Fat prolonged distributions of type (4,6)

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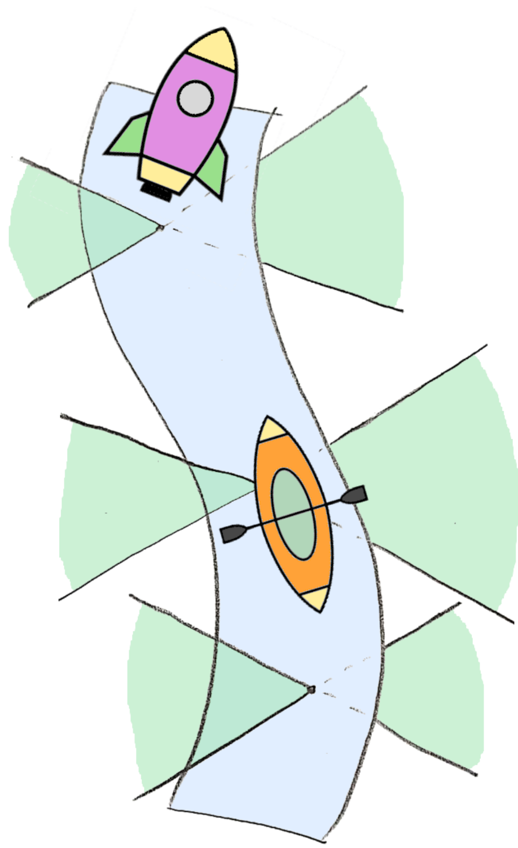

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Abstract

A distribution is a smooth sub-bundle of the tangent bundle of a given manifold. It can represent a physical system with restrictions on the degrees of freedom. Bracket generating distributions are a distinguished class of interest in control theory: when the restriction on the system is given by a bracket generating distribution, any configuration (e.g. position and orientation) can be obtained using the restricted directions only.

This thesis focuses on fat distributions (also called strongly bracket generating distributions), which are, in a sense, the most extreme case of bracket generating distribution. A lot is known about co-rank 1 fat distributions (also called contact distributions), but much less is known for higher co-ranks.

We focus on co-rank 2 distributions that are induced by the canonical distribution on the Grassmann bundle of 2-planes of a manifold. We define this class of distributions and refer to them as prolonged distributions. To be precise, we look at co-rank 2 sub-bundles of the Grassmann bundle $\mathrm{Gr}_2(TX)$ of a 4-dimensional manifold X. We consider the canonical distribution on $\mathrm{Gr}_2(TX)$ and restrict it to the given sub-bundle. The main question we investigate is under what conditions this restriction defines a fat distribution on the sub-bundle manifold.

Our contributions go in two directions.

First, we assume the 4-dimensional base manifold X to be endowed with an almost complex structure J. We consider the rank-2 sub-bundle of the Grassmann bundle consisting of the 2-planes invariant under the almost complex structure J. This sub-bundle forms a 6-dimensional manifold and the fibers are in fact complex Grassmannians. We show that the prolonged distribution of this sub-bundle is a fat distribution of co-rank 2.

Furthermore, we consider a rank-2 fiber bundle M over a 4-dimensional base manifold X and a bundle map that maps M into the Grassmann bundle $\mathrm{Gr}_2(TX)$; we identify necessary and sufficient local conditions for the bundle map to induce a fat prolonged distribution $\mathcal{D}$ of co-rank 2 on the fiber bundle M. More precisely, we show that requiring the prolonged distribution $\mathcal{D}$ on M to be fat is equivalent to requiring that the fibers of M –that map into the corresponding Grassmannian-fiber via the bundle map– are transverse to what we call the infinitesimal cone field on the Grassmannian. As a consequence, we show that, in this case, if M is closed, the fibers of M are either 2-spheres or projective planes, which is the main result of this thesis.

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Introduction

Distributions (also called tangent distributions) are smooth sub-bundles of the tangent bundle of a manifold. More precisely,

> **Definition 1** Let M be an n-dimensional manifold. A (smooth) *distribution (or tangent distribution)* $\mathcal{D}$ of *rank* r and *co-rank* k on M of dimension is a (smooth) rank-r sub-bundle of the tangent bundle TM, i.e. $n = r + k$.

They can represent a physical system with restrictions on the degrees of freedom. Consider parallel parking a car, canoe or spaceship: common sense tells us we can park even though we can not move sideways directly. In geometrical terms, the fact that any configuration (e.g. position and orientation) can be obtained using the restricted directions only, would correspond to a bracket generating distribution on the configuration space.

> **Definition 2** A distribution $\mathcal{D}$ on a manifold M is called *bracket generating at* $x \in M$, if the tangent space T_xM is spanned by
>
> $$T_xM = \langle \mathcal{D}_x, [\mathcal{D}, \mathcal{D}]_x, [\mathcal{D}, [\mathcal{D}, \mathcal{D}]]_x, \dots \rangle.$$
>
> Here $[\mathcal{D}, \mathcal{D}]_x$ is the subspace defined by
>
> $$[\mathcal{D}, \mathcal{D}]_x = \{[V, W]_x \mid V, W \in \Gamma\mathcal{D}\}.$$
>
> The other spanning elements are given analogously. The distribution is *bracket generating* if it is bracket generating at every point $x \in M$.

Even though distributions are well-studied objects, only some specific classes are well understood.

Classification problem

The classification of bracket generating distributions up to homotopy is known to be a challenging problem, see for example [4]. However, there are classification results for specific classes of bracket generating distributions and specific families therein, often for fixed step, dimension and rank.

An important tool used to classify many classes of distributions is the h-principle introduced in 1973 by Gromov, see [7].In particular, the h-principle provided a classification for distributions –and in fact many more geometric structures– on open manifolds. That leaves us with the classification problem up to homotopy on closed manifolds for these geometric strutures.

In the literature is suggested that bracket generating distributions satisfy the h-principle when they are not *maximally non-integrable*. Evidence in this direction is provided in [9]. Intuitively speaking, a distribution is maximally non-integrable when it has as many non-trivial Lie brackets as possible (implying that it is bracket generating). Giving a more precise definition (which depends on rank and dimension) is rather involved and out of the scope of this thesis; here it suffices to say that the class of distributions we focus on is maximally non-integrable.

Contact and even-contact distributions

The most famous class of non-integrable bracket generating distributions is probably the class of contact distributions.

> **Definition 3** A distribution $\mathcal{D}$ of co-rank 1 on an n-dimensional manifold M is called *contact* at $x \in M$ if for every non-zero vector field V in $\Gamma\mathcal{D}$ there exists a vector field W in $\Gamma\mathcal{D}$ such that T_xM is given by the span
> $$T_xM = \langle \mathcal{D}_x, [V, W]_x \rangle.$$
> If $\mathcal{D}$ is contact at all $x \in M$ we call $\mathcal{D}$ a *contact distribution*.

Contact distributions appear on manifolds of odd dimension only. They can be seen as homogeneous versions of symplectic structures. Therefore a lot of machinery from the symplectic context can be applied. In the contact setting there exists an equivalent for the Darboux Theorem, that provides a local normal form for symplectic structures. Moreover, contact distributions present global stability due to Gray's Theorem, which is analogous to Moser's stability Theorem in symplectic geometry.

Contact distributions on open manifolds were classified in 1973 as a particular case of the aforementioned h-principle. However, there was also progress in the classification for closed manifolds. In 1989 Eliashberg defined and classified 3-dimensional overtwisted contact manifolds, see [6]. He used the h-principle ideas, but specific to the contact setting. Later in the 2010's rapid developments took place regarding contact structures in dimension at least 5. Overtwisted contact manifolds were defined and classified.

Next to contact distributions we have the so-called even-contact distributions which appear on even manifolds only. A complete classification via the h-principle is given by McDuff, see [10].

Distributions of higher co-rank

Up to dimension 6 there are already quite some classification results for bracket generating distributions of co-rank greater than 1. Due to the properties of the Lie bracket, the only types of maximally non-integrable bracket generating distributions that can appear for dimensions 3 to 6 are: $(2, 3, 4)$, $(3, 5)$,

$(2,3,5)$, $(3,6)$, $(4,6)$, and $(2,3,5,6)$.

For type $(2,3,4)$ distributions, also called Engel structures, the sub-classes loose and overtwisted are introduced and classified by the h-principle, in [3] and [5] respectively. It is still an open question if there exist more classes of maximally non-integrable bracket generating distributions of this type and this question seems hard to address.

For the $(2,3,5)$ distributions, called Cartan structures, the overtwisted class has been defined recently and it has been shown that the h-principle holds for this class, see [13]. Similarly to Engel's structures, it is unclear whether there exist more classes of maximally non-integrable distributions of this type next to the overtwisted class.

The full classification for type $(3,5), (3,6)$ are presented in [9], also via the h-principle.

The bracket generating distributions of type $(4,6)$ divide into two classes: hyperbolic and elliptic. The terminology originates from quadratic forms that can be defined for this specific type (see [11]). The hyperbolic class was classified by means of h-principle type techniques in the same work [9]. The elliptic class is in fact precisely the class of fat distributions of type $(4,6)$.

Definition 4 A distribution $\mathcal{D}$ on a manifold M is called *fat* (or *strongly bracket generating) at* $x \in M$, if for every choice of vector field $V \in \Gamma\mathcal{D}$ that is non-zero at x we have that T_xM is equal to the span

$$T_xM = \langle \mathcal{D}_x, [V,\mathcal{D}]_x \rangle.$$

Here $[V,\mathcal{D}]_x$ is the subspace defined by

$$[V,\mathcal{D}]_x = \{[V,W]_x \mid W \in \Gamma\mathcal{D}\}.$$

The distribution is *fat* if it is fat at every point $x \in M$.

In this same work [9], the authors conjecture that the h-principle does not hold for the elliptic class, i.e. the fat class. This implies that techniques of a different nature than the h-principle should be used for their classification. In this thesis we explore exactly that area, concerning the fat distributions of type $(4,6)$.

Structure of the thesis

In Chapter 1 we introduce several sub-classes of distributions, in particular fat distributions, the central objects of this text, and state Rayner's theorem about the admissible ranks for fat distributions. Then we introduce curvature and define some equivalent formulations for fatness in co-rank 1 and 2.

Chapter 2 provides a brief recap of the Grassmannian, the Grassmann bundle and coordinates on them.

In Chapter 3, we first review the canonical distribution on the Grassmann bundle of a manifold, Then we look at co-rank 2 fiber bundles M over a 4-dimensional manifold X with a bundle map into Grassmann bundle $\mathrm{Gr}_2(TX)$; we consider the canonical distribution on $\mathrm{Gr}_2(TX)$ which induces a distribution on M via the bundle map, which results into the so-called prolonged distributions. Within this framework, we state and prove one of the main results of this thesis, Theorem 3.11.

Theorem 1 Let (X, J) be 4-dimensional manifold endowed with an almost complex structure J. Then, the canonical distribution on $\mathrm{Gr}_2(TX)$ restricts to a fat distribution on the complex Grassmannian bundle $\mathrm{Gr}_2(TX, J)$ of J-invariant 2-planes.

In Chapter 4 we introduce the so-called infinitesimal cone field; we identify characterizing local properties for fat prolonged distributions of type $(4, 6)$ in terms of the infinitesimal cone field. The main result of this chapter, Theorem 4.15, states that the fibers of the fiber bundle M have to be transverse to the infinitesimal cone field in order to induce a fat prolonged distribution.

Theorem 2 Let M be a rank 2 fiber bundle over a 4-dimensional manifold X with a bundle map that maps into $\mathrm{Gr}_2(TX)$. The canonical distribution on $\mathrm{Gr}_2(TX)$ induces a fat distribution on M if and only the fibers of M map into the Grassmannian-fibers of $\mathrm{Gr}_2(TX)$, transversely to the infinitesimal cone field $\mathcal{C}$ from Definition 4.9.

In Chapter 5, we use the characterization of fatness in terms of the infinitesimal cone field to obtain topological constraints on the fibers of a rank 2 fiber bundle over a 4-dimensional manifold X with a bundle map that maps into $\mathrm{Gr}_2(TX)$, such that its prolonged distribution is fat – Theorem 5.8, which is the main result of this thesis.

Theorem 3 Let M be a co-rank 2 sub-bundle of $\mathrm{Gr}_2(TX)$ that is closed as a manifold. If the canonical distribution on $\mathrm{Gr}_2(TX)$ restricts to a fat distribution on M, then the fibers of M are homeomorphic to either spheres or projective planes.

Chapter 1

Distributions

In this chapter we introduce several of the many well studied sub-classes of distribution, in particular the class of fat distributions, that are the central objects of this text. In computations later in this text, it is convenient to use alternative descriptions for fatness. Therefore, after introducing the needed notions, we state several equivalent definitions. Explicitly for co-rank 1 and 2 we give even simpler characterizations.

1.1 Distributions and integrability

We start with a fundamental sub-class of distributions that is most opposite to the distributions of concern in this text. It is the class of distributions that are induced by foliations. A foliation on a manifold M consists of a smooth partition of M into sub-manifolds, whose tangent spaces define a distribution on M. More precisely,

> **Definition 1.1** Let M be an n-dimensional manifold. A *foliation* $\mathcal{F}$ of *rank* r on M is a partition $\{L_i\}_{i\in I}$ of M into disjoint connected sub-manifolds L_i, $i \in I$, called the *leaves* of $\mathcal{F}$, such that the following holds. For all $x \in M$ there exists an open neighborhood U and a submersion
>
> $$f : U \to \mathbb{R}^{n-r},$$
>
> with the property that, for all i such that $L_i \cap U$ is non-empty, there exists $x_i \in \mathbb{R}^{n-r}$ such that
>
> $$L_i \cap U = f^{-1}(x_i).$$

For any rank-r foliation $\mathcal{F}$, one has the induced rank-r distribution $T\mathcal{F}$, consisting of the tangent spaces to the leaves of $\mathcal{F}$. In Figure 1.1 we illustrate an example of a distribution induced by a foliation.

Foliations, and their tangent distributions, play a central role in a variety of fields, such as the theory of integrable systems, non-commutative geometry, and Poisson geometry. Closer to the spirit of this thesis, important results concerning the classification up to homotopy of foliations can be found in work by A. Haefliger, see [8], and W. Thurston, [14].

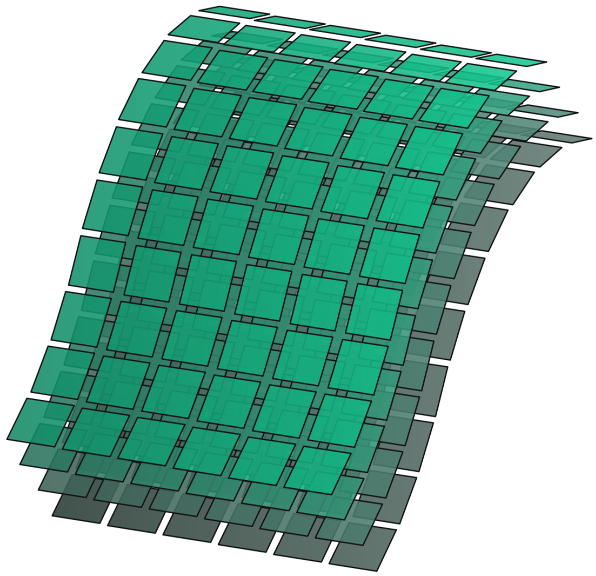

Figure 1.1: Example of a distribution induced by a foliation.

This leads to one of the simplest questions we can ask about distributions: whether they are tangent to foliations. This is also known as the integrability problem.

Definition 1.2 Let $\mathcal{D}$ be a distribution on M. We say that $\mathcal{D}$ is *integrable* if it is induced by a foliation on M. That is, for every point x in M there exists a leaf L_i of $\mathcal{F}$ such that

$$\mathcal{D}_x = T_x L_i$$

The famous Frobenius theorem relates this to the following property.

Definition 1.3 Let $\mathcal{D}$ be a distribution on M. We say that $\mathcal{D}$ is *involutive* if it is closed with respect to the Lie bracket of vector fields, i.e.

$$[X,Y] \in \Gamma(\mathcal{D}), \quad \text{for all } X,Y \in \Gamma(\mathcal{D}).$$

Here, $\Gamma(\mathcal{D})$ denotes the set of sections of the vector bundle $\mathcal{D}$ on M, that is, vector fields on M tangent to $\mathcal{D}$.

Now the afore mentioned Frobenius theorem states that involutivity of a distribution $\mathcal{D}$ is equivalent to the requirement for $\mathcal{D}$ to be tangent to a foliation.

Theorem 1.4 (Frobenius) A distribution $\mathcal{D}$ is tangent to a foliaton $\mathcal{F}$, $\mathcal{D} = T\mathcal{F}$ if and only if it is involutive.

In the general setting, taking Lie brackets of vector fields tangent to a distribution $\mathcal{D}$ results in a flag of linear sub-spaces of the tangent space TM. Rather than looking at the set $\Gamma(\mathcal{D})$ of sections of $\mathcal{D}$, one looks at the sheaf $\Gamma_{\mathcal{D}}$ of local sections of $\mathcal{D}$. Taking iterated Lie brackets results in a flag

$$\Gamma^1_{\mathcal{D}} \hookrightarrow \Gamma^2_{\mathcal{D}} \hookrightarrow \Gamma^3_{\mathcal{D}} \hookrightarrow \dots$$

of sub-sheaves of the sheaf of vector fields on M, where $\Gamma^1_{\mathcal{D}} := \Gamma_{\mathcal{D}}$ and

$$\Gamma^{i+1}_{\mathcal{D}} := \Gamma^i_{\mathcal{D}} + [\Gamma_{\mathcal{D}}, \Gamma^i_{\mathcal{D}}], \quad i \geq 1.$$

$\mathcal{D}$ is called regular if the sheaves $\Gamma^i_{\mathcal{D}}$ are sheaves of sections of vector sub-bundles of TM. That is,

Definition 1.5 A distribution $\mathcal{D}$ on M is called *regular* if there exists a flag of vector sub-bundles of TM

$$\mathcal{D} \hookrightarrow \mathcal{D}^2 \hookrightarrow \mathcal{D}^3 \hookrightarrow \dots$$

such that the set of global sections of $\mathcal{D}_i$ is precisely the set of global sections of the sheaf $\Gamma^i_{\mathcal{D}}$, that is

$$\Gamma(\mathcal{D}^i) = \Gamma^i_{\mathcal{D}}(M).$$

The corresponding integer list of the dimensions of these sub-sheafs at a given point is called the growth vector, a fundamental numerical invariant associated to the distribution.

Definition 1.6 The *growth vector* of a regular distribution $\mathcal{D}$ is the vector

$$(\text{rank}(\mathcal{D}), \text{rank}(\mathcal{D}^2), \dots, \text{rank}(\mathcal{D}^r))$$

where r is the minimal integer such that $\text{rank}(\mathcal{D}^r) = \text{rank}(\mathcal{D}^{r+1})$.

Definition 1.7 A distribution $\mathcal{D}$ on M is called *regular* when its growth vector is constant on M.

An elegant introduction to these notions and related concepts is provided in the book [11] by R. Montgomery.

1.2 Bracket generating distributions

We focus on a sub-class of regular distributions that are so-called bracket generating; these are the distributions for which the growth vector reaches the dimension of M. This means that eventually the entire tangent space is spanned by taking $k-1$ nested Lie brackets of horizontal vector fields, i.e. vector fields tangent to the distribution.

Definition 1.8 A regular distribution $\mathcal{D}$ on M is called *step-k bracket generating* if

$$\mathcal{D}^k = TM.$$

For a bracket generating distribution $\mathcal{D}$ the growth vector denotes the *type* of the distribution.

Note that, for a bracket generating distribution, the last integer of the growth vector indicates the dimension of the manifold. From the properties of the Lie bracket follows that a necessary condition for a manifold M to admit a bracket generating distributions of step higher than 1 is that the dimension of M must be at least 3. A basic example is given below.

