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Overtwisted $(2,3,5)$–structures and the h–principle

Master's Thesis

F.G.J. ter Haar

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Supervisor:

Dr. Á. del Pino Gómez

Second reader:

Prof. Dr. G. Cavalcanti

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Contents

Introduction

In the field of differential topology, people are often interested in manifolds endowed with some geometric structure. In this thesis we will look at geometric structures called *distributions*. In particular, we are interested in a special type of distributions which are called *bracket-generating*. In a moment, we will start off with giving a brief introduction to these objects. As one investigates distributions, several questions frequently arise:

What manifolds admit such a structure?
Which of these structures are homotopic to each other?
What is the topology of the space of such structures?

A useful tool to try and answer these questions is the *$h-$principle*. Here the 'h' represents *homotopy*. By showing that *the $h-$principle holds* for a certain family of geometric structures, we can gain insights into the existence of these structures and the topology of the space of geometric structures. In this introduction we will comment on this principle and discuss previous research conducted in this field. Finally, we will discuss the outline of the thesis and discuss the main result it presents.

Distributions

Mathematically speaking, we define a distribution as a subbundle of the tangent bundle. So what does this mean? At every point in our manifold, we can look at the tangent space which consists of all vectors which are tangent to the manifold at that specific point. Taking a linear subspace of these vectors at each point in the manifold, defines a distribution. As an example, suppose we pick a 2-dimensional subspace, i.e. a plane, in a smooth manner at every point in the manifold. In this case, we say a distribution is of *rank 2*. We can then investigate whether this plane turns when moving into different directions, or if the plane stays still.

A more intuitive interpretation of a distribution is the following. One can see the vectors in the distribution as the "*allowed directions of motion*". To illustrate this concept, consider the example of a coin rolling on a table. In Figure 1, the $x-$ and $y-$coordinate indicate the position of the coin, and θ indicates the direction the coin points in. Hence, the state space of the coin can be described by the manifold $\mathbb{R}^2 \times \mathbb{S}^1$.

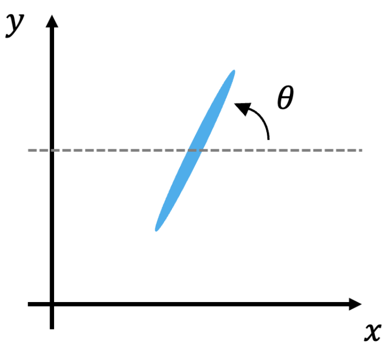

Figure 1: Illustration of the rolling coin distribution D_{coin}.

We observe that the motions of the coin form a rank 2 distribution, D_{coin}, which is spanned by the following two vector fields:

$$D_{\text{coin}} = \langle \partial_\theta, \cos(\theta)\partial_x + \sin(\theta)\partial_y \rangle.$$

Intuitively, the first vector field allows us to rotate the coin in any direction, and the second vector field enables us to roll the coin in the direction it is pointing in. Note that by moving along these vector fields, we can place the coin in any desired position and direction!

Keeping up with this philosophy, a question one can ask then is: *given two points on the manifold, can we find a path from one to the other, following the allowed directions of motion?* This brings us to the definition of *bracket-generating* distributions. We define these as distributions for which at every point one can move in any direction by moving along infinitesimal directions tangent to the distribution. Observe that the rolling coin example is indeed a bracket-generating distribution.

A way of mathematically measuring whether one can move in certain directions along a distribution, is the Lie bracket. Namely, the Lie bracket of two vector fields measures their failure to commute. Thus, applying the Lie bracket to two vector fields tangent to a distribution, can give us a third direction, which we can apparently also move in, while staying tangent to the distribution. Therefore, formally we define bracket-generating distributions as the following.

Definition 0.1. *Let M be a manifold and let $D \subseteq TM$ be a distribution on M. We can look at the following sequence:*

$$D^{(1)} := D \subseteq D^{(2)} := [D, D] \subseteq D^{(3)} := [D, [D, D]] \subseteq \cdots \subseteq TM.$$

We say D is ***bracket-generating*** *if $D^{(l)} = TM$ for some $l \geq 1$.*

Contact structures are a classic example of a widely-studied class of bracket-generating distributions. They are defined on odd-dimensional manifolds and generate the whole tangent bundle by a single application of the Lie bracket. *Engel structures* are bracket-generating distributions defined on 4-dimensional manifolds, which generate the whole tangent bundle after applying the Lie bracket twice. This thesis will focus on bracket-generating distributions on $5-$manifolds called $(2,3,5)-$*structures.* The name is quite suggestive: these are distributions of rank 2, which after applying the Lie bracket once, generate a rank 3 distribution and after two applications, generate the whole tangent bundle. In 1910, the French mathematician Élie Cartan gave a solution to the (local) equivalence problem for maximally bracket-generating 2-distributions on 5-manifolds, i.e. $(2,3,5)-$structures [4]. In doing so, he made a surprising connection with the complex Lie algebra of type G_2. Therefore, in the literature, these structures are also referred to as *distributions of Cartan type.* Many of the original contributions of this thesis are inspired by known results on contact and Engel structures, and in a number of cases we shall therefore discuss these first, in order to justify and clarify the findings on $(2,3,5)-$structures.

Let us take a look at a real-life example of a $(2,3,5)-$manifold (i.e. a manifold that admits a $(2,3,5)-$structure). Take a stationary ball of radius R and a second ball of radius r which rolls on the first. See also Figure 2. We can identify the configuration space Q with the manifold $(g, \mathbf{x}) \in SO_3 \times \mathbb{S}^2$, where the first term g describes the rotation of the second ball relative to its initial position and $R\mathbf{x}$ describes the point of contact between the two balls. Note that this configuration space is 5-dimensional. It turns out that Q admits a $(2,3,5)-$structure! Namely, the distribution which describes the motions of the ball of radius r, where it rolls without twisting or slipping, is a rank 2 distribution. We call it the *rolling distribution*, and denote it by D_ρ, where $\rho = \frac{R}{r}$ is the ratio of the two radii. This distribution is a $(2,3,5)-$structure, provided the values of the Gaussian curvatures are disjoint, i.e. if $R \neq r$ [2].

The $h-$principle

A PDR, short for *partial differential relation*, is a condition one can impose on a section and its derivatives. For example, requiring that the derivative $df : TM \to TN$ of a map $f : M \to N$ is injective, implies that f is an immersion. Sometimes we want to investigate whether an immersion exists between two manifolds. A necessary condition, is that there exists a map $f : M \to N$ and a lift $F : TM \to f^*TN$. We call a pair of such maps a *formal solution*, and an actual immersion a *solution* to the PDR. Thus, looking at the space of formal solutions, can help us in proving the existence of actual solutions.

However, just proving that a formal solution exists is not sufficient. Hence, a natural question to ask is:

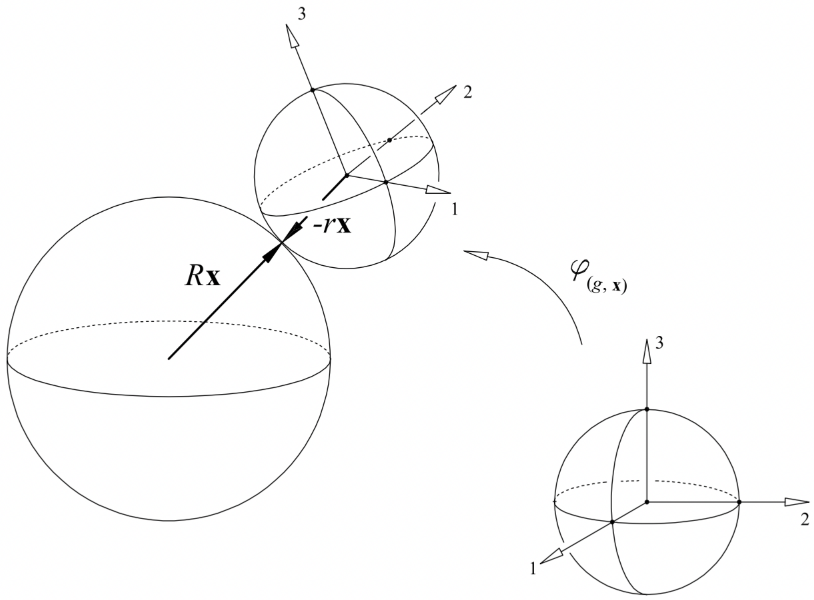

Figure 2: Illustration of the rolling distribution [2, p.7].

Given a formal solution, can we transform it into a genuine solution?

This is where the $h-$principle comes into play, which we define as follows.

Definition 0.2. *Let $\mathcal{R}$ be a PDR. We say $\mathcal{R}$ satisfies the $h-$**principle** if the inclusion*

$$i : \{solutions\ of\ \mathcal{R}\} \hookrightarrow \{formal\ solutions\ of\ \mathcal{R}\}$$

is a weak homotopy equivalence.

Let us elaborate a little on this definition. Namely, showing that the inclusion above is surjective on π_0-level means that we can homotope every formal solution into a genuine one. Moreover, injectivity of the inclusion map on π_0-level shows that any two solutions which are formally homotopic, are also homotopic through genuine solutions. Thus, showing that the $h-$principle holds for a PDR can shed light on the existence and classification up to homotopy of certain geometric structures. Let us give a brief history overview of this mathematical principle.

The concept of the $h-$principle first appeared in [25] by Gromov and [28] by Gromov and Eliashberg under the name of $w.h.e$-principle. The name $h-$principle was first introduced in 1969 by Gromov in his book [27]. Before the $h-$principle (or $w.h.e.$-principle) was officially defined, people already proved that it holds for certain spaces of geometric structures. For example, Smale [39] (1958) and Hirsch [29] (1959) proved that the inclusion

$$\mathrm{Imm}^f(M,N) \hookrightarrow \mathrm{Imm}(M,N)$$

is a weak homotopy equivalence if $\dim(M) < \dim(N)$. Here $\mathrm{Imm}^f(M,N)$ is the space of formal immersions, which contains the pairs of maps as discussed above. In 1969, Gromov showed that there are more general methods for proving $h-$principles [24]. Namely, using these methods, he proved the following powerful result. Note that we use $Sol_{\mathcal{R}}(M)$ and $Sol^f_{\mathcal{R}}(M)$ below, to denote the space of solutions and the space of formal solutions of the partial differential relation $\mathcal{R}$, respectively.

Theorem 0.3. *[Gromov, 1969] Let M be an open manifold and $E \to M$ a vector bundle. Then any open, Diff-invariant differential relation $\mathcal{R} \subset J^r(E)$ satisfies the $h-$principle, i.e.*

$$Sol_{\mathcal{R}}(M) \hookrightarrow Sol^f_{\mathcal{R}}(M)$$

is a weak homotopy equivalence.

We say a differential relation is open, if it is open in the topology endowed on the jet bundle, and it is Diff-invariant, if it is invariant under diffeomorphisms of the base manifold M. For more on jet bundles and differential relations we refer to [15]. It turns out that there are a lot of geometric structures which can be expressed as an open and Diff-invariant PDR. This is because operators such as the Lie bracket, the wedge product, the de Rham differential and forms are preserved by diffeomorphisms of M. Therefore, for open manifolds, Gromov's theorem is a very useful tool. Also contact, Engel and $(2,3,5)$-structures can be expressed as open, Diff-invariant PDR's. A natural question at this point is then the following: *What about closed manifolds?*

In 1983, Bennequin proved that the $h-$principle fails in general for contact structures on closed manifolds [1]. However, it turns out that the $h-$principle on closed manifolds does hold for a special family of contact structures, which are called *overtwisted*. For this, we go back to the 70's, when Lutz [37] and Martinet [30] showed that any closed 3-dimensional manifold admits a contact structure. They used a construction which we call *Lutz twisting*. This turns out to be a very useful construction, not only to prove existence, but also to prove that the $h-$principle holds for these overtwisted contact structures. This was proven in dimension 3 by Eliashberg in 1989 [13]. Most of the research conducted on contact structures in the past years has been on *tight* contact structures, which are contact structures which are not overtwisted. Quite recently, in 2018, del Pino and Vogel proved in their paper that on closed manifolds there is also an $h-$principle for *overtwisted* Engel structures [12]. A main open question in the study of Engel structures is if the $h-$principle holds or fails on closed manifolds in general.

After contact and Engel structures, $(2,3,5)-$structures are an interesting class of distributions to investigate, for dimensional reasons. As mentioned before, we know by Gromov that the $h-$principle holds for them on open manifolds. However, on closed manifolds we still know very little. Whether the $h-$principle holds or fails in general is an open question! One can also wonder if, similarly to contact and Engel structures, we can define a special class of *overtwisted* $(2,3,5)-$structures, for which we could prove an $h-$principle on closed manifolds. This brings us to the contents of this thesis.

What is the thesis about?

The goal of the thesis is to define a special class of $(2,3,5)-$structures, which are called *overtwisted*, and to prove that the $h-$principle holds for these structures on closed manifolds. The idea will be to define the overtwisted $(2,3,5)-$structures in such a way that we precisely have all the ingredients in order to prove that the $h-$principle indeed holds for them. The main result of this thesis is stated in the following theorem.

Theorem 0.4. *Let M be a closed 5-dimensional manifold. Then the inclusion*

$$\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta) \hookrightarrow \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta),$$

is a weak homotopy equivalence.

Here, $\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$ and $\mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$ denote the spaces of overtwisted and formal overtwisted $(2,3,5)-$structures, respectively, both with a fixed embedding Δ of the *overtwisted disc*. These concepts will be explained in more detail later on in the thesis. This result is an original contribution to the research field. Since our focus is on proving Theorem 0.4, the structure of the thesis is also adjusted accordingly. Let us give an outline of the contents.

The first two chapters will serve as introductory chapters. In Chapter 1 we discuss bracket-generating distributions, and pay special attention to contact and Engel structures. As mentioned before, we will often refer back to these structures, as our results on $(2,3,5)-$structures will be inspired by known results on contact and Engel structures. In Chapter 2 we introduce $(2,3,5)-$structures, discuss a local description and prove that they can indeed be expressed as an open, Diff-invariant relation.

In Chapter 3 we prove Theorem 3.8, which shows that we can construct a $(2,3,5)-$structure on mapping tori admitting a suitable formal $(2,3,5)-$structure. This is a new result, and it already highlights some of the constructions we will present in the proof of Theorem 0.4. Subsequently, we will discuss Lutz twisting in contact and Engel manifolds in Chapter 4. In Chapter 5 we will define Lutz twisting in $(2,3,5)-$manifolds, and this will resemble the constructions in Chapter 4. After having defined Lutz twisting, we can then define the notion of an overtwisted disc and thereafter overtwisted $(2,3,5)-$structures in Chapter 6. Additionally, we will prove in this chapter, that in the presence of one overtwisted disc, one can produce extra copies by homotopy. This will ensure that we preserve the fixed embedding Δ of the overtwisted disc in the proof of Theorem 0.4. In Chapter 6 we will also discuss Conjecture 6.4, which will not be proved in this thesis, but we will use it to prove the main result.

In several cases, the proof of that an $h-$principle holds follows a particular scheme. For instance, del Pino and Vogel used this scheme in [12] to prove the $h-$principle for overtwisted Engel structures on closed manifolds. The idea is the following. First we triangulate the manifold, and then we obtain the required condition in the codimension-1 skeleton. What is left, is to obtain the desired structure in the top-simplices. This last step is also called the *extension problem.* In Chapter 7 we reduce the proof of Theorem 0.4 to the extension problem, and lastly in Chapter 8, we finish the proof of Theorem 0.4 by performing the extension.

To conclude, we will summarise the proof of Theorem 0.4, and discuss some open questions and ideas for potential future research. The thesis also contains three appendices, which entail several interesting topics in differential/contact topology I have researched during the making of this thesis. They can provide more background into topics in the field, but in order to preserve the flow of the story, they are not included in the main body.

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Chapter 1

Bracket-generating distributions

This chapter will be a brief introduction to distributions, and in particular bracket-generating distributions. Smooth distributions are locally spanned by vector fields, to which we can apply the Lie bracket. Applying this operation iteratively can yield new, i.e. linearly independent, vector fields. If eventually we can span the whole tangent bundle with these vector fields, we call a distribution *bracket-generating.* Recall that the rolling coin distribution discussed in the introduction, is an example of a bracket-generating distribution. The object of interest in this thesis is also a type of bracket-generating distributions. In the first section of this chapter we will go over some basic definitions regarding distributions and the Lie bracket, and in the second and third section we will discuss two types of bracket-generating distributions, called contact and Engel structures.

1.1 Distributions

Let us formally define distributions.

Definition 1.1. *Let M be a smooth manifold. A* ***distribution of rank*** k ***on*** M *is a rank-k subbundle of TM. A distribution is said to be* ***smooth*** *if it is a smooth subbundle of TM.*

We write $\mathfrak{Dist}(M)$ for the space of all smooth distributions on M, and we endow it with the C^0-topology.

Remark 1.2. We note that there are two types of Whitney C^r-topologies, the weak and the strong C^r-topology. See [21] for more on these topologies. When trying to prove an h-principle, one should be careful with these topologies. Namely, differential relations are often open in the strong, but not in the weak, because they involve constraints all the way up to infinity. Therefore, when perturbing distributions in an $\epsilon-$close manner for $\epsilon > 0$ very small, we want to do this in the strong topology. However, if we then construct homotopies between these distributions (by for example linear interpolation), these will only be continuous in the weak topology. Therefore, we are interested in the homotopy type of certain subspaces of $\mathfrak{Dist}(M)$ with respect to the weak topology. ▶

If ξ is a distribution, we can write

$$\xi = \cup_{p\in M}\xi_p$$

where ξ_p is a $k-$dimensional linear subspace of T_pM. Being a smooth distribution means that at every point in the manifold there is a neighbourhood U, and smooth vector fields $X_1, ..., X_k : U \to TM$, such that ξ_p is spanned by $X_1|_p, ..., X_k|_p$ for every $p \in U$ [36, p. 491]. All the distributions we will be considering in this thesis are smooth.

Given a distribution, we can look at vector fields which are tangent to it. This gives us the following correspondence:

$$\{\text{Distributions on } M\} \leftrightarrow \left\{C^\infty - \text{modules of vector fields of point wise constant rank on } M\right\}.$$

Here we identify a distribution ξ with the space $\Gamma(\xi) \subset \mathfrak{X}(M)$ consisting of vector fields tangent to ξ. We can recover ξ from $\Gamma(\xi)$ by evaluating evaluating the vector fields at every point in M. The space $\Gamma(\xi)$ is in fact a $C^\infty-$module, which satisfies the sheaf condition.

Lemma 1.3. *Let M be a smooth manifold and let ξ be a distribution on M. Then $\Gamma(\xi)$ is a C^∞-module.*

Proof. Let $X, Y \in \Gamma(\xi)$ and let $f \in C^\infty(M)$. We note that $\mathfrak{X}(M)$ is a $C^\infty-$module with the operations

$$(X+Y)_p = X_p + Y_p$$
$$(fX)_p = f(p)X_p.$$

Since $\xi_p \subseteq T_pM$ is a smooth linear subspace and $X_p, Y_p \in \xi_p$, we have that $X_p + Y_p, f(p)X_p \in \xi_p$. The module structure is immediately inherited from $\mathfrak{X}(M)$. □

Remark 1.4. We introduce this correspondence as in the rest of the thesis we will often identify a distribution with its sheaf of sections, i.e. the vector fields which are tangent to it. We will use this identification from now on without explicitly pointing out the correspondence. ▶

1.1.1 Involutive and non-integrable distributions

The Lie bracket is an operator on vector fields, and since smooth distributions are locally spanned by vector fields, we can apply this operator on them. The Lie bracket of two vector fields measures their failure to commute. If the Lie bracket $[X, Y]$ of two vector fields X and Y tangent to a distribution ξ, is linearly independent of the initial two, it means that we can also move in the direction $[X, Y]$, while staying tangent to the distribution. This brings us to the next definition.

Definition 1.5. *A distribution ξ is said to be **involutive** if for every two vector fields X and Y tangent to ξ, their Lie bracket $[X, Y]$ is also tangent to ξ [36, p. 492].*

If $\xi = \langle X_i \mid i \in I\rangle$ for vector fields X_i on M, then we can define the following distribution:

$$[\xi, \xi] := \langle X_i, [X_j, X_k] \mid i, j, k \in I\rangle.$$

If ξ is an involutive distribution, we sometimes use the shorthand notation $[\xi, \xi] \subset \xi$.

Example 1.6. Let $\xi = \langle X, Y\rangle$ be a distribution of rank 2 on $\mathbb{R}^3$ spanned by the following two vector fields

$$X = x\partial_x + \partial_y + x(y+1)\partial_z \quad \text{and} \quad Y = \partial_x + y\partial_z.$$

We note that

$$[X, Y] = \partial_z - \partial_x - (y+1)\partial_z = -Y.$$

Therefore $[\xi, \xi] \subset \xi$, and thus ξ is involutive [36, p. 498-499]. △

We have seen that a distribution ξ is involutive if $[\xi, \xi] \subset \xi$. However, it can also be the case that applying the Lie bracket yields a "new", i.e. linearly independent, vector field. This motivates the following definition for distributions of rank 2.

Definition 1.7. *Let M be a smooth manifold of dimension n and let ξ be a rank 2 distribution on M. We say ξ is **non-integrable** if $[\xi, \xi]$ is a rank 3 distribution.*

We denote by $\mathfrak{Dist}_{n.i.}(M)$ the space of non-integrable 2-distributions on M, and we endow it with the C^1-topology. We endow this space with the C^1-topology, as the key property of non-integrable distributions has to do with first derivatives. We can also define a formal analogue of these distributions.

Definition 1.8. *Let M be a smooth manifold of dimension n. A **formal non-integrable** 2-distribution is a pair $\xi \subset \mathcal{E}$ where ξ is a rank 2 distribution and $\mathcal{E}$ a rank 3 distribution on M, together with an isomorphism*

$$\xi \wedge \xi \cong \mathcal{E}/\xi.$$

We denote by $\mathcal{F}\mathfrak{Dist}_{n.i.}(M)$ the space of formal non-integrable distributions and we endow it with the C^0-topology, see also Remark 1.9. We note that non-integrable distributions are also formally non-integrable, where the isomorphism is induced by the Lie bracket. I.e. we have the inclusion

$$\mathfrak{Dist}_{n.i.}(M) \hookrightarrow \mathcal{F}\mathfrak{Dist}_{n.i.}(M), \quad \xi \mapsto (\xi, [\xi, \xi], [.,.]).$$

Remark 1.9. Let us clarify what we mean when we endow the space $\mathcal{F}\mathfrak{Dist}(M)$ with the C^0-topology. First of all, we note that $\mathcal{F}\mathfrak{Dist}(M)$ is a Serre fibration over a subset V of $\mathfrak{Dist}(M) \times \mathfrak{Dist}(M)$ consisting of pairs $(\xi^2, \mathcal{E}^3)$ such that $\xi \subset \mathcal{E}$, and where the projection map is given by projecting to the distributions. Let $(\xi, \mathcal{E}, \varphi), (\tilde{\xi}, \tilde{\mathcal{E}}, \tilde{\varphi}) \in \mathcal{F}\mathfrak{Dist}(M)$. We want to define what it means for $(\xi, \mathcal{E}, \varphi)$ and $(\tilde{\xi}, \tilde{\mathcal{E}}, \tilde{\varphi})$ to be close. We can endow the space $\mathfrak{Dist}(M) \times \mathfrak{Dist}(M)$ with the C^0-topology, and the space of linear maps can also be topologised by for example the topology induced by the Euclidean metric by identifying linear maps with matrices.

We note that if $\tilde{\xi}$ is C^0-close to ξ, we have that $\tilde{\xi}$ is transverse to TM/ξ. Let $\pi : TM \to \xi$ be a projection onto ξ. We can choose this projection by for example choosing a metric on M. Let $\eta : TM \to TM/\xi$ be a quotient map such that $\tilde{\xi}$ is precisely the complement of ξ, i.e. $\eta^{-1}(TM/\xi) = \tilde{\xi}$. We note that $\eta^{-1} \circ \varphi \circ (\pi|_{\tilde{\xi}} \wedge \pi|_{\tilde{\xi}})$ is a linear map from $\tilde{\xi} \wedge \tilde{\xi}$ to $TM/\tilde{\xi}$.

We then define $(\xi, \mathcal{E}, \varphi)$ and $(\tilde{\xi}, \tilde{\mathcal{E}}, \tilde{\varphi})$ to be C^0-close, if and only if ξ and $\tilde{\xi}$ and $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are C^0-close as distributions, and $\tilde{\varphi}$ and $\eta^{-1} \circ \varphi \circ (\pi|_{\tilde{\xi}} \wedge \pi|_{\tilde{\xi}})$ are close as linear maps. One can show that this topology on $\mathcal{F}\mathfrak{Dist}(M)$ is also the coarsest topology such that the projection map $\mathcal{F}\mathfrak{Dist}(M) \to \mathfrak{Dist}(M) \times \mathfrak{Dist}(M)$ is continuous. ▶

There exists in fact an $h-$principle for this type of distributions.

Theorem 1.10. *Let M be an n-dimensional manifold with $n > 3$. Then the inclusion*

$$\mathfrak{Dist}_{n.i.}(M) \hookrightarrow \mathcal{F}\mathfrak{Dist}_{n.i.}(M)$$

is a weak homotopy equivalence.

A proof of Theorem 1.10 can be found in [10, p. 77] and it essentially uses the Smale–Hirsch theorem for the h-principle for immersions. Since Smale-Hirsch also holds in a parametric and relative setting, Theorem 1.10 holds in a parametric and relative setting as well. This $h-$principle will be a useful tool in this thesis to prove other $h-$principles.

1.1.2 Bracket-generating distributions

Since taking the Lie bracket of two vector fields yields another vector field, one can apply the Lie bracket iteratively on vector fields tangent to a distribution. This motivates the following definitions.

Definition 1.11. *We call a string "a", where a is a formal variable, a bracket expression of length 1. We call*

$$[A(a_1, ..., a_i), B(a_{i+1}, ..., a_{i+j})]$$

a ***bracket expression of length*** $i + j + 1$ *where $A(-)$ and $B(-)$ are bracket expressions of length i and j respectively [11, p. 5].*

Definition 1.12. *Let ξ be a distribution. We define its* ***associated Lie flag*** *as the sequence of C^∞-modules*

$$\Gamma^{(1)}(\xi) \subset \Gamma^{(2)}(\xi) \subset \cdots \subset \Gamma^{(i)}(\xi) \subset \Gamma^{(i+1)}(\xi) \subset \dots$$

where $\Gamma^{(i)}(\xi) = \langle A(a_1, ..., a_i) | a_1, ..., a_i \in \xi \rangle_{C^\infty}$ is the C^∞-span of bracket-expressions of length $\leq i$ [11, p. 5].

We note that $\Gamma^{(1)}(\xi) = \Gamma(\xi)$.

Example 1.13. Let $\xi = \langle \partial_x + z^2\partial_y, \partial_z \rangle$ a rank 2 distribution on $\mathbb{R}^3$. This distribution is also called the *Martinet distribution*. We can compute the following Lie brackets,

$$[\partial_x + z^2\partial_y, \partial_z] = -2z\partial_y,$$
$$[\partial_z, -2z\partial_x] = -2\partial_y.$$

From this follows that

$$\Gamma(\xi) = \langle \partial_x + z^2\partial_y, \partial_z \rangle_{C^\infty}$$
$$\Gamma^{(2)}(\xi) = \langle \partial_x + z^2\partial_y, \partial_z, -2z\partial_y \rangle_{C^\infty}$$
$$\Gamma^{(3)}(\xi) = \langle \partial_x + z^2\partial_y, \partial_z, -2z\partial_y, -2\partial_y \rangle_{C^\infty} = \mathfrak{X}(\mathbb{R}^3).$$

Here $\mathfrak{X}(M)$ denotes the module of all vector fields on M. We note that the corresponding distribution of $\Gamma^{(2)}(\xi)$ is of rank 2 if $z = 0$ and of rank 3 if $z \neq 0$. Therefore, when we evaluate $\Gamma^{(2)}(\xi)$ at every point, this does not give us a well-defined distribution. This motivates the following definition. ([11, p. 6]) △

Definition 1.14. *We say a distribution ξ is **weakly regular** if all the $\Gamma^{(i)}(\xi)$ in its associated Lie flag correspond to genuine distributions. In this case we write $\Gamma^{(i)}(\xi) = \xi^{(i)}$ [11, p. 6].*

We note that the Martinet distribution in Example 1.13 is *not* weakly regular.

Lemma 1.15. *Let ξ be a weakly regular distribution on M. Then there exists an integer m such that $\xi^{(i)} = \xi^{(m)}$ for all $i \geq m$ [11, p. 6].*

Proof. Since ξ is weakly regular, we know that all the $\xi^{(i)}$ are distributions. If $\xi^{(i+1)} \neq \xi^{(i)}$ then we must have that $\text{rank}(\xi^{(i+1)}) > \text{rank}(\xi^{(i)})$, since $\xi^{(i)} \subset \xi^{(i+1)}$. However, the rank of all the distributions in the Lie flag are bounded by the rank of TM. Therefore, eventually the flag must stabilise. □

This result justifies the following definition ([11, p. 6]):

Definition 1.16. *Let ξ be a weakly regular distribution which stabilises in the m^{th} step. The **growth vector** of ξ is the vector given by*

$$\left(\text{rank}(\xi^{(1)}), \text{rank}(\xi^{(2)}), ..., \text{rank}(\xi^{(m)})\right).$$

All the distributions we will be considering in this thesis from now on will be weakly regular.

Definition 1.17. *We say a distribution ξ is **bracket-generating** if $\xi^{(m)} = \mathfrak{X}(M)$ for some integer $m \geq 1$ [11, p. 8].*

If a distribution is bracket-generating, then infinitesimally, we are able to move in all directions of the tangent space. This result is also known as Chow's theorem [7]. We note that the Martinet distribution in Example 1.13 is bracket-generating as $\xi^{(3)} = \mathfrak{X}(M)$. In the following two sections we will discuss contact and Engel structures, which are both special types of bracket-generating distributions. Also the structures we are mainly focusing on in this thesis, $(2,3,5)-$structures, are bracket-generating distributions on $5-$manifolds, with growth vector $(2,3,5)$, as the name suggests.

1.2 Contact structures

In this section we will study a special type of bracket-generating distributions called contact structures.

Definition 1.18. *Let ξ be a rank-2 distribution on a 3-manifold M such that locally $\xi = \langle X, Y \rangle$ on an open $U \subset M$. We say ξ is a **contact structure** on U if $\langle X, Y, [X, Y] \rangle = TM$ on U.*

We denote the space of contact structures on a 3-manifold M by $Cont(M)$ and we endow it with the C^1-topology. We endow this space with the C^1-topology because when working with contact structures, we are interested in first derivatives.

Example 1.19. Let ξ be the rank 2 distribution on $\mathbb{R}^3$ spanned by the vector fields ∂_y and $\partial_x + y\partial_z$. We note that

$$[\partial_y, \partial_x + y\partial_z] = \partial_z,$$

and thus ξ is a contact structure. In fact, ξ is called the **standard contact structure** on $\mathbb{R}^3$, and it is depicted in Figure 1.1 [11, p. 24]. △

Remark 1.20. Contact structures have a unique local model. This was proven by the French mathematician Darboux [8], and the neighbourhoods which admit this model are also called *Darboux balls.* See Section A.2 for a proof of this result using a technique called Moser's trick. ▶

Remark 1.21. A contact structure can also be defined as the kernel of a 1-form α such that $\alpha \wedge d\alpha \neq 0$. See [11, p. 22] for a proof that this is indeed an equivalent definition. We will sometimes use this equivalent definition in proofs. Furthermore, contact structures can in fact be defined on all manifolds M with dimension $2n + 1$ for $n \geq 1$. Then, a contact structure is defined as the kernel of a 1-form α such that $\alpha \wedge d\alpha^n \neq 0$. This corresponds to a maximally non-integrable hyperplane distribution. When we prove that contact structures have a unique local model in Section A.2, we use this definition based on the 1-form. ▶

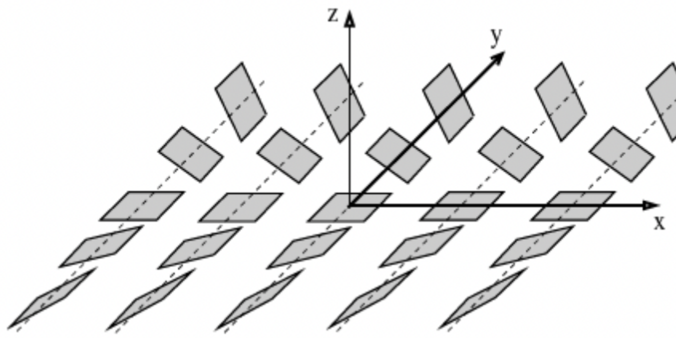

Figure 1.1: Illustration of the standard contact structure on $\mathbb{R}^3$ [15, p. 88].

Example 1.22. Let $\mathbb{T}^3 \simeq \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \ni (x, y, z)$ denote the 3-torus and let $k \in \mathbb{Z}^+$ be an integer. We can define the following distribution on $\mathbb{T}^3$:

$$\xi_k = \langle \partial_z, \sin(\pi k z)\partial_x - \cos(\pi k z)\partial_y \rangle.$$

We note that

$$[\partial_z, \sin(\pi k z)\partial_x - \cos(\pi k z)\partial_y] = \pi k \cos(\pi k z)\partial_x + \pi k \sin(\pi k z)\partial_y \notin \xi_k,$$

as $\sin(\pi k z)\partial_x - \cos(\pi k z)\partial_y$ and $\pi k \cos(\pi k z)\partial_x + \pi k \sin(\pi k z)\partial_y$ are linearly independent for all $z \in \mathbb{S}^1$. We conclude that ξ_k is a contact structure on $\mathbb{T}^3$ [11, p. 24]. △

The next lemma tells us more about when distributions on $\mathbb{R}^3$ are contact. For rank-2 distributions on 3-manifolds, one can also interpret these as local conditions for being contact.

Lemma 1.23. *Let $\xi = \langle \partial_x, \cos(f)\partial_y + \sin(f)\partial_z \rangle$ be a distribution on $\mathbb{R}^3$, with $f : \mathbb{R}^3 \to \mathbb{R}$ a smooth function. Then*

- *is contact if and only if $\frac{\partial f}{\partial x} \neq 0$,*
- *is involutive if and only if $\frac{\partial f}{\partial x} = 0$.*

Proof. We note that

$$[\partial_x, \cos(f)\partial_y + \sin(f)\partial_z] = -\sin(f) f_x \partial_y + \cos(f) f_x \partial_z.$$

This is linearly independent of ∂_x and $\cos(f)\partial_y + \sin(f)\partial_z$ if and only if $f_x = \frac{\partial f}{\partial x} \neq 0$. And thus ξ is contact if and only if $\frac{\partial f}{\partial x} \neq 0$, and ξ is involutive if and only if $\frac{\partial f}{\partial x} = 0$. □

Remark 1.24. When we consider ξ as a plane in $\mathbb{R}^3$, we note that ξ is contact if and only if ξ is turning in the $x-$direction, whereas ξ is involutive if and only if the plane is still. Furthermore, we note that the set $\left\{(x, y, z) \in \mathbb{R}^3 | \frac{\partial f}{\partial x} \neq 0\right\}$ is an open set, and thus being contact is an open property, whereas being involutive is a closed property. I.e. if ξ is contact in a point, it is also contact in a small neighbourhood around that point. This is not necessarily the case with involutive. ▶

1.2.1 The h-principle for contact structures on open manifolds

The formal analogue of a contact structure is defined in the following way.

Definition 1.25. *A **formal contact structure** is a rank-2 distribution ξ on a 3-manifold M together with an isomorphism*

$$\xi \wedge \xi \cong TM/\xi.$$

We denote the space of formal contact structures by $\mathcal{F}Cont(M)$ and we endow it with the C^0-topology. We note that any contact structure is also a formal contact structure, where the isomorphism is induced by the Lie bracket [11, p. 22-23]. I.e. we have the inclusion

$$Cont(M) \hookrightarrow \mathcal{F}Cont(M), \quad \xi \mapsto (\xi, [.,.]).$$

Remark 1.26. Just like the space of formal non-integrable distributions, the space of formal contact structures also forms a Serre fibration over $\mathfrak{Dist}(M)$. When we say that we endow $\mathcal{F}Cont(M)$ with the C^0-topology, we mean this in the same way as we endowed the space $\mathcal{F}\mathfrak{Dist}(M)$ with the C^0-topology, i.e. we endow the space $\mathfrak{Dist}(M)$ with the C^0-topology and we endow $\mathcal{F}Cont(M)$ with the coarsest topology such that the projection map $\mathcal{F}Cont(M) \to \mathfrak{Dist}(M)$ is continuous. ▶

The contact condition can also be expressed as a differential relation, and the following lemma shows that this relation is in fact open and Diff-invariant.

Lemma 1.27. *Being a contact structure is an open and Diff-invariant partial differential relation.*

Proof. Let M be a 3-dimensional manifold. Then the following differential relation is equivalent to the contact condition

$$\mathcal{R}_{contact} = \{j^1(X_1, X_2)(M) \mid X_1, X_2, [X_1, X_2] \text{ are linearly independent}\} \subset J^1(TM \times TM)$$

where X_1 and X_2 are vector fields on M. Note that vectors being linearly independent is equivalent to some determinant being non-zero. From this follows that $\mathcal{R}_{contact}$ is an open subset of $J^1(TM{\times}TM)$. Also, diffeomorphisms of M preserve the Lie bracket and determinants being non-zero, and thus $\mathcal{R}_{contact}$ is indeed Diff-invariant. □

The following theorem states that the $h-$principle for contact structures holds on open manifolds. This is a consequence of Theorem 0.3 by Gromov, and the previous lemma.

Theorem 1.28. *Let M be an open manifold. Then the inclusion*

$$Cont(M) \hookrightarrow \mathcal{F}Cont(M),$$

is a weak homotopy equivalence.

Naturally, one can ask whether the $h-$principle for contact structures also holds on closed manifolds. In the early 80's, Bennequin [1] showed that the answer in general is no. He proved the following statement:

Theorem 1.29 (Bennequin)**.** *In the space of plane fields in $\mathbb{S}^3$ there is a connected component that contains two different connected components of positive contact structures.*

This result shows that there is a connected component in $\mathcal{F}Cont(\mathbb{S}^3)$ which contains two contact structures which are not homotopic. I.e. this shows that the map $\pi_0(Cont(\mathbb{S}^3)) \to \pi_0(\mathcal{F}Cont(\mathbb{S}^3))$ is not injective. Therefore, the $h-$principle for contact structures cannot hold in general for closed manifolds. However, it does hold for a particular family of contact structures, called *overtwisted* contact structures. We will discuss these objects in Section 4.1.

1.3 Engel structures

In the last section of this chapter, we will take a look at another type of bracket-generating distributions, namely Engel structures.

Definition 1.30. *Let D be a rank-2 distribution on a 4-manifold M. We say D is an* ***Engel structure*** *if*

- $\mathcal{E} := [D, D]$ *is a rank 3 distribution,*
- $[\mathcal{E}, \mathcal{E}] = TM$.

We denote the space of Engel structures on a manifold M by $Engel(M)$ and we endow it with the C^2-topology. We note that when we are working with Engel structures, we are interested in the first and second derivatives, which is why we endow it with the C^2-topology.

Remark 1.31. Let $D = \langle X_1, X_2\rangle$ be an Engel structure. Then

- $\mathcal{E} = \langle X_1, X_2, [X_1, X_2]\rangle$,
- $[\mathcal{E}, \mathcal{E}] = \langle X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]], [X_2, [X_1, X_2]]\rangle = TM$.

We note that we must have that either $[X_1, [X_1, X_2]] \in \mathcal{E}$ or $[X_2, [X_1, X_2]] \in \mathcal{E}$. I.e. the map

$$D \times \mathcal{E}/D \xrightarrow{[.,.]} TM/\mathcal{E}$$

induced by the Lie bracket has a kernel $\mathcal{W} \subset D$, which we call the **characteristic line field** [12, p. 6]. ▶

Remark 1.32. Just like contact structures, Engel structures also have a unique local model. This was first proven by Engel in 1889 [16]. This is also where the name *Engel* structure comes from. The result states that given an Engel manifold (M, D), with local coordinates $(t, x, y, z) \in M$, we can (locally) write

$$D = \ker(dz - ydx, dy - tdx) = \langle \partial_t, \partial_x + y\partial_x + t\partial_y \rangle.$$

▶

If D is an Engel structure, then $\mathcal{E} = [D, D]$ is also a special type of distribution, called an *even-contact structure.*

Definition 1.33. *Let M be a 4-manifold and let $\mathcal{E}$ be a rank 3 distribution. We say $\mathcal{E}$ is an* ***even-contact structure*** *if $[\mathcal{E}, \mathcal{E}] = TM$.*

It follows that D is an Engel structure if and only if $\mathcal{E} = [D, D]$ is an even-contact structure.

Remark 1.34. We note that if $\mathcal{E}$ is an even-contact structure then the map

$$\mathcal{E} \times \mathcal{E} \xrightarrow{[.,.]} TM$$

induced by the Lie bracket has a kernel $\mathcal{W} \subset \mathcal{E}$. This kernel is characterised by the property that $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$, which means that all flows tangent to $\mathcal{W}$ preserve the even-contact structure $\mathcal{E}$. ▶

Let us now look at some examples of Engel structures.

Example 1.35. Let D be a rank-2 distribution on $\mathbb{R}^4$ given by $D = \langle \partial_w, \partial_x + z\partial_y + w\partial_z \rangle$. We note that

$$[\partial_w, \partial_x + z\partial_y + w\partial_z] = \partial_z,$$
$$[\partial_z, \partial_x + z\partial_y + w\partial_z] = \partial_y.$$

Therefore

$$\mathcal{E} := [D, D] = \langle \partial_w, \partial_x + z\partial_y + w\partial_z, \partial_z \rangle$$

is a rank-3 distribution, and

$$[\mathcal{E}, \mathcal{E}] = \langle \partial_w, \partial_x + z\partial_y + w\partial_z, \partial_z, \partial_y \rangle = \langle \partial_x, \partial_y, \partial_z, \partial_w \rangle = T\mathbb{R}^4.$$

We conclude that D is an Engel structure. Furthermore, since

$$[\partial_w, \partial_z] = 0,$$

we note that $\langle \partial_w \rangle \subset D$ is the characteristic line field [11, p. 17]. △

Example 1.36. Let (M, ξ) be a contact 3-manifold, such that $\xi = \langle X, Y \rangle$. We then define the following distribution on $M \times [0, 1]$:

$$D_{(p,t)} := \langle \partial_t, \cos(t)X_p + \sin(t)Y_p \rangle.$$

We note that

$$[\partial_t, \cos(t)X_p + \sin(t)Y_p] = -\sin(t)X_p + \cos(t)Y_p$$
$$[\cos(t)X_p + \sin(t)Y_p, -\sin(t)X_p + \cos(t)Y_p] = [X_p, Y_p].$$

Since ξ is a contact structure, $[X_p, Y_p]$ is linearly independent of X_p and Y_p. Therefore D is an Engel structure on $M \times [0, 1]$ with the characteristic line field $\langle \partial_t \rangle$. The Engel structure D is also called the **contact prolongation** of ξ [6, p. 5]. △

Just like in the contact case, we want to know when distributions on $\mathbb{R}^4$ are Engel. Again, one can also view these as local conditions for rank-2 distributions on 4-manifolds for being Engel. Before we can look at these conditions, we need to define the concept of *convexity*.

Definition 1.37. *Let $f : \mathbb{R} \to \mathbb{S}^n$ with $n \geq 2$ be a map. We say f is* ***convex*** *at t if $f(t)$, $\dot{f}(t)$ and $\ddot{f}(t)$ are linearly independent, i.e. the velocity and acceleration at time t are linearly independent vectors.*

Using this definition we can describe local conditions for being Engel in the next lemma [6, p. 5]:

Lemma 1.38. *Let $D = \langle \partial_t, H = f\partial_x + g\partial_y + h\partial_z \rangle$ be a rank-2 distribution on $\mathbb{R}^4$, where $f, g, h : \mathbb{R}^4 \to \mathbb{R}$ are smooth functions. We may assume that H has unit length, and thus for each (x, y, z) we can look at the map*

$$H_{(x,y,z)} : \mathbb{R} \to \mathbb{S}^2.$$

Then D is Engel at (x, y, z, t) if and only if at least one the following conditions holds:

- *the map $H_{(x,y,z)}$ is convex,*
- *the vectors $H_{(x,y,z)}(t)$ and $\dot{H}_{(x,y,z)}(t)$ span a contact structure in a neighbourhood $\mathcal{O}p((x, y, z)) \times \{t\}$.*

Proof. First of all, for $\mathcal{E}$ to be a 3-distribution, we must have that $[\partial_t, H] =: \dot{H}$ produces a new linearly independent vector field. I.e. the map $H_{(x,y,z)}$ is an immersion. It follows that $\mathcal{E} = \langle \partial_t, H, \dot{H} \rangle$. Then either $\ddot{H} := [\partial_t, \dot{H}]$ or $[H, \dot{H}]$ must give us the fourth linearly independent vector field. This is equivalent to $H_{(x,y,z)}$ being convex or the vectors $H_{(x,y,z)}(t)$ and $\dot{H}_{(x,y,z)}(t)$ spanning a contact structure in a neighbourhood $\mathcal{O}p(x, y, z) \times \{t\}$, respectively. □

1.3.1 The h-principle for Engel structures on open manifolds

The formal analogue of an Engel structure is defined in the following way.

