On self-similar groups of intermediate growth

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Abstract

We study several examples of self-similar groups of subexponential growth: the (generalized) Grigorchuk groups and the kneading automata groups induced by the sequences $1(10)^\omega$, $11(0)^\omega$ and $0(011)^\omega$. According to Nekrashevych, the last three groups appear as iterated monodromy groups for some complex post-critically finite quadratic polynomials. In particular, they support Nekrashevych's conjecture on the intermediate growth of iterated monodromy groups.

For each of the cases we implement the method of incompressible elements. We conclude that the set of incompressible elements shares a common trait for all examples: the automaton described by the alternating patterns of incompressible elements consists of disjoint circles.

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Chapter 1

Introduction

The study of finitely generated groups involves classifying them using *word growth*. This quasi-isometry invariant was introduced during the 1950s [16], awakening significant interest among mathematicians for much of the 20th century. Notably, by 1968, it had become evident that all known classes of groups exhibited either polynomial growth (e.g., abelian groups) or exponential growth (e.g., free groups). An intriguing question was posed by Milnor : Do there exist groups that display neither of these two types of growth? In other words, are there any groups of *intermediate* growth? This question persisted until 1983 when Rostislav Grigorchuk provided a remarkable answer by constructing the now-famous first Grigorchuk group of intermediate growth [12]. This example of his gave rise to the concept of *self-similarity* in groups.

Since Grigorchuk's discovery, a considerable number of researchers immersed themselves in this new subject, exploring and improving methods to demonstrate intermediate growth. Most groups of intermediate growth are constructed using self-similar automorphisms acting on rooted trees. The general idea to prove intermediate growth is to show that there exists length contraction on some level. Notably, Bartholdi and Pochon introduced the concept of incompressible elements [4]. These elements within the group essentially resist any reduction in their length, proving to be significant factors that determine the growth type of the entire group. Indeed, their work revealed that the type of growth is closely connected to how these specified elements behave.

Although significant progress has been made in understanding groups with polynomial or exponential growth, much remains to be explored regarding groups of intermediate growth. These exotic groups appear naturally in the context of complex dynamics. Specifically, Volodymyr Nekrashevych came up with the idea of assigning to a post-critically finite polynomial a group that encodes the combinatorics and symbolic dynamics of the map [14]. This group is called the *iterated monodromy group* and it acts on an infinite d-regular rooted tree, where d is the degree of the polynomial.

The following conjecture, attributed to Nekrashevych, appeared in [5] : If $p : \mathbb{C} \to \mathbb{C}$ is a post-critically finite quadratic polynomial with a pre-periodic kneading sequence, then the iterated monodromy group associated to it possesses intermediate growth. Early on, this conjecture found validation through the pioneering work of Bux and Perez, who managed to demonstrate that the iterated monodromy group of the polynomial z^2+i exhibits subexponential growth (implying intermediate growth since the proof for superpolynomial growth is quite direct). Despite the initial successes, the quest to establish the conjecture encountered permanent obstacles in the form of counterexamples. Grigorchuk and Zuk [11] showed that the iterated monodromy group of z^2-1 has exponential growth. This discovery fundamentally challenged Nekrashevych's conjecture. The tuning [6] of z^2-1 by z^2+i results in a post-critically finite quadratic polynomial $h(z) = z^2+c$ with pre-periodic kneading sequence such that the iterated monodromy group of z^2-1 embeds into the one of h. As a consequence, it was established that the iterated monodromy group of h has exponential growth, making it an authentic counterexample to Nekrashevych's conjecture.

In addition to the previous conjecture, Nekrashevych put forth another proposition. He conjectured:

Conjecture 1.0.1. If $p : \mathbb{C} \to \mathbb{C}$ is a non-renormalizable post-critically finite quadratic polynomial with a pre-periodic kneading sequence, then the associated iterated monodromy group exhibits intermediate growth.

Notably, the hypothesis of non-renormalizability serves to eliminate any counterexamples that may arise from tuning, offering a more robust foundation for the initial conjecture.

In this thesis we discuss examples of groups of intermediate growth within the class of self-similar groups, some of which naturally arise as iterated monodromy groups. We begin by providing illustrative examples of such groups, like the Grigorchuk groups. Furthermore, we present examples of groups that support Nekrashevych's conjecture. At this point we must emphasize that, as of now, the conjecture remains unproven. Finally, throughout our study, we identify correlations and patterns among the mentioned examples that may contribute to a potential proof of the Nekrasevych conjecture.

The thesis is divided into two main chapters and is structured as follows. The first chapter provides an overview of the essential background material necessary for describing the required algebraic structures, primarily relying on [14]. Section 2.1 covers automorphisms of rooted trees and self-similar groups. Section 2.2 delves into the structure of the group of automorphisms and discusses wreath decomposition. Section 2.3 and 2.4 explore automata and kneading automata, respectively, their connection to the concept of self-similarity and the two forms one may distinguish kneading automata into. Section 2.5 focuses on group growth and describes the method we will mainly use to determine whether a group exhibits subexponential growth, the method of incompressible elements. Section 2.6 involves the construction of the iterated monodromy group of a quadratic polynomial and, more generally, a rational function, followed by the statement of the conjecture.

The second chapter commences by providing a proof of intermediate growth for the Grigorchuk group, followed by the case of the second Grigorchuk group. It then proceeds to address the generalized scenario, where we let the group have n generators for every natural number n. Additionally, the chapter establishes subexponential growth for groups that satisfy the hypotheses of the conjecture. The initial investigation focuses on the iterated monodromy group associated with the quadratic polynomial $z^2 + i$, which happens to be the group generated by the automaton with kneading sequence $1(10)^\omega$. Subsequently, attention turns to two groups generated by kneading automata: one with the kneading sequence $11(0)^\omega$ and the other with $0(011)^\omega$. In both instances, we demonstrate subexponential growth. To achieve all the above, we implement the method of incompressible elements, either directly or by making suitable adjustments or restricting ourselves to appropriate subsets. One of our primary objectives is to apply a uniform methodology, utilizing the same propositions and methods across all the examples, thereby ensuring our approach is candidate for a potential general solution.

Following the presentation of the examples, we draw conclusions that directly relate to Nekrasevych's conjecture. Our primary finding suggests a common characteristic among all the instances when demonstrating subexponential growth using the method of incompressible elements. Particularly, we observe that elements in these examples, which resist length reduction, share a specific pattern. Indeed, if one considers the automaton generated by patterns of letters who alternate with each other, the ones who end up being incompressible create disjoint circles . Although this observation may not be the decisive factor in proving the conjecture, it provides valuable insights into potential approaches or directions that could lead to a potential proof.

Chapter 2

Background

Throughout this chapter we will mainly follow [14] along with the following sources: [3], [4], [5], [7], [8], [10].

2.1 Rooted trees and self-similar groups

Let X denote a finite set, which we call alphabet, and X^* denote the set

$$\{x_1 x_2 \cdots x_n \mid x_i \in X\}$$

of all finite words over the alphabet X, including the empty word $\varnothing$. In particular, X^* is the free monoid generated by X, having concatenation of words as operation. We can naturally associate it with the vertex set of a regular rooted tree, in which two words are connected by an edge if and only if they are of the form v and vx, where $v \in X^*$ and $x \in X$. The empty word is the root of the tree X^*.

The set $X^n \subseteq X^*$ is called the *nth level* of the tree. A map $f : X^* \to X^*$ is an *endomorphism* of the tree if it preserves the root and adjacency of vertices, meaning that for any two adjacent vertices $v, vx \in X^*$ there exists a $y \in X$ such that $f(vx) = f(v)y$. It is easy to prove by induction on n that $f(X^n) \subseteq X^n$, so f preserves each level of the tree. If f is also a bijection, then we call f an *automorphism* of the tree X^*. Let us denote by $\operatorname{Aut} X^*$ the group of all automorphisms of the rooted tree X^* and by id the identity automorphism.

Remark. We are using right actions in all cases. So, the image of a vertex $v \in X^*$ under action of an element $g \in \operatorname{Aut} X^*$ is denoted v^g and in the product $g_1 g_2$, where $g_1, g_2 \in \operatorname{Aut} X^*$, the element g_1 acts first.

Definition 2.1.1. A group $G \leq \operatorname{Aut}(X^*)$ is called *level-transitive* if it acts transitively on every level X^n of the tree.

Definition 2.1.2. Let $G \leq \operatorname{Aut} X^*$ be an automorphism group of the rooted tree X^*.

1. A *vertex stabilizer* is the subgroup $G_v = \{g \in G \mid v^g = v\}$, where $v \in X^*$ is a vertex.
2. The *nth level stabilizer* is the subgroup $\operatorname{St}_G(n) = \bigcap_{v \in X^n} G_v$. (We will just write $\operatorname{St}(n)$ if it is clear which G is under consideration.)

The following properties of the stabilizer subgroups are straightforward from the definitions.

Proposition 2.1.1. [14, Proposition 1.2.3] Let G be a level-transitive automorphism group of the rooted tree X^*. Then the following statements are true:

1. The vertex stabilizer G_v for $v \in X^n$ is a subgroup of index $|X|^n$ in G.
2. For every $x \in X^*$ and $g \in G$ we have $g^{-1} \cdot G_v \cdot g = G_{v^g}$.

3. The level stabilizers $\mathrm{St}_G(n)$ are normal finite index subgroups of G and $\bigcap_{n\geq 1} \mathrm{St}_G(n) = \{\mathrm{id}\}$.

Let $g : X^* \to X^*$ be an automorphism of the rooted tree X^*. Consider a vertex $v \in X^*$ and the subtrees vX^* and $v^g X^*$. Here vX^* denotes the subtree with the root v and with the set of vertices equal to the set of words starting with v. Then, g restricts to a map $g : vX^* \to v^g X^*$, which is a morphism of rooted trees. Now, the subtree vX^* is naturally isomorphic to the whole tree X^*. The isomorphism is the map

$$\begin{aligned} s_v : vX^* &\to X^* \\ vw &\mapsto w. \end{aligned}$$

The same holds for $v^g X^*$. Identifying vX^* and $v^g X^*$ with X^* we get an automorphism $g|_v : X^* \to X^*$, specifically $g|_v = s_{v^g} \circ g \circ s_v^{-1}$. We will call this automorphism *the restriction of g in v* and it is uniquely determined by the condition

$$(vw)^g = v^g (w)^{g|_v} \text{ for all } w \in X^*.$$

Remark. Here we have defined restrictions for automorphisms of the rooted tree. However, we may define them for endomorphisms in a similar way.

Definition 2.1.3. A group G acting faithfully on X^* is said to be *self-similar* if for every $g \in G$ and every $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$(xw)^g = y(w^h)$$

for every $w \in X^*$.

Applying the above equation several times we see that for every finite word $v \in X^n$ and every $g \in G$ there exist $h \in G$ and a word $u \in X^n$ such that $(vw)^g = u(w^h)$ for all $w \in X^*$, meaning that the word u is the image of v under the action of g. Since G acts faithfully, the group element h is defined uniquely. It is also called the restriction of g in the word v and is written $h = g|_v$. If we consider in the above definition G as a subgroup of $\mathrm{Aut}\, X^*$, then the two notions of restriction $g|_v$ coincide. We are also able to define equivalently self-similarity by stating that G is self-similar if each restriction $g|_v$ is in G for all $g \in G$ and $v \in X^*$.

It follows directly that for any $v, v_1, v_2 \in X^*$ and $g, g_1, g_2 \in G$ we have:

- $g|_{v_1 v_2} = (g|_{v_1})|_{v_2}$
- $(g_1 g_2)|_v = (g_1|_v)(g_2|_{v^{g_1}})$

We proceed with a few examples of self-similar groups [2, Section 3.4]:

1. **The adding machine** Let a be the automorphism of the binary tree $\{0,1\}^*$ defined by the following recursive formulas:

$$(0w)^a = 1w \qquad (1w)^a = 0w^a,$$

   where $w \in \{0,1\}^*$ is arbitrary. The automorphism a generates an infinite cyclic group of automorphisms of the tree $\{0,1\}^*$. Thus we get an action of the group $\mathbb{Z}$ on $\{0,1\}^*$, which will be also called the adding machine action.

2. **The dihedral group** Let a and b be the automorphisms of the tree $\{0,1\}^*$ defined by the rules:

$$\begin{aligned} (0w)^a &= 1w \qquad & (1w)^a &= 0w \\ (0w)^b &= 0w^a \qquad & (1w)^b &= 1w^b, \end{aligned}$$

   where $w \in \{0,1\}^*$ is arbitrary. The group generated by the automorphisms a and b is isomorphic to the infinite dihedral group $\mathbf{D}_\infty$.

3. **The first Grigorchuk group**. The Grigorchuk group is generated by the automorphisms a, b, c, d of the tree $\{0,1\}^*$ defined by the rules:

$$\begin{aligned} (0w)^a &= 1w & (1w)^a &= 0w \\ (0w)^b &= 1w^a & (1w)^b &= 0w^c \\ (0w)^c &= 1w^a & (1w)^c &= 0w^d \\ (0w)^d &= 1w & (1w)^d &= 0w^b. \end{aligned}$$

   In [9] it is proven that it is an infinite finitely generated torsion group.

2.2 Wreath Product

Definition 2.2.1. Let H be a group acting (from the right) by permutations on a set X and let G be an arbitrary group. Then, the *(permutational) wreath product* $G \wr H$ is the semi-direct product $G^X \rtimes H$, where H acts on the direct power G^X by the respective permutations of the direct factors.

Every element of the wreath product $G \wr H$ can be written in the form $g \cdot h$, where $g \in G^X$ and $h \in H$. If we fix some indexing $\{x_1, \dots, x_d\}$ of the set X, then g can be written as $\langle\langle g_1, \dots, g_d \rangle\rangle$ for $g_i \in G$. Here g_i is the coordinate of g corresponding to x_i. Then the multiplication rule for elements $\langle\langle g_1, \dots, g_d \rangle\rangle \alpha$ and $\langle\langle f_1, \dots, f_d \rangle\rangle \beta$ in $G \wr H$ is given by the formula

$$\langle\langle g_1, \dots, g_d \rangle\rangle \alpha \cdot \langle\langle f_1, \dots, f_d \rangle\rangle \beta = \langle\langle g_1 f_{1^\alpha}, \dots, g_d f_{d^\alpha} \rangle\rangle \alpha\beta, \tag{2.2.1}$$

where i^α is the image of i under the action of α, meaning that $(x_i)^\alpha = x_{i^\alpha}$.

We have the following canonical representation of $\operatorname{Aut} X^*$ as a permutational wreath product.

Proposition 2.2.1. [14, Proposition 1.4.2] Consider the automorphism group $\operatorname{Aut} X^*$ of the rooted tree X^* and the symmetric group $\mathcal{S}(X)$ of all permutations of X. Fix some indexing $\{x_1, \dots, x_d\}$ of X. Then, we have an isomorphism

$$\begin{aligned} \psi : \operatorname{Aut} X^* &\to \operatorname{Aut} X^* \wr \mathcal{S}(X) \\ \psi(g) &\mapsto \langle\langle g|_{x_1}, \dots, g|_{x_d} \rangle\rangle \alpha, \end{aligned}$$

where α is the permutation equal to the action of g on $X \subseteq X^*$.

We will usually identify $g \in \operatorname{Aut} X^*$ with its image $\psi(g) \in \mathcal{S}(X) \wr \operatorname{Aut} X^*$, so that we write

$$g = \langle\langle g|_{x_1}, \dots, g|_{x_d} \rangle\rangle \alpha. \tag{2.2.2}$$

According to this convention, we have $\operatorname{Aut} X^* = \operatorname{Aut} X^* \wr \mathcal{S}(X)$. The subgroup $(\operatorname{Aut} X^*)^X \le \operatorname{Aut} X^* \wr \mathcal{S}(X)$ is the first level stabilizer $\operatorname{St}(1)$ and it acts on the tree X^* in the natural way

$$(x_i v)^{\langle\langle g_1, \dots, g_d \rangle\rangle} = x_i (v^{g_i}),$$

namely, the ith coordinate of $\langle\langle g_1, \dots, g_d \rangle\rangle$ acts on the ith subtree $x_i X^*$. The subgroup $\mathcal{S}(X) \le \operatorname{Aut} X^* \wr \mathcal{S}(X)$ is identified with the group of *rooted automorphisms* $\alpha = \langle\langle \mathrm{id}, \dots, \mathrm{id} \rangle\rangle \alpha$ acting by the rule

$$(xv)^\alpha = (x^\alpha) v.$$

Relation (2.2.2) is called *wreath recursion*. It is a compact way to define recursively automorphisms of the rooted tree X^*. In general, every self-similar group which has $\{g_1, \dots, g_n\}$ as set of generators is described by recurrent formulas:

$$
\begin{aligned}
g_1 &= \langle\!\langle h_{11}, h_{12}, \ldots, h_{1d} \rangle\!\rangle \alpha_1 \\
g_2 &= \langle\!\langle h_{21}, h_{22}, \ldots, h_{2d} \rangle\!\rangle \alpha_2 \\
&\vdots \\
g_n &= \langle\!\langle h_{n1}, h_{n2}, \ldots, h_{nd} \rangle\!\rangle \alpha_n,
\end{aligned}
\tag{2.2.3}
$$

where $h_{ij} = g_i|_{x_j}$ and α_i is the action of g_i on X. Conversely, any set of formulas (2.2.3), for which α_i are in $\mathcal{S}(X)$ and each h_{ij} is a word in the set $\{g_1, \ldots, g_n\}$, uniquely defines a self-similar group with generators $g_1, \ldots, g_n$.

In the binary case, where $X = \{0,1\}$, the infinite binary tree $\{0,1\}^*$ has two subtrees connecting to the root vertex, the left one and the right one. Both subtrees are binary infinite rooted trees in their own right. The wreath decomposition of $\mathrm{Aut}\{0,1\}^*$ is written as

$$\mathrm{Aut}\{0,1\}^* = (\mathrm{Aut}\{0,1\}^* \times \mathrm{Aut}\{0,1\}^*) \rtimes \langle (0\ 1) \rangle.$$

So, each element $g \in \mathrm{Aut}\{0,1\}^*$ can be seen as $g = \langle\!\langle g_0, g_1 \rangle\!\rangle \alpha$, where $g_0 = g|_0$ and $g_1 = g|_1$ are in $\mathrm{Aut}\{0,1\}^*$ and α is $(0\ 1)$ or the identity in $\mathcal{S}(\{0,1\})$. Each automorphism g of the tree acts as follows. The automorphism g will act on the left subtree as g_0 and on the right one as g_1 and then it will either interchange the left and the right subtree or not, depending on α in its wreath decomposition.

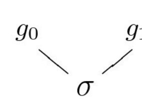
The permutation $(0\ 1) \in \mathcal{S}(\{0,1\})$ induces a specific rooted automorphism $\sigma \in \mathrm{Aut}\{0,1\}^*$ given by

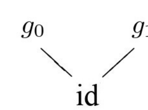
$$\sigma = \langle\!\langle \mathrm{id}, \mathrm{id} \rangle\!\rangle (0\ 1),$$

which is called the *swap*. The swap, basically, interchanges the left subtree and the right one. Note that the swap interacts nicely with the pair notation from above:

$$\sigma \langle\!\langle g_0, g_1 \rangle\!\rangle \sigma = \langle\!\langle g_1, g_0 \rangle\!\rangle.$$

Finally, the wreath decomposition allows us to depict any automorphism of the binary tree pictorially:

$\langle\!\langle g_0, g_1 \rangle\!\rangle \sigma =$ 

$\langle\!\langle g_0, g_1 \rangle\!\rangle =$ 

2.3 Automata

Definition 2.3.1. An *automaton* A over an alphabet X is given by

1. a set of *states*, usually denoted by A too;
2. a map $\tau : A \times X \to X \times A$.

If $\tau(q,x) = (y,p)$, then y and p as functions of (q,x) are called the *output* and *transition functions*, respectively. An automaton is said to be *finite*, if the alphabet X and the set of states of A are finite. If we want to emphasize that A is an automaton over the alphabet X, we denote it (A,X) and if $\tau(q,x) = (y,p)$, then we write

- $y = x^q$

- $p = q|_x$

If q is the current state of the automaton A and it gets on input a finite word $v \in X^*$, then A processes it letter by letter: it reads the first letter x of v, gives the letter x^q on output, goes to the state $q|_x$ and it is ready to process the word v further. In the end it will give as output some word of the same length as v and it will stop at some state of A.

Let (A, X) be an automaton and $q \in A$. Since the prefix of length n of the image w^q depends only on the prefix of length n of the word $w \in X^*$, the map $q : X^* \to X^*$ defined by q is an endomorphism of the rooted tree X^*. On the other hand, let us denote by $Q(g) = \{g|_v : v \in X^*\}$ the set of restrictions of an endomorphism g of the tree X^*. Then, $Q(g)$ can be interpreted as a set of internal states of an automaton, which being in a state $g|_v$ and reading on the input tape a letter x, types on the output tape the letter $x^{g|_v}$ and goes to the state $g|_v|_x = g|_{vx}$. It follows directly from the properties of the restriction that the transformation $q : X^* \to X^*$ defined by a state q of this automaton coincides with the original action of q. In particular, this shows that every endomorphism of X^* is defined by a state of an automaton.

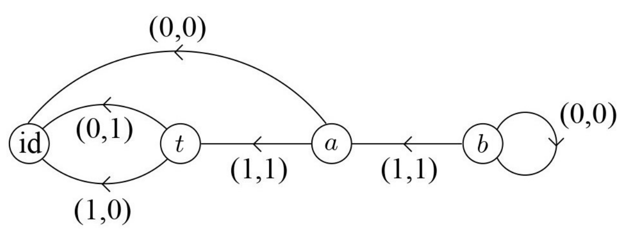
Definition 2.3.2. An automaton (A, X) is said to be *reduced* if different states of A induce different endomorphisms of X^*.

It is convenient to define automata using their *Moore diagrams*. It is a directed labeled graph whose vertices are identified with the states of the automaton. If $\tau(q, x) = (y, p)$, then we have an arrow starting in q and ending in p, labeled by (x, y).

Example 2.3.1. [7, Example 2.3] Consider the alphabet $X = \{0, 1\}$ and set $A = \{\mathrm{id}, t, a, b\}$. Next, define $\tau : A \times X \to X \times A$ by the rules:

$$\begin{aligned} \tau(\mathrm{id}, 0) &= (0, \mathrm{id}) & \tau(\mathrm{id}, 1) &= (1, \mathrm{id}) \\ \tau(t, 0) &= (1, \mathrm{id}) & \tau(t, 1) &= (0, \mathrm{id}) \\ \tau(a, 0) &= (0, \mathrm{id}) & \tau(a, 1) &= (1, t) \\ \tau(b, 0) &= (0, b) & \tau(b, 1) &= (1, a). \end{aligned}$$

Then A becomes an automaton. One can easily check that id is the identity and t is the swap as endomorphisms of $\{0, 1\}^*$. The Moore diagram for A is depicted by the figure [7, Figure 1] below:



For each state $q \in A$ of an automaton (A, X) we define a function $\tau_q : X \to X$ by the rule $\tau_q(x) = \pi_1(\tau(q, x))$, where $\pi_1 : X \times A \to X$ is projection on the first coordinate. If $\tau_q \neq \mathrm{id}_X$, we say that q is an *active state*. In Example 2.3.1 the state t is the only active state.

An automaton A is said to be *invertible* if every one of its states defines an invertible transformation of X^*. It is easy to prove that an automaton is invertible if and only if the map τ_q is a bijection for every $q \in A$.

Definition 2.3.3. Let (A, X) be an invertible automaton. The *group generated by the automaton* A is the group $\langle A \rangle = \langle q : X^* \to X^* \mid q \in A \rangle \leq \mathrm{Aut}\, X^*$ generated by the transformations defined by all states of A.

The group generated by a finite automaton is obviously self-similar and finitely generated. Conversely, if a group $G \le \operatorname{Aut} X^*$ is finitely generated, self-similar, and the set $\{g|_v \mid v \in X^*\}$ is finite for every generator g of G, then G is generated by a finite automaton. One can take all the automata defining the generators of the group G and next take their disjoint union.

2.4 Kneading automata

In this section, we provide the definition of a kneading automaton that is notably simpler compared to those found in Nekrasevych's book [14]. Our choice to work with the binary alphabet is the reason behind that. It is important to emphasize that the definition of the kneading automaton we use would not be correct if we were to employ a larger alphabet. The reader interested in the general case may find it in [14, Section 6.6.1].

Definition 2.4.1. Let A be an invertible reduced automaton over the alphabet $\{0,1\}$. We say that A is a *kneading automaton* if the following holds:

1. there is only one active state;
2. in the Moore diagram of A, each non-identity state has exactly one incoming arrow;
3. at most one outgoing arrow from the active state leads to a non-identity state.

An example of a kneading automaton is actually the automaton from Example 2.3.1 with the active state τ.

If A is a kneading automaton over the binary alphabet, it can be categorized into two general forms. To clarify this distinction, consider the graph obtained after removing the identity state and all arrows leading to it in the Moore diagram of A. From a topological perspective, it is clear that the resulting directed graph Γ_A falls into either of two configurations: a circle or a circle with a sticker $[0,1]$ attached to one of its ends (in the latter case, the active state corresponds to the unique vertex of degree 1). Moreover, one is able to recover the Moore diagram of A from Γ_A since what is missing are solely the identity state and all the arrows leading to it.

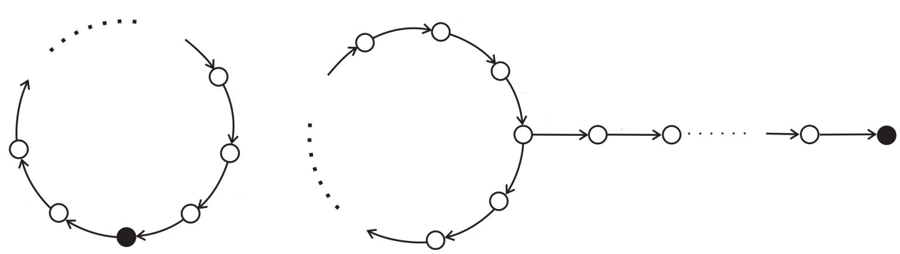


Figure 2.1: The general forms of a kneading automaton.

Remark. In the following, we abbreviate the four possible edge labels $(0,0)$, $(1,1)$, $(0,1)$, and $(1,0)$ in the graph Γ_A by 0, 1, $*_0$, and $*_1$, respectively.

Definition 2.4.2. Let A be a kneading automaton. We define the *kneading sequence* of A as follows:

1. In the case where Γ_A is topologically a circle, we trace the arrows backwards from the active state, while recording the labels in the order they are encountered. Specifically, if Γ_A has n vertices $a_0, \dots, a_{n-1}$, where a_0 is the active state, and each arrow from a_i to a_{i-1} has label l_i, then we start from the active state and by following the arrows backwards we write down the sequence. Formally, our arrows l_i are labeled by pairs. However, since we made the abbreviation in the remark above, when we reach the active state again, we record a string $v = \ell_1\ell_2\cdots\ell_{n-1}\ell_0$. We define the kneading sequence to be v^ω and in this case we say that the kneading sequence is *periodic*.

2. If Γ_A is topologically a circle with a sticker, then we similarly trace the arrows backward from the active state (which is necessarily the unique vertex of degree 1 in Γ_A) and record the labels. The kneading sequence takes the form $u(v)^\omega$, where $u \in \{0,1\}^*$ is the label of the sticker and $v \in \{0,1\}^*$ is the label of the circle, both of which are non-trivial strings. The latter label v is recovered from Γ_A in the way we described in the first part. In this case we say that the kneading sequence is *pre-periodic*.

Let it be noted that through its kneading sequence one is able to figure out the kneading automaton A by constructing first Γ_A and subsequently its Moore diagram. Another observation is that different kneading sequences clearly induce different automata. For example, the automaton derived from $1(0)^\omega$ is not the same as the one with sequence $1(00)^\omega$.

Definition 2.4.3. A kneading automaton A over $\{0,1\}$ is *planar* if there is some circular ordering $a_1, \cdots, a_m$ of the non-trivial states of A such that $(a_1 \cdots a_m)^2|_x$ is a cyclic shift of $a_1 \cdots a_m$ for each $x \in \{0,1\}$.

Remark. It is sufficient to check the condition of Definition 2.4.3 for one letter x.

The following definition is based on the discussion from [14, Section 6.10.1, page 184]:

Definition 2.4.4. A kneading automaton over $\{0,1\}$ is said to *have bad isotropy groups* if its kneading sequence is of the form $u(v)^\omega$, where u is non-trivial and v is a proper power.

For instance, the automaton described in Example 2.3.1 is planar. This is clear from the fact that $(abt)^2|_0 = bta$ and $(abt)^2|_1 = tab$. In addition, it does not have bad isotropy groups.

2.5 Growth of groups

Given two non-decreasing functions $f, g : \mathbb{N} \to \mathbb{R}_0^+$ we say f *is dominated by* g, we write $f \precsim g$, if there exists a natural number $C \geq 1$ such that $f(n) \leq g(Cn)$ for all $n \geq 1$. The functions f and g are said to be *equivalent*, written $f \sim g$, if $f \precsim g$ and $g \precsim f$. Note that all exponential functions b^n are equivalent and polynomial functions of different degree are inequivalent; the same holds for the subexponential functions e^{n^a}. We have

$$0 \precnsim n \precnsim n^2 \precnsim \cdots \precnsim e^{\sqrt{n}} \precnsim \cdots \precnsim e^n.$$

Note also that the ordering $\precsim$ is not linear.

We say that $f : \mathbb{N} \to \mathbb{R}_0^+$ is of

1. *polynomial growth* if there exists $d \in \mathbb{N}$ such that $f \precsim n^d$ (if $d = 1$, we say growth is *linear*);
2. *superpolynomial growth* if $n^d \precsim f$ for all $d \in \mathbb{N}$;
3. *exponential growth* if $f \sim e^n$;
4. *subexponential growth* if $f \precnsim e^n$;
5. f is of *intermediate growth* if f is of superpolynomial growth and of subexponential growth.

Definition 2.5.1. Consider a group G with a fixed finite and symmetric generating set Σ. Any map

$$\tilde{\ell} : \Sigma \to \mathbb{R}^+,$$

assigning a strictly positive weight to each generator will be called *weight* on (G, Σ). It extends to a *length function* on the set Σ^* of words over the alphabet Σ:

$$\begin{aligned} \tilde{\ell} : \Sigma^* &\to \mathbb{R}_0^+ \\ w = x_1 \cdots x_r &\mapsto \sum_{i=1}^r \tilde{\ell}(x_i). \end{aligned} \tag{2.5.1}$$

This length descends to a *length function* l on G as folows:

$$\begin{aligned}\ell : G &\to \mathbb{R}_0^+\\ g &\mapsto \min\{\tilde{\ell}(w) \mid w \in \Sigma^* \text{ represents } g\}.\end{aligned}$$

The map $\ell : G \to \mathbb{R}_0^+$ satisfies the following properties:

1. For any group element $g \in G$ we have $\ell(g) \geq 0$.
2. We have $\ell(g) = 0$ if and only if $g = 1$.
3. For any two group elements $g, h \in G$ the inequality $\ell(gh) \leq \ell(g) + \ell(h)$ holds.
4. For any radius r, the set $B_{G,\Sigma,\tilde{\ell}}(r) = \{g \in G \mid \ell(g) \leq r\}$ is finite. We call it the ball of radius r in G.

Definition 2.5.2. A word $w \in \Sigma^*$ of minimal length is *reduced* if it contains no subword of the form xx^{-1}, where $x \in \Sigma$.

Definition 2.5.3. The *growth function* gr associated to Σ and $\tilde{\ell} : \Sigma \to \mathbb{R}^+$ is defined as the "combinatorial volume" of the ball of radius r in G:

$$\mathrm{gr}(r) = \mathrm{gr}_{G,\Sigma,\tilde{\ell}}(r) = \#\{g \in G \mid \ell(g) \leq r\}.$$

Lemma 2.5.1. [1, Lemma 2] Let Σ and Σ' be two finite generating sets for the group G and let ℓ and ℓ' be weights on (G, Σ) and (G, Σ') respectively. Then, $\mathrm{gr}_{G,\Sigma,\tilde{\ell}} \sim \mathrm{gr}_{G,\Sigma',\tilde{\ell}'}$.

Remark. The latter implies that even though gr depends on the choices of Σ and $\tilde{\ell} : \Sigma \to \mathbb{R}^+$, different generating sets and weights induce equivalent growth functions. In particular, any other choice of Σ and $\tilde{\ell}$ will yield a Lipschitz equivalent length function on G.

Definition 2.5.4. The *growth type* of a finitely generated group G is defined to be the growth type (polynomial, exponential or intermediate) of its growth function gr.

The following proposition is modeled upon the standard proof of subexponential growth for the First Grigorchuk group.

Proposition 2.5.1. [5, Proposition 10] Let H be a finite index subgroup of a finitely generated group G and let l be a length function on G as above. Suppose that there exist $\eta \in [0, 1)$, $p \in (0, 1]$, $K \geq 0$ and an injective homomorphism

$$\begin{aligned}\varphi : H &\to \overbrace{G \times \cdots \times G}^{n \text{ factors}}\\ h &\mapsto (\varphi_1(h), \ldots, \varphi_n(h))\end{aligned}$$

such that the following holds:

$$\text{For each } r, \text{ the proportion of elements in } \{h \in H \mid \ell(h) \leq r\} \text{ satisfying } \sum_{i=1}^{n} \ell(\varphi_i(h)) \leq \eta r + K \text{ is at least } p.$$

Then, G has subexponential growth.

Let us assume now that $G \leq \operatorname{Aut} X^*$ is a self-similar group. Fix some indexing $\{x_1, \cdots, x_d\}$ of X. We know that the first level stabilizer $\mathrm{St}(1)$ is a finite index subgroup of G and from Proposition 2.2.1 we get an injective homomorphism

$$\begin{aligned}\psi : \mathrm{St}(1) &\to \overbrace{G \times \cdots \times G}^{n \text{ factors}}\\ \psi(g) &\mapsto \langle\!\langle g|_{x_1}, \ldots, g|_{x_d} \rangle\!\rangle.\end{aligned}$$

Corollary 2.5.1. Let G be a self-similar group acting on the rooted tree X^*. Using the notation above, suppose that there exist $\eta \in [0,1)$, $p \in (0,1]$ and $K \geq 0$ such that for each r, the proportion of elements in $\{g \in \mathrm{St}(1) \mid \ell(g) \leq r\}$ satisfying $\sum_{i=1}^{d} \ell(g_{x_i}) \leq \eta r + K$ is at least p. Then, G has subexponential growth.

When we defined the weights on the generating set of G, we set $\tilde{\ell}$ to take only positive values. Nevertheless, as we will see later, it is often convenient to allow some generators to have zero length. Given a finite symmetric generating set of a group G, we are able to define a metric on G called the word metric. For our needs, it is our advantage to consider the general notion of a word pseudometric.

Definition 2.5.5. Let G be a finitely generated group and Σ be a symmetric finite set generating G. A map $|\cdot| : \Sigma \to \{0,1\}$ which associates to every generator a length of 0 or 1 will be called a *(pseudo)weight* on Σ. A (pseudo)weight can be extended to a map

$$\begin{aligned} |\cdot| : G &\to \mathbb{N} \\ g &\mapsto \min\{\sum_{i=1}^{k} |s_i| \mid g = s_1 \dots s_k \text{ with } s_i \in \Sigma\} \end{aligned}$$

called the *word pseudonorm* of G (associated to $(\Sigma, |\cdot|)$). The corresponding pseudometric

$$\begin{aligned} d : G \times G &\to \mathbb{N} \\ (g,h) &\mapsto |g^{-1}h| \end{aligned}$$

is the *word pseudometric* of G (associated to $(\Sigma, |\cdot|)$).

If every generator is assigned a length of 1, then the word pseudometric is in fact a metric, called the *word metric*. If there is only a finite number of elements with length 0, one can define a growth function for the group with regards to the given pseudometric. The growth function thus obtained is in fact equivalent to the usual growth function.

Proposition 2.5.2. [8, Proposition 2.3] Let G be a group generated by a finite symmetric set Σ and $|\cdot| : \Sigma \to \{0,1\}$ be a (pseudo)weight on Σ. If the subgroup

$$G_0 = \langle \{s \in \Sigma \mid |s| = 0\} \rangle$$

is finite, then the growth function

$$\begin{aligned} \gamma_{G,\Sigma,|\cdot|} : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto |B_{G,\Sigma,|\cdot|}(n)|, \end{aligned}$$

where $B_{G,\Sigma,|\cdot|} = \{g \in G \mid |g| \leq n\}$, is well-defined. Furthermore, $\gamma_{G,\Sigma,|\cdot|} \sim \gamma_{G,\Sigma}$, where $\gamma_{G,\Sigma}$ is the usual growth function obtained by giving length 1 to each generator.

Remark. A word pseudonorm of G whose words of zero length create a finite subgroup will be called *proper* word pseudonorm. Proposition 2.5.2 ensures that the growth function induced by a proper word pseudonorm is equivalent to the one induced by a word metric. Therefore, we will treat the two cases as indistinguishable.

If ones wants to establish subexponential growth for a group acting on a rooted tree, a common approach is to find a suitable subset which has the same growth as the initial one and show that if we project its elements to some level, we end up having a notable percentage of elements that exhibit length reduction. In particular, in Proposition 2.5.1 we look for a subgroup of finite index, we project its elements into the product of as many factors of G as necessary so that for no matter how big the length of an element inside the subgroup, the projection yields a significant length reduction. In what follows, we will work with a similar idea, meaning that we will look at the growth rate of incompressible elements of G.

Definition 2.5.6. Let G be a finitely generated group acting on a d-regular rooted tree and let $|\cdot|$ be a proper pseudonorm on G. We say that $|\cdot|$ is a *non-ℓ_1-expanding* word pseudonorm on G if for every $g = \langle\!\langle g_1, \dots, g_d \rangle\!\rangle\alpha$ in G we have $\sum_{i=1}^{d} |g_i| \leq |g|$.

Suppose that G is a finitely generated group acting on a d-regular rooted tree with a non-ℓ_1-expanding proper pseudonorm $|\cdot|$. Let $\mathcal{I}_n$ denote the subset of G of elements that have no length reduction up to the nth level of the tree. It is defined recursively by $\mathcal{I}_0 = G$ and

$$\mathcal{I}_n = \{g = \langle\!\langle g_1, \dots, g_d \rangle\!\rangle\alpha \in G \mid \sum_{i=1}^{d} |g_i| = |g| \text{ and } g_i \in \mathcal{I}_{n-1} \text{ for every } 1 \leq i \leq d\}.$$

Then, $\mathcal{I} = \bigcap_{n\geq 0} \mathcal{I}_n$ is the set of words that have no length reduction on any level of the tree. We call the elements of $\mathcal{I}$ *incompressible*. The proposition below originates from [4, Proposition 5], but a stronger statements of it exists in [8].

Proposition 2.5.3. [8, Theorem 3.13] Let $G = \langle \Sigma \rangle$ be a group as above with Σ finite and $\Sigma \subseteq \mathcal{I}$ and let $\Omega(n)$ be the sphere of radius $n \geq 1$ in G with respect to $|\cdot|$. If there exists a function $\delta : \mathbb{N} \to \mathbb{R}_0^+$ of subexponential growth with $\ln(\delta)$ concave such that $\mathcal{I} \cap \Omega(n) \leq \delta(n)$ for all $n \geq 1$, then G has subexponential growth. Moroever, if $\mathcal{I}_k$ has linear growth for some k, then the growth of G is bounded in the following way:

$$\gamma_{G,\Sigma}(n) \precsim e^{n\frac{(\log\log n)^2}{\log n}}.$$

Remark. In all of our examples $\mathcal{I}$ grows polynomially.

Apart from tools to prove subexponential growth, we are in need of one which determines whether we have superpolynomial growth as well.

Definition 2.5.7. Two groups G_1 and G_2 are called *commensurable*, denoted $G_1 \approx G_2$, if they contain isomorphic groups of finite index. A group G is called *multilateral*, if G is infinite and $G \approx G^m$ for some $m \geq 2$.

Proposition 2.5.4. [10, Lemma 6.4] Every multilateral group G has superpolynomial growth. Moreover, we have that $\exp(n^a) \precsim \mathrm{gr}(n)$ for some $a > 0$.

2.6 Iterated monodromy group

For the construction of the iterated monodromy group, we follow [2, Section 5.1]. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and suppose we have a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. If d is the degree of the rational function, then the map f is a d-fold branched covering, meaning that it locally behaves as $z \mapsto z^{d_0}$ for some $d_0 \in \mathbb{N}$ in some orientation-preserving coordinate charts.

Definition 2.6.1. A rational map $f : \mathbb{C} \to \mathbb{C}$ is called *post-critically finite* if the post-critical set $P = \{f^n(c) \mid c \text{ is critical point and } n \geq 1\}$ is finite.

From now, we work only with post-critically finite rational maps. In this case, we get that $f^{-1}(\hat{\mathbb{C}} \setminus P)$ is a subset of $\hat{\mathbb{C}} \setminus P$ and f becomes a local homeomorphism at every $t \in \hat{\mathbb{C}} \setminus P$.

Consider an arbitrary $t \in \hat{\mathbb{C}} \setminus P$. Then, from what we mentioned, it is clear that for every $n \in \mathbb{N}$ the point has d^n preimages under f^n. Let T denote the disjoint union of the sets $f^{-n}(t)$ for $n \geq 0$, where we set $f^{-0}(t)$ to be $\{t\}$, i.e.

$$T = \bigcup_{n=0}^{\infty} f^{-n}(t) \times \{n\}.$$

Note that we see an element (z, n) as one in $f^{-n}(t)$. We identify T with the vertex set of a d-regular rooted tree with root $(t, 0)$ in which a vertex (z, n) is connected with the vertex $(f(z), n-1)$. We call this tree the *preimage tree* of the point t.

Consider γ a loop in $\hat{\mathbb{C}} \setminus P$ based at t. Then, since $f^n : \hat{\mathbb{C}} \setminus f^{-n}(P) \to \hat{\mathbb{C}} \setminus P$ becomes a covering, for every element $v = (z, n)$ in the preimage tree T there exists a path γ_v starting in z such that $f^n(\gamma_v) = \gamma$. Then, it is direct that $f^n(z^\gamma) = t$, hence the element $v^\gamma = (z^\gamma, n)$ is in T.

Proposition 2.6.1. [2, Proposition 5.2] The map $v \mapsto v^\gamma$ is an automorphism of the preimage tree which depends only on the homotopy class of γ in $\hat{\mathbb{C}} \setminus P$, that is we have an action of the fundamental group $\pi_1(\hat{\mathbb{C}} \setminus P, t)$ on the preimage tree T. The set of all such automorphisms is a group which is the quotient of the fundamental group of the space $\hat{\mathbb{C}} \setminus P$ with the kernel of the action. Up to isomorphism, this group does not depend on the choice of the base point t.

Definition 2.6.2. The group from Proposition 2.6.1 is called the *iterated monodromy group* of the map f and is written $\mathrm{IMG}(f)$.

The preimage tree T is d-regular, so we are able to identify it with the tree X^* for an alphabet X with d letters. Namely, consider an arbitrary bijection $\Lambda : X \to f^{-1}(t)$. For every $x \in X$ consider also a connecting path ℓ_x in $\hat{\mathbb{C}} \setminus P$ from the basepoint t to $\Lambda(x)$. We are able to extend the bijection $\Lambda : X \to f^{-1}(t)$ to a map $\Lambda : X^* \to T$ of rooted trees inductively by the rules:

1. $\Lambda(\varnothing) = (t, 0)$;
2. for each $v \in X^n$ and $x \in X$, we set $\Lambda(xv)$ to be the endpoint of the (unique) path ℓ_v starting at $\Lambda(v) \in f^{-n}(t)$ such that $f^n(\ell_v) = \ell_x$.

Proposition 2.6.2. [2, Proposition 5.3] The map $\Lambda : X^* \to T$ is an isomoprhism of the rooted trees.

Definition 2.6.3. The *standard action* of the iterated monodromy group on the tree X^* is obtained from its action on the preimage tree T by conjugating it with the isomorphism $\Lambda : X^* \to T$.

The following proposition gives a way to compute this standard action.

Proposition 2.6.3. [2, Proposition 5.4] The standard action of an iterated monodromy group is self-similar. More precisely, if γ is a loop at t and $x \in X$ is a letter then, respectively to the standard action, for every $v \in X^*$ we have

$$(xv)^\gamma = y(v^{\ell_x \gamma_x \ell_y^{-1}}),$$

where γ_x is the preimage of γ starting at $\Lambda(x)$ and y is such that $\Lambda(y)$ is the end of γ_x (i.e., $x^\gamma = y$).

Note that $l_x \gamma_x l_y^{-1}$ is a loop based at t.

Example 2.6.1. Consider the polynomial $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with $f(z) = z^2$. The set of critical points is $C = \{0, \infty\}$, so the post-critical set is $P = \{0, \infty\}$. We also calculate $f^{-1}(P) = \{0, \infty\}$. Then, the map

$$f : \hat{\mathbb{C}} \setminus f^{-1}(P) \to \hat{\mathbb{C}} \setminus P$$

becomes a covering map. We have to choose a basepoint. Since 0 is not in the domain of the covering, we set $t = 1$. Next, we calculate $f^{-1}(1) = \{-1, 1\}$ and we set $x = 1$ and $y = -1$. We have

$$\pi_1(\hat{\mathbb{C}} \setminus f^{-1}(C), t) \cong \langle a \rangle,$$

where a is the simple loop based at t around the critical point 0 going counterclockwise. Next, take ℓ_x equal to the constant path at $t = x$ and define ℓ_y to be the upper semicircle from t to y. Denote by a_x, a_y the lifts of a starting at x and y, respectively. The lift of a loop around a critical point should go around exactly one $v \in f^{-1}(0)$. It remains a loop if v does not belong to the critical set and it opens otherwise. Since a is a loop around the critical point $0 \in f^{-1}(0)$, the corresponding lifts are not loops. Using the standard action of the iterated monodromy group, we have that

- $(xw)^a = x^a w^{\ell_x a_x \ell_{x^a}^{-1}} = yw^{\ell_x a_x \ell_y^{-1}} = yw^{\mathrm{id}}$
- $(yw)^a = y^a w^{\ell_y a_y \ell_{y^a}^{-1}} = xw^{\ell_y a_y \ell_x^{-1}} = xw^a$

So, the standard action of the iterated monodrmoy group of z^2 is described by $a = \langle\!\langle \mathrm{id}, a \rangle\!\rangle (0\ 1)$, meaning that a acts on the binary tree $\{0,1\}^*$ as the adding machine.

Let $f(z) = z^2 + c$ be a post-critically finite quadratic polynomial. We can naturally associate to f a *kneading sequence* describing the symbolic dynamics of the critical point 0; see for example [14, Section 6.10.3].

Theorem 2.6.1. [14, Theorem 6.9.6] Let A be an invertible reduced kneading automaton over X, where $|X| = 2$. If A is planar and does not have bad isotropy groups, then $\langle A \rangle = \langle q : X^* \to X^* \mid q \in A \rangle$ is the iterated monodromy group of a post-critically finite quadratic polynomial $p : \mathbb{C} \to \mathbb{C}$ and the kneading sequence of A is also the kneading sequence of p (up to relabeling of X).

After all the notions we described, it is only fair to state the conjecture once again. Before that, we are in need of the definition of non-renormalizability. We derive the latter from [13, Section 7.1].

Definition 2.6.4. Let U, V be a pair of disks. A *proper map between disks* $f : U \to V$ is a holomorphic map such tath $f^{-1}(K)$ is compact for every compact set $K \subseteq V$. Then, $f^{-1}(x)$ is finite for all $x \in V$ and the cardinality of the inverse image of a point (counted with multiplicity) is the *degree* of f.

Definition 2.6.5. Let $f(z) = z^2 + c$ be a complex quadratic polynomial. We say that f is *renormalizable* if there exist open disks U and V in $\mathbb{C}$ such that U is compactly contained in V, the critical point $0 \in U$, and for $f^n : U \to V$ we have that f^n is proper of degree 2 and $f^{nk}(0) \in U$ for all $k \geq 0$. Otherwise, we say that f is *non-renormalizable.*

Conjecture 2.6.2. If $f : \mathbb{C} \to \mathbb{C}$ is a non-renormalizable post-critically finite quadratic polynomial with pre-periodic kneading sequence, then $\mathrm{IMG}(f)$ has intermediate growth.

Chapter 3

Examples

3.1 First Grigorchuk group

We start with the first example in history to establish the existence of groups of intermediate growth. The first Grigorchuk group, named after Rostislav Grigorchuk, is defined as $\mathcal{G} = \langle \sigma, \alpha, \beta, \gamma \rangle$, where $\sigma, \alpha, \beta, \gamma$ are binary tree automorphisms defined by

- $\sigma = \langle\!\langle \mathrm{id}, \mathrm{id} \rangle\!\rangle (0\ 1)$
- $\alpha = \langle\!\langle \sigma, \beta \rangle\!\rangle$
- $\beta = \langle\!\langle \sigma, \gamma \rangle\!\rangle$
- $\gamma = \langle\!\langle \mathrm{id}, \alpha \rangle\!\rangle$

Lemma 3.1.1. [5, Lemma 11] The set $\{\mathrm{id}, \alpha, \beta, \gamma\}$ is a subgroup of $\mathcal{G}$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. The system

$$\begin{aligned} x &= \langle\!\langle \mathrm{id}, y \rangle\!\rangle \\ y &= \langle\!\langle \mathrm{id}, z \rangle\!\rangle \\ z &= \langle\!\langle \mathrm{id}, x \rangle\!\rangle, \end{aligned}$$

where $x, y, z \in \mathrm{Aut}\{0,1\}^*$, clearly defines $x = y = z = \mathrm{id}$. On the other side, $(\alpha^2, \beta^2, \gamma^2)$ and $(\alpha\beta\gamma, \beta\gamma\alpha, \gamma\alpha\beta)$ also verify the above system, so all generators of G have order 2 and commute, while the product of two of them equals the third one. As a result, $\{\mathrm{id}, \alpha, \beta, \gamma\}$ is indeed a subgroup isomorphic to a quotient of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The proof becomes complete by realising that $\alpha = \langle\!\langle \sigma, \beta \rangle\!\rangle \neq \mathrm{id}$, since it acts non-trivially on the left subtree, implying that $\gamma \neq \mathrm{id}$ too, while $\gamma \neq \alpha$. $\square$

Given Lemma 3.1.1, we get that $(\sigma\gamma)^4 = (\sigma\beta)^8 = (\sigma\alpha)^{16} = \mathrm{id}$. Indeed, $(\sigma\gamma)^2 = \sigma\gamma\sigma\gamma = \langle\!\langle a, a \rangle\!\rangle$. So, $(\sigma\gamma)^4 = \mathrm{id}$ and the rest are proven similarly. Taking into consideration these properties we deduce that every reduced word in $\mathcal{G}$ must have the following minimal form

$$[\sigma] * \sigma * \cdots * \sigma * [\sigma] \text{ , where } * \in \{\alpha, \beta, \gamma\}.$$

Here and below, the notation $[\sigma]$ corresponds to either σ or id. To avoid confusion, note that if $*$ is found inside a word, then $* \in \{\alpha, \beta, \gamma\}$, while if $*$ is inside the left or right coordinate of the wreath decomposition, then $*$ can be also σ.

In order to show that $\mathcal{G}$ has intermediate growth, we must show that $\mathcal{G}$ has both superpolynomial and subexponential growth. We begin with the first task.

Proposition 3.1.1. [10, Section 6] The group $\mathcal{G}$ has superpolynomial growth.

Proof. According to Proposition 2.5.4, it is enough to show that $\mathcal{G}$ is multilateral. To prove that it is infinite, consider the subgroup $\mathrm{St}(1)$ of $\mathcal{G}$. Note that a word (in generators) is in this subgroup if and only if the number of occurrences of σ is even. This means that $\mathcal{G} = \mathrm{St}(1) \coprod \sigma\,\mathrm{St}(1)$, making the first level stabilizer a subgroup of index 2, hence it is a normal subgroup.

Next, if we look at the minimal form of a word in $\mathrm{St}(1)$, we may see it as a product of $*$ and $\sigma * \sigma = *^{\sigma}$, where $* \in \{\alpha, \beta, \gamma\}$, so

$$\mathrm{St}(1) = \langle \alpha, \beta, \gamma, \alpha^{\sigma}, \beta^{\sigma}, \gamma^{\sigma} \rangle.$$

By definition, $\mathrm{St}(1) \leq \mathcal{G} \leq \mathrm{Aut}(\{0,1\}^*)$. Next, the image $\psi(\mathrm{St}(1))$, where ψ is the canonical isomorphism from Proposition 2.2.1, is in $\mathrm{Aut}(\{0,1\}^*) \times \mathrm{Aut}(\{0,1\}^*)$ and, in fact, $\psi(\mathrm{St}(1)) \leq \mathcal{G} \times \mathcal{G}$ from self-similarity. On the other hand, the projection of the image onto each component contains all generators $\sigma, \alpha, \beta, \gamma \in \mathcal{G}$ and therefore it is surjective. To close this argument, suppose that $\mathcal{G}$ is a finite group. Then, since $\mathrm{St}(1)$ is a proper subgroup of $\mathcal{G}$ which is mapped surjectively onto it, we have

$$|\mathcal{G}| \leq |\mathrm{St}(1)| < |\mathcal{G}|.$$

This is a contradiction, and thus $\mathcal{G}$ is infinite.

Moving on, we will show that $\mathcal{G} \approx \mathcal{G} \times \mathcal{G}$, making $\mathcal{G}$ multilateral. Consider the subgroups $\mathrm{St}(1) \leq \mathcal{G}$ and $\psi(\mathrm{St}(1)) \leq \mathcal{G} \times \mathcal{G}$ which are isomorphic, because ψ is an isomorphism. The subgroup $\mathrm{St}(1)$ has finite index, so we only need to verify that $[\mathcal{G} \times \mathcal{G}, \psi(\mathrm{St}(1))] < \infty$. With this intention consider the normal closure

$$A = \langle g^{-1} \alpha g | g \in \mathcal{G} \rangle$$

of $\alpha \in \mathcal{G}$. Then, A is a normal subgroup of $\mathcal{G}$. Since $\beta = \alpha\gamma$, we get that $\mathcal{G} = \langle \sigma, \alpha, \gamma \rangle$ and $\mathcal{G}/A$ is a quotient of $\langle \sigma, \gamma \rangle$. As $\sigma^2 = \gamma^2 = (\sigma\gamma)^4 = \mathrm{id}$, we deduce that $[\mathcal{G} : A] \leq |\langle \sigma, \gamma \rangle| = 8$. Note that $A \times A$ is in $\psi(\mathrm{St}(1))$. Indeed, let $x \in \mathrm{St}(1)$ and write it as $x = \langle\!\langle g_0, g_1 \rangle\!\rangle$ with $g_i \in \mathcal{G}$. Then, $x^{-1}\gamma x = \langle\!\langle \mathrm{id}, g_1^{-1}\alpha g_1 \rangle\!\rangle$. Since each projection of $\psi(\mathrm{St}(1))$ is mapped surjectively onto $\mathcal{G}$, for any $g \in \mathcal{G}$ we can choose $x \in G$ so that $g_1 = g$. Hence, $\psi(\mathrm{St}(1))$ contains all elements of the form $\langle\!\langle \mathrm{id}, g^{-1}\alpha g \rangle\!\rangle$ with $g \in \mathcal{G}$. Similarly, by looking at $x^{-1}\gamma^{\sigma} x$ instead of γ, we get that $\langle\!\langle g^{-1}\alpha g, \mathrm{id} \rangle\!\rangle \in \psi(\mathrm{St}(1))$ for all $g \in \mathcal{G}$.

Hence $A \times A \leq \psi(\mathrm{St}(1))$ and

$$[\mathcal{G} \times \mathcal{G}, \psi(\mathrm{St}(1))] \leq [\mathcal{G} \times \mathcal{G}, A \times A] = [G : A]^2 \leq 8^2 = 64.$$

So, $\psi(\mathrm{St}(1))$ has finite index in $\mathcal{G} \times \mathcal{G}$, making $\mathcal{G}$ and $\mathcal{G} \times \mathcal{G}$ commeasurable.

□

Remark. The method employed here to prove superpolynomial growth is analogous for the subsequent cases. Therefore, we will omit it from now on and completely turn our attention to the discussion of subexponential growth.

We will provide two "different" proofs, meaning that we will implement both Propositions 2.5.1 and 2.5.3.

Proposition 3.1.2. The group $\mathcal{G}$ has subexponential growth.

First proof. [5, Theorem 12] Consider the injective homomorphism

$$\psi : \mathrm{St}(1) \hookrightarrow \mathcal{G} \times \mathcal{G}.$$

Let ℓ be the length function on $\mathcal{G}$ induced by the weights:

$$\tilde{\ell}(\sigma) = 3 \quad \tilde{\ell}(\alpha) = 5 \quad \tilde{\ell}(\beta) = 4 \quad \tilde{\ell}(\gamma) = 3.$$

We know that each word $w \in \mathrm{St}(1)$ is written as $w = [\sigma] * \sigma * \cdots * \sigma * [\sigma]$ with an even number of σ letters. Split w into blocks of four letters, possibly followed by a single shorter block in the end. One can check that the homomorphism ψ will reduce the length of each four letter word $u \in \{\sigma * \sigma *, *\sigma * \sigma\}$ by a factor of at least $\frac{7}{8}$, that is: $\ell(u|_0) + \ell(u|_1) \le \frac{7}{8}\ell(u)$. For example, $\sigma\alpha\sigma\gamma = \langle\!\langle \beta, \sigma\alpha \rangle\!\rangle$ corresponds to a reduction from length 14 to 12. The worst case is attained by the block $\sigma\alpha\sigma\alpha = \langle\!\langle \beta\sigma, \sigma\beta \rangle\!\rangle$ which yields a reduction from 16 to 14. We obtain that for every $g \in \mathcal{G}$

$$l(g|_0)) + l(g|_1) \le \frac{7}{8} l(g) + 3.$$

Proposition 2.5.1 with parameters $\eta = \frac{7}{8}, p = 1$ and $K = 3$ guarantees that $\mathcal{G}$ has subexponential growth. Moving forward, we provide another proof using Proposition 2.5.3.

Second proof. We are now looking for the incompressible elements of the group, i.e., the elements which admit no length reduction at any level. In order to do so, we will change the weights to

$$|\sigma| = 0 \quad |\alpha| = 1 \quad |\beta| = 1 \quad |\gamma| = 1.$$

As a result, our pseudonorm becomes proper and non-l_1-expanding, while $\{\sigma, \alpha, \beta, \gamma\} \subseteq \mathcal{I}$.

We start by describing the elements of $\mathcal{G}$ which have no length reduction up to level 1. Note that

$$*\sigma\gamma\sigma* = \langle\!\langle *\alpha*, ** \rangle\!\rangle.$$

Since the product of two of the generators equals the third, we deduce that in $\mathcal{I}_1$ patterns of the form $[\sigma] * \sigma\gamma\sigma * [\sigma]$ cannot occur. For instance, $\sigma\alpha\sigma\gamma\sigma\beta\sigma = \langle\!\langle \beta\gamma, \sigma\alpha\sigma \rangle\!\rangle = \langle\!\langle \alpha, \sigma\alpha\sigma \rangle\!\rangle \notin \mathcal{I}_1$. Therefore, for a word $[\sigma] * \sigma * \cdots * \sigma * [\sigma]$ in $\mathcal{I}_1$

- the letter γ can only occur in the first or last $*$.

Moving on to the next level, we get that in $\mathcal{I}_2$ we cannot have patterns such as $[\sigma] * \sigma * \sigma\beta\sigma * \sigma * [\sigma]$, as there is going to appear either a length reduction or the pattern which got excluded in the previous level. For example,

$$\sigma\alpha\sigma\alpha\sigma\beta\sigma\alpha\sigma\alpha\sigma = \langle\!\langle \beta\sigma\gamma\sigma\beta, \sigma\beta\sigma\beta\sigma \rangle\!\rangle \notin \mathcal{I}_2, \text{ since } \beta\sigma\gamma\sigma\beta \notin \mathcal{I}_1.$$

By definition the sequence $(\mathcal{I}_n)_n$ is decreasing. So, for a word in $\mathcal{I}_2$ we know that

- γ can only occur in the first or last $*$;
- β can only occur in the first two or last two $*$'s.

Continuing, in $\mathcal{I}_3$ there can be no pattern of the form $[\sigma] * \sigma * \sigma * \sigma\alpha\sigma * \sigma * \sigma * [\sigma]$. So, for a word in $\mathcal{I}_3$ we have the following rules

- γ can only occur in the first or last $*$;
- β can only occur in the first two or last two $*$'s;
- a can only occur in the first three or last three $*$'s.

From these rules we deduce that in $\mathcal{I}_3$ we cannot have a word with pseudonorm greater than 6. So, $\mathcal{I}_3$ is bounded. By Proposition 2.5.3 the group $\mathcal{G}$ has subexponential growth.

3.2 Second Grigorchuk group

The second Grigorchuk group is described as $\mathcal{H} = \langle \sigma, \alpha, \beta \rangle$, where the generators are defined by the following equations:

- $\sigma = \langle\!\langle \mathrm{id}, \mathrm{id} \rangle\!\rangle (0\ 1)$
- $\alpha = \langle\!\langle \sigma, \beta \rangle\!\rangle$
- $\beta = \langle\!\langle \mathrm{id}, \alpha \rangle\!\rangle$

As mentioned in [5], this group is less manageable than $\mathcal{G}$, as it contains a "self-replicating element" $\sigma\alpha\beta$ of infinite order.

Set $\gamma = \alpha\beta$. We will consider γ as a generator, meaning that we set $\mathcal{H} = \langle \sigma, \alpha, \beta, \gamma \rangle$. Similarly to the previous case, we bring up the properties of the generators we need.

Lemma 3.2.1. [5, Lemma 14] The elements α, β generate a copy of $\mathbb{Z}_2 \times \mathbb{Z}_2$ inside $\mathcal{H}$. In addition, $\sigma\alpha$ and $\sigma\beta$ have order 8 and 4 respectively, while $\sigma\gamma$ has infinite order.

Proof. Consider two systems:

$$\alpha^2 = \langle\!\langle \mathrm{id}, \beta^2 \rangle\!\rangle$$
$$\beta^2 = \langle\!\langle \mathrm{id}, \alpha^2 \rangle\!\rangle$$

and

$$\alpha\beta\alpha\beta = \langle\!\langle \mathrm{id}, \beta\alpha\beta\alpha \rangle\!\rangle$$
$$\beta\alpha\beta\alpha = \langle\!\langle \mathrm{id}, \alpha\beta\alpha\beta \rangle\!\rangle.$$

Therefore, $\alpha^2 = \beta^2 = (\alpha\beta)^2 = \mathrm{id}$ and α, β generate a quotient of $\mathbb{Z}_2 \times \mathbb{Z}_2$. However, we see that $\alpha \neq \mathrm{id}$, making $\beta \neq \mathrm{id}$, while $\alpha \neq \beta$. As a result, $\langle \alpha, \beta \rangle$ has more than two elements and the quotient is actually a copy of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Continuing, observe that

$$(\sigma\beta)^2 = \sigma\beta\sigma\beta = \langle\!\langle \alpha, \alpha \rangle\!\rangle \text{ and } (\sigma\alpha)^2 = \sigma\alpha\sigma\alpha = \langle\!\langle \beta\sigma, \sigma\beta \rangle\!\rangle.$$

Thus, it is direct that $\sigma\beta$ has order 4 and $\sigma\alpha$ has order 8. On the contrary, $\sigma\gamma$ replicates itself:

$$(\sigma\gamma)^2 = \langle\!\langle \gamma\sigma, \sigma\gamma \rangle\!\rangle, \text{ hence } (\sigma\gamma)^{2n} = \langle\!\langle (\gamma\sigma)^n, (\sigma\gamma)^n \rangle\!\rangle.$$

Since $(\sigma\gamma)^{2n} = \mathrm{id}$ implies $(\sigma\gamma)^n = \mathrm{id}$, the order of $\sigma\gamma$ is either odd or infinite. But any odd power of $\sigma\gamma$ acts non-trivially on the tree, as it performs a swap at the root vertex. So, $\sigma\gamma$ has infinite order. □

The lemma above implies that every element in $\mathcal{H}$ has a minimal form

$$[\sigma] * \sigma * \cdots * \sigma * [\sigma] \text{ , where } * \in \{\alpha, \beta, \gamma\}.$$

Proposition 3.2.1. The group $\mathcal{H}$ has subexponential growth.

Proof. We will chase down the incompressible elements in $\mathcal{H}$. Set the weights of the generators to be

$$|\sigma| = 0 \quad |\alpha| = 1 \quad |\beta| = 1 \quad |\gamma| = 1.$$

Note that the induced pseudonorm on $\mathcal{H}$ is proper and non-l_1-expanding, while $\{\sigma, \alpha, \beta, \gamma\} \subseteq \mathcal{I}$.

Next, we observe that in $\mathcal{I}_1$, we cannot have patterns of the form

$$*\sigma\beta\sigma* = \langle\!\langle *\alpha*, ** \rangle\!\rangle$$

due to length reduction in the right subtree. Moving up one level, in $\mathcal{I}_2$ consider the pattern

$$*\sigma * \sigma * \sigma\alpha\sigma * \sigma * \sigma* = \langle\!\langle * * *\beta * **, * * *\sigma * ** \rangle\!\rangle.$$

If the star either before of after α is β, then the word is going to contain the forbidden subword $*\sigma\beta\sigma*$. However, if both of them are not equal to β, then the latter will appear in the left coordinate (the second and second to last stars are not β by $\mathcal{I}_1$). As a result, we have to exclude the pattern above containing α and in $\mathcal{I}_2$ the only possible reduced form of words is

$$[\sigma] *_1 \sigma *_2 \sigma *_3 \sigma\gamma\sigma\gamma\sigma\cdots\sigma\gamma\sigma\gamma\sigma *_3 \sigma *_2 \sigma *_1 [\sigma] \text{ , where } *_1 \in \{\alpha, \beta, \gamma\}, *_2 \in \{\alpha, \gamma\} \text{ and } *_3 \in \{\alpha, \gamma\}.$$

(Note that not every combination of $*_1, *_2, *_3$ is possible in $\mathcal{I}_2$.) It follows that the growth of $\mathcal{I}_2$ is dominated by n, hence it has subexponential growth. Proposition 2.5.3 now implies that $\mathcal{H}$ also has subexponential growth. □

3.3 Generalized Grigorchuk group

Let $n \in \mathbb{N}$. Consider the group $\mathcal{G}_n = \mathcal{G}(\epsilon_1, \dots, \epsilon_n) = \langle \sigma, a_1, a_2, \dots, a_n \rangle$ where the generators are defined as

- $a_i = \langle\!\langle \sigma^{\epsilon_i}, a_{i+1} \rangle\!\rangle$ for $i = 1, \dots, n-1$
- $a_n = \langle\!\langle \sigma^{\epsilon_n}, a_1 \rangle\!\rangle$,

where each $\epsilon_i = 0, 1$ for each $1 \le i \le n$. Just like in the previous two cases, each a_i has order 2 and they commute with each other, implying that $\{a_1, \dots, a_n\}$ generate a quotient of $(\mathbb{Z}_2)^n$.

Note that the first Grigorchuk group corresponds to $\mathcal{G}(1,1,0)$, while the second one to $\mathcal{G}(1,0)$. Set $S = \{a_{i_1}a_{i_2}\cdots a_{i_k} | 1 \le k \le n \text{ and } \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}\}$ to be the set of all possible combinations created by the generators except the swap. Then, $\mathcal{G}_n = \langle \{\sigma\} \cup S \rangle$ and

$$|\{\sigma\} \cup S| = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Based on what is written, each reduced word in $\mathcal{G}_n$ has the following minimal form

$$[\sigma] * \sigma * \cdots * \sigma * [\sigma] \text{ , where } * \in S.$$

We will once again look for the incompressible elements. In particular, we will assign to σ weight 0 and to each element in S weight 1. Then, the pseudonorm of $\mathcal{G}_g = \langle \{\sigma\} \cup S \rangle$ induced by these weights becomes proper and non-l_1-expanding.

Before we proceed with proving subexponential growth, we make an observation. Let $D = \{j_1, j_2, \dots j_k\} \subseteq \{1, \dots, n\}$ $(k \le n)$ be the biggest set such that $\epsilon_{j_i} = 0$, meaning that D contains exactly the indices of the (initial) generators which act as the identity on the left subtree. Now, if we want to find how many elements in S which consist of two letters have the identity as restriction to the left, we notice that it holds for

1. every product $a_{j_\lambda} \cdot a_{j_\mu}$ with $j_\lambda, j_\mu \in D$;
2. every product $a_p \cdot a_q$ with $p, q \in \{1, \dots, n\} \setminus D$.

As a result, we have $\binom{k}{2} + \binom{n-k}{2}$ choices of generators of two letters. Doing the same for generators of three letters, we see that it holds for

1. every product $a_{j_\lambda} \cdot a_{j_\mu} \cdot a_{j_\nu}$ with $j_\lambda, j_\mu, j_\nu \in D$;
2. every product $a_{j_\lambda} \cdot a_p \cdot a_q$ with $j_\lambda \in D$ and $p, q \in \{1, \dots, n\} \smallsetminus D$.

Therefore, we have $\binom{k}{3} + k\binom{n-k}{2}$ choices of generators of three letters. In general, setting $\binom{x}{y} = 0$ for $x < y$ we have for a generator in S which consists of m letters ($m \le n$)

$$\binom{k}{m} + \binom{k}{m-2}\binom{n-k}{2} + \binom{k}{m-4}\binom{n-k}{4} + \dots + \binom{k}{[m]}\binom{n-k}{m-[m]}$$

choices to have identity on the left where $[m]$ is the projection of m into $\mathbb{Z}/\mathbb{Z}_2$. Set F to be the set of all such generators of S.

Proposition 3.3.1. The group $\mathcal{G}_n$ has subexponential growth.

Proof. From now on, we will not mention anymore which pattern is forbidden at each level, since there will certainly exist a $k \in \mathbb{N}$ such that $\mathcal{I}_k$ does not contain any of the patterns we excluded.

Let us begin by observing that for each $g \in F$ we have

$$*\sigma g \sigma * = \langle\!\langle *g|_1 *, ** \rangle\!\rangle.$$

Suppose that on the left side of the equation in the position of each star there is a generator. Then, on the right side the product $**$ will make a generator in S, to which we have assigned length 1. Hence, there is going to be length reduction and every pattern of the form $*\sigma g \sigma *$ with $g \in F$ is forbidden.

Next, observe that a product of generators $a_1, \dots, a_n$ will act on the right subtree as the product of those whose indices are (cyclically) forward-shifted. For example, $a_1 a_2 = (\sigma^{\epsilon_1 + \epsilon_2}, a_2 a_3)$. In particular, we see the following shift patterns (note that they may be shorter than n):

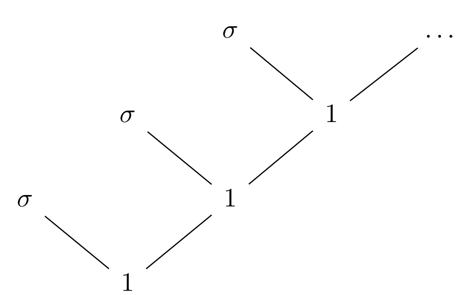
$$\begin{tikzcd}
a_1 \arrow[r] & a_2 \arrow[d] & a_1 a_2 \arrow[r] & a_2 a_3 \arrow[d] & a_1 a_2 a_3 \arrow[r] & a_2 a_3 a_4 \arrow[d] & \cdots \\
a_n \arrow[u] & \cdots \arrow[l] & a_n a_1 \arrow[u] & \cdots \arrow[l] & a_n a_1 a_2 \arrow[u] & \cdots \arrow[l] & \cdots
\end{tikzcd}$$

Consider an element $g \in S$ such that a pattern of the form $\cdots * \sigma * \sigma g \sigma * \sigma * \cdots$ is forbidden. Let $s^{-1}(g) \in S$ be the element given by applying the backward shift to the indices in g (see the diagrams above). Suppose that $s^{-1}(g)$ acts as the swap on the right subtree, otherwise it induces a forbidden pattern too. We have

$$\cdots * \sigma x \sigma s^{-1}(g) \sigma y \sigma * \cdots = \langle\!\langle \cdots * x|_0 g y|_0 * \cdots, \cdots * x|_1 \sigma y|_1 * \cdots \rangle\!\rangle. \tag{3.3.1}$$

If $x|_0 = y|_0 = \sigma$, then the pattern above is excluded given our assumption. Suppose then that $y|_0 = \mathrm{id}$ (the case when $x|_0 = \mathrm{id}$ is similar). Then, $y \in F$ and on the left of (3.3.1) the subword $*\sigma y \sigma *$ appears which is forbidden. It follows that every backward shift of an element of F generates a forbidden pattern as well.

The only elements of S left to check are those whose orbits under the index shift do not contain any elements in F. Every such orbit consists of elements $g \in S$ such that $g|_0 = \sigma$. By induction on the word length, through self-similarity, it is direct that all such elements are the same and equal to $\gamma \in \mathcal{H}$. One could also see this pictorially :



So, every word of the form $\sigma * \sigma \cdots \sigma * \sigma$, where each $*$ is an element of an orbit that does not contain any element in F under the index shift, is essentially $(\sigma\gamma)^\omega$. The latter implies that there exists $\kappa \in \mathbb{N}$ such that $\mathcal{I}_\kappa$ has linear growth, hence $\mathcal{G}_n$ has subexponential growth. ☐

3.4 The automaton $1(10)^\omega$

The group generated by the kneading sequence $1(10)^\omega$ is $G(1(10)^\omega) = \langle \sigma, \alpha, \beta \rangle$ where

- $\sigma = \langle\langle \mathrm{id}, \mathrm{id} \rangle\rangle (0\ 1)$
- $\alpha = \langle\langle \sigma, \beta \rangle\rangle$
- $\beta = \langle\langle \alpha, \mathrm{id} \rangle\rangle$

Remark. The group $G(1(10)^\omega)$ is the iterated monodromy group of $z^2 + i$.

Lemma 3.4.1. [5, Lemma 20] Any two of the generators span a finite dihedral group inside $\mathcal{I}$:

$$D_4(\sigma, \beta) = \langle \sigma, \beta \rangle = \langle \sigma, \beta \mid \sigma^2 = \beta^2 = (\sigma\beta)^4 = \mathrm{id} \rangle$$
$$D_8(\sigma, \alpha) = \langle \sigma, \alpha \rangle = \langle \sigma, \alpha \mid \sigma^2 = \alpha^2 = (\sigma\alpha)^8 = \mathrm{id} \rangle$$
$$D_8(\alpha, \beta) = \langle \alpha, \beta \rangle = \langle \alpha, \beta \mid \alpha^2 = \beta^2 = (\alpha\beta)^8 = \mathrm{id} \rangle.$$

Proof. To begin with, $\alpha^2 = \langle\langle \mathrm{id}, \beta^2 \rangle\rangle$ and $\beta^2 = \langle\langle \alpha^2, \mathrm{id} \rangle\rangle$. So, it is easy to see that both α and β are involutions. Next, $(\sigma\beta)^2 = \sigma\beta\sigma\beta = \langle\langle \alpha, \alpha \rangle\rangle$. So, $(\sigma\beta)^4 = \mathrm{id}$. Similarly, we obtain that $(\sigma\alpha)^8 = (\alpha\beta)^8 = \mathrm{id}$. ☐

Since $\alpha\beta$ has order 8, we have that $\alpha\beta = \beta\alpha\beta\alpha\beta\alpha \neq \beta\alpha$. That means that α, β do not commute. Just like in the previous cases, we will attempt to implement Proposition 2.5.3.

Proposition 3.4.1. The group $G(1(10)^\omega)$ has subexponential growth.

Proof. We will follow the proof written by Nekrasevych in [15]. Set the weights as follows:

$$|\sigma| = 0 \quad |\alpha| = 1 \quad |\beta| = 1.$$

The pseudonorm becomes proper and non-l_1-expanding and the generators become incompressible elements. If we want to consider the minimal form for a word as before, we first look at which elememts in $\langle \alpha, \beta \rangle$ are eligible. Note that $\alpha\beta\alpha = \langle\langle \sigma\alpha\sigma, 1 \rangle\rangle$, so the eligible elements are

$$\begin{aligned} \alpha &= \langle\langle \sigma, \beta \rangle\rangle & \beta &= \langle\langle \alpha, \mathrm{id} \rangle\rangle \\ \alpha\beta &= \langle\langle \sigma\alpha, \beta \rangle\rangle & \beta\alpha &= \langle\langle \alpha\sigma, \mathrm{id} \rangle\rangle \\ & & \beta\alpha\beta &= \langle\langle \alpha\sigma\alpha, \beta \rangle\rangle. \end{aligned}$$

So, we assume that every word is written as

$$[\sigma] * \sigma * \cdots * \sigma * [\sigma], \text{ where } * \in \{\alpha, \beta, \alpha\beta, \beta\alpha, \beta\alpha\beta\}.$$

First, the subword $\beta\sigma * \sigma\beta = \langle\langle \alpha * \alpha, * \rangle\rangle$ is forbidden, because putting any letter either induces length reduction or makes $\alpha\beta\alpha$ appear. In addition, the subword $\beta\alpha\sigma\beta\sigma\alpha\beta = \langle\langle \mathrm{id}, \beta\alpha\beta \rangle\rangle$ is clearly excluded, due to length reduction. Let $*_\alpha$ denote any incompressible element in $\langle \alpha, \beta \rangle$ that contains α, meaning that $*_\alpha \in \{\alpha, \alpha\beta, \beta\alpha, \beta\alpha\beta\}$, and consider the subwords:

$$*_\alpha\sigma\alpha\sigma*\sigma\alpha\sigma*_\alpha \ , \ *_\alpha\sigma\alpha\sigma*\sigma\beta\alpha\sigma*_\alpha \ , \ *_\alpha\sigma\alpha\beta\sigma*\sigma\alpha\sigma*_\alpha \ , \ *_\alpha\sigma\alpha\beta\sigma*\sigma\beta\alpha\sigma*_\alpha \ , \ *_\alpha\sigma\beta\alpha\sigma*\sigma\alpha\beta\sigma*_\alpha. \quad (3.4.1)$$

Each of the above will have in its second coordinate either the pattern $\beta\sigma * \sigma\beta$ or $\beta\alpha\sigma\beta\sigma\alpha\beta$ or $\beta\beta$, so we exclude them as well. Note now that if $[x]$ appears multiple times in a word, its occurrences are not strictly all x or all id. The following subwords are not allowed:

$$\alpha\beta\alpha \ \ , \ \ \beta\sigma * \sigma\beta \ \ , \ \ \beta\alpha\sigma\beta\sigma\alpha\beta \ \ , \ \ \alpha\sigma\alpha[\beta]\sigma * \sigma[\beta]\alpha\sigma\alpha. \tag{3.4.2}$$

Note that by already combining lists (3.4.1) and (3.4.2) we are able to exclude specific subwords. For example, consider $\sigma\alpha[\beta]\sigma\beta\sigma[\beta]\alpha\sigma$. If we place β on the left or on the right, the pattern $\beta\sigma * \sigma\beta$ appears. So, the subword should have both on the left and on the right side $*_a$, which is not allowed due to (3.4.1). Therefore, we exclude $\sigma\alpha[\beta]\sigma\beta\sigma[\beta]\alpha\sigma$. Using the exact same argument, we get that $\sigma\alpha[\beta]\sigma\beta\alpha\beta\sigma[\beta]\alpha\sigma$ is not allowed either. The problem we encounter at this point is that we cannot yet deduce subexponential growth for $\mathcal{I}$ just by excluding all these patterns. So, we have to try a different approach. We will restrict ourselves to a smaller set for which establishing polynomial growth implies that $\mathcal{I}$ has polynomial growth as well.

Let $\mathcal{I}'$ denote the set of the restrictions of elements in $\mathcal{I}$, i.e.

$$\mathcal{I}' = \{g|_x \mid g \in \mathcal{I} \text{ and } x = 0, 1\}.$$

Since $\mathcal{I}' \subseteq \mathcal{I}$, all patterns excluded in $\mathcal{I}$ are already excluded in $\mathcal{I}'$. In particular, as $|g| = |g|_0| + |g|_1|$ for all $g \in \mathcal{I}$, proving polynomial growth for $\mathcal{I}'$ implies the same for $\mathcal{I}$. Observe that $\sigma\alpha\sigma\alpha\sigma \notin \mathcal{I}'$. Indeed, since each reduced word is of the form $\sigma * \sigma * \sigma \cdots \sigma * \sigma * \sigma$ and β is the only letter such that $\beta|_1 = \mathrm{id}$, a word which contains on the left or on the right coordinate $\sigma\alpha\sigma\alpha\sigma$ must have a subword in which β appears at every other star. For example, some cases with the least β are

- $\alpha\sigma\beta\sigma\beta\alpha\beta\sigma\beta\sigma\alpha = \langle\!\langle \sigma\alpha\sigma\alpha\sigma, \dots \rangle\!\rangle$
- $\alpha\beta\sigma\beta\sigma\alpha\beta\sigma\beta\sigma\alpha = \langle\!\langle \sigma\alpha\sigma\alpha\sigma, \dots \rangle\!\rangle$

However, $\beta\sigma * \sigma\beta$ is not allowed, hence $\sigma\alpha\sigma\alpha\sigma \notin \mathcal{I}'$. Now, consider the subword $\alpha[\beta]\sigma\alpha[\beta]\sigma\alpha\beta\sigma\alpha$. If the next letter is σ, then the one after that should be α. But $\alpha[\beta]\sigma\alpha[\beta]\sigma\alpha\beta\sigma\alpha\sigma\alpha$ is forbidden due to (3.4.1). So, the next letter has to be β and (3.4.2) dictates that the word continue as

$$\alpha[\beta]\sigma\alpha[\beta]\sigma\alpha\beta\sigma\alpha\beta\sigma\alpha \cdots = \alpha[\beta]\sigma\alpha[\beta](\sigma\alpha\beta)^\omega.$$

So, there is only one possible continuation to the right for the following words

$$\alpha\sigma\alpha\sigma\alpha\beta\sigma\alpha \ \ , \ \ \alpha\sigma\alpha\beta\sigma\alpha\beta \ \ , \ \ \alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\alpha \ \ , \ \ \alpha\beta\sigma\alpha\beta\sigma\alpha\beta. \tag{3.4.3}$$

Furthermore, if $g \in \mathcal{I}'$ starts with a word ν from (3.4.3), then g is determined by its length and ν. Set

$$\mathcal{L} = \{g \in \mathcal{I} \mid g \text{ does not contain subwords from (3.4.2), (3.4.3), } \sigma\alpha\sigma\alpha\sigma \text{ and their inverses}\}.$$

Establishing polynomial growth for $\mathcal{L}$ implies that $\mathcal{I}$ has polynomial growth too, because every element in $\mathcal{I}'$ will be of the form $\nu_1\nu_2\nu_2$, where

1. $\nu_1 \in \mathcal{L}$;
2. ν_2 is from (3.4.3);
3. ν_3 is uniquely determined by its length and ν_2.

Consider a word in $\mathcal{L}$ starting with $\alpha\beta\sigma\alpha\beta$. According to the above, the only possible continuation is $\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta$. Now, we have two different options for the next letter: σ or a. In the first case $\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\sigma$ goes on as $\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\sigma\alpha\beta\sigma\alpha$ due to (3.4.1). If the next letter is σ, then the only way to proceed is by adding $\alpha\beta\sigma$. Observe that

$$\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma = \langle\!\langle \sigma\alpha\beta\sigma\beta\alpha\beta\sigma\beta, \beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta \rangle\!\rangle.$$

The left coordinate of the word contains $\beta\sigma*\sigma\beta$, so it is not allowed. We deduce that $\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\sigma\alpha\beta\sigma\alpha$ continues as

$$\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\sigma\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta = \alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta \cdot \sigma \cdot \alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta.$$

In the second case the initial word turns into $\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\alpha$. If the next letter is σ, then we cannot proceed further, as it will either appear $\beta\sigma * \sigma\beta$ or a pattern from (3.4.1). So, the word continues as $\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\alpha\beta\sigma\alpha$. Now, if the next letter is σ, then by putting α or β a pattern from (3.4.1) or $\beta\sigma * \sigma\beta$ will emerge again. The only possible continuation is

$$\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta = \alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta \cdot \alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta.$$

As a result, a word in $\mathcal{L}$ starting with $\alpha\beta\sigma\alpha\beta$ must be a prefix of $\nu[\sigma]\nu[\sigma]\cdots$, where $\nu = \alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta$. Since $\nu = \langle\!\langle \sigma\alpha\beta\sigma\beta\alpha, \beta\sigma\alpha\beta\sigma\alpha \rangle\!\rangle$, we get that $\nu\nu\sigma\nu|_1 = \beta\sigma\alpha\beta\sigma\alpha\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\alpha$ contains $\alpha\beta\sigma\alpha\beta\sigma\alpha\beta$, hence it is not allowed. So, if a word starts with $\alpha\beta\sigma\alpha\beta$, it is either $(\nu\sigma)^\omega$ or $(\nu)^\omega$. Consider

$$\mathcal{L}' = \{g \in \mathcal{L} \mid g \text{ does not contain } \alpha\beta\sigma\alpha\beta \text{ or its inverse } \beta\alpha\sigma\beta\alpha\}.$$

Based on the above, it is clear that if $\mathcal{L}'$ has polynomial growth, then so does $\mathcal{L}$. Suppose that an element in $\mathcal{L}'$ starts with $\alpha\beta\sigma\alpha$. Then, it may only go on like $\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta$. If the next letter is σ, then the word continues like $\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha$, whose wreath decomposition is

$$\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\sigma\alpha\beta\sigma\alpha\sigma\alpha = \langle\!\langle \sigma\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma, \beta\sigma\beta\alpha\beta\sigma\beta \rangle\!\rangle.$$

Since the right coordinate is of the form $\beta\sigma*\sigma\beta$, the word is not allowed. Therefore, the initial word proceeds as $\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\alpha$. If the next letter is σ, then by putting α or β a subword from (3.4.1) or $\beta\sigma * \sigma\beta$ will emerge. Therefore, the word continues as $\alpha\beta\sigma\alpha\sigma\alpha\beta\sigma\beta\alpha\beta\sigma\alpha$. If we add σ, then the subword $\sigma\alpha\beta\sigma\beta\alpha\beta\sigma\alpha\sigma$ appears, while if we add β, then $\alpha\beta\sigma\alpha\beta$ appears. Both of them are forbidden, so the subword $\alpha\beta\sigma\alpha$ is not allowed in $\mathcal{L}'$. Similarly, by taking extensions to the left, we see that $\alpha\sigma\beta\alpha$ is not allowed either.

Moving on, the subword $\sigma\beta\sigma\beta$ in $\mathcal{L}'$ can only have on the right the letter α. In addition, since $\sigma\beta$ has order 4 and $\sigma\beta\sigma\beta\sigma = \beta\sigma\beta$, the word $\alpha\sigma\beta\sigma\beta\sigma$ is basically $\alpha\beta\sigma\beta$. So, we may assume that $\sigma\beta\sigma\beta$ can only be contained in $\alpha\sigma\beta\sigma\beta\alpha$. If we add σ on both sides, then the word $\sigma\alpha\sigma\beta\sigma\beta\alpha\sigma$ appears which is forbidden. Then, without loss of generality, we suppose that $\sigma\beta\sigma\beta$ must be a subword of $\alpha\sigma\beta\sigma\beta\alpha\beta$ which continues either as $\alpha\sigma\beta\sigma\beta\alpha\beta\sigma\alpha$ or $\alpha\sigma\beta\sigma\beta\alpha\beta\sigma\beta$. Both cases contain either $\alpha\beta\sigma\alpha$ or $\beta\sigma * \sigma\beta$, so $\sigma\beta\sigma\beta$ is not allowed. Similarly, it follows that $\beta\sigma\beta\sigma$ is excluded in $\mathcal{L}'$.

Consider an element in $\mathcal{L}'$ starting with $\beta\sigma\beta$. The next letter has to be α and if we proceed with β, then we will have $\beta\sigma\beta\alpha\beta\sigma\alpha$ or $\beta\sigma\beta\alpha\beta\sigma\beta$, both of which are not allowed. So, $\beta\sigma\beta$ continues as $\beta\sigma\beta\alpha\sigma\alpha$. At this point we distinguish two cases: the next letter is either σ or β. In the first case, we have $\beta\sigma\beta\alpha\sigma\alpha\sigma$. If the next letter is β, then the word will continue as $\beta\sigma\beta\alpha\sigma\alpha\sigma\beta\sigma\alpha$, because $\alpha\sigma\beta\alpha$ and $\sigma\beta\sigma\beta$ are not allowed. Continuing this exercise will appear either $\beta\sigma\beta\alpha\sigma\alpha\sigma\beta\sigma\alpha\beta\sigma\{\alpha,\beta\}$ or $\beta\sigma\beta\alpha\sigma\alpha\sigma\beta\sigma\alpha\sigma$, both of which are not allowed. As a result, the next letter of $\beta\sigma\beta\alpha\sigma\alpha\sigma$ is α and the word goes on like

$$\beta\sigma\beta\alpha\sigma\alpha\sigma\alpha\beta\sigma\beta\alpha\sigma\alpha\sigma\alpha\cdots$$

In the second case, we have $\beta\sigma\beta\alpha\sigma\alpha\beta$ which proceeds as $\beta\sigma\beta\alpha\sigma\alpha\beta\sigma\beta\alpha\sigma\alpha\cdots$. Therefore, a word in $\mathcal{L}'$ starting with $\beta\sigma\beta$ continues as

$$\beta\sigma\beta\{\alpha\sigma\alpha, \alpha\sigma\alpha\sigma\alpha\}\beta\sigma\beta\{\alpha\sigma\alpha, \alpha\sigma\alpha\sigma\alpha\}\beta\sigma\beta\cdots$$

Observe that $\alpha\sigma\alpha\sigma\alpha\beta\sigma\beta\alpha\sigma\alpha$ is not valid due to (3.4.2). So, $\beta\sigma\beta$ can only be the prefix of $(\beta\sigma\beta\alpha\sigma\alpha)^\omega$. Finally, define

$$\mathcal{L}'' = \{g \in \mathcal{L}' \mid \beta\sigma\beta \text{ is not a subword of } g\}.$$

Once again, according to what we have just proved, if $\mathcal{L}''$ has polynomial growth, then that implies that $\mathcal{L}'$ has the same growth. Let an element in $\mathcal{L}''$ starting with $\beta\sigma$. The next letter has to be a, meaning that we have $\beta\sigma\alpha$. If the next letter is σ, then we can only obtain $\beta\sigma\alpha\sigma\alpha\beta\sigma\{\alpha,\beta\}$, which is not allowed, since $\alpha\beta\sigma\alpha$ and $\beta\sigma\beta$ are forbidden in $\mathcal{L}''$. If the next letter is β, then we get $\beta\sigma\alpha\beta\sigma\{\alpha,\beta\}$ which is not allowed either, because we have excluded $\alpha\beta\sigma\alpha$ and $\beta\sigma * \sigma\beta$. So, $\beta\sigma$ cannot exist in $\mathcal{L}''$, hence β has to be followed by α. Moreover, $\alpha\beta$ cannot exist, otherwise from the above we create the word $\alpha\beta\alpha$ which is forbidden. So, α should be followed by σ.

To close this argument, all words in $\mathcal{L}''$ starting with β should continue as $\beta\alpha\sigma$. If the next letter is α, then the word proceeds as $\beta\alpha\sigma\alpha\sigma\{\alpha\sigma,\beta\alpha\}$ which is impossible as we do not allow $\sigma\alpha\sigma\alpha\sigma$ or $\alpha\sigma\beta\alpha$. As a result, all words that start with β are of the form $\beta\alpha\sigma\beta\alpha\sigma\cdots = (\beta\alpha\sigma)^\omega$. The latter implies that the growth of $\mathcal{L}''$ is bounded, hence $\mathcal{I}$ has polynomial growth and our proof here is done.

□

3.5 The automaton $11(0)^\omega$

The group generated by the kneading sequence $11(0)^\omega$ is $G(11(0)^\omega) = \langle t, a, b\rangle$, where

- $t = \langle\langle \mathrm{id}, \mathrm{id}\rangle\rangle(0\ 1)$
- $a = \langle\langle \mathrm{id}, t\rangle\rangle$
- $b = \langle\langle b, a\rangle\rangle$

Let it be noted that the automaton induced by this kneading sequence is the one described in Example 2.3.1.

Lemma 3.5.1. [7, Lemma 4.7] $\langle a, b\rangle = \langle a, b \mid a^2 = b^2 = (ab)^4 = \mathrm{id}\rangle$. In addition, $(at)^4 = \mathrm{id}$.

Proposition 3.5.1. The group $G(11(0)^\omega)$ has subexponential growth.

Proof. Define the weights:

$$|t| = 0 \quad |a| = 1 \quad |b| = 1.$$

The pseudonorm becomes proper and non-ℓ_1-expanding, while $\{t, a, b\} \subseteq \mathcal{I}$. Since $(ab)^4 = \mathrm{id}$, we do not need to take into consideration all products of a and b. We observe that $bab = \langle\langle \mathrm{id}, ata\rangle\rangle$ and $abab = \langle\langle \mathrm{id}, tata\rangle\rangle$ so each word has the following minimal form

$$[t] * t * \cdots * t * [t], \text{ where } * \in \{a, b, ab, ba, aba\}.$$

For this case, we will look inside $\mathcal{I}$ at the alternating patterns, namely patterns of the form $txt * tyt \in \mathcal{I}$, where x, y are allowed elements in $\langle a, b\rangle$. If we show that the automaton generated by these alternating patterns consists of pairwise disjoint circles, then $\mathcal{I}$ has subexponential growth. The reader interested in the reason behind the latter may find it in [17]. Before we start looking at which patterns are allowed, note that as growth represents the combinatorial volume of a ball of radius n, equivalent words should not be counted twice.

Let $*_b$ be a letter which contains at least b, meaning that $*_b \in \{b, ab, ba, aba\}$. We start by considering the subword

$$*_b tbt *_b = \langle\langle bab, *b*\rangle\rangle. \tag{3.5.1}$$

In the previous lemma we have shown that bab admits length reduction. Therefore, it is not allowed and the subword $*_b tbt *_b$ is forbidden. Next, consider the pattern

$$[t] * tat * tat * t * [t] = \langle\!\langle *t * t*, * * * \rangle\!\rangle \tag{3.5.2}$$

As we said before, equivalent words should not be counted twice. If we substitute any $*$ with the letter a, then the word is not in its minimal form, meaning that it corresponds to another word inside the group with smaller length. For example,

- $*tatatat* = *a*$
- $\cdots tatat * tat* = \cdots ata * tat*$.

As a result, we may assume that each star in the latter should be a letter which contains b, hence

$$[t] * tat * tat * t * [t] = [t] *_b tat *_b tat *_b t * [t] = \langle\!\langle btbtb, * * * \rangle\!\rangle.$$

The word $btbtb$ is not allowed, so the pattern (3.5.2) is forbidden as well. This makes us exclude the pattern

$$*tabat * tabat* = \langle\!\langle *tat * tat*, *b * b* \rangle\!\rangle.$$

Now, consider the subword $tbtatbt$. If we add to the left or the right the subword ta or at, respectively, then the word is not allowed due to (3.5.2). So, the subword must be continued as

$$*_b tbtatbt *_b = \langle\!\langle baab, *btb* \rangle\!\rangle = \langle\!\langle \mathrm{id}, *btb* \rangle\!\rangle.$$

There is clearly length reduction, so we end up excluding $tbtatbt$. In a similar way, let us look at the subword $abtatba$. Since bab and (3.5.2) are not allowed, the only way to expand it is

$$*_b tabtatbat *_b \langle\!\langle btaatb, *btb* \rangle\!\rangle = \langle\!\langle \mathrm{id}, *btb* \rangle\!\rangle.$$

We observe again reduction in its length, hence $abtatba$ is not allowed. Moving on, consider the subword $tbtbtatb$. According to our previous results, the word should be contained inside $tatbtbtatbat$. If we add on the left the subword ta, then the word will not remain in its minimal form without letters getting cancelled, because $tatat = ata$. We assume that every subword we treat is always in minimal form. Therefore, we assume that the subword continues on the left as $*_b tatbtbtatbat$. and due to (3.5.2) the initial subword should be part of

$$t *_b t *_b tatbtbtatbat *_b = \langle\!\langle *btbtb, b * abtb* \rangle\!\rangle.$$

We do not allow $btbtb$, hence the subword $tbtbtatb$ is forbidden. Using a symmetric argument, we are able to exclude $btatbtbt$. Next, we look at the subword $tbtabtbt$. Due to (3.5.1) we can only add the ta both on the left. The only way to continue this word so that it will be of minimal form is

$$t *_b tatbtabtbt = \langle\!\langle *aba, btbtab \rangle\!\rangle.$$

The above is not allowed due to (3.5.1), so $tbtabtbt$ is excluded. By expanding similarly on the right, we also exclude $tbtbatbt$. As for $tbtabatbt$, it can only be contained in

$$atbtabatbt = \langle\!\langle aba, tbtatbt \rangle\!\rangle$$

which we exclude, because it has $tbtatbt$ in the right coordinate. We deduce that the pattern $tbt * tbt$ is forbidden. Next, consider the subword $tbtabtatb$. Given (3.5.1) and (3.5.2), it must be contained in

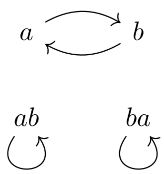
$$*_b t *_b tatbtabtatb[a]t *_b = \langle\!\langle b * abtb*, *btbt[t]b \rangle\!\rangle.$$

We observe in the right coordinate either $btbtb$ or bb. So, $tbtabtatb$ is not allowed. Using again a symmetric argument, we exclude $btatbatbt$. Finally, consider $tbtabtb$. The next letter from the right cannot be t, otherwise we would have $tbt * tbt$. Then, it continues as $tbtabtba$. If we add tat, meaning $tbtabtbatat$, then this word is equivalent to $tbtabtbtata$, which is not allowed. Therefore, the only possible way to expand the initial word is

$$*_b t *_b tatbtabtbat*_b = \langle\langle b * abatb, *btbtab* \rangle\rangle.$$

The pattern $*_b tbt *_b$ appears once again, so $tbtabtb$ is not allowed. Similarly, $btbatbt$ is not allowed either.

Using all the forbidden subwords and patters we discovered, we created an algorithm on the computer. The algorithm simply runs all the possible words for every alternating pattern, eliminates the ones that contain forbidden subwords and gives as an output whether the pattern is allowed or not. The results are that, except reduced words which are equivalent to ones corresponding to allowed patterns, the automaton generated by the alternating patterns with no length reduction is

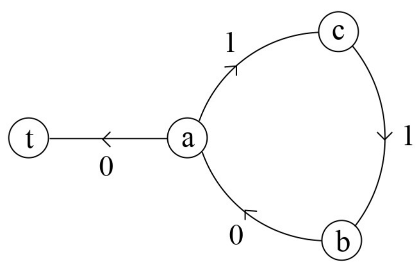


Therefore, $\mathcal{I}$ has subexponential growth.

□

3.6 The automaton $0(011)^\omega$

The automaton with kneading sequence $0(011)^\omega$ is the one described by the following Moore diagram



The group generated by this automaton is $G(0(011)^\omega) = \langle t, a, b, c \rangle$, where t is the swap and the other generators are defined by

- $a = \langle\langle t, c \rangle\rangle$
- $b = \langle\langle a, \mathrm{id} \rangle\rangle$
- $c = \langle\langle \mathrm{id}, b \rangle\rangle$

Lemma 3.6.1 (4.10)**.** [7] $\langle a, b, c \rangle \cong \langle a, b \rangle \times \langle c \rangle \cong D_8(a, b) \times \mathbb{Z}/2\mathbb{Z}$, where $D_8(a, b) = \langle a, b \mid a^2 = b^2 = (ab)^8 = \mathrm{id} \rangle$.

We will give in this case length 1 to each generator except the swap, namely

$$|t| = 0 \quad |a| = 1 \quad |b| = 1 \quad |c| = 1.$$

Proposition 3.6.1. The group $G(0(011)^\omega)$ has subexponential growth.

Proof. We are going to use again the method of incompressible elements. As before, we can write each element of $G(0(011)^\omega)$ as an alternating word in t and elements of $\langle a,b,c\rangle$. Observe that once again we are able to decrease the number of letters. If we take into account that

1. c commutes with a and b;
2. all generators have order 2;
3. $aba = \langle\langle tat, \mathrm{id}\rangle\rangle$;
4. $baba = \langle\langle atat, \mathrm{id}\rangle\rangle$,

then the only letters that do not admit length reduction are

$$V = \{a, ab, b, ba, bab, c, ca, cab, cb, cba, cbab\}.$$

Therefore, we may assume that each word is of the form

$$[t] * t * \cdots * t * [t] \text{ where } * \in V.$$

Similarly to the previous example, we will look at the patterns of alternating words. At this point we have to inform the reader that due to the substantial number of cases that required examination, we used again the algorithm that assisted us in the process by eliminating big amounts of subwords through brute force. However, we will still present the correct sequence of subwords we excluded and give justification of why they are not allowed. To begin with, note that except for b, c, cb all other letters have on the right coordinate either b or cb. Also, only the letters b, c, cb do not contain in their left coordinate any t. Consider the set of patterns

$$\{a, ab, ba, bab\}t\{b,c\}tx = \langle\langle \cdots, c[a]c\rangle\rangle \text{ for } x \in \{a, ab, ba, bab, ca, cab, cba, cbab\} = V \setminus \{b, c, cb\}. \tag{3.6.1}$$

We deduce that each subword of the form we have just written is excluded. Note again that the letters $ca, cab, cba, cbab$ have bc for left coordinate, therefore all paterns

$$\{ca, cab, cba, cbab\}tcty = \langle\langle \cdots, bc\{c, b, cb\}\rangle\rangle \text{ with } y \in V \setminus \{b\}. \tag{3.6.2}$$

are also excluded. Moving on, let $bt * tb = \langle\langle a * a, *\rangle\rangle$. We know that the subwords $aba = \langle\langle tat, \mathrm{id}\rangle\rangle$ and $aca = c$ admit length reduction, therefore we do not allow them. As a result, $bt * tb$ is not allowed either. By using the exact same argument, we exclude all patterns of the form $*_1 t * t *_2$ where

1. $*_1$ is in $\{b, ab, bab, cab, cb, cbab\}$;
2. $*_2$ is in $\{b, ba, bab, cb, cba, cbab\}$.

At this point we stop finding which subwords are forbidden with brute force and we proceed to get assistance for the computer. We checked the following pattern

$$\cdots * tct * tct * \cdots = \langle\langle \cdots * b * b * \cdots, \cdots * * * \cdots\rangle\rangle$$

After making sure to exclude all subwords from (3.6.1), (3.6.2) and of the form $*_1 t * t *_2$, the algorithm ensures that the patter of c alternating with itself gets excluded as well. The next patterns that get eliminated are

$$\cdots * t * t\{ba, cba\}t * t\{ab, cab\}t * t * \cdots = \langle\langle \cdots * c[b] * [b]c * \cdots, \cdots * at * ta * \cdots\rangle\rangle$$

When giving as input all the previous result to the compute and running again the algorithm, it turns out that the only subwords which we allow are

1. *cab* alternating with itself;
2. *cba* alternating with itself;
3. c alternating with b.

As a result, the automaton generated by the alternating patters, after excluded the subwords we showed that admit length reduction, is described by the following figure:

$$\begin{tikzcd} b \arrow[r, bend left] & c \arrow[l, bend left] \\ cab \arrow[loop below] & cba \arrow[loop below] \end{tikzcd}$$

We conclude that $\mathcal{I}$ has subexponential growth and so does the initial group.

$\square$

Chapter 4

Conclusion

Let us revisit our primary objective which involved exploring self-similar groups of intermediate growth and providing examples that validate the Nekrashevych conjecture. This conjecture states that the iterated monodromy group assigned to a non-renormalizable complex quadratic polynomial with pre-periodic kneading sequence exhibits intermediate growth. To pursue this aim, we focused our attention on the method of incompressible elements. During this process, we ensured that the pseudonorm was both proper and non-ℓ_1-expanding and we guaranteed the incompressibility of each generator. Subsequently, we sought various approaches to demonstrate that the set $\mathcal{I}$ exhibits, at most, subexponential growth by excluding specific subwords or patterns.

Our investigations across all cases, including those that do not support the conjecture, such as the Grigorchuk groups, have yielded a common underlying pattern. When examining the incompressible elements, we observed that the automaton generated by alternating patterns consists, at most, exclusively of disjoint circles. In the first Grigorchuk group, we ended up excluding all patterns after a certain length of words, leading to a bounded set of incompressible elements. In the second and generalized groups, only one or none element of the form $(\sigma\gamma)^\omega$ remains incompressible. Turning to the iterated monodromy group of z^2+i, although the set $\mathcal{I}$ initially lacked a straightforward description, our analysis of smaller sets, achieved by eliminating specific prefixes such as (3.4.3), $\sigma\alpha\sigma\alpha\sigma$, $\alpha\beta\sigma\alpha\beta$, or $\beta\sigma\beta$, revealed that the remaining words are of the form $(\beta\alpha\sigma)^\omega$. Continuing our exploration, in the automaton generated by the sequence $11(0)^\omega$, permissible patterns alternate between a and b, in addition to including $(tab)^\omega$ and $(tba)^\omega$. Lastly, the automaton generated by $0(011)^\omega$ allows patterns to alternate between b and c along with the words $(tcab)^\omega$ and $(tcba)^\omega$.

The examples we have discussed in this research project have already been explored in previous works by authors such as Bux and Perez [5], Nekrasevych [15], and Ashley S. Dougherty [7]. However, our intention was to address them collectively using a consistent approach: the method of incompressible elements. To be candid, implementing this proposition was not always a straightforward task. While the methodology worked smoothly for the initial examples, we encountered challenges when dealing, for example, with the case of $G(1(10)^\omega)$. It necessitated repeated restriction to smaller sets to ascertain whether $\mathcal{I}$ exhibits subexponential growth. In general, despite employing brute force to exclude patterns or specific subwords, there is no fixed methodology to follow. It is essential to acknowledge that the problem we tackled is relatively young, with a history of less than 40 years, and thus, there exists a limited number of examples available for study. Moreover, the existing examples do not share a uniform methodology, adding to the complexity of the research landscape.

Despite not successfully proving the conjecture, our research efforts have yielded results that hold relevance to our primary objective. The main conclusion of this thesis is that all the examined examples share a common trait. This observation leads us to speculate that the key to proving the conjecture lies in demonstrating that the automaton described by the alternating patterns of incompressible elements consists of disjoint circles. Such a breakthrough would essentially verify our conjecture, as we have previously mentioned that this property alone is sufficient to establish a subexponential estimate for the growth of $\mathcal{I}$. It is crucial to acknowledge that this endeavor presents considerable challenges. We have already encountered difficulties in implementing Proposition 2.5.3, even when examining specific examples, let alone in the general case. Nevertheless, considering that no other methodology suitable for all these examples has been established thus far, we believe that this avenue of exploration holds significant potential and may ultimately lead us closer to resolving the conjecture.

I would like to extend my heartfelt gratitude to my family and friends for their support throughout this research endeavor, providing both practical assistance and emotional encouragement. Additionally, I am deeply grateful to my two supervisors. First, to Mikhail, my primary one, whose guidance and readiness to address my queries have been invaluable throughout this research journey. Second, to Gunther, my second one, for dedicating time, energy, and enthusiasm in reviewing my proposal, attending my presentation, and providing feedback for this project. As I close this chapter of my academic journey, I look forward to the future and the potential impact of our research findings in furthering the understanding of self-similar groups and contributing to the broader mathematical community.

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