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PSEUDO-ORTHOGONAL YANG-MILLS THEORIES AND CONNECTIONS TO GRAVITY

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This thesis is dedicated to the efforts and sacrifices of
Antonino Mistretta and Domenica Militano, my wonderful parents.

Questa tesi è dedicata agli sforzi e ai sacrifici di
Antonino Mistretta e Domenica Militano, i miei meravigliosi genitori.

Abstract

The goal of this thesis is to study pseudo-orthogonal Yang-Mills actions and understand under which conditions they contain the Hilbert-Einstein action free of instabilities. The physical motivation is to construct a renormalizable theory of quantum gravity. We first provide the mathematical background necessary to introduce gauge theories in curved spacetime and General Relativity. Subsequently, we develop the concept of geometrical Yang-Mills theory, i.e. Yang-Mills theories for which part of the gauge connection takes also the role of the cotetrad fields. The resulting theory retains only a part of the original gauge group as its group of symmetry. We show that for some pseudo-orthogonal groups one obtains the Hilbert action (and consequently Einstein's equations) as part of the theory. Then, specializing to a coordinate system, we derive the Hamiltonian of such theories and we analyze the constraints that arise in phase space due to the redundancy of the Yang-Mills Lagrangian. We study the class of such constraints and it turns out that the theory possesses both first and second class constraints. Finally, we establish the conditions under which the constraints are preserved by the evolution. The next steps – left for future work – would include the study of dynamics, stability and symmetry breaking to the theory that at low energies would be equivalent to Einstein's general relativity.

Contents

Introduction

The first gravitational theory saw the birth of Theoretical Physics, perhaps the last one one will see its end. For more than a century, researchers from all over the world tried to properly quantize the gravitational field. Up to now, no undoubtedly convincing solution to this problem has been found. Starting from general relativity in 1915, the theory has been extended and generalized in many different ways (see [1] for a review). Of these several theories, string theory is one of the only examples for which Einstein's field equations are obtained without adding by hand the Hilbert action to the theory, or by introducing it as a counterterm as in some induced gravity theories. The goal of this thesis is to show that there exists another class of theories obtained by a suitable generalization of the well-known Yang-Mills theories that contains Einstein theory in the torsionless low energy limit.

As the Standard Model should teach us, Fundamental Nature seems to be described (at least at some scales) by gauge theories of the Yang-Mills type. These theories are proved to be renormalizable and thus it motivates us to look in this kind of theories for a possible solution to the renormalizability problem of quantum gravity. Moreover, it is well-known that the renormalization of flat spacetime theories generically induces higher order geometric scalar with respect to the Ricci scalar present in the Hilbert action. Usually this terms provide instabilities of the Ostrogradski kind[10], due to the presence in the action of quadratic second-order time derivatives with respect to some components of the metric tensor. Since the Yang-Mills theory fundamentally provides a theory of a field strength squared, it seems natural to use an appropriate gauge group in order to reproduce at least these counterterms in a stable way. This is possible due to the fact that a Yang-Mills theory is a first order formalism similar to the Palatini formalism in GR (i.e. the variational principle of Hilbert action for which Christoffel symbols are considered free and they are not the Levi-Civita connection[5]). Another reason why we think Yang-Mills theories have a chance of explaining gravity is the geometrical structure that lies underneath them. General relativity and Yang-Mills theories are two of the most important examples of Differential Geometry in Theoretical Physics. We will then address the question whether one can use the geometry of gauge theories to derive general relativity.

Guided by the results of [3], that shows linear GR as a spin-2 field theory, and [2], that explains gravity as an effective field theory, we are inspired to use spin-1 Yang-Mills fields (which is known that can be put in product to form the spin-2 rep) to describe a theory that reduces to GR in some low-energy limit. It is worth noticing that it was proved by Edward WItten[9] that one can recover 2+1 dimensional general relativity from a Chern-Simons theory (i.e. a "topological" Yang-Mills theory), we will then attempt to define a fully "geometrical" Yang-Mills theory to obtain the same results in four spacetime dimensions.

Following the work by James T. Wheeler and Juan Trujillo[8], we studied the possibility of obtaining a gravitational theory such as Weyl squared gravity from a Yang-Mills theory of the conformal group. Soon we realized that, in order to twist the geometry of spacetime with the geometrical structure of their gauge theory, we needed a way to define the metric tensor in a non-trivial gauge theoretical way. We developed a new class of Yang-Mills theories, called geometrical Yang-Mills theory. We will show that it's possible to define a metric for spacetime using part of the gauge fields as cotetrad fields. Pseudo-orthogonal groups will result as the best choice for our gauge groups. In particular, de Sitter theory (SO(1,4)) will reduce, in its torsionless low energy limit, to general relativity with the appearance of a gauge theoretical Planck mass and cosmological constant.

The thesis is divided as follows. In Chapter 1 we introduce all the mathematical preliminaries necessary to understand standard Yang-Mills theories in arbitrary pseudo-Riemannian manifolds. If the reader is already familiar with these concepts we advise them to proceed directly to the second chapter and come back to the first only if needed. Chapter 2 focuses on a general introduction to gravitational theories and GR. Introducing the tetrad fields, we will see that Einstein's theory already contains a gauge description with respect to local Lorentz transformations. Some examples of generalization of Einstein's theory will be provided to introduce the renormalizability and instability issues and to make a connection with what will be found in our theories. The main chapter, Chapter 3, is dedicated to the definition of geometrical Yang-Mills theory showing the necessity of studying pseudo-orthogonal gauge groups. We will present two theories which have connections with gravity. The most important is the de Sitter gauge theory we've already mentioned. It's the easiest consistent example of geometrical Yang-Mills theory and contains the Ricci scalar as part of the action. The second example is the revised Wheeler-Trujillo theory for the conformal group that motivated us in the first place. Finally, in Chapter 4 we introduce an Hamiltonian formalisms for generic Yang-Mills theories in curved dynamical spacetime. In particular we focus on the constraints of the theory establishing their class and their self-consistency conditions. We conclude addressing the missing steps of a proper instability analysis and giving some outlooks for the theories we developed.

Chapter 1

Mathematical introduction to Yang-Mills theories

We will leave for granted the definition and basic applications of smooth mainfold, exterior or Lie derivatives, tensors, and so on. We will start with a brief introduction on group theory. Then we will explain the concepts of principal and vector bundles, with the related notions of connection (gauge field) and curvature (field strength). In the end we will explain the mathematical background of gauge theories and we will introduce the Yang-Mills action. We advise the reader without a proper differential geometry background to keep [4] or [6] as reference throughout the following chapter. We personally suggest to use [4] and references therein for proofs of the mathematical claims we will make. (Everything will be supposed to be smooth).

1.1 A rough introduction to group theory

Def 1.1.1 (Lie group)**.** *A set G, equipped with an operation $g \star h, g, h \in G$, is called a group if:*

1 $\forall g, h \in G, g \star h \in G$;

1 $\exists!\, e \in G$ *such that* $e \star g = g \star e = g, \forall g \in G$;

1 $\forall g \in G, \exists g^{-1} \in G$ *such that* $g \star g^{-1} = g^{-1} \star g = e$.

A group G is called a Lie group if the product

$$\begin{aligned} p : G \times G &\to G, \\ (g, h) &\mapsto g \star h, \end{aligned}$$

and the inverse

$$\begin{aligned} i : G &\to G, \\ g &\mapsto g^{-1}, \end{aligned}$$

are smooth functions on $G \times G$ and G, respectively. This gives to G a manifold smooth structure.

In the following we will be considering matrix Lie group, i.e. $G \subset GL(r)$, for some r. For simplicity we will denote the group product as $g \star h = gh$.

Def 1.1.2 (Lie algebra)**.** *Given a Lie group G, we define the Lie algebra of G, denoted by $\mathfrak{g}$, as:*

$$\mathfrak{g} = \left\{ A \in Mat_{r\times r} \text{ such that } e^A \in G \right\},$$

where $e^A = \sum_n \frac{A^n}{n!}$.

Remark. *Some properties of the Lie algebra:*

- $\left(e^A\right)^{-1} = e^{-A}$*;*
- *Given* $A, B \in \mathfrak{g}, \alpha A + \beta B \in \mathfrak{g}, \forall \alpha, \beta \in \mathbb{R}$*;*
- *Given* $g \in G$ *and* $A \in \mathfrak{g}$*, we have* $gAg^{-1} \in \mathfrak{g}$*.*

With this structure the Lie algebra is a vector space over $\mathbb{R}$, the algebra structure is given by the commutator:

$$[A, B] = AB - BA \in \mathfrak{g}. \tag{1.1}$$

Notice that the commutator is $\mathbb{R}$-linear. Since the Lie algebra is a vector space, we can choose a basis for it, say $\{G_a\}_{a=1,\dots,n}$, where n is the dimensionality of $\mathfrak{g}$. We know that $[G_i, G_j] \in \mathfrak{g}$, so it can be expandend in the basis:

$$[G_i, G_j] = c_{ij}^{\ k} G_k, \tag{1.2}$$

the elements of the basis are usually called *generators* of the Lie group G, while the $c_{ij}^{\ k}$ are called *structure constants* of the Lie algebra $\mathfrak{g}$. We will now give a classification of Lie algebras. Consider subsets $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$. We introduce the notation $[\mathfrak{a}, \mathfrak{b}] \subset g$ as the set of all finite sums of elements of the form $[X, Y]$, with $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$.

Def 1.1.3 (Ideal)**.** *Let $\mathfrak{g}$ be a Lie algebra. An* ideal *in $\mathfrak{g}$ is a vector subspace $\mathfrak{a} \subset \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$.*

Def 1.1.4 (Lie algebra classification)**.** *Let $\mathfrak{g}$ be a Lie algebra.*

1. *The Lie algebra $\mathfrak{g}$ is called* simple *if $\mathfrak{g}$ is non-abelian and $\mathfrak{g}$ has no non-trivial ideals (different from 0 and $\mathfrak{g}$).*
2. *The Lie algebra $\mathfrak{g}$ is called* semisimple *if $\mathfrak{g}$ has no non-zero abelian ideals.*

Def 1.1.5 (Lie group representations)**.** *Let G be a Lie group and V a vector space. A representation of G on V is given by a smooth map:*

$$\rho : G \to GL(V),$$

such that ($\forall g, h \in G$):

1. $\rho(gh) = \rho(g)\rho(h)$*,*
2. $\rho_e = e_{GL(V)}$*,*

here $GL(V)$ is the general linear group of V.

Def 1.1.6 (Lie algebra representations)**.** *Let $\mathfrak{g}$ be a Lie algebra and V a vector space. Then a* representation *of $\mathfrak{g}$ on V is a Lie algebra homomorphism*

$$\phi : \mathfrak{g} \to \mathfrak{gl}(V) = End(V)$$
$$\phi([X, Y]) = \phi(X) \circ \phi(Y) - \phi(Y) \circ \phi(X) \quad \forall X, Y \in \mathfrak{g}.$$

In particular, there is one representation that will be extremely important for us. In the following fix a Lie group G and its Lie algebra $\mathfrak{g}$.

Def 1.1.7 (Conjugation)**.** *For an element $g \in G$ we define the inner automorphism* conjugation *to be:*

$$c_g = L_g \circ R_{g^{-1}} : G \to G,$$
$$x \mapsto gxg^{-1}.$$

We indicate with $(c_g)_* : \mathfrak{g} \to \mathfrak{g}$ its differential.

Theorem 1.1.8 (Adjoint representation of a Lie group)**.** *The map*

$$Ad : G \to GL(\mathfrak{g}),$$
$$g \mapsto Ad(g) \equiv Ad_g = (c_g)_*,$$

is a Lie group homomorphism, i.e. a representation of the Lie group G on the vector space $\mathfrak{g}$, called the adjoint representation *of the Lie group G. In particular we have ($X \in \mathfrak{g}, g \in G$):*

$$Ad_g X = gXg^{-1}.$$

Theorem 1.1.9 (Adjoint representation of a Lie algebra)**.** *The map*

$$ad : \mathfrak{g} \to End(\mathfrak{g}),$$

given by

$$ad = Ad_*,$$

is a Lie algebra homomorphim, i.e. a representation of the Lie algebra $\mathfrak{g}$ on itself, called the adjoint representation *of the Lie algebra $\mathfrak{g}$. The map satisfies the formula:*

$$(ad)(X)(Y) = ad_X Y = [X, Y] \quad \forall X, Y \in \mathfrak{g}.$$

Def 1.1.10 (Action of a Lie group on a manifold)**.** *Let G be a Lie group and M a smooth manifold. We define the smooth right action of G on M to be:*

$$R : M \times G \to M,$$
$$(x, g) \mapsto xg.$$

Keeping $g \in G$ fixed, we also define the right g-action to be:

$$R_g : M \to M,$$
$$x \mapsto xg.$$

Def 1.1.11 (Fundamental vector field)**.** *Given $A \in \mathfrak{g}$, $R : M \times G \to M$ a right action and f a function on M, we define:*

$$\tilde{X}(f)(x) = \frac{d}{dt}\bigg|_{t=0} f(xe^{tX}).$$

Notice that $\tilde{X}$ is a derivation (i.e. it obeys Leibniz rule and $\mathbb{R}$-linearity), then it defines a vector field on M. The vector field so defined (that we will keep denoted as $\tilde{X}$) is the fundamental vector field *associated with $X \in \mathfrak{g}$.*

Considering $M = G$, it is well-known that there is a Lie algebra isomorphism:

$$\mathfrak{g} \cong T_e G,$$
$$X \leftrightarrow \tilde{X},$$

where $T_e G$ is the tangent space of G at the identity e, in particular we have $\widetilde{[A, B]} = \left[\tilde{A}, \tilde{B}\right]$. This means that $T_e G$ is spanned by the fundamental vector fields associated to the generators of G. Let's look at the cotangent space $T_e^* G$ and choose the dual basis $\left\{\theta^i\right\}_{i=1,\dots,n}$ such that $\theta^i(\tilde{G}_j) = \delta^i{}_j$. Consider the exterior derivative (on G) of the dual form θ^i:

$$\begin{aligned} d\theta^i(\tilde{G}_j, \tilde{G}_k) &= \frac{1}{2}\left(\tilde{G}_j\left(\theta^i(\tilde{G}_k)\right) - \tilde{G}_k\left(\theta^i(\tilde{G}_j)\right) - \theta^i(\left[\tilde{G}_j, \tilde{G}_k\right])\right) = \\ &= -\frac{1}{2}\theta^i(\widetilde{[G_j, G_k]}) = \\ &= -\frac{1}{2}c_{jk}{}^l \delta^i{}_l = \\ &= -\frac{1}{2}c_{jk}{}^i, \end{aligned}$$

where we used Eq.(1.2). Noticing $c_{jk}{}^i = -c_{kj}{}^i$ and applying duality, we find:

$$d\theta^i = -\frac{1}{2}c_{jk}{}^i \theta^j \wedge \theta^k. \tag{1.3}$$

These are the *Cartan structure equations* for a Lie group G.

Now we'll state without proof an important theorem on the quotient space.

Theorem 1.1.12 (Manifold structure on G/H). *Let G be a Lie group and $H \subset G$ a closed subgroup. Then G/H has a unique structure of a smooth manifold such that $\pi : G \to G/H$ is a submersion.*

1.2 Principal and vector bundles

In the following consider G to be a Lie group and M to be a fixed manifold.

Def 1.2.1 (Principal G-Bundle). *A principal G-bundle is given by a manifold P (the total space) equipped with the following:*

1. *a smooth right action $R_g : P \to P, g \in G$ as defined in Subsection 1.1;*
2. *a smooth submersion $\pi : P \to M$ such that $\pi(pg) = \pi(p), p \in P$;*
3. *$\pi : P \to M$ is* locally trivial, *i.e*

$$\forall\, x \in M,\ \exists U \subset M \text{ (open neighborhood of } x) \Big| \ \pi^{-1}(U) \overset{G-equivariant}{\cong} U \times G.$$

The trivial bundle is $\pi : M \times G \to M$.

It follows from Theorem 1.1.12 that $\pi : G \to G/H$ has the structure of a principal H-bundle. Analogously we define a vector bundle in the following way.

Def 1.2.2 (Vector Bundles)**.** *A $\mathbb{R}$-vector bundle (or for a generic field $\mathbb{K}$) of rank r over M is given by a smooth manifold E (the total space) equipped with the following:*

1. *a smooth surjective map $\pi : E \to M$;*
2. *$\forall x \in M$ (the fiber at x) $E_x := \pi^{-1}(x) \cong \mathbb{R}^r$;*
3. *$\forall x \in M$, $\exists U \subset M, x \in U$, such that $\pi^{-1}(U) \cong U \times \mathbb{R}^r$.*

The trivial bundle is $\pi : M \times V \to M$, where V is an r-dimensional vector space.

Def 1.2.3 (Section)**.** *Let $\pi : P \to M$ be a principal G-bundle. A section is given by a smooth map $s : M \to P$ such that $\pi \circ s = id_M$. A local section is defined on an open subset $U \subset M$. We will denote the space of sections as $\Gamma(M; P)$.*

Let $\pi : E \to M$ be a vector bundle. A section is given by a smooth map $s : M \to E$ such that $\pi \circ s = id_M$. A local section is defined on an open subset $U \subset M$. We will denote the space of sections as $\Gamma(M; E)$.

Def 1.2.4 (Local frame)**.** *Let $\pi : E \to M$ be a vector bundle of rank r. A local frame is given by r sections $\{e_i\}_{i=1,\dots,r}$ defined over some $U \subset M$ such that $\{e_i\}_{i=1,\dots,r}$ forms a basis for E_x, $\forall x \in U$. A frame induces a diffeomorphism:*

$$E(U) = \pi^{-1}(U) \cong U \times \mathbb{R}^r.$$

Notice that there is no concept of frame for principal bundles. This reflects the property that the structure group G acts transitively on the fiber. Thus, once we have a section $s(x) = p \in P_x$ we can use it to induce a diffeomorphism as follows.

$$\begin{aligned} \phi : P(U) &\overset{\cong}{\to} U \times G, \\ s(x) &\mapsto (x, e), \\ p = s(x)g &\mapsto (x, g), \ p \in P, g \in G. \end{aligned}$$

Def 1.2.5 (Associated vector bundle)**.** *Let $\pi : P \to M$ be a principal G-bundle and (ρ, V) a representation of G. The vector bundle associated to V is given by:*

$$E(P; V) \equiv P \times_\rho V := (P \times V)/G \to M,$$

where the quotient is taken as:

$$(p, v) \sim (p', v') \Leftrightarrow (p', v') = (pg, \rho(g^{-1})v), p \in P, v \in V, g \in G.$$

Proposition 1.2.6 (Local sections of associated vector bundles)**.** *Let $\pi : P \to M$ be a principal G-bundle and $E = P \times_\rho V$ an associated vector bundles. Let $s : U \to P$ be a local section for P. Then there is a 1-to-1 relation between smooth sections $\tau : U \to E$ and smooth maps $f : U \to V$, given by:*

$$\tau(x) = [s(x), f(x)], \quad \forall\, x \in U.$$

In particular, there is an associated vector bundle which will be important in the following. This is the case when $V = \mathfrak{g}$ and $\rho(g)X = g^{-1}Xg$, i.e. ρ is the adjoint representation of G. This bundle is called the *adjoint bundle* and it is indicated as $\mathrm{Ad}(P) := E(P, \mathfrak{g})$.

Def 1.2.7 (Basic differential forms)**.** *Let $\pi : P \to M$ be a principal G-bundle, $E(P; V)$ the vector bundle associated with the representation ρ and $\underline{V} := P \times V$, i.e. a trivial vector bundle over P with fiber V. A differential form $\alpha \in \Omega^k(P; \underline{V}) = \Omega^k(P) \otimes V$ is basic if:*

1. $R_g^*\alpha = \rho(g^{-1})\alpha, \forall g \in G$;

2. $\alpha(\tilde{X}) = 0, \forall X \in \mathfrak{g}$.

Remark. *Let $\pi : P \to M$ be a principal G-bundle and ρ a rep of G on V. Then the pull-back (π^*) of differential forms induces an isomorphism:*

$$\Omega^k(M; E(P; V)) \cong \Omega^k_{basic}(P; \underline{V}).$$

Def 1.2.8 (Horizontal distribution)**.** *Let $\pi : P \to M$ be a principal G-bundle. A vector subbundle $\mathcal{H} \subset TP$ is called a distribution, here TP is the Tangent Bundle of P. If the map $T_p\pi : \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism for every $p \in P$, then $\mathcal{H}$ is called an horizontal distribution.*

Def 1.2.9 (Connection on principal G-bundles)**.** *Let $\pi : P \to M$ be a principal G-bundle. A connection on P is a horizontal distribution $\mathcal{H}$ that is G-equivariant, i.e.*

$$\mathcal{H}_{pg} = T_pR_g(\mathcal{H}_p),$$

where T_pR_g is the tangent map induced by the right action.

Equivalently, let $\pi : P \to M$ be a principal G-bundle. A connection on P is given by a $\mathfrak{g}$-valued 1-form $\boldsymbol{\omega} \in \Omega^1(P, \mathfrak{g})$ such that:

1. $R_g^*\boldsymbol{\omega} = Ad_{g^{-1}}(\boldsymbol{\omega})$, *for all* $g \in G$;

2. $\boldsymbol{\omega}(\tilde{X}) = X$, *for all* $X \in \mathfrak{g}$,

where $Ad_{g^{-1}}$ denotes the adjoint action of G on $\mathfrak{g}$.

The two definitions are related as follows:

$$Ker(\boldsymbol{\omega})_p = \mathcal{H}_p.$$

Def 1.2.10 (Connection on vector bundles)**.** *Let $\pi : E \to M$ be a vector bundle. A connection is a bilinear map:*

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(M; E) &\to \Gamma(M; E), \\ (X, s) &\mapsto \nabla_X(s), \; X \in \mathfrak{X}(M), s \in \Gamma(M; E), \end{aligned}$$

such that ($\forall f \in C^\infty(M)$):

1. $\nabla_{fX}(s) = f\nabla_X(s)$;

2. $\nabla_X(fs) = f\nabla_X(s) + X(f)s$.

In physics literature one usually finds *covariant derivative* instead of connection.

Notice that the connection 1-form of a principal bundle is not basic. However it is straightforward to see that the difference between two connection one forms (ω and ω') is a basic 1-form (then, it's isomorphic to a form living on the base manifold). Analogously, the connection ∇ of a vector bundle doesn't define a 1-form, but the difference between two ∇, ∇' does.

Def 1.2.11 (Curvature on principal bundles)**.** *Let $\pi : P \to M$ be a principal G-bundle with connection 1-form $\boldsymbol{\omega} \in \Omega^1(P; \mathfrak{g})$. The curvature F of a connection ω is defined as:*

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) := d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}] \in \Omega^2(P; \mathfrak{g}).$$

Remark. *The curvature* $\boldsymbol{\Omega}(\boldsymbol{\omega})$ *is a basic form, i.e.* $\boldsymbol{\Omega}(\boldsymbol{\omega}) \in \Omega^2_{basic}(P; \underline{\mathfrak{g}}) \cong \Omega^2(M; Ad(P))$.

Since $\boldsymbol{\Omega}$ defines a 2-form with values in $\mathrm{Ad}(P)$, it seems reasoning to see what connection $\nabla^{\boldsymbol{\omega}}$ is induced on $\mathrm{Ad}(P)$ by $\boldsymbol{\omega}$ on P. Consider $\pi : P \to M$ a principal G-bundle and $E = P \times_\rho V$ the vector bundle associated to the rep (ρ, V). Notice that we have already shown, Def.1.2.7, that basic forms on P are isomorphic to E-valued differential forms. For 0-forms this implies:

$$\Gamma(M, E) \cong \left\{ f : P \to V, f(pg) = g^{-1} f(p), \forall\, g \in G \right\}. \tag{1.4}$$

Moreover, the representation $\rho : G \to GL(V)$ induces a morphism of Lie algebras $\tilde{\rho} : \mathfrak{g} \to End(V)$ (i.e. the tangent map of ρ):

$$\tilde{\rho}(X) := \frac{d}{dt}\Big|_{t=0} \rho(e^{tX})$$

.

Def 1.2.12 (Connection on associated vector bundle)**.** *We define the connection on* $E = P \times_\rho V$ *to be:*

$$\begin{aligned} \nabla^{\boldsymbol{\omega}} : \Omega^0_{basic}(P; V) &\longrightarrow \Omega^1_{basic}(P; V), \\ \nabla^{\boldsymbol{\omega}}(f) &= df + \tilde{\rho}(\boldsymbol{\omega}) f, \\ \nabla^{\boldsymbol{\omega}}_X(f) &= X(f) + \tilde{\rho}(\boldsymbol{\omega}(X)) f. \end{aligned}$$

In particular, for $\mathrm{Ad}(P)$, we have $\tilde{\rho}(X)Y = [X, Y]\,,\ \ X, Y \in \mathfrak{g}$. Then the induced connection on $\mathrm{Ad}(P)$ is given by:

$$\nabla^{\boldsymbol{\omega}}_{\mathrm{Ad}(P)} = d + [\boldsymbol{\omega}, \cdot]\,. \tag{1.5}$$

Theorem 1.2.13 (Bianchi identity for principal bundles)**.** *Let* $\pi : P \to M$ *be a principal* G*-bundle with connection* $\boldsymbol{\omega}$*. Then*

$$\nabla^{\boldsymbol{\omega}}_{Ad(P)} \boldsymbol{\Omega}(\boldsymbol{\omega}) = d\boldsymbol{\Omega}(\boldsymbol{\omega}) + [\boldsymbol{\omega}, \boldsymbol{\Omega}(\boldsymbol{\omega})] = 0.$$

As it can be easily proved using the definition Def.1.2.11. Indeed we find ($\{e_i\}_i$ basis for $\mathfrak{g}$, $\omega^i \in \Omega(P)$, $\Omega^i \in \Omega^2(P)$):

$$\begin{aligned} \nabla^{\boldsymbol{\omega}} \Omega(\omega) =& d\left(d\omega + \frac{1}{2} \omega^i \wedge \omega^j \otimes [e_i, e_j] \right) + [\omega, \Omega(\omega)] = \\ =& \frac{1}{2} \left(d\omega^i \wedge \omega^j \otimes [e_i, e_j] - \omega^i \wedge d\omega^j \otimes [e_i, e_j] \right) + \\ &+ \omega^i \wedge d\omega^j \otimes [e_i, e_j] + \omega^i \wedge \omega^l \wedge \omega^m \otimes [e_i, [e_l, e_m]] = 0, \end{aligned} \tag{1.6}$$

where we used the Jacobi identity to kill the last term. Notice that we proved it using only algebraic relations and the definition of the curvature. This implies that the Bianchi identities are algebraic relations and they do not depend on the geometrical structure.

Def 1.2.14 (Curvature on vector bundles)**.** *Let* $\pi : E \to M$ *be a vector bundle with connection* ∇*. The curvature* $F(\nabla) : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(M; E) \to \Gamma(M; E)$ *is defined as:*

$$F(\nabla)(X, Y)(s) := \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \right)(s), \quad X, Y \in \mathfrak{X}(M), s \in \Gamma(M; E).$$

Remark. *The curvature* $F(\nabla)$ *is clearly antisymmetric in the vector entries and it can be easily checked that it is* $C^\infty(M)$*-linear in all entries and fiber preserving (i.e. it sends sections to sections).*

$$\Rightarrow F(\nabla) \in \Omega^2(M; End(E)),$$

where End(E) denotes the vector bundle $\pi : End(E) = \bigcup_{x\in M} End(E_x) \to M$*, End*$(E_x)$ *denotes the endomorphism vector space of the fiber at* $x \in M$*.*

Def 1.2.15 (Exterior covariant derivative for vector bundles)**.** *Let* $\pi : E \to M$ *be a vector bundle with connection* ∇*. The exterior covariant derivative* $d_\nabla : \Omega^k(M;E) \to \Omega^{k+1}(M;E)$ *is the unique extension of* $\nabla : \Gamma(M;E) \to \Omega^1(M;E)$ *to higher degrees satisfying*

$$d_\nabla(\gamma \wedge \omega) = d\gamma \wedge \omega + (-1)^k \gamma \wedge d_\nabla(\omega), \;\; \gamma \in \Omega^k(M), \omega \in \Omega^l(M;E).$$

More concretely we have ($\alpha \in \Omega^k(M;E)$)*:*

$$\begin{aligned} d_\nabla \alpha(X_0, ..., X_k) = &\sum_{i=0}^{p} (-1)^i \nabla_{X_i} \Big(\alpha(X_0, ..., \hat{X}_i, ..., X_k) \Big) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k), \end{aligned}$$

where the hat on vectors means that they have been removed from there.

Remark. $d^2_\nabla(s) = (d_\nabla \circ d_\nabla)(s) = F(\nabla)(s)$.

Since the curvature takes value in the End(E)-bundle, it is interesting to see which connection ∇^{End} on End(E) is induced by ∇ on E.

Def 1.2.16 (Connection on the End(E)-bundle)**.** *We define the connection on End*(E) *to be:*

$$\nabla^{End(E)}(f)(s) := \nabla(f(s)) - f(\nabla(s)), \quad \forall s \in \Gamma(M, E).$$

From this connection we can build the exterior covariant derivative $d_{\nabla^{End(E)}}$ on End(E) as in Def.(1.2.15).

Theorem 1.2.17 (Bianchi identity for vector bundles)**.** *Let* $\pi : E \to M$ *be a vector bundle with connection* ∇*. Then*

$$d_{\nabla^{End(E)}}(F(\nabla)) = 0.$$

Now we will present the local description of connection and curvature. Let $\pi : P \to M$ be a principal G-bundle and consider $U, V \subset M$, together with the isomorphisms $\phi_U : \pi^{-1}(U) \to U \times G$ and $\phi_V : \pi^{-1}(V) \to V \times G$ such that $U \cap V \neq \emptyset$. Then we have the following situation in the overlap:

$$\begin{aligned} \pi^{-1}(U\cap V) &\xrightarrow{\phi_U} U \cap V \times G, \\ \pi^{-1}(U\cap V) &\xrightarrow{\phi_V} U \cap V \times G, \\ \Rightarrow \phi_U \circ \phi_V^{-1} &: U \cap V \times G \to U \cap V \times G, \\ \phi_U \circ \phi_V^{-1}(x, g) &= (x, g\phi_{UV}(x)), \end{aligned}$$

where $\phi_{UV} : U \cap V \to G$. Functions constructed as ϕ_{UV} are called *transition functions* and, together with their domain of definition, they define uniquely the bundle. We will call the set of domains and transition functions $\{U_\alpha, \phi_{\alpha\beta}\}_{\alpha,\beta}$ a *trivializing cover* of the principal bundle.

The same conclusion holds for vector bundles. In this case in the overlap we have:

$$
\begin{aligned}
\pi^{-1}(U \cap V) &\xrightarrow{\phi_U} U \cap V \times \mathbb{R}^r, \\
\pi^{-1}(U \cap V) &\xrightarrow{\phi_V} U \cap V \times \mathbb{R}^r, \\
\Rightarrow \phi_U \circ \phi_V^{-1} : U \cap V \times \mathbb{R}^r &\to U \cap V \times \mathbb{R}^r, \\
\phi_U \circ \phi_V^{-1}(x, v) &= (x, \phi_{UV}(x)v),
\end{aligned}
$$

where $\phi_{UV} : U \cap V \to GL(r, \mathbb{R})$.

Let $\pi : P \to M$ be a principal G-bundle equipped with a connection $\boldsymbol{\omega} \in \Omega^1(P; \mathfrak{g})$ and fix a trivializing cover for P. On any U_α in the covering we have a local section $s_\alpha : U_\alpha \to P\big|_{U_\alpha}$ and we can use it to pull-back $\boldsymbol{\omega}$. We then define the *local connection 1-form* as:

$$\boldsymbol{\omega}_\alpha = s_\alpha^* \boldsymbol{\omega} \in \Omega^1(U_\alpha; \mathfrak{g}). \tag{1.7}$$

Moreover, if we give a chart to the base manifold, $\{x^\mu\}_\mu$, and a basis to the Lie algebra $\mathfrak{g}$, $\{e_a\}_a$, we can write:

$$\boldsymbol{\omega}_{(\alpha)\mu} := \boldsymbol{\omega}_\alpha(\partial_\mu), \tag{1.8}$$

$$\boldsymbol{\omega}_{(\alpha)\mu} = \sum_a \omega^a_{(\alpha)\mu} e_a. \tag{1.9}$$

Once again, consider $U_\alpha \cap U_\beta \neq \emptyset$, then we have:

$$s_\alpha = s_\beta \phi_{\alpha\beta} \tag{1.10}$$

and for the local connection 1-form we find the following transformation:

$$\boldsymbol{\omega}_\beta = \phi_{\alpha\beta}^{-1} \boldsymbol{\omega}_\alpha \phi_{\alpha\beta} + \phi_{\alpha\beta}^{-1} d\phi_{\alpha\beta}. \tag{1.11}$$

Let $\pi : E \to M$ be a vector bundle with connection ∇ and fix a trivializing cover as before. On any U_α in the covering we have a local frame $\{e_i\}_i$ that gives an isomorphism $\pi^{-1}(U_\alpha) \overset{\phi_\alpha}{\to} U_\alpha \times \mathbb{R}^r$. On $U_\alpha \times \mathbb{R}^r$ the standard exterior derivative defines a connection, as can be easily verified. However we have also the ∇-induced connection by ϕ_α. Since the difference between two connections is a 1-form, we define the *local connection 1-form* as:

$$\nabla\big|_\alpha = d + A_\alpha, \;\; A_\alpha \in \Omega^1(U_\alpha, \mathrm{Mat}_{r\times r}(\mathbb{R})) \tag{1.12}$$

and we find the transformation rule:

$$A_\beta = \phi_{\alpha\beta}^{-1} A_\alpha \phi_{\alpha\beta} + \phi_{\alpha\beta}^{-1} d\phi_{\alpha\beta}. \tag{1.13}$$

Moreover, if we give a chart to the base manifold, $\{x^\mu\}_\mu$, and a local frame to E, $\{e_i\}_i$, we can write ($s \in \Gamma(M; E)$, $s = \sum_i s_i e_i$):

$$A_\mu := A_\alpha(\partial_\mu), \tag{1.14}$$

$$A_\mu(s) = \sum_i s_i A_\mu(e_i) := \sum_{i,j} s_i \left(A_\mu\right)_{ij} e_j. \tag{1.15}$$

From Def.(1.2.11) and Eq.(1.7) it is easy to see that the local expression for the curvature of a principal G-bundle is:

$$\boldsymbol{\Omega}_\alpha = d\boldsymbol{\omega}_\alpha + \frac{1}{2} \left[\boldsymbol{\omega}_\alpha, \boldsymbol{\omega}_\alpha\right] = s_\alpha^* \boldsymbol{\Omega}(\boldsymbol{\omega}) \tag{1.16}$$

and we have the transformation rule:

$$\mathbf{\Omega}_\alpha = \mathrm{Ad}_{\phi^{-1}_{\alpha\beta}}(\mathbf{\Omega}_\beta). \tag{1.17}$$

Notice that $\mathrm{Ad}_{\phi^{-1}_{\alpha\beta}}$ are the transition functions of the adjoint bundle $\mathrm{Ad}(P)$.

For vector bundles we know that $d^2_\nabla(s) = F(\nabla)(s)$, for all $s \in \Gamma(M;E)$. Since in a local trivialization (one of the U_α in the cover) we have $d_\nabla = d + A_\alpha$, the local expression for the curvature of a vector bundle is:

$$F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha, \tag{1.18}$$

with transformation rule:

$$F_\beta = \phi^{-1}_{\alpha\beta} F_\alpha \phi_{\alpha\beta}. \tag{1.19}$$

We will now introduce the concept of *metric* for vector bundles. It will give us an essential tool to write scalar actions out of our curvatures.