Definition 1.39. *A* ***formal Engel structure*** *is a rank-2 distribution D on a 4-manifold M together with a flag $\mathcal{W} \subset D \subset \mathcal{E} \subset TM$, where $\mathcal{W}$ is a line field and $\mathcal{E}$ a rank-3 distribution, and isomorphisms*

$$D \wedge D \cong \mathcal{E}/D \quad \text{and} \quad D/\mathcal{W} \wedge \mathcal{E}/D \cong TM/\mathcal{E}.$$

We denote the space of formal Engel structures by $\mathcal{F}Engel(M)$ and we endow it with the C^0-topology. We note that any genuine Engel structure is also a formal Engel structure, where the isomorphisms are induced by the Lie bracket [12, p. 6]. I.e. we have the inclusion

$$Engel(M) \hookrightarrow \mathcal{F}Engel(M), \quad D \mapsto (\mathcal{W}, D, [D, D], [.,.], [.,.]).$$

Remark 1.40. Just like formal non-integrable distributions and formal contact structures, we do not literally endow $\mathcal{F}Engel(M)$ with the C^0-topology, but by this we mean the following. We note that $\mathcal{F}Engel(M) \to \mathfrak{Dist}(M) \times \mathfrak{Dist}(M) \times \mathfrak{Dist}(M)$ is a Serre fibration where the projection is given by projecting to the flag of distributions. We endow $\mathfrak{Dist}(M)$ with the C^0-topology, and we topologise $\mathcal{F}Engel(M)$ with the coarsest topology such that this projection map is continuous. ▶

The Engel condition can also be expressed as a differential relation, and the following lemma proves that this relation is in fact open and Diff-invariant.

Lemma 1.41. *Being an Engel structure is an open and Diff-invariant relation.*

Proof. Let M be a 4-dimensional manifold. The the following differential relation is equivalent to the Engel condition:

$$\mathcal{R}_{Engel} = \{j^2(X_1, X_2)(M) \mid X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]] \text{ are linearly independent}\},$$

where X_1 and X_2 are vector fields on M. Using the exact same reasoning as in Lemma 1.27, it follows that $\mathcal{R}_{Engel} \subset J^2(TM \times TM)$ is indeed open and Diff-invariant. □

The following theorem states that the $h-$principle for Engel structures on open manifolds holds. Just like in the contact case, this is a consequence of Theorem 0.3 by Gromov and the previous lemma.

Theorem 1.42. *Let M be an open manifold. Then the inclusion*

$$Engel(M) \hookrightarrow \mathcal{F}Engel(M)$$

is a weak homotopy equivalence.

Naturally, one can wonder if the h-principle for Engel structures also holds on closed manifolds. Unlike in the contact case, we do not know the answer to this question. Whether this is the case is one of the main open questions in this field. However, we do know that there is a special class of Engel structures, called *overtwisted* Engel structure, for which the $h-$principle does hold on closed manifolds. This was proven by del Pino and Vogel in 2018 [12], as also mentioned in the introduction.

Chapter 2

(2,3,5)-structures

In this chapter we will discuss the main object of interest of this thesis, $(2,3,5)-$structures. Just like contact and Engel structures, these are a type of bracket-generating distributions. We will first discuss the definition and go over some local $(2,3,5)-$conditions. Thereafter, we will prove that there exists a (non-unique) local model for $(2,3,5)$-structures and we will define what we call the *standard* $(2,3,5)-$structure. Lastly, we will look at *formal* $(2,3,5)-$structures and as a corollary of Gromov's h-principle, prove that the h-principle holds for $(2,3,5)-$structures on open manifolds.

2.1 Definition and local conditions

The name of a $(2,3,5)-$structure is certainly suggestive. The formal definition is the following.

Definition 2.1. *Let M be a 5-dimensional manifold, and let D be a smooth rank 2 distribution on M. We say D is a $(2,3,5)$-**structure** if D has growth vector $(2,3,5)$, i.e. we have isomorphisms*

$$D \times D \xrightarrow{[.,.]} D^{(2)}/D$$
$$D \times D^{(2)}/D \xrightarrow{[.,.]} TM/D^{(2)}.$$

We denote the space of $(2,3,5)$-structures by $\mathfrak{Dist}_{(2,3,5)}(M)$ and we endow it with the C^2-topology. Just like with Engel structures, when working with $(2,3,5)-$structures we are interested in the first and second derivatives, since we are applying the Lie bracket twice. This is why we endow it with the C^2-topology.

Remark 2.2. Recall that locally, smooth distributions can be described as a span of vector fields. Let $D = \langle X_1, X_2\rangle$ with X_1 and X_2 vector fields on M. Then D is a $(2,3,5)$-structure if

$$\langle X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]], [X_2, [X_1, X_2]]\rangle = TM.$$

If D is oriented and $\{X_1, X_2\}$ is a framing of D compatible with this orientation, then we call $\{X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]], [X_2, [X_1, X_2]]\}$ a $(2,3,5)$-**framing**. ▶

Example 2.3. Let

$$D = \langle \partial_t, X := \partial_x + y\partial_z + t\partial_y + t^2\partial_w\rangle,$$

be a rank 2 distribution on $\mathbb{R}^5$, with coordinates (t, x, y, z, w). We see that

$$[\partial_t, X] = \partial_y + 2t\partial_w,$$
$$[\partial_t, [\partial_t, X]] = 2\partial_w,$$
$$[X, [\partial_t, X]] = -\partial_z.$$

Every Lie bracket gives a new vector field which is linearly independent from the previous vector fields. From this follows that the growth vector of D is $(2,3,5)$ and thus D is a $(2,3,5)-$structure. △

We now want to describe conditions for when a 2-distribution D is locally a $(2,3,5)-$structure. These local conditions will help us several times later on in this thesis to prove that certain distributions are $(2,3,5)$.

Fix coordinates $(p,t) \in \mathbb{D}^4 \times \mathbb{R}$, and let $\partial_t \in D$. Let $\{W(p), X(p), Y(p), Z(p)\}$ be a framing of $T_p\mathbb{D}^4$, then we can identify the vector field

$$(p,t) \mapsto (H_p(t))_1 \cdot W + (H_p(t))_2 \cdot X + (H_p(t))_3 \cdot Y + (H_p(t))_4 \cdot Z,$$

with maps $(H_p : \mathbb{R} \to \mathbb{S}^3)_{p\in\mathbb{D}^4}$. Thus D can be described in terms of a smooth $\mathbb{D}^4$-family of curves $(H_p)_{p\in\mathbb{D}^4}$. We write $D = \langle \partial_t, H\rangle$.

Proposition 2.4. *Let D be a 2-distribution on a 5-dimensional manifold, locally described by $\mathbb{D}^4-$family of maps $(H_p)_{p\in\mathbb{D}^4}$. Then D is a $(2,3,5)$-structure near $(p,t) \in \mathbb{D}^4 \times \mathbb{R}$ if and only if the following conditions hold:*

(i) H_p is an immersion at time t.

(ii) $H_p(t), \dot{H}_p(t), \ddot{H}_p(t)$ are linearly independent, i.e. the map H_p is convex at t,

(iii) $\langle H_q(t), \dot{H}_q(t)\rangle$ is a non-integrable 2-distribution in the level set $Op(p) \times \{t\}$,

(iv) $[H_p(t), \dot{H}_p(t)]$ is linearly independent of $H_p(t), \dot{H}_p(t)$ and $\ddot{H}_p(t)$.

Proof. First of all, condition *(i)* is equivalent to D being non-integrable. Namely, we note that $H_p(t)$ is tangent to $\mathbb{D}^4 \times \{t_0\}$ for $t_0 \in \mathbb{R}$, thus

$$[\partial_t, H_p(t)] = \partial_t(H_p(t)) - H_p(t)(\partial_t) = \partial_t(H_p(t)) - 0 = \dot{H}_p(t).$$

We also note that $[\partial_t, H_p(t)]$ is tangent to $\mathbb{D}^4 \times \{t_0\}$ for $t_0 \in \mathbb{R}$, thus the condition of D being non-integrable is equivalent to $H_p(t)$ and $\dot{H}_p(t)$ being linearly independent. In turn, this is equivalent to the curve H_p being an immersion.

Secondly, for D to be a $(2,3,5)$-distribution, the distribution $[D,D]$ must be bracket-generating, i.e. we must have that

$$\langle \partial_t, H_p(t), \dot{H}_p(t), [\partial_t, \dot{H}_p(t)], [H_p(t), \dot{H}_p(t)]\rangle = TM.$$

Since $\dot{H}_p(t)$ is tangent to the foliation by level sets $\mathbb{D}^4 \times \{t_0\}$ for $t_0 \in \mathbb{R}$, we have that $[\partial_t, \dot{H}_p(t)] = \ddot{H}_p(t)$. Furthermore, $[H_q(t), \dot{H}_q(t)] \notin \langle H_q(t), \dot{H}_q(t)\rangle$ if and only if $\langle H_q(t), \dot{H}_q(t)\rangle$ is non-integrable. Therefore, condition *(i)-(iv)* are then equivalent to

$$\langle \partial_t, H_p(t), \dot{H}_p(t), \ddot{H}_p(t), [H_p(t), \dot{H}_p(t)]\rangle = TM.$$

□

2.2 The Monge normal form

For contact and Engel structures we know that locally there exists a unique model, see Remark 1.20 and 1.32. For $(2,3,5)-$structures we also have a local description, called the *Monge normal form* [38, p. 1]. However, contrary to the contact and Engel case, this model is non-unique. In this section we will define the notion of the Monge normal form, and we will prove that locally any $(2,3,5)-$structure can be put in such form. Lastly, we will define the *standard* $(2,3,5)-$structure, which has a specific Monge normal form.

A *Monge equation* is a first order partial differential equation of the form

$$w'(x) = F(x, z(x), z'(x), z''(x), w(x)). \tag{2.1}$$

It turns out that it can be convenient to encode $(2,3,5)-$structures by such Monge equations. We note that any curve $(x, z(x), w(x)) \in \mathbb{R}^3$ can be extended to a curve $(x, z(x), z'(x), z''(x), w(x)) \in J^2(\mathbb{R},\mathbb{R}) \times \mathbb{R} \cong \mathbb{R}^5$. By construction, such a curve is tangent to the distribution given by

$$D_F = \ker(dz - ydx, dy - tdx, dw - F(x,z,y,t,w)dx) = \langle \partial_t, \partial_x + y\partial_z + t\partial_y + F(x,z,y,t,w)\partial_w\rangle.$$

Here we use the coordinates $(x,z,y,t,w) \in \mathbb{R}^5$. On the other hand, the projection of any curve which is tangent to the distribution D_F (for which the pullback of dx is nowhere zero) to the $(x,z,w)-$plane is also a solution to the differential equation (2.1). Thus, there is a clear correspondence between Monge equations and distributions of the form D_F.

Definition 2.5. *We say the distribution*

$$D_F := \langle \partial_t, \partial_x + y\partial_z + t\partial_y + F\partial_w \rangle$$

with $F : \mathbb{R}^5 \to \mathbb{R}$ *is in **Monge normal form**.*

We see that

$$\begin{aligned}
[\partial_t, \partial_x + y\partial_z + t\partial_y + F\partial_w] &= \partial_y + (\partial_t F)\partial_w,\\
[\partial_t, \partial_y + (\partial_t F)\partial_w] &= (\partial_{tt} F)\partial_w,\\
[\partial_y + (\partial_t F)\partial_w, \partial_x + y\partial_z + t\partial_y + F\partial_w] &= \partial_z,
\end{aligned}$$

and thus D_F is $(2,3,5)$ precisely when $\partial_{tt}F \neq 0$. It turns out that any $(2,3,5)-$structure can locally be described by such a Monge equation for which $\partial_{tt}F$ is nowhere zero, which we prove in the next proposition [22].

Proposition 2.6. *Let* (M, D) *be a* $(2,3,5)-$*manifold, and let* $B \subset M$ *be a ball in* M *with local coordinates* (x, z, y, t, w)*. Then there is a function* $F : \mathbb{R}^5 \to \mathbb{R}$ *with* $\partial_{tt}F \neq 0$ *such that*

$$D = D_F := \langle \partial_t, \partial_x + y\partial_z + t\partial_y + F(x, z, y, t, w)\partial_w \rangle.$$

Proof. Locally, we can pick the following vector fields:

$$T, X \in D, \quad Y \in \mathcal{E}/D, \quad Z := [X, Y], \quad W \text{ linearly independent of } X, Y \text{ and } Z.$$

Thus, locally $\{T, X, Y, Z, W\}$ is a frame for $T\mathbb{R}^5$. Using the flow of the vector field T, we can see $T = \partial_t$ as one of the coordinate directions, and we can assume that the vector fields X, Y, Z and W are tangential to the fibres induced by this flow box. Then, for every value of t we see that the distribution

$$\tilde{\mathcal{E}} := \langle X, Y, W \rangle$$

is an even-contact structure. There exists a Darboux theorem for even-contact structures [11, p. 18], and from this follows that there are coordinates (x, y, z, w) such that we can write

$$\tilde{\mathcal{E}} = \ker(dz - ydx) = \langle \partial_x + y\partial_z, \partial_y, \partial_w \rangle.$$

We can then view the vector field X as a function of t, for which we have

$$X_{(x,y,w,z)}(t) = \partial_x + y\partial_z + G\partial_y + F\partial_w,$$

with functions $G, F : \mathbb{R}^5 \to \mathbb{R}$. Since D is non-integrable, X_p must be an immersion. From this follows that $\partial_t G \neq 0$, and using the implicit function theorem we can replace G by t, i.e. $X_p(t)$ is of the form

$$X_p(t) = \partial_x + y\partial_z + t\partial_y + F\partial_w.$$

Lastly, we must have that $\partial_{tt}F$ is non-zero, which ensures that $[T, [T, X]]$ is linearly independent of $\{T, X, [T, X], [X, [T, X]]\}$. □

As mentioned in the beginning of this section, this Monge normal form is not unique. Different functions F can yield equivalent 2-plane fields D_F. Monge equations have been used to classify $(2,3,5)-$structures up to local diffeomorphism, see for example [43]. In the next definition we single out the case where $F = t^2$. Note that this is precisely the $(2,3,5)-$structure from Example 2.3. We will use this local model later on when we define overtwisted discs in $(2,3,5)-$manifolds in Chapter 6. Since $F = t^2$ is probably the most simple function one can think of, for which $\partial_{tt}F \neq 0$, we refer to it as the *standard* $(2,3,5)-$structure.

Definition 2.7. *Let* $(\mathbb{R}^5, D_{st})$ *be the* $(2,3,5)-$*manifold with*

$$D_{st} = \langle \partial_t, \partial_x + y\partial_z + t\partial_y + t^2\partial_w \rangle.$$

We call D_{st} *the **standard*** $(2,3,5)-$***structure*** *on* $\mathbb{D}^5$.

Example 2.8. In the introduction of this thesis we discussed the example of the rolling distribution. Very recently, in February 2023, Randall proved in his paper [38] that this distribution has the following Monge normal form:

$$D_\rho = \ker\left(dz - ydx, dy - tdx, dw - \left(ty^2 + \frac{1}{\alpha^2 - 1}\left(\sqrt{t}w - \frac{1}{2\sqrt{t}x}\right)^2\right)dx\right)$$

where α is a constant depending on the radii R and r. △

In Section 1.3, we showed that given a contact manifold (M, ξ), we can construct an Engel structure on the manifold $M \times [0,1]$. We call this Engel structure a *contact prolongation*. Analogously, in the next proposition we show that given an Engel manifold (M, D), we can construct a $(2,3,5)-$structure on the manifold $M \times [0,1]$. Similarly, we call this $(2,3,5)-$structure the *Engel prolongation*.

Proposition 2.9. *Let (M, D_E) be an Engel manifold. Then there exists a $(2,3,5)-$structure $D_{(2,3,5)}$ on $M \times [0,1]$, such that on every fibre, $D_{(2,3,5)}$ restricts to the given Engel structure D_E.*

Proof. We know that for Engel structures, there exists a local canonical form [16]. I.e., there are local coordinates $(t, x, y, z) \in M$ such that

$$D_E = \ker(dz - ydx, dy - tdx) = \langle \partial_t, \partial_x + y\partial_z + t\partial_y \rangle.$$

Now let (t, x, y, z, w) denote local coordinates in $M \times [0,1]$. We can define the distribution

$$D_{(2,3,5)} := \ker(dz - ydx, dy - tdx, dw - t^2 dx) = \langle \partial_t, \partial_x + y\partial_z + t\partial_y + t^2\partial_w \rangle.$$

We note that this is indeed a $(2,3,5)-$structure, which restricts to D_E on each fibre $M \times \{w\}$. □

2.3 The h-principle for $(2,3,5)-$structures on open manifolds

We can also define a formal analogue of a $(2,3,5)-$structure.

Definition 2.10. *A **formal** $(2,3,5)$**-structure** is a 2-distribution D on a 5-dimensional manifold M together with a flag $D \subset \mathcal{E} \subset TM$, where $\mathcal{E}$ is a rank-3 distribution, and isomorphisms*

$$D \wedge D \simeq \mathcal{E}/D \quad \text{and} \quad D \wedge \mathcal{E}/D \simeq TM/\mathcal{E}.$$

We denote the space of formal $(2,3,5)-$structures by $\mathcal{F}\mathfrak{Dist}_{(2,3,5)}(M)$ and we endow it with the C^0-topology. We note that any genuine $(2,3,5)-$structure is also a formal $(2,3,5)-$structure, where the isomorphisms are induced by the Lie bracket. I.e. we have the inclusion

$$\mathfrak{Dist}_{(2,3,5)}(M) \hookrightarrow \mathcal{F}\mathfrak{Dist}_{(2,3,5)}(M), \quad D \mapsto (D, [D,D], [.,.], [.,.]).$$

Remark 2.11. Just as with the other spaces of formal objects we have seen thus far, let us briefly clarify what we mean when we endow $\mathcal{F}\mathfrak{Dist}_{(2,3,5)}(M)$ with the C^0-topology. We note that $\mathcal{F}\mathfrak{Dist}_{(2,3,5)}(M) \to \mathfrak{Dist}(M) \times \mathfrak{Dist}(M)$ is a Serre fibration where the projection is given by projecting to the distributions. We endow $\mathfrak{Dist}(M)$ with the C^0-topology and we topologise $\mathcal{F}\mathfrak{Dist}_{(2,3,5)}(M)$ with the coarsest topology such that this projection map is continuous. ▶

The $(2,3,5)-$condition can also be expressed as a differential relation, and the following lemma proves that this relation is open and Diff-invariant.

Lemma 2.12. *Being a $(2,3,5)-$structure is an open and Diff-invariant relation.*

Proof. Let M be a $5-$dimensional manifold. Then the following differential relation is equivalent to the $(2,3,5)-$condition:

$$\mathcal{R}_{(2,3,5)} = \{j^2(X_1, X_2)(M) \mid X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]], [X_2, [X_1, X_2]] \text{ are linearly independent}\},$$

where X_1 and X_2 are vector fields on M. Using the exact same reasoning as in Lemma 1.27, it follows that $\mathcal{R}_{(2,3,5)} \subset J^2(TM \times TM)$ is indeed open and Diff-invariant. □

The following theorem states that the $h-$principle for $(2,3,5)-$structures holds on open manifolds. Just like for contact and Engel structures, this is a consequence of Theorem 0.3 by Gromov and Lemma 2.12.

Theorem 2.13. *Let M be an open manifold. Then the following inclusion*

$$\mathfrak{Dist}_{(2,3,5)}(M) \hookrightarrow \mathcal{F}\mathfrak{Dist}_{(2,3,5)}(M),$$

is a weak homotopy equivalence.

The main result of this thesis, as also mentioned in the introduction, is that the $h-$principle also holds on closed manifolds, but for a special family of $(2,3,5)-$structures called *overtwisted* $(2,3,5)-$structures. We will define these objects in Chapter 6. Whether the $h-$principle holds on closed manifolds for $(2,3,5)-$structures in general, is unknown.

Remark 2.14. Recall that for a manifold to admit a $(2,3,5)-$structure, it must certainly admit a formal $(2,3,5)-$structure. In [9] Dave and Haller address the existence problem of orientable $(2,3,5)-$structures, i.e. which manifolds admit an orientable rank 2 distribution with growth vector $(2,3,5)$? Their approach is split into two steps. First some immediate topological obstructions are observed, which has partly to do with Proposition C.18. For example, for a manifold to admit an orientable $(2,3,5)-$structure it must always be spinnable.

On open manifolds, if the topological requirements are met, then an orientable $(2,3,5)-$structure exists. The proof of this result uses the h-principle for $(2,3,5)-$structures on open manifolds, i.e. it is enough to show that the manifold admits a formal $(2,3,5)-$structure. However, we do not know whether this $h-$principle holds for closed manifolds, and thus it is also not immediately clear if every closed manifold admits an orientable $(2,3,5)-$structure. Nevertheless, in [9] some examples of $(2,3,5)-$structures on closed manifolds are constructed. ▶

Chapter 3

Bracket-generating distributions on the mapping torus

In this chapter we construct explicit bracket-generating distributions on mapping tori admitting a suitable formal analogue of these distributions. For example, in the last section of this chapter we will construct an explicit $(2,3,5)-$structure on all mapping tori admitting a suitable formal $(2,3,5)-$structure. This is in fact a new result, as it was not known whether this was possible up until now. Next to this being an original contribution of the thesis, this particular construction is also quite interesting because it already highlights some of the key issues we are going to encounter when proving the h-principle for overtwisted $(2,3,5)-$structures on closed manifolds. Namely, it uses the idea of adding more turning into a structure, in order to achieve the $(2,3,5)-$condition. See also the end of Section 3.3 for more remarks on this. In the first two sections we construct an explicit contact structure and an explicit Engel structure on the mapping torus. These constructions are more simple than the $(2,3,5)-$construction, and are thus enlightening to look at first.

3.1 A contact structure on the mapping torus

In this section we construct an explicit contact structure on mapping tori admitting a suitable formal contact structure. We will first prove the following preparatory lemma, which allows us to write distributions on the mapping torus in a certain form such that they are easier to work with.

Lemma 3.1. *Let N be a parallelizable $n-$manifold with $n \geq 1$ and let $\varphi : N \to N$ be a diffeomorphism. Let D be a distribution of rank $1 \leq k \leq n$ on the mapping torus*

$$M_\varphi := \frac{N \times [0,1]}{(x,1) \sim (\varphi(x),0)},$$

which is parallelizable and transverse to the fibres. Then there exist vector fields $X_1, ..., X_{k-1}$ on $N \times \mathbb{R}$ tangential to the fibres, and a map $f : N \times \mathbb{R} \to M_\varphi$ invariant under the $\mathbb{Z}-$action given by

$$\mathbb{Z} \times N \times \mathbb{R} \to N \times \mathbb{R}, \quad (z,x,t) \mapsto (\varphi^z(x), z+t)$$

such that

$$f^*D = \langle \partial_t, X_1, ..., X_{k-1} \rangle.$$

Proof. We pick a vector field $T \in D$ which is transverse to the fibres in M_φ. Then let

$$f : N \times \mathbb{R} \to M_\varphi$$

be the map given by $f(x,t) = \phi_t^T(x)$, where ϕ_t^T is the flow of T at time t. From this follows that

$$f^*T = \partial_t.$$

Furthermore, since f^*D is a rank k distribution, we can pick vector fields $X_1, ..., X_{k-1}$ on $N \times \mathbb{R}$ tangential to the fibres such that

$$f^*D = \langle \partial_t, X_1, ..., X_{k-1} \rangle.$$

We note that f is automatically invariant under the $\mathbb{Z}-$action because the vector field T is defined on the mapping torus M_φ. □

Remark 3.2. Note that the map f in the lemma above is a quotient map from $N\times\mathbb{R}$ to the mapping torus M_φ. The flow of the vector field $T\in D$ is in fact homotopic to the map φ. Therefore, when we quotient using the map f or φ, we obtain the same structure M_φ. However, the map f has the desired property that we can express (the pull-back of) D in the explicit form as above, which is why we proved this lemma. The invariance under the $\mathbb{Z}-$action will not be used in this section, but it will become important in the last section where we construct a $(2,3,5)-$structure on a mapping torus. ▶

From now on, we will not explicitly write down the pull-back of the map we have constructed in the lemma above, but we will just write down the formal contact structure in the desired form. Note that we are thus implicitly using this pull-back in the proof of the next theorem.

Theorem 3.3. *Let N be a parallelizable 2-dimensional manifold and let $\varphi : N\to N$ be a diffeomorphism. Let ξ be a formal contact structure on the mapping torus M_φ which is orientable and transverse to the fibres. Then M_φ admits a genuine contact structure.*

Proof. We note that Lemma 3.1 allows us to write $\xi=\langle\partial_t,X\rangle$ where X is a vector field tangent to the fibres. Let $Y\in TM/\xi$ be a vector field on M_φ linearly independent of ∂_t and X, which is tangential to the fibres. We then define the following vector fields on $N\times\{t\}$:

$$\begin{aligned}\overline{X}_t &:= \cos(kt)X_t+\sin(kt)Y_t\\ \overline{Y}_t &= \dot{\overline{X}}_t.\end{aligned}$$

Here $k>0$ is a constant. We note that

$$\overline{Y}_t=\dot{\overline{X}}_t=k\left(-\sin(kt)X_t+\cos(kt)Y_t\right)+\cos(kt)\dot{X}_t+\sin(kt)\dot{Y}_t.$$

For large values of k, the contribution of the derivatives of the frame $\{X_t,Y_t\}$ becomes arbitrarily small, and the vector fields $\overline{X}_t$ and $\overline{Y}_t$ become linearly independent. From this follows that $\overline{\xi}:=\langle\partial_t,\overline{X}\rangle$ is a genuine contact structure. □

Remark 3.4. Let us give some intuition behind the proof of Theorem 3.3. Imagine we reparameterise the $t-$coordinate such that it becomes very stretched out. Then the derivatives of the frame $\{X_t,Y_t\}$ with respect to t become almost zero. Therefore, if we then turn in X_t and Y_t with constant speed, we obtain the contact condition. If we then parameterise the $t-$coordinate back to the starting situation, we obtain the very fast turning which we see in the vector field $\overline{X}_t$ defined above. ▶

3.2 An Engel structure on the mapping torus

Similarly as in the contact case, we are now going to construct an explicit Engel structure on all mapping tori admitting a suitable formal Engel structure. Just as in the previous section, we use Lemma 3.1 to write a formal Engel structure in a form which is easier to work with. The following theorem was first proven by Geiges [17], as an additional comment in a review on [33].

Theorem 3.5. *Let N be a parallelizable 3-dimensional manifold and let $\varphi : N\to N$ a diffeomorphism. Let $\mathcal{W}\subset D\subset\mathcal{E}\subset TM$ be a formal Engel structure on the mapping torus*

$$M_\varphi := \frac{N\times[0,1]}{(x,1)\sim(\varphi(x),0)},$$

which is orientable and transverse to the fibres $N\times\{t\}$ for all $t\in[0,1]$. Then M_φ also admits a genuine Engel structure.

Proof. From Lemma 3.1 we know that we can write $D=\langle\partial_t,X\rangle$ and $\mathcal{E}=\langle\partial_t,X,Y\rangle$ with X and Y vector fields tangential to the fibres. Let W be a vector field also tangent to the fibres, and linearly independent of X and Y. We then define the following vector fields

$$\begin{aligned}\overline{X}_t &:= \cos(kt)X_t+\sin(kt)Y_t+k^\delta W_t,\\ \overline{Y}_t &:= \dot{\overline{X}}_t,\\ \overline{W}_t &:= \ddot{\overline{X}}_t.\end{aligned}$$

Here $k > 0$ and $\delta > 0$ are positive constants. We note that

$$\overline{Y}_t = \dot{\overline{X}}_t = k(-\sin(kt)X_t + \cos(kt)Y_t) + \cos(kt)\dot{X}_t + \sin(kt)\dot{Y}_t + k^\delta \dot{W}_t,$$

and

$$\overline{W}_t = \ddot{\overline{X}}_t = -k^2\left(\cos(kt)X_t + \sin(kt)Y_t\right) + 2k\left(-\sin(kt)\dot{X}_t + \cos(kt)\dot{Y}_t\right) + \cos(kt)\ddot{X}_t + \sin(kt)\ddot{Y}_t + k^\delta \ddot{W}_t.$$

We will show that for appropriate values of k and δ, these three vectors $\overline{X}_t$, $\overline{Y}_t$ and $\overline{W}_t$ are linearly independent. Note that this will mean that $\langle \partial_t, \overline{X}\rangle$ is in fact an Engel structure on M_φ (see also Lemma 1.38). We look at the following matrix

$$M := \begin{pmatrix} \cos(kt) & -k\sin(t) + \mathcal{O}(k^\delta) & -k^2\cos(kt) + \mathcal{O}(k) \\ \sin(kt) & k\cos(kt) + \mathcal{O}(k^\delta) & -k^2\sin(kt) + \mathcal{O}(k) \\ k^\delta & \mathcal{O}(k^\delta) & \mathcal{O}(k) \end{pmatrix}.$$

We note that this is a matrix of the vectors $\overline{X}_t$, $\overline{Y}_t$ and $\overline{W}_t$ as columns, with respect to the basis $\{X_t, Y_t, W_t\}$. Here we have used to notation $\mathcal{O}(k^a)$ to indicate the terms of order at most k^a.

We can work out the determinant of this matrix and we obtain

$$\det(M) = k^{3+\delta} + \mathcal{O}(k^{2+2\delta}).$$

Again, $\mathcal{O}(k^{2+2\delta})$ indicates terms of order at most $k^{2+2\delta}$. We note that if $0 < \delta < 1$ and $k > 1$, then we have that $k^{3+\delta} > k^{2+2\delta}$. Thus choosing $\delta < 1$ and k sufficiently large enough, means that the leading term will overcome the coefficients which appear in the error term. We conclude that there exist k and δ such that the determinant of M is non-zero, and thus values for k and δ such that the vector fields $\overline{X}_t$, $\overline{Y}_t$ and $\overline{W}_t$ are linearly independent. We conclude that $\overline{D} := \langle \partial_t, \overline{X}\rangle$ is an Engel structure on M_φ. □

Remark 3.6. Again, let us provide some intuition behind the construction above. The idea is very similar to idea behind the construction in the previous section. Suppose we reparameterise the $t-$coordinate such that it becomes very stretched out. Then the derivatives of the frame $\{X_t, Y_t, W_t\}$ with respect to t are almost zero. Thus by turning in X_t and Y_t with constant speed and adding a small contribution of W_t for convexity, we obtain the Engel condition. If we then reparameterise the $t-$coordinate back to the original situation, we obtain the very fast turning we see in the vector field $\overline{X}_t$. ▶

3.3 A $(2, 3, 5)$-structure on the mapping torus

In this section we construct a $(2, 3, 5)$-structure on all mapping tori admitting a suitable formal $(2, 3, 5)-$structure. Up to now it was not known whether this was possible, and thus this construction is an original result. Furthermore, the construction already highlights some of the key issues one has to overcome when proving Theorem 0.4, the main result of this thesis, on the h-principle of overtwisted $(2, 3, 5)-$structures. To construct this $(2, 3, 5)-$structure, we first prove the following preparatory lemma. Note that we again use Lemma 3.1, to write a formal $(2, 3, 5)-$structure into a form which is easier to work with.

Lemma 3.7. *Let N be a parallelizable $4-$dimensional manifold and let $\varphi : N \to N$ a diffeomorphism. Let $D \subset \mathcal{E} \subset TM$ be a formal $(2, 3, 5)-$structure on the mapping torus*

$$M_\varphi := \frac{N \times [0, 1]}{(x, 1) \sim (\varphi(x), 0)},$$

which is orientable and transverse to the fibres. Then there exists a rank 3 distribution $\tilde{\mathcal{E}}$ on M_φ such that

- *$\tilde{\mathcal{E}}$ is formally homotopic to $\mathcal{E}$,*
- *$f^*\tilde{\mathcal{E}} = \langle \partial_t, \tilde{X}, \tilde{Y}\rangle$ for vector fields $\tilde{X}$ and $\tilde{Y}$ on $N \times \mathbb{R}$ tangential to the fibres,*
- *$\tilde{\mathcal{E}} \cap (N \times \{t\})$ is non-integrable for all $t \in [0, 1]$.*

Proof. From Lemma 3.1 we know that we can write $f^*D = \langle \partial_t, X\rangle$ and $f^*\mathcal{E} = \langle \partial_t, X, Y\rangle$ for vector fields X and Y on $N \times \mathbb{R}$ tangential to the fibres. We write $\xi_t := \langle X_t, Y_t\rangle$, and we note that this is a distribution of rank 2 on the 4-manifold $N \times \{t\}$. From the isomorphism $D \wedge \mathcal{E}/D \cong TM/\mathcal{E}$ we obtain the isomorphism

$$\xi_t \wedge \xi_t \cong TM/\xi_t.$$

We then invoke Theorem 1.10, first on the subsets $N \times [n-\delta, n+\delta] \subset N \times \mathbb{R}$ for $n \in \mathbb{Z}$ and for some $0 < \delta < 1$, relative to the endpoints of the intervals. From this we obtain a non-integrable distribution $\tilde{\xi}_t$ on $N \times \{t\}$ for $t \in [n-\delta, n+\delta]$ homotopic to ξ_t. Note that on every interval, the distribution f^*D looks the same (as it is a pull-back of a distribution by a quotient map), and thus on every interval we can also choose $\tilde{\xi}_t$ the same. This will be helpful when we quotient again by the map f to construct a distribution on the mapping torus.

We then do the same for the distributions ξ_t for $t \in [n+\delta-\epsilon, n+1-\delta+\epsilon]$ with $0 < \epsilon < \delta$, also relative to the end-points. This ensures that $\tilde{\xi}_t$ is non-integrable for all $t \in [0,1]$. Because the $h-$principle in Theorem 1.10 holds in a parametric setting, we can find vector fields $\tilde{X}$ and $\tilde{Y}$ such that $\tilde{X}_t = \tilde{X} \cap (N \times \{t\})$ and $\tilde{Y}_t = \tilde{Y} \cap (N \times \{t\})$ and $\tilde{\xi}_t = \langle \tilde{X}_t, \tilde{Y}_t\rangle$. Then the push-forward of the distribution $\langle \partial_t, \tilde{X}, \tilde{Y}\rangle$ by the map f is the desired distribution $\tilde{\mathcal{E}}$. □

From now on we will use Lemma 3.1 and 3.7, but for simplicity we will not explicitly write down the pull-back by f. We will for example write $D = \langle \partial_t, X\rangle$ for the distribution on M_φ, but the reader should note that implicitly we are using a pull-back of a quotient map which makes use of the flow lines of the vector field transverse to the fibres and tangent to D.

Theorem 3.8. *Let N be a parallelizable 4-dimensional manifold and let $\varphi : N \to N$ a diffeomorphism. Let $D \subset \mathcal{E} \subset TM$ be a formal $(2,3,5)-$structure on the mapping torus*

$$M_\varphi := \frac{N \times [0,1]}{(x,1) \sim (\varphi(x), 0)},$$

which is orientable and transverse to the fibres $N \times \{t\}$ for all $t \in [0,1]$. Then M_φ also admits a genuine $(2,3,5)-$structure.

Proof. From Lemma 3.1 we know that we can write $D = \langle \partial_t, X\rangle$ and $\mathcal{E} = \langle \partial_t, X, Y\rangle$. From Lemma 3.7 we know there exists a rank 3 distribution $\tilde{\mathcal{E}} = \langle \partial_t, \tilde{X}, \tilde{Y}\rangle$ formally homotopic to $\mathcal{E}$ such that $\tilde{\xi}_t = \langle \tilde{X}_t, \tilde{Y}_t\rangle$ is non-integrable.

We now pick a vector field $\tilde{Z} \in \langle [\tilde{X}, \tilde{Y}]\rangle$ on M_φ, tangential to the fibres which is then linearly independent of $\tilde{X}$ and $\tilde{Y}$ by construction. Furthermore, we pick a vector field $\tilde{W}$ on M_φ tangential to the fibres, which is linearly independent of $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$.

We then define the following vector fields on $N \times \{t\}$:

$$\begin{aligned}
\overline{X}_t &:= \cos(kt)\tilde{X}_t + \sin(kt)\tilde{Y}_t + k^\delta \tilde{W}_t \\
\overline{Y}_t &:= \dot{\overline{X}}_t \\
\overline{Z}_t &:= [\overline{X}_t, \overline{Y}_t] \\
\overline{W}_t &:= \ddot{\overline{X}}_t.
\end{aligned}$$

Here $k > 0$ and $\delta < 0$ are constants. Note that showing these four vector fields are linearly independent, implies that $\langle \partial_t, \overline{X}\rangle$ is a $(2,3,5)$-structure on M_φ. We will show that for appropriate values of k and δ, this is indeed the case.

We see that

$$\overline{Y}_t = \dot{\overline{X}}_t = k\left(-\sin(kt)\tilde{X}_t + \cos(kt)\tilde{Y}_t\right) + k^\delta \dot{\tilde{W}}_t + \cos(kt)\dot{\tilde{X}}_t + \sin(kt)\dot{\tilde{Y}}_t,$$

and

$$\begin{aligned}
\overline{Z}_t &= [\overline{X}_t, \overline{Y}_t] \\
&= k^{1+\delta}\left(\cos(kt)[\tilde{W}_t, \tilde{Y}_t] - \sin(kt)[\tilde{W}_t, \tilde{X}_t]\right) + k[\tilde{X}_t, \tilde{Y}_t] + k^{2\delta}[\tilde{W}_t, \dot{\tilde{W}}_t] \\
&\quad + k^\delta\left(\cos(kt)[\tilde{X}_t, \dot{\tilde{W}}_t] + \sin(kt)[\tilde{Y}_t, \dot{\tilde{W}}_t] + \cos(kt)[\tilde{W}_t, \dot{\tilde{X}}_t] + \sin(kt)[\tilde{W}_t, \dot{\tilde{Y}}_t]\right) \\
&\quad + \cos^2(kt)[\tilde{X}_t, \dot{\tilde{X}}_t] + \cos(kt)\sin(kt)\left([\tilde{X}_t, \dot{\tilde{Y}}_t] + [\tilde{Y}_t, \dot{\tilde{X}}_t]\right) + \sin^2(kt)[\tilde{Y}_t, \dot{\tilde{Y}}_t].
\end{aligned}$$

Lastly we see that

$$\begin{aligned}\overline{W}_t &= \ddot{\overline{X}}_t \\ &= -k^2\left(\cos(kt)\tilde{X}_t + \sin(kt)\tilde{Y}_t\right) + 2k\left(-\sin(kt)\dot{\tilde{X}}_t + \cos(kt)\dot{\tilde{Y}}_t\right) + k^\delta \ddot{\tilde{W}}_t + \cos(kt)\ddot{\tilde{X}}_t + \sin(kt)\ddot{\tilde{Y}}_t.\end{aligned}$$

We then look at the following matrix

$$M := \begin{pmatrix} \cos(kt) & -k\sin(kt) + \mathcal{O}(1) & \mathcal{O}(k^{1+\delta}) & -k^2\cos(kt) + \mathcal{O}(k) \\ \sin(kt) & k\cos(kt) + \mathcal{O}(1) & \mathcal{O}(k^{1+\delta}) & -k^2\sin(kt) + \mathcal{O}(k) \\ 0 & \mathcal{O}(1) & k + \mathcal{O}(k^{1+\delta}) & \mathcal{O}(k) \\ k^\delta & \mathcal{O}(1) & \mathcal{O}(k^{1+\delta}) & \mathcal{O}(k) \end{pmatrix}.$$

We note that this is a matrix of the vectors $\overline{X}_t$, $\overline{Y}_t$, $\overline{Z}_t$ and $\overline{W}_t$ as columns, with respect to the basis $\{\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t, \tilde{W}_t\}$. Here we have used the notation $\mathcal{O}(k^a)$ to indicate the terms of order at most k^a.

One can compute the determinant of this matrix and we obtain

$$\det(M) = k^{4+\delta} + \mathcal{O}(k^3) + \mathcal{O}(k^{4+2\delta}).$$

We note that for all $-1 < \delta < 0$ we have that $4 + \delta > 3, 4 + 2\delta$. Thus, choosing k sufficiently large, means that the leading term will overcome the coefficients which appear in the error term. We conclude that there exist $k > 0$ and $-1 < \delta < 0$, such that the determinant of M is non-zero, and thus values for k and δ such that the vector fields $\overline{X}_t$, $\overline{Y}_t$, $\overline{Z}_t$ and $\overline{W}_t$ are linearly independent. We conclude that $\overline{D} := \langle \partial_t, \overline{X} \rangle$ is a $(2,3,5)-$structure on M_φ. □

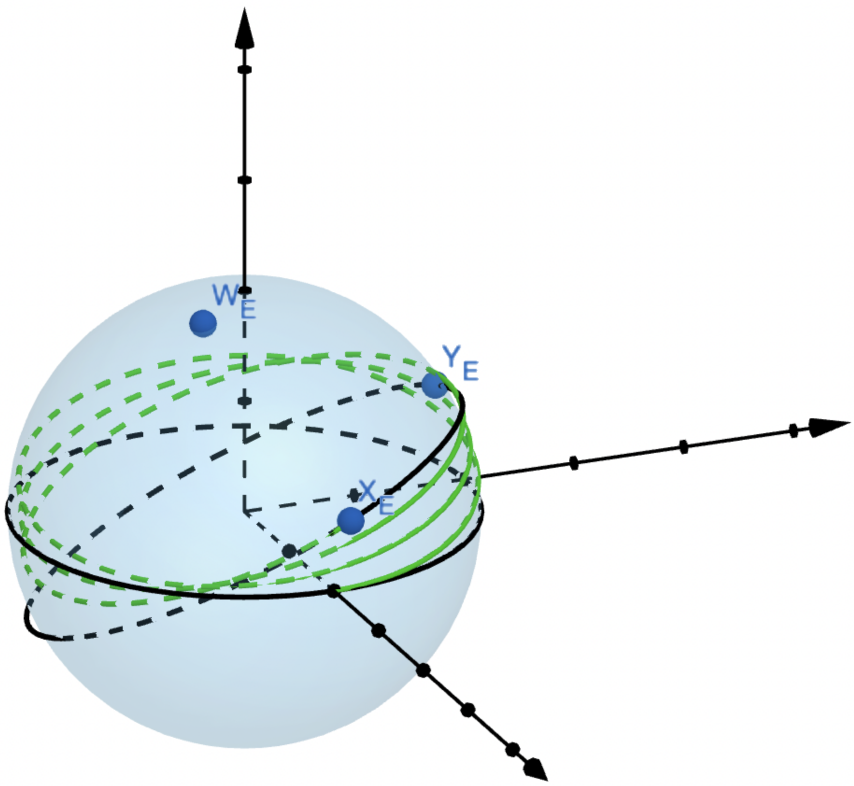

Figure 3.1: Illustration of the vector field $\overline{X}_t$ from the proof of Theorem 3.8. Here the black horizontal axes represent the vector fields $\tilde{X}_0$ and $\tilde{Y}_0$ and the vertical axis represents $\tilde{W}_0$. The blue dots indicate the tilted vectors $\tilde{X}_\epsilon$, $\tilde{Y}_\epsilon$ and $\tilde{W}_\epsilon$. We see that the vector field $\overline{X}_t$, indicated here by the green line, turns quite aggressively in the $\tilde{X}_t$ and $\tilde{Y}_t$ direction, and there is a slight contribution in the $\tilde{W}_t$ direction for convexity.

Remark 3.9. Let us make some remarks regarding Theorem 3.8 and its proof.

(i) First of all, let us give some intuition behind the construction above. A way of thinking about it is the following. Imagine we reparameterise the t−coordinate such that it becomes very stretched out. Then this will make the family of non-integrable distributions $\tilde{\xi}_t$ more constant in t. In this way, the first and second derivatives of the frame $\{\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t, \tilde{W}_t\}$ can be made arbitrarily small. If we then turn in $\tilde{X}_t$ and $\tilde{Y}_t$ with constant speed, and add a little $\tilde{W}_t$ for convexity, we obtain the $(2,3,5)$−condition. If we now parameterise the t−coordinate back to $[0,1]$, we obtain the very fast turning which we see in the vector field $\overline{X}_t$ defined above.

