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Dynamic Disciplines

On the nineteenth-century history of applied mathematics

Master's Thesis

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Abstract

The field of mathematics is usually divided into two branches: pure and applied mathematics. This division dates back to the early nineteenth century and resulted from many different developments in mathematics and academics. In this thesis, we trace back these developments to investigate the position and development of applied mathematics as an increasingly self-contained discipline. We examine the relationship between pure and applied mathematics as scientific fields and provide a case study into one of the branches of nineteenth-century applied mathematics: hydrodynamics. We show that applied mathematics developed into a scientific field when pure mathematics separated itself increasingly from the physical world. As mathematics became separated from nature, applied mathematics became increasingly separated from pure mathematics.

Prologue

The field of mathematics, with its many subfields and disciplines, is usually divided into two main branches: pure mathematics and applied mathematics. For the past few years, while studying applied mathematics, I have tried several times to explain this distinction to non-mathematicians and found myself not being able to explain it to my own satisfaction. I have asked mathematicians and found myself not completely satisfied with the answers. It made me wonder: what distinguishes pure from applied mathematics? Is pure mathematics more abstract than applied? Applied more real than pure? What does it mean to do pure mathematics and is applied not pure enough to be part of it? If we take the terms literally, one could say that in applied mathematics we use pure mathematics and apply it to some problem or scientific field. In practice, however, it simply does not seem as straightforward as this.

While the pure/applied division seemed to be taken for granted during my Master's Applied Mathematics, my Master's History and Philosophy of Science taught me to question these kinds of demarcations. Why do we have this division of pure versus applied? Is it relevant to have it? When did mathematicians first start defining it? With these questions in mind, I chose to focus my thesis project on the history of applied mathematics.

There is still a discussion about the role of the pure/applied division in mathematics and before diving into history, I want to draw some attention to this. One of the often-referenced sources in this discussion is Hardy's *A Mathematician's Apology*, which was meant as a defence of pure mathematics. To accomplish this, Hardy criticised applied mathematics in this essay:

> *But is not the position of an ordinary applied mathematician in some ways a little pathetic? If he wants to be useful, he must work in a humdrum way, and he cannot give full play to his fancy even when he wishes to rise to the heights. 'Imaginary' universes are so much more beautiful than this stupidly constructed 'real' one; and most of the finest products of an applied mathematician's fancy must be rejected, as soon as they have been created, for the brutal but sufficient reason that they do not fit the facts.*[1]

This criticism of applied mathematics is quite harsh and, I think, not grounded, but it shows how passionate Hardy was about defending pure over applied mathematics. Hardy's essay is said to exemplify the view of pure mathematicians that "mathematics would suffer from a too utilitarian point of view" during the time of the world wars and after.[2] Hardy himself clearly aims to promote pure mathematics over applied and claims "I have never done anything 'useful'."[3]

More recent examples of a discussion around pure and applied mathematics are not hard to find. David Wilson criticises applied mathematicians for "all too often not applying mathematics to anything in particular at all."[4] He distinguishes applied mathematics, the discipline that according to him has often little to do with real-world problems, from *applying* mathematics, which is using mathematics for particular (physical) problems and, therefore, useful. "The field of applied mathematics in academic settings tends

[1]Godfrey Harold Hardy and Charles Percy Snow. *A Mathematician's Apology. Reprinted, with a Foreword by CP Snow.* Cambridge University Press (original published in 1940), 1967, p. 135.

[2]Nicholas J Higham et al. *Princeton companion to applied mathematics.* Princeton University Press, 2015, p. 72.

[3]Godfrey Harold Hardy and Charles Percy Snow. *A Mathematician's Apology. Reprinted, with a Foreword by CP Snow.* Cambridge University Press (original published in 1940), 1967, p. 150. Ironically, Hardy is now well known for the Hardy-Weinberg equation, which is frequently used in population genetics.

[4]David P Wilson. "Mathematics is applied by everyone except applied mathematicians". In: *Applied mathematics letters* 22.5 (2009), pp. 636–637, p. 636.

not to be about applying mathematics directly but often about exploring 'mathematically interesting' phenomena of complex systems of equations which may somewhat describe the dynamics or processes of a real-world system. The mathematical exploration does not advance, or provide insight into, the application and nor does it advance fundamental mathematical theory."[5]

In defence of applied mathematics, Bruce Henry directly reacted to Wilson's criticism and stated that "[t]he academic discipline of Applied Mathematics sits somewhere between, and across, the academic discipline of Pure Mathematics and the pragmatism of applications. Much of the domain of Applied Mathematics is abstract and may not appear to be useful for real-world applications. However it is through such abstractions that new mathematics is created and the rich mapping between the physical universe and mathematics, which is necessary for applications, is advanced."[6]

Others argue about the positions of pure and applied mathematics within mathematics as a whole. In an interesting news article from 1997, two Dutch mathematicians articulate their worries about the reputation of mathematics as a scientific field. G.Y. Nieuwland argues that mathematics is becoming less popular and mathematicians should showcase its usefulness more often, promoting applied mathematics and its connection to the physical world. F. Den Hollander claims otherwise: "The appreciation for applied mathematics has grown in recent decades and has gone too far."[7] His more conservative view is that mathematics is simply not for everyone and it should not need to sell itself.

The same G.Y. Nieuwland wrote his oration in 1969 on whether applied mathematics really is mathematics. His conclusion was that applied mathematics is not mathematics: he mentions that applied mathematics is interdisciplinary, but always defined by the non-mathematical discipline and not by mathematics.[8] The importance of results in applied mathematics depends on their importance in the non-mathematical fields, he argues.

Another objection against the terminology of pure and applied is given by Ferdinand Verhulst. He argues that there exists a clear description of neither pure nor applied mathematics in his article *Forget 'pure and applied' mathematics*.[9] According to Verhulst, the distinction is not even relevant. He proposes a different way of dividing mathematics: into problem solvers and conceptualists.[10]

Whether the discussion is about the position of pure versus applied, the right interpretation of applied mathematics, or the distinction as a whole, the terms pure and applied mathematics are often interpreted as problematic. In this thesis, I attempt to give a historical account of the distinction between pure and applied mathematics and how it developed. The history I present here will not be complete or exhaustive, as much more can be said than is possible in one thesis, but I hope it can contribute to the current literature. I also hope that this history can be used to reflect on the present-day situation.

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I would like to give many, many thanks to Viktor, for supervising my thesis, giving advice and honest criticism, but never worrying and always concluding a meeting with "it's going to be interesting". I always felt supported in my decision to try and write a broad history and felt reassured in my attempt. Special thanks for introducing the term 'wokkel' when describing a vortex.

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[5] David P Wilson. "Just a little analysis". In: *Journal and Proceedings of the Royal Society of New South Wales*. Vol. 147. 453/454. 2014, pp. 91–93, p. 92.

[6] Bruce Henry. "An applied mathematician's apology". In: *Journal and Proceedings of the Royal Society of New South Wales*. Vol. 147. 453/454. 2014, pp. 94–100, p. 95.

[7] "De waardering voor de toegepaste wiskunde groeide de afgelopen decennia en is te ver doorgeschoten." Martine Zuidweg. "De Confrontatie". In: *Trouw* (June 25, 1997). (Visited on 09/11/2023).

[8] Gerke Yke Nieuwland. *Is toegepaste wiskunde ook wiskunde?* H.J. Paris, 1969, p. 13.

[9] Ferdinand Verhulst. "Vergeet 'zuivere en toegepaste' wiskunde". In: *Nieuw Archief voor de Wiskunde* 4 (2016), pp. 295–297, p. 295.

[10] Problem solvers occupy themselves with more specific problems, whereas conceptualists look more for mathematical structures.

Thanks to the history of mathematics group, who welcomed me during their Friday lunch (even if I wasn't there that often) and explained the ways of academic life to me, or presented riddles to think about.

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Chapter 1

Introduction

The nineteenth century is often portrayed as the century in which our modern scientific disciplines were formed. An example of this is physics, but science as a whole became a new "general social category."[1] Mathematics is no exception: many of the fields in mathematics as we know them today originate from the nineteenth century.

The branching of mathematics into pure and applied is said to have emerged in the late eighteenth or early nineteenth century as well, and both fields developed increasingly into autonomous fields of science in the course of the century. The very term *applied* was new: it replaced the term *mixed* mathematics, which was used before as the counterpart of pure mathematics. In the eighteenth century, pure and mixed mathematics both served to describe nature. When the term mixed was replaced by applied, this started to change. Amid the development of new, highly abstract mathematical fields and the increasing professionalisation of science, pure and applied became more separate than they had been before. Pure mathematics became detached from nature, while the final goal of applied mathematics remained to describe the physical world. The result was two separate fields, with their own research topics and reputations.

When looking at this history, it seems remarkable that one field of mathematics became divided into two fields that were autonomous and independent — at least in theory, in practice there were still a lot of connections and interactions between the two. This leads to many questions, but one in particular:

> How and why did applied mathematics become an increasingly self-contained field of mathematics in the nineteenth century?

This thesis is an attempt to answer this question. We have taken applied mathematics as the main focus of this thesis. Historians of mathematics have often focused on pure mathematics and the theoretical developments within mathematics.[2] The history of applied mathematics has gradually gained more interest and we would like to add to this history. The development from mixed to applied is recognised by some secondary sources, but not as many as might be expected. The *Oberwolfach Report* provides abstracts of talks on "the ways in which historically boundaries were drawn between 'pure' mathematics on the one hand and 'mixed' or 'applied' mathematics on the other from about 1500 until today."[3] These talks, performed during a workshop in 2013, elaborate on different aspects of mixed or applied mathematics in

[1] "These new labels and categories reflected the fact that science had both delimited itself more fully from philosophy, theology, and other types of traditional learning and culture and differentiated itself internally into increasingly specialized regions of knowledge." David Cahan. *From natural philosophy to the sciences: Writing the history of nineteenth-century science*. University of Chicago Press, 2003, p. 4.

[2] Moritz Epple, Tinne Hoff Kjeldsen, and Reinhard Siegmund-Schultze. "From "mixed" to "applied" mathematics: Tracing an important dimension of mathematics and its history". In: *Oberwolfach Reports* 10.1 (2013), pp. 657–733, p. 657; David Cahan. *From natural philosophy to the sciences: Writing the history of nineteenth-century science*. University of Chicago Press, 2003, p. 135; Tom Archibald. "Images of applied mathematics in the German mathematical community". In: *Changing images in mathematics: From the French Revolution to the new millenium*. Ed. by Umberto Bottazini and Amy Dahan Dalmedico. Routledge, 2013. Chap. 3.

[3] Epple, Kjeldsen, and Siegmund-Schultze, "From "mixed" to "applied" mathematics: Tracing an important dimension of mathematics and its history", p. 657.

a variety of topics and case studies. They are a great source of inspiration for investigating the history of applied mathematics but are also relatively brief in their content since they are summarised in the report. Another important source on the history of applied mathematics is the *Princeton Companion to Applied Mathematics*, which starts with a concise history of the field of applied mathematics.[4] It discusses the general developments and tendencies throughout the centuries and also recognises the development from mixed to applied mathematics. The article *The Changing Perception of Mathematics through History* by Henry Mulder also discusses the developments from mixed to applied, but also concisely.[5]

Other well-known works in the history of mathematics often do not focus on the history of applied mathematics in itself: Struik's *Concise History of Mathematics* mentions applied mathematics, but not its development into a discipline. The same can be said for Cooke's *The History of Mathematics: a Brief Course*.[6] Morris Kline discusses the developments of mathematics separating from nature and mentions that "the gradual rise and acceptance of the view that mathematics should embrace arbitrary structures that need have to bearing, immediate or ultimate, on the study of nature led to a schism that is described today as pure versus applied mathematics."[7]

We used *Epistemological and Social Problems of the Sciences in the Early Nineteenth Century*, edited by Jahnke and Otte, and *Social History of Nineteenth Century Mathematics*, edited by Mehrtens, Bos and Schneider, as accounts of nineteenth-century developments in mathematics. These works mention some changes in applied mathematics but mostly focus on pure mathematics. They add social context to the developments in mathematics, which we mean to include as well.

We hope to add to this literature by investigating the changes, and the reasons behind them, in applied mathematics in the nineteenth century. This is not only a history of applied mathematics: it is about the relation between mathematics and other fields of science. About the relation between theory and practice, and between pure and applied. It puts the history of mathematics as a whole into perspective and connects it to developments outside of mathematics. Taking applied mathematics as the main focus will, hopefully, shine a new light on the relationship between pure and applied mathematics and the history of mathematics altogether. In investigating the reasons behind the changes in applied mathematics, we consider changes in pure mathematics. This approach allows us to look at the effect that pure and applied have on each other.

The aim of this thesis is twofold. On the one hand, we want to write a general history. A general history of applied mathematics as a discipline can be extremely comprehensive and comprise several centuries and countries. This thesis will cover a period of roughly one century — the nineteenth — and a bit of the eighteenth century.[8] We will focus on European countries, mainly France, Germany and England. Some others, however, are included when relevant. This thesis has, therefore, a Eurocentric topic, mostly Western Europe. Of course, a complete account of nineteenth-century applied mathematics cannot be given based on Germany, France and England. The focus on these countries, however, does enable us to consider the collection of secondary literature that is already known and add to the general story that has already been written on the history of mathematics.

In writing a general history of applied mathematics in the nineteenth century, we need to consider the changes in pure mathematics as well. This idea served as the main approach in writing this thesis: we do not consider applied mathematics as an isolated field, but as one that is strongly influenced by its counterpart. The nineteenth century saw some fundamental changes in mathematics that, according to the general narrative, greatly influenced pure mathematics. We will argue that these developments also exerted their impact on applied mathematics. So the history of applied mathematics is considered in relation to the overall structure of the mathematical sciences in the nineteenth century. This thesis will be an attempt to paint a general picture of the discipline of applied mathematics in the nineteenth century

[4] Higham et al., *Princeton companion to applied mathematics*.

[5] Henry Martyn Mulder. "The Changing Perception of Mathematics through History". In: *Nieuw Archief voor Wiskunde, 4e serie, deel* 8 (1990), pp. 27–42.

[6] Roger L Cooke. *The history of mathematics: A brief course*. John Wiley & Sons, 2011.

[7] Morris Kline. *Mathematical Thought from Ancient to Modern Times: Volume 3*. Vol. 3. Oxford university press, 1990, p. 1036.

[8] The twentieth century can be said to be equally relevant, as it became more common to have professors of applied mathematics then. We will, however, not pay attention to this century because we want to investigate the origin of applied mathematics as a category, and because this would make the scope of this thesis too big.

and its connections to pure mathematics. It will not be complete, but rather a starting point for further research.

On the other hand, we will provide an example of nineteenth-century applied mathematics. This enables us to investigate the reflection of the broad developments in actual mathematical practice and to examine whether we can see these developments in mathematical works. We will do a comparison of three case studies in hydrodynamics: the study of the motion of fluids. Nowadays, hydrodynamics is considered a part of physics, but in the nineteenth century, it was part of applied mathematics. Hydrodynamics provides an interesting field when investigating the relation between theory and practice; between mathematics and its applications. Our three case studies in hydrodynamics will be about three different methods of stability analysis.

Following the two aims of this thesis, the project is largely divided into two parts. Chapter 2 describes the developments, intrinsic and extrinsic, that accompanied the semantic change from mixed to applied mathematics. We will see that these developments contributed to an increasing separation between pure and applied mathematics and them becoming independent fields of science. Sections 2.1 and 2.2 describe the semantic and conceptual change from mixed mathematics to applied mathematics. The connotations of these terms differ greatly, even though they can both be seen as the counterpart of pure mathematics. These sections describe the changes in the terms and their meaning, but not yet the reasons for this: we do this in the sections following these.

In section 2.3 we discuss the 'rise of pure mathematics' and the causes for it. As abstraction became the dominant requirement of mathematics, mathematics started to detach itself from the physical world and the physical sciences. The focus on foundational research, specialisation, and the founding of new mathematical fields only strengthened this detachment. We will provide different reasons for the rise of pure mathematics from secondary literature and aim to critically examine these.

Section 2.4 focuses on the extrinsic developments. Being a mathematician increasingly became a profession that was practised at universities. A combined teaching and research role at universities, with an ensured income, enabled mathematicians to focus on mathematics that was not immediately useful. The founding of journals contributed to a growing mathematical community and international knowledge-sharing.

Chapter 3 discusses the three case studies we have carried out. We investigated three methods for analysing hydrostatic and hydrodynamic stability. These three methods — that were selected approximately from the early, mid and late nineteenth century — exemplify some of the developments described in chapter 2. We will see that the methods become increasingly more abstract and decreasingly physical.

In the final conclusions, we will return to the research question and discuss our findings. In general, we will argue that applied mathematics as a category within mathematics became increasingly self-contained because of general developments in mathematics as well as society. Mathematics as a whole became a profession and pure mathematics became increasingly pursued for its own sake. Applied mathematics as a term became more restricted than it had been in the eighteenth century. At the same time, it became a more demarcated subject within mathematics.

We need some terminological reflections. First, we will employ different meanings for the terms applied mathematics and applying mathematics. Applied mathematics will refer to the discipline: a category within mathematics. We will be investigating the position and development of applied mathematics in this thesis. When we refer to applying mathematics, this is simply referred to as the act of applying a mathematical theory to something, be it a natural phenomenon or another part of mathematics. This means that we can speak of applying mathematics even when the category of applied mathematics was not yet established.

The terms scientific field and discipline are used interchangeably in this thesis. With both terms, we mean a body of knowledge that is about a certain topic. Pure and applied mathematics can be seen as two different fields of science, but each consists of many subfields that are in themselves disciplines.

Lastly, the term abstraction may be confusing. What we mean by abstraction of mathematics is the development of mathematics becoming increasingly distanced from the physical world; being less and less about concrete objects. This does not mean that mathematics was not abstract or rigorous before the development of abstraction. It means that the connection to the physical world is increasingly lost.

By investigating the development of applied mathematics into a field of science, we hope to contribute to the existing literature on the history of mathematics. We do this by balancing a broad history with a specific case study and therefore focusing on both the general developments and the specific mathematics. Additionally, we hope to contextualise the terms pure and applied mathematics: to provide historical context for these terms. We hope that such considerations help to reflect on the present-day distinction in mathematics.

Chapter 2

From mixed to applied

2.1 Mixing mathematics

> *Mathematics is the science which has as its object the properties of magnitude insofar as that they are calculable and measurable. [...] Mathematics is divided into two classes; the first is called pure mathematics and considers the properties of magnitude in an abstract way since the point of view of magnitude is either calculable or measurable: in the first case it is represented by numbers, in the second by magnitude; in the former case, pure mathematics is called Arithmetic and in the latter, Geometry. The second class is called mixed Mathematics; it has as an aim the properties of objects of concrete size, that is to say of the magnitude as seen in certain bodies or particular subjects. Counted among mixed Mathematics are Mechanics, Optics, Astronomy, Geography, Chronology, Military architecture, Hydrostatics, Hydraulics, Hydrography or Navigation etc.*[1]

This 1776 definition in Diderot's and d'Alembert's *Encyclopédie* is a leading example for examining the eighteenth-century view on mathematical sciences. Mathematics was seen as the science of magnitudes, either abstract or concrete. Abstract mathematics, called pure, was the science of magnitudes and relations between quantities, irrespective of natural objects. It consisted of arithmetic and geometry.[2] The counterpart of pure mathematics was mixed mathematics and it consisted of different fields that perhaps partly defined the term. Mixed mathematics was about concrete objects. This conception of mathematics and its division into pure and mixed was generally shared by contemporaries.[3]

The term *mixed mathematics* implies something different from applied mathematics — even though mixed served as the counterpart of pure mathematics, just as applied does nowadays. Mixed mathematics was, quite literally, mathematics mixed with other fields of science, combined with them instead of applied to them.[4] Mathematics was seen as an inherent part of the fields it was mixed with: those were mathematical sciences. Nowadays these fields would largely be a part of physics, but in the eighteenth and nineteenth centuries, these belonged to mathematics: they "were inextricably linked with the production of mathematical knowledge."[5] This shows how connected the natural sciences were; a more holistic view of the mathematical sciences was maintained.

This may partly explain the fluidity between mathematics and its application that, despite the distinction that existed on paper, is usually recognized in the eighteenth century: the distinction between pure and mixed mathematics that can be found in encyclopedias did not correspond to a sharp division

[1] Denis Diderot and Jean Le Rond d'Alembert. *Encyclopédie, ou, Dictionnaire raisonné des sciences, des arts et des métiers.* Vol. 10. Pergamon Press, 1765, p. 188, 189.

[2] Many fields within pure mathematics as we now know them originate from the nineteenth century or later.

[3] Henk JM Bos. "Mathematics and rational mechanics". In: *The ferment of knowledge: Studies in the historiography of eighteenth-century science* (1980), p. 329.

[4] Epple, Kjeldsen, and Siegmund-Schultze, "From "mixed" to "applied" mathematics: Tracing an important dimension of mathematics and its history", p. 665.

[5] Ibid., p. 657.

in practice. The division between pure and mixed was maintained as a useful way of dichotomising mathematics into two branches, but mathematics as a whole served mostly as a means to describe nature. Mathematicians mainly focused on "calculus and its application to mechanics."[6] There was no clear division between mathematics and its application, between mathematician and (mathematical) physicist.[7] Structural foundational work in mathematics originated in the nineteenth century when mathematicians focused more and more on pure mathematics as a research field.[8] Eighteenth-century mathematicians were not indifferent to foundational work, but the main focus of mathematics was producing results that could be connected to the physical world. Theoretical as well as practical mathematics were meant to describe nature.[9]

There are several signs of this dominating role of nature in eighteenth-century mathematics. In *The History of Applied Mathematics*, Barrow-Green and Siegmund-Schultze discuss the different themes of the prize competitions in mathematics: "the optimum arrangement of ship masts (1727), the motion of the moon (1764/68), the motion of the satellites of Jupiter (1766), the three-body problem (1770/72), the secular perturbations of the moon (1774), the perturbations of comet orbits (1776/78/80), the perturbation of the orbit of Pallas (in the beginning of the nineteenth century), the question of heat conduction (1810/12), and the propagation of sound waves in liquids (1816)."[10] These illustrate the focus on applications in the eighteenth century; topics on pure mathematics are seen more often in the nineteenth century. Felix Klein additionally argues that the starting point of mathematics was "the contemplation of nature."[11]

The term mixed mathematics originates from earlier centuries and is often connected to Francis Bacon and his division of the sciences.[12] The encyclopedia of Diderot and d'Alembert is an important example of an explicit mention of mixed mathematics. Also, Montucla's *Histoire des Mathématiques* mentions "Mathématiques mixtes" as opposite to pure.[13]

Important mathematical works from the eighteenth century, however, often do not mention mixed mathematics: we did not find such references in Jakob Bernoulli's *Ars Conjectandi* (1713), Johann Bernoulli's *Discours sur lex Loix de la Communication du Mouvement* (1727), Euler's *Mechanica* (1736), Daniel Bernoulli's *Hydrodynamica* (1738), and d'Alember's *Traité de Dynamique* (1743).[14] Some works do mention mixed mathematics, one of them being Bossut's *Course de Mathématiques*. Bossut explicitly mentions "mathématiques mixtes" as a category within mathematics.[15] He provides a definition for it:

> *Mixed mathematics borrows from physics, or from the essence of matter, some primordial property, from which it draws, with the aid of pure mathematics, all the other properties which relate to the subject with which it treats: this class includes mechanics, hydrodynamics, astronomy, optics and acoustics.*[16]

[6] D.J. Struik. *A concise history of mathematics.* Courier Corporation, 2012, p. 163.

[7] Judith Grabiner. "Mathematics around 1800". In: *The transformation of science in Germany at the beginning of the nineteenth century.* Ed. by Olaf Breidbach and Roswitha Burwick. The Edwin Mellen Press, 2013. Chap. 4, pp.125–183, p. 159.

[8] Judith Grabiner. "Changing Attitudes Toward Mathematical Rigor: Lagrange and Analysis in the Eighteenth and Nineteenth Centuries". In: *Epistemological and social problems of the sciences in the early nineteenth century.* Ed. by Hans Niels Jahnke and Michael Otte. Reidel Publishing Company, 1979, pp. 311–330, p. 312.

[9] Archibald, "Images of applied mathematics in the German mathematical community".

[10] Higham et al., *Princeton companion to applied mathematics*, p. 62.

[11] Felix Klein. "The arithmetizing of mathematics". In: *Bulletin of the American Mathematical Society* 2.8 (1896), pp. 241–249, p. 241.

[12] Nicholas J Higham et al. *Princeton companion to applied mathematics.* Princeton University Press, 2015, p. 60; Gary I Brown. "The Evolution of the Term" Mixed Mathematics"". In: *Journal of the History of Ideas* 52.1 (1991), pp. 81–102, p. 82.

[13] Jean-Étienne Montucla. *Histoire des mathématiques.* Vol. 2. chez Ch.-Ant. Jombert, 1758, p. 3/84.

[14] These works were selected somewhat randomly, but the selection of authors is based on them being referred to as the "principal actors" of eighteenth-century mathematics by Barrow-Green and Siegmunt-Schultze. Nicholas J Higham et al. *Princeton companion to applied mathematics.* Princeton University Press, 2015, p. 62. Jakob Bernoulli. *Ars coniectandi.* Impensis Thurnisiorum, fratrum, 1713; Johann Bernoulli. *Discours sur les loix de la communication du Mouvement.* Jombert, 1727; Leonhard Euler. *Mechanica.* Petropoli ex typographia Academiae scientarum, 1736; Daniel Bernoulli. *Hydrodynamica: sive de viribus et motibus fluidorum commentarii.* 1738; Jean Le Rond d'Alembert. *Traité de dynamique.* David l'aîné, 1743.

[15] Charles Bossut. *Cours de mathématiques.* Vol. 1. F. Didot, 1800, p. 2.

[16] "Les mathématiques mixtes empruntent de la physique, ou de l essence de la matière, quelque propriété primordiale, d'où elles tirent, à l'aide des mathématiques pures, toutes les autres propriétés qui se rapportent au sujet dont elles traitent:

The absence of the term mixed mathematics in many mathematical works may suggest that the term itself was not as relevant to mathematicians as it may have been to philosophers or historians. Perhaps the fact that the whole body of mathematics was to explain nature made the distinction between pure and mixed redundant to mathematicians. Even when they were occupying themselves with topics considered mixed mathematics, like (hydro)dynamics or mechanics.

In their *Map of Knowledge*, part of which is depicted in Figure 2.1, Diderot and d'Alembert represent the division of mathematics into its different sub-disciplines. Interestingly, and not mentioned in the written definition of mathematics in the encyclopedia, there seem to be three main branches of mathematics: pure, mixed, and the *physicomathematiques*, or mathematical physics. Diderot and d'Alembert explain in the description of their map:

> *Quantity, the object of mathematics, could be considered either alone and independent of real and abstract individual things from which one gained knowledge of it, or it could be considered in these real and abstract beings, or it could be considered in their effects investigated according to real or supposed causes; this reflection leads to the division of mathematics into pure mathematics, mixed mathematics, physicomathematics.*[17]

What they mean with this exactly remains rather obscure: mixed mathematics and physico-mathematics are said to consist of the same disciplines and their exact distinction is not discussed. Mathematical physics is said to bring together "observation and experiment with mathematical calculation, [applying] this calculation to the phenomena of nature."[18] This seems quite similar to the notion of mixed mathematics: mixing mathematical and physical theories.

2.2 Nineteenth-century applied mathematics

In the nineteenth century, the term mixed mathematics was gradually replaced by the term applied mathematics. In Britain, this took a bit longer than other European countries, which is illustrated in the eighth edition of the *Encyclopedia Britannica* (1857) that still uses the term mixed mathematics.

> *Although in every department of mathematics, results are obtained by strict logical deduction from a few first principles explicitly assumed, the science may be regarded as consisting of two distinct branches according to the nature of the evidence on which the truth of its first principles is admitted. In the one, which constitutes pure mathematics, the first principles require no special inductive process to convince us of their truth, and scarcely, indeed, demand the evidence of our senses. [...] Thus pure mathematics is founded upon definitions of necessary truth and is pre-eminently the most certain of all the sciences. Mixed mathematics denotes the application of pure mathematics to natural objects; and presupposes some knowledge of their properties derived from the senses, or of general laws obtained by induction from a sufficient number of observations. The logical, or strictly mathematical processes of deduction in pure and mixed mathematics are identical; the difference between the two sciences being, that in the one, the first principles are self-evident, while in the other, laws and facts, which are not necessarily self-evident, but derived from observation, are admitted, along with the axioms and definitions of pure mathematics, as fundamental principles of the science.*[19]

cette classe comprend la mécanique, l'hydrodynamique, l'astronomie, l'optique et l'acoustique." Charles Bossut. *Cours de mathématiques.* Vol. 1. F. Didot, 1800, p. 2.

[17]La quantité , objet des Mathématiques , pouvoit être considérée , ou seule et indépendamment des individus réels , & des individus abstraits dont on en tenoit la connoissance ; ou dans ces individus réels & abstraits ; ou dans leurs effets recherchés d'après des causes réelles ou supposées ; & cette séconde vue de la réflexion a distribué les Mathématiques en Mathématiques pures , Mathématiques mixtes , Physiicomathématiques. Denis Diderot and Jean Le Rond d'Alembert. *Encyclopédie, ou, Dictionnaire raisonné des sciences, des arts et des métiers.* Vol. 10. Pergamon Press, 1765, p. xlix

[18]"On appelle ainsi les parties de la Physique, dans lesquelles on réunit l'observation & l'expérience au calcul mathématique , & où l'on applique ce calcul aux phénomènes de la nature." Denis Diderot and Jean Le Rond d'Alembert. *Encyclopédie, ou, Dictionnaire raisonné des sciences, des arts et des métiers.* Vol. 10. Pergamon Press, 1765, p. 536.

[19]*Encyclopedia Britannica.* 8th ed. Vol. 14. Adam and Charles Black, 1857, p. 353.

Figure 2.1: Part of the *Map of Knowledge* from Diderot and d'Alembert. Mathematics is divided into pure, mixed, and physico-mathematics.

The reason for the use of the term mixed mathematics comes, according to the author in this encyclopedia, from the "mixture of mathematical deduction with experimental processes." This definition of mixed mathematics is, however, different from Diderot's and d'Alembert's. We saw that Diderot and d'Alembert focus on the objects of pure and mixed mathematics, which are abstract and concrete, respectively. The Encyclopedia Britannica describe mathematics in terms of its methods and first principles. We can interpret this as a shift in thinking about mathematics. Putting emphasis on the objects of pure and mixed mathematics does not suggest restrictions, or superiority, of the methods used for either. The Encyclopedia Britannica states that, in mixed mathematics, pure mathematics is applied to natural objects. This suggests a superior position of pure mathematics: pure comes first and applied later. The focus on methods means a focus on pure mathematics as the main method in both pure and mixed mathematics. In the ninth edition of this encyclopedia, the term mixed is replaced by applied, but the connotation is very similar: pure mathematics is applied to natural objects.[20] This eighth edition of the Encyclopedia thus already seems to define mixed mathematics as 'applied mathematics', but still uses the old term.

The same kind of definition is employed in different encyclopedias and mathematical works of the nineteenth century. We already saw Bossut's definition from his *Course de mathématiques* (1800). In the Dutch *Encyclopaedie* of Winkler Prins (1881) we see the same: in applied mathematics the theorems from pure mathematics are applied to existing objects.[21]

In this shift from mixed to applied, we can detect a different attitude towards pure mathematics as well as applied. Pure mathematics tends to receive a superior position compared to applied: it became the norm within mathematics, whereas applied seems to have meant the first principles were to arise from nature. At the end of the nineteenth century, while the term applied mathematics became increasingly adopted, the disciplines that had belonged to it became increasingly a part of physics. Applied mathematics came to be a more restricted, more narrow term than mixed mathematics had been, at least semantically, while pure mathematics started to reign. The change from mixed to applied mathematics was thus not only semantic but conceptual. By emphasising the role of pure mathematics within applied mathematics, the term applied mathematics suggests that fully finished mathematical theory is applied to other fields of science. It disregards the dialectical nature of mathematics and the fields it is applied to.[22] Daston argues bluntly that "the very term applied mathematics tells all: in order to be applied, the mathematics must already exist in its own right, just as theory is "applied" to practice."[23]

At the same time, a more explicit distinction between pure and applied mathematics was increasingly adopted. Whereas the distinction between pure and mixed mathematics in the eighteenth century did not create two different disciplines, this did happen increasingly during the nineteenth century. So the use of the term applied mathematics was accompanied by a stronger separation between pure mathematics and its application.[24]

Next to the discontinuity in the term mixed or applied mathematics and their connotations, there

[20] *Encyclopedia Britannica*. 9th ed. Vol. 14. Adam and Charles Black, 1883, p. 629.

[21] "Wiskunde (De) of mathesis ontwikkelt den zamenhang der verschillende grootheden, welke op de eene of andere wijze met elkander verbonden zijn. Zij ontdekt de verschillende vormen, waaronder dezelfde grootheid kan voorkomen, en leert, hoe men uit zekere bekende grootheden de onbekenden kan vinden, welke daarmede in verband staan. Men onderscheidt de wiskunde in zuivere en toegepaste, naar gelang men de grootheden op zich zelve of in verband met andere eigenschappen beschouwt. Men kan de zuivere wiskunde beschouwen a1s theorie en de toegepaste a1s de aanwending daarvan op bestaande voorwerpen. De zuivere wiskunde wordt verdeeld in rekenkunde, die de getallen, en in meetkunde, die de uitgebreidheden behandelt (zie onder Rekenkunde en Meetkunde). De eerste bevat de cijferkunst, de algebra, de hoogere arithmetica en de analysis van eindige en oneindige grootheden. Toegepaste wiskunde noemt men die wetenschappen welke een eigenaardigen grondslag hebben, maar hare bepalingen en bewijzen aan de stellingen der zuivere wiskunde ontleenen." A. Winkler Prins. *Geïllustreerde Encylopaedie. Woordenboek voor Wetenschap en Kunst, Beschaving en Nijverheid.* Vol. 14. C.L. Brinkman, 1881, p. 645.

[22] I agree here with, amongst others Bos in his article *Rational Mechanics*. Henk JM Bos. "Mathematics and rational mechanics". In: *The ferment of knowledge: Studies in the historiography of eighteenth-century science* (1980), p. 329

[23] Lorraine J Daston. *Fitting numbers to the world: The case of probability theory.* Ed. by William Aspray and Philip Kitcher. 1988, p. 221.

[24] These general developments are described, amongst other, by Moritz Epple, Tinne Hoff Kjeldsen, and Reinhard Siegmund-Schultze. "From "mixed" to "applied" mathematics: Tracing an important dimension of mathematics and its history". In: *Oberwolfach Reports* 10.1 (2013), pp. 657–733, p. 658; and Gert Schubring. "The Conception of Pure Mathematics as an Instrument in the Professionalization of Mathematics". In: *Social history of nineteenth century mathematics.* Ed. by Herbert Mehrtens, Henk JM Bos, and Ivo Schneider. Springer, 1981, pp. 111–134, p. 118. It should be noted here that an institutional separation between pure and applied mathematics is largely a twentieth-century development.

were also continuities in the concept of applied mathematics compared to mixed. In general, the main focus of either was natural phenomena. Mixed as well applied mathematics was seen as a science of nature, connected to the physical world, whereas pure mathematics was more abstract, not about concrete objects. Mixed and applied also consisted largely of the same fields: mechanics, optics, astronomy, hydrodynamics and many more were said to belong to mixed and later to applied mathematics.[25] Thus, while the overarching term changed, the same disciplines were said to be part of mixed as well as applied mathematics.

When these fields were increasingly considered to be physics, mathematics and physics increasingly became two separate fields. Mathematics became "primarily pure, and only secondarily concerned with 'application'," whereas the study of nature increasingly belonged to physics.[26] Throughout the nineteenth century, applied mathematics remained largely defined in terms of the fields it consisted of, including those mentioned above. Woodward writes, in his article *The Century's Progress in Applied Mathematics* (1900) that "The most important of these branches [of applied mathematics] appear to be analytical mechanics, geodesy, dynamical astronomy, spherical or observational astronomy, the theory of elasticity, and hydromechanics."[27]

The change from mixed to applied is said to have started in Germany and somewhat later in France and Britain.[28] It was accompanied by many different changes in mathematics as well as other sciences and society. We want to explore the main causes of applied mathematics becoming increasingly self-contained. These causes can be intrinsic — coming from mathematics itself, with new problems and new fields arising — or extrinsic — the new teaching role mathematicians gained and the new institutions that were founded strongly impacted mathematics as a scientific field. These developments will be discussed in the next sections.

2.3 Towards abstraction

> *The popular conception of mathematics is that of a strictly logically coordinated system, complete in itself, such as we meet with, for instance, in Euclid's geometry; but as a matter of fact, modern mathematics in its origin was of a totally different character. With the contemplation of nature as starting point, and its interpretation as object, a philosophical principle, the principle of continuity, was made fundamental; and the use of this principle characterizes the work of the great pioneers, Newton and Leibnitz, and the mathematicians of the whole of the eighteenth century — a century of discoveries in the evolution of mathematics. Gradually, however, a more critical spirit asserted itself and demanded a logical justification for the innovations made with such assurance, the establishment, as it were, of law and order after the long and victorious campaign.*[29]

This section will outline the intrinsic developments in mathematics that contributed to the separation of applied mathematics as an increasingly self-contained discipline. When trying to trace back the reasons for this, we need to consider general developments in mathematics as a whole. Secondary literature presents several causes for the 'rise' of pure mathematics in the nineteenth century, which we will discuss and connect to the developments in applied mathematics. These developments are: specialisation within mathematics, the increasing focus on foundational research, and the rise of new fields such as non-Euclidean geometry. These developments are strongly connected. The general tendency we will see is that mathematics as a whole starts to distance itself more and more from the physical world. As

[25] R. Chambers W. Chambers. *Chambers's Encyclopædia: A Dictionary of Universal Knowledge for the People*. Vol. 7. J.B. Lippincott company, 1901, p. 91; Denis Diderot and Jean Le Rond d'Alembert. *Encyclopédie, ou, Dictionnaire raisonné des sciences, des arts et des métiers*. Vol. 10. Pergamon Press, 1765, p. 10.

[26] Umberto Bottazini and Amy Dahan Dalmedico. *Changing images in mathematics: From the French Revolution to the new millennium*. Routledge, 2013.

[27] Robert S Woodward. "The century's progress in applied mathematics". In: *Science* 11.264 (1900), pp. 81–92, p. 133.

[28] Epple, Kjeldsen, and Siegmund-Schultze, "From "mixed" to "applied" mathematics: Tracing an important dimension of mathematics and its history", p. 664.

[29] Klein, "The arithmetizing of mathematics", p 241.

mathematics becomes increasingly abstract, applied mathematics becomes increasingly distant from its pure counterpart.

What is usually meant by the rise of pure mathematics, is that mathematics as a whole ceases to be a science connected to nature: "The new mathematical research gradually emancipated itself from the ancient tendency to see in mechanics and astronomy the final goal of the exact sciences."[30] In the nineteenth century, pure mathematics "increasingly gets systematic public support for the first time", even with the industrial revolution as historical background.[31] It was pure mathematics rather than applied mathematics that deviated from the usual practice during the nineteenth century — and pure mathematics started to become the norm. The increasing focus on foundations took the attention away from natural phenomena and towards abstraction.[32] While pure mathematics separated itself from nature, the "widespread 19th-century view [was] that the purpose of applied mathematics was primarily to reveal the physical causes of natural phenomena."[33]

Most eighteenth-century mathematics served to describe the natural world. This did not mean, however, that the foundations of mathematics were completely neglected. They were discussed as introductory information on mathematical topics, rather than "used to justify the full complement of results of the calculus."[34] Foundations were rarely a research subject, but a part of the prerequisite knowledge of the mathematician. Towards the end of the eighteenth century, this changed and mathematicians started to focus increasingly on the (logical) foundations of mathematics as a research subject. Grabiner and Struik identify mathematicians feeling that their research field was almost finished; mathematics was regarded as complete. Struik analyses that this feeling of completeness was due to the idea that mathematical progress was progress in mechanics and astronomy. The developments in these fields seemed to be at their maximal height and with them the developments in mathematics.[35] This feeling led mathematicians to consider the foundations of their science; as the fields of mechanics and astronomy were thought to be near complete, there was still much work to be done in the foundations of mathematics. This focus on foundations led mathematicians away from the applications of their science.

Another reason for the widening gap between pure and applied mathematics is said to be increasing specialisation in mathematics. Struik, for example, writes: "a division between 'pure' and 'applied' mathematics accompanied the growth of specialisation."[36] Archibald writes: "Considerable mathematical sophistication was required to create the necessary tools or generalisations, fine-tune them, and understand clearly when they were valid. Such efforts were increasingly a full-time job."[37] He argues that mathematics and physics separated because the amount of work in either field simply became too much to combine. Mathematicians spent their time on problems which were less and less connected to the physical world and left the applications to others.

A different intrinsic reason for the rise of pure mathematics is put forward by Scharlau. He argues that theoretical research was increasingly important because of the new mathematical problems that were posed.[38] Scharlau emphasizes that the need for pure mathematical research came from mathematics

[30] Struik, *A concise history of mathematics*, p. 201.

[31] Epple, Kjeldsen, and Siegmund-Schultze, "From "mixed" to "applied" mathematics: Tracing an important dimension of mathematics and its history", p. 664.

[32] David Cahan. *From natural philosophy to the sciences: Writing the history of nineteenth-century science*. University of Chicago Press, 2003, p. 132; Umberto Bottazini and Amy Dahan Dalmedico. *Changing images in mathematics: From the French Revolution to the new millennium*. Routledge, 2013.

[33] Epple, Kjeldsen, and Siegmund-Schultze, "From "mixed" to "applied" mathematics: Tracing an important dimension of mathematics and its history", p. 688.

[34] Grabiner, "Changing Attitudes Toward Mathematical Rigor: Lagrange and Analysis in the Eighteenth and Nineteenth Centuries", p. 313.

[35] This feeling seems to be expressed by, amongst others, Lagrange. Judith Grabiner. "Changing Attitudes Toward Mathematical Rigor: Lagrange and Analysis in the Eighteenth and Nineteenth Centuries". In: *Epistemological and social problems of the sciences in the early nineteenth century*. Ed. by Hans Niels Jahnke and Michael Otte. Reidel Publishing Company, 1979, pp. 311–330, p. 315; D.J. Struik. *A concise history of mathematics*. Courier Corporation, 2012, p. 199

[36] Ibid., p. 202.

[37] Bottazini and Dalmedico, *Changing images in mathematics: From the French Revolution to the new millennium*.

[38] Winfried Scharlau. "The Origins of Pure Mathematics". In: *Epistemological and social problems of the sciences in the early nineteenth century*. Ed. by Hans Niels Jahnke and Michael Otte. Reidel Publishing Company, 1979, pp. 331–348, p. 338.

itself: mathematical problems that needed solving, especially when connections between different, first regarded as unrelated, mathematical fields were found.[39] "I would like to advance the thesis that the decisive condition for the origin of pure mathematics was the fact that for the first time in the history of mathematics a large number of connections were discovered between seemingly different problem areas and results."[40] Eighteenth-century mathematics consisted of largely separate topics, but in the nineteenth century, mathematicians started to find connections between these fields. It was the connection between different problem areas within mathematics that yielded more research into the underlying structures of those connections.[41] Scharlau opposes that the reason for more foundational work was specialisation, but argues that it was rather the "transcending of special viewpoints".[42]

Scharlau provides several examples of mathematicians finding connections between different fields of research. We will repeat one of those, to illustrate his line of reasoning. He presents "one of the greatest results of Euler" as being the series[43]

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

Then he states that Fourier obtained this calculation as a result of his theory of Fourier series, which was a result of the developments of his theory of heat.[44]. So indeed we see that one calculation is recognised in several instances of mathematics. Scharlau provides some other, similar examples and regards this to sufficiently prove his claim.

Another development within mathematics that sparked the focus on abstraction was the forming of new fields within pure mathematics, most notably non-Euclidean geometries. Richards writes that, until the end of the eighteenth century, geometry had been the study of space. The self-evidence of Euclidean geometry was challenged by the rise of several non-Euclidean geometries.[45] This led to the formalisation of geometry as a mathematical science with fewer ties to space as a real-world phenomenon. Mathematics as a whole became further removed from the physical world: "Euclidean geometry, once unblushingly regarded as the true theory of physical space, became the study of one among many abstract mathematical spaces."[46] The forming of new fields in pure mathematics had at least two consequences. First, it is part of the specialisation in mathematics; as more fields in pure mathematics arose, more work was to be done in those fields, leaving less time for research in their applications. Second, the new non-Euclidean geometries proved that there was not just one possible mathematical description of our world. Euclidean geometry lost its reputation as this true mathematical description. We thus see the practical as well as conceptual consequences of these nineteenth-century developments.