Def 1.2.18 (Bundle metric on vector bundles)**.** *Let $E \to M$ be a $\mathbb{K}$-vector bundle over M. A (Euclidean, $\mathbb{K} = \mathbb{R}$, or Hermitian $\mathbb{K} = \mathbb{C}$) bundle metric is a metric on each fibre E_x that varies smoothly with $x \in M$. Namely, it is a section:*

$$\langle \cdot, \cdot \rangle \in \Gamma(E^* \otimes E^*) \quad (\mathbb{K} = \mathbb{R}),$$
$$\langle \cdot, \cdot \rangle \in \Gamma(\bar{E}^* \otimes E^*) \quad (\mathbb{K} = \mathbb{C}),$$

which defines in each point $x \in M$ a non-degenerate symmetric or Hermitian form:

$$\langle \cdot, \cdot \rangle_x : E_x \times E_x \to \mathbb{K}.$$

Def 1.2.19 (Metric compatible connection)**.** *Let E be a vector bundle equipped with the bundle metric g. Let ∇ be a connection on E. ∇ is said to be* metric compatible *if and only if ($X, Y \in \Gamma(M;E)$):*

$$dg(X,Y) = g(\nabla X, Y) + g(X, \nabla Y).$$

As we have already mentioned, the adjoint bundle is the vector bundle we will be studying the most throughout this thesis. We will then give it an important bundle metric. Recall that the the adjoint bundle is a vector bundle of a Lie algebra on which we have the adjoint action of its Lie group. We will then look for a metric which is invariant under the group action.

Def 1.2.20 (Killing form)**.** *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. The Killing form $B_\mathfrak{g}$ on $\mathfrak{g}$ is defined by:*

$$B_g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K},$$
$$(X,Y) \mapsto tr(ad_X \circ ad_Y),$$

where, given f a linear endomorphism in a vector space V with basis $\{v_i\}_i$, we define the matrix f_{ij} as:

$$f(v_j) = \sum_i f_{ij} v_i \tag{1.20}$$

and the trace as:

$$\mathrm{tr}(f) = \sum_i f_{ii} \tag{1.21}$$

and it doesn't depend on the choice of the basis. One can prove that the Killing form defines a $\mathbb{K}$-bilinear and symmetric form on $\mathfrak{g}$.

Theorem 1.2.21 (Invariance of Killing form under automorphisms)**.** *Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra automorphism of $\mathfrak{g}$. Then the Killing form $B_\mathfrak{g}$ satisfies:*

$$B_\mathfrak{g}(\sigma X, \sigma Y) = B_\mathfrak{g}(X, Y) \quad \forall X, Y \in \mathfrak{g}.$$

In particular this holds for $\sigma = Ad_g$, with $g \in G$ arbitrary.

Theorem 1.2.22 (Cartan's criterion for semisimplicity)**.** *A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form $B_\mathfrak{g}$ is non-degenerate.*

Which shows that for semisimple Lie algebras we can use the Killing form as an adjoint invariant metric on each fiber of the adjoint-bundle. Extending it smoothly we get an invariant bundle metric $\langle \cdot, \cdot \rangle_{\mathrm{Ad}}$. Since pseudo-orthogonal algebras are simple Lie algebra, we will use the Killing form to generate our bundle metric as explained above. Given a local frame for the adjoint bundle $\{e_i\}_i$, we introduce the following notation:

$$G_{ij} \equiv \langle e_i, e_j \rangle_{\mathrm{Ad}} = c_{il}{}^m c_{jm}{}^l, \tag{1.22}$$

where we used Def.1.2.20. Given two sections of the adjoint bundle F and H, we can then write:

$$\langle F, H \rangle_{\mathrm{Ad}} = F^i G^j \langle e_i, e_j \rangle_{\mathrm{Ad}} = F^i H^j G_{ij}. \tag{1.23}$$

1.3 Gauge theories

Throughout the following fix a principal G-bundle and an associated vector bundle $E = P \times_\rho V$.

Def 1.3.1 (Gauge)**.** *A global gauge is a global section $s : M \to P$, a local gauge is a local section $s : U \to P$.*

Def 1.3.2 (Mathematical gauge transformation (Bundle Automorphism))**.** *A gauge transformation is given by a fiber-preserving G-equivariant diffeomorphism $f : P \to P$ (a bundle automorphism of P), i.e.*

1. *$\pi \circ f = \pi$,*
2. *$f(pg) = f(p)g$, $\forall\, p \in P, \forall\, g \in G$.*

The set of all such trasformation is called the *automorphism group* or the *gauge group* of P and it is indicated as $\mathcal{G}(P)$ or $\mathrm{Aut}(P)$. Usually, in Physics literature, the term gauge group is used to indicate the structure group G. There's in an intimate connection between the two formalism that we will explain in the following.

Def 1.3.3 (G-valued maps on P)**.**

$$C^\infty(P, G)^G := \left\{ \sigma : P \to G \middle| \sigma(pg) = g^{-1}\sigma(p)g \right\}.$$

This set is a group with pointwise multiplication.

Proposition 1.3.4 (Correspondence between bundle automorphism and G-valued maps)**.** *The map*

$$\mathcal{G}(P) \to C^\infty(P, G)^G,$$
$$f \mapsto \sigma_f,$$

with σ_f defined by:

$$f(p) = p\sigma_f(p),$$

is a group isomorphism.

Def 1.3.5 (Physical gauge transformation)**.** *A physical gauge transformation is a smooth map $\tau : U \to G$, $U \subset M$. The set of all physical gauge transformations forms a group $C^\infty(U, G)$ under pointwise multiplication.*

A rigid physical transformation is a constant map $\tau : U \to G$. Again this set has a group structure and it is isomorphic to G

Proposition 1.3.6 (Physical vs Mathematical gauge transformations)**.** *Let $s : U \to P$ be a local section. Then s defines a group isomorphism*

$$\begin{aligned} C^\infty(P_U, G)^G &\to C^\infty(U, G), \\ \sigma &\mapsto \tau_\sigma = \sigma \circ s, \end{aligned}$$

where the inverse is given by:

$$\begin{aligned} C^\infty(U, G) &\to C^\infty(P_U, G)^G, \\ \tau &\mapsto \sigma_\tau, \end{aligned}$$

where

$$\sigma_\tau(s(x)g) = g^{-1}\tau(x)g, \quad \forall\, x \in U, \forall\, g \in G.$$

Then, after we have chosen a section, i.e. a local trivialization for P, we can identify mathematical and physical gauge transformations.

Theorem 1.3.7 (Action of mathematical gauge transformations on associated bundles)**.** *The group of mathematical gauge transformations of the principal bundle acts on the associated vector bundle through bundle isomorphisms via*

$$\begin{aligned} &\mathcal{G}(P) \times E \longrightarrow E, \\ &(f, [p, v]) \mapsto f \cdot [p, v] = [f(p), v] = [p \cdot \sigma_f(p), v]. \end{aligned}$$

Theorem 1.3.8 (Action of physical gauge transformations on associated bundles)**.** *Let $s : U \to P$ be a local gauge and $\Phi : U \to E$ a local section. Following Prop.(1.2.6) we can write:*

$$\Phi(x) = [s(x), \phi(x)]\,, \;\; \forall\, x \in U, \;\; \phi : U \to V.$$

Suppose f is a local mathematical gauge transformation and $\tau_f : U \to G$ the corresponding physical transformation (as in Prop.(1.3.6)).

$$\Rightarrow (f \cdot \Phi)(x) = \left[s(x), \rho^{-1}(\tau_f(x))\phi(x)\right].$$

In the context of gauge theories, we will call the ω^a_μ, introduced in Eq.(1.8), the *(local) gauge fields*. We have already shown (Eq.(1.11)) how the gauge fields change after a change of gauge (i.e. a change of local section).

Theorem 1.3.9 (Transformation of connections under mathematical gauge transformations)**.** *Suppose that $f \in \mathcal{G}(P)$ is a global mathematical gauge transformation. Then $f^*\boldsymbol{\omega}$ is a connection 1-form on P and:*

$$f^*\boldsymbol{\omega} = Ad_{\sigma_f^{-1}} \circ \boldsymbol{\omega} + \sigma_f^* \mu_G,$$

where μ_G is the Maurer-Cartan 1-form defined as: $\mu_G\big|_g = g^{-1}dg$.

Remark. $f^*\boldsymbol{\Omega}(\boldsymbol{\omega}) = Ad_{\sigma_f^{-1}}\boldsymbol{\Omega}(\boldsymbol{\omega})$.

In order to build the Yang-Mills action for a gauge theory we need more structure on the base manifold. Throughout the following we will consider M to be an n-dimensional, oriented, pseudo-Riemaniann manifold (M, g), where g is the pseudo-Riemaniann metric.

Def 1.3.10 (Canonical volume form on (M, g)). *Let $\{e_i\}_{i=0,...,n-1}$ be an oriented, orthonormal basis of T_pM. Then the volume form is defined as:*

$$dvol_g\,(e_0, ..., e_{n-1}) = +1.$$

If $\{x^\mu\}_{\mu=0,...,n-1}$ is a local chart on M, we have:

$$dvol_g = \sqrt{|g|}dx^0 \wedge ... \wedge dx^{n-1},$$

where g is the determinant of the matrix whose entries are:

$$g_{\mu\nu} := g(\partial_\mu, \partial_\nu).$$

Def 1.3.11 (Scalar product of forms). *Given $\omega, \eta \in \Omega^k(M)$ we define the scalar product on k-forms (in local coordinates) as:*

$$\begin{aligned} \langle\cdot,\cdot\rangle : \Omega^k(M) \times \Omega^k(M) &\longrightarrow C^\infty(M), \\ (\omega, \eta) &\mapsto \sum_{\mu_1<...<\mu_k} \omega_{\mu_1...\mu_k}\eta^{\mu_1...\mu_k} = \\ &= \frac{1}{k!}\sum_{\mu_1...\mu_k} \omega_{\mu_1...\mu_k}\eta^{\mu_1...\mu_k} = \\ &= \frac{1}{k!}\omega_{\mu_1...\mu_k}\eta^{\mu_1...\mu_k}, \end{aligned}$$

where again $\omega_{\mu_1...\mu_k} = \omega\left(\partial_{\mu_1}, ..., \partial_{\mu_k}\right)$.

Remark. *This definition is independent of the choice of the local chart.*

Both the component notation and the Einstein summation convention (as in the last step of the previous equality) will be considered understood unless stated otherwise.

Def 1.3.12 (Hodge star operator). *The Hodge star operator,*

$$* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M),$$

is the linear map defined by:

$$\langle\omega, \eta\rangle\, dvol_g = \omega \wedge *\eta.$$

Remark. *1. In local coordinates we have:[6]*

$$*(dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}) = \frac{\sqrt{|g|}}{(n-k)!}g^{\nu_1\mu_1}...g^{\nu_k\mu_k}\epsilon_{\nu_1...\nu_k\nu_{k+1}...\nu_n}dx^{\nu_{k+1}} \wedge ... \wedge dx^{\nu_n},$$

where ϵ is the total antisymmetric symbol such that $\epsilon_{0...n-1} = +1$;

2. $*1 = dvol_g$;

3. If the signature of g is (s, t), we have $*^2\omega = (-1)^t\omega$.

Def 1.3.13 (L^2-scalar product of forms)**.** *Given $\omega, \eta \in \Omega^k(M)$ we define the L^2-scalar product on k-forms as:*

$$\langle \cdot, \cdot \rangle_{L^2} : \Omega^k(M) \times \Omega^k(M) \longrightarrow \mathbb{R},$$
$$\langle \omega, \eta \rangle_{L^2} = \int_M \langle \omega, \eta \rangle \, dvol_g.$$

Analogously, for twisted forms $\omega \in \Omega^k(M;E)$, E a vector bundle over (M,g), we have the following generalization. Fix a local frame $\{e_i\}_{i=1,\dots,r}$ for E, let $F, G \in \Omega^k(M;E)$ and expand:

$$F = \sum_{i=1}^{r} F_i \otimes e_i, \qquad G = \sum_{j=1}^{r} G_j \otimes e_j.$$

Then we define:

1. $\langle F, G \rangle_E = \sum_{i,j} \langle F_i, G_j \rangle \langle e_i, e_j \rangle_E$, where $\langle \cdot, \cdot \rangle_E$ denotes the scalar product induced by the metric chosen on E;
2. $*F = \sum_i (*F_i) \otimes e_i$;
3. $\langle F, G \rangle_{E,L^2} = \int_M F^i \wedge *G^j \langle e_i, e_j \rangle_E$.

Consider the curvature $\boldsymbol{\Omega}(\boldsymbol{\omega})$ of P. We have then that it defines a twisted 2-form $\Omega(\omega)_M \in \Omega^2(M, \mathrm{Ad}(P))$. We give to $\mathrm{Ad}(P)$ the same scalar product as in Eq.(1.22).

Remark. *This scalar product is Ad-invariant so that the scalar product of twisted forms it induces is Ad-invariant as well*

Def 1.3.14 (Covariant codifferential)**.** *We define the* covariant codifferential d^*_∇ *to be:*

$$d^*_\nabla : \Omega^{k+1}(M,E) \longrightarrow \Omega^k(M,E),$$
$$d^*_\nabla = (-1)^{t+nk+1} * d_\nabla * .$$

Theorem 1.3.15 (The covariant codifferential on twisted forms is formal adjoint of covariant differential)**.** *Let M be a manifold. The covariant codifferential d^*_∇ is the formal adjoint of the exterior covariant differential d_∇ with respect to the L^2-scalar product (provided that ∇ is compatible with the scalar product on E) on forms with compact support χ such that $\chi \cap \partial M = 0$, i.e.*

$$\langle d_\nabla \omega, \eta \rangle_{E,L^2} = \langle \omega, d^*_\nabla \eta \rangle_{E,L^2} \,,$$

for all $\omega \in \Omega^k_0(M,E)$, $\eta \in \Omega^{k+1}_0(M,E)$ with support as explained above.

Consider the adjoint bundle. The covariant derivative is given by $d_\nabla = d + [\omega, -]$ and the inner product is given by G_{ij} as in Eq.(1.22). Let $\{e_i\}_i$ be a local frame for Ad-bundle. The condition for d_∇ to be compatible with the inner product is given by (compare with Def.1.2.19):

$$\begin{aligned}\langle \nabla e_i, e_j \rangle_{\mathrm{Ad}} + \langle e_i, \nabla e_j \rangle_{\mathrm{Ad}} = \langle de_i, e_j \rangle_{\mathrm{Ad}} + \langle e_i, de_j \rangle_{\mathrm{Ad}} + \langle e_k, e_j \rangle_{\mathrm{Ad}} \, \omega^l {c_{li}}^k + \omega^l {c_{lj}}^k \langle e_i, e_k \rangle_{\mathrm{Ad}} = \\ = d\left(\langle e_i, e_j \rangle_{\mathrm{Ad}}\right) + \omega^l \left[G_{ik} {c_{lj}}^k + G_{kj} {c_{li}}^k\right],\end{aligned} \tag{1.24}$$

which shows that the covariant derivative is compatible with the metric if and only if:

$$c_{lij} + c_{lji} = 0, \tag{1.25}$$

where we raise and lower Lie algebra index with the Ad-invariant metric tensor G_{ij}.

Def 1.3.16 (Yang-Mills action)**.** *We define the Yang-Mills action as ($\alpha \in \mathbb{R}$):*

$$\begin{aligned} S_{YM}[\boldsymbol{\omega}] &= \alpha \left\langle \boldsymbol{\Omega}(\boldsymbol{\omega}), \boldsymbol{\Omega}(\boldsymbol{\omega}) \right\rangle_{Ad(P),L^2} \\ &= \alpha \int_M \Omega^i(\omega) \wedge *\Omega^j(\omega) G_{ij}. \end{aligned}$$

Remark. *The Yang-Mills action is invariant under the gauge group $\mathcal{G}(P)$.*

The equations of motion for a Yang-Mills theory are obtained by taking the functional variation of the action with respect to the connection $\boldsymbol{\omega}$. Notice first that since $\boldsymbol{\Omega}(\boldsymbol{\omega}) = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}]$ we immediately have:

$$\boldsymbol{\omega} \mapsto \boldsymbol{\omega} + \delta\boldsymbol{\omega} \qquad \boldsymbol{\Omega}(\boldsymbol{\omega}) \mapsto \boldsymbol{\Omega}(\boldsymbol{\omega}) + d_{\boldsymbol{\omega}}\delta\boldsymbol{\omega}. \tag{1.26}$$

The variation of the action in Def.1.3.16 yelds:

$$\begin{aligned} S_{YM}[\boldsymbol{\omega} + \delta\boldsymbol{\omega}] &= \langle \boldsymbol{\Omega} + \delta\boldsymbol{\Omega}, \boldsymbol{\Omega} + \delta\boldsymbol{\Omega} \rangle_{\mathrm{Ad}(P),L^2} = \\ &= S_{YM}[\boldsymbol{\omega}] + 2 \langle d_{\boldsymbol{\omega}}\delta\boldsymbol{\omega}, \boldsymbol{\Omega} \rangle_{\mathrm{Ad}(P),L^2} = \\ &= S_{YM}[\boldsymbol{\omega}] + 2 \langle \delta\boldsymbol{\omega}, d^*_{\boldsymbol{\omega}}\boldsymbol{\Omega} \rangle_{\mathrm{Ad}(P),L^2} \end{aligned} \tag{1.27}$$

and the equations of motion are:

$$d^*_{\boldsymbol{\omega}}\boldsymbol{\Omega} = 0. \tag{1.28}$$

Chapter 2

General relativity and extensions

In this chapter we are going to study the relevant properties of gravitational theories. The first part is dedicated to an introduction to general relativity both in standard coordinate basis notation and in tetrad formalism. The latter will be well-suited to the introduction of a Lorentz gauge symmetry in the theory. In the second part we will consider and motivate generalizations of Einstein's theory. In the end, we will provide a prove of Ostrogradsky theorem showing how these generalizations lead to classical instabilities if one treats them naively.

2.1 Standard formulation of GR

In this section we are going to introduce the basic concepts needed to construct the Einstein-Hilbert action for General Relativity. This theory is based on two assumptions:

- (General covariance) The laws of physics are the same in all reference frames (for all observers);
- (Equivalence principles) In an arbitrary gravitational field no local experiment can distinguish a freely falling nonrotating system (*local inertial system*) from a uniformly moving system in the absence of a gravitational field.

They imply:

- Physical laws must be written in a coordinate-free fashion;
- For any point x in our configuration space M (a topological set) there exist an open neighborhood $U \subset M$ (this is why we need a topology on M), such that U is diffeomorphic (because dynamics requires derivatives) to $\mathbb{R}^{1,n-1}$ equipped with the canonical Minkowski metric η (this is the mathematical definition of absence of a gravitational field) and orientation. Notice that this gives to M the structure of a pseudo-Riemannian manifold.

Einstein gravity is then a geometric theory based on the *tangent bundle* (here is where the metric is defined) of some manifold. We will present both the standard formalism and the tetrad notation. Throughout this section we will fix an oriented, n-dimensional, pseudo-Riemaniann manifold (M, g) with Lorentzian signature. This manifold will be called *spacetime.*

2.1.1 Standard formulation

The tangent bundle TM is a vector bundle $\pi : TM \to M$, with $\pi^{-1}(x) = T_xM$, $x \in M$. In order to take derivative of sections of TM, i.e. vector fields on M (such as 4-velocities), we need to choose a connection.

Def 2.1.1 (Levi-Civita connection)**.** *The Levi-Civita connection is the unique connection* $\overset{\circ}{\nabla}$ *on* TM *that satisfies:*

$$dg(X,Y) = g(\overset{\circ}{\nabla}X, Y) + g(X, \overset{\circ}{\nabla}Y),\ \forall X, Y \in \mathfrak{X}(M) \qquad \textit{(metric compatibility)};$$
$$T(\overset{\circ}{\nabla})(X,Y) = \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X - [X,Y] = 0\ \forall X, Y \in \mathfrak{X}(M) \qquad \textit{(torsion-free)}.$$

Here we introduced the *torsion* associated to a connection on the tangent bundle, i.e.

$$T(\nabla) : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M), \tag{2.1}$$
$$T(\nabla)(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]\,. \tag{2.2}$$

As in Eq.(1.14) we can write in local coordinates:

$$\overset{\circ}{\nabla}(X) = dX + \overset{\circ}{\Gamma}(X^\alpha e_\alpha) = \left[dX^\alpha + X^\beta (\overset{\circ}{\Gamma})^\alpha{}_\beta\right] e_\alpha, \tag{2.3}$$

$$\overset{\circ}{\nabla}_\mu(X) = \left[\partial_\mu X^\alpha + X^\beta (\overset{\circ}{\Gamma}_\mu)^\alpha{}_\beta\right] e_\alpha, \tag{2.4}$$

where

$$(\overset{\circ}{\Gamma}_\mu)^\alpha{}_\beta =: \overset{\circ}{\Gamma}{}^\alpha_{\mu\beta} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\mu g_{\delta\beta} + \partial_\beta g_{\delta\mu} - \partial_\delta g_{\mu\beta}\right) \tag{2.5}$$

are called the *Christoffel symbols.* Notice that

$$\overset{\circ}{\Gamma}{}^\alpha_{\beta\gamma} e_\alpha = \overset{\circ}{\nabla}_\beta e_\gamma, \tag{2.6}$$

where $\nabla_\beta \equiv \nabla_{e_\beta}$.

The curvature associated with the Levi-Civita connection is called the *Riemann curvature*, $\mathbf{R}$, and is given by, in local coordinates:

$$\overset{\circ}{\mathbf{R}} = d\overset{\circ}{\Gamma} + \overset{\circ}{\Gamma} \wedge \overset{\circ}{\Gamma}, \tag{2.7}$$

$$(\overset{\circ}{\mathbf{R}})^\alpha{}_\beta = (d\overset{\circ}{\Gamma})^\alpha{}_\beta + (\overset{\circ}{\Gamma})^\alpha{}_\gamma \wedge (\overset{\circ}{\Gamma})^\gamma{}_\beta, \tag{2.8}$$

$$(\overset{\circ}{\mathbf{R}}(\partial_\mu, \partial_\nu))^\alpha{}_\beta = \partial_\mu \overset{\circ}{\Gamma}{}^\alpha_{\beta\nu} - \partial_\nu \overset{\circ}{\Gamma}{}^\alpha_{\beta\mu} + \overset{\circ}{\Gamma}{}^\alpha_{\gamma\mu} \overset{\circ}{\Gamma}{}^\gamma_{\beta\nu} - \overset{\circ}{\Gamma}{}^\alpha_{\gamma\nu} \overset{\circ}{\Gamma}{}^\gamma_{\beta\mu} =: R^\alpha{}_{\beta\mu\nu}. \tag{2.9}$$

The symbols $R^\alpha{}_{\beta\mu\nu}$ are the components of the *Riemann curvature tensor.*

From the Riemann curvature tensor we get:

- (Ricci tensor) $\overset{\circ}{R}_{\mu\nu} := \overset{\circ}{R^\alpha}{}_{\mu\alpha\nu}$;
- (Ricci scalar) $\overset{\circ}{R} := \overset{\circ}{R^\mu}{}_\mu = g^{\mu\nu} \overset{\circ}{R}_{\mu\nu}$.

With the Ricci scalar we can define the Einstein-Hilbert action:

Def 2.1.2 (Einstein-Hilbert action)**.** *The Einstein-Hilbert action is given by:*

$$S_{EH}[g] = \alpha \int_M *\overset{\circ}{R} = \alpha \int_M \overset{\circ}{R}\, dvol_g = \alpha \int_M d^n x \sqrt{|g|} \overset{\circ}{R},$$

where α is a constant and we used $d^n x\sqrt{|g|} \equiv \sqrt{|g|} dx^0 \wedge ... \wedge dx^{n-1}$.

The equations of motion obtained with the variational principle are the well-known vacuum Einstein's equations:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0. \tag{2.10}$$

2.1.2 Tetrad formulation of GR

We have seen in Subsection 1.2 that, given a point $x \in M$, we can always find $U \subset M$, open neighborhood of x, such that: $TM\big|_U \cong M \times \mathbb{R}^n$. Notice though that TM is not an arbitrary vector bundle over M. Indeed, once we have a local chart (ϕ, U) for M, this induces a local frame for TM (which is the tangent map of ϕ), i.e.

$$\phi : U \longrightarrow \mathbb{R}^{1,n-1}, \tag{2.11}$$

$$x \mapsto x^\mu(x); \tag{2.12}$$

$$\tilde{\phi} : TM\Big|_U \longrightarrow U \times T\mathbb{R}^{1,n-1} = U \times \mathbb{R}^{1,n-1}, \tag{2.13}$$

$$Y_x \mapsto (x, Y^\mu(x)\partial_\mu). \tag{2.14}$$

This means that we can choose simultaneously coordinates on M and a frame on TM, from now on we will consider this correspondence understood and we will just say coordinates for both M and TM. Notice that if we use the notation $\{x^\mu\}_\mu \equiv \{x^\mu\}_{\mu=0,\dots,n-1}$ for the local chart, then we use $\{\partial_\mu\}_\mu \equiv \{\partial_\mu\}_{\mu=0,\dots,n-1}$ for the local frame it induces. This is what we meant by ∂_μ in, for instance, Eq.(2.7). In the literature, this frame is called *coordinate basis*.

In this section we will formulate GR in a *non-coordinate basis* for TM made of orthonormal vector fields $\{e_a\}_a$ (with respect to g), i.e.

$$g(e_a, e_b) = \eta_{ab}. \tag{2.15}$$

We will call the set of this vector fields *tetrad*, even when $n \neq 4$, or *vielbeins* (vierbeins if $n = 4$). Notice that non-coordinates stands from the fact that these vector fields have non-vanishing Lie Bracket:

$$[e_a, e_b]\,\Big|_p = c_{ab}{}^c(p) e_c, \tag{2.16}$$

$$c_{ab}{}^c = e^c{}_\nu \left[e_a{}^\mu \partial_\mu e_b{}^\nu - e_b{}^\mu \partial_\mu e_a{}^\nu \right](p). \tag{2.17}$$

Taking the dual basis as a frame for the cotangent bundle T^*M and exploiting duality, we can write:

$$g = \eta_{ab}\, \theta^a \otimes \theta^b. \tag{2.18}$$

The dual forms to the tetrad are called *solder forms*.

We can expand the tetrad in the coordinate basis introduced above:

$$e_a = e_a{}^\mu \partial_\mu. \tag{2.19}$$

The symbols ${e_a}^\mu$ are the components of the tetrad in the coordinate basis, they will still be called *vielbeins* (or vierbeins when $n = 4$). Analogously, we have:

$$\theta^a = {e^a}_\mu dx^\mu. \tag{2.20}$$

Throughout this subsection, we will use Latin letters for the non-coordinate indices and Greek letters for coordinate indices. Notice that non-coordinate indices are raised/lowered by η and coordinate indices are raised/lowered by g, as it is well known that the metric induces a canonical isomorphism $V \cong V^*$. As in the previous subsection, in order to take derivatives, we need to introduce a connection. Notice that giving a connection to a bundle always implies making a choice. Einstein's choice, as we have anticipated before, is the Levi-Civita connection. One should think of this connection as the canonical connection for a (pseudo-)Riemannian manifold. We indicated this connection as $\overset{\circ}{\nabla}$ and its coefficients in local coordinates (and in coordinate basis) as $\overset{\circ}{\Gamma}{}^\alpha_{\beta\gamma}$. Notice that Def.2.1.1 is independent of coordinates and thus it still defines a connection also when we use the tetrad formalism. The only difference is that the Christoffel symbols will be different from the usual ones. Indeed we will indicate them as Γ^a_{bc}. We define them to be:

$$\overset{\circ}{\nabla}_a e_b = \overset{\circ}{\Gamma}{}^c_{ba} e_c. \tag{2.21}$$

Substituting Eq.(2.19) into Eq.(2.21) we immediately get:

$$\overset{\circ}{\Gamma}{}^c_{ba} = {e^c}_\nu {e_b}^\mu \left(\partial_\mu {e_a}^\nu + {e_a}^\lambda \overset{\circ}{\Gamma}{}^\nu_{\lambda\nu} \right) = {e^c}_\nu {e_b}^\mu \overset{\circ}{\nabla}_\mu {e_a}^\nu. \tag{2.22}$$

Notice that in the last equality $\overset{\circ}{\nabla}$ acts only on the coordinate index of the bein, this is just a convenient notation, though the second step is formally more correct. This means we can write the Leci-Civita connection as:

$$\overset{\circ}{\nabla}(V = V^a e_a) = (dV^a) e_a + V^b \theta^c \overset{\circ}{\Gamma}{}^b_{ac} e_a =: (dV^a) e_a + V^a {\omega^b}_a e_b, \tag{2.23}$$

where we have introduced the *spin connection*, i.e. ${\omega^a}_b = \overset{\circ}{\Gamma}{}^a_{bc} \theta^c$, here ω is a matrix valued, *local*, 1-form. We know that the Levi-Civita is the unique connection that satisfies both metric-compatibility and torsion free. To see how these two conditions are expressed in terms of the spin connection, we look back to Def.2.1.1 and we replace $X \to e_a$, $Y \to e_b$. From metricity we have:

$$\begin{aligned} dg(e_a, e_b) = 0 &= g(\overset{\circ}{\nabla} e_a, e_b) + g(e_a, \overset{\circ}{\nabla} e_b) \\ 0 &= {\omega^c}_a g(e_c, e_b) + {\omega^c}_b g(e_a, e_c) \\ 0 &= \omega_{ba} + \omega_{ab}, \end{aligned} \tag{2.24}$$

$$\omega_{ab} = -\omega_{ba}. \tag{2.25}$$

From torsion-free we find:

$$\begin{aligned} \overset{\circ}{\nabla}_a e_b - \overset{\circ}{\nabla}_b e_a - [e_a, e_b] &= 0 \\ \left[{\omega^c}_b(e_a) - {\omega^c}_a(e_b) - {c_{ab}}^c\right] e_c &= 0 \\ \overset{\circ}{\Gamma}{}^c_{ba} - \overset{\circ}{\Gamma}{}^c_{ab} - {c_{ab}}^c &= 0. \end{aligned} \tag{2.26}$$

Notice that, since torsion is a map that takes two vectors to give one, we can write its components (in tetrad basis) as:

$$T^a e_a = \left[d\theta^a + {\omega^a}_b \wedge \theta^b \right] e_a. \tag{2.27}$$

Replacing $X \to e_a$, $Y \to e_b$ and $Z \to e_c$ in Def.1.2.14 we get the curvature in the tetrad formalism:

$$\begin{aligned}
R^a{}_{bcd} e_a &= \mathbf{R}(e_c, e_d) e_b && (2.28)\\
&= \overset{\circ}{\nabla}_c \overset{\circ}{\nabla}_d e_b - \overset{\circ}{\nabla}_d \overset{\circ}{\nabla}_c e_b - \overset{\circ}{\nabla}_{[e_c, e_d]} e_b = && (2.29)\\
&= \overset{\circ}{\nabla}_c \left(\omega^l{}_b(e_d) e_l\right) - \overset{\circ}{\nabla}_d \left(\omega^l{}_b(e_c) e_l\right) - c_{cd}{}^f \overset{\circ}{\nabla}_f e_b = && (2.30)\\
&= \left(\partial_c \omega^l{}_b(e_d)\right) e_l + \omega^l{}_b(e_d)\omega^m_l(e_c) e_m - \left(\partial_d \omega^l{}_b(e_c)\right) e_l + \omega^l{}_b(e_c)\omega^m_l(e_d) e_m - c_{cd}{}^f \omega^m_b(e_f) e_m = && (2.31)\\
&= \left[\partial_c \omega^a{}_b(e_d) + \omega^l{}_b(e_d)\omega^a{}_l(e_c) - \partial_d \omega^a{}_b(e_c) + \omega^l{}_b(e_c)\omega^m_l(e_d) e_m - c_{cd}{}^f \omega^a{}_b(e_f)\right] e_a = && (2.32)\\
&= \left[d\omega^a{}_b + \omega^a{}_l \wedge \omega^l{}_b\right](e_c, e_d) e_a, && (2.33)
\end{aligned}$$

where in the end we showed:

$$\overset{\circ}{\mathbf{R}}{}^a{}_b = d\omega^a{}_b + \omega^a{}_l \wedge \omega^l{}_b. \tag{2.34}$$

Notice that Eq.(2.25) is somewhat surprising. We know from Eq.(1.12) that the local connection 1-form of a vector bundle takes value in the space of matrices, which we can see as the Lie algebra of $GL(n)$, i.e.

$$\mathfrak{gl}(n) \cong \mathrm{Mat}_{n\times n}(\mathbb{R}). \tag{2.35}$$

However, since $\omega_{ab} = -\omega_{ba}$, we have that the spin connection is a local 1-form that takes value not in $\mathrm{Mat}_{n\times n}$ but in a subalgebra of it, the algebra of the Lorentz group, i.e.

$$\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{so}(1, n-1). \tag{2.36}$$

This is why we called ω the spin connection. Comparing Def.1.2.12 with Eq.(2.25) a question arises naturally. Is TM the vector bundle on M associated to a principal $SO(1, n-1)$-bundle?

Let us first give a definition for general vector bundles.

Def 2.1.3 (Reduction of structure groups)**.** *Let $\pi : E \to M$ be a vector bundle of rank r. We say that its* structure group *can be reduced to $G \subset GL(r, \mathbb{R})$ if there exists a principal G-bundle $\pi : P \to M$ together with an isomorphism of vector bundles:*

$$E \cong E(P, \mathbb{R}^r)$$

Proposition 2.1.4. *The structure group of a vector bundle can be reduced to $G \subset GL(r, \mathbb{R})$ if and only if there exists a trivializing cover $\{U_\alpha, \phi_\alpha\}_{\alpha\in A}$, $M \subseteq \bigcup_{\alpha\in A} U_\alpha$, with the property that all transition functions $\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ take values in G.*

Proposition 2.1.5. *The reduction of the structure group to $O(r)$ is equivalent to choosing a metric on E. If E is also orientable, then the structure group can be reduced to $SO(r)$.*

Specializing to TM, this implies we can find a principal $SO(1, n-1)$-bundle P, such that:

$$TM \cong P \times_\rho \mathbb{R}^n, \tag{2.37}$$

where ρ is the fundamental representation of the Lorentz group in n dimensions (we will then imply $g = \rho(g)$ in the following). This shows how the tangent bundle can be seen as a vector bundle associated to a principal $SO(1,n-1)$-bundle L_{TM}. The connection 1-form ω that we defined in this chapter is then the connection on TM induced by

$$\omega : TL_{TM} \longrightarrow \mathfrak{so}(1,n-1),$$

the connection 1-form on L_{TM}. When we are dealing with a principal bundle P whose fiber is isomorphic to $SO(1,n-1)$, we call P a *principal Lorentz bundle*.

We have already seen that principal bundle arise naturally in physics when one is dealing with gauge theories. Since we have found the principal bundle associated to general relativity, it seems natural to construct its gauge group and see if we can define GR as a gauge theory of the Lorentz group. We will see that we can construct a gauge theory for Einstein gravity, but it will be special since the action in Def.2.1.2 is not a Yang-Mills action.

A gauge symmetry is not like "normal" symmetries, i.e. transformation of symmetry that depends on some fixed, *constant*, parameters. A gauge symmetry is signaling the presence of redundancy in our description of reality. We should then ask ourselves, what is the redundancy in the tetrad formalisms? We introduced a set of local orthonormal vector fields, i.e. the tetrad, and throughout this section we only used the orthonormality property of this set. The problem is that this property doesn't uniquely specify the tetrad. Consider a tetrad at a point $x \in M$, $\{e_a\}_a\big|_x$. If we act on it with an element $g \in SO(1,n-1)$ we get to a new set $\{e'_a\}_a\big|_x$ which still satisfies Eq.(2.15) at x. This is the redundancy we were looking for. The set of all tetrads related by a Lorentz transformation is the set of *local inertial frames*. When we move from the point $x \in M$ to an open subset $U \in M$, we see that the Lorentz transformations become local, i.e. they depend on parameters which are functions of spacetime. We then identified the gauge group of general relativity, or at least part of it. Mathematical gauge transformations are given by $f : L_{TM} \to L_{TM}$ and, once we choose a local section $s : U \to L_{TM}$, they correspond to physical gauge transformations $\tau_f : U \to SO(1,n-1)$ as given by Proposition 1.3.6, when one is dealing with Lorentz transformation, one usually writes Λ instead of τ_f, in the following we will stick to the former. We have already shown how the connection changes under gauge transformations (see Theorem 1.3.9), i.e.

$$\omega \xrightarrow{\Lambda} \Lambda^{-1}\omega\Lambda + \Lambda^{-1}d\Lambda. \tag{2.38}$$

Notice that this remains true for the induced connection on the associated vector bundle TM since ρ is a representation, i.e. a group homomorphism. The tetrad, as any other element of $\mathfrak{X}(M)$, is a section of this associated vector bundle. These objects change as given by Theorems 1.3.7-1.3.8, in particular:

$$e_a \xrightarrow{\Lambda} \Lambda e_a, \tag{2.39}$$

where again, since ρ is the fundamental rep of $SO(1,n-1)$, we simply wrote $\Lambda = \rho(\Lambda)$.