(ii) In the proof above, we can actually also choose $\delta > 0$, but we should however be more careful. Let us explain. Let A be a vector field linearly independent of $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$, then we note that $\tilde{\xi}_t \oplus \langle A_t \rangle$ is an even-contact structure. We can choose $\tilde{W}_t$ to be in the kernel $\mathcal{W}_t$ of $\tilde{\xi}_t \oplus \langle A_t \rangle$. This means that $[\mathcal{W}_t, \tilde{\xi}_t \oplus \langle A_t \rangle] \subset \tilde{\xi}_t \oplus \langle A_t \rangle$. This ensures that the $\tilde{Z}$ component in the vector field $\overline{Z}_t$ does not get killed by the $k^{1+\delta}$-term. One can compute a similar matrix M and its determinant, and one will see that for $0 < \delta < \frac{1}{2}$, this matrix has non-zero determinant.

(iii) We have shown in this chapter that in a quite similar manner, one can construct contact structures, Engel structures and $(2,3,5)$-structures on all mapping tori admitting a suitable formal analogue of these structures. An open question at this point is the following: does a similar construction also work for other rank-2 distributions?

(iv) As also mentioned in the beginning of this chapter, the construction of the $(2,3,5)$−structure on the mapping torus already highlights some of the key issues we are going to encounter when proving the h−principle for overtwisted $(2,3,5)$−structures on closed manifolds. Namely, when we reduce the problem to the extension problem in Chapter 7, we use a very similar construction in order to achieve the $(2,3,5)$−condition in a neighbourhood of the codimension-1 skeleton. There, we also use this idea of dilating the parameter t, such that the derivatives of the frame with respect to t become very little, and introducing the turning in order to achieve the $(2,3,5)$−condition.

▶

Chapter 4

Contact and Engel Lutz twisting

In this chapter we will focus on modifications of distributions called *Lutz twisting.* Lutz twisting is a surgery on a distribution where part of it is replaced by a distribution which "twists" more. For contact and Engel structures, Lutz twisting has already been introduced before in the literature, see [20] and [12], and we will discuss these in the next two sections. In order to define Lutz twisting, we first discuss the concept of adding *torsion* into a structure. This is a more general construction, and Lutz twisting will be a specific case of adding torsion along certain hypersurfaces.

So why are we interested in these surgeries? Well, using the Lutz twist, one can define the notion of an *overtwisted disc* in a contact or Engel manifold. Structures containing an embedding of an overtwisted disc are so called *overtwisted.* For overtwisted contact and Engel structures there exist $h-$principles on closed manifolds. This makes the Lutz twist an interesting concept to investigate. In the next chapter, we will define the notion of a $(2,3,5)-$Lutz twist, which will very much resemble the twisting in the contact and Engel case. Thus, it is insightful to treat the contact and Engel twisting first. Eventually, we will use the $(2,3,5)-$Lutz twist, to prove the $h-$principle for *overtwisted* $(2,3,5)-$structures, i.e. Theorem 0.4.

4.1 Contact Lutz twisting

In this section we will discuss torsion and Lutz twisting in contact structures. Torsion in contact manifolds was first introduced by Giroux in [20], and is therefore often referred to as *Giroux torsion* in the literature. Adding torsion or Lutz twisting, is a surgery performed on contact structures, which again yields a contact structure containing more "twisting". Lutz twisting was first discussed in R. Lutz's thesis [37] (which was never published), and thus the surgery is named after this French mathematician. Lutz introduced this procedure in order to construct examples of contact structures in manifolds other than $\mathbb{S}^3$. At the time, only few examples were known, and this allowed him to construct contact structures in all closed 3-manifolds. The result that every orientable manifold admits a contact structure, was shown and published by Martinet in [30], in which he used/altered the construction of Lutz.

4.1.1 Local model along a hypersurface

Giroux torsion is a surgery in a contact manifold which is performed along hypersurfaces. Therefore, we are first going to describe a local model for a contact structure along a hypersurface in this subsection.

Let (M,ξ) be a contact manifold and let N be an orientable transverse hypersurface. In order to define torsion, we want to describe ξ in local coordinates along N. Therefore, let T be a vector field tangent to ξ and transverse to N, and let ϕ_t denote its flow. Taking $\epsilon > 0$ small, we can use the flow of T to describe a tubular neighbourhood $\mathcal{U}(N) \cong N \times (-\epsilon,\epsilon)$ of $N \cong N \times \{0\}$. This allows us to take coordinates $(p,t) \in N \times (-\epsilon,\epsilon)$, such that $\partial_t = T$.

We can then pick a vector field H on $N \times (-\epsilon,\epsilon)$ such that $\xi = \langle \partial_t, H\rangle$. This means that we can describe ξ by maps $(H_p : (-\epsilon,\epsilon) \to \mathbb{S}^1)_{p\in N}$, by normalizing the vector field H, and we can look at the vector field

$$\dot{H} := [\partial_t, H].$$

We are now ready to prove the following proposition. It gives us an explicit description of ξ in the tubular neighbourhood of N.

Proposition 4.1. *Let (M, ξ) be a contact manifold, and let $N \subset M$ be a transverse hypersurface. Let $\mathcal{U}(N) \cong N \times (-\epsilon, \epsilon)$ be the tubular neighbourhood induced by the vector field $T \in D$, as described above. Then there is a framing $\{X \in \xi \cap TN, Y\}$ of TN such that ξ in $\mathcal{U}(N)$ can be described as*

$$\xi(p,t) = \langle \partial_t, H = \cos(t)X_p + \sin(t)Y_p \rangle.$$

Proof. Because of how we constructed $\mathcal{U}(N)$ we know that we can describe ξ in this neighbourhood as

$$\xi = \langle \partial_t, H \rangle$$

where

$$H_p(t) = (H_p(t))_1 \cdot X_p + (H_p(t))_2 \cdot Y_p,$$

for vector fields X and Y on $\mathcal{U}(N)$. If we choose the vector field $X \in \xi \cap TN$, then we must have that $\partial_t (H_p(t))_2 \neq 0$, as ξ is a contact structure. Therefore, using the implicit function theorem, and rescaling the vector fields, we get that ξ is of the form

$$\xi = \langle \partial_t, H = X_p + tY_p \rangle.$$

Then, by reparameterising the $t-$coordinate and scaling the vector fields, we get that there are vector fields $X \in \xi \cap TN$ and Y such that ξ is of the desired form

$$\xi = \langle \partial_t, H = \cos(t)X_p + \sin(t)Y_p \rangle$$

on $\mathcal{U}(N)$. □

4.1.2 Construction of the contact Lutz twist

In this section we will describe the construction of adding Giroux torsion along a hypersurface, and thereafter define Lutz twisting. Let (M, ξ) be a contact manifold and let N be a hypersurface transverse to ξ. From Proposition 4.1 we know that there is a tubular neighbourhood $\mathcal{U}(N) \cong N \times (-\epsilon, \epsilon)$ where we have an explicit description of what ξ looks like, namely

$$\xi(p,t) = \langle \partial_t, H = \cos(t)X_p + \sin(t)Y_p \rangle.$$

Therefore, we can describe the structure ξ by a family of maps $H_p : (-\epsilon, \epsilon) \to \mathbb{S}^1$ given by

$$H_p(t) = (\cos(t), \sin(t)).$$

Note that the coordinates of these maps are determined by the directions of the vector fields X_p and Y_p, and thus depend on the point $p \in N$.

To introduce torsion in the contact structure, we are going to replace the maps $(H_p)_{p\in N}$, by maps $(G_p)_{p\in N}$. The maps $(G_p)_{p\in N} : (\epsilon, \epsilon) \to \mathbb{S}^1$ consist of three parts:

(1) The first part of the curve is equal to $H_p((-\epsilon, 0])$.

(2) The middle part of the curve is a loop in $\mathbb{S}^1$ starting and ending at $H_p(0)$.

(3) The last part of the curve is equal to $H_p([0, \epsilon))$.

The explicit expression for the curves $(G_p)_{p\in N} : (\epsilon, \epsilon) \to \mathbb{S}^1$ is then given by

$$\left(\cos\left(\frac{\pi + \epsilon}{\epsilon} t + \pi\right), \sin\left(\frac{\pi + \epsilon}{\epsilon} t + \pi\right)\right)$$

for $t \in (-\epsilon, \epsilon)$. Note that we have adjusted the speed and have translated the variable t, such that the curves are defined on the interval $(-\epsilon, \epsilon)$.

By replacing the maps $(H_p)_{p\in N}$ by $(G_p)_{p\in N}$, we obtain a new structure which we will denote by $\mathcal{L}(\xi)$. We note that $\mathcal{L}(\xi)$ has the following properties:

- $\mathcal{L}(\xi)$ is the same as ξ outside of the tubular neighbourhood $N \times (-\epsilon, \epsilon)$,

- $\mathcal{L}(\xi)$ is tangent to the vector field T on $N \times (-\epsilon, \epsilon)$,
- the distribution $\mathcal{L}(\xi)$ makes an additional turn along ∂_t with respect to ξ.

We claim that the resulting structure is again a contact structure.

Lemma 4.2. *The structure $(M, \mathcal{L}(\xi))$ is a contact manifold.*

Proof. First of all, we note that we have only altered the contact structure inside $N \times (-\epsilon, \epsilon)$, thus we only have to prove something inside this neighbourhood. We consider the curves $(G_p)_{p \in N}$ from the construction above. We note that these curves are immersions, which is a sufficient condition for being contact. □

The construction we described above, is the process of adding torsion along the hypersurface N. Lutz twisting is a special case of adding torsion, which we specify in the next definition.

Definition 4.3. *Let (M, ξ) be a contact manifold, and let $N \subset M$ be a transverse hypersurface. Then, we say the contact structure $\mathcal{L}(\xi)$ is obtained from ξ by the construction described above, by adding **Giroux torsion** along N.*

*If N the boundary of a tubular neighbourhood of a transverse knot, then we say $\mathcal{L}(\xi)$ is obtained from ξ by adding a **Lutz twist** along N.*

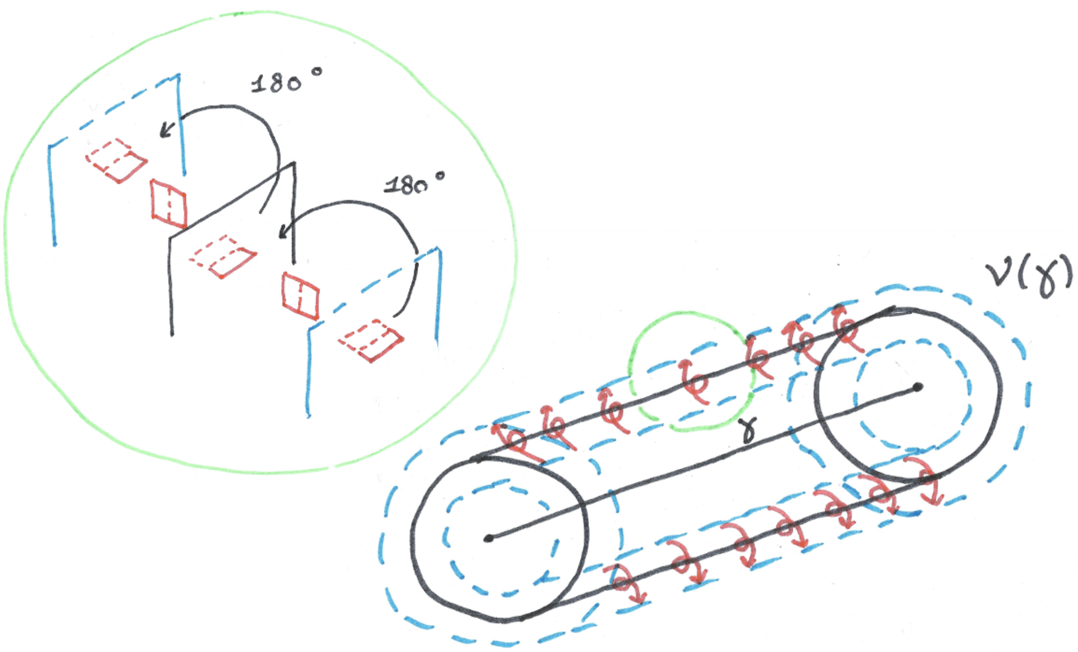

Figure 4.1: An illustration of a contact Lutz twist added along the boundary of the tubular neighbourhood of a transverse knot $\gamma : \mathbb{S}^1 \to M$. Here the tubular neighbourhood is indicated in blue, and the distribution $\mathcal{L}(\xi)$ in red. We see the $\mathcal{L}(\xi)$ makes a full turn around $\partial\nu(L)$, in the radial direction (which is the $t-$direction in the construction we have seen).

Remark 4.4. As mentioned in the introduction of this chapter, Lutz twisting allows us to define the notion of an *overtwisted disc* in a contact manifold. We will not formally define this here, but here is is the general idea. Recall, that a Lutz twist contains a neighbourhood where the contact structure makes a full turn, i.e. a $2\pi-$turn. We can look at a subset of this contact manifold, where the contact structure only makes half a turn, i.e. a $\pi-$turn. This contact submanifold is called an *overtwisted disc.* In Figure 4.2 we see an illustration of the overtwisted disc. It portrays a slice of the tubular neighbourhood of the transverse knot along which we Lutz twist. Note that in this picture, the Lutz twist starts all the way in the center, i.e. at the knot. This structure is homotopic to the structure we originally defined as the Lutz twist.

A contact structure is called *overtwisted* if it contains an embedding of the overtwisted disc. It turns out that the $h-$principle holds for overtwisted contact structures on closed manifolds. This was first proven by Eliashberg in 1989 [13].

We point out that in the Engel and the $(2,3,5)-$case, the overtwisted disc will be defined as adding a *full* Lutz twist along some hypersurface. In fact, we could also define the overtwisted disc in a contact manifold as a full Lutz twist, because it turns out that these definitions are equivalent. I.e. a contact structure which contains an overtwisted disc defined by half a Lutz twist, is homotopic to a contact structure with an overtwisted disc defined by a full Lutz twist, and vice versa. However, the half-Lutz twist definition is more common in the literature, which is why we also introduce it here. ▶

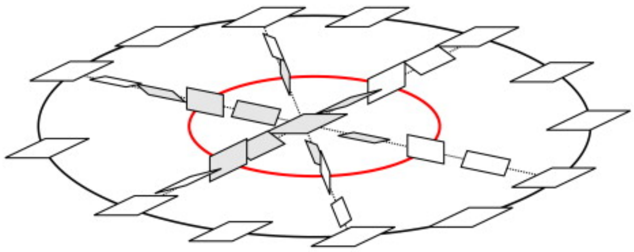

Figure 4.2: An illustration of the overtwisted disc [31, p. 148]. Here the planes portray the distribution at that point, and we see that moving radially outward, the distribution makes half a turn.

4.2 Engel Lutz twisting

In this section we will discuss Engel torsion and Lutz twisting. Just like in the contact case, adding Engel torsion is a surgery on Engel structures, and Engel Lutz twisting is a specific case of adding torsion along certain hypersurfaces. The surgery was first introduced by del Pino and Vogel in [12], and it allowed them to prove an h-principle for overtwisted Engel structures on closed manifolds. The material in this section is based on their paper, but the construction given here is more explicit. This will also be insightful for the next chapter, as in Section 5.2 we will see how to construct a $(2,3,5)-$Lutz twist, which will be done in a very similar manner.

4.2.1 Local model along a hypersurface

Similarly as in the contact case, Engel torsion is added along a hypersurface transverse to the distribution. Therefore, we first want to describe an Engel structure along such a transverse hypersurface. We will construct a local model in this subsection.

Before we can describe this local model, we need to do some preliminary work. Let (M,D) be an Engel manifold and let N be an orientable transverse hypersurface. Let $\mathcal{E}$ denote the distribution $[D,D]$. We note that we can also define the following distributions:

- a line field $L_N := D \cap TN$,
- a plane field $\xi_N := \mathcal{E} \cap TN$ which contains L_N.

$$\begin{aligned} N^+ &= \{x \in N \mid \xi_N \text{ is a positive contact structure close to } x\} \\ N^- &= \{x \in N \mid \xi_N \text{ is a negative contact structure close to } x\} \\ N^0 &= N - (N^+ \cup N^-). \end{aligned}$$

We note that N is transverse to $\mathcal{W}$ at all $x \in N^+ \cup N^-$, and $\mathcal{W}$ is tangent to N at all points $x \in N^0$. Since being contact is an open condition, we note that the sets N^+ and N^- are open and by construction N^0 is closed. See Figure 4.3 for an illustration of these spaces.

In order to define adding torsion, we want to describe D in local coordinates along N. Therefore, let T be a vector field tangent to D and transverse to N, and let ϕ_t denote its flow. Taking $\epsilon > 0$ small, we can use the flow ϕ_t of T to describe a tubular neighbourhood $\mathcal{U}(N) \cong N \times (-\epsilon,\epsilon)$ of $N \cong N \times \{0\}$. This allows us to take coordinates $(p,t) \in N \times (-\epsilon,\epsilon)$, such that $\partial_t = T$. We want to choose this vector field T such that the neighbourhood $N \times (-\epsilon,\epsilon)$ has certain nice properties. Namely,

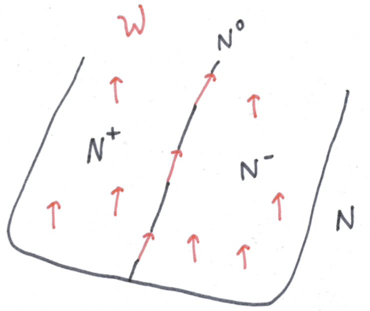

Figure 4.3: An illustration of D along a hypersurface N.

- Outside a little neighbourhood of $N^0 \times (\epsilon, \epsilon)$ we choose T such that it is tangent to $\mathcal{W}$,
- If T is tangent to $\mathcal{W}$ at $(p, 0)$, then the orbit of T through p for $t \in (-\epsilon, \epsilon)$ is part of the leaf of $\mathcal{W}$ at p,
- If T is not tangent to $\mathcal{W}$ at $(p, 0)$, the orbit of T through p for $t \in (-\epsilon, \epsilon)$ is never tangent to $\mathcal{W}$.

These assumptions on T ensure that the properties ∂_t has at a point $(p, 0) \in N \times \{0\}$ also hold at the entire fibre $\{p\} \times (-\epsilon, \epsilon)$.

Since we have chosen T tangent to D, we can pick a vector field H on $N \times (-\epsilon, \epsilon)$ such that $D = \langle \partial_t, H \rangle$, and $D \cap T(N \times \{t\})$ is spanned by the restriction of H to $N \times \{t\}$. In fact we can then describe D by maps $(H_p : (-\epsilon, \epsilon) \to \mathbb{S}^2)_{p \in N}$, and we can look at the vector fields

$$\dot{H} := [\partial_t, H] \quad \text{and} \quad \ddot{H} := [\partial_t, \dot{H}].$$

On N^0 we know that $\mathcal{W}$ is tangent to N, and thus ∂_t is not tangent to $\mathcal{W}$. Therefore, $\langle H, \dot{H}, \ddot{H} \rangle$ is a framing of TN. We have already fixed an orientation of N, and thus this framing is either positive or negative. This gives us reason to define the following spaces:

$$N^{0,+} = \{x \in N \mid \langle H, \dot{H}, \ddot{H} \rangle \text{ is a positive framing of } TN \text{ at } x\}$$
$$N^{0,-} = \{x \in N \mid \langle H, \dot{H}, \ddot{H} \rangle \text{ is a negative framing of } TN \text{ at } x\}.$$

In order to construct a local model of an Engel structure along a hypersurface, we need to pick a framing of the hypersurface in a careful way. This motivates the following definition.

Definition 4.5. *Let (M, D) be an Engel manifold and let $N \subset M$ be a hypersurface transverse to D. We say that $\{X, Y, Z\}$, where X, Y and Z are vector fields on N, is **a frame associated to** (D, N) if $X \in D \cap TN$, $Y \in \mathcal{E} \cap TN$ and $\{X_p, Y_p, Z_p\}$ is a frame of T_pN for $p \in N$.*

Using such framings, the following proposition gives us a more explicit description of D in $\mathcal{U}(N)$.

Proposition 4.6. *Let (M, D) be an Engel manifold and let $N \subset M$ be a hypersurface transverse to D. Let $\mathcal{U}(N) \cong N \times (-\epsilon, \epsilon)$ be the tubular neighbourhood induced by the vector field T as described above. Then there is a frame $\{X, Y, Z\}$ associated to (D, N) such that D in $\mathcal{U}(N)$ can be described as*

$$D(p, t) = \langle \partial_t, H = \cos(t)X_p + \sin(t)Y_p + g(p, t)Z_p \rangle$$

where $g : N \times (-\epsilon, \epsilon) \to \mathbb{R}$ such that

- *$g(p, t) = 0$ if ∂_t is tangent to $\mathcal{W}$ at $(p, 0)$, i.e. if $p \in N^+ \cup N^-$,*
- *$g(p, t)$ is convex if $p \in N^{0,+}$,*
- *$g(p, t)$ is concave if $p \in N^{0,-}$.*

Proof. As discussed above, we have that $D = \langle \partial_t, H \rangle$ where $(H_p : (-\epsilon, \epsilon) \to \mathbb{S}^2)_{p \in N}$ with

$$H_p(t) = (H_p(t))_1 \cdot X_p + (H_p(t))_2 \cdot Y_p + (H_p(t))_3 \cdot Z_p.$$

Since D is an Engel structure at least one of the two conditions must hold

(i) the vectors $H_p(t)$, $\dot{H}_p(t)$ and $\ddot{H}_p(t)$ are linearly independent,

(ii) $\langle H_p(t), \dot{H}_p(t)\rangle$ span a contact structure in a neighbourhood of (p,t).

This means that the curves $H_p : (-\epsilon,\epsilon) \to \mathbb{S}^2$ must be immersions, and thus by choosing $X \in D \cap TN$ and $Y \in \mathcal{E} \cap TN$ appropriately, and invoking the implicit function theorem, we get that the Taylor expansion of H_p looks like

$$H_p(t) = X_p + tY_p + \mathcal{O}(t^2).$$

By scaling the vectors and reparameterising the $t-$coordinate, we can write

$$H_p(t) = \cos(t)X_p + \sin(t)Y_p + \mathcal{O}(t^2).$$

Note that these vector fields X and Y are not the exact same as before, but for simplicity we still denote them like this.

Since D is an Engel structure we must have that the image of H_p is either convex or concave (in the case of condition (i)), or that H_p is contained in a maximal circle on the sphere (in the case of condition (ii)). In both cases we have that H_p is graphical over the equator spanned by the vector fields X and Y. Therefore, using the implicit function theorem we get that there is a vector field Z on N such that

$$H_p(t) = \cos(t)X_p + \sin(t)Y_p + g(p,t)Z_p$$

for a function $g : N \times (-\epsilon,\epsilon) \to \mathbb{R}$. On the subspace $N^0 \subset N$ we have that $\langle H, \dot{H}, \ddot{H}\rangle$ forms a framing of TN, which means that the function g must be either convex or concave here. Since this framing is positive on $N^{0,+}$ we must have that g is convex here, and on $N^{0,-}$ we must have that g is concave. If ∂_t is tangent to $\mathcal{W}$ at $(p,0)$, we are in the situation of condition (ii). Therefore, g can be chosen to be 0 at these points. □

The last thing we will show in this subsection, is that we can homotope an Engel structure such that it has an even more explicit local model around a transverse hypersurface. This will help us in the next subsection to define Engel torsion in a very explicit way.

Lemma 4.7. *Let (M,D) be an Engel manifold and let $N \subset M$ be a hypersurface transverse to D. Let $\mathcal{U}(N) \cong N \times (-\epsilon,\epsilon)$ be the tubular neighbourhood induced by the vector field T as described above. Then there is a frame $\{X,Y,Z\}$ associated to (D,N) such that D is homotopic as Engel structures to $\overline{D}$, which in a neighbourhood $N \times (-\delta,\delta)$ with $0 < \delta < \epsilon$, can be described as*

$$\overline{D} = \langle \partial_t, \overline{H} = \cos(t)X_p + \sin(t)Y_p + \alpha(p)t^2 Z_p\rangle,$$

where $\alpha : N \to \mathbb{R}$ is a function such that

- $\alpha(p) = 0$ *if* $p \in N^+ \cup N^-$,
- $\alpha(p) > 0$ *if* $p \in N^{0,+}$,
- $\alpha(p) < 0$ *if* $p \in N^{0,-}$.

Proof. From Proposition 4.6 we know that in $\mathcal{U}(N)$ the Engel structure D can be described as

$$D(p,t) = \langle \partial_t, H = \cos(t)X_p + \sin(t)Y_p + g(p,t)Z_p\rangle$$

with g as in Proposition 4.6. From the properties listed in the proposition, we know that g is of the form

$$g(p,t) = \alpha(p)t + \mathcal{O}(t^3),$$

where $\alpha : N \to \mathbb{R}$ is as in the statement. We then define the following vector fields for $s \in [0,1]$:

$$F_s(p,t) := \cos(t)X_p + \sin(t)Y_p + g_s(p,t)Z_p$$

with

$$g_s(p,t) := sg(p,t) + (1-s)\left(\chi(t)g(p,t) + (1-\chi(t))\alpha(p)t^2\right).$$

Here χ is a bump function which is 0 in a very small interval around $t = 0$, as depicted in Figure 4.4. We note that $\dot{\chi}$ is of order $\frac{1}{\epsilon}$ and $\ddot{\chi}$ is of order $\frac{1}{\epsilon^2}$.

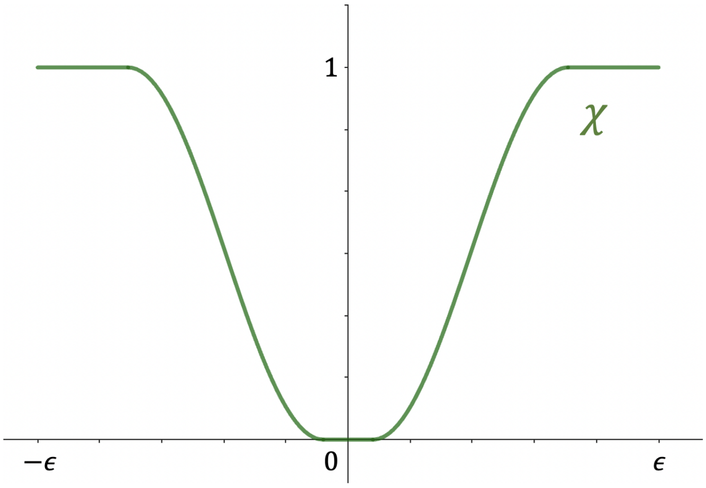

Figure 4.4: Illustration of the bump function used in the proof of Lemma 4.7 (and later on also in the proof of Lemma 5.4).

Let $D_s(p,t) := \langle \partial_t, F_s(p,t)\rangle$. We see that D_0 is of the form

$$D_0 = \langle \partial_t, \cos(t)X_p + \sin(t)Y_p + \alpha(p)t^2 Z_p\rangle$$

in a small neighbourhood around $t = 0$, and that $D_1 = D$. We want to show that the distributions D_s are in fact Engel structures in $\mathcal{U}(N)$. We see that

$$\dot{g}_s(p,t) = s\dot{g}(p,t) + (1-s)\left(\dot{\chi}(t)(g(p,t) - \alpha(p)t^2) + \chi(t)(\dot{g}(p,t) - 2\alpha(p)t) + 2\alpha(p)t\right)$$
$$\ddot{g}_s(p,t) = s\ddot{g}(p,t) + (1-s)\left(\ddot{\chi}(t)(g(p,t) - \alpha(p)t^2) + 2\dot{\chi}(t)\left(\dot{g}(p,t) - 2\alpha(p)t\right) + \chi(t)(\ddot{g}(p,t) - 2\alpha(p)) + 2\alpha(p)\right).$$

Furthermore,

$$\dot{F}_{s,p}(t) = -\sin(t)X_p + \cos(t)Y_p + \dot{g}_s(p,t)Z_p$$
$$\ddot{F}_{s,p}(t) = -\cos(t)X_p - \sin(t)Y_p + \ddot{g}_s(p,t)Z_p.$$

We want to show that $F_{s,p}(t)$, $\dot{F}_{s,p}(t)$ and $\ddot{F}_{s,p}(t)$ are linearly independent. From our assumptions on χ we see that

$$\dot{g}_s(p,t) = s\left(2\alpha(p)t + \mathcal{O}(t^2)\right) + (1-s)\left(\mathcal{O}\left(\frac{1}{\epsilon}\right)\mathcal{O}(t^3) + \mathcal{O}(t)\mathcal{O}(t^2) + 2\alpha(p)t\right)$$
$$\ddot{g}_s(p,t) = s\left(2\alpha(p) + \mathcal{O}(t)\right) + (1-s)\left(\mathcal{O}\left(\frac{1}{\epsilon^2}\right)\mathcal{O}(t^3) + 2\mathcal{O}\left(\frac{1}{\epsilon}\right)\mathcal{O}(t^2) + \mathcal{O}(t)\mathcal{O}(t) + 2\alpha(p)\right).$$

We note that for $t \in (-\epsilon, \epsilon)$ and ϵ small, that the vector field $F_{s,p}(t)$ is close to X_p, $\dot{F}_{s,p}(t)$ close to Y_p and $\ddot{F}_{s,p}(t)$ close to $-X_p + 2\alpha(p)Z_p$. Then by Lemma 1.38 it follows that indeed for $p \in N^0$ the distributions D_s are Engel.

Now for $p \in N^+ \cup N^-$ we have that $\alpha(p) = 0$, and thus $\ddot{F}_{s,p}(t)$ is not necessarily linearly independent of $F_{s,p}(t)$ and $\dot{F}_{s,p}(t)$. We then compute

$$[F_{s,p}(t), \dot{F}_{s,p}(t)] = [X_p, Y_p] + (1 - \mathcal{O}(t^2))\dot{g}_s(p,t)[X_p, Z_p] + \mathcal{O}(t)\dot{g}_s(p,t)[Y_p, Z_p],$$

where we used the fact that $g = 0$ for $p \in N^+ \cup N^-$. For p in the interior of $N^+ \cup N^-$ we have that g is zero, and thus g_s is constantly zero. It follows that on the boundary of $N^+ \cup N^-$, the derivative $\dot{g}_s$ must also be zero. From this follows that $[F_{s,p}(t), \dot{F}_{s,p}(t)] = [X_p, Y_p]$, and since $\{X_p, Y_p\}$ is non-integrable for $p \in N^+ \cup N^-$, we note that $F_{s,p}(t)$, $\dot{F}_{s,p}(t)$ and $[F_{s,p}(t), \dot{F}_{s,p}(t)]$ are linearly independent on $N^+ \cup N^-$. We conclude that D_s is also Engel on $N^+ \cup N^-$, which finishes the proof. □

4.2.2 Construction of the Engel Lutz twist

In this section we will describe the construction of adding Engel torsion along a hypersurface. As mentioned before, the idea of this construction is based on the construction in [12], but in this section we will additionally give a more explicit formula for the torsion. Thereafter, we will define Lutz twisting as a specific case of adding torsion along certain hypersurfaces. In Section 5.2 we will see how to construct a $(2,3,5)-$Lutz twist, which will be done almost analogously.

Let (M, D) be an Engel manifold and let N be a hypersurface transverse to D. From Proposition 4.6 we know that there is a tubular neighbourhood $\mathcal{U}(N) \cong N \times (-\epsilon, \epsilon)$ where we have an explicit description of what D looks like, namely

$$D(p,t) = \langle \partial_t, H = \cos(t)X_p + \sin(t)Y_p + g(p,t)Z_p \rangle$$

with $g : N \times (-\epsilon, \epsilon) \to \mathbb{R}$ as in Proposition 4.6. From Lemma 4.7 we know that D is homotopic through Engel structures to $\overline{D}$, which in a neighbourhood $N \times (-\delta, \delta)$ with $\delta < \epsilon$ can be described as

$$\overline{D} = \langle \partial_t, \overline{H} = \cos(t)X_p + \sin(t)Y_p + \alpha(p)t^2 Z_p \rangle$$

with $\alpha(p)$ as in Lemma 4.7. We can describe the structure $\overline{D}$ on $N \times (-\delta, \delta)$ by a family of maps $\overline{H}_p : (-\delta, \delta) \to \mathbb{S}^2$ given by

$$\overline{H}_p(t) = \frac{1}{\sqrt{1 + \alpha(p)^2 t^4}} (\cos(t), \sin(t), \alpha(p)t^2).$$

Note that the coordinates of these maps are determined by the directions of the vector fields X_p, Y_p and Z_p, and thus depend on the point $p \in N$.

To introduce torsion in the Engel structure, we are going to replace the maps $(\overline{H}_p)_{p\in N}$ by maps $(G_p)_{p\in N}$. First, we construct a C^2-family of maps $(F_p)_{p\in N}$, and then we smooth this family to obtain $(G_p)_{p\in N}$.

The curves $(F_p)_{p\in N} : (-\delta, \delta) \to \mathbb{S}^2$ consist of three parts:

(1) The first part of the curve is equal to $\overline{H}_p((-\delta, 0])$,

(2) The middle part of the curve is the unique circle in $\mathbb{S}^2$ through $\overline{H}_p(0)$, tangent to the equator spanned by X_p and Y_p, and which has the same curvature as $\overline{H}_p$ at $t = 0$,

(3) The last part of the curve is equal to $\overline{H}_p([0, \delta))$.

To explicitly construct these curves we first need to find the curvature of $\overline{H}_p$ at $t = 0$.

Lemma 4.8. *The curvature of the map $\overline{H}_p : (-\delta, \delta) \to \mathbb{S}^2$ given by*

$$\overline{H}_p(t) = \frac{1}{\sqrt{1 + \alpha(p)^2 t^4}} (\cos(t), \sin(t), \alpha(p)t^2)$$

at $t = 0$ is equal to $\sqrt{1 + 4\alpha(p)^2}$.

Proof. Let $T(t) = \frac{\overline{H}_p'(t)}{|\overline{H}_p'(t)|}$, then the curvature of $\overline{H}_p$ at t is given by

$$\kappa(t) = \frac{|T'(t)|}{|\overline{H}_p'(t)|}.$$

For shorthand notation we use $\alpha_p := \alpha(p)$. We note that

$$\overline{H}_p'(t) = (1 + \alpha_p^2 t^4)^{-3/2} \left(-(1 + \alpha_p^2 t^4)\sin(t) - 2\alpha_p^2 t^3 \cos(t), (1 + \alpha_p^2 t^4)\cos(t) - 2\alpha_p^2 t^3 \sin(t), 2\alpha_p t \right).$$

It follows that

$$|\overline{H}_p'(t)| = \frac{\sqrt{(1 + \alpha_p^2 t^4)^2 + 4\alpha_p^4 t^6 + 4\alpha_p^2 t^2}}{(1 + \alpha_p^2 t^4)^{3/2}},$$

and thus

$$T(t)=\frac{1}{\sqrt{(1+\alpha_p^2t^4)^2+4\alpha_p^4t^6+4\alpha_p^2t^2}}\left(-(1+\alpha_p^2t^4)\sin(t)-2\alpha_p^2t^3\cos(t),(1+\alpha_p^2t^4)\cos(t)-2\alpha_p^2t^3\sin(t),2\alpha_pt\right).$$

Differentiating this expression and filling in $t=0$ we get

$$T'(0)=(-1,0,2\alpha_p),$$

and thus

$$\kappa(0)=\frac{|T'(0)|}{|\overline{H}_p'(0)|}=\frac{\sqrt{1+4\alpha_p^2}}{1}=\sqrt{1+4\alpha_p^2}.$$

□

From this follows that the middle part of the curve F_p is a circle of radius $\frac{1}{\sqrt{1+4\alpha_p^2}}$, tangent at the point $(1,0,0)$ to the equator spanned by X_p and Y_p. Such a circle can be described by the following formula:

$$t\mapsto\left(\frac{\cos(t)}{1+4\alpha_p^2}+\frac{4\alpha_p^2}{1+4\alpha_p^2},\frac{\sin(t)}{\sqrt{1+4\alpha_p^2}},\frac{2\alpha_p}{1+4\alpha_p^2}-\frac{2\alpha_p\cos(t)}{1+4\alpha_p^2}\right),\quad\text{for } t\in[0,2\pi].$$

Note that when $\alpha(p)=0$, this formula describes the equator spanned by X_p and Y_p.

Combining this curve with the map $\overline{H}_p(-\delta,\delta)$, we can find an explicit expression for the curves $(F_p)_{p\in N}:(-\delta,\delta)\to\mathbb{S}^2$, consisting of three parts:

(1) For $t\in\left(-\delta,-\frac{\pi\cdot\delta}{\pi+\sqrt{1+4\alpha_p^2}\delta}\right]$ it is given by

$$\frac{1}{\sqrt{1+\alpha_p^2\left(\frac{\pi+\sqrt{1+4\alpha_p^2}\delta}{\sqrt{1+4\alpha_p^2}\delta}t+\frac{\pi}{\sqrt{1+4\alpha_p^2}}\right)^4}}\left(\cos\left(\frac{\pi+\sqrt{1+4\alpha_p^2}\delta}{\sqrt{1+4\alpha_p^2}\delta}t+\frac{\pi}{\sqrt{1+4\alpha_p^2}}\right),\sin\left(\frac{\pi+\sqrt{1+4\alpha_p^2}\delta}{\sqrt{1+4\alpha_p^2}\delta}t+\frac{\pi}{\sqrt{1+4\alpha_p^2}}\right),\right.$$
$$\left.\alpha_p\left(\frac{\pi+\sqrt{1+4\alpha_p^2}\delta}{\sqrt{1+4\alpha_p^2}\delta}t+\frac{\pi}{\sqrt{1+4\alpha_p^2}}\right)^2\right).$$

(2) For $t\in\left[-\frac{\pi\cdot\delta}{\pi+\sqrt{1+4\alpha_p^2}\delta},\frac{\pi\cdot\delta}{\pi+\sqrt{1+4\alpha_p^2}\delta}\right]$ it is given by

$$\left(\frac{\cos\left(\frac{\pi+\delta\sqrt{1+4\alpha_p^2}}{\delta}t+\pi\right)}{1+4\alpha_p^2}+\frac{4\alpha_p^2}{1+4\alpha_p^2},\frac{\sin\left(\frac{\pi+\delta\sqrt{1+4\alpha_p^2}}{\delta}t+\pi\right)}{\sqrt{1+4\alpha_p^2}},\frac{2\alpha_p}{1+4\alpha_p^2}-\frac{2\alpha_p\cos\left(\frac{\pi+\delta\sqrt{1+4\alpha_p^2}}{\delta}t+\pi\right)}{1+4\alpha_p^2}\right).$$

(3) For $t\in\left[\frac{\pi\cdot\delta}{\pi+\sqrt{1+4\alpha_p^2}\delta},\delta\right)$ it is given by

$$\frac{1}{\sqrt{1+\alpha_p^2\left(\frac{\pi+\sqrt{1+4\alpha_p^2}\delta}{\sqrt{1+4\alpha_p^2}\delta}t-\frac{\pi}{\sqrt{1+4\alpha_p^2}}\right)^4}}\left(\cos\left(\frac{\pi+\sqrt{1+4\alpha_p^2}\delta}{\sqrt{1+4\alpha_p^2}\delta}t-\frac{\pi}{\sqrt{1+4\alpha_p^2}}\right),\sin\left(\frac{\pi+\sqrt{1+4\alpha_p^2}\delta}{\sqrt{1+4\alpha_p^2}\delta}t-\frac{\pi}{\sqrt{1+4\alpha_p^2}}\right),\right.$$
$$\left.\alpha_p\left(\frac{\pi+\sqrt{1+4\alpha_p^2}\delta}{\sqrt{1+4\alpha_p^2}\delta}t-\frac{\pi}{\sqrt{1+4\alpha_p^2}}\right)^2\right).$$

Note that we have adjusted the speed and have translated the variable t such that the curves are defined on $(-\delta,\delta)$ and are C^2. In Figure 4.5 you can see what the maps $(F_p)_{p\in N}$ look like.

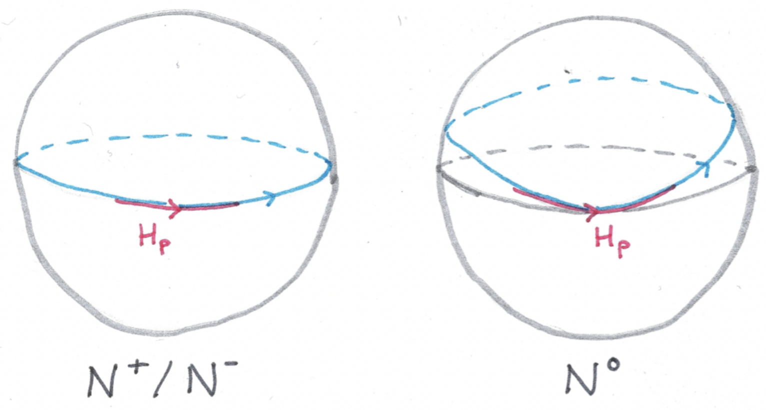

Figure 4.5: Illustration of the maps $(F_p)_{p\in N}$, which are formed by adding an extra loop, indicated in blue, to the maps $\overline{H}_p$, with the same convexity as $\overline{H}_p$ at $t = 0$. On the subsets N^+ or N^- we know from Proposition 4.6 that the image of $\overline{H}_p$ is a piece of the equator through $(1,0,0)$ and $(0,1,0)$. The map F_p is this curve $\overline{H}_p$ with an additional loop around the equator. On the subset $N^{0,+}$ we know that the map $\overline{H}_p$ is convex, and on $N^{0,-}$ is concave. The maps F_p are in this case the same but with an extra convex or concave loop in the middle.

We now want to smooth $(F_p)_{p\in N}$ to obtain the desired maps $(G_p)_{p\in N}$. The relevant properties of the curves $(F_p)_{p\in N}$ depend only on their 2−jet, as we are only interested in first and second derivatives when working with Engel structures. The three segments of the curves are already C^2-smooth, and any C^2-small smoothing yields therefore a smooth family of curves $(G_p)_{p\in N}$ with the same properties regarding convexity as $(F_p)_{p\in N}$.

By replacing the maps $(\overline{H}_p)_{p\in N}$ by $(G_p)_{p\in N}$, we obtain a new structure which we will denote by $\mathcal{L}(\overline{D})$. We note that $\mathcal{L}(\overline{D})$ has the following properties:

- $\mathcal{L}(\overline{D})$ is the same as $\overline{D}$ outside of the tubular neighbourhood $N \times (-\delta, \delta)$,
- $\mathcal{L}(\overline{D})$ is tangent to the vector field T on $N \times (-\delta, \delta)$,
- on $N^+ \cup N^-$ the even-contact structures associated to $\mathcal{L}(\overline{D})$ and $\overline{D}$ are the same, but $\mathcal{L}(\overline{D})$ make an additional turn along ∂_t.

We claim that the resulting structure $\mathcal{L}(\overline{D})$ is still an Engel structure.

Lemma 4.9. *The structure $(M, \mathcal{L}(\overline{D}))$ is an Engel manifold.*

Proof. First of all, we note that we have only altered the Engel structure inside $N \times (-\delta, \delta)$, thus we only have to prove something inside this neighbourhood. We consider the curves $(F_p)_{p\in N}$ from the construction above. We note that whenever $\overline{H}_p$ is a convex or a concave curve, so if F_p. This is a sufficient condition for $\mathcal{L}(\overline{D})$ to be an Engel structure.

Let U be the set of $p \in N$ such that $\overline{H}_p$ is everywhere tangent to the equator through $(1,0,0)$ and $(0,1,0)$. By construction, the maps F_p are also tangent to this equator, see also Figure 4.5. However, we cannot immediately conclude that $\mathcal{L}(\overline{D})$ is Engel from this. This is because the set U is closed and the contact condition (also condition (ii) in the proof of Proposition 4.6) must hold in a neighbourhood of the point. To fix this problem, we note that the maximal circles tangent to the curves F_p, C^∞-converge to the equator as (p,t) approaches U. Therefore, the contact condition, i.e. condition (ii), must also hold in a small neighbourhood of U. We conclude that $\mathcal{L}(\overline{D})$ is indeed an Engel structure everywhere. □

Definition 4.10. *Let (M, D) be an Engel manifold and $N \subset M$ a transverse hypersurface. By Lemma 4.7 we know that we can homotope D such that it is of the form*

$$\overline{D} = \langle \partial_t, \overline{H} = \cos(t)X_p + \sin(t)Y_p + \alpha(p)t^2 Z_p \rangle,$$

in a neighbourhood $N \times (-\delta, \delta)$*, with* $\delta > 0$ *and* $\{X, Y, Z\}$ *a frame associated to* (D, N)*, as in Lemma 4.7.*

Then, we say the Engel structure $\mathcal{L}(\overline{D})$ *is obtained from* D *by the construction described above, by adding* ***Engel torsion*** *along* N*.*

If N *is a 3-torus, then we say* $\mathcal{L}(\overline{D})$ *is obtained from* D *by adding an* ***Engel-Lutz twist*** *along* N *[12, p. 22-23].*

Remark 4.11. Similarly as in the contact case, the Engel-Lutz twist is also used to define the notion of an *overtwisted disc* in an Engel manifold. This is introduced in [12]. In this paper, Engel-Lutz twisting is defined as adding Engel torsion along a 3-torus which is obtained from a transverse knot. The fact that the hypersurface is constructed from a transverse knot, is used in the proof of the $h-$principle for overtwisted Engel structures on closed manifolds. The definition of the overtwisted disc also contains several additional technical conditions. Here we do not discuss this result or proof, and thus we have slightly simplified the definition of the Lutz twist above. ▶

Remark 4.12. To conclude this chapter let us briefly recap the constructions we have seen. In the contact case we have seen how we can replace the structure in a neighbourhood of a transverse hypersurface with a structure which "twists" more. As seen in Figure 4.1, this literally meant that the contact structure would make an additional loop in $\mathbb{S}^1$, with respect to the $t-$direction. Similarly, in the Engel case we have seen how we can replace the structure in a neighbourhood of a transverse hypersurface with a structure which also contains more twisting. The resulting structure makes an additional loop in $\mathbb{S}^2$, parameterised by the $t-$coordinate, with the same convexity as the original structure. Both resulting distributions are again contact or Engel. In the next chapter we will introduce the notion of $(2, 3, 5)-$torsion and the $(2, 3, 5)-$Lutz twist. This construction will very much resemble the constructions we have seen in this chapter. ▶

Chapter 5

$(2,3,5)$-Lutz twisting

After having discussed contact and Engel Lutz twisting in the previous chapter, we are now going to discuss Lutz twisting in $(2,3,5)-$manifolds. The material in this chapter is inspired by the Engel-Lutz twist construction and we follow a very similar scheme as before: we first introduce local models of a $(2,3,5)-$structure along a hypersurface, and then introduce the torsion in such a model. The $(2,3,5)-$Lutz twist is a specific case of adding torsion, and it will be important for defining overtwisted $(2,3,5)-$structures and proving that the h-principle holds for this family of $(2,3,5)$-structures on closed manifolds. The results in this chapter have not been introduced before in the literature, and thus this chapter is an original contribution of the thesis.

5.1 Local model along a transverse hypersurface

Let (M,D) be a $(2,3,5)$-manifold and let N be an orientable transverse hypersurface. Just like in the previous chapter, we first want to find out what the $(2,3,5)-$structure looks like in a neighbourhood of N.

First of all, let $\mathcal{E}$ denote the distribution $[D,D]$. We then note that we can also define the following distributions:

- a line field $L_N := D \cap TN$,
- a plane field $\xi_N := \mathcal{E} \cap TN$ which contains L_N,
- a 3-dimensional subspace $F_N := [\xi_N, \xi_N]$ of TN which contains ξ_N.

In order to describe D along N, we choose a vector field T tangent to D and transverse to N, and we let ϕ_t denote its flow. We can use the flow ϕ_t to describe a tubular neighbourhood $\mathcal{U}(N) \cong N \times (-\epsilon, \epsilon)$. This leads to the following definition.

Definition 5.1. *Let (M,D) be a $(2,3,5)-$manifold and $N \subset M$ a hypersurface transverse to D. We say that the tubular neighbourhood $\mathcal{U}(N)$ as described above, is the* ***tubular neighbourhood associated to*** *T.*

On this tubular neighbourhood we have that $D = \langle \partial_t, H \rangle$ where H is a vector field tangent to N. From this follows that we can describe D as a family of maps $(H_p : (-\epsilon,\epsilon) \to \mathbb{S}^3)_{p\in N}$. We can then look at the vector fields

$$\dot{H} := [\partial_t, H] \quad \text{and} \quad \ddot{H} := [\partial_t, \dot{H}].$$

Since D is a $(2,3,5)$-structure, we must have that $\langle H, \dot{H}, \ddot{H}, [H,\dot{H}] \rangle$ is a framing of TN (see also Proposition 2.4). By fixing an orientation on N, this framing can be either positive or negative.

In order to construct a local model of a $(2,3,5)-$structure along a hypersurface, we need to pick a framing of the hypersurface in a careful way. This motivates the following definition.

Definition 5.2. *Let (M,D) be a $(2,3,5)-$manifold and let $N \subset M$ be a hypersurface transverse to D. We say that $\{X,Y,Z,W\}$, where X, Y, Z and W are vector fields on N, is* ***a frame associated to*** *(D,N) if $X \in D \cap TN$, $Y \in \mathcal{E} \cap TN$, $Z \in F_N$ and $\{X_p, Y_p, Z_p, W_p\}$ is a frame of T_pN for $p \in N$.*

Using such framings, the following proposition gives us a more explicit description of D in $\mathcal{U}(N)$.

Proposition 5.3. *Let (M, D) be a $(2,3,5)-$manifold and let $N \subset M$ be a hypersurface transverse to D such that L_N is a trivial line field. Let $\mathcal{U}(N)$ be the tubular neighbourhood associated to T. Then there exists a frame $\{X, Y, Z, W\}$ associated to (D, N), such that D in $\mathcal{U}(N)$ can be described by the model*

$$D(p,t) = \langle \partial_t, H = \cos(t)X_p + \sin(t)Y_p + g(p,t)Z_p + h(p,t)W_p \rangle$$

where

- $g : N \times (-\epsilon, \epsilon) \to \mathbb{R}$ *a function which is cubic in* t*,*
- $h : N \times (-\epsilon, \epsilon) \to \mathbb{R}$ *a function which is quadratic in* t*.*

Proof. Let $\{T, X, Y, Z, W\}$ be a framing of TM such that $\{X, Y, Z, W\}$ is a framing of TN associated to (D, N). As also discussed above, we get that $D = \langle \partial_t, H \rangle$ where $(H_p : (-\epsilon, \epsilon) \to \mathbb{S}^3)_{p \in N}$ with

$$H_p(t) = (H_p(t))_1 \cdot X + (H_p(t))_2 \cdot Y + (H_p(t))_3 \cdot Z + (H_p(t))_4 \cdot W.$$

Since D is a $(2,3,5)$-structure the following conditions must hold (see also Proposition 2.4):

(i) H_p is an immersion at time t.

(ii) $H_p(t), \dot{H}_p(t), \ddot{H}_p(t)$ are linearly independent, i.e. the map H_p is convex at t,

(iii) $\langle H_q(t), \dot{H}_q(t) \rangle$ is a non-integrable 2-distribution in the level set $Op(p) \times \{t\}$,

(iv) $[H_p(t), \dot{H}_p(t)]$ is linearly independent of $H_p(t), \dot{H}_p(t)$ and $\ddot{H}_p(t)$.

We note that we can rescale the vector field X, such that the coefficient in front of it becomes 1. Then let $a(p,t)$ be the coefficient in front of Y. Since H_p is an immersion, we know that $\partial_t a \neq 0$, and thus we can apply the implicit function theorem parametrically on t, and replace a by t. I.e. we now have that $H_p(t)$ is of the form

$$H_p(t) = X_p + tY_p + B(p,t)$$

where $B(p,t)$ is a vector in the span of Z and W. Rescaling the vector fields X and Y and reparameterising in t, we can write

$$H_p(t) = \cos(t)X_p + \sin(t)Y_p + \tilde{B}(p,t)$$

where again $\tilde{B}(p,t)$ is a vector in the span of Z and W. We can Taylor expand $\tilde{B}(p,t)$ as follows

$$\tilde{B}(p,t) = t^2 F_p + G(p,t)$$

where $\langle F, G(-,t) \rangle = \langle Z, W \rangle$, and $G(p,t)$ is of third order in t. This follows from the convexity of the curve H_p. We can also expand $G(p,t)$ as

$$u(p,t)F_p + v(p,t)Z_p,$$

with functions u and v both vanishing up to order three. From this follows that we can write

$$H_p(t) = \cos(t)X_p + \sin(t)Y_p + g(p,t)F_p + h(p,t)Z_p,$$

where $g(p,t) = t^2 + u(p,t)$ is quadratic in t and $h(p,t) = v(p,t)$ is cubic in t, which was the desired result. □

In order to construct an explicit formula for torsion in $(2,3,5)-$manifolds, we want to simplify the local model that we have seen in the previous proposition. Thus, in the next lemma we prove that any $(2,3,5)$-structure is homotopic to a $(2,3,5)$-structure which has a more simple description in a neighbourhood of a given transverse hypersurface. Note that we are replacing the distribution with a distribution with the same 2-jet.

Lemma 5.4. *Let (M, D) be a $(2,3,5)-$manifold and let $N \subset M$ be a hypersurface transverse to D. Let $\mathcal{U}(N)$ be the tubular neighbourhood associated to T. Then there exist a frame $\{X, Y, Z, W\}$ associated to (D, N) such that D is homotopic as $(2,3,5)-$structures to $\tilde{D}$ which in a neighbourhood $N \times (-\delta, \delta)$ with $\delta < \epsilon$ can be described as*

$$\tilde{D}(p,t) = \langle \partial_t, \tilde{H} = \cos(t)X_p + \sin(t)Y_p + t^2 W_p \rangle.$$

Proof. From Proposition 5.3 we know that in $\mathcal{U}(N)$ the $(2,3,5)$-structure D can be described as

$$D(p,t) = \langle \partial_t, H = \cos(t)X_p + \sin(t)Y_p + g(p,t)Z_p + h(p,t)W_p \rangle$$

with g cubic and h quadratic in t, i.e. we can write

$$g(p,t) = t^3 + \mathcal{O}(t^4) \quad \text{and} \quad h(p,t) = t^2 + \mathcal{O}(t^3).$$

Note that with a rescaling of Z_p and W_p, we can indeed assume that the coefficients in front of t^3 and t^2 in g and h above are 1. We then define the following vector fields for $s \in [0,1]$:

$$F_s(p,t) := \cos(t)X_p + \sin(t)Y_p + g_s(p,t)Z_p + h_s(p,t)W_p$$

with

$$g_s(p,t) := sg(p,t) + (1-s)\chi(t)g(p,t)$$

and

$$h_s(p,t) := sh(p,t) + (1-s)(\chi(t)h(p,t) + (1-\chi(t))t^2)$$

where χ is a bump function which is 0 in a very small interval around $t = 0$, as depicted in Figure 4.4. We note that $\dot{\chi}$ is of order $\frac{1}{\epsilon}$ and $\ddot{\chi}$ is of order $\frac{1}{\epsilon^2}$.

Let $D_s(p,t) := \langle \partial_t, F_s(p,t) \rangle$. We see that D_0 is of the form

$$D_0 = \langle \partial_t, \cos(t)X_p + \sin(t)Y_p + t^2 W_p \rangle$$

in a small neighbourhood around $t = 0$, and that $D_1 = D$. We want to show that the distributions D_s are in fact $(2,3,5)$-structures in $\mathcal{U}(N)$. Recall the following conditions for being a $(2,3,5)$-structure (see also Proposition 2.4):

(i) $F_{s,p}$ is an immersion at time t.

(ii) $F_{s,p}(t), \dot{F}_{s,p}(t), \ddot{F}_{s,p}(t)$ are linearly independent, i.e. the map $F_{s,p}$ is convex at t,

(iii) $\langle F_{s,p}(t), \dot{F}_{s,p}(t) \rangle$ is a non-integrable 2-distribution in the level set $Op(p) \times \{t\}$,

(iv) $[F_{s,p}(t), \dot{F}_{s,p}(t)]$ is linearly independent of $F_{s,p}(t), \dot{F}_{s,p}(t)$ and $\ddot{F}_{s,p}(t)$.

Since the coefficients in front of X_p and Y_p have not changed, we note that $F_{s,p}$ and $\dot{F}_{s,p}$ are still linearly independent, and thus $F_{s,p}$ is an immersion (condition (i)).

For condition (ii) we see that

$$\begin{aligned}
F_{s,p}(t) &= \cos(t)X_p + \sin(t)Y_p + g_s(p,t)Z_p + h_s(p,t)W_p \\
\dot{F}_{s,p}(t) &= -\sin(t)X_p + \cos(t)Y_p + \dot{g}_s(p,t)Z_p + \dot{h}_s(p,t)W_p \\
\ddot{F}_{s,p}(t) &= -\cos(t)X_p - \sin(t)Y_p + \ddot{g}_s(p,t)Z_p + \ddot{h}_s(p,t)W_p.
\end{aligned}$$

We note that

$$\begin{aligned}
\dot{g}_s(p,t) &= s\dot{g}(p,t) + (1-s)\dot{\chi}(t)g(p,t) + (1-s)\chi(t)\dot{g}(p,t) \\
\ddot{g}_s(p,t) &= s\ddot{g}(p,t) + (1-s)\ddot{\chi}(t)g(p,t) + 2(1-s)\dot{\chi}(t)\dot{g}(p,t) + (1-s)\chi(t)\ddot{g}(p,t) \\
\dot{h}_s(p,t) &= s\dot{h}(p,t) + (1-s)\left(\dot{\chi}(t)(h(p,t) - t^2) + \chi(t)\dot{h}(p,t) + 2(1-\chi(t))t\right) \\
\ddot{h}_s(p,t) &= s\ddot{h}(p,t) + (1-s)\left(\ddot{\chi}(t)(h(p,t) - t^2) + 2\dot{\chi}(t)(\dot{h}(p,t) - 2t) + \chi(t)\ddot{h}(p,t) + 2(1-\chi(t))\right).
\end{aligned}$$

As a result of our assumptions on χ, we have the approximations

$$\dot{g}_s(p,t) = s\left(3t^2 + \mathcal{O}(t^3)\right) + (1-s)\mathcal{O}\left(\frac{1}{\epsilon}\right)\left(t^3 + \mathcal{O}(t^4)\right) + (1-s)\mathcal{O}(t)\left(3t^2 + \mathcal{O}(t^3)\right)$$

$$\ddot{g}_s(p,t) = s\left(6t + \mathcal{O}(t^2)\right) + (1-s)\mathcal{O}\left(\frac{1}{\epsilon^2}\right)\left(t^3 + \mathcal{O}(t^4)\right) + 2(1-s)\mathcal{O}\left(\frac{1}{\epsilon}\right)\left(3t^2 + \mathcal{O}(t^3)\right) + (1-s)\mathcal{O}(t)\left(6t + \mathcal{O}(t^2)\right)$$

$$\dot{h}_s(p,t) = s\left(2t + \mathcal{O}(t^2)\right) + (1-s)\left(\mathcal{O}\left(\frac{1}{\epsilon}\right)\mathcal{O}(t^3) + \mathcal{O}(t)\left(2t + \mathcal{O}(t^2)\right) + 2\left(1 - \mathcal{O}(t)\right)t\right)$$

$$\begin{aligned}
\ddot{h}_s(p,t) &= s\left(2 + \mathcal{O}(t)\right) + (1-s)\left(\mathcal{O}\left(\frac{1}{\epsilon^2}\right)\mathcal{O}(t^3) + 2\mathcal{O}\left(\frac{1}{\epsilon}\right)\mathcal{O}(t^2) + \chi(t)\left(2 + \mathcal{O}(t)\right) + 2\left(1 - \chi(t)\right)\right) \\
&= s\left(2 + \mathcal{O}(t)\right) + (1-s)\left(\mathcal{O}\left(\frac{1}{\epsilon^2}\right)\mathcal{O}(t^3) + 2\mathcal{O}\left(\frac{1}{\epsilon}\right)\mathcal{O}(t^2) + \chi(t)\mathcal{O}(t) + 2\right) \\
&= 2 + s\mathcal{O}(t) + (1-s)\left(\mathcal{O}\left(\frac{1}{\epsilon^2}\right)\mathcal{O}(t^3) + 2\mathcal{O}\left(\frac{1}{\epsilon}\right)\mathcal{O}(t^2) + \mathcal{O}(t)\mathcal{O}(t)\right).
\end{aligned}$$

Furthermore we have that

$$\cos(t) = 1 + \mathcal{O}(t^2), \quad \sin(t) \sim t, \quad g_s(p,t) \sim t^3, \quad h_s(p,t) \sim t^2.$$

Therefore, for t close enough to 0, we have that $F_{s,p}$ lies very close to the vector field X_p, $\dot{F}_{s,p}$ very close to the vector field Y_p and $\ddot{F}_{s,p}$ very close to the vector field $-X_p + 2W_p$. From this follows that indeed for t close enough to 0, we have that $F_{s,p}$ is convex.

For condition (iii) we compute

$$\begin{aligned}
[F_{s,p}(t), \dot{F}_{s,p}(t)] = [X_p, Y_p] &+ (\cos(t)\dot{g}_s(p,t) + \sin(t)g_s(p,t))[X_p, Z_p] \\
&+ (\cos(t)\dot{h}_s(p,t) + \sin(t)h_s(p,t))[X_p, W_p] + (\sin(t)\dot{g}_s(p,t) - \cos(t)g_s(p,t))[Y_p, Z_p] \\
&+ (\sin(t)\dot{h}_s(p,t) - \cos(t)h_s(p,t))[Y_p, W_p] + (g_s(p,t)\dot{h}_s(p,t) - h_s(p,t)\dot{g}_s(p,t))[Z_p, W_p].
\end{aligned}$$

For t close to 0, we have that $[F_{s,p}(t), \dot{F}_{s,p}(t)]$ is close to the vector field $[X_p, Y_p]$. Since near 0, the vector fields $F_{s,p}(t)$, $\dot{F}_{s,p}$ and $\ddot{F}_{s,p}(t)$ lie very close to the vector fields X_p, Y_p and $-X_p + 2W_p$ respectively, it follows that $\langle F_{s,p}(t), \dot{F}_{s,p}(t)\rangle$ is indeed non-integrable, and $[F_{s,p}(t), \dot{F}_{s,p}(t)]$ is linearly independent of $F_{s,p}(t)$, $\dot{F}_{s,p}$ and $\ddot{F}_{s,p}(t)$.

We conclude that D is homotopic through $(2,3,5)-$structures D_s to D_0, which on $N \times (-\delta, \delta)$, for a $\delta < \epsilon$, is of the form

$$D_0 = \langle \partial_t, \cos(t)X_p + \sin(t)Y_p + t^2 W_p \rangle.$$

This was the desired result. □

5.1.1 Reducing convexity

As mentioned before, in order to explicitly define torsion in a $(2,3,5)-$manifold, we need to work with a local model around a transverse hypersurface. In the previous subsection we have shown that there is indeed a local model (Proposition 5.3), and that we can simplify it by homotopy (Lemma 5.4). In this subsection we show that we can further homotope the distribution in Lemma 5.4, while keeping the frame fixed, such that it becomes "less convex". This will ensure that once we add $(2,3,5)$-torsion, the resulting structure is still $(2,3,5)$. For the proof of this result we will need a bump function of which we can control its second derivative. Let us give the construction of such a bump function.

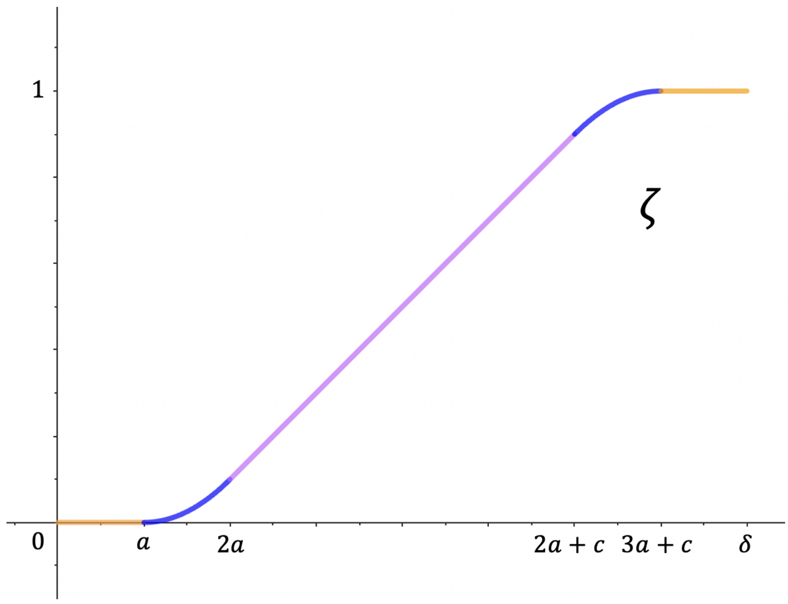

Figure 5.1: Illustration of the step function ζ, which is build up out of 5 parts, as given in equation (5.1).

First take a look at Figure 5.1. Here we portray a step function ζ. We see that it is formed out of 5 parts, which are explicitly given by

$$\zeta(t) = \begin{cases} 0 & \text{if } t \in [0, a], \\ q \cdot (t-a)^2 & \text{if } t \in [a, 2a] \\ \frac{1-2qa^2}{c} t + qa^2 - \frac{2a-4qa^3}{c} & \text{if } t \in [2a, 2a+c] \\ -q \cdot (t-3a-c)^2 + 1 & \text{if } t \in [2a+c, 3a+c] \\ 1 & \text{if } t \in [3a+c, \delta) \end{cases} . \tag{5.1}$$

We note that at the points $t = a, 3a + c$ the function ζ is automatically C^1. For the function ζ to be C^1 at the points $t = 2a, 2a + c$, we need to demand that

$$a = -\frac{c}{2} + \sqrt{\frac{1}{4}c^2 + \frac{1}{2q}}.$$

From this follows that we can choose q freely, and still construct a C^1 function ζ. Furthermore, we note that ζ is not C^2. Namely, in the first, third and last part of the function, the second derivative is 0, whereas the second derivative in the second part is $2q$, and in the fourth part is $-2q$. However, we can take a C^2−smoothing of the function ζ, which we will still denote by ζ, which has an arbitrarily close 1−jet, and is C^2−smooth. We note that we can pick this smoothing such that $\zeta''(t) \in [-2q, 2q]$ for all $t \in [0, \delta)$.

By mirroring the function ζ in the y−axis, we get a C^2−smooth bump function, which we will denote by $\overline{\zeta}$, see Figure 5.2. Its second derivative $\overline{\zeta}''(t)$ lies in $[-2q, 2q]$ for all $t \in (-\delta, \delta)$.

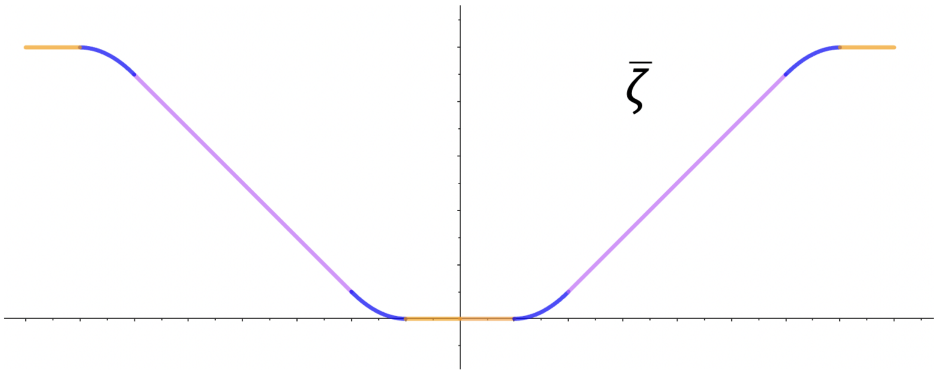

Figure 5.2: Illustration of the bump function $\overline{\zeta}$, which arises by mirroring the C^2-smooth step function ζ in the y−axis. We will use this bump function in the proof of Lemma 5.5.

Using this bump function, we are now going to homotope the local model from Lemma 5.4, to a model which is less convex.

Lemma 5.5. *Let (M, D) be a $(2,3,5)$−manifold and let $N \subset M$ be a hypersurface transverse to D. Let $\mathcal{U}(N) \cong N \times (-\epsilon, \epsilon)$ be the tubular neighbourhood associated to T. Let $\{X, Y, Z, W\}$ be a frame associated to (D, N). By Lemma 5.4 we know that D is homotopic to $\tilde{D}$, which on a neighbourhood $N \times (-\delta, \delta)$, for $\delta < \epsilon$, is given by*

$$\tilde{D} = \langle \partial_t, \tilde{H} = \cos(t)X_p + \sin(t)Y_p + t^2 W_p \rangle.$$

Then for every $0 < \alpha < 1$, there is a homotopy between $\tilde{D}$ and $\overline{D}$ through $(2,3,5)$−structures, where $\overline{D}$ on a neighbourhood $N \times (-\mu, \mu)$ with $\mu < \epsilon$ can be described as

$$\overline{D}(p,t) = \langle \partial_t, \overline{H} = \cos(t)X_p + \sin(t)Y_p + \alpha t^2 W_p \rangle.$$

Proof. From Lemma 5.4 we know indeed that D is homotopic to $\tilde{D}$ through $(2,3,5)$−structures, and $\tilde{D}$ can be described in a neighbourhood $N \times (-\delta, \delta)$ as

$$\tilde{D}(p,t) = \langle \partial_t, \overline{H} = \cos(t)X_p + \sin(t)Y_p + t^2 W_p \rangle.$$

We then define the following distributions on $N \times (-\delta, \delta)$,

$$D_s := \langle \partial_t, F_{s,p}(t) := \langle \cos(t) X_p + \sin(t) Y_p + f_s(t) W_p \rangle,$$

with

$$f_s(t) := (1-s)t^2 + s(\overline{\zeta}(t)t^2 + (1-\overline{\zeta}(t))\alpha t^2),$$

where $\overline{\zeta}$ is the bump function as constructed above, and $0 < \alpha < 1$. We note that $D_0 = D$ and that

$$D_1 = \langle \partial_t, \cos(t) X_p + \sin(t) Y_p + \alpha t^2 W_p \rangle$$

in a neighbourhood $N \times (-\mu, \mu)$ for $\mu < \delta$. We want to show that the distributions D_s are $(2,3,5)-$structures by showing that the conditions from Proposition 2.4 hold for the maps $F_{s,p}$.

We note that condition (i) is satisfied since we have not changed the coordinates in front of X_p and Y_p, which ensure that $F_{s,p}(t)$ and $\dot{F}_{s,p}(t)$ are linearly independent. For condition (ii) we compute:

$$\begin{aligned} F_{s,p}(t) &= \cos(t) X_p + \sin(t) Y_p + f_s(t) W_p \\ \dot{F}_{s,p}(t) &= -\sin(t) X_p + \cos(t) Y_p + \dot{f}_s(t) W_p \\ \ddot{F}_{s,p}(t) &= -\cos(t) X_p - \sin(t) Y_p + \ddot{f}_s(t) W_p. \end{aligned}$$

Since $f_s(t)$ is quadratic in t, we note that the vector field $F_{s,p}(t)$ lies very close to the vector field X_p. Furthermore, we see that

$$\begin{aligned} \dot{f}_s(t) &= 2(1-s)t + s\left(\dot{\overline{\zeta}}(t)\left(t^2 - \alpha t^2\right) + \overline{\zeta}(t)\left(2t - 2\alpha t\right) + 2\alpha t\right) \\ \ddot{f}_s(t) &= 2(1-s) + s\left(\ddot{\overline{\zeta}}(t)\left(t^2 - \alpha t^2\right) + 2\dot{\overline{\zeta}}(t)\left(2t - 2\alpha t\right) + \overline{\zeta}(t)\left(2 - 2\alpha\right) + 2\alpha\right). \end{aligned}$$

We can then make the following approximation:

$$\dot{f}_s(t) = 2(1-s)t + s\left(\mathcal{O}\left(\frac{1}{\delta}\right)\left(t^2 - \alpha t^2\right) + \mathcal{O}(t)(2t - 2\alpha t) + 2\alpha t\right).$$

From this follows that the vector field $\dot{F}_{s,p}(t)$ lies very close to the vector field Y_p. Now let us unpack all the terms in the expression $\ddot{f}_s(t)$. We know that $\ddot{\overline{\zeta}}(t) \in [-2q, 2q]$, and that $\overline{\zeta}(t)$ is of order t. We then see that

$$\ddot{f}_s(t) = 2 - 2s(1-\alpha) + s\ddot{\overline{\zeta}}(t)(t^2 - \alpha t^2) + 4st\dot{\overline{\zeta}}(t)(1-\alpha) + s\mathcal{O}(t)(2 - 2\alpha).$$

We note that the term $4st\dot{\overline{\zeta}}(t)(1-\alpha)$ is positive for all $t \in (-\delta, \delta)$, that the term $s\mathcal{O}(t)(2-2\alpha)$ is small, and that also the term $s\ddot{\overline{\zeta}}(t)(t^2 - \alpha t^2)$ is small if we choose q small. Therefore, we can make sure that $\ddot{f}_s(t) > 0$ for all $t \in (-\delta, \delta)$. This means that the vector field $\ddot{F}_{s,p}(t)$ lies very close to the vector field $-X_p + c(p,s)W_p$, where $c : N \times [0,1] \to \mathbb{R}_{>0}$ is a function depending on p and the homotopy parameter s. From this we may conclude that $F_{s,p}(t)$, $\dot{F}_{s,p}(t)$ and $\ddot{F}_{s,p}(t)$ are linearly independent.

For condition (iii) we compute

$$\begin{aligned} [F_{s,p}(t), \dot{F}_{s,p}(t)] =& [X_p, Y_p] + (\cos(t)\dot{f}_s(t) + \sin(t) f_s(t))[X_p, W_p] \\ &+ (\sin(t)\dot{f}_s(t) - \cos(t) f_s(t))[Y_p, W_p]. \end{aligned}$$

Using the approximations above, we note that for t close to 0, the vector field $[F_{s,p}(t), \dot{F}_{s,p}(t)]$ lies very close to the vector field $[X_p, Y_p]$. Using the fact that $\langle X_p, Y_p \rangle$ is a non-integrable distribution, we can conclude that for t close enough to 0, $\langle F_{s,p}(t), \dot{F}_{s,p}(t) \rangle$ is also a non-integrable distribution, and $[F_{s,p}(t), \dot{F}_{s,p}(t)]$ is linearly independent of $F_{s,p}(t)$, $\dot{F}_{s,p}(t)$ and $\ddot{F}_{s,p}(t)$.

It follows that for t close enough to 0, D is homotopic through D_1 through $(2,3,5)$-structures, which proves the desired result. □

Remark 5.6. We note that the homotopy in Lemma 5.5 keeps the frame fixed, and reduces the convexity of the map $\tilde{H}$. Therefore, this homotopy is only C^1−small, whereas the homotopy in Lemma 5.4 is C^2−small. In the next chapter we explain how to add a $(2,3,5)$−Lutz twist in this particular model. For the resulting structure to still be $(2,3,5)$, we needed to reduce the convexity of $\tilde{H}$, which is why we introduce this particular homotopy in Lemma 5.5. Just like in the Engel case, the $(2,3,5)$−Lutz twist will be constructed by adding an extra loop to maps $\overline{H}_p : (-\mu,\mu) \to \mathbb{S}^2$ for $p \in N$. Here the equator of $\mathbb{S}^2$ is spanned by the vector fields X and Y as in Lemma 5.5. We need the loop to be relatively close to this equator, as the properties of the vector fields X and Y will ensure that property (iii) of Proposition 2.4 holds for the resulting structure. Reducing the convexity of the map $\tilde{H}$ will ensure this, although it may seem counter-intuitive at first. ▶

5.2 Construction of the $(2,3,5)$-Lutz twist

In this section we will describe the construction of adding $(2,3,5)$−torsion along a hypersurface. As mentioned before, this construction will resemble the construction of the Engel-Lutz twist. After we have described how to add torsion, we will define $(2,3,5)$−Lutz twisting.

Let (M,D) be a $(2,3,5)$-manifold and let N be a hypersurface transverse to D. From Proposition 5.3 we know that there is a tubular neighbourhood $\mathcal{U}(N) \cong N \times (-\epsilon,\epsilon)$ where we have an explicit description of what D looks like, namely

$$D(p,t) = \langle \partial_t, H_p(t) = \cos(t)X_p + \sin(t)Y_p + g(p,t)Z_p + h(p,t)W_p \rangle$$

with g and h as in Proposition 5.3. From Lemma 5.4 and 5.5 we know that D is homotopic through $(2,3,5)$-structures to $\overline{D}$ which in a neighbourhood $N \times (-\mu,\mu)$ with $\mu < \epsilon$ can be described as

$$\overline{D}(p,t) = \langle \partial_t, \overline{H}_p(t) = \cos(t)X_p + \sin(t)Y_p + \alpha t^2 W_p \rangle,$$

with $0 < \alpha < 1$, as in Lemma 5.5. We can describe the structure $\overline{D}$ by a family of maps $(\overline{H}_p) : (-\mu,\mu) \to \mathbb{S}^2$ given by

$$\overline{H}_p(t) = \frac{1}{\sqrt{1+\alpha^2 t^4}}(\cos(t), \sin(t), \alpha t^2).$$

Note that the coordinates of these maps are determined by the directions of the vector fields X_p, Y_p and W_p, and thus depend on $p \in N$.

In this section we explain how to introduce $(2,3,5)$−torsion in $\overline{D}$. We do this by replacing the family $(\overline{H}_p)_{p\in N}$ by maps $(G_p)_{p\in N}$. Just like in the Engel case, we first construct a C^2-family of maps $(F_p)_{p\in N}$ and then we smooth this family to obtain $(G_p)_{p\in N}$. The maps $(F_p)_{p\in N} : (-\mu,\mu) \to \mathbb{S}^2$ consist of three parts:

(1) The first part of the curve is equal to $\overline{H}_p((-\mu,0])$,

(2) The middle part of the curve is the unique circle through $\overline{H}_p(0)$, tangent to the equator spanned by X_p and Y_p, and which has the same curvature as $\overline{H}_p$ at $t = 0$,

(3) The last part of the curve is equal to $\overline{H}_p([0,\mu))$.

From Lemma 4.8 we know that the curvature of $\overline{H}_p$ at $t = 0$ is equal $\sqrt{1+4\alpha^2}$. From this follows that the middle part of the map F_p is a circle of radius $\frac{1}{\sqrt{1+4\alpha^2}}$, tangent at the point $(1,0,0)$ to the equator spanned by X_p and Y_p. Such a circle can be described by the following formula:

$$t \mapsto \left(\frac{\cos(t)}{1+4\alpha^2} + \frac{4\alpha^2}{1+4\alpha^2}, \frac{\sin(t)}{\sqrt{1+4\alpha^2}}, \frac{2\alpha}{1+4\alpha^2} - \frac{2\alpha\cos(t)}{1+4\alpha^2} \right), \quad \text{for } t \in [0,2\pi].$$

Combining this map with the curve $\overline{H}_p(-\mu,\mu)$, we can find an explicit expression for the curves $(F_p)_{p\in N} : (-\mu,\mu) \to \mathbb{S}^3$ consisting of three parts:

(1) For $t \in \left(-\mu, -\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}\right]$ it is given by

$$\frac{1}{\sqrt{1+\alpha^2\left(\frac{\pi+\sqrt{1+4\alpha^2}\mu}{\sqrt{1+4\alpha^2}\mu}t+\frac{\pi}{\sqrt{1+4\alpha^2}}\right)^4}}\left(\cos\left(\frac{\pi+\sqrt{1+4\alpha^2}\mu}{\sqrt{1+4\alpha^2}\mu}t+\frac{\pi}{\sqrt{1+4\alpha^2}}\right),\sin\left(\frac{\pi+\sqrt{1+4\alpha^2}\mu}{\sqrt{1+4\alpha^2}\mu}t+\frac{\pi}{\sqrt{1+4\alpha^2}}\right),\right.$$
$$\left.\alpha\left(\frac{\pi+\sqrt{1+4\alpha^2}\mu}{\sqrt{1+4\alpha^2}\mu}t+\frac{\pi}{\sqrt{1+4\alpha^2}}\right)^2\right).$$

(2) For $t \in \left[-\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}, \frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}\right]$ it is given by

$$\left(\frac{\cos\left(\frac{\pi+\mu\sqrt{1+4\alpha^2}}{\mu}t+\pi\right)}{1+4\alpha^2}+\frac{4\alpha^2}{1+4\alpha^2}, \frac{\sin\left(\frac{\pi+\mu\sqrt{1+4\alpha^2}}{\mu}t+\pi\right)}{\sqrt{1+4\alpha^2}}, \frac{2\alpha}{1+4\alpha^2}-\frac{2\alpha\cos\left(\frac{\pi+\mu\sqrt{1+4\alpha^2}}{\mu}t+\pi\right)}{1+4\alpha^2}\right).$$

(3) For $t \in \left[\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}, \mu\right)$ it is given by

$$\frac{1}{\sqrt{1+\alpha^2\left(\frac{\pi+\sqrt{1+4\alpha^2}\mu}{\sqrt{1+4\alpha^2}\mu}t-\frac{\pi}{\sqrt{1+4\alpha^2}}\right)^4}}\left(\cos\left(\frac{\pi+\sqrt{1+4\alpha^2}\mu}{\sqrt{1+4\alpha^2}\mu}t-\frac{\pi}{\sqrt{1+4\alpha^2}}\right),\sin\left(\frac{\pi+\sqrt{1+4\alpha^2}\mu}{\sqrt{1+4\alpha^2}\mu}t-\frac{\pi}{\sqrt{1+4\alpha^2}}\right),\right.$$
$$\left.\alpha\left(\frac{\pi+\sqrt{1+4\alpha^2}\mu}{\sqrt{1+4\alpha^2}\mu}t-\frac{\pi}{\sqrt{1+4\alpha^2}}\right)^2\right).$$

Note that we have adjusted the speed and have translated the variable t such that the curves are defined on $(-\mu, \mu)$ and are C^2. In Figure 5.3 you can see an illustration of F_p. We see indeed that the second part of the curve is an extra loop, tangent to $\overline{H}_p$ at $t = 0$, with the same curvature.

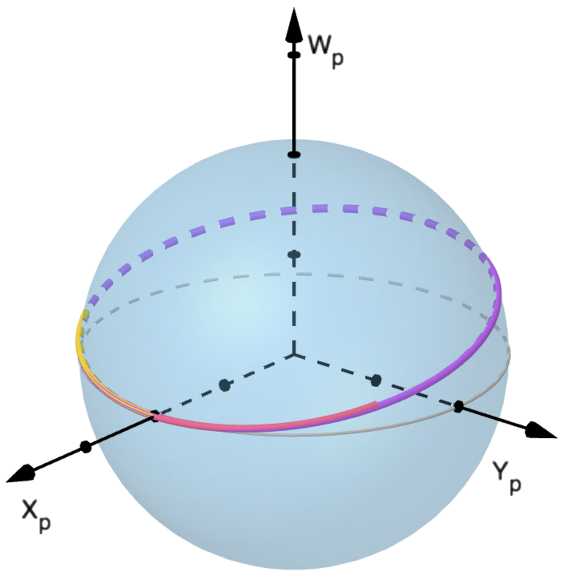

Figure 5.3: Illustration of the maps $(F_p)_{p\in N}$ from the $(2,3,5)-$torsion construction. Here the yellow curve represents $\overline{H}_p((-\mu, 0])$, the purple circle is the middle part of the curve F_p and the pink curve represents the curve $\overline{H}_p([0, \mu))$.

We now want to smooth the family $(F_p)_{p\in N}$ to obtain the desired maps $(G_p)_{p\in N}$. The relevant properties of the curves $(F_p)_{p\in N}$ regarding convexity, depend only on their 2-jet. Any C^2-small smoothing yields therefore a smooth family of curves $(G_p)_{p\in N}$ with the same properties $(F_p)_{p\in N}$.

By replacing the maps $(\overline{H}_p)_{p\in N}$ by $(G_p)_{p\in N}$ we obtain a new structure $\mathcal{L}(\overline{D})$. We note that $\mathcal{L}(\overline{D})$ has the following properties

- $\mathcal{L}(\overline{D})$ is the same as $\overline{D}$ outside of the tubular neighbourhood $N \times (-\mu, \mu)$,
- $\mathcal{L}(\overline{D})$ is tangent to the vector field T on $N \times (-\mu, \mu)$,
- the structure $\mathcal{L}(\overline{D})$ makes an extra turn in the W-direction with respect to the original structure $\overline{D}$.

We claim that for appropriate values of α, the resulting structure $\mathcal{L}(\overline{D})$ is still a $(2,3,5)$-structure.

Lemma 5.7. *Let $\overline{D}$ be defined as in Lemma 5.5. Then for every $\alpha > 0$ small enough, the structure $(M, \mathcal{L}(\overline{D}))$ as described above, is a $(2,3,5)-$manifold.*

Proof. First of all, we note that we have only altered the $(2,3,5)$-structure inside the tubular neighbourhood $N \times (-\mu, \mu)$, thus we only have to prove it is $(2,3,5)$ in this region. We consider the curves $(F_p)_{p\in N}$ from the construction above. To prove that $\mathcal{L}(\overline{D})$ is $(2,3,5)$ we should prove the following conditions from Proposition 2.4:

(i) F_p is an immersion at time t.

(ii) $F_p(t), \dot{F}_p(t), \ddot{F}_p(t)$ are linearly independent, i.e. the map $F_{s,p}$ is convex at t,

(iii) $\langle F_p(t), \dot{F}_p(t)\rangle$ is a non-integrable 2-distribution in the level set $Op(p) \times \{t\}$,

(iv) $[F_p(t), \dot{F}_p(t)]$ is linearly independent of $F_p(t), \dot{F}_p(t)$ and $\ddot{F}_p(t)$.

Since $\overline{H}_p$ is (locally) an immersion everywhere, and the added turn in the middle part of F_p is also an immersion, we conclude that F_p is a local immersion for every $t \in (-\mu, \mu)$. Furthermore, since $\overline{H}_p$ is convex or concave, F_p is also convex or concave, and thus $F_p(t)$, $\dot{F}_p(t)$ and $\ddot{F}_p(t)$ are linearly independent vectors. This ensures conditions (i) and (ii).

For condition (iii) first let $t \in \left[-\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}, \frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}\right]$, then we see that

$$F_p(t) \xrightarrow{\alpha\to 0} \left(\cos\left(\frac{\pi+\mu}{\mu}t+\pi\right), \sin\left(\frac{\pi+\mu}{\mu}t+\pi\right), 0\right),$$

and

$$\dot{F}_p(t) = \frac{\pi+\mu\sqrt{1+4\alpha^2}}{\mu}\left(\frac{-\sin\left(\frac{\pi+\mu\sqrt{1+4\alpha^2}}{\mu}t+\pi\right)}{1+4\alpha^2}, \frac{\cos\left(\frac{\pi+\mu\sqrt{1+4\alpha^2}}{\mu}t+\pi\right)}{\sqrt{1+4\alpha^2}}, \frac{2\alpha\sin\left(\frac{\pi+\mu\sqrt{1+4\alpha^2}}{\mu}t+\pi\right)}{1+4\alpha^2}\right)$$

thus

$$\dot{F}_p(t) \xrightarrow{\alpha\to 0} \frac{\pi+\mu}{\mu}\left(-\sin\left(\frac{\pi+\mu}{\mu}t+\pi\right), \cos\left(\frac{\pi+\mu}{\mu}t+\pi\right), 0\right).$$

Therefore, we see that

$$[F_p(t), \dot{F}_p(t)] \xrightarrow{\alpha\to 0} \frac{\pi+\mu}{\mu}\left(\cos^2\left(\frac{\pi+\mu}{\mu}t+\pi\right)[X_p, Y_p] - \sin^2\left(\frac{\pi+\mu}{\mu}t+\pi\right)[Y_p, X_p]\right) = \frac{\pi+\mu}{\mu}[X_p, Y_p].$$

From this we see that

$$\langle F_p(t), \dot{F}_p(t)\rangle \xrightarrow{\alpha\to 0} \langle X_p, Y_p\rangle$$

and combining this with

$$[F_p(t), \dot{F}_p(t)] \xrightarrow{\alpha\to 0} \frac{\pi+\mu}{\mu}[X_p, Y_p],$$

we can indeed conclude that condition (iii) holds for $t \in \left[-\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}, \frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}\right]$.

For $t \in \left(-\mu, -\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}\right] \cup \left[\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}, \mu\right)$ the curves F_p are essentially the same as the curves $\overline{H}_p$, up to a translation of the variable t. Since $\langle \overline{H}_p(t), \dot{\overline{H}}_p(t)\rangle$ spans a non-integrable distribution for $t \in (-\mu, \mu)$, we can conclude that $\langle F_p(t), \dot{F}_p(t)\rangle$ also spans a non-integrable distribution for

$t \in \left(-\mu, -\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}\right] \cup \left[\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}, \mu\right)$.

For the last condition, first let $t \in \left[-\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}, \frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}\right]$. We see that

$$\ddot{F}_p(t) = \left(\frac{\pi + \mu\sqrt{1+4\alpha^2}}{\mu}\right)^2 \left(\frac{-\cos\left(\frac{\pi+\mu\sqrt{1+4\alpha^2}}{\mu}t + \pi\right)}{1+4\alpha^2}, \frac{-\sin\left(\frac{\pi+\mu\sqrt{1+4\alpha^2}}{\mu}t + \pi\right)}{\sqrt{1+4\alpha^2}}, \frac{2\alpha\cos\left(\frac{\pi+\mu\sqrt{1+4\alpha^2}}{\mu}t + \pi\right)}{1+4\alpha^2}\right).$$

We note that $[F_p(t), \dot{F}_p(t)]$ has a Z_p-component whereas $F_p(t)$, $\dot{F}_p(t)$ and $\ddot{F}_p(t)$ do not. Therefore, condition (iv) indeed holds for $t \in \left[-\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}, \frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}\right]$.

Again, for $t \in \left(-\mu, -\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}\right] \cup \left[\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}, \mu\right)$ the curves F_p are essentially the same as the curves $\overline{H}_p$, up to a translation of the variable t. Therefore, for $t \in \left(-\mu, -\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}\right] \cup \left[\frac{\pi\cdot\mu}{\pi+\sqrt{1+4\alpha^2}\mu}, \mu\right)$, condition (iv) already holds for the curves F_p.

Since these conditions only depend on the 2-jets of the curves $(F_p)_{p\in N}$, they automatically also hold for the maps $(G_p)_{p\in N}$. From this we may conclude that $\mathcal{L}(\overline{D})$ is a $(2,3,5)$-structure. □

Remark 5.8. In the proof above it becomes clear why we needed to reduce the convexity of the functions $\tilde{H}$ in Lemma 5.5. Namely, we see that only for small values of $\alpha > 0$, the distribution $\mathcal{L}(\overline{D})$ is a $(2,3,5)$−structure. Intuitively, if $\alpha > 0$ is large, the W_p-component in part (2) of the function F_p becomes quite large, which means that the extra loop moves further away from the equator spanned by X_p and Y_p. This then could influence the non-integrability of the distribution $\langle F_p(t), \dot{F}_p(t)\rangle$. ▶

We are now ready to introduce the definitions of $(2,3,5)$−torsion and Lutz twisting.

Definition 5.9. *Let (M, D) be a $(2,3,5)$−manifold, and let $N \subset M$ be a transverse hypersurface. From Lemma 5.5 we know that we can homotope D such that it is of the form*

$$\overline{D} = \langle \partial_t, \overline{H} = \cos(t)X_p + \sin(t)Y_p + \alpha t^2 Z_p\rangle$$

in a neighbourhood $N \times (-\mu, \mu)$ with $\mu > 0$ and $\{X, Y, Z, W\}$ a frame associated to (D, N), as in Lemma 5.5. Let $0 < \alpha < 1$ be small enough such that $\mathcal{L}(\overline{D})$ is a $(2,3,5)$−structure.

*Then, we say the $(2,3,5)$−structure $\mathcal{L}(\overline{D})$ is obtained from D, by the construction described above, by adding $(2,3,5)$−**torsion** along N.*

*If N is the boundary of the tubular neighbourhood of a transverse 3-manifold, then we say $\mathcal{L}(\overline{D})$ is obtained from D by adding a $(2,3,5)$−**Lutz twist** along N.*

Remark 5.10. Let us shed some light on why we define $(2,3,5)$−Lutz twisting along the boundary of the tubular neighbourhood of a 3-dimensional submanifold. In the next chapter we will define the overtwisted disc in a $(2,3,5)$−manifold by Lutz twisting in a particular setting. In the extension part in Chapter 8, we will need to move the overtwisted disc around in the manifold, as we will use it to obtain the $(2,3,5)$−condition in the top-simplices. It turns out that it will be easier to move around 3-dimensional transverse submanifolds, and thereby moving around the overtwisted disc. This is why we define Lutz twisting along these particular hypersurfaces. ▶

Remark 5.11. Let us comment on some notation. Let (M, D) be a $(2,3,5)$-manifold and let $N \subset M$ be a transverse hypersurface. From now on, we will write $\mathcal{L}(D)$ for the structure which is obtained from D by adding a $(2,3,5)$−Lutz twist along N. Note that this does not change the definition. We first still implicitly carry out the homotopy from Lemma 5.5 before we add the torsion, but we do not denote this. ▶

Chapter 6

The overtwisted disc

Recall that in Chapter 5, we have defined $(2,3,5)-$torsion and Lutz twisting. The goal of this chapter is to define the notion of an *overtwisted disc* in a (formal) $(2,3,5)-$manifold. Also recall that the main result we are trying to prove, Theorem 0.4, is an $h-$principle for overtwisted $(2,3,5)-$structures, i.e. $(2,3,5)-$structures which contain an embedding of the overtwisted disc.

In the previous chapter we have seen that Lutz twisting in $(2,3,5)-$manifolds is defined as adding torsion along the boundary of the tubular neighbourhood of a transverse 3-manifold. In the first section of this chapter, we will look at the existence of such transverse 3-manifolds in $(2,3,5)-$manifolds. Thereafter, in Section 6.2, we will show how to shrink the tubular neighbourhood of such a submanifold, which will secure some desired properties of these objects. In Section 6.3, we will discuss a result on transverse embeddings, Conjecture 6.4, which will introduce another property the overtwisted disc will have to possess. In the fourth section, we will then be ready to formally define the overtwisted disc.

Last but not least, we will show that we can replicate these discs via homotopy such that extra copies appear. See also Proposition 6.13. We call this the self-replicating property of the overtwisted disc. This will be useful in the final step of the proof of Theorem 0.4, the extension, where we use the overtwisted disc to obtain the $(2,3,5)-$condition in the top-simplices.

The material in this chapter has not been published before, and thus this is an original contribution of the thesis.

6.1 Transverse submanifolds

In this section we will show that transverse $3-$submanifolds exist in $(2,3,5)-$manifolds. As mentioned before, Lutz twists are added along the boundaries of tubular neighbourhoods of transverse 3-manifolds. To prove that these submanifolds exist, we use the fact that there is a (non-unique) local model for $(2,3,5)-$structures, the Monge normal form, as explained in Section 2.2.

Lemma 6.1. *Let (M,D) be a $(2,3,5)-$manifold, and let $B \subset M$ be a ball in M with local coordinates (t,x,y,z,w) such that*

$$D = \langle \partial_t, \partial_x + y\partial_z + t\partial_y + F(t,x,y,z,w)\partial_w \rangle$$

with $F : \mathbb{R} \to \mathbb{R}$ a function with $\partial_{tt}F \neq 0$. Then there is an embedding

$$g : \mathbb{D}^3 \hookrightarrow B \subset M$$

such that $L := g(\mathbb{D}^3)$ is transverse to D.

Proof. We denote $\mathbb{D}^3 := \{(x_1,x_2,x_3) \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$, and we map

$$g(x_1,x_2,x_3) = (0,0,lx_1,lx_2,lx_3),$$

where l is chosen such that indeed $g(\mathbb{D}^3) \subset B$. We note that $L = g(\mathbb{D}^3) = \langle \partial_y, \partial_z, \partial_w \rangle$, and thus by construction, L is transverse to D. □

This lemma tells us that there exist transverse 3-dimensional submanifolds L in a $(2,3,5)-$manifold. We note that the boundary of the normal bundle, $\partial\nu(L)$, is a hypersurface along which one can add torsion.

6.2 Shrinking the tubular neighbourhood

In the previous section we have seen that transverse 3-manifolds exist in $(2,3,5)-$manifolds. The boundary of the tubular neighbourhood of such a submanifold is a hypersurface. In this section we will introduce some notation which will allow us to introduce some properties for these hypersurfaces. The goal of this, is to specify a condition which allows us to shrink the tubular neighbourhood as much as we want. This will help us later on in showing that in the presence of an overtwisted disc, new ones appear via homotopy.

Let us work in the same setting as in Lemma 6.1. By the tubular neighbourhood theorem we know that

$$\partial\nu(L) \cong \partial(L \times \mathbb{D}^2_h),$$

where h is the radius of the 2-disc which is small enough such that $\nu(L) \subset B$ and that $\partial\nu(L)$ is transverse to D. Recall from the proof of Lemma 6.1 that we constructed L such that it has radius l. From now on, we say

$$l \text{ is the } \textit{radius} \text{ of } \nu(L) \quad \text{and} \quad h \text{ is the } \textit{height} \text{ of } \nu(L).$$

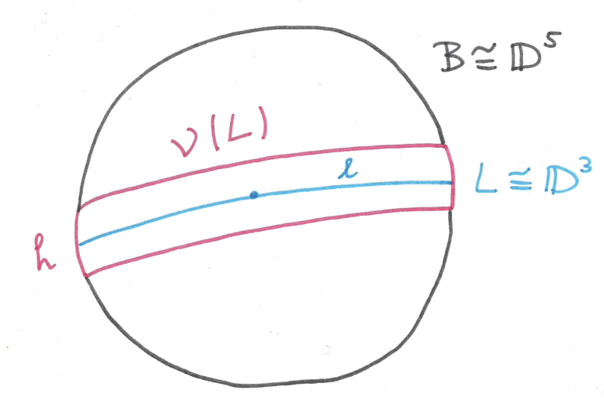

Figure 6.1: Illustration of a transverse $3-$manifold in a ball $B \subset M$. Here we see that the normal bundle $\nu(L)$, indicated in red, has height h, and that the 3-manifold L, indicated in blue, has radius l.

In the next section, we will define the overtwisted disc in a $(2,3,5)-$manifold as a Lutz twist which is added along a hypersurface of the form $\partial\nu(L)$ (along with some other properties). A crucial property of the overtwisted disc will be that we can shrink the tubular neighbourhood of L, such that the middle part of it becomes as thin as we want, while also staying transverse to the distribution. See Figure 6.2 for an illustration of this. We need this property later on, for moving pieces of the overtwisted disc around in the manifold, while also staying transverse to the $(2,3,5)$-structure, and for showing that in the presence of an overtwisted disc, new ones appear via homotopy.

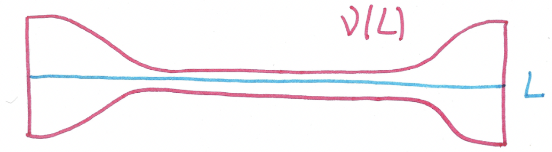

Figure 6.2: Illustration of a 3-manifold L, indicated in blue, and its normal bundle $\nu(L)$, indicated in red, which has been shrunk such that it becomes very thin in the middle part of L.

The reason we shrink it in this hourglass shape, is because at the edges we want the structure to match up nicely with the rest of the structure outside B. We should however, be careful, because we still want the red band to be transverse to D. We note that the radius l of L is given by the embedding, but that we are free to choose the height h of $\nu(L)$. We will now explain how to choose

h small enough such that we can indeed shrink $\partial\nu(L)$ in this way, while preserving transversality.

First of all, we note that there are local coordinates, such that D cuts L vertically, i.e. it makes a 90 degree angle with respect to L. However, moving away from L, this angle will change. Take a look at Figure 6.3. Here we denote this angle by θ and it changes roughly linearly in h. We want the angle that the red line makes to be less than the angle that the distribution makes, i.e. less than θ. This gives us the following bound:

$$\frac{h}{l/4} < \tan\left(\frac{\pi}{2} \pm C \cdot h\right).$$

If we choose h small enough, we note that $\frac{\pi}{2} \pm C \cdot h \geq \frac{\pi}{4}$, and thus the following bound is sufficient:

$$h < \min\left(\frac{l}{4}, \frac{\pi}{4C}\right) =: \tau_0.$$

We call τ_0 the *shrinking constant* of L ,as setting this bound for h will give us a lot of freedom in shrinking the tubular neighbourhood. As mentioned before, this will be a very important property of the overtwisted disc later on in the extension part, for moving pieces of the overtwisted disc around in the manifold, while also staying transverse to the distribution, and for showing the self-replicating property of the overtwisted disc.

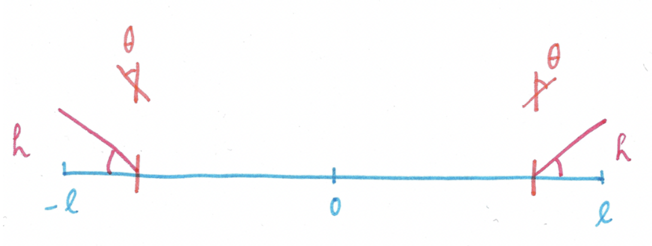

Figure 6.3: In blue we see a transverse 3-manifold L with radius l. Along L the distribution is vertical, which is illustrated by the vertical orange lines, but moving away from L, the distribution will tilt. The angle it makes is given by θ. We want this angle to be less than the red angle indicated in the figure, which ultimately defines the shrinking constant τ_0 as defined above.

6.3 Transverse embeddings of 3-manifolds

As mentioned before, the overtwisted disc will be defined as a Lutz twist which is added along the boundary of the tubular neighbourhood of a transverse 3-manifold. For the extension part of the proof of Theorem 0.4, we will need to move the overtwisted disc towards the top-simplices. Recall from the introduction that we can reduce the h-principle from Theorem 0.4 to an extension problem, where we only need to obtain the desired $(2,3,5)-$condition in the top-simplices. We will do this by moving the 3-manifold, while also keeping it transverse to the distribution. This brings us to the following definition.

Definition 6.2. *Let M be manifold, D a distribution on M and N a manifold with* $\dim(N) \leq \dim(M)$. *We say an embedding $f : N \hookrightarrow M$ is* ***transverse to*** *D, if*

$$d_p f(T_p N) \oplus D_{f(p)} = T_p M$$

for every $p \in N$.

We can also define a formal analogue of these types of embeddings.

Definition 6.3. *Let M be manifold, D a distribution on M and N a manifold with* $\dim(N) \leq \dim(M)$. *A pair $(f, (F_s)_{s\in[0,1]})$ is called a* ***formal transverse embedding*** *if*

- *$f : N \hookrightarrow M$ is an embedding,*

- *for every $s \in [0,1]$, $F_s : TN \to TM$ is a monomorphisms which lifts f, i.e.*

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ \uparrow & & \uparrow \\ TN & \xrightarrow{F_s} & TM \end{array}$$

commutes,

- $F_0 = dg$ *and* $F_s(T_pN) \oplus D_{f(p)} = T_{f(p)}M$ *for every* $p \in N$.

For moving transverse 3-manifolds around in our manifold, we would like to have an $h-$principle for these maps. This brings us to the next statement. Note that this statement contains the term "loose", which we will comment on in Remark 6.5.

Conjecture 6.4. *Let M be a $n-$dimensional manifold, and D a bracket-generating distribution on M. Let N be a manifold such that codim(N) = rank(D). Then, there exists an $h-$principle for transverse embeddings of N with respect to D if*

- *codim$(N) \geq 3$,*
- *codim$(N) = 2$, and if the embeddings of N into M are loose.*

Remark 6.5. Let us briefly comment on the term "loose" in Conjecture 6.4. In [35], Murphy proves an $h-$principle for a class of Legendrian embeddings in contact manifolds of dimension greater or equal to 5. She calls these Legendrian embeddings *loose.* The idea is very similar to that of proving an $h-$principle for distributions which are overtwisted. Namely, the class of embeddings is defined in such a way, that all the ingredients are present to prove that the $h-$principle indeed holds. This is also how we want to define these loose transverse embeddings, but since the statement has not been proven yet, the explicit definition of loose is not immediately clear at this moment.

We are not going to prove this result in the thesis, due to its complexity and lack of time. Unfortunately, there has not been a proof written down in the literature yet, which is why we denote it as a conjecture. However, for now, we will think of loose transverse embeddings as a class of transverse embeddings, for which the $h-$principle above indeed holds. ▶

As a corollary of Conjecture 6.4, we have the following result.

Corollary 6.6. *Let (M, D) be a $(2,3,5)-$manifold and let*

$$f_0 : L^3 \hookrightarrow M, \quad \tilde{f}_1 : U \subset L \hookrightarrow M$$

be embeddings transverse to D, with $U \cong \mathbb{D}^3$ and f_0 loose. Then there exist maps

$$f_s : L \hookrightarrow M$$

transverse to D for all $s \in [0,1]$ with $f_s|_{\partial L} = f_0|_{\partial L}$ and $f_1|_U = \tilde{f}_1$.

This corollary will help us in Chapter 8 to attach pieces of the overtwisted disc to the top-simplices. Since we need to use this result when moving the overtwisted disc, we will need the embedding of the 3-manifold which we use for the Lutz twisting, to be loose. Therefore, when we formally define the overtwisted disc, the term "loose" will also appear.

6.4 The overtwisted disc

In this section we will formally define the notion of an overtwisted disc in a $(2,3,5)-$manifold. We will define it by adding a Lutz twist along the boundary of the tubular neighbourhood of a transverse 3-manifold. The definition given here is very similar to the definition in [12] by del Pino and Vogel for overtwisted discs in Engel manifolds. Just like the overtwisted disc in Engel manifolds, we will require the overtwisted disc in a $(2,3,5)-$manifold to satisfy an additional quantitative property. Note that the definition below contains a looseness assumption, which does not appear in the definition in [12].

Definition 6.7. *Consider the* $(2,3,5)-$*manifold* $(\mathbb{D}^5, D_{st})$ *and let* L *be a 3-dimensional submanifold as in Section 6.1, with shrinking constant* τ_0*, which is embedded in* M *by a loose embedding. Let* $\partial\nu(L)$ *denote the boundary of the tubular neighbourhood of* L*, and let* $\partial\tilde{\nu}(L)$ *denote the boundary of a slightly smaller tubular neighbourhood of* L*. Let* $\mathcal{L}(D_{st})$ *be the* $(2,3,5)-$*structure obtained from* D_{st} *by adding a* $(2,3,5)-$*Lutz twist along* $\partial\tilde{\nu}(L)$*. Then*

$$\Delta_{OT} = (\mathbb{D}^5, D_{OT} = \mathcal{L}(D_{st})) \subset (\mathbb{D}^5, D_{st})$$

is an ***overtwisted disc****.*

We say a $(2,3,5)-$*manifold* (M,D) *is* ***overtwisted*** *if there is an embedding*

$$\psi : \Delta_{OT} \hookrightarrow (M,D)$$

which pulls back the $(2,3,5)-$*structure, i.e.* $\psi^*(D) = D_{OT}$*.*

Remark 6.8. Let us explain why we add the Lutz twist in the definition above, not along $\partial\nu(L)$, but along the boundary of a slightly smaller tubular neighbourhood $\partial\tilde{\nu}(L)$. This ensures that the overtwisted disc is entirely contained in the tubular neighbourhood $\nu(L)$. This will help us later on when we shrink the overtwisted disc in showing the self-replicating property in Section 6.5. See also Figure 6.4 for an illustration of the overtwisted disc. ▶

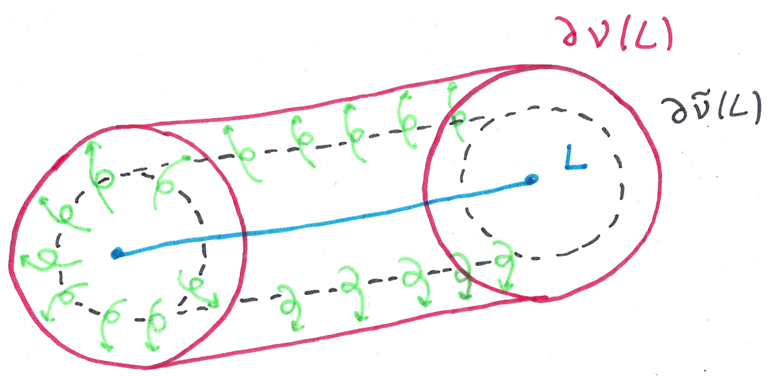

Figure 6.4: An illustration of the overtwisted disc as in Definition 6.7. Here the 3-dimensional submanifold L is indicated in blue, and the boundary of the tubular neighbourhood $\nu(L)$ in red. Furthermore, we note that the Lutz twist, which is schematically indicated in green, is added along the boundary of the slightly smaller tubular neighbourhood $\tilde{\nu}(L)$.

We denote the space of overtwisted $(2,3,5)-$structures by $\mathfrak{Dist}^{OT}_{(2,3,5)}(M)$, and we can endow it with the subspace topology coming from $\mathfrak{Dist}_{(2,3,5)}(M)$. The statement we are trying to prove, Theorem 0.4, is on the space of $(2,3,5)-$structures with a *fixed* overtwisted disc. By this we mean that we are essentially establishing a specific model for the overtwisted disc, by fixing an embedding.

Let $\Delta : \mathbb{D}^5 \to M$ be a smooth embedding. We define $\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta) \subset \mathfrak{Dist}_{(2,3,5)}(M)$ to be the space of those $(2,3,5)-$structures such that the pullback by Δ is D_{OT}. Similarly, we define $\mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta) \subset \mathcal{F}\mathfrak{Dist}_{(2,3,5)}(M)$ to be the space of formal overtwisted $(2,3,5)-$structures, which are genuinely $(2,3,5)$ on $\mathrm{im}(\Delta)$, and the pullback by Δ is D_{OT}. We endow both spaces with the subspace topology.

Remark 6.9. In the next section we will show that in the presence of an overtwisted disc, we can produce new ones by homotopy. This allows us to use/alter one of these discs in the proof of Theorem 0.4, while there is also still a disc present of the fixed model. In the extension part of the proof of Theorem 0.4, we will move pieces of the overtwisted disc around in the manifold. As explained before, we shall need to shrink the tubular neighbourhood $\nu(L)$ for this. The upshot of defining the overtwisted disc in the standard $(2,3,5)-$structure D_{st}, is that the latter is preserved under the scaling we will perform (see also Lemma 6.10). ▶

6.5 Self-replication of the overtwisted disc

In Chapter 8 we are going to use the overtwisted disc to prove Theorem 0.4. However, as mentioned earlier in this chapter, the spaces $\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$ and $\mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$ contain structures with a *fixed* embedding of the overtwisted disc. When we use the disc in the extension part, how do we know if it is still the same as the initial model? In this section we are going to show that in the presence of an overtwisted disc, we can produce new ones by homotopy. We call this the *self-replicating property* of the overtwisted disc. It allows us to leave one of these discs (in the fixed model) for what it is. So when we talk about self-replication in this section, we mean that we are going to homotope the structure, such that another copy of the overtwisted disc will appear. In this section it will also become clear that the shrinking constant in Definition 6.7 will be a key ingredient in the self-replication.

Let us briefly outline how we are going to show this self-replicating property. In the first subsection we are going to shrink the tubular neighbourhood $\nu(L)$ by using a scaling map. Thereafter, we are going to construct a bump function, which will allow us to construct a hypersurface along which we can Lutz twist. In the last subsection, we will show that Lutz twisting along this newly constructed hypersurface, will produce two overtwisted discs in the initial model.

6.5.1 Shrinking the tubular neighbourhood

For the self-replication, we first want to shrink the original overtwisted disc. We do this by scaling the tubular neighbourhood $\nu(L)$ by the following map:

$$\psi_\lambda : \mathbb{D}^5 \to \mathbb{D}^5, \quad (x, z, y, t, w) \mapsto (\lambda x, \lambda^3 z, \lambda^2 y, \lambda t, \lambda^3 w),$$

where $0 < \lambda < 1$ is a constant. See Figure 6.5 for an illustration of this. We write $\psi^*(\nu(L))$ for the scaled tubular neighbourhood.

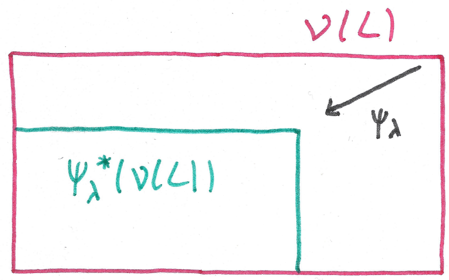

Figure 6.5: An illustration of the scaling function ψ_λ applied to the tubular neighbourhood $\nu(L)$, where L is a 3-dimensional submanifold transverse to the $(2,3,5)-$structure. We denote the shrunken tubular neighbourhood by $\psi_\lambda^*(\nu(L))$.

An important property of the map ψ_λ, is that it preserves the $(2,3,5)-$structure D_{st} along $\nu(L)$. This is proven in the following lemma.

Lemma 6.10. *Let $(\mathbb{D}^5, D_{st})$ be the standard $(2,3,5)-$manifold with*

$$D_{st} = \langle \partial_t, \partial_x + y\partial_z + t\partial_y + t^2\partial_w \rangle.$$

Then the diffeomorphism ψ_λ preserves D_{st}.

Proof. We note that $D_{st} = \ker(\alpha, \beta, \gamma)$ where

$$\begin{aligned} \alpha &= dy - tdx \\ \beta &= dz - ydx \\ \gamma &= dw - t^2 dx. \end{aligned}$$

We see that

$$\begin{aligned} \psi_\lambda^*\alpha &= \lambda^2 dy - \lambda^2 t dx = \lambda^2 \alpha \\ \psi_\lambda^*\beta &= \lambda^3 dz - \lambda^3 y dx = \lambda^3 \beta \\ \psi_\lambda^*\gamma &= \lambda^3 dw - \lambda^3 t^2 dx = \lambda^3 \gamma. \end{aligned}$$

Since $\ker(\lambda^2\alpha, \lambda^3\beta, \lambda^3\gamma) = \ker(\alpha, \beta, \gamma)$, the diffeomorphism ψ_λ indeed preserves D_{st}. □

6.5.2 A suitable bump function

In this subsection we construct a bump function which will connect the smaller tubular neighbourhood $\psi_\lambda^*(\nu(L))$ to the edge of the ball B, see Figure 6.6. This will ensure that when we Lutz twist along this newly-formed hypersurface, it all matches up nicely to the rest of the $(2,3,5)-$structure. If we view the radius of L as the $x-$axis, with $x = 0$ as its center, we note that the figures in this section portray the positive $x-$values. For the negative $x-$values, we perform the processes described in this section analogously, but mirrored in $x = 0$.

We need the bump function to have the following properties:

- The slope of the function should be at most $\frac{h}{l/4}$.
- As λ gets closer to 0, the bump function should contain a larger horizontal part at height λh.
- At the upper corner the function should contain a small horizontal part at height h.

Let us shed light on why we want these properties for the bump function. In Definition 6.7 we required L to have shrinking constant τ_0. In Section 6.1 we explained that requiring $h < \tau_0$ means that we can cut off the tubular neighbourhood $\nu(L)$ with a slope of $\frac{h}{l/4}$ as in Figure 6.3, while the distribution stays transverse to $\nu(L)$. Therefore, the first condition above, ensures that the distribution is still transverse to the bump function. The second condition will become more clear in the next section. Namely, in this horizontal part at height λh, we will eventually see the (scaled) original overtwisted disc, plus an extra copy next to it. This is why we require the bump function to be more horizontal when λ gets closer to 0. Lastly, we require the function to contain a small part of height h at the upper corner, because when we Lutz twist along the bump function, we want it to match up with the Lutz twisting at the boundary of $\nu(L)$ in a smooth manner.

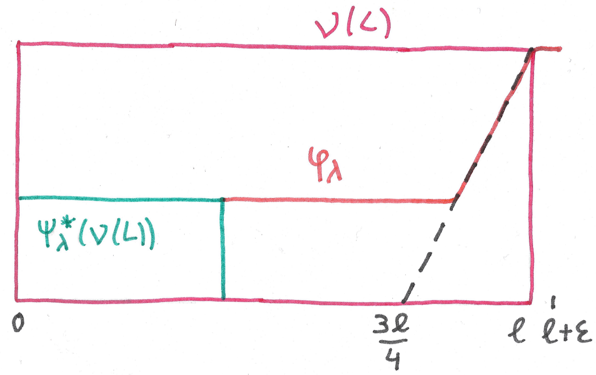

Figure 6.6: An illustration of the bump function φ_λ which connects $\psi_\lambda^*(\nu(L))$ to the edge of the ball B. Note that φ_λ partly runs along the dotted line with slope $\frac{4h}{l}$.

Let us give a more explicit expression for this bump function to show that it indeed exists. We can express the dotted line in Figure 6.6 by a map $f : \mathbb{R} \to \mathbb{R}$ with

$$f(m) = \frac{4h}{l}m - 3h.$$

Note that for simplicity we parameterise the length and height of the overtwisted disc by a single parameter, while in fact h depends on two and l on three variables. However, this construction still clearly illustrates the purpose and properties of the bump function.

We can then construct the map $\varphi_\lambda : [0, l + \epsilon] \to \mathbb{R}$ with

$$\varphi_\lambda(m) := \begin{cases} \lambda h & \text{if } m \in [0, \frac{\lambda+3}{4}l] \\ \frac{4h}{l}m - 3h & \text{if } m \in [\frac{\lambda+3}{4}l, l] \\ h & \text{if } m \in [l, l+\epsilon] \end{cases},$$

which has all the desired properties as discussed above. We then take a C^1-small smoothing of φ_λ (which we will still denote by φ_λ), which will give us a smooth function with the same desired properties. The bump function $\varphi_\lambda([0, l + \epsilon])$ together with the scaled tubular neighbourhood $\psi_\lambda^*(\nu(L))$, form a hypersurface along which we can Lutz twist. In the next section we will see that when we do this carefully, two copies of the overtwisted disc will appear.

6.5.3 An extra copy of the overtwisted disc appearing

In the previous subsection we showed that we can construct a bump function φ_λ which connects $\psi_\lambda^*(\nu(L))$ with the edge of the ball B. We note that $\partial(\psi_\lambda^*(\nu(L)))$ and φ_λ together form a hypersurface, which we denote by H_λ. See also Figure 6.7. We denote the boundary of the tubular neighbourhood $\nu(L)$ by H_1. In this subsection we will show that along H_λ we have an isomorphic copy of the (original) overtwisted disc in D along H_1, and thereafter we will show that next to the overtwisted disc along H_λ, there exists another isomorphic copy. Recall that this was the purpose of this section, and that we call this the self-replicating property of the overtwisted disc.

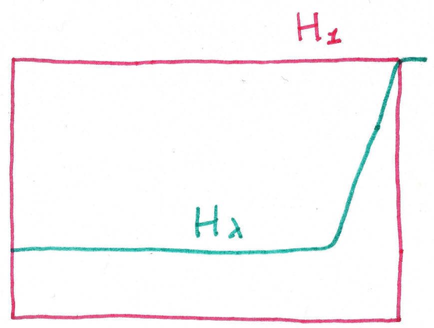

Figure 6.7: Illustration of the hypersurfaces H_1 and H_λ. The hypersurface H_λ is formed by the scaled boundary $\partial(\psi_\lambda^*(\nu(L)))$ and the bump function φ_λ.

First of all, suppose we 'undo' the Lutz twist of D along H_1. We denote the resulting structure by $\mathcal{L}^{-1}(D)$. Recall from Definition 6.7 that $\mathcal{L}^{-1}(D)$ looks like D_{st} over B. From Lemma 6.10 we know that this local model is preserved under the scaling of ψ_λ. We then want to add a Lutz twist along H_λ, in such a way, such that there appears an overtwisted disc along H_λ which is isomorphic to the initial overtwisted disc along H_1. This is shown in the next lemma.

Lemma 6.11. *Let $\mathcal{L}_\lambda(\mathcal{L}^{-1}(D))$ denote the structure which arises from adding a Lutz twist along H_λ. Then, for every λ, there exist an isomorphism*

$$\Phi_\lambda : \mathcal{L}_\lambda(\mathcal{L}^{-1}(D))|_{\partial(\psi_\lambda^*(\nu(L)))} \to \mathcal{L}_1(\mathcal{L}^{-1}(D)),$$

from an overtwisted disc along $\partial(\psi_\lambda^*(\nu(L)))$ to the initial overtwisted disc.

Proof. Recall from Section 5.2 that we constructed the Lutz twist using a vector field $T \in D$ transverse to the hypersurface. On H_1 we denote this vector field by T_1. On $\partial(\psi_\lambda^*(\nu(L)))$ we have the vector field $\psi_\lambda^*(T_1)$. On the bump function φ_λ we connect the vector field $\psi_\lambda^*(T_1)$ to T_1 in a continuous way. Note that this is possible because H_λ is a hypersurface transverse to D. We denote the resulting vector field by T_λ. See also Figure 6.8 below.

We use this T_λ to Lutz twist along H_λ and we denote the resulting distribution by $\mathcal{L}_\lambda(\mathcal{L}^{-1}(D))$. Note that the structure $\mathcal{L}_\lambda(\mathcal{L}^{-1}(D))$ then matches up nicely at the edges, and that we automatically have an isomorphism:

$$\Phi_\lambda : \mathcal{L}_\lambda(\mathcal{L}^{-1}(D))|_{\partial(\psi_\lambda^*(\nu(L)))} \to \mathcal{L}_1(\mathcal{L}^{-1}(D)),$$

from the overtwisted disc in $\mathcal{L}_\lambda(\mathcal{L}^{-1}(D))$ along $\partial(\psi_\lambda^*(\nu(L)))$ to the initial overtwisted disc in $\mathcal{L}_1(\mathcal{L}^{-1}(D)) = D$. □

To prove that an extra copy of the overtwisted disc appears next to the overtwisted disc along $\partial(\psi_\lambda^*(\nu(L)))$, we first look at the following lemma.

Lemma 6.12. *Let G be a Lie group,*

$$\sigma : G \times N \to N, \quad (g, x) \mapsto g \cdot x$$

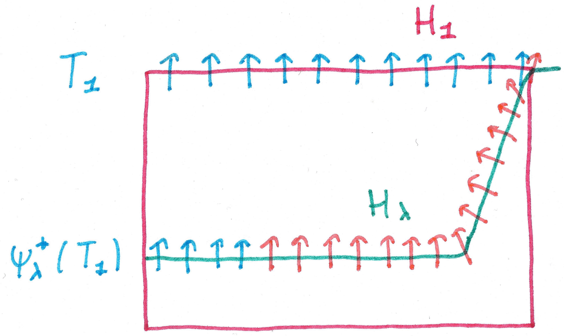

Figure 6.8: Illustration of vector fields transverse to the hypersurfaces H_1 and H_λ used in the proof of Lemma 6.11. Here T_1 and $\psi_\lambda^*(T_1)$ are indicated in blue. The vector field T_λ is formed by $\psi_\lambda^*(T_1)$ and the orange vectors which connect $\psi_\lambda^*(T_1)$ to T_1. We choose T_λ in such a way that it is a smooth vector field tangent to the distribution and transverse to H_λ, which agrees with $\psi_\lambda^*(T_1)$ on $\partial(\psi_\lambda^*(\nu(L)))$, and agrees with T_1 on the small horizontal part in the top right corner. Note that one could also in fact take T_λ to be vertical on the entire hypersurface H_λ, as H_λ itself is never vertical.

an action of G on the manifold N, and let D be a $(2,3,5)-$structure on the manifold $N \times (-\epsilon, \epsilon)$. Suppose that D is preserved under the action

$$\tilde{\sigma} : G \times N \times (-\epsilon, \epsilon) \to N \times (-\epsilon, \epsilon), \quad (g, x, t) \mapsto (g \cdot x, t),$$

then $\mathcal{L}(D)$ is also preserved under $\tilde{\sigma}$.

Proof. Recall from the construction of the Lutz twist in Section 5.2 we use the model

$$\bar{D}(p,t) = \langle \partial_t, \bar{H}_p(t) = \cos(t)X_p + \sin(t)Y_p + \alpha t^2 W_p \rangle,$$

with $0 < \alpha < 1$, coming from Lemma 5.5, to add the extra loop. Also recall from Lemma 5.4 that the frame $\{X, Y, Z, W\}$ is associated to (D, N), meaning that $X \in D \cap TN$, $Y \in \mathcal{E} \cap TN$ and $Z \in F_N$. If a Lie group action preserves D, these vector fields are preserved, and thus this local model is also preserved. Meaning if we Lutz twist using this local model, the structure $\mathcal{L}(D)$ is also preserved under the Lie group action. □

Using this lemma we can prove that next to the overtwisted disc along $\partial(\psi_\lambda^*(\nu(L)))$, another (isomorphic) copy of the overtwisted disc exists.

Proposition 6.13. *In the presence of one overtwisted disc with a given model (i.e. a given embedding $\Delta : \Delta_{OT} \to (M, D)$), we can produce another one with the same model by homotopy, which preserves the formal data. More specifically, in the setting of this section, next to the overtwisted disc along $\partial(\psi_\lambda^*(\nu(L)))$, there exists another isomorphic copy along H_λ.*

Proof. First of all, we note that the standard $(2,3,5)-$structure D_{st} can be given as the kernel of three 1-forms:

$$D_{st} = \ker(dy - tdx, dz - ydx, dw - t^2dx).$$

This kernel is invariant under translations in the $z-$ and $w-$direction, and thus by Lemma 6.12 the structure $\mathcal{L}(D_{st})$ is also invariant under translations in the $z-$ and $w-$direction.

Recall from Section 6.1 that the transverse 3-manifold L was spanned by the directions ∂_y, ∂_z and ∂_w. Therefore, the structure $\mathcal{L}_\lambda(\mathcal{L}^{-1}(D))|_{\partial(\psi_\lambda^*(\nu(L)))}$ is preserved when moving in the horizontal part of H_λ at height λh along the ∂_z and ∂_w direction. From this follows that an isomorphic copy of the overtwisted disc appears next to the overtwisted disc along $\partial(\psi_\lambda^*(\nu(L)))$.

By Lemma 6.11 we know that the overtwisted disc along $\partial(\psi_\lambda^*(\nu(L)))$ is isomorphic to the initial overtwisted disc. Therefore, we have indeed shown that in the presence of one overtwisted disc with a given model/embedding, we can produce another one with the same model by homotopy. We note that the homotopy preserves the formal data, as it is a homotopy of genuine $(2,3,5)-$structures. □

Chapter 7

Reducing to the extension problem

In this chapter we will discuss several results and techniques which will help us to reduce the proof of Theorem 0.4 to a simplified problem, namely an *extension problem*, as also discussed in the introduction of this thesis. This means that after this chapter, we will only need to achieve the $(2,3,5)-$condition in the top-simplices in order to prove Theorem 0.4. In the first subsection we will discuss the setting of the problem we are trying to reduce. Thereafter, we will look at adapted triangulations, which will help us achieve the $(2,3,5)-$condition in the codimension-1 skeleton in the third section in a controlled manner. As a consequence of this result, we will prove the $h-$principle for $(2,3,5)-$structures on open manifolds in the last section. Recall that this last result is also an immediate consequence of Gromov, and the proof we present also uses similar techniques as in the proof of Theorem 0.3.

7.1 Setup

Let M be a 5-dimensional closed manifold, and let

$$D \subset \mathcal{E} \subset TM$$

be a formal overtwisted $(2,3,5)-$structure. We will from now on sometimes also just write D for the formal $(2,3,5)-$structure, which implicitly comes with this flag and the isomorphisms from Definition 2.10. Furthermore, we require that D contains an embedding Δ of the overtwisted disc. As a corollary of Theorem 0.4, we will also prove that the h-principle is relative in the domain. Therefore, let $U \subset M$ be a closed submanifold such that:

- $D|_U$ is $(2,3,5)$,
- the manifold $M\backslash U$ is connected,
- Δ is contained in $M\backslash U$.

Lastly, in order to talk about angles and distances we fix a Riemannian metric on M.

The goal of this chapter is to homotope the formal $(2,3,5)-$structure D such that it becomes genuinely $(2,3,5)$ in the complement of some nice-looking open subsets of M. These subsets are called *shells*, and on these we will ultimately perform the extension.

Definition 7.1. *Let $D \subset \mathcal{E} \subset TM$ be a formal $(2,3,5)-$structure on M such that $D = \langle T, X\rangle$ for two vector fields T and X on M. Let $c > 0$ be a given constant. We define a* ***shell*** *to be an open subset $V \subset M$ such that*

- *D is $(2,3,5)$ on a neighbourhood of the boundary,*
- *V is a flow box of T. Let ∂_t denote the vector field T in the coordinates of V.*
- *The angle between $D|_{t=0}$ and $D|_{t=t_0}$ is bounded (in absolute value) by $c > 0$, for every t_0. The same holds for $\mathcal{E}$.*

Remark 7.2. Let us briefly clarify the third condition in the definition above. A shell is a flow box of a vector field T, and thus we can express T as ∂_t in V. This means that $D = \langle \partial_t, X \rangle$, and we can view the vector field X as a map

$$X : (-\epsilon, \epsilon) \to \mathbb{S}^3, \quad t \mapsto X(t).$$

The third condition in Definition 7.1 ensures that when we move in the $t-$coordinate from one side to the other side of V, this map does not change that much. We will need this later on in the extension part where we "fill up" the top-simplices with pieces of the overtwisted disc. ▶

The following statement will be the main result of this chapter:

Proposition 7.3. *Let $D \subset \mathcal{E} \subset TM$ a formal overtwisted $(2,3,5)$-structure. Then there is a formal overtwisted $(2,3,5)$-structure*

$$\overline{D} \subset \overline{\mathcal{E}} \subset TM$$

satisfying the following conditions:

(i) It is homotopic to the original formal data, relative to Δ and U.

(ii) $\overline{D}$ is $(2,3,5)$ in the complement of a finite collection of shells.

We can immediately carry out a first reduction of the problem, by applying Gromov (Theorem 0.3). Namely, let B be a closed ball in M such that $\Delta \subset \mathring{B}$. We note that $M \backslash B$ is an open manifold. We can therefore use Gromov to homotope the formal structure to a genuine $(2,3,5)-$structure on $M \backslash B$, relative to the boundary ∂B and relative to U. This allows us the rest of the proof to assume that $M = \mathbb{D}^5$, and that the formal $(2,3,5)-$structure is genuinely $(2,3,5)$ near the boundary $\partial \mathbb{D}^5$. Since now M is orientable, we know by Proposition C.18 that D is also orientable. This allows us to write $D = \langle T, X \rangle$ for vector fields T and X on M, which form a positively oriented basis for D. In the next section we will therefore often write $T \subset D \subset \mathcal{E}$ for the flag of the formal $(2,3,5)-$structure D. We note that this first reduction step preserves the formal data coming from the formal $(2,3,5)-$structure, as it is an application of an $h-$principle.

7.2 Adapted triangulations

An important ingredient in the proof of the h-principle is a triangulation $\mathcal{T}$ of our manifold M, which is adapted to some particular distributions on M. Namely, we want the simplices to be small enough such that these distributions are almost constant with respect to the affine coordinates of the simplices. The definitions and lemmas which we will discuss in this section are very similar to, and inspired by, the definitions and lemmas in Section 6.2.2. from [12]. In [12] the definitions and results are focused on the parametric h-principle for Engel structures, but since the h-principle we are trying to prove is not parametric, the definitions and statements here are simplified.

7.2.1 General position and adapted triangulations

We start with some definitions on simplices and triangulations.

Definition 7.4. *Let M be a manifold of dimension m and ξ a smooth distribution of codimension q on M. A top-simplex $\sigma \subset M$ is in* ***general position*** *with respect to ξ if the linear projection $\sigma \to \mathbb{R}^m/\xi_p$, for all $p \in M$, maps each q-dimensional subsimplex of $\partial\sigma$ to a non-degenerate simplex of $\mathbb{R}^q \cong \mathbb{R}^m/\xi_p$. We note that this linear projection is defined in the affine coordinates provided by the simplex σ.*

We say a triangulation $\mathcal{T}$ of M is ***in general position****, if every top-simplex is in general position.*

If $\mathcal{T}$ is a triangulation which is in general position with respect to a distribution ξ, then it follows that every simplex of $\mathcal{T}$ is in fact transverse to ξ. Now let D be a formal $(2,3,5)$-structure, with a vector field $T \subset D$ tangent to D. Recall that a formal $(2,3,5)$-structure comes with a flag $D \subset \mathcal{E} \subset TM$. For our construction we want a triangulation with simplices transverse to these distributions, which leads to the following definition.

Definition 7.5. *A top-simplex $\sigma \subset M$ is* ***adapted*** *to $T \subset D \subset \mathcal{E}$ if σ is in general position with respect to the distributions T, D and $\mathcal{E}$.*

A triangulation $\mathcal{T}$ is ***adapted*** *to $T \subset D \subset \mathcal{E}$ if*

- *every top-simplex $\sigma \in \mathcal{T}$ is adapted to $T \subset D \subset \mathcal{E}$,*
- *it is a triangulation of the pair $(M, \mathcal{O}p(U))$, where $\mathcal{O}p(U)$ is a neighbourhood of U.*

We then introduce some notation which will help us point out some useful properties of adapted triangulations. Let σ be an adapted top-simplex. We write $T_\sigma \subset D_\sigma \subset \mathcal{E}_\sigma$ for the restrictions of the spaces $T \subset D \subset \mathcal{E}$ to σ. Since σ is adapted, T_σ, D_σ and $\mathcal{E}_\sigma$ are transverse to the subsimplices of σ. Therefore, given a codimension-1 subsimplex of σ, T_σ is nowhere tangent to it, D_σ intersects it in a line field and $\mathcal{E}_\sigma$ intersects it in a plane field.

We then fix orientations of T_σ and D_σ, which allows us to define the following space:

$$\sigma_- := \text{union of those faces where } T_\sigma \text{ points into } \sigma.$$

As stated above, D_σ cuts codimension-1 subsimplices in a line field, and thus we can also define the following space:

$$\mathcal{H} := T\sigma_- \cap D_\sigma.$$

We note that $\mathcal{H}$ divides the boundary of σ_- into two parts:

$$\partial_-\sigma_- := \text{subspace of } \sigma_- \text{ where } \mathcal{H} \text{ points inwards}$$
$$\partial_+\sigma_- := \text{subspace of } \sigma_- \text{ where } \mathcal{H} \text{ points outwards.}$$

Both these spaces are codimension-2 complexes and homeomorphic to closed balls.

Remark 7.6. In Figure 7.1 you can see an illustration of these spaces. Note that σ is here represented as a 3-simplex, as a 5-simplex is hard to draw. A useful property of these adapted triangulations becomes apparent in this figure. Namely, we note that the line field $\mathcal{H}$, informally speaking, starts at one side of the face and moves towards the other side of the face. Later on, in the extension part, we will use the flow of this line field to move a piece of the overtwisted disc, along the bottom of the simplex. ▶

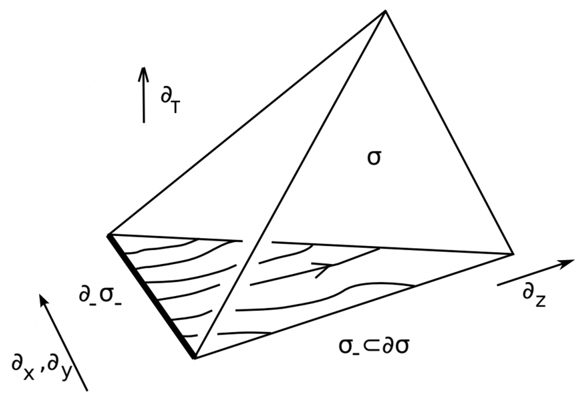

Figure 7.1: Illustrations of the different subsets of σ as defined above [12, p. 36]. The vector field T is given by the vertical direction, the bottom face represents σ_- and the thickened line the subspace $\partial_-\sigma_-$. The lines on the bottom face represent the line field $\mathcal{H}$ which is tangent to D.

7.2.2 Existence of adapted triangulations

In this section we show that adapted triangulations actually exist, and also certain covers associated to these triangulations.

Lemma 7.7. *There exists*

- *a cover $\{U_i\}$ of $M \backslash \mathcal{O}p(U \cup \Delta)$ by balls that are flow boxes of T, and*
- *an adapted triangulation $\mathcal{T}$,*

such that every top-simplex $\sigma \in \mathcal{T}$ is contained in some U_i.

Proof. By compactness of M we know that there is a finite covering of $M \backslash \mathcal{O}p(U \cup \Delta)$ by flow boxes of T. We can then fix a triangulation subordinate to this covering. We then apply to this triangulation what is called Thurston's jiggling. This is the carefully perturbing and subdividing of the simplices using crystalline subdivision. Thurston's jiggling yields a triangulation in which the simplices are indeed in general position with respect to a certain distribution, and the argument can be applied to all the distributions involved. See also [40] for the original argument by Thurston. □

Definition 7.8. *Let $\mathcal{T} = \{\sigma\}$ be an adapted triangulation. A covering $\{\mathcal{U}(\sigma)\}_{\sigma \in \mathcal{T}}$ of $M \backslash \mathcal{O}p(U \cup \Delta)$ is* ***associated to*** $\mathcal{T}$ *if:*

(i) We have $\sigma \subset \cup_{\sigma' \subseteq \sigma} \mathcal{U}(\sigma')$ for every $\sigma \in \mathcal{T}$.

(ii) Each $\mathcal{U}(\sigma)$ is a flow box of T. From this follows that we can divide the boundary of $\mathcal{U}(\sigma)$ into two parts, the horizontal and vertical part:

$$\partial^h \mathcal{U}(\sigma) := \textit{subspace of } \partial \mathcal{U}(\sigma) \textit{ which is transverse to } T,$$
$$\partial^v \mathcal{U}(\sigma) := \textit{subspace of } \partial \mathcal{U}(\sigma) \textit{ which is tangent to } T.$$

(iii) $\mathcal{U}(\sigma)$ and $\mathcal{U}(\sigma')$ only intersect if $\sigma' \subset \sigma$ or $\sigma \subset \sigma'$. Furthermore, if $\sigma' \subset \sigma$, then $\mathcal{U}(\sigma)$ intersects $\partial \mathcal{U}(\sigma')$ in its vertical part.

Furthermore, if $\partial_t = T$ in an open $\mathcal{U}(\sigma)$, then we call $\mathcal{U}(\sigma)|_{t=t_0}$ the ***fibre*** *of $\mathcal{U}(\sigma)$ at $t = t_0$. We also use the notation $\mathcal{U}(\sigma)_t$ for the fibre of $\mathcal{U}(\sigma)$ at t.*

In the following lemma we prove that given a certain adapted triangulation, one can actually construct such an associated covering.

Lemma 7.9. *Given an adapted triangulation $\mathcal{T}$ and a covering $\{U_i\}$ as in Lemma 7.7, there exists a covering $\{\mathcal{U}(\sigma)\}_{\sigma \in \mathcal{T}}$ of $M \backslash \mathcal{O}p(U \cup \Delta)$ associated to it.*

Proof. Every top-simplex $\sigma \in \mathcal{T}$ is contained in some flow box U_i of T. We use induction on the dimension of σ to prove the result.

First, suppose $\dim(\sigma) = 0$. We fix a neighbourhood $\mathcal{U}(\sigma)$ of σ which is a flow box of T. We fix its length along the flow lines of T, but shrink the neighbourhood in all other directions. Since all simplices are transverse to T, it follows that all incident 1-simplices will enter through $\partial^v \mathcal{U}(\sigma)$.

Suppose the statement holds for all simplices of dimension $\dim(\sigma) = n < \dim(m)$. Now let $\dim(\sigma) = n + 1$. In order to construct the neighbourhood $\mathcal{U}(\sigma)$, we first slightly shrink the simplex σ, which yields a simplex σ', such that the boundary of σ' is still contained in $\cup_{\tau \subsetneq \sigma} \mathcal{U}(\tau)$. Thereafter, we thicken the slightly smaller simplex σ' to a neighbourhood $\mathcal{U}(\sigma)$ such that the thickening along T is much greater than in the other directions. This will give the desired property that simplices containing σ will enter $\mathcal{U}(\sigma)$ through $\partial^v \mathcal{U}(\sigma)$. □

7.3 Achieving the $(2,3,5)$-condition in the codimension-1 skeleton

In this section we will show that we can homotope a formal $(2,3,5)$-structure such that it becomes genuinely $(2,3,5)$ in a neighbourhood of the codimension-1 skeleton. Since this neighbourhood is open, we could achieve this by applying Gromov. However, when we use Gromov, we do not know precisely what the resulting structure will look like. In this section, we will achieve the $(2,3,5)-$condition around the codimension-1 skeleton in a controlled way. Namely, after we have proven Proposition 7.3, we know quite well what the distribution looks like in the shells. This will help us in Chapter 8, when we perform the extension. The results in this section are new and thus this section is an original contribution of the thesis. Proposition 7.3 will follow as a corollary of this result. The proofs in this section are very similar to the construction of the $(2,3,5)-$structure on the mapping torus in Section 3.3. We first prove the following preparatory lemma of which the statement and proof resemble the statement and proof of Lemma 3.7.

Lemma 7.10. *Let $T \subset D \subset \mathcal{E}$ be a formal $(2,3,5)-$structure, and let $\mathcal{T}$ be an adapted triangulation with $\{\mathcal{U}(\sigma)\}_{\sigma \in \mathcal{T}}$ an associated covering. Then there is a rank 3 distribution $\tilde{\mathcal{E}}$ such that*

- *$\tilde{\mathcal{E}}$ and $\mathcal{E}$ agree on $U \cup \Delta$,*
- *$\tilde{\mathcal{E}} = \langle T, \tilde{X}, \tilde{Y}\rangle$ for vector fields $\tilde{X}$ and $\tilde{Y}$ tangential to the fibres of $\mathcal{U}(\sigma)$,*
- *$\tilde{\xi}_t := \tilde{\mathcal{E}} \cap (\mathcal{U}(\sigma) \cap \{t\})$ is non-integrable for all t,*
- *$\tilde{D} := \langle \partial_t, \tilde{X}\rangle$ is formally homotopic to D.*

Proof. Recall that $\mathcal{T}$ being an adapted triangulation also means that there is a subcomplex $\mathcal{T}_\partial \subset \mathcal{T}$ which triangulates the boundary $\partial \mathcal{O}p(U \cup \Delta)$. Since D is already $(2,3,5)$ on U and Δ, we are only going to perform modifications on $\mathcal{T}\backslash\mathcal{T}_\partial$. This will ensure the first property in the statement. We will be working inductively on the dimensions of the simplices $\sigma \in \mathcal{T}\backslash\mathcal{T}_\partial$.

First let σ be a $0-$simplex, and let $\mathcal{U}(\sigma)$ be the corresponding neighbourhood coming from the associated covering of $\mathcal{T}$. Recall that $\mathcal{U}(\sigma)$ is a flow box of T, and we denote by ∂_t the local coordinate direction of this vector field. Note that we can parameterise t such that σ lies in the fibre $t = 0$. We then pick a vector field X which lies in D and is tangential to the fibres, i.e. we have

$$D = \langle \partial_t, X\rangle.$$

We also pick a vector field $Y \in \mathcal{E}$ tangential to the fibres, and linearly independent of X, i.e. we have

$$\mathcal{E} = \langle \partial_t, X, Y\rangle.$$

We note that on every fibre, we can view $\xi_t := \langle X_t, Y_t\rangle$, where $X_t := X \cap \mathcal{U}(\sigma)_t$ and $Y_t := Y \cap \mathcal{U}(\sigma)_t$, as a 2-distribution on a 4-manifold.

From the isomorphism $D \wedge \mathcal{E}/D \cong TM/\mathcal{E}$ we obtain the isomorphism

$$\xi_t \wedge \xi_t \cong TM/\xi_t.$$

We then invoke Theorem 1.10 parametrically in t. From this we obtain a non-integrable distribution $\tilde{\xi}_t$ on $\mathcal{U}(\sigma)_t$, homotopic to ξ_t. This means that we can find vector fields $\tilde{X}$ and $\tilde{Y}$ such that $\tilde{\xi}_t = \langle \tilde{X}_t, \tilde{Y}_t\rangle$, where $\tilde{X}_t := \tilde{X} \cap \mathcal{U}(\sigma)_t$ and $\tilde{Y}_t := \tilde{Y} \cap \mathcal{U}(\sigma)_t$. We then define the rank-3 distribution $\tilde{\mathcal{E}} = \langle \partial_t, \tilde{X}, \tilde{Y}\rangle$ on $\mathcal{U}(\sigma)$, which has the desired properties.

Now let σ be a 1-simplex and $\mathcal{U}(\sigma)$ the corresponding neighbourhood. Again, we have that $\mathcal{U}(\sigma)$ is a flow box of T, and thus we can denote one of the coordinate directions by ∂_t. We want to carry out the same process as above to yield the desired distribution $\tilde{\mathcal{E}}$. However, we should be careful about overlapping neighbourhoods $\mathcal{U}(\sigma')$ for $\sigma' \subset \sigma$. Using the $h-$principle from Theorem 1.10 relatively to these overlapping neighbourhoods, ensures that we do not alter the distributions we have already constructed around the 0-simplices.

We can iterate this process until we have dealt with all the simplices. The resulting distribution $\tilde{\mathcal{E}}$ has the desired properties. We note that D and $\tilde{D}$ are formally homotopic, as the main ingredient we use in this proof is an $h-$principle (and thus the formal data is preserved). □

Using this lemma we will now show that we can homotope the formal $(2,3,5)-$structure such that it becomes genuinely $(2,3,5)$ in a neighbourhood of the codimension-1 skeleton.

Lemma 7.11. *Let $T \subset D \subset \mathcal{E}$ be a formal overtwisted $(2,3,5)-$structure. Let $\mathcal{T}$ be an adapted triangulation with $\{\mathcal{U}(\sigma)\}_{\sigma\in\mathcal{T}}$ an associated covering. Then, there is a plane field $\overline{D}$ such that*

(i) $\overline{D}$ is formally homotopic to D through plane fields containing T,

(ii) $\overline{D}$ and D agree on $U \cup \Delta$

(iii) $\overline{D}$ is $(2,3,5)$ in $\mathcal{U}(\sigma)$ for every $\sigma \in \mathcal{T}$ with $\dim(\sigma) < \dim(M)$.

Proof. Just as in the previous lemma, we are only going to modify D on $\mathcal{T}\backslash\mathcal{T}_\partial$, and we will be working on the dimension of the simplices $\sigma \in \mathcal{T}\backslash\mathcal{T}_\partial$.

From Lemma 7.10 we know that there exists a rank-3 distribution $\tilde{\mathcal{E}}$ with all the properties as listed in the lemma. We now pick a vector field $\tilde{Z} \in \langle[\tilde{X}, \tilde{Y}]\rangle$ tangential to the fibres of $\mathcal{U}(\sigma)$, which is linearly independent of $\tilde{X}$ and $\tilde{Y}$ by construction, and we pick a vector field $\tilde{W}$ tangential to the fibres such that locally we have $TM = \langle \partial_t, \tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \rangle$.

Now let σ be a 0-simplex, and let $\mathcal{U}(\sigma)$ be the corresponding neighbourhood coming from the associated covering of $\mathcal{T}$. Note that $\mathcal{U}(\sigma)$ is a flow box of $T = \partial_t$, and assume that $t \in (-\delta, \delta)$ for some $\delta > 0$. Now take a look at a function $\rho : (-\delta, \delta) \to (-\delta, \delta)$ depicted in Figure 7.2.

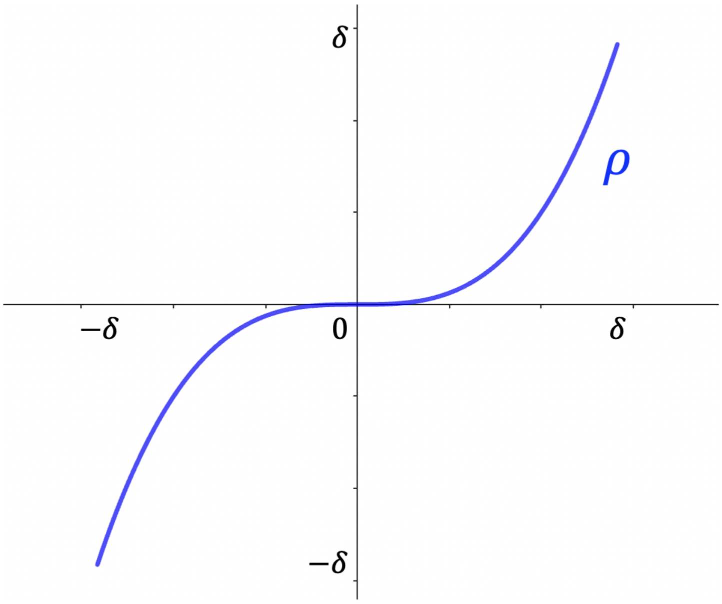

Figure 7.2: Illustration of the function $\rho : (-\delta, \delta) \to (-\delta, \delta)$ used in the proof of Lemma 7.11. Note that in a neighbourhood around 0, ρ is almost constant.

We then look at the pull-back of the vector fields $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and $\tilde{W}$ by ρ. We note that by doing this, there is a neighbourhood around the simplex σ where the vector fields $\rho^*(\tilde{X})$, $\rho^*(\tilde{Y})$, $\rho^*(\tilde{Z})$ and $\rho^*(\tilde{W})$ depend very little on t. I.e. we can assume that the derivatives of the frame $\{\rho^*(\tilde{X}_t), \rho^*(\tilde{Y}_t), \rho^*(\tilde{Z}_t), \rho^*(\tilde{W}_t)\}$ with respect to t are arbitrarily small. Additionally, we note that the dilation ρ preserves the formal data, i.e. $\tilde{D}$ is formally homotopic to $\rho^*(\tilde{D})$.

Let $\overline{\zeta}$ be the bump function from Figure 5.2, as constructed in Subsection 5.1.1. We then define the vector field $\overline{X}$ by

$$\overline{X}_t := ((1 - \overline{\zeta}(t)) \cos(t) + \overline{\zeta}(t)) \rho^*(\tilde{X}_t) + (1 - \overline{\zeta}(t)) \sin(t) \rho^*(\tilde{Y}_t) + (1 - \overline{\zeta}(t)) t^2 \rho^*(\tilde{W}_t).$$

We are going to show that $\overline{D} = \langle \partial_t, \overline{X} \rangle$ is $(2, 3, 5)$ in a neighbourhood of σ. Therefore, we want to take a closer look at the vector fields $\dot{\overline{X}}_t$, $\ddot{\overline{X}}_t$ and $[\overline{X}_t, \dot{\overline{X}}_t]$. We will do this in the neighbourhood around σ where the vector fields $\rho^*(\tilde{X})$, $\rho^*(\tilde{Y})$, $\rho^*(\tilde{Z})$ and $\rho^*(\tilde{W})$ depend very little on t. Assuming here that the derivatives $\{\rho^*(\tilde{X}_t), \rho^*(\tilde{Y}_t), \rho^*(\tilde{Z}_t), \rho^*(\tilde{W}_t)\}$ with respect to t are arbitrarily small, and thus ignoring these terms for now, we have the following:

$$\begin{aligned}
\dot{\overline{X}}_t &= \Big((1 - \cos(t)) \dot{\overline{\zeta}}(t) - \big(1 - \overline{\zeta}(t)\big) \sin(t) \Big) \rho^*(\tilde{X}_t) + \Big(-\sin(t) \dot{\overline{\zeta}}(t) + \big(1 - \overline{\zeta}(t)\big) \cos(t) \Big) \rho^*(\tilde{Y}_t) \\
&\qquad + \Big(-t^2 \dot{\overline{\zeta}}(t) + \big(1 - \overline{\zeta}(t)\big) 2t \Big) \rho^*(\tilde{W}_t) \\
\ddot{\overline{X}}_t &= \Big((1 - \cos(t)) \ddot{\overline{\zeta}}(t) + 2\sin(t) \dot{\overline{\zeta}}(t) - (1 - \overline{\zeta}(t)) \cos(t) \Big) \rho^*(\tilde{X}_t) \\
&\quad + \Big(-\sin(t) \ddot{\overline{\zeta}}(t) - 2\cos(t) \dot{\overline{\zeta}}(t) - (1 - \overline{\zeta}(t)) \sin(t) \Big) \rho^*(\tilde{Y}_t) \\
&\quad + \Big(-t^2 \ddot{\overline{\zeta}}(t) - 4t \dot{\overline{\zeta}}(t) + 2(1 - \overline{\zeta}(t)) \Big) \rho^*(\tilde{W}_t) \\
[\overline{X}_t, \dot{\overline{X}}_t] &= \Big((1 - \overline{\zeta}(t))^2 + \cos(t) \overline{\zeta}(t)(1 - \overline{\zeta}(t)) - \sin(t) \dot{\overline{\zeta}}(t) \Big) \rho^*([\tilde{X}_t, \tilde{Y}_t]) + \\
&\quad \Big(-t^2 \overline{\zeta}(t) \dot{\overline{\zeta}}(t) + (\cos(t) 2t + \sin(t) t^2)(1 - \overline{\zeta}(t))^2 + (2t \overline{\zeta}(t) - t^2 \dot{\overline{\zeta}}(t))(1 - \overline{\zeta}(t)) \Big) \rho^*([\tilde{X}_t, \tilde{W}_t]) \\
&\quad + \Big((\sin(t) 2t - \cos(t) t^2)(1 - \overline{\zeta}(t))^2 \Big) \rho^*([\tilde{Y}_t, \rho^* \tilde{W}_t]).
\end{aligned}$$

By construction we have that close to 0, the bump function $\overline{\zeta}$ is either 0 or of order t^2, its derivative is either 0 or of order t, and its second derivative is either 0 or of constant value between $-2q$ and $2q$. In the case where $\overline{\zeta} = 0$, the vector fields $\overline{X}_t$, $\dot{\overline{X}}_t$, $\ddot{\overline{X}}_t$ and $[\overline{X}_t, \dot{\overline{X}}_t]$ are automatically linearly independent. In the other case, we can look at the following approximations:

$$\begin{aligned}
\overline{X}_t &= \left((1-\mathcal{O}(t^2))(1+\mathcal{O}(t^2)) + \mathcal{O}(t^2)\right)\rho^*(\tilde{X}_t) + (1-\mathcal{O}(t^2))\mathcal{O}(t)\rho^*(\tilde{Y}_t) + (1-\mathcal{O}(t^2))t^2\rho^*(\tilde{W}_t),\\
\dot{\overline{X}}_t &= \left(\mathcal{O}(t^2)\mathcal{O}(t)) - \left(1-\mathcal{O}(t^2)\right)\mathcal{O}(t)\right)\rho^*(\tilde{X}_t) + \left(\mathcal{O}(t)\mathcal{O}(t) + \left(1-\mathcal{O}(t^2)\right)\left(1+\mathcal{O}(t^2)\right)\right)\rho^*(\tilde{Y}_t)\\
&\quad + \left(-t^2\mathcal{O}(t) + \left(1-\mathcal{O}(t^2)\right)2t\right)\rho^*(\tilde{W}_t)\\
\ddot{\overline{X}}_t &= \left(\mathcal{O}(t^2)\cdot \pm 2s + 2\mathcal{O}(t)\mathcal{O}(t) - (1-\mathcal{O}(t^2))\left(1+\mathcal{O}(t^2)\right)\right)\rho^*(\tilde{X}_t)\\
&\quad + \left(O(t)\cdot \pm 2s - 2\left(1+\mathcal{O}(t^2)\right)\mathcal{O}(t) - (1-\mathcal{O}(t^2))\mathcal{O}(t)\right)\rho^*(\tilde{Y}_t)\\
&\quad + \left(-t^2\cdot \pm 2s - 4t\mathcal{O}(t) + 2(1-\mathcal{O}(t^2))\right)\rho^*(\tilde{W}_t)\\
[\overline{X}_t, \dot{\overline{X}}_t] &= \left((1-\mathcal{O}(t^2))^2 + \left(1+\mathcal{O}(t^2)\right)\mathcal{O}(t^2)(1-\mathcal{O}(t^2)) - \mathcal{O}(t)\mathcal{O}(t)\right)\rho^*([\tilde{X}_t, \tilde{Y}_t]) +\\
&\quad \left(-t^2\mathcal{O}(t^2)\mathcal{O}(t) + \left(\left(1+\mathcal{O}(t^2)\right)2t + \mathcal{O}(t)t^2\right)(1-\mathcal{O}(t^2))^2 + \left(2t\mathcal{O}(t^2) - t^2\mathcal{O}(t)\right)(1-\mathcal{O}(t^2))\right)\\
&\quad \cdot \rho^*\left([\tilde{X}_t, \tilde{W}_t]\right) + \left((\mathcal{O}(t)2t - \left(1+\mathcal{O}(t^2)\right)t^2)(1-\mathcal{O}(t^2))^2\right)\rho^*\left([\tilde{Y}_t, \tilde{W}_t]\right).
\end{aligned}$$

From this follows that for t close to 0, and thus close to σ, the vector field $\overline{X}_t$ is close to the vector field $\rho^*(\tilde{X}_t)$, the vector field $\dot{\overline{X}}_t$ is close to $\rho^*(\tilde{Y}_t)$, the vector field $\ddot{\overline{X}}_t$ is close to $-\rho^*(\tilde{X}_t) + 2\rho^*(\tilde{W}_t)$, and the vector field $[\overline{X}_t, \dot{\overline{X}}_t]$ is close to $\rho^*([\tilde{X}_t, \tilde{Y}_t])$. Therefore, $\overline{D} = \langle \partial_t, \overline{X}\rangle$ is indeed $(2,3,5)$ in a neighbourhood of σ. Since these vector fields are so close to each other, we note that $\overline{D}$ is also formally homotopic to $\rho^*(\tilde{D})$.

Now let σ be a 1-simplex and $\mathcal{U}(\sigma)$ the corresponding neighbourhood. Again, we have that $\mathcal{U}(\sigma)$ is a flow box of T, and thus we can denote one of the coordinate directions by ∂_t. We want to carry out the same process as above to yield the desired plane field $\overline{D}$. However, just like in the previous lemma, we should be careful about overlapping neighbourhoods $\mathcal{U}(\sigma')$ for $\sigma' \subset \sigma$. We know from Definition 7.8 that $\mathcal{U}(\sigma)$ intersects $\partial\mathcal{U}(\sigma')$ in its vertical part, i.e. the part where T is tangent to $\partial\mathcal{U}(\sigma')$. This means that along these fibres, we have already constructed the desired plane field $\overline{D}$. Using Lemma 5.4, we know we can homotope $\overline{D}$ along these fibres such that it has the form

$$\langle \partial_t, \cos(t)X_p + \sin(t)Y_p + t^2 Z_p\rangle,$$

and we cut off away from σ', with an appropriate bump function, to yield a smooth distribution. We note that this is a C^0-small change, and thus this distribution is homotopic to the distribution before. We can then construct $\overline{X}_t$ around σ in the same way as before, and interpolate between the distribution at the 0-simplices. This constructs a distribution, which we will also denote by $\overline{D}$, which is $(2,3,5)$ in a neighbourhood around σ.

We can iterate this process until we have dealt with all the simplices of positive codimension. The resulting distribution $\overline{D}$ is then indeed $(2,3,5)$ in a neighbourhood of the codimension-1 skeleton, $D = \overline{D}$ on $U \cup \Delta$ and $\overline{D}$ is formally homotopic to D. □

Remark 7.12. In the proof above, the idea is the following. We dilate the $t-$coordinate such that it becomes very stretched out. This makes the family of non-integrable distributions $\tilde{\xi}_t$ more constant in t. Therefore, the first and second derivatives of the frame $\{\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t, \tilde{W}_t\}$ are made arbitrarily small. We then turn in the $\tilde{X}_t$ and $\tilde{Y}_t$ direction and add a little $\tilde{W}_t$ for convexity, and we obtain the $(2,3,5)-$condition. We note that this is very similar to the idea behind the mapping torus construction in Section 3.3. There we did not explicitly carry out the dilation, but the idea was the same. See also Remark 3.9(i). ▶

We are now ready to prove the main result of this chapter.

Proof of Proposition 7.3. From Lemma 7.7 we know that there exists an adapted triangulation, and from Lemma 7.9 we know that there exists a covering associated to this triangulation. Then by Lemma 7.11 there exists a formal $(2,3,5)-$structure $\overline{D}$ which is formally homotopic to D, relative to Δ and U, and which is genuinely $(2,3,5)$ in a neighbourhood of the codimension-1 skeleton.

What is left is to construct the shells. Let $\mathcal{U}(\sigma)$ be a neighbourhood from the associated covering of a top-simplex σ. We know that $\mathcal{U}(\sigma)$ is a flow box of T, and by possibly shrinking this neighbourhood, we have that $\overline{D}$ is $(2,3,5)$ on the boundary. Furthermore, we want the angle that the distributions $\overline{D}$ and $\overline{\mathcal{E}}$ make with every fibre $t = t_0$ to be bounded by a given constant $c > 0$. We can achieve this by possibly dividing the constructed neighbourhood into several separate neighbourhoods. The resulting neighbourhoods are shells. Since the manifold we are working on is closed and thus compact, there are finitely many top-simplices, and thus finitely many shells. We conclude that $\overline{D}$ is $(2,3,5)$ in the complement of a finite collection of shells. □

7.4 The h-principle for $(2,3,5)-$structures on open manifolds

In this section we will discuss an immediate consequence of Lemma 7.11. Namely, that the inclusion of the space of $(2,3,5)-$structures into the space of formal $(2,3,5)-$structures on open manifolds, induces a surjection on π_0-level. See also Lemma 7.16. We have seen before in Section 2.3 that the $h-$principle holds for $(2,3,5)-$structures on open manifolds, and Lemma 7.16 is also a corollary of this result. However, in this section we will prove it using Lemma 7.11 and the fact that the handle-dimension of an open manifold is strictly smaller than its dimension. This will also highlight why in the closed manifold case we in fact still need to perform the extension. Let us first look at the concept of handle-dimension.

Definition 7.13. *Let M be a smooth manifold. Suppose there exists a simplicial complex K of dimension n, with a deformation retract*

$$(\psi_s)_{s\in[0,1]} : M \to K$$

*onto K, such that ψ_s is a diffeomorphism for all $s \neq 1$, but there is no simplicial complex K' of dimension $n-1$ with the same property, we say the **handle-dimension** of M is n.*

Theorem 7.14. *The handle-dimension of an open smooth manifold M, is smaller than its dimension.*

Proof. We will give a brief sketch of the proof as it is quite technical. The idea is the following. Pick a triangulation $\mathcal{T}$ of M. We then construct a graph $\varphi : G \hookrightarrow M$ with the following properties:

- φ does not intersect simplices of codimension greater of equal to 2,
- φ contains the barycenters of all top-simplices,
- $\varphi(G)$ is a union of affine simplices connecting each barycenter to the boundary of some codimension-1 face,
- G is a tree (a connected graph with no cycles).

This graph allows us to define an isotopy $(\psi_t)_{t\in[0,1]}$, using the isotopy extension theorem, such that the image of ψ_1 lies in the complement of the barycenters of the top-simplices. □

Remark 7.15. The proof of Gromov's $h-$principle, Theorem 0.3, is in fact also a consequence of this theorem. Therefore, using this theorem to prove the $h-$principle for $(2,3,5)-$structures on open manifolds, is in fact very similar to redoing the proof of Gromov applied to this specific case. ▶

Combining this theorem with Lemma 7.11, we can now (re)prove the following result.

Lemma 7.16. *Let M be an open manifold. Then the following inclusion*

$$\mathfrak{Dist}_{(2,3,5)}(M) \hookrightarrow \mathcal{F}\mathfrak{Dist}_{(2,3,5)}(M)$$

induces a surjection on π_0-level.

Proof. Let D be a formal $(2,3,5)-$structure on an open manifold M. By Lemma 7.11 we know that there exists a formal $(2,3,5)-$structure $\overline{D}$ which is homotopic to D, and which is $(2,3,5)$ along the codimension-1 skeleton. Since M is open, we know that the handle dimension is strictly smaller than $\dim(M)$. From this follows that there exists a deformation retraction of M to a neighbourhood which lies in the complement of the barycenters of the top-simplices. Let $(\psi_s)_{s\in[0,1]}$

denote this deformation retraction. Now let $(\phi_t)_{t\in[1,2]}$ be an isotopy pushing the top-simplices minus barycenter radially out towards their boundaries. We conclude that

$$(\phi_1 \circ \psi_1)^*(\overline{D})$$

is a $(2,3,5)-$structure on the whole of M, which is homotopic to $\overline{D}$ by the homotopy $(\psi_s^*(\overline{D}))_{s\in[0,1]}$. □

Remark 7.17. The proof of Lemma 7.16 relies heavily on this deformation retraction described above. Since the handle dimension of a closed manifold is not necessarily strictly smaller than its dimension, we cannot invoke this trick. Therefore, we need to work a bit harder to prove the h-principle for overtwisted $(2,3,5)-$structures on closed manifolds. Namely, we need to still obtain the $(2,3,5)$-condition in the top-simplices. Recall from the introduction, that this is precisely what is called the extension problem, and that we will use the overtwisted disc to achieve the $(2,3,5)-$condition on the whole manifold. ▶

Chapter 8

The extension

In this chapter we will prove the main result of this thesis. Let us repeat the statement.

Theorem 0.4. *Let M be a closed 5-dimensional manifold. Then the inclusion*

$$\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta) \hookrightarrow \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta),$$

is a weak homotopy equivalence.

We will do this by first proving that this inclusion induces a bijection on π_0-level, i.e. the first sections of this chapter will be devoted to proving the following theorem.

Theorem 8.1. *Let M be a closed 5-dimensional manifold. Then the following map*

$$\iota_* : \pi_0\left(\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)\right) \to \pi_0\left(\mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)\right),$$

induced by the inclusion $\iota : \mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta) \hookrightarrow \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$, is a bijection.

In Chapter 7, we reduced the surjectivity part of this problem to the extension problem. I.e. given a formal $(2,3,5)-$structure, we found a way to homotope it to a formal $(2,3,5)-$structure which is $(2,3,5)$ in the complement of a finite collection of shells. What is left is to obtain the $(2,3,5)-$condition in these shells. We do so by first remodeling these neighbourhoods such that they are easier to work with. Next, we will recall a result from Chapter 6, which will help us to a move piece of the overtwisted disc towards a shell. Thereafter, we will show how we can use the overtwisted disc to "fill up" the shell, and thereby obtain the $(2,3,5)-$condition. This finishes the surjectivity part of the proof of Theorem 8.1. After that, we will prove that the map is also injective, by using very similar techniques as we used for the surjectivity part. The last section of this chapter focuses on proving that the inclusion is also an isomorphism on higher homotopy groups, which will finish the proof of Theorem 0.4.

The material presented in this chapter has not been published before, and thus this is an original contribution of the thesis. As a small remark, the proofs presented in this chapter are slightly less detailed than the proofs we have seen before. We invite the reader to think about, and fill in these details.

8.1 Set up

Let us quickly recap the setting we are working in and what we have achieved in Chapter 7. Let M be a closed 5-dimensional manifold and let

$$D \subset \mathcal{E} \subset TM$$

be a formal overtwisted $(2,3,5)-$structure with a vector field $T \in D$. Let Δ be an embedding of the overtwisted disc into (M,D), and let $U \subset M$ be a submanifold such that

- $D|_U$ is $(2,3,5)$,
- $M \backslash U$ is connected,

- Δ is contained in $M \backslash U$.

In Chapter 7 we proved that there is a formal $(2,3,5)$−structure

$$\overline{D} \subset \overline{\mathcal{E}} \subset TM$$

which satisfies the following conditions:

(i) It is homotopic to the original formal data through plane fields containing T, relative to Δ and U.

(ii) $\overline{D}$ is $(2,3,5)$ in the complement of a finite collection of shells.

For the surjectivity part of Theorem 8.1, what is left is to homotope $\overline{D}$ such that it becomes $(2,3,5)$ also in the shells. The proof of Proposition 7.3 uses a triangulation $\mathcal{T}$ adapted to $T \subset D \subset \mathcal{E}$ with associated covering $\{\mathcal{U}(\sigma)\}_{\sigma \in \mathcal{T}}$. We will continue to use this same triangulation and covering in this chapter.

8.2 Remodelling the shells

In this section we explain how we want to remodel (or view) the shells, such that they are easier to work with once we perform the extension. Recall from Section 7.1 that a shell is an open subset $V \subset M$ such that

- D is $(2,3,5)$ on a neighbourhood of the boundary,
- V is a flow box of T, so we can denote $T = \partial_t$ in V,
- the angle between $D|_{t=0}$ and $D|_{t=t_0}$ is bounded (in absolute value) by a given constant $c > 0$, for every t_0. The same holds for $\mathcal{E}$.

We want the opens which we are going to perform the extension on, to have the following additional properties:

(i) They are of (smooth) cubic form, meaning their boundaries have a (4-dimensional) bottom and top face, connected by (4-dimensional) side faces in a smooth manner,

(ii) They are flow boxes of T, and the bottom and top face are transverse to T,

(iii) On the bottom face there should be flow lines tangent to $\overline{D}$ going from one side face to the opposite side face.

See also Figure 8.1 for an illustration of this. We can in fact easily construct these opens. Namely, in Chapter 7 we constructed the distribution $\overline{D}$ by making it $(2,3,5)$ in a neighbourhood of the codimension-1 skeleton, and we chose the shells by looking at slightly smaller copies of the top-simplices. Within this neighbourhood of the codimension-1 skeleton, we have enough freedom to choose such a neighbourhood of cubic form, as described above. Note that the initial properties of a shell are then still satisfied.

Furthermore, recall that the triangulation we are working with is adapted to the vector field T, meaning that the faces of a top-simplex are transverse to T. If we choose the faces of cube close enough to the boundary of the top-simplex, this will mean that also the bottom and top face are transverse to T. Lastly, recall from Remark 7.6, that if a triangulation is adapted to a distribution $\overline{D}$, then on the bottom of each top-simplex, there are flow lines tangent to $\overline{D}$ flowing from one side to the other. See also Figure 7.1. This is a property which follows from the fact that the distributions T and $\overline{D}$ are transverse to all simplices in $\mathcal{T}$. Since we have not altered T, this is still the case. So, again, choosing the cube close enough to the boundary of the top-simplex will yield the same property, and thus property (iii).

In the rest of this chapter, we shall work with these opens, and refer to them as *cubical shells*.

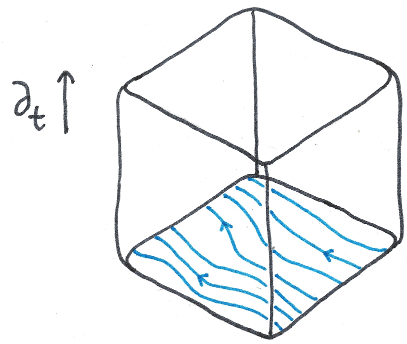

Figure 8.1: Illustration of a shell in cubical form. Here the vector field $T = \partial_t$ is the vertical direction, and the blue flow lines on the bottom are tangent to $\overline{D}$.

8.3 Moving the overtwisted disc

As mentioned before, we want to use the overtwisted disc to fill up the cubical shells. This requires that we find a way to move the overtwisted disc towards the shells. For this we recall the following result from Chapter 6.

Corollary 6.6. *Let* (M, D) *be a* $(2,3,5)-$*manifold and let*

$$f_0 : L^3 \hookrightarrow M, \quad \tilde{f}_1 : U \subset L \hookrightarrow M$$

be embeddings transverse to D*, with* $U \cong \mathbb{D}^3$ *and* f_0 *loose. Then there exist maps*

$$f_s : L \hookrightarrow M$$

transverse to D *for all* $s \in [0,1]$ *with* $f_s|_{\partial L} = f_0|_{\partial L}$ *and* $f_1|_U = \tilde{f}_1$.

From Definition 6.7 we know that we define an overtwisted disc in a manifold as a Lutz twist added along the boundary of the tubular neighbourhood of a transverse 3-manifold L, which is embedded by a loose, transverse embedding. Corollary 6.6 tells us how to move this 3-manifold, and thus we have a way to move the overtwisted disc to the cubical shells. See also Figure 8.2. We note that for every shell, we only move a piece $\mathbb{D}^3 \subset L$ towards it.

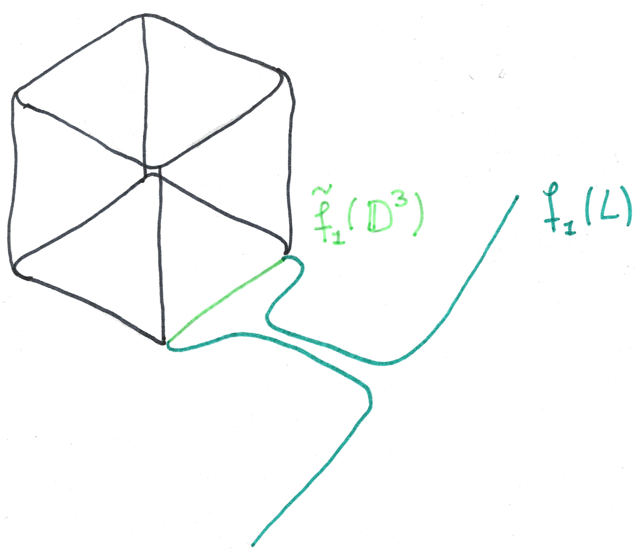

Figure 8.2: Illustration of how we move a piece of the overtwisted disc towards a cubical shell, using Proposition 6.6 to move the 3-manifold L. Recall that the Lutz twist is added along $\nu(L)$.

8.4 Filling up the shells

We will obtain the $(2,3,5)-$condition in the cubical shells in two steps. First, we stretch out the tubular neighbourhood $\nu(L)$ along the bottom of the cube. Secondly, we stretch out the Lutz twist coming from the overtwisted disc all the way to the top of the cube. Doing so, we need to be careful that the resulting distribution is still smooth and matches up nicely with the neighbourhood of the boundary of the cube where the distribution is already $(2,3,5)$. If we have achieved the $(2,3,5)-$condition in the cubical shells, we can use this in the next section to show that we can homotope a formal overtwisted $(2,3,5)-$structure, into a genuine overtwisted $(2,3,5)-$structure.

8.4.1 Moving along the bottom of the cube

We first take a look at the following lemma. It states that we can isotope transverse hypersurfaces, through transverse hypersurfaces, using flows of transverse vector fields.

Lemma 8.2. *Let D be a rank 2 distribution on a manifold M. Let N be a hypersurface which is transverse to D. In particular, let X be a vector field tangent to D and transverse to N. Let φ_t denote the flow of X. Then $\varphi_t(N)$ is also transverse to X, and thus to D, for all time t.*

Proof. Let $q \in \varphi_t(N)$ such that $\varphi_t(p) = q$ for $p \in N$. Since N is transverse to X, we know that

$$T_pN \oplus X_p = T_pM.$$

We then note that

$$\begin{aligned} T_q(\varphi_t(N)) \oplus X_q &= (d_p\varphi_t)(T_pN) \oplus (X \circ \varphi_t)_q \\ &= (d_p\varphi_t)(T_pN) \oplus (d_p\varphi_t)(X_p) \\ &= (d_p\varphi_t)(T_pN \oplus X_p) \\ &= (d_p\varphi_t)(T_pM) \\ &= T_qM. \end{aligned}$$

The last step is justified by the fact that φ_t is a diffeomorphism and thus $d_p\varphi_t : T_pM \to T_qM$ is a bijection between tangent spaces. This shows that indeed $\varphi_t(N)$ is transverse to X, and thus to D, for all time t. ☐

Recall that we can use Corollary 6.6 to move (piece of) the overtwisted disc towards a cubical shell. We now want to use this idea of moving transverse hypersurfaces along flows of vector fields, to push the boundary of the tubular neighbourhood $\partial\nu(L)$ along the bottom of the cubical shells. Recall that on the bottom faces of the shells we have flow lines tangent to $\overline{D}$. We will move part of $\partial\nu(L)$ along these flow lines to obtain a neighbourhood of L which covers the whole bottom face. This idea is written down formally in the next lemma. The idea of the proof is very similar to that of Lemma 8.2.

Lemma 8.3. *Let $B \subset M$ be a 4-dimensional submanifold which is isomorphic to the space $[0,1]^4$. Let L be a 3-dimensional manifold which we can embed in B as $[0,1]^3 \times \{0\}$, and let $\nu(L)$ denote its normal bundle. Let D be a rank 2 distribution on M. Let X be a vector field tangent to D with flow φ_t, for which there exists a map $f : [0,1]^3 \times \{0\} \to \mathbb{R}$ such that*

$$\varphi_f([0,1]^3 \times \{0\}) = [0,1]^3 \times \{1\}.$$

I.e. the flow lines of X on B, flow from one side to the other. See also Figure 8.3 for an illustration of this setting. We can then use the flow of X to isotope the boundary $\partial\nu(L)$ in the direction of $[0,1]^3 \times \{1\}$ such that the resulting neighbourhood around L covers all of B, and the boundary is transverse to D. See Figure 8.4 for an illustration of this result.

Proof. Let N be a subset of $\partial\nu(L)$ which is a thickening of the intersection of $\partial\nu(L)$ with B. We note that we can choose N such that N is transverse to X. See also Figure 8.5. Let $\tilde{N}$ be a slightly smaller thickening inside N. We then construct the following function:

$$g : N \to \mathbb{R}, \quad p \mapsto \chi(p)f(p),$$

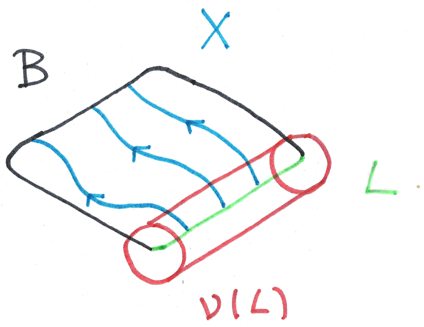

Figure 8.3: Illustration of the setting of Lemma 8.3.

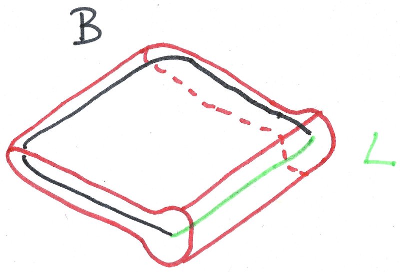

Figure 8.4: Illustration of the result of Lemma 8.3.

where $\chi : N \to [0,1]$ is a smooth bump function which is 1 on $\tilde{N}$ and 0 near the boundary of N, see also Figure 8.6. We can then define the map

$$\varphi_g : N \to M, \quad p \mapsto \varphi_{g(p)}(p).$$

Note that φ_g is a diffeomorphism onto its image. We then want to show that $\varphi_g(N)$ is still transverse to the vector field X.

Let $q \in \varphi_g(N)$ such that $\varphi_g(p) = q$ for $p \in N$. Since N is transverse to X, we know that

$$T_pN \oplus X_p = T_pM.$$

We then note that

$$\begin{aligned} T_q(\varphi_g(N)) \oplus X_q &= (d_p\varphi_g)(T_pN) \oplus (X \circ \varphi_g)_q \\ &= (d_p\varphi_g)(T_pN) \oplus (d_p\varphi_g)(X_p) \\ &= (d_p\varphi_g)(T_pN \oplus X_p) \\ &= (d_p\varphi_g)(T_pM) \\ &= T_qM. \end{aligned}$$

Here in the last step, we use that $d\varphi_g|_N$ is a bijection on tangent spaces. From this we can conclude that $\varphi_g(N)$ is still transverse to the vector field X. We note that $(\partial\nu(L) - N) \cup \varphi_g(N)$ is a smooth hypersurface transverse to D, of which its inside is a neighbourhood of L which covers B. □

Using the homotopy of the tubular neighbourhood $\nu(L)$ above, we can stretch out the piece of overtwisted disc along the bottom face of the cubical shell. This is shown in the next lemma.

Lemma 8.4. *Let M be a manifold and let $D \subset \mathcal{E} \subset TM$ be an overtwisted formal $(2,3,5)-$structure on M. Let C be a cubical shell disjoint from the overtwisted disc, and let D be genuinely $(2,3,5)$ in a neighbourhood of the boundary ∂C. Suppose we have used Corollary 6.6 to move a piece of the overtwisted disc towards C. Let D' denote this distribution. We can then homotope D', such that the resulting structure is overtwisted on the whole bottom face of the shell C. The resulting distribution is formally homotopic to D'.*

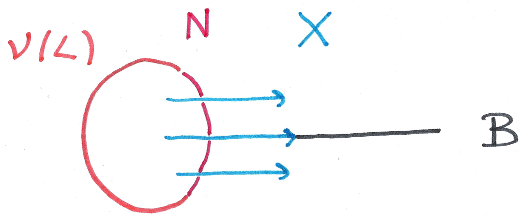

Figure 8.5: Illustration of the proof of Lemma 8.3.

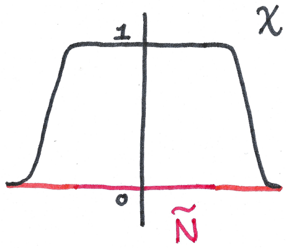

Figure 8.6: Illustration the bump function χ used in the proof of Lemma 8.3.

Proof. First of all, we can use Lemma 8.3 to push the boundary $\partial\nu(L)$ along the bottom face as also portrayed in Figure 8.3 and 8.4. We do this carefully, such that $\partial\nu(L)$ stays in the neighbourhood of ∂C where $D^{'}$ is $(2,3,5)$. To obtain a distribution which is overtwisted along the bottom of the shell, we stretch the Lutz twist along the bottom face as we push the boundary $\partial\nu(L)$. Note that essentially, we are just reparameterising in the $X-$direction. The resulting distribution is formally homotopic to $D^{'}$, as this proof just uses homotopies of distributions which are genuinely $(2,3,5)$. □

8.4.2 Moving the overtwisted disc vertically

In the previous subsection we have shown how we can stretch out the overtwisted disc along the bottom of the cubical shell. What is left, is to use this overtwisted disc to fill up the shell in the vertical direction, i.e. from bottom to top. This is formulated in the following lemma.

Lemma 8.5. *Let M be a manifold and let $D \subset \mathcal{E} \subset TM$ be an overtwisted formal $(2,3,5)-$structure on M. Let C be a cubical shell disjoint from the overtwisted disc, and let D be genuinely $(2,3,5)$ in a neighbourhood of the boundary ∂C. Suppose we have used Corollary 6.6 to move a piece of the overtwisted disc towards C, and we have used Lemma 8.4 to push the disc along the bottom of the shell. Let $D^{'}$ denote this resulting distribution. Then there exists a constant $c > 0$ (as in Definition 7.1), such that we can homotope $D^{'}$, using the overtwisted disc on the bottom, making it $(2,3,5)$ on whole of C. The resulting structure is formally homotopic to $D^{'}$.*

Proof. First of all, recall from the definition of a shell that the vertical direction is parameterised by the $t-$coordinate, and that we have the following property. Given a constant $c > 0$, the angle between $D|_{t=0}$ and $D|_{t=t_0}$ is bounded (in absolute value) by $c > 0$, for every t_0, and the same holds for $\mathcal{E}$. At the end of this proof, it will become clear how we should choose c.

Take a look at Figure 8.7, where we portray the side face of the shell C. Along the bottom, there is an overtwisted disc, and in orange we have indicated the hypersurface along which the Lutz twist is added. Since D is $(2,3,5)$ in a neighbourhood of ∂C, we can find another hypersurface, indicated in blue in Figure 8.7, along which one could Lutz twist. Note that the blue and orange hypersurface overlap near the bottom. We then use Corollary 6.6 to move a piece of overtwisted disc towards C, such that we also obtain a Lutz twist along blue. We place this Lutz twist in such a way that it matches with the Lutz twist along orange at the overlapping areas.

Now take a look at Figure 8.8. We know that the vertical direction of a shell is parameterised by the $t-$coordinate, and thus we can reparameterise t, such that the bottom of the shell is at level $t = 0$, and the top at level $t = 1$. Figure 8.8 shows a homotopy, which brings the orange hypersurface up,

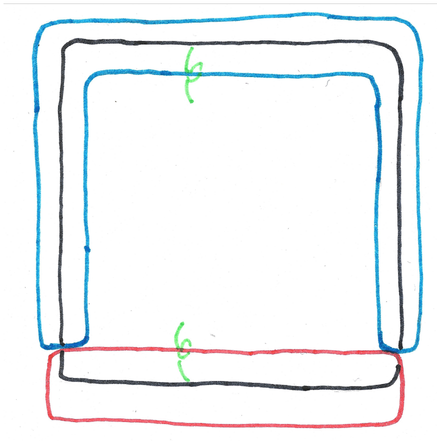

Figure 8.7: Illustration of the side face of the shell C. In orange and blue we see two hypersurfaces of tubular neighbourhoods around the boundary of the shell. Along these hypersurfaces, there are Lutz twists, which we indicate by the small green twist.

such that the middle part is situated at $t = \frac{1}{2}$. Note that we construct this homotopy using similar techniques as in Lemma 8.2. When performing this homotopy, we stretch out the Lutz twist along orange such that it fills up the whole shell. Note that the resulting distribution is $(2, 3, 5)$ in C, and during this process, we have pushed some area which is not $(2, 3, 5)$, above the shell. We denote the resulting distribution by D_0.

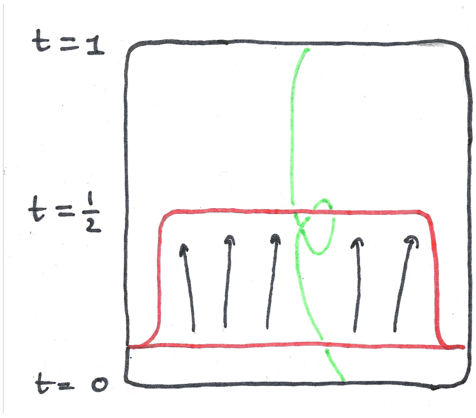

Figure 8.8: In this illustration we portray the homotopy of the orange hypersurface as used in the proof of Lemma 8.5. The hypersurface is pushed to the level $t = \frac{1}{2}$, and during this process, the Lutz twist is stretched and eventually fills the whole shell.

Now take a look at Figure 8.9. Here we see a very similar homotopy, where we push the blue hypersurface down to half way the shell (i.e. $t = \frac{1}{2}$). Again, when we perform this homotopy, we stretch out the Lutz twist such that it fills the whole shell. Note that again, the resulting distribution is $(2, 3, 5)$ on C, but we have pushed some region where the distribution is not $(2, 3, 5)$, below the shell.

To obtain the desired distribution, we interpolate from D_0 to D_1 in the $t-$coordinate, when moving from the bottom to the top of the shell. We denote the resulting distribution by D'. So why is D' $(2, 3, 5)$ on C (and a neighbourhood of C)? Recall that when moving from bottom to top in a shell, in the direction of the $t-$coordinate, the distributions D and $\mathcal{E}$ do not change drastically. Namely, this was bounded by a constant c. When we choose c small enough, the distribution at the bottom and the top will look very similar. Therefore, the Lutz twist around orange and blue will also look very alike. This means that the linear interpolation between D_0 and D_1 will not be that aggressive, and thus the resulting structure will be $(2, 3, 5)$. Also, note that by linearly interpolating from D_0 to D_1, the regions which were not $(2, 3, 5)$ which we pushed out of C in the homotopies of Figures 8.8 and 8.9, are not present in the resulting distribution D' anymore.

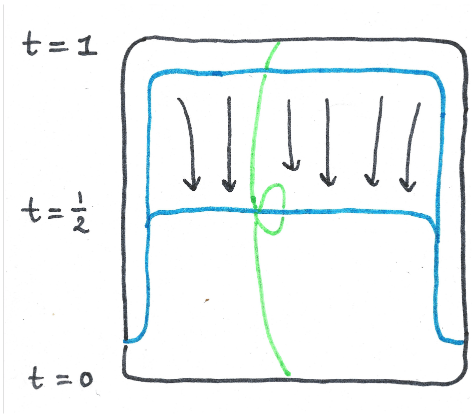

Figure 8.9: In this illustration we portray the homotopy of the blue hypersurface as used in the proof of Lemma 8.5. The hypersurface is pushed down to the level $t = \frac{1}{2}$, and during this process, the Lutz twist is stretched and eventually fills the whole shell.

We note that D is homotopic to D_0, and by linear interpolation, it is homotopic to D'. Lastly, we note that this homotopy is also a formal homotopy, as we chose the constant $c > 0$ small. Namely, because of this, D and $\mathcal{E}$ change very little when moving from bottom to top, and also D_0 and D_1 differ only very little. Therefore, all the homotopies in this proof preserve the given formal data. □

8.5 Proof of Theorem 8.1

We have now gathered all the ingredients in order to prove the surjectivity part of Theorem 8.1. In this section we will combine all these results, also from previous chapters, to prove that the inclusion indeed induces a surjection on path components. Thereafter, it turns out that we can use many of those same techniques and results in order to prove the injectivity on π_0-level. This finishes the proof of Theorem 8.1.

8.5.1 Proof of π_0-surjectivity

Theorem 8.6. *Let M be a closed 5-dimensional manifold. Then the following map, induced by the inclusion, is a surjection:*

$$\pi_0\left(\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)\right) \to \pi_0\left(\mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)\right).$$

Proof. Let $D \subset \mathcal{E} \subset TM$ be a formal overtwisted $(2,3,5)-$structure with embedding $\Delta : \Delta_{OT} \hookrightarrow (M, D)$, which is already $(2,3,5)$ on a subset $U \subseteq M$. From Proposition 7.3 we know that there exists a formal $(2,3,5)-$structure

$$\overline{D} \subset \overline{\mathcal{E}} \subset TM$$

which is homotopic to the original formal data, relative to Δ and U, and $\overline{D}$ is $(2,3,5)$ in the complement of a finite collection of shells.

From Proposition 6.13 we know that in the presence of an overtwisted disc, we can produce another one with the same model, by homotopy, which respects the formal data. We still denote the resulting structure, after carrying out this homotopy, by $\overline{D} \subset \overline{\mathcal{E}} \subset TM$. We then pick one of the overtwisted discs, and we focus on a single shell. As discussed in Section 8.2, we can view this shell as a *cubical.* We then use Corollary 6.6 to move a piece of the overtwisted disc towards the cubical shell. This corollary respects the formal data, as it is an $h-$principle. Thereafter, we use Lemma 8.4 to push the overtwisted disc along the bottom face. Lastly, we use Lemma 8.5 to stretch out the overtwisted disc in the vertical direction, and obtain a distribution with the $(2,3,5)-$condition in the whole shell.

From Proposition 7.3 we know that there are only finitely many shells, and thus we can choose a constant $c > 0$, such that we can perform this homotopy on every shell. This procedure yields a genuine overtwisted $(2,3,5)-$structure, which we will denote by $D' \subset \mathcal{E}' \subset TM$, which contains an (original) embedding $\Delta : \Delta_{OT} \hookrightarrow (M, D)$ of the overtwisted disc, is not altered on U (see also Corollary 8.9), and is homotopic to the initial formal $(2,3,5)-$structure $D \subset \mathcal{E} \subset TM$. We note that all the homotopies carried out in this proof respect the original formal data coming from the formal $(2,3,5)-$structure $D \subset \mathcal{E} \subset TM$, and thus this proves indeed that the map in the statement is surjective. $\square$

8.5.2 Proof of π_0-injectivity

In this subsection we will prove that the map in Theorem 8.1 is also injective. This finishes the proof of this theorem, as we have already showed that it is surjective. As mentioned before, the injectivity proof uses many of the same techniques and results as the surjectivity proof. However, since we are working with a *homotopy* of formal $(2,3,5)-$structures which we want to turn into genuine $(2,3,5)-$structures, we need to apply several of these results in a parametric setting. We do not explicitly prove that these results also hold in a parametric setting, which is why the proof below is given as a sketch.

Theorem 8.1. *Let M be a closed 5-dimensional manifold. Then the following map*

$$\iota_* : \pi_0\left(\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)\right) \to \pi_0\left(\mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)\right),$$

induced by the inclusion $\iota : \mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta) \hookrightarrow \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$, is a bijection.

Sketch of proof. From Theorem 8.6 we know that the map in the statement is a surjection, so what is left to show is that is it also an injection. Therefore, let D_0 and D_1 be two overtwisted $(2,3,5)-$structures with a fixed embedding of the overtwisted disc $\Delta : \Delta_{OT} \hookrightarrow (M,D)$, that are homotopic as formal overtwisted $(2,3,5)-$ structures. Let $(D_s)_{s\in[0,1]} \subset \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$ denote this homotopy.

Consider the 6-dimensional manifold $M\times[0,1]$, where $[0,1]$ is parameterised by s. From Lemma 7.7 and 7.9, we know there exists an adapted triangulation together with an associated covering. Using Thurston's jiggling [40], we can ensure that this adapted triangulation is also transverse to the fibres $M\times\{s\}$. Shortly, it will become clear why we put this extra condition on the triangulation.

We note that the distributions D_s are all transverse to simplices of positive codimension. Therefore, we can use the same techniques as in the proofs of Lemma 7.10, Lemma 7.11 and Proposition 7.3, but in a parametric setting, to obtain a homotopy of distributions $(\overline{D}_s)_{s\in[0,1]}$ with $\overline{D}_0 = D_0$ and $\overline{D}_1 = D_1$, such that the $\overline{D}_s$ are $(2,3,5)$ in the complement of a finite collection of shells. We can do this relatively to $s = 0,1$, as D_0 and D_1 are already genuinely $(2,3,5)$. We note that in this case, shells are open sets of $M\times[0,1]$, and thus 6-dimensional. We will refer to these shells as 6-shells.

What is left is to obtain the $(2,3,5)-$condition for the distributions on the fibres of the 6-shells. For this we will need the overtwisted discs, but since we want to keep a copy of the initial embedding, we need to replicate it. From Proposition 6.13 we know that in the presence of an overtwisted disc, we can produce another one with the same model by homotopy. We carry out this process for all the distributions $\overline{D}_s$, parametrically in s, to yield a homotopy of distributions with each two copies (or more if necessary) of the initial embedding $\Delta : \Delta_{OT} \hookrightarrow (M,D)$. We will still denote this homotopy by $(\overline{D}_s)_{s\in[0,1]}$.

We have constructed the triangulation in such a way that all the simplices are also transverse to the fibres $M\times\{s\}$. From this follows that the fibres of the 6-shells are either 0-simplices, or 5-shells. See also Remark 8.7 for more comments on this. Now, as described in Section 8.2 we can view 5-dimensional shells as cubical. Similarly, we view 6-shells as *cubical*, if each fibre over s is a 0-simplex or a 5-dimensional cubical shell, in the sense of Section 8.2.

We then apply Corollary 6.6 in a parametric sense, i.e. we move a piece of the overtwisted disc times a piece of the interval $[0,1]$ and attach it to a cubical 6-shell, by attaching it on every fibre over s to the cubical 5-shells in the same way as in Figure 8.2, but smoothly with respect to s. We then apply Lemma 8.4 to push the overtwisted disc along the bottom of every 5-dimensional shell, which we again do in a smooth manner with respect to parameter s. Lastly, we apply Lemma 8.5, parametrically in s, to stretch out the overtwisted disc in the vertical direction of every 5-shell, and by this we obtain distributions on the fibres, which are $(2,3,5)$ on the 5-shells.

From Proposition 7.3 we know that there are only finitely many 6-shells, and thus we can carry out this process finitely many times. This procedure yields a homotopy of $(2,3,5)-$structures, which we will denote by $(D'_s)_{s\in[0,1]}$, with $D'_0 = D_0$ and $D'_1 = D_1$, and which contain an (original) embedding $\Delta : \Delta_{OT} \hookrightarrow (M, D)$ of the overtwisted disc. □

Remark 8.7. In the proof of Theorem 8.1 above, we triangulate the manifold $M \times [0,1]$ such that the simplices are also transverse to the fibres $M \times \{s\}$. We do this such that on each fibre, a 6-dimensional simplex either looks like a 0-simplex or a 5-simplex. This makes it easier to apply the results we have already proven for 5-simplices, like Corollary 6.6, Lemma 8.4 and Lemma 8.5. In the next section, we will prove that the inclusion

$$\iota : \mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta) \hookrightarrow \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta),$$

also induces a bijection on higher homotopy groups. For this, we will need to look at $\mathbb{S}^n-$families of formal $(2,3,5)-$structures. We will then triangulate the manifold $\mathbb{S}^n \times M$ and require that the simplices are also transverse to the fibres $\{x\} \times M$, for $x \in \mathbb{S}^n$. This is for precisely the same reason. Then the cut of the top-simplices with the fibres $\{x\} \times M$ will either look like 0- or 5-simplices, which makes it easier to apply results we have already seen for 5-simplices. ▶

8.6 Higher homotopy groups

Recall that a weak homotopy equivalence means that a map induces an isomorphism on not just path-components, but also on higher homotopy groups. Therefore, to prove Theorem 0.4, we need still need to prove the theorem below. Just like the π_0-injectivity part, the proof of this theorem uses a lot of the same results and techniques that we used for the π_0-surjectivity part. However, now we need to adapt it to $\mathbb{S}^n-$families of (formal) $(2,3,5)-$structures. The proof given below is more of a sketch, to provide the idea of the proof, but the reader is welcome to fill in the technical details.

Theorem 8.8. *Let M be a closed 5-dimensional manifold. Let D_0 be an element in $\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$. Then the following map,*

$$\iota_* : \pi_n(\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta), D_0) \to \pi_n(\mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta), \iota(D_0))$$

induced by the inclusion $\iota : \mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta) \hookrightarrow \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$, is a bijection for all $n \geq 1$.

Sketch of proof. First, we want to show that the map ι_* is surjective. Let $f : \mathbb{S}^n \to \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$ be a pointed map. We can also view this map as an $\mathbb{S}^n-$family of formal $(2,3,5)-$structures $(D_x)_{x\in\mathbb{S}^n}$. Let us give an outline of the proof. Note that the structure is very similar to the proof of Theorem 8.1.

- First, we construct an adapted triangulation, which is also transverse to the fibres $\{x\} \times M$ for $x \in \mathbb{S}^n$.
- Thereafter, we use the same techniques as in Proposition 7.3, but parametrically in x, to obtain an $\mathbb{S}^n-$family of formal $(2,3,5)-$structures $(\overline{D}_x)_{x\in\mathbb{S}^n}$, where each $\overline{D}_x$ is $(2,3,5)$ along the codimension-1 skeleton.
- Hereafter, we move a region $O \times \Delta(\Delta_{OT})$, where O is an open in $\mathbb{S}^n$, towards every top-simplex. We do this by applying Corollary 6.6, but then in a parametric setting.
- Lastly, we use this overtwisted region to fill up the top-simplex and obtain the $(2,3,5)-$condition. We note that for every top-simplex, its cut with the fibre $\{x\} \times M$ is transverse, and thus of the form of a 5-simplex. Therefore, on every fibre, we can use Lemma 8.4 and Lemma 8.5 parametrically, to obtain the $(2,3,5)-$condition in the top-simplices. See also Remark 8.7

From this we obtain an $\mathbb{S}^n-$family of $(2,3,5)-$structure $(D'_x)_{x\in\mathbb{S}^n}$ which are homotopic to the original family $(D_x)_{x\in\mathbb{S}^n}$ we started with. This proves that the map ι_* is surjective.

For the injectivity, let $f_0, f_1 : \mathbb{S}^n \to \mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$ be two pointed maps, and let $(h_s)_{s\in[0,1]} : \mathbb{S}^n \to \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$ be homotopy with $h_0 = f_0$ and $h_1 = f_1$. We then want to construct a homotopy $(g_s)_{s\in[0,1]} : \mathbb{S}^n \to \mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$ with $g_0 = f_0$ and $g_1 = f_1$. Note that this boils down to homotoping a $\mathbb{S}^n \times [0,1]-$family of formal $(2,3,5)-$structures $(D_{(x,s)})_{(x,s)\in\mathbb{S}^n\times[0,1]}$ into a $\mathbb{S}^n \times [0,1]-$family of genuine $(2,3,5)-$structures, relative to 0 and 1. To construct this homotopy, we use a very similar strategy as in the proof of Theorem 8.1. Let us again give an outline of the proof.

- We first construct an adapted triangulation of $\mathbb{S}^n \times M \times [0,1]$, such that the simplices are also transverse to the fibres $\{x\} \times M \times [0,1]$ for $x \in \mathbb{S}^n$, and to the fibres $\mathbb{S}^n \times M \times \{s\}$ for $s \in [0,1]$.
- We then use the same techniques as in Proposition 7.3, but parametrically in (x,s), to obtain an $\mathbb{S}^n \times [0,1]-$family of formal $(2,3,5)-$structures $(\overline{D}_{(x,s)})_{(x,s)\in\mathbb{S}^n\times[0,1]}$, where each $D_{(x,s)}$ is $(2,3,5)$ along the codimension-1 skeleton.
- Thereafter, we move a region $O \times \Delta(\Delta_{OT}) \times (a,b)$, for an open $O \subseteq \mathbb{S}^n$ and an open interval $(a,b) \subset [0,1]$, towards every top-simplex. Again, we do this by applying Corollary 6.6, but then in a parametric setting.
- Lastly, we use this overtwisted region to fill up the top-simplex, and obtain the $(2,3,5)-$condition. Again, we note that for every top-simplex, its cut with a fibre $\{x\} \times M \times \{s\}$ is transverse, and thus of the form of a 5-simplex. Therefore, on all those fibres, we can use Lemma 8.4 and Lemma 8.5, in a smooth manner with respect to the parameters x and s, to obtain the $(2,3,5)-$condition in the top-simplices.

From this we obtain an $\mathbb{S}^n \times [0,1]-$family of $(2,3,5)-$structures $(D'_{(x,s)})_{(x,s)\in\mathbb{S}^n\times[0,1]}$, which is homotopic to the original family $(D_{(x,s)})_{(x,s)\in\mathbb{S}^n\times[0,1]}$. This proves that the map ι_* is injective. □

The main result of this thesis, Theorem 0.4, follows from Theorem 8.1 and Theorem 8.8. I.e. this proves the $h-$principle for overtwisted $(2,3,5)-$structures on closed manifolds!

Recall, that we reduced the statement of Theorem 0.4 in Chapter 7 to an extension problem. In fact, all the results we presented in Chapter 7 were relative to a region U, where the formal $(2,3,5)-$structure was already $(2,3,5)$. Moreover, the results and constructions in Chapter 6 and Chapter 8, do not alter the distribution on this region U. Therefore, to conclude this final chapter, and as a corollary of Theorem 0.4, we have the following result.

Corollary 8.9. *Let M be a closed 5-manifold, and let $U \subseteq M$ be a closed subset such that $M\backslash U$ is connected. Suppose $D \in \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$, with $D|_U$ is $(2,3,5)$ and Δ is contained in $M\backslash U$. Then there exists a homotopy $(D_s)_{s\in[0,1]} \subset \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$ such that*

- *$D_0 = D$,*
- *$D_s = D$ on $U \cup \Delta$,*
- *$D_1 \in \mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$.*

Furthermore, let D_0 and D_1 be two elements in $\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$, such that $D_0|_U$ and $D_1|_U$ are $(2,3,5)$, and let $(D_s)_{s\in[0,1]}$ be a homotopy in $\mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$ such that $D_s|_U$ is also $(2,3,5)$ for every $s \in [0,1]$. Then there exists a homotopy $(D'_s)_{s\in[0,1]} \subset \mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta)$, such that

- *$D'_0 = D_0$ and $D'_1 = D_1$,*
- *$D'_s = D_s$ on $U \cup \Delta$.*

We note that this statement is a result on path-components, but that a similar result can be formulated for the higher homotopy groups.

Conclusion

In this thesis, we looked at a specific type of bracket-generating distributions, called $(2,3,5)$-structures. The primary goal was to prove an $h-$principle on closed manifolds, for a particular family of $(2,3,5)-$structures, which are called *overtwisted*. As mentioned in the introduction, proving an $h-$principle for a certain type of geometric structure, can tell us more about the existence and topology of the space of such structures. Let us repeat the main result of the thesis.

Theorem 0.4. *Let M be a closed 5-dimensional manifold. Then the inclusion*

$$\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta) \hookrightarrow \mathcal{F}\mathfrak{Dist}^{OT}_{(2,3,5)}(M,\Delta),$$

is a weak homotopy equivalence.

This is a new result, and thus an original contribution of the thesis to the research field.

So how did we prove this theorem? The general idea for proving an $h-$principle for overtwisted structures, as also seen in the contact [13] and Engel case [12], is that we define the overtwistedness in such a way, that we have all the necessary ingredients for proving the $h-$principle. This is also the approach we used in this thesis. We defined Lutz twisting in $(2,3,5)-$manifolds, using the contact and Engel case as a source of inspiration. This allowed us to define the overtwisted disc, for which we proved a self-replicating property. To establish the surjectivity on π_0-level, we then reduced Theorem 0.4 to an extension problem, i.e. we homotoped a formal overtwisted $(2,3,5)-$structure, to a distribution which is already $(2,3,5)$ in a neighbourhood of the codimension-1 skeleton. Hereafter, we used the (extra copy of the) overtwisted disc, in order to "fill up" the top-simplices, and obtain the $(2,3,5)-$condition on the whole manifold. We defined the overtwisted disc in such a way that it has all the properties we need in order to perform this extension. Using many of the same techniques, we were also able to prove the injectivity on π_0-level, and the isomorphisms on higher homotopy groups.

Now, let us reflect on some limitations of this thesis. First of all, the proof of Theorem 0.4 uses a statement we did not prove within the thesis, namely Conjecture 6.4. We did justify why we can use it, but due to a lack of time, we did not provide a proof. Additionally, the proofs in Chapter 8 are less detailed than the proofs presented in the chapters before. Especially when we apply results in a parametric setting, we do not always explicitly show how we do this. Both of these limitations are due to time constraints, and we invite the reader to think about these details.

To conclude, we are going to discuss some open questions and ideas for potential further investigation, which arise from the material presented in this thesis. These questions and ideas could be used as inspiration for future research. Note that the limitations described above, form the first items in the list below.

- First of all, as mentioned above, we did not prove Conjecture 6.4 in this thesis. This is a statement on an $h-$principle for a special class of transverse embeddings, which we call *loose*. A similar statement has been proven by Murphy in [35], for loose Legendrian embeddings in contact manifolds. Similarly, one could explicitly define the definition of a loose transverse embedding in a manifold endowed with a bracket-generating distribution, and prove that the $h-$principle holds.
- Many of the results and proofs in this thesis are inspired by results and proofs in [12]. In this paper, del Pino and Vogel proved the $h-$principle for overtwisted Engel structures on closed manifolds, in a parametric setting. In this thesis we also stated a parametric $h-$principle, and we proved, although at some points a bit more sketchily, the full weak homotopy equivalence. One could try and use the more explicit proofs presented in [12] to also make some of the proofs in this thesis, regarding parametricity, more detailed.

- The next four items all entail possible generalisations of the results in this thesis. First of all, $(2, 3, 5)-$structures were a natural choice of object to investigate after contact and Engel structures, because of dimensional reasons. Therefore, the next object of interest could be distributions with growth vector $(2, 3, 5, 6)$, as these are maximally non-integrable distributions of rank 2 on $6-$manifolds. One can define what it means for $(2, 3, 5, 6)-$structures to be overtwisted, and prove an $h-$principle for them on closed manifolds. However, one could also try a more general approach, and define overtwisted maximally non-integrable distributions of rank 2, on manifolds of arbitrary dimension! Many of the techniques used for the contact, Engel and now the $(2, 3, 5)-$case, are very much related, and this makes it plausible that we can indeed generalise to arbitrary maximally non-integrable distributions of rank 2.

- The previous generalisation will probably also make it easier, to possibly generalise the result to arbitrary maximally non-integrable distributions (so not necessarily of rank 2). One would have to think about what an overtwisted disc would look like in such a structure, and how we would use it to achieve the $h-$principle. A first thing to investigate, is whether we can construct a similar reduction argument as has been done for contact, Engel and $(2, 3, 5)-$structures.

- A different way of generalising to higher dimensional manifolds, is the following. Recall that we can also define contact structures on higher odd-dimensional manifolds as maximally non-integrable hypersurfaces. In [3] Borman, Eliashberg and Murphy establish a parametric $h-$principle for overtwisted contact structures on manifolds of *all* dimensions. It would be interesting to investigate how their approach is related to the approach we follow in this thesis, and to perhaps use this to generalize the methods used to prove Theorem 0.4 to higher dimensional manifolds.

- For the last generalisation, note that in this thesis, we focused on distributions of rank 2 on 5-manifolds, which are everywhere maximally non-integrable. However, suppose we consider distributions of rank 2 which are non-integrable, but not everywhere *maximally* non-integrable? For example, suppose we look at overtwisted distributions which have growth vector $(2, 3, 5)$ almost everywhere, but which are $(2, 3, 4, 5)$ along a hypersurface? Could we also prove an $h-$principle on closed manifolds for those structures? And what about even less regular behaviour? We note that when we allow very irregular behaviour, too far from maximally non-integrable, then an $h-$principle would be immediate, as we have a lot more flexibility. However, there are a lot of cases where an $h-$principle is not immediate, and thus are definitely interesting to investigate.

- In this thesis, we have defined overtwisted $(2, 3, 5)-$structures as $(2, 3, 5)-$structures which admit an embedding of an overtwisted disc. This is a quite formal definition. A question which arises, is if there are more conceptual ways of thinking about and defining an overtwisted disc. This could help us in finding more intuitive examples of overtwisted $(2, 3, 5)-$structures. For contact structures, there are several equivalent criteria to decide whether a structure is overtwisted. For example, the standard Legendrian unknot is loose, and also, there is an open book compatible with the contact structure, which is negatively stabilised. See [5] for more equivalent criteria. Can we find such equivalent criteria for overtwisted $(2, 3, 5)-$structures?

- In [9], Dave and Haller discuss the existence question for $(2, 3, 5)-$structures on orientable manifolds. They obtain a complete answer in the open case, but not entirely for the closed case. Thus, this is a first open question. Another question would be, what closed manifolds admit an *overtwisted* $(2, 3, 5)-$structure? Using the $h-$principle proven in this thesis, we can already reduce this question to investigating which manifolds admit a formal overtwisted $(2, 3, 5)-$structure. For contact structures, we already know that there are some criteria that obstruct overtwistedness. For example, an overtwisted contact manifold cannot be the boundary of a symplectic manifold, which was proven by Gromov [26] and Eliashberg [14]. Can we find similar criteria for overtwisted $(2, 3, 5)-$structures?

- Often, it is interesting to investigate whether there are any intuitive or physically motivated examples of mathematical objects. In the introduction of this thesis we saw a physical example of a $(2, 3, 5)-$structure. Namely, the motion of a ball rolling on a stationary ball without twisting or slipping. This raises the question, are there any physical examples of *overtwisted* $(2, 3, 5)-$structures? In the contact case, we do have a physical example. Namely, the velocity field of an inviscid, incompressible fluid flow on a 3-dimensional Riemannian manifold, corresponds to a contact 1-form. In [19], Ghrist and Komendarczyk show that the so called "energy minimizing flow" is not necessarily a tight contact structure, but can also be overtwisted. Is there a similar example for overtwisted Engel or $(2,3,5)-$structures?

- Last, but certainly not least, whether the $h-$principle holds in general for $(2,3,5)-$structures on closed manifolds, is of course still an open question. This will probably be a harder question to answer compared to the ones posed above. Namely, also for Engel structures, this is still one of the main open questions in the field.

Appendix A

Moser's trick

In this appendix we are going to prove two strong results regarding (contact) forms. The proofs of both statements use a technique called *Moser's trick*. The trick is named after J. Moser, a German-American mathematician, who used the trick in 1965 to prove a statement on volume forms and when they are conjugate by a diffeomorphism of M [34]. It turns out that the trick can be used to prove many other results, of which we will discuss two in this appendix.

A.1 Gray Stability

In this section we will prove the well-known result of Gray's stability [23]. From this statement follows that any two contact structures which are homotopic to each other, in fact only differ by a diffeomorphism of the manifold. This makes working with contact structures really flexible, as invariants we define will remain unchanged when we homotope a contact structure. We first look at two preparatory lemmas before we prove the result.

Lemma A.1. *[18, p. 404-406] Let α be a differential k-form on a manifold M, and $(\psi_t)_{t\in[0,1]}$ an isotopy of M. Define a vector field X_t on M by $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ (i.e. ψ_t is the flow of X_t). Then*

$$\frac{d}{dt}\left(\psi_t^*\alpha\right) = \psi_t^*\left(\mathcal{L}_{X_t}\alpha\right).$$

Proof. The statement follows from observing that:

(i) the formula holds for functions,

$$\begin{aligned}\frac{d}{dt}(\psi_t^* f) &= \frac{d}{dt}\left(f \circ \psi_t\right)\\ &= df \circ \frac{d}{dt}\psi_t\\ &= df \circ \left(X_t \circ \psi_t\right)\\ &= \psi_t^*\left(\iota_{X_t} df\right)\\ &= \psi_t^*\left(\mathcal{L}_{X_t} f\right).\end{aligned}$$

(ii) if it holds for differential forms α and β, then it also holds for $\alpha \wedge \beta$,

$$\begin{aligned}\frac{d}{dt}\psi_t^*(\alpha\wedge\beta) &= \frac{d}{dt}(\psi_t^*(\alpha)\wedge\psi_t^*(\beta))\\ &= \frac{d}{dt}\psi_t^*(\alpha)\wedge\psi_t^*(\beta) + \psi_t^*(\alpha)\wedge\frac{d}{dt}\psi_t^*(\beta)\\ &= \psi_t^*(\mathcal{L}_{X_t}\alpha)\wedge\psi_t^*(\beta) + \psi_t^*(\alpha)\wedge\psi_t^*(\mathcal{L}_{X_t}\beta)\\ &= \psi_t^*(\mathcal{L}_{X_t}\alpha\wedge\beta + \alpha\wedge\mathcal{L}_{X_t}\beta)\\ &= \psi_t^*(\mathcal{L}_{X_t}(\alpha\wedge\beta)).\end{aligned}$$

(iii) if it holds for α, then also for $d\alpha$,

$$\begin{aligned}\frac{d}{dt}(\psi_t^* d\alpha) &= \frac{d}{dt}(d\psi_t^* \alpha)\\ &= d\frac{d}{dt}(\psi_t^* \alpha)\\ &= d\psi_t^*(\mathcal{L}_{X_t}\alpha)\\ &= \psi_t^*(d\mathcal{L}_{X_t}\alpha)\\ &= \psi_t^*(\mathcal{L}_{X_t}d\alpha).\end{aligned}$$

(iv) the algebra of differential forms is locally generated by functions and their differentials.

□

The next lemma is also proven in [18, p. 59-60], and we use the same structure of proof here.

Lemma A.2. *Let $(\alpha_t)_{t\in[0,1]}$ be a smooth family of differential k-forms on a manifold M, and $(\psi_t)_{t\in[0,1]}$ an isotopy of M. Define a vector field X_t on M by $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ (i.e. ψ_t is the flow of X_t). Then*

$$\frac{d}{dt}\left(\psi_t^* \alpha_t\right) = \psi_t^*\left(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t\right).$$

Proof. We know from Lemma A.1 that for a time-independent k-form α we have

$$\frac{d}{dt}\left(\psi_t^* \alpha\right) = \psi_t^*\left(\mathcal{L}_{X_t}\alpha\right).$$

Using this, we see that

$$\begin{aligned}\frac{d}{dt}\left(\psi_t^* \alpha_t\right) &= \lim_{h\to 0}\frac{\psi_{t+h}^*\alpha_{t+h} - \psi_t^*\alpha_t}{h}\\ &= \lim_{h\to 0}\frac{\psi_{t+h}^*\alpha_{t+h} - \psi_{t+h}^*\alpha_t + \psi_{t+h}^*\alpha_t - \psi_t^*\alpha_t}{h}\\ &= \lim_{h\to 0}\psi_{t+h}^*\left(\frac{\alpha_{t+h}-\alpha_t}{h}\right) + \lim_{h\to 0}\frac{\psi_{t+h}^*\alpha_t - \psi_t^*\alpha_t}{h}\\ &= \psi_t^*\left(\frac{d}{dt}\alpha_t\right) + \psi_t^*\left(\mathcal{L}_{X_t}\alpha_t\right)\\ &= \frac{d}{dt}\left(\psi_t^*\alpha_t\right) = \psi_t^*\left(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t\right).\end{aligned}$$

□

The next theorem states the result of Gray's stability. For the original proof we refer the reader to [23]. The proof worked out below is based on the proof in [18, p. 60-61], and as mentioned before, it uses Moser's trick.

Theorem A.3 (Gray's stability theorem)**.** *Let M be a closed manifold and let $(\alpha_t)_{t\in[0,1]}$ be a smooth family of contact forms on M. There there is a smooth family $(\psi_t)_{t\in[0,1]} \subset Diff(M)$ and a smooth family of non-vanishing functions $(f_t)_{t\in[0,1]}$ on M such that*

$$\psi_0 = id \quad \textit{and} \quad \psi_t^*\alpha_t = f_t\alpha_0.$$

In particular, if we set $\xi_t := \ker(\alpha_t)$, then the ξ_t are diffeomorphic under ψ_t to ξ_0.

Remark A.4. As stated before, we shall prove this result using Moser's trick. The idea is the following. We assume that ψ_t is the flow of a time-dependent vector field X_t. Then the result we are trying to prove can be reformulated as a result concerning X_t. If this result can be proven, we can then integrate X_t to find ψ_t. Note that we are working on a closed manifold, which is convenient, because then the flow of X_t will be defined on the whole manifold. [18, p. 60] ▶

Proof. We shall obtain ψ_t by finding the corresponding vector fields X_t such that

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \psi_0 = id.$$

We now differentiate both sides of the desired equation with respect to t, which gives us:

$$\frac{d}{dt}\left(\psi_t^* \alpha_t\right) = \left(\frac{d}{dt} f_t\right) \cdot \alpha_0 = \left(\frac{d}{dt} f_t\right) \cdot (f_t)^{-1} \psi_t^* \alpha_t = g_t \cdot \psi_t^* \alpha_t$$

where $g_t := \left(\frac{d}{dt} f_t\right) \cdot (f_t)^{-1}$.

Note that we are precisely in the setting of Lemma A.2. Therefore we have the identity

$$\frac{d}{dt}\left(\psi_t^* \alpha_t\right) = \psi_t^* \left(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t\right).$$

Using this we get

$$\psi_t^* \left(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t\right) = g_t \cdot \psi_t^* \alpha_t.$$

By removing ψ_t^* on both sides of the equation we get

$$\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t = \left(g_t \circ \psi_t^{-1}\right) \cdot \alpha_t = h_t \cdot \alpha_t,$$

where $h_t := g_t \circ \psi_t^{-1}$. We thus want to find vector fields X_t such that the identity above holds.

Using the Cartan formula the desired equation becomes

$$\frac{d}{dt}\alpha_t + d\iota_{X_t}\alpha_t + \iota_{X_t} d\alpha_t = h_t \cdot \alpha_t.$$

If we choose $X_t \in \xi_t = \ker \alpha_t$ then the equation becomes

$$\frac{d}{dt}\alpha_t + \iota_{X_t} d\alpha_t = h_t \cdot \alpha_t. \tag{A.1}$$

Now let R_t be the *Reeb vector field* associated to α_t (see also Remark A.5). Then R_t satisfies $R_t \in \ker(\alpha_t)$ and $\alpha_t(R_t) = 1$. Plugging in R_t in equation A.1 gives us the following:

$$\frac{d}{dt}\alpha(R_t) = h_t.$$

Now let $Z_t \in \xi_t = \ker(\alpha_t)$, and let us plug in Z_t into equation A.1, this gives us

$$\frac{d}{dt}\alpha_t(Z_t) + \iota_{X_t} d\alpha_t(Z_t) = 0,$$

so

$$\iota_{X_t} d\alpha_t = -\dot{\alpha}_t|_{\ker \alpha_t}.$$

Since $d\alpha_t$ is non-degenerate, this determines X_t uniquely. Then we can find ψ_t by integrating X_t. The functions f_t are uniquely determined from h_t and ψ_t with the initial value $f_0 = 1$. □

Remark A.5. In the proof of Theorem A.3 we use a vector field called the *Reeb vector field*. Given a contact form α, the Reeb vector field is defined as the unique vector field R_α which has the properties

$$d\alpha(R_\alpha, -) \equiv 0, \quad \alpha(R_\alpha) \equiv 1.$$

We note that the Reeb vector field is really connected to the contact form, and not to the contact structure the form defines. Namely, two different contact forms which define the same structure, may have very different Reeb vector fields. ▶

A.2 Darboux's theorem

In this section we will prove another important result on contact structures using Moser's trick. In the previous section we saw that contact structures are very flexible, as there are no local invariants. Darboux's theorem states that we can describe any contact structure on a manifold locally by the same model, called the standard contact structure (see also Figure A.1).

Darboux's theorem is quite useful when working with contact structures, as for any point in the manifold, one can pick a neighbourhood with local coordinates (which is also called a *Darboux ball*) such that the contact structure is of standard form. Let us formally state the result. The proof below is based on the proof in [18, p. 67-68], and for the original proof we refer the reader to [8].

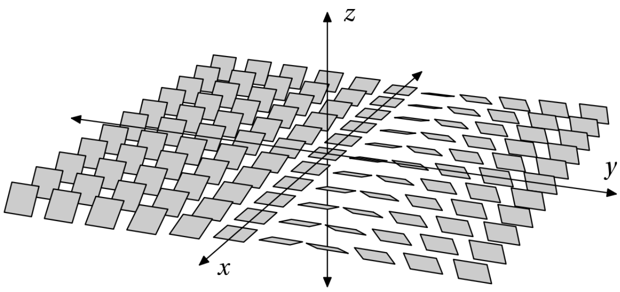

Figure A.1: The standard contact structure on $\mathbb{R}^3$ spanned by the vector fields ∂_y and $\partial_x + y\partial_z$ [42].

Theorem A.6 (Darboux's theorem)**.** *Let α be a contact form on a manifold M of dimension $2n+1$, and let $p \in M$. Then there are coordinates $x_1, ..., x_n, y_1, ..., y_n, z$ on a neighbourhood $U \subset M$ of p such that*

$$\alpha|_U = dz + \sum_{j=1}^{n} x_j dy_j.$$

Proof. We pick a neighbourhood V of p such that $V \cong \mathbb{R}^{2n+1}$ and $p = 0$. We then choose linear coordinates $x_1, ..., x_n, y_1, ..., y_n, z$ on $\mathbb{R}^{2n+1}$ such that on $T_0\mathbb{R}^{2n+1}$ we have that

$$\begin{cases} \alpha(\partial_z) = 1, \\ \iota_{\partial_z} d\alpha = 0, \\ \partial_{x_j}, \partial_{y_j} \in \ker\alpha (j = 1, ..., n), \\ d\alpha = \sum_{j=1}^{n} dx_j \wedge dy_j. \end{cases}$$

You can find these local coordinates by applying the normal form theorem for skew-symmetric forms on a vector space. Let $\alpha_0 := dz + \sum_{j=1}^{n} x_j dy_j$. Note that α_0 is of the desired form. Let

$$\alpha_t := (1-t)\alpha_0 + t\alpha$$

denote a homotopy between α_0 and α. At 0 we have that

$$\alpha_t = (1-t)(dz) + t(dz) = dz = \alpha$$

and

$$d\alpha_t = (1-t)\left(\sum_{j=1}^{n} dx_j \wedge dy_j\right) + t\left(\sum_{j=1}^{n} dx_j \wedge dy_j\right) = \sum_{j=1}^{n} dx_j \wedge dy_j = d\alpha.$$

Therefore, because α is a contact form, all the α_t are also contact forms at 0. Since being a contact form is an open condition, there is a small neighbourhood around 0 such that α_t is a contact form for $t \in [0,1]$.

We now want to use Moser's trick to find an isotopy $(\psi_t)_{t\in[0,1]}$ of a neighbourhood of 0 such that $\psi_t^*\alpha_t = \alpha_0$. As before, Moser's trick entails differentiating the desired equation with respect to t and assuming ψ_t is the flow of a time-dependent vector field X_t. Doing the former we get

$$\frac{d}{dt}\psi_t^*\alpha_t = 0.$$

Using Lemma A.2 we get

$$\psi_t^* \left(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t \right) = 0.$$

Using that ψ_t is a diffeomorphism and the Cartan formula, we get

$$\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t = \frac{d}{dt}\alpha_t + d\iota_{X_t}\alpha_t + \iota_{X_t}d\alpha_t = 0.$$

We now write $X_t = H_t R_t + Y_t$ where R_t is the Reeb vector field of α_t, $Y_t \in \xi_t := \ker \alpha_t$ and H_t is a function. Writing X_t in this way gives us the following,

$$\begin{aligned}\frac{d}{dt}\alpha_t + d\left(\alpha_t\left(H_t R_t + Y_t\right)\right) + \left(d\alpha_t\right)\left(H_t R_t + Y_t\right) = 0,\\ \frac{d}{dt}\alpha_t + dH_t + d\alpha_t\left(Y_t\right) = 0,\\ \frac{d}{dt}\alpha_t + dH_t + \iota_{Y_t}d\alpha_t = 0.\end{aligned}$$

We now use the same trick as in the proof of Theorem A.3. We plug a vector field into the equation above, from which we obtain a different equation. Plugging in the Reeb vector field R_t gives us

$$\frac{d}{dt}\alpha_t(R_t) + dH_t(R_t) + \iota_{Y_t}d\alpha_t(R_t) = \frac{d}{dt}\alpha_t(R_t) + dH_t(R_t) = 0,$$

so

$$dH_t(R_t) = -\frac{d}{dt}\alpha_t(R_t).$$

We can now find the function H_t in a small neighbourhood of the origin by integrating this equation. Note that this neighbourhood must be small enough such that none of the Reeb vector fields R_t have closed orbits, as otherwise integrating would give 0. As usual with finding functions by integration, we can set that $H_t(0) = 0$. Furthermore, since $\alpha_t = \alpha$ at the origin, we have that $\frac{d}{dt}\alpha_t = 0$ at the origin. From this follows that $dH_t|_0 = 0$.

Now that we have found the functions H_t, we can use the equation

$$\frac{d}{dt}\alpha_t + dH_t + \iota_{Y_t}d\alpha_t = 0,$$

to find Y_t. We have that

$$\iota_{Y_t}d\alpha_t = -\frac{d}{dt}\alpha_t - dH_t,$$

and since $d\alpha_t$ is non-degenerate we can determine Y_t uniquely from this, with $Y_t(0) = 0$. Since we set $X_t = H_t R_t + Y_t$, we have now found the desired vector fields X_t. We note that $X_t(0) = 0$ for all t, because $H_t(0) = 0$ and $Y_t(0) = 0$.

We define ψ_t to be the local flow of X_t. This flow is not necessarily everywhere defined. However, since $X_t(0) = 0$ for all $t \in [0,1]$, the flow is defined in the origin. Since the domain of definition of a local flow on a manifold is always open, the domain of definition of ψ_t is open. Therefore, ψ_t is in fact defined on a small neighbourhood of 0. And thus, we have found a neighbourhood U around p with $U \cong \mathbb{R}^{2n+1}$ and $p = 0$, such that

$$\psi_1^* \alpha|_U = \alpha_0.$$

This means that

$$\begin{cases} \tilde{x}_j := x_j \circ \psi_1^{-1} & \text{for } j = 1, ..., n, \\ \tilde{y}_j := y_j \circ \psi_1^{-1} & \text{for } j = 1, ..., n, \\ \tilde{z} := z \circ \psi_1^{-1} \end{cases}$$

are the desired coordinates such that

$$\alpha|_U = d\tilde{z} + \sum_{j=1}^{n} \tilde{x}_j d\tilde{y}_j.$$

□

Appendix B

Contact embeddings

In this appendix, we will be interested in certain embeddings into overtwisted contact manifolds. In the first section we will define a formal analogue of a contact embedding, and see that any such object in an overtwisted contact manifold, can be homotoped into a genuine contact embedding. Moreover, in the second section, we will define the concept of a formal Legendrian, and similarly show that any formal Legendrian in an overtwisted contact manifold can be homotoped into a genuine Legendrian. These results highlight that the existence of an overtwisted disc may be very helpful for proving $h-$principles for certain structures on contact manifolds.

B.1 Constructing contact embeddings

In this section, our goal will be to construct a genuine contact embedding on an overtwisted contact manifold, using a formal analogue of such a map. This is a first step into the direction of proving an $h-$principle for such embeddings. When proving this, the overtwistedness will be a key ingredient. Let us first look at the definition of a contact embedding.

Definition B.1. *Let (M, ξ) and (N, ξ_N) be contact manifolds. A map $f : (N, \xi_N) \to (M, \xi)$ is called a **contact embedding** if it is an embedding, and it pulls back the contact structure ξ to ξ_N, i.e. $f^*\xi = \xi_N$.*

There is also a formal analogue of this definition, which is defined as follows.

Definition B.2. *Let (M, ξ) and (N, ξ_N) be contact manifolds. We say $(f : N \to M, (\xi_t)_{t\in[0,1]})$ is a **formal contact embedding** if $f : N \to M$ is an embedding and $(\xi_t)_{t\in[0,1]}$ is a homotopy of distributions such that $\xi_0 = f^*\xi$ and $\xi_1 = \xi_N$.*

The theorem below tells us that given a formal contact embedding we can use the formal data to turn the embedding into a genuine contact embedding. The proof of the theorem below was first written down by Borman, Eliashberg and Murphy in [3, Corollary 1.4], but the ideas used in the proof were already thought of by Gromov much earlier.

Theorem B.3. *Let (M, ξ) be an overtwisted contact manifold, and (N, ξ_N) a contact manifold where N is compact with boundary. Furthermore, let $(f : (N, \xi_N) \to (M, \xi), (\xi_t)_{t\in[0,1]})$ be a formal contact embedding, such that $f(N)$ is disjoint from an overtwisted disc in M. Then there is a contact embedding*

$$\tilde{f} : (\tilde{N}, \xi_{\tilde{N}}) \to (M, \xi)$$

where $\tilde{N} = N - (\partial N \times [0, \delta])$, for $\delta > 0$ small, is a slightly smaller copy of N, and $\xi_{\tilde{N}} = \xi_N|_{\tilde{N}}$, with $\tilde{f}$ homotopic to $f|_{\tilde{N}}$.

Proof. We define the following distributions on M for $t \in [0, 1]$:

$$\tilde{\xi}_t = \begin{cases} \xi_t & \text{on } f(N)^c \\ (f \cdot \chi(x))_*\xi_t & \text{on } x \in f(N), \end{cases}$$

where $\chi : M \to [0, 1]$ is a bump function which is 0 on $f(N)^c$ and 1 on a slightly smaller open U in $f(N)$, see also Figure B.1. This gives us a homotopy of distributions $(\tilde{\xi}_t)_{t\in[0,1]}$ on M. We note that $\tilde{\xi}_0 = \xi$ and $f^*(\tilde{\xi}_1) = \xi_N$ on U due to the cut-off function.

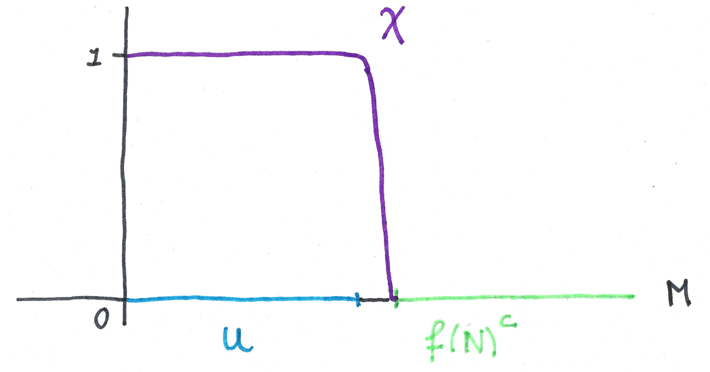

Figure B.1: A schematic drawing of the bump function χ used in the proof of Theorem B.3.

In [11, p. 35-37] it is shown how we can deform the homotopy $(\tilde{\xi}_t)_{t\in[0,1]}$ into a homotopy of genuine contact structures on M, which we will still denote by $(\tilde{\xi}_t)_{t\in[0,1]}$. Recall that this process uses the overtwistedness of M. On $M \times \{1\}$ we have that $f^*(\tilde{\xi}_1|_U) = \xi_N|_{f^{-1}(U)}$, which is already contact. We note that this process of making the homotopy contact, changes the contact structure $\tilde{\xi}_1|_U$ on a small region near the boundary of U. However, there is still a slightly smaller copy of $\tilde{N}$ of U, and thus of N, such that $f^*(\tilde{\xi}_1|_{f(\tilde{N})}) = \xi_1|_{\tilde{N}} = \xi_N|_{\tilde{N}}$.

Then by Gray's stability (Theorem A.3) there exist diffeomorphisms $(\psi_t : M \to M)_{t\in[0,1]}$ with $\psi_0 = \text{id}$ and $\psi_t^*(\tilde{\xi}_0) = \tilde{\xi}_t$. We note that $\psi_t \circ f : (\tilde{N}, \xi_{\tilde{N}}) \to (M, \xi)$ is an embedding and we see that on $\tilde{N}$ we have

$$(\psi_1 \circ f)^*(\xi)|_{\tilde{N}} = f^*(\psi_1^*(\tilde{\xi}_0))|_{\tilde{N}} = f^*(\tilde{\xi}_1)|_{\tilde{N}} = f^*((f \cdot \chi(x))_*\xi_1)|_{\tilde{N}} = f^*(f_*(\xi_N))|_{\tilde{N}} = \xi_N|_{\tilde{N}}.$$

We set $\tilde{f} := \psi_1 \circ f$ and conclude that

$$\tilde{f} : (\tilde{N}, \xi_{\tilde{N}}) \to (M, \xi)$$

is a contact embedding with $\xi_{\tilde{N}} = \xi_N|_{\tilde{N}}$, which is homotopic to f by the homotopy of embeddings

$$(\psi_t \circ f)_{t\in[0,1]}.$$

□

Remark B.4. By the tubular neighbourhood theorem we know that there exists a neighbourhood of ∂N which is of the form $\partial N \times [0, \epsilon)$ for some $\epsilon > 0$. Therefore, in the statement above, we can write the slightly smaller copy $\tilde{N}$ of N as $\tilde{N} = N - (\partial N \times [0, \delta])$, for $\delta > 0$ small. ▶

B.1.1 Corollary on convex boundary

In Theorem B.3 we have seen that we can homotope a formal contact embedding in an overtwisted contact manifold, into a genuine contact embedding. However, there is one catch, we need to slightly shrink the domain manifold N into a smaller copy $\tilde{N}$, as the structure ξ_N has been altered along the boundary during the proof. In this subsection, we shall discuss a particular setting where we can avoid this technicality. For this, we first need to look at the following two definitions.

Definition B.5. *Let (M, ξ) be a contact manifold, and let X be a vector field on M. Let φ_t denote the flow of X. We say X is a* ***contact vector field*** *if for all $t \in \mathbb{R}$, the flow φ_t is a contact diffeomorphism, i.e. φ_t is a diffeomorphism such that*

$$\varphi_t^*\xi = \xi.$$

Definition B.6. *Let (M, ξ) be a contact manifold. We say a hypersurface $W \subset M$ is* ***convex*** *if there exists a contact vector field X in (M, ξ), such that $X \pitchfork W$.*

Requiring the boundary of the domain manifold N to be convex, will give us a more clean result as a corollary of Theorem B.3.

Corollary B.7. *Let (M, ξ) be an overtwisted contact manifold, and (N, ξ_N) a compact contact manifold with boundary, and ∂N convex. Furthermore, let $(f : (N, \xi_N) \to (M, \xi), (\xi_t)_{t \in [0,1]})$ be a formal contact embedding such that $f(N)$ is disjoint from an overtwisted disc in M. Then there is a contact embedding*

$$g : (N, \xi_N) \to (M, \xi)$$

homotopic to f.

Proof. Since ∂N is convex, we know there exists a contact vector field X which is transverse to ∂N. We can then use the flow φ_t of this vector field, to "push in" the contact structure. Namely, there exists an $s \in \mathbb{R}$ such that $\varphi_s(N) \subset \tilde{N}$, where $\tilde{N}$ is the space from Theorem B.3. Then

$$\Phi_t : (N, \xi_N) \to (\varphi_t(N), \xi_N)$$

for $t \in [0, s]$ is a homotopy of contact embeddings. From Theorem B.3 we also know there exists a contact embedding

$$\tilde{f} : (\varphi_s(N), \xi_{\tilde{N}}) \to (M, \xi).$$

Then

$$\tilde{f} \circ \Phi_s : (N, \xi_N) \to (M, \xi)$$

is a contact embedding which is homotopic to f (by using the homotopy of embeddings Φ_t from above, and the homotopy stated in the proof of Theorem B.3). □

B.2 Constructing Legendrian knots

In this section we will look at formal Legendrian knots, and how we can homotope these into genuine Legendrian knots. We will prove a statement on overtwisted contact manifolds which will use Theorem B.3. Let us first define these objects.

Definition B.8. *A* ***formal Legendrian knot*** *is a pair $(g, (G_s)_{s \in [0,1]})$ where $g : \mathbb{S}^1 \to (M, \xi)$ is a a smooth knot, (M, ξ) an overtwisted contact structure and for every $s \in [0, 1]$, $G_s : T\mathbb{S}^1 \to TM$ is a monomorphism which lifts g, i.e.*

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{\;g\;} & M \\ \uparrow & & \uparrow \\ T\mathbb{S}^1 & \xrightarrow{\;G_s\;} & TM \end{array}$$

commutes, with $G_0 = dg$ and $G_1 : T\mathbb{S}^1 \to \xi$.

To construct a genuine Legendrian out of such a structure, we will first need to take a look at the following two preparatory lemmas.

Lemma B.9. *Let E be an oriented vector bundle over $\mathbb{S}^1$. Then E is trivial.*

Proof. Consider the map

$$f : [0, 1] \to \mathbb{S}^1, \quad t \mapsto e^{2\pi i t}.$$

Then f^*E is a vector bundle over $[0, 1]$. Since $[0, 1]$ is contractible we know that $f^*E \cong [0, 1] \times \mathbb{R}^n$ where $n = \text{rank}(E)$. By definition of the pull-back there exists a continuous bundle map $F : f^*E \to E$, i.e. a continuous map F such that the following diagram commutes

$$\begin{array}{ccc} [0,1] \times \mathbb{R}^n & \xrightarrow{\;F\;} & E \\ {\scriptstyle \pi_{[0,1]}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathbb{S}^1}} \\ [0,1] & \xrightarrow{\;f\;} & \mathbb{S}^1 \end{array}$$

and F is an isomorphism on the fibres. Therefore, we can construct a unique linear isomorphism $A_f : \mathbb{R}^n \to \mathbb{R}^n$, such that $\det(A_f) > 0$ and $F(0, v) = F(1, A_f(v))$. From this follows that all vector bundles over $\mathbb{S}^1$ are obtained from a trivial vector bundle over $[0, 1]$ by identifying $(0, v) \sim (1, A(v))$ for an $A \in GL^+(n, \mathbb{R})$. Note that $A \in GL^+(n, \mathbb{R})$, precisely because E is oriented, and thus A must be orientation preserving. We note that $GL^+(n,\mathbb{R})$ has only one path-component, and thus there exists a path A_t for $t \in [0,1]$ such that $A_0 = A^{-1}$ and $A_1 = \mathrm{id}$. Then we see that

$$[0,1]\times\mathbb{R}^n/\sim_{\mathrm{id}} \to [0,1]\times\mathbb{R}^n/\sim_A, \quad (t,v)\mapsto(t,A_t\circ A(v))$$

is a bundle isomorphism. Note that $[0,1]\times\mathbb{R}^n/\sim_{\mathrm{id}}$ is just the trivial bundle. We can conclude that indeed any orientable vector bundle over $\mathbb{S}^1$ is trivial. □

Lemma B.10. *There exists a retract from* $\mathbb{R}^2\times[0,1]$ *to*

$$\left(\mathbb{R}^2\times\{0,1\}\right)\cup(\{(0,0)\}\times[0,1]).$$

Proof. We note that we can identify $\mathbb{R}^2$ with the space $(-1,1)^2$. Using this identification, we are first going to construct a retract from $A := (-1,1)^2\times[0,1]$ to

$$B := \left((-1,1)^2\times\{0,1\}\right)\cup((-1,1)\times\{0\}\times[0,1]).$$

We do this by viewing A as a subspace of $\mathbb{R}^3$, and picking a point a above it, and a point b below it. See also Figure B.2. Let A_+ denote points $(x_1,x_2,x_3)\in A$ for which $x_2\geq 0$, and A_- those points for which $x_2\leq 0$. We then construct a retract

$$r_1 : A\to B$$

by sending a point $x\in A_+$ to the first point of intersection of the line through x and a with the subspace $B\subset A$, and by sending a point $x\in A_-$ to the first point of intersection of the line through x and b with the subspace $B\subset A$. We note that this map is continuous, and well-defined on $A_+\cap A_-$. Furthermore, note that for $y\in B$, we have $r_1(y)=y$, and thus we can conclude that r_1 is indeed a retract.

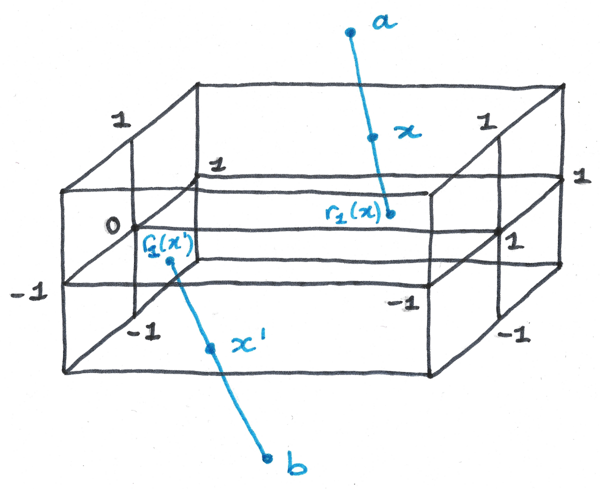

Figure B.2: Illustration of the retract $r_1 : A\to B$ used in the proof of Lemma B.10.

Secondly, we are going to construct a retract from

$$C := (-1,1)\times(0,1)$$

to

$$D := ((-1,1)\times\{0,1\})\cup(\{(0,0)\}\times[0,1]).$$

We do this by viewing C as a subspace of $\mathbb{R}^2$ and picking a point c above it and a point d below it. See also Figure B.3. Let C_+ denote the points $(x_1,x_2)\in C$ for which $x_1\geq 0$ and C_- denote the points for which $x_1\leq 0$. We then construct a retract

$$r_2 : C\to D$$

by sending a point $x\in C_+$ to the first point of intersection of the line through z and c with the subspace $D\subset C$, and by sending a point $x\in C_-$ to the first point of intersection of the line through

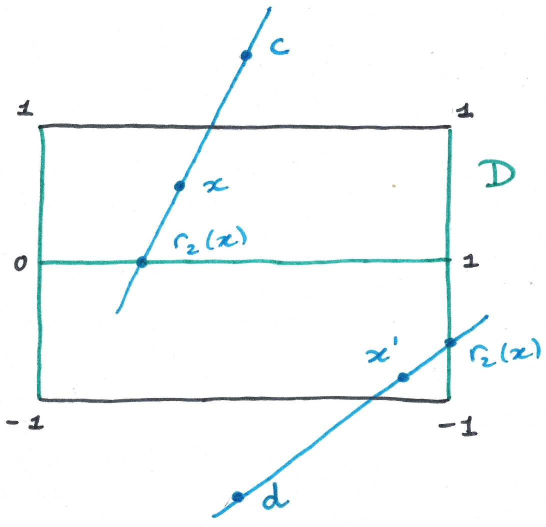

Figure B.3: Illustration of the retract $r_2 : C \to D$ used in the proof of Lemma B.10.

x and d with the subspace $D \subset C$. We note that this map is continuous and well-defined on $C_+ \cap C_-$. Furthermore, note that for $y \in D$ we have $r_2(y) = y$, and thus we can conclude that r_2 is indeed a retract.

Note that we can view D and C as subsets of B, as also indicated in Figure B.4. We can then construct a retract from A to the space

$$E := \left((-1,1)^2 \times \{0,1\}\right) \cup (\{(0,0)\} \times [0,1])$$

using the retracts r_1 and r_2. Namely, we first apply $r_1 : A \to B$, and then apply the map

$$B \to E, \quad x \mapsto \begin{cases} r_2(x) & \text{if } x \in C \\ x & \text{else.} \end{cases}$$

This is a continuous map for which $r(x) = x$ if $x \in E$. Therefore, r is the desired retract.

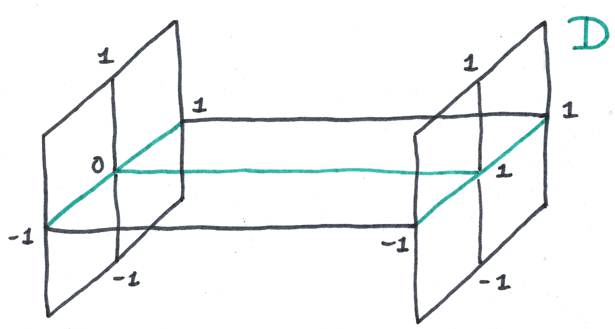

Figure B.4: Illustration of the space D as a subset of the space B, which we use in the proof of Lemma B.10.

□

Remark B.11. We note that the proof of Lemma B.10 can be generalised to produce a retract from $\mathbb{R}^n \times [0,1]$ to $\left(\mathbb{R}^n \times \{0,1\}\right) \cup (\mathbf{0} \times [0,1])$. Namely, one can construct and compose similar retracts as in the proof of Lemma B.10 to get the desired map. We will use this fact to prove the main result of this section. ▶

We are now ready to prove that every formal Legendrian knot in an overtwisted contact manifold, is in fact homotopic to a genuine Legendrian knot. We note that the statement below makes use of

Theorem B.3, and that this theorem uses the overtwistedness of the contact structure. See also [35, Theorem 1.2] for a proof by Murphy of this result.

Theorem B.12. *Let $(g,(G_s)_{s\in[0,1]})$ be a formal Legendrian knot in an overtwisted contact manifold (M,ξ). Then g is homotopic to a Legendrian knot.*

Proof. First of all, we isotope g such that it is disjoint from the overtwisted disc $D_{OT} \subset M$. We can do this by for example picking a point in the D_{OT} and pushing the knot out radially, see also Figure B.5.

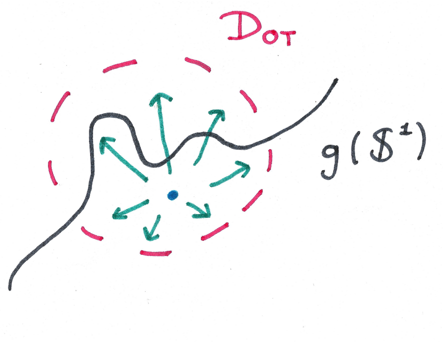

Figure B.5: Pushing the knot g radially out of the $D_{OT} \subset M$, as done in the proof of Theorem B.12.

We then define normal bundles

$$\nu_s := TM/G_s(T\mathbb{S}^1).$$

We note that ν_0 is the normal bundle of $g(\mathbb{S}^1)$ and that ν_0 is homotopic to ν_s for every $s \in [0,1]$. We denote this homotopy by ρ_s.

We have that

$$TM|_{\mathbb{S}^1} = G_1(T\mathbb{S}^1) \oplus \nu_1$$

and we know that G_1 takes values in ξ. From this follows that ν_1 is framed by ξ. Since each ν_s is homotopic to ν_1, each ν_s also receives a framing.

By Lemma B.9 we know that each ν_s is trivial, so $\nu_s \cong \mathbb{S}^1 \times \mathbb{R}^2$ and $\mathbb{D}\nu_s \cong \mathbb{S}^1 \times \mathbb{D}^2$. We then look at the bundles $T\mathbb{D}\nu_s$ and we note that we have a splitting at $\mathbb{S}^1$,

$$T\mathbb{D}\nu_s|^1_{\mathbb{S}} \cong T\mathbb{S}^1 \oplus L_0 \oplus L_1.$$

Here we define L_0 to be the orthogonal complement of $T\mathbb{S}^1$ inside ξ, and L_1 to be the orthogonal complement of $T\mathbb{S}^1 \oplus L_0$.

We then look at the following contact manifolds

$$(\mathbb{D}\nu_s, \xi_{leg})$$

where ξ_{leg} is given by one of the following contact structures:

$$\xi_{leg} := \ker(dy + xdt) = \langle \partial_t + x\partial_y, \partial_x \rangle,$$

or

$$\xi_{leg} = \ker(\cos(t)dx + \sin(t)dy) = \langle \partial_t, \sin(t)\partial_x - \cos(t)\partial_y \rangle.$$

These two models are depicted in Figure B.6.

We note that in both models, $\mathbb{S}^1$ is tangent to ξ_{leg}. We note that these are the two possible models for Legendrian knots, where in the first model ξ_{leg} is coorientable and in the second it is not. Furthermore, we note that in the first model we have

$$T\mathbb{S}^1 = \langle \partial_t + x\partial_y \rangle, \quad L_0 = \langle \partial_x \rangle, \quad L_1 = \langle \partial_y \rangle,$$

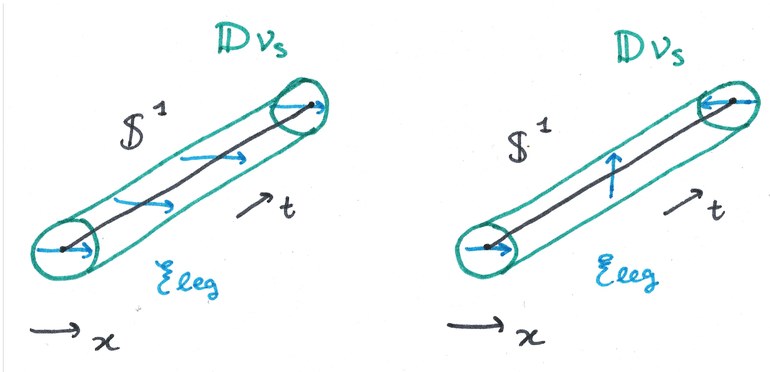

Figure B.6: Illustration of the two models for $(\mathbb{D}\nu_s(g), \xi_{leg})$, used in the proof of Theorem B.12.

and in the second model

$$T\mathbb{S}^1 = \langle \partial_t \rangle, \quad L_0 = \langle \sin(t)\partial_x - \cos(t)\partial_y \rangle, \quad L_1 = \langle \cos(t)\partial_x + \sin(t)\partial_y \rangle.$$

We then define the following maps

$$F_s : T\mathbb{D}\nu_s|^1_{\mathbb{S}} \cong T\mathbb{S}^1 \oplus L_0 \oplus L_1 \to TM|^1_{\mathbb{S}}$$

by $F_s := (G_s, \rho_s)$. Note that $F_0 = (dg, id)$, and $F_1 = (G_1, \rho_1)$ thus $F_1(\xi_{leg}) = \xi$.

We then want to construct maps

$$H_s : T\mathbb{D}\nu_s \to TM$$

such that $H_s|_{\mathbb{S}^1} = F_s$, $H_0 = id$ and $H_1(\xi_{leg}) = \xi$. Note that this is a homotopy extension problem. We see that

$$\begin{aligned} \cup_{s\in[0,1]} T\mathbb{D}\nu_s &\cong \cup_{s\in[0,1]} T(\mathbb{S}^1 \times \mathbb{D}^2) \\ &\cong T\mathbb{S}^1 \times T\mathbb{D}^2 \times [0,1] \\ &\cong \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times [0,1]. \end{aligned}$$

We note that the homotopy extension problem comes down to showing that

$$\left(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \{0,1\}\right) \cup \left(\mathbb{S}^1 \times \mathbb{R} \times \{(0,0)\} \times \{(0,0)\} \times [0,1]\right)$$

is a retract of $\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times [0,1]$. This is equivalent to showing that

$$\left(\mathbb{R}^4 \times \{0,1\}\right) \cup (\mathbf{0} \times [0,1])$$

is a retract of $\mathbb{R}^4 \times [0,1]$. This follows from Remark B.11.

We then define a formal contact embedding $(f, (\xi_s)_{s\in[0,1]})$ by

$$f : (N, \xi_N) \to (M, \xi_M), \quad \text{where } N := \mathbb{S}^1 \times \mathbb{D}(L_0 \oplus L_1) \simeq \mathbb{D}\nu_0(g) \text{ and } \xi_N = \xi_{leg}$$

together with the formal data

$$\xi_s := H_s(\xi_{leg}).$$

We note that $H_0(\xi_{leg}) = \xi_{leg}$ and $\xi_1 = H_1(\xi_{leg}) = \xi_M|_{\mathbb{D}\nu_0(g)}$.

We then apply Corollary B.7 and conclude that f is homotopic to a genuine contact embedding

$$\tilde{f} : (N, \xi_N) \mapsto (M, \xi).$$

This means that $\tilde{f}^*\xi_M = \xi_N = \xi_{leg}$, and thus $\tilde{f}|_{\mathbb{S}^1}$ is a Legendrian knot which is homotopic to g. □

Remark B.13. In the proof above we identify $\mathbb{D}\nu_0$ as $\mathbb{S}^1 \times \mathbb{D}^2$. Note that explicitly we are looking at $g(\mathbb{S}^1) \subset M$, but we just identify it with a copy of $\mathbb{S}^1$. This is the reason why we want $H_0 = \text{id}$ and why in the formal data for the formal contact embedding we have $\xi_1 = \xi_M|_{\mathbb{D}\nu_0(g)} = f^*\xi_M$. ▶

Appendix C

Orientations

In this appendix we shall discuss a property of smooth manifolds called *orientation.* Intuitively, an orientation is a choice between two different ways in which objects can be situated with respect to their surroundings. For 2-dimensional manifolds, choosing an orientation is the same as choosing what should be a clockwise and what should be a counterclockwise rotation. Not all manifolds are *orientable*! See also Figure C.1. Here we see that the sphere can be given an orientation, but the Möbius band cannot. Namely, it is impossible to choose a consistent way of defining a clockwise turning on the band, because as we move along the manifold this choice will be flipped.

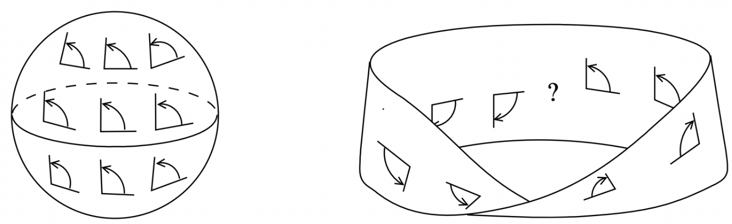

Figure C.1: A sphere is orientable but a Möbius band is not [36, p. 378].

In this appendix we will first discuss some definitions and basic results regarding orientations. After that, we will look at orientations in relation to contact, Engel and $(2,3,5)-$structures.

C.1 Definitions and basic results

This first section will be a brief introduction to orientations. First, we will look at orientations on vector spaces. This will then allow us to also define this concept on manifolds, and thereafter on vector bundles. Many of the statements in the next two subsections are sourced from [36], but at times some of the proofs which were left out have been filled in (namely, the proofs of Lemma C.2 and Lemma C.7).

C.1.1 Orientations on vector spaces

The goal of this subsection is to define orientations on vector spaces. We first look at the following definition.

Definition C.1. *Let V be a vector space of dimension $n \geq 1$, and let $(v_1, ..., v_n)$ and $(w_1, ..., w_n)$ be ordered bases of V. Let (B_i^j) denote the transition matrix, i.e.*

$$v_i = B_i^j w_j.$$

We say $(v_1, ..., v_n)$ and $(w_1, ..., w_n)$ are ***consistently oriented*** *if (B_i^j) has positive determinant.*

Lemma C.2. *Let V be a vector space. Being consistently oriented is an equivalence relation of the set of all ordered bases of V. Furthermore, there are exactly two equivalence classes.*

Proof. We note that this result is a consequence of properties of the determinant. Namely, reflexivity follows from the fact that the identity matrix has positive determinant. Furthermore, symmetry follows from the fact that $\det(B^{-1}) = \frac{1}{\det B}$, and lastly, transitivity follows from the fact that $\det(B \cdot A) = \det(B) \cdot \det(A)$. Thus, being consistently oriented is indeed an equivalence relation.

Now let $[u_i]$, $[v_i]$ and $[w_i]$ be equivalence classes of ordered bases, and suppose that $[u_i] \neq [v_i]$ and $[u_i] \neq [w_i]$. Let

$$u_i = A_i^j v_j, \quad u_i = B_i^k w_k \quad \text{and} \quad v_j = C_j^k w_k.$$

It must hold that $\det(A_i^j) < 0$ and $\det(B_i^k) < 0$, and we see that

$$u_i = A_i^j v_j = A_i^j C_j^k w_k,$$

thus $B_i^k = A_i^j C_j^k$. From this must follow that $\det(C_j^k) > 0$. Therefore, $[v_i] = [w_i]$. □

This result allows us to define the following concepts.

Definition C.3. *Let V be a vector space of dimension $n \geq 1$. We define an **orientation on** V to be an equivalence class of ordered bases. If $(v_1, ..., v_n)$ is an ordered basis, we write $[v_1, ..., v_n]$ for its equivalence class of consistently oriented bases, and $-[v_1, ..., v_n]$ for its opposite orientation.*

*Let V a vector space with $\dim(V) = n \geq 1$, together with a choice of orientation. Then we say V is **oriented**. Any basis $(v_1, ..., v_n)$ which is in the equivalence class of the orientation is said to be **positively oriented**, and any basis which is not, is said to be **negatively oriented**.*

For a vector space V with $\dim(V) = 0$, we define an orientation on V to be a choice between ± 1.

Example C.4. Let $(e_1, ..., e_n)$ be the standard basis of $\mathbb{R}^n$. We call the orientation $[e_1, ..., e_n]$ the **standard orientation** on $\mathbb{R}^n$. △

C.1.2 Orientations on manifolds

Now that we have defined orientations on vector fields, we can move on to manifolds.

Definition C.5. *Let M be a smooth manifold. A choice of orientation on every tangent space of M, is called a **point wise orientation**.*

Observe that this definition is not very useful yet. Namely, note that the orientation could switch to its opposite when moving through the manifold. We need to define conditions, such that nearby tangent spaces carry the same orientation. This motivates the next definition.

Definition C.6. *Let M be a smooth manifold of dimension n. Let $(X_1, ..., X_n)$ be a local frame for TM on an open $U \subset M$. We say $(X_1, ..., X_n)$ is **positively oriented** if for all $p \in U$, $(X_1|_p, ..., X_n|_p)$ is a positively oriented basis for T_pM. Analogously, we can define **negatively oriented** local frames.*

*A point wise orientation of M is called **continuous**, if every $p \in M$ is in a domain of an oriented local frame. A continuous point wise orientation on M, is called an **orientation on** M.*

*We say M is **orientable**, if there exists an orientation we can define on it. Analogously, M is **non-orientable**, if we cannot define an orientation on it.*

Intuitively, it would be logical if connected components of manifolds have only two possible orientations. The following proposition proves this result.

Proposition C.7. *Let M be a smooth, connected and orientable manifold. Then M has exactly two possible orientations.*

Proof. First of all, we note that if $\mathcal{O}$ is an orientation on M, then $-\mathcal{O}$ is also an orientation on M different from $\mathcal{O}$.

Now let $\mathcal{O}$ and $\tilde{\mathcal{O}}$ be two orientations on M. We use the notation $\mathcal{O}_p$ for the orientation of T_pM induced by $\mathcal{O}$. We then define the following set:

$$\mathcal{A} := \{p \in M | \mathcal{O}_p = \tilde{\mathcal{O}}_p\}.$$

We want to show that this set is open in M. Let $p \in \mathcal{A}$. Then there exists a local frame (X_i) on an open $U \subset M$ containing p, which is positively oriented with respect to $\mathcal{O}$. Similarly, there is another local frame $(\tilde{X}_i)$ on an open $\tilde{U} \subset M$ containing p, which is positively oriented with respect to $\tilde{\mathcal{O}}$. We can assume U to be connected, and by possibly reducing it to a smaller neighbourhood of p, we can also assume that $U \subset \tilde{U}$.

Since $p \in \mathcal{A}$, we have that $\mathcal{O}_p = \tilde{\mathcal{O}}_p$. Therefore, the transition matrix $(B_i^j(p))$ such that

$$X_i|_p = B_i^j(p)\tilde{X}_j|_p,$$

has positive determinant. Furthermore, we note that the map

$$\det_B : U \to \mathbb{R}, \quad q \mapsto \det(B(q))$$

is continuous, $\det_B$ is nowhere vanishing and $\det_B(p) > 0$. From this follows that $\det(B)|_U > 0$. Therefore, $U \subset \mathcal{A}$, and thus $\mathcal{A}$ is indeed open.

Using the exact same argumentation as above, one can also show that $M - \mathcal{A}$ is also open in M. Since M is connected, it must follow that $\mathcal{A} = M$ or $\mathcal{A} = \emptyset$. In the first case, we have that $\mathcal{O} = \tilde{\mathcal{O}}$. In the second case we have that $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are two distinct orientations with $\mathcal{O}_p \neq \tilde{\mathcal{O}}_p$ for $p \in M$. For any other orientation $\overline{\mathcal{O}}$ we must have that $\overline{\mathcal{O}}_p = \mathcal{O}_p$ or $\overline{\mathcal{O}}_p = \tilde{\mathcal{O}}_p$ for all $p \in M$. Using the same argumentation as above, it must follow that $\overline{\mathcal{O}} = \mathcal{O}$ or $\overline{\mathcal{O}} = \tilde{\mathcal{O}}$. This proves that there are exactly two possible orientations for M. □

To conclude this subsection, we look at one more result which gives us a class of manifolds which are always orientable.

Lemma C.8. *Let M be a parallelizable smooth manifold. Then M is orientable.*

Proof. Since M is parallelizable, there exists a global smooth frame (X_i). We define a point wise orientation on M by setting $(X_i|_p)$ to be positively oriented. This point wise orientation is continuous, as every $p \in M$ is in its domain. □

Example C.9. From Lemma C.8 we can conclude that several manifolds are orientable. For example, $\mathbb{R}^n$, the $n-$torus $\mathbb{T}^n$, $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ are all parallelizable, and thus orientable. Furthermore, all Lie groups are also parallelizable and thus orientable. We also know that products of parallelizable manifolds are again parallelizable manifolds, and thus also orientable. △

C.1.3 Orientations on vector bundles

To conclude this section, we look at orientations on vector bundles. We shall see that these are in fact a generalization of orientations on manifolds. First, we need to define the notion of an orientation-preserving map [36, p. 383]:

Definition C.10. *Let M and N be oriented smooth manifolds, and let $F : M \to N$ be a local diffeomorphism. We say F is **orientation-preserving**, if the isomorphism d_pF maps positively oriented bases of T_pM to positively oriented bases of $T_{F(p)}N$. We say F is **orientation-reversing**, if d_pF maps positively oriented bases of T_pM to negatively oriented bases of $T_{F(p)}N$.*

We then define orientations on vector bundles.

Definition C.11. *Let $\pi : E \to M$ be a smooth vector bundle. An **orientation on** E is*

(i) an orientation on each fibre E_p for $p \in M$, such that,

(ii) for each point $p \in M$, the local trivialisation

$$\phi_U : U \times \mathbb{R}^n \to \pi^{-1}(U),$$

with $p \in U$, is fibrewise orientation-preserving, i.e.

$$\phi_U(q, -) : \mathbb{R}^n \to \pi^{-1}(U), \quad t \mapsto \phi_U(q,t)$$

is orientation-preserving for all $q \in U$.

Here $\mathbb{R}^n$ is endowed with the standard orientation, as described in Example C.4 [32, p. 96].

This definition is in fact compatible with the definition of an orientation on a smooth manifold, as we will see in the next lemma.

Lemma C.12. *Let M be a smooth manifold. Then M is orientable if and only if TM is orientable.*

Proof. We note that the second condition in Definition C.11 is equivalent to there being sections $s_1, ..., s_n : U \to \pi^{-1}(U)$ such that the basis $(s_1(q), ..., s_n(q))$ is positively oriented for each $q \in U$. Since we are working with the tangent bundle, sections are vector fields. And thus, this conditions is equivalent to a point wise orientation being continuous. □

We will now prove a lemma which will be useful in the remaining sections of this appendix.

Lemma C.13. *A vector bundle E is orientable if and only if its determinant bundle $\det(E)$ is orientable.*

Proof. Suppose $\det(E)$ is orientable. Let (φ_i) be a trivialisation of $\det(E)$ with cover (U_i) of M. Let (ψ_j) be a trivialisation of E with cover (V_j) of M. We can assume that each V_j is connected. Note that for every ψ_j, $\det(\psi_j)$ is a trivialisation of $\det(E)$ on V_j. We note that condition (ii) in Definition C.11 is equivalent to the transition maps between trivialisations having positive determinant. See also [36, p. 381-382].

Now let $p \in V_j \cap U_i$. We can consider the transition map between $\det(\psi_j)$ and φ_i. Since $\det(E)$ is oriented, the sign of the transition map between $\det(\psi_j)$ and φ_i, which we will denote by ρ_j, is independent of the U_i such that $p \in U_i$. Therefore, the following map is well-defined:

$$V_j \to \{\pm 1\}, \quad q \mapsto \operatorname{sign}(\rho_j).$$

Since V_j is connected, this map is constant on V_j. Then we can modify the map ψ_j if necessary, by composing with a reflection, such that we may assume that the transition map is positive.

We will now show that these modified trivialisations (ψ_j) ensure that E is orientable. We note that the transition map between ψ_j and ψ_k has positive determinant if and only if the transition map between $\det(\psi_j)$ and $\det(\psi_k)$ is positive. Both ψ_j and ψ_k have positive transition maps with ψ_i for any i such that $p \in U_i$. Therefore, their transition map must also be positive.

Conversely, assume that E is oriented. Again, let (ψ_j) be a trivialisation of E with cover (V_j) of M. Then $(\det(\psi_j))$ is a trivialisation of $\det(E)$ with cover (V_j) of M. The transition maps are positive, as the transition maps of (ψ_j) have positive determinant. □

C.2 Contact structures

Having defined orientations on vector spaces, manifolds and vector bundles, we now shift our focus to distributions again. In this section we will focus on contact structures, and we will see that any manifold which admits a (formal) contact structure, is in fact canonically oriented.

Lemma C.14. *Let M be a 3-manifold which admits a formal contact structure ξ. Then M is canonically oriented [11, p. 23].*

Proof. Since ξ is a formal contact structure we have the following isomorphism

$$\xi \wedge \xi \cong \det(\xi) \cong TM/\xi.$$

We see that

$$TM = \xi \oplus TM/\xi,$$

and thus

$$\det(TM) \cong \det(\xi) \otimes TM/\xi \cong TM/\xi \otimes TM/\xi.$$

We note that TM/ξ is a line bundle and it is known that the tensor product of a line bundle with itself is in fact trivial. It then follows from Lemma C.13 that M is indeed naturally oriented. □

C.3 Engel structures

In this section we will discuss some results regarding (formal) Engel structures and orientability. The results from this section are sourced from Chapter 2 in [41]. Recall that an Engel structure comes with a flag $\mathcal{W} \subset D \subset \mathcal{E} \subset TM$, where $\mathcal{W}$ is the characteristic line field and $\mathcal{E}$ is an even-contact structure. The first result we will discuss in this section has to do with even-contact structures.

Proposition C.15. *Let $\mathcal{E}$ be an even-contact structure on a 4-dimensional manifold M. Then an orientation of the characteristic line field $\mathcal{W}$ induces an orientation on M, and an orientation on M induces an orientation on $\mathcal{W}$.*

Proof. Locally, we can look at a hypersurface N which is transverse to $\mathcal{W}$. We note that $TN \cap \mathcal{E}$ is a contact structure, and thus by Lemma C.14, N is canonically oriented. Therefore, an orientation of $\mathcal{W}$ yields an orientation on M, and also the other way around. □

The next result shows that if a manifold admits an Engel structure, then the even-contact structure has a canonical orientation.

Proposition C.16. *Let (M, D) be an Engel manifold. Then the even-contact structure $\mathcal{E} