that the following diagrams commute

$$\begin{tikzcd} A \otimes (I \otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "1_A \otimes \lambda_B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes 1_B"] \\ & A \otimes B & \end{tikzcd}$$

$$\begin{tikzcd} A \otimes (B \otimes (C \otimes D)) \arrow[rr, "1_A \otimes \alpha_{B,C,D}"] \arrow[d, "\alpha_{A,B,C\otimes D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[d, "\alpha_{A,B\otimes C,D}"] \\ (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A\otimes B,C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[dl, "\alpha_{A,B,C} \otimes 1_D"] \\ & ((A \otimes B) \otimes C) \otimes D & \end{tikzcd}$$

The monoidal category is **symmetric** if there is is a natural isomorphism in two variables σ with $\sigma_A, B : A \otimes B \to B \otimes A$ expressing commutativity. It has to satisfy the following coherence conditions

- $\rho_A = \lambda_A \circ \sigma_{A,I}$,
- $1_B \otimes \sigma_{A,C} \circ \alpha_{B,A,C} \circ \sigma_{A,B} \otimes 1_C = \alpha_{B,C,A} \circ \sigma_{A,B\otimes C} \circ \alpha_{A,B,C}$ as maps $(A \otimes B) \otimes C \to B \otimes (C \otimes A)$,
- $1_{A\otimes B} = \sigma_{B,A} \circ \sigma_{A,B}$.

Remark 3.20. The coherence conditions are necessary to make sure that whichever way brackets are moved, the result is the same. It has been shown by Mac Lane [14] that these conditions are enough to guarantee that every way of applying associativity is the same.

Remark 3.21. In higher category theory, for a suitable notion of monoidal ∞-categories we expect that all coherence equations for associativity hold only up to homotopy. This induces an infinite sequence of coherence polyhedra, called the Stasheff associahedra. As this method is unwieldy in practice, a different definition of monoidal ∞-categories is given by generalizing a defining property of monoidal categories.

Definition 3.22. Let $\mathcal{C}$ be a monoidal category with tensor product $\otimes$. We define a new category $\mathcal{C}^{\otimes}$.

1. The objects of $\mathcal{C}^{\otimes}$ are, possibly empty, sequences of objects of $\mathcal{C}$, denoted $[C_1, \ldots, C_n]$.
2. The morphisms of $\mathcal{C}^{\otimes}$ from $[C_1, \ldots, C_n]$ to $[C'_1, \ldots, C'_m]$ consist of a non-strictly order-preserving map $f : [m] \to [n]$ and a corresponding (possibly empty) collection of morphisms $C_{f(i-1)+1} \otimes \cdots \otimes C_{f(i)} \to C_i$ for all $1 \leq i \leq m$
3. Composition follows from composition of order preserving maps, composition in $\mathcal{C}$ and the constraints given by the monoidal structure on $\mathcal{C}$.

We can define a forgetful functor $p : \mathcal{C}^{\otimes} \to \Delta^{op}$ by $p([C_1, \ldots, C_n]) = [n]$, which turns out to be an op-fibration of categories. Moreover, if we denote $\mathcal{C}^{\otimes}_{[n]}$ to be the fiber of p over $[n] \in \Delta^{op}$ then $\mathcal{C}^{\otimes}_{[1]} \simeq \mathcal{C}$, and the inclusions $\{i-1, i\} \subseteq [n]$ for all $1 \leq i \leq n$ induce an equivalence $\mathcal{C}^{\otimes}_{[n]} \simeq (\mathcal{C})^{\times n}$. These two properties actually capture all the information of the monoidal structure [10, p. 5]. We choose this notion as our starting point for a definition.

The correct notion of fibration that is needed is the coCartesian fibration. These are introduced in Appendix A.1

Definition 3.23. A **monoidal ∞-category** is a coCartesian fibration $p : \mathcal{C}^{\otimes} \to N(\Delta^{\mathrm{op}})$ such that for all $n \geq 0$ the functors $\mathcal{C}^{\otimes}_{n} \to \mathcal{C}^{\otimes}_{\{i,i+1\}}$ determine an equivalence

$$\mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{\{0,1\}} \times \cdots \times \mathcal{C}^{\otimes}_{\{n-1,n\}} \simeq \left(\mathcal{C}^{\otimes}_{[1]}\right)^{\times n}.$$

The fiber $\mathcal{C} = \mathcal{C}^{\otimes}_{[1]}$ is the underlying ∞-category of the monoidal ∞-category,

The projection $[1] \to [0]$ induces a functor $0 \simeq \mathcal{C}^{\otimes}_{[0]} \to \mathcal{C}^{\otimes}_{[1]} \simeq \mathcal{C}$, which determines the unit object in $\mathcal{C}$, unique up to equivalence.

The tensor product is given by the diagram

$$\mathcal{C} \times \mathcal{C} \simeq \mathcal{C}^{\otimes}_{\{0,1\}} \times \mathcal{C}^{\otimes}_{\{1,2\}} \xleftarrow{\theta} \mathcal{C}^{\otimes}_{[2]} \to \mathcal{C}^{\otimes}_{\{0,2\}} \simeq \mathcal{C}$$

where θ is an equivalence. Taking a homotopy inverse, we obtain a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, defined up to equivalence.

We now want to define a symmetric monoidal ∞-category. The lack of commutativity comes from the order-preserving structure in Δ^{op}. To remedy this, we introduce an alternative base for the fibration.

Definition 3.24. Let $\mathcal{F}$in denote the category of pointed finite sets. We denote an element by $\langle n \rangle = \{0 < 1 < \cdots < n\}$, with 0 the chosen basepoint. Morphisms from $\langle n \rangle$ to $\langle m \rangle$ only preserve the basepoint, they do not need to be order-preserving.

Denote by $\rho^{j,\langle n \rangle} : \langle n \rangle \to \langle 1 \rangle$ the unique pointed map with

$$\rho^{j,\langle n \rangle}(i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.25. A **symmetric monoidal ∞-category** is a coCartesian fibration $p : \mathcal{C}^{\otimes} \to N(\mathcal{F}\text{in})$ such that for all $n \geq 0$ the functors $\rho_!^{j,\langle n\rangle} : \mathcal{C}^{\otimes}_{\langle n\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle}$ determine an equivalence

$$\mathcal{C}^{\otimes}_{\langle n\rangle} \to \left(\mathcal{C}^{\otimes}_{\langle 1\rangle}\right)^{\times n}.$$

From here on, we will focus on the symmetric case. However, all concepts also have a pure monoidal variant. For these results, we refer to [5] and [10].

One handy example is that the identity functor $N(\mathcal{F}\text{in}) \to N(\mathcal{F}\text{in})$ is a symmetric monoidal ∞-category [5]. The next example is the tensor product we will use the most.

Example 3.26. The ∞-category of spectra Sp admits a symmetric monoidal structure, that is characterized by the two properties [11, p. 6.10]

1. The bifunctor $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ preserves small colimits in each variable.

2. The unit object is the sphere spectrum $\mathbb{S}$.

Example 3.27. The ∞-category of pointed spaces $\mathcal{S}_*$ also admits a monoidal structure. Analogous to the ∞-category of spectra it conserves colimits in each variable.[10, p. 4.2.9]

Example 3.28. Let $p : \mathcal{C}^{\otimes} \to N(\mathcal{F}\text{in})$ be a symmetric monoidal ∞-category.

1. Let K be a simplicial set. Then $\mathrm{Fun}(K, \mathcal{C}^{\otimes})$ is a symmetric monoidal ∞-category where the tensor product is defined pointwise [11, Remark 1.24].

2. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory that is stable under equivalence. Define $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ to be the full subcategory on the following objects: $C \in \mathcal{C}^{\otimes}_{\langle n\rangle}$ belongs to $\mathcal{D}^{\otimes}$ if and only if $\rho_!^{j,\langle n\rangle} : \mathcal{C}^{\otimes}_{\langle n\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle}$ sends it to an element of $\mathcal{D}$. Then $\mathcal{D}^{\otimes}$ is a symmetric monoidal ∞-category and the inclusion $\mathcal{D}^{\otimes} \subseteq \mathcal{C}^{\otimes}$ is a symmetric monoidal functor.

3.2.2 Monoidal functors

Symmetric monoidal ∞-categories can be compared using a suitable notion of functor between them. Intuitively, such a functor $F : \mathcal{C} \to \mathcal{D}$ has to send products to products, so there is a map

$$F(C_1 \otimes \cdots \otimes C_n) \to F(C_1) \otimes \cdots \otimes F(C_n).$$

Loosely speaking, this gives us three types of monoidal functors

- **strict** if the map is an equality,
- **strong** if the map is an equivalence,
- **lax** if this map exists.

For completeness we give the exact definition of these functors. We call a morphism $\alpha : \langle m\rangle \to \langle n\rangle$ in $\mathcal{F}\text{in}$ **inert** or **collapsing** if $\alpha^{-1}(i)$ is a singleton for every $1 \leq i \leq n$. Note that the ρ^j are inert.

Definition 3.29. Let $p : \mathcal{C}^\otimes \to N(\mathcal{F}\text{in})$ and $q : \mathcal{D}^\otimes \to N(\mathcal{F}\text{in})$ be symmetric monoidal ∞-categories. Let $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ be a functor such that the diagram commutes

$$\begin{tikzcd} \mathcal{C}^\otimes \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{D}^\otimes \arrow[dl, "q"] \\ & N(\mathcal{F}\text{in}) & \end{tikzcd}$$

We say that F is a

- **(strong) symmetric monoidal functor** if it sends p-coCartesian morphisms to q-coCartesian morphisms,
- **lax symmetric monoidal functor** if it sends p-coCartesian lifts of inert morphisms to q-coCartesian morphisms.

We can organize these in ∞-categories, namely as the full subcategories in $\mathrm{Map}_{N(\mathcal{F}\text{in})}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$, that is, the ∞-category of functors that satisfy the above diagram. We denote these by $\mathrm{Fun}^{\mathrm{Mon}}(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \subseteq \mathrm{Fun}^{\mathrm{Lax}}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$. The opposite $F^{op} : (\mathcal{C}^\otimes)^{op} \to (\mathcal{D}^\otimes)^{op}$

It will turn out that having a lax symmetric monoidal functor will be enough in most cases. This is due to the fact that algebra objects play nicely with these functors.
We now sum up some useful properties of lax symmetric monoidal functors. The opposite category of a symmetric monoidal ∞-category also has a symmetric monoidal structure. Given a lax symmetric monoidal functor, its opposite functor is called oplax symmetric monoidal, which is lax symmetric monoidal on the opposite categories.

Proposition 3.30.

1. *The composition of two symmetric monoidal functors is again symmetric monoidal.*
2. *The extraction of adjoints gives inverse equivalences between lax symmetric monoidal right adjoints and oplax symmetric monoidal left adjoints. [7, Proposition A]*

Proof. 1. Let $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ be a lax symmetric monoidal functor of symmetric monoidal ∞-categories $p : \mathcal{C}^\otimes \to N(\mathcal{F}\text{in})$ and $q : \mathcal{D}^\otimes \to N(\mathcal{F}\text{in})$. Let α be a p-coCartesian morphism such that $p(\alpha)$ is inert. By commutativity of $p = q \circ F$, we know that $F(\alpha)$ is also projected to $p(\alpha)$ by q, and is therefore a q-coCartesian lift of an inert morphism. Another lax symmetric monoidal functor G will hence send $p(\alpha)$ to another coCartesian morphism. $\square$

We now look at the interaction between t-structures and symmetric monoidal structures on stable ∞-categories.

Definition 3.31. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories. It is **exact** if it carries fiber sequences to fiber sequences.

Definition 3.32. Let $\mathcal{C}$ be a stable symmetric monoidal ∞-category with a t-structure. The t-structure is **compatible** with the symmetric monoidal structure if for every $C \in \mathcal{C}$ the functor $C \otimes -$ is exact, and $\mathcal{C}_{\geq 0}$ contains the unit and is closed under tensor products.

Example 3.33. The standard t-structure on the ∞-category of spectra is compatible with the smash product [10, Lemma 4.3.5].

When a t-structure is compatible with the symmetric monoidal structure, the full subcategory $\mathcal{C}_{\geq 0}$ is also a symmetric monoidal ∞-category.
The heart inherits a monoidal structure [10, Proposition 1.3.12 and Lemma 4.3.5]

Proposition 3.34. *The homotopy group functor*

$$\pi_* : \mathrm{Sp} \to \prod_{\mathbb{Z}} \mathrm{Ab}$$

is lax symmetric monoidal.

Intuition. We give some intuition behind this statement. The homotopy groups of a spectrum E are given by homotopy classes of maps $[\mathbb{S}^k, E]$. These fit together to a map

$$[\mathbb{S}^k, E] \otimes [\mathbb{S}^l, F] \to [\mathbb{S}^{k+l}, E \otimes F]$$

□

3.2.3 Algebra objects

To do algebra in category theory we need more multiplicative structure than just a tensor product on a category. It turns out to be useful to look at objects that possess their own multiplication. Multiple structures, like rings and R-algebras can be described in this way. We generalize this concept to ∞-categories, and conclude with the definition of an E_∞-ring.

Definition 3.35. An algebra object in a monoidal category $\mathcal{C}$ is an object X together with a multiplication map $\mu : X \otimes X \to X$ and an identity map $\eta : I \to X$, that satisfy the associativity and identity conditions. It is a commutative algebra object if this multiplication commutes.

Example 3.36. The category of abelian groups with tensor product over $\mathbb{Z}$ is a monoidal category with unit $\mathbb{Z}$. An algebra object in this category is a ring.

We want to define this notion for the context of symmetric monoidal ∞-categories using coCartesian fibrations. The idea of the definition is that a section of p picks out a tuple of objects of $\mathcal{C}$. Maps in $N(\mathcal{F}\mathrm{in})$ lift to maps between different tuples. We then add conditions to ensure that all these objects are isomorphic, and the maps between tuples describe the algebra structure.

Definition 3.37. Let $p : \mathcal{C}^\otimes \to N(\mathcal{F}\mathrm{in})$ be a symmetric monoidal ∞-category, with underlying ∞-category $\mathcal{C}$. A **commutative algebra object** of $\mathcal{C}$ is a section $A : N(\mathcal{F}\mathrm{in}) \to \mathcal{C}^\otimes$ of p, such that it sends inert morphisms to p-coCartesian morphisms of $\mathcal{C}^\otimes$.

The definition resembles that of a lax monoidal functor closely. We can even define the ∞-category of commutative algebra objects to be $\mathrm{CAlg}(\mathcal{C}^\otimes) = \mathrm{Fun}^{\mathrm{Lax}}(N(\mathcal{F}\mathrm{in}), \mathcal{C}^\otimes)$.

As we showed before, lax monoidal functors can be composed. Combined with the previous statement this implies that lax monoidal functors conserve commutative algebra objects. Let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a lax monoidal functor and $A : N(\mathcal{F}\mathrm{in}) \to \mathcal{C}^{\otimes}$ be an algebra object of $\mathcal{C}$. Then $F \circ A : N(\mathcal{F}\mathrm{in}) \to \mathcal{D}^{\otimes}$ is an algebra object of $\mathcal{D}^{\otimes}$.

Just as an algebra object is a lax monoidal functor, we can define a **coalgebra** object to be an oplax monoidal functor $N(\mathcal{F}\mathrm{in}) \to \mathcal{C}^{\otimes}$.
Algebra objects in the stable ∞-category of spectra have their own name.

Definition 3.38. An E_∞-ring is a commutative algebra object in the ∞-category of spectra.

4 From towers of spectra to spectral sequences

In this section we explore how the theory of ∞-categories that we gave an exposure of in the previous section can be used to construct spectral sequences.

The general idea is that we start with a sequence of spectra

$$\cdots \to X^{i+1} \to X^i \to X^{i-1} \to \ldots$$

We will use the tower convention in this thesis, that means that the morphisms go in decreasing order.

Definition 4.1. A **tower** or **filtration** on an ∞-category $\mathcal{C}$ is the functor category $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$.

From such a tower we construct a spectral sequence. This spectral sequence starts at the homotopy groups of the cofibers $\pi_{p+q}(\mathrm{cofib}\, f^p)$ and converges (under suitable conditions) to the homotopy group of the colimit $\pi_{p+q} \operatorname{colim} X^i$.

There are multiple general methods of going from a filtration on a space to an associated spectral sequence. In Section 2.2 we used the theory of exact couples. Another approach is that of Cartan-Eilenberg systems, this is the approach that Lurie generalizes in [13, §1.2.2]. We will sometimes take inspiration from this approach.

We will mostly focus, however, on giving a summary of the approach of [8] using 'décalage'. In this method, we construct the next page of our spectral sequence by constructing a new filtration, called the décalée. Loosely speaking, we turn the page already on the level of the filtration. The approach is in some places similar to the approach with Cartan-Eilenberg systems, but is more functorial. This makes it easier to show multiplicativity, one 'only' needs to show that the used functors are lax symmetric monoidal.

The end-goal of this section is the following statement:

Theorem 4.2. *There exists a functor* $E_*^{*,*} : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{SSeq}$ *which preserves the multiplicative structure.*

This will be made explicit in Theorems 4.22 and 4.24.

4.1 Towers of spectra

As we have seen in Section 2, the concept of the associated graded is very useful in talking about spectral sequences. We generalize the notion of the associated graded to a functor.

Definition 4.3. Let $X \in \mathrm{Tow}(\mathrm{Sp})$ be a filtration. We denote for $p < q$ the cofibers by $X(p)/X(q) := \mathrm{cofib}(X(q) \to X(p))$.

The q**th associated graded functor** is $\mathrm{Gr}^q : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Sp}$ defined by $\mathrm{Gr}^q(X) = X(q)/X(q+1)$. The **(total) associated graded functor** $\mathrm{Gr} : \mathrm{Tow}(\mathrm{Sp}) \to \prod_{\mathbb{Z}} \mathrm{Sp}$ is given by $\mathrm{Gr}(X) = (X(n)/X(n+1))_{n\in\mathbb{Z}}$

We will often encounter the associated graded in a pushout diagram together with other cofibers $X(p)/X(q)$. These are related to each other in the following way.

Proposition 4.4. *Let $X \in \mathrm{Tow}(\mathrm{Sp})$ be a filtration and let $i \geq j \geq k \geq l$. Then the square*

$$\begin{tikzcd}
X(k)/X(i) \arrow[r] \arrow[d] & X(l)/X(i) \arrow[d] \\
X(k)/X(j) \arrow[r] & X(l)/X(j)
\end{tikzcd}$$

is a pushout square.

Proof. Let $X \in \mathrm{Tow}(\mathrm{Sp})$ be a filtration. Let $i \geq j \geq k$ and denote $f : X(i) \to X(j)$ and $g : X(j) \to X(k)$. From the fact that we can 'compose' triangles in a stable ∞-category [13, Theorem 1.1.2.14], we have a triangle $\mathrm{cofib}(f) \to \mathrm{cofib}(g \circ f) \to \mathrm{cofib}(g)$. Using the shorthand notation $\mathrm{cofib}(f) = X(j)/X(i)$, this means that the diagram

$$\begin{tikzcd}
X(j)/X(i) \arrow[r] \arrow[d] & X(k)/X(i) \arrow[d] \\
0 \arrow[r] & X(k)/X(j) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout. Let $l \leq k$, and apply the procedure again to construct the diagram

$$\begin{tikzcd}
X(j)/X(i) \arrow[r] \arrow[d] & X(k)/X(i) \arrow[r] \arrow[d] & X(l)/X(i) \arrow[d] \\
0 \arrow[r] & X(k)/X(j) \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & X(l)/X(j)
\end{tikzcd}$$

where the left square and the outer rectangle are pushouts. By the pasting lemma for pushouts, the right square is also a pushout.

□

We now want to look at multiplicative properties of the associated graded. First we need to define a symmetric monoidal structure on $\mathrm{Tow}(\mathrm{Sp})$.

Definition 4.5. [8, p. 197] Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit I. The **Day convolution product** of two filtrations $X, Y \in \mathrm{Tow}(\mathcal{C})$ is degree-wise given by

$$(X \otimes Y)(n) \simeq \mathrm{colim}_{i+j \geq n} X(i) \otimes Y(j).$$

The unit is the tower

$$I_{\mathrm{Tow}(\mathcal{C})}(n) = \begin{cases} I & \text{if } n \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

For more details on the coherence of the Day convolution, we refer to [17].

We will mostly be interested in the information that exists in the homotopy category, as that is what factors through to the category of spectral sequences. To that end, we will define functors that preserve this structure.

Definition 4.6. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $X, Y, Z \in \mathrm{Tow}(\mathrm{Sp})$. A map $X \otimes Y \to Z$ is called a **pairing of filtered objects.**

We can now state how the associated graded interacts with Day convolution.

Proposition 4.7. *[8, Proposition II.1.13 and pp. 198-199] The total associated graded functor* $\mathrm{Gr} : \mathrm{Tow}(\mathrm{Sp}) \to \prod_{\mathbb{Z}} \mathrm{Sp}$ *is strong symmetric monoidal. Here we equip every degree of* $\prod_{\mathbb{Z}} \mathrm{Sp}$ *with the symmetric monoidal structure of* Sp*. Degreewise, there is an equivalence*

$$\mathrm{Gr}^q(X \otimes Y) \simeq \bigoplus_{i+j=q} \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y).$$

Moreover, let $\phi : X \otimes Y \to Z$ *be a map of filtrations. This induces pairings* $\mathrm{Gr}^{i,j}(\phi) : \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y) \to \mathrm{Gr}^{i+j}(Z)$ *for all integers* i, j*.*

There is also a notion of differential up to homotopy on the associated graded. This will later be used to define the differentials in our spectral sequence.

Proposition 4.8. *Let* $Y \in \mathrm{Tow}(\mathrm{Sp})$ *be a filtration. We then define the map* δ^r *to be the map in the following diagram of pushouts*

$$\begin{tikzcd}
\mathrm{Gr}^{i+1}(Y) \arrow[r] \arrow[d] & \frac{Y(i)}{Y(i+2)} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Gr}^i(Y) \arrow[r, "\delta^r"] & \mathrm{Gr}^{i+1}(Y[1]
\end{tikzcd}$$

Then $\delta^r \circ \delta^r$ *is zero up to homotopy.*

Proof. Using Proposition 4.4 we extend the defining diagram to the four central squares in the following diagram

$$\begin{tikzcd}
\frac{Y(i+1)}{Y(i+3)} \arrow[r] \arrow[d] & \frac{Y(i)}{Y(i+3)} \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[ddr, bend left, "\alpha_2"] & \\
\frac{Y(i+1)}{Y(i+2)} \arrow[r] \arrow[d] & \frac{Y(i)}{Y(i+2)} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, "\epsilon"] \arrow[dr, bend right, "\alpha_1"'] & \frac{Y(i)}{Y(i+1)} \arrow[r, "\delta^r"] \arrow[ddr, dashed, "\gamma"] & \frac{Y(i+1)}{Y(i+2)}[1] & \\
 & & & \frac{Y(i+1)}{Y(i+3)}[1] \arrow[ul, dotted, "\beta"']
\end{tikzcd}$$

. Taking the pushout of $0 \leftarrow \frac{Y(i+1)}{Y(i+3)} \rightarrow 0$, we get $\frac{Y(i+1)}{Y(i+3)}[1]$, together with the maps α_1, α_2 and β.
The left rectangle is also a pushout square, inducing by the universal property the map γ with $\gamma \circ \epsilon \simeq \alpha_1$.
We shift the entries in this diagram around to draw attention to the outer pushout square.

$$\begin{tikzcd}
\frac{Y(i+1)}{Y(i+3)} \arrow[r] \arrow[d] & \frac{Y(i)}{Y(i+3)} \arrow[r] \arrow[d] & 0 \arrow[d, "\alpha_2"'] \arrow[ddr] & \\
0 \arrow[r, "\epsilon"] \arrow[ur, phantom, "\ulcorner"] \arrow[rr, bend right, "\alpha_1"'] & \frac{Y(i)}{Y(i+1)} \arrow[r, dashed, "\gamma"] \arrow[drr, bend right, "\delta^r"'] & \frac{Y(i+1)}{Y(i+3)}[1] \arrow[dr, dotted, "\beta"] & \\
 & & & \frac{Y(i+1)}{Y(i+2)}[1]
\end{tikzcd}$$

Note that the whole rectangle is the outer pushout diagram from the previous picture. By the pasting lemma for pushouts, the right square is also a pushout. This implies that $\beta \circ \gamma \simeq \delta^r$.
We fit this triangle in the following diagram

$$\begin{tikzcd}
\frac{Y(i)}{Y(i+1)} \arrow[r, "\gamma"] \arrow[dr, "\delta"'] & \frac{Y(i+1)}{Y(i+3)}[1] \arrow[r] \arrow[d, "\beta"'] & \frac{Y(i+3)}{Y(i+3)}[2] \simeq 0 \arrow[d] \\
 & \frac{Y(i+1)}{Y(i+3)}[1] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \frac{Y(i+1)}{Y(i+3)}[2]
\end{tikzcd}$$

which is constructed using Proposition 4.4. We see that $\delta \circ \delta$ is zero up to homotopy. Passing to homotopy groups, it follows that $d^r \circ d^r = 0$.

$\square$

The pairings and differentials satisfy the following Leibniz rule.

Proposition 4.9 (Leibniz rule)**.** *[8, Theorem II.1.21] Given a pairing of filtrations $\phi : X \otimes Y \to Z$, the diagram*

$$\begin{tikzcd}
\mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y) \arrow[r, "\delta^r_X \otimes 1 \oplus 1 \otimes \delta^r_Y"] \arrow[d, "\mathrm{Gr}^{i,j}(\phi)"] & \left(\mathrm{Gr}^{i+1}(X) \otimes \mathrm{Gr}^j(Y) \oplus \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^{j+1}(Y)\right)[1] \arrow[d, "\mathrm{Gr}^{i+1,j}(\phi) \oplus \mathrm{Gr}^{i,j+1}(\phi)"] \\
\mathrm{Gr}^{i+j}(Z) \arrow[r, "\delta^r_Z"] & \mathrm{Gr}^{i+j+1}(Z)[1]
\end{tikzcd}$$

commutes for all integers i, j.

Given a spectrum, we can construct a tower by killing homotopy groups over or under a varying degree. These are called the Postnikov and Whitehead tower respectively. We will only make use of the latter one. To describe this tower, we introduce the induced t-structure on filtrations. Later, we define another t-structure on Tow(Sp), namely the Beilinson t-structure. Therefore, we call this the canonical t-structure.

Definition 4.10. Let $\mathcal{C}$ be a stable ∞-category with a t-structure. Then there is a functor $\tau^{\mathrm{can}}_{\geq 0} : \mathrm{Tow}(\mathcal{C}) \to \mathrm{Tow}(\mathcal{C})$ given by $\tau^{\mathrm{can}}_{\geq 0}(X)(n) = \tau_{\geq n}X(n)$.
The essential image of this functor is $\mathrm{Tow}(\mathcal{C})^{\mathrm{can}}_{\geq 0} = \{X \in \mathrm{Tow}(\mathcal{C}) \mid X(n) \in \mathcal{C}_{\geq n}\}$ This determines a t-structure on $\mathrm{Tow}(\mathcal{C})$, called the **canonical t-structure**.

This t-structure is well-defined, for the proof we refer to [8, Proposition II.1.22]. The essence is that any (co)localization functor satisfying certain stability properties determines a t-structure, see also [13, Proposition 1.2.1.16]. We use this t-structure to define the Whitehead filtration.

Definition 4.11. Let $X \in \mathrm{Sp}$ be a spectrum. The **Whitehead filtration** of X is a filtration $\tau_{\geq \bullet}X \in \mathrm{Tow}(\mathrm{Sp})$ given in each degree by $(\tau_{\geq \bullet}X)(n) \simeq \tau_{\geq n}X$.

It can be shown that the canonical t-structure is compatible with the Day convolution product [8, Proposition II.1.23]. This implies the lax monoidality of the Whitehead filtration.

Proposition 4.12. *[8, p. II.1.26] The Whitehead filtration $\tau_{\geq \bullet} : \mathcal{C} \to \mathrm{Tow}(\mathcal{C})$ is lax symmetric monoidal.*

Proof. As the t-structure is compatible, by [11, Propositions 1.26 and 1.31] not only the truncation functors are lax symmetric monoidal, but also the inclusion $I : \mathrm{Tow}(\mathrm{Sp})_{\geq 0} \to \mathrm{Tow}(\mathrm{Sp})$ is symmetric monoidal. By [1, Proposition 2.21], the colimit functor $\mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Sp}$ is left adjoint to the symmetric monoidal functor $\mathrm{Sp} \to \mathrm{Tow}(\mathrm{Sp})$ sending each spectra to a constant tower on that spectra. Therefore, the colimit functor is also lax monoidal. The Whitehead filtration is the right adjoint of $\mathrm{colim} \circ I$, so it is lax monoidal by part 2 of Proposition 3.30. $\square$

Definition 4.13. The **Beilinson t-structure** on $\mathrm{Tow}(\mathrm{Sp})$ is defined by

$$\begin{aligned}\mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei}}_{\geq n} &:= \{X \in \mathrm{Tow}(\mathrm{Sp}) \mid \mathrm{Gr}^q(X) \in \mathrm{Sp}_{\geq n-q}\, \forall q\} \\ \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei}}_{\leq n} &:= \{X \in \mathrm{Tow}(\mathrm{Sp}) \mid X(q) \in \mathrm{Sp}_{\leq n-q}\, \forall q\}\end{aligned}$$

The Beilinson t-structure is well-defined [8, Proposition II.2.1] and is also compatible with Day convolution, see [1, Proposition 6.13].

Theorem 4.14. *[8, Theorems II.2.10 and 11] There is a equivalence of categories $\mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit} \simeq \mathrm{Ch}(\mathrm{Ab})$. Explicitly, a filtration $X \in \mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit}$ is sent to the chain complex*

$$\cdots \longrightarrow \pi_1\left(\mathrm{Gr}^{-1}(X)\right) \longrightarrow \pi_0\left(\mathrm{Gr}^{0}(X)\right) \longrightarrow \pi_{-1}\left(\mathrm{Gr}^{1}(X)\right) \longrightarrow \cdots$$

Moreover, this equivalence is symmetric monoidal, where $\mathrm{Ch}(\mathrm{Ab})$ has the standard tensor product and $\mathrm{Tow}(\mathrm{Sp})^{\mathrm{Bei},\heartsuit}$ has the symmetric monoidal structure given by $X \otimes_{\mathrm{Bei},\heartsuit} Y = \tau^{\mathrm{Bei}}_{\leq 0}(X \otimes Y)$.

Definition 4.15. Let $X \in \mathrm{Tow}(\mathrm{Sp})$. The **décalée of** X is the filtration

$$\cdots \to \mathrm{colim}_i(\tau^{\mathrm{Bei}}_{\geq n+1}X)(i) \to \mathrm{colim}_i(\tau^{\mathrm{Bei}}_{\geq n}X)(i) \to \mathrm{colim}_i(\tau^{\mathrm{Bei}}_{\geq n-1}X)(i) \to \dots$$

obtained by applying the colimit to every term of the Beilinson-Whitehead tower of X. This gives a functor $\mathrm{Déc} : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp})$. Degree wise this reduces to

$$\mathrm{Déc}(X)(n) = \mathrm{colim}_i(\tau^{\mathrm{Bei}}_{\geq n}X)(i).$$

Proposition 4.16. *[8, Proposition II.2.19] The functor* Déc *is lax symmetric monoidal.*

Proof. The Beilinson-Whitehead filtration $\tau^{\mathrm{Bei}}_{\geq \bullet} : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Tow}(\mathrm{Sp}))$ is lax symmetric monoidal, similarly to the proof of Proposition 4.12. By [10, Proposition 1.5.10] the whole functor is lax symmetric monoidal. □

The next construction needs the definition of coherent cochain complexes.

Definition 4.17. [1, Definition 2.1] Define Ch to be the ordinary category with as objects $\mathbb{Z} \cup \{*\}$ where $\{*\}$ is a zero object and as morphisms

$$\mathrm{Ch}(n, m) = \begin{cases} \{\partial_n, 0\} & \text{if } m = n - 1 \\ \{id, 0\} & \text{if} m = n \\ \{0\} & \text{otherwise} \end{cases}$$

Let $\mathcal{C}$ be a pointed ∞-category. The ∞-category of **coherent cochain complexes** is the full subcategory $\mathrm{Ch}^*(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Ch}^{op}, \mathcal{C})$ spanned by pointed functors.

Construction 4.18. We construct an Eilenberg-Mac Lane functor $H : \mathrm{Ch}(\mathrm{Ab}) \to \mathrm{Sp}$ according to the theory of [1]. There is a symmetric monoidal functor $\mathcal{I} : \mathrm{Ch}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp})$ defined as $\mathcal{I}C^\bullet = \mathrm{Map}_{\mathrm{Ch}(\mathrm{Sp})}(\mathbb{S}^\bullet_{[\bullet]}, C)$, where $\mathbb{S}^n_{[n]}(m) = \mathbb{S}[n]$ if $m = n$ and 0 otherwise [1, pp. 2, 13–14].
Then define the **Eilenberg-Mac Lane functor** $H : \mathrm{Ch}(\mathrm{Ab}) \to \mathrm{Sp}$ as $H := \mathrm{colim} \circ \mathcal{I}$.

We will need the following two properties of the Eilenberg-Mac Lane functor.

Proposition 4.19. *The Eilenberg-Mac Lane functor is lax symmetric monoidal*

Proof. As stated in the construction, $\mathcal{I}$ is symmetric monoidal. We showed in the proof of Proposition 4.12 that the colimit functor is lax symmetric monoidal. The result follows. □

Proposition 4.20. *Homotopy groups of Eilenberg-Mac Lane complexes give cohomology groups [1, Proposition 8.10]. Let $A^\bullet$ be a cochain complex of abelian groups, then*

$$\pi_{-n} H A^\bullet \simeq H^n(A^\bullet)$$

Theorem 4.21. *[8, Theorem II.2.20 and II.2.22] There is an equivalence*

$$\mathrm{Gr} \circ \mathrm{Déc} \simeq H \circ \Sigma^{tot} \circ \pi^{\mathrm{Bei}}_*$$

of functors $\mathrm{Tow}(\mathrm{Sp}) \to \prod_{\mathbb{Z}} \mathrm{Sp}$*, where we apply* $H : \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab}) \to \prod_{\mathbb{Z}} \mathrm{Sp}$ *levelwise and* $\Sigma^{tot} : \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab}) \to \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab})$ *is defined by* $\Sigma^{tot}(C^\bullet_n)_{n\in\mathbb{Z}} = (C^\bullet_n[n])_{n\in\mathbb{Z}}$*. Moreover, this equivalence is lax symmetric monoidal.*
Degreewise, this equivalence is expressed as $\mathrm{Gr}^q \, \mathrm{Déc}(X) \simeq H(\pi^{\mathrm{Bei}}_q(X)[q])$*.*

4.2 Spectral sequences

We have now defined all necessary ingredients for the construction of the spectral sequence of a filtration. Moreover, all involved functors are lax symmetric monoidal. In this section we combine all these results to show that a filtration gives rise to a multiplicative spectral sequence.

Theorem 4.22. *Let $X \in \mathrm{Tow}(\mathrm{Sp})$ be a filtration. There is a spectral sequence where the elements of each page are determined by*

$$E^r_{n,s} = \pi_n \operatorname{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X))$$

The differential is induced by the indicated connecting homomorphism in the following pushout diagram:

$$\begin{tikzcd}
\operatorname{Gr}^{i+1}(\mathrm{Déc}^{r-1}(X)) \arrow[r] \arrow[d] & \frac{\mathrm{Déc}^{r-1}(X)(i)}{\mathrm{Déc}^{r-1}(X)(i+2)} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \operatorname{Gr}^{i}(\mathrm{Déc}^{r-1}(X)) \arrow[r, "\delta^r"] & \operatorname{Gr}^{i+1}(\mathrm{Déc}^{r-1}(X)[1]
\end{tikzcd}$$

where $i = (r-1)n + s$. Taking homotopy groups π_n gives a homological Adams graded differential $d^r : E^r_{n,s} \to E^r_{n-1,s+r}$ of bidegree $(-1, r)$.

Proof. We first check that $d^r \circ d^r = 0$. This is the result of Proposition 4.8, applied to $Y = \mathrm{Déc}^{r-1}(X)$.

We now show the existence of the required isomorphisms $E^{r+1}_{n,s} \cong H(E^r_{n,s}, d^r)$.
By Theorem 4.21 it holds that $\operatorname{Gr}^q \mathrm{Déc}(X) \simeq H(\pi^{\mathrm{Bei}}_q(X)[q])$. We use Proposition 4.20 to compute the homotopy groups of this formula. Note that we are now applying H to a chain complex instead of a cochain complex, which reverses the indexing. Taking the homotopy groups we get

$$\begin{aligned}
\pi_n(\operatorname{Gr}^q \mathrm{Déc}(X)) &= \pi_n(H(\pi^{\mathrm{Bei}}_q(X)[q])) \\
&= \frac{\ker \delta : \pi_n(\operatorname{Gr}^{q-n}(X)) \to \pi_{n-1}(\operatorname{Gr}^{q-n+1}(X)))}{\operatorname{im}(\delta : \pi_{n+1}(\operatorname{Gr}^{q-n-1}(X)) \to \pi_n(\operatorname{Gr}^{q-n}(X)))}
\end{aligned}$$

We apply this formula to an element of the E^{r+1}-page.

$$\begin{aligned}
E^{r+1}_{n,s} &= \pi_n \operatorname{Gr}^{rn+s}(\mathrm{Déc}^r(X)) \\
&= \frac{\ker(\delta : \pi_n(\operatorname{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X))) \to \pi_{n-1}(\operatorname{Gr}^{(r-1)n+s+1}(\mathrm{Déc}^{r-1}(X))))}{\operatorname{im}(\delta : \pi_{n+1}(\operatorname{Gr}^{(r-1)n+s-1}(\mathrm{Déc}^{r-1}(X))) \to \pi_n(\operatorname{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X))))} \\
&= \frac{\ker d^r : E^r_{n,s} \to E^r_{n-1,s+r}}{\operatorname{im} d^r : E^r_{n+1,s-r} \to E^r_{n,s}} \\
&= H(E^r_{n,s}, d^r)
\end{aligned}$$

$\square$

Note that $E^1_{n,s} = \pi_n \operatorname{Gr}^s(X)$. Up to grading, this is the same first page as is constructed in [13, §1.2.2]. We can therefore expect that these two constructions of spectral sequences actually lead to the same result. In that same section it is shown that, under certain conditions, the spectral sequence converges. This leads us to conjecture that this spectral sequence also converges. We mirror the statement [13, Proposition 1.2.2.14], and compare with the convergence statements in [4]. A very similar convergence statement is shown in [1, Theorem 9.1], and in [1, Remark 9.2] it is said that future work by Hedenlund-Krause-Nikolaus might shine more light on convergence statements.

Conjecture 4.23. *Let $X \in \operatorname{Tow}(\mathrm{Sp})$ be a filtration such that $X(n) \simeq 0$ for $n \ll 0$. Then the spectral sequence converges strongly*

$$E^r_{n,s} \implies \pi_{n+s}(\operatorname{colim} X)$$

We now show that the spectral sequence is multiplicative, in the sense of Definition 2.10.

Theorem 4.24. *Let $\phi : X \otimes Y \to Z$ be a pairing of filtrations. Then there exists an induced pairing on spectral sequences*

$$\phi : E(X) \otimes E(Y) \to E(Z)$$

Proof. Let $\phi : X \otimes Y \to Z$ be a pairing of filtrations. By Proposition 4.7, this induces a map $\operatorname{Gr}^i(X) \otimes \operatorname{Gr}^j(Y) \to \operatorname{Gr}^{i+j}(Z)$. We then apply the homotopy group functor, which is lax symmetric monoidal by Proposition 3.34. This results in the pairing $\pi_n \operatorname{Gr}^i(X) \otimes \pi_{n'} \operatorname{Gr}^j(Y) \to \pi_{n+n'} \operatorname{Gr}^{i+j}(Z)$.
Recall that Déc is a lax monoidal functor by Proposition 4.16, so when we apply it repeatedly to the pairing ϕ we get a new pairing $\text{Déc}^{r-1}(\phi) : \text{Déc}^{r-1}(X) \otimes \text{Déc}^{r-1}(Y) \to \text{Déc}^{r-1}(Z)$. We substitute this in place of our original pairing, and let $i = (r-1)n + s$, $j = (r-1)n' + s'$, to get a pairing

$$\begin{aligned} \psi^r : \pi_n \operatorname{Gr}^{(r-1)n+s}(\text{Déc}^{r-1}(X)) \otimes \pi_{n'} \operatorname{Gr}^{(r-1)n'+s'}(\text{Déc}^{r-1}(Y)) \\ \to \pi_{n+n'} \operatorname{Gr}^{(r-1)(n+n')n+s+s'}(\text{Déc}^{r-1}(Z)) \end{aligned} \tag{4.1}$$

which is a pairing of our spectral sequence.

We now check that the pairing satisfies the required conditions from Definition 2.10. First we check that $d^r\psi = \psi(d^r \otimes 1 + 1 \otimes d^r)$. Applying Proposition 4.9 to the pairing $\text{Déc}^{r-1}(\phi)$, and afterwards applying the lax monoidal homotopy group functor π_n from Proposition 3.34 results in the wanted equation.

The second condition is that

$$\begin{tikzcd}
E(X)^{r+1} \otimes E(Y)^{r+1} \arrow[r, "\psi^{r+1}"] \arrow[d, "\phi^r_E(X)\otimes\phi^r_E(Y)"'] & E(Z)^{r+1} \arrow[dd, "\phi^r_E"] \\
H(E(X)^r) \otimes H(E(Y)^r) \arrow[d] & \\
H(E(X)^r \otimes E(Y)^r) \arrow[r, "H(\psi^r)"'] & H(E(Z)^r)
\end{tikzcd}$$

commutes.

Applying the lax symmetric monoidality of π_* 3.34, the equivalence $\mathrm{Gr} \circ \mathrm{Déc} \simeq H \circ \Sigma^{Tot} \circ \pi_*^{\mathrm{Bei}}$ 4.21 and of the Eilenberg-Mac Lane functor H 4.19 to the pairing ψ^{r+1} 4.1

$$\begin{tikzcd}
\pi_n(\mathrm{Gr}^p \mathrm{Déc}^r X) \otimes \pi_j(\mathrm{Gr}^q \mathrm{Déc}^r Y) \arrow[r] \arrow[d] & \pi_{n+n'}(\mathrm{Gr}^{p+q} \mathrm{Déc}^r Z) \arrow[d, "\cong"] \\
\pi_{n+n'}(H(\pi_p^{\mathrm{Bei}} \mathrm{Déc}^{r-1} X \otimes \pi_q^{\mathrm{Bei}} \mathrm{Déc}^{r-1} Y)[p+q]) \arrow[r] & \pi_{n+n'}(H(\pi_{p+q}^{\mathrm{Bei}} \mathrm{Déc}^{r-1} Z)[p+q])
\end{tikzcd}$$

where $p = rn + s$ and $q = rn' + s'$ $\square$

Remark 4.25. The information of this theorem can be neatly packaged in the context of ∞-operads. Every (symmetric) monoidal ∞-category is an ∞-operad, and the category of spectral sequences can be thought of as a multicategory, which is also an operad. In this setting, the theorem would be as follows: the functor $E^*_{*,*} : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{SSeq}$ admits the structure of a map of ∞-operads. As SSeq is a 1-category, it is not necessary to check the higher coherences. As Tow(Sp) is symmetric monoidal, every multilinear map is a composite of bilinear map. Hence, this statement reduces to the theorem we have just proven.

5 Multiplicative generalized Serre spectral sequence

One of the classical examples of spectral sequences has been the Serre spectral sequence. This spectral sequence is used to describe the singular (co)homology of a the total space of a (Serre) fibration in terms of the base space and the fiber

$$E_2^{p,q} = H^p\left(B, H^q(F)\right) \implies H^{p+q}(X)$$

The spectral sequence we want to construct is a common generalization of the Serre spectral sequence and the Atiyah-Hirzebruch spectral sequence that we constructed in Theorem 2.22. We will do this using the theory of the previous sections. We already stated this result in the classical context before Remark 2.23, but now we are equipped to prove that it has a multiplicative structure.

5.1 Some definitions

To describe this spectral sequence, we first need to understand all the ingredients. In this section we look at the coalgebra structure on a space and cohomology with values in spectra. The starting point of our spectral sequence is the ∞-category of spaces. We explore how every space is a coalgebra object.

Definition 5.1. A monoidal structure on $\mathcal{C}$ is **Cartesian** if the unit object is final and $C \otimes 1 \leftarrow C \otimes D \rightarrow 1 \otimes D$ exhibits $C \otimes D$ as the product of C and D for all objects $C, D \in \mathcal{C}$

Existence of Cartesian structures is guaranteed in categories with finite products.

Proposition 5.2. *[11, Proposition 2.8 (5)] Let $\mathcal{C}$ be an ∞-category that admits finite products. Then $\mathcal{C}$ has a Cartesian monoidal structure that is symmetric.*

A Cartesian structure on $\mathcal{C}$ induces a coCartesian structure on $\mathcal{C}^{op}$. Then [11, Corollary 2.18] implies that the forgetful functor $\mathrm{CAlg}(\mathcal{C}^{\mathrm{op}}) \rightarrow \mathcal{C}^{op}$ is a trivial Kan fibration.

Example 5.3. The ∞-category of spaces $\mathcal{S}$ admits finite products. It follows that we can see every space as a coalgebra object. Intuitively, we can see the diagonal map as this coalgebra structure.

The following concept is explored more in [12, §4.4.4]. Note that the ∞-category of spectra Sp admits small colimits [13, Corollary 1.4.4.2 and Proposition 1.4.4.4].

Definition 5.4. Let $h \in \mathrm{Sp}$ be a spectrum and $X \in \mathcal{S}$ a space. We define the **tensor product** to be $h \otimes X_+ = \mathrm{colim}_X \, \mathrm{const}_h$

We will mostly be interested in the adjoint of this operation.

Proposition 5.5. *Let $X \in \mathcal{S}$ be a space. Then the tensor operation $- \otimes X_+ : \mathrm{Sp} \rightarrow \mathrm{Sp}$ has a right adjoint, which we denote by $(-)^{X_+}$.*

Proof. This is by the Adjoint functor theorem, [12, Corollary 5.5.2.9]. Both $\mathcal{S}$ and Sp are presentable.
Note that we can write

$$\begin{aligned} h \otimes X_+ &= \operatorname{colim}_X \operatorname{const}_h = \operatorname{colim}_X \operatorname{const}_{h\otimes\mathbb{S}} \\ &= \operatorname{colim}_X h \otimes \operatorname{const}_{\mathbb{S}} \\ &= \operatorname{colim}_X h \otimes \Sigma^\infty_+ \mathbb{S} = h \otimes \Sigma^\infty_+ X \end{aligned}$$

by the generating properties of the sphere spectrum $\mathbb{S}$, see [13, Corollary 1.4.4.6]. We can therefore factor the tensor operation as follows

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, "-\otimes X_+"] \arrow[d, "id_{\mathrm{Sp}}\times\{X\}"'] & \mathrm{Sp} \\ \mathrm{Sp}\times\mathcal{S} \arrow[r, "id_{\mathrm{Sp}}\times\Sigma^\infty_+"'] & \mathrm{Sp}\times\mathrm{Sp} \arrow[u, "-\otimes-"'] \end{tikzcd}$$

All these functors preserve colimits, and hence $-\otimes X_+$ preserves colimits. By applying the Adjoint functor theorem we get the result.

□

We can think of $(-)^{X_+}$ as $\hom(X,-)$, just as in the ordinary tensor-hom adjunction. We want to use this construction to define cohomology. Recall from Brown's theorem 2.16 that reduced generalized cohomology theories have the shape $\tilde{h}^s(-) \simeq [-, h_s]$, so cohomology is like the contravariant hom-functor.

We now define $h^{(-)_+} : \mathcal{S}^{op} \to \mathrm{Sp}$ and show it is lax monoidal if h is a ring spectrum.

Construction 5.6. Note that taking mapping spaces forms a functor $\mathrm{Map}_{\mathrm{Sp}}(-,-) : \mathrm{Sp}^{op}\times\mathrm{Sp} \to \mathcal{S}$. We fix a spectrum $h \in \mathrm{Sp}$ for the second component, and take the n-fold suspension. Then we precompose with $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$ to form the functor $\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+(-), \Sigma^n h) : \mathcal{S}^{op} \to \mathcal{S}$ for each n. Taken together, they form a functor to spectra, as for each $X \in \mathcal{S}$

$$\begin{aligned} \Omega \operatorname{Map}_{\mathrm{Sp}}(\Sigma^\infty_+(X), \Sigma^n h) &\simeq \operatorname{Map}_{\mathrm{Sp}}(\Sigma\Sigma^\infty_+(X), \Sigma^n h) \\ &\simeq \operatorname{Map}_{\mathrm{Sp}}(\Sigma^\infty_+(X), \Sigma^{n-1} h) \end{aligned}$$

We denote this functor as $h^{(-)_+} := \lim_n \operatorname{Map}_{\mathrm{Sp}}(\Sigma^\infty_+(-), \Sigma^n h)$.

Proposition 5.7. *Let $h \in \mathrm{Sp}$ be an E_∞-ring spectrum. We view the ∞-category of spaces with the coalgebra structure that comes from the product. Let $\Delta : X \to X \times X$ be the diagonal map on X. Then there exists a map $h^{(\Delta)_+} : h^{X_+} \otimes h^{X_+} \to h^{(X\times X)}$.*

Proof. As $h \in \mathrm{Sp}$ is an E_∞-ring spectrum, we have a map $h \otimes h \to h$. When we apply the adjunction $-\otimes X_+ \dashv (-)^{X_+}$ to the identity $h^{X_+} \to h^{X_+}$, we get a map $h^{X_+} \otimes X_+ \to h$. Applying this map two times, we get

$$h^{X_+} \otimes h^{X_+} \otimes (X \times X)_+ \to h \otimes h \to h$$

where the last map comes from the ring spectrum. When we take the adjoint of this composite, we get a map

$$h^{X_+} \otimes h^{X_+} \to h^{(X \times X)_+}$$

□

Proposition 5.8. *Let $X \in \mathcal{S}$ be a space and $h \in \mathrm{Sp}$ be a spectrum. Then $\pi_n h^{X_+} = h^n(X)$ is a reduced generalized cohomology theory.*

Proof. By the adjunction $\Omega^\infty \vdash \Sigma^\infty_+$, it holds for every n that

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+(X), \Sigma^n h) \simeq \mathrm{Map}_{\mathcal{S}}(X, \Omega^\infty \Sigma^n h)$$

Now

$$\begin{aligned} \pi_n h^{X_+} &\simeq \pi_0 \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+(X), \Sigma^n h) \\ &\simeq \pi_0 \mathrm{Map}_{\mathcal{S}}(X, \Omega^\infty \Sigma^n h) \end{aligned}$$

By Brown's representability theorem 2.16, this is a reduced generalized cohomology theory.

□

Remark 5.9. As a last preparation, we look at cohomology with local coefficients in this setting. Let $X \in \mathcal{S}$ be a space and $F : X \to \mathrm{Sp}$ be a functor. Note that $\mathrm{Fun}(X, \mathrm{Sp})^\heartsuit \simeq \mathrm{Fun}(X, \mathrm{Sp}^\heartsuit) \simeq \mathrm{Fun}(X, \mathrm{Ab})$. Since Ab is a 1-category, higher coherences are suppressed, and we can identify this with $\mathrm{Fun}(\tau_{\leq 1} X, \mathrm{Ab})$ and $\tau_{\leq 1} X$ can be identified with $\Pi_1 X$. Under these identifications, we recover the original definition of a local coefficient system.

We can also describe cohomology with local coefficients in a categorical way. Let $X \in \mathcal{S}$ be a space and $G : X \to \mathrm{Ab}$ a functor. Then we define the **cohomology of X with local coefficients in** G to be $H^s(X; G) := \pi_{-s}(\lim_X G)$.
Note that we can form a local coefficient system by applying the homotopy group to the composition $h^{(-)_+} \circ F$ for $h \in \mathrm{Sp}$ and $F \in \mathrm{Fun}(X, \mathcal{S})$. It is precisely this local coefficient system that we will use in the next section. We write $h^s(F) := \pi_{-s}(h^{(-)_+} \circ F)$.

5.2 The spectral sequence

We construct the Leray-Serre-Atiyah-Hirzebruch spectral sequence using the tools developed until this point.

Theorem 5.10. *Let $B \in \mathcal{S}$ be a space, $F \in \mathrm{Fun}(B, \mathcal{S}^{\mathrm{op}})$ be a functor classifying a right fibration $p : E \to B$ in $\mathcal{S}$. Furthermore, let $h \in \mathrm{Sp}$ be a spectrum. If the filtration*

$$Y(\bullet) = \lim_B (\tau_{\geq \bullet} \circ h^{(-)_+} \circ F)$$

exists, then it induces a spectral sequence

$$E_1^{p,q} = H^p(B; h^q(F))$$

Proof. By Theorem 4.22, a filtration induces a spectral sequence

$$E^r_{n,s} = \pi_n \operatorname{Gr}^{(r-1)n+s}(\text{Déc}^{r-1}(Y)).$$

Taking $r = 1$ we get

$$\begin{aligned} E^1_{n,s} &= \pi_n \operatorname{Gr}^s Y \\ &= \pi_n(\operatorname{cofib}(Y(s+1) \to Y(s))) \end{aligned}$$

We calculate the cofibers. Note that for a general object X, we can construct a fiber sequence around $\tau_{\geq s}X$ using Remark 3.15. This fiber sequence is $\tau_{\geq s+1}X = \tau_{\geq s+1}\tau_{\geq s}X \to \tau_{\geq s}X \to \tau_{\leq s}\tau_{\geq s}X$. Moreover $\tau_{\leq s}\tau_{\geq s}X \simeq \tau_{\leq s}(\tau_{\geq 0}(X[-s])[s]) \simeq (\tau_{\leq 0}\tau_{\geq 0}X[-s]) \simeq \pi_s X[s]$. These facts imply that,

$$\begin{aligned} &\operatorname{cofib}(\lim_B(\tau_{\geq s+1} \circ h^{(-)+} \circ F) \to \lim_B(\tau_{\geq s} \circ h^{(-)+} \circ F)) \\ &= \lim_B(\operatorname{cofib}(\tau_{\geq s+1} \circ h^{(-)+} \circ F \to \tau_{\geq s} \circ h^{(-)+} \circ F)) \\ &= \lim_B \pi_s(h^{(-)+} \circ F)[s] \end{aligned}$$

So the E^1-page becomes

$$E^1_{n,s} = \pi_n \lim_B \pi_s(h^{(-)+} \circ F)[s] = \pi_{n-s} \lim_B \pi_s(h^{(-)+} \circ F)$$

Now $h^{(-)+} \circ F : B \to \mathrm{Sp}$ is a local system.

$$\begin{aligned} E^1_{n,s} &= \pi_{n-s} \lim_B \pi_s(h^{(-)+} \circ F) \\ &= H^{s-n}(B; \pi_s(h^{(-)+} \circ F)) \\ &= H^{s-n}(B; h^{-s}(F)) \end{aligned}$$

with differentials of bidegree $(-1, r)$. We regrade this spectral sequence as $p = s - n + 1$ and $q = -s - 1$, to get a cohomologically graded spectral sequence $E_1^{p,q} = H^p(B; h^q(F))$. ∎

5.3 Multiplicativity

Theorem 5.11. *Let $B \in \mathcal{S}$ be a space, $F \in \operatorname{Fun}(B, \mathcal{S}^{op})$ be a right fibration and let $h \in \mathrm{Sp}$ be a ring spectrum. Then the spectral sequence is multiplicative.*

Proof. The spectral sequence is induced by the filtration $Y = \lim_B(\tau_{\geq \bullet} \circ h^{(-)+} \circ F) \in \operatorname{Tow}(\mathrm{Sp})$. By Theorem 4.24 we need to exhibit a pairing $Y \otimes Y \to Y$. The first step will be to show that $\tau_{\geq \bullet} \circ h^{(-)+} \circ F$ is a diagram valued in $\operatorname{Alg}(\operatorname{Tow}(\mathrm{Sp}))$.

By Example 5.3 it holds that every space in $\mathcal{S}$ is a coalgebra object in $\operatorname{CAlg}(\mathcal{S}^{\mathrm{op}})$. This means that $F : B \to \mathcal{S}^{op}$ is valued in coalgebra objects $\operatorname{Alg}(\mathcal{S}^{op})$. We can define $F \otimes F$ pointwise, and the coalgebra structure on the image of F gives a map $F(x) \to F(x) \otimes F(x)$ for every

$x \in B$. Now h is a ring spectrum, and if we apply $h^{(-)_+} : \mathcal{S}^{op} \to \mathrm{Sp}$ to this map, then by Proposition 5.7 we get a pairing $h^{F(x)_+} \otimes h^{F(x)_+} \to h^{(F(x)\times F(x))_+}$.
By Proposition 4.12, $\tau_{\geq \bullet} : \mathrm{Sp} \to \mathrm{Tow}(\mathrm{Sp})$ is lax symmetric monoidal, so it preserves the pairing.
In Theorem 5.10 we have seen that the limit exists in Tow(Sp). By [11, Proposition 4.17] it also factors over algebra objects. This means that Y has a pairing, and thus we get a multiplicative spectral sequence. $\square$

A Some results on fibrations

In this appendix we state some results that we use in the main text on (co)Cartesian fibrations, and left/right fibrations. The theory is mostly taken from [12], chapters 2-4. We remark already that every left fibration is a coCartesian fibration, so all results that are stated for the first also hold for the latter.

In higher category theory it is often difficult to give a concrete description of an ∞-category. Therefore, we often use certain fibrations to prove higher categorical statements. The notion of (co)Cartesian fibrations will allow us to talk about fibers that are ∞-categories, while left and right fibrations have fibers in spaces.

A.1 coCartesian and Cartesian fibrations

coCartesian fibrations are also inner fibrations, so we first define these.

Definition A.1. An **inner fibration** of simplicial sets is a morphism $f : X \to S$ that has the right lifting property with respect to all inner horn inclusions $\Lambda^n_i \subseteq \Delta^n$ for all $0 < i < n$.

We now define (co)Cartesian fibrations, this is done in two steps.

Definition A.2. Let $q : X \to S$ be an inner fibration of simplicial sets, and $e : x \to y$ an edge in X. The edge e is **q-cartesian** if for all $n \geq 2$ there exists a lift in all diagrams of the form

$$\begin{tikzcd} \Delta^1 \arrow[d, "i"] \arrow[dr, "e"] & \\ \Lambda^n_n \arrow[r, "\sigma"] \arrow[d] & X \arrow[d, "q"] \\ \Delta^n \arrow[r, "\sigma'"'] \arrow[ur, dashed] & S \end{tikzcd}$$

where i is the inclusion into the $n-1, n$ edge.

Dually, the edge e is **q-cocartesian** when we take the left outer horn inclusion Λ^n_0 and include e on the first edge.

Remark A.3. Looking at the bottom square, this would be the property that q has the right lifting property with respect to right outer horn inclusions. The top triangle adds the restriction that the 'last' edge of the horn is send to e. The condition that $n \geq 2$ comes from the fact that otherwise we would can not include the edge into the horn.

Definition A.4. Let $q : X \to S$ be an inner fibration of simplicial sets, it is a **(co)Cartesian fibration** if for every edge $e : s \to t$ in S and every lift $\hat{t}$ of t (ie $q(\hat{t}) = t$) there is a q-(co)cartesian edge $\hat{e} : \hat{s} \to \hat{t}$ which is a lift of e (ie $q(\hat{e}) = e$).

These notions are dual, an inner fibration $q : X \to S$ is a Cartesian fibration if and only if q^{op} is a coCartesian fibration.

Given a coCartesian fibration $p : X \to S$, we have for each point $s \in S$ a fiber $X_s := X \times_S \{s\}$, which is an ∞-category. We now also want that each edge $s \to t$ determines a functor $X_s \to X_t$ on the fibers. This is made explicit in the straightening theorem, which is one of the most important uses of (co)Cartesian fibrations.

Theorem A.5. *Let $\mathcal{C}$ be a small ∞-category. There is an equivalence of categories*

$$\mathrm{coCart}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty)$$

By $\mathrm{coCart}(\mathcal{C})$ we mean the full subcategory of $sSet/\mathcal{C}$ spanned by coCartesian fibrations. Fibers in the context of geometry have an important application. Given a curve on a manifold, one can lift this to a map of fibers. An example is the parallel transport on a vector bundle along a smooth curve. A similar lifting problem can also be solved for (co)Cartesian fibrations.

Proposition A.6. *[**Kerodon**][Proposition 5.2.2.4]*
Let $q : X \to S$ be a a coCartesian fibration, and let $e : s \to t$ in $\mathcal{C}$ be an edge. Then there exists a functor $e_! : X_s \to X_t$, called the covariant transport.

Dually, if $q : X \to S$ is a Cartesian fibration, there exists a contravariant transport functor $e^* : X_t \to X_s$. These functors occur in the definition of a monoidal ∞-category in section 3.2

A.2 Left and right fibrations

The theory of (co)Cartesian fibrations is most usefull in its generality, and is the prefered notion of fibration in Cat_∞. However, this generality can also make them more difficult to work with. When working in the ∞-category of spaces $\mathcal{S}$, we can make use of left and right fibrations instead. While most results that we state also hold for (co)Cartesian fibrations, the statements often have a simpler form for left and right fibrations.

Definition A.7. A **left fibration** of simplicial sets is a morphism $f : X \to S$ that has the right lifting property with respect to all horn inclusions $\Lambda^n_i \subseteq \Delta^n$ for all $0 \leq i < n$. A **right fibration** has this property for all $0 < i \leq n$.

We denote for a simplicial set $S \in \mathrm{sSet}$ by $\mathrm{LFib}(S)$ the full subcategory in $\mathrm{sSet}\,/S$ spanned by left fibrations $X \to S$, and similarly $\mathrm{Rfib}(S)$. Note that by duality $\mathrm{LFib}(S) \simeq \mathrm{RFib}(S^{op})$.

We remark that if q is a right fibration, then every edge is q-cartesian, so q is a Cartesian fibration. Dually, every left fibration is a coCartesian fibration. In this sense, the notion of right fibration is stronger than that of a Cartesian fibration, which in turn is stronger than that of an inner fibration.

The straightening theorem becomes easier in the setting of left fibrations.

Corollary A.8. *Let $\mathcal{C}$ be a small ∞-category. There is an equivalence of categories*

$$\mathrm{LFib}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{S})$$

Now left fibrations over spaces are actually Kan fibrations

Proposition A.9. *[12][2.1.3.3] Let X be a simplicial set and $S \in \mathcal{S}$ a space. Then for a morphism of simplicial sets $f : X \to S$, the following are equivalent*

1. *f is a left fibration,*
2. *f is a right fibration,*
3. *f is a Kan fibration.*

As every morphism in $\mathcal{S}$ factors as an equivalence and a Kan fibration, the full subcategory of left fibrations over $B \in \mathcal{S}$ are just all morphisms in $\mathcal{S}/B$. The straightening theorem in this setting is as follows.

Corollary A.10. *Let $B \in \mathcal{S}$ be a space. There is an equivalence of categories*

$$\mathcal{S}/B \xrightarrow{\sim} \mathrm{Fun}(B^{op}, \mathcal{S})$$

The straightening theorem allows us to define a universal left fibration $q : \mathcal{S}_* \to \mathcal{S}$.
For the precise definition we refer to [12][3.3.2]. The main point is that we can write every left fibration as a pullback of this universal fibration along a functor.

Proposition A.11. *[12][3.3.2.8] Let $p : X \to S$ be a left fibration of simplicial sets. Then there exists a map $F : S \to \mathcal{S}$ such that p is the pullback of q along F.*

$$\begin{tikzcd} X \arrow[r, "\tilde{F}"] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{S}_* \arrow[d, "q"] \\ S \arrow[r, "F"] & \mathcal{S} \end{tikzcd}$$

*We say that p is **classified** by F.*

Remark A.12. The notion of universal fibrations holds more generally in the context of (co)Cartesian fibrations, with classifying functors $F : S \to \mathrm{Cat}_\infty$. If we look at classifying functors $F : S \to \mathcal{S}$, then by [12][Proposition 3.3.2.5], these only classify left fibrations, so we do not need to concern ourselves here with (co)Cartesian fibrations.

There is a nice relation between limits and colimits of diagrams in spaces and the left fibrations these diagrams classify.

Proposition A.13. *[12][3.3.3.4 and 3.3.4.6] Let K be a simlicial set, $p : K \to \mathcal{S}$ a diagram and $X \to K$ a left fibration classified by p. Then there are the following natural isomorphisms in the homotopy category of spaces $h\mathcal{S}$:*

- $\lim p \simeq \mathrm{Map}_K(K, X)$
- $\mathrm{colim}\, p \simeq X$

References

[1] Stefano Ariotta. *Coherent cochain complexes and Beilinson t-structures, with an appendix by Achim Krause*. 2021. arXiv: 2109.01017 [math.AT].

[2] J. Michael Boardman. "Conditionally convergent spectral sequences". In: 239 (1999), pp. 49–84. DOI: 10.1090/conm/239/03597.

[3] James F Davis and Paul Kirk. *Lecture notes in algebraic topology*. Vol. 35. American Mathematical Society Providence, 2001.

[4] Rok Gregoric. *Grothendieck Spectral Sequence in Higher Algebra*. Available online at https://web.ma.utexas.edu/users/gregoric/Notes.html.

[5] Moritz Groth. *A short course on ∞-categories*. 2015. arXiv: 1007.2925 [math.AT].

[6] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[7] Rune Haugseng et al. *Lax monoidal adjunctions, two-variable fibrations and the calculus of mates*. 2023. arXiv: 2011.08808 [math.CT].

[8] A. P. Hedenlund. "Multiplicative Tate Spectral Sequences". Available online at https://www.mn.uio.no/math/personer/vit/rognes/theses/hedenlund-thesis.pdf. PhD thesis. 2021.

[9] Yiannis Loizides. *The Atiyah-Hirzebruch Spectral Sequence*. Available online at https://math.gmu.edu/~yloizide/.

[10] Jacob Lurie. *Derived Algebraic Geometry II: Noncommutative Algebra*. 2007. arXiv: math/0702299 [math.CT].

[11] Jacob Lurie. *Derived Algebraic Geometry III: Commutative Algebra*. Version 3. 2007. arXiv: math/0703204 [math.CT].

[12] Jacob Lurie. *Higher Topos Theory (AM-170)*. Available online at https://www.math.ias.edu/~lurie/. Princeton University Press, 2009. ISBN: 9780691140490. URL: http://www.jstor.org/stable/j.ctt7s47v.

[13] Jacob Lurie. "Higher algebra". Unpublished. Available online at https://www.math.ias.edu/~lurie/. Sept. 2017.

[14] Saunders MacLane. "Natural associativity and commutativity". In: *Rice Institute Pamphlet-Rice University Studies* 49.4 (1963).

[15] John McCleary. *A User's Guide to Spectral Sequences*. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2000. DOI: 10.1017/CBO9780511626289.

[16] John McCleary. *A History of Spectral Sequences*. Available online at https://www.uio.no/studier/emner/matnat/math/MAT9580/v21/dokumenter/. 2013.

[17] Thomas Nikolaus. *Stable ∞-Operads and the multiplicative Yoneda lemma*. 2016. arXiv: 1608.02901 [math.AT].

[18] nLab authors. *Atiyah-Hirzebruch spectral sequence*. https://ncatlab.org/nlab/show/Atiyah%E2%80%93Hirzebruch+spectral+sequence. Revision 42. Sept. 2023.

[19] G.W. Whitehead. *Elements of Homotopy Theory*. Graduate Texts in Mathematics. Springer New York, 2012. ISBN: 9781461263180. URL: `https://books.google.nl/books?id=xEDaBwAAQBAJ`.

[20] Kirsten Graham Wickelgren. *Lecture 13: Atiyah-Hirzebruch Spectral Sequence*. Available online at `https://services.math.duke.edu/~kg[w/8803_Stable/`. 2015.