Now we are ready to tackle the last goal of this section: rewrite the Einstein-Hilbert action using the tetrad and the solder forms. Notice that we have cheated a bit in this section. The tetrad we introduced can be defined only locally. If we could find a global tetrad frame for TM, this frame would induce $TM \overset{\cong}{\to} M \times \mathbb{R}^n$ which means that the manifold is *parallelizable* and then not so interesting, geometrically speaking. The important thing to keep in mind is that, even if tetrads and solder forms are not global objects, we can build global objects out of them. Give an open cover to M, $\{U_\alpha\}_\alpha$ in such a way that we have a tetrad frame on TM for any α, $\{\mathbf{e}_\alpha\}_\alpha$ (here we used $\mathbf{e}_\alpha$ as a symbol for all the n vector fields that form the tetrad in the open U_α). We have already studied the situation on the overlap with the appearance of the transition functions. In this case however, we are passing from an orthonormal frame to another one, thus the transition functions take value in $SO(1,n-1)$, as one would expect since TM is a Lorentz associated vector bundle. This means that building global object out of the tetrad and the solder forms is the same as building gauge invariant object, with the gauge group introduced above. In the tetrad formalism one can write the Hilbert action in Def.2.1.2 as:

$$S[e^a] = \int_M e^a \wedge e^b \wedge *\boldsymbol{R}_{ab} = \int_M \sqrt{-g} d^4x\, R, \tag{2.40}$$

where we used Eq.(B.7). The goal of this thesis is to obtain this action as part of a Yang-Mills action of the form in Def.1.3.16.

2.2 Generalizing general relativity

The goal of this section is to generalize the action in Def.2.1.2 by considering all the other possible scalar terms of canonical dimension four that could be added to the Hilbert action without spoiling its symmetry. We will follow the work of [1]. In Theoretical Physics one usually works in natural units, i.e. setting $\hbar = c = 1$. This implies that any dimensionful quantity has the dimension of a mass (or, equivalently, energy) to some power, this is what we mean by *canonical dimension* or *mass dimension* of a dimensionful quantity. Setting $\hbar = 1$ immediately implies that the action must be dimensionless (since $\hbar$ is the quantum of action). Noticing that $c = 1$ implies $[d^n x] = -n$ and $[\partial_\mu] = +1$, where we write $[X]$ meaning *canonical dimension of X in units of mass.* In this section we fix $n = 4$.

2.2.1 Higher derivatives theory

The most general action we could write respecting the prescriptions above is:[1]

$$S[g] = \int_M \sqrt{-g} d^4x \left[\frac{1}{2\lambda}\mathbf{C}^2 - \frac{1}{\rho}E_4 + \frac{1}{\xi}R^2 + \tau\Box R + \frac{1}{\kappa^2}(R - 2\Lambda)\right], \tag{2.41}$$

where λ, ρ, ξ, τ are independent parameters. We will now define all the terms present in the action.

The first term appearing in the action is the *Weyl tensor* $\mathbf{C} \in \Omega^2(M, \mathrm{End}(TM))$. Given a set of local coordinates we can define (we re-introduce n as dimensionality in order to give a more general definition):

$$\begin{aligned} C_{\alpha\beta\mu\nu} :=& R_{\alpha\beta\mu\nu} + \frac{1}{n-2}\left(g_{\beta\mu}R_{\alpha\nu} - g_{\alpha\mu}R_{\beta\nu} + g_{\alpha\nu}R_{\beta\mu} - g_{\beta\nu}R_{\alpha\mu}\right) + \\ &+ \frac{1}{(n-1)(n-2)}R\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right), \\ \Rightarrow \mathbf{C}^\alpha{}_\beta =& C^\alpha{}_{\beta\mu\nu}\, dx^\mu \wedge dx^\nu, \end{aligned} \tag{2.42}$$

where $R^\alpha{}_{\beta\mu\nu}$, $R_{\mu\nu}$ and R are the components of the Riemann tensor, Ricci tensor and the Ricci scalar respectively. In Eq.(2.41) we wrote:

$$\begin{aligned} \mathbf{C}^2 = C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{4}{n-2}R_{\alpha\beta}R^{\alpha\beta} + \frac{2}{(n-1)(n-2)}R^2, \\ \int_M *\mathbf{C}^2 = -\int_M \mathrm{Tr}\left[\mathbf{C} \wedge *\mathbf{C}\right]. \end{aligned} \tag{2.43}$$

Notice that this term depends non trivially on the fourth derivatives of the metric.

The second summand in the action, which is valid only in even dimension $n = 2m$, is E_4 and it is defined as:

$$\int_M *E_4 = \int_M e(TM) := \int_M \mathrm{Pf}\left(\frac{1}{2\pi}\mathbf{R}\right), \tag{2.44}$$

where Pf denotes the *Pfaffian*, $e(TM)$ denotes the *Euler class* of the tangent bundle and $\mathbf{R} \in \Omega^2(M, End(TM))$ is the usual curvature tensor.

Def 2.2.1 (Pfaffian)**.** *The* Pfaffian *of a* $2m \times 2m$ *skew-symmetric matrix* X *is defined as the* polynomial *Pf*(X)*, such that:*

$$det(X) = (Pf(X))^2\,.$$

Notice that the determinant of a skew-symmetric matrix is always a perfect square, but it is only in even dimension that its square root is itself a polynomial in the matrix entries. The *Euler class* is defined for oriented bundles as:

$$e(TM) = \mathrm{Pf}(\frac{1}{2\pi}\mathbf{R}) \tag{2.45}$$

and it can be shown that it is independent from the choice of a connection on TM. Moreover, the following theorem proves that its contribution to the action is a topological invariant, and thus, it is also metric independent.

Theorem 2.2.2 (Chern-Gauss-Bonnet Theorem)**.** *Let* M *be a compact oriented Riemaniann manifold* M *of dimension* $2m$ *and* ∇ *a metric connection on its tangent bundle* TM *with curvature* $\mathbf{R}$ *relative to a positively oriented orthonormal frame. Then*

$$\int_M e(TM) = \int_M Pf\left(\frac{1}{2\pi}\mathbf{R}\right) = \chi(M),$$

where $\chi(M)$ *is the* Euler characteristic *and its a topological invariant of* M*.*

In four dimension we have:

$$E_4 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2. \tag{2.46}$$

The last three terms requires less introduction since they are just the square and the d'Alambertian of the Ricci scalar plus the usual Hilbert action, supplemented by a cosmological constant Λ. The action in Eq.(2.41) contains two terms that don't affect the classical equation of motion. The first one is clearly the Euler contribution, since it is a topological invariant. The second one is $\Box R$ since it gives rise to a contribution on the boundary, which is kept fixed by the variational principle. Indeed, let $y \in \Omega^1(M)$, $y = y_\mu dx^\mu$ with $y_\mu = \partial_\mu R$:

$$\Box R = \mathring{\nabla}_\mu \partial^\mu R = \mathring{\nabla}_\mu y^\mu \Rightarrow \int_M *\Box R = \int_M *\mathring{\nabla}_\mu y^\mu = \int_M d\,(*y) = \int_{\partial M} *y. \tag{2.47}$$

Eventhough the Euler term is a topological invariant we can use Eqs.(2.42-2.46) to rewrite the action in the totally equivalent form:

$$S[g] = \int_M \sqrt{-g}d^4x \left[xR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + yR_{\mu\nu}R^{\mu\nu} + zR^2 + \tau\Box R + \frac{1}{\kappa^2}\left(R - 2\Lambda\right)\right], \tag{2.48}$$

where the parameters x, y, z are related to ρ, λ, ξ. Notice that in both expression for the action there is only one dimensionful parameter: $\frac{1}{\kappa^2} = \frac{m_P^2}{16\pi}$, where $m_P \sim 10^{19}$GeV is the *Planck mass.* Thus one would expect, in the regime $|\partial_\lambda g_{\alpha\beta}| << m_p$, the higher derivatives contribution small when compared with the Hilbert term in the action. This is called *Planck suppression.* However, since the lagrangian contains higher derivatives, these terms are potentially unstable and whether or not Planck suppression applies is still unknown and requires careful investigation.

2.2.2 Semiclassical gravity and renormalization-induced higher derivatives

The reason why we are so interested in an extended action like in Eq.(2.41) is its *semiclassical* implications. Our goal is to study quantum field theories in the presence of a classical background gravitational field $g_{\mu\nu}$. We will see that even if the QFT under consideration is renormalizable in flat space, the same is not necessarily true in curved spacetime. In particular, quantum corrections can generate all the *vacuum*[1] terms present in Eq.(2.41) and non-minimal couplings between the quantum fields and the gravitational field. In the following we will be using the functional picture of quantum field theory.

One of the most fundamental object in a QFT is the *generating functional* which is usually given by:

$$Z[J] = \frac{1}{\mathcal{N}} \int \mathcal{D}\varphi \, e^{(iS[g,\varphi]+J\varphi)},$$

where $J\varphi$ symbolically denotes all the sources J coupled to the quantum fields φ of the theory and $\mathcal{N}$ is a normalization constant. Since we are dealing with a classical metric field we can rewrite the generating functional as:

$$\begin{aligned} Z[g,J] &= \frac{1}{\mathcal{N}_0} e^{iS_{vacuum}[g]} \int \mathcal{D}\varphi \, e^{i(S_m[\varphi,g]+J\varphi)}, \\ \mathcal{N}_0 &= \int \mathcal{D}\varphi e^{iS[g,\varphi]} \Big|_{g_{\mu\nu}=\eta_{\mu\nu}}, \end{aligned} \tag{2.49}$$

where the normalization factor $\mathcal{N}_0$ is chosen in such a way that quantum induced corrections can source classical gravity. In order to briefly explain how the higher derivative terms are generated by quantum effects, we consider the background field method for the classical metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. This means that we will assume we can expand any quantity, such as the generating functional as well as Feynman diagrams, in a power series in $h_{\mu\nu}$. Thus any diagram in the theory will generate infinitely many more due to this expansion. This $h_{\mu\nu}$-proportional contributions can come or from existing vertices (e.g. $\bar{\Psi} A_\mu \gamma^\mu \Psi$) or from new vertices that simply weren't there in the flat space theory (e.g. $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$). All these contributions can only decrease or keep invariant the degree of divergence of a given diagram, but they can never increase it since these contributions may only add vertices but never remove them.

Let's now turn our attention on theories that are renormalizable in flat space. Given the argument above, we see that such theories have good chances to be renormalizable also in curved spacetimes. First, we summarize some of the most important properties of a renormalizable QFT in flat space:

1. The divergences that occur in the theory are local;
2. The divergences can be removed by local, gauge invariant counterterms;
3. Since the counterterms need to be added to the action, they must be geometrical scalars of mass dimension 4;
4. The predictions of the theory are finite after the inclusion of a finite set of such counterterms.

[1]In this subsection the term *vacuum* will refer to pure gravitational contribution to any quantity such as the action. This distinctions may not be trivial depending on the theory understudy, especially when symmetry breaking occurs. In this subsection we will assume that such distinction is possible and done.

Notice that the third property implies that all divergent diagrams in a renormalizable theory have mass dimensions 4. This is extremely important. Consider, for instance, a vacuum bubble, i.e. a connected diagram with no external lines. Suppose that we are working with a theory in which such a vacuum bubble has quartic divergence. If we look into the infinite series of diagrams that are generated from it, one can see that it will give rise to quadratic and logarithmically divergent diagrams. This happens because the insertion of $h_{\mu\nu}$-contribution can only produce new vertices in a vacuum bubble. The first vertex produced this way doesn't change the degree of divergence since it doesn't add any propagator to the diagram. However, the second vertex already generates a propagator, which in the most general case goes as $\left(k^2+m^2\right)^\alpha$, where k is the four-momentum of the field appearing in the propagator, m is its mass (when non-vanishing) and α is some (possibly half-integer) power. For the sake of practical simplicity we will focus on the specific case in which the field in the propagator is a scalar field, so that we can take $\alpha = -1$. Notice though that our considerations will still be generic since we may just go on to another vertex contribution and we will still obtain the same result.

Going back to our diagram, which is supposed to be quartically divergent, we see that the second and the third vertex contributions give a quadratically and a logarithmically divergent diagram. However, as pointed out before, in a renormalizable flat space theory all divergent diagrams must have mass dimension 4. Now, let's us assume that there are no other dimensionful quantity in the theory except the fields. Since $h_{\mu\nu}$ is dimensionless, the quadratically and logarithmically divergent diagram will be proportional to the second and fourth derivatives of $h_{\mu\nu}$, respectively. However, if the theory is to be renormalizable also in curved space time, the counterterms needed to remove these divergences must be geometric scalars. However, we have already listed all geometric scalars built from the metric, its first, second and fourth derivatives, in particular the only geometric scalar one can build out of the metric and its first and (linear in) second derivatives is the Ricci scalar. In this way, we see how we really needed to include the higher derivatives terms into the action in Eq.(2.41), otherwise we wouldn't be able to absorb the infinities coming from these new diagrams. In the same way, non-minimal coupling between fields and gravity is required in order to absorb the infinities coming from new diagrams contributing to the self-energy or vacuum-polarization of the fields.

Eventhough we needed a theory without dimensionful parameters in order to uniquely generate such high derivatives counterterms, more formal techniques imply the same results for theories that violets this restriction, e.g. massive theories. There is a remarkable lesson to learn here. Renormalizability in curved spacetimes requires the use of the most general action (Eq.(2.41)) for $S_{vac}[g]$ and the non-minimal coupling between fields and gravity. However, the higher derivative terms in Eq.2.41 may spoil the classical stability of the theory due to the Ostrogradsky instability, which will be the subject of the following subsection.

2.2.3 Ostrogradsky Instability

In this subsection we will study the Ostrogradsky theorem in the context in which it was originally formulated, classical mechanics, following the work of Woodard[10]. For simplicity we will consider a one-dimensional problem, so that $M \subset \mathbb{R}$.

Theorem 2.2.3 (Ostrogradsky Theorem). *Let $L[x, \dot{x}, \ddot{x}]$ be a lagrangian that non-degenerately*[2] *depends on $\ddot{x}$, i.e.*

$$\frac{\partial^2 L[x, \dot{x}, \ddot{x}]}{\partial \ddot{x}^2} \neq 0.$$

[2]Notice that this property implies that the second order time derivative can not be removed by means of integration by parts

Then the hamiltonian obtained from it is unbounded from below, with consequential instability of the system.

Let's first analyse the equations of motion (the *Euler-Lagrange equations*) for a lagrangian that depends non trivially on the acceleration. The action is given as usual:

$$S[x] = \int_M dt L[x, \dot{x}, \ddot{x}] \tag{2.50}$$

and its functional variation under $x \mapsto x + \delta x$ is given by:

$$\begin{aligned}
S + \delta S &= \int_M dt\, L[x + \delta x, \dot{x} + \delta \dot{x}, \ddot{x} + \delta \ddot{x}] = \\
&= \int_M dt \left[L[x, \dot{x}, \ddot{x}] + \delta x \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial x} + \delta \dot{x} \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \dot{x}} + \delta \ddot{x} \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \ddot{x}} \right] = \\
&= S + \int_M dt\, \delta x \left[\frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial x} - \frac{d}{dt} \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \ddot{x}} \right] \\
\delta S &= \int_M dt\, \delta x \left[\frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial x} - \frac{d}{dt} \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \ddot{x}} \right],
\end{aligned} \tag{2.51}$$

where we used integration by part exploiting the fact that the variations die off at the boundary when one is deriving the equations of motions. From Eq.2.51 we see that the Euler-Lagrange equation is:

$$\frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial x} - \frac{d}{dt} \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \ddot{x}} = 0, \tag{2.52}$$

which can be recast as:

$$\ddot{\ddot{x}} = \mathcal{F}\left(x, \dot{x}, \ddot{x}, \dddot{x}\right). \tag{2.53}$$

Since this equation is a fourth-order differential equation, we need to specify four initial conditions, which means that the Hamiltonian of this system will have four canonical variables. Ostrogradsky's choice for these four variables is:

$$\begin{aligned}
X_1 &= x, \quad P_1 = \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \ddot{x}}, \\
X_2 &= \dot{x}, \qquad\qquad P_2 = \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \ddot{x}}.
\end{aligned} \tag{2.54}$$

Non-degeneracy in the lagrangian implies that we can invert these relations into an equation for $\ddot{x}$, i.e. we can find an acceleration $A(X_1, X_2, P_2)$ such that:

$$\left. \frac{\partial L[x, \dot{x}, \ddot{x}]}{\partial \ddot{x}} \right|_{x = X_1,\, \dot{x} = X_2,\, \ddot{x} = A} = P_2. \tag{2.55}$$

We can then Legendre transform the lagrangian to finally get the Hamiltonian:

$$\begin{aligned}
H\left[X_1, X_2, P_1, P_2\right] &= P_1 \dot{X}_1 + P_2 \dot{X}_2 - L[x, \dot{x}, \ddot{x}] = \\
&= P_1 X_2 + P_2 A - L[x, \dot{x}, \ddot{x}].
\end{aligned} \tag{2.56}$$

The canonical equation of motion are given by:

$$\dot{X}_i = \frac{\partial H\left[X_1, X_2, P_1, P_2\right]}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial H\left[X_1, X_2, P_1, P_2\right]}{\partial X_i}. \tag{2.57}$$

It is easy to check that this hamiltonian generates time evolution. The phase space representation in Eq.(2.54) comes from the evolution equations for X_1, X_2 and P_1.

$$\dot{X}_1 = \frac{\partial H\left[X_1, X_2, P_1, P_2\right]}{\partial P_1} = X_2 \quad \Rightarrow X_2 = \dot{x}, \tag{2.58}$$

$$\dot{X}_2 = \frac{\partial H\left[X_1, X_2, P_1, P_2\right]}{\partial P_2} = A + \frac{\partial A}{\partial P_2}\left[P_2 - \frac{\partial L[x,\dot{x},\ddot{x}]}{\partial \ddot{x}}\right] = A \quad \Rightarrow P_2 = \frac{\partial L[x,\dot{x},\ddot{x}]}{\partial \ddot{x}}, \tag{2.59}$$

$$\dot{P}_2 = -\frac{\partial H\left[X_1, X_2, P_1, P_2\right]}{\partial X_2} = -P_1 - P_2\frac{\partial A}{\partial X_2} + \frac{\partial L[x,\dot{x},\ddot{x}]}{\partial \dot{x}} + \frac{\partial L[x,\dot{x},\ddot{x}]}{\partial \ddot{x}}\frac{\partial A}{\partial X_[2]} = -P_1 + \frac{\partial L[x,\dot{x},\ddot{x}]}{\partial \dot{x}},$$
$$\Rightarrow P_1 = \frac{\partial L[x,\dot{x},\ddot{x}]}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L[x,\dot{x},\ddot{x}]}{\partial \ddot{x}}. \tag{2.60}$$

Finally, the time evolution is given by the canonical equation for P_1:

$$\dot{P}_1 = -\frac{\partial H\left[X_1, X_2, P_1, P_2\right]}{\partial X_1} = -P_2\frac{\partial A}{\partial X_1} + \frac{\partial L[x,\dot{x},\ddot{x}]}{\partial x} + \frac{\partial L[x,\dot{x},\ddot{x}]}{\partial \ddot{x}}\frac{\partial A}{\partial X_1} = \frac{\partial L[x,\dot{x},\ddot{x}]}{\partial x},$$
$$\Rightarrow \frac{d}{dt}\frac{\partial L[x,\dot{x},\ddot{x}]}{\partial \dot{x}} - \frac{d^2}{dt^2}\frac{\partial L[x,\dot{x},\ddot{x}]}{\partial \ddot{x}} = \frac{\partial L[x,\dot{x},\ddot{x}]}{\partial x}. \tag{2.61}$$

The most important thing to notice in Eq.(2.56) is that it is *linear* in P_1 and thus it is unbounded from below. When the Hamiltonian doesn't posses a global minimum we say that the system understudy is *unstable*. This happens because every configuration which corresponds to a local minimum of the Hamiltonian, can be perturbed with a finite energy in such a way that the next semi-stable configuration will have lower energy. Since the energy doesn't have any lower bound this process could go on forever and thus generate what we call *instability* in the system.

The same analysis we did for two time derivatives can be repeated for higher derivatives, in particular, if the lagrangian is non-degenerate in the N-th order time derivative, $N > 1$, then the Hamiltonian will be bounded from below only in one of the momenta, while the others $N-1$ will appear linearly as P_1 in Eq.(2.56). In this way we see that there are unstable directions in phase space, i.e. those directions for which the linear term $P_1 X_2$ is negative. The only way for a theory to avoid Ostrogradsky instability is to violate its only hypothesis: the non-degeneracy of the lagrangian. This can be done explicitly, e.g. the Hilbert action Def.2.1.2, where R is linear in the second derivatives of the metric. Another way to avoid the instability is to have (or impose) constraints in phase space such that the unstable directions are unphysical, for example this could be done in gauge theories.

Notice that Ostrogradsky theorem implies that the theories of Subsection 2.2.1 contains classical instabilities if one consider them as only metric dependent. We will see that the Yang-Mills formalism will allow us to introduce quantities as the curvature squared in a first order formalism which doesn't generate Ostrogradsky instabilities.

Chapter 3

Geometrical Yang-Mills theories

This chapter is dedicated to the mathematical construction of some special Yang-Mills theories that we will call *geometrical*. The reason for this name is that we will study the implications of using part of the gauge connection as a tetrad. Throughout the following fix a manifold M and a princpal G-bundle $P \to M$ (G being a Lie group). The last two sections are dedicated to physical applications of the formalism at hand. In particular we will show the connection between de Sitter theory and GR and between conformal theory and Weyl gravity.

3.1 Gauge theoretical metric and geometrical action

The goal of this section is to define the metric on M ($g \in T^*M \otimes T^*M$) using gauge connections on P, ($\boldsymbol{\omega} \in T^*P \otimes \mathfrak{g}$). Then we will show how to write a geometrical Yang-Mills action and we will provide its particular variation.

3.1.1 Metric properties and tetrad candidates

Throughout this subsection we will identify the most important properties of the metric and we will then find some gauge fields that could define it. First of all, the metric tensor lies in the symmetric $\binom{0}{2}$ representation of the frame bundle of M that we introduced in Subsection 2.1.2. Since our goal is to define it through the gauge connections (which can be considered $\binom{0}{1}$), we will need the symmetric direct product rep, exactly as one usually does with the tetrad i.e. $g_{\mu\nu} = \eta_{ab} e^a \otimes e^b$, where η is (for the moment) an arbitrary symmetric constant matrix. Let's consider the gauge connection to be $\boldsymbol{\omega} = \tilde{\boldsymbol{\alpha}} + \boldsymbol{\beta}$, where $\tilde{\boldsymbol{\alpha}}, \boldsymbol{\beta} \in T^*P$. We consider $\tilde{\boldsymbol{\alpha}}$ to be the part of the gauge connection that defines the metric, while the latter is independent from $\boldsymbol{\beta}$. The distinction in $\tilde{\boldsymbol{\alpha}}$ and $\boldsymbol{\beta}$ generates a distinction also in the gauge algebra. This happens since this forms take value in the Lie algebra of our gauge group. We will then introduce $\{a_i\}_{i=1,\ldots,N_A}$ and $\{b_j\}_{j=1,\ldots,N_B}$ with $N_A + N_B = N = \dim(\mathfrak{g})$ such that we can write:

$$\boldsymbol{\omega} = \omega^i \otimes e_i = \tilde{\boldsymbol{\alpha}} + \boldsymbol{\beta} = \tilde{\alpha}^i \otimes a_i + \beta^j \otimes b_j, \tag{3.1}$$

where $\{e_i\}_{i=1,\ldots,N}$ is a basis for the Lie algebra $\mathfrak{g}$. We are then tempted to define the metric tensor as:

$$g \equiv \eta_{ij} \tilde{\alpha}^i \otimes \tilde{\alpha}^j. \tag{3.2}$$

Notice that if we intend to consider the fields $\tilde{\alpha}^i$ equivalent to standard tetrad fields we need more properties. In particular, N_A must be equal to n, dimensionality of M. Moreover, we will show in Subsection 3.1.3 that dimensional analysis gives a dimensionless connection 1-form (for the non-tetrad fields) and the dimension of an inverse mass for the tetrad 1-form. Since the connection 1-forms and the tetrad fields are part of the same connection 1-form they need to have the same mass dimension. We will then write $\alpha^i = M^{-1}\tilde{\alpha}^i$, where M is a constant with the dimension of a mass, and define the dimensionless metric the same as in the previous expression.

$$g = \eta_{ab}\alpha^a \otimes \alpha^b = M^{-2}\eta_{ab}\tilde{\alpha}^a \otimes \tilde{\alpha}^b. \tag{3.3}$$

However, this equation doesn't actually define a metric on M since, strictly speaking, the tensor constructed in this way lies in $T^*P \otimes_{\text{symm}} T^*P$. We have seen already how to pullback 1-forms from P to M in order to get local connection 1-form. In particular, we know that we need a section $s \in \Gamma(M, P)$. We will then consider α^i in Eq.(3.3) to be the local connection 1-form (times M) associated with $\boldsymbol{\alpha}$ and the section s. Notice that making another choice for s changes the metric definition, unless the change of gauge (in the language of Subsection 1.3) leaves Eq.(3.3) invariant. In general we will show that one in general loses part of the original gauge symmetry defining the metric in this way. With the construction above we turned M into a (pseudo-)Riemannian manifold (M, g).

3.1.2 The necessity of pseudo-orthogonal gauge groups

We have established in Subsection 2.1.2 that the choice of a metric is equivalent to the reduction of the Frame bundle of M to a principal O(s,t)-bundle, with (s,t) being the signature of the metric. Following the principle of special relativity and the 'mostly-plus' convention, we declare that the signature of our metric is (1,3). This implies that the Frame bundle is given by a Lorentz bundle as we have explained before. Consider for the moment the trivial situation for which (M, g) is just Minkowski spacetime, in particular the Riemann tensor vanishes everywhere. One can show that in this situation, one can use the gauge-tetrad local 1-form (in the case of contractible spacetimes, as Minkowski is, one can pullback the connections globally) as coordinates. the metric tensor in this coordinates is then given by $g_{\mu\nu} = \eta_{\mu\nu}$, showing the physical interpretation of the matrix η. Indeed, we can see that it represents the metric of flat (or also asymptotically flat) spacetime. Since we defined the signature to be (1,3), this turns η into the Minkowski metric, i.e. $\eta = \text{diag}(-1, 1, 1, 1)$. Going back to arbitrary (M, g), one can now see what kind of gauge transformations leave the metric on M invariant. We know that under a physical gauge transformation ($h(x) \in G$) a local connection 1-form changes as:

$$\boldsymbol{\omega} \to h^{-1}\boldsymbol{\omega} h + h^{-1}dh. \tag{3.4}$$

Recall that the first term in the transformation rule above is the adjoint representation (i.e. a *linear* action) of the Lie group G on its Lie algebra $\mathfrak{g}$.

Remark. *Every linear map between finite dimensional vector spaces can be represented in matrix form.*

In the case at hand we have:

$$h^{-1}\boldsymbol{\omega} h = \omega^i \otimes h^{-1}e_i h \equiv \omega^i H^k{}_i e_k. \tag{3.5}$$

Calling $A, B \subset \mathfrak{g}$, $A = \text{span}\{a_i\}_i$ and $B = \text{span}\{b_j\}_j$, we infer the transformation on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$:

$$\begin{aligned} \boldsymbol{\alpha} &\to h^{-1}\boldsymbol{\omega} h\Big|_{\in A} + h^{-1}dh\Big|_{\in A} \equiv H^i{}_k\omega^k a_i + (h^{-1}dh)^i a_i, \\ \boldsymbol{\beta} &\to h^{-1}\boldsymbol{\omega} h\Big|_{\in B} + h^{-1}dh\Big|_{\in B} \equiv H^j{}_k\omega^k b_j + (h^{-1}dh)^j b_j, \end{aligned} \tag{3.6}$$

which shows that the metric tensor is generically non-invariant with respect to the gauge group action. Indeed we find:

$$g \to \eta_{ab} \left(H^a{}_k \omega^k * (h^{-1}dh)^a \right) \otimes \left(H^b{}_k \omega^k * (h^{-1}dh)^b \right), \tag{3.7}$$

with $a, b = 1, ..., N_A$ and $k = 1, ..., N$.. The previous equation shows that in general the only gauge transformations that leave the metric (and, consequently, the pseudo-Riemannian manifold) invariant are the transformations for which $(h^{-1}dh)^a = 0$, $H^a{}_k$ is non-zero only for $k = 1, ..., N_A$ and the matrix $H \in O(1, 3)$, (i.e. it mixes the tetrad fields among them in a pseudo-orthogonal fashion). Introducing an orientation on (M, g) and demanding invariance of the orientation under the gauge action we can reduce the group to $SO(1, 3)$. It seems natural to conclude that in order to retain part of the original gauge symmetry after introducing the gauge theoretical metric introduced in Subsection 3.1.1 one needs $SO(1, 3) \subset G$. Notice that using only the Lorentz group is not enough. As we have explained in Subsection 2.1.2 the gauge connection of a Lorentz bundle is associated with the covariant derivative on the tangent bundle of M induced by the diffeomoerphism $TM \cong Fr(M) \times_\rho \mathbb{R}^n$, with ρ the fundamental representation of the Lorentz group. In particular, we have shown that demanding torsionless of a Lorentz connection is the same as fixing the Levi-CIvita connection on TM. Geometric fields like the Riemann curvature will be determined by the $SO(1, 3)$ connection as in Eq.(2.34). We will then need to extend the Lorentz algebra adding n or more generators. We will use n of this as gauge-tetrad fields and, as we have shown, they have to satisfy ($\Lambda \in SO(1, 3)$):

$$\boldsymbol{\alpha} \to h^{-1}\boldsymbol{\omega} h\Big|_{\in A} + h^{-1}dh\Big|_{\in A} = \Lambda^i{}_k \tilde{\alpha^k} a_i. \tag{3.8}$$

3.1.3 Geometrical action

In the following we will use the standard notation for the tetrad, i.e. $\alpha^a \equiv e^a$. As we have explained before, in order to define a gauge theoretical metric on M, we need $SO(1, 3) \subset G$. Considering the standard Yang-Mills action in Def.1.3.16. It is evident that in general the theory will be Lorentz invariant and not G-invariant. This happens because the Hodge-star operator introduces in the action a non-trivial metric dependence. We will then give a non-degenerate inner product to $\mathfrak{g}$ that is at least Lorentz invariant, G_{ij}. The action reads the same as for standard Yang-Mills theories:

$$S[\boldsymbol{\omega}] = \int_M \Omega^i \wedge *\Omega^j G_{ij}. \tag{3.9}$$

Before providing its variation we will first do a dimensional analysis of the action identifying the mass dimension of our gauge fields. First notice that we can write the volume V of a 4-dimensional compact submanifold $U \in M$ as:

$$V = \int_U *1 = \int_U \frac{\epsilon_{abcd}}{4!} e^a \wedge e^b \wedge e^c \wedge e^d. \tag{3.10}$$

Since a 4-volume has mass dimension -4 we will define the mass dimension of the tetrad fields (the same for any coordinate 1-form dx^μ) to be -1. Consider now a finite one-dimensional path γ on M and a dimensionful function $f : \gamma \to \mathbb{R}$ with mass dimension M_f. We have:

$$f(x_f) - f(x_i) = \int_\gamma df = \int_\gamma e^a \partial_a f. \tag{3.11}$$

This immediately implies that $\partial_a f$ has mass dimension equal to $M_f + 1$ (since the tetrad has dimension -1). Since the metric components are defined as $g_{ab} = \eta_{cd} e^c(\tilde{e}_a) \otimes e^d(\tilde{e}_b)$ we see that they are dimensionless. From now on we will write the mass dimension of a quantity ϕ as $[\phi]$. The action must be dimensionless, i.e. $[S] = 0$ since we fix $\hbar = 1$ (and we also fix $c = 1$ so that we can measure everything in unit of mass). We know that the curvature 2-form is given by (compare with Def.1.2.11):

$$\Omega^i = d\omega^i + \frac{1}{2} c_{jk}{}^i \omega^j \wedge \omega^k. \tag{3.12}$$

Usually one finds the mass dimension of fields by demanding adimensionality for their kinetic terms. As shown before, since the metric components are dimensionless, we have already fixed the dimension of the tetrad field to be -1. Indeed we have $[e^a] = [e^a{}_\mu dx^\mu] = 0 - 1 = -1$. The situation is different for the part of the connection that doesn't define the metric ($\boldsymbol{\beta}$). In this case we have for its curvature ($\boldsymbol{B}$):

$$B^j = d\beta^j + \frac{1}{2} c_{lm}{}^j \omega^l \wedge \omega^m. \tag{3.13}$$

Assuming orthogonality of $A, B \subset \mathfrak{g}$ with respect to the bundle-metric on the adjoint bundle (as will be for the theories we will study in the following), we find that the kinetic term corresponding to the $\boldsymbol{\beta}$ fields is given by:

$$\int_M d\beta_i \wedge * d\beta^i. \tag{3.14}$$

Remark. *In four spacetime dimensions the hodge of a 2-form has the same mass dimension of the original 2-form.*

This happens because, on a four dimensional manifold, the Hodge star operator maps 2-forms into 2-forms multiplying by dimensionless quantities (the Levi-Civita symbol and the metric tensor). We then have to compute the mass dimension of $d\beta^i$.

$$\begin{aligned} [d\beta^i] &= [\partial_\mu \beta^i{}_\nu] + [dx^\mu \wedge dx^\nu] = \\ &= [\beta^i{}_\nu] + 1 - 2 = \\ &= [\beta^i{}_\nu] - 1. \end{aligned} \tag{3.15}$$

Comparing with Eq.(3.14) we find:

$$\begin{gathered} 0 = \left[\int_M d\beta^i \wedge * d\beta_i \right] = 2 \left([\beta^i{}_\nu] - 1 \right) \\ [\beta^i{}_\nu] = 1. \end{gathered} \tag{3.16}$$

Notice that the connection 1-form (not its coordinates) are dimensionless, in contrast with the gauge tetrad fields which are fixed by the geometry to be of mass dimension -1. Thus as we have shown in Subsection 3.1.2 we have to consider $e^a = M^{-1} \tilde{e}^a$ where $\tilde{e}^a$ is the true dimensionless gauge connection. In particular, exploiting the orthogonality of G_{ij} expressed above, we have that the kinetic term for the tetrad fields reads:

$$\int_M d\tilde{e}^a \wedge * d\tilde{e}_a = M^2 \int_M de^a \wedge * e_a. \tag{3.17}$$

The same for the interaction terms in the action, which are given by:

$$\int_M \left[c_{jk}{}^l d\omega^i \wedge * \omega^j \omega^k + \frac{1}{4} c_{mn}{}^i c_{jk}{}^l \omega^m \wedge \omega^n \wedge * \left(\omega^j \wedge \omega^k \right) \right] G_{il}. \tag{3.18}$$

We have thus shown that twisting gauge connection and tetrad fields requires a dimensionful coupling costant (since it isn't an overall factor) in order to have a consistent Yang-Mills theory.