Lastly, Maddy argues that the understanding of mathematical theories about our physical world changed in the nineteenth century. Newton, for example, saw the mathematical description of physical phenomena as literal truths. In the nineteenth century, however, mathematicians started to see these mathematical descriptions as models for our world instead of the actual description of it. Mathematics became a tool to describe the physical world. "We have seen how our best mathematical accounts of physical phenomena are not the literal truths Newton took them for but freestanding abstract models that resemble the world in ways that are complex and sometimes not fully understood. Paradoxical as it may sound, it now appears that even applied mathematics is pure."[47]

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[39] Scharlau, "The Origins of Pure Mathematics", p. 339.

[40] Winfried Scharlau. "The Origins of Pure Mathematics". In: *Epistemological and social problems of the sciences in the early nineteenth century*. Ed. by Hans Niels Jahnke and Michael Otte. Reidel Publishing Company, 1979, pp. 331–348, p. 339. It is interesting that Scharlau mentions the 'origin' of pure mathematics rather than the 'rise' of pure mathematics. It could be argued that, indeed, our modern conception of pure mathematics arose from the nineteenth century and, therefore, this could be called the origin rather than the rise of pure mathematics. Since, however, the term already existed, we prefer the 'rise' of pure mathematics to describe the nineteenth-century developments.

[41] Ibid., p. 339.

[42] Ibid., p. 339.

[43] Ibid., p. 340.

[44] Ibid., p. 340.

[45] Joan Richards. "The geometrical tradition: mathematics, space, and reason in the nineteenth century". In: *The modern physical and mathematical sciences* 5 (2003), pp. 447–467, p. 450.

[46] Penelope Maddy. "How applied mathematics became pure". In: *The Review of Symbolic Logic* 1.1 (2008), pp. 16–41, p. 33.

[47] Ibid., p. 33.

One of the important figures in the development towards foundational research was Lagrange. Not only was he one of the first to systematically investigate the foundations of mathematics as would be adopted by many nineteenth-century mathematicians, but he also paid much attention to making calculus more rigorous by "abandoning appeals to geometry in favour of algebraic arguments."[48] He sparked the focus on rigour and foundations as we now know it, breaking with a centuries-long tradition, and thus became a predecessor for the main developments in mathematics that would characterise the nineteenth century. Lagrange himself put it as follows: "No figures will be found in this work. The methods I present require neither constructions nor geometrical or mechanical arguments, but solely algebraic operations subject to a regular and uniform procedure. Those who appreciate mathematical analysis will see with pleasure mechanics becoming a new branch of it and hence, will recognize that I have enlarged its domain"[49]

This detachment of mathematics from the physical world is recognized by many nineteenth-century mathematicians. In a letter to Legendre from 1830, Jacobi writes:

> *It is true that Fourier had the opinion that mathematics should be useful and explain natural phenomena, but a philosopher like him should know that the only purpose of this science is to honour the human mind.*[50]

The Dutch mathematician Korteweg states in his oration:

> *It was mathematics that as the first [of the sciences] disconnected itself from the requirement of immediate usefulness. One can rightly say, in my opinion, that she [mathematics] is aware of her position as oldest sister and will remain aware of this.* [51]

At the end of the nineteenth century, Woodward reflects on the progress of applied mathematics during the century and concludes that there were also voices that argued against the 'purification' of mathematics.

> *Lagrange had, as he supposed, reduced mechanics to pure mathematics. Geometrical reasonings and diagrammatic illustrations were triumphantly banished from this science and replaced by the systematic and unerring processes of algebra. [...] The mathematical world has not only accepted Lagrange's estimate of his work, but has gone further, and considers his achievement one of the most brilliant and important in the whole range of mathematical science. [...] As we can now see without much difficulty, Lagrange and most of his contemporaries in their eagerness to put mechanics on a sound analytical basis overlooked to a serious extent its more important physical basis. The prevailing mathematical opinion was that a science is finished as soon as it is expressed in equations. One of the first to protest against this view was Poinsot, though the preëminent importance of the physical aspect of mechanics did not come to be adequately appreciated until the latter half of the present century.*[52]

So mathematics was distanced, partly consciously, from the physical world and even in applied mathematics this goal was pursued. The mathematics that was used by Lagrange to express physical processes was aimed to be pure mathematics. At the same time, others voiced arguments in favour of physical interpretation and understanding.

Felix Klein is one of the mathematicians who expressed criticism of the rise of pure mathematics. In his article *The Arithmetizing of Mathematics*, Klein argues that logical reasoning cannot replace intuition.[53] In

[48]Grabiner, "Changing Attitudes Toward Mathematical Rigor: Lagrange and Analysis in the Eighteenth and Nineteenth Centuries", p. 316.

[49]Joseph Louis de Lagrange. *Mécanique analytique.* Vol. 1. Courcier, 1811, p. 8.

[50]"Il est vrai que M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain." C.G.J. Jacobi and A.M. Legendre. "Correspondance mathématique entre Legendre et Jacobi". In: *Journal für die Reine und Angewandte Mathematik* 1875.80 (1875), pp. 205–279, p. 273.

[51]"Het was [...] de wiskunde die zich als eerste losmaakte van den eisch van het onmiddelijk nuttige. [...] Men kan van haar, naar mijne meening, met recht getuigen, dat ze hare roeping als oudste zuster zich bewust is geweest en ongetwijfeld bewust zal blijven." Diederik Johannes Korteweg. *De wiskunde als hulpwetenschap.* Vol. 1. JH & G. van Heteren, 1881, p. 5.

[52]Woodward, "The century's progress in applied mathematics", p. 135.

[53]Klein, "The arithmetizing of mathematics", p. 247.

his *Vorlesung uber die Entwicklung der Mathematik im 19 Jahrhundert*, Klein expresses his discontent with the purification of mathematics. He criticises the current state of mathematics by stating its "sense for the concrete special case remains undeveloped."[54] He writes, beautifully and perhaps slightly dramatically,

> *The mathematics of our day appears to me like a large weapon shop in peace time. The store window is filled with showpieces whose ingenious, artful and pleasing design enchants the connoisseur. The real origin and purpose of these things, to attack and defeat the enemy, has retreated so far into the background of consciousness as to be forgotten.*[55]

Such reflections on the state of mathematics in the nineteenth century are informative: mathematicians were at the time aware of the changes towards abstraction and foundations and were critical towards the state of their own field.

Several reasons for the rise of pure mathematics are mentioned in this section. First, the feeling of mathematics being a finished science. Second, the increasing specialisation in mathematics: as the body of mathematical knowledge simply became too large to comprehend, mathematicians started to focus on one small part of mathematics, which was often pure mathematics. Furthermore, the connections that were found between several mathematical fields are said to have stimulated the tendency towards pure mathematical research. Fourth, the founding of new mathematical fields, like non-Euclidean geometry, distanced mathematics from the physical world and towards abstraction and purification. Last, the interpretation of mathematical theories of the physical world changed: they became descriptions instead of true representations. We will come back to these developments in chapter 4, where we will connect them to our case studies.

The developments we have outlined in this section present reasons for applied mathematics to become increasingly self-contained: as pure mathematics deviates from its usual practice, applied mathematics consequently changes. Gradually, the idea that we have pure mathematics first and only secondly apply this to phenomena became the prevailing thought. "By the 1870s the majority of mathematicians saw their principal task in the elaboration of the fundamental theoretical principles of mathematics. [...] The lecture programs organized by the local mathematical societies reflect the dominant role of these activities. But, at the same time, there were always mathematicians who also engaged in solving problems posed by physics and astronomy. However, a certain arrogant attitude spread that looked upon applications of mathematics to concrete practical problems as a somewhat primitive exercise."[56]

2.4 Professionalisation and institutionalisation

In the previous section, we discussed the intrinsic developments that sent mathematics towards abstraction and foundational research and away from the physical world. In this section, we will consider the extrinsic developments that influenced mathematics to become a more autonomous field of science; the development of professionalisation. We will see that the development of mathematics into a scientific profession went hand in hand with the rise of pure mathematics discussed in the previous section: the intrinsic and extrinsic developments strengthened each other.

During the eighteenth century, scientists, including mathematicians, were members of societies or employed at royal courts. Universities were places of teaching, but hardly places of innovation. Mathematical research took place in academies and societies. During the eighteenth century, this gap between "research-oriented academies and the teaching-oriented universities" became one to be bridged.[57] Mathematicians were employed at the academies, but the next generations started to work at universities, simultaneously

[54] Felix Klein, Robert Hermann, and Gerald M Ackerman. *Development of Mathematics in the 19th Century.* 1979, p. 195.

[55] Ibid., p. 65.

[56] Renate Tobies. "On the Contribution of Mathematical Societies To Promoting Applications of Mathematics in Germany". In: *The History of Modern Mathematics: Institutions and Applications.* Ed. by D.E. Rowe and J. McCleary. Academic Press, 1988, pp. 223–244, p. 225.

[57] Ivo Schneider. "Forms of professional activity in mathematics before the nineteenth century". In: *Social history of nineteenth century mathematics.* Ed. by Herbert Mehrtens, Henk JM Bos, and Ivo Schneider. Springer, 1981, pp. 89–110, p. 106.

as researchers and teachers.[58] "University systems were modernised and technical institutions founded, professionalism and specialisation encouraged."[59]

At the same time, new social classes had a new outlook on life, with an interest in science and education. "Mathematics as a science and as a teaching subject received an institutional boost both at Technical Universities and later at traditional Universities."[60] Due to these new positions at schools and universities, doing mathematical research and teaching mathematics became a full-time profession: the "universal scholar" was replaced by a specialised researcher.[61]

An important milestone at the end of the eighteenth century was the opening of the École Polytechnique in Paris, in 1794. Driven by the belief in progress that dominated the French Revolution, this new school set "the example for technical teaching over the Western World and by its stress on the mathematical sciences also deeply influenced university instruction and research."[62] "The revolutionary ideology promoted the idea of educated citizens for the Republic and of careers open to the talented."[63] Mathematical education was encouraged even more, because of an increasing awareness that "academic mathematics could be applied to technology."[64] This boosted the place of mathematics in general education. These educational reforms in France influenced other European countries. In Germany, this happened during the first half of the nineteenth century and was accompanied by a "neohumanistic movement which secured a new form and status for pure mathematics."[65]

Mathematicians, increasingly protected by their professional positions at technical schools and universities, "no longer felt a need to maintain [their] claims to universality and its ties to other subjects. As their professional identity was increasingly tied to their particular mathematical knowledge, the value of that knowledge came to lie more in its separateness and inaccessibility than in its universal appeal."[66] The professionalisation of mathematics enabled mathematicians to solely pursue pure mathematics and pay less attention to connecting mathematics to the physical world. Especially in Germany, "scientists seldom faced the problem of having to justify their work to theological and political authorities."[67]

At the same time, mathematical research gradually separated itself from the practical tendency that dominated the eighteenth century. Moreover, scientific research as a whole became increasingly separated from practical use; science was pursued for its own sake.[68] As this 'purification' of science expanded, mathematics became an autonomous field of research, supported and recognised publicly.[69] Mathematicians emphasised that one could do mathematical research for the "honour of the human mind."[70] So the rise of pure mathematics that we discussed went hand in hand with academic reforms that boosted intrinsic mathematical developments even more.

Some historians of mathematics connect the new teaching role that mathematicians now had gained, to the increasing interest in the foundations of mathematics. As most active mathematicians became teachers as well as researchers, they were stimulated to convey the foundations of their subject clearly to students.[71]

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[58] Schubring, "The Conception of Pure Mathematics as an Instrument in the Professionalization of Mathematics", p. 116.

[59] D.J Struik. "Mathematics in the Early Part of the Nineteenth Century". In: *Social history of nineteenth century mathematics*. Ed. by Herbert Mehrtens, Henk JM Bos, and Ivo Schneider. Springer, 1981, pp. 6–20, p. 19.

[60] Epple, Kjeldsen, and Siegmund-Schultze, "From "mixed" to "applied" mathematics: Tracing an important dimension of mathematics and its history", p. 664.

[61] Schubring, "The Conception of Pure Mathematics as an Instrument in the Professionalization of Mathematics", p. 111.

[62] Struik, "Mathematics in the Early Part of the Nineteenth Century", p. 8, 9.

[63] Grabiner, "Mathematics around 1800", p. 159.

[64] Schneider, "Forms of professional activity in mathematics before the nineteenth century", p. 107.

[65] Ibid., p. 107.

[66] Richards, "The geometrical tradition: mathematics, space, and reason in the nineteenth century", p. 457.

[67] David Rowe. "Mathematical schools, communities, and networks". In: *The modern physical and mathematical sciences* 5 (2003), pp. 111–132, p. 119.

[68] Struik, *A concise history of mathematics*, p. 201.

[69] Epple, Kjeldsen, and Siegmund-Schultze, "From "mixed" to "applied" mathematics: Tracing an important dimension of mathematics and its history", p. 665.

[70] Ibid., p. 688.

[71] Judith Grabiner. "Changing Attitudes Toward Mathematical Rigor: Lagrange and Analysis in the Eighteenth and Nineteenth Centuries". In: *Epistemological and social problems of the sciences in the early nineteenth century*. Ed. by Hans Niels Jahnke and Michael Otte. Reidel Publishing Company, 1979, pp. 311–330, p. 316; Joan Richards. "The geometrical tradition: mathematics, space, and reason in the nineteenth century". In: *The modern physical and mathematical sciences* 5 (2003), pp. 447–467, p. 455.

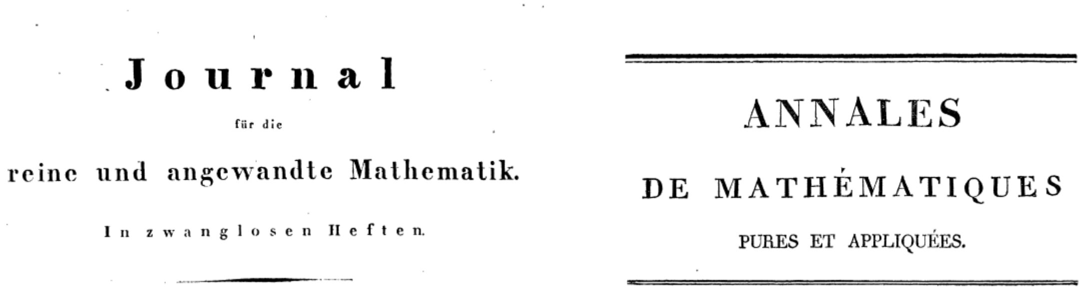


Figure 2.2: Crelle's journal on the left and Gergonne's on the right.

Schubring argues that pure mathematics was to serve as a basis for practical study within teaching. He writes that the separation between pure and applied mathematics did not "place pure mathematics in opposition to its application; rather pure mathematics was considered a prerequisite for the latter".[72]

2.4.1 Journals

In the early nineteenth century, several journals for pure and applied mathematics were founded. Two very important ones included the German *Journal für die reine und angewandte Mathematik* and the French *Annales de Mathématiques Pures et Appliquées.* In Britain, the *Cambridge Mathematics Journal* was founded. These journals are yet another sign of the professionalisation of mathematics in the nineteenth century. Their founding shows the existence of a community and a need to share knowledge. We will focus mostly on the German journal because it is most often mentioned in the literature and, therefore, seems the most influential. We will, however, discuss the journals in order of appearance.

The first French journal specifically aimed at mathematics was the *Journal de l'école polytechnique — Mathématiques* (1794) and, as the title shows, it was connected to the École Polytechnique.[73]. Founded in 1810 by Joseph Gergonne, the Annales de Mathématiques was the next French journal. Famous names that come across when reading the issues of this second journal are Gergonne, Poncelet, Sturm, Ampère and Poisson. Gergonne writes in the 'prospectus' of this journal that the journal will mainly focus on pure mathematics, but will not exclude the applications of mathematics.[74] This shows that, despite wanting to include all parts of mathematics, articles on pure mathematics were thought most important to publish. After 22 volumes, in 1832, however, the journal stopped publishing. Four years later the Journal de Mathématiques Pures et Appliquées was founded by the mathematician Liouville, as a journal connected to the École Polytechnique, and this journal continues to publish mathematical articles today.

The Journal für die reine und angewandte Mathematik was founded in 1826 and still publishes issues today. Leopold Crelle, its first editor, was a German mathematician and engineer and provided the journal with its nickname Crelle's Journal. Crelle was inspired by the French journal — which partly explains the title of 'pure and applied' — and founded a German one to give German mathematicians a place to share their theories. He succeeded in putting "German mathematics on the world map."[75] Many well-known mathematicians have written articles for the journal, among others Abel, Jacobi, Poncelet, Möbius, Dirichlet, Weierstrass, Dedekind and Riemann. In the "vorrede" (preface) of the journal, Crelle writes: "There is hardly an important object of knowledge that does not have its German journal. Only the

[72] Gert Schubring. "On education as a mediating element between development and application: the plans for the Berlin Polytechnical Institute (1817-1850)". In: *Epistemological and social problems of the sciences in the early nineteenth century.* Ed. by Hans Niels Jahnke and Michael Otte. Reidel Publishing Company, 1979, pp. 269–286, p. 277.

[73] Grabiner, "Mathematics around 1800", p. 162.

[74] "Ces Annales seront principalement consacrées aux Mathématiques pures, et sur-tout aux recherches qui auront pour objet d'en perfectionner et d'en simplifier l'enseignement. Le titre de l'ouvrage annonce assez d'ailleurs que , si l'on n'y doit rien rencontrer d'absolument étranger au Calcul, à la Géométrie et à la Mécanique rationnelle, les rédacteurs sont néanmoins dans l'intention de n'en rien exclure de ce qui pourra donner lieu à des applications de ces diverses branches des sciences exactes." Joseph Gergonne. "Prospectus". In: *Annales de Mathématiques pures et appliquées* 1.1 (1810), pp. i–iv, p. ij

[75] Grabiner, "Mathematics around 1800", p. 162.

great, unlimited mathematics [...], which of all sciences is perhaps most associated with truth, currently has none."[76]

Crelle discusses the topics the journal will contain. Firstly, there will be articles on pure mathematics: on "analysis, geometry and the whole of the theory of mechanics."[77] Secondly, the journal will consist of articles on applied mathematics, "applications of all kinds of mathematics, meaning the theory of light, the theory of heat, the theory of sound, of probability; also hydraulics, machine theory, mathematical geography, geodesy, etc."[78] Astronomy will also not be excluded, but Crelle specifically states that this science will not be the main focus of the journal since this topic is large enough for a journal of its own. The goal of the journal is twofold: communicate theorems that are already known but not yet published, but also republish theorems that deserve more attention. Until the publication of issue 53, in 1857, the articles in each issue of Crelle's journal were subdivided into analysis, geometry, mechanics and applied mathematics, the larger part of the articles being classified as analysis or geometry.

Crelle also mentions that he wants his journal to be as elaborate as possible and not exclude important topics or provide a journal only for mathematicians: "Above all, it [a journal] must avoid all one-sidedness. Nothing harms the development of a science more than repeated one-sided preference for this or that method."[79] Klein emphasises, some 100 years after the founding of the journal, that this goal was not fulfilled: the "intellectual current [of pure science as the ideal science] overwhelmed the journal, originally dedicated to all branches of mathematics, and made it into an organ of abstract and specialized mathematics of the most rigorous kind."[80]

In England, in 1837, the *Cambridge Mathematics Journal* was founded, later called the *Quarterly Journal of Pure and Applied Mathematics*. Interestingly, these journals are all said to be journals of pure as well as applied mathematics, while the focus of the journals was not always equally divided. It is well known that Crelle's journal had the nickname of the journal of *reine unangewandte mathematik*.[81] Gergonne stated in his prospectus that the focus would be on pure mathematics. So even though these journals did publish articles on many of the subjects that were considered applied, the majority of the articles were on pure mathematics.[82] It seems that to give a complete account of mathematics, one needed to address pure as well as applied mathematics. It also seems that the founders of the journal thought this important since they explicitly state that neither will be excluded. At the same time, however, pure mathematics received more attention, indicating that publishing articles on pure mathematics was, in practice, regarded as more important.

It is interesting to note that, although applied was a rather new term in the nineteenth century, these journals all employ this new term in their titles. This may show the relevance of the term applied mathematics over mixed mathematics for mathematicians. The term mixed mathematics was also still in circulation, so it is not unlikely that the choice for the term applied rather than mixed was a conscious one. We did not, however, find sources that support this. What we can conclude from the titles of the journals is that the distinction between pure and applied already existed in the early nineteenth century and that the distinction was relevant for the mathematical community.

2.5 Conclusions

Let us summarise what we have discussed so far, and see how the different developments can be connected. We have seen that eighteenth-century mathematics was strongly connected to the physical world. Not

[76] August Leopold Crelle. "Vorrede". In: *Journal für die reine und angewandte Mathematik* 1.1 (1826), pp. 1–4.

[77] Ibid.

[78] Ibid.

[79] "Vorzüglich muss sie alle Einseitigkeit vermeiden. Nichts schadet der Entwickelung einer Wissenschaft mehrmals einseitige Vorliebe für diese oder jene Methode." August Leopold Crelle. "Vorrede". In: *Journal für die reine und angewandte Mathematik* 1.1 (1826), pp. 1–4, p. 2.

[80] Klein, Hermann, and Ackerman, *Development of Mathematics in the 19th Century*, p. 86.

[81] 'Pure unapplied mathematics.' D.J Struik. "Mathematics in the Early Part of the Nineteenth Century". In: *Social history of nineteenth century mathematics.* Ed. by Herbert Mehrtens, Henk JM Bos, and Ivo Schneider. Springer, 1981, pp. 6–20, p. 19; Felix Klein, Robert Hermann, and Gerald M Ackerman. *Development of Mathematics in the 19th Century.* 1979, p. 86.

[82] On what was considered pure at the time.

only mixed mathematics, but mathematics as a whole was a natural science, even though pure and mixed mathematics were said to have different objects of research. The term mixed mathematics seems to not be as relevant to mathematicians as one might expect. It is mostly mentioned in historical works or encyclopedias, implying the term was more important philosophically than in practice. The distinction pure/mixed did not seem to create two different fields within mathematics.

In the nineteenth century, various developments contributed to fundamental changes in mathematics as a scientific field. We discussed intrinsic developments: changes within mathematics that led to the rise of pure mathematics. Several reasons for this have been presented. The increasing focus on foundations, the feeling of mathematics being a finished field, specialisation within mathematics, and new connections that were found between different fields within mathematics. The founding of completely new mathematical fields, most notably non-Euclidean geometries, also contributed to an increasing focus on foundations and abstraction. Geometry lost its character as the true theory of the physical world. The extent to which mathematical descriptions of the physical world were seen as the truth also changed: from the true description to a model of the world. Mathematics started to slowly detach from the physical world and drifted towards a science in which abstraction and rigour were the norm.

These intrinsic developments went hand in hand with the increasing professionalisation of mathematics. Not only were mathematicians interested in new mathematical developments, they had the opportunity to work on abstract mathematics because they could do their research at universities. As mathematics became a profession, mathematicians did not have to account for their work as being useful or lucrative. The teaching positions that mathematicians had next to their research positions also influenced the focus on foundations in mathematics. On the one hand, it enabled mathematicians to immediately introduce new developments to students. On the other hand, trying to convey mathematics to students is said to have led mathematicians to increasingly investigate the foundations of their science for teaching purposes.

The concurrence of these intrinsic and extrinsic developments can hardly be seen as incidental. They were both necessary for the nineteenth-century developments towards the purification of mathematics. Together, these developments enabled mathematics to become an increasingly autonomous field of science, without ties to other fields or our natural world. The intrinsic and extrinsic developments reinforced each other. Their mutual occurrence contributed to the emancipation of pure mathematics. The question that then remains is: what happened to applied mathematics?

We have seen that parallel to these developments, the term mixed mathematics was replaced by applied mathematics. This term was more restricted than mixed mathematics had been: applied mathematics now became the part of mathematics that was about real-world problems to which pure mathematics was applied. So while mathematics was largely distanced from nature, the ties to the physical world were kept intact through applied mathematics. The physical world thus seems to be a critical concept in the separation of pure and applied mathematics.

This did not mean that the connection between applied mathematics and the physical world did not change. In applied mathematics — more separate from pure but nevertheless a part of mathematics — we can also see the tendency towards abstraction and rigour. It is argued, as we have seen, that mathematical theories start to be abstract models instead of true descriptions of the physical world. In this sense, even applied mathematics starts to purify.

As pure mathematics became the norm within mathematics, the prevailing opinion was that we have pure mathematics first and can apply it, in the literal sense, to real-world phenomena. This puts applied mathematics in a secondary position compared to pure mathematics: pure mathematics is developed and used in a finished form in physical sciences. The two-way influence between pure mathematics and its applications that was embraced in the eighteenth century, was increasingly neglected in the nineteenth century.

At the same time, applied mathematics seems to have been a more relevant category to mathematicians than mixed mathematics had been. Mathematicians acknowledged it as a field, used it as a category in the mathematical journals and advertised their work as applied or pure. We think that this is due to the developments in pure mathematics and that it contributed to the formation of the two separate fields of pure and applied mathematics that became even more apparent in the twentieth century. It is interesting that the distinction between pure and applied became significant in the nineteenth century: it shows that pure and applied were indeed growing apart and that mathematicians were aware of this.

Despite the general tendency towards abstraction and pure mathematics, applied mathematics remained a field that received attention from many mathematicians. Many mathematicians still occupied themselves with physical problems and used mathematics to analyse those. "Gauss (1777-1855), who with Lagrange and Cauchy (1789-1857) must be ranked among the founders of modern pure mathematics," contributed greatly to astronomy, mechanics and (hydro)dynamics.[83] Let us turn to one of these fields.

[83]Woodward, "The century's progress in applied mathematics", p. 134.

Chapter 3

Hydrodynamics and stability analysis

There is scarcely any question in dynamics more important for Natural Philosophy than the stability or instability of motion.[1]

In the second part of this thesis, we will focus on one of the sub-fields of applied mathematics in the nineteenth century: hydrodynamics. Zooming in on the mathematics itself is important for several reasons, not the least to see what mathematics looked like in practice, what methods were used and how they changed over time. From a modern point of view, hydrodynamics may not seem an obvious choice for a case study in applied mathematics; today hydrodynamics is a branch of physics. In the eighteenth and nineteenth centuries, however, a clear distinction between physics and mathematics had not been established and many branches that are considered physics today were part of mathematics then. We chose hydrodynamics because of this position. It is one of those fields that serves as a crossroad between mathematical and physical problems, and between practice and theory.

Hydrodynamics is the discipline within mechanics that studies the motion of fluids. The subject is usually divided into two branches: hydrostatics and hydrodynamics. The former is the study of fluids in a stationary position, the latter of fluids moving. Darrigol describes hydrodynamics as "the art of subjecting flow to the general principles of dynamics."[2] Indeed, hydrodynamics was seen as a part of mechanics in which laws of dynamics were applied to fluids. The modern field of hydrodynamics is said to have begun with the investigations of Bernoulli but took flight when d'Alembert and Euler started their research. They applied dynamical principles to hydrodynamics and "thereby discovered the general equations of the motion of a perfect fluid, and placed the object on a satisfactory basis."[3] The French mathematicians Laplace, Lagrange and Poisson took up the investigations, followed by British (Stokes, W. Thomson, J.J. Thomson, Rayleigh) and German (Kirchhoff, Helmholtz) mathematicians after the first half of the nineteenth century.[4]

Despite the obvious practical applications of the study of the motion of fluids, a large part of its research in the nineteenth century was highly mathematical and not immediately suitable for practical application at all. Very often fluids are idealised or situations simplified to fit them into mathematical equations. As such, hydrodynamics was largely divided into two categories: one mathematical, comprised of mathematicians in academies or universities, and the other the more practical — often called hydraulics — consisting of engineers and instrument makers.[5] While there was a desire to unite these worlds, this did not fully succeed until the twentieth century:[6] "at the beginning of the nineteenth century the consensus was

[1]William Thomson Baron Kelvin and Peter Guthrie Tait. *Treatise on natural philosophy*. Vol. 1. Claredon Press, 1867, p. 282.

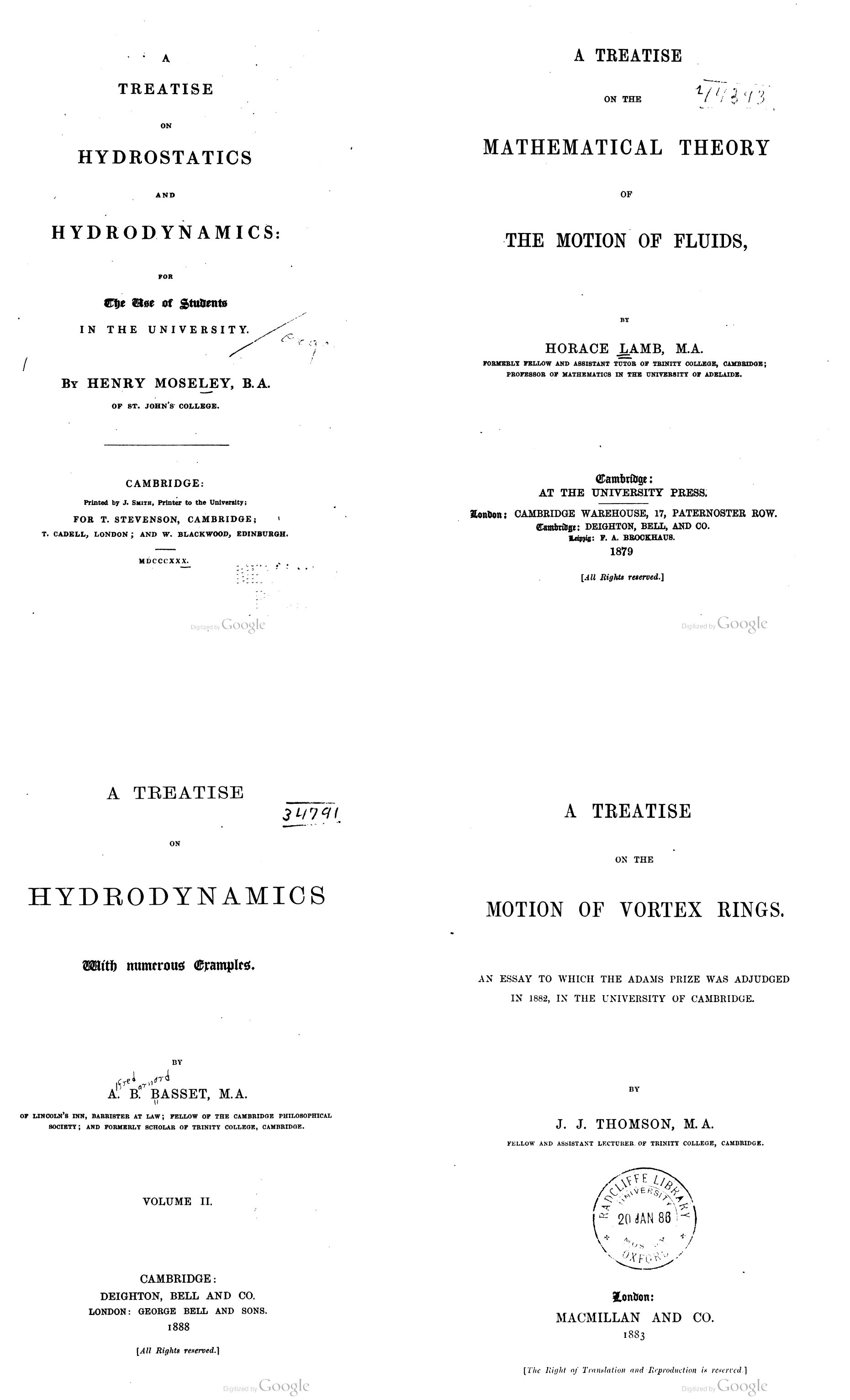
[2]Olivier Darrigol. *Worlds of flow: A history of hydrodynamics from the Bernoullis to Prandtl*. Oxford University Press, 2005, p. v.

[3]Alfred Barnard Basset. *A treatise on hydrodynamics: with numerous examples*. Vol. 1. Bell and Company, 1888, p. 1.

[4]For an elaborate eighteenth- and nineteenth-century history of hydrodynamics see Darrigol's *Worlds of Flow*. Olivier Darrigol. *Worlds of flow: A history of hydrodynamics from the Bernoullis to Prandtl*. Oxford University Press, 2005.

[5]Ibid., p. vi.

[6]A division between the theoretical and practical parts remained, but the theoretical results became feasible for practical



A TREATISE ON HYDROSTATICS AND HYDRODYNAMICS: FOR The Use of Students IN THE UNIVERSITY.

By HENRY MOSELEY, B.A. OF ST. JOHN'S COLLEGE.

CAMBRIDGE: Printed by J. Smith, Printer to the University; FOR T. STEVENSON, CAMBRIDGE; T. CADELL, LONDON; AND W. BLACKWOOD, EDINBURGH. MDCCCXXX.

A TREATISE ON THE MATHEMATICAL THEORY OF THE MOTION OF FLUIDS,

BY HORACE LAMB, M.A. FORMERLY FELLOW AND ASSISTANT TUTOR OF TRINITY COLLEGE, CAMBRIDGE; PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF ADELAIDE.

Cambridge: AT THE UNIVERSITY PRESS. London: CAMBRIDGE WAREHOUSE, 17, PATERNOSTER ROW. Cambridge: DEIGHTON, BELL, AND CO. Leipzig: F. A. BROCKHAUS. 1879

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A TREATISE ON HYDRODYNAMICS With numerous Examples.

BY A. B. BASSET, M.A. OF LINCOLN'S INN, BARRISTER AT LAW; FELLOW OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY; AND FORMERLY SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE.

VOLUME II.

CAMBRIDGE: DEIGHTON, BELL AND CO. LONDON: GEORGE BELL AND SONS. 1888

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A TREATISE ON THE MOTION OF VORTEX RINGS.

AN ESSAY TO WHICH THE ADAMS PRIZE WAS ADJUDGED IN 1882, IN THE UNIVERSITY OF CAMBRIDGE.

BY J. J. THOMSON, M.A. FELLOW AND ASSISTANT LECTURER OF TRINITY COLLEGE, CAMBRIDGE.

London: MACMILLAN AND CO. 1883

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Figure 3.2: Four of the textbooks that have been investigated.

that rational fluid dynamics could not explain practically important phenomena such as fluid resistance and flow retardation. Most knowledge of these phenomena was empirical and derived from the observations and measurements accumulated by hydraulic engineers. We will focus on the mathematical, and theoretical, part of hydrodynamics and the relation between physical interpretation and mathematical application in it."[7] Nevertheless, there were many attempts to "make hydrodynamics more relevant to the practical problems of flow."[8]

Mathematicians were aware of the practical limitations of hydrodynamics in their time. Brown describes this in his article *On recent progress toward the solution of problems in hydrodynamics*.[9] He writes: "The differential equations of motion may, perhaps, be written down, but the limitations which have to be imposed before a solution can be discovered are so numerous that the solution, when found, often gives no approximation at all to the real circumstances."[10] So in mathematics, some situations are simplified in such a way that the final solution does not apply to real-world situations anymore. Brown pleads for a stronger connection between pure and applied mathematics:

> *Pure mathematicians will not find their knowledge useless here [in hydrodynamics], and students will not be backward in following the footsteps of such men as Laplace, Stokes, Kelvin, von Helmholtz and Rayleigh. The tendency towards the separation of pure mathematics from their applications to physical problems has already been arrested. The future progress of Hydrodynamics appears to demand a closer union of these two branches of science.*[11]

In the coming sections, we will look at nineteenth-century textbooks on hydrodynamics. The nineteenth century saw a "wealth of new textbooks."[12] These textbooks were often based on lectures that were given at universities or academies and were therefore aimed at students. Textbooks usually represent established knowledge of a scientific field; those things considered general knowledge are summed up in a textbook, often joined with references to important authors and influential works. Representing the contemporary situation of a scientific field is historically illuminating in several ways. First, it shows the existing knowledge in a clear way: what was known and what was not known. It does not necessarily show the newest inventions and innovations, but the accepted ones. Consequently, it also represents what was considered interesting and what was not, and the important questions that were asked. Lastly, textbooks require prerequisite knowledge from their reader and show what mathematical tools or methods were assumed to be known. By using textbooks as the primary source of our case studies, we can assume a certain generality of the methods that are introduced. In other words, we would like to draw conclusions on the commonly used methods. The different textbooks that we have used will be discussed whenever they are relevant in the next section, as well as their authors.

Within these textbooks, we will specifically focus on stability analysis. The stability of the motion of fluids was a central question in nineteenth-century hydrodynamics.[13] Darrigol provides two reasons for this:

> *Firstly, the discrepancy between actual fluid behavior and known solutions of the hydrodynamic equations suggested the instability of these solutions. Secondly, the British endeavor to reduce all physics to the motion of a perfect liquid presupposed the stability of the forms of motion used to describe matter and ether. Instability in the former case and stability in the latter case needed to be proved.*

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situations and theoretical results became useful as a guide for engineers.

[7] Olivier Darrigol. "Between hydrodynamics and elasticity theory: the first five births of the Navier-Stokes equation". In: *Archive for History of Exact Sciences* 56.2 (2002), pp. 95–150, p. 105.

[8] Darrigol, *Worlds of flow: A history of hydrodynamics from the Bernoullis to Prandtl*, p. vi.

[9] Ernest W Brown. "On Recent Progress Toward the Solution of Problems in Hydrodynamics". In: *Science* 8.202 (1898), pp. 641–651.

[10] Ibid., p. 642.

[11] It remains unclear where the separation of pure mathematics from its applications has been arrested, but the fact that Brown mentions it shows it was worth discussing. Ernest W Brown. "On Recent Progress Toward the Solution of Problems in Hydrodynamics". In: *Science* 8.202 (1898), pp. 641–651, p. 651.

[12] Grabiner, "Mathematics around 1800", p. 163.

[13] Olivier Darrigol. "Stability and instability in nineteenth-century fluid mechanics". In: *Revue d'histoire des mathématiques* 8.1 (2002), pp. 5–65, p. 6.

For our purpose, the first reason is most interesting. As mathematicians were trying to bridge the gap between theory and practice, the problem of (in)stability had to be solved. De Prony states in his textbook:

> *The determination of the positions of equilibrium would only be an object of pure curiosity if it were not combined with the examination of the stability or non-stability of these positions.*[14]

This quote suggests that stability added (physical) meaning to an otherwise theoretical situation of equilibrium. Within the case studies on hydrodynamics, the focus on stability analysis throughout the nineteenth century provides an opportunity to say something about the development of mathematical methods in hydrodynamics. What methods were used? How did they develop? To what extent was pure mathematics 'applied' to hydrodynamical stability?

The concept of stability was used with different interpretations and in different situations. Without a rigorous mathematical definition of stability, investigations on this topic were often based on "intuition, past experience or experiment, and improvised mathematics."[15] It is interesting to investigate whether stability analysis became progressively systematic and rigorously defined, in the light of the abstraction of mathematics. We should keep in mind, however, that "[e]xact mathematical definitions of stability for a dynamical system, as well as general stability theorems for nonlinear systems, were first formulated by Russian scientists at the end of the nineteenth century."[16]

For our case studies, we will look at three different methods that were used in the nineteenth century for analysing the stability of fluids. The first one will be about analysing the stability of floating bodies by means of the concept of the metacentre. We will see how physical interpretation is central in this case. To investigate this method, we will look at the textbooks *l'Equilibre et le Mouvement des Corps* written by De Prony and *A Treatise on Hydrostatics and Hydrodynamics* by Moseley. The second method is used for analysing the stability of a body moving through a fluid, which is done by the derivation of an equation that is equivalent to a known one. We will see that the physical interpretation of the mathematics used is still important, but less apparent. Additionally, the derivation of the equation is largely analytical and had no immediate interpretation or this was not important. We will base the discussion of this method on four textbooks: *A Treatise on the Mathematical Theory of the Motion of Fluids* by Lamb, *A Treatise on Hydrodynamics: with numerous examples* by Basset, *Vorlesung über Mathematische Physik* by Kirchhoff, and *Treatise on Natural Philosophy* by Thomson and Tait. The third method can be interpreted as being mostly applied mathematics, more than the other two methods. The stability of several vortices in a fluid is analysed with the use of the theory of matrices and determinants. This last method is described in *A Treatise on the Motion of Vortex Rings*, written by J.J. Thomson. The method is also mentioned in Basset's textbook. The three different methods are discussed in the same order as they appeared in nineteenth-century textbooks: the first is early-nineteenth century, the second mid-nineteenth century, and the third method late-nineteenth century. These three methods thus provide snapshots of stability analysis throughout the nineteenth century. They will be discussed chronologically.

Analysing the different methods for stability analysis enables us to investigate the ways pure mathematics was applied to problems in hydrodynamics. We can examine to what extent physical interpretation and intuition play a role. Rigorously defined methods are sometimes used, but authors also employ their understanding of physical objects to conclude whether motion is stable. We will see that sometimes prerequisite knowledge is assumed that does not seem straightforward. In this chapter, we will first discuss the different methods separately, and conclude with a comparative section.

The selection of the textbooks is based on a few requirements. First, the textbooks need to perform stability analysis. Some early nineteenth-century textbooks do not discuss stability and are therefore not used in our case studies. Second, we would prefer the textbooks to have been relevant ones for hydrodynamics. Darrigol mentions that the textbooks of Basset and Lamb, which we will discuss in section

[14] "La détermination des positions d'équilibre ne seroit qu'un objet de pure curiosité si l'on n'y réunissoit l'examen de la stabilité ou non-stabilité de ces positions." Riche Prony. *Plan raisonné de la partie de l'enseignement de l'École polytechnique qui a pour objet l'équilibre et le mouvement des corps.* De l'Imprimerie de Courcier, imprimeur pour les mathématiques, 1801, p. 103.

[15] Darrigol, "Stability and instability in nineteenth-century fluid mechanics", p. 218.

[16] Remco I Leine. "The historical development of classical stability concepts: Lagrange, Poisson and Lyapunov stability". In: *Nonlinear Dynamics* 59 (2010), pp. 173–182, p. 179.

3.2, were among the best-known ones in nineteenth-century hydrodynamics.[17] In these two textbooks, references are made to the textbooks of Kirchhoff, Thomson and Tait, and J.J. Thomson, making these textbooks obvious choices for further research. The textbooks of De Prony and Moseley were chosen for their analysis with the use of the metacentre.

It should be noted that all the methods we discuss are specifically used to solve physical problems: the stability of a floating body, of a body moving through a fluid, and of several vortices rotating in a fluid. Consequently, there is at least some physical interpretation of the solutions in each of the textbooks we discuss, simply because the authors are dealing with physical problems. To draw conclusions on the way mathematics is used in its fields of application, we need to have a certain (even if artificial) distinction between pure and applied mathematics. We will draw this distinction based on the encyclopedias and journals from the nineteenth century, as described in the previous sections. Whenever we say that pure mathematics is used, we specifically mean mathematics that was developed for its own sake, separately from the field of (hydro)dynamics. Additionally, the references given by the authors of the textbooks will be determinative in concluding what kind of mathematics was used. We will see that there are some textbooks that are often referred to. We will discuss those when they appear in our story.

3.1 The metacentre

The earliest textbooks we investigate, from the first third of the nineteenth century, discuss stability in terms of the metacentre. This concept is used to analyse the hydrostatic stability of floating bodies — most practically: ships. The physical principles of the stability of floating bodies were already known for a long time and date back to the time of Archimedes, but the first analytical expression for stability analysis using the metacentre is said to have been defined in the eighteenth century.[18]

One of the textbooks was written by Gaspard Riche de Prony (1755-1839) in 1801. De Prony was a French mathematician and engineer. He studied mathematics and was a professor at the École Polytechnique as well as the École des Ponts et Chaussées. His work on hydrodynamics, *l'Equilibre et le Mouvement des Corps*, was written after he taught this subject at the École Polytechnique and published by *l'Imprimerie de Courcier*, which was the publisher for mathematical works.[19] Grattan-Guinness counts De Prony among the group of scientists that occupied themselves mostly with mechanics and were "strongly driven by the needs of engineering."[20] Darrigol writes that "Until the 1830s at least, the production of advanced mathematical physics in an engineering context remained a uniquely French phenomenon, largely depending on the creation of the Ecole Polytechnique."[21] De Prony can be seen as one of the mathematicians who occupied themselves with engineering and practical mathematics.

Another textbook that discusses the metacentre is written by the British mathematician Henry Moseley. Henry Moseley (1801-1872) studied mathematics at Cambridge, St. John's College, and completed the Mathematical Tripos in 1826.[22] Before entering Cambridge, Moseley spent some time in France where he was taught the French mathematics that had not yet come through to Britain. Here he became familiar with the works of Laplace. When Moseley started his mathematical studies around 1820, discussion on university reforms on applications of mathematics had just begun. Those in favour of reform urged "the creation of other honours degrees and the inclusion of analytical techniques in the subject matter of the

[17]Darrigol, *Worlds of flow: A history of hydrodynamics from the Bernoullis to Prandtl*, p. xi.

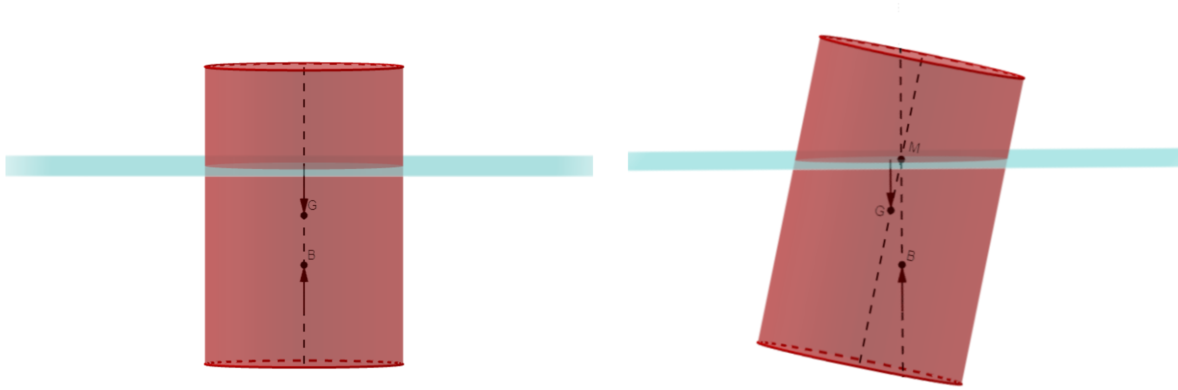
[18]Horst Nowacki and Larrie D Ferreiro. "Historical roots of the theory of hydrostatic stability of ships". In: *Contemporary Ideas on Ship Stability and Capsizing in Waves.* Springer, 2011, pp. 141–180, p. 142.

[19]Prony, *Plan raisonné de la partie de l'enseignement de l'École polytechnique qui a pour objet l'équilibre et le mouvement des corps.*

[20]Ivor Grattan-Guinnes. "Modes and Manners of Applied Mathematics: The Case of Mechanics". In: *The History of Modern Mathematics: Institutions and Applications.* Ed. by D.E. Rowe and J. McCleary. Academic Press, 1988, pp. 109–126, p. 112.

[21]Darrigol, *Worlds of flow: A history of hydrodynamics from the Bernoullis to Prandtl*, p. 135.

[22]Henry Moseley is no unfamiliar name in the history of nineteenth-century science. Moseley's son, also called Henry Moseley, was a physiologist and biologist. The third Henry Moseley, the grandson, is known for his contributions to physics, amongst others on radioactivity and the theory of atomic structure. The mathematical Tripos was a week-long examination after an education in mathematics. The students would be ranked according to their achievements in the Tripos. John Lewis Heilbron, Henry Gwyn Jeffreys Moseley, et al. *HGJ Moseley: the life and letters of an English physicist, 1887-1915.* Univ of California Press, 1974, pp. 2-5.



(a) A position of equilibrium: G and B are on the same vertical line, so the forces are not causing the body to rotate.

(b) The body slightly rotated: M is above G, so there will be a restoring force back to the initial position of equilibrium.

Figure 3.3: A cylinder floating in a fluid. The metacentric height is GM and depends on the rotation of the body. After the rotation depicted on the right, the body will return to the initial position of equilibrium on the left. This is thus a stable equilibrium.

tripos."[23] During those reforms, French mathematics was increasingly introduced into British education.[24] Analysis and its applications in, amongst others, hydrostatics and hydrodynamics, quickly entered the tripos, enabling Moseley to attend lectures on these topics.

After his studies, Moseley joined the church and became a priest because his position on the ranking list after the Tripos gave him little perspective on a professorship. However, this did not stop him from pursuing a scientific career. Only a few years later, in 1830, he wrote *A Treatise on Hydrostatics and Hydrodynamics: for the use of Students in the University*. This work was to provide interested students with applied analysis. With it, he intended to lay down the first principles of hydrostatics and hydrodynamics.[25] After the founding of London's King's College, Moseley was appointed as a chaplain and professor of natural philosophy and astronomy. He worked here for twelve years, teaching and writing on many applied topics.[26]

The concept of the metacentre is strongly based on forces. When a body is floating in a fluid, we have a downward force, its weight, and an upward force, called the buoyancy force. The whole body has its centre of mass, which we call G, while the centre of the displaced fluid is called the buoyancy centre B. If the density of the floating body is homogeneous, the centre of buoyancy is equal to the centre of mass of the part of the body that is immersed in the fluid. If the density is not uniform, however, this is not the case.[27] Whenever the two centres G and B are on the same vertical line and the upward and downward forces are equal, as in figure 3.3a, the floating body is in a position of equilibrium.

Now suppose we perform a small rotation around the centre of mass of the body. This will change the position of the centre of buoyancy since the part that is immersed in the fluid will become larger or smaller. The metacentre M is defined as the intersection of the line through G that rotates along with the body and the vertical line through B, see figure 3.3b. The distance GM is called the metacentric

[23] Heilbron, Moseley, et al., *HGJ Moseley: the life and letters of an English physicist, 1887-1915*, p. 3.

[24] Darrigol, *Worlds of flow: A history of hydrodynamics from the Bernoullis to Prandtl*, p. 135.

[25] Henry Moseley. *A Treatise on Hydrostatics and Hydrodynamics: For the Use of Students in the University*. Stevenson, 1830, p. iiv.

[26] Heilbron, Moseley, et al., *HGJ Moseley: the life and letters of an English physicist, 1887-1915*, p. 4.

[27] It is likely that Moseley was aware of this, as he devotes a chapter in his textbook on density, but does not mention it explicitly in his chapter on stability.

height. When M is located above G, the distance GM is considered positive, and there will be a restoring force that makes the body return to its initial position of equilibrium. This is the case in figure 3.3b. The equilibrium is said to be stable. The metacentric height is negative when M is below G, and in this case, there will be an overturning force, rotating the body even further away from the equilibrium. The equilibrium will thus be unstable. If, after the rotation, G and B are both on the same vertical line again, the body will be in a new position of equilibrium.

The stability analysis discussed by De Prony and Moseley is based on a formula that describes the metacentric height. We will use Moseley's textbook as the principal perspective, and discuss De Prony's when necessary. We are mostly interested in the way they conduct their stability analysis but will discuss small parts of their explanation of equilibria.

The chapters we are interested in are Moseley's chapter VI, *On the Stability of Floating Bodies*, and De Prony's *Recherche des conditions de la stabilité de l'équilibre des corps flottans*. Both first provide concepts necessary for understanding the definitions of equilibria and stability and then elaborate on different forms of stability of floating bodies. Moseley derives the formula with which one can investigate the stability of equilibria, before concluding his discussion with multiple examples of different equilibria for different bodies. De Prony introduces the formula but does not provide a justification for it. Perhaps he considered it less relevant because of his engineering background: for doing practical calculations, only the formula is necessary. Throughout both works, the physical interpretation of concepts is central. Moseley's first introduction to stability is explained entirely in words and using intuitive concepts. Moseley adds figures to illustrate bodies in fluids and their rotation: no analytical expressions are used to define or explain concepts. Whenever mathematics is introduced, the different components are directly related to the physical world.

For an equilibrium to exist, we need (1) the weight of the body to be equal to the weight of the displaced fluid, and (2) the centre of mass G of the body and the centre buoyancy B to be on the same vertical line. Whenever one of the conditions is unsatisfied — for example when the body is moved up or down or is rotated around a point — "the equilibrium will manifestly be destroyed."[28] Moseley states:

> *The question of stability consists in determining whether the body, when left to itself under these circumstances, will continually recede further from its position of equilibrium, or oscillate about and eventually recover it.*[29]

Before defining an analytical expression for the stability analysis of equilibria, Moseley discusses the physical situation as we have also discussed. He uses figure 3.4, in which we can interpret M as the centre of mass of the body, n as the centre of buoyancy, and μ as the metacentre. These specific terms are not used by Moseley, though they were known and used at the time, for example by De Prony. We look at PVQ as a floating body in equilibrium that is rotated about its centre of mass. The equilibrium is said to be stable if, when the body is rotated slightly, the body returns to the same position of equilibrium. Moseley says this happens when the point n moves in the opposite direction to the rotation of the body. Whenever the point n moves along the rotational motion of the body, the motion will be "continued and continually accelerated."[30] In this case, the equilibrium is unstable. This way of explaining may seem vague, but it is similar to our explanation above. Moseley assumes the reader understands the physical insight that is needed for his explanation, namely the understanding of the rotation of a body and how the centre of buoyancy changes after a rotation.

Moseley's definitions of (un)stable equilibria do not differ much from the modern concept of stability: we investigate what happens when a certain system is disturbed slightly. Moseley adds two additional forms of stability. An equilibrium is one of indifference if the point n remains on the vertical MK. In this case, a slight disturbance does not influence the equilibrium. An equilibrium is of mixed stability if the point n will move in the same direction, despite the direction of the rotation of the body. In this case, the

[28]There are situations, however, in which the equilibrium is not "manifestly destroyed", for example when a body rotates about its centre axis. See the cylinder in figure 3.3: if it rotates around its centre axis, the points G and B will not deviate from the line they are on. Henry Moseley. *A Treatise on Hydrostatics and Hydrodynamics: For the Use of Students in the University.* Stevenson, 1830, p. 78

[29]Ibid., p. 78.

[30]Ibid., p. 80.

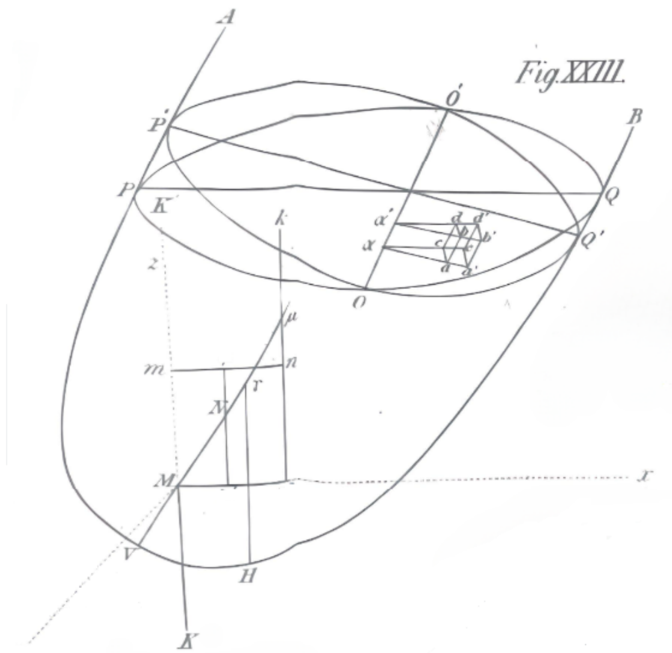


Figure 3.4: Figure 23 as depicted in Moseley's book. The body PVQ rotates around the centre of mass M, with n the centre of buoyancy and μ the metacentre.

rotational movement of the body will completely determine whether the equilibrium is stable or unstable since it can be moving in the same direction as n or the opposite.

The analytical expression that Moseley and De Prony define for analysing the stability of an equilibrium is the following:

$$M\mu = \frac{I}{m} \mp a.$$

The metacentric height is $M\mu$. This is then equated to the ratio of I, the moment of inertia of the plane of flotation, and m, the mass of the part immersed. The plane of flotation is the section of the floating body made by the horizontal surface of the fluid in which it floats. Moseley does not mention it, but it is assumed that there are no waves and we have a smooth surface. The parameter a is the distance between the centre of mass and the buoyancy centre. The sign in the formula can be chosen to be positive or negative according to whether the centre of gravity of the part immersed lies below or above the centre of gravity of the body, respectively. So when using this formula, one needs to understand the physical situation to some extent, in order to determine the sign. Moseley states that an equilibrium is stable, unstable, or of indifference when $M\mu$ is, respectively, positive, negative, or equal to zero. De Prony uses the same conditions. When analysing the stability of a floating body, using the formula for the metacentric height, it is assumed that the body is rotated with a small angle.

Let us look at an example Moseley gives: "To determine the stability of a cone when floating vertically."[31] Consider figure 3.5 and assume that this cone has a constant density. We let α and α_I be the radii of the whole cone and the immersed part, respectively.[32] Also, β and β_I are the distances from the bases to the vertex. To use the formula $M\mu = \frac{I}{m} \mp a$ we first need to find a: the distance from the centre of gravity of the whole cone to the centre of gravity of the part immersed. The distances of these centres to the vertex of the cone are, respectively, $\frac{3}{4}\beta$ and $\frac{3}{4}\beta_I$. Moseley considers this a known property. We find that $a = \frac{3}{4}(\beta - \beta_I)$.

[31] Moseley, *A Treatise on Hydrostatics and Hydrodynamics: For the Use of Students in the University*, p. 85.

[32] I have slightly adjusted Moseley's notation to prevent confusion. Moseley uses a to refer to the formula as well as the radii of the cones, which is why I chose to use α for the radii.

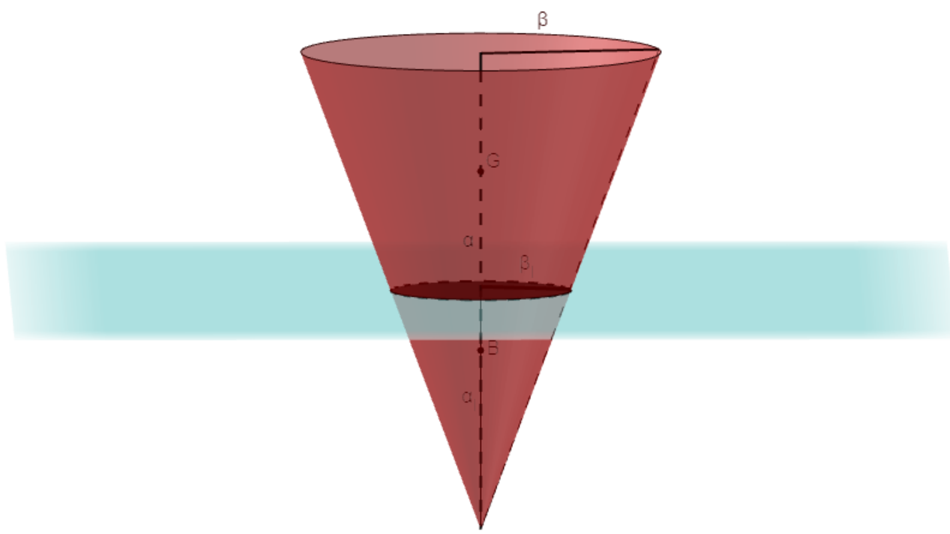


Figure 3.5: A cone floating vertically in a fluid.

Since the whole cone and the part that is immersed are two similar cones, Moseley states that

$$\alpha_I^3 = \sigma\alpha^3 \text{ and } \beta_I^3 = \sigma\beta^3,$$

where σ is a constant that describes the ratio between the whole cone and the part immersed in the fluid.

The mass of the immersed part of the cone is $M = \frac{1}{3}\pi\alpha_I^2\beta_I$. This is also treated as known by Moseley. The moment of inertia of the plane of flotation, which is a circle, is $I = \frac{1}{4}\pi\alpha_I^4$. So we have

$$M\mu = \frac{\frac{1}{4}\pi\alpha_I^4}{\frac{1}{3}\pi\alpha_I^2\beta_I} - \frac{3}{4}(\beta - \beta_I).$$

We take the minus sign in the formula because the centre of gravity of the cone lies above the centre of the part immersed.[33] We can simplify the formula and substitute α and β for α_I and β_I, using the similarity of the cones. This ultimately gives us

$$\begin{aligned} M\mu &= \frac{3}{4}\frac{\alpha_I^2}{\beta_I} - \frac{3}{4}(\beta - \beta_I) \\ &= \frac{3}{4}\left(\sigma^{\frac{1}{3}}\frac{\alpha}{\beta} - \beta(1 - \sigma^{\frac{1}{3}})\right). \end{aligned}$$

Moseley supposes that $\alpha = \beta$. He probably makes this assumption to simplify the example. Then we obtain

$$M\mu = \frac{3}{4}\left(2\sigma^{\frac{1}{3}} - 1\right).$$

Hence the equilibrium is stable when $\sigma > \frac{1}{8}$, unstable when $\sigma < \frac{1}{8}$ and of indifference when $\sigma = \frac{1}{8}$.

There is a natural way to interpret this result. Since σ is the ratio between the whole cone and the cone beneath the surface, the stability depends on its value. If σ is large enough, it means that the part of the cone immersed is a large part of the whole cone and the cone is, therefore, in a stable equilibrium. The smaller the part immersed, the more likely the cone is to topple when pushed. So if σ becomes small enough, the equilibrium will become unstable. Moseley does not add this interpretation.

[33]Moseley, *A Treatise on Hydrostatics and Hydrodynamics: For the Use of Students in the University*, p. 86.

Stability analysis using the metacentre is strongly based on a physical interpretation: when is the centre of mass above or below the centre of buoyancy? Even the analytical expression for the metacentric height, which can be derived mathematically, is used on physical grounds: we have stability when the metacentric height is positive because in that situation we have a restoring force that returns the body to its initial equilibrium. This is a rather intuitive way of doing stability analysis. Moseley's and De Prony's work can therefore be readily interpreted as mixed mathematics: there is an immediate connection between the mathematical elements and physical objects. The formula of the metacentric height is based on the geometrical properties of a floating body.

It can thus be said that Moseley and De Prony published this stability analysis within the framework of mixed mathematics and with strong connections to practical problems. Considering especially De Prony's background as an engineer, it makes sense that his textbooks focus on practical mathematics.

3.2 The motion of an ellipsoid

Our second case study will be based on a different kind of stability analysis: deriving a formula analogous to one that is already known. Several textbooks in the second half of the nineteenth century treat the stability of an object moving through a liquid, which is a dynamic problem rather than a static one and requires a different way of stability analysis. We will focus on one kind of object in particular: the ellipsoid. The motion of an ellipsoid through a fluid will prove to be similar to the motion of a swinging pendulum. Before discussing the specific problem and analysis, we will introduce the authors who used the same method in their textbooks.

Horace Lamb (1849-1934) is the first author and a well-known British mathematician. He studied at Trinity College in Cambridge and stayed there for three years after his studies as a lecturer, before moving to Australia in 1875 —at the age of 26 — and becoming the first professor of mathematics at the University of Adelaide.[34] In 1879 he wrote *A Treatise on the Mathematical Theory of the Motion of Fluids*, which was published in Cambridge and based on the lectures he gave on hydrodynamics in 1874.[35] This textbook was translated into German by Richard Reiff, a mathematics professor in Tübingen, in 1884.[36] Other German textbooks refer to this translation, indicating that it was a well-known work in Germany as well as Britain. After a reprint, the literary magazine The Athenaeum wrote "This is an able and carefully written work on one of the most thorny subjects in applied mathematics."[37]

Lamb returned to England in 1885 and became a professor of pure mathematics at Owens College in Manchester. The committee that selected him for this position noted that his publications so far, the book on hydrodynamics and several papers on electricity, were all focused on applied mathematics.[38] They emphasised, however, that these applied works imply "a high position in Pure Mathematics also."[39] Two points can be taken from this. First, the committee saw the distinction between pure and applied mathematics as a relevant one and considered this division for a position as a 'pure mathematician'. Second, the position was granted to Lamb, implying he was seen as suitable for the position. Shortly after this appointment, however, he became a professor of pure and applied mathematics at the same university and he held this position until his retirement in 1920.[40] The chair of applied mathematics was already occupied by Arthur Schuster, but he took over the chair in physics. Lamb was the one to propose this,

[34]C. C. Gillispie, F. L. Holmes, and N. Koertge. "Lamb, Horace". In: *Complete dictionary of scientific biography* 7 (2008), pp. 594–595.

[35]Horace Lamb. *A Treatise on the Mathematical Theory of the Motion of Fluids.* Cambridge University Press, 1879. Darrigol mentions that this treatise and its revised reprint are among the best-known nineteenth-century treatises on hydrodynamics. Olivier Darrigol. *Worlds of flow: A history of hydrodynamics from the Bernoullis to Prandtl.* Oxford University Press, 2005, p. ix.

[36]Horace Lamb and Richard August Reiff. *Einleitung in die Hydrodynamik.* Akademische Verlagsbuchhandlung von JCB Mohr, 1884.

[37]"Mathematical literature: Hydrodynamics by Horace Lamb". In: *The Athenaeum Journal of literature, science, the fine arts, music and the drama.* january to june (1896), p. 90.

[38]Brian Launder. "Horace Lamb and the circumstances of his appointment at Owens College". In: *Notes and Records of the Royal Society* 67.2 (2013), pp. 139–158, p. 146.

[39]Ibid., p. 146.

[40]Gillispie, Holmes, and Koertge, "Lamb, Horace".

and his own appointment as professor in pure and applied mathematics.[41] Lamb thus actively put himself forward as the professor of pure and applied mathematics, indicating that he saw himself fit as an applied mathematician as well as a pure one. The obituary notes on Lamb, from the Royal Society, state that, after Lamb's appointment, the teaching of applied mathematics was done by the mathematical staff.[42] "Thus, as the Beyer Professor of Pure and Applied Mathematics, Lamb's title and authority finally properly reflected his expertise and interests and provided the scope to develop in Manchester the broad spectrum of mathematics teaching and research that was to place the university among the top establishments in these fields."[43]

Alfred Barnard Basset (1854-1930) wrote his textbook *A Treatise on Hydrodynamics: with numerous examples* in 1888.[44] This work consists of two volumes, with which he intended to provide "the results of the most important investigations in the mathematical theory of hydrodynamics."[45] His obituary states that, not needing to adopt a profession due to his wealthy descent, Basset was able to devote himself completely to mathematical research. He "produced a succession of papers on applied mathematics, mainly on subjects suggested by current discussions. The 'classical' hydrodynamics had at that time a great fascination for a number of rising mathematicians."[46] He became a fellow of the Royal Society in 1889. Basset occupied himself with many topics in applied mathematics: not only hydrodynamics, but elasticity, electrostatics, and the electromagnetic theory of light.[47] Later in his mathematical career, Basset "turned his attention to pure mathematics."

After the two volumes on hydrodynamics, Basset published a second textbook, *An Elementary Treatise on Hydrodynamics*, in which he "abstained from introducing any of the more advanced methods of analysis."[48] He wrote this hydrodynamics treatise for those who were not "acquainted with the higher branches of mathematics."[49] Basset states that one can know parts of the branches of mathematical physics even without extensive knowledge of mathematics, although extensive mathematical knowledge is required for possessing "exhaustive knowledge" in mathematical physics.[50] Basset showed awareness of a difference in knowledge in those interested in hydrodynamics and the relevance of differentiating between those with and those without higher mathematical skills by publishing a separate treatise for each.

Lamb and Basset both base several of their discussions on the work of the German Gustav Robert Kirchhoff (1824-1884). Kirchhoff was a German physicist and mathematician. He was known for his experimental work but accepted the chair of theoretical physics in Berlin when his failing health prevented him from experimenting. He wrote many articles on mathematical physics, many of which were published in Crelle's journal. His famous volume, *Vorlesung über Mathematische Physik* (1876), was meant to arrive "at the general equations of mechanics through purely mathematical considerations."[51] During his studies, Kirchhoff came under the influence of Franz Neumann, who "introduced and further developed the ideas of the French school of mathematical physics in Germany."[52]

Another famous and often-cited textbook is the *Treatise on Natural Philosophy* (1867) by William Thomson (also known as Lord Kelvin) and Peter Guthrie Tait.[53] When other works mention the similarity between the motion of an ellipsoid and a pendulum, they usually refer to this work, suggesting Thomson and Tait may have been the first to establish this connection and present it to a broader audience. Thomson and Tait were both professors of Natural Philosophy at different universities in Britain. In his *The century's*

[41] Launder, "Horace Lamb and the circumstances of his appointment at Owens College", p. 152.

[42] Augustus Edward Hough Love and Richard Tetley Glazebrook. "Sir Horace Lamb, 1849-1934". In: *Obituary Notices of Fellows of the Royal Society* (1935), pp. 375–392, p. 378.

[43] Launder, "Horace Lamb and the circumstances of his appointment at Owens College", p. 152.

[44] Alfred Barnard Basset. *A treatise on hydrodynamics: with numerous examples.* Vol. 1. Bell and Company, 1888; Alfred Barnard Basset. *A treatise on hydrodynamics: with numerous examples.* Vol. 2. Bell and Company, 1888.

[45] Basset, *A treatise on hydrodynamics: with numerous examples*, preface.

[46] H. L. "Mr. A.B. Basset, F.R.S.". In: *Nature* 127.3198 (1931), p. 244, p. 244.

[47] H.L. "Alfred Barnard Basset, 1854-1930". In: *Obituary Notices of Fellows of the Royal Society* (1935), pp. i–ii, p. i.

[48] Alfred Barnard Basset. *An elementary treatise on hydrodynamics and sound.* Deighton, Bell, 1890, p. iii.

[49] Ibid., p. iii.

[50] Ibid., p. iii.

[51] "so gelangt man durch rein mathematische Betrachtungen zu den allgemeinen Gleichungen der Mechanik." Gustav Kirchhoff. *Vorlesungen über mathematische Physik.* Vol. 1. Leipzig, 1876, preface

[52] C. C. Gillispie, F. L. Holmes, and N. Koertge. "Kirchhoff, Gustav Robert". In: *Complete dictionary of scientific biography* 7 (2008), pp. 379–393, p. 380.

[53] Kelvin and Tait, *Treatise on natural philosophy.*

progress in applied mathematics, Woodward illustrates the relevance of Thomson's and Tait's treatise.

> *Another powerful impulse was given to hydrokinetics, and to all other branches of mathematical physics as well, by Kelvin and Tait's Natural Philosophy—the Principia of the nineteenth century—the first edition of which appeared in 1867. From this great work have sprung most of the ideas and methods appertaining to the theory of motion of solids in fluids, a theory which has yielded many interesting and surprising results through the researches of Kirchhoff, Clebsch, Bjerknes, Greenhill, Lamb, and others.*[54]

The body of textbooks we discussed above represents the core of hydrodynamics textbooks in the nineteenth century. All textbooks mentioned were highly influential and Lamb, Basset, Kirchhoff, Thomson and Tait are well-known names in hydrodynamics. The mutual references of these authors in their textbooks show that we can speak of a European body of knowledge in hydrodynamics, most notably in Britain and Germany, but also in France.[55] Authors were aware of new textbooks and used developments described in those to write their own treatises.

These textbooks all discuss the stability of a solid that is moving through a fluid and examine whether the motion of this object is stable. The structure of the argument in each of the textbooks is the same: derive an equation for the movement of the object, conclude that this equation is similar to a known one, and use the known properties to draw conclusions about the stability of the motion of an object. We will take as an object the ellipsoid, of which an example is depicted in figure 3.6. An ellipsoid is a solid of revolution of an ellipse. A solid of revolution is a solid that is obtained by rotating a planar curve about an axis. Such a solid is symmetrical about this axis. An ellipsoid is a body of revolution with respect to its minor or major axis. One gets an oblate (UFO-shaped) or prolate (oval-shaped) ellipsoid, respectively (see figure 3.6). This section will use Lamb's textbook as the principal perspective.

The motion of an ellipsoid through a liquid is brought back to a known equation: the equation of the motion of a pendulum. The equation of the pendulum was frequently used in hydrodynamics for the analysis of the motion of an ellipsoid through a fluid.[56] The equation of a pendulum is $m\ddot{s} = -gm\sin(\theta)$, in which θ is the angle with which the pendulum is 'swinging', m the mass of the swinging object, g the gravitational constant, and s the arc length elevation of the pendulum.[57] On the right-hand side, we have a negative sign because gravity acts to decrease s. We can rewrite s as $l\theta$, with l the length of the pendulum, obtaining

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta).$$

We will see that this is exactly the equation that is derived for the motion of an ellipsoid through a fluid.

To get to the discussion on stability, we will trace back some of the steps in Lamb's chapter *On the motion of solids through a liquid*.[58] Lamb defines equations of the motion of a solid through a liquid that has been "produced instantaneously from rest by the action of a properly chosen set of impulsive forces applied to the solid."[59] These are expressed in terms of the total kinetic energy T of the solid and the fluid together, the angular (rotational) velocities p, q, r, and the translational (linear) velocities u, v, w, all at any instant t. The external forces often taken to be zero are denoted by X, Y, Z for the translational,

[54] Woodward, "The century's progress in applied mathematics", p. 160.

[55] This does not mean the community of hydrodynamics was restricted to Europe or these three countries. In The Netherlands and Austria, for example, we have also encountered mathematicians working on hydrodynamics. These are, however, not discussed in this thesis. Examples of textbooks by Austrian professors are: Georg von Vega. *Anleitung zur Hydrodynamik*. Vol. 4. Beck, 1819 and Viktor von Lang. *Einleitung in die theoretische Physik*. F. Vieweg und sohn, 1891. The Dutch mathematician D.J. Korteweg also wrote on hydrodynamics in, for example, Diederik Johannes Korteweg and Gustav De Vries. "XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 39.240 (1895), pp. 422–443.

[56] Basset, *A treatise on hydrodynamics: with numerous examples*, p. 191.

[57] Viktor Blåsjö. "Intuitive Infinitesimal Calculus". In: *Intellectual Mathematics* (2015), p. 46.

[58] Lamb, *A Treatise on the Mathematical Theory of the Motion of Fluids*, Ch. V.

[59] Ibid., p. 121.

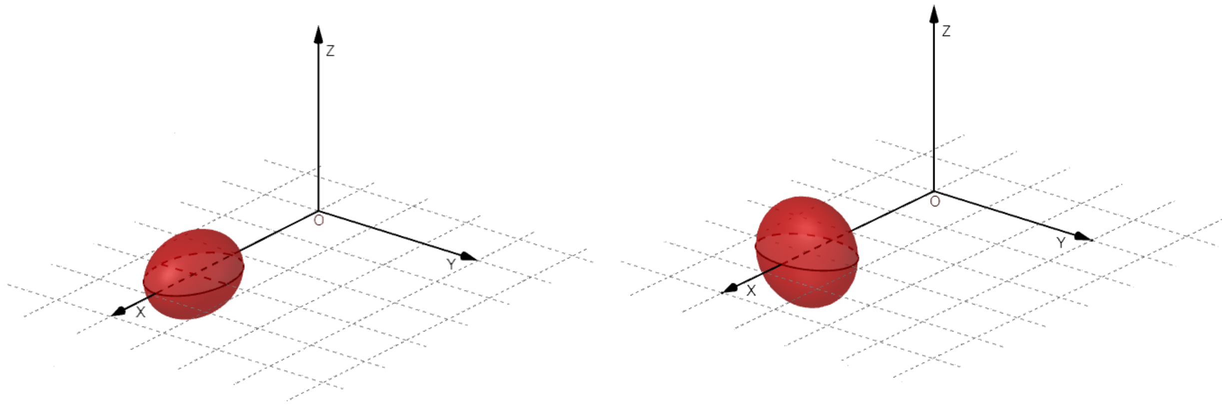


Figure 3.6: A prolate (left) and oblate (right) ellipsoid. In both cases, we assume the ellipsoid is moving in the x-direction.

and L, M, N for the rotational forces. The equations for the motion are:[60]

$$\begin{aligned}\frac{d}{dt}\frac{dT}{du} &= r\frac{dT}{dv} - q\frac{dT}{dw} + X & \qquad \frac{d}{dt}\frac{dT}{dp} &= w\frac{dT}{dv} - v\frac{dT}{dw} + r\frac{dT}{dq} - q\frac{dT}{dr} + L\\ \frac{d}{dt}\frac{dT}{dv} &= p\frac{dT}{dw} - r\frac{dT}{du} + Y & \frac{d}{dt}\frac{dT}{dq} &= u\frac{dT}{dw} - w\frac{dT}{du} + p\frac{dT}{dr} - r\frac{dT}{dp} + M\\ \frac{d}{dt}\frac{dT}{dw} &= q\frac{dT}{du} - p\frac{dT}{dv} + Z & \frac{d}{dt}\frac{dT}{dr} &= v\frac{dT}{du} - u\frac{dT}{dv} + q\frac{dT}{dp} - p\frac{dT}{dq} + N.\end{aligned}$$

These equations are also called Kirchhoff's equations since he proposed them in this form. Say we consider an ellipsoid as in figure 3.6. The translational velocities u, v, w are in the directions of x, y, z, respectively. The angular velocities p, q, r are rotations about the x, y, z axis, respectively.[61] The total kinetic energy T for a solid of revolution is

$$2T = Au^2 + Bv^2 + Cw^2 + Pp^2 + Qq^2 + Rr^2.$$

Here u, v, w, p, q, r are velocities as described above and the capital letters are coefficients. For our purpose we do not need to have a complete understanding of the physics behind this equation, only how this formula is used.

We have the following situation. We have a solid of revolution that is moving in the direction of its axis of symmetry, "without rotation about its axis of symmetry, and with this axis always in one plane."[62] The axis of symmetry can either be the minor or major axis, depending on whether we have an oblate or prolate ellipsoid, respectively. Consider the oblate ellipsoid in figure 3.6 and say it is moving in the x direction. The axis of symmetry (its major axis) in this direction will remain in the xy plane and there will thus be no upward or downward movement of the ellipsoid. We also have no rotation about this symmetry axis.[63] In this way, the physical situation is reduced to a two-dimensional one and mathematically easier to analyse.

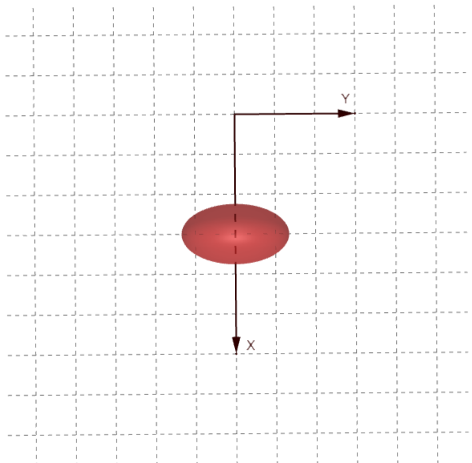
If we use the expression for the kinetic energy for a body of revolution and substitute this in the general expression for the motion of a solid through a liquid, we find the following equations for a solid of

[60] Lamb, *A Treatise on the Mathematical Theory of the Motion of Fluids*, p. 126.

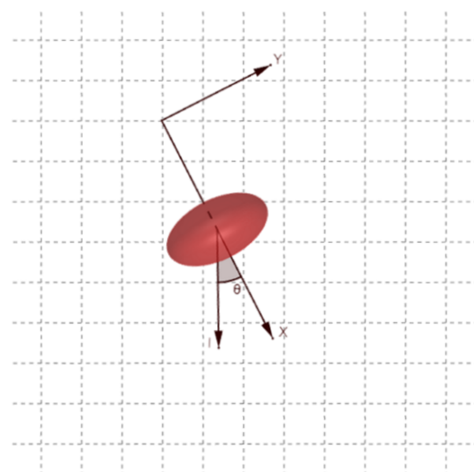
[61] It could also be that these rotations are about the axes of the ellipsoid, which are in the same direction as the x, y, z axis but do not have to be on these axes. Lamb does not explain this.

[62] Lamb, *A Treatise on the Mathematical Theory of the Motion of Fluids*, p. 133.

[63] In general, of course, such a body of revolution may move in any direction, not necessarily parallel to its axis of symmetry. For this analysis, however, it is assumed that this is the case. We put the axis of symmetry on the same line as the x-axis in figure 3.6 to conform it to the situation sketched by Lamb and the other authors. Lamb did not add pictures to his analysis, but this depiction of the situation is, we believe, close to his interpretation of the situation.



(a) The oblate ellipsoid from above. The impulse force is causing the ellipsoid to move in the x-direction.



(b) The ellipsoid is slightly displaced. The I-direction is the initial direction of the impulse and the axes rotate along with the displacement. The angle θ is defined as the disturbance of the motion.

revolution moving through a liquid, respecting the mentioned restrictions.

$$A\frac{du}{dt} = rBv$$
$$B\frac{dv}{dt} = -rAu$$
$$R\frac{dr}{dt} = (A-B)uv.$$

We assume that the ellipsoid is moved from a position of rest by an impulse force I in the direction of its symmetry axis, the x direction in our case. So the ellipsoid is pushed forward through the fluid by an impulse force and no other (external) forces are moving it. It is also assumed that the ellipsoid moves in an infinite fluid, so the location of the ellipsoid in the fluid does not influence its motion. Now say the ellipsoid is generally moving in a straight line since there are no forces other than the impulse, but it is slightly pushed out of its way. We want to investigate what happens to the motion of the ellipsoid. We take θ as the angle that the centre of the ellipsoid makes with its initial axis of direction, see figure 3.7b.[64] We have

$$Au = I\cos(\theta), \quad Bv = -I\sin(\theta), \quad r = \dot{\theta}.$$

So $I\cos(\theta)$ is the velocity in the u direction, which coincides with the x direction, and $-I\sin(\theta)$ is the velocity in the v (y) direction. These expressions follow from basic trigonometric properties. The angular velocity r is simply the rate of change in θ.

Substituting these into our equations for the motion of the solid of revolution, we obtain:

$$-AI\sin(\theta)\dot{\theta} = -I\sin(\theta)\dot{\theta}$$
$$-BI\cos(\theta)\dot{\theta} = -I\cos(\theta)\dot{\theta}$$
$$R\ddot{\theta} + \frac{A-B}{AB}I^2\sin(\theta)\cos(\theta) = 0.$$

[64]Interestingly, Lamb does not explicitly state that θ is the disturbance of the otherwise straight motion. For stability analysis, however, we want some small displacement of the motion to investigate what happens after this displacement.

This last equation is the only important one. According to Lamb, the first two only describe the "fixity of the direction of the impulse."[65] The third equation can be rewritten, when taking $2\theta = \phi$, as follows:

$$\ddot{\phi} + \frac{A-B}{ABR} I^2 \sin(\phi) = 0.$$

Lamb recognises this as the equation of the common pendulum, which it is indeed if we take $\frac{A-B}{ABR} I = \frac{g}{l}$. He refers to Thomson and Tait's textbook for a more extensive discussion on the similarity between the motion of an ellipsoid through a liquid and a pendulum. This discussion is rather vague at points, although Thomson and Tait seem to arrive at the same equation and the same conclusions about stability. Lamb immediately concludes that "it appears from [this equation] that the motion of the solid parallel to its axis is stable or unstable according as $A \lessgtr B$."[66] He does not seem interested in the exact solution of the equation; it seems subordinate to the stability of the motion.[67] Lamb does not, however, explain whether the motion is stable when $A < B$ or $B > A$ and why this is the case. He seems to assume the reader has knowledge of the motion and equation of a pendulum and does not need an additional explanation for analysing the stability of the motion of a solid through a liquid.

Our interpretation, based on the motion and equation of the pendulum, is as follows. When considering the equation of a pendulum, in modern notation $\ddot{\theta} = -\frac{g}{l}\sin(\theta)$, the right-hand side is negative, because the force of gravity acts to decrease θ. This means that the force acting on the pendulum is directed toward its lowest position. If the right-hand side were positive, those forces would enlarge θ, meaning the pendulum would swing upwards without returning.

A similar situation can be sketched for Lamb's equation, which we can rewrite as

$$\ddot{\phi} = -\frac{A-B}{ABR} I^2 \sin(\phi).$$

When $A < B$, the constant $\frac{A-B}{ABR} I^2$ will be negative, causing the right-hand side to be positive and the motion to be unstable.[68] When $A > B$, we thus have stable motion.[69] Lamb concludes that the ellipsoid is prolate when $A < B$, so a prolate ellipsoid moving in the direction of its major axis will be in an unstable equilibrium. When $A > B$ the ellipsoid will be an oblate one: an oblate ellipsoid moving in the direction of its minor axis will be stable.[70] Stable motion means that, after a disturbance of the motion with angle θ, the angle will diminish. When an ellipsoid in a fluid is thus pushed slightly out of its direction, with angle θ, it will tend to return to its previous motion, after oscillating around the x-axis. If the motion is unstable, the ellipsoid will not return to its previous direction of motion, but completely move away from its initial direction. It makes sense to assume that Lamb would have such a physical interpretation in mind. At the same time, it is telling that he did not need an additional justification for the stability of motion. It was sufficient to derive the equation of the pendulum and then simply conclude that the motion is stable when $A > B$ and unstable when $A < B$. The physical interpretation of the equation of a pendulum did not require additional explanation, even though it seems the only way to justify the motion being (un)stable.

Lamb does not explain how the body's motion through a liquid connects to the pendulum's physical motion. He only derives the formula for the motion of a solid through a liquid and states that it is similar to the formula for the motion of a pendulum. Here, Lamb neglects the physical interpretation and only shows the connection mathematically. On the one hand, Lamb's work seems highly dependent, as Moseley's

[65] Lamb, *A Treatise on the Mathematical Theory of the Motion of Fluids*, p. 133.

[66] Lamb uses the $\lessgtr$ sign to denote A being larger than B or vice versa. Horace Lamb. *A Treatise on the Mathematical Theory of the Motion of Fluids*. Cambridge University Press, 1879, p. 134.

[67] He does mention that the solution can be found using elliptic functions and refers to this solution in Kirchhoff's work.

[68] The coefficients A, B, R, and I are all positive.

[69] Note that Lamb's notation, using $A \gtrless B$, may confuse. We are accustomed to reading "the motion is stable or unstable according as $A \gtrless B$" as 'the motion is stable or unstable when $A < B$ or $A > B$', which is the opposite of what we argue. Comparing Lamb's quote to other sections of his book in which stability is discussed, however, leads to the conclusion that we should read this reversed.

[70] Unfortunately, Lamb does not explain why we have a prolate ellipsoid when $A < B$ and an oblate ellipsoid when $A > B$. This probably has to do with the kinetic energy of the object, since A and B are coefficients of the kinetic energy of the ellipsoid.

book, on physical interpretations and explanations. On the other hand, he uses mathematical derivations without adding a physical interpretation. He uses simplifications in his mathematical analysis: assuming there is no upward or downward motion, for example, allows for a two-dimensional analysis rather than a three-dimensional one. Hence this analysis works for specific situations with specific bodies, such as the ellipsoid. An interesting addition to Lamb's analysis is provided by Thomson and Tait. They state that "it must be remembered that the real circumstances differ greatly [...] from those of the abstract problem, of which we take leave for the present."[71]

Thomson and Tait also discuss the practical situations that can now be explained.

> *The tendency of a body to turn its flat side, or its observed length (as the case may be) across the direction of its motion through a liquid [...] is closely connected with the dynamical explanation of many curious observations well known in practical mechanics, among which may be mentioned; that the towing-rope of a canal boat, when the rudder is left straight, takes a position in a vertical plane cutting the axis before its middle point; that a boat sculled rapidly across the direction of the wind, always (unless it is extraordinarily unsymmetrical in its draught of water, and in the amounts of surface exposed to the wind, towards its two ends) requires the weather oar to be worked hardest to prevent it from running up on the wind, and that a sailing vessel generally "carries a weather helm" for the same reason; that in a heavy gale it is exceedingly difficult, and often found impossible, to get a ship out of "the trough of the sea," and that it cannot be done at all without rapid motion ahead, whether by steam or sails; that an elongated rifle-bullet requires rapid rotation about its axis to keep its point foremost. The curious motions of a flat disc, oyster-shell, or the like, when dropped obliquely into Water, resemble, no doubt, to some extent those described in § 333.*[72]

So the fact that an ellipsoid is in stable motion when it moves in the direction of its minor axis — or, as one could say, in the direction of its flat or longest side — is in accordance with some "curious observations well known in practical mechanics."

3.3 The motion of vortex rings

The last work we will investigate is *A Treatise on the Motion of Vortex Rings*, written by Joseph John Thomson.[73] The book is no textbook, it was not primarily written for educational purposes, but Thomson's method is mentioned in Basset's treatises. We examine Thomson's own work because it is more elaborate than Basset's and certainly relevant for the field of hydrodynamics.[74]

Thomson was educated at Trinity College in mathematics — pure as well as applied, as was common in British mathematics education —, and stayed at this college as a professor after his studies. He immediately started doing research, mostly on applied mathematics, but he also did experimental work.[75] At the end of 1884, Thomson was elected for the Cavendish Professorship at Cambridge. This was a professorship in physics. Thomson is said to have not taken his election very seriously: he was seen, and saw himself, "as more a mathematician than an experimental physicist."[76] He nevertheless was very successful in this position and he pursued a career as an experimentalist.

For his treatise on vortex motion, he received an Adams Prize; the goal of the treatise was to provide "a general investigation of the action upon each other of two closed vortices in a perfect incompressible fluid."[77] Thomson worked mostly on topics in mathematical physics, like hydrodynamics and electrody-

[71] Kelvin and Tait, *Treatise on natural philosophy*, p. 269.

[72] Ibid., p. 269.

[73] One should not confuse this Thomson with the well-known William Thomson, from the Thomson and Tait textbook. Both made important contributions to hydrodynamics.

[74] Alfred Barnard Basset. *An elementary treatise on hydrodynamics and sound.* Deighton, Bell, 1890, p. 112; Alfred Barnard Basset. *A treatise on hydrodynamics: with numerous examples.* Vol. 2. Bell and Company, 1888, p. 79, 80

[75] Rayleigh. "Joseph John Thomson, 1856-1940". In: *Obituary Notices of Fellows of the Royal Society* (1941), pp. 587–609, p. 588.

[76] Ibid., p. 588.

[77] C. C. Gillispie, F. L. Holmes, and N. Koertge. "Thomson, Joseph John". In: *Complete dictionary of scientific biography* 13 (2008), pp. 362–373, p. 362.

namics.

Thomson investigates whether several vortices, in proximity to each other, remain in a stable motion when they move simultaneously along a circle in a fluid. The vortices rotate themselves and generate motion in the fluid: they influence each other and generate a rotating motion of the system of vortices along the circle.[78] Thomson's starting point is the following.

> *The problem we are about to investigate is this: A system consisting of n equal straight cylindrical vortices, arranged at equal intervals round the circumference of a circle, is slightly displaced; what is the subsequent motion? We suppose the radius of a cross-section of a vortex to be small compared with the distance between two vortices.*[79]

So we have a system of n vortices, situated around a circle in a fluid, at equal distances from one another, see figure 3.8 for an example. These vortices rotate around each other along the circle. Thomson investigates the stability of such a system for $2 \leq n \leq 7$, so up to seven vortices. We are interested in a horizontal cross-section of the vortices, such as depicted in figure 3.8. This means that, even though the physical problem is a three-dimensional one, we can approach it as a two-dimensional problem; we do not consider upward or downward motion, only inward or outward from the centre of the circle, or a circular motion about the centre of the circle.

To illustrate his method, we need some of the equations that Thomson uses. For our purpose, it is not necessary to give a detailed description of their derivation. It will suffice to provide the equations and explain their components.

We will start by considering the position of a single vortex. Thomson uses a radius vector

$$\vec{r_k} = (r + x_k)$$

for the k^{th} vortex, where r is the radial coordinate of the undisturbed position of the vortex and x_k is the disturbance. This is a position vector: it represents the position of the k^{th} vortex in relation to a reference point. In our case, this reference point is the centre of the circle. Thomson does not explain it, but r is the distance from the centre of the circle to the position of the k^{th} vortex on the circle. This distance does not depend on a specific vortex but is the same for all vortices. We also have the angle

$$\phi_k = (w_k + \theta_k),$$

that describes the position of the k^{th} vortex on the circle. Here w_k is the angle of the undisturbed position, so when all vortices are at equal distances from one another. The disturbance in the angles is called θ_k. We assume that x_k and θ_k are small. The strength of the vortices is taken to be m. Thomson does not elaborate on the physical meaning of the strength of a vortex, but as we understand it, it has to do with the rotation of the vortex itself and the influence it exerts on the fluid around it; a vortex creates motion in the fluid around it and this motion is dependent on the strength of the vortex. This is also the motion that incites the system of vortices to rotate along the circle. Thomson can provide a stream function that describes the fluid flow due to the vortices for a certain point in the fluid.[80]

As in the previous two case studies, we have a small disturbance of a steady state and are interested in the behaviour of the fluid and objects in it after this disturbance. In this case, the disturbance is in the direction of the radius vector. We can imagine the vortices being pushed slightly away from the centre of the circle and wonder what happens to the motion of all the vortices together. All vortices may be disturbed: Thomson finds expressions for all x_n and then analyses the stability of the motion.

Two equations provide the necessary mathematical expressions to carry out the calculations for the stability analysis. Thomson derives these equations himself but for reasons of brevity, we do not include

[78] There is thus no external force that moves the vortices. Thomson derives the period of 'one vibration' in his chapter on a system of vortices. Joseph John Thomson. *A Treatise on the Motion of Vortex Rings: an essay to which the Adams prize was adjudged in 1882, in the University of Cambridge.* Macmillan, 1883, p. 85.

[79] Ibid., p. 94.

[80] He uses this function in his derivation of 3.3.1 and 3.3.2. We do not provide this derivation, but for the physical understanding of the equations, it is useful to know that the rotation of the vortices influences the fluid the vortices are in.

this derivation. First, the equation for the radial velocity, the velocity along the radius vector, of the k^{th} vortex is

$$\frac{dx_k}{dt} = \frac{m}{2\pi r} \sum_{i=1, i\neq k}^{n} \left(\frac{\theta_k - \theta_i}{1 - \cos(w_k - w_i)} \right). \tag{3.3.1}$$

The radial velocity is the object's motion towards or away from the middle of the circle.

Secondly, the velocity perpendicular to the radius vector is described by the equation:[81]

$$\frac{rd\theta_k}{dt} = -\frac{m}{2\pi r^2} \left(-\frac{x_k(n-1)(n-11)}{6} + \sum_{i=1, i\neq k}^{n} \frac{x_i}{1 - \cos(w_k - w_i)} \right). \tag{3.3.2}$$

This is thus the circular velocity. Note that in either equation we do not want $i = k$ in the summation. A vortex is not acting to change its own disturbance, this disturbance is only due to the motion of the other vortices. In equation 3.3.1 this term would vanish but in equation 3.3.2 we would also obtain a value of 0 in the denominator. Thomson derived these equations to be functions of x_k and θ_k. When undisturbed, the vortices have no velocity along the radius vector, so the radial velocity needs to be a function that goes to zero when x_k and θ_k do. This is now the case.

Let us look at the example of 3 vortices.[82] These are arranged as the points of an equilateral triangle. Therefore, $w_2 - w_1 = w_3 - w_2 = 120°$. We can use this to obtain the equation for the radial velocity, following equation 3.3.1, for the first vortex, with $n = 3$:

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{m}{2\pi r} \left(\frac{\theta_1 - \theta_2}{\frac{3}{2}} + \frac{\theta_1 - \theta_3}{\frac{3}{2}} \right) \\ &= \frac{m}{2\pi r} \left(\frac{2\theta_1 - \theta_2 - \theta_3}{\frac{3}{2}} \right). \end{aligned}$$

With similar expressions for the other two vortices. Thomson states that $\theta_1 + \theta_2 + \theta_3 = 0$. He does not explain why we have this property. It seems to be a result of the influence the vortices exert on each other; when one vortex moves towards its neighbouring vortex it may attract this neighbour and thus cause a disturbance of this vortex in the opposite direction. In total, then, the disturbances cancel out. Using this to eliminate θ_2 and θ_3 from the equation for x_1 we find

$$\frac{dx_1}{dt} = \frac{m}{\pi r}\theta_1.$$

These are the equations for the radial velocities of the vortices. Using equation 3.3.2 we obtain

$$\begin{aligned} \frac{rd\theta_1}{dt} &= -\frac{m}{2\pi r^2} \left(-\frac{x_1(3-1)(3-11)}{6} + \frac{x_2}{1 - \cos(w_1 - w_2)} + \frac{x_3}{1 - \cos(w_1 - w_3)} \right) \\ &= \frac{rd\theta_1}{dt} = -\frac{m}{2\pi r^2} \left(\frac{8x_1}{3} + \frac{2x_2}{3} + \frac{2x_3}{3} \right) \end{aligned}$$

for the first vortex, and similar expressions for the other two. Thomson now states that $x_1 + x_2 + x_3 = 0$, "since $\sum mr^2$ is constant".[83] Thomson does not explain this property, but it seems to be a conservation law; somehow the vortices influence each other in such a way that the radial disturbances cancel out. Thomson finds

$$\begin{aligned} \frac{rd\theta_1}{dt} &= -\frac{m}{2\pi r^2} \left(\frac{8x_1 - 2x_1}{3} \right) \\ &= -\frac{m}{\pi r^2} x_1. \end{aligned}$$

[81] Nowadays this is better known as the tangential velocity.

[82] Thomson, *A Treatise on the Motion of Vortex Rings: an essay to which the Adams prize was adjudged in 1882, in the University of Cambridge*, p. 98.

[83] Ibid., p. 98.

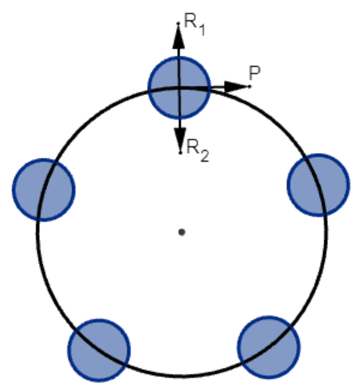


Figure 3.8: A two-dimensional, schematic depiction of the 5 cylinders along a circle. The direction of the radial velocity is either R_1 or R_2 and the direction of the velocity perpendicular to the radial velocity is denoted by P.

If we take the expressions $\frac{dx_1}{dt} = \frac{m}{\pi r}\theta_1$ and $\frac{rd\theta_1}{dt} = -\frac{m}{\pi r^2}x_1$, we can combine these two and deduce that

$$\frac{d^2x_1}{dt^2} + \frac{m^2}{\pi^2 r^4}x_1 = 0.$$

And finally, when solving this differential equation, Thomson obtains

$$x_1 = A\sin(\frac{m}{\pi r^2}t + a)$$
$$r\theta_1 = A\cos(\frac{m}{\pi r^2}t + a).$$

Similar expressions can be derived for $x_2, x_3, r\theta_2$ and $r\theta_3$. Thomson now concludes: "thus the motion in this case is stable, and the time of a small oscillation $= \frac{2\pi^2 r^2}{m}$, the same as the period of rotation of the undisturbed system."[84] Since the disturbance of the motion is outward or inward, we can interpret one oscillation of a vortex as moving away from the centre of the circle and back towards it. According to Thomson, the time it takes to make such an oscillation is the same as the period of rotation of the vortex system. So the vortices are rotating in their circular shape and at the same time moving inwards and outwards. The motion is stable in this case because the vortices oscillate about the stable position on the circle and are not completely pushed out of their position along the circle without returning.

The case of four vortices is similar to the case of three vortices. When Thomson considers five, however, he starts using matrices for his stability analysis and we will look into this. Let us first investigate how Thomson uses equations 3.3.1 and 3.3.2 for the five vortices. We will again provide the derivation for the first vortex and conclude that the others are obtained in a similar matter. Thomson does this as well.

[84]Thomson, *A Treatise on the Motion of Vortex Rings: an essay to which the Adams prize was adjudged in 1882, in the University of Cambridge*, p. 99.

From equation 3.3.1 we find

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{m}{2\pi r}\left(\frac{\theta_1-\theta_2}{1-\cos(\frac{2}{5}\pi)}+\frac{\theta_1-\theta_3}{1-\cos(\frac{4}{5}\pi)}+\frac{\theta_1-\theta_4}{1-\cos(\frac{6}{5}\pi)}+\frac{\theta_1-\theta_5}{1-\cos(\frac{8}{5}\pi)}\right)\\ &= \frac{m}{2\pi r}\left(\theta_1\left(\frac{1}{1-\cos(\frac{2}{5}\pi)}+\frac{1}{1-\cos(\frac{4}{5}\pi)}+\frac{1}{1-\cos(\frac{6}{5}\pi)}+\frac{1}{1-\cos(\frac{8}{5}\pi)}\right)-\frac{\theta_2+\theta_5}{1-\cos(\frac{2}{5}\pi)}-\frac{\theta_3+\theta_4}{1-\cos(\frac{4}{5}\pi)}\right).\end{aligned}$$

Here we used that $\cos(\frac{2}{5}\pi)=\cos(\frac{8}{5}\pi)$ and $\cos(\frac{4}{5}\pi)=\cos(\frac{6}{5}\pi)$. We can also rewrite $\frac{1}{1-\cos(\frac{2}{5}\pi)}+\frac{1}{1-\cos(\frac{4}{5}\pi)}+\frac{1}{1-\cos(\frac{6}{5}\pi)}+\frac{1}{1-\cos(\frac{8}{5}\pi)}=4$. And since $\theta_1+\theta_2+\theta_3+\theta_4+\theta_5=0$, and thus $\theta_3+\theta_4=-\theta_1-\theta_2-\theta_5$, we obtain

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{m}{2\pi r}\left(4\theta_1-\frac{\theta_2+\theta_5}{1-\cos(\frac{2}{5}\pi)}-\frac{-\theta_1-\theta_2-\theta_5}{1-\cos(\frac{4}{5}\pi)}\right)\\ &= \frac{m}{2\pi r}\left(4\theta_1-\frac{\theta_1}{1-\cos(\frac{4}{5}\pi)}+(\theta_2+\theta_5)\left(-\frac{1}{1-\cos(\frac{2}{5}\pi)}+\frac{1}{1-\cos(\frac{4}{5}\pi)}\right)\right)\\ &= \frac{m}{2\pi r}\left(a\theta_1+b(\theta_2+\theta_5)\right).\end{aligned}$$

Thomson does not provide these steps, but immediately gives us this last equation. He is able to eliminate the terms θ_3 and θ_4 in the equation for x_1. In similar ways, he obtains equations for x_2 to x_5. We have that $a=\frac{25-\sqrt{5}}{5}$ and $b=-\frac{2}{\sqrt{5}}$

In a similar way, by using equation 3.3.2 and that $x_1+x_2+x_3+x_4+x_5=0$, Thomson finds that

$$\begin{aligned}\frac{d\theta_1}{dt} &= -\frac{m}{2\pi r^2}\left(4x_1+\frac{x_2}{1-\cos(\frac{2}{5}\pi)}+\frac{x_3}{1-\cos(\frac{4}{5}\pi)}+\frac{x_4}{1-\cos(\frac{6}{5}\pi)}+\frac{x_5}{1-\cos(\frac{8}{5}\pi)}\right)\\ &= -\frac{m}{2\pi r^2}\left(x_1\left(4-\frac{1}{1-\cos(\frac{4}{5\pi})}\right)+(x_2+x_5)\left(\frac{1}{a-\cos(\frac{2}{5}\pi)}-\frac{1}{1-\cos(\frac{4}{5}\pi)}\right)\right)\\ &= -\frac{m}{2\pi r^2}\left(cx_1-b(x_2+x_5)\right).\end{aligned}$$

With $c=\frac{15+\sqrt{5}}{5}$ and with similar expressions for x_2 to x_5. We see that Thomson expresses the motion of vortex 1 in terms of its own position and the position of its neighbouring vortices. The initial equations, however, show that the motion depends on all vortices. In principle, Thomson is able to choose in what manner he uses the properties of $x_1+x_2+x_3+x_4+x_5=0$ and $\theta_1+\theta_2+\theta_3+\theta_4+\theta_5=0$ and to choose which terms he wants to eliminate.

Finally, by combining the expressions for θ_1 and x_1, just as we did for three vortices, we obtain complex expressions for the radial disturbance. When we replace the constants in front of the variables by a' and b' we obtain expressions that are easier to work with.[85] Thomson thus finds[86]

$$\begin{aligned}\frac{d^2x_1}{dt^2} &= -\left(a'x_1+b'(x_2+x_5)\right)\\ \frac{d^2x_2}{dt^2} &= -\left(a'x_2+b'(x_3+x_1)\right)\\ \frac{d^2x_3}{dt^2} &= -\left(a'x_3+b'(x_4+x_2)\right)\\ \frac{d^2x_4}{dt^2} &= -\left(a'x_4+b'(x_5+x_3)\right)\\ \frac{d^2x_5}{dt^2} &= -\left(a'x_5+b'(x_1+x_4)\right).\end{aligned}$$

[85] We follow Thomson's notation when we use a' and b'. We have $a'=ac-b^2$ and $b'=bc-ab+b^2$.

[86] Thomson, *A Treatise on the Motion of Vortex Rings: an essay to which the Adams prize was adjudged in 1882, in the University of Cambridge*, p. 101.

Thomson now states: "$x_1, x_2, \ldots$ vary as $e^{\lambda t}$."[87] Thomson thus assumes that the radial disturbances x_1, x_2, x_3, x_4, x_5 are of the form $e^{\lambda t}$, probably with different coefficients, so $x_1 = A_1 e^{\lambda t}, \ldots, x_5 = A_5 e^{\lambda t}$. When we use this, the differential equations become

$$\begin{aligned}
\lambda^2 x_1 &= -\left(a' x_1 + b'(x_2 + x_5)\right) \\
\lambda^2 x_2 &= -\left(a' x_2 + b'(x_3 + x_1)\right) \\
\lambda^2 x_3 &= -\left(a' x_3 + b'(x_4 + x_2)\right) \\
\lambda^2 x_4 &= -\left(a' x_4 + b'(x_5 + x_3)\right) \\
\lambda^2 x_5 &= -\left(a' x_5 + b'(x_1 + x_4)\right).
\end{aligned}$$

Now, Thomson states, the equation to determine λ is

$$\begin{vmatrix}
a' + \lambda^2 & b' & 0 & 0 & b' \\
b' & a' + \lambda^2 & b' & 0 & 0 \\
0 & b' & a' + \lambda^2 & b' & 0 \\
0 & 0 & b' & a' + \lambda^2 & b' \\
b' & 0 & 0 & b' & a' + \lambda^2
\end{vmatrix} = 0.$$

Thomson thus uses the fact that the determinant is equal to 0 to determine λ. We know that the system 3.3.3 has non-trivial solutions if and only if the determinant equals 0. Thomson must have known that there are non-trivial solutions of this system — in fact, we are looking for the non-trivial solutions since there will be no disturbance of the vortices in the trivial case. We are interested in the situation in which there is a small disturbance and the stability of the vortices.

For finding the determinant, Thomson uses a theorem from *A Treatise on the Theory of Determinants*, written by Robert Forsyth Scott.[88] Scott provides a general formula for the determinant of a matrix of the form

$$\begin{vmatrix}
a_1 & a_2 & \ldots & a_n \\
a_n & a_1 & \ldots & a_{n-1} \\
a_{n-1} & a_n & \ldots & a_{n-2} \\
\ldots & & & \\
a_2 & a_3 & \ldots & a_1
\end{vmatrix}$$

The determinant of this matrix, where the same value a_1 is on the diagonal, is of the form

$$(a_1 + a_2 + a_3 + \cdots + a_n) \prod (a_1 + a_2\omega + a_2\omega^2 + \cdots + a_n\omega^{n-1}), \qquad (3.3.3)$$

where ω "is one of the roots of the equation $x^n - 1 = 0$, unity being excepted."[89]

Using this, and taking $1, \omega, \omega^2, \omega^3, \omega^4$ to be the fifth roots of unity, Thomson can conclude that the equation to be solved for λ^2 is

$$(a' + \lambda^2 + 2b')(a' + \lambda^2 + 2\cos(\frac{2\pi}{5})b')^2(a' + \lambda^2 + 2\cos(\frac{4\pi}{5})b')^2 = 0.$$

The "sole condition of stability" is, according to Thomson, that the values of λ^2 are negative.[90] He refers to Thomson and Tait, who state that "If the roots (λ^2) of the determinantal equation [...] are all real and negative, the equilibrium is stable: in every other case it is unstable."[91]

[87]Thomson, *A Treatise on the Motion of Vortex Rings: an essay to which the Adams prize was adjudged in 1882, in the University of Cambridge*, p. 101.

[88]Robert Forsyth Scott. *Treatise on the Theory of Determinants and their Applications in Analysis and Geometry.* University Press, 1880, p. 82.

[89]Joseph John Thomson. *A Treatise on the Motion of Vortex Rings: an essay to which the Adams prize was adjudged in 1882, in the University of Cambridge*. Macmillan, 1883, p. 102; Robert Forsyth Scott. *Treatise on the Theory of Determinants and their Applications in Analysis and Geometry.* University Press, 1880, p. 83. A full explanation of this property is given by Scott.

[90]Thomson, *A Treatise on the Motion of Vortex Rings: an essay to which the Adams prize was adjudged in 1882, in the University of Cambridge*, p. 102.

[91]Kelvin and Tait, *Treatise on natural philosophy*, p. 280.

The values of λ^2 are

$$-(a' + 2b')$$
$$-(a' + 2\cos(\frac{2\pi}{5})b')$$
$$-(a' + 2\cos(\frac{4\pi}{5})b').$$

These are all negative, which follows from the values of a' and b'. When λ^2 is negative, λ will be imaginary. Since the solutions for x_1 to x_5 are of the form $e^{\lambda t}$, we will, again, find a periodical solution for the disturbance of each vortex. In this case, the periods of vibration are

$$\frac{4\pi^2 r^2}{m\sqrt{14+2\sqrt{5}}}, \frac{\pi^2 r^2}{m}, \frac{2\pi^2 r^2}{m\sqrt{3}}.$$

We see in Thomson's work that for each additional vortex, the time of an oscillation becomes smaller.[92] For three vortices, the time of an oscillation was the same as a rotation of the system of vortices. For each additional vortex, the vibrations are faster. When we have seven vortices, at least one value of λ^2 is positive, leading to a real value for λ and unstable motion of the vortices. Thomson thus concludes that "six is the greatest number of vortices which can be arranged at equal intervals round the circumference of the circle."[93]

Much more than the two previous case studies, Thomson is using pure mathematics and applying it to a natural phenomenon. Scott's treatise, to which Thomson refers, deals with pure mathematics: Scott discusses the theory of determinants without motivation from their application to some kind of physical object. Matrices and their determinants are dealt with in their own right. The mathematics discussed is abstract and not immediately intuitive. In this sense, we could call Thomson's treatise, most literally, applied mathematics.

On the other hand, Thomson considers six different situations, namely $2 \leq n \leq 7$, and separately discusses those. This still is quite practical and it shows that Thomson's highest goal was not to be as general as possible but to show how the physical situations are mathematically different. The physical interpretation of the periodical solutions of the radial velocity is also important: to explain why the motion is stable, we need to understand that the vortices oscillate about a steady position. Thomson, however, does not give this physical interpretation, but immediately concludes whether the motion is stable or not.

3.4 Comparing the methods

In the previous sections, we have looked at three different types of stability analysis of three different physical situations. We have seen how nineteenth-century textbooks dealt with the static situation of a floating body and with the dynamic situations of one object or several vortices moving in a fluid. In this concluding section of this chapter, we will compare these different methods. We will focus on a few different threads to discuss the similarities and differences. First, the importance of physical interpretation in each method. By following the three case studies, we can reflect on the role of physical understanding and how this role seems to have changed over time. Second, we will compare the mathematical tools and methods that were used in each case study, to examine the use of pure mathematics, the role of diagrams, and the importance of solutions. Third, we will consider the way in which the concept of stability is different or the same and how this influences the analyses. Lastly, we want to discuss the knowledge that a reader of the textbooks was assumed to have. From this, we can conclude what authors thought important to explain and what not.

These different threads are not isolated but strongly connected. Therefore, we have structured this comparative section as follows. We discuss the four different threads by first considering the different

[92]Compare these values to the one for three vortices: $\frac{2\pi^2 r^2}{m}$.

[93]Thomson does not add that one can, in principle, arrange seven vortices on a circle, but not in a stable way. Joseph John Thomson. *A Treatise on the Motion of Vortex Rings: an essay to which the Adams prize was adjudged in 1882, in the University of Cambridge.* Macmillan, 1883, p. 106.

problems that were to be solved: what are the first principles used, the simplifications and idealisations, and how is the problem defined? Hereafter, we discuss the calculations: what mathematical tools are used and how are these explained? Last, we discuss the solutions and the translation from the mathematical results back to the physical: when is the motion stable and why? And how are these conclusions drawn?

We have already mentioned that, in each of the three case studies, physical interpretation plays a role, albeit in different ways. It should be noted beforehand, that the problems that the textbooks deal with are in the first place physical ones. A certain level of physical interpretation is inevitable because the mathematical investigations arise from a physical motivation. We will focus on the differences between the case studies.

3.4.1 Posing the problem: what has to be solved?

For our three case studies, we have considered three different situations: a body floating in a fluid, an ellipsoid moving through a fluid, and a number of vortices rotating in a fluid. The extent to which each of these relates to a practical situation differs. The problem of the stability of a floating body has an immediate physical motivation: we can think of the stability of a ship and use the metacentric height to analyse it. When we know the required properties of the floating body, we can, in principle, analyse the stability of any object floating. The analysis of the motion of an ellipsoid through a fluid also presents itself from a practical point of view: we can wonder how a submarine moves through water or how a bullet flies through the air. However, the fact that we analyse the stability of an ellipsoid rather than a general object, makes the problem a bit less practical. The stability of the vortices, though still a physical problem, is evidently the least practical of the three case studies. It is not clear how the analysis leads to a solution that could be used in some practical setting.

We can see an increasing dependence on simplifications and idealisations in our case studies. In the case study of the metacentre, it is clear that first principles are obtained directly from physical knowledge. We need to understand the centres of mass and buoyance and the forces acting on those. We do not need many simplifications or idealisations to translate the physical problem to the analytical one: the terms in the expression $M\mu = \frac{I}{m} \mp a$ have a direct connection to the physical properties of the floating body.

For the motion of an ellipsoid and the vortices in a fluid, the situations that are analysed seem not as rooted in the physical world as for the metacentre. Even though they are all physical situations, the dynamic situations need more idealisations and simplifications, compared to the static. In the analysis of the ellipsoid, we have to assume that the ellipsoid moves in an infinite liquid, for example. But also the choice of investigating an ellipsoid instead of a general object is a simplification: the ellipsoid is a body of revolution and its symmetrical properties simplify the equations. In addition, it is assumed that there is no upward or downward motion, which makes it possible to analyse the situation as if it were a two-dimensional one instead of a three-dimensional one.

When analysing the linked vortices, we need even stronger simplifications and idealisations to make the physical situation mathematically feasible. The cylindrical shape of the vortices, for example, is a simplification, but also them being infinitely long. The distribution of the vortices on a circle at equal distances to one another is a clear idealisation as well. Again, we can analyse the situation as if it were a two-dimensional one because we are interested in the inward- and outward motion of the vortices. We can thus see that each case study is a bit less grounded in physical observation.

3.4.2 Doing the math

We have seen that the problems that are solved in our case studies are less and less grounded in the physical world. The same can be said for the calculations: they become increasingly more abstract. The physical interpretation is crucial for the metacentre because the mathematics used strongly depends on and connects to the concrete objects it describes. There are few steps in the first textbooks that cannot be readily translated into a concrete situation.

To a lesser extent, we see this in the derivation of the equation of the pendulum. Many mathematical expressions are still connected to physical properties or objects, but abstract derivation is used to obtain the desired formula. We can still see a two-way interaction between mathematics and its application:

there is some intuitive interpretation, but also a more abstract use of trigonometry and mathematical substitution. The physical similarity between the motion of an ellipsoid and the motion of a pendulum is not explained, but only mathematically established.

The analysis of the linked vortices is, without doubt, the most abstract and has the least physical interpretation. The use of the determinant for solving the system of equations for λ^2, for example, has no immediate connection to the physical world. It is, most literally, pure mathematical applied to a physical phenomenon. But also the equations that are used have less clear connections to physical objects. The techniques used come from abstract mathematics and not from the physics of the situation.

We want to draw attention to the use of geometric arguments in each of the case studies. In explaining the problem of the stability of a floating body, Moseley directly refers to the rotation of an actual body in a fluid and bases this on a figure, albeit a schematic rather than a realistic depiction of the physical situation. The reliance on geometric properties is therefore most apparent in the metacentre, compared to the other two case studies. This may partly be due to the metacentre simply being a geometric concept. In the other case studies, however, we may also see the influence of Lagrange in the abandoning of geometric arguments. In the case study on the motion of the ellipsoid, we can connect the mathematical terms to geometric objects, but in the textbooks, this is not frequently done. In the case study on vortices, the connection to geometric objects is the least apparent: it does not seem important to Thomson to elaborate on it and most of the mathematical computations are strictly reduced to algebraic ones.

We thus see that the techniques that are used in our case studies are increasingly imported from theoretical works outside of hydrodynamics. The physical intuition of the stability analysis changes into a marginal one instead of a substantial or even essential one. In our case studies, there is no general method for stability analysis yet. We see that the mathematics that is used for each situation is, although imported from theoretical works, specifically introduced to analyse the situation at hand.

3.4.3 Stable or not?

Interestingly, the physical interpretation that the authors add to their stability conclusions is minimal in most textbooks. Often, the authors conclude that an equilibrium is "thus stable or unstable", but they do not explain why. We need to add the physical interpretation ourselves. For the metacentre, this is rather straightforward, as we have seen in the example of the floating cone.

For the motion of an ellipsoid through a fluid, however, the physical interpretation is less obvious for those who are unfamiliar with the motion and equation of a pendulum. Understanding why that motion is (un)stable requires some investigation into this equation. The stability of the vortices is simply concluded, without any additional explanation or interpretation. Understanding why the motion is (un)stable is not straightforward and seems to require some clarification that is not given by Thomson.

We wonder why physical interpretations of the results of the stability analysis were so often omitted. Was a reader expected to have sufficient knowledge of the physical situation to do this for themselves? Especially for the first two case studies, this makes sense. The equation of the pendulum was a well-known one and it may have been conform to the general knowledge of the time to exclude an explanation. This could also be the case for the vortices, but, to our knowledge, we are not dealing with equations that were generally known. Perhaps the physical interpretation was supposed to be self-explanatory due to another known property of vortices.

The actual motion of the objects discussed in the case studies seems subordinate to the stability of the motion. This makes sense since the goal of the authors was to do a stability analysis. It is still interesting, however, that they did not feel the need to include the solutions of the equations they used. We see this most clearly in the case study on the ellipsoid: the motion of the ellipsoid seems unimportant and remains unsolved. But also in Thomson's work, the actual motion of the vortices is largely left out. We only analyse the disturbance of the motion.

The solutions of the stability analysis of our case studies thus remain quite abstract. In the last two case studies, the (in)stability is not connected to some physical or practical situation. Here we may recognise the development of seeing mathematical equations as an abstract description of the world, rather than the true description. We have seen that Thomson and Tait concluded that the abstract problem of the ellipsoid differs from the actual circumstances. So it was acknowledged that the mathematical and physical situations were not one and the same thing. In the case study on the vortices, this is not explicitly stated, but it is obvious that the mathematical and physical situations are not the same. Mathematicians were aware of the fact that their mathematical solutions were not always easily translatable to practical ones.

Chapter 4

Conclusions

We have, in the first two parts of this thesis, discussed the broad developments in nineteenth-century mathematics as well as the specific case studies. Let us now return to the research question we started with.

> *How and why did applied mathematics become an increasingly self-contained field of mathematics in the nineteenth century?*

In these last concluding sections of this thesis, we want to connect the two parts of the thesis to answer our research question. We will first discuss which developments described in the first part, chapter 2, are also apparent in our case studies. We will consider the different reasons for the rise of pure mathematics from section 2.3 again and see how these are reflected in the textbooks. Second, we will go back to the authors of the textbooks: where did they do their research? What was their research focus? Can we call them applied mathematicians? We will conclude our thesis with some final reflections on the distinction between pure and applied mathematics.

4.1 The rise of applied mathematics

Let us recall the different developments discussed in section 2.3. These are specialisation, finding connections between mathematical fields, the founding of new fields, mathematics being considered a finished field, and the changing attitude towards mathematical descriptions of physical phenomena. These developments influenced the rise of pure mathematics; pure mathematics became increasingly abstract and detached from the natural world. What about applied mathematics?

The nineteenth century was a century of specialisation in many fields of science, mathematics being no exception. Through specialisation in mathematics, mathematicians increasingly focused on a restricted part of mathematics, developing their own small research topic and not needing to pay attention to other topics or applications. If we look at our case studies in the light of specialisation, two points can be made. First, in the case studies, we can detect some increasing specialisation in mathematics. Especially in the linked vortices and the reference to Scott's *Theory of Determinants*. We see that the mathematical theory of determinants is described by Scott solely for theoretical purposes, not for practical ones.[1]

Second, the authors of our textbooks seem to spend time on a broad variety of topics in applied mathematics. Here, specialisation is less apparent. The works of Kirchhoff and Thomson and Tait, for example, aim to give a general account of different fields within mechanics, of which hydrodynamics is just one part. Basset and J.J. Thomson are known to have researched many different topics in mechanics. We see that the work of mathematicians in applied mathematics was not restricted to one field within applied mathematics. This did not mean, of course, that these mathematicians were not highly specialised in their mathematics: their works were very technical and complicated. They did, however, pursue a variety of research topics.

[1]Scott only discusses the applications of determinants in analysis and geometry, not in practical situations.

We do see that connections between fields in applied mathematics were found and investigated. Examples of this are connections between hydrodynamics and elasticity, or hydrodynamics and electromagnetism.[2] But also the similarity between the motion of an ellipsoid through a fluid and a swinging pendulum is an analogy between two, initially unrelated, phenomena. Mathematicians were aware of mathematical analogies between physical phenomena and used these to analyse new situations. Our case studies do not, however, show that the connections that were found between different fields led to fundamental research. New connections were useful and interesting because they provided mathematicians with tools for analysing new phenomena. [3]

In our case studies, we can only detect an indirect impact of the founding of new fields on applied mathematics. We chose to do a case study on hydrodynamics, which was already an established field of science, so here we will not see the development of new fields. In J.J. Thomson's work, however, we have seen that matrices and determinants came into use in the nineteenth century and that they were useful tools in applied mathematics. Some vector notation also started to be employed at the end of the century. The use of abstract mathematical theories in applied mathematics influences the level of abstraction as well as physical intuition in applied mathematics. In this sense, we do see that more abstract methods applied to physical phenomena change the physical interpretation of the applied works.

The changing attitude towards mathematical descriptions of natural phenomena is very clear in the case studies. Thomson and Tait even explicitly stated that the mathematical description is very different from the physical situation: an explicit difference between abstract mathematics and concrete phenomenon was recognised. We believe that this recognition is important for several reasons. First, it strongly illustrates the separation of mathematics and the physical world. In applied mathematics, this means that the physical phenomena described are more distant from the mathematics that is used to describe them than they were before. This can be interpreted as a separation of pure mathematics and its application: it is not hard to imagine that pure mathematics was increasingly seen as a useful tool for application but at the same time as distant from those applications.

Second, and consequently, when such an attitude towards mathematical theories is maintained, it makes more sense to talk about applying mathematics rather than mixing it. We can recognise the different connotations of mixed and applied mathematics in the different attitudes towards mathematical descriptions of nature. Using mathematics as a tool to describe a phenomenon suggests that mathematics is applied; whereas seeing mathematics as the true description of a phenomenon suggests much more that mathematics and physical phenomena are mixed.

4.2 The applied mathematician

For this thesis project, we have looked at different nineteenth-century textbooks on hydrodynamics. In doing so, we met eight different authors who wrote on hydrodynamics for teaching purposes but also did research in this field and often other fields as well. In light of nineteenth-century professionalisation, we will consider these authors and their professions.

In the nineteenth century, universities became important places for teaching as well as research, enabling mathematicians to pursue mathematics as a profession. All authors that we have discussed, with the exception of two of them, were employed at universities. The exceptions are De Prony, who was employed at the technical École de Polytechnique, and Basset, who was simply rich enough to not need a position as a professor to do his research. Moseley already pursued his career as a mathematician when he was a priest, but was employed later in his life as a professor at King's College.

If we compare the positions that the authors held, we see that they vary a lot. Moseley, W. Thomson, and Tait were all professors of natural philosophy, which is the broad terminology that was used for natural sciences and was employed in Britain for a longer time than in other European countries. Kirchhoff and

[2] Olivier Darrigol. "Between hydrodynamics and elasticity theory: the first five births of the Navier-Stokes equation". In: *Archive for History of Exact Sciences* 56.2 (2002), pp. 95–150, p. 95; Olivier Darrigol. *Worlds of flow: A history of hydrodynamics from the Bernoullis to Prandtl.* Oxford University Press, 2005, p. v.

[3] This does not mean that connections between fields never led to fundamental research. For example, certain kinds of systems of equations could have shown up in different situations, requiring a general theory of matrices.

J.J. Thomson were both professors of physics, though in Thomson's case this was initially seen as an inappropriate appointment.

Lamb's appointment as a professor of pure and applied mathematics is perhaps the most interesting to consider. The fact that he was suited for a position as a pure mathematician but actively went after the position of applied mathematician shows two things. It indicates that a mathematician mostly occupied with applied mathematics was suitable for a position as a pure mathematician. It also illustrates Lamb's own feeling of being an applied mathematician and his wanting to be employed as such.

Even though all mathematicians worked in fields that were usually recognised as applied mathematics, these mathematicians occupied different positions at universities. At their universities, they were able to do research and to teach — and to write textbooks for their students. The great variety of positions may suggest that applied mathematics was still developing into a demarcated and well-defined field. Applied mathematics, mathematical physics, and physics were not always distinct fields.

It seems that the authors were aware of a division between pure and applied mathematics, or at least between highly theoretical and more practical mathematics, and acted on account of this division. Lamb's position as a pure and applied mathematician shows this. Additionally, Basset wrote different treatises for different audiences: one with extensive knowledge of pure mathematics and one without.

In hydrodynamics, there existed a clear community of mathematicians who were aware of each other's work and communicated about important and interesting problems. This community was international: we saw connections between different European countries. Many of the authors referred to each other's work, indicating that they were aware of the body of knowledge that already existed. As the main goal of textbooks usually is to give an account of the established body of knowledge of a field, they show nicely to what extent authors were aware of works in other European countries.

We cannot easily generalise this to a community in applied mathematics. We know that mathematicians were aware of the division between pure and applied and historical literature states that these turned increasingly into separate fields. We can see that hydrodynamics was such a field, but cannot conclude from this that there was a community of applied mathematicians.

4.3 Dynamic disciplines

In the nineteenth century, mathematicians started to use the term applied mathematics to describe the counterpart of pure mathematics. Not only was the introduction of this new term accompanied by many developments within and outside of mathematics, but it would increasingly describe an autonomous field of mathematical research. The approach in this thesis was to investigate the developments in applied mathematics by turning to the rise of pure mathematics. The underlying idea was that, when we want to analyse the development of applied mathematics, we need to consider the changing connection and interaction between pure and applied. We found that, compared to the eighteenth century, it was pure mathematics rather than applied that deviated from the earlier practices in mathematics.

The purification of mathematics in the nineteenth century played an enormous role in the relation between pure and applied: mathematics became an abstract science and its applications got a secondary reputation. The primary position of either pure or applied mathematics seemed new at the time. The holistic view of mathematics in the eighteenth century may have made any judgement of value irrelevant. In the nineteenth century, however, mathematicians started to identify a primary position of pure mathematics. This greatly influenced the further development of mathematics as a field of science, as well as its division into pure and applied.

Pure and applied mathematics became increasingly separate and autonomous fields due to several developments. The fact that many of these developments were consciously initiated — we can think of Lagrange distancing himself from geometric arguments — shows that mathematicians were, at least partly, aiming for more pure mathematics. On the other hand, the founding of new fields can be unforeseen events and changing attitudes may present themselves somewhat randomly. The rise of pure mathematics happened at least partially unconsciously. This shows that pure and applied mathematics as scientific categories are subject to developments that do not always come solely from the actors within these categories. They arise from many developments from within as well as outside of mathematics.

Hence the title of this thesis: pure and applied mathematics were, and are, dynamic disciplines. They are formed due to conscious and unconscious developments and they are subject to change. They are dependent on their historical and local contexts. Contextualising these widely used terms creates awareness of their dynamic aspects and may help us reflect on the way they are used nowadays.

4.4 Limitations and future work

The history of applied mathematics is a broad history and by no means complete. In fact, many different directions for new research are possible. This thesis focused on the relationship between pure and applied mathematics to analyse the formation of applied mathematics. There are, however, other ways of approaching this history that might also be very interesting.

We have not focused on the role of engineers in the nineteenth-century applications of mathematics. Engineers may play an interesting role in the history of applied mathematics, because, whereas applied mathematics may still be highly theoretical, engineers focus on science that is immediately practical. Engineers are usually interested in the parts of mathematics that can be useful in the most direct sense. The role of engineers would therefore be an interesting addition to the history of applied mathematics.

We have briefly discussed the positions at universities of the authors of the textbooks, but the role of universities in the formation of pure and applied mathematics would be very interesting and relevant. An investigation into curriculums could, for example, add to the history of applied mathematics as an academic category. We have seen that there were positions for applied mathematicians at the end of the nineteenth century, but not how these originated and were promoted.

The case studies we have done are illustrative of the general developments in (applied) mathematics, but also too restricted to conclusively represent the general developments. We see tendencies and can connect the broader developments to what we see in our case studies, but more research is needed to decisively confirm the general developments we have seen. Ideally, we would have many more case studies, in many different fields (pure as well as applied), and on many different topics.

Epilogue

I would like to end this thesis with some last reflections. During the past year, I have been considering pure and applied mathematics increasingly as historical categories, influenced by social, academic and mathematical developments. I have come to believe that this perspective is useful for reflecting on the present-day division of mathematics into pure and applied. By admitting that the division resulted from many different historical circumstances, we can, if we want, devaluate the importance of the division.

Of course, it is useful to have some kind of classification of mathematical fields. It is useful to convey to others what kind of work one is doing and dividing mathematics into different subfields contributes to this. But here I am talking about fields such as number theory or geometry. It seems to me that pure and applied mathematics are somewhat artificial terms. They refer to an overarching division of mathematics into pure mathematics and its applications but I wonder how relevant this division is.

In my thesis project, I have attempted to write a history of applied mathematics. One of the things I have learned is that applied mathematics was a new category in the nineteenth century and that it appeared as such in a period of purification of mathematics. The prevailing opinion came to be that there is pure mathematics first en that this can be applied, secondarily, to physical phenomena. This is also what the term applied mathematics, in its most literal sense, suggests.

Another thing I have learned is that even though the division of mathematics into pure and applied seems to be traditional, it is actually rather new in the history of mathematics. Before the nineteenth century, such a division of mathematics did not seem as relevant as it became in the nineteenth century. Why is the term applied mathematics nowadays used to describe a part of mathematics that seems so much more intricate and comprehensive than this restricted term suggests?

I believe that the distinction between pure and applied mathematics is a problematic one, which does not do justice to the body of mathematical knowledge as a whole. Not only do we overlook the interaction between mathematics and its applications when adopting the terms pure and applied, but I think that the research that is done within applied mathematics is far from 'unpure'. The primary and secondary connotation of pure and applied, respectively, seems unfounded and one-sided.

Moreover, the definitions that were adopted in the nineteenth-century division between pure and applied do not seem to apply to the current situation: applied mathematics is not about the physical world anymore, not even in principle. Who, then, decides what is pure and what is applied? Letting go of this dichotomy, I think, may lead to a less restricted view of pure and applied —but mostly of applied —mathematics. It may also solve the discussions I have touched upon in the prologue of this thesis.

I do not know whether to argue for new terms that describe the modern mathematical sciences in a better way or for a complete disregard of any division of mathematics into two overarching fields. I do argue, however, for a more holistic view of mathematics such as the one adopted before the nineteenth century. I would like to finish this thesis with a quote that summarises the view I have come to adhere to:

> *The science of mathematics may be compared to a tree thrusting its roots deeper and deeper into the earth and freely spreading out its shady branches to the air. Are we to consider the roots or the branches as its essential part? Botanists tell us that the question is badly framed, and that the life of the organism depends on the mutual action of its different parts.*[4]

[4]Klein, "The arithmetizing of mathematics", p 248.

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