Example 1.9. We consider $\mathbb{R}^3$ with coordinates $\{x, y, z\}$. An example of a bracket generating distribution $\mathcal{D}$ on $\mathbb{R}^3$ is point-wise given by the span

$$\mathcal{D}_p = \langle \partial_y, \partial_x + y\partial_z \rangle, \quad p \in \mathbb{R}^3.$$

This distributions is illustrated in Figure 1.2. Note that this distribution is of type $(2, 3)$. △

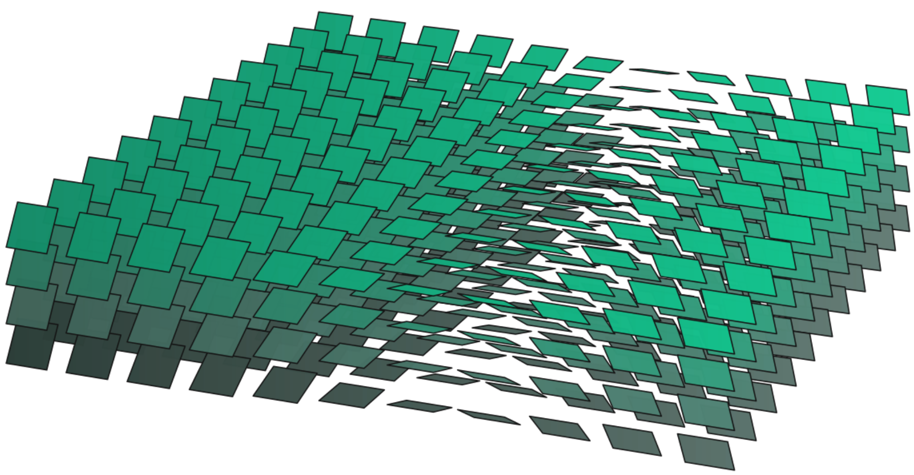

Figure 1.2: Bracket generating distribution $\mathcal{D}$ on $\mathbb{R}^3$ is point-wise given by the span $\mathcal{D}_p = \langle \partial_y, \partial_x + y\partial_z \rangle, \quad p \in \mathbb{R}^3$.

An important property of bracket generating distributions is given by Chow's theorem.

Theorem 1.10 (Chow) Let $\mathcal{D}$ be a bracket generating distribution on M. Then, for all $p, q \in M$ there exists a path

$$\gamma : [0, 1] \to M,$$

such that $\gamma(0) = p$, $\gamma(1) = q$ and γ is tangent to $\mathcal{D}$.

Returning to applications in control theory: a system with restricted degrees of freedom corresponding to a bracket generating distribution has the property that every two points in the configuration space can be reached using the restricted directions only.

1.3 The canoe versus the spaceship

Fat distributions can be considered the most extreme class of maximally non-integrable distributions, since it is as far away from being integrable as possible.

Definition 1.11 A distribution $\mathcal{D}$ on a manifold M is called *fat* (or *strongly bracket generating) at* $x \in M$, if for every choice of vector field $V \in \Gamma\mathcal{D}$ that is non-zero at x we have that T_xM is equal to the span

$$T_xM = \langle \mathcal{D}_x, [V, \mathcal{D}]_x \rangle.$$

Here $[V, \mathcal{D}]_x$ is the subspace defined by

$$[V, \mathcal{D}]_x = \{[V, W]_x \mid W \in \Gamma\mathcal{D}\}.$$

The distribution is *fat* if it is fat at every point $x \in M$.

We properly introduce fat distributions and tools to study them in the next chapter.

Fat distributions do not appear with type $(4, 6)$ only. In fact, all contact distributions are fat. They form exactly the set of fat distributions with co-rank 1. For higher co-rank fat distributions, the type $(4, 6)$ class is naturally the first candidate to look at, due to the following result by Rayner.

Theorem 1.12 (Rayner) Suppose $\mathcal{D}$ is a rank r-distribution on M with dim $M = n$. If $\mathcal{D}$ is fat then r is divisible by 2 and if $r < n - 1$, then r is divisible by 4. Suppose $\mathcal{D}$ is a rank k-distribution on M with dim $M = n$. If D is fat then the following numeric constraints hold

- k is divisible by 2 and if $k < n - 1$, then k *is divisable by* 4.
- $k \geq (n - k) + 1$
- The sphere S^{k-1} admits $n - k$ linearly independent vector fields.

Conversely, given any pair (k, n) satisfying the above, there is a germ of fat distribution of type (k, n)

Indeed, for a non-contact, fat distribution $\mathcal{D}$ on M, we have that the rank r must be divisible by 4. The first co-rank above 1 is 2, so $(4, 6)$ is the first type for which a fat distribution can be encountered. Unlike in the contact setting, only very few examples of fat distributions with co-rank 1 are known, even fewer for co-rank greater than 2. A known family of examples is induced by holomorphic contact distributions, see [1, Example 2.3, p. 4].

Example 1.13. A hololomorphic contact distribution of type $(2r, 2r + 1)$ on a manifold of complex dimension $2r + 1$ induce a real fat distribution of type $(4r, 2r + 2)$ on the underlying real manifold. △

In particular this implies that a holomorphic contact distribution of type $(2, 3)$ induces a fat distribution of type $(4, 6)$. A natural question that arises now is if there are also examples of fat distributions that are

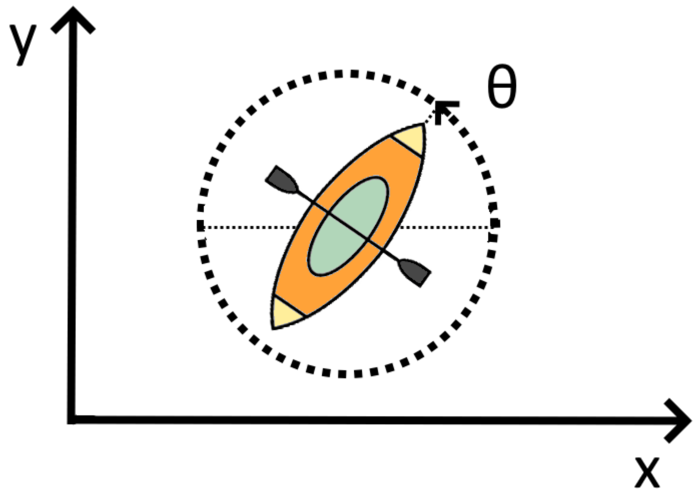

Figure 1.3: Configuration space of the canoe: $\mathbb{R}^2 \times S^1$

not associated to holomorphic contact distributions. In this thesis we provide a new concrete family of such examples, which are natural candidates to answer this question.

Example 1.14. Assume we paddle a canoe on the water. Our configuration space consists of the positions of the canoe on the 2-dimensional water surface and its orientations, see Figure 1.3. Together this forms the 3-dimensional manifold $M = \mathbb{R}^2 \times S^1$ for which we use the (local) coordinates $p = (x, y, \theta)$. We have control over (combinations of) the following two movements: we can rotate the canoe, changing the orientation, and we can paddle the canoe forward and backward in the direction of its orientation, changing the position. This corresponds to a type (2,3) distribution $\mathcal{D}^{\text{canoe}}$ on TM spanned by the vector fields V, W given by

$$V = \partial_\theta$$
$$W = \cos(\theta)\partial_x + \sin(\theta)\partial_y$$

In order to check if the distribution $\mathcal{D}^{\text{canoe}}$ is fat, we compute

$$[V, W] = -\sin(\theta)\partial_x + \cos(\theta)\partial_y.$$

Note that for any $p \in M$, indeed

$$T_pM = \langle V_p, W_p, [V, W]_p \rangle.$$

Hence $\mathcal{D}^{\text{canoe}}$ is fat. In particular, since D is a co-rank 1 distribution, it is contact. $\triangle$

Other well-known examples of distributions appear in the context of principal bundles. Let M be the total space of a principal bundle and let the distribution $\mathcal{D}$ be given by the horizontal space for a connection ∇ on M. Note that if ∇ is a flat connection,

$$[X, Y] = 0, \quad \text{for all } X, Y \in \Gamma\mathcal{D}.$$

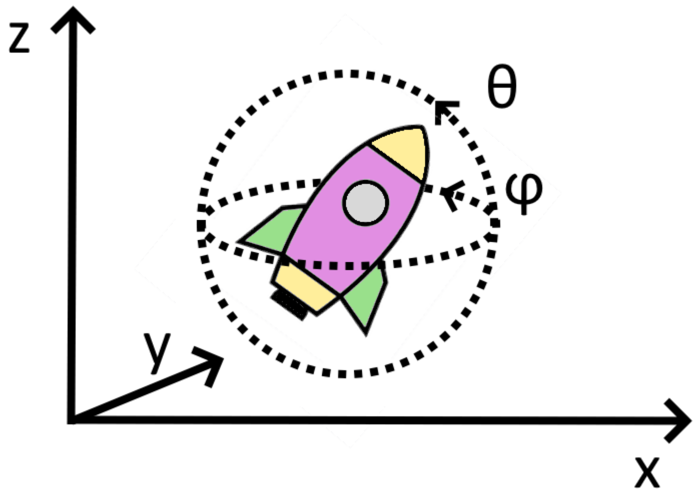

Figure 1.4: Configuration space of the spaceship: $\mathbb{R}^3 \times S^2$

Hence $\mathcal{D}$ is very far from being fat, in fact it is a foliation. In fact, the terminology "fat" has its origin from this context: in [15] a connection on a principal bundle is called fat when it is very far from being flat, i.e. it has "a lot of curvature". However, the definition of fat is more subtle than just not being flat. This is illustrated in the next example, where we take the canoe from Example 1.14 to a higher dimensional setting.

Example 1.15. Analogous to Example 1.14, assume we drive a spaceship in the galaxy. The configuration space consists of the positions of the ship in the 3-dimensional galaxy and its orientations, see Example 1.15. This forms the 5 dimensional manifold $M = \mathbb{R}^3 \times S^2$ for which we use the local coordinates $p = (x, y, z, \theta, \varphi)$. We have control over (combinations of) the following movements: rotation in both angles, changing the orientation of the spaceship, and moving forward an backward, changing the position. This corresponds to a type (3,5) distribution $\mathcal{D}^{\text{ship}}$ on TM spanned by the vector fields V_1, V_2, V_3 given by

$$\begin{aligned} V_1 &= \partial_\theta \\ V_2 &= \partial_\varphi \\ V_3 &= \cos(\varphi)(\cos(\theta)\partial_x + \sin(\theta)\partial_y) + \sin(\varphi)\partial_z \end{aligned}$$

In order to see that $\mathcal{D}^{\text{ship}}$ is not fat it suffices to show that, for the choice of vector V_1 at any point p, the tangent space T_pM can not be spanned by $\mathcal{D}_p, [V_1, V_2]_p, [V_1, V_3]_p$. Indeed, at any point p we have that

$$[V_1, V_2]_p = 0.$$

△

1.4 Curvature

In this section we make precise what is meant by curvature. In order to do so we start with the following lemma.

Lemma 1.16 Let $\mathcal{D}$ be a distribution on a manifold M. Then the map $\hat{F}\colon \Gamma(\mathcal{D})\times\Gamma(\mathcal{D}) \to \Gamma(TM/\mathcal{D})$ given by

$$(V, W) \in \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \mapsto [V, W] \mod \Gamma(\mathcal{D})$$

is $C^\infty(M)$ linear in both entries.

Proof. The map $\hat{F}$ is skew symmetric, so it is enough to show

$$[fV, W] = f[V, W] \mod \Gamma(\mathcal{D})$$

for all $f \in C^\infty(M)$ and $V, W \in \Gamma(\mathcal{D})$. We have that

$$[fV, W] = -W(f) \cdot V + f \cdot [V, W]$$

and $V \in \Gamma(\mathcal{D})$, which implies the statement. □

Because of the lemma above, the map $\hat{F}$ is induced by a bundle map.

Definition 1.17 Let $\mathcal{D}$ be a distribution on a manifold M. The *curvature* of $\mathcal{D}$ is the linear bundle map $F\colon \bigwedge^2 \mathcal{D} \to TM/\mathcal{D}$ point-wise given by

$$F_x(v, w) = -[V, W],$$

where $x \in M$, $v, w \in \mathcal{D}_x$ and where $V, W \in \Gamma(\mathcal{D})$ are vector fields extending v, w respectively.

Remark 1.18. A distribution $\mathcal{D}$ is involutive if and only if its curvature vanishes everywhere. On the contrary, a distribution $\mathcal{D}$ is fat if and only if $F(v, \cdot)\colon \mathcal{D} \to TM/\mathcal{D}$ is surjective everywhere for all $v \in \Gamma(\mathcal{D})$.

The simpler characterizations of fat distributions we promised at the start of this chapter are closely related to the dual of the curvature.

Definition 1.19 Let $\mathcal{D}$ be a distribution on a manifold M. We define the *annihilator bundle* $\mathcal{D}^\perp \in T^*M$ of $\mathcal{D}$ as the bundle of co-vectors annihilating $\mathcal{D}$.

Remark 1.20. The annihilator bundle $\mathcal{D}^\perp$ is canonically dual to $TM/\mathcal{D}$. Hence we can express the dual of the curvature map as $F^*\colon \mathcal{D}^\perp \to \bigwedge^2 \mathcal{D}^*$.

Definition 1.21 The linear bundle map $F^* : \mathcal{D}^\perp \to \Lambda^2\mathcal{D}^*$ given by the dual of the curvature F is called the *dual curvature map*. Explicitly F^* is given by

$$F^*(\alpha)(v, w) = \alpha(-[V, W])$$

for $V, W \in \Gamma\mathcal{D}$.

Proposition 1.22 The dual curvature map is given by

$$F^*(\alpha) = \mathrm{d}\alpha|_{\mathcal{D}}.$$

Proof. Let $\alpha \in \mathcal{D}^\perp$ and let $X, Y \in \Gamma\mathcal{D}$. We have Cartan's formula

$$\mathrm{d}\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]). \tag{1.1}$$

Since α annihilates $\mathcal{D}$, $\alpha(X) = \alpha(Y) = 0$ and it follows that $\alpha(-[X, Y]) = \mathrm{d}\alpha(X, Y)$. □

Proposition 1.23 Suppose $\mathcal{D}$ is a co-rank k distribution on a manifold M. Then $\mathcal{D}$ is fat at $x \in M$ if and only if

$$\omega(\alpha) = \mathrm{d}\alpha|_{\mathcal{D}}$$

is a non-degenerate 2-form on $\mathcal{D}$ at x, for all $\alpha \in \mathcal{D}^\perp$ that are non-vanishing at x.

Proof. If we assume $\mathcal{D}$ is fat at $x \in M$, i.e. for all $X \in \Gamma\mathcal{D}$ and for all $W \in TM/\mathcal{D}$, both non-zero at x, there exists a $Y \in \Gamma\mathcal{D}$ such that

$$[X, Y] = W.$$

This is equivalent to requiring that there exists a $Y \in \Gamma\mathcal{D}$ such that

$$\alpha([X, Y]) = 1$$

for all $X \in \Gamma\mathcal{D}$ and for all $\alpha \in \mathcal{D}^\perp$, both non-zero at x. Via the same reasoning in the proof of Proposition 1.22, this is equivalent to requiring that there exists a $Y \in \Gamma\mathcal{D}$ such that

$$-\mathrm{d}\alpha(X, Y) = 1$$

for all $X \in \Gamma\mathcal{D}$ and for all $\alpha \in \mathcal{D}^\perp$, both non-zero at x. I.e., $\mathrm{d}\alpha|_{\mathcal{D}}$ is non-degenerate at x. □

We have that $\mathcal{D}^\perp$ is a k-dimensional vector sub-bundle of T^*M. Hence, locally, $\mathcal{D}$ can be described as the intersected kernels of some k (independent) 1-forms.

Definition 1.24 Let $\mathcal{D}$ be a co-rank k distribution on M. A collection $\alpha_1, \ldots, \alpha_k$ of 1-forms in $\mathcal{D}^\perp$ satisfying

$$\mathcal{D} = \ker \alpha_1 \cap \cdots \cap \ker \alpha_k$$

on an open subset U of M is called a set of *(locally) defining 1-forms* for $\mathcal{D}$ on U.

Note that a set of locally defining 1-forms locally generate $\mathcal{D}^\perp$. This description allows for a more workable equivalent definition of fat. For co-rank 1 the alternative description is straightforward.

Corollary 1.25 Suppose $\mathcal{D}$ is a co-rank 1 distribution on an n-dimensional manifold M. Let $x \in M$ and let $\mathcal{D}$ be defined by a 1-form α in a neighborhood U of x. Then $\mathcal{D}$ is contact (i.e. fat) if and only if $\omega = \mathrm{d}\alpha|_{\mathcal{D}}$ is non-degenerate.

Proof. Since $\mathcal{D}$ is of co-rank 1, $\mathcal{D}^\perp$ is of rank 1 and α is non-vanishing on U. Hence $\mathcal{D}_x^\perp$ is given by the linear span of α_x. Let $\alpha' \in \mathcal{D}^\perp$. Then $\alpha'_x = c\alpha_x$ for some $c \in \mathbb{R}$. Now since

$$\mathrm{d}(c\alpha)|_{\mathcal{D}} = c\mathrm{d}\alpha|_{\mathcal{D}},$$

if follows from Proposition 1.23 that $\mathcal{D}$ is contact if and only if $\omega = \mathrm{d}\alpha|_{\mathcal{D}}$ is non-degenerate at x. □

Example 1.26. We consider the distribution $\mathcal{D}^{\text{canoe}}$ corresponding to the canoe of Example 1.14. Note that $\mathcal{D}^{\text{canoe}}$ can be expressed via the locally defining 1-form

$$\alpha = \sin(\theta)\mathrm{d}x - \cos(\theta)\mathrm{d}y.$$

I.e. we have that $\mathcal{D}^{\text{canoe}} = \ker \alpha$. Now we compute

$$\mathrm{d}\alpha = \cos \mathrm{d}\theta \wedge \mathrm{d}x + \sin \mathrm{d}\theta \wedge \mathrm{d}y,$$

and we see that $\mathrm{d}\alpha|_{\mathcal{D}^{\text{canoe}}}$ is indeed non-degenerate on $\mathcal{D}^{\text{canoe}}$. This shows once again that $\mathcal{D}^{\text{canoe}}$ is contact. △

A more general equivalent description of fatness is the following.

Proposition 1.27 Suppose $\mathcal{D}$ is a co-rank k distribution on an n-dimensional manifold M defined locally by a pair of 1-forms $\alpha_1, \dots, \alpha_k$. Then $\mathcal{D}$ is fat at if and only if the following conditions hold.

1. Each $\omega_i = \mathrm{d}\alpha_i|_{\mathcal{D}}$ is non-degenerate, i.e. symplectic, for $i = 1, \dots, k$.
2. At any point x and for any $0 \neq v \in \mathcal{D}_x$ we have that

$$\text{co-rank}(v^{\perp_1} \cap \cdots \cap v^{\perp_2}) = k,$$

where $v^{\perp_i}$ denotes the symplectic complement of $\{v\}$ with respect to ω_i.

It follows as an immediate generalisation of the description given in [1, p. 4] for co-rank 2.

Definition 1.28 Suppose $\mathcal{D}$ is a distribution of co-rank k on manifold M. Let U be an open in M and let ω_1, ω_2 be two non-degenerate 2-forms defined on $\mathcal{D}|_U$. The *rank measure* of the pair ω_1, ω_2 is the unique local automorphism $R^{1,2} \colon \mathcal{D}|_U \to \mathcal{D}|_U$ defined by

$$\omega_1(\cdot, R\cdot) = \omega_2(\cdot, \cdot).$$

For a distribution $\mathcal{D}$ of co-rank $k \geq 2$, any pair of annihilating 1-forms α_1 and α_2 generates a sub-bundle $\langle \alpha_1, \alpha_2 \rangle \subset \mathcal{D}^\perp$. However, if we have that $\omega_i = \mathrm{d}\alpha_i|_D$ is non-degenerate for $i = 1, 2$, it is not guaranteed that all non-zero 1-forms in $\langle \alpha_1, \alpha_2 \rangle$ satisfy this property, which is required for $\mathcal{D}$ to be fat. We need an additional condition on the pair α_1, α_2. This results in the corollary below for co-rank 2 distributions, as shown in [1, Prop. 2.1, p. 4].

Corollary 1.29 Suppose $\mathcal{D}$ is a co-rank 2 distribution on an n-dimensional manifold M defined locally by a pair of 1-forms α_1, α_2. Then $\mathcal{D}$ is fat if and only if the following conditions hold.

1. $\omega_i = \mathrm{d}\alpha_i|_D$ is non-degenerate for $i = 1, 2$.
2. The rank measure $R^{1,2} : D \to D$ relating ω_1 to ω_2 has no real eigenvalues.

Proof. Locally the annihilator bundle $\mathcal{D}^\perp$ is generated by α_1 and α_2.

($\Rightarrow$) Suppose $\mathcal{D}$ to be fat. Then, by Proposition 1.23, we have that $\omega(\alpha) = \mathrm{d}\alpha|_\mathcal{D}$ is a non-degenerate 2-form on $\mathcal{D}$ for all $0 \neq \alpha$ in the annihilator bundle $\mathcal{D}^\perp$. In particular this holds for $\omega_1 = \mathrm{d}\alpha_1|_\mathcal{D}$ and $\omega_2 = \mathrm{d}\alpha_2|_\mathcal{D}$, hence the first property is satisfied. We prove the second property by contradiction. Assume that at a point $x \in M$ the rank measure R^{12} has real eigenvalue λ with corresponding eigenvector X. Then we have that

$$\begin{aligned}\omega_1(\cdot, X) &= \omega_2(\cdot, R^{12}X)\\ &= \omega_2(\cdot, \lambda X)\\ &= \lambda\omega_2(\cdot, X).\end{aligned}$$

This implies that $\omega_1 - \lambda\omega_2$ is a degenerate 2-form. Now we observe that

$$\begin{aligned}\omega_1 - \lambda\omega_2 &= \mathrm{d}\alpha_1|_\mathcal{D} - \lambda \mathrm{d}\alpha_2|_\mathcal{D}\\ &= \mathrm{d}(\alpha_1 - \lambda\alpha_2)|_\mathcal{D}.\end{aligned}$$

Hence $\mathrm{d}(\alpha_1 - \lambda\alpha_2)|_\mathcal{D}$ is degenerate. But $\alpha_1 - \lambda\alpha_2$ is an element in the annihilator bundle $\mathcal{D}^\perp$, so this contradicts Proposition 1.23. Hence R^{12} does not have a real eigenvalue.

($\Leftarrow$) Suppose conditions 1) and 2) hold. It suffices to show that every element β in the annihilator bundle $\mathcal{D}^\perp$ satisfies that $\mathrm{d}\beta|_\mathcal{D}$ is non-degenerate. Let β be an element of the annihilator bundle $\mathcal{D}^\perp$. Then $\beta = c_1\alpha_1 + c_2\alpha_2$, for some scalars $c_1, c_2 \in \mathbb{R}$. If either c_1 or c_2 is equal to zero we have that $\mathrm{d}\beta|_\mathcal{D}$ is non-degenerate by condition 1) and linearity. If both $c_1, c_2 \neq 0$, we have that

$$\begin{aligned}\mathrm{d}\beta|_\mathcal{D} &= \mathrm{d}(c_1\alpha_1 + c_2\alpha_2)|_\mathcal{D}\\ &= c_1\mathrm{d}\alpha_1|_\mathcal{D} + c_2\mathrm{d}\alpha_2|_\mathcal{D}\\ &= c_1\omega_1 + c_2\omega_2.\end{aligned}$$

Again, we prove by contradiction. Assume d$\beta|_{\mathcal{D}}$ is degenerate. Then $c_1\omega_1 + c_2\omega_2$ is degenerate. Hence there exists a point x and a vector X in $\mathcal{D}_x$ such that

$$c_1\omega_1(\cdot, X) + c_2\omega_2(\cdot, X) = 0.$$

Denote by R^{12} be the rank measure relating ω_1 and ω_2. Then we have that

$$c_1\omega_1(\cdot, X) + c_2\omega_1(\cdot, R^{12}X) = 0.$$

This implies that

$$\omega_1(\cdot, \lambda X) = \omega_1(\cdot, R^{12}X),$$

where $\lambda = -\frac{c_1}{c_2}$. Note λ is real. Since both ω_1 and ω_2 are non-degenerate, we have that $\lambda X = R^{12}X$. Hence λ is a real eigenvalue of R^{12}. This contradicts condition 2), hence d$\beta|_{\mathcal{D}}$ must be non-degenerate. The statement follows. □

Remark 1.30. In the co-rank 2 case, the characterization for fatness from Corollary 1.29 can alternatively be described in terms of quadratic forms, see [9].

Chapter 2

Distributions and the Grassmann bundle

As we have seen in Chapter 1, a rank-r distribution on a manifold M is defined by a choice of rank-r linear subspace of of the tangent space T_xM for each point x, smoothly varying with respect to x. At a point x, the set of all possible choices of rank-r linear subspaces of the tangent space is described by the rank-r Grassmannian of T_xM. All the Grassmannians of the tangent spaces for the points in M together form a fiber bundle, which is called Grassmann bundle of M. It turns out that every rank-r distribution on a manifold M can interpreted as a global section of the rank-r Grassmann bundle of M. This makes the Grassmann bundle of a given manifold M the universal object for distributions on M.

Furthermore, we will see in Chapter 3 that the Grassmann bundle itself is a manifold that comes with a canonical distribution defined on it. In fact, this is consistent with the Grassmannian being a universal object for distributions. This canonical distribution plays a central role in the construction of prolonged distributions, which form the class of distributions that we focus on later in this text.

We first introduce the Grassmannian of a given vector space with its homogeneous coordinates and we show how to construct an atlas for this manifold. After that, we define the Grassmann bundle of a given manifold. Then we observe that a choice of distribution on a manifold M corresponds to a choice of global section of the Grassmann bundle of M.

2.1 Grassmannian

2.1.1 Spanning vectors

We first introduce the Grassmannian in the classical way, i.e. we describe linear sub-spaces in terms of spanning sets of vectors. This description provides us with some intuition for these spaces. However, as we have seen in Section 1.4, it is convenient to be able to use a description of linear sub-spaces using co-vectors, which we will do afterwards.

> **Definition 2.1** Let V be a vector space of dimension n. The rank-r *Grassmannian* of V is the space of all linear subspaces of dimension r in V. It is denoted by $\mathrm{Gr}^r(V)$ or by $\mathrm{Gr}_k(V)$, where $k = n - r$ indicates the co-rank.

The Grassmannian $\mathrm{Gr}^r(V)$ carries the structure of a smooth manifold of dimension rk. We first introduce homogeneous coordinates for this space $\mathrm{Gr}^r(V)$, then we show how to construct an atlas.

Homogeneous coordinates

Choose a basis $\{x_1, \ldots, x_n\}$ for the n-dimensional vector space V. The spanning set of r linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ defines an element

$$\langle \mathbf{v}_1, \ldots, \mathbf{v}_r \rangle$$

in $\mathrm{Gr}^r(V)$. Conversely, a point $H \in \mathrm{Gr}^r(V)$ is described by a (non-unique) spanning set of r linearly independent vectors

$$H = \langle \mathbf{v}_1, \ldots, \mathbf{v}_r \rangle.$$

Now consider a second spanning set of vectors for H

$$H = \langle \mathbf{w}_1, \ldots, \mathbf{w}_r \rangle.$$

For these two spanning sets that define H we denote by A and B the rank r matrices

$$A = \begin{pmatrix} v_{11} & \cdots & v_{r1} \\ \vdots & \ddots & \vdots \\ v_{1n} & \cdots & v_{rn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} w_{11} & \cdots & w_{r1} \\ \vdots & \ddots & \vdots \\ w_{1n} & \cdots & w_{rn} \end{pmatrix}$$

defining the respective spanning sets with respect to the chosen basis. Observe that the two spanning sets define the same linear subspace H if and only if there exists a $Q \in GL_r$ such that

$$A = B \cdot Q.$$

The discussion above exhibits $\mathrm{Gr}^r(V)$ as the quotient of the vector space of $r \times n$ matrices of rank r [1] under the right action of GL_r. [2]

From this perspective, it is natural to denote $H \in \mathrm{Gr}^r(V)$ as the equivalence class

$$H = [A]$$

where A is any suitable $n \times r$ matrix of rank r that serves as representative for H.

Atlas

Here we recall how to construct an smooth atlas for the Grassmannian. We use the notation $\mathcal{L} = (l_1, \ldots, l_r)$, with $l_i \in \{1, \ldots, n\}$, to indicate the choice of r vectors $\phi_{\mathcal{L}} = x_{l_1}, \ldots, x_{l_r}$ from the chosen basis $\{x_1, \ldots, x_n\}$. We use the notations $\mathcal{L}^\perp = (l_{r+1}, \ldots, l_n)$ and $\phi_{\mathcal{L}^\perp}$ to denote the remaining numbers in $\{1, \ldots, n\}$ and the corresponding elements $x_{l_{r+1}}, \ldots, x_{l_n}$ of the basis for V, respectively.

[1] In fact this space forms the *Stiefel manifold* St_r, it is the manifold of r-tuples of linearly independent vectors in V.

[2] In fact, the action of GL_r on this manifold is free and proper. This implies directly that the quotient, the grassmanian Gr_r, is a smooth manifold.

Consider the subset Gr$^r(V)_{\mathcal{L}}$ of all $H \in$ Gr$^r(V)$ such that $x_{l_1}, \ldots, x_{l_r}$ project to a basis for H – with respect to the standard inner product associated to the previously chosen basis $\{x_1, \ldots, x_n\}$. Note that Gr$^r(V)_{\mathcal{L}}$ is open in Gr$^r(V)$. For every $H \in$ Gr$^r(V)_{\mathcal{L}}$ there exist unique numbers m_{ij}, with $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, k\}$ such that H is given by the span of the vectors

$$\begin{aligned} \mathbf{v}_1 &= x_{l_1} + m_{11}x_{l_{r+1}} + \cdots + m_{1k}x_{l_n}, \\ &\vdots \\ \mathbf{v}_r &= x_{l_r} + m_{r1}x_{l_{r+1}} + \cdots + m_{rk}x_{l_n}. \end{aligned}$$

As an example, if we set $\mathcal{L} = (1, \ldots, r)$, an element H in Gr$^r(V)_{\mathcal{L}}$ is given by the span of the vectors

$$\begin{aligned} \mathbf{v}_1 &= x_1 + m_{11}x_{r+1} + \cdots + m_{1k}x_n, \\ &\vdots \\ \mathbf{v}_r &= x_r + m_{r1}x_{r+1} + \cdots + m_{rk}x_n. \end{aligned}$$

The numbers m_{ij} form the $r \times k$ matrix

$$M = \begin{pmatrix} m_{11} & \cdots & m_{r1} \\ \vdots & \ddots & \vdots \\ m_{1k} & \cdots & m_{rk} \end{pmatrix}.$$

The corresponding homogeneous coordinates for H are given by

$$\begin{bmatrix} I \\ M \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ m_{11} & \cdots & m_{r1} \\ \vdots & \ddots & \vdots \\ m_{1k} & \cdots & m_{rk} \end{bmatrix}$$

We can then define a chart $\chi_{\mathcal{L}}\colon$ Gr$^r(V)_{\mathcal{L}} \to \mathbb{R}^{rk}$ via

$$\begin{bmatrix} I \\ M \end{bmatrix} \mapsto (m_{11}, ..., m_{rk}).$$

The inverse map $\chi_{\mathcal{L}}^{-1}$ is given by

$$M \mapsto \begin{bmatrix} I \\ M \end{bmatrix}, \quad M \in \mathbb{R}^{rk}.$$

$\chi_{\mathcal{L}}^{-1}$ can be interpreted as taking M to the span of its graph as illustrated in Section 2.1.1. Seen from this perspective it is clear that all elements in Gr$^r(V)_{\mathcal{L}}$ must be transversal to the linear subspace of V spanned by $\phi_{\mathcal{L}^\perp}$. In this fashion the chart $\chi_{\mathcal{L}}$ can be seen as a projection of the graph.

By varying $\mathcal{L}$ we obtain a cover of Gr^r, and this construction provides the desired atlas on the Grassmanian. Note that, for $\mathcal{L} = (l_1, \ldots, l_r)$ and $\mathcal{L} = (l'_1, \ldots, l'_r)$, the intersection Gr$^r(V)_{\mathcal{L}} \cap$ Gr$^r(V)_{\mathcal{L}'}$ is given by

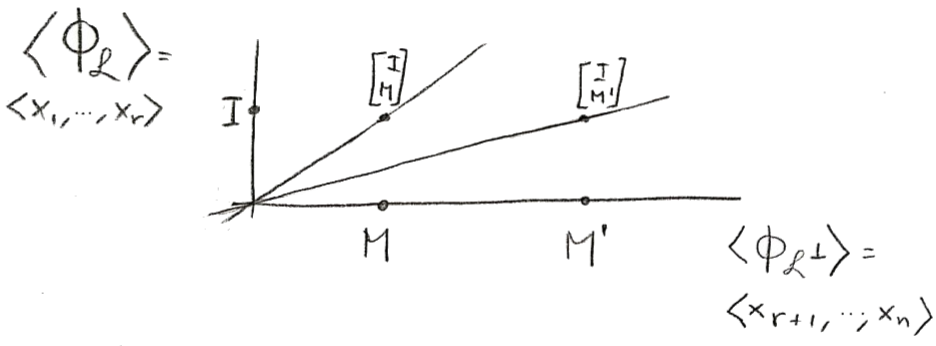

Figure 2.1: Graph of elements $M, M' \in \mathbb{R}^{rk}$ under $\chi_{\mathcal{L}}^{-1}$

the points $H \in \mathrm{Gr}^r(V)$ such that both frames $\phi_{\mathcal{L}}$ and $\phi_{\mathcal{L}'}$ project to a basis of H. The resulting transition function can be described point-wise in terms of the linear map sending the basis

$$\begin{aligned} \mathbf{v}_1 &= x_{l_1} + m_{11} x_{l_{r+1}} + \cdots + m_{1k} x_{l_n}, \\ &\vdots \\ \mathbf{v}_r &= x_{l_r} + m_{r1} x_{l_{r+1}} + \cdots + m_{rk} x_{l_n} \end{aligned}$$

to the basis

$$\begin{aligned} \mathbf{v}'_1 &= x_{l'_1} + m'_{11} x_{l'_{r+1}} + \cdots + m'_{1k} x_{l'_n}, \\ &\vdots \\ \mathbf{v}'_r &= x_{l'_r} + m'_{r1} x_{l'_{r+1}} + \cdots + m'_{rk} x_{l'_n}. \end{aligned}$$

of H, and is smooth.

2.1.2 Switching to co-vectors

Since it is often feasible to describe a distribution $\mathcal{D}$ in terms of a set of locally defining 1-forms, it is convenient also to describe the elements in the Grassmannian and the Grassmann bundle in this way.

Homogeneous coordinates with co-vectors

Sticking to the notation introduced in Section 2.1.1, we construct a second set of homogeneous coordinates for the Grassmannian.

We choose the same dual basis $\{x^1, \ldots, x^n\}$ for the n-dimensional vector space V. Consider a set of k

linearly independent 1-forms

$$\alpha^1 = \mu_1^1 x^1 + \mu_2^1 x^2 \cdots + \mu_n^1 x^n,$$
$$\vdots$$
$$\alpha^k = \mu_1^k x^1 + \mu_2^k x^2 \cdots + \mu_n^k x^n.$$

The system of equations

$$\alpha^1 = 0, \ldots, \alpha^k = 0$$

defines a point $H \in \mathrm{Gr}_k(V)$, that is, $H = \ker(\alpha^1) \cap \cdots \cap \ker(\alpha^k)$. Conversely, every point $H \in \mathrm{Gr}_k(V)$ can be described – in a non-unique way – as the zero locus

$$\bigcap_{j=1}^{k} \ker(\alpha^j)$$

for a set of k suitably chosen forms.

We denote by $\mathcal{A}$ the matrix

$$\mathcal{A} = \begin{pmatrix} \mu_1^1 & \cdots & \mu_k^1 \\ \vdots & \ddots & \vdots \\ \mu_1^k & \cdots & \mu_n^k \end{pmatrix}$$

defining the respective system. In a similar way to the description with a spanning set of vectors, we observe that two systems of equations defined by matrices $\mathcal{A}$ and $\mathcal{B}$ have the same zero locus if and only if there exists a $Q \in GL_k$ such that

$$\mathcal{A} = Q \cdot \mathcal{B}.$$

In turn, the Grassmannian $\mathrm{Gr}_k(V)$ can be identified with the quotient of the vector space of $k \times n$ matrices of rank k under the left action of GL_k. In this fashion, it is natural to denote $H \in \mathrm{Gr}_k(V)$ as the equivalence class

$$H = [\mathcal{A}]$$

where $\mathcal{A}$ is any suitable $k \times n$ matrix of rank k that serves as representative for the system defining H.

Atlas with co-vectors

Again, we use the notations $\mathcal{L} = (l_1, \ldots, l_r)$, with $l_i \in \{1, \ldots, n\}$, and $\phi_{\mathcal{L}}$ to indicate the choice of r out of n co-vectors from the chosen basis $\{x^1, \ldots, x^n\}$; and the notations $\mathcal{L}^\perp = (l_{r+1}, \ldots, l_n)$ and $\phi_{\mathcal{L}^\perp}$ to denote the remaining numbers in $\{1, \ldots, n\}$ and the corresponding remaining elements of the basis frame, respectively.

We consider the subset $\mathrm{Gr}_k(V)_{\mathcal{L}}$ of all $H \in \mathrm{Gr}_2(V)$ such that $\{x^{l_1}, \ldots, x^{l_r}\}$ are non-vanishing 1-forms when restricted to H. Note that this subset in $\mathrm{Gr}_k(V)$ is open. For every $H \in \mathrm{Gr}_k(V)_{\mathcal{L}}$ there exist unique numbers μ_i^j, with $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, k\}$ such that H is given by the relations

$$\alpha_1 = x^{l_{r+1}} + \mu_1^1 x^{l_1} + \cdots + \mu_r^1 x^{l_r},$$
$$\vdots$$
$$\alpha_k = x^{l_n} + \mu_1^k x^{l_1} + \cdots + \mu_r^k x^{l_r}.$$

Again we go over the example where we set $\mathcal{L} = (1, \dots, r)$. An element H in Gr$_k(V)_\mathcal{L}$ is given by the relations

$$\alpha_1 = x^{r+1} + \mu_1^1 x^1 + \cdots + \mu_r^1 x^r,$$
$$\vdots$$
$$\alpha_k = x^n + \mu_1^k x^1 + \cdots + \mu_r^k x^r.$$

The numbers μ_i^j form the $k \times r$ matrix

$$\mathcal{M} = \begin{pmatrix} \mu_1^1 & \cdots & \mu_r^1 \\ \vdots & \ddots & \vdots \\ \mu_1^k & \cdots & \mu_r^k \end{pmatrix}$$

The corresponding global coordinates of H are given by

$$[\mathcal{M} \colon I] = \begin{bmatrix} \mu_1^1 & \cdots & \mu_r^1 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mu_1^k & \cdots & \mu_r^k & 0 & \cdots & 1 \end{bmatrix}$$

We then define a chart $\chi_\mathcal{L} \colon$ Gr$_k(V)_\mathcal{L} \to \mathbb{R}^{k \times r}$ by setting

$$\chi_\mathcal{L} \colon [\mathcal{M} \colon I] \mapsto (\mu_1^1, ..., \mu_r^k).$$

By varying $\mathcal{L}$ we obtain a cover of Gr$_k(V)$, and this construction provides us with an atlas on the Grassmanian.

Remark 2.2. Note that this atlas is compatible with the previously constructed atlas using vectors. Indeed, note that the transition function $\mathcal{T} \colon$ Mat$_{2\times 2} \to$ Mat$_{2\times 2}$ for two charts defined on the same open Gr$^r(V)_\mathcal{L}$ (in the two respective atlases) is given by

$$M \mapsto \mathcal{M} = -M^T.$$

2.2 Grassmann bundle

Grassmannians as fibers

We now turn to the Grassmann bundle over a manifold X. As the name suggests, this is a fiber bundle over X, whose fiber over a point x is the Grassmanian of the tangent space $T_x X$.

> **Definition 2.3** The rank-r Grassmann bundle of an n-dimensional manifold X is the set
>
> $$\{(x, H) : \; x \in X, \; H \in Gr^r(T_x X)\}.$$
>
> It is denoted by Gr$^r(TX)$ or Gr$_k(TX)$, where $k = n - r$ indicates the co-rank.

The bundle projection π^G: $\mathrm{Gr}^r(TX) \mapsto X$ is given by

$$(x, H) \to x.$$

A trivializing atlas for $\mathrm{Gr}^r(TX)$ over X can be obtained right out of a coordinate atlas for X and the atlas for $\mathrm{Gr}^r(V)$ introduced earlier. We briefly recall the constructions of trivializations and fibered coordinates below.

Let $U \subset X$ be an open set together with a chart $\psi : U \to \mathbb{R}^n$ that maps

$$x \mapsto (x^1, ..., x^n).$$

Observe that there is a canonical isomorphism

$$\mathrm{Gr}^r(TX)|_U \cong \mathrm{Gr}^r(TU);$$

Moreover, the coordinate vector fields

$$\partial_{x_1}, \ldots, \partial_{x_n}$$

corresponding to ψ provide a basis for T_xX, for all $x \in U$, inducing the isomorphisms

$$\varphi_x\colon\ \mathrm{Gr}^r(T_xX) \to \mathrm{Gr}^r(\mathbb{R}^n), \quad x \in X.$$

Now we can define the trivialization Ψ : $\mathrm{Gr}^r(TU) \to \mathbb{R}^n \times \mathrm{Gr}^r(\mathbb{R}^n)$ by

$$(x, H) \to (\psi(x), \varphi_x(H)).$$

As anticipated, this essentially proves the following well known fact.

Proposition 2.4 The projection

$$\pi^G\colon\ \mathrm{Gr}^r(TX) \to X$$

is a fiber bundle with fiber $\mathrm{Gr}^r(\mathbb{R}^n)$

For later use, we recall how to construct fibered coordinates for $\mathrm{Gr}^r(TX)$ using the charts for $Gr^r(V)$ discussed at the beginning of this section. Over an open $U \subset X$ together with a chart $\psi : U \to \mathbb{R}^n$ that maps

$$x \to (x^1, ..., x^n)$$

we consider the coordinate frame

$$\partial_{x_1}, \ldots \partial_{x_n}.$$

Similar to the Grassmannian setting, we use the notation $\mathcal{L} = (l_1, \ldots, l_r)$, with $l_i \in \{1, \ldots, n\}$, to indicate the choice of r coordinate vector fields

$$\phi_{\mathcal{L}} = \partial_{x_{l_1}}, \ldots, \partial_{x_{l_r}}$$

from the n coordinate vector fields $\partial_{x_1}, \ldots, \partial_{x_n}$. we denote by $\mathsf{Gr}^r(TU)_{\mathcal{L}}$ the open set

$$\{H \in \mathsf{Gr}^r(T_xX)_{\mathcal{L}} \colon x \in U\}.$$

Now we can define fibered coordinate chart $\Psi_{\mathcal{L}} \colon \mathsf{Gr}^r(TU)_{\mathcal{L}} \to \mathbb{R}^n \times \mathbb{R}^{rk})$ by

$$(x, H) \to (\psi(x), \chi_{\mathcal{L}}(H)).$$

Varying the choice $\mathcal{L}$ of coordinate vectors and the chart (ψ, U) provides us with a fibered atlas for $\mathsf{Gr}^r(TX) \to X$.

Distributions as sections

Since a rank-r distribution $\mathcal{D}$ on a manifold X is defined by the choice of an r-dimensional subspace H of T_xX for every point $x \in X$, it can be identified with the section σ of the rank-r Grassmann bundle $\mathsf{Gr}^r(TX)$ of X given by

$$x \mapsto H \coloneqq \mathcal{D}_x, \quad x \in X$$

where H is interpreted as a point in $\mathsf{Gr}^r(T_xX)$.

Questions concerning the space of distributions with rank r on a given manifold X can now be rephrased in terms of the space (sheaf) of sections $\Gamma\,\mathsf{Gr}^r(TX)$ of the corresponding Grassmann bundle over X.

Chapter 3

Fat prolongation construction

In this chapter we ultimately introduce the prolongation construction and define the class of prolonged distributions. Prolongations in this context were used by Cartan, for the previously mentioned famous $(2, 3, 5)$ structures he investigated, see [2]. For a gentle introduction concerning prolongations of distributions and related concepts we refer to [1].

The prolonged distributions form the class for which we investigate fatness for type $(4, 6)$ distributions in this text. In order to do so, we look at co-rank 2 fiber bundles M over a 4-dimensional manifold X with a bundle map into Grassmann bundle $\mathrm{Gr}_2(TX)$. Then we consider the canonical distribution on $\mathrm{Gr}_2(TX)$ which induces a distribution on M via the bundle map, the so-called prolonged distribution on M. The main question we investigate is under what conditions this restriction defines a fat distribution on the fiber bundle manifold M.

Moreover, we provide a family of examples of fat prolonged $(4, 6)$ distributions: we consider the rank-2 sub-bundle of the Grassmann bundle consisting of the 2-planes invariant under the almost complex structure J. This sub-bundle forms a 6-dimensional manifold and the fibers are in fact complex Grassmannians. We show that the prolonged distribution of this sub-bundle is a fat distribution of co-rank 2.

3.1 Prolongation

In this section we first define the canonical distribution on the Grassmann bundle of a given manifold that was hinted at in the previous chapter. It provides the setup for the prolonged distributions that we define after that.

Canonical distribution

Now we are setup to define the canonical distribution $\mathcal{D}^{\text{can}}$ on $\mathrm{Gr}_k(TX)$ for a given manifold X.

> **Definition 3.1** Let X be an n-dimensional manifold and consider its Grassmann bundle $\pi^G : \mathrm{Gr}_k(TX) \to$

X. The *canonical distribution* $\mathcal{D}^{\mathsf{can}}$ on $\mathsf{Gr}_k(TX)$ is the distribution point-wise given by

$$\mathcal{D}^{\mathsf{can}}_H = (\mathsf{d}\pi^G)^{-1}(H) \quad \subset T_H\,\mathsf{Gr}_k(TX),$$

where the pre-image is with respect to the linear subspace H of the tangent space of X.

An overview of the maps involved is given in Figure 3.1.

$$\begin{array}{ccc} & & \mathcal{D}^{\mathsf{can}} \\ & & \big\downarrow \iota \\ \mathsf{Gr}_k(TX) & \xleftarrow{\tau^G} & T\,\mathsf{Gr}_k(TX) \\ \pi^G \big\downarrow & & \big\downarrow \mathsf{d}\pi^G \\ X & & TX \end{array}$$

Figure 3.1: Canonical distribution on the Grassmann bundle

Remark 3.2. Note that $\mathcal{D}^{\mathsf{can}}$ yields a smooth sub-bundle of $T\,\mathsf{Gr}_k(TX)$. Hence it defines a smooth distribution. Indeed, let U be an open subset of X with coordinates $(x_1, \ldots, x_n)$. Let $(x, H) \in \mathsf{Gr}_k(TU)_{\mathcal{L}}$ with $\mathcal{L} = (l_1, \ldots, l_k)$ (see Chapter 2 for the notation), i.e. the linear subspace H in T_xX is given by the span of the vectors

$$\begin{aligned} \mathbf{v}_1 &= \partial_{x_{l_1}} + m_{11}\partial_{x_{l_{r+1}}} + \cdots + m_{1k}\partial_{x_{l_n}}, \\ &\vdots \\ \mathbf{v}_r &= \partial_{x_{l_r}} + m_{r1}\partial_{x_{l_{r+1}}} + \cdots + m_{rk}\partial_{x_{l_n}}. \end{aligned}$$

Then the linear subspace $\mathcal{D}^{\mathsf{can}}_H = (\mathsf{d}\pi^G)^{-1}(H)$ is spanned by the tangent vectors in $T_H\,\mathsf{Gr}_k(TX)$ given by

$$\begin{aligned} &\partial_{x_{l_1}} + m_{11}\partial_{x_{l_{r+1}}} + \cdots + m_{1k}\partial_{x_{l_n}} \in T_H\,\mathsf{Gr}_k(TX) \\ &\vdots \\ &\partial_{x_{l_r}} + m_{r1}\partial_{x_{l_{r+1}}} + \cdots + m_{rk}\partial_{x_{l_n}} \in T_H\,\mathsf{Gr}_k(TX). \end{aligned}$$

Prolonged distribution

Let X be an n-dimensional manifold and consider its Grassmann bundle $\pi^G\colon\ \mathsf{Gr}_k(TX) \to X$ Now let $\pi^M\colon\ M \to X$ be a fiber bundle over X together with bundle map $\varphi\colon\ M \to Gr_k(TX)$, i.e. the following diagram commutes.

$$\begin{array}{ccc} \mathsf{Gr}_k(TX) & \xleftarrow{\varphi} & M \\ & \searrow^{\pi^G} & \big\downarrow \pi^M \\ & & X \end{array}$$

Definition 3.3 The *prolonged distribution* $\mathcal{D}^{prol}$ *on* M *with respect to* φ is pointwise given by

$$\mathcal{D}^{\text{prol}}_y := (\mathrm{d}\pi^M)^{-1}(\varphi(y)) \quad \subset T_yM.$$

where we interpret $\varphi(y)$ as a subspace of the tangent space of X. We denote by (M, π^M, φ) the *defining triple* for $\mathcal{D}^{\text{prol}}$.

An overview of the maps involved is given in Figure 3.2.

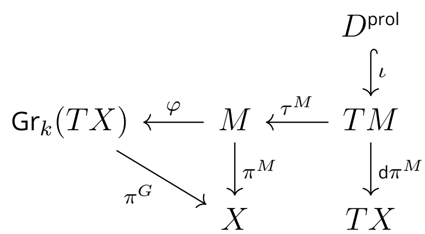

Figure 3.2: Prolonged distribution on fiber bundle M

Example 3.4. If we set $\varphi := \text{id}_{\text{Gr}_k(TX)}$, we see that the prolonged distribution $\mathcal{D}^{\textbf{prol}}$ on $\text{Gr}_k(TX)$ is in fact the canonical distribution $\mathcal{D}^{\textbf{can}}$ on $\text{Gr}_k(TX)$. △

We revisit Example 1.14 –but now the orientations of the canoe are projectivised– and show that the contact distribution $\mathcal{D}^{\text{canoe}}$ is in fact a prolongation.

Example 3.5. Consider the configuration space of the canoe $M = \mathbb{R}^2 \times \mathbb{R}P^1$ and the plane $X = \mathbb{R}^2$ indicating solely the position of the canoe on the water. We define the map $\pi^M : M \to X$ as the projection given by

$$(x, y, \theta) \mapsto (x, y).$$

Consider the Grassmann bundle $\text{Gr}^1(T\mathbb{R}^2)$ of $\mathbb{R}^2$ and the bundle map $\varphi : M \to \text{Gr}^1(T\mathbb{R}^2)$ given by

$$(x, y, \theta) \mapsto H = \langle \cos(\theta)\partial_x + \sin(\theta)\partial_y \rangle.$$

Then the prolonged distribution $\mathcal{D}^{\text{prol}}$ is point-wise given by

$$\begin{aligned} \mathcal{D}^{\text{prol}}_{(x,y,\theta)} &= (\mathrm{d}\pi^M)^{-1}(H) \\ &= \langle \cos(\theta)\partial_x + \sin(\theta)\partial_y, \partial_\theta \rangle, \end{aligned}$$

which is precisely to the contact distribution $\mathcal{D}^{\text{canoe}}$ as defined previously. △

Also the space ship from Example 1.15 –again with projectivised orientations– turns out to be associated to a prolongation.

Example 3.6. Consider the configuration space of the spaceship $M = \mathbb{R}^3 \times \mathbb{R}P^2$ and the space $X = \mathbb{R}^3$ indicating the position of the space ship in the galaxy. We define the projection map $\pi^M : M \to X$ as

$$(x, y, z, \theta, \varphi) \mapsto (x, y, z).$$

Consider the Grassmann bundle $\text{Gr}^1(T\mathbb{R}^3)$ of $\mathbb{R}^3$ and the bundle map $\psi\colon M \to \text{Gr}^2(T\mathbb{R}^3)$ given by

$$(x, y, z, \theta, \varphi) \mapsto H = \langle \cos(\varphi)(\cos(\theta)\partial_x + \sin(\theta)\partial_y) + \sin(\varphi)\partial_z \rangle.$$

The prolonged distribution $\mathcal{D}^{\text{prol}}$ is point-wise given by

$$\begin{aligned}\mathcal{D}^{\text{prol}}_{(x,y,z,\theta,\varphi)} &= (\mathrm{d}\pi^M)^{-1}(H) \\ &= \langle \cos(\varphi)(\cos(\theta)\partial_x + \sin(\theta)\partial_y) + \sin(\varphi)\partial_z, \partial_\theta, \partial_\varphi \rangle,\end{aligned}$$

which corresponds to the distribution $\mathcal{D}^{\text{ship}}$ from earlier. △

In fact, Example 3.5 can be generalized to arbitrary dimensions in such a way that it defines a contact distribution.

Proposition 3.7 Let X be an n-dimensional manifold and consider its Grassmann bundle $\text{Gr}_{n-1}(TX)$. The prolonged distribution $\mathcal{D}^{\text{can}}$ on $\text{Gr}_{n-1}(TX)$ is a contact distribution.

Proof. Let $U \subset X$ be an open set with coordinates $(x_1, \ldots, x_n)$, and let $H \in \text{Gr}_{n-1}(T_xU)$. We have that $H \in \text{Gr}_{n-1}(T_xU)_{\mathcal{L}}$ for some $\mathcal{L}$. Without loss of generality, we assume $\mathcal{L} = (1, \ldots, n-1)$. Then the elements H in $\text{Gr}_{n-1}(T_xU)_{\mathcal{L}}$ are given by the kernel of the linear 1-form

$$\alpha_1 = dx_n - \sum_{i=1}^{n-1} m_{i1}\mathrm{d}x_i$$

in T_x^*X, smoothly depending on m_{i1}, as discussed in Chapter 2. It follows that, on the open $\text{Gr}_{n-1}(TU)_{\mathcal{L}}$, $\mathcal{D}^{\text{can}}$ is given by the kernel of the one-form

$$\beta = dx_n - \sum_{i=1}^{n-1} m_{i1}\mathrm{d}x_i.$$

Now we compute

$$\mathrm{d}\beta = -\sum_{i=1}^{n-1} m_{i1} \wedge dx_i.$$

In this chart $\text{Gr}_{n-1}(TU)_{\mathcal{L}}$, we have that $\mathrm{d}\beta$ restricts to a non-degenerate 2-form on $\mathcal{D}^{\text{can}}$. Hence, by Corollary 1.25, $\mathcal{D}^{\text{can}}$ is contact here. It follows that $\mathcal{D}^{\text{can}}$ is contact everywhere. □

Remark 3.8. Like in Example 3.5, we can identify $\text{Gr}_{n-1}(TX)$ with the projectivization PT^*X.

3.2 Complex Grassmannian

In this section, we provide a concrete family of examples fat distributions of type $(4, 6)$ using the prolongation construction.

The general framework is the following. Let (X, J) be an almost complex 4-dimensional manifold. Consider the Grassmann bundle $\mathsf{Gr}_2(TX)$ over X and let M be the sub-bundle

$$M \coloneqq \mathsf{Gr}_2(TX, J) \subset \mathsf{Gr}_2(TX)$$

given by the (almost) complex grassmanian on X,

$$\mathsf{Gr}_2(TX, J) = \{H \in \mathsf{Gr}_2(TX) | JH = H\},$$

consisting of planes $H \in \mathsf{Gr}_2(TX)$ which are J-invariant.

Note that the fibers of M have co-dimension 2 within the fibers of $\mathsf{Gr}_2(TX)$, hence M has dimension $4 + 2 = 6$. We consider the prolonged distribution $\mathcal{D}^{\mathsf{prol}}$ on M with respect to the inclusion $i \colon M \hookrightarrow \mathsf{Gr}_2(TX)$, i.e. we look at the defining triple $(M, \pi^G|_M, i)$.

Let $U \subset X$ be an open subset of X together with a local frame $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ of X such that

$$J|_U = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.1}$$

with respect to ϕ. We now prove the following lemma, which gives a local description of the inclusion $i \colon M \hookrightarrow \mathsf{Gr}_2(TX)$.

Lemma 3.9 With respect to the coordinates $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for $\mathsf{Gr}(T_xX)$ induced by ϕ at x (see Chapter 2), the submanifold $M_x \subset \mathsf{Gr}_2(T_xX)$ is described by the equations

$$\lambda_1 = \lambda_4, \quad \lambda_3 = -\lambda_2.$$

Proof. In the same spirit as that of Chapter 2, denote by $\mathsf{Gr}_2(TU)_{\mathcal{L}}$, where $\mathcal{L} = (1, 2)$, the open set of all 2-planes to which the pair $\{\phi_{l_1}, \phi_{l_2}\}$ projects to a basis, for $l_1 \neq l_2$ integers between 1 and 4.

Let $H \in M_x$. Then H is preserved by the complex structure J given by

$$J|_U = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with respect to ϕ. Hence the pair $\{\phi_1, \phi_2\}$ or the pair $\{\phi_3, \phi_4\}$ projects to a basis for H (or both). Assume without loss of generality that the pair $\{\phi_1, \phi_2\}$ projects to a basis for H, i.e. $\mathcal{L} = (1, 2)$.

We have that $H \in \mathsf{Gr}_2(TU)_{\mathcal{L}}$ is spanned by

$$V_1 = \phi_1 + \lambda_1\phi_3 + \lambda_2\phi_4$$
$$V_2 = \phi_2 + \lambda_3\phi_3 + \lambda_4\phi_4$$

for $\lambda_i \in \mathbb{R}$ smoothly depending on x. Note that $J|_U$ maps $\phi_1 \to \phi_2$. Then if H is preserved under $J|_U$, it also maps $V_1 \mapsto V_2$.It follows that $H \in M$ if and only if H is spanned by

$$\begin{aligned} V_1 &= \phi_1 + \lambda_1\phi_3 + \lambda_2\phi_4 \\ V_2 &= \phi_2 - \lambda_2\phi_3 + \lambda_1\phi_4, \end{aligned}$$

or equivalently, it is given by the intersected kernels of the forms

$$\begin{aligned} \alpha_1 &= \phi_3^* - \lambda_1\phi_1^* + \lambda_2\phi_2^* \\ \alpha_2 &= \phi_4^* - \lambda_2\phi_1^* - \lambda_1\phi_2^*, \end{aligned}$$

where ϕ_i^* denotes the dual of ϕ_i.

This leads to the equations $\lambda_1 = \lambda_4$ and $\lambda_2 = -\lambda_3$. □

The lemma above gives a sub-manifold chart $M_x \cap \mathrm{Gr}_2(TU)_{\mathcal{L}} \to \mathbb{R}^2$ that maps

$$H \mapsto (\lambda_1, \lambda_2)$$

for the fiber M_x over x.

Remark 3.10. Note that if J is integrable, then there exists a local *coordinate frame* ϕ such that J is of the form of eq. (3.1). In this setting, the construction above fits in the coordinate description of the Grassmanian given in Chapter 2. In that case we can extend the chart for the fiber M_x to the neighborhood U, forming fibered coordinates.

Theorem 3.11 The prolonged distribution $\mathcal{D}^{\mathrm{prol}}$ on M is fat.

Proof. Fatness is a local property, hence it suffices to show that $\mathcal{D}^{\mathrm{prol}}$ is fat at an arbitrary point $(x, H) \in M$. As in Lemma 3.9, there exists an open neighbourhood U of x and a local frame $\phi = \{\phi_1, \phi_2, \phi_3, \phi_4\}$ on U such that

$$J|_U = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$H' \in M \cap \mathrm{Gr}_2(TU)_{\mathcal{L}}$ is given by the intersected kernels of the forms

$$\begin{aligned} \alpha_1 &= \phi_3^* - \lambda_1\phi_1^* + \lambda_2\phi_2^* \\ \alpha_2 &= \phi_4^* - \lambda_2\phi_1^* - \lambda_1\phi_2^*. \end{aligned}$$

Recall from Corollary 1.29 that in order to show the distribution $\mathcal{D}^{\mathrm{prol}}$ is fat on $\mathrm{Gr}_2(TU)_{\mathcal{L}}$, we have to show that

1. the two forms $\omega_1 = \mathrm{d}\alpha_1|_{\mathcal{D}}$ and $\omega_2 = \mathrm{d}\alpha_2|_{\mathcal{D}}$ are non-degenerate;

2. the rank measure $R^{1,2}$ of $\mathcal{D}^{\text{prol}}$ relating ω_1 to ω_2 via

$$\omega_1(\cdot, R^{1,2}\cdot) = \omega_2(\cdot, \cdot),$$

has no real eigenvalues.

For 1. we compute

$$\begin{aligned} \mathrm{d}\alpha_1 &= \mathrm{d}\phi_3^* - \mathrm{d}\lambda_1 \wedge \phi_1^* - \lambda_1 \mathrm{d}\phi_1^* + \mathrm{d}\lambda_2 \wedge \phi_2^* + \lambda_2 \mathrm{d}\phi_2^* \\ \mathrm{d}\alpha_2 &= \mathrm{d}\phi_4^* - \mathrm{d}\lambda_2 \wedge \phi_1^* - \lambda_2 \mathrm{d}\phi_1^* - \mathrm{d}\lambda_1 \wedge \phi_2^* - \lambda_1 \mathrm{d}\phi_2^*. \end{aligned}$$

Note that the ϕ_i's do not depend on λ_1 and λ_2. In order to check non-degeneracy we compute

$$\begin{aligned} \omega_1 \wedge \omega_1 &= 2\phi_1^* \wedge \phi_2^* \wedge \mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 \\ \omega_2 \wedge \omega_2 &= 2\phi_1^* \wedge \phi_2^* \wedge \mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2. \end{aligned}$$

Note here that the resulting terms of 4-forms not including both $\mathrm{d}\lambda_1$ and $\mathrm{d}\lambda_2$ in the wedge product vanish on $\mathcal{D}^{\text{prol}}$. Both $\omega_i \wedge \omega_i$ are volume forms on $\mathcal{D}^{\text{prol}}$ which implies that both ω_1 and ω_2 are non-degenerate on $\mathcal{D}^{\text{prol}}$.

Now we show 2. At the point $H \in M$ we have that

$$\mathcal{D}_H^{\text{prol}} = H + T_H M_x \subset T\,\mathrm{Gr}_2(TM),$$

where H on the right hand side is interpreted as a subspace $H \subset T_x X$. With this in mind we can choose the following basis for $\mathcal{D}_H^{\text{prol}}$:

$$\{V_1, V_2, \partial_{\lambda_1}, \partial_{\lambda_2}\},$$

where $V_1, V_2 \in TM$ are the vectors

$$\begin{aligned} V_1 &= \phi_1 + \lambda_1 \phi_3 + \lambda_2 \phi_4, \\ V_2 &= \phi_2 - \lambda_2 \phi_3 + \lambda_1 \phi_4. \end{aligned}$$

Note they are indeed linearly independent and of in the kernel of the defining 1-forms.

Then we have that the rank measure R^{12} at H satisfies

$$\begin{aligned} V_1 &\mapsto V_2 \\ V_2 &\mapsto -V_1 \\ \partial_{\lambda_1} &\mapsto \partial_{\lambda_2} \\ \partial_{\lambda_2} &\mapsto -\partial_{\lambda_1}. \end{aligned}$$

Hence, with respect to the basis $\{V_1, V_2, \partial_{\lambda_1}, \partial_{\lambda_2}\}$ of $D \in \mathcal{D}^{\text{prol}}$ the rank measure $R^{1,2}$ at H is given

$$R^{1,2} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Note $R^{1,2}$ has no real eigenvalues (which is independent of the chosen frame); hence the prolonged distribution $\mathcal{D}^{\text{prol}}$ is fat at (x, H). Since (x, H) is chosen arbitrarily, this completes the proof. □

Chapter 4

Fatness and the infinitesimal cone field

In this chapter we introduce the degenerate cones: for each 2-plane H in the Grassmannian, we consider the 2-planes that intersect H in a subspace with dimension at least 1. We show that locally, this forms a cone centered at the 2-plane H itself.

Taking the infinitesimal analogue of this cone allows us to define what we call the infinitesimal cone field on the Grassmannian. We then identify characterizing local properties for fat prolonged distributions of type $(4, 6)$ in terms of the infinitesimal cone field. Namely, we show that requiring the prolonged distribution $\mathcal{D}$ on fiber bundle M to be fat is equivalent to requiring that the fibers of M –that map into the corresponding Grassmannian-fiber via the given bundle map– are transverse to the infinitesimal cone field.

4.1 Transversality

The tool we use to measure the dimension of intersection of two 2-planes is transversality, which we will introduce in various forms. The first two definitions below are classical and we use them to define the degenerate cone for a 2-plane H in the Grassmannian.

Definition 4.1 Let V be a vector space and let W and W' be linear subspaces of V. We say that W and W' are *transverse* if

$$V = W + W'.$$

In this case we write $W \pitchfork W'$. If W and W' are not transverse we write $W \not\pitchfork W'$.

Definition 4.2 Let M be a manifold and let N and N' be sub-manifolds of M. We say that N and N' are *transverse at a point* x in the intersection $N \cap N'$ if

$$T_xM = T_xN + T_xN'.$$

If N and N' are transverse at all points in the intersection $N \cap N'$ we say N and N' are *transverse*; in that case we write $N \pitchfork N'$. If N and N' are not transverse we write $N \not\pitchfork N'$.

Definition 4.3 Let V be a 4-dimensional vector space and consider the co-rank 2 Grassmannian $\mathrm{Gr}_2(V)$. Let H be a point in $\mathrm{Gr}_2(V)$. The *degenerate cone* $\mathfrak{C}_H$ of H in the Grassmannian $\mathrm{Gr}_2(V)$ is a subspace of $\mathrm{Gr}_2(V)$ given by

$$\mathfrak{C}_H = \{H' \in \mathrm{Gr}_2(V) \colon H' \not\pitchfork H\}$$

The reason why this space is called a cone is explained by Proposition 4.5. To arrive there we need the following lemma.

Lemma 4.4 The degenerate cone $\mathfrak{C}_H$ in the chart $(\mathrm{Gr}^r(V)_\mathcal{L}, \chi_\mathcal{L})$, where $\chi_\mathcal{L}(H) = M \in \mathrm{Mat}_{2x2}$, is given by

$$\chi_\mathcal{L}(\mathfrak{C}_H) = \{M + Q \in \mathrm{Mat}_{2x2} \colon Q \in \mathrm{Mat}_{2x2}, \text{rank}\, Q < 2\}. \tag{4.1}$$

Proof. An element $H' \in \mathfrak{C}_H \cap \mathrm{Gr}_2(V)_\mathcal{L} \setminus \{H\}$ is a 2-plane in V that has a 1-dimensional intersection with H with the additional requirement that $\Phi_\mathcal{L}$ projects to a basis for H'. Hence H' can be written as

$$H' = \langle u, v + w \rangle,$$

where v is a vector in $\langle \Phi_{\mathcal{L}^\perp} \rangle$, which is complementary to H, w is a vector in H, and u is a non-zero vector in H such that $u \perp w$. Here the orthogonal complement is with respect to the standard inner product associated to the chosen basis. We illustrate the choice of vectors in Figure 4.1 and the plane that is defined by them in Figure 4.2. Note that the choice of v and w is sufficient to define H', since after fixing these two, every choice of non-zero vector u in $w^\perp \cap H$ defines the same 2-plane, i.e. the element H' is also given by

$$H' = \langle v + w \rangle + w^\perp \cap H. \tag{4.2}$$

For simplicity, without loss of generality, we assume $\mathcal{L} = (1, 2)$. Then the unique spanning vectors in V for H corresponding to the chart $\mathrm{Gr}_2(V)_\mathcal{L}$ are given by

$$x_1 + m_{11}x_3 + m_{12}x_4, \tag{4.3}$$

$$x_2 + m_{21}x_3 + m_{12}x_4; \tag{4.4}$$

and the chosen vectors are expressed by $v = v_1x_3 + v_2x_4$ and $w = w_1x_1 + w_2x_2$. The unique pair of spanning vectors in V for H' corresponding to the chart $\mathrm{Gr}_2(V)_\mathcal{L}$ are then given by adding the vector $v + w$, decomposed with respect to this basis as $w_1 \cdot v$ and $w_2 \cdot v$, to the respective spanning vectors for H given in Equation (4.3) and Equation (4.4), see again Figure 4.2. I.e., H' is given by

$$x_1 + (m_{11} + w_1v_1)x_3 + (m_{12} + w_1v_2)x_4,$$
$$x_2 + (m_{21} + w_2v_1)x_3 + (m_{12} + w_2v_2)x_4.$$

In the chart of $\mathrm{Gr}_2(V)_\mathcal{L}$, that corresponds to the matrix $M + Q \in \mathrm{Mat}_{2x2}$, where Q is given by

$$Q = \begin{pmatrix} w_1v_1 & w_1v_2 \\ w_2v_1 & w_2v_2 \end{pmatrix},$$

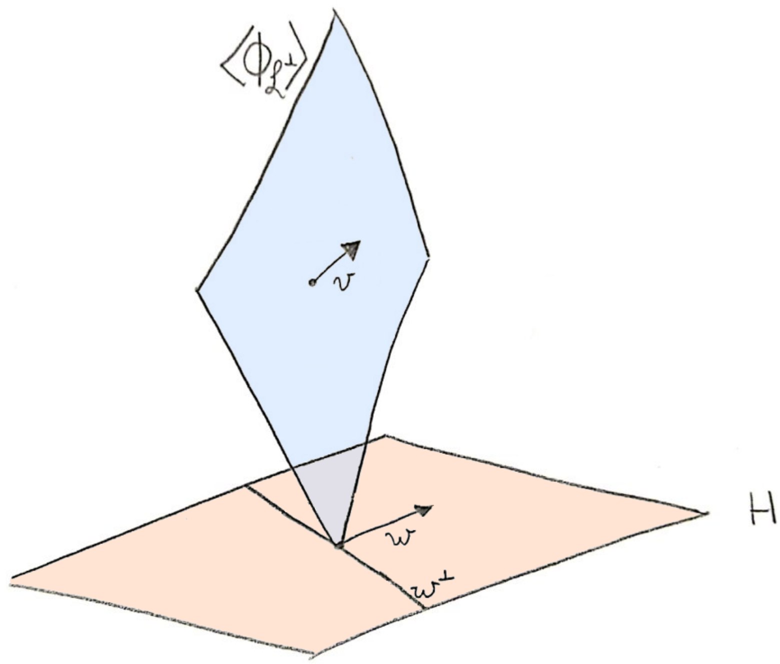

Figure 4.1: Choice of vectors $v \in \langle \Phi_{\mathcal{L}^\perp} \rangle$, complementary to H, and $w \in H$.

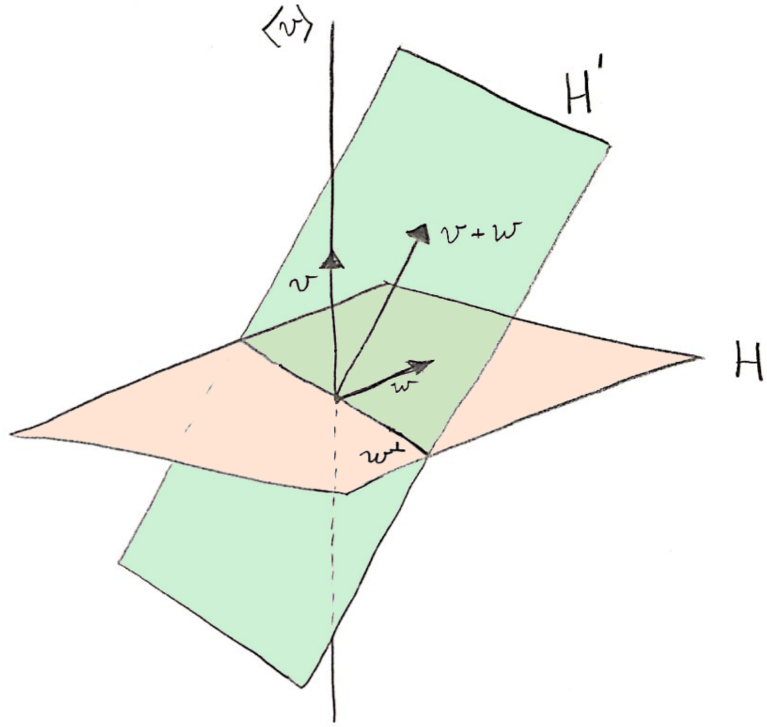

Figure 4.2: The plane $H' = \langle v + w \rangle + w^\perp \cap H$.

which is equal to the outer product $Q = w^T v$ of the vectors w and v. Therefore Q is a matrix of rank < 2. On the other hand, every matrix $Q \in \text{Mat}_{2x2}$ with rank maximally 1 can be written –in a non-unique way– as an outer product $Q = w^T v$. Two such vectors v, w define an element $H' \in \mathfrak{C}_H \cap \text{Gr}_2(V)_{\mathcal{L}}$ via the span given in Equation (4.2). The statement follows. □

Now we consider the intersection $\mathfrak{C}_H \cap \text{Gr}_2(V)_{\mathcal{L}}$. In the chart $(\text{Gr}^r(V)_{\mathcal{L}}, \chi_{\mathcal{L}})$ we fix a norm $\|\cdot\|$ on the space $\text{Mat}_{2\times 2} \cong \mathbb{R}^4$. Without loss of generality we take the standard norm on $\mathbb{R}^4$. We consider the space $T \subset \text{Mat}_{2\times 2}$ of matrices Q of norm 1 with rank lower than 2,

$$T = \{Q \in \text{Mat}_{2\times 2} \colon \|Q\| = 1; \text{rank}\, Q < 2\}.$$

Proposition 4.5 The intersection $\mathfrak{C}_H \cap \text{Gr}_2(V)_{\mathcal{L}}$ is a cone over T. Moreover, the space T is an embedded torus T^2.

Proof. We rewrite $\chi_{\mathcal{L}}(\mathfrak{C}_H)$ from eq. (4.1) as a cone over T as follows

$$\chi_{\mathcal{L}}(\mathfrak{C}_H) = \{M + c \cdot Q \colon c \in \mathbb{R}; Q \in T\}.$$

Now it suffices to show that the space T is homeomorphic to the torus T^2.

To see this, note that an element Q in T must be a matrix of rank 1 in particular, since a matrix of norm 1 has rank at least 1. We claim it can be written – still in a non-unique way– as the outer product of $v, w \in S^1 \subset \mathbb{R}^2$. Namely, as in the proof of Lemma 4.4, we have that any rank 1 matrix can be written as the outer product of two non-zero vectors $v, w \in \mathbb{R}^2$ given by

$$Q = wv^T = \begin{pmatrix} w_1 v_1 & w_1 v_2 \\ w_2 v_1 & w_2 v_2 \end{pmatrix}.$$

Since the pairs (v, w) and $(cv, w/c)$ generate the same matrix Q via the outer product, for any non-zero scalar $c \in \mathbb{R}$, we can choose v to have norm 1, i.e.

$$\|v\|^2 = v_1^2 + v_2^2 = 1.$$

Moreover, since Q has norm 1, we have that

$$\|Q\|^2 = w_1^2 v_1^2 + w_1^2 v_2^2 + w_2^2 v_1^2 + w_2^2 v_2^2 = 1,$$

which can be rewritten as

$$\begin{aligned} \|Q\|^2 &= (w_1^2 + w_2^2)(v_1^2 + v_2^2) \\ &= \|w\|^2 \|v\|^2 = 1. \end{aligned}$$

Hence, we have that w has norm 1 as well

$$\|w\|^2 = w_1^2 + w_2^2 = 1.$$

Now we observe that two elements $(v,w),(v',w') \in S^1 \times S^1 \cong T^2$ define the same matrix Q if an only if $(v',w') = (-v,-w)$. Hence the space T is equal to the quotient

$$T = S^1 \times S^1/\mathbb{Z}_2 \cong T^2/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts on T^2 via $(v,w) \mapsto (-1)^{i-1}(v,w), \quad i \in \mathbb{Z}_2$. This is a free, orientation preserving action and $\mathbb{Z}_2$ is finite. Since the quotient of the torus given by a free, orientation preserving action of a finite group is homeomorphic to a torus of the same dimension, we have that T is homeomorphic to T^2. $\square$

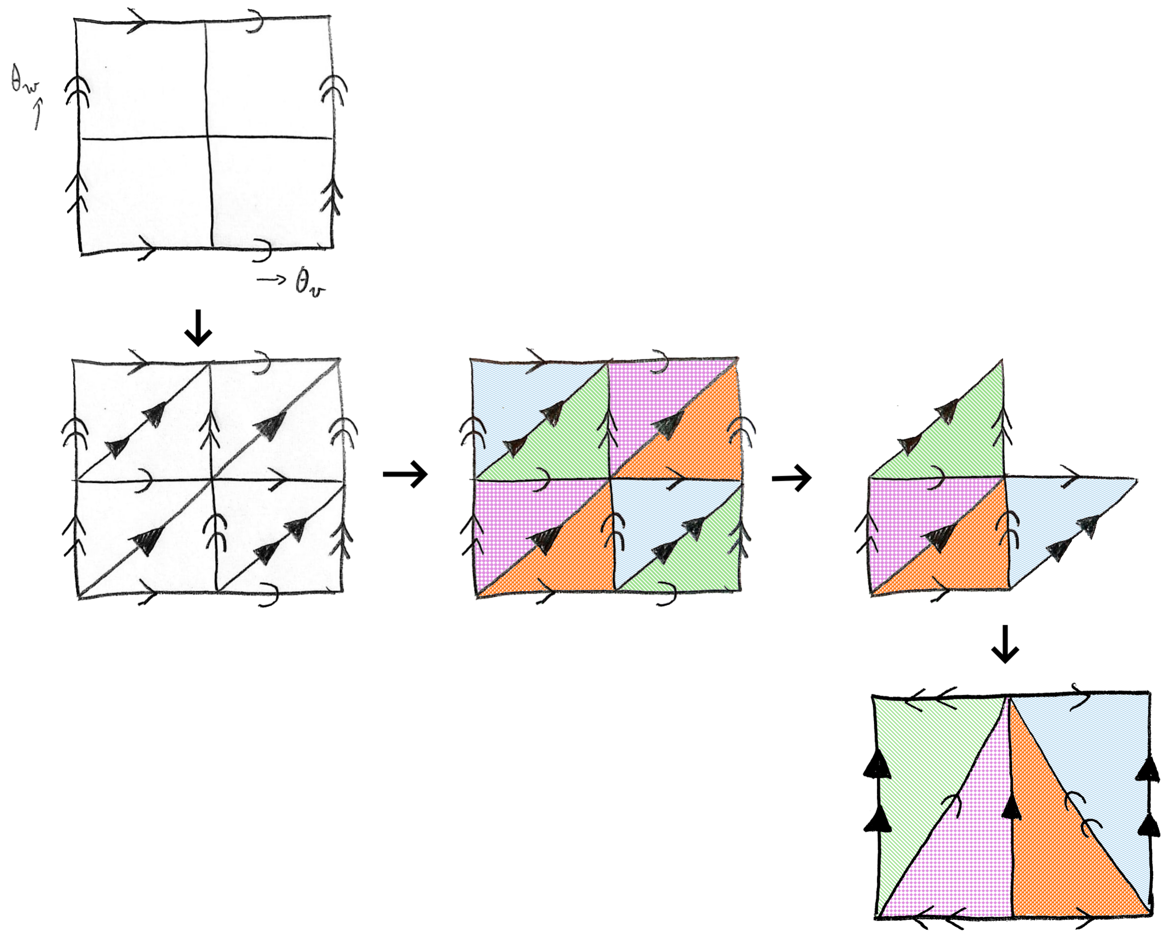

Figure 4.3: Quotient of the torus by the action of $\mathbb{Z}^2$ given by $(\theta_v, \theta_w) \mapsto (\theta_v + \pi, \theta_w + \pi)$ is again a torus.

Remark 4.6. If we describe v and w by the angles $\theta_v, \theta_w \in S^1$ then the action of $\mathbb{Z}_2$ on the torus $T^2 = S^1 \times S^1$ is given by $(\theta_v, \theta_w) \mapsto (\theta_v + i\pi, \theta_w + i\pi)$, $i \in \mathbb{Z}_2$. Now it can also be seen from the sequence shown in Figure 4.3 that this quotient of T^2 is again homeomorphic to a copy of the torus T^2.

Remark 4.7. Interpreting the vectors v and w as elements in vector space V, as done in the proof of Lemma 4.4, gives more intuition behind the reasoning above. The choice of vectors $v \in S^1 \subset \Phi_{\mathcal{L}^\perp}$ and $w \in S^1 \subset H$ are illustrated in Figure 4.4. The element $H' = \langle v + w \rangle + w^\perp \cap H$ is illustrated in Figure 4.5. Note that indeed (v,w) and $(-v,-w)$ define the same plane H'.

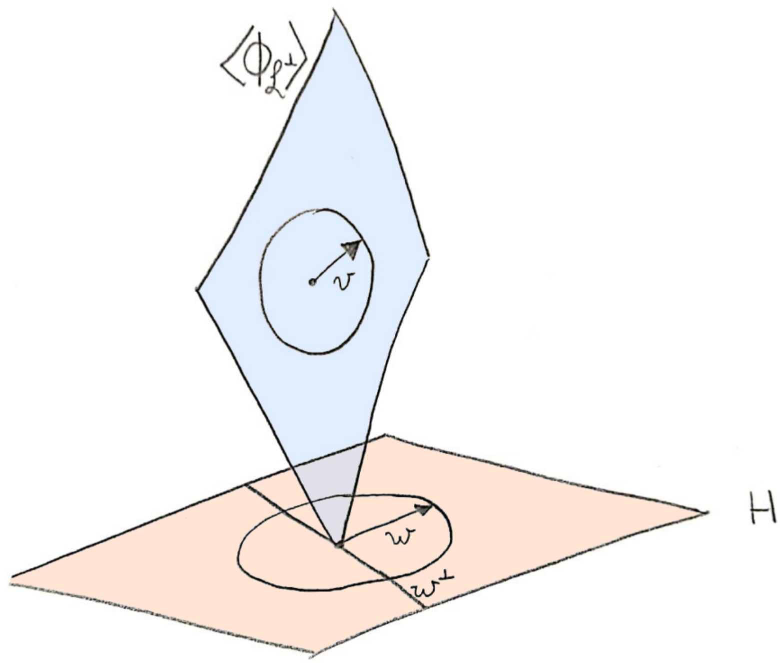

Figure 4.4: Choice of vectors $v \in S^1 \subset \langle \Phi_{\mathcal{L}^\perp} \rangle$, complementary to H, and $w \in S^1 \subset H$.

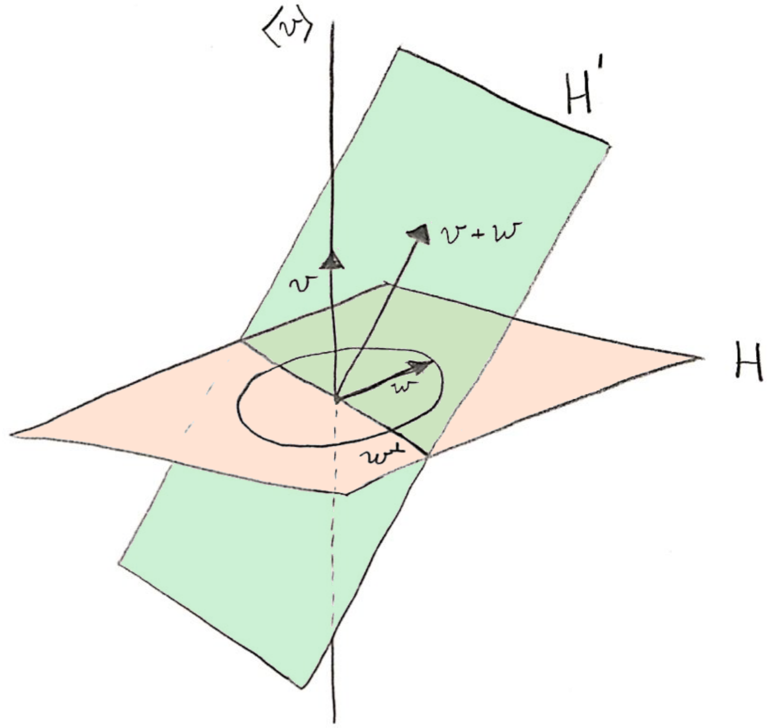

Figure 4.5: The plane $H' = \langle v + w \rangle + w^\perp \cap H$.

4.2 Infinitesimal cone field

In this section we introduce another notion of transversality that fits well for the purpose of this text: it is used to phrase Theorem 4.15, the main result of this chapter.

In the intersection with $\mathrm{Gr}_2(V)_{\mathcal{L}}$, we can write the degenerate cone $\mathfrak{C}_H$ of H as the union over the elements Q in the torus T given by

$$\mathfrak{C}_H \cap \mathrm{Gr}_2(V)_{\mathcal{L}} = \bigcup_{Q \in T} \mathrm{Image}(\gamma_{\mathcal{L}}^Q), \tag{4.5}$$

where the path $\gamma_{\mathcal{L}}^Q : \mathbb{R} \to (\mathrm{Gr}_2(T_x X))$ is given by

$$t \mapsto \chi_{\mathcal{L}}^{-1}(M + tQ).$$

The tangent vector of such curve $\gamma_{\mathcal{L}}^Q$ is given by

$$\mathrm{d}\chi_{\mathcal{L}}^{-1}(Q) \in T_H \,\mathrm{Gr}_2(V),$$

where the matrix Q is now considered as element in the tangent space $T_M \,\mathrm{Mat}_{2\times 2}$. In homogeneous coordinates the path $\gamma_{\mathcal{L}}^Q$ is given by

$$t \mapsto \begin{bmatrix} I \\ M + tQ \end{bmatrix} = \left[\begin{pmatrix} I \\ M \end{pmatrix} + t \begin{pmatrix} 0 \\ Q \end{pmatrix} \right]. \tag{4.6}$$

In what follows, for simplicity, we identify the tangent spaces $T_M \,\mathrm{Mat}_{2\times 2}$ and $T_H \,\mathrm{Gr}_2(V)$, i.e. a matrix Q is identified with the tangent vector $\mathrm{d}\chi_{\mathcal{L}}^{-1}(Q)$.

Remark 4.8. It follows from the expression of paths given in eq. (4.6) that the rank of Q is well defined. Namely, after multiplying $\begin{pmatrix} 0 \\ Q \end{pmatrix}$ with an element in GL_2, we obtain another matrix of rank 1. The rank of Q is independent of the choice of basis and the choice of chart containing H.

For each parametrized curve $\gamma_{\mathcal{L}}^Q$ in eq. (4.5), given by

$$t \mapsto \chi_{\mathcal{L}}^{-1}(M + tQ), \quad Q \in T,$$

its tangent vector $\dot{\gamma}_{\mathcal{L}}^Q|_{t=0}$ is represented by the same matrix Q that appears in the torus T given by

$$T = \{Q \in \mathrm{Mat}_{2\times 2} \colon \|Q\| = 1; \mathrm{rank}\, Q < 2\}.$$

Hence the torus T appears another time in the tangent space $T_H \,\mathrm{Gr}_2(V)$, generating another infinitesimal cone $\mathcal{C}_H$. (Note that the elements in the linear span $\{cQ \colon c \in \mathbb{R}\}$ that appears in the cone $\mathcal{C}_H$ are associated to the tangent vectors of reparemetrizations of the curve $\gamma_{\mathcal{L}}^Q$.) Considering such cone in the tangent space $T_H \,\mathrm{Gr}_2(V)$, for every $H \in \mathrm{Gr}_2(V)$, gives rise to a T^2-cone field on the Grassmannian.

Definition 4.9 Let V be a 4-dimensional vector space and consider the co-rank 2 Grassmannian $\mathrm{Gr}_2(V)$. The *infinitesimal cone field* $\mathcal{C}$ on the Grassmannian $\mathrm{Gr}_2(V)$ is a field of sub-spaces (cones, i.e. non-linear sub-spaces) of the tangent space, point-wise given by

$$\mathcal{C}_H = \{Q \in T_H \,\mathrm{Gr}_k(V) \colon \ \mathrm{rank}\, Q < 2\}.$$

We notice that $\mathcal{C}_H$ is well defined thanks to Remark 4.8.

Let Y be a 2-dimensional manifold and let $\varphi_Y : Y \to \mathrm{Gr}_2(V)$ be a smooth map. Denote the image $\varphi(Y)$ by Σ.

Definition 4.10 Let y be a point in Y and let y_1, y_2 be local coordinates in a chart around y. Denote by $H = \varphi(y)$ the image of y in Σ. We say that φ is *(elliptically) transverse to the infinitesimal cone field $\mathcal{C}$ at H* if φ is immersive at H and

$$\mathcal{C}_H \cap \mathrm{Image}(\mathrm{d}\varphi)_y = 0.$$

If φ is transverse to the infinitesimal cone field $\mathcal{C}$ at all points H in Σ we say that φ is *transverse to the infinitesimal cone field $\mathcal{C}$*; in that case we write $\Sigma \pitchfork \mathcal{C}$. If Σ is not transverse to the infinitesimal cone field $\mathcal{C}$ we write $\Sigma \not\pitchfork \mathcal{C}$.

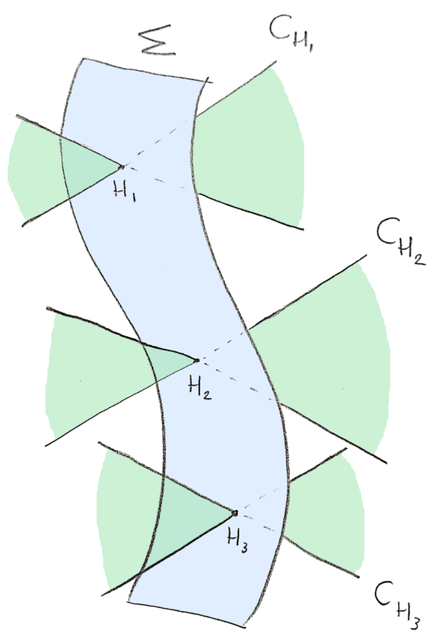

Figure 4.6: Σ transverse to the infinitesimal cone field $\mathcal{C}$, illustrated for the points $H_1, H_2, H_3 \in \Sigma$ and the respective cones $\mathcal{C}_{H_1}, \mathcal{C}_{H_2}, \mathcal{C}_{H_3}$.

If φ is an embedding, its image Σ is a sub-manifold. For sub-manifolds there is a more elegant way of describing transversality.

Definition 4.11 Let N be a 2-dimensional sub-manifold of $\mathrm{Gr}_2(V)$. We say that N is *(elliptically) transverse to the infinitesimal cone field $\mathcal{C}$ at H* if

$$\mathcal{C}_H \cap T_H\Sigma = 0.$$

If N is transverse to the infinitesimal cone field $\mathcal{C}$ at all points H in N we say that N is *transverse to the infinitesimal cone field $\mathcal{C}$*; in that case we write $N \pitchfork \mathcal{C}$. If N is not transverse to the infinitesimal cone field $\mathcal{C}$ we write $N \not\pitchfork \mathcal{C}$.

For any 2-dimensional sub-manifold N of $\mathrm{Gr}_2(V)$ we can consider the inclusion map $\iota_N\colon N \to \mathrm{Gr}_2(V)$ which is an embedding; in particular it is an immersion. Note that N being transverse to the infinitesimal cone field as a sub-manifold is consistent with the meaning of transverse to the infinitesimal cone field for the immersion ι_N according to Definition 4.10.

We work out what the property in Definition 4.10 is in terms of tangent vectors.

Lemma 4.12 Denote by μ_i^j the coordinate expressions of φ with respect to the chart $\mathrm{Gr}_2(V)_{\mathcal{L}}$ of H. Let Σ be transverse to the infinitesimal cone field $\mathcal{C}$ at $H = \varphi(y)$. Then for all non-zero tangent vectors $v \in T_yY$ with components (v_1, v_2) in the chosen chart, the matrix

$$v_1 \begin{pmatrix} \frac{\partial \mu_1}{\partial y_1}(y) & \frac{\partial \mu_2}{\partial y_1}(y) \\ \frac{\partial \mu_3}{\partial y_1}(y) & \frac{\partial \mu_4}{\partial y_1}(y) \end{pmatrix} + v_2 \begin{pmatrix} \frac{\partial \mu_1}{\partial y_2}(y) & \frac{\partial \mu_2}{\partial y_2}(y) \\ \frac{\partial \mu_3}{\partial y_2}(y) & \frac{\partial \mu_4}{\partial y_2}(y) \end{pmatrix}. \tag{4.7}$$

has non-zero determinant.

Proof. The proof is a computation. We have that $(\mathrm{d}\varphi)_y\colon T_yY \to T_H\Sigma$ is given by

$$\begin{aligned} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \quad & (\varphi_{y_1}(y), \varphi_{y_2}(y)) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial \mu_1}{\partial y_1}(y) & \frac{\partial \mu_1}{\partial y_2}(y) \\ \frac{\partial \mu_2}{\partial y_1}(y) & \frac{\partial \mu_2}{\partial y_2}(y) \\ \frac{\partial \mu_3}{\partial y_1}(y) & \frac{\partial \mu_3}{\partial y_2}(y) \\ \frac{\partial \mu_4}{\partial y_1}(y) & \frac{\partial \mu_4}{\partial y_2}(y) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} & (4.8) \\ &= v_1 \begin{pmatrix} \frac{\partial \mu_1}{\partial y_1}(y) & \frac{\partial \mu_2}{\partial y_1}(y) \\ \frac{\partial \mu_3}{\partial y_1}(y) & \frac{\partial \mu_4}{\partial y_1}(y) \end{pmatrix} + v_2 \begin{pmatrix} \frac{\partial \mu_1}{\partial y_2}(y) & \frac{\partial \mu_2}{\partial y_2}(y) \\ \frac{\partial \mu_3}{\partial y_2}(y) & \frac{\partial \mu_4}{\partial y_2}(y) \end{pmatrix}. & (4.9) \end{aligned}$$

□

Let Y be a 2-dimensional manifold and let $\varphi_Y\colon Y \to \mathrm{Gr}_2(V)$ be a smooth map. Denote the image $\varphi(Y)$ by Σ.

Lemma 4.13 We have that φ is transverse to the infinitesimal cone field $\mathcal{C}$ at $H = \varphi(y)$ if and only if the following two conditions hold.

1. The Jacobians of the first and second pair of coordinate maps, respectively given by

$$B_1 = \begin{pmatrix} \frac{\partial \mu_1^1}{\partial y_1}(y) & \frac{\partial \mu_1^1}{\partial y_2}(y) \\ \frac{\partial \mu_2^1}{\partial y_1}(y) & \frac{\partial \mu_2^1}{\partial y_2}(y) \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} \frac{\partial \mu_1^2}{\partial y_1}(y) & \frac{\partial \mu_1^2}{\partial y_2}(y) \\ \frac{\partial \mu_2^2}{\partial y_1}(y) & \frac{\partial \mu_2^2}{\partial y_2}(y) \end{pmatrix},$$

   are non-singular matrices.

2. The isomorphism $A \in GL_2(\mathbb{R})$ given by $B_1 A = B_2$ has non-real eigenvalues.

Proof. $(\Leftarrow)$ We prove the contrapositive of the statement. Assume Σ is not transverse to the infinitesimal cone field $\mathcal{C}$ at H. First we suppose that φ is not immersive at H. We use the coordinate descriptions from Lemma 4.12. Then the 2×4 matrix in eq. (4.8) is of rank 1. Note that this matrix is precisely

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

therefore, in this case, B_1 and B_2 must be of rank 1, so condition 2 cannot hold.

Now suppose that φ is immersive at H. Since Σ is not transverse to the cone, there exists an element $(v, w) \in T_y\mathbb{R}^2 \cong \mathbb{R}^2$ such that the resulting matrix in the righthandside of eq. (4.7) is in C_H, and hence singular. This implies in particular that the rows are linearly dependent, i.e. there exists a scalar $a \in \mathbb{R}$ such that such that

$$\begin{pmatrix} \frac{\partial \mu_1^2}{\partial y_1}(y) \\ \frac{\partial \mu_2^2}{\partial y_1}(y) \end{pmatrix} v_1 + \begin{pmatrix} \frac{\partial \mu_1^2}{\partial y_2}(y) \\ \frac{\partial \mu_2^2}{\partial y_2}(y) \end{pmatrix} v_2 = a \left(\begin{pmatrix} \frac{\partial \mu_1^1}{\partial y_1}(y) \\ \frac{\partial \mu_2^1}{\partial y_1}(y) \end{pmatrix} v_1 + \begin{pmatrix} \frac{\partial \mu_1^1}{\partial y_2}(y) \\ \frac{\partial \mu_2^1}{\partial y_2}(y) \end{pmatrix} v_2 \right).$$

Now this is equivalent to

$$B_1^{-1} B_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = a \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

so a is an the eigenvalue of $A = B_1^{-1} B_2$, with corresponding eigenvector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. This contradicts condition 2.

$(\Rightarrow)$ Again we prove the contrapositive. First assume condition 1 does not hold. We assume without loss of generality that B_1 is singular. Then one of its eigenvalues must be zero. Let (w_1, w_2) be the corresponding eigenvector, i.e.

$$\begin{pmatrix} \frac{\partial \mu_1^1}{\partial y_1}(y) \\ \frac{\partial \mu_2^1}{\partial y_1}(y) \end{pmatrix} w_1 + \begin{pmatrix} \frac{\partial \mu_1^1}{\partial y_2}(y) \\ \frac{\partial \mu_2^1}{\partial y_2}(y) \end{pmatrix} w_2 = 0$$

Then the first row of

$$w_1 \begin{pmatrix} \frac{\partial \mu_1}{\partial y_1}(y) & \frac{\partial \mu_2}{\partial y_1}(y) \\ \frac{\partial \mu_3}{\partial y_1}(y) & \frac{\partial \mu_4}{\partial y_1}(y) \end{pmatrix} + w_2 \begin{pmatrix} \frac{\partial \mu_1}{\partial y_2}(y) & \frac{\partial \mu_2}{\partial y_2}(y) \\ \frac{\partial \mu_3}{\partial y_2}(y) & \frac{\partial \mu_4}{\partial y_2}(y) \end{pmatrix}$$

vanishes, hence the resulting matrix is of rank lower than 2. If it is the zero matrix, it follows from eq. (4.8) that φ is not immersive at H. If it is a matrix of rank 1, the intersection $\mathcal{C}_H \cap T_H\Sigma \neq 0$, since this matrix is contained in the intersection. This shows Σ is not tangent to the infinitesimal cone field $\mathcal{C}$ at H.

Now assume condition 1 holds, but condition 2 does not hold. Then $A = B_1^{-1}B_2$ has a real eigenvalue $a \in \mathbb{R}$. Let $w = (w_1, w_2)$ be the corresponding eigenvector, i.e. $B_1^{-1}B_2 w = aw$. This is equivalent to $B_2 w = aB_1 w$ which we can write as

$$\begin{pmatrix} \frac{\partial \mu_1^2}{\partial y_1}(y) \\ \frac{\partial \mu_2^2}{\partial y_1}(y) \end{pmatrix} w_1 + \begin{pmatrix} \frac{\partial \mu_1^2}{\partial y_2}(y) \\ \frac{\partial \mu_2^2}{\partial y_2}(y) \end{pmatrix} w_2 = a \left(\begin{pmatrix} \frac{\partial \mu_1^1}{\partial y_1}(y) \\ \frac{\partial \mu_2^1}{\partial y_1}(y) \end{pmatrix} w_1 + \begin{pmatrix} \frac{\partial \mu_1^1}{\partial y_2}(y) \\ \frac{\partial \mu_2^1}{\partial y_2}(y) \end{pmatrix} w_2 \right).$$

This implies that the rows of

$$w_1 \begin{pmatrix} \frac{\partial \mu_1}{\partial y_1}(y) & \frac{\partial \mu_2}{\partial y_1}(y) \\ \frac{\partial \mu_3}{\partial y_1}(y) & \frac{\partial \mu_4}{\partial y_1}(y) \end{pmatrix} + w_2 \begin{pmatrix} \frac{\partial \mu_1}{\partial y_2}(y) & \frac{\partial \mu_2}{\partial y_2}(y) \\ \frac{\partial \mu_3}{\partial y_2}(y) & \frac{\partial \mu_4}{\partial y_2}(y) \end{pmatrix}$$

are linearly dependent, hence the resulting matrix is of rank lower than 2. Just as above, this implies Σ is not tangent to the infinitesimal cone field $\mathcal{C}$ at H. The statement follows. □

Remark 4.14. Condition 2 implies that the determinants of the non-singular matrices B_1 and B_2 must have the same sign. Namely, since A has non-real eigenvalues, the eigenvalues must be a conjugate pair in particular. Hence the determinant of A is positive. Then it follows from

$$\det B_1 \cdot \det A = \det B_2$$

that det B_1 and det B_2 have the same sign. Combining this with the fact that B_1 and B_2 have non-zero determinant, we deduce that the immersions as in Definition 4.10 are locally divided into two disjoint families, one with positive Jacobians and one with negative Jacobians at a point y. Namely, let φ_+ and φ_- be two immersions, transverse to the infinitesimal cone field $\mathcal{C}$, such that $\varphi_+(y) = \varphi_-(y) = H$ and the Jacobian of φ_+ and φ_- at a point $y \in Y$ are positive and negative respectively. Then the images of $(\mathrm{d}\varphi_+)_y$ and $(\mathrm{d}\varphi_-)_y$ in $T_H\,\mathrm{Gr}(V)$ are transversal. These two ways to be be transverse to the infinitesimal cone field are shown schematically in Figure 4.7.

Let $\mathcal{D}^{\text{prol}}$ be a prolonged distribution of type (4,6) with defining triple $(M, \pi^M\colon M \to X, \phi\colon M \to \mathrm{Gr}(TX))$, such that M is a rank-2 bundle over 4-dimensional manifold X. Then M is a 6-dimensional manifold; The Grassmann bundle $\mathrm{Gr}(TX)$ is a rank-4 bundle over X.

Theorem 4.15 The prolonged distribution $\mathcal{D}^{\text{prol}}$ is fat if and only if for every $x \in X$, the image $\Sigma_x = \varphi(M_x)$ of the fiber M_x is transverse to the infinitesimal cone field $\mathcal{C}$ defined on the Grassmannian $\mathrm{Gr}_2(T_xM)$.

Proof. It suffices that this holds for an arbitrarily chosen point $x \in X$ and another arbitrarily chosen point in its fiber M_x. Let $x \in X$. Let $y \in M$ such that $\pi^M(y) = x$ and let $(x, H) = \varphi(y)$ be its image under the bundle map φ. Consider the restriction $\varphi|_{M_x}\colon M_x \to \Sigma_x \subset \mathrm{Gr}_2(T_xX)$ of φ to the fiber M_x that maps into the Grassmannian $\mathrm{Gr}_2(T_xX)$.

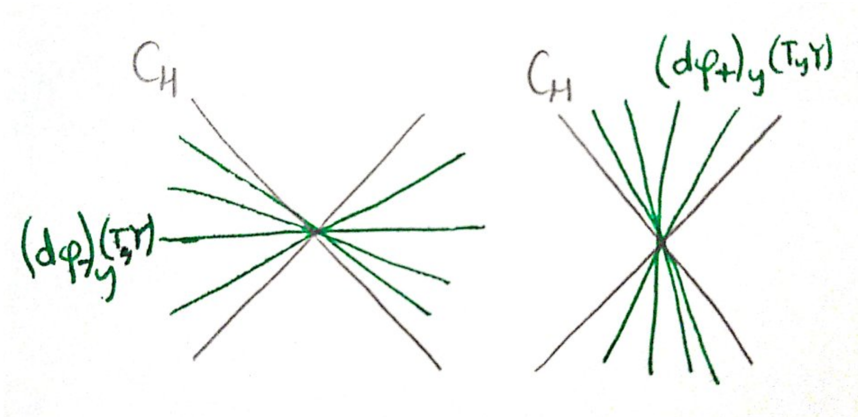

Figure 4.7: Schematic images $(\mathrm{d}\varphi_-)_y$ and $(\mathrm{d}\varphi_+)_y$ from the two disjoint families of immersions, one with negative Jacobians and one with positive Jacobians at a point y.

We have to show that the prolonged distribution $\mathcal{D}^{\text{prol}}$ is fat at $y \in M$ if and only if the image Σ_x of $\varphi|_{M_x}$ is transverse to the infinitesimal cone field $\mathcal{C}_H$ defined on the Grassmannian $\text{Gr}_2(T_xM)$.

Let $U \subset X$ be an open neighborhood around x together with a chart $\psi : U \to \mathbb{R}^4$ that maps

$$x \mapsto (x^1, ..., x^4).$$

We use the coordinates on the Grassmann bundle induced by these local coordinates on X using co-vectors, see Chapter 2, and we assume without loss of generality that the choice $\mathcal{L} = (1, 2)$ induces a chart around H. Then the map $\varphi|_{M_x}$ expressed in these coordinates in the coordinates (y_1, y_2) of the chart around y, is given by

$$(y_1, y_2) \mapsto= \begin{pmatrix} \mu_1^1 & \mu_2^1 \\ \mu_1^2 & \mu_2^2 \end{pmatrix}.$$

This means that the μ_i^j are the coordinate functions depending on y_1, y_2 so that

$$\alpha_1 = x^3 + \mu_1^1 x^1 + \mu_2^1 x^2,$$
$$\alpha_k = x^4 + \mu_1^2 x^1 + \mu_2^2 x^2.$$

define the plane $H \subset T_xX$ locally. It suffices to show, that the prolonged distribution $\mathcal{D}^{\text{prol}}$ is fat at $y \in M$ if and only if the two conditions from Lemma 4.13 hold for Σ_x, the image of $\varphi|_{M_x}$. Those conditions are

1. The Jacobians of the first and second pair of coordinate functions of φ, respectively given by

$$B_1 = \begin{pmatrix} \frac{\partial \mu_1^1}{\partial y_1}(y) & \frac{\partial \mu_1^1}{\partial y_2}(y) \\ \frac{\partial \mu_2^1}{\partial y_1}(y) & \frac{\partial \mu_2^1}{\partial y_2}(y) \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} \frac{\partial \mu_1^2}{\partial y_1}(y) & \frac{\partial \mu_1^2}{\partial y_2}(y) \\ \frac{\partial \mu_2^2}{\partial y_1}(y) & \frac{\partial \mu_2^2}{\partial y_2}(y) \end{pmatrix},$$

are non-singular matrices.

2. The isomorphism $A \in GL_2(\mathbb{R})$ given by $B_1 A = B_2$ has non-real eigenvalues.

Note that the first and second pair of coordinate maps of φ correspond to the coordinate functions describing the defining 1-forms α_1, α_2. Hence the B_i correspond to the Jacobians J^{α_i} of the coordinate functions of the locally defining 1-forms.

First we show $\mathrm{d}\alpha_i|_{\mathcal{D}^{\text{prol}}}$ is non-degenerate if and only if $B_i = J^{\alpha_i}$ is non-degenerate for $i = 1, 2$. We compute

$$\begin{aligned}\mathrm{d}\alpha_1 &= \mathrm{d}\mu_1 \wedge \mathrm{d}x_1 + \mathrm{d}\mu_2 \wedge \mathrm{d}x_2 \\ &= \left(\frac{\partial \mu_1^1}{\partial x_2}\mathrm{d}x_2 + \frac{\partial \mu_1^1}{\partial x_3}\mathrm{d}x_3 + \frac{\partial \mu_1^1}{\partial x_4}\mathrm{d}x_4 + \frac{\partial \mu_1^1}{\partial y_1}\mathrm{d}y_1 + \frac{\partial \mu_1^1}{\partial y_2}\mathrm{d}y_2\right) \wedge \mathrm{d}x_1 \\ &\quad + \left(\frac{\partial \mu_2^1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial \mu_2^1}{\partial x_3}\mathrm{d}x_3 + \frac{\partial \mu_2^1}{\partial x_4}\mathrm{d}x_4 + \frac{\partial \mu_2^1}{\partial y_1}\mathrm{d}y_1 + \frac{\partial \mu_2^1}{\partial y_2}\mathrm{d}y_2\right) \wedge \mathrm{d}x_2.\end{aligned}$$

We have that $\mathrm{d}\alpha_1|_{\mathcal{D}^{\text{prol}}}$ is non-degenerate if and only if $\mathrm{d}\alpha_1 \wedge \mathrm{d}\alpha_1|_{\mathcal{D}^{\text{prol}}}$ is a (non-degenerate) volume form, so we compute

$$\begin{aligned}\mathrm{d}\alpha_1 \wedge \mathrm{d}\alpha_1|_{\mathcal{D}^{\text{prol}}} &= -2\left(\frac{\partial \mu_1^1}{\partial y_1}\frac{\partial \mu_2^1}{\partial y_2} - \frac{\partial \mu_1^1}{\partial y_2}\frac{\partial \mu_2^1}{\partial y_1}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2 \\ &= -2 \det J^{\alpha_1} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2.\end{aligned}$$

Note here that the resulting terms of 4-forms not including both $\mathrm{d}a$ and $\mathrm{d}b$ in the wedge product vanish on $\mathcal{D}^{\text{prol}}$, since $\partial_{y_1}, \partial_{y_2} \in \mathcal{D}^{\text{prol}}$, and that indeed only $\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2$ restricts to a volume form on $\mathcal{D}^{\text{prol}}$. Hence $\mathrm{d}\alpha_1|_{\mathcal{D}^{\text{prol}}}$ is non-degenerate if and only if J^{α_1} is non-degenerate.

Similarly, we compute

$$\begin{aligned}\mathrm{d}\alpha_2 &= \mathrm{d}\mu_1^2 \wedge \mathrm{d}x_1 + \mathrm{d}\mu_2^2 \wedge \mathrm{d}x_2 \\ &= \left(\frac{\partial \mu_1^2}{\partial x_2}\mathrm{d}x_2 + \frac{\partial \mu_1^2}{\partial x_3}\mathrm{d}x_3 + \frac{\partial \mu_1^2}{\partial x_4}\mathrm{d}x_4 + \frac{\partial \mu_1^2}{\partial y_1}\mathrm{d}y_1 + \frac{\partial \mu_1^2}{\partial y_2}\mathrm{d}y_2\right) \wedge \mathrm{d}x_1 \\ &\quad + \left(\frac{\partial \mu_2^2}{\partial x_1}\mathrm{d}x_1 + \frac{\partial \mu_2^2}{\partial x_3}\mathrm{d}x_3 + \frac{\partial \mu_2^2}{\partial x_4}\mathrm{d}x_4 + \frac{\partial \mu_2^2}{\partial y_1}\mathrm{d}y_1 + \frac{\partial \mu_2^2}{\partial y_2}\mathrm{d}y_2\right) \wedge \mathrm{d}x_2.\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}\alpha_2 \wedge \mathrm{d}\alpha_2|_{\mathcal{D}^{\text{prol}}} &= -2\left(\frac{\partial \mu_1^2}{\partial y_1}\frac{\partial \mu_2^2}{\partial y_2} - \frac{\partial \mu_1^2}{\partial y_2}\frac{\partial \mu_2^2}{\partial y_1}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2 \\ &= -2 \det J^{\alpha_2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2,\end{aligned}$$

implying $\mathrm{d}\alpha_2|_{\mathcal{D}^{\text{prol}}}$ is non-degenerate if and only if J^{α_2} is non-degenerate. Now assume that both J^{α_1} and J^{α_2} are non-degenerate, or equivalently that $\mathrm{d}\alpha_1|_{\mathcal{D}^{\text{prol}}}$ and $\mathrm{d}\alpha_2|_{\mathcal{D}^{\text{prol}}}$ are. Note that we can choose a basis for $H \in \mathcal{D}^{\text{prol}}$ given by

$$\{v_1, v_2, \partial_{y_1}, \partial_{y_1}\},$$

where the v_i have a non-zero component in ∂_{x_i} and no component in ∂_{x_j}, for $(i,j) \in \{(1,2), 2, 1)\}$. Since we may rescale, we choose the non-zero component in ∂_{x_i} equal to 1, i.e. v_i is of the form $\partial_{x_i} + \cdots + 0 \cdot \partial_{x_j}$. Then from the just computed expressions for $\mathrm{d}\alpha_1$ and $\mathrm{d}\alpha_2$ we derive that the rank measure

R^{12}_H satisfies

$$\begin{aligned} v_1 &\mapsto Av_1 \\ v_2 &\mapsto Av_2 \\ \partial_{y_1} &\mapsto A\partial_{y_1} \\ \partial_{y_2} &\mapsto A\partial_{y_2}. \end{aligned}$$

Note that at the point H we have that

$$\mathcal{D}^{\text{prol}}_H = H + T_H M_x \subset T\,\mathsf{Gr}_2(TM),$$

where on the righthandside H is interpreted as the subspace $H \subset T_x X$. The rank measure $R^{12}_{(x,H)}$ at (x, H) is the linear map $H + T_H M_x \to HTM_x$ given by

$$R^{12}_{(x,H)} \colon (w_X, w_{M_x}) \mapsto (A(w_X), A(w_{M_x}))$$

with respect to the basis $\{v_1, v_2\}$ for $H \in T_x X$ and the basis $\{\partial_{y_1}, \partial_{y_1}\}$ for $T_H M_x$. Hence R^{12}_H has non-real eigenvalues if and only if A has. This implies that the prolonged distribution $\mathcal{D}^{\text{prol}}$ is fat at $y \in M$ if and only if Σ_x is transverse to the infinitesimal cone field $\mathcal{C}_H$ at H.

Since y was chosen arbitrarily, this completes the proof. □

Remark 4.16. Recall result in Theorem 3.11, stating that for the (almost) complex Grassmannian that associated to a 4-dimensional almost complex manifold X the prolonged distribution $\mathcal{D}^{\text{prol}}$ is fat. We see now that this result follows directly from Theorem 4.15. Indeed, the local parametrization $\varphi \colon \mathbb{R}^2 \to \mathsf{Gr}_2(T_x X)$ –which is implicitly used there– is given by

$$(\lambda_1, \lambda_2) \mapsto \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$$

The corresponding Jacobians, are given by

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and, from Lemma 4.13, we see that the almost complex Grassmanians are transverse to the infinitesimal cone.

In fact, we can replace the complex structure J in the proof of in Theorem 3.11 by $J' = FJ$ – which again defines a complex structure on $T_x X$. That means that we consider the complex Grassmannian $\mathsf{Gr}_2(TX, J') = \{H \in \mathsf{Gr}_2(TX) | J'H = H\}$ instead, where

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Via the same computations as in the proof of Theorem 3.11 we now obtain that the Jacobians of the corresponding (lightly adjusted) local parametrization are now given by

$$B_1' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B_2' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so, again by Lemma 4.13, it follows that $\mathsf{Gr}_2(TX, J')$ is transverse to the infinitesimal cone.

Finally, we observe that the determinants of B_1, B_2 are positive and the determinants of B_1', B_2' are negative. From the similarity of the charts it follows that this holds in fact for every point in $\mathsf{Gr}_2(TX)$. From Remark 4.14, the two complex Grassmanians

$$\mathsf{Gr}_2(TX, J) = \{H \in \mathsf{Gr}_2(TX) | JH = H\} \quad \text{and} \quad \mathsf{Gr}_2(TX, J') = \{H \in \mathsf{Gr}_2(TX) | J'H = H\},$$

associated to the complex structures J and J' on X are transverse.

Chapter 5

Fibers inducing fat prolongations

With the machinery developed in Chapter 4, we can deduce some properties of the class of fat prolonged distributions of type $(4, 6)$ under consideration. The main result of this chapter is showing that for the family of fat prolonged distributions on a closed fiber bundle manifold M, the fibers of $M \to X$ are forced to be 2-spheres or projective planes.

Again, we consider a 4-dimensional vector space V and H_0 be a 2-plane in V. We fix a basis $\{x_1, \dots, x_4\}$ for V such that x_1, x_2 span H_0 and x_3, x_4 span the orthogonal complement $H^\perp$ of H_0 with respect to this basis.

We consider the two realizations of the quaternions in $GL_4(V)$ given by the spans

$$\mathbb{H}_1 = \langle I, \mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1 \rangle \quad \text{and} \quad \mathbb{H}_2 = \langle I, \mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2 \rangle,$$

where I is the identity map on V and the imaginary elements are the complex structures on V given by

$$\mathbf{i}_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \mathbf{j}_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \mathbf{k}_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{i}_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \mathbf{j}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathbf{k}_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

with respect to the fixed basis.

The unit spheres of imaginary quaternions in $\mathbb{H}_1$ and $\mathbb{H}_2$ are given by

$$S(\operatorname{Im} \mathbb{H}_1) = \{J \in \mathbb{H}_1 \colon \det J = 1\}$$

and

$$S(\operatorname{Im} \mathbb{H}_2) = \{J \in \mathbb{H}_2 \colon \det J = 1\}.$$

We denote them by S_1 and S_2 respectively.

Remark 5.1. Both S_1 and S_2 are homeomorphic to S^2. However, they are disjoint sub-spaces of $GL_4(V)$.

For a 2-plane $H \in Gr_2(V)$ there exist precisely two pairs of elements, i.e. complex structures, in these imaginary unit quaternion spheres such that H is preserved under their action on V. The action of the rotations that these four H-preserving complex structures induce are illustrated in Figure 5.1. They are

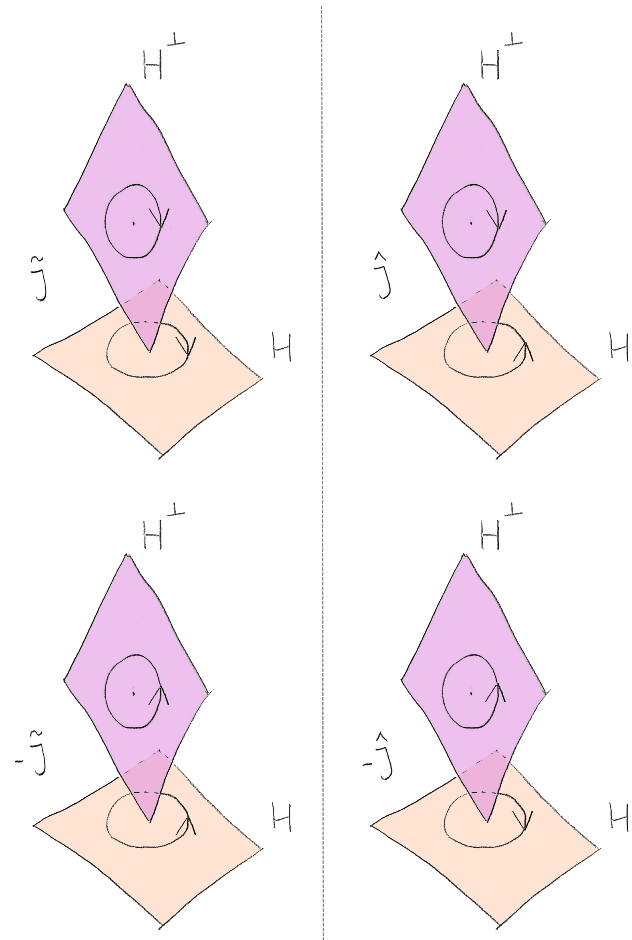

Figure 5.1: Rotations induced by 4 versions of unit quaternions preserving H, two given by $J_1(H), -J_1(H) \in S_1$ and two given by $J_2(H), -J_2(H) \in S_2$.

distributed evenly over the two spheres, one pair in S_1 and one pair in S_2. More precisely, consider the isotropy group Iso(H) of H with respect to the left action of $GL_4(V)$ on Gr$_2(V)$ given by

$$\text{Iso}(H) = \{A \in GL_4 : AH \subset H\}.$$

We have that Iso(H) intersects both unit quaternion spheres in the two pairs of antipodal elements that also preserve the orthogonal complement $H^{\perp}$, i.e.

$$\text{Iso}(H) \cap S_1 = \{J_1(H), -J_1(H)\}, \quad \text{Iso}(H) \cap S_2 = \{J_2(H), -J_2(H)\}.$$

Here the orthogonal complement is with respect to the fixed basis $\{x_1, ...x_4\}$. In fact, $J_i(H)$ and $J_i(H)$ represent a well defined element in the projectification of the imaginary quaternions

$$[J_i(H)] \in PS_i \cong RP^2$$

for $i \in \{1, 2\}$. This allows to introduce two maps

$$\pi_i : \mathrm{Gr}_2(V) \to PS_i, \quad i \in \{1, 2\}$$

sending a 2-plane H to the projective class of a complex structures on V preserving it:

$$\pi_i : H \in Gr_2(V) \mapsto [J_i(H)].$$

Lemma 5.2 The maps π_i are surjective and smooth.

Proof. For every complex structure on V there is a 2-plane in V invariant under it, so the maps π_i are surjective.

Now we show that π_i is smooth. It suffices to show that π_i restricts to a smooth map on the open sets $\mathrm{Gr}_2(V)_{\mathcal{L}}$. Without loss of generality, we show this for $\mathcal{L} = (1, 2)$. The calculations are analogous for the other charts $\mathrm{Gr}(V)_{\mathcal{L}'}$ that cover $\mathrm{Gr}_2(V)$.

Recall that $\chi_{\mathcal{L}}\colon\ \mathrm{Gr}_2(V)_{\mathcal{L}} \to \mathbb{R}^4 \cong \mathrm{Mat}_{2\times 2}$ maps For all $H \in \mathrm{Gr}_2(V)_{\mathcal{L}}$,

$$H \mapsto M = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}.$$

Finally, we denote by $B(H)_{\mathcal{L}}$ is the element in $GL_4(V)$ given by

$$\begin{pmatrix} 1 & 0 & m_{11} & m_{21} \\ 0 & 1 & m_{12} & m_{22} \\ -m_{11} & -m_{12} & 1 & 0 \\ -m_{21} & -m_{22} & 0 & 1 \end{pmatrix}.$$

Then $\pi_i \circ \chi_{\mathcal{L}}^{-1}$ is given by

$$M = \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix} \mapsto [B(H)_{\mathcal{L}} \cdot J_i(H_0) \cdot B^{-1}(H)_{\mathcal{L}}], \quad H_0 = \langle x_1, x_2 \rangle$$

Indeed, $B(H)_{\mathcal{L}} J_i(H_0) B^{-1}(H)_{\mathcal{L}}$ is a complex structure on V preserving H and its complement, hence its projective class is precisely $[J(H)]$. We then see that $\pi_i \circ \xi_{\mathcal{L}}^{-1}$ is smooth, because it involves only smooth operations in $GL_4(V)$ and the map $B\colon\ Gr(V)_{\mathcal{L}} \to GL_4(V)$ which is given by the m_{ij}, and hence smooth.

□

We now look at the fibers of the map π_i.

Proposition 5.3 The fiber of π_i over $[J_i(H)]$, $H \in Gr_2(TX)$, is given by the complex Grassmanian $\mathrm{Gr}_2(TX, J_i(H))$.

Proof. A point $\hat{H} \in \mathrm{Gr}_2(TX)$ is in $\pi_i^{-1}(H)$, for $H \in Gr_2(TX)$, if and only if

$$[J_i(\hat{H})] = [J_i(H)],$$

which happens if and only if

$$J_i(H) = \pm J_i(\hat{H}),$$

By definition, both $J_i(\hat{H})$ and its opposite preserve $\hat{H}$ (and its orthogonal complement), so $J_i(H)$ also preserves $\hat{H}$ (and its orthogonal complement), and we have

$$\hat{H} \in \mathrm{Gr}_2(TX, J_i(H)) = \{H' \in \mathrm{Gr}_2(TX) | J_i(H)H' = H'\}.$$

□

From the proposition above, we obtain the following two corollaries.

Corollary 5.4 The map π_i is submersive.

Proof. It follows from Proposition 5.3 from dimension counting. In fact, Proposition 5.3 implies that the fibers of π_i are embedded submanifolds of real dimension 2. Since π_i is a smooth map from a 4-dimensional manifold with 2-dimensional fibers, the rank of $d\pi_i$ is everywhere equal to 2. Finally, since $PS_i \cong RP^2$ is 2-dimensional, this implies that π_i is a submersion. □

Corollary 5.5 The fibers of π_1 are transverse to the fibers of π_2.

Proof. The fibers of π_1 and π_2 are given by the complex Grassmannians associated to the class of complex structures $[J_1]$ and $[J_2]$ in S_1 and S_2 respectively. Similar to the complex structures J and $J' = FJ$ in Remark 4.16, the two pairs of Jacobians of their respective parametrizations at a point of intersection have opposite sign, which implies that they are transverse since they are transverse to the infinitesimal cone field. □

In the following, let $\mathcal{D}^{\mathrm{prol}}$ be a prolonged distribution of type (4,6) with defining triple $(M, \pi^M \colon M \to X, \varphi \colon M \to \mathrm{Gr}(TX))$, such that M is a closed rank-2 bundle over the 4-dimensional manifold X. For a point $x \in X$ we denote the image $\varphi(M_x)$ by Σ_x, like before. We consider the π_i now as maps $\mathrm{Gr}_2(TxX) \to PS_i$.

Corollary 5.6 Assume the prolonged distribution $\mathcal{D}^{\mathrm{prol}}$ induced by φ is fat. Then the immersed fibers $\Sigma_x \in \mathrm{Gr}_2(T_xX)$ are transverse to the fibers of either π_1 or π_2.

Proof. By Theorem 4.15, φ_x is transverse to the infinitesimal cone field $\mathcal{C}$. Now the statement follows from the observations in Remark 4.14. An immersion φ has to be in one of the two families that is transverse to the infinitesimal cone field, i.e. its Jacobians are positive or negative. Then it has to be transverse to one of the two complex Grassmannians the two respective pairs of Jacobians have both positive and both negative determinants. □

The intuition behind the previous two corollaries is given in Figure 5.2.

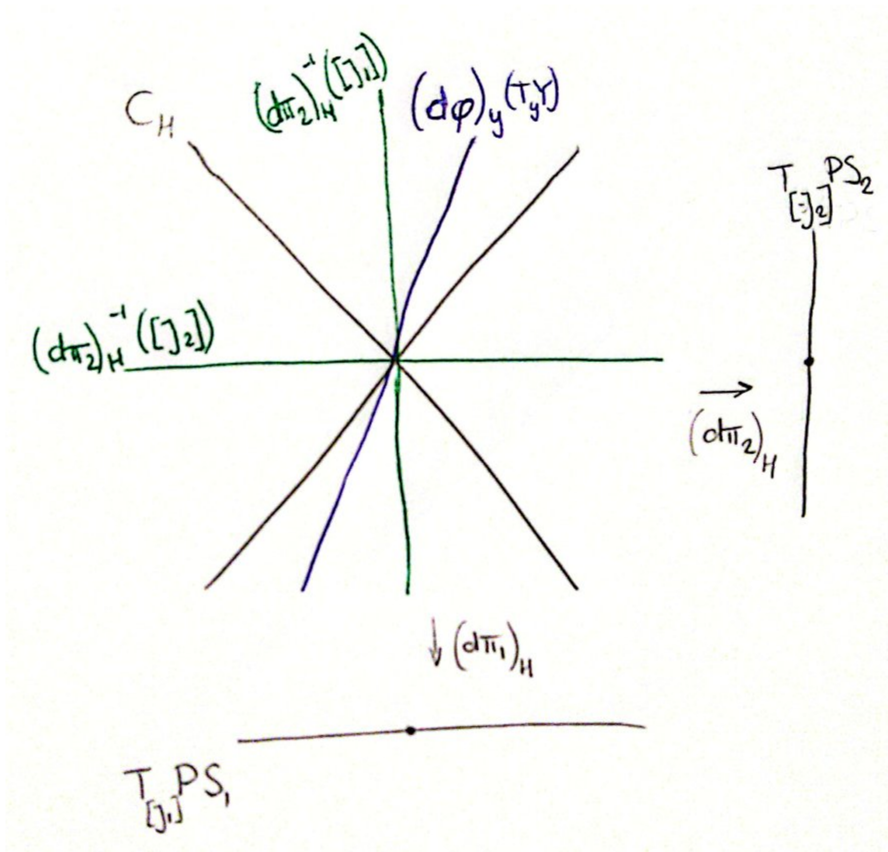

Figure 5.2: The fibers of π_1 and π_2 are the complex Grassmannians associated to the base point in the respective spheres S_1 and S_2, of which we saw in Remark 4.16 that they are transverse to the cone field (each in a different way) and in particular they are transverse to each other. In this picture their tangent spaces are represented the horizontal and vertical lines. The immersed fiber given by $(\mathrm{d}\varphi)_y(T_yY) \in \mathrm{Gr}_2(T_xX)$ that is transverse to the cone field must be transverse to the fibers of either π_1 or π_2. In this picture it is represented by the blue line transverse to the fibers of $\mathrm{d}\pi_2$.

Proposition 5.7 Assume the prolonged distribution $\mathcal{D}^{\mathrm{prol}}$ induced by φ is fat. Consider the immersed fiber Σ_x. Then one of the two restricted maps $\pi_1|_{\Sigma_x}$ and $\pi_2|_{\Sigma_x}$ is a surjective submersion.

Proof. We have that Σ_x has to be transverse to the fibers of one of the two projections. Without loss of generality, say it is transverse to the fibers of π_2. Since M is closed, also its fibers are closed, implying that

Σ_x is closed as well. Recall that φ is an immersion of a 2-dimensional manifold into the 4-dimensional manifold $\text{Gr}_2(T_xX)$. Then since π_2 is a surjective submersion and its fibers are 2-dimensional $\pi_2|_{\Sigma_x}$, $\pi_2|_{\Sigma_x}$ is a surjective submersion. □

Theorem 5.8 The immersed fiber Σ_x is homeomorphic to $\mathbb{R}P^2$ or S^2.

Proof. One of the projections π_i is a surjective submersion. Σ is 2-dimensional and so is $\mathbb{R}P^2$. Hence π_i is a covering map. The only two covering spaces of $\mathbb{R}P^2$ are $\mathbb{R}P^2$ and S^2, hence Σ must be homeomorpic to one of them. □

Chapter 6

Conclusion

In this thesis, we focused on co-rank 2 distributions that are induced by the canonical distribution on the Grassmann bundle of 2-planes of a manifold. We defined this class of distributions and refer to them as prolonged distributions. We restricted further to distributions of type $(4, 6)$ and investigated the necessary and sufficient conditions for distributions in this class to be fat.

We first looked at rank-2 fiber bundles M with a 4-dimensional almost complex base manifold (X, J). We considered the sub-bundle of complex Grassmannians $\mathrm{Gr}_2(TX, J)$ in $\mathrm{Gr}_2(TX)$. We showed that the prolonged distribution $\mathcal{D}^{\mathrm{prol}}$ of this sub-bundle is a fat distribution of co-rank 2. In order to prove this, we used a local frame adapted to J on X; we expressed the defining 1-forms in terms of this frame and showed that their differentials restrict to non-degenerate 2-forms on $\mathcal{D}$ and the rank measure relating them has non-real eigenvalues.

Then we introduced the degenerate cones corresponding to a point in the Grassmannian and the (related) infinitesimal cone field. We showed that for a rank-2 fiber bundle M over a 4-dimensional manifold X, the infinitesimal cone field on a fiber of $\mathrm{Gr}_2(TX)$ detects fatness for the prolonged distribution $\mathcal{D}^{\mathrm{can}}$ of type $(4, 6)$ on M in the sense of Theorem 4.15: $\mathcal{D}^{\mathrm{prol}}$ is fat if and only if the immersed fibers –via the bundle map– of M are transverse to the cone field on the associated fiber of $\mathrm{Gr}_2(TX)$. In order to show this local characterization of fatness, we first proved a technical result, Lemma 4.13, relating tranversality to the cone field to properties of the Jacobians of the defining 1-forms: the eigenvalues of the linear map relating them should have non-real eigenvalues in particular. These eigenvalues re-appeared as the eigenvalues of the rank measure in the proof of Theorem 4.15.

This Theorem has strong topological consequences for the admitted fibers of the bundle M in case it is closed as a manifold and its prolonged distribution $\mathcal{D}^{\mathrm{prol}}$ is fat. Namely, in case M is closed and its prolonged distribution is fat, the fibers of M are either 2-spheres or projective planes. This is stated in Theorem 5.8, the main result of this thesis. We showed this by defining two surjective submersions from a fiber of M onto $\mathbb{R}P^2$, where for each such map $\mathbb{R}P^2$ is represented by the projectification of one of the two copies of the imaginary quaternion unit sphere in GL_4 respectively. We proceeded by showing that the immersed fiber of M is transversal to the fibers of one of the two projections, using that they are transversal to the infinitesimal cone field, by Lemma 4.13. From this we deduced that either one or the other projection restricts to be a surjective submersion from the fiber of M onto $\mathbb{R}P^2$, implying that the fiber is a covering map for $\mathbb{R}P^2$, which implies the result.

Chapter 7

Outlook

There are several questions that arise after obtaining these results, just as interesting directions to look into.

Concluding the findings in this thesis

One of the aims we discussed at the start of this thesis was to find concrete examples of fat distributions of co-rank greater than 1, not associated to holomorphic contact structures. We provided new families for which this definitely seems the case, but it remains to show this formally. It would be interesting to work this out for manifolds not allowing a complex structure for example.

More specifically, it would be interesting to see if there exist indeed examples of fiber bundles M inducing fat prolonged distributions with projective planes as fibers. In that case it is even more unlikely that the distribution is associated to a holomorphic contact structure. We have evidence that if an immersed fiber Σ starting at a center point in one of the charts of the the Grassmannian passes through the center point of its antipodal chart, it is a sphere. Namely, we suspect this can be shown using a Morse function: on chart $\mathrm{Gr}_2(T_xX)_{\mathcal{L}} \cup \mathrm{Gr}_2(T_xX)_{\mathcal{L}^\perp}$ given by $f : \mathrm{Gr}_2(T_xX)_{\mathcal{L}} \cup \mathrm{Gr}_2(T_xX)_{\mathcal{L}^\perp} \to \mathbb{R}$, such that

$$f\colon [\mathcal{N}\colon \mathcal{M}] \mapsto \frac{\det \mathcal{M}}{\det \mathcal{N} + \det \mathcal{M}}.$$

Moreover, we suspect that if we do not cross the antipodal point, we intersect the antipodal degenerate cone (at the antipodal point) in a circle, which would hint at the possibility for fibers that are indeed homeomorphic to $\mathbb{R}P^2$ as it seems like one could glue the disc formed by Σ of the first chart over this circle.

Next to this it would be interesting to investigate if we can deduce what the homotopy classes of fat prolonged distributions are with the tools introduced in this thesis.

Higher dimensional analogues

We think that classification results of a similar nature as for the $(4, 6)$ case we focused on this thesis can be found for co-rank 2 fat prolonged distributions of rank greater than 4 using generalized techniques. In particular, we suspect that there is an analogue for the relation of the degenerate cone and the induced infinitesimal cone field with fat prolonged distributions for higher dimensions.

The definition for the degenerate cone generalizes to higher dimensions.

Definition 7.1 Let V be a vector space and consider the co-rank k Grassmannian $\mathsf{Gr}_k(V)$. Let H be a point in $\mathsf{Gr}_k(V)$. The *degenerate cone* $\mathfrak{C}_H$ of H in the Grassmannian $\mathsf{Gr}_k(V)$ is the subvariety of $\mathsf{Gr}_k(V)$ given by

$$\{H' \in \mathsf{Gr}_k(V) \colon H' \not\pitchfork H\}$$

In the same way, an infinitesimal cone field is defined.

Definition 7.2 Let V be a vector space and consider the co-rank k Grassmannian $\mathsf{Gr}_k(V)$. The *infinitesimal cone field* $\mathcal{C}$ on the Grassmannian $\mathsf{Gr}_k(V)$ is a field of subverieties of the tangent space, point-wise given by

$$\mathcal{C}_H = \{Q \in T_H \mathsf{Gr}_k(V) \colon \operatorname{rank} Q < k\}$$

The approach to prove a higher dimensional version of Theorem 4.15, would be to fist generalize the technical Lemma 4.13.

Higher co-rank fat distributions

We believe that a new class of fat distributions of co-rank greater than 2 can be identified: One can construct fat distributions of co-rank k on a manifold M admitting $k-1$ transversal fibrations with almost complex structures on the fibers.

The approach we use for this is based on the proof of Rayner's Theorem (from [12]), where one of the key ingredients for the existence of formal fat distributions is the existence of several distinct representations of Clifford algebras, which correspond to linear complex structures on the tangent spaces. The idea of the construction that would define the new class of examples is that the almost complex structures on the fibers of the transversal fibrations could be a global realization of a distribution for which at each point the complex structures on the fibers correspond to the representations of Clifford algebras from Rayner's Theorem. We intend to use the prolongation techniques used in this thesis in this generalized setting to generate a fat prolonged distributions of higher co-rank.

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