Master's Thesis
Connecting arithmetic functions and continuous distribution functions

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Abstract

In this thesis, we discuss some classical results in probabilistic number theory, focusing on when the outputs of an arithmetic function, usually multiplicative, attain a continuous distribution function. We study the inception of these theories around the early 20th century in the work of Schoenberg, who inspired Davenport to show that abundant numbers have a continuous distribution. It was not until 2013 that Jennings, Pollack and Thompson looked at this problem from a different perspective and generalized the result on abundant numbers in a new direction. Moreover, Jennings managed to generalize this to other functions besides the sum of divisors function. While Schoenberg gave necessary and sufficient conditions, those given by Jennings are purely sufficient. Jennings' result has the advantage of being easier to apply. In this thesis, we find a function that satisfies Schoenberg's but not Jennings', conditions. We also compare Schoenberg's conditions with conditions from modern probability theory.

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1 Introduction

Number theory is the field of study which focuses on the structure of integers and arithmetic functions. On the other hand, probability theory is the field of study which is interested in studying the distribution of random variables and their properties like mean and variance. Probabilistic number theory then is the field which studies the probabilistic properties of arithmetic functions $f(n)$, as the output of $f(n)$ for any arbitrary n can be hard to predict, especially for large n. One of the most well-known theorems in probabilistic number theory has to be attributed to Paul Erdős and Mark Kac [6].

Theorem 1.1 (Erdős-Kac, 1940) *Let $f(n)$ be a strongly additive function with $|f(p)| \leq 1$.*
Let

$$A(n) = \sum_{p \leq n} \frac{f(p)}{p},$$

and

$$B(n) = \sqrt{\sum_{p \leq n} \frac{f(p)^2}{p}}.$$

Then

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leq x \mid \frac{f(n) - A(n)}{B(n)} \leq u \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{t^2}{2}} \mathrm{d}t.$$

Hence $A(n)$ can be interpreted as the normal order of f and $B(n)$ as the variance.

Before this, results about arithmetic functions had mostly involved estimating the asymptotic growth average growth of arithmetic functions. Thus, this result was the first to not only establish the average growth, but also pin down asymptotically how large the variance from this average is. Moreover, they managed to connect the behaviour of certain arithmetic functions with one of the more central distributions in probability, the normal distribution. The inception of probabilistic number theory is probably best attributed to Schoenberg's thesis under Issay Schur in 1928 [14]. In his thesis, Schoenberg gives a few examples of numbers which are continuously distributed along the unit interval. The main example which caught Davenport's eye was the choice of the multiplicative function

$$g(n) := \frac{\varphi(n)}{n}.$$

Let the moments μ_k, for $k > 0$, of the function above be given by

$$\mu_k := \lim_{x\to\infty} \frac{1}{n} \sum_{n \leq x} g(n)^k.$$

Then the analytic continuation of these moments for Re $s > 0$, $\Phi(s)$, is given by

$$\Phi(s) = \prod_p \left\{ 1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s \right\},$$

a formula which he obtained from his doctoral advisor Issay Schur. Schoenberg established necessary and sufficient conditions on $\Phi(it)$ so that the numbers

$$\frac{\varphi(1)}{1}, \frac{\varphi(2)}{2}, \dots, \frac{\varphi(n)}{n},$$

are continuously distributed along the unit interval. An observation we made was that instead of computing $\Phi(s)$ through this analytic continuation, one can compute $\Phi(it)$ directly as the characteristic function of $\log g(n)$. Not much is known about how Schur established his representation of $\Phi(s)$. However, soon after Schoenberg's publication, Davenport applied the same conditions for the numbers

$$\frac{1}{\sigma(1)}, \frac{2}{\sigma(2)}, \ldots, \frac{n}{\sigma(n)}. \tag{1}$$

Davenport's paper [2] includes a derivation of $\Phi(s)$ through Möbius inversion, which we will discuss in 2.2. It is worth mentioning that this derivation would have been simpler in Schoenberg's case, as we have that

$$g(p^k) = \frac{\varphi(p^k)}{p^k} = \frac{p^k - p^{k-1}}{p^k} = 1 - \frac{1}{p},$$

independent of k. Hence when applying Möbius inversion to $g(n)$ one sees that for $k \geq 2$ the terms vanish, as $g(p^k) - g(p^{k-1}) = 0$. The direct implication is that the product representation of this $\Phi(s)$ is without an infinite sum, whereas this is not generally the case. Davenport's interest in the numbers (1) arose from his interest in abundant numbers and its density in the natural numbers. Recall that a natural number n is called *perfect* if $\sigma(n) = 2n$, *abundant* if $\sigma(n) > 2n$ and *deficient* if $\sigma(n) < 2n$. A natural number n is then called χ-*abundant* if $\sigma(n) \geq \chi n$, and define $A(\chi, n)$ to be the number of χ-abundant numbers up to and including n. Notice that the two definitions differ in whether the inequality is strict or not. As Davenport managed to show that the distribution is in fact continuous, this discrepancy does not matter. He proved that the density

$$A(\chi) \coloneqq \lim_{n\to\infty} \frac{A(\chi, n)}{n} \tag{2}$$

exists for every χ and is continuous in χ.
Thus the claim that the limit in (2) exists for every χ and is continuous in χ, is equivalent to the claim that asymptotically $\frac{n}{\sigma(n)}$ is continuously distributed along the unit interval; then $A(\chi, n)$ equals the number of numbers $m \leq n$ for which $\frac{m}{\sigma(m)} \leq x = \frac{1}{\chi}$. The paper of Davenport's not only discussed the continuity of the distribution function, but also contained an inclusion-exclusion method to approximate the proportion of numbers which are χ-abundant, going one step beyond proving the distribution exists, making attempts to approximate the distribution function.

To clarify, we call $z(t)$ a distribution function for $0 \leq t \leq 1$ when

1. $z(0) = 0$ and $z(1) = 1$,
2. z is non-decreasing,
3. and z is right-continuous.

Let $\phi(t)$ (or $\phi_X(t)$) denote the characteristic function (of the random variable X).
Further, let $d = (a, b)$ denote the greatest common divisor of integers a and b and $\varphi(n)$ denote the Euler totient function. The variable p shall always denote a prime number and sums or products of the form $\sum\limits_{p}$ or $\sum\limits_{p\geq 7}$ will always be taken over the primes.

2 Historical survey

In this section we will discuss some of the historical advances in probabilistic number theory and discuss proofs of Schoenberg, Davenport and Jennings, Pollack and Thompson.

2.1 Schoenberg's thesis

Now we will discuss some of Schoenberg's results of his doctoral thesis pertaining to the distribution of a countable set of numbers in the unit interval and dissect the proof of his main result. Schoenberg's paper [14] on the distribution of numbers was inspired by Weyl's paper of a similar name [17], whose paper discussed when numbers

$$\alpha_1, \alpha_2, \ldots$$

are equidistributed modulo 1. Schoenberg instead proved theorems on the distribution of arbitrary numbers

$$x_{1n}, x_{2n}, \ldots, x_{nn} \pmod 1,$$

including the relation between numbers $f(x_{jn})$ and the asymptotic distribution of these numbers modulo 1 for a class of functions f, and the continuity of the aforementioned distribution function. Throughout this section, let the numbers

$$x_{1n}, x_{2n}, \ldots, x_{nn} \tag{3}$$

be in the unit interval. For k a positive integer we let their k-th *Stieltjes moments* be given by

$$\mu_k := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} x_{jn}^k \tag{4}$$

and the k-th *Fourier moments* be given by

$$\omega_k := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i k x_{jn}}. \tag{5}$$

Let it be noted that Fourier moments allowed Schoenberg to obtain results about numbers when taking them mod 1. We also let $z(t)$ for $0 \leq t \leq 1$ be the distribution function of the numbers (3), i.e., for $0 < t < 1$ this is given by

$$z(t) := \lim_{n\to\infty} \frac{1}{n} \cdot |\{1 \leq j \leq n \mid x_{jn} \leq t\}|,$$

with $z(0) = 0$ and $z(1) = 1$. Before we go any further into his thesis, in Schoenberg's Third Theorem a beautiful theorem is applied to extend the domain of a function, and we wish to state it in full here.
[11, p.111]

Theorem 2.1 (Le Roy, Lindelöf) *Given a Taylor series*

$$f(x) = \sum_{k=0}^{\infty} \Phi(k) x^k$$

such that for $\Phi(k)$

1. *$\Phi(s)$ is holomorphic on Re $s > \alpha$ for some $\alpha \in \mathbb{R}$*

2. *There exists an angle $0 \leq \theta < \pi$ so that for any $\epsilon > 0$ and $r > 0$ we have*

$$\left|\Phi(\alpha + re^{i\psi})\right| < e^{r(\theta+\epsilon)} \qquad \text{for } -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$$

then $f(x)$ is holomorphic on the set of all complex numbers with argument in $(\theta, 2\pi - \theta)$. Moreover for every positive integer m with $m - 1 < \alpha < m$ we have that

$$\sum_{k=m}^{\infty} \Phi(k)x^k = \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\Phi(z)x^z}{1 - e^{2\pi iz}} \mathrm{d}z.$$

Idea of the proof: To intuitively understand this theorem, it is important to see that this theorem relies on the residue theorem. One can see that the denominator has a zero of order 1 at every integer, hence all the poles and residues will occur when z is a positive integer inside the path of integration. For the path of integration, the path along a semicircle centered at $z = \alpha$ with radius $r \to \infty$ is taken as can be seen in the image below. The conditions of the theorem are then shown to be sufficient so that the integral purely along the semicircle approaches zero, hence the integral along the vertical path equals the sum of the residues.

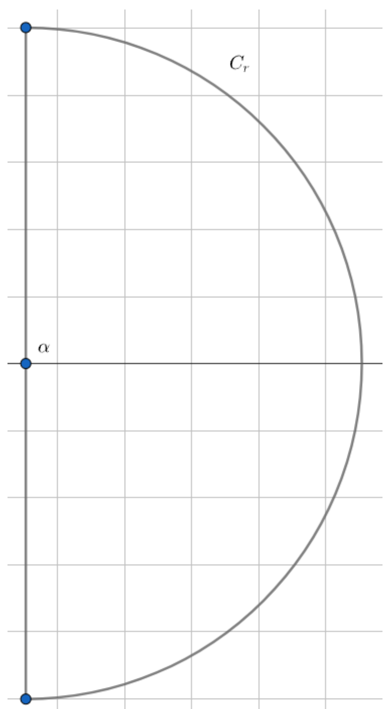

Figure 1: Contour path

2.1.1 Schoenberg's first two theorems

To properly be able to use both moments of the numbers (3), the following theorem is central and applied implicitly throughout the whole paper:

Theorem 2.2 (Schoenberg's First Theorem) *Let $f(x)$ be a function of bounded variance for $0 \leq x \leq 1$ and $z(t)$ be the distribution function of the numbers* (3). *Assuming $z(t)$ is continuous, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(x_{jn}) = \int_0^1 f(t)\mathrm{d}z(t),$$

where the integral above is to be interpreted as a Riemann-Stieltjes integral.

Now one can see that for all positive k,

$$\int_0^1 t^k \mathrm{d}z(t) = \mu_k \tag{6}$$

$$\int_0^1 e^{2\pi i t k} \mathrm{d}z(t) = \omega_k. \tag{7}$$

This allowed Schoenberg to establish the following result.

Theorem 2.3 (Schoenberg's Second Theorem) *Let the k-th moments* (4) *(or* (5)*) be given and let $z(t)$, $z(0) = 0$, be a continuous and monotonic non-decreasing function so that the equation* (6) *(resp,* (7)*,) holds. Then the numbers* (3) *are asymptotically continuously distributed, with $z(t)$ as their distribution function.*

This theorem is not of much help in establishing whether any arbitrary set of numbers is asymptotically distributed mod 1. However, together with the first theorem, this allows us to extend one result with another.

Corollary 2.3.1 *Let the numbers in the unit interval*

$$x_{1n}, x_{2n}, \ldots, x_{nn}$$

and their distribution function $z(t)$ be given. Let $\psi(t)$ be another distribution function. Then the numbers

$$\psi(x_{1n}), \psi(x_{2n}), \ldots, \psi(x_{nn})$$

are also asymptotically distributed along the unit interval with distribution function $z(\psi^{-1}(t))$.

One example for this would be

$$x_{in} := \frac{i}{n},$$
$$z(t) = t,$$
$$\psi(t) = \sin\left(\frac{1}{2}\pi t\right),$$

which satisfy the conditions to be a continuous distribution function. Then also the numbers

$$\sin\left(\frac{\pi}{2n}\right), \sin\left(\frac{2\pi}{2n}\right), \ldots, \sin\left(\frac{n\pi}{2n}\right),$$

are asymptotically distributed with their distribution function given by

$$z(\psi^{-1}(t)) = \frac{2\arcsin(t)}{\pi}.$$

Another such interesting result is the following corollary:

Corollary 2.3.2 *Let*

$$x_{1n}, x_{2n}, \ldots, x_{nn}$$

be in $(0, 1]$ *and suppose that their distribution function* $z(t)$ *is given. Then the numbers*

$$\frac{1}{x_{1n}}, \frac{1}{x_{2n}}, \ldots, \frac{1}{x_{nn}} \bmod 1$$

also are asymptotically continuously distributed, with their distribution function given by

$$z^{\dagger}(t) = \sum_{j=1}^{\infty} \left(z\left(\frac{1}{j}\right) - z\left(\frac{1}{j+t}\right) \right).$$

This, for example, allows him to verify and extend a result from Pólya: the numbers

$$\frac{n}{1}, \frac{n}{2}, \ldots, \frac{n}{n} \bmod 1$$

are asymptotically distributed, with distribution function

$$z^{\dagger}(t) = \int_0^1 \frac{1-x^t}{1-x} dx.$$

For $k > 1$, the numbers

$$\sqrt[k]{\frac{n}{1}}, \sqrt[k]{\frac{n}{2}}, \ldots, \sqrt[k]{\frac{n}{n}} \bmod 1,$$

are asymptotically distributed with distribution function

$$z_k^{\dagger}(t) = \sum_{j=1}^{\infty} \frac{1}{j^k} + \frac{(-1)^{k-1}}{(k-1)!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \log \Gamma(x+1).$$

2.1.2 Schoenberg's main theorems

The goal of Schoenberg's third and fourth theorems is to give conditions for the continuity of $z(t)$, given the k-th Stieltjes moments, respectively Fourier moments, of the numbers (3). We will merely mention the fourth theorem and dissect the proof of the third theorem in depth. This is because the third theorem gives an insightful connection to probability theory and it is more closely related to current results as it can be applied to multiplicative functions.

Theorem 2.4 (Schoenberg's Fourth Theorem) *Necessary and sufficient conditions for any set of numbers*

$$x_{1n}, x_{2n}, \ldots, x_{nn}$$

to be asymptotically continuously distributed when taken mod 1 are the existence of the k*-th Fourier moments*

$$\omega_1, \ldots, \omega_n,$$

along with the condition that for these Fourier moments

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} |\omega_j| = 0.$$

The fourth theorem seems to use the same tools as discussed in the paper by Weyl; however, this is no longer the case for the third. We cannot immediately state the third theorem, as its conditions are not directly on the Stieltjes moments but on an analytic function derived from them. For the sake of brevity we will refer to Stieltjes moments simply by moments. For numbers in the unit interval

$$x_{1n}, x_{2n}, \ldots, x_{nn}, \tag{3}$$

Schoenberg assumes that all these moments exist. However as can also be seen in modern probability theory, this is not a trivial assumption; we will discuss this problem in more detail in Section 4. To show the connection between the moments to the continuity of $z(t)$, we will extend the moments to the open right half complex plane, so that for Re $s > 0$,

$$\Phi(s) = \int_0^1 t^s \mathrm{d}z(t) \quad \text{and} \quad \Phi(k) = \mu_k. \tag{8}$$

Further, the limit

$$\Phi(iv) := \lim_{u \to 0^+} \Phi(u + iv)$$

exists uniformly for $-\infty < v < \infty$. Then the theorem is as follows.

Theorem 2.5 (Schoenberg's Third Theorem) *Assume that the k-th moments of the numbers* (3) *exist for all $k \geq 0$. The distribution function $z(t)$ is continuous if and only if the following two conditions are met for $\Phi(s)$:*

$$\Phi(0) = 1 \quad \text{and} \quad \lim_{x \to \infty} \frac{1}{x} \int_0^x |\Phi(i\lambda)| \mathrm{d}\lambda = 0. \tag{9}$$

Then the numbers in (3) *are asymptotically continuously distributed.*

In order to apply the continuity of the unknown $z(t)$, we will use the following function

$$f(x) := \int_0^1 \frac{\mathrm{d}z(t)}{1 - xt},$$

which is analytic and holomorphic on $\mathbb{C} \setminus [1, \infty]$. This function connects $z(t)$ with μ_k. In particular, for $|x| < 1$ we have

$$f(x) = 1 + \sum_{k=1}^{\infty} \mu_k x^k.$$

Before we can relate the final result depending on $\Phi(s)$ to this function f, we will need the following lemma which is an important step towards the second condition in (9).