There is another term that could be added to the action without violating any of the prescriptions in Subsection 2.2.2.

$$\int_M \Omega^i \wedge \Omega^j G_{ij}. \tag{3.19}$$

Notice that there is no presence of the Hodge-star operator, thus this term doesn't depend on the metric tensor. In particular, if G_{ij} is bi-invariant with respect to the adjoint action, Eq.(3.19) is invariant with respect to the whole gauge group and not only the Lorentz subgroup. However, one can show that this term is proportional to a boundary integral which gives no contribution to the classical dynamics since the boundary is kept fixed by the variational principle. Nonetheless, it could give rise to interesting phenomena at the quantum level.

3.1.4 Variation of the geometrical action and the equations of motion

The goal of this subsection is to provide the functional variation of the geometrical Yang-Mills action in Eq.(3.9) under an infinitesimal variation of all the gauge connections. We already know the variation of the curvature (cfr. Eq.(1.26)), i.e.

$$\boldsymbol{\omega} \mapsto \boldsymbol{\omega} + \delta\boldsymbol{\omega} \qquad \boldsymbol{\Omega}(\boldsymbol{\omega}) \mapsto \boldsymbol{\Omega}(\boldsymbol{\omega}) + d_{\boldsymbol{\omega}} \delta\boldsymbol{\omega}, \tag{3.20}$$

so that the last piece we need (peculiar of a geometric Yang-Mills theory) is the variation of the Hodge star operator. Consider a 2-form $A \in \Omega^2(M)$ and let $*'$ denote the Hodge star operator that depends on the metric $g' = \eta_{ab}(\alpha^a + \delta\alpha^a) \otimes (\alpha^b + \delta\alpha^b) \equiv g + \delta g$, where α^a are the components of the gauge field $\boldsymbol{\omega}$ that defines the metric. (Notice that we will compute $*'A$ and not $*'A'$, the reason will be clear later.).

$$\begin{aligned} *'A &= *'\left(\frac{A_{\mu\nu}}{2} dx^\mu \wedge dx^\nu\right) = \frac{\sqrt{-g'}}{2(n-2)!} A_{\mu\nu} g'^{\mu\alpha} g'^{\nu\beta} \epsilon_{\alpha\beta\rho\sigma} dx^\rho \wedge dx^\sigma = \\ &= *A + \frac{\delta\sqrt{-g}}{2(n-2)!} A_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} \epsilon_{\alpha\beta\rho\sigma} dx^{\rho \wedge dx^\sigma} + \frac{\sqrt{-g}}{(n-2)!} A_{\mu\nu} \left(\delta g^{\mu\alpha}\right) g^{\nu\beta} \epsilon_{\alpha\beta\rho\sigma} dx^\rho \wedge dx^\sigma + O((\delta g)^2), \end{aligned} \tag{3.21}$$

where, for notational simplicity, we dropped the delta Diracs (notice that the theory is local so that most of the time the presence of the Dirac distribution doesn't give any interesting contribution) From Eq.(3.3) we find:

$$\begin{aligned} \frac{\delta\sqrt{-g}}{\delta e^a{}_\gamma} &= e_a{}^\gamma \sqrt{-g}, \\ \frac{\delta g^{\mu\nu}}{\delta e^a{}_\gamma} &= -g^{\mu\gamma} e_a{}^\nu - g^{\nu\gamma} e_a{}^\mu. \end{aligned} \tag{3.22}$$

Plugging these results back in Eq.(3.21) we get:

$$*'\, A = *A + e_a{}^\gamma * A - \frac{\sqrt{-g}}{(n-2)!} \left(g^{\mu\gamma} e_a{}^\alpha + g^{\alpha\gamma} e_a{}^\mu\right) A_{\mu\nu} g^{\nu\beta} \epsilon_{\alpha\beta\rho\sigma} dx^\rho \wedge dx^\sigma. \tag{3.23}$$

Finally we can give the action variation. Provided that G_{ij} satisfies Eq.(1.25), we can write (in analogy with Eq.(1.27)):

$$
\begin{aligned}
S \to & S + 2\left\langle \delta\boldsymbol{\omega}, d^*_{\boldsymbol{\omega}}\boldsymbol{\Omega}\right\rangle_{Ad(P),L^2} + \\
& + \int \left[e_a{}^{\gamma}\Omega^i \wedge *\Omega^j - \frac{\sqrt{-g}}{(n-2)!}\Omega^i \wedge \Omega^j{}_{\mu\nu}\left(g^{\mu\gamma}e_a{}^{\alpha} + g^{\alpha\gamma}e_a{}^{\mu}\right) g^{\nu\beta}\epsilon_{\alpha\beta\rho\sigma}dx^{\rho}\wedge dx^{\sigma}\right] G_{ij}\delta e^a{}_{\gamma} = \\
& = S + 2\left\langle \delta\boldsymbol{\omega}, d^*_{\boldsymbol{\omega}}\boldsymbol{\Omega}\right\rangle_{Ad(P),L^2} - 2G_{ij}\int_M \left[\Omega^{i\gamma\nu}\Omega^j{}_{\mu\nu}e_a{}^{\mu} - \frac{e_a{}^{\gamma}}{4}\Omega^i{}_{\mu\nu}\Omega^{j\mu\nu}\right]\delta e^a{}_{\gamma}\sqrt{-g}d^4x\,,
\end{aligned}
\tag{3.24}
$$

which gives as equations of motion:

$$
\begin{aligned}
& G_{aj}\left[\frac{1}{\sqrt{-g}}\partial_{\gamma}\left(\sqrt{-g}\Omega^{j\delta\gamma}\right) + c_{lm}^{\ j}\omega^l{}_{\gamma}\Omega^{m\delta\gamma}\right] = G_{ij}\left[\Omega^{i\delta\nu}\Omega^j{}_{\mu\nu}e_a{}^{\mu} - \frac{e_a{}^{\delta}}{4}\Omega^i{}_{\mu\nu}\Omega^{j\mu\nu}\right]\Bigg|_{\text{if } a \text{ is a tetrad index}}, \\
& G_{aj}\left[\frac{1}{\sqrt{-g}}\partial_{\gamma}\left(\sqrt{-g}\Omega^{j\delta\gamma}\right) + c_{lm}^{\ j}\omega^l{}_{\gamma}\Omega^{m\delta\gamma}\right] = 0\Bigg|_{\text{if } a \text{ isn't a tetrad index}}.
\end{aligned}
\tag{3.25}
$$

We found the peculiarity of a geometric Yang-Mills theory. The gauge fields that take the role of the tetrad are not source-free in vacuum, yet they are sourced by the *energy-momentum tensor* of the theory.

3.1.5 Outlook: interacting matter

In this subsection we are going to briefly explain how to introduce gauge charged matter for geometrical Yang-Mills theory. In usual Yang-Mills theory, charged fields live in associated vector bundle as in Def.1.2.5 $P \times_{\rho} V$. For instance, for chromodynamics P is given by an $SU(3)$-bundle and (ρ, V) is given by the fundamental representation (i.e. the color multiplets). Since we have already established that geometrical theories are invariant, in general, only with respect to the Lorentz subgroup, it seems natural to charge the fields only with respect to this latter. We can then use the standard representation theory of SO(1,3) to define our fields. The trivial example is the scalar spin-0 theory. In this case we have $(\rho, V) = (\rho_{\text{trivial}}, \mathbb{R})$, where $\rho_{\text{trivial}}(g) = 1, \forall g \in \text{SO}(1,3)$. As usual, the connection on the principal bundle induces a covariant derivative on the associated vector bundle (compare with Def. 1.2.12). In the case of the scalar field this is simply given by the standard differential as it can be easily checked.

More interesting is the case of the fundamental representation (i.e. the multiplets of the Lorentz group). In this case, since the Lorentz-subbundle is related to the frame bundle of M (as explained in Subsection 3.1.2), the fields correspond to spacetime vector fields. As usual they can be represented by:

$$
A = (A_0, A_1, A_2, A_3)\,, \tag{3.26}
$$

where the components are referred to a tetrad frame since we reduced the frame bundle to its Lorentz subbundle. The transformations of these fields is given by local Lorentz transformations.

$$
A_a \to \Lambda_a{}^b A_b. \tag{3.27}
$$

In particular, we notice that one can always find a transformation for which two of the three spatial components of the vector field vanish. This could perhaps help if one intends to study possible simmetry breaking scenarios of the theory. We will see that for the de Sitter theory these vector fields could turn the spacetime torsion into a non-dynamical field.

3.2 de Sitter gauge theory

In this section we are going to study the geometrical Yang-Mills theory for the de Sitter group (i.e. $SO(1,4)$). We will show that the geometrical Yang-Mills action contains the Hilbert action in the presence of a cosmological constant. We advice the reader to go through Appendix A to get more familiar with the techniques we will be using.

3.2.1 de Sitter group and de Sitter bundle

The de Sitter group is defined as the isometry group (i.e. the set of diffeomorphisms $\phi : M \to M$ for which $\phi_* g(\phi^* X, \phi^* Y) = g(X,Y)$, $X, Y \in \Gamma(TM)$) of the *de Sitter space.* This latter corresponds to the following vacuum solution of the Einstein's equation with cosmological constant Λ:

$$ds^2 = -dt^2 + e^{2\left(\frac{\Lambda}{3}\right)^{\frac{1}{2}} t} \left[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{3.28}$$

This space can be mapped into the four dimensional surface in $\mathbb{R}^5$ given by:

$$-(z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 = \frac{3}{\Lambda}, \tag{3.29}$$

with five-dimensional metric:

$$\eta = \operatorname{diag}(-1,1,1,1,1). \tag{3.30}$$

The last expression shows that the isometry group of de Sitter space is the same as the one for $\mathbb{R}^{1,4}$, i.e de Sitter group is given by $SO(1,4)$ and it clearly contains the Lorentz group as a subgroup. As we have shown in Subsection 3.1.2, this makes de Sitter group a good candidate to define a geometrical Yang-Mills theory. Notice that with respect to the Lorentz group we have exactly four more generators and we will use them to define the tetrad fields on M. In the following we will use lower-case latin letter for the Lie algebra indeces corresponding to the Lorentz generators, i.e. $M_{AB}|_{A,B=0,\ldots,3} \equiv M_{[ab]}$. The others four generator $M_{[a4]}$ will be called $\tilde{P}_a$. Comparing with Eq.(A.4), we get the commutators in the Lie algebra basis with this new notation:

$$\begin{aligned} \left[M_{[ab]}, M_{[cd]}\right] &= \eta_{bc} M_{[ad]} + \eta_{ad} M_{[bc]} + \eta_{db} M_{[ca]} + \eta_{ac} M_{[db]}, \\ \left[M_{[ab]}, \tilde{P}_c\right] &= \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b, \\ \left[\tilde{P}_a, \tilde{P}_c\right] &= M_{ca}, \end{aligned} \tag{3.31}$$

which gives for the Killing metric (compare with Eq.(A.6)):

$$\begin{aligned} &G_{[ab][cd]} = 2(N-2)\left[\eta_{[bc]}\eta_{[da]} - \eta_{bd}\eta_{ca}\right], \\ &G_{[ab][c4]} = 0, \\ &G_{[a4][c4]} = -2(N-2)\eta_{ac}. \end{aligned} \tag{3.32}$$

We now consider the particular pseudo-orthogonal bundle for which the structure group is given by de Sitter group. As usual we introduce a connection:

$$\boldsymbol{\omega} = \frac{1}{2}\omega^{[AB]} \otimes M_{[AB]} = \frac{1}{2}\omega^{[ab]} \otimes M_{[ab]} + e^a \otimes \tilde{P}_a, \tag{3.33}$$

which gives for the curvature (compare with Eq.(A.12)):

$$\begin{aligned}\mathbf{\Omega} &= d\boldsymbol{\omega} + \frac{1}{2}\left[\boldsymbol{\omega}, \boldsymbol{\omega}\right] = \\ &= \frac{1}{2}\left[d\omega^{[ab]} + \omega^{[a}{}_{c]} \wedge \omega^{[cb]} - e^a \wedge e^b\right] \otimes M_{ab} + \left[de^a + \omega^{[a}{}_{b]} \wedge e^b\right] \otimes \tilde{P}_a \equiv \\ &\equiv \frac{1}{2}\Omega^{[ab]} \otimes M_{[ab]} + T^a \otimes \tilde{P}_a.\end{aligned} \tag{3.34}$$

3.2.2 de Sitter Yang-Mills theory

In this subsection we are going to introduce a de Sitter geometrical Yang-Mills theory. We will show that the torsionless configurations of the theory corresponds to General Relativity, supplemented by a Riemann squared term.

As we have mentioned above we will use the fields $\{e^a\}_a$ as tetrad fields. We will then define the metric as in Subsection 3.1.1:

$$g = \eta_{ab} e^a \otimes e^b. \tag{3.35}$$

The other part of the connection 1-form is related to the Lorenz generators and as such it corresponds to the covariant derivative on the tangent bundle of M as explained in Subsection 3.1.2. Comparison between Eq.(2.27) and Eq.(3.34) shows that, under the interpretation explained above, the curvature related to the tetrad fields is given by the torsion on M related to the covariant derivative inherited by $\omega^{[ab]}$. We introduce the notation:

$$R^{[ab]} = d\omega^{[ab]} + \omega^{[a}{}_{c]} \wedge \omega^{[cb]}, \tag{3.36}$$

so that we can write:

$$\Omega^{[ab]} = R^{[ab]} - \tilde{e}^a \wedge \tilde{e}^b = R^{[ab]} - M^2 e^a \wedge e^b. \tag{3.37}$$

We have already shown in Subsection 2.1.1 that demanding tosionless on a metric-compatible connection automatically fixes it to be the Levi-Civita connection. Recalling Eq.(2.34), we see that for the configurations for which $T^a = 0$, we have:

$$\overset{\circ}{R}{}^{ab} = R^{[ab]}. \tag{3.38}$$

This equivalence will be extremely important now that we will build the action.

Following the recipe of Subsection 3.1.3 we provide the action for the geometrical Yang-Mills theory for the de Sitter group (α is an arbitrary dimensionless constant):

$$\begin{aligned}S[\omega^{[ab]}, e^a] &= \int_M \frac{1}{4}\Omega^{[ab]} \wedge *\Omega^{[cd]} G_{[ab][cd]} + \tilde{T}^a \wedge *\tilde{T}^b G_{[a4][b4]} = \\ &= \alpha \int_M \frac{1}{2}\Omega^{[a}{}_{c]} \wedge *\Omega^{[c}{}_{a]} - M^2 T^a \wedge *T_a = \\ &= \alpha \int_M \frac{1}{2} R^{[a}{}_{c]} \wedge *R^{[c}{}_{a]} - M^2 T^a \wedge *T_a - M^2 e^a \wedge e^c \wedge *R_{ca} + M^4 \frac{1}{2} e^a \wedge e^c \wedge *(e_c \wedge e_a) = \\ &= \alpha \int_M \sqrt{-g}d^4x \left[-\frac{1}{4}R^{[ac]\mu\nu}R_{[ac]\mu\nu} + \frac{M^2}{2}T^{[a]\mu\nu}T_{[a]\mu\nu} + M^2(R - 2\Lambda)\right],\end{aligned} \tag{3.39}$$

where $\Lambda = \frac{n(n-1)}{4}M^2 = 3M^2$ is the *gauge theoretical cosmological constant* coming from the last term in the third line of the previous equation. Notice that we introduced the mass parameter

M in accordance with the dimensional analysis provided in Subsection 3.1.3. The importance of this action is its torsionless limit. Indeed for torsionless configuration we find:

$$\begin{aligned} S[\omega^{[ab]}, e^a] &= \alpha \int_M \frac{1}{2} \overset{\circ}{R}{}^{[a}{}_{c]} \wedge * \overset{\circ}{R}{}^{[c}{}_{a]} - M^2 e^a \wedge e^c \wedge * \overset{\circ}{R}_{ca} + M^4 \frac{1}{4} e^a \wedge e^c \wedge * (e_c \wedge e_a) = \\ &= \alpha \int_M \sqrt{-g} d^4x \left[-\frac{1}{4} \overset{\circ}{R}{}^{[ac]\mu\nu} \overset{\circ}{R}_{[ac]\mu\nu} + M^2 (\overset{\circ}{R} - 2\Lambda) \right], \end{aligned} \tag{3.40}$$

which gives, as mentioned above, the Einstein-Hilbert action supplemented by a gauge theoretical cosmological constant and a Riemann squared term. Notice that the latter is not multiplied by M, and this constant needs to be of the same order as the Planck mass to give a true correspondence with general relativity. This means that the effect of this interaction on the Hilbert action is suppressed by a Planck mass squared.

Using the results of Subsection 3.1.4, we will now provide the equations of motion of the action in Eq.(3.39). For the Lorentz connections we find:

$$\begin{aligned} 0 =& G_{[ab][cd]} \left[\frac{1}{2\sqrt{-g}} \partial_\gamma \left(\sqrt{-g} \Omega^{[cd]\delta\gamma} \right) + \frac{1}{4} c_{[ef][lm]}{}^{[cd]} \omega^{[ef]}_{\gamma} \Omega^{[lm]\delta\gamma} + M^2 c_{[4f][4m]}{}^{[cd]} e^f{}_\gamma T^{m\delta\gamma} \right] = \\ =& \frac{\alpha}{2\sqrt{-g}} \partial_\gamma \left[\sqrt{-g} \left(R_{[ab]}{}^{\delta\gamma} - M^2 (e_a{}^\delta e_b{}^\gamma - e_a{}^\gamma e_b{}^\delta) \right) \right] + \frac{1}{2} \Delta_{[ab][cd]} \left(\omega^{[c}_{l]\gamma} \Omega^{[ld]\delta\gamma} - \omega^{[d}_{l]\gamma} \Omega^{[lc]\delta\gamma} + e^d{}_\gamma T^{c\delta\gamma} - e^c{}_\gamma T^{d\delta\gamma} \right) = \\ =& \frac{\alpha}{2\sqrt{-g}} \partial_\gamma \left[\sqrt{-g} \left(R_{[ab]}{}^{\delta\gamma} - M^2 (e_a{}^\delta e_b{}^\gamma - e_a{}^\gamma e_b{}^\delta) \right) \right] + \frac{1}{2} M^2 \left(e_{b\gamma} T_a{}^{\delta\gamma} - e_{a\gamma} T_b{}^{\delta\gamma} \right) \\ &+ \frac{1}{2} \left[\omega_{[al]\gamma} \left(R^{[l}{}_{b]}{}^{\delta\gamma} - M^2 (e^{l\delta} e_b{}^\gamma - e^{l\gamma} e_b{}^\delta) \right) - \omega_{[bl]\gamma} \left(R^{[l}{}_{a]}{}^{\delta\gamma} - M^2 (e^{l\delta} e_a{}^\gamma - e^{l\gamma} e_a{}^\delta) \right) \right] = 0. \end{aligned} \tag{3.41}$$

This equation corresponds to the generalization of flat-space Maxwell equations $\partial_\nu F^{\mu\nu} = 0$ to the case of non-Abelian gauge theory in curved spacetimes. For tetrad fields the situation is different. From Eq.(3.25) we can see that for the tetrad fields we have the contribution of the Hodge-star operator, a simil energy momentum tensor that appears on the right hand side. We first look at the LHS, i.e.

$$\begin{aligned} & - 2\alpha\eta_{aj} M^2 \left[\frac{1}{\sqrt{-g}} \partial_\gamma \left(\sqrt{-g} T^{j\delta\gamma} \right) + \frac{1}{2} c_{[l4][mn]}{}^{j} \left(e^l{}_\gamma \Omega^{[mn]\delta\gamma} - \omega^{[mn]}_{\gamma} T^{l\delta\gamma} \right) \right] = \\ =& - 2\alpha\eta_{aj} M^2 \left[\frac{1}{\sqrt{-g}} \partial_\gamma \left(\sqrt{-g} T^{j\delta\gamma} \right) + \frac{\eta_{nl}\delta^j{}_m - \eta_{ml}\delta^j{}_n}{2} \left(e^l{}_\gamma \Omega^{[mn]\delta\gamma} - \omega^{[mn]}_{\gamma} T^{l\delta\gamma} \right) \right] = \\ =& - 2\alpha\eta_{aj} M^2 \left[\frac{1}{\sqrt{-g}} \partial_\gamma \left(\sqrt{-g} T^{j\delta\gamma} \right) - e_{n\gamma} \Omega^{[jn]\delta\gamma} + \omega^{[jn]}_{\gamma} T_n{}^{\delta\gamma} \right] = \\ =& - 2\alpha\eta_{aj} M^2 \left[\frac{1}{\sqrt{-g}} \partial_\gamma \left(\sqrt{-g} T^{j\delta\gamma} \right) + \omega^{[jn]}_{\gamma} T_n{}^{\delta\gamma} - e_{n\gamma} \left(R^{[jn]\delta\gamma} - e^{j\delta} e^{n\gamma} + e^{j\gamma} e^{n\delta} \right) \right] = \\ =& - 2\alpha\eta_{aj} M^2 \left[\frac{1}{\sqrt{-g}} \partial_\gamma \left(\sqrt{-g} T^{j\delta\gamma} \right) + \omega^{[jn]}_{\gamma} T_n{}^{\delta\gamma} - e_{n\gamma} R^{[jn]\delta\gamma} + M^2 (n-1) e^{j\delta} \right] \end{aligned} \tag{3.42}$$

and for the RHS:

$$2\Big\{G_{[EF][CD]}\Omega^{[EF]\delta\nu}\Omega^{[CD]}_{\mu\nu}e_a^{\ \mu}-\frac{e_a^{\ \delta}}{4}G_{[EF][CD]}\Omega^{[EF]}_{\mu\nu}\Omega^{[CD]\mu\nu}\Big\}=$$
$$=-\frac{\alpha}{2}e_a^{\ \delta}\left[\frac{1}{2}R^{[dc]\mu\nu}R_{[cd]\mu\nu}-M^2T^{d\mu\nu}T_{d\mu\nu}-M^2\left(e^d_{\ \mu}e^c_{\ \nu}-e^c_{\ \mu}e^d_{\ \nu}\right)R_{[cd]}^{\ \ \mu\nu}+\right.$$
$$\left.+\frac{M^4}{2}\left(e^{d\mu}e^{c\nu}-e^{c\mu}e^{d\nu}\right)\left(e_{c\mu}e_{d\nu}-e_{d\mu}e_{c\nu}\right)\right]+2\alpha e_a^{\ \mu}\left[\frac{1}{2}R^{[dc]\delta\nu}R_{[cd]\mu\nu}-M^2T^{d\delta\nu}T_{d\mu\nu}+\right.$$
$$\left.-M^2\left(e_{c\mu}e_{d\nu}-e_{d\mu}e_{c\nu}\right)R^{[dc]\delta\nu}+\frac{M^4}{2}\left(e^{d\delta}e^{c\nu}-e^{c\delta}e^{d\nu}\right)\left(e_{c\mu}e_{d\nu}-e_{d\mu}e_{c\nu}\right)\right]=$$
$$=-\frac{\alpha}{2}e_a^{\ \delta}\left[\frac{1}{2}R^{[dc]\mu\nu}R_{[cd]\mu\nu}-M^2T^{d\mu\nu}T_{d\mu\nu}+2M^2R+n(1-n)M^4\right]+$$
$$+2\alpha\left[e_a^{\ \mu}\frac{1}{2}R^{[dc]\delta\nu}R_{[cd]\mu\nu}-M^2T^{d\delta\nu}T_{d\mu\nu}e_a^{\ \mu}+2M^2R^{[cd]\delta\nu}\eta_{ac}e_{d\nu}+(1-n)M^4e_a^{\ \delta}\right],\tag{3.43}$$

where we called $R\equiv e^c_{\ \mu}e^d_{\ \nu}R_{[cd]}^{\ \ \mu\nu}$. The last two expressions give the equations of motion for the tetrad fields:

$$\frac{1}{\sqrt{-g}}\partial_\gamma\left(\sqrt{-g}T_a^{\ \delta\gamma}\right)+\omega_{[an]\gamma}T^{n\delta\gamma}+\mathrm{Ric}_a^{\ \delta}-\frac{e_a^{\ \delta}}{2}\left(R-2\Lambda\right)=\frac{1}{2M^2}\left(\Theta_{\mathrm{lorentz}}\right)_a^{\ \delta}+\left(\Theta_{\mathrm{torsion}}\right)_a^{\ \delta},\tag{3.44}$$

here we identified:

$$\begin{aligned}\mathrm{Ric}_a^{\ \delta}&=R^{[cd]\delta\nu}\eta_{ac}e_{d\nu},\\ \left(\Theta_{\mathrm{lorentz}}\right)_a^{\ \delta}&=e_a^{\ \mu}R^{[cd]\delta\nu}R_{[cd]\mu\nu}-\frac{1}{4}e_a^{\ \delta}R^{[cd]\mu\nu}R_{[cd]\mu\nu},\\ \left(\Theta_{\mathrm{torsion}}\right)_a^{\ \delta}&=e_a^{\ \mu}T^{d\delta\nu}T_{d\mu\nu}-\frac{1}{4}e_a^{\ \delta}T^{d\mu\nu}T_{d\mu\nu}.\end{aligned}\tag{3.45}$$

Once again, it is interesting to study torsionless solutions to the equations of motion. We see that they correspond to Einstein's field equations supplemented by a 'geometrical' energy momentum tensor and the corresponding equations for the Lorentz connection, namely:

$$\begin{aligned}\mathring{\mathrm{Ric}}_a^{\ \delta}-\frac{e_a^{\ \delta}}{2}\left(\mathring{R}-2\Lambda\right)&=\frac{1}{2M^2}\left(\Theta_{\mathrm{lorentz}}\right)_a^{\ \delta},\\ G_{[ab][cd]}\left[\frac{1}{2\sqrt{-g}}\partial_\gamma\left(\sqrt{-g}\Omega^{[cd]\delta\gamma}\right)+\frac{1}{4}c_{[ef][lm]}^{\quad\ \ [cd]}\omega^{[ef]}_{\ \gamma}\Omega^{[lm]\delta\gamma}\right]&=0.\end{aligned}\tag{3.46}$$

This result is somewhat surprising. Indeed, we obtained Einstein's equations and a cosmological constant from the standard Yang-Mills action. In other words, we derived the equations of the gravitational field from a theory more similar to QCD or the Electro-weak interaction, and in general to the Standard Model physics. Moreover, notice that the difference between proper GR and Eq.(3.46) is a factor which is of second order in the Riemann curvature tensor but is also suppressed by an inverse Planck mass squared. The contribution coming from a non-vanishing RHS of vacuum Einstein's equation will then be relevant only when the curvature is of the same order of magnitude as the Planck mass. This is the situation one usually finds when considering portion of spacetime very close to the well-known singularities in some of the most famous solutions to GR equations of motion. We expect this term to give tangible contributions in such scenarios, perhaps, but this is just an educated guess, preventing or taming the presence of singularities. More likely this would effectively modify the structure of some event horizons in an analogy with what happens with the Reissner-Nordstrom solution. It is worth mentioning that adding matter to the theory will result in a contribution on the RHSs of Eq.(3.46). For the tetrad equation we would find the energy-momentum tensor of the matter field, as it can be easily check since it would appear from the Hodge-star variation again. For the Lorentz connection we would have the contribution coming from the gauge current (as in standard Yang-Mills theory) which for us will be represented by the angular momentum of the matter fields (since the gauge symmetry group is given by local Lorentz transformations).

An important aspect of this theory is the appearance of a gauge theoretical cosmological costant. This costant is positive and it's proportional to the Planck mass squared. Having such a big cosmological costant could be thought as a problem, expecially in cosmology, where one of the most important model predicts a very small Λ. However, eventhough the classical value is fixed by the structure constant of the de Sitter group to be proportional to the Planck mass, there is no way to say whether (after providing a quantization scheme) renormalization would keep this correspondence satisfied. Indeed, looking at the action, we see that, taking the factor M^2 inside the overall constant α, the bare value Λ will absorb the infinities coming from the *zero-point energy* of the fields (one can think of a scalar field theory to convince themselves that this is the situation). On the other hand, the bare Planck mass appears as a coupling costant in front of the other term in the action (the Riemann squared term), thus we expect it would receive quantum corrections associated with counterterms related to the field strength renormalization parameter Z. With all these considerations in mind, we expect that the physical Planck mass and the physical cosmological costant will be in general independent without spoiling the symmetry of the action (i.e. local Lorentz symmetry).

Consider again the complete de Sitter geometrical action in Eq.(3.39). We would like to give an intuitive scheme that one could follow in order to constraint dynamically the second term (torsion squared) to be zero. Let $A = A_\mu dx^\mu = A_a e^a$ be a 1-form vector field on M, not necessarily a gauge boson. Here, as in Subsection 3.1.5 we exploited the interpretation of part the connection field as tetrad fields to express the coordinates of the A field in this orthonormal basis. Recalling that the tetrad generators correspond to $M_{[a4]}$ one can see from Eq.(A.13) that the covariant derivative acting on the tetrad fields is given by:

$$d_{\boldsymbol{\omega}} e^a = de^a + \omega^{[a}{}_{b]} e^b = T^a, \tag{3.47}$$

so that charging the covector fields A as a Lorentz multiplet[1], we find:

$$d_{\boldsymbol{\omega}} A = dA_a \wedge e^a + A_a T^a. \tag{3.48}$$

It clearly provides a gauge invariant expression (compare with Eq.(3.27)) and it's different from the standard exterior derivative only if torsion is non-vanishing (as it is the case when one consider the general covariance principle applied to electro-magnetism). The easiest term to include in the action for such a field would be the standard kinetic term:

$$\begin{aligned}\int_M F \wedge *F &\equiv \int_M d_{\boldsymbol{\omega}} A \wedge *d_{\boldsymbol{\omega}} A = \\ &= \int_M \left[dA_a \wedge e^a \wedge *\left(dA_b \wedge e^b\right) + 2A_a T^a \wedge *\left(dA_b \wedge e^b\right) + A_a A_b T^a \wedge *T^b\right].\end{aligned} \tag{3.49}$$

[1]This means that we are applying the principle of general covariance passing from rigid to local Lorentz transformation acting on the covector. We need Lorentz and not the entire general linear group, as would be the case for general coordinate invariance, since as we studied in Subsection 2.1.2 we can always cover a Lorentzian manifold with local orthonormal frames for the tangent bundle.

We can then see that if one introduces a suitable potential for the A field, there is the possibility of a condensation of the field such to give $\langle A_a, A_b\rangle = -\frac{1}{2}M^2\eta_{ab}$. The semiclassical limit would then correspond to the torsionless action in Eq.(3.40). We notice that since we don't observe any other massless vector fields in the universe, we think this vector field should be massive enough to not make it detectable with modern experiments. Furthermore, since there is no dependence of the action on the derivative of the tetrad fields, we argue that without the torsion squared term the torsion field becomes non-dynamical and once we fix the initial conditions to give a vanishing torsion this will remain true throughout evolution.

In conclusion, using the de Sitter group as gauge group for a geometrical Yang-Mills theory we are able to obtain Einstein's theory of gravity as a low energy torsionless limit of our theory. The Yang-Mills formulation and in particular the structure constants of the de Sitter algebra give the Hilbert action supplemented with a Riemann squared term, which is suppressed by a Planck mass squared, a torsion squared factor, which could be possibly removed dynamically as we have shown in Eq.(3.49), and a gauge theoretical cosmological costant (as well as the usual mass parameter expected in all geometrical Yang-Mils theories). In Chapter 4 we will establish a Hamiltonian formalism suited for geometrical Yang-Mills theories. In particular we will consider the constraints arising in phase space and we will provide their analysis.

It is worth noticing that replacing de Sitter group SO(1,4) with anti-de Sitter group SO(2,3) one would find the same reults we have found for de Sitter since the algebra of the two groups is very similar, in particular the theory will still contain the Einstein-Hilbert action. The difference lies in the cosmological constant which would be negative for the case of AdS gauge theory. This comes from some relative sign between the structure constants of $\mathfrak{so}(1,4)$ and $\mathfrak{so}(2,3)$.

3.3 Conformal gauge theory

In this section we will be studying a *conformal gauge theory*, i.e a gauge theory with gauge group given by the *conformal group* of flat space $SO(2,4)$. First, we will introduce the conformal algebra and the conformal bundle. Then, we will be giving a consistent geometrical Yang-Mills action as in Section 3.1 and we will study its equations of motion.

3.3.1 Conformal group and conformal bundles

In this subsection we will define the conformal group and its algebra, mapping it to the pseudo-orthogonal group SO(2,4). Then we will introduce a principal $SO(2,4)$-bundle with relative connections and curvatures.

Def 3.3.1 (Conformal transformation). *Let (M, g) be a pseudo-Riemaniann manifold. A diffeomorphism $f : M \to M$ is called a* conformal transformation *if it preserves the metric tensor up to a local scale, i.e.*

$$f^* g_{f(p)} = e^{2\sigma(p)} g_p \qquad p \in M,\ \sigma \in C^\infty(M),$$

in a local chart $\{x_\mu\}_\mu$:

$$\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)) = e^{2\sigma(p)} g_{\mu\nu}(p),$$

where $y^\mu = f^* x^\mu \equiv x^\mu \circ f$.

The set of all such transformation clearly constitues a group which we denote as $\mathrm{Conf}(M, g)$.

Consider $X \in \mathfrak{X}(M)$, if the infinitesimal flow of X at $x \in M$ (in coordinates $x^\mu(x) \to x^\mu(x) + \epsilon X^\mu(x)$, $\epsilon << 1$) generates a conformal transformation we call X a *conformal Killing* *vector* (CKV) at x. The condition in Def.3.3.1 translates to:

$$\frac{\partial\left(x^{\alpha}+\epsilon X^{\alpha}\right)}{\partial x^{\mu}} \frac{\partial\left(x^{\beta}+\epsilon X^{\beta}\right)}{\partial x^{\nu}} g_{\alpha\beta}(x+\epsilon X)=e^{2\sigma(x)} g_{\mu\nu}(x), \tag{3.50}$$

which implies that g and X need to satisfy:

$$\mathcal{L}_X g_{\mu\nu}=X^{\alpha}\partial_{\alpha} g_{\mu\nu}+2\left(\partial_{(\mu} X^{\alpha}\right) g_{\nu)\alpha}=\psi g_{\mu\nu}, \tag{3.51}$$

where, noting that $\sigma \propto \epsilon$ we defined $\frac{\epsilon\psi}{2} := \sigma$ and it is easy to see that it is given by:

$$\psi=\frac{X^{\alpha} g^{\mu\nu}\partial_{\alpha} g_{\mu\nu}+2\partial_{\mu} X^{\mu}}{n}. \tag{3.52}$$

Proposition 3.3.2.
- *A linear combination of CKVs is still a CKV;*
- *The Lie bracket of two CKVs is again a CKV.*

Thus we see that the set of conformal Killing vectors constitues an algebra which is called the *conformal algebra*.

Through the rest of this section we will focus on the conformal group of flat space, i.e. Conf(M,η), where η is the Minkowski metric. The conformal algebra of flat space is given by, $a,b=0,1,2,3$:

$$\begin{aligned}
&\text{[Lorentz transformations]} && M_{ab}=x_a\partial_b-x_b\partial_a,\\
&\text{[Translations]} && P_a=\partial_a,\\
&\text{[Special conformal transformations (SCTs)]} && K_a=2x_a x^b\partial_b-x^2\partial_a,\\
&\text{[Dilations]} && D=x^a\partial_a,
\end{aligned} \tag{3.53}$$

with Lie brakets:

$$\begin{aligned}
[M_{ab},M_{cd}]&=\eta_{bc}M_{ad}+\eta_{ad}M_{bc}+\eta_{db}M_{ca}+\eta_{ac}M_{db},\\
[M_{ab},P_c]&=\eta_{cb}P_a-\eta_{ca}P_b,\\
[M_{ab},K_c]&=\eta_{bc}K_a-\eta_{ac}K_b,\\
[P_a,K_b]&=2(\eta_{ab}D-M_{ab}),\\
[D,K_b]&=K_b,\\
[D,P_a]&=-P_a.
\end{aligned} \tag{3.54}$$

We consider the following change of basis in $\mathfrak{conf}(M,\eta)$.

$$\begin{aligned}
&M_{a4}\mapsto\frac{P_a+K_a}{2},\\
&M_{-1a}\mapsto\frac{P_a-K_a}{2},\\
&M_{-14}\mapsto D,\\
&M_{ab}\mapsto M_{ab}
\end{aligned} \tag{3.55}$$

and they satisfy the algebra of $\mathfrak{so}(2,4)$ (the six dimensional Lorentz group with $\eta'=\mathrm{diag}(-1,-1,1,1,1,1)$), i.e.

$$[M_{AB},M_{CD}]=\eta'_{BC}M_{AD}+\eta'_{AD}M_{BC}+\eta'_{DB}M_{CA}+\eta'_{AC}M_{DB}, \tag{3.56}$$

where capital latin indices run from -1 to 4; -1 and 0 are the time indices and 1,2,3 and 4 are the spatial indices. This allows us to use the results of Appendix A, in particular we will represent the generators with the usual choice:

$$(M_{AB})^I{}_J = \delta^I{}_A \eta_{BJ} - \delta^I{}_B \eta_{AJ}. \tag{3.57}$$

We will use the Killing form as metric tensor in Lie algebra space, in accordance with Eq.(A.6).

We call *conformal bundle* a principal bundle $\pi : P \to M$ such that its structure group is given by SO(2,4). As usual we can introduce a $\mathfrak{so}(2,4)$-valued connection 1-form $\boldsymbol{\omega}$ which can be expanded in the generators in Eq.(3.53) as follows:

$$\begin{aligned}\boldsymbol{\omega} &= \frac{1}{2}\omega^{ab} \otimes M_{ab} + e^a \otimes P_a + f_a \otimes K^a + \omega \otimes D \\ &= \frac{1}{2}\omega^{AB} \otimes M_{AB},\end{aligned} \tag{3.58}$$

where ω^{ab}, e^a, f_a, ω, $\omega^{AB} \in \Omega^1(P)$. Before reading further, we suggest the reader to go through Appendix B in orther to get more acquainted with the techniques used in this chapter. Recalling Def.1.2.11 and the formula in Eq.(B.5), we can write the curvature $\boldsymbol{\Omega}$ associated with this connection:

$$\begin{aligned}\boldsymbol{\Omega} =& \frac{1}{2}\left[d\omega^a{}_b + \omega^a{}_c \wedge \omega^{cb} + 2\left(f^b \wedge e^a + e^b \wedge f^a\right)\right] \otimes M_{ab} + \\ &+ \left[de^a + \omega^{ab} \wedge e_b + e^a \wedge \omega\right] \otimes P_a + \\ &+ \left[df^a + \omega^{ab} \wedge f_b + \omega \wedge f^a\right] \otimes K_a + \\ &+ \left[d\omega + 2e^a \wedge f_a\right] \otimes D = \\ \equiv& \frac{1}{2}\Omega^{ab} \otimes M_{ab} + T^a \otimes P_a + S^a \otimes K_a + \Omega \otimes D,\end{aligned} \tag{3.59}$$

or, in the $\mathfrak{so}(2,4)$ basis:

$$\boldsymbol{\Omega} = \frac{1}{2}\Omega^A{}_B \otimes M^A{}_B, \tag{3.60}$$

with (compare with Eq.(B.4)):

$$\begin{aligned}&\Omega^{AB} = \Omega^{ab} \quad A, B = 0, 1, 2, 3, \\ &\Omega^{a\,-1} = -2\left(T^a - S^a\right), \\ &\Omega^{4a} = -2\left(T^a + S^a\right), \\ &\Omega^{-14} = \Omega.\end{aligned} \tag{3.61}$$

Moreover, we have the Bianchi identity $d_\omega \boldsymbol{\Omega} = 0$ from Theorem 1.2.13:

$$\begin{aligned}\frac{1}{2}d\Omega^{ab} + \frac{\omega^a{}_c \wedge \Omega^{cb} - \omega^b{}_c \wedge \Omega^{ca}}{2} + f^b \wedge T^a - f^a \wedge T^b - e^a \wedge S^b + e^b \wedge S^a = 0, \\ dT^a + \omega^{ab} \wedge T_b + e_b \wedge \Omega^{ba} + e^a \wedge \Omega - \omega \wedge T^a = 0, \\ dS^a + \omega^{ab} \wedge S_b + f_b \wedge \Omega^{ba} - f^a \wedge \Omega + \omega \wedge S^a = 0, \\ d\Omega + 2\left(e_a \wedge S^a - f_a \wedge T^a\right) = 0,\end{aligned} \tag{3.62}$$

or, in components:

$$\begin{aligned}\left[\partial_\gamma \Omega^{ab}{}_{\mu\nu} + 4e^a{}_\gamma S^b{}_{\mu\nu} + 4f^a{}_\gamma T^b{}_{\mu\nu} + 2\omega^{ac}_\gamma \Omega^b{}_{c\mu\nu}\right] dx^\gamma \wedge dx^\mu \wedge dx^\nu = 0, \\ \left[\partial_\gamma T^b{}_{\mu\nu} + 2\left(e^a{}_\gamma \Omega^b{}_{a\mu\nu} - \omega_\gamma T^b{}_{\mu\nu} + e^b{}_\gamma \Omega_{\mu\nu} + \omega^{ab}_\gamma T_{a\mu\nu}\right)\right] dx^\gamma \wedge dx^\mu \wedge dx^\nu = 0, \\ \left[\partial_\gamma S^b{}_{\mu\nu} + 2\left(f^b{}_\gamma \Omega_{\mu\nu} - \omega_\gamma S^b{}_{\mu\nu} - f^a{}_\gamma \Omega^b{}_{a\mu\nu} - \omega^{ab}_\gamma S_{a\mu\nu}\right)\right] dx^\gamma \wedge dx^\mu \wedge dx^\nu = 0, \\ \left[\partial_\gamma \Omega_{\mu\nu} + 2e_{a\gamma} S^a{}_{\mu\nu} - 2f_{a\gamma} T^a{}_{\mu\nu}\right] dx^\gamma \wedge dx^\mu \wedge dx^\nu = 0.\end{aligned} \tag{3.63}$$

It is interesting to look explicitly at the transformations of the connection and the curvature under conformal gauge transformations. In accordance with the general transformation rules in Eqs.(1.11-1.17) we have:

$$\begin{aligned} \boldsymbol{\omega} &\mapsto \boldsymbol{\omega}' = a^{-1}\boldsymbol{\omega} a + a^{-1} da, \\ \boldsymbol{\Omega} &\mapsto \boldsymbol{\Omega}' = a^{-1}\boldsymbol{\Omega} a, \end{aligned} \tag{3.64}$$

where $a \in \mathrm{SO}(2,4)$. For transformation infinitesimally close to the identity we can write $a \sim \mathbb{1} + \epsilon \mathbf{a} + O(\epsilon^2)$, where ϵ is a small parameter and $\mathbf{a} = \frac{a^{AB}}{2} M_{BA} \in \Omega^0(M, \mathfrak{so}(2,4))$, a^{AB} are functions. The transformation rules in Eq.(3.64) becomes:

$$\begin{aligned} \boldsymbol{\omega}' &= \boldsymbol{\omega} + \epsilon d_{\boldsymbol{\omega}} \mathbf{a} + O(\epsilon^2), \\ \boldsymbol{\Omega}' &= \boldsymbol{\Omega} + \epsilon \left[\boldsymbol{\Omega}, \mathbf{a}\right] + O(\epsilon^2), \end{aligned} \tag{3.65}$$

or, in a local trivialization:

$$\begin{aligned} \omega'^{ab}_{\ \ \gamma} &= \omega^{ab}_{\ \ \gamma} + \epsilon \left[\partial_\gamma a^{ab} + 4\left(a^b{}_K e^a{}_\gamma + a^b{}_P f^a{}_\gamma\right) + a^b{}_c \omega^{ac}{}_\gamma\right] + O(\epsilon^2), \\ e'^b{}_\gamma &= e^b{}_\gamma + \epsilon \left[\partial_\gamma a^b{}_P + 2\left(a^b{}_a e^a{}_\gamma - a^b{}_P \omega_\gamma + a e^b{}_\gamma + a_{aP}\omega^{ab}{}_\gamma\right)\right] + O(\epsilon^2), \\ f'^b{}_\gamma &= f^b{}_\gamma + \epsilon \left[\partial_\gamma a^b{}_K + 2\left(a f^b{}_\gamma - a^b{}_K \omega_\gamma - a^b{}_a f^a{}_\gamma - a_{aK}\omega^{ab}{}_\gamma\right)\right] + O(\epsilon^2), \\ \omega'_\gamma &= \omega_\gamma + \epsilon \left[\partial_\gamma a + 2a^a{}_K e_{a\gamma} - 2a^a{}_P f_{a\gamma}\right] + O(\epsilon^2); \\ \\ \Omega'^{bc}_{\ \ \mu\nu} &= \Omega^{bc}_{\ \ \mu\nu} + \epsilon \left[4\left(T^b{}_{\mu\nu} a^c{}_K + a^c{}_P S^b{}_{\mu\nu}\right) - 2a^c{}_a \Omega^{ab}{}_{\mu\nu}\right] + O(\epsilon^2), \\ T'^b{}_{\mu\nu} &= T^b{}_{\mu\nu} + \epsilon \left[2\left(a^b{}_a T^a{}_{\mu\nu} - a^b{}_P \Omega_{\mu\nu} + a T^b{}_{\mu\nu} + a_{aP}\Omega^{ab}{}_{\mu\nu}\right)\right] + O(\epsilon^2), \\ S'^b{}_{\mu\nu} &= S^b{}_{\mu\nu} + \epsilon \left[2\left(a S^b{}_{\mu\nu} - a^b{}_K \Omega_{\mu\nu} - a^b{}_a S^a{}_{\mu\nu} - a_{aK}\Omega^{ab}{}_{\mu\nu}\right)\right] + O(\epsilon^2), \\ \Omega'_{\mu\nu} &= \Omega_{\mu\nu} + \epsilon \left[2\left(a^a{}_K T_{a\mu\nu} - a_{aP} S^a{}_{\mu\nu}\right)\right] + O(\epsilon^2). \end{aligned} \tag{3.66}$$

3.3.2 Connection to Weyl Gravity

In this subsection we will show how to recover Weyl gravity and Bach equations from a geometrical Yang-Mills theories with gauge group given by the conformal group. This theory was first developed by James T. Wheeler and his Ph.D. student Juan Trujillo[8]. Their idea was based on a different structure with respect to what we have defined as geometrical Yang-Mills theories in Section 3.1. They consider the manifold structure of the conformal group and they consider the quotient with respect to the *inhomogenous Weyl group* IW, i.e. Lorentz transformations, dilations and SCTs. The group one obtains is the translation group with manifold structure diffeomorphic to $\mathbb{R}^4$. They then use Theorem 1.1.12 and Def.1.2.1 to define a principal IW-bundle over the translation group. Eventhough the gauge group is given (before geometrization) by IW they still expand the connection and curvature in the whole conformal basis as we did. This is how they solder the geometry of Yang-Mills theory with the geometry of spacetime. In contrast, we start from a principal pseudo-orthogonal bundle with a connection that is as usual expanded on the whole Lie algebra of the gauge group. We then need to reduce the symmetry in order to consider part of this connection as tetrad fields defining the geometry of spacetime. On the other hand, Wheeler's idea is to take the quotient of the gauge group with respect to the tetrad gauge generator ($\{a_i\}_i$ in the language of Subsection 3.1.1). In this way the fundamental vector fields associated with them (compare with Def.1.1.11) already define vector fields on M by construction and the gauge symmetry is given by the whole IW group (and not only its Lorentz subgroup as in our theories). However, from Eq.(1.3) we see that the Lie bracket of the tetrad fields as vector fields can't be different from the commutator of the generators associated with them. Since the translation group is an Abelian ideal of the conformal algebra, this means that the Lie bracket of the tetrad on M is trivial and we can use them as coordinates. It is straightforward to prove that if one then fixes the torsion on M to be zero, we can always find a global coordinate system with both Levi-Civita connection and Riemann curvature tensor identically zero. Naturally, in order to get any comparison with GR, they need to set torsion to zero in their theory and thus the geometry of spacetime becomes trivial and there's no gravitational field in any solution. We will now show that we can use our concept of geometrical Yang-Mills theory to save the conceptual mistake we mentioned and obtain Weyl squared theory for torsionless configurations.

The action we will start with is the conformal version of the general geometrical action in Eq.(3.9), and it's given by:

$$\begin{aligned} S_{YM}[\boldsymbol{\omega}] &= \frac{\alpha}{2}\int_M \Omega^A{}_B \wedge *\Omega^B{}_A = \\ &= \alpha \int_M \left[\frac{1}{2}\Omega^a{}_b \wedge *\Omega^b{}_a + \Omega \wedge *\Omega - 8T^a \wedge *S_a\right] = \\ &= \frac{\alpha}{2}\int_M \sqrt{-g}d^4x \left[\frac{1}{2}\Omega_{ab\mu\nu}\Omega^{ba\mu\nu} + \Omega_{\mu\nu}\Omega^{\mu\nu} - 8T_{a\mu\nu}S^{a\mu\nu}\right]. \end{aligned} \tag{3.67}$$

Notice that in this particular geometrical Yang-Mills theory there is no gauge theoretical mass parameter appearing in the action, and consequently in the equations of motion. This is due to the fact that the generators for translations and special conformal transformations are not orthogonal with respect to the metric in Lie algebra space. Indeed from the last term in the action we see that eventhough we know that the torsion form has mass dimension -1 it will be compensated by the mass dimension of the SC curvature. In this way we find that the mass dimension of the gauge field f^a needs to be +1.

1st step: change the metric in algebra space

We impose the following two constraints:

$$\begin{aligned} T^a &= 0, \\ S^a &= 0. \end{aligned} \tag{3.68}$$

They allow us to change the action in Eq.(3.67) into:[2]

$$S = \int_M \alpha\Omega^a{}_b \wedge *\Omega^b{}_a + \beta\Omega \wedge *\Omega. \tag{3.69}$$

Notice that the metric tensor we use here in Lie algebra space is not necessarily the restriction of the Killing form to Lorentz and dilational components only, as it can be checked comparing with Eq.(3.67). Indeed, we would have $\alpha = 1/2$ and $\beta = 1$. However, as we discussed in Subsection 3.1.3, the metric in algebra space doesn't need to be fully gauge invariant. In this case, one can show (using the transformations in Eq.(3.64)) that the action is invariant with respect to local Lorentz transformation as well as local dilations for any value of α and β. Moreover, there is no symmetry in the geometrical action preventing α and β to run differently due to renormalization. Thus, even starting with the action as in Eq.(3.67) and the constraints in Eq.(3.68), the most general theory that takes into account possible radiative corrections is the action in Eq.(3.69).

[2] Notice that this action is of course Lorentz-invariant as it should be for a geometrical Yang-Mills action, moreover one can prove using Eq.(3.66) and the metric definition that the theory is also symmetric with respect to gauge dilations.

We will then study this theory and we will show how to obtain Weyl gravity as a gauge-fixed version of the geometrical conformal theory.

The equations of motion are taken from variations of the action with respect to the connection fields. We start with the Lorentz connection. In accordance with Eq.(3.59), we see that ω_{ab} appears only in the Lorentz curvature Ω_{ab}. Its variation gives:

$$\begin{aligned} \delta\Omega^{ab} &= d\delta\omega^{ab} + \delta\omega^{ac} \wedge \omega_c{}^b + \omega^a{}_c \wedge \delta\omega^{cb}, \\ \delta S &= 2\alpha \int_M \left[-\delta\omega^{ab} \wedge d * \Omega_{ba} + \delta\omega^{ac} \wedge \omega_c{}^b \wedge *\Omega_{ba} - \delta\omega^{cg} \wedge \omega^a{}_c \wedge *\Omega_{ga}\right]. \end{aligned} \tag{3.70}$$

The equation of motion is then:

$$d * \Omega_{fe} + \omega_f{}^b \wedge *\Omega_{be} - \omega^a{}_e \wedge *\Omega_{fa} \equiv D * \Omega_{fe} = 0. \tag{3.71}$$

The variation with respect to the dilational connection ω involves only its curvature Ω and immediately provides the following equation of motion (compare with Eq.(3.59)):

$$d * \Omega = 0. \tag{3.72}$$

The special conformal connection appears in both curvatures and its variation gives:

$$\begin{aligned} \delta\Omega^{ab} &= 2\left(\delta f^b \wedge e^a - \delta f^a \wedge e^b\right), \\ \delta\Omega &= -2\delta f^a \wedge e_a, \\ \Rightarrow \delta S &= \int_M \delta f^c \wedge \left[4\alpha\left(e^a \wedge *\Omega_{ca} - e^b \wedge *\Omega_{bc}\right) - 4\beta e_c \wedge *\Omega\right], \end{aligned} \tag{3.73}$$

which gives the following equation of motion:

$$2\alpha e^a \wedge *\Omega_{ca} = \beta e_c \wedge *\Omega. \tag{3.74}$$

Finally, we study the equations of motion for the tetrad/translation connections. We will take variations with respect to the tetrad components in the tetrad basis, i.e. $\delta e^a \equiv b e^a{}_b\, e^b$. As we have studied in Subsection 3.1.4, we know that this equation will contain contributions coming from the variation of the Hodge-star operator as in Eq.(3.23). The other part of the variation comes from the explicit dependence of both $\Omega^{[ab]}$ and Ω on the tetrad fields. Indeed, we have:

$$\begin{aligned} \delta\Omega^{[ab]} &= -2\left(\delta e^a \wedge f^b - \delta e^b \wedge f^a\right), \\ \delta\Omega &= 2\delta e^a \wedge f_a, \end{aligned} \tag{3.75}$$

which gives as equations of motion: (compare with Eq.(3.24))

$$\begin{aligned} 2\alpha f^b{}_m \Omega^{\ \ lm}_{[cb]} + \beta f_{cm}\Omega^{lm} &= \alpha\left(\Omega^{[ab]lr}\Omega_{[ba]cr} - \frac{\delta_c{}^l}{4}\Omega^{[ab]rs}\Omega_{[ba]rs}\right) + \beta\left(\Omega^{lr}\Omega_{cr} - \frac{\delta_c{}^l}{4}\Omega^{rs}\Omega_{rs}\right), \\ 2\alpha f^b{}_m \Omega^{\ \ lm}_{[cb]} + \beta f_{cm}\Omega^{lm} &= \alpha\left(\Theta_L\right)_c{}^l + \beta\left(\Theta_D\right)_c{}^l, \end{aligned} \tag{3.76}$$

where we indentified the Lorentz and dilational *energy-momentum tensors*. Using Eq.(B.7) we can write also the other equations of motion in components (in this section we use a tetrad basis also in form space in order to have nicer equations to read):

$$D_a \Omega^a{}_{bcd} = 0, \tag{3.77}$$

$$\partial_a \Omega^{ab} = 0, \tag{3.78}$$

$$2\alpha\Omega^d{}_{cde} + \beta\Omega_{ce} = 0. \tag{3.79}$$

Notice that symmetrizing the last equation we get the following constraint:

$$\Omega^d{}_{(c|d|e)} = 0. \tag{3.80}$$

2nd step: vanishing dilational curvature

We will now solve the equations of motion, in particular we will show how the dilational curvature dinamically vanishes. The Bianchi identity for a vanishing torsion reads (compare with Eq.(3.62)):

$$e_b \wedge \Omega^{ba} + e^a \wedge \Omega = 0, \tag{3.81}$$

or, in components:

$$\Omega^a{}_{[bcd]} = \delta^a{}_{[b}\Omega_{cd]}. \tag{3.82}$$

Taking the trace in a and c gives:

$$\Omega^a{}_{bad} - \Omega^a{}_{dab} = -(N-2)\Omega_{bd} = -2\Omega_{bd}, \tag{3.83}$$

while, antisymmetrizing Eq.(3.79) and comparing with the Bianchi identity, we get:

$$\begin{aligned} -\alpha \left[\Omega^d{}_{cde} - \Omega^d{}_{edc}\right] &= \beta\Omega_{ce}, \\ \Rightarrow \text{(BIANCHI)}\, (2\alpha - \beta)\, \Omega_{ce} &= 0. \end{aligned} \tag{3.84}$$

Then, provided $2\alpha \neq \beta$, we have:

$$\Omega_{ab} = 0 \tag{3.85}$$

and Eq.(3.79) becomes:

$$\Omega^d{}_{cde} = 0. \tag{3.86}$$

Notice that now both Eq.(3.80) and Eq.(3.78) are trivially solved. The Bianchi identity in Eq.(3.82) now gives:

$$\Omega^a{}_{[bcd]} = 0. \tag{3.87}$$

Now we define the *Riemann curvature* $R^a{}_b$ in accordance with Eq.(2.34):

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{3.88}$$

We can write the Lorentz curvature components as: (compare with Eq.(3.59))

$$\begin{aligned} \Omega^a{}_b &= \frac{1}{2}\Omega^a{}_{bcd} e^c \wedge e^d, \\ \Omega^a{}_{bcd} &= R^a{}_{bcd} + 2\left[f_{bc}\delta^a{}_d - f_{bd}\delta^a{}_c - f^a{}_c\eta_{bd} + f^a{}_d\eta_{cb}\right]. \end{aligned} \tag{3.89}$$

Plugging it in Eq.(3.86) we find ($f = f^a{}_a$):

$$\begin{aligned} R_{bd} &= 2\left[(N-2)\, f_{bd} + f\eta_{bd}\right], \\ R &= 4\,(N-1)\, f. \end{aligned} \tag{3.90}$$

which can be inverted to give:

$$f_{ab} = \frac{1}{2\,(N-2)}\left[R_{ab} - \frac{R\eta_{ab}}{2\,(N-1)}\right] = \frac{1}{2}\mathcal{R}_{ab}, \tag{3.91}$$

where we introduced the *Schouten tensor* $\mathcal{R}_{ab}$. Inserting Eq.(3.91) in Eq.(3.89) we finally obtain:

$$\begin{aligned}
\Omega^a{}_{bcd} &= R^a{}_{bcd} + \mathcal{R}_{bc}\delta^a{}_d - \mathcal{R}_{bd}\delta^a{}_c - \mathcal{R}^a{}_c\eta_{bd} + \mathcal{R}^a{}_d\eta_{bc} = \\
&= R^a{}_{bcd} + \frac{1}{(N-2)}\left[\left(R_{bc} - \frac{R\eta_{bc}}{2(N-1)}\right)\delta^a{}_d - \left(R_{bd} - \frac{R\eta_{bd}}{2(N-1)}\right)\delta^a{}_c - \left(R^a{}_c - \frac{R\delta^a{}_c}{2(N-1)}\right)\eta_{bd} + \right. \\
&\left. + \left(R^a{}_d - \frac{R\delta^a{}_d}{2(N-1)}\right)\eta_{bc}\right] = \\
&= R^a{}_{bcd} - \frac{1}{N-2}\left[R^a{}_c\eta_{bd} - R^a{}_d\eta_{bc} - R_{bc}\delta^a{}_d + R_{bd}\delta^a{}_c\right] - \frac{R}{(N-1)(N-2)}\left[\delta^a{}_d\eta_{bc} - \eta_{bd}\delta^a{}_c\right] = \\
&\equiv C^a{}_{bcd}.
\end{aligned} \tag{3.92}$$

We have shown how the equations of motion force the Lorentz curvature to be equivalent to the Weyl tensor. With all the considerations of this subsection we can rewrite the action as:

$$S[e^a, \omega^b{}_c] = \alpha \int_M C^a{}_b \wedge *C^b{}_a = -\alpha \int_M \sqrt{-g} d^4x\, C^{abcd} C_{abcd} \tag{3.93}$$

and the equation of motion for the spin connection is now:

$$D_a C^{abcd} = 0. \tag{3.94}$$

Consider then Eq.(3.76). Since we have seen that $f_{[a]b} = \mathcal{R}_{ab}$, $\Omega_{[ab]cd} = C_{abcd}$ and $\Omega = 0$, we can rewrite the equation as:

$$R_{ac} C^{abcd} = \alpha \left(\Theta_L\right)^{bd}. \tag{3.95}$$

Taking the derivative of Eq.(3.94) and combining with the previous expression we obtain:

$$D_c D_a C^{abcd} + R_{ac} C^{abcd} = \alpha \left(\Theta_L\right)^{bd}, \tag{3.96}$$

which are extended *Bach's equations* for which the RHS would be zero. Once again, in the geometrical theory we find a source in the RHS, coming from the contribution of the Hodge-star operator.

Eventhough our action is given by Eq.(3.93) and we have found Bach's equations, the theory is not Weyl gravity yet. The reason lies in the expression for the gauge theoretical torsion in Eq.(3.59), which is:

$$T^{[a]} = de^a + \omega^{[ab]} \wedge e_b + e^a \wedge \omega. \tag{3.97}$$

Since the last term is non-vanishing in general, this expression is not equivalent to the geometrical torsion on the tangent bundle of spacetime (which would be given by the first two terms). Looking at Eq.(3.85) we find:

$$\begin{aligned}
0 = \Omega &= d\omega + 2e^a \wedge f_a = \\
&= d\omega + \mathcal{R}_{ab} e^a \wedge e^b \\
0 &= d\omega,
\end{aligned} \tag{3.98}$$

which shows that ω is a closed 1-form. Supposing that we can apply Poincare Lemma, we have:

$$\omega = d\phi \qquad \phi \in C^\infty(M). \tag{3.99}$$

We know how the gauge fields change under a gauge dilation (which we have noticed still gives a gauge invariant action), i.e.

$$\begin{aligned}
\omega^a{}_b &\mapsto \omega^a{}_b, \\
e^a &\mapsto e^\alpha e^a, \\
f^a &\mapsto e^{-\alpha} f^a, \\
\omega &\mapsto \omega + d\alpha.
\end{aligned} \tag{3.100}$$

Choosing $\alpha = -\phi$ this shows we can find a gauge in which the dilational connection identically vanishes. This gauge is called *Riemannian gauge.* Notice that in this gauge torsion is:

$$T^a = 0 = de^a + \omega^a{}_b \wedge e^b, \tag{3.101}$$

which is the standard geometrical torsion as in Eq.(2.1). This implies that the spin-connection (which is already metric compatible by definition) is also torsion-free turning it in the Levi-Civita connection and $R^{[ab]}$ in the true Riemann curvature tensor, the same for all geometrical quantities derived from it.

In conclusion, we have seen that another important example of a geometric Yang-Mills theory is the Wheeler-Trujillo torsionless conformal gauge theory. The reduced gauge group is given by the Lorentz subgroup (as it should always be for geometric theories) and dilations. The Lie algebra metric in Eq.(3.69) is invariant with repect to the transformations mentioned above for any value of α and β. For $2\alpha \neq \beta$ we have shown how to reduce the theory to a Cartan-Weyl gravity (i.e. a $\boldsymbol{C}^2$ theory for which the Riemann tensor depends both on a non-vanishing torsion and on the usual Levi-Civita connection). Exploiting the dilational gauge invariance, there exist a gauge called *Riemannian gauge* for which the gauge theoretical and the geometrical torsion are equivalent. Here the geometry of spacetime is actually Riemannian and the on-shell equivalence with Weyl gravity is enstablished.

Notice that since both de Sitter and anti-de Sitter group are subgroups of SO(2,4), we can reinterpret the conformal gauge field in such a way to obtain the same gravitational theory as in Section 3.2 as part of the conformal Yang-Mills theory. The results don't really differ from the de Sitter case so we didn't provide them explicitly but it's really easy to convince ourselves that calling $\omega^{[a4]} = e^a$ this is actually the case.

Chapter 4

Hamiltonian analysis

In this chapter we are going to show how to write the Hamiltonian of such a pseudo-orthogonal Yang-Mills Lagrangian. We will then use this Hamiltonian to analyse the constraints that arise in phase space thanks to the degeneracy of the Lagrangian's Hessian (i.e. det $\left(\frac{\delta L}{\delta\dot{\omega}_i\delta\dot{\omega}_j}\right)$=0).

4.1 Building the Hamiltonian

This section is dedicated to the construction of a self-consistent Hamiltonian. The first thing we need to do is to identify the Lagrangian of our theory. If this was a classical mechanics problem, it would be trivial since we could write the action as:

$$S = \int_{\gamma\subset M} dt\, L, \tag{4.1}$$

where t is the evolution parameter that parametrizes the classical path γ. When one considers field theory the situation is different. The integration in the action is extended to the whole spacetime manifold and, in general, there is no unique and coordinate-independent way to break the integral into space and time directions. We will then accept that in order to do so we need to work in a coordinate frame, which we will call the *researcher frame*. The Hamiltonian will be a frame dependent structure of the theory. Please notice that this has nothing to do with the possible non-zero curvature of M. Rigid Lorentz transformations mix time and space in a non-trivial way, which shows that also the Standard Model Hamiltonian would be frame dependent.

Our reasoning proceeds as follow. We will consider a local chart (U,φ)[1] for M so that we can write the Yang-Mills action as:

$$\begin{aligned} S &= -\alpha\int_U \Omega_{[AB]}\wedge *\Omega^{[AB]} = \\ &= -\alpha'\int_{\varphi(U)} d^4x\sqrt{-g}\left[g^{\mu\alpha}g^{\nu\beta}\Omega_{[AB]\mu\nu}\Omega^{[AB]}_{\alpha\beta}\right]. \end{aligned} \tag{4.2}$$

Notice that, up to now, the action is still independent from the coordinate frame we chose on U thanks to the invariant volume element. This is because we still didn't specify a researcher frame. We define a researcher frame to be: *a local coordinates system with a specified time (read*

[1] We will consider this chart as covering the whole region of space-time we intend to study.

evolution) direction. Mathematically speaking, it means that we change the integration mesure as follows:

$$\int_{\varphi(U)\cong\mathbb{R}^4} \sqrt{-g} d^4x \longrightarrow \int_{\varphi(U)\cong\mathbb{R}\times\mathbb{R}^3} \sqrt{-g} dt dx^3. \tag{4.3}$$

Investigation on both the dependence of the *time choice* and the possibility to extend it globally are ongoing. We will call the time derivatives of our fields *researcher velocities* and they will be denoted by $\dot{\omega}^{AB}$. Analogously, we will call the space derivatives *researcher derivations* and they will be denoted by $\partial_i \omega^{AB}$. The action can be now written as:

$$S[\omega^{AB}, \dot{\omega}^{[AB]}, \partial_i \omega^{[AB]}] = -\alpha' \int_{\mathbb{R}\subset\varphi(U)} dt \left[\int_{\mathbb{R}^3\subset\varphi(U)} \sqrt{-g} d^3x \left(g^{\mu\alpha} g^{\nu\beta} \Omega_{[AB]\mu\nu} \Omega^{[AB]}_{\ \ \alpha\beta} \right) \right]. \tag{4.4}$$

In analogy with Eq.(4.1) we define the *researcher Lagrangian* of the pseudo-orthogonal Yang-Mills theory to be:

$$L[\omega^{[AB]}, \dot{\omega}^{[AB]}, \partial_i \omega^{[AB]}, t] = -\alpha' \int_{\mathbb{R}^3\subset\varphi(U)} \sqrt{-g} d^3x \left(g^{\mu\alpha} g^{\nu\beta} \Omega_{[AB]\mu\nu} \Omega^{[AB]}_{\ \ \alpha\beta} \right). \tag{4.5}$$

In order to have a clearer notation we will now write only $\mathbb{R}^n$ for the integration domain, of course one always has to keep in mind the overall dependence on the researcher frame φ. In the same way, we will omit the "researcher" prefix when referring to our fields or their derivatives.

We will now explicitly show the dependence of the Lagrangian on velocities and derivations of the fields. Notice that the only components of the curvatures two-forms that depend on the velocities are $\Omega_{(AB)0i}$. The computation proceeds as follows:

$$\begin{aligned}
L = &- \alpha' \int_{\mathbb{R}^3} \sqrt{-g} d^3x \left[g^{\mu 0} g^{\nu\beta} \Omega_{[AB]\mu\nu} \Omega^{[AB]}_{\ \ 0\beta} + g^{\mu i} g^{\nu\beta} \Omega_{[AB]\mu\nu} \Omega^{[AB]}_{\ \ i\beta} \right] = \\
= &- \alpha' \int_{\mathbb{R}^3} \sqrt{-g} d^3x \Big[g^{\mu 0} g^{0j} \Omega_{[AB]\mu 0} \Omega^{[AB]}_{\ \ 0j} + g^{\mu 0} g^{ij} \Omega_{[AB]\mu i} \Omega^{[AB]}_{\ \ 0j} + \\
&+ g^{0i} g^{\nu\beta} \Omega_{[AB]0\nu} \Omega^{[AB]}_{\ \ i\beta} + g^{ji} g^{\nu\beta} \Omega_{[AB]j\nu} \Omega^{[AB]}_{\ \ i\beta} \Big] = \\
= &- \alpha' \int_{\mathbb{R}^3} \sqrt{-g} d^3x \Big[g^{i0} g^{j0} \Omega_{[AB]i0} \Omega^{[AB]}_{\ \ 0j} + g^{00} g^{ij} \Omega_{[AB]0i} \Omega^{[AB]}_{\ \ 0j} + \\
&+ g^{k0} g^{ij} \Omega_{[AB]ki} \Omega^{[AB]}_{\ \ 0j} + g^{0i} g^{j0} \Omega_{[AB]0j} \Omega^{[AB]}_{\ \ i0} + g^{0i} g^{jk} \Omega_{[AB]0j} \Omega^{[AB]}_{\ \ ik} + \\
&+ g^{ji} g^{0\beta} \Omega_{[AB]j0} \Omega^{[AB]}_{\ \ i\beta} + g^{ji} g^{k\beta} \Omega_{[AB]jk} \Omega^{[AB]}_{\ \ i\beta} \Big] = \\
= &- \alpha' \int_{\mathbb{R}^3} \sqrt{-g} d^3x \Big[2 g^{i0} g^{j0} \Omega_{(AB)i0} \Omega^{(AB)}_{\ \ 0j} + 2 g^{k0} g^{ij} \Omega_{(AB)ki} \Omega^{(AB)}_{\ \ 0j} + \\
&+ g^{00} g^{ij} \Omega_{[AB]0i} \Omega^{[AB]}_{\ \ 0j} + g^{ji} g^{00} \Omega_{[AB]j0} \Omega^{[AB]}_{\ \ i0} + g^{ji} g^{0k} \Omega_{[AB]j0} \Omega^{[AB]}_{\ \ ik} + \\
&+ g^{ji} g^{k0} \Omega_{[AB]jk} \Omega^{[AB]}_{\ \ i0} + g^{ji} g^{kl} \Omega_{[AB]jk} \Omega^{[AB]}_{\ \ il} \Big] = \\
= &- \alpha' \int_{\mathbb{R}^3} \sqrt{-g} d^3x \Big[2 (g^{00} g^{ij} - g^{i0} g^{j0}) \Omega_{[AB]0i} \Omega^{[AB]}_{\ \ 0j} + 4 g^{k0} g^{ij} \Omega_{[AB]ki} \Omega^{[AB]}_{\ \ 0j} + \\
&+ g^{ij} g^{kl} \Omega_{[AB]jk} \Omega^{[AB]}_{\ \ il} \Big].
\end{aligned} \tag{4.6}$$

Introducing the notation $M^{ij} \equiv (g^{00}g^{ij} - g^{i0}g^{j0})$ we can write the Lagrangian in the more compact form:

$$L = -\alpha' \int_{\mathbb{R}^3} \sqrt{-g} d^3x \left[2M^{ij}\Omega_{[AB]0i}\Omega^{[AB]}_{\ \ 0j} + 4g^{k0}g^{ij}\Omega_{[AB]ki}\Omega^{[AB]}_{\ \ 0j} + g^{ij}g^{kl}\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ il} \right]. \quad (4.7)$$

We can now find the canonical momenta Π associated to our fields. In order to do so we need to take functional derivatives of the Lagrangian with respect to the velocities. Since the metric tensor doesn't depend on the velocities, we find:

$$\frac{\delta L}{\delta \dot{\omega}^{[CD]}_{\ \ \mu}} = \int_{\mathbb{R}^3} d^3x \left(-4\alpha'\sqrt{-g}\right) \frac{\delta \Omega^{[AB]}_{\ \ 0j}}{\delta \dot{\omega}^{[CD]}_{\ \ \mu}} \left[M^{ij}\Omega_{[AB]0i} + g^{k0}g^{ij}\Omega_{[AB]ki}\right]. \quad (4.8)$$

Since we have:

$$\Omega^{[AB]}_{\ \ 0j} = \dot{\omega}^{[AB]}_{\ \ j} - \partial_j \omega^{[AB]}_{\ \ 0} + \omega_{[AC]0}\omega^{[C}_{\ \ B]j} - \omega_{[AC]j}\omega^{[C}_{\ \ B]0}, \quad (4.9)$$

we find:

$$\frac{\delta \Omega^{[AB]}_{\ \ 0j}(x)}{\delta \dot{\omega}^{[CD]}_{\ \ \mu}(y)} = \Delta^{[AB]}_{\ \ [CD]}\delta^{\mu}_{\ j}\delta(x-y) = \Delta^{[AB]}_{\ \ [CD]}\delta^{\mu}_{\ j}\delta^3(\vec{x}-\vec{y})\delta(x^0-y^0). \quad (4.10)$$

Plugging it in Eq.(4.8) we get the canonical momenta, namely:

$$\Pi_{[CD]}^{\ \ l} := \frac{\delta L}{\delta \dot{\omega}^{[CD]}_{\ \ l}} = \left(-4\alpha'\sqrt{-g}\right)\left[M^{il}\Omega_{[CD]0i} + g^{k0}g^{il}\Omega_{[CD]ki}\right], \quad (4.11)$$

$$\Pi_{[CD]}^{\ \ 0} := \frac{\delta L}{\delta \dot{\omega}^{[CD]}_{\ \ 0}} = 0. \quad (4.12)$$

The second equation is the *primary constraint* of our theory. As we shall see in a moment, we won't be able to invert the previous set of equations in order to find $\dot{\omega}$ as a function of canonical fields and momenta.

Let's study the Hessian of the Lagrangian, i.e.

$$T_{[AB]\ [CD]}^{\ \ \alpha\ \ \beta} := \frac{\delta^2 L}{\delta \dot{\omega}^{[AB]}_{\ \ \alpha}\delta \dot{\omega}^{[CD]}_{\ \ \beta}} = \begin{cases} 0 & \text{for } (\alpha,\beta) = (0,\beta) \text{ or } (\alpha,0), \\ \left(-4\alpha'\sqrt{-g}\right) M^{ij}\Delta_{[AB][CD]} & \text{for } (\alpha,\beta) = (i,j), \end{cases} \quad (4.13)$$

where we used Eqs.(4.10-4.11-4.12). Notice that one can write the Hessian as the functional derivative of the momenta with respect to the velocities. Since it doesnt depend on the velocities, it is reasonable to assume the following ansatz for the momenta:

$$\Pi_{[CD]}^{\ \ \beta} = T_{(AB)\ (CD)}^{\ \ \alpha\ \ \beta}\dot{\omega}^{[AB]}_{\ \ \alpha} + P_{[CD]}^{\ \ \beta}(\omega, x). \quad (4.14)$$

Comparison with Eqs.(4.11-4.12) shows:

$$\begin{aligned} P_{[CD]}^{\ \ 0} =& 0, \\ P_{[CD]}^{\ \ l} =& \left(4\alpha'\sqrt{-g}\right)\Big[\Delta_{[AB][CD]}M^{ml}\dot{\omega}^{[AB]}_{\ \ m} - \left(\dot{\omega}_{[CD]i} - \partial_i\omega_{[CD]0} + \omega_{[CA]0}\omega^{[A}_{\ \ D]i} - \omega_{[CA]i}\omega^{[A}_{\ \ D]0}\right)M^{il} + \\ & - g^{k0}g^{il}\Omega_{[CD]ki}\Big] \\ =& \left(-4\alpha'\sqrt{-g}\right)\left[\left(-\partial_i\omega_{[CD]0} + \omega_{[CA]0}\omega^{[A}_{\ \ D]i} - \omega_{[CA]i}\omega^{[A}_{\ \ D]0}\right)M^{il} + g^{k0}g^{il}\Omega_{[CD]ki}\right]. \end{aligned} \quad (4.15)$$

The last step we need to do in order to get $\dot{\omega}(\omega, \Pi, x)$ is to find the inverse of the Hessian. Notice that this matrix is proportional to the identity in every entry except for the spacetime indeces[2]. We need to find the inverse of the matrix M^{ml}. For this purpose, the following identities will be useful:

$$\begin{aligned} g^{\mu\alpha} g_{\alpha\nu} &= \delta^{\mu}{}_{\nu}, \\ 0 = \delta^{0}{}_{j} &= g^{0\alpha} g_{\alpha j} = g^{00} g_{0j} + g^{0k} g_{kj}, \\ &\Rightarrow g^{0k} g_{kj} = -g^{00} g_{0j}; \\ \delta^{i}{}_{j} &= g^{i\alpha} g_{\alpha j} = g^{i0} g_{0j} + g^{ik} g_{kj}, \\ &\Rightarrow g^{ik} g_{kj} = \delta^{i}{}_{j} - g^{i0} g_{0j}. \end{aligned} \tag{4.16}$$

We will now prove that the inverse of M^{ij} is given by $\frac{g_{jl}}{g^{00}}$.

$$\begin{aligned} M^{ml} \frac{g_{lk}}{g^{00}} &= g^{00} g^{ml} \frac{g_{lk}}{g^{00}} - g^{m0} g^{l0} \frac{g_{lk}}{g^{00}} = \\ &= \frac{1}{g^{00}} \left[g^{00} (\delta^{m}_{\ k} - g^{m0} g_{0k}) - g^{m0} (-g^{00} g_{0k}) \right] = \delta^{m}_{\ k}, \end{aligned} \tag{4.17}$$

which implies that we can express the velocities (clearly except for $\dot{\omega}^{[AB]}_{\ 0}$) in terms of the momenta (compare with Eq.(4.14)):

$$\dot{\omega}_{[CD]k} = \left(-4\alpha' \sqrt{-g}\right)^{-1} \frac{g_{lk}}{g^{00}} \left(\Pi_{[CD]}^{\ l} - P_{[CD]}^{\ l}\right). \tag{4.18}$$

Notice that plugging Eq.(4.17) into Eq.(4.11) we find:

$$\Omega_{[CD]0n} = \left(-4\alpha' \sqrt{-g}\right)^{-1} \frac{g_{ln}}{g^{00}} \left(\Pi_{[CD]}^{\ l} + 4\alpha' \sqrt{-g} g^{k0} g^{il} \Omega_{[CD]ki}\right), \tag{4.19}$$

which is the only composite field in the Lagrangian that depends on the velocities.

We are now going to rewrite the Lagrangian in Eq.(4.7) as a function of fields and momenta. For the sake of clarity we are going to work it out step by step.

$$\begin{aligned} M^{ij} \Omega_{[AB]0i} =& M^{ij} \frac{g_{li}}{g^{00}} \left(-4\alpha' \sqrt{-g}\right)^{-1} \left(\Pi_{[AB]}^{\ l} + 4\alpha' \sqrt{-g} g^{k0} g^{sl} \Omega_{[AB]ks}\right) = \\ =& \left(-4\alpha' \sqrt{-g}\right)^{-1} \left(\Pi_{[AB]}^{\ j} + 4\alpha' \sqrt{-g} g^{k0} g^{sj} \Omega_{[AB]ks}\right), \\ M^{ij} \Omega_{[AB]0i} \Omega^{[AB]}_{\ 0j} =& \left(-4\alpha' \sqrt{-g}\right)^{-2} \left(\Pi_{[AB]}^{\ j} + 4\alpha' \sqrt{-g} g^{k0} g^{sj} \Omega_{[AB]ks}\right) \frac{g_{lj}}{g^{00}} \left(\Pi^{[AB]l} + 4\alpha' \sqrt{-g} g^{p0} g^{tl} \Omega^{[AB]}_{\ pt}\right) = \\ =& \left(-4\alpha' \sqrt{-g}\right)^{-2} \Big[\Pi_{[AB]}^{\ j} \Pi^{[AB]l} + 8\alpha' \sqrt{-g} \Pi_{[AB]}^{\ j} g^{k0} g^{sl} \Omega^{[AB]}_{\ ks} + \\ &+ \left(-4\alpha' \sqrt{-g}\right)^{2} \left(g^{k0} g^{sj} \Omega_{[AB]ks}\right) \left(g^{p0} g^{tl} \Omega^{[AB]}_{\ pt}\right)\Big] \frac{g_{lj}}{g^{00}}; \end{aligned} \tag{4.20}$$

$$\Omega_{[AB]ki} \Omega^{[AB]}_{\ 0j} = \left(-4\alpha' \sqrt{-g}\right)^{-1} \frac{g_{lj}}{g^{00}} \Omega_{[AB]ki} \left(\Pi^{[AB]l} + 4\alpha' \sqrt{-g} g^{s0} g^{tl} \Omega^{[AB]}_{\ st}\right). \tag{4.21}$$

[2] Notice that in Eq.(4.13) we omitted the Dirac-delta in local coordinates space eventhough technically it would be there.

The Lagrangian can be rewritten as follows:

$$\begin{aligned}
L =& \int_{\mathbb{R}^3} d^3x \Big\{ \left(-2\alpha' \sqrt{-g}\right) \frac{g_{lj}}{g^{00}} \left(-4\alpha' \sqrt{-g}\right)^{-2} \Big[\Pi^{\ j}_{[AB]} \Pi^{[AB]l} + 8\alpha' \sqrt{-g} \Pi^{\ j}_{[AB]} g^{k0} g^{sl} \Omega^{[AB]}_{\ \ ks} + \\
&+ \left(-4\alpha' \sqrt{-g}\right)^2 \left(g^{k0} g^{sj} \Omega_{[AB]ks}\right) \left(g^{p0} g^{tl} \Omega^{[AB]}_{\ \ pt}\right) \Big] + \\
&+ \left(-4\alpha' \sqrt{-g}\right) \left[\left(-4\alpha' \sqrt{-g}\right)^{-1} g^{k0} g^{ij} \Omega_{[AB]ki} \left(\Pi^{[AB]l} + 4\alpha' \sqrt{-g} g^{s0} g^{tl} \Omega^{[AB]}_{\ \ st}\right)\right] \frac{g_{lj}}{g^{00}} + \\
&- \alpha' \sqrt{-g} \left[g^{ij} g^{kl} \Omega_{[AB]jk} \Omega^{[AB]}_{\ \ il} \right] \Big\} = \\
=& \int_{\mathbb{R}^3} d^3x \Big\{ \left[\frac{1}{2} \left(-4\alpha' \sqrt{-g}\right)^{-1} \frac{g_{lj}}{g^{00}} \Pi^{\ j}_{[AB]} \Pi^{[AB]l} \right] + 2\alpha' \sqrt{-g} \frac{g_{lj}}{g^{00}} \left[\left(g^{k0} g^{ij} \Omega_{[AB]ki}\right) \left(g^{s0} g^{tl} \Omega^{[AB]}_{\ \ st}\right)\right] + \\
&- \alpha' \sqrt{-g} \left[g^{ij} g^{kl} \Omega_{[AB]jk} \Omega^{[AB]}_{\ \ il} \right] \Big\}
\end{aligned} \tag{4.22}$$

and it doesn't depend on the velocities anymore. We can now do the Legendre transformation to get the Hamiltonian, i.e. $H = \Pi\dot{\omega} - L$.

$$\begin{aligned}
H =& \int_{\mathbb{R}^3} d^3x \Big\{ \left(-4\alpha' \sqrt{-g}\right)^{-1} \Pi^{\ l}_{[CD]} \frac{g_{lk}}{g^{00}} \left(\Pi^{[CD]k} - P^{[CD]k} \right) + \dot{\omega}^{[CD]}_{\ \ 0} \Pi^{\ 0}_{[CD]} \\
&- \left[\frac{\left(-4\alpha' \sqrt{-g}\right)^{-1}}{2} \Pi^{\ j}_{[AB]} \Pi^{[AB]l} + 2\alpha' \sqrt{-g} \left(g^{k0} g^{ij} \omega_{[AB]ki}\right) \left(g^{s0} g^{tl} \Omega^{[AB]}_{\ \ st}\right) \right] \frac{g_{lj}}{g^{00}} + \\
&+ \alpha' \sqrt{-g} g^{ij} g^{kl} \Omega_{[AB]jk} \Omega^{[AB]}_{\ \ il} \Big\} = \\
=& \int_{\mathbb{R}^3} d^3x \Big\{ \frac{\left(-4\alpha' \sqrt{-g}\right)^{-1}}{2} \Pi^{\ l}_{[CD]} \Pi^{[CD]k} \frac{g_{lk}}{g^{00}} - \left(-4\alpha' \sqrt{-g}\right)^{-1} \Pi^{\ l}_{[CD]} P^{[CD]k} \frac{g_{lk}}{g^{00}} + \dot{\omega}^{[CD]}_{\ \ 0} \Pi^{\ 0}_{[CD]} + \\
&- 2\alpha' \sqrt{-g} \left(g^{k0} g^{ij} \Omega_{[AB]ki}\right) \left(g^{s0} g^{tl} \Omega^{[AB]}_{\ \ st}\right) \frac{g_{lj}}{g^{00}} + \alpha' \sqrt{-g} g^{ij} g^{kl} \Omega_{[AB]jk} \Omega^{[AB]}_{\ \ il} \Big\}.
\end{aligned} \tag{4.23}$$

Since H still depends on some velocities, it is not a proper Hamiltonian. We will now follow Dirac's procedure to show that H doesn't depend on the velocities on-shell and we will define then a generalized Hamiltonian. Dirac's reasoning goes as follows[7]. Let's consider the variation of the functional $\Pi\dot{\omega} - L$ under an infinitesimal change in all its argument:

$$\begin{aligned}
\delta(\Pi^{\ \mu}_{[CD]} \dot{\omega}^{[CD]}_{\ \ \mu} - L(\omega, \dot{\omega})) &= \left(\delta \Pi^{\ \mu}_{[CD]}\right) \dot{\omega}^{[CD]}_{\ \ \mu} + \Pi^{\ \mu}_{[CD]} \left(\delta \dot{\omega}^{[CD]}_{\ \ \mu}\right) - \frac{\delta L}{\delta \dot{\omega}^{[CD]}_{\ \ \mu}} \delta \dot{\omega}^{[CD]}_{\ \ \mu} - \frac{\delta L}{\delta \omega^{[CD]}_{\ \ \mu}} \delta \omega^{[CD]}_{\ \ \mu} \\
&= \delta(\Pi^{\ \mu}_{[CD]}) \dot{\omega}^{[CD]}_{\ \ \mu} - \dot{\Pi}^{\ \mu}_{[CD]} \delta \omega^{[CD]}_{\ \ \mu},
\end{aligned} \tag{4.24}$$

where we used both the momenta definition and the Euler-lagrange equations $\frac{d}{dt} \frac{\delta L}{\delta \dot{\omega}^{[CD]}_{\ \ \mu}} = \frac{\delta L}{\delta \omega^{[CD]}_{\ \ \mu}}$. The previous equation shows that H doesn't depend on the velocities when the equations of motion are satisfied. In non-constrained systems one would then match the terms in Eq.(4.24) with the (functional) derivatives of the Hamiltonian to get the canonical Hamilton's equations. However, when constraints are present, the variation of the fields are no longer independent. Indeed, consider a general constraint of the general form $\phi(\omega, \Pi) = 0$. Then it follows:

$$\frac{\delta \phi}{\delta \omega^{[CD]}_{\ \ \mu}} \delta \omega^{[CD]}_{\ \ \mu} + \frac{\delta \phi}{\delta \Pi^{\ \mu}_{[CD]}} \delta \Pi^{\ \mu}_{[CD]} = 0, \tag{4.25}$$

which shows that the variation are no longer independent. Now we can compare the variation in Eq.(4.24) with the variation of the Hamiltonian plus an arbitrary function of spacetime u (actually it will be defined on the researcher frame) multypling the identity in Eq.(4.25). We find:

$$\delta(\Pi^{\ \mu}_{[CD]})\dot{\omega}^{[CD]}_{\ \ \mu} - \dot{\Pi}^{\ \mu}_{[CD]}\delta\omega^{[CD]}_{\ \ \mu} = \delta H + u\left(\frac{\delta\phi}{\delta\omega^{[CD]}_{\ \ \mu}}\delta\omega^{[CD]}_{\ \ \mu} + \frac{\delta\phi}{\delta\Pi^{\ \mu}_{[CD]}}\delta\Pi^{\ \mu}_{[CD]}\right), \tag{4.26}$$

which shows that the Hamilton's equations of motion are given by:

$$\begin{aligned}\dot{\omega}^{[CD]}_{\ \ \mu} &= \frac{\delta H}{\delta\Pi^{\ \mu}_{[CD]}} + u\frac{\delta\phi}{\delta\Pi^{\ \mu}_{[CD]}},\\ \dot{\Pi}^{\ \mu}_{(CD)} &= -\frac{\delta H}{\delta\omega^{[CD]}_{\ \ \mu}} - u\frac{\delta\phi}{\delta\omega^{\ \mu}_{[CD]}}.\end{aligned} \tag{4.27}$$

We then define the *total Hamiltonian* to be:

$$H_T := H(\omega, \Pi^k) + u\phi(\omega, \Pi), \tag{4.28}$$

where Π^k stands for the uncostrained momenta. Notice that H_T is on-shell equivalent to the original Hamiltonian that depends on the velocities. We can now rewrite the equations of motion in terms of Poisson brackets, namely:

$$\dot{\omega}^{[CD]}_{\ \ \mu} = \left\{\omega^{[CD]}_{\ \ \mu}, H_T\right\}, \qquad \dot{\Pi}^{\ \mu}_{[CD]} = \left\{\Pi^{\ \mu}_{[CD]}, H_T\right\}. \tag{4.29}$$

For our theory the constraints are $\phi_{[CD]} = \Pi^{\ 0}_{[CD]} = 0$, which shows that our total Hamiltonian is given by:

$$\begin{aligned}H_T = \int_{\mathbb{R}^3} d^3x\Bigg\{&\frac{\left(-4\alpha'\sqrt{-g}\right)^{-1}}{2}\Pi^{\ l}_{[CD]}\Pi^{[CD]k}\frac{g_{lk}}{g^{00}} - \left(-4\alpha'\sqrt{-g}\right)^{-1}\Pi^{\ l}_{[CD]}P^{[CD]k}\frac{g_{lk}}{g^{00}}+\\ &- 2\alpha'\sqrt{-g}\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ \ st}\right)\frac{g_{lj}}{g^{00}} + \alpha'\sqrt{-g}g^{ij}g^{kl}\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ \ il}\Bigg\}+\\ &u^{[CD]}\cdot\phi_{[CD]},\end{aligned} \tag{4.30}$$

where the last term stands for all the Lagrange multipliers and constraints that will arise during the analysis of the primary constraints above. The e.o.m. are given by Eq.(4.29). From the Hamiltonian one can already see a potential problem of our theory. Notice that we can rewrite the first two terms in Eq.(4.30) as:

$$\int_{\mathbb{R}^3}\left(-4\alpha'\sqrt{-g}\right)^{-1}\frac{g_{lk}}{g^{00}}\left[\left(\Pi^{\ l}_{[CD]} - P_{[CD]l}\right)\left(\Pi^{[CD]k} - P^{[CD]k}\right) - P_{[CD]l}P^{[CD]}_{\ \ \ \ k}\right], \tag{4.31}$$

which is the kinetic energy (first term) and the left over from completing the square (second term). Since for pseudo-orthogonal groups the metric in Lie algebra space we use is often a indefinite inner product, we will have *kinetic instabilities* in the theory, i.e. fields for which the kinetic energy comes with the wrong sign in the total hamiltonian. Notice that there is no indefiniteness coming from the inner product in spacetime indeces. This is due to the presence of the primary constraints that force the timelike component of the momenta to vanish. In the constraints analysis we will give we won't find any constraint able to declare the kinetic instabilities unphysical. However, the presence of second-class constraints suggests that the simplectic structure of phase space is not canonical and thus not all hope is gone that after one properly identifies the physical phase space of the theory there won't be this kind of issues. After all, we have shown in Section 3.2 that the torsionless low energy limit of de Sitter theory coincides with general relativity so we expect at least a portion of physical phase space to be stable and possibly closed with respect to the evolution.

4.2 Constraints analysis

In this section we are going to study the self-consistency of the theory. In particular we are going to verify the conditions under which the evolution preserves the primary constraints we've found in the previous section. Since the constraints are given by $\phi_{[CD]} = \Pi_{[CD]}^{\ 0} = 0$, the self-consistency condition is given by $\dot{\phi}_{[CD]} = 0$, or, equivalently:

$$0 \approx \dot{\phi}_{[CD]} = \left\{ \phi_{[CD]}, H_T \right\} = \left\{ \Pi_{[CD]}^{\ 0}, H_T \right\} = -\frac{\delta H_T}{\delta \omega^{[CD]}_{\ \ 0}}. \tag{4.32}$$

We will solve the previous equation first for non-tetrad fields and then for the tetrad.

4.2.1 Secondary constraints for non-tetrad fields

Focusing on the non-tetrad fields, it is evident that the only contribution to Eq.(4.32) comes from the second term in Eq.(4.30), i.e.

$$\begin{aligned}\left(-4\alpha'\sqrt{-g}\right)^{-1} \Pi_{[CD]}^{\ l} \frac{g_{lk}}{g^{00}} P^{[CD]k} =& \Pi_{[CD]}^{\ i} \left(-\partial_i \omega^{[CD]}_{\ \ 0} + \omega^{[C}_{\ \ A]0} \omega^{[AD]}_{\ \ i} - \omega^{[C}_{\ \ A]i} \omega^{[AD]}_{\ \ 0} \right) + \\ &+ g^{m0} g^{ik} \Omega^{[CD]}_{\ \ mi} \frac{g_{lk}}{g^{00}} \Pi_{[CD]}^{\ l}.\end{aligned} \tag{4.33}$$

Here, and everywhere in this subsection, the indeces $[C, D]$ stand for non-tetrad fields only. Since $\Omega^{[CD]}_{\ \ ki}$ doesn't depend on $\omega^{[CD]}_{\ \ 0}$, only the first line actually contributes. The secondary constraint for non tetrad fields is then (after one integration by part and dismissing boundary terms):

$$0 \approx -\frac{\delta H_T}{\delta \omega^{[CD]}_{\ \ 0}} = D_i \Pi_{[CD]}^{\ i} \equiv \left[\partial_i \Pi_{[CD]}^{\ i} + \Pi_{[CA]}^{\ i} \omega_{[D\ \ i}^{\ \ A]} - \omega^{[A}_{\ \ C]i} \Pi_{[AD]}^{\ i} \right], \tag{4.34}$$

which corresponds to a *generalized Gauss law.* Notice that the adjective generalized stands both for the generalized "electric field" ($\Omega_{(CD)0i}$), that one has because the gauge theory is not abelian, and the generalized canonical momenta which is a non-trivial combination of generalized electric and magnetic fields if the metric tensor has non-zero components of the kind g^{i0} (for example Kerr's Black Holes), and it is this combination that is source-free in the vacuum. Notice that this is not a consequence of the geometrical nature of the theory but it comes from the fact that our fields have support on arbitrary Lorentzian manifold and not on Minkowski spacetime. It will follow a second constraints analysis as in Eq.(4.32).

4.2.2 Secondary constraints for tetrad fields

Since the metric tensor depends on the tetrad fields, the identification of the secondary constraints will take more efforts than for the non-tetrad case. Notice that we can decompose the variation of the Hamiltonian with respect to the tetrad as follows:

$$\frac{\delta H_T}{\delta e^a{}_\gamma} = \frac{\delta H_T}{\delta g^{\mu\nu}} \frac{\partial g^{\mu\nu}}{\partial e^a{}_\gamma} + \left.\frac{\delta H_T}{\delta e^a{}_\gamma}\right|_{g^{\mu\nu}\ \text{fixed}}. \tag{4.35}$$

The second term is the same one would have for the other gauge connection (in the following we assume $e^a = \omega^{[\cdot a]}$). It follows that we have:

$$\left.\frac{\delta H_T}{\delta e^a{}_0}\right|_{g^{\mu\nu}\ \text{fixed}} = -D_i \Pi_{[\cdot a]}{}^i. \tag{4.36}$$

The variation with respect to $g^{\mu\nu}$ is quite involved, we will tackle the problem in several steps. Let us introduce the following notation (compare with Eq.(4.15)):

$$\tilde{P}_{[CD]}{}^l = \left(-4\alpha'\sqrt{-g}\right)^{-1} P_{[CD]}{}^l = M^{il}\left(-\partial_i\omega_{[CD]0} + \omega_{[CA]0}\omega^{[A}{}_{D]i} - \omega_{[CA]i}\omega^{[A}{}_{D]0}\right) + g^{k0}g^{il}\Omega_{[CD]ki}. \tag{4.37}$$

Then we have:

$$\begin{aligned}
\frac{\delta H_T}{\delta g^{\mu\nu}} = &\int_{\mathbb{R}^3} d^3x \Bigg\{ \frac{\delta\left(-4\alpha'\sqrt{-g}\right)^{-1}}{\delta g^{\mu\nu}} \frac{1}{2}\frac{g_{lk}}{g^{00}} \Pi_{[CD]}{}^l \Pi^{[CD]k} + \left(-4\alpha'\sqrt{-g}\right)^{-1} \frac{1}{2}\frac{\delta \frac{g_{lk}}{g^{00}}}{\delta g^{\mu\nu}} \Pi_{[CD]}{}^l \Pi^{[CD]k} + \\
&- \Pi_{[CD]}{}^l \frac{\delta \frac{g_{lk}}{g^{00}}}{\delta g^{\mu\nu}} \tilde{P}^{[CD]k} - \Pi_{[CD]}{}^l \frac{g_{lk}}{g^{00}} \frac{\delta \tilde{P}^{[CD]k}}{\delta g^{\mu\nu}} - \frac{\delta\left(2\alpha'\sqrt{-g}\right)}{\delta g^{\mu\nu}} \left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\left(g^{s0}g^{tl}\Omega^{[AB]}{}_{st}\right)\frac{g_{lj}}{g^{00}} + \\
&- \left(2\alpha'\sqrt{-g}\right) 2\frac{\delta\left(g^{k0}g^{ij}\right)}{\delta g^{\mu\nu}} \Omega_{[AB]ki}\left(g^{s0}g^{tl}\Omega^{[AB]}{}_{st}\right)\frac{g_{lj}}{g^{00}} - \left(2\alpha'\sqrt{-g}\right)\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\left(g^{s0}g^{tl}\Omega^{[AB]}{}_{st}\right)\frac{\delta\frac{g_{lj}}{g^{00}}}{\delta g^{\mu\nu}} + \\
&+ \frac{\delta\alpha'\sqrt{-g}}{\delta g^{\mu\nu}} g^{ij}g^{kl}\Omega_{[AB]jk}\Omega^{[AB]}{}_{il} + \alpha'\sqrt{-g}\frac{\delta\left(g^{ij}g^{kl}\right)}{\delta g^{\mu\nu}}\Omega_{[AB]jk}\Omega^{[AB]}{}_{il}\Bigg\} = \\
= &\int_{\mathbb{R}^3} d^3x \Bigg\{ \Bigg[\frac{\left(-4\alpha'\sqrt{-g}\right)^{-2}(-4\alpha')}{-2}\frac{g_{lk}}{g^{00}}\Pi_{[CD]}{}^l\Pi^{[CD]k} - 2\alpha'\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\left(g^{s0}g^{tl}\Omega^{[AB]}{}_{st}\right)\frac{g_{lj}}{g^{00}} + \\
&+ \alpha' g^{ij}g^{kl}\Omega_{[AB]jk}\Omega^{[AB]}{}_{il}\Bigg]\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \\
&+ \left[\frac{\left(-4\alpha'\sqrt{-g}\right)^{-1}}{2}\Pi_{[CD]}{}^l\Pi^{[CD]j} - \Pi_{[CD]}{}^l\tilde{P}^{[CD]j} - 2\alpha'\sqrt{-g}\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\left(g^{s0}g^{tl}\Omega^{[AB]}{}_{st}\right)\right]\frac{\delta\frac{g_{lj}}{g^{00}}}{\delta g^{\mu\nu}} + \\
&- \Pi_{[CD]}{}^l\frac{g_{lk}}{g^{00}}\frac{\delta\tilde{P}^{[CD]k}}{\delta g^{\mu\nu}} + \left(-4\alpha'\sqrt{-g}\right)\frac{\delta\left(g^{k0}g^{ij}\right)}{\delta g^{\mu\nu}}\Omega_{[AB]ki}\left(g^{s0}g^{tl}\Omega^{[AB]}{}_{st}\right)\frac{g_{lj}}{g^{00}} + \\
&+ \alpha'\sqrt{-g}\frac{\delta\left(g^{ij}g^{kl}\right)}{\delta g^{\mu\nu}}\Omega_{[AB]jk}\Omega^{[AB]}{}_{il}\Bigg\}.
\end{aligned} \tag{4.38}$$

We will now provide some useful intermediate formulas. Introduce the notation $D^{\rho\sigma}_{\mu\nu} \equiv \delta^\rho{}_{(\mu}\delta^\sigma{}_{\nu)}$ so that we can write:

$$\frac{\delta g^{\alpha\beta}(x)}{\delta g^{\mu\nu}(y)} = D^{\alpha\beta}_{\mu\nu}\delta(x-y). \tag{4.39}$$

We omit the delta Dirac for convenience.

$$\begin{aligned}
\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} &= -\frac{1}{2}g_{\mu\nu}\sqrt{-g},\\
\frac{\delta M^{il}}{\delta g^{\mu\nu}} &= g^{il}D^{00}_{\ \ \mu\nu} + g^{00}D^{il}_{\ \ \mu\nu} - g^{l0}D^{i0}_{\ \ \mu\nu} - g^{i0}D^{l0}_{\ \ \mu\nu},\\
\frac{\delta\left(g^{\alpha\beta}g^{\rho\sigma}\right)}{\delta g^{\mu\nu}} &= D^{\alpha\beta}_{\ \ \mu\nu}g^{\rho\sigma} + D^{\rho\sigma}_{\ \ \mu\nu}g^{\alpha\beta},\\
\frac{\delta\frac{g_{lk}}{g^{00}}}{\delta g^{\mu\nu}} &= -\frac{g_{l(\mu}g_{\nu)k}}{g^{00}} - \frac{g_{lk}}{\left(g^{00}\right)^2}D^{00}_{\ \ \mu\nu}.
\end{aligned} \tag{4.40}$$

Plugging this formulas into Eq.(4.38) one gets the variation of H_T with respect to the metric tensor in a completely generic pseudo-orthogonal Yang-Mills theory (not even geometric). However, let us first contract Eq.(4.40) with the variations of the metric tensor with respect to the tetrad which is:

$$\frac{\delta g^{\mu\nu}}{\delta e^a_{\ \gamma}} = -g^{\mu\gamma}e_a^{\ \nu} - g^{\nu\gamma}e_a^{\ \mu}. \tag{4.41}$$

In this way the computation will be a little bit shorter. We get:

$$\begin{aligned}
\frac{\delta\sqrt{-g}}{\delta e_a^{\ 0}} =& \frac{1}{2}\sqrt{-g}g_{\mu\nu}\left(2g^{\mu 0}e_a^{\ \nu}\right) = \sqrt{-g}e_a^{\ 0},\\
\frac{\delta M^{il}}{\delta e_a^{\ 0}} =& -\left(g^{il}D^{00}_{\ \ \mu\nu} + g^{00}D^{il}_{\ \ \mu\nu} - g^{l0}D^{i0}_{\ \ \mu\nu} - g^{i0}D^{l0}_{\ \ \mu\nu}\right)\left(g^{\mu 0}e_a^{\ \nu} + g^{\nu 0}e_a^{\ \mu}\right) =\\
=& -\left[g^{il}\left(2g^{00}e_a^{\ 0}\right) + g^{00}\left(g^{i0}e_a^{\ l} + g^{l0}e_a^{\ i}\right) - g^{l0}\left(g^{i0}e_a^{\ 0} + g^{00}e_a^{\ i}\right) - g^{i0}\left(g^{l0}e_a^{\ 0} + g^{00}e_a^{\ l}\right)\right] =\\
=& -2\left[g^{00}g^{il} - g^{l0}g^{i0}\right]e_a^{\ 0} - \left(g^{00}g^{i0} - g^{i0}g^{00}\right)e_a^{\ l} + \left(g^{l0}g^{00} - g^{l0}g^{00}\right)e_a^{\ i} =\\
=& -2M^{il}e_a^{\ 0},\\
\frac{\delta\left(g^{\alpha\beta}g^{\rho\sigma}\right)}{\delta e_a^{\ 0}} =& -\left[\left(g^{\alpha 0}e_a^{\ \beta} + g^{\beta 0}e_a^{\ \alpha}\right)g^{\rho\sigma} + \left(g^{\rho 0}e_a^{\ \sigma} + g^{\sigma 0}e_a^{\ \rho}\right)g^{\alpha\beta}\right],\\
\frac{\delta\frac{g_{lk}}{g^{00}}}{\delta e_a^{\ 0}} =& +2g^{00}e_a^{\ 0}\frac{g_{lk}}{\left(g^{00}\right)^2} + \frac{1}{2}\left(g_{l\mu}g_{\nu k} + g_{l\nu}g_{\mu k}\right)\left(g^{\mu 0}e_a^{\ \nu} + g^{\nu 0}e_a^{\ \mu}\right) = 2\frac{g_{lk}}{g^{00}}e_a^{\ 0}.
\end{aligned} \tag{4.42}$$

Comparing this formula with Eq.(4.37) we get:

$$\begin{aligned}
\frac{\delta\tilde{P}^{\ \ l}_{[CD]}}{\delta g^{\mu\nu}}\frac{\delta g^{\mu\nu}}{\delta e_a^{\ 0}} =& \frac{\delta M^{il}}{\delta e_a^{\ 0}}\left(-\partial_i\omega_{[CD]0} + \omega_{[CA]0}\omega^{[A}_{\ \ D]i} - \omega_{[CA]i}\omega^{[A}_{\ \ D]0}\right) + \frac{\delta g^{k0}g^{il}}{\delta e_a^{\ 0}}\Omega_{[CD]ki} =\\
=& \left(-\partial_i\omega^{[CD]}_{\ \ \ 0} + \omega_{[CA]0}\omega^{[A}_{\ \ D]i} - \omega_{[CA]i}\omega^{[A}_{\ \ D]0}\right)\left(-2M^{il}e_a^{\ 0}\right) +\\
& -\Omega_{[CD]ki}\left(g^{k0}g^{il}e_a^{\ 0} + g^{00}g^{il}e_a^{\ k} + g^{k0}g^{i0}e_a^{\ l} + g^{k0}g^{l0}e_a^{\ i}\right)\\
=& \left(-\partial_i\omega^{[CD]}_{\ \ \ 0} + \omega_{[CA]0}\omega^{[A}_{\ \ D]i} - \omega_{[CA]i}\omega^{[A}_{\ \ D]0}\right)\left(-2M^{il}e_a^{\ 0}\right) +\\
& -\Omega_{[CD]ki}\left(g^{k0}g^{il}e_a^{\ 0} + g^{00}g^{il}e_a^{\ k} + g^{k0}g^{l0}e_a^{\ i}\right),
\end{aligned} \tag{4.43}$$

where we used the antysymmetry of Ω in the spacetime indeces. Before contracting with $\frac{g_{lm}}{g^{00}}$ we will show these useful identities (compare with Eq.(4.16)):

$$\begin{aligned} g_{lm}g^{k0}g^{il} &= \delta^i{}_m g^{k0} - g^{i0}g_{0m}g^{k0}, \\ g^{00}g^{il}g_{lm} &= g^{00}\delta^i{}_m - g^{00}g^{i0}g_{0m}, \\ g^{k0}g_{lm}g^{l0} &= -g^{k0}g^{00}g_{0m}. \end{aligned} \tag{4.44}$$

We find:

$$\begin{aligned} \frac{g_{lm}}{g^{00}}\frac{\delta\tilde{P}^{\ l}_{[CD]}}{\delta g^{\mu\nu}}\frac{\delta g^{\mu\nu}}{\delta e_a{}^0} =& 2\delta^i{}_m\left(\partial_i\omega^{[CD]}_{\ \ 0} - \omega_{[CA]0}\omega^{[A}_{\ \ D]i} + \omega_{[CA]i}\omega^{[A}_{\ \ D]0}\right)e_a{}^0 + \\ &+ \frac{\Omega^{[CD]}_{\ \ ki}}{-g^{00}}\left[\left(\delta^i{}_m g^{k0} - g^{i0}g_{0m}g^{k0}\right)e_a{}^0 + g^{00}\left(\delta^i{}_m - g^{i0}g_{0m}\right)e_a{}^k - g^{00}g^{k0}g_{0m}e_a{}^i\right] = \\ =& 2\delta^i{}_m\left(\partial_i\omega^{[CD]}_{\ \ 0} - \omega_{[CA]0}\omega^{[A}_{\ \ D]i} + \omega_{[CA]i}\omega^{[A}_{\ \ D]0}\right)e_a{}^0 - \frac{\Omega^{[CD]}_{\ \ ki}}{g^{00}}\left[\left(g^{k0}e_a{}^0 + g^{00}e_a{}^k\right)\delta^i{}_m\right]; \end{aligned} \tag{4.45}$$

$$\Pi^{\ m}_{[CD]}\frac{g_{lm}}{g^{00}}\frac{\delta\tilde{P}^{\ l}_{[CD]}}{\delta g^{\mu\nu}}\frac{\delta g^{\mu\nu}}{\delta e_a{}^0} = 2\Pi^{\ i}_{[CD]}\left(\partial_i\omega^{[CD]}_{\ \ 0} - \omega_{[CA]0}\omega^{[A}_{\ \ D]i} + \omega_{[CA]i}\omega^{[A}_{\ \ D]}\right)e_a{}^0 - \frac{\Omega^{[CD]}_{\ \ ki}\Pi^{\ i}_{[CD]}}{g^{00}}\left(g^{k0}e_a{}^0 + g^{00}e_a{}^k\right). \tag{4.46}$$

Now we can plug all the intermediate steps into Eq.(4.38) to get:

$$
\begin{aligned}
\frac{\delta H_T}{\delta e^a{}_0} =& \Bigg\{ \Bigg[\frac{\left(-4\alpha'\sqrt{-g}\right)^{-2}\left(-4\alpha'\sqrt{-g}\right)}{-2} \frac{g_{lk}}{g^{00}} \Pi^{\ \ l}_{[CD]} \Pi^{[CD]k} - 2\alpha'\sqrt{-g}\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\right)\frac{g_{lj}}{g^{00}} + \\
&+ \alpha'\sqrt{-g} g^{ij}g^{kl}\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ il} \Bigg] e_a{}^0 + \\
&+ \Bigg[\frac{\left(-4\alpha'\sqrt{-g}\right)^{-1}}{2} \Pi^{\ \ l}_{[CD]}\Pi^{[CD]j} - \Pi^{\ \ l}_{[CD]}\tilde{P}^{[CD]j} - 2\alpha'\sqrt{-g}\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\right)\Bigg]\left(2\frac{g_{lj}}{g^{00}}e_a{}^0\right) + \\
&- D_i\Pi^{\ \ i}_{[\cdot a]} - 2\Pi^{\ \ i}_{[CD]}\left(\partial_i\omega^{[CD]}_{\ \ \ \ 0} - \omega^{[C}_{\ \ \ A]0}\omega^{[A}_{\ \ \ D]i} + \omega^{[C}_{\ \ \ A]i}\omega^{[A}_{\ \ \ D]0}\right)e_a{}^0 + \frac{\Omega^{[CD]}_{\ \ \ \ ki}\Pi^{\ \ i}_{[CD]}}{g^{00}}\left(g^{k0}e_a{}^0 + g^{00}e_a{}^k\right) + \\
&- \left(-4\alpha'\sqrt{-g}\right)\Omega_{[AB]ki}\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\right)\left(g^{k0}g^{ij}e_a{}^0 + g^{00}g^{ij}e_a{}^k + g^{k0}g^{i0}e_a{}^j + g^{k0}g^{j0}e_a{}^i\right)\frac{g_{lj}}{g^{00}} + \\
&- \alpha'\sqrt{-g}\left[\left(g^{i0}e_a{}^j + g^{j0}e_a{}^i\right)g^{kl} + \left(g^{k0}e_a{}^l + g^{l0}e_a{}^k\right)g^{ij}\right]\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ il}\Bigg\} = \\
=& \Bigg\{ \Bigg[-\frac{\left(-4\alpha'\sqrt{-g}\right)^{-1}}{2}\Pi^{\ \ l}_{[CD]}\Pi^{[CD]k}\frac{g_{lk}}{g^{00}} - 2\alpha'\sqrt{-g}\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\right)\frac{g_{lj}}{g^{00}} + \\
&+ \alpha'\sqrt{-g}g^{ij}g^{kl}\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ il} + 2\frac{g_{lj}}{g^{00}}\Bigg(\frac{\left(-4\alpha'\sqrt{-g}\right)^{-1}}{2}\Pi^{\ \ l}_{[CD]}\Pi^{[CD]j} - \Pi^{\ \ l}_{[CD]}\tilde{P}^{[CD]j} + \\
&- 2\alpha'\sqrt{-g}\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\right)\Bigg) - 2\Pi^{\ \ i}_{[CD]}\left(\partial_i\omega^{[CD]}_{\ \ \ \ 0} - \omega^{[C}_{\ \ \ A]0}\omega^{[AD]}_{\ \ \ \ i} + \omega^{[C}_{\ \ \ A]i}\omega^{[AD]}_{\ \ \ \ 0}\right) + \\
&+ \Omega^{[CD]}_{\ \ \ \ ki}\Pi^{\ \ i}_{[CD]}\frac{g^{k0}}{g^{00}} - \left(-4\alpha'\sqrt{-g}\right)g^{k0}\Omega_{[AB]kl}\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\right)\Bigg]e_a{}^0 + \\
&- D_i\Pi^{\ \ i}_{[\cdot a]} + \Omega^{[CD]}_{\ \ \ \ ki}\Pi^{\ \ i}_{[CD]}e_a{}^k - \frac{\left(-4\alpha'\sqrt{-g}\right)}{g^{00}}\Omega_{[AB]ki}\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\right)\left[e_a{}^k g^{00}\left(\delta^i{}_l - g^{i0}g_{0l}\right) - g^{k0}g^{00}g_{0l}e_a{}^i\right] + \\
&- \alpha'\sqrt{-g}\left[\left(g^{i0}e_a{}^j + g^{j0}e_a{}^i\right)g^{kl} + \left(g^{k0}e_a{}^l + g^{l0}e_a{}^k\right)g^{ij}\right]\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ il}\Bigg\} = \\
=& \Bigg\{ - D_i\Pi^{\ \ i}_{[\cdot a]} + \left[\Omega^{[CD]}_{\ \ \ \ ki}\Pi^{\ \ i}_{[CD]} - \left(-4\alpha'\sqrt{-g}\right)\Omega_{[AB]kl}\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\right)\right]e_a{}^k + \\
&- \alpha'\sqrt{-g}\left[\left(g^{i0}e_a{}^j + g^{j0}e_a{}^i\right)g^{kl} + \left(g^{k0}e_a{}^l + g^{l0}e_a{}^k\right)g^{ij}\right]\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ il} + \\
&+ e_a{}^0\Bigg[\frac{\left(-4\alpha'\sqrt{-g}\right)^{-1}}{2}\Pi^{\ \ l}_{[CD]}\Pi^{[CD]j}\frac{g_{lj}}{g^{00}} - 6\alpha'\sqrt{-g}\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\right)\frac{g_{lj}}{g^{00}} + \\
&+ \alpha'\sqrt{-g}g^{ij}g^{kl}\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ il} - \Omega^{[CD]}_{\ \ \ \ ki}\Pi^{\ \ i}_{[CD]}\frac{g^{k0}}{g^{00}} - \left(-4\alpha'\sqrt{-g}\right)g^{k0}\Omega_{[AB]kl}\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\right)\Bigg]\Bigg\} = \\
=& \Bigg\{ - D_i\Pi^{\ \ i}_{(\cdot a)} + \left[\Omega^{(CD)}_{\ \ \ \ ki}\Pi^{\ \ i}_{(CD)} - \left(-4\alpha'\sqrt{-g}\right)\Omega_{(AB)kl}\left(g^{s0}g^{tl}\Omega^{(AB)}_{\ \ st}\right)\right]e_a{}^k + \\
&- \alpha'\sqrt{-g}\left[\left(g^{i0}e_a{}^j + g^{j0}e_a{}^i\right)g^{kl} + \left(g^{k0}e_a{}^l + g^{l0}e_a{}^k\right)g^{ij}\right]\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ il} + \\
&+ e_a{}^0\Bigg[\frac{\left(-4\alpha'\sqrt{-g}\right)^{-1}}{2}\Pi^{\ \ l}_{[CD]}\Pi^{[CD]j}\frac{g_{lj}}{g^{00}} - 2\alpha'\sqrt{-g}\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\left(g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\right)\frac{g_{lj}}{g^{00}} + \\
&+ \alpha'\sqrt{-g}g^{ij}g^{kl}\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ il} - \Omega^{[CD]}_{\ \ \ \ ki}\Pi^{\ \ i}_{[CD]}\frac{g^{k0}}{g^{00}}\Bigg]\Bigg\} \approx 0.
\end{aligned}
\tag{4.47}
$$

These are the secondary constraints associated to the tetrad fields. The generalized electric field for the tetrad obeys a *very generalized Gauss law* and it is not source-free in the vacuum. In deriving the previous equation we used the following result:

$$\Pi^{\ m}_{[CD]}\frac{g_{lk}}{g^{00}}\tilde{P}^{[CD]l} = \Pi^{\ i}_{[CD]}\left(-\partial_i\omega^{[CD]}_{\ \ \ 0} + \omega^{[C}_{\ \ A]0}\omega^{[AD]}_{\ \ \ i} - \omega^{[C}_{\ \ A]i}\omega^{[AD]}_{\ \ \ 0}\right) + \frac{g^{k0}\left(\delta^i_{\ m} - g^{i0}g_{0m}\right)}{g^{00}}\Pi^{\ m}_{[CD]}\omega^{[CD]}_{\ \ \ ki} =$$
$$= \Pi^{\ i}_{[CD]}\left(-\partial_i\omega^{[CD]}_{\ \ \ 0} + \omega^{[C}_{\ \ A]0}\omega^{[AD]}_{\ \ \ i} - \omega^{[C}_{\ \ A]i}\omega^{[AD]}_{\ \ \ 0}\right) + \frac{g^{k0}}{g^{00}}\Pi^{\ i}_{[CD]}\Omega^{[CD]}_{\ \ \ ki}. \tag{4.48}$$

4.2.3 Analysis of secondary constraints

The goal of this subsection is to study the secondary constraints of the theory, establish their class and see if they bring new constraints or not. We start with non-tetrad fields. Notice that we can rewrite the secondary constraints in Eq.(4.34) as:

$$\phi^{(2)}_{[AB]} = \partial_i\Pi^{\ i}_{[AB]} + \frac{1}{2}c_{[LM][NQ][AB]}\omega^{[LM]}_{\ \ \ i}\Pi^{[NQ]i}, \tag{4.49}$$

where we used:

$$\frac{1}{2}c_{[CD][EF][AB]}A^{[CD]}B^{[EF]} = A_{[AC]}B^{[C}_{\ \ B]} - A_{[BC]}B^{[C}_{\ \ A]}. \tag{4.50}$$

Before computing the Poisson brackets as in Eq.(4.32), we give two important property concerning the delta Dirac and its derivatives:

$$\left[f(y) - f(x)\right]\partial_x\delta(x-y) = f'(x)\delta(x-y), \tag{4.51}$$
$$\partial_x\delta(x-y) + \partial_y\delta(x-y) = 0, \tag{4.52}$$

which are to be intendend in the distributional sense. We will know prove them against well-behaved test functions of both x and y.

$$\int dy\, g(y)\left[f(y) - f(x)\right]\partial_x\delta(x-y) = \partial_x\left(\int dy\, g(y)\left[f(y) - f(x)\right]\delta(x-y)\right) + \int dy\, g(y)f'(x)\delta(x-y) =$$
$$= \int dy\, g(y)f'(x)\delta(x-y); \tag{4.53}$$

$$\int dx\, g(x)\left[f(y) - f(x)\right]\partial_x\delta(x-y) = \int dx\, \left\{\partial_x\left[g(x)\left(f(y) - f(x)\right)\delta(x-y)\right] - \partial_x\left[g(x)\left(f(y) - f(x)\right)\right]\delta(x-y)\right\} =$$
$$= \int dx f'(x)g(x)\delta(x-y) - \int dx g'(x)\left(f(y) - f(x)\right)\delta(x-y) =$$
$$= \int dx f'(x)g(x)\delta(x-y). \tag{4.54}$$

In this way we proved Eq.(4.51), now we go on with the second property. Notice that since Eq.(4.52) is symmetric in x and y, we just need to prove it once.

$$\int dx\, f(x)\left[\partial_x\delta(x-y) + \partial_y\delta(x-y)\right] = -\int dx\, f'(x)\delta(x-y) + \partial_y\int dx\, f(x)\delta(x-y) =$$
$$= -f'(y) + f'(y) = 0, \tag{4.55}$$

so that we concluded our proof.

We will now establish the class of these constraints. In order to do so we need to compute the Poisson brackets between the constraints. It is evident that Eq.(4.49) doesn't depend on $\omega^{[AB]}_{\ \ 0}$. This implies that the Poisson brackets between primary and secondary constraints vanish. We will now compute the Poisson brackets between secondary constraints:

$$
\begin{aligned}
\left\{\phi^{(2)}_{[AB]}(x), \phi^{(2)}_{[CD]}(y)\right\} =& \left\{\partial^x_i \Pi_{[AB]}^{\ \ i}(x), \partial^y_j \Pi_{[CD]}^{\ \ j}(y)\right\} + \frac{1}{2} c_{[LM][NQ][CD]} \left\{\partial^x_i \Pi_{[AB]}^{\ \ i}(x), \omega^{[LM]}_{\ \ j}(y) \Pi^{[NQ]j}(y)\right\} + \\
&+ \frac{1}{2} c_{[LM][NQ][AB]} \left\{\omega^{[LM]}_{\ \ i}(x) \Pi^{[NQ]i}(x), \partial^y_j \Pi_{[CD]}^{\ \ j}(y)\right\} + \\
&+ \frac{1}{4} c_{[LM][NQ][AB]} c_{[PR][ST][CD]} \left\{\omega^{[LM]}_{\ \ i}(x) \Pi^{[NQ]i}(x), \omega^{[PR]}_{\ \ j}(y) \Pi^{[ST]j}(y)\right\} = \\
=& - \frac{1}{2} c_{[AB][NQ][CD]} \partial^x_i \left(\delta(x-y)\right) \Pi^{[NQ]i}(y) + \frac{1}{2} c_{[CD][NQ][AB]} \partial^y_i \left(\delta(x-y)\right) \Pi^{[NQ]i}(x) + \\
&+ \frac{1}{4} \left(-c_{[LM][PR][AB]} c^{[PR]}_{\ \ [ST][CD]} + c^{[PR]}_{\ \ [ST][AB]} c_{[LM][PR][CD]}\right) \omega^{[LM]}_{\ \ i}(x) \Pi^{[NQ]i}(x) \delta(x-y) = \\
=& \frac{1}{2} c_{[AB][CD]}^{\ \ \ \ \ \ \ [NQ]} \left\{ \left[\Pi_{[NQ]}^{\ \ i}(y) - \Pi_{[NQ]}^{\ \ i}(x)\right] \partial^x_i \left(\delta(x-y)\right) + \right. \\
&\left. + \frac{1}{2} c_{[LM][ST][NQ]} \omega^{[LM]}_{\ \ i}(x) \Pi^{[ST]i}(x) \delta(x-y)\right\} = \\
=& \frac{1}{2} c_{[AB][CD]}^{\ \ \ \ \ \ \ [NQ]} \left[\partial^x_i \Pi_{[NQ]}^{\ \ i}(x) + \frac{1}{2} c_{[LM][ST][NQ]} \omega^{[LM]}_{\ \ i}(x) \Pi^{[ST]i}(x)\right] \delta(x-y) = \\
=& \frac{1}{2} c_{[AB][CD]}^{\ \ \ \ \ \ \ [NQ]} \phi^{(2)}_{[NQ]}(x)\Big|_{[NQ]\ \text{non-tetrad}} \delta(x-y) + \frac{1}{2} c_{[AB][CD]}^{\ \ \ \ \ \ \ [\cdot q]} D^x_i \Pi_{[\cdot q]}^{\ \ i}(x) \delta(x-y) \approx \\
\approx& \frac{1}{2} c_{[AB][CD]}^{\ \ \ \ \ \ \ [\cdot q]} D^x_i \Pi_{[\cdot q]}^{\ \ i}(x) \delta(x-y),
\end{aligned}
\tag{4.56}
$$

where we used the total antisymmetry of the structure constants, the Jacobi identity and Eqs(4.51)-(4.52).

It follows that the secondary constraints for non-tetrad fields are second-class in general. In particular, we find that Eq.(4.56) doesn't vanish in the following cases:

- if $[AB] = [-1, b]$ and $[CD] = [-1, 4]$;
- if $[AB] = [ab]$ and $[CD] = [\cdot d]$.

Before computing the self-consistency condition for tetrad fields, we provide the following notation:

$$
\phi^{(2)\ \text{Gauss}}_{[\cdot a]} \equiv D_i \Pi_{[\cdot a]}^{\ \ i},
\tag{4.57}
$$

$$
\begin{aligned}
\phi^{(2)\ \text{source}}_{[\cdot a]} \equiv& \left[\Omega^{(CD)}_{\ \ ki} \Pi_{(CD)}^{\ \ i} - \left(-4\alpha' \sqrt{-g}\right) \Omega_{(AB)kl} \left(g^{s0} g^{tl} \Omega^{(AB)}_{\ \ st}\right)\right] e_a^{\ k} + \\
&- \alpha' \sqrt{-g} \left[\left(g^{i0} e_a^{\ j} + g^{j0} e_a^{\ i}\right) g^{kl} + \left(g^{k0} e_a^{\ l} + g^{l0} e_a^{\ k}\right) g^{ij}\right] \Omega_{(AB)jk} \Omega^{(AB)}_{\ \ il} + \\
&+ e_a^{\ 0} \left[\frac{\left(-4\alpha' \sqrt{-g}\right)^{-1}}{2} \Pi_{(CD)}^{\ \ l} \Pi^{(CD)j} \frac{g_{lj}}{g^{00}} - 2\alpha' \sqrt{-g} \left(g^{k0} g^{ij} \Omega_{(AB)ki}\right) \left(g^{s0} g^{tl} \Omega^{(AB)}_{\ \ st}\right) \frac{g_{lj}}{g^{00}} + \right. \\
&\left. + \alpha' \sqrt{-g} g^{ij} g^{kl} \Omega_{(AB)jk} \Omega^{(AB)}_{\ \ il} - \Omega^{(CD)}_{\ \ ki} \Pi_{(CD)}^{\ \ i} \frac{g^{k0}}{g^{00}}\right].
\end{aligned}
\tag{4.58}
$$

The self-consistency conditions for non-tetrad secondary constraints are:

$$\begin{aligned}\left\{\phi^{(2)}_{[PR]}, H_T\right\} =&\partial_i\left\{\Pi_{[PR]}^{\;\;i}, H\right\} + \frac{1}{2}c_{[LM][NQ][PR]}\left[\left\{\Pi^{[NQ]i}, H\right\}\omega^{[LM]}_{\;\;i} + \Pi^{[NQ]i}\left\{\omega^{[LM]}_{\;\;i}, H\right\}\right] + \\ &+ \int_{\mathbb{R}^3} d^3x \left[u^{(2)[CD]}\left\{\phi^{(2)}_{[PR]}, \phi^{(2)}_{[CD]}\right\} + \left\{\phi^{(2)}_{[PR]}, u^{(2)[CD]}\right\}\phi^{(2)}_{[CD]} + \left\{\phi^{(2)}_{[PR]}, u^{(1)[CD]}\phi^{(1)}_{[CD]}\right\}\right] \approx \\ \approx &- \partial_i\frac{\delta H}{\delta\omega^{[PR]}_{\;\;i}} + \frac{1}{2}c_{[LM][NQ][PR]}\left[-\frac{\delta H}{\delta\omega_{[NQ]i}}\bigg|_{g^{\mu\nu}\text{ fixed}}\omega^{[LM]}_{\;\;i} + \Pi^{[NQ]i}\frac{\delta H}{\delta\Pi_{[LM]}^{\;\;i}}\right] + \\ &- \frac{1}{2}c_{[LM][\cdot q][PR]}\frac{\delta H}{\delta g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial\omega_{[\cdot q]i}}\omega^{[LM]}_{\;\;i} + u^{(2)[CD]}\frac{1}{2}c_{[PR][CD]}^{\;\;[\cdot q]}D_i\Pi_{[\cdot q]}^{\;\;i} + \\ &+ \int_{\mathbb{R}^3} d^3x\, u^{(2)[\cdot d]}\left\{\phi^{(2)}_{[PR]}, \phi^{(2)source}_{[\cdot d]}\right\} \approx 0.\end{aligned} \tag{4.59}$$

We will carry on the computation in several steps for the sake of clarity. We find (compare with Eq.(B.9)):

$$\begin{aligned}\partial_a\frac{\delta H}{\delta\omega^{[PR]}_{\;\;a}} =&\partial_a\partial_m\left[\frac{g^{m0}}{g^{00}}\left(\Pi_{[PR]}^{\;\;a} + 4\alpha'\sqrt{-g}g^{k0}g^{pa}\Omega_{[PR]kp}\right)\right] - \partial_a\partial_i\left[\frac{g^{a0}}{g^{00}}\left(\Pi_{[PR]}^{\;\;i} + 4\alpha'\sqrt{-g}g^{k0}g^{pi}\Omega_{[PR]kp}\right)\right] + \\ &- \partial_a\partial_i\left(4\alpha'\sqrt{-g}g^{aj}g^{ki}\Omega_{[PR]jk}\right) + \\ &+ \frac{1}{2}c_{[LM][NQ][PR]}\bigg\{ - \partial_a\left(\Pi^{[LM]a}\omega^{[NQ]}_{\;\;0}\right) + \partial_a\left[\frac{g^{a0}}{g^{00}}\left(\Pi^{[LM]i}\omega^{[NQ]}_{\;\;i} + 4\alpha'\sqrt{-g}g^{k0}g^{pi}\Omega^{[LM]}_{\;\;kp}\omega^{[NQ]}_{\;\;i}\right)\right] + \\ &- \partial_a\left[\frac{g^{m0}}{g^{00}}\left(\Pi^{[LM]a}\omega^{[NQ]}_{\;\;m} + 4\alpha'\sqrt{-g}g^{k0}g^{pa}\Omega^{[LM]}_{\;\;kp}\omega^{[NQ]}_{\;\;m}\right)\right] - \partial_a\left(4\alpha'\sqrt{-g}g^{aj}g^{kl}\Omega^{[LM]}_{\;\;jk}\omega^{[NQ]}_{\;\;l}\right)\bigg\} = \\ =&\frac{1}{2}c_{[LM][NQ][PR]}\bigg\{ - \partial_a\left(\Pi^{[LM]a}\omega^{[NQ]}_{\;\;0}\right) + \partial_a\left[\frac{g^{a0}}{g^{00}}\left(\Pi^{[LM]i}\omega^{[NQ]}_{\;\;i} + 4\alpha'\sqrt{-g}g^{k0}g^{pi}\Omega^{[LM]}_{\;\;kp}\omega^{[NQ]}_{\;\;i}\right)\right] + \\ &- \partial_a\left[\frac{g^{m0}}{g^{00}}\left(\Pi^{[LM]a}\omega^{[NQ]}_{\;\;m} + 4\alpha'\sqrt{-g}g^{k0}g^{pa}\Omega^{[LM]}_{\;\;kp}\omega^{[NQ]}_{\;\;m}\right)\right] - \partial_a\left(4\alpha'\sqrt{-g}g^{aj}g^{kl}\Omega^{[LM]}_{\;\;jk}\omega^{[NQ]}_{\;\;l}\right)\bigg\};\end{aligned} \tag{4.60}$$

$$\begin{aligned}\frac{1}{2}c_{[LM][NQ][PR]}\frac{\delta H}{\delta\omega_{[NQ]i}}\bigg|_{g^{\mu\nu}\text{ fixed}}\omega^{[LM]}_{\;\;i} =&\frac{1}{2}c_{[LM][NQ][PR]}\omega^{[LM]}_{\;\;i}\bigg\{\partial_m\left[\frac{g^{m0}}{g^{00}}\left(\Pi^{[NQ]i} + 4\alpha'\sqrt{-g}g^{k0}g^{pi}\omega^{[NQ]}_{\;\;kp}\right)\right] + \\ &- \partial_m\left[\frac{g^{i0}}{g^{00}}\left(\Pi^{[NQ]m} + 4\alpha'\sqrt{-g}g^{k0}g^{pm}\Omega^{[NQ]}_{\;\;kp}\right)\right] + \partial_l\left(4\alpha'\sqrt{-g}g^{ij}g^{kl}\omega^{[NQ]}_{\;\;jk}\right)\bigg\} + \\ &+ \frac{1}{4}c_{[EF][ST][PR]}c_{[LM][NQ]}^{\;\;[ST]}\omega^{[EF]}_{\;\;i}\bigg\{ - \Pi^{[LM]i}\omega^{[NQ]}_{\;\;0} + \\ &+ \frac{g^{i0}}{g^{00}}\left(\Pi^{[LM]l}\omega^{[NQ]}_{\;\;l} + 4\alpha'\sqrt{-g}g^{k0}g^{pl}\Omega^{[LM]}_{\;\;kp}\omega^{[NQ]}_{\;\;l}\right) + \\ &- \frac{g^{m0}}{g^{00}}\left(\Pi^{[LM]i}\omega^{[NQ]}_{\;\;m} + 4\alpha'\sqrt{-g}g^{k0}g^{pi}\Omega^{[LM]}_{\;\;kp}\omega_{[NQ]}^{\;\;m}\right) + \\ &- 4\alpha'\sqrt{-g}g^{ij}g^{kl}\Omega^{[LM]}_{\;\;jk}\omega^{[NQ]}_{\;\;l}\bigg\};\end{aligned} \tag{4.61}$$

$$
\begin{aligned}
\frac{1}{2}c_{[LM][NQ][PR]}\Pi^{[NQ]i}\frac{\delta H}{\delta \Pi^{\ i}_{[LM]}} =& \frac{1}{2}c_{[LM][NQ][PR]}\Pi^{NQi}\partial_i\omega^{[LM]}_{\ 0} + \frac{1}{2}c_{EF[ST][PR]}\Pi^{[ST]i}\Big\{(-8\alpha'\sqrt{-g})^{-1}\frac{g_{ik}}{g^{00}}\Pi^{[EF]k}+\\
&- \frac{1}{2}c_{[LM][NQ]}^{\qquad\ \ [EF]}\omega^{[LM]}_{\ 0}\omega^{[NQ]}_{\ i} - \frac{g^{m0}}{g^{00}}\Omega^{[EF]}_{\ mi}\Big\} =\\
=& \frac{1}{2}c_{[LM][NQ][PR]}\Pi^{NQi}\partial_i\omega^{[LM]}_{\ 0} + \frac{1}{2}c_{EF[ST][PR]}\Pi^{[ST]i}\Big\{ - \frac{1}{2}c_{[LM][NQ]}^{\qquad\ \ [EF]}\omega^{[LM]}_{\ 0}\omega^{[NQ]}_{\ i}+\\
&- \frac{g^{m0}}{g^{00}}\Omega^{[EF]}_{\ mi}\Big\}.
\end{aligned}
\tag{4.62}
$$

Putting all together we finally find:

$$
\begin{aligned}
\Big\{\phi^{(2)}_{[PR]}, H_T\Big\} \approx& \frac{1}{2}c_{[LM][NQ][PR]}\Big\{ \left(\partial_m\omega^{[LM]}_{\ i} - \partial_i\omega^{[LM]}_{\ m}\right)\left[\frac{g^{m0}}{g^{00}}\left(\Pi^{[NQ]i} + 4\alpha'\sqrt{-g}g^{k0}g^{pi}\Omega^{[NQ]}_{\ kp}\right)\right] - \left(\partial_i\Pi^{[NQ]i}\right)\omega^{\ 0}_{[LM]}+\\
&+ \left(4\alpha'\sqrt{-g}g^{aj}g^{kl}\Omega^{[LM]}_{\ jk}\right)\partial_a\omega^{[NQ]}_{\ l}\Big\}+\\
&+ \frac{1}{4}\omega^{[EF]}_{\ i}\Pi^{[LM]i}\omega^{[NQ]}_{\ 0}\left[c_{[EF][ST][PR]}c_{[LM][NQ]}^{\qquad\ \ [ST]} - c_{[ST][LM][PR]}c_{[NQ][EF]}^{\qquad\ \ [ST]}\right]+\\
&+ \frac{1}{8}\left(c_{[EF][ST][PR]}c_{[LM][AB]}^{\qquad\ \ [NQ]} - c_{[AB][NQ][PR]}c_{[LM][EF]}^{\qquad\ \ [NQ]}\right)(4\alpha'\sqrt{-g}g^{ij}g^{kl}\Omega^{[LM]}_{\ jk}\omega^{[NQ]}_{\ l})+\\
&+ \frac{1}{4}\left(c_{[EF][NQ][PR]}c_{[LM][AB]}^{\qquad\ \ [NQ]} - c_{[AB][NQ][PR]}c_{[LM][EF]}^{\qquad\ \ [NQ]}\right)\cdot\\
&\cdot\left[\omega^{[EF]}_{\ i}\omega^{[AB]}_{\ m}\frac{g^{m0}}{g^{00}}\left(\Pi^{[LM]i} + 4\alpha'\sqrt{-g}g^{k0}g^{pi}\Omega^{[LM]}_{\ kp}\right)\right] - \frac{1}{2}c_{[EF][ST][PR]}\frac{g^{m0}}{g^{00}}\Pi^{[ST]i}\Omega^{[EF]}_{\ mi}+\\
&- \frac{1}{2}c_{[LM][\cdot q][PR]}\frac{\delta H}{\delta g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial\omega_{[\cdot q]i}}\omega^{[LM]}_{\ i} + u^{(2)[CD]}\frac{1}{2}c_{[PR][CD]}^{\qquad\ [\cdot q]}D_i\Pi^{\ i}_{[\cdot q]} \approx\\
\approx& - \frac{1}{2}c_{[LM][NQ][PR]}\omega^{[LM]}_{\ 0}D_i\Pi^{[NQ]i} + c_{[LM][NQ][PR]}\alpha'\sqrt{-g}g^{ij}g^{kl}\Omega^{[LM]}_{\ jk}\Omega^{[NQ]}_{\ il}+\\
&+ \frac{1}{2}c_{[LM][NQ][PR]}\Omega^{[LM]}_{\ mi}\frac{g^{m0}}{g^{00}}\left(\Pi^{[NQ]i} + 4\alpha'\sqrt{-g}g^{k0}g^{pi}\Omega^{[NQ]}_{\ kp}\right) - \frac{1}{2}c_{[EF][ST][PR]}\frac{g^{m0}}{g^{00}}\Pi^{[ST]i}\Omega^{[EF]}_{\ mi}+\\
&- \frac{1}{2}c_{[LM][\cdot q][PR]}\frac{\delta H}{\delta g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial\omega_{[\cdot q]i}}\omega^{[LM]}_{\ i} + u^{(2)[CD]}\frac{1}{2}c_{[PR][CD]}^{\qquad\ [\cdot q]}D_i\Pi^{\ i}_{[\cdot q]} \approx\\
\approx& \left(u^{(2)[CD]} - \omega^{[CD]}_{\ 0}\right)\frac{1}{2}c_{[PR][CD]}^{\qquad\ [\cdot q]}D_i\Pi^{\ i}_{[\cdot q]} - \frac{1}{2}c_{[LM][\cdot q][PR]}\frac{\delta H}{\delta g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial\omega_{[\cdot q]i}}\omega^{[LM]}_{\ i}+\\
&+ \int_{\mathbb{R}^3} d^3x\, u^{(2)[\cdot d]}\Big\{\phi^{(2)}_{[PR]}, \phi^{(2)source}_{[\cdot d]}\Big\} \approx 0.
\end{aligned}
\tag{4.63}
$$

There are no new constraints associated with the non-tetrad fields. Indeed, they provide restrictions on the choice of the arbitrary functions $u^{(2)[CD]}$ for $[CD]$ being as mentioned above. Notice that the arbitrary functions of spacetime related to the Lorentzian Gauss Law are still free. It happens because, as one can see from the structure constants of pseudo-orthogonal groups and Eq.(4.63), their self-consistency condition fixes the functions $u^{(2)[\cdot d]}$. This reflects the property of geometric gauge theories which are in general symmetric only with respect to the subgroup of the total gauge group that leaves the metric tensor invariant. In our case it is clear from the gauge transformation of the tetrad-connection fields that this subgroup corresponds to the Lorentz group.

We now proceed with the tetrad secondary constraints. Since $\phi^{(2)\ \mathrm{source}}_{[\cdot a]}$ depends on the metric, it also depends on $\omega^{[\cdot a]}_{\ \ 0} \equiv e^a_{\ 0}$. In particular, we find:

$$\left\{\phi^{(2)}_{[\cdot a]}, \phi^{(1)}_{[\cdot b]}\right\} \not\approx 0, \tag{4.64}$$

which gives for their self-consistency conditions:

$$\int_{\mathbb{R}^3} d^3x \left[u^{(1)[\cdot b]}\left\{\phi^2_{[\cdot a]}, \phi^{(1)}_{[\cdot b]}\right\} + \left\{\phi^2_{[\cdot a]}, u^{(2)[CD]}\phi^{(2)}_{[CD]}\right\}\right] + \left\{\phi^{(2)}_{[\cdot a]}, H\right\} \approx 0. \tag{4.65}$$

It follows that these four equations will fix the four arbitrary functions $u^{(1)[\cdot a]}$.

In conclusion, we have shown that the secondary constraints of the theory don't generate any new constraint on phase space. The self-consistency conditions provide equations for the arbitrary functions u in the Hamiltonian in the following sense:

- Eq.(4.63) for $[-1, 4]$ relates the functions $u^{(2)[-1a]}$ (or $u^{(2)[4a]}$ if $[\cdot a] = [-1a]$);
- Eq.(4.63) for $[-1a]$ (or $[4a]$ if $[\cdot a] = [-1a]$) fixes the function $u^{(2)[-1,4]}$;
- Eq.(4.63) for $[ab]$ fixes the functions $u^{(2)[\cdot a]}$;
- Eq.(4.65) for $[\cdot a]$ fixes the functions $u^{(1)[\cdot b]}$.

The only arbitrary Lagrange multipliers we are left with are the ones corresponding to the Lorentz group. Once again we find that the gauge symmetry of our theory, and in particular of its phase space, is given in general by the Lorentz subgroup. We can now consider the Hamiltonian definition as complete and self-consistent with respect to the action formalism provided in Section 3.1. We have shown that the Hamiltonian formalism breaks coordinate invariance of the theory but not of the dynamics, since it can be proved (by means of an exhausting computation) that the equations of motion obtained by variation of the action in Eq.(3.9) are the same one gets from the simplectic structure in Eq(4.29), in particular they are covariant with respect to change of coordinates. The kinetic energy in Eq.(4.31) shows that there are no Ostrogradski kind of instabilities but the non-compact nature of pseudo-orthogonal groups gives rise to kinetic instabilities. We have established the presence of primary and secondary constraints on phase space which are both first and second-class. In order to solve the constraints[3] it will be necessary the introduction of the Dirac bracket. After the Dirac procedure is complete, one will have defined the physical phase space of the theory on which the Dirac brackets will impose a non-canonical simplectic geometry. The next step would then be to start the process of canonical quantization. The classical kinetic instabilities should turn in unitarity problems of the quantum theory. Perhaps, constraining the Lie algebra norm of the physical momenta to be definite positive throughout evolution could show that at least a portion of phase space is stable.

Notice that most of the reasoning of this chapter could be used, by properly adapting the notation, to describe the Hamiltonian of any Yang-Mills theory in curved spacetime. Indeed, a lot of the formulas we used as intermediate steps don't require the presence of the gauge theoretical metric. It is the introduction of the gauge tetrad fields that complicates the structure of phase space. Even in curved spacetime and with a dynamical metric, one can use our formulas to convince themselves that the primary and secondary constraints would be all first class. In the case of compact Yang-Mills theories (i.e. with compact gauge group G), the Hamiltonian formalism we provided is already complete. This is because the compactness of the gauge group

[3]By *solving* the constraints we mean using the relations $\phi_{[CD]}$ in order to express the dynamics only in terms of physical variables

allows us to define a gauge invariant and positive definite inner product in Lie algebra space. Moreover, since the constraints would all be first class, they can be immediately solved and taking the quotient of what's left of phase space with respect to the gauge transformations one can immediately define the physical phase space of the theory. In particular, one can do it also in the presence of the Hilbert action in the Palatini formalism by adapting the notation we used throughout this chapter. We believe that the reasoning applied throughout this final chapter could be very useful in the general setting of being able to provide a Hamiltonian treatment for generic Yang-Mills theories (as the Standard Model) in dynamical spacetime without using the well-known ADM decomposition.

Conclusion

The main goal of this thesis was to reformulate general relativity in the context of Yang-Mills theories. Indeed, it is well-known that Einstein's theory suffers from singularity problems (see [5]) and it is been proved that the theory is not renormalizable (see [1] for a review). Knowing that gravity can be formulated as an effective field theory[2], we showed that it is possible to formulate GR as a constrained version of some more general theory in its low energy limit.

Throughout this thesis we have shown the intimate relation between pseudo-orthogonal Yang-Mills gauge fields and gravitational theories. In order to introduce a gauge theoretical cotetrad fields, we have developed a new class of theories, *geometrical Yang-Mills theories*. Defining the metric through these particular gauge fields requires the introduction of a mass parameter which will take the role of a gauge theoretical Planck mass. The geometrical action one obtains will be generically invariant only with respect to the Lorentz subgroup. However, we have shown that the invariant subgroup can be extended, for instance for Wheeler-Trujillo gravity[8] this also includes local dilations. The equations of motion one obtains are the standard curved spacetime generalization of the well-known Yang-Mills equations. In the case of the tetrad fields, one sees that these fields are sourced in the vacuum by a energy-momentum tensor related to the field strength of our gauge connection.

The main result of the thesis is the de Sitter gauge theory we developed in Section 3.2. The tosionless low energy limit of the theory coincides with general relativity with the appearance of a positive gauge theoretical cosmological constant. This result allows us to consider Einstein's theory of gravity as part of a more general Lorentz invariant de Sitter Yang-Mills theory. We argue that the theory might be renormalizable since the vertex structure of the theory is essentially the same as ordinary flat-space Yang-Mills theories. However, since we studied arbitrary curved spacetimes, there could be new counterterms associated to the non-trivial geometry of the spacetime as in Subsection 2.2.2. We believe that this counterterms could be added to the theory using our geometrical Yang-Mills formalism, so that they shouldn't provide any Ostrogradsky instability as we have shown in Chapter 4. Using the conformal group as gauge group, we have shown that one can use our formalism to recover Weyl squared gravity, which is needed to provide the necessary counterterms to the curved spacetime generalizations of standard renormalizable flat-space theories (such as quartic scalar field theory). These results inspire us to say that geometrical Yang-Mills theories could be an useful formulation of gravitational theories in general.

We provided the first steps towards canonical quantization ($[,] = i\hbar\{,\}$) building the Hamiltonian and studying the phase space of the theory. Since pseudo-orthogonal groups are non-compact Lie groups, the Killing form we use in Lie algebra space gives rise to an indefinite inner product. This means that some fields will pick the wrong sign in the kinetic energy and they could generate instability problems in the theory. However, as one can see from the kinetic energy of the theory, there is no sign of Ostrogradsky instabilities in the theory at hand, as we originally expected since Yang-Mills theories consider the Riemann squared term in the first order formalism. As it is usually the case with gauge theories, we find both primary and secondary constraints. They are both first and second-class constraints and their self-consistency conditions reduce the gauge redundancy in phase space to the Lorentz subgroup of our original gauge group. Once again we see that the introduction of the gauge cotetrad fields reduces the original gauge symmetry of the theory. There are no new constraints arising in phase space and thus the Hamiltonian we provide is complete.

The natural next step of this research would be to try to solve the kinetic issue we have discussed above. The presence of second-class constraints shows that one need the formalism of Dirac brackets to properly solve the constraints without changing the dynamics. The simplectic geometry one would find at the end of this process will in general be different from the standard one given by canonical Poisson brackets. The hope is that, after one is able to solve the constraints using Dirac's procedure and to properly identify physical phase-space taking the quotient with respect to the residual gauge symmetry, the unstable fields would turn unphysical or harmless at least. The hope is that gauge symmetry will prevent these classical instabilities from developing. Another possible solution would be to start the process of canonical quantization using the Hamiltonian and simplectic structure we provided. On the Hilbert space of states one can then find a way to keep the ghost fields (i.e. the quantum analogous of the classical kinetic instability) away from the dynamics resulting in a unitary S-matrix. If this is done without spoiling the symmetry of the theory, the procedure should provide a self-consistent gauge invariant quantum theory for which the correspondence with general relativity is established at the classical level. In the end, one could also study the effect of the Riemann squared term in de Sitter gauge gravity in situations for which it is natural to expect a curvature of the same order of magnitude of the Planck mass. This could give interesting contributions expecially in regions of spacetime close to GR singularities, perhaps preventing their formation or modifying the structure of event horizons.

In conclusion, we provided a new formalism for which interesting results can be found in the context of gravitational theoires. We have showed another way of deriving general relativity out of the geometrical gauge principle and we have provided a consistent Hamiltonian framework suitable for any Yang-Mills theory in dynamical curved spacetime.

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Appendix A

Pseudo-orthogonal groups and bundles

A.1 Pseudo-orthogonal groups

In this section we are going to define pseudo-orthogonal groups and study their properties. Consider the vector space $\mathbb{R}^N$ equipped with the following metric (in the canonical cartesian basis):

$$\eta = \text{diag}(\underbrace{-1, .., -1}_{S}, \underbrace{+1, ..., +1}_{T}), \tag{A.1}$$

with $S + T = N$. This inner product turns $\mathbb{R}^N$ into the (pseudo-)normed vector space $\mathbb{R}^{S,T}$ (for $S = 1$ and $T = 3$ we get Minkowski space time). We define the (fundamental rep of the) pseudo-orthogonal group $O(S,T)$ as the set of transformations on $\mathbb{R}^{S,T}$ that leave the inner product $\eta(X,Y)$ invariant, $X, Y \in \mathbb{R}^{S,T}$, i.e. $\eta(\Lambda \cdot X, \Lambda \cdot Y) = \eta(X,Y)$, $\Lambda \in O(S,T)$. It can be proved that pseudo-orthogonal groups are Lie groups. It seems then natural to look at transformations infinitesimally close to the identity in order to identify their generators, i.e. ${\Lambda^a}_b = {\delta^a}_b + \alpha {M^a}_b + O(\alpha^2)$. We get:

$$\begin{aligned} \eta(X,Y) = \eta_{AB} X^A Y^B &\to \eta_{AB} X^A Y^B + \alpha \left[\eta_{AB} M^A_{\ C} X^C Y^B + \eta_{AB} X^A M^B_{\ C} Y^C\right] + O(\alpha^2) = \\ &= \eta_{AB} X^A Y^B + \alpha X^C Y^B \left[M_{BC} + M_{CB}\right] + O(\alpha^2), \\ &\Rightarrow M_{BC} = -M_{CB}. \end{aligned} \tag{A.2}$$

Here and throughout this appendix capital latin indeces run from $-S{+}1, ..., 0, ..., T$. The previous result shows that the generators of the pseudo-orthogonal group $O(S,T)$ are given by $N \times N$ antisymmetric matrices (when one index is lowered as in Eq.(A.2)). There are then $\frac{N(N-1)}{2}$ linearly independent generators which are given by:

$$\left(M_{AB}\right)^I_{\ J} = {\delta^I}_A \eta_{BJ} - {\delta^I}_B \eta_{AJ}. \tag{A.3}$$

Notice that $M_{AB} = -M_{BA}$ so that from now on we will write $M_{[AB]}$ for the pseudo-orthogonal generators. In the following we will consider only the proper-orthochronus pseudo-orthogonal group (i.e. the part of $O(S,T)$ which is connected to the identity), so that with the exponential map (compare with Def.1.1.2) we can recover the whole group.

Now we are ready to study the commutators between the elements of the pseudo-orthogonal Lie algebra. We compute:

$$
\begin{aligned}
\left(\left[M_{[AB]}, M_{[CD]}\right]\right)^{I}{}_{K} = & \left(M_{[AB]}\right)^{I}{}_{J}\left(M_{[CD]}\right)^{J}{}_{K} - \left(M_{[CD]}\right)^{I}{}_{J}\left(M_{[AB]}\right)^{J}{}_{K} = \\
= & \left(\delta^{I}{}_{A}\eta_{BJ} - \delta^{I}{}_{B}\eta_{AJ}\right)\left(\delta^{J}{}_{C}\eta_{DK} - \delta^{J}{}_{D}\eta_{CK}\right) - \left(\delta^{I}{}_{C}\eta_{DJ} - \delta^{I}{}_{D}\eta_{CJ}\right)\left(\delta^{J}{}_{A}\eta_{BK} - \delta^{J}{}_{B}\eta_{AK}\right) = \\
= & \left(\delta^{I}{}_{A}\eta_{BC}\eta_{DK} - \delta^{I}{}_{A}\eta_{BD}\eta_{CK} - \delta^{I}{}_{B}\eta_{AC}\eta_{DK} + \delta^{I}{}_{B}\eta_{AD}\eta_{CK}\right) + \\
& - \left(\delta^{I}{}_{C}\eta_{DA}\eta_{BK} - \delta^{I}{}_{C}\eta_{DB}\eta_{AK} - \delta^{I}{}_{D}\eta_{CA}\eta_{BK} + \delta^{I}{}_{D}\eta_{CB}\eta_{AK}\right) = \\
= & \left(M_{[AD]}\right)^{I}{}_{K}\eta_{BC} - \left(M_{[AC]}\right)^{I}{}_{K}\eta_{BD} - \left(M_{[BD]}\right)^{I}{}_{K}\eta_{AC} + \left(M_{[BC]}\right)^{I}{}_{K}\eta_{AD} = \\
= & \left[\Delta^{[EF]}{}_{[AD]}\eta_{BC} - \Delta^{[EF]}{}_{[AC]}\eta_{BD} - \Delta^{[EF]}{}_{[BD]}\eta_{AC} + \Delta^{[EF]}{}_{[BC]}\eta_{AD}\right]\left(M_{EF}\right)^{I}{}_{K} \equiv \\
\equiv & c_{[AB][CD]}{}^{[EF]}\left(M_{[EF]}\right)^{I}{}_{K},
\end{aligned}
\tag{A.4}
$$

where we introduced the identity in antisymmetric $\binom{0}{2}$ tensor space, i.e.

$$
\Delta^{[AB]}{}_{[CD]} = \frac{1}{2}\left(\delta^{A}{}_{C}\delta^{B}{}_{D} - \delta^{A}{}_{D}\delta^{B}{}_{C}\right). \tag{A.5}
$$

Having identified the structure constants of the pseudo-orthogonal algebra we can now compute the Killing metric for this Lie algebra as in Eq.(1.22), namely:

$$
\begin{aligned}
G_{[AB][CD]} = & c_{[AB][LM]}{}^{[EF]}c_{[CD][EF]}{}^{[LM]} = \\
= & \left[\eta_{BL}\Delta^{[EF]}{}_{[AM]} + \eta_{AM}\Delta^{[EF]}{}_{[BL]} + \eta_{BM}\Delta^{[EF]}{}_{[LA]} + \eta_{AL}\Delta^{[EF]}{}_{[MB]}\right]\cdot \\
& \cdot\left[\eta_{DE}\Delta^{[LM]}{}_{[CF]} + \eta_{CF}\Delta^{[LM]}{}_{[DE]} + \eta_{DF}\Delta^{[LM]}{}_{[EC]} + \eta_{CE}\Delta^{[LM]}{}_{[FD]}\right] = \\
= & 2(N-2)\left[\eta_{BC}\eta_{DA} - \eta_{BD}\eta_{CA}\right]
\end{aligned}
\tag{A.6}
$$

Moreover we can also verify that this metric satisfies Eq.(1.24), indeed we have:

$$
c_{[AB][CD]}{}^{[EF]}G_{[EF][LM]} = -c_{[AB][LM]}{}^{[EF]}G_{[EF][CD]}. \tag{A.7}
$$

To prove it we first inspect the LHS:

$$
\begin{aligned}
c_{[AB][CD]}{}^{[EF]}G_{[EF][LM]} = & 2(N-2)\left(\eta_{BC}\Delta^{[EF]}{}_{[AD]} + \eta_{AD}\Delta^{[EF]}{}_{[BC]} + \eta_{BD}\Delta^{[EF]}{}_{[CA]} + \eta_{AC}\Delta^{[EF]}{}_{[DB]}\right)\cdot \\
& \cdot\left(\eta_{FL}\eta_{EM} - \eta_{FM}\eta_{EL}\right) = \\
= & 2(N-2)\Big[\eta_{BC}\left(\eta_{MA}\eta_{LD} - \eta_{MD}\eta_{LA}\right) + \eta_{AD}\left(\eta_{MB}\eta_{LC} - \eta_{MC}\eta_{LB}\right) + \\
& - \eta_{BD}\left(\eta_{MA}\eta_{LC} - \eta_{MC}\eta_{LA}\right) - \eta_{AC}\left(\eta_{MB}\eta_{LD} - \eta_{MD}\eta_{LB}\right)\Big] = \\
= & 2(N-2)\Big[\eta_{BC}\eta_{MA}\eta_{LD} - \eta_{BC}\eta_{MD}\eta_{LA} + \eta_{AD}\eta_{MB}\eta_{LC} - \eta_{AD}\eta_{MC}\eta_{LB} + \\
& - \eta_{BD}\eta_{MA}\eta_{LC} + \eta_{BD}\eta_{MC}\eta_{LA} - \eta_{AC}\eta_{MB}\eta_{LD} + \eta_{AC}\eta_{MD}\eta_{LB}\Big].
\end{aligned}
\tag{A.8}
$$

The RHS is found by swapping $[LM] \leftrightarrow [CD]$ in the previous equation.

$$\begin{aligned}
c_{[AB][LM]}{}^{[EF]} G_{[EF][CD]} =& 2(N-2)\Big[\eta_{BL}\eta_{DA}\eta_{CM} - \eta_{BL}\eta_{DM}\eta_{CA} + \eta_{AM}\eta_{DB}\eta_{CL} - \eta_{AM}\eta_{DL}\eta_{CB} + \\
& - \eta_{BM}\eta_{DA}\eta_{CL} + \eta_{BM}\eta_{DL}\eta_{CA} - \eta_{AL}\eta_{DB}\eta_{CM} + \eta_{AL}\eta_{DM}\eta_{CB}\Big] = \\
=& - c_{[AB][CD]}{}^{[EF]} G_{[EF][LM]},
\end{aligned} \tag{A.9}$$

which proves the claim.

A.2 Pseudo-orthogonal bundles

Throughout this section we will fix a manifold M and a principal $SO(S,T)$-bundle $P \to M$. We introduce a connection 1-form $\boldsymbol{\omega}$ on P and we expand it in the basis of $\mathfrak{so}(S,T)$ in Eq.(A.3).

$$\boldsymbol{\omega} = \frac{1}{2}\omega^{[AB]} \otimes M_{[AB]}, \tag{A.10}$$

where the 1/2 in front is necessary to avoid overcounting. As in Subsection1.2 we consider the adjoint bundle $\mathrm{Ad}(P)$. In particular, we will consider the commutator between twisted differential forms ($\boldsymbol{A} = \frac{1}{2}A^{[AB]} \otimes M_{AB}$, $A^{[AB]} \in \Omega^k(P)$). We find:

$$\begin{aligned}
[\boldsymbol{A}, \boldsymbol{B}] =& \frac{1}{4}A^{[AB]} \wedge B^{[CD]} \otimes \left[M_{[AB]}, M_{[CD]}\right] = \\
=& \frac{1}{4}A^{[AB]} \wedge B^{[CD]} \otimes \left(\eta_{[BC]}M_{[AD]} + \eta_{[AD]}M_{[BC]} + \eta_{DB}M_{CA} + \eta_{AC}M_{DB}\right) = \\
=& \frac{1}{2}\left(A^{[A}{}_{C]} \wedge B^{[CB]} - A^{[B}{}_{C]} \wedge B^{[CA]}\right) \otimes M_{[AB]}.
\end{aligned} \tag{A.11}$$

Using Def.1.2.11 we find the curvature associated with $\boldsymbol{\omega}$.

$$\begin{aligned}
\boldsymbol{\Omega} &= d\boldsymbol{\omega} + \frac{1}{2}\left[\boldsymbol{\omega}, \boldsymbol{\omega}\right] = \\
&= \frac{1}{2}\left[d\omega^{[AB]} + \omega^{[A}{}_{C]} \wedge \omega^{[CB]}\right] \otimes M_{[AB]}.
\end{aligned} \tag{A.12}$$

We also compute the covariant derivative on twisted forms.

$$\begin{aligned}
d_{\boldsymbol{\omega}}\boldsymbol{A} &= d\boldsymbol{A} + [\boldsymbol{\omega}, \boldsymbol{A}] = \\
&= \frac{1}{2}\left[dA^{[AB]} + \omega^{[A}{}_{C]} \wedge A^{[CB]} - \omega^{[B}{}_{C]} \wedge A^{[CA]}\right].
\end{aligned} \tag{A.13}$$

Appendix B

Useful formulas

B.1 Commutators and covariant exterior derivatives for the conformal group

Here we will give some of the formulas that we will frequently be using in Section 3.3.

Often we will need to compute the commutator of differential forms twisted with elements of $\mathfrak{so}(2,4)$, i.e. ($\mathbf{A} = A^i \otimes e_i$, $\mathbf{B} = B^j \otimes e_j$)

$$[\mathbf{A}, \mathbf{B}] := A^i \wedge B^j \otimes [e_i, e_j]\,. \tag{B.1}$$

In the following we will write:

$$\begin{aligned} \mathbf{A} &= \frac{1}{2} A^a{}_b \otimes M^a{}_b + A^a{}_P \otimes P_a + A^a{}_K \otimes K_a + A \otimes D = \\ &= \frac{1}{2} A^{AB} \otimes M_{AB} \end{aligned} \tag{B.2}$$

and analogously for $\mathbf{B}$. Comparing Eq.(B.2) with Eq.(3.55) we can write:

$$\begin{aligned} \frac{1}{2} A^{AB} \otimes M_{AB} &= \frac{1}{2} \left[A^{ab} \otimes M_{ab} + A^{-14} \otimes D + A^{-1b} \otimes \left(\frac{P_b - K_b}{2} \right) - A^{4\,-1} \otimes D - A^{4b} \otimes \left(\frac{P_b + K_b}{2} \right) \right] = \\ &= \frac{1}{2} A^{ab} \otimes M_{ab} + A^{-14} \otimes D + \frac{A^{-1b} - A^{4b}}{4} \otimes P_b - \frac{A^{-1b} + A^{4b}}{4} \otimes K_b, \end{aligned} \tag{B.3}$$

which gives:

$$A^{AB} = \begin{cases} A = a, B = b, & A^{ab}, \\ A = -1, B = 4 & A, \\ A = 4, B = a & -2\,(A^a{}_K + A^a{}_P)\,, \\ A = -1, B = a & 2\,(A^a{}_P - A^a{}_K)\,. \end{cases} \tag{B.4}$$

We can now compute:

$$
\begin{aligned}
[\boldsymbol{A},\boldsymbol{B}] =&\frac{1}{4}A^{ab}\wedge B^{cd}\otimes[M_{ab},M_{cd}]+\frac{1}{2}A^{ab}\wedge B^{c}{}_{P}\otimes[M_{ab},P_c]+\frac{1}{2}A^{ab}\wedge B^{c}{}_{K}\otimes[M_{ab},K_c]+\\
&+\frac{1}{2}A^{a}{}_{P}\wedge B^{cd}\otimes[P_a,M_{cd}]+A^{a}{}_{P}\wedge B^{b}{}_{K}\otimes[P_a,K_b]+A^{a}{}_{P}\wedge B\otimes[P_a,D]+\\
&+\frac{1}{2}A^{a}{}_{K}\wedge B^{cd}\otimes[K_a,M_{cd}]+A^{a}{}_{K}\wedge B^{b}{}_{P}\otimes[K_a,P_b]+A^{a}{}_{K}\wedge B\otimes[K_a,D]+\\
&+A\wedge B^{b}{}_{P}\otimes[D,P_b]+A\wedge B^{b}{}_{K}\otimes[D,K_b]=\\
=&\frac{1}{4}A^{ab}\wedge B^{cd}\otimes(\eta_{bc}M_{ad}+\eta_{ad}M_{bc}+\eta_{db}M_{ca}+\eta_{ac}M_{db})+\frac{1}{2}\left(A^{ab}\wedge B^{c}{}_{P}-A^{c}{}_{P}\wedge B^{ab}\right)\otimes(\eta_{cb}P_a-\eta_{ca}P_b)+\\
&+\frac{1}{2}\left(A^{ab}\wedge B^{c}{}_{K}-A^{c}{}_{K}\wedge B^{ab}\right)\otimes(\eta_{bc}K_a-\eta_{ac}K_b)+2\left(A^{a}{}_{P}\wedge B^{b}{}_{K}-A^{b}{}_{K}\wedge B^{a}{}_{P}\right)\otimes(\eta_{ab}D-M_{ab})+\\
&+(A^{a}{}_{P}\wedge B-A\wedge B^{a}{}_{P})\otimes P_a-(A^{a}{}_{K}\wedge B-A\wedge B^{a}{}_{K})\otimes K_a=\\
=&A^{a}{}_{c}\wedge B^{cb}\otimes M_{ab}+A^{ab}\wedge B_{bP}\otimes P_a+A_{aP}\wedge B^{ab}\otimes P_b+\left(A^{ab}\wedge B_{bK}+A_{bK}\wedge B^{ba}\right)\otimes K_a+\\
&+2\left(A^{a}{}_{P}\wedge B_{aK}-A^{a}{}_{K}\wedge B_{aP}\right)\otimes D+2\left(A^{b}{}_{K}\wedge B^{a}{}_{P}-A^{a}{}_{P}\wedge B^{b}{}_{K}\right)\otimes M_{ab}+\\
&+(A^{a}{}_{P}\wedge B-A\wedge B^{a}{}_{P})\otimes P_a-(A^{a}{}_{K}\wedge B-A\wedge B^{a}{}_{K})\otimes K_a=\\
=&\left[\frac{A^{a}{}_{c}\wedge B^{cb}-A^{b}{}_{c}\wedge B^{ca}}{2}+A^{b}{}_{K}\wedge B^{a}{}_{P}-A^{a}{}_{K}\wedge B^{b}{}_{P}-A^{a}{}_{P}\wedge B^{b}{}_{K}+A^{b}{}_{P}\wedge B^{a}{}_{K}\right]\otimes M_{ab}+\\
&+\left[A^{ab}\wedge B_{bP}+A_{bP}\wedge B^{ba}+A^{a}{}_{P}\wedge B-A\wedge B^{a}{}_{P}\right]\otimes P_a+\\
&+\left[A^{ab}\wedge B_{bK}+A_{bK}\wedge B^{ba}-A^{a}{}_{K}\wedge B+A\wedge B^{a}{}_{K}\right]\otimes K_a+\\
&+2\left[A^{a}{}_{P}\wedge B_{aK}-A^{a}{}_{K}\wedge B_{aP}\right]\otimes D,
\end{aligned}
\tag{B.5}
$$

where we used the commutation relations in Eq.(3.54).

We compute also the exterior covariant derivative acting on $\mathbf{A}$.

$$
\begin{aligned}
d_{\boldsymbol{\omega}}\mathbf{A} =&d\mathbf{A}+[\boldsymbol{\omega},\mathbf{A}]=\\
=&\left[\frac{1}{2}dA^{ab}+\frac{\omega^{a}{}_{c}\wedge A^{cb}-\omega^{b}{}_{c}\wedge A^{ca}}{2}+f^{b}\wedge A^{a}{}_{P}-f^{a}\wedge A^{b}{}_{P}-e^{a}\wedge A^{b}{}_{K}+e^{b}\wedge A^{a}{}_{K}\right]\otimes M_{ab}+\\
&+\left[dA^{a}{}_{P}+\omega^{ab}\wedge A_{bP}+e_b\wedge A^{ba}+e^{a}\wedge A-\omega\wedge A^{a}{}_{P}\right]\otimes P_a+\\
&+\left[dA^{a}{}_{K}+\omega^{ab}\wedge A_{bK}+f_b\wedge A^{ba}-f^{a}\wedge A+\omega\wedge A^{a}{}_{K}\right]\otimes K_a+\\
&+\left[dA+2\left(e^{a}\wedge A_{aK}-f^{a}\wedge A_{aP}\right)\right]\otimes D.
\end{aligned}
\tag{B.6}
$$

B.2 From forms to coordinates (local) representation

In the following section fix a pseudo-Riemannian 4-dimensional manifold (M,g) and a (possibly local) basis for T^*M. Often we will need to write in coordinates expressions like $\delta e^a\wedge e^c\wedge *\Omega_{ac}$ or $\delta\omega\wedge d*\Omega$. We can compute then the more general formula for the wedge product between a

2-form ($B \in \Omega^2(M)$) and the hodge star of a 2-form ($A \in \Omega^2(M)$):

$$\begin{aligned} B \wedge *A &= \frac{\sqrt{-g}}{4(n-2)!} A^{\mu\nu} B_{\rho\sigma} \epsilon_{\mu\nu\alpha\beta} dx^\rho \wedge dx^\sigma \wedge dx^\alpha \wedge dx^\beta = \\ &= \frac{1}{4(n-2)!} A^{\mu\nu} B_{\rho\sigma} \epsilon_{\mu\nu\alpha\beta} \tilde{\epsilon}^{\rho\sigma\alpha\beta} \sqrt{-g} d^4x = \\ &= \frac{1}{2} A^{\mu\nu} B_{\mu\nu} \sqrt{-g} d^4x \,, \end{aligned} \tag{B.7}$$

where we introduced the total anti-symmetric contravariant symbol $\tilde{\epsilon}^{\alpha\beta\rho\sigma}$ (which is equal to 1 if $(\alpha, \beta, \rho, \sigma)$ is an even permutation of $(0, 1, 2, 3)$). It is straightforward to prove that $\tilde{\epsilon}^{\alpha\beta\rho\sigma} \epsilon_{\mu\nu\rho\sigma} = 2(n-2)! \Delta^{\alpha\beta}_{\mu\nu}$. The precedent formula shows how to pass from forms notation to the more physics familiar coordinate framework.

B.3 Hamiltonian derivatives

In this appendix we will give some useful formula for Chapter 4. First of all we give the functional derivatives of the Hamiltonian with respect to fields and momenta. For the momenta we get:

$$\begin{aligned} \frac{\delta H_T}{\delta \Pi^{\ \ a}_{[EF]}} &= (-8\alpha' \sqrt{-g})^{-1} \Pi^{[EF]k} \frac{g_{ak}}{g^{00}} - \frac{g_{ak}}{g^{00}} \tilde{P}^{[EF]k} = \\ &= (-8\alpha' \sqrt{-g})^{-1} \frac{g_{ak}}{g^{00}} \Pi^{[EF]k} + \partial_a \omega^{[EF]}_{\ \ 0} - \frac{1}{2} c_{[LM][NQ]}^{\qquad\ \ [EF]} \omega^{[LM]}_{\ \ 0} \omega^{[NQ]}_{\ \ a} - \frac{g^{m0}}{g^{00}} \Omega^{[EF]}_{\ \ ma}, \end{aligned} \tag{B.8}$$

while, for non-tetrad fields, we obtain:

$$\begin{aligned} \frac{\delta H_T}{\delta \omega^{[EF]}_{\ \ a}} =& \partial_m \left[\left(\Pi^{\ \ a}_{[EF]} + 4\alpha' \sqrt{-g} g^{k0} g^{pa} \omega_{[EF]kp} \right) \frac{g^{m0}}{g^{00}} \right] - \partial_i \left[\left(\Pi^{\ \ i}_{[EF]} + 4\alpha' \sqrt{-g} g^{k0} g^{pi} \Omega_{[EF]kp} \right) \frac{g^{a0}}{g^{00}} \right] + \\ &+ \partial_i \left(4\alpha' \sqrt{-g} g^{aj} g^{ki} \Omega_{[EF]jk} \right) + \\ &+ \frac{1}{2} c_{[LM][NQ][EF]} \Bigg\{ - \Pi^{[LM]a} \omega^{[NQ]}_{\ \ 0} + \frac{g^{a0}}{g^{00}} \left[\Pi^{[LM]i} \omega^{[NQ]}_{\ \ i} + 4\alpha' \sqrt{-g} g^{k0} g^{pi} \Omega^{[LM]}_{\ \ kp} \omega^{[NQ]}_{\ \ i} \right] + \\ &- \frac{g^{m0}}{g^{00}} \left[\Pi^{[LM]a} \omega^{[NQ]}_{\ \ m} + 4\alpha' \sqrt{-g} g^{k0} g^{pa} \Omega^{[LM]}_{\ \ kp} \omega^{[NQ]}_{\ \ m} \right] - 4\alpha' \sqrt{-g} g^{aj} g^{ki} \Omega^{[LM]}_{\ \ jk} \omega^{[NQ]}_{\ \ i} \Bigg\}. \end{aligned} \tag{B.9}$$

The functional deerivative with respect to the timelike component of the gauge fields is already given in Eq.(4.61). For tetrad fields we know that the situation is differernt. Since the Hamiltonian depends non-trivially on the metric the functional derivative will get another term with respect to standard Yang-Mills fields. We introduce the following notation:

$$\begin{aligned} \left. \frac{\delta H}{\delta \omega^{[\cdot a]}_{\ \ \gamma}} \right|_{\text{geometric}} &\equiv \frac{\delta H}{\delta g^{\mu\nu}} \frac{\partial g^{\mu\nu}}{\partial \omega^{[\cdot a]}_{\ \ \gamma}}, \\ \left. \frac{\delta H}{\delta \omega^{[\cdot a]}_{\ \ \gamma}} \right|_{\text{YM}} &\equiv \frac{\delta H}{\delta \omega^{[\cdot a]}_{\ \ \gamma}} - \left. \frac{\delta H}{\delta \omega^{[\cdot a]}_{\ \ \gamma}} \right|_{\text{geometric}}. \end{aligned} \tag{B.10}$$

Clearly we find that the Yang-Mills part of the variation (YM) is the same one we found for non-tetrad fields in Eq.(B.9). Comparing with Eq.(4.38) and the formulas given after, we can compute the geometric part:

$$\left.\frac{\delta H}{\delta \omega^{[\cdot a]}_{\ \gamma}}\right|_{\text{geometric}} =\Bigg\{ \frac{\left(-4\alpha'\sqrt{-g}\right)^{-1}}{-2}\frac{g_{lk}}{g^{00}}\Pi^{\ \ l}_{[CD]}\Pi^{[CD]k} - 2\alpha'\sqrt{-g}\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)\frac{g^{s0}}{g^{00}}\Omega^{[AB]}_{\ \ sj}\alpha'\sqrt{-g}g^{ij}g^{kl}\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ il}\Bigg\}\cdot$$
$$\cdot\,\omega^{\ \ \gamma}_{[\cdot a]}+$$
$$+\Bigg\{\frac{\left(-4\alpha'\sqrt{-g}\right)^{-1}}{2}\Pi^{\ \ l}_{[CD]}\Pi^{[CD]j} - \Pi^{\ \ l}_{[CD]}\tilde{P}^{[CD]j} - 2\alpha'\sqrt{-g}\left(g^{k0}g^{ij}\Omega_{[AB]ki}\right)g^{s0}g^{tl}\Omega^{[AB]}_{\ \ st}\Bigg\}\cdot$$
$$\cdot\left[\frac{1}{g^{00}}\left(\delta^{\gamma}_{\ l}\omega_{[\cdot a]j}+\delta^{\gamma}_{\ j}\omega_{[\cdot a]l}\right)+2\frac{g_{lj}g^{0\gamma}\omega^{\ \ 0}_{[\cdot a]}}{\left(g^{00}\right)^2}\right]+$$
$$-\Pi^{\ \ k}_{[CD]}\frac{g_{kl}}{g^{00}}\Bigg\{\left[M^{l\gamma}\omega^{\ \ i}_{[\cdot a]}+M^{i\gamma}\omega^{\ \ l}_{[\cdot a]}+2\left(g^{il}g^{0\gamma}-g^{l0}g^{i\gamma}-g^{i0}g^{l\gamma}\right)\omega^{\ \ 0}_{[\cdot a]}\right]\cdot$$
$$\cdot\left(\partial_i\omega^{[CD]}_{\ \ \ 0}-\omega^{[C}_{\ \ A]0}\omega^{[AD]}_{\ \ \ i}+\omega^{[C}_{\ \ A]i}\omega^{[AD]}_{\ \ \ 0}\right)+$$
$$-\left(g^{p\gamma}g^{il}\omega^{\ \ 0}_{[\cdot a]}+g^{0\gamma}g^{il}\omega^{\ \ p}_{[\cdot a]}+g^{p0}g^{i\gamma}\omega^{\ \ l}_{[\cdot a]}+g^{l\gamma}g^{p0}\omega^{\ \ i}_{[\cdot a]}\right)\Omega^{[CD]}_{\ \ \ pi}\Bigg\}+$$
$$-\left(-4\alpha'\sqrt{-g}\right)\Omega_{[AB]ki}\frac{g^{s0}}{g^{00}}\Omega^{[AB]}_{\ \ sj}\Bigg\{g^{k\gamma}g^{ij}\omega^{\ \ 0}_{[\cdot a]}+g^{0\gamma}g^{ij}\omega^{\ \ k}_{[\cdot a]}+g^{i\gamma}g^{k0}\omega^{\ \ j}_{[\cdot a]}+g^{j\gamma}g^{k0}\omega^{\ \ i}_{[\cdot a]}\Bigg\}+$$
$$-\alpha'\sqrt{-g}\Omega_{[AB]jk}\Omega^{[AB]}_{\ \ il}\Bigg\{g^{i\gamma}g^{kl}\omega^{\ \ j}_{[\cdot a]}+g^{j\gamma}g^{kl}\omega^{\ \ i}_{[\cdot a]}+g^{k\gamma}g^{ij}\omega^{\ \ l}_{[\cdot a]}+g^{l\gamma}g^{ij}\omega^{\ \ k}_{[\cdot a]}\Bigg\}. \tag{B.11}$$

Fom the expressions given in this appendix one immediately obtains the Poisson brackets between the Hamiltonian and the fields and momenta, thus one gets the Hamiltonian equations of motion in the simplectic formalism. It can be verified that the latter are equivalent to the Lagrangian equations, in particular they are covariant. However, the computation is extremely long and it doesn't give any new insights on the theory at hand. As such, we decided to not put it here but to leave the computation as a very mean exercise for the reader.

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