Lemma 2.6 *Let $z(t)$ be given as above. Then the following two statements are equivalent:*

(i) $z(t)$ is continuous for $t \in (0, 1]$;

(ii) For fixed $r \geq 1$ we have that

$$\lim_{x \to r} (x - r) f(x) = 0$$

where x approaches r on any path with a fixed angle so the path does not intersect the interval $[1, \infty)$.

Proof: Let $\theta = \frac{1}{r}$. Then statement (ii) is the same as saying

$$\lim_{\frac{1}{x}\to\theta}(1-x\theta)f(x) = \lim_{\frac{1}{x}\to\theta}\int_0^1 \frac{1-x\theta}{1-xt}\mathrm{d}z(t) = 0$$

Notice for the integrand that

$$\left|\frac{1-x\theta}{1-xt}\right| = \left|\frac{\frac{1}{x}-\theta}{\frac{1}{x}-t}\right| < C,$$

for some $C \in \mathbb{R}$ and that for t outside the interval $(\theta-\epsilon, \theta+\epsilon)$ we have

$$\lim_{\frac{1}{x}\to\theta}\left|\frac{\frac{1}{x}-\theta}{\frac{1}{x}-t}\right| = 0.$$

This means that for the integral above

$$\begin{aligned}\lim_{\frac{1}{x}\to\theta}\int_0^1 \frac{1-x\theta}{1-xt}\mathrm{d}z(t) &= \lim_{\frac{1}{x}\to\theta}\int_0^{\theta-\epsilon} \frac{1-x\theta}{1-xt}\mathrm{d}z(t) + \int_{\theta-\epsilon}^{\theta+\epsilon} \frac{1-x\theta}{1-xt}\mathrm{d}z(t) + \int_{\theta+\epsilon}^1 \frac{1-x\theta}{1-xt}\mathrm{d}z(t)\\ &= \lim_{\frac{1}{x}\to\theta}\int_{\theta-\epsilon}^{\theta+\epsilon} \frac{1-x\theta}{1-xt}\mathrm{d}z(t).\end{aligned}$$

The implication $(i) \implies (ii)$ now follows quickly. Assuming (i) means that $z(t)$ is continuous at $t = \theta$. Then

$$\lim_{\frac{1}{x}\to\theta}\int_0^1 \frac{1-x\theta}{1-xt}\mathrm{d}z(t) \leq C\cdot(z(\theta+\epsilon) - z(\theta-\epsilon)) = 0.$$

Note that this implication also works for $\theta = 1$, with the small adjustment that the important integration domain becomes $(1-\epsilon, 1]$.

The opposite direction we will prove through contraposition. Thus assume that $z(t)$ is not continuous. Any discontinuities outside of $(\theta-\epsilon, \theta+\epsilon)$ in fact do not affect the result as we have seen that outside this domain, the integrand itself approaches zero. This means that we may focus on the situation where $z(t)$ exhibits a single jump discontinuity of $\delta > 0$ at $t = \theta$ in a neighbourhood of θ. Then we define a new function $z_1(t)$ where

$$z_1(t) = \begin{cases} z(t) & \text{if } 0 \leq t < \theta \\ z(t) - \delta & \text{if } \theta \leq t \leq 1. \end{cases}$$

Similar to above we have

$$\lim_{\frac{1}{x}\to\theta}\int_0^1 \frac{1-x\theta}{1-xt}\mathrm{d}z(t) = \lim_{\frac{1}{x}\to\theta}\int_0^1 \frac{1-x\theta}{1-xt}\mathrm{d}z_1(t) + \left(\frac{1-x\theta}{1-x\theta}\right)\cdot\delta$$

and as $z_1(t)$ is now continuous at $t = \theta$ we see that in fact the integral on the right-hand side again equals zero. Hence

$$\lim_{\frac{1}{x}\to\theta}(1-x\theta)f(x) = \delta \neq 0.$$

Thus the two statements are equivalent. □

Proposition 2.1 *The analytic continuation of the moments, $\Phi(s)$, is uniquely defined and holomorphic on* Re $s > 0$.

The fact that $\Phi(s)$ is holomorphic is a required condition for Theorem 2.1.
Proof: Recall from (8) that for Re $s > 0$

$$\Phi(s) = \int_0^1 t^s \mathrm{d}z(t).$$

Firstly, to check it is holomorphic we claim that its complex derivative is given by

$$\Phi'(s) = \int_0^1 t^s \log t \mathrm{d}z(t).$$

Let $s = u + iv$ be fixed with $u > 0$ and $0 < |h| < \frac{u}{2}$, so that $\mathrm{Re}(s+h) > 0$. Then

$$\begin{aligned}\left|\frac{\Phi(s+h)-\Phi(s)}{h} - \int_0^1 t^s \log t \mathrm{d}z(t)\right| &= \left|\int_0^1 \left(\frac{t^h - 1}{h} - \log t\right) t^{u+iv} \mathrm{d}z(t)\right| \\ &\leq \int_0^1 \left|\frac{t^h-1}{h} - \log t\right| t^u \mathrm{d}z(t).\end{aligned}$$

To estimate the quantity in the modulus, we will use the power series of the exponential function

$$\begin{aligned}\left|\frac{t^h-1}{h} - \log t\right| &= \left|\frac{e^{h\log t}-1}{h} - \log t\right| = \left|\sum_{k=0}^{\infty} \frac{\frac{(h\log t)^k}{k!} - 1}{h} - \log t\right| \\ &= \left|\sum_{k=1}^{\infty} \frac{\frac{(h\log t)^k}{k!}}{h} - \log t\right| = \left|\sum_{k=1}^{\infty} \frac{h^{k-1}\log^k t}{k!} - \log t\right| \\ &= \left|\sum_{k=2}^{\infty} \frac{h^{k-1}\log^k t}{k!}\right| \leq |h| \log^2 t \sum_{k=0}^{\infty} \frac{|h|^k |\log t|^k}{(k+2)!} \\ &\leq |h| \log^2 t \sum_{k=0}^{\infty} \frac{\left(|h| \log \frac{1}{t}\right)^k}{k!} = |h| \log^2 t e^{|h| \log \frac{1}{t}}.\end{aligned}$$

In the last steps we used that $0 < t < 1$ so that $|\log t| = \log \frac{1}{t}$. Now as $|h| < \frac{u}{2}$ and taking $h \to 0$ we see that

$$\left|\frac{\Phi(s+h)-\Phi(s)}{h} - \int_0^1 t^s \log t \mathrm{d}z(t)\right| \leq |h| \int_0^1 \log^2 t \cdot t^{-|h|} \cdot t^u \mathrm{d}z(t) \leq |h| \int_0^1 t^{\frac{u}{2}} \log^2 t \mathrm{d}z(t) \to 0.$$

To show that the integrals are well-defined at the lower bound, we may use l'Hôpital's rule. We will demonstrate this now for the final integral above.

$$\lim_{t\to 0^+} \frac{\log^2 t}{t^{-\frac{u}{2}}} = \lim_{t\to 0^+} \frac{2t^{-1}\log t}{-\frac{u}{2} t^{-\frac{u}{2}-1}} = \frac{-4}{u} \cdot \lim_{t\to 0^+} \frac{\log t}{t^{-\frac{u}{2}}} = \frac{-4}{u} \cdot \lim_{t\to 0^+} \frac{t^{-1}}{-\frac{u}{2} t^{-\frac{u}{2}-1}} = \frac{8}{u^2} \cdot \lim_{t\to 0^+} t^{u/2} = 0.$$

This function is now unique by a theorem discussed by Hardy in [8]. Again let $s = u + iv$. Then

$$|\Phi(s)| = \left|\int_0^1 t^s \mathrm{d}z(t)\right| \leq \int_0^1 t^u \mathrm{d}z(t) \leq \int_0^1 \mathrm{d}z(t) = z(1) - z(0) = 1,$$

so that $\Phi(s)$ is bounded on Re $s > 0$. As we have seen above in the proof we will need some knowledge of the function $\Phi(s)$ on the complex axis. To this end we define

$$\Phi(iv) := \lim_{\epsilon\to 0^+} \int_\epsilon^1 t^{iv} \mathrm{d}z(t), \tag{10}$$

and wish to show that

$$\lim_{u\to 0^+} \Phi(u+iv) = \Phi(iv).$$

The reason this needs care is that at $v=0$ one would obtain

$$\Phi(0) = \int_0^1 t^0 \mathrm{d}z(t),$$

at the origin, which is improper if we don't let the lower bound of the integral approach 0 instead. To show that one can naturally extend Φ to the imaginary axis we have

$$\begin{aligned}
|\Phi(u+iv) - \Phi(iv)| &= \lim_{\epsilon\to 0^+} \left| \int_0^1 t^{u+iv}\mathrm{d}z(t) - \int_\epsilon^1 t^{iv}\mathrm{d}z(t) \right| \\
&= \lim_{\epsilon\to 0^+} \left| \int_0^\epsilon t^{u+iv}\mathrm{d}z(t) + \int_\epsilon^1 (t^u - 1)t^{iv}\mathrm{d}z(t) \right| \\
&\le \lim_{\epsilon\to 0^+} \left| \int_0^\epsilon t^{u}\mathrm{d}z(t) \right| + \int_\epsilon^1 (1-t^u)\mathrm{d}z(t) \\
&\le \lim_{\epsilon\to 0^+} \epsilon\cdot 1\cdot (z(\epsilon) - z(0)) + \int_\epsilon^1 (1-t^u)\mathrm{d}z(t),
\end{aligned}$$

which does not rely on v. As $u \to 0^+$ we see that the right-hand side approaches zero as $z(t)$ is bounded.

Having gained more insight into the function $\Phi(s)$, we can now see where the first condition of (9) comes into play.

Corollary 2.6.1 *The following two are equivalent*

(i) $\Phi(0) = 1$*;*

(ii) $z(t)$ *is continuous at* $t = 0$*.*

Proof: Taking $t = 0$ in definition (10) we see that

$$\Phi(0) = \lim_{\epsilon\to 0^+} \int_\epsilon^1 \mathrm{d}z(t) = \lim_{\epsilon\to 0^+} z(1) - z(\epsilon) = 1 - \lim_{\epsilon\to 0^+} z(\epsilon)$$

so that $\Phi(0) = 1$ implies

$$\lim_{\epsilon\to 0^+} z(\epsilon) = 0,$$

i.e., $z(t)$ is continuous at $t = 0$. □

Now as

$$f(x) = 1 + \sum_{k=1} \mu_k x^k = 1 + \sum_{k=1}^{\infty} \Phi(k) x^k \quad \text{for } |x| < 1,$$

we wish to connect the continuity result of Lemma 2.6 to our function Φ. However to apply Lemma 2.6, we require that $x > 1$, so we need to extend the representation of f as a function of Φ. To this end we will use a result stated in full at the start of this section as Theorem 2.1, one which implies that

$$f(x) = 1 + \sum_{k=1}^{\infty} \Phi(k) x^k = 1 + \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{\Phi(z)x^z}{1-\exp 2\pi i z}\mathrm{d}z \quad \text{for } 0 < \alpha < 1$$

for $x \neq 0$ with the argument of x so that $0 < \arg x < 2\pi$, as then we can properly denote

$$x^z = \exp z \log x = \exp z(\log |x| + i \arg x).$$

Importantly, we may now apply the conditions on f for the continuity of $z(t)$ to our function Φ, without having to deal with a Riemann-Stieltjes integral where we integrate with respect to an unknown distribution function $z(t)$.

As we have shown that $\Phi(iv)$ is well-defined, we may take the path of integration along the imaginary axis, i.e., let α approach 0, except for the pole at the origin. Hence we will let

$$f(x) = 1 + \int_C \frac{\Phi(z)x^z}{1 - \exp 2\pi i z} \mathrm{d}z,$$

where C denotes the path along the imaginary axis but with a semi-circle centered at the origin, with radius $0 < \chi < 1$. In the future we will denote the path along only the semi-circle with S.

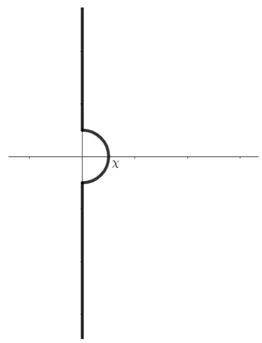

Figure 2: Path of integration

Now in the final step we will adapt the condition stated in Lemma 2.6:

$$\lim_{x \to r} (x - r) f(x) = 0.$$

Let $x = re^{i\theta}$ with r fixed in $[1, \infty)$ and $0 < \theta < 2\pi$. By l'Hôpital's rule

$$\lim_{\theta \to 0^+} \frac{|x - r|}{\theta} = r \lim_{\theta \to 0^+} \frac{|e^{i\theta} - 1|}{\theta} = r \lim_{\theta \to 0^+} \frac{|ie^{i\theta}|}{1} = r$$

Then we have

$$\begin{aligned} 0 &= \lim_{x \to r} (x - r) f(x) \\ &= \lim_{\theta \to 0^+} (re^{i\theta} - r) f(re^{i\theta}) \\ &= r \lim_{\theta \to 0^+} \frac{|e^{i\theta} - 1|}{\theta} \cdot \theta f(re^{i\theta}) \\ &= \lim_{\theta \to 0^+} \theta f(re^{i\theta}) \\ &= \lim_{\theta \to 0^+} \theta + \theta \int_{-i\infty}^{-i\chi} \frac{\Phi(z) e^{z(\log r + i\theta)}}{1 - \exp 2\pi i z} \mathrm{d}z + \theta \int_S \frac{\Phi(z) e^{z(\log r + i\theta)}}{1 - \exp 2\pi i z} \mathrm{d}z + \theta \int_{i\chi}^{i\infty} \frac{\Phi(z) e^{z(\log r + i\theta)}}{1 - \exp 2\pi i z} \mathrm{d}z. \end{aligned} \tag{11}$$

As it turns out, we can approximate these first two integrals. Recall that $|\Phi(s)| < 1$. For the first:

$$\left|\int_{-i\infty}^{-i\chi} \frac{\Phi(z)e^{z(\log r+i\theta)}}{1-\exp(2\pi iz)}\mathrm{d}z\right| = \left|\int_{-\infty}^{-\chi} \frac{\Phi(iv)e^{iv(\log r+i\theta)}}{1-\exp(-2\pi v)}\mathrm{d}v\right| \le \int_{\chi}^{\infty} \frac{\left|e^{-iv\log r}\cdot e^{v\theta}\right|}{\exp(2\pi v)-1}\mathrm{d}v$$
$$\le \int_{\chi}^{\infty} \frac{e^{v\theta}}{\exp(2\pi v)-1}\mathrm{d}v < \int_{\chi}^{\infty} \frac{e^{v\pi/2}}{\exp(2\pi v)-1}\mathrm{d}v,$$

which is independent of θ, hence its limit in (11) will be 0. For the second, note that as z follows the path around a semi-circle of radius χ, that

$$|x^z| = e^{|\log|x|+i\theta|\cdot|z|} \le e^{\chi\cdot(r+\theta)},$$

so that the second integral can be approximated as such:

$$\left|\int_S \frac{\Phi(z)e^{z(\log r+i\theta)}}{1-\exp(2\pi iz)}\mathrm{d}z\right| \le \int_S \frac{e^{\chi(\log r+\theta)}}{|1-\exp(2\pi iz)|}\mathrm{d}z < \int_S \frac{e^{\chi(\log r+\pi/2)}}{|1-\exp(2\pi iz)|}\mathrm{d}z,$$

again independent of θ. These two approximations imply that the condition of (11) in fact reduces to

$$0 = \lim_{\theta\to 0^+} \theta \int_{i\chi}^{i\infty} \frac{\Phi(z)e^{z(\log r+i\theta)}}{1-\exp(2\pi iz)}\mathrm{d}z = \lim_{\theta\to 0^+} \theta \int_{\chi}^{\infty} \frac{\Phi(i\lambda)e^{i\lambda\log r-\lambda\theta}}{1-\exp(-2\pi\lambda)}\mathrm{d}\lambda.$$

To clean up this equation we will apply a temporary substitution $\lambda = vk$, where $k = \theta^{-1}$:

$$\left|\theta\int_{\chi}^{\infty} \frac{\Phi(i\lambda)e^{i\lambda\log r-\lambda\theta)}}{1-\exp(-2\pi\lambda)}\mathrm{d}\lambda\right| \le \int_{\chi/k}^{\infty} \frac{|\Phi(ivk)|\,e^{-v}}{1-\exp(-2\pi vk)}\mathrm{d}v \le \frac{1}{1-\exp(-2\pi\chi)}\int_{\chi/k}^{\infty} |\Phi(ivk)|\,e^{-v}\mathrm{d}v$$
$$\le \frac{1}{1-\exp(-2\pi\chi)}\int_0^{\infty} |\Phi(ivk)|e^{-v}\mathrm{d}v.$$

Hence the condition now becomes

$$\lim_{k\to\infty} \int_0^{\infty} |\Phi(ivk)|e^{-v}\mathrm{d}v = 0.$$

Notice that the integrand is non-negative. Moreover, as $|\Phi(iv)| \le 1$ and as $e^{-v} > 0$, the condition that this limit approaches zero wholly depends on k, hence for any fixed $m > 0$ the following is sufficient

$$\lim_{k\to\infty} \int_0^{m} |\Phi(ivk)|\mathrm{d}v = 0.$$

Now we reverse the previous substitution $v = \lambda k^{-1}$ so

$$\lim_{k\to\infty} \frac{1}{k}\int_0^{km} |\Phi(i\lambda)|\mathrm{d}\lambda = \lim_{k\to\infty} \frac{1}{km}\int_0^{km} |\Phi(i\lambda)|\mathrm{d}\lambda = 0.$$

Now with one final substitution $x = km$ the second condition of Theorem 2.5 too has been proven:

$$\lim_{x\to\infty} \frac{1}{x}\int_0^{x} |\Phi(i\lambda)|\mathrm{d}\lambda = 0.$$

$\square$

2.2 Davenport's example function

Davenport applied Schoenberg's Third Theorem to the ratio of n to its sum of divisors. In order to prove the continuity of the distribution function of the values $g(n) = \frac{n}{\sigma(n)}$, we must study

$$\Phi(s) \coloneqq \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \left(\frac{m}{\sigma(m)}\right)^s. \tag{12}$$

To evaluate this function we will use a product representation of $\Phi(s)$, which we will establish now. Let

$$\varrho_s(m) \coloneqq \sum_{d|m} \mu\left(\frac{m}{d}\right)\left(\frac{d}{\sigma(d)}\right)^s.$$

Then by the Möbius inversion formula we also have that

$$\left(\frac{m}{\sigma(m)}\right)^s = \sum_{d|m} \varrho_s(d)$$

and

$$\frac{1}{n}\sum_{m=1}^{n}\left(\frac{m}{\sigma(m)}\right)^s = \frac{1}{n}\sum_{m=1}^{n} \varrho_s(m)\left\lfloor\frac{n}{m}\right\rfloor. \tag{13}$$

The reason that ϱ is more convenient is because of its representation as a product:

$$\varrho_s(m) = \prod_{p^k\|m}\left(\left(\frac{1-p^{-1}}{1-p^{-k-1}}\right)^s - \left(\frac{1-p^{-1}}{1-p^{-k}}\right)^s\right) = \prod_{p^k\|m}\left(\left(\frac{1-p^{-k-1}}{1-p^{-1}}\right)^{-s} - \left(\frac{1-p^{-k}}{1-p^{-1}}\right)^{-s}\right). \tag{14}$$

This representation follows from the representation as a sum because of the following. Let $m = p_1^{r_1}p_2^{r_2}\dots p_k^{r_k}$ be the prime factorization of m. Consider the quantity $\mu\left(\frac{m}{d}\right)$. This is equal to zero if and only if $\frac{m}{d}$ is divisible by a square. This means that to contribute to the sum, $\frac{m}{d}$ must have prime factorization $\frac{m}{d} = p_1^{s_1}p_2^{s_2}\dots p_k^{s_k}$, where each s_i must equal either 0 or 1, or $s_i \in \{0,1\}$. In short we will denote $\overrightarrow{s} \coloneqq (s_1, s_2, \dots, s_k)$. This means that we can start rewriting ϱ_s using this fact about $\frac{m}{d}$ and the fact that $\mu(n), \sigma(n)$ are multiplicative. Then

$$\begin{aligned}\varrho_s(m) &= \sum_{d|m}\mu\left(\frac{m}{d}\right)\left(\frac{d}{\sigma(d)}\right)^s \\ &= \sum_{\overrightarrow{s}\in\{0,1\}^k}\prod_{i=1}^{k}\left\{\mu\left(p_i^{s_i}\right)\left(\frac{p_i^{r_i-s_i}}{\sigma(p_i^{r_i-s_i})}\right)^s\right\} \\ &= \prod_{i=1}^{k}\left\{\left(\frac{p_i^{r_i}}{\sigma\left(p_i^{r_i}\right)}\right)^s - \left(\frac{p_i^{r_i-1}}{\sigma\left(p_i^{r_i-1}\right)}\right)^s\right\}.\end{aligned}$$

This is now equal to (14) because

$$\frac{p^k}{\sigma(p^k)} = \frac{p^k}{\frac{p^{k+1}-1}{p-1}} = \frac{p-1}{p-p^{-k}} = \frac{1-p^{-1}}{1-p^{-k-1}}.$$

To show the limit in (12) equals the product representation we are working towards, we will analyze the product above. Note that for any $1 \leq v < u < 2$ we have that

$$\begin{aligned}|u^{-s} - v^{-s}| &= \left| s \int_v^u x^{-s-1} \mathrm{d}x \right| \\ &\leq |s|(u-v) \cdot \max\left(u^{-\sigma-1}, v^{-\sigma-1}\right) \qquad \sigma = \mathrm{Re}\ s \\ &\leq |s|(u-v)2^{|\sigma|}.\end{aligned}$$

In the product our u, v are dependent only on $p_i^{r_i}$, hence

$$u - v = \left| \left(\frac{1-p^{-k-1}}{1-p^{-1}} \right) - \left(\frac{1-p^{-k}}{1-p^{-1}} \right) \right| = \left| \frac{p^{-k} - p^{-k-1}}{1-p^{-1}} \right| = p^{-k}.$$

Now we can start estimating ϱ_s:

$$\begin{aligned}|\varrho_s(m)| &= \left| \prod_{p^k \| m} \left(\left(\frac{1-p^{-k-1}}{1-p^{-1}} \right)^{-s} - \left(\frac{1-p^{-k}}{1-p^{-1}} \right)^{-s} \right) \right| \\ &= \prod_{p^k \| m} \left| \left(\frac{1-p^{-k-1}}{1-p^{-1}} \right)^{-s} - \left(\frac{1-p^{-k}}{1-p^{-1}} \right)^{-s} \right| \\ &\leq \prod_{p^k \| m} |s| 2^{|\sigma|} p^{-k} \leq C_s^{\Theta(m)} m^{-1},\end{aligned}$$

where C_s depends only on s and $\Theta(m)$ equals the number of prime divisors of m. Due to Ramanujan [13, 16, p.83] it is known that

$$\Theta(m) = \mathcal{O}\left(\frac{\log m}{\log \log m} \right).$$

Then, for $|s|$ bounded, we see that

$$\varrho_s(m) = \mathcal{O}\left(m^{-1+\epsilon}\right).$$

Then from 13 we obtain

$$\begin{aligned}\frac{1}{n} \sum_{m=1}^{n} \left(\frac{m}{\sigma(m)} \right)^s &= \frac{1}{n} \sum_{m=1}^{n} \varrho_s(m) \frac{n}{m} - \frac{1}{n} \sum_{m=1}^{n} \varrho_s(m) \left(\frac{n}{m} - \left\lfloor \frac{n}{m} \right\rfloor \right) \\ &= \left(\sum_{m=1}^{\infty} \varrho_s(m) \frac{1}{m} - \sum_{m=n+1}^{\infty} \varrho_s(m) \frac{1}{m} \right) + \mathcal{O}\left(\frac{1}{n} \sum_{m=1}^{n} |\varrho_s(m)| \right) \\ &= \sum_{m=1}^{\infty} \frac{\varrho_s(m)}{m} + \mathcal{O}\left(\sum_{m=n+1}^{\infty} \frac{|\varrho_s(m)|}{m} \right) + \mathcal{O}\left(\frac{1}{n} \sum_{m=1}^{n} |\varrho_s(m)| \right).\end{aligned}$$

Now for bounded $|s|$ we see that as n approaches ∞, the two error terms approach zero. Hence the limit in 12 exists and equals

$$\begin{aligned}
\Phi(s) &= \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \left(\frac{m}{\sigma(m)} \right)^{s} \\
&= \sum_{m=1}^{\infty} \frac{\varrho_s(m)}{m} \\
&= \sum_{m=1}^{\infty} \frac{1}{m} \prod_{p^k \| m} \left(\left(\frac{1-p^{-k-1}}{1-p^{-1}} \right)^{-s} - \left(\frac{1-p^{-k}}{1-p^{-1}} \right)^{-s} \right) \\
&= \sum_{m=1}^{\infty} \prod_{p^k \| m} p^{-k} \left(\left(\frac{1-p^{-k-1}}{1-p^{-1}} \right)^{-s} - \left(\frac{1-p^{-k}}{1-p^{-1}} \right)^{-s} \right) \\
&= \prod_{p} \left\{ \sum_{k=0}^{\infty} p^{-k} \left(\left(\frac{1-p^{-k-1}}{1-p^{-1}} \right)^{-s} - \left(\frac{1-p^{-k}}{1-p^{-1}} \right)^{-s} \right) \right\} \\
&= \prod_{p} \left\{ 1 + \sum_{k=1}^{\infty} p^{-k} \left(\left(\frac{1-p^{-k-1}}{1-p^{-1}} \right)^{-s} - \left(\frac{1-p^{-k}}{1-p^{-1}} \right)^{-s} \right) \right\}.
\end{aligned}$$

We can verify that in fact $\Phi(0) = 1$ and that $|\Phi(s)| \le 1$ for $\sigma \ge 0$.

2.2.1 Applying Schoenberg's Theorem to Davenport's $\Phi(s)$

Theorem 2.5 states that it is sufficient to prove that

$$\Phi(0) = 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{x} \int_0^x |\Phi(i\lambda)| \mathrm{d}\lambda = 0, \tag{15}$$

with Φ as defined in (12). Notice that the second is equivalent to

$$\int_0^x |\Phi(i\lambda)| \mathrm{d}\lambda = o(x).$$

We mention this only as it is easier to state. For the arguments of the product we see that

$$\begin{aligned}
&1 + \sum_{k=1}^{\infty} p^{-k} \left(\left(\frac{1-p^{-k-1}}{1-p^{-1}} \right)^{-it} - \left(\frac{1-p^{-k}}{1-p^{-1}} \right)^{-it} \right) \\
&= 1 + p^{-1} \left(\left(\frac{1-p^{-2}}{1-p^{-1}} \right)^{-it} - 1 \right) + \sum_{k=2}^{\infty} p^{-k} \left(\left(\frac{1-p^{-k-1}}{1-p^{-1}} \right)^{-it} - \left(\frac{1-p^{-k}}{1-p^{-1}} \right)^{-it} \right) \\
&= 1 + p^{-1} \left(\left(\frac{(1-p^{-1})(1+p^{-1})}{1-p^{-1}} \right)^{-it} - 1 \right) + p^{-2} \sum_{k=2}^{\infty} p^{-k+2} \left(\left(\frac{1-p^{-k-1}}{1-p^{-1}} \right)^{-it} - \left(\frac{1-p^{-k}}{1-p^{-1}} \right)^{-it} \right) \\
&= 1 - p^{-1} + p^{-1} \left(1 + p^{-1}\right)^{-it} + \vartheta_p p^{-2},
\end{aligned}$$

where

$$\vartheta_p \coloneqq \sum_{k=0}^{\infty} p^{-k} \left(\left(\frac{1-p^{-k-3}}{1-p^{-1}} \right)^{-it} - \left(\frac{1-p^{-k-2}}{1-p^{-1}} \right)^{-it} \right).$$

For $p \geq 5$, ϑ_p satisfies the upper bound

$$\begin{aligned}|\vartheta_p| &\leq \sum_{k=0}^{\infty} p^{-k} \left| \left(\frac{1-p^{-k-3}}{1-p^{-1}} \right)^{-it} - \left(\frac{1-p^{-k-2}}{1-p^{-1}} \right)^{-it} \right| \\ &\leq \sum_{k=0}^{\infty} p^{-k} \cdot 2 = 2 \cdot \frac{1}{1-p^{-1}} \leq \frac{5}{2},\end{aligned}$$

and as such

$$|\vartheta_p p^{-2}| \leq \frac{1}{10}.$$

To prove 15 we require the following steps.

$$\begin{aligned}|\Phi(it)| &= \prod_p \left| 1 - p^{-1} + p^{-1} \exp\left(-it \log \frac{p+1}{p} \right) + \vartheta_p p^{-2} \right|, \\ \log|\Phi(it)| &= \sum_p \log \left| 1 - p^{-1} + p^{-1} \exp\left(-it \log \frac{p+1}{p} \right) + \vartheta_p p^{-2} \right| \\ &= \operatorname{Re} \sum_p \log \left(1 - p^{-1} + p^{-1} \exp\left(it \log \frac{p+1}{p} \right) + \vartheta_p p^{-2} \right).\end{aligned}$$

Let $z := p^{-1} - p^{-1} \exp\left(it \log \frac{p+1}{p} \right) - \vartheta_p p^{-2}$. If $|z| < 1$, then we know that $\log(1-z) = -z + \mathcal{O}\left(z^2\right).$ Furthermore, for $|z| \leq \frac{2}{3}$ the implied constant is less than 2 in absolute value. This in fact holds for $p \geq 5$, as by the bounds found above we have

$$|z| \leq |p^{-1}| + |p^{-1}| + |\vartheta_p p^{-2}| \leq \frac{1}{5} + \frac{1}{5} + \frac{1}{10} \leq \frac{2}{3}.$$

Now we can further work out the integrand

$$\begin{aligned}\log|\Phi(it)| =& \operatorname{Re} \sum_p \log \left(1 - p^{-1} + p^{-1} \exp\left(it \log \frac{p+1}{p} \right) + \vartheta_p p^{-2} \right) \\ =& \log \left| \frac{1}{2} + \frac{1}{2} \exp\left(it \log \frac{3}{2} \right) + \frac{\vartheta_2}{4} \right| + \log \left| \frac{2}{3} + \frac{1}{3} \exp\left(it \log \frac{4}{3} \right) + \frac{\vartheta_3}{9} \right| + \operatorname{Re} \sum_{p \geq 5} \log(1-z) \\ =& C_t - \operatorname{Re} \sum_{p \geq 5} \left(z + \mathcal{O}\left(z^2\right) \right) \\ =& C_t - \operatorname{Re} \sum_{p \geq 5} \frac{1 - \exp\left(it \log \frac{p+1}{p} \right)}{p} - \operatorname{Re} \sum_{p \geq 5} \vartheta_p p^{-2} + \operatorname{Re} \sum_{p \geq 5} \mathcal{O}\left(p^{-2}\right).\end{aligned}$$

As we know that $|\vartheta_p| \leq \frac{5}{2}$ for $p \geq 5$, that the implied constant of the last asymptotic is less than 2, that $\sum_p p^{-2}$ is bounded, and that $|C_t|$ is in fact bounded, we in fact have

$$\begin{aligned}\log|\Phi(it)| &= -\sum_{p \geq 5} \frac{1 - \cos\left(t \log \frac{p+1}{p} \right)}{p} + \mathcal{O}(1) \\ &= -\sum_{p \geq 5} \frac{1}{p} \left(2 \sin^2 \left(\frac{t}{2} \log \frac{p+1}{p} \right) \right) + \mathcal{O}(1).\end{aligned}$$

As we wish to show the integral in (15) is $o(x)$, all constant multiples do not matter. Hence for the integral it remains to show that the following is $o(x)$

$$\begin{aligned}
\int_0^x |\Phi(it)|\mathrm{d}t &= \int_0^x \exp\log|\Phi(it)|\mathrm{d}t \\
&= \int_0^x \exp\left(-2\sum_{p\geq 5}\frac{1}{p}\sin^2\left(\frac{t}{2}\log\frac{p+1}{p}\right)\right)\mathrm{d}t \\
&= \int_0^x \exp\left(-2\sum_{p}\frac{1}{p}\sin^2\left(\lambda\log\frac{p+1}{p}\right)\right)\mathrm{d}\lambda,
\end{aligned}$$

or equivalently that

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x \exp\left(-\sum_{p}\frac{2}{p}\sin^2\left(\lambda\log\frac{p+1}{p}\right)\right)\mathrm{d}\lambda = 0. \tag{16}$$

A careful reader comparing the original paper with this exposition may notice that in Davenport's paper he omitted the 2 in front of the sum. We chose to keep it in as it doesn't affect the result. For the final step we will make use of the following lemma:

Lemma 2.7 *The real numbers* $\log\frac{3}{2}, \log\frac{4}{3}, \log\frac{6}{5}, \ldots, \log\frac{p+1}{p}, \ldots$ *are linearly independent over* $\mathbb{Q}$*. That is to say, any equation*

$$\sum c_p \log\frac{p+1}{p} = 0$$

with $c_p \in \mathbb{Q}$ *and where the* p *are any finite collection of prime numbers, is only possible if* $c_p = 0$ *for all* p*.*

Proof: As the p are any finite collection of prime numbers, we may assume $c_p \neq 0$ for all c_p. Then we rewrite the equation as such:

$$\begin{aligned}
\exp\left(\sum c_p\log\frac{p+1}{p}\right) &= 1 \\
\prod\left(\frac{p+1}{p}\right)^{c_p} &= 1 \\
\prod (p+1)^{c_p} &= \prod p^{c_p}.
\end{aligned}$$

Let q be the largest prime number in the collection of primes. Then q divides the right-hand side of the equation. This q however must divide the left-hand side too, which is only possible if $q = 3$. However then we would have the equality

$$4^{c_3}3^{c_2} = 3^{c_3}2^{c_2}$$

which obviously cannot hold. □

Now to prove (16) we will do so with the following two claims. Let $\psi_P(\lambda) := \exp\left(-\sum_{p\leq P}\frac{2}{p}\sin^2\left(\lambda\log\frac{p+1}{p}\right)\right)$. Note that the integrand of (16) is upper bounded by $\psi_P(\lambda)$. Then

(*i*) The limit

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x \psi_P(\lambda)\mathrm{d}\lambda =: A^{(P)}$$

exists for every prime P.

(*ii*) We have that $A^{(P)} \to 0$ as $P \to \infty$.

To prove (*i*) we will look at the Fourier series of our functions. Our function is even, so we have that

$$\exp\left(-\frac{2}{p}\sin^2 z\right) = \sum_{-\infty}^{\infty} a_n^{(p)} e^{izn} = a_0^{(p)} + \sum_{n=1}^{\infty} a_n^{(p)} \left(e^{izn} + e^{-izn}\right)$$

where $z = \lambda \log \frac{p+1}{p}$ and $\sum_{-\infty}^{\infty} |a_n^{(p)}| < \infty$ as the second derivative of our function is continuous. Then

$$\begin{aligned}\psi_P(\lambda) &= \prod_{p \le P} \left(a_0^{(p)} + \sum_{n=1}^{\infty} a_n^{(p)} \left(e^{izn} + e^{-izn}\right)\right) \\ &= \prod_{p \le P} a_0^{(p)} + \sum_{k=1}^{\infty} A_k e^{ib_k \lambda},\end{aligned}$$

where the b_k are sums of the quantities $n \log \frac{p+1}{p}$. By Lemma 2.7 these are all non-zero. This is important because for any non-zero b we have that

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x} \int_0^x e^{ib\lambda} \mathrm{d}\lambda &= \lim_{x\to\infty} \frac{1}{x} \left[\frac{e^{ib\lambda}}{ib}\right]_{\lambda=0}^{\lambda=x} \\ &= \lim_{x\to\infty} \frac{1}{x} \frac{e^{ibx} - 1}{ib} = 0,\end{aligned}$$

hence

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x \sum_{k=1}^{\infty} A_k e^{ib_k\lambda} \mathrm{d}\lambda = 0.$$

This means for (*i*) that the limit indeed exists:

$$\begin{aligned}A^{(P)} &= \lim_{x\to\infty} \frac{1}{x} \int_0^x \psi_P(\lambda) \mathrm{d}\lambda \\ &= \lim_{x\to\infty} \frac{1}{x} \int_0^x \prod_{p \le P} a_0^{(p)} + \sum_{k=1}^{\infty} A_k e^{ib_k\lambda} \mathrm{d}\lambda \\ &= \prod_{p \le P} a_0^{(p)}.\end{aligned}$$

Now to prove (*ii*) we must show that the product of $A^{(P)}$ diverges to zero as P approaches ∞. To this end, as

$$a_0^{(p)} = \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{2}{p}\sin^2 z} \mathrm{d}z \le \frac{1}{2\pi} \cdot 2\pi \cdot 1 = 1,$$

it suffices to show that

$$\sum_p \left(1 - a_0^{(p)}\right)$$

diverges. We will apply that for $0 \leq x \leq 1$ we have that $1 - e^{-x} \geq \frac{x}{2}$.

$$\begin{aligned}
\sum_{p\leq P}\left(1 - a_0^{(p)}\right) &= \sum_{p\leq P}\left(1 - \frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{2}{p}\sin^2 z}\mathrm{d}z\right) \\
&= \frac{1}{2\pi}\sum_{p\leq P}\int_0^{2\pi}\left(1 - e^{-\frac{2}{p}\sin^2 z}\right)\mathrm{d}z \\
&\geq \frac{1}{2\pi}\sum_{p\leq P}\int_0^{2\pi}\frac{2\sin^2 z}{2p}\mathrm{d}z \geq \frac{1}{2\pi}\int_0^{2\pi}\sin^2 z\mathrm{d}z \cdot \sum_{p\leq P}\frac{1}{p}
\end{aligned}$$

which diverges as $P \to \infty$. Thus we have indeed proven that

$$0 \leq \lim_{x\to\infty}\frac{1}{x}\int_0^x |\Phi(it)|\mathrm{d}t \leq \lim_{P\to\infty}\lim_{x\to\infty}\frac{1}{x}\int_0^x \psi_p(\lambda)\mathrm{d}\lambda = 0,$$

implying that asymptotically the quantities $\frac{n}{\sigma(n)}$ are continuously distributed along $(0,1]$. □

2.3 Extending Davenport's result

Instead of stating that a certain sequence of numbers attains a continuous distribution, Jennings, Pollack and Thompson, hereafter JPT, [10] reformulated the result stating that the limit

$$D(u) := \lim_{x\to\infty}\frac{1}{x}\sum_{\substack{n\leq x\\ n/\sigma(n)\leq u}} 1 \tag{17}$$

exists for $u \in [0,1]$ and is continuous in terms of u. This reformulation showed the possibility of further generalizing this result by replacing the summand with another multiplicative function $f(n)$. The first theorem is as follows.

Theorem 2.8 (JPT) *Let f be a multiplicative function so that*

$$\limsup_{x\to\infty}\frac{1}{x}\sum_{n\leq x}|f(n)|^2 < \infty. \tag{18}$$

If one of the following conditions holds

i) for every integer $k \geq 0$,

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x} f(n)\left(\frac{n}{\sigma(n)}\right)^k \qquad \text{exists,} \tag{19}$$

ii) both of the following are bounded

$$\sum_p \frac{|f(p)-1|}{p} \qquad \text{and} \qquad \sum_p\sum_{j\geq 2}\frac{|f(p^j)|}{p^j},$$

iii) if $|f(n)| \leq 1$ and the following converges

$$\sum_p \frac{f(p)-1}{p},$$

then for $u \in [0,1]$*, the following*

$$D_f(u) := \lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\le x\\ n/\sigma(n)\le u}} f(n) \tag{20}$$

exists and $D_f(u)$ *is continuous.*

The reason (18) is required is to be able to apply Cauchy-Schwarz in the proof. For the first condition, the outline of the proof is as follows.
Outline of the proof: The condition in (20) of $n/\sigma(n) \le u$ can be replaced by instead summing with the indicator function $\mathbf{1}_u(x)$, which equals 1 if and only if $x \le u$ and 0 otherwise:

$$D_f(u) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) \cdot \mathbf{1}_u\left(\frac{n}{\sigma(n)}\right).$$

This is nice as one can find a Cauchy sequence of continuous functions of which the indicator function is its limit. As $u \in [0,1]$, then in turn each continuous function may be approximated via the Weierstrass Approximation Theorem by another Cauchy sequence of polynomials in $\frac{n}{\sigma(n)}$. As the $\left(\frac{n}{\sigma(n)}\right)^k$ form a basis for the polynomials in terms of $\frac{n}{\sigma(n)}$, it then follows that all the limits exists, hence the conclusion follows.

This theorem may be applied to many multiplicative functions. With other results like Wirsing's Theorem in mind, JPT looked to extend this result to nonnegative functions f. The reason the theorem above cannot be applied to some functions f is that already for a relatively 'small' function $\tau(n)$, which denotes the number of divisors of n, this result fails to hold. This is because for $k = 0$, we have that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \tau(n) \sim \log x,$$

which is unbounded. Hence came the idea to loosen the condition of being bounded in mean value. Moreover, instead of dividing only by x, they instead divided by x and the average order of the function $f(n)$:

Theorem 2.9 (JPT) *Suppose that* f *is a nonnegative multiplicative function with the property that as* $x \to \infty$*,*

$$\sum_{p\le x} f(p)\frac{\log p}{p} \sim \kappa \log x \tag{21}$$

for some $\kappa > 0$*. Suppose also that* $f(p)$ *is bounded for primes* p *and that*

$$\sum_p \sum_{j\ge 2} \frac{f(p^j)}{p^j} < \infty. \tag{22}$$

If $\kappa \le 1$*, suppose further that*

$$\sum_{p^j\le x} f(p^j) \ll_f \frac{x}{\log x} \quad (\text{for } x \ge 2). \tag{23}$$

Let

$$S(f;x) := \sum_{n\le x} f(n).$$

Then the function

$$\widetilde{D}_f(u) = \lim_{x\to\infty} \frac{1}{S(f;x)} \sum_{\substack{n\le x \\ n/\sigma(n)\le u}} f(n)$$

exists for $u \in [0,1]$, is continuous in terms of u and strictly increasing.

The fact that this distribution function is strictly increasing is a new result and can, with the choice $f(n) = 1$, be applied to extend Davenport's finding that $D(u)$ also is strictly increasing. To familiarise oneself with the proof of Theorem 2.9 we will follow the proof step by step for the example function $f(n) = \tau(n)$. These conditions on $f(p)$ are required to apply a theorem by Wirsing later in the proof.

Theorem 2.10 (JPT, $f(n) = \tau(n)$ fixed) *Let τ be the arithmetic function for the number of divisors of a natural number n, or $\tau \coloneqq \sigma_0$. Then the following*

$$\widetilde{D}_\tau(u) \coloneqq \lim_{x\to\infty} \frac{1}{S(\tau;x)} \sum_{\substack{n\le x \\ n/\sigma(n)\le u}} \tau(n)$$

exists, is continuous and strictly increasing.

For $x \ge 1$, we define

$$F_x(u) = \frac{1}{S(\tau;x)} \sum_{\substack{n\le x \\ \log(n/\sigma(n))\le u}} \tau(n).$$

The introduction of the log into the summation is because it will simplify the characteristic function ϕ_x of F_x, as we will use the characteristic function of $F_x(u)$ to prove both the existence and continuity of the limit. Notice that with this existence we have $\widetilde{D}_\tau(e^u) = \lim_{x\to\infty} F_x(u)$.

2.3.1 Proof of existence

To prove the convergence of the F_x as $x \to \infty$ we will apply Lévy's Convergence Theorem.

Proposition 2.2 (Lévy) *Suppose that $\{F_x\}$ is a collection of distribution functions indexed by real numbers $x \ge 1$. For each $x \ge 1$, let $\phi_x(t)$ be the characteristic function of F_x. Then the following are equivalent.*

i) The F_x converge weakly to a distribution function F, as $x \to \infty$;

ii) As $x \to \infty$, the ϕ_x converge pointwise on all of $\mathbb{R}$ to a function ψ that is continuous at 0.

Then ψ is the characteristic function of F.

To check that our F_x are indeed distribution functions, notice that

$$\lim_{v\to-\infty} F_x(v) = \lim_{v\to-\infty} \frac{1}{S(\tau;x)} \sum_{\substack{n\le x \\ \log(n/\sigma(n))\le v}} \tau(n) = 0$$

and that for $u \geq 0$, the condition in the summation becomes $\frac{n}{\sigma(n)} \leq 1$, which is trivially true as $\sigma(n) \geq n$. Then indeed we see that for $u \geq 0$, $F_x(u) = \frac{1}{S(\tau;x)} \sum_{n \leq x} \tau(n) = 1$. For fixed x it follows that $F_x(u)$ acts as a step function as u increases. Let W_x be the set of all the values attained by $\frac{n}{\sigma(n)}$ for $n \leq x$, and let $I_{x,w}$ be the set of those integers $k \leq x$ such that $\frac{k}{\sigma(k)} = w$, with $w \in W_x$. Then we can see that as u increases, $F_x(u)$ will have a saltus of $\sum_{k \in I_{x,w}} \tau(k)$ when $u = \log w$. Hence the F_x are right-continuous and never decreasing and indeed distribution functions. To compute the characteristic function ϕ_x of F_x, notice that for a fixed x, the F_x correspond to a discrete random variable that take on the value $\log \frac{k}{\sigma(k)}$ with probability

$$\frac{1}{S(\tau;x)} \sum_{\substack{n \leq x \\ n \in I_{x,k/\sigma(k)}}} \tau(n).$$

Hence,

$$\begin{aligned} \phi_x(t) &= \sum_{n \leq x} e^{it(\log n/\sigma(n))} \cdot \frac{1}{S(\tau;x)} \tau(n) \\ &= \frac{1}{S(\tau;x)} \sum_{n \leq x} \tau(n) \left(\frac{n}{\sigma(n)} \right)^{it}. \end{aligned}$$

To evaluate the existence of the limit ψ we will need the following theorem by Wirsing.

Proposition 2.3 (Wirsing) *Suppose that f is a complex-valued multiplicative function with the following properties. As $x \to \infty$, there exists some $\kappa > 0$ such that*

$$\sum_{p \leq x} f(p) \frac{\log p}{p} \sim \kappa \log x.$$

Suppose also that $f(p)$ is bounded and that

$$\sum_{p} \sum_{j \geq 2} \frac{|f(p^j)|}{p^j} < \infty.$$

In the case that $\kappa \leq 1$ it is required that for $x \geq 2$

$$\sum_{p^j \leq x} |f(p^j)| \ll_f \frac{x}{\log x}.$$

Lastly, suppose that

$$\sum_{p} \frac{1}{p} (|f(p)| - \mathrm{Re}(f(p))) < \infty.$$

Then as $x \to \infty$

$$\sum_{n \leq x} f(n) \sim \frac{e^{-\gamma\kappa}}{\Gamma(\kappa)} \frac{x}{\log x} \prod_{p \leq x} \left(\sum_{j=0}^{\infty} \frac{f(p^j)}{p^j} \right).$$

Let us now check the conditions for this proposition with the choice $\tau(n)$, which then also shows that $\tau(n)$ fulfills the conditions of the main theorem. Notice that for $\tau(n)$, we have that $\tau(p^j) = j+1$ for $j \geq 0$. Hence the first condition becomes

$$2 \cdot \sum_{p \leq x} \frac{\log p}{p} \sim 2 \log x,$$

where we obtain $\kappa = 2$ by Mertens' First Theorem. The final condition follows immediately as τ is not only a real, but also a nonnegative function, i.e., $|\tau(p)| = \mathrm{Re}(\tau(p))$. To prove the second condition, we will first compute a closed form of the partial sums of the inner sum. Let $S_{p,n} \coloneqq \sum_{j=2}^{n} \frac{j+1}{p^j}$. Then

$$\begin{aligned}
pS_{p,n} &= \sum_{j=2}^{n} \frac{j+1}{p^{j-1}} = \frac{3}{p} + \sum_{j=2}^{n-1} \frac{j+2}{p^j} \\
pS_{p,n} - S_{p,n} &= \frac{3}{p} - \frac{n+1}{p^n} + \sum_{j=2}^{n-1} \left(\frac{1}{p}\right)^j = \frac{3}{p} - \frac{n+1}{p^n} + \frac{1}{p} \cdot \frac{1-(1/p)^{n-2}}{p-1} \\
S_{p,n} &= (p-1)^{-1} \left(\frac{3}{p} - \frac{n+1}{p^n} + \frac{1}{p} \cdot \frac{1-(1/p)^{n-2}}{p-1} \right) \\
S_p \coloneqq \lim_{n\to\infty} S_{p,n} &= (p-1)^{-1} \left(\frac{3}{p} + \frac{1}{p} \cdot \frac{1}{p-1} \right) = \frac{3p-2}{p(p-1)^2}.
\end{aligned}$$

To show this is bounded, we will compute the double sum not over the primes but over the natural numbers.

$$\begin{aligned}
\sum_p \sum_{j\geq 2} \frac{j+1}{p^j} &= \sum_p S_p \leq \sum_{k\geq 2} S_k = \sum_{k\geq 1} \frac{3k+1}{k^2(k+1)} \\
&= 2 \cdot \sum_{k\geq 1} \frac{k}{k^2(k+1)} + \sum_{k\geq 1} \frac{k+1}{k^2(k+1)} = 2 \sum_{k\geq 1} \frac{1}{k(k+1)} + \sum_{k\geq 1} \frac{1}{k^2} = 2 + \zeta(2) < \infty.
\end{aligned}$$

Having checked all the conditions for 2.3, we thus obtain the following asymptotic formula

$$S(\tau; x) \sim e^{-2\gamma} \frac{x}{\log x} \prod_{p\leq x} \left(\sum_{j=0}^{\infty} \frac{j+1}{p^j} \right).$$

Importantly, we may also use 2.3 for the sum in ϕ_x. This follows for two reasons, that $\left| \left(\frac{n}{\sigma(n)}\right)^{it} \right| = 1$ and

$$\left| \left(\frac{p}{\sigma(p)}\right)^{it} - 1 \right| \leq \left| \exp\left(it \log \frac{p}{p+1} \right) - 1 \right| = \left| \sum_{k=1}^{\infty} \frac{(it \log \frac{p}{p+1})^k}{k!} \right| \leq |t| \cdot \log \frac{p+1}{p} = |t| \int_p^{p+1} x^{-1} \mathrm{d}x \leq \frac{|t|}{p}.$$

This implies that $\tau(p) \left(\frac{p}{\sigma(p)}\right)^{it} = \tau(p) + \mathcal{O}(|t|/p)$. This fact will be used again when verifying the existence of the infinite product that is the characteristic function. For now, Wirsing's formula gives us that

$$\sum_{n\leq x} \tau(n) \left(\frac{n}{\sigma(n)}\right)^{it} \sim e^{-2\gamma} \frac{x}{\log x} \prod_{p\leq x} \left(\sum_{j=0}^{\infty} \frac{j+1}{p^j} \left(\frac{p^j}{\sigma(p^j)}\right)^{it} \right).$$

Combining the asymptotic formulae we see that for t fixed,

$$\phi_x(t) \sim \prod_{p\leq x} \left(\left(\sum_{j=0}^{\infty} \frac{j+1}{p^j} \left(\frac{p^j}{\sigma(p^j)}\right)^{it} \right) \cdot \left(\sum_{j=0}^{\infty} \frac{j+1}{p^j} \right)^{-1} \right).$$

For brevity we will write

$$\alpha_p(t) = \sum_{j=0}^{\infty} \frac{j+1}{p^j} \left(\frac{p^j}{\sigma(p^j)}\right)^{it}, \quad \Delta_p = \sum_{j=0}^{\infty} \frac{j+1}{p^j} \quad \text{and} \quad \eta_p = \sum_{j=2}^{\infty} \frac{j+1}{p^j}.$$

Important facts about these terms are that $\Delta_p = 1 + \frac{2}{p} + \eta_p$ and that one of the conditions of 2.3 was that $\sum_p \eta_p < \infty$, hence $\eta_p \to 0$ as $p \to \infty$. To show that the limit $\phi_x(t)$ exists for fixed t, we need to find estimates for the arguments in the product, i.e., $\alpha_p(t)\Delta_p^{-1}$. As $\tau(p)\left(\frac{p}{\sigma(p)}\right)^{it} = \tau(p) + \mathcal{O}(|t|/p)$, we see that

$$\begin{aligned}\alpha_p(t) &= 1 + \frac{2}{p}\left(\frac{p}{\sigma(p)}\right)^{it} + \mathcal{O}(\eta_p)\\ &= 1 + \frac{2}{p} + \mathcal{O}\left(\frac{|t|}{p^2} + \eta_p\right).\end{aligned}$$

To estimate Δ_p^{-1}, we know that $\frac{2}{p} + \eta_p \to 0$ as $p \to \infty$, hence there exists $p_0 > 0$ such that for all $p > p_0$, $0 \le \Delta_p - 1 \le \frac{1}{2}$. This allows us to find an estimate for Δ_p^{-1} by using that $\frac{1}{z+1} = 1 - z + \mathcal{O}(z^2)$, so

$$\begin{aligned}\Delta_p^{-1} &= 1 - (\Delta_p - 1) + \mathcal{O}((\Delta_p - 1)^2)\\ &= 1 - \frac{2}{p} - \eta_p + \mathcal{O}\left(\frac{1}{p^2} + \eta_p\right)\\ &= 1 - \frac{2}{p} + \mathcal{O}\left(\frac{1}{p^2} + \eta_p\right).\end{aligned}$$

Now we may estimate the arguments of the product for $p > p_0$:

$$\begin{aligned}\alpha_p(t)\Delta_p^{-1} &= \left(1 + \frac{2}{p} + \mathcal{O}\left(\frac{|t|}{p^2} + \eta_p\right)\right)\left(1 - \frac{2}{p} + \mathcal{O}\left(\frac{1}{p^2} + \eta_p\right)\right)\\ &= 1 - \frac{4}{p^2} + \mathcal{O}\left(\frac{|t|}{p^2} + \frac{1}{p^2} + \eta_p\right)\\ &= 1 + \mathcal{O}\left(\frac{|t|+1}{p^2} + \eta_p\right).\end{aligned}$$

As

$$\sum_p \frac{1}{p^2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

we know that the series

$$\sum_{p>p_0} |\alpha_p(t)\Delta_p^{-1} - 1|$$

converges uniformly on any $[-T, T]$. This implies that the corresponding infinite product

$$\prod_{p>p_0} \alpha_p(t)\Delta_p^{-1}$$

converges to a continuous function of t. As extending this product to all primes instead of just those $p > p_0$ only adds a finite number of terms, the limit of ϕ_x as $x \to \infty$ must also be continuous, so that

$$\psi(t) := \prod_p \left(\left(\sum_{j=0}^{\infty} \frac{j+1}{p^j}\left(\frac{p^j}{\sigma(p^j)}\right)^{it}\right) \cdot \left(\sum_{j=0}^{\infty} \frac{j+1}{p^j}\right)^{-1}\right) \tag{24}$$

exists. This result allows us to use 2.2 to show that indeed the F_x converge weakly to a distribution function F whose characteristic function is ψ.

2.3.2 Proof of continuity

To prove the continuity of F, we will do so with another result of Lévy's.

Proposition 2.4 (Lévy) *Suppose that Y is an infinite convolution of purely discontinuous distribution functions $X_1, X_2, \dots$, i.e., $Y = X_1 * X_2 * \cdots$. Let d_k be the maximal jump in probability for each X_k. If $\sum_{k=1}^{\infty}(1-d_k)$ diverges, then the limit distribution is continuous.*

To show that the limit F indeed is an infinite convolution, we will instead show that its characteristic function is an infinite product of characteristic functions. This comes from the fact that for two independent distribution functions X, Y, we have that $\phi_{X*Y} = \phi_X \cdot \phi_Y$. We see from 24 that ψ indeed is an infinite product, so let

$$\phi_{X_p}(t) := \left(\sum_{j=0}^{\infty} \frac{j+1}{p^j}\left(\frac{p^j}{\sigma(p^j)}\right)^{it}\right) \cdot \left(\sum_{j=0}^{\infty} \frac{j+1}{p^j}\right)^{-1}.$$

To prove that F is continuous, we need to determine the maximal jump d_p of the X_p. Let X_p be the discrete random value which takes the value $\log \frac{p^j}{\sigma(p^j)}$ with probability $\frac{1}{\Delta_p} \cdot \frac{\tau(p^j)}{p^j} = \frac{1}{\Delta_p} \cdot \frac{j+1}{p^j}$. Then we see that indeed ϕ_{X_p} is the characteristic function of X_p:

$$\begin{aligned}
\phi_{X_p}(t) &= \sum_{j=0}^{\infty} \exp\left(it \log \frac{p^j}{\sigma(p^j)}\right) \cdot \mathbb{P}\left(X_p = \log \frac{p^j}{\sigma(p^j)}\right) \\
&= \sum_{j=0}^{\infty} \left(\frac{p^j}{\sigma(p^j)}\right)^{it} \frac{j+1}{p^j} \cdot \frac{1}{\Delta_p} = \left(\sum_{j=0}^{\infty} \frac{j+1}{p^j}\left(\frac{p^j}{\sigma(p^j)}\right)^{it}\right) \cdot \left(\sum_{j=0}^{\infty} \frac{j+1}{p^j}\right)^{-1}.
\end{aligned}$$

Now what's left is to determine the maximal jumps for each of the X_p. As the X_p are defined to have probabilities equal to $\frac{1}{\Delta_p} \cdot \frac{j+1}{p^j}$ for $j \geq 0$, we see that for $j = 0$ the probability equals $\frac{1}{\Delta_p}$. As $p \geq 2$, notice that for $j > 0$ we will always have that $\frac{j+1}{p^j} < 1$, hence the maximal jump for each X_p is in fact $d_p = \frac{1}{\Delta_p}$. To prove the infinite sum diverges, it is sufficient to show that the sum diverges for a subset of p. As we have seen before in the proof of existence, $\Delta_p = 1 + \frac{2}{p} + \eta_p$ and both $\frac{2}{p}$ and $\eta_p \to 0$ as $p \to \infty$, so we may choose a $p_0 > 0$ such that for each $p > p_0, \Delta_p < 2$. This means that the sum in the proof can be bounded from below as follows

$$\sum_p (1-d_p) \geq \sum_{p>p_0}(1-d_p) = \sum_{p>p_0} \frac{\Delta_p - 1}{\Delta_p} \geq \frac{1}{2}\sum_{p>p_0}(\Delta_p - 1) \geq \frac{1}{2}\sum_{p>p_0}\frac{2}{p} = \sum_{p>p_0}\frac{1}{p}.$$

As we know that the sum of the reciprocals of primes diverges, the summation above must also diverge. This means that the $\sum_p (1-d_p)$ indeed diverge and thus $\psi(t)$ and F must be continuous.

2.3.3 Proof of strict monotonicity

We have used F to prove existence and continuity, as the log in the summation condition kept the characteristic function less 'messy'. To prove strict monotonicity this is no longer required, thus we will directly show that $\widetilde{D}_f$ is strictly increasing on $[0,1]$. We already know that it is never decreasing as τ is nonnegative.

Thus what remains to show is that for each $u, v \in [0,1]$ with $v < u$ that

$$\widetilde{D}_\tau(u) - \widetilde{D}_\tau(v) \geq \liminf_{x\to\infty} \frac{1}{x\log x} \sum_{\substack{n\leq x \\ v<n/\sigma(n)\leq u}} \tau(n) > 0. \tag{25}$$

To estimate this it will be useful to see that τ is supported on squarefree integers. To observe this we will again need the theorem by Wirsing as stated in 2.3 but applied to the multiplicative function $\tau(n)\cdot\mu^2(n)$, where μ is the Möbius function defined as follows

$$\mu(n) = \begin{cases} 0 & \text{if} \quad n \text{ is not squarefree} \\ (-1)^k & \text{if} \quad n \text{ is squarefree and } k \text{ is the number of prime divisors of } n. \end{cases}$$

To apply 2.3, notice that for every prime p, $\tau(p) = \tau(p)\cdot\mu^2(p) = \tau(p)\cdot 1$. As the requirements for the theorem are only dependent on the behaviour of the function for the primes, we see that we can indeed use the theorem to find an asymptotic formula for summing $\tau(n)\mu^2(n)$:

$$S(\tau\mu^2; x) \sim e^{-2\gamma}\cdot\frac{x}{\log x}\prod_{p\leq x}\left(1+\frac{2}{p}\right).$$

To compare the order of magnitude of τ and $\tau\mu^2$, recall that for $S(\tau;x)$ we have

$$S(\tau;x) \sim e^{-2\gamma}\frac{x}{\log x}\prod_{p\leq x}\left(\sum_{j=0}^{\infty}\frac{j+1}{p^j}\right) = e^{-2\gamma}\frac{x}{\log x}\prod_{p\leq x}\left(1+\frac{2}{p}+\eta_p\right).$$

Hence we see that

$$S(\tau;x) \sim S(\tau\mu^2;x)\prod_{p\leq x}\left(\frac{1+\frac{2}{p}+\eta_p}{1+\frac{2}{p}}\right) = S(\tau\mu^2;x)\prod_{p\leq x}\left(1+\mathcal{O}(\eta_p)\right).$$

As we have already shown that $\sum_p \eta_p < \infty$ we see that the product converges as $x\to\infty$, hence the two asymptotic formulae are of the same order of magnitude. This is useful as the infinite product of the squarefree asymptotic formula does not have an infinite sum. We will now start analyzing the sum in (25). To work with the extra condition under the sum, recall that $|\log\frac{p}{\sigma(p)}| = |\log\frac{p}{p+1}| = \log\frac{p+1}{p} \asymp \frac{1}{p}$. This implies that we may choose a squarefree natural number m such that $v < \frac{m}{\sigma(m)} \leq u$. Because the continuity of the distribution function had already been proven, that would have sufficed to show there indeed exists such an m, however this method allows one to construct such an m when given u and v. Let $y > 0$ to be determined later, but for now large enough such that $p < y$ for each prime p dividing m. As we essentially need to show that the limit in (25) is not zero, it will be sufficient to show this is not the case for just a subset of the $n \leq x$. Namely the n that can be written as the product $n = mq$, where m is as chosen previously, q is squarefree and its prime divisors are larger than y, i.e., $(q, \prod_{p\leq y} p) = 1$, where for brevity we shall from now on use $\Pi_y := \prod_{p\leq y} p$. For this subset of n the contribution to the sum in (25) can be written as

$$\sum_{\substack{q\leq x/m \\ (q,\Pi_y)=1 \\ v<mq/\sigma(mq)\leq u}} \tau(mq) = \tau(m)\sum_{\substack{q\leq x/m \\ (q,\Pi_y)=1 \\ v<mq/\sigma(mq)\leq u}} \tau(q).$$

We wish to analyze these sums by the same asymptotic formulas as above; however, this implies that we need to simplify the conditions under the sum. To this end we will need the following observations. The first is that when rewriting the third condition, we obtain that

$$v\frac{\sigma(m)}{m} < \frac{q}{\sigma(q)} \leq u\frac{\sigma(m)}{m}.$$

As we have chosen m such that $\frac{m}{\sigma(m)} \leq u$, we have that $u\frac{\sigma(m)}{m} \geq 1$. As $\frac{q}{\sigma(q)}$ is always less than 1, we see that the right-hand-side of the inequalities can be dropped. Now we can further rewrite the left-hand inequality to be $0 < 1 - v\frac{\sigma(m)}{m}\frac{\sigma(q)}{q} < 1$. The second observation is that we can in fact take this inequality out of the conditions by introducing it to the argument, i.e., sum the quantity $\tau(q)\left(1 - v\frac{\sigma(m)}{m}\frac{\sigma(q)}{q}\right)$ instead of just $\tau(q)$. Due to the nature of the original inequality, integers q such that the conditions were met now, have their contribution to the sum diminished by this new factor. On the other hand, integers q such that the conditions were not met, are now in fact subtracted from the sum. Hence we have that

$$\sum_{\substack{q \leq x/m \\ (q,\Pi_y)=1 \\ v<mq/\sigma(mq)\leq u}} \tau(mq) = \tau(m) \sum_{\substack{q\leq x/m \\ (q,\Pi_y)=1 \\ v\sigma(m)/m<q/\sigma(q)}} \tau(q)$$
$$\geq \tau(m) \sum_{\substack{q\leq x/m \\ (q,\Pi_y)=1}} \tau(q)\left(1 - v\frac{\sigma(m)}{m}\frac{\sigma(q)}{q}\right).$$

To finally remove the condition for q to be coprime to Π_y, we will use an indicator function. Let $\mathbf{1}_y$ be defined such that

$$\mathbf{1}_y(n) = \begin{cases} 1 & \text{if } (n,\Pi_y) = 1 \\ 0 & \text{if } (n,\Pi_y) > 1. \end{cases}$$

To further simplify the sum above, we define $a_y(n) = \tau(n)\mathbf{1}_y(n)$ and $b_y(n) = \tau(n)\frac{\sigma(n)}{n}\mathbf{1}_y(n)$. This shortens said sum to be

$$S(a_y; x/m) - v\frac{\sigma(m)}{m}S(b_y; x/m).$$

Now that as we have rewritten the entire sum in terms of $S(-;x/m)$ we see again that we may use Wirsing's theorem to asymptotically estimate the $S(-;x/m)$ as these modified formulas still satisfy the conditions for 2.3. An important note beforehand is that we may modify the asymptotic formula for $S(\tau;x)$; instead of taking the product over all primes up to x, we may take the product over all primes up to x/m as the term $1+\frac{2}{p}$ is bounded. This means for $S(\tau;x)$ we have the following asymptotic formula

$$S(\tau;x) \sim e^{-2\gamma}\frac{x}{\log x}\prod_{p\leq x/m}\left(1+\frac{2}{p}\right).$$

Likewise, for a_y we have

$$S(a_y;x/m) \sim e^{-2\gamma}\frac{x/m}{\log x/m}\prod_{p\leq x/m}\left(1+\frac{2}{p}\mathbf{1}_y(p)\right)$$
$$\sim \frac{e^{-2\gamma}}{m}\frac{x}{\log x}\prod_{y<p\leq x/m}\left(1+\frac{2}{p}\right).$$

Combining these two we see that

$$S(a_y;x/m) \sim S(\tau;x)\frac{1}{m}\prod_{p\leq y}\left(1+\frac{2}{p}\right)^{-1}.$$

For b_y we see that

$$S(b_y; x/m) \sim e^{-2\gamma}\frac{x/m}{\log x/m}\prod_{p\le x/m}\left(1+\frac{2}{p}\cdot\frac{\sigma(p)}{p}\mathbf{1}_y(p)\right)$$
$$\sim \frac{e^{-2\gamma}}{m}\frac{x}{\log x}\prod_{y<p\le x/m}\left(1+\frac{2}{p}\cdot\frac{p+1}{p}\right).$$

Again, comparing this to the formula for τ we see

$$S(b_y; x/m) \sim S(\tau; x)\frac{1}{m}\prod_{p\le x/m}\left(1+\frac{2}{p}\right)^{-1}\prod_{y<p\le x/m}\left(1+\frac{2}{p}\left(1+\frac{1}{p}\right)\right)$$
$$\sim S(\tau; x)\frac{1}{m}\prod_{p\le y}\left(1+\frac{2}{p}\right)^{-1}\prod_{y<p\le x/m}\left(1+\frac{2}{p}\right)^{-1}\prod_{y<p\le x/m}\left(1+\frac{2}{p}\left(1+\frac{1}{p}\right)\right)$$
$$\sim S(\tau; x)\frac{1}{m}\prod_{p\le y}\left(1+\frac{2}{p}\right)^{-1}\prod_{y<p\le x/m}\left(1+\frac{2}{p^2+2p}\right).$$

Recall that the aim was to prove that

$$\liminf_{x\to\infty}\frac{1}{S(\tau; x)}\sum_{\substack{n\le x\\ v<n/\sigma(n)\le u}}\tau(n) > 0.$$

And after restricting to summing over an ever smaller subset of n we obtained that

$$\liminf_{x\to\infty}\frac{1}{S(\tau; x)}\sum_{\substack{n\le x\\ v<n/\sigma(n)\le u}}\tau(n) \ge \liminf_{x\to\infty}\frac{1}{S(\tau; x)}\tau(m)\left(S(a_y; x/m) - v\frac{\sigma(m)}{m}S(b_y; x/m)\right).$$

We will now finish the proof by using the asymptotic formulae to show that the right-hand-side indeed is always positive.

$$\liminf_{x\to\infty}\frac{1}{S(\tau; x)}\tau(m)\left(S(a_y; x/m) - v\frac{\sigma(m)}{m}S(b_y; x/m)\right)$$
$$=\tau(m)\liminf_{x\to\infty}\left(\frac{S(a_y; x/m)}{S(\tau; x)} - v\frac{\sigma(m)}{m}\frac{S(b_y; x/m)}{S(\tau; x)}\right)$$
$$=\tau(m)\liminf_{x\to\infty}\left(\frac{1}{m}\prod_{p\le y}\left(1+\frac{2}{p}\right)^{-1} - v\frac{\sigma(m)}{m}\frac{1}{m}\prod_{p\le y}\left(1+\frac{2}{p}\right)^{-1}\prod_{y<p\le x/m}\left(1+\frac{2}{p^2+2p}\right)\right)$$
$$=\frac{\tau(m)}{m}\prod_{p\le y}\left(1+\frac{2}{p}\right)^{-1}\left(1 - v\frac{\sigma(m)}{m}\prod_{y<p}\left(1+\frac{2}{p^2+2p}\right)\right).$$

To prove this is positive, we only need to show that

$$1 - v\frac{\sigma(m)}{m}\prod_{y<p}\left(1+\frac{2}{p^2+2p}\right)$$

is positive. For this final step, recall that the choice of m requires that $v < \frac{m}{\sigma(m)}$ such that $v\frac{\sigma(m)}{m} < 1$. Lastly, we see that for the terms in the product, we have that $1 < 1 + \frac{2}{p^2+2p} \le 1 + \mathcal{O}(\frac{1}{p^2})$. Hence as $y \to \infty$, we see that this product tends to 1 from above. We can now choose y to be large enough such that $v\frac{\sigma(m)}{m}\prod_{y<p}\left(1+\frac{2}{p^2+2p}\right)$ too remains less than 1 so that everything remains positive. Thus, $\widetilde{D}_\tau(u) - \widetilde{D}_\tau(v) > 0$ for all $u, v \in [0, 1]$ with $v < u$. □

2.4 Jennings' further generalization

After the work by JPT to generalize the summand of (17), Jennings [9] showed that one may take this one step further, which is to generalize the condition of the sum. By following Davenport, it had been natural to keep the argument $n/\sigma(n) \le u$; however, Jennings in her work found conditions for a multiplicative function g so that

$$\widetilde{D}_{f,g}(u) := \lim_{x\to\infty} \frac{1}{S(f;x)} \sum_{\substack{n\le x \\ g(n)\le u}} f(n) \tag{26}$$

also exists and is continuous. Her main result is as follows:

Theorem 2.11 (Jennings) *Suppose that $f(n)$ is a nonnegative multiplicative function such that as $x \to \infty$,*

$$\sum_{p\le x} f(p)\frac{\log p}{p} \sim \kappa \log x \tag{27}$$

for some $\kappa > 0$. Suppose also that $f(p)$ is bounded for primes p and that

$$\sum_p \sum_{j\ge 2} \frac{f(p^j)}{p^j} < \infty. \tag{28}$$

If $\kappa \le 1$, suppose further that

$$\sum_{p^j \le x} f(p^j) \ll_f \frac{x}{\log x} \quad \text{(for } x \ge 2). \tag{29}$$

As for $g(n)$, let $g(n)$ be a multiplicative function with image in $(0,1]$ such that for all $j \ge 1$, $g(p^j)$ is bounded away from zero and the series

$$\sum_p \frac{1}{p} ||\log g(p)||, \tag{30}$$

converges, where $||\cdot|| := \min(1, |\cdot|)$. Also suppose that

$$\sum_{\substack{p \\ g(p)\ne 1}} \frac{f(p)}{p} \tag{31}$$

diverges. Then the limit (26) exists for all u in $[0,1]$ is continuous. Let $S := \{n \in \mathbb{N} \mid f(n) > 0\}$. Then $\widetilde{D}_{f,g}(u)$ is strictly increasing on the interior of the closure of $g(S)$.

A difference between this theorem and 2.9 is that Jennings specified exactly where the distribution function would be strictly increasing, whereas 2.9 stated it to be strictly increasing everywhere on $[0,1]$. This is because this is necessarily true for $g(n) = \frac{n}{\sigma(n)}$. As condition (27) requires $\kappa > 0$, we know that there must be infinitely many prime p so that $f(p) \ne 0$, as otherwise the sum would be finite. Moreover by condition (31) $f(p)$ is bounded and $\sum_p \frac{f(p)}{p}$ diverges. Now for $\frac{n}{\sigma(n)}$, we know that $\frac{p}{\sigma(n)} = \frac{p+1}{p} = 1 - \frac{1}{p+1}$. Hence for any neighbourhood $(1-\delta, 1)$, there are infinitely many p with $g(p)$ in that neighbourhood and therefore there cannot be a p which is closest to 1. Hence with this set of p that have $f(p) \ne 0$, we can approximate any value ν in the unit interval arbitrarily well because we can find a sequence of $\{p_i\}_{i=1}^{\infty}$ with $f(p_i) \ne 0$ such that

$$\nu \le \prod_{i=1}^{k} \left(1 - \frac{1}{p_i+1}\right) \quad \text{and} \quad \lim_{k\to\infty} \prod_{i=1}^{k} \left(1 - \frac{1}{p_i+1}\right) = \nu.$$

Hence this condition follows directly for $g(n) = \frac{n}{\sigma(n)}$.

3 Completing the diagram

In this section we discuss some of our findings on if, and if so how, we can extend Jennings' result. As we have seen historically, first the summand of the distribution function

$$D_{f,g}(u) = \lim_{x\to\infty} \frac{1}{S(f;x)} \sum_{\substack{n\le x\\ g(n)\le u}} f(n)$$

was generalized with a fixed g, and later Jennings was able to make both f and g be general functions satisfying a set of conditions. This caught our attention as Schoenberg's necessary and sufficient conditions can be seen as a complete classification for when

$$D_{1,g}(u) = \lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\le x\\ g(n)\le u}} 1$$

exists and is continuous.

3.1 Understanding the conditions on $g(n)$

We now discuss our findings in trying to find the gaps between Schoenberg's necessary and sufficient conditions, and Jennings' sufficient conditions. We attempted to follow the steps of Schoenberg's example but with the function

$$h(n) = \frac{\varphi(n)}{\sigma(n)}. \tag{32}$$

We chose this function for two reasons. The first is that we can use Jennings' conditions to show that this function does in fact admit a limit law. The second is that examples of multiplicative functions which are also bounded away from zero for $g(p^k)$ seem to be limited. Another possible choice could be $\frac{p^{rk}}{\sigma_r(p^k)}$, which will run into the same issues as our $h(n)$. Our choice naturally fulfills condition 1 as both $\varphi(n)$ and $\sigma(n)$ are multiplicative and the fact that $\varphi(n) < n < \sigma(n)$ by definition. We can see it fulfills condition 2 as

$$h(p^k) = p^k \frac{p-1}{p} \frac{p-1}{p^{k+1}-1} = \frac{(1-p^{-1})^2}{1-p^{-k-1}}.$$

For $k = 1$ we see that

$$h(p) = \frac{(1-p^{-1})^2}{1-p^{-2}} = \frac{1-p^{-1}}{1+p^{-1}}$$

never equals 1 for all primes p, hence condition 4 is also fulfilled. Lastly, notice that

$$|\log h(p)| = \left|\log \frac{p-1}{p+1}\right| = \int_{p-1}^{p+1} w^{-1} \mathrm{d}w \asymp \frac{2}{p},$$

so that

$$\sum_p \frac{\min\left(1, |\log h(p)|\right)}{p} \asymp \sum_p \frac{2}{p^2}.$$

Thus to check the conditions by Schoenberg we will follow the same steps as taken by Davenport. Recall that the first big step was to establish a multiplicative representation of $\Phi(s)$ through Möbius inversion, so that

$$\Phi(s) = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} g(m)^s = \sum_{m=1}^{\infty} \frac{\varrho_s(m)}{m} + \lim_{n\to\infty} \mathcal{O}\left(\frac{1}{n} \sum_{m=1}^{n} |\varrho_s(m)| \right) + \mathcal{O}\left(\sum_{m=n+1}^{\infty} \frac{|\varrho_s(m)|}{m} \right).$$

The last error term naturally vanishes as $n \to \infty$, however for the first one we need more careful estimates. Recall that

$$\varrho_s(m) = \prod_{p^k || m} \left\{ h(p^k)^s - h(p^{k-1})^s \right\},$$

where Davenport bounded the argument of the product by

$$\begin{aligned} \left| \left(\frac{1}{h(p^k)} \right)^{-s} - \left(\frac{1}{h(p^{k-1})} \right)^{-s} \right| &= \left| s \int_{h(p^k)^{-1}}^{h(p^{k-1})^{-1}} w^{-s-1} \mathrm{d}w \right| \qquad (33) \\ &\le |s| \cdot |h(p^k)^{-1} - h(p^{k-1})^{-1}| \cdot \max(h(p^k)^{-1}, h(p^{k-1})^{-1})^{-u-1}. \end{aligned}$$

Here

$$|h(p^k)^{-1} - h(p^{k-1})^{-1}| = \left| \frac{p^{-k} - p^{-k-1}}{(1-p^{-1})^2} \right| = p^{-k} \left| \frac{1}{1-p^{-1}} \right| \le 2p^{-k}$$

and

$$h(p^k)^{-1} = \frac{1 - p^{-k-1}}{(1-p^{-1})^2},$$
$$\text{so } 1 \le h(p^k)^{-1} \le 4.$$

Then (33) becomes

$$|h(p^k)^{-1} - h(p^{k-1})^{-1}| \le |s| 2p^{-k} \cdot 4^{|\sigma+1|},$$

and thus

$$|\varrho_s(m)| \le C_s^{\Theta(m)} m^{-1},$$

where the C_s depends solely on s. Together with the assumption that $|s|$ is bounded, we obtain

$$\varrho_s(m) = \mathcal{O}\left(m^{-1+\epsilon}\right),$$

a sufficient estimate so that the error terms vanish. Notice that of course to obtain a sensible upper bound on $h(p^k)^{-1}$ one requires a lower bound on $h(p^k)$ away from zero. We have thus found the product representation of $\Phi(s)$, so the characteristic function of $\log h(n)$ is

$$\Phi(it) = \prod_p \left\{ 1 + \sum_{j\ge1} p^{-j} \left(h(p^j)^{it} - h(p^{j-1})^{it} \right) \right\}.$$

Then to establish whether the distribution is continuous we needed

$$\lim_{x\to\infty} \int_0^x |\Phi(it)| \mathrm{d}t = 0,$$

to which the same estimates Davenport applied lead to the condition that

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x \exp\left(-\sum_p \frac{1-\cos(t\log h(p))}{p}\right) \mathrm{d}t = 0. \tag{34}$$

Here we can see that the continuity of the distribution function of the numbers $h(n)$ does not depend on the values of $h(p^j)$ for $j \geq 2$. However, this step is where problems arise for our function. As $h(p) = \frac{p-1}{p+1}$, showing that the quantities $\log(h(p))$ are linearly independent over $\mathbb{Q}$ is not only hard, it is simply not true. Already for the set of primes $\{2, 3, 5, 7\}$, we see that solving for the linear dependence:

$$\begin{aligned} c_2 \log h(2) + c_3 \log h(3) + c_5 \log h(5) + c_7 \log h(7) = 0 \quad \text{implies} \\ 2^{c_3} 4^{c_5} 6^{c_7} &= 3^{c_2} 4^{c_3} 6^{c_5} 8^{c_7} \\ 2^{c_3+2c_5+c_7} 3^{c_7} &= 2^{2c_3+c_5+3c_7} 3^{c_2+c_5}, \end{aligned}$$

so one obtains a 2-dimensional subspace of solutions of $\mathbb{Z}^4$. Even excluding these primes, one sees that as both $p-1$ and $p+1$ are both even, all the primes dividing these quantities are less than $p/2$. Thus it might be possible to show that from some point forward these quantities will be linearly independent, however that seems unlikely. Hence we see that even though Schoenberg's conditions are complete, they are sadly not easily applicable. Further attempts to show that this integral equals zero have led to nothing. We see that in the exponent, we have an infinite sum of $\frac{1}{p}$ which is offset by $\frac{-\cos(t\log h(p))}{p}$. It seems true that this sum will diverge as for arbitrary prime p, $\cos(t \log h(p))$ should not be close to 1 too often. Another reason to believe this is as follows. Assume that the limit does not equal 0. Then as $t \to \infty$ there must be a large enough/not sparse subset of t for which all of the $\cos(t \log h(p))$ lie arbitrarily close to 1. However as $t \to \infty$ it feels only natural that the cosines behave like independent random variables so that the cosines should 'sync up' to equal 1 only on a set of density 0. By all the reasons above we believe the integrand will approach zero when $t \to \infty$, a result of $x \to \infty$. However quantifying these statement has been fruitless. For the nice example functions regarding $\frac{n}{\sigma(n)}$ and $\frac{\varphi(n)}{n}$ we saw that we could apply linear independence; however this seems to have been a lucky break.

3.2 Conditions for multiplicative $g(n)$

As the example above shows, checking Schoenberg's conditions is far from a trivial task and historically, this had also been noticed. In Davenport's paper, he also listed several conditions which he applied to $\frac{n}{\sigma(n)}$ and were sufficient to obtain his result:

(1) $0 < g(n) \leq 1$,

(2) $g(n)$ is multiplicative,

(3) $|g(p^k) - g(p^{k-1})| \leq Cp^{-kc}$,

(4) There exists a prime p_0 so that for all primes $p > p_0$, the quantities $\log g(p)$ are linearly independent over $\mathbb{Q}$.

Here, (1) comes from Schoenberg's theorem, (2) is necessary to apply Möbius inversion and compute the analytic continuation of the moments, (3) was necessary to show that the error term in $\Phi(s)$ on $\varrho_s(m)$ was $\mathcal{O}\left(m^{-1+\epsilon}\right)$ and (4) had been applied to show that the integral (34) equals zero. Notice that these conditions miss that he needed $g(p)$ to be bounded away from 0.

After Davenport's paper, Schoenberg saw he could improve Davenport's conditions by instead looking at Fourier transforms of the distribution function, more commonly known nowadays as the characteristic function [15]. For multiplicative functions, he established the following sufficient conditions

(1) $g(p^k) > 0$,

(2) $\sum_p \frac{1}{p} \min(1, |\log g(p)|)$ converges,

(3) there exists an infinite increasing sequence of primes $\{p_i\}_{k=1}^{\infty}$ with $g(p_i) \neq g(p_j)$ if $i \neq j$ such that $\sum_{k=1}^{\infty} \frac{1}{p_i}$ diverges.

Then the first two conditions establish the existence of the limiting distribution and the last is required to establish its continuity. One important difference from Davenport is that this no longer requires all $g(p^k)$ to be bounded away from zero, only that the $g(p)$'s which are close to zero are sufficiently sparse so that for those primes, $\sum_{\text{those primes } p} \frac{1}{p} < \infty$. At first glance this seems to be a boundary case between Schoenberg's conditions and Jennings' conditions as Jennings truly needs all the $g(p^k)$ to be bounded away from 0; however this result is only really required to show the function is strictly increasing on a specific subset, on top of the continuity. This result too allows for $g(n)$ to extend beyond the unit disk, as long as again, those $g(p)$ with $g(p) > e$ must be sparse enough that the sum of their reciprocals also converges. Moreover, Schoenberg lets go of any other condition for $g(p^k)$ with $k \geq 2$. Simultaneously to Schoenberg, Erdős [5] realized the following conditions for multiplicative g to attain a continuous distribution

(1) $g(n) \geq 1$,

(2) $\sum_p \frac{1}{p} \min(1, |\log g(p)|)$ converges,

(3) $g(p) \neq g(q)$ for all pairs of primes.

Notice that these results are a bit more limited than Schoenberg's; however Erdős proved this result without any Fourier transformations. The progress for conditions on multiplicative $g(n)$ then concluded with the Erdős-Wintner Theorem:

Theorem 3.1 (Erdős-Wintner, 1939) *Necessary and sufficient conditions for an additive arithmetic function $f(n)$ to attain a limiting distribution is that the following three series converge*

$$\sum_{|f(p)|>1} \frac{1}{p} \tag{35}$$

$$\sum_{|f(p)|\leq 1} \frac{f(p)}{p} \tag{36}$$

$$\sum_{|f(p)|\leq 1} \frac{f(p)^2}{p}. \tag{37}$$

The characteristic function of the limiting distribution will have the following representation

$$\phi(t) = \prod_p (1 - p^{-1}) \sum_{k=0} \frac{e^{itf(p^k)}}{p^k}. \tag{38}$$

The limiting distribution will be continuous if and only if

$$\sum_{f(p)\neq 0} \frac{1}{p} \tag{39}$$

diverges.

This is in fact the theorem on which Jennings' conditions are based. Applying Wirsing's Theorem must be the reason why she lost the "necessary" direction of the "necessary and sufficient" condition. Beforehand it had seemed to us that one should be able to 'complete a commutative diagram', where first generalizing g then f or first generalizing f then g should give new insights to when arithmetic functions attain a limiting distribution. It turns out that, through careful application of the Erdős-Wintner Theorem, Jennings did not generalize g after generalizing f, but instead she generalized both simultaneously as demonstrated in the diagram below. This means that when regarding only multiplicative functions, we will not be able to improve on Jennings' results by completing the diagram.

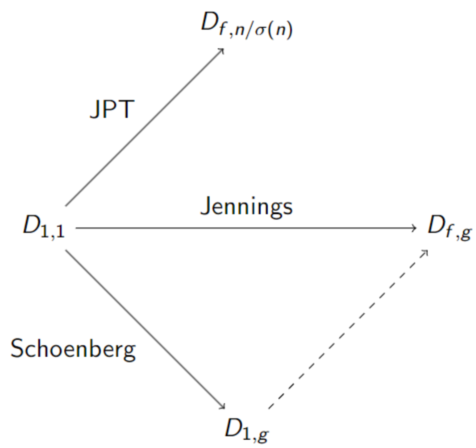

Figure 3: Diagram of results

But, multiplicative functions are uncommon, especially in probability theory, so perhaps we might be able to extend this diagram by instead considering non-multiplicative functions. Schoenberg's Third Theorem could allow for such an extension, as his theorem does not require any multiplicativity. In fact, after more careful inspection of JPT's Theorem 2.8, we noticed two key elements of the proof which allow for extension of the results. The first is that the proof solely uses the fact that the existence and continuity of the limit

$$D(u) = \lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\le x \\ n/\sigma(n)\le u}} 1$$

has been established, without any other knowledge of its nature. The second realization is that in the proof, f being multiplicative is only required for the second and third conditions. These two observations allow for the following theorem, which is a modification of JPT's first theorem.

Theorem 3.2 *Let $f(n)$ be an arithmetic function that is bounded in mean square, i.e.,*

$$\limsup_{x\to\infty} \frac{1}{x} \sum_{n\le x} |f(n)|^2 < \infty.$$

Suppose also that $g(n)$ is an arithmetic function for which the limit

$$\widetilde{D}_{1,g} = \lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\le x \\ g(n)\le u}} 1$$

exists and is continuous, i.e., g attains a continuous distribution. Further suppose that for every nonnegative k, the limiting mean value

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n)\left(g(n)\right)^k$$

exists. Then the limit

$$\widetilde{D}_{f,g} = \lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\le x\\ g(n)\le u}} f(n)$$

also exists and is continuous.

4 Schoenberg and modern Probability Theory

One may observe that Schoenberg's Third Theorem assumes the existence of the moments μ_k, and those familiar with probability theory may recognize that their existence is neither immediate nor implicit, even when given a distribution function. However, in probability theory, for a random variable X the boundedness of the absolute moments does imply the existence of the moments of a distribution, in this case:

$$\begin{aligned}\mathbb{E}|X|^k &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{\infty}x_{jn}^k \leq \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{\infty}1 = 1, \text{ or} \\ &= \int_{-\infty}^{\infty}t^k\mathrm{d}z(t) = \int_0^1 t^k\mathrm{d}z(t) \leq |1-0|\cdot(z(1)-z(0)) = 1.\end{aligned}$$

However, a requirement for this to be consistent is that X is a measurable function. Whereas this doesn't take away from Schoenberg's Third Theorem, what this does imply is that both proofs by Schoenberg and Davenport were, at the time, incomplete. This is because they both assumed the moments exist without ever checking this fact. This is an important oversight. Define the function

$$g(n) = \begin{cases}\frac{1}{3} & \text{if} \quad n\in[2^k,2^{k+1}-1], k\geq 0 \text{ even} \\ \frac{2}{3} & \text{if} \quad n\in[2^k,2^{k+1}-1], k\geq 1 \text{ odd.}\end{cases}$$

Then this function fails to even have a mean value, its first moment. As we have seen in Section 3.2, the Erdős-Wintner Theorem establishes necessary and sufficient conditions for additive functions to possess a limiting distribution, essentially fixing the oversight from both proofs as the theorem applies to both $\log\frac{\varphi(n)}{n}$ and $\log\frac{n}{\sigma(n)}$. What can be said however is that the characteristic function of a distribution function $F_X(x)$ does always exist:

$$\phi_X(t) \coloneqq \mathbb{E}[e^{itX}] = \int_{-\infty}^{\infty}e^{itx}\mathrm{d}F_X(x),$$

and hence one may wonder why Schoenberg's Third Theorem depends on the analytic continuation of the moments. It seems that the main reason for this is that, looking back in time, the study of moments of a sequence or distribution predates the study of the characteristic function. This is further exemplified in the works of Schoenberg, as it does not seem he made the connection that the function $\Phi(it)$ in his first paper on the topic [14] and the characteristic function $L(t)$ in his second paper [15] are one and the same. Perhaps he did make the connection afterwards without publishing. However, in our search of the literature we could not find the result that the characteristic function can be derived as an 'analytic continuation' of the moments in the same manner that the Gamma function is an 'analytic continuation' of the factorial. In fact applications of the characteristic function only started after Kolmogorov applied measure theory to lay the axiomatic foundations for modern probability theory, by means of "About the Analytical Methods of Probability Theory" (1931) and "Foundation of the Theory of Probability" (1933). Of course, this goes even further to solidify Schoenberg's result, as his ideas were ahead of probability theory at the time. One of the main results from the characteristic function follows from the inversion formula

$$F_X(b)-F_X(a)+\frac{1}{2}\mathbb{P}(X=a)-\mathbb{P}(X=b) = \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-ita}-e^{-itb}}{it}\phi_X(t)\mathrm{d}t,$$

from which one can show that the distribution function is absolutely continuous if the characteristic function is absolutely integrable, i.e.,

$$\lim_{T\to\infty}\int_{-T}^{T}|\phi_X(t)|\mathrm{d}t < \infty.$$

There is another result on the characteristic function which is in fact stronger. This is the result that if $\phi_X(t) \to 0$ as $t \to \infty$, then there are no point masses in the distribution [3, p.130–131]. This result, relating continuity to absolute integrability, in hindsight seems awfully familiar to the work of Schoenberg. In fact, it can be shown that;

Proposition 4.1 *If $g(n)$ attains a distribution function, then Schoenberg's $\Phi(it)$ is the characteristic function of the additive arithmetic function $f(n) = \log g(n)$.*

Proof: As in the proof of Erdős-Wintner Theorem in Tenenbaum's book [16, p. 326], Delange's Theorem is a key step:

Theorem 4.1 (Delange) *Let g be a multiplicative function with $|g| \leq 1$.*
Part one. If the following limit is non-zero and exists

$$M(g) := \lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} g(n).$$

Then

i) $\sum_p \frac{1-g(p)}{p}$ *converges*

ii) there exists some positive integer k such that $g(2^k) \neq 1$.

Part two. If i) is satisfied, then g has a mean value given by

$$M(g) = \prod_p (1 - p^{-1}) \sum_{j=0}^{\infty} \frac{g(p^j)}{p^j}.$$

This means that

$$\phi(t) = M(g^{it}) = \prod_p (1 - p^{-1}) \sum_{j=0}^{\infty} \frac{g(p^j)^{it}}{p^j}.$$

Recall that by following Davenport's method of establishing $\Phi(s)$, for a multiplicative function g, if it exists, $\Phi(it)$ is given by

$$\Phi(it) = \prod_p \left\{ 1 + \sum_{j \geq 1} p^{-j} \Big(g(p^j)^{it} - g(p^{j-1})^{it} \Big) \right\}.$$

To show that these representations are equal, notice that the sum is absolutely convergent:

$$|g(p^j)^{it} - g(p^{j-1})^{it}| \leq 2,$$

hence the infinite sum is bounded by a geometric series. Thus we may rearrange the terms of the sum without changing its limit.

For $\phi(t)$ we see that the argument of the product becomes

$$
\begin{aligned}
(1-p^{-1})\sum_{j=0}^{\infty}\frac{g(p^j)^{it}}{p^j} &= \sum_{j=0}^{\infty}\frac{g(p^j)^{it}}{p^j} - \sum_{j=0}^{\infty}\frac{g(p^j)^{it}}{p^{j+1}} \\
&= \left(1+\sum_{j=1}^{\infty}\frac{g(p^j)^{it}}{p^j}\right) - \left(\sum_{j=1}^{\infty}\frac{g(p^{j-1})^{it}}{p^j}\right) \\
&= 1+\sum_{j=1}^{\infty}p^{-j}\Big(g(p^j)^{it}-g(p^{j-1})^{it}\Big),
\end{aligned}
$$

hence the representations are the same. □

4.1 Schoenberg's theorem applied to the Cantor distribution

Now that we have proven that Schoenberg's Third Theorem essentially gives conditions for the characteristic function of a distribution, we wish to show its usefulness by applying his theorem to the aforementioned Cantor distribution. The continuity is well-established by analysis of the distribution function and can also be proven by applying Lévy's Theorem 2.4 to its characteristic function. We will now show that Schoenberg's Third Theorem can in fact be used to show the continuity of the Cantor distribution, purely by looking at the integral of the characteristic function. Thus we need to prove that

$$
\lim_{x\to\infty}\frac{1}{x}\int_0^x\prod_{j\geq 1}\left|\cos\left(\frac{t}{3^j}\right)\right|\mathrm{d}t = 0.
$$

By making some substitutions and by upper bounding the integrand we obtain the following:

$$
\begin{aligned}
\lim_{x\to\infty}\frac{1}{x}\int_0^x\prod_{j\geq 1}\left|\cos\left(\frac{t}{3^j}\right)\right|\mathrm{d}t &= \lim_{x\to\infty}\frac{1}{x}\int_0^{\pi}\prod_{j\geq 1}\left|\cos\left(\frac{t}{3^j}\right)\right|\mathrm{d}t + \frac{1}{x}\int_{\pi}^{x}\prod_{j\geq 1}\left|\cos\left(\frac{t}{3^j}\right)\right|\mathrm{d}t \\
&\leq \lim_{x\to\infty}\frac{1}{x}\int_0^{\pi}1\mathrm{d}t + \frac{1}{x}\int_{\pi}^{x}\prod_{j\geq 1}\left|\cos\left(\frac{t}{3^j}\right)\right|\mathrm{d}t.
\end{aligned}
$$

Here, the first integral has both a bounded interval and integrand, hence its integral will be bounded. This means that its contribution will approach zero as $x\to\infty$. In the next step we will change the limiting upper bound of x to $3^{N+1}\pi$. This change is inspired by [3, p.133, Exercise 3.3.11], as the $3^k\pi$, for $k=1,2,\dots,$ form a subsequence of $\phi(t)$ for which $\phi(3^k\pi)\not\to 0$ as $k\to\infty$. Then

$$
\begin{aligned}
\lim_{x\to\infty}\frac{1}{x}\int_0^x\prod_{j\geq 1}\left|\cos\left(\frac{t}{3^j}\right)\right|\mathrm{d}t &\leq \lim_{N\to\infty}\frac{1}{3^{N+1}\pi}\int_{\pi}^{3^{N+1}\pi}\prod_{j\geq 1}\left|\cos\left(\frac{t}{3^j}\right)\right|\mathrm{d}t \\
&= \lim_{N\to\infty}\frac{1}{3^{N+1}\pi}\sum_{k=0}^{N}\int_{3^k\pi}^{3^{k+1}\pi}\prod_{j\geq 1}\left|\cos\left(\frac{t}{3^j}\right)\right|\mathrm{d}t \\
&= \lim_{N\to\infty}\frac{1}{3^{N+1}\pi}\sum_{k=0}^{N}3^k\pi\int_1^3\prod_{j\geq 1}\left|\cos\left(\frac{3^k\pi u}{3^j}\right)\right|\mathrm{d}u.
\end{aligned}
$$

In the step above we made the substitution $3^k\pi u = t$ as this takes the variables out of the bounds of integration. Next we will split the terms of the infinite product in two, the first will be those cosines where the power of 3 is non-negative, and the second will be those cosines where the power of 3 is negative. Then we may further simplify the expression by upper bounding the second product by 1, as all terms are bounded by 1:

$$\begin{aligned}
\lim_{x\to\infty}\frac{1}{x}\int_0^x\prod_{j\geq 1}\left|\cos\left(\frac{t}{3^j}\right)\right|\mathrm{d}t &\leq \lim_{N\to\infty}\frac{1}{3^{N+1}\pi}\sum_{k=0}^{N}3^k\pi\int_1^3\prod_{m=0}^{k-1}\left|\cos\left(3^m\pi u\right)\right|\prod_{j\geq 1}\left|\cos\left(\frac{\pi u}{3^j}\right)\right|\mathrm{d}u\\
&\leq \lim_{N\to\infty}\sum_{k=0}^{N}3^{k-N-1}\int_1^3\prod_{m=0}^{k-1}\left|\cos\left(3^m\pi u\right)\right|\mathrm{d}u\\
&= \lim_{N\to\infty}\sum_{c=0}^{N}3^{-c-1}\int_1^3\prod_{m=0}^{N-c-1}\left|\cos\left(3^m\pi u\right)\right|\mathrm{d}u,\quad \text{where } c=N-k.
\end{aligned}$$

Now let

$$f_N \coloneqq \sum_{c=0}^{N}3^{-c-1}\int_1^3\prod_{m=0}^{N-c-1}\left|\cos\left(3^m\pi u\right)\right|\mathrm{d}u,$$

where the empty product equals 1. Then we can find the recurrence relation

$$\begin{aligned}
f_{N+1} &= \sum_{c=0}^{N+1}3^{-c-1}\int_1^3\prod_{m=0}^{N-c}\left|\cos\left(3^m\pi u\right)\right|\mathrm{d}u\\
&= 3^{-1}\int_1^3\prod_{m=0}^{N}|\cos(3^m\pi u)|\mathrm{d}u+\sum_{c=1}^{N+1}3^{-c-1}\int_1^3\prod_{m=0}^{N-c}\left|\cos\left(3^m\pi u\right)\right|\mathrm{d}u\\
&= 3^{-1}\int_1^3\prod_{m=0}^{N}|\cos(3^m\pi u)|\mathrm{d}u+3^{-1}\sum_{c'=0}^{N}3^{-c'-1}\int_1^3\prod_{m=0}^{N-c'-1}\left|\cos\left(3^m\pi u\right)\right|\mathrm{d}u\\
&= \delta_N+\frac{1}{3}f_N,
\end{aligned}$$

where

$$\delta_N \coloneqq 3^{-1}\int_1^3\prod_{m=0}^{N}|\cos(3^m\pi u)|\mathrm{d}u.$$

Now let

$$\delta \coloneqq \lim_{N\to\infty}\delta_N.$$

If this limit exists, then

$$\lim_{N\to\infty}f_N=\frac{3}{2}\delta,$$

hence it is sufficient to show that $\delta_N\to 0$ as $N\to\infty$. We will show the following: for all but a sparse set (density 0) of numbers x in the unit interval the inequality

$$\cos\left(3^m\pi x\right)\leq\frac{1}{2}$$

holds for infinitely many $m\geq 0$. Our integral is from 1 to 3, and $|\cos(\pi x)|$ has period 1 (and for larger m the functions $|\cos(3^m\pi x)|$ are definitely also 1-periodic), hence it is sufficient to show this for the unit interval. Then for our inequality above, for $m=1$ we have

$$\cos(\pi x)\leq\frac{1}{2}\quad\text{implies}\quad x\in\left[\frac{1}{3},\frac{2}{3}\right].$$

We can also restate the right-hand side to be those x for which the first digit in the ternary expansion is a 1, as it is precisely the middle third of the unit interval. This leaves out the boundary point $x = \frac{2}{3}$, but for our purposes it is sufficient to consider only the aforementioned x. For $m = 1$, the inequality is

$$\cos(3\pi x) \leq \frac{1}{2} \quad \text{implies} \quad x \in \left[\frac{1}{9}, \frac{2}{9}\right] \cup \left[\frac{4}{9}, \frac{5}{9}\right] \cup \left[\frac{7}{9}, \frac{8}{9}\right].$$

Now these are those x for which the second digit in the ternary expansion is a 1, as these are again the middle thirds of the intervals $\left[0, \frac{1}{3}\right]$, $\left[\frac{1}{3}, \frac{2}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. Hence for the m-th inequality, we see that the inequality is satisfied for those x which have a 1 at the $m+1$-st digit of the ternary expansion. Thus showing that the inequality holds infinitely often for almost all x in the unit interval follows from saying that almost all x have infinitely many 1's in their ternary expansion, which is true. But in fact if that inequality holds for infinitely many m for all x except for a set of density 0, then that means that

$$\int_0^1 \lim_{N\to\infty} \prod_{m=0}^{N} |\cos(3^m \pi u)| \mathrm{d}u \leq \int_0^1 \lim_{k\to\infty} \prod_{m=0}^{k} \left(\frac{1}{2}\right) \mathrm{d}u = 0.$$

Hence for δ_N we have

$$\begin{aligned}
\lim_{N\to\infty} \delta_N &= \lim_{N\to\infty} 3^{-1} \int_1^3 \prod_{m=0}^{N} |\cos(3^m \pi u)| \mathrm{d}u, \\
&= \frac{2}{3} \int_0^1 \lim_{N\to\infty} \prod_{m=0}^{N} |\cos(3^m \pi u)| \mathrm{d}u, \\
&\leq 0.
\end{aligned}$$

As δ_N is an integral of a non-negative function, we see that

$$0 \leq \lim_{N\to\infty} \delta_N \leq 0.$$

Thus, the Cantor distribution is continuous. □

5 Polynomial representations for $f(n)$

The second theorem 2.9 by JPT applies to many multiplicative functions f. One of the arithmetic functions that works as well is the function that counts the number of representations a number has as a sum of two squares, up to some symmetry:

$$r(n) := \frac{1}{4}\#\{(x_1, x_2) \in \mathbb{Z}^2 \setminus \{0\} \mid x_1^2 + x_2^2 = n\}.$$

The reason this is interesting is that, even though $r(n)$ is an arithmetic function, we can also interpret this result as asking, "in how many ways can n be represented by the integer polynomial or quadratic form $f(x_1, x_2) = x_1^2 + x_2^2$." Let us first show that this function indeed satisfies the conditions of the theorem. A classic results states that for p prime, we have that

$$r(p) = \begin{cases} 1 & \text{if} \quad p = 2, \\ 2 & \text{if} \quad p \equiv 1 \mod 4, \\ 0 & \text{if} \quad p \equiv 3 \mod 4. \end{cases}$$

Thus the first condition of the theorem becomes finding $\kappa > 0$ such that

$$\sum_{p \le x} r(p) \frac{\log p}{p} = 1 + 2 \sum_{\substack{p \le x \\ p \equiv 1(4)}} \frac{\log p}{p} \sim \kappa \log x.$$

This follows from the more general result that for $(l, k) = 1$,

$$\sum_{\substack{p \le x \\ p \equiv l(k)}} \frac{\log p}{p} \sim \frac{1}{\varphi(k)} \log x.$$

This result is one that lies deep in number theory, as in [1, p.148–154], where it is used as an intermediary step to prove Dirichlet's result on primes in arithmetic progressions. From this, we see that

$$\sum_{p \le x} r(p) \frac{\log p}{p} \sim \log x.$$

Further values for primes powers are

$$r(p^k) = \begin{cases} 1 & \text{if} \quad p = 2, \\ k+1 & \text{if} \quad p \equiv 1 \mod 4, \\ 1 & \text{if} \quad p \equiv 3 \mod 4 \text{ and } 2 \mid k, \\ 0 & \text{else}. \end{cases}$$

One can see that the second condition is also fulfilled, as $r(p^k) \le \tau(p^k)$ and we already checked that the double sum of the latter converges. In this case, we have that $\kappa \le 1$, or in fact $\kappa = 1$, hence we need to check that

$$\sum_{p^j \le x} r(p^j) \ll_r \frac{x}{\log x},$$

holds. Observe that

$$\begin{aligned}
\sum_{p^j \le x} r(p^j) &= \sum_{p \le x} r(p) + \sum_{j \ge 2} \sum_{p \le x^{1/j}} r(p^j) \\
&\le \sum_{p \le x} 2 + \sum_{j \ge 2} \sum_{p \le x^{1/j}} (j+1) \\
&\le 2\frac{x}{\log x} + \sum_{j=2}^{\lfloor \frac{\log x}{\log 2} \rfloor} \sum_{p \le x^{1/2}} (j+1) \\
&\le 2\frac{x}{\log x} + \sum_{j=2}^{\lfloor \frac{\log x}{\log 2} \rfloor} (j+1) \cdot \frac{x^{1/2}}{\log x^{1/2}} \\
&\le 2\frac{x}{\log x} + \frac{(\lfloor \frac{\log x}{\log 2} \rfloor + 4)(\lfloor \frac{\log x}{\log 2} \rfloor - 1)}{2} \frac{x^{1/2}}{\frac{1}{2}\log x} \\
&= \mathcal{O}\left(\frac{x}{\log x}\right) + \mathcal{O}\left(\frac{(\log x)^2 x^{1/2}}{\log x}\right) \\
&= \mathcal{O}\left(\frac{x}{\log x}\right).
\end{aligned}$$

Now we see that indeed, all the conditions are satisfied for the multiplicative function $r(n)$, thus the distribution function

$$\widetilde{D}_r(u) = \lim_{x \to \infty} \frac{1}{S(r;x)} \sum_{\substack{n \le x \\ n/\sigma(n) \le u}} r(n)$$

exists and is continuous. By geometric interpretations, we see that $4 \cdot r(n)$ is a function that counts the number of lattice points inside a circle of radius $\sqrt{n}$, hence the above becomes

$$\widetilde{D}_r(u) = \lim_{n \to \infty} \frac{1}{\pi n} \left\{ (x_1, x_2) \in \mathbb{Z}^2 \mid 0 < x_1^2 + x_2^2 \le n \text{ and } \frac{x_1^2 + x_2^2}{\sigma(x_1^2 + x_2^2)} \le u \right\}. \tag{40}$$

One might ask what other quadratic forms or even other types of polynomials also mimic this behaviour. Considering the arithmetic function that counts the number of representations of a number n as a sum of k squares

$$r_k(n) := \# \left\{ \sum_{i=1}^{k} x_i^2 = n \mid x_i \in \mathbb{Z} \right\},$$

then it is known [7, p.131–132] that only for $k = 1, 2, 4, 8$, the function

$$f_k(n) := \frac{r_k(n)}{2k},$$

is multiplicative. Hence these are the first functions for which we hopefully can get a similar result through Theorem 2.9. Notice that when $k = 2$ we have that $f_2(n) = r(n)$, our function from earlier. Before, we computed $S(r;x)$ by geometrically interpreting the sum of squares, and in fact we can do the same for the other r_k:

$$\begin{aligned}
S(f_1;x) &\sim \sqrt{x}, \\
S(f_4;x) &\sim \frac{1}{8} \cdot \frac{\pi^2}{2} x^2, \\
S(f_8;x) &\sim \frac{1}{16} \cdot \frac{\pi^4}{24} x^4.
\end{aligned}$$

Sadly, $f_1(n)$ does not satisfy the conditions for Theorem 2.9 as the first conditions fails because $f_1(p) = 0$. The functions f_4 and f_8 also fail the first conditions. We have that

$$\begin{aligned} r_4(n) &= 8 \sum_{4 \nmid d | n} d, \\ r_8(n) &= (-1)^n 16 \sum_{d|n} (-1)^d d^3. \end{aligned}$$

Hence for prime values we have for f_k:

$$\begin{aligned} f_4(p) &= p + 1, \\ f_8(p) &= p^3 + 1. \end{aligned}$$

Thus the first conditions fails in both cases

$$\sum_{p \le x} \frac{p+1}{p} \log p = \sum_{p \le x} \log p + \sum_{p \le x} \frac{\log p}{p} = x + \mathcal{O}\left(\frac{x}{\log x}\right),$$
$$\sum_{p \le x} \frac{p^3+1}{p} \log p.$$

The first result comes from Chebyshev's first function, and the second clearly cannot be $\mathcal{O}(\log x)$. Hence these representations, even though multiplicative, fail to be approximated by Wirsing's Theorem. Perhaps quicker to verify this fact is that f_4 and f_8 are in fact not bounded for primes. This seems to happen due to the amount of variables allowing for too many representations, hence it might be helpful to only consider binary quadratic forms. A problem is that most of the results in this field do not always have that the number of representations of n is a multiplicative function, but only that the set of represented numbers is closed under multiplication. And most cases don't even have that. Nonetheless, we conclude with an open question we did not have time for in our thesis. We believe that the multiplicative condition is only necessary to allow the application of our current known methods, and is not necessary for the result to hold. In fact, the result (40) can be interpreted to be a statement on the integer points on the interior of the circle.

Let $f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$ be a binary quadratic form, with $a, b, c \in \mathbb{Z}$, discriminant $D = b^2 - ac < 0$ and $a > 0$. The condition on the discriminant is to ensure that f represents only positive or negative numbers, and the condition on a is to ensure f only represents positive integers. This is to exclude instances such as Pell's equation, where there are infinitely many representations for certain integers. These conditions can also be geometrically interpreted so that the condition $f(x_1, x_2) \le n$ results in a compact subset of $\mathbb{R}^2$. Now let $R_f(n)$ denote the number of representations of n, i.e.,

$$R_f(n) = \#\{(x_1, x_2) \in \mathbb{Z}^2 \mid f(x_1, x_2) = n\}.$$

Before we obtained $S(f; x)$ by geometric interpretations, and for these binary quadratic forms we can do the same. Then

$$\begin{aligned} S(f; n) &:= Vol\{f(x_1, x_2) \le x \mid (x_1, x_2) \in \mathbb{R}^2\} \\ &= \iint_{f(x_1, x_2) \le n} \mathrm{d}x_1 \mathrm{d}x_2. \end{aligned}$$

This area is that of a skew ellipse, hence we will use the following substitution to compute it.

$$\begin{aligned} n &\geq ax_1^2 + bx_1x_2 + cx_2^2 \\ 1 &\geq \frac{1}{4an}\left((2ax_1 + bx_2)^2 + (\sqrt{-D}x_2)^2\right) \\ 1 &\geq \frac{1}{4an}\left(u^2 + v^2\right) \\ 4an &\geq u^2 + v^2. \end{aligned}$$

This inequality gives us a circle with radius $\sqrt{4an}$, hence the area is $4\pi an$. For the Jacobian of this transformation from the ellipse to the circle, we have

$$\begin{vmatrix} \frac{\partial x_1}{u} & \frac{\partial x_1}{v} \\ \frac{\partial x_2}{u} & \frac{\partial x_2}{v} \end{vmatrix} = \begin{vmatrix} \frac{u}{\partial x_1} & \frac{u}{\partial x_2} \\ \frac{v}{\partial x_1} & \frac{v}{\partial x_2} \end{vmatrix}^{-1} = \begin{vmatrix} 2a & b \\ 0 & \sqrt{-D} \end{vmatrix}^{-1} = \frac{1}{2a\sqrt{-D}}.$$

Thus for $S(f;n)$ we obtain

$$\begin{aligned} S(f;n) &= \iint_{u^2+v^2 \leq 4an} \frac{1}{2a\sqrt{-D}} \mathrm{d}u\mathrm{d}v \\ &= \frac{4\pi an}{2a\sqrt{-D}} \\ &= \frac{2\pi n}{\sqrt{-D}}. \end{aligned}$$

Thus our two conjectures, one only for f and another for both f and g are as follows.

Conjecture 5.1 *Let $f(x_1, x_2)$ be an integer binary quadratic form. If it is also a positive definite form, the limit*

$$\widetilde{D}_f(u) = \lim_{x\to\infty} \frac{\sqrt{-D}}{2\pi x} \sum_{\substack{n \leq x \\ n/\sigma(n) \leq u}} R_f(n) = \lim_{n\to\infty} \frac{\sqrt{-D}}{2\pi n} \left\{ (x_1, x_2) \in \mathbb{Z}^2 \mid 0 < f(x_1, x_2) \leq n \text{ and } \frac{f(x_1, x_2)}{\sigma(f(x_1, x_2))} \leq u \right\}$$

exists for every $u \in [0, 1]$ and is continuous in terms of u.

Conjecture 5.2 *Let $f(x_1, x_2)$ be an integer binary quadratic form. If it is also a positive definite form, and $g(n)$ satisfies the conditions of Schoenberg, then the limit*

$$\widetilde{D}_{f,g}(u) = \lim_{x\to\infty} \frac{\sqrt{-D}}{2\pi x} \sum_{\substack{n \leq x \\ g(n) \leq u}} R_f(n) = \lim_{n\to\infty} \frac{\sqrt{-D}}{2\pi n} \left\{ (x_1, x_2) \in \mathbb{Z}^2 \mid 0 < f(x_1, x_2) \leq n \text{ and } g(f(x_1, x_2)) \leq u \right\}.$$

exists and is continuous for every $u \in [0, 1]$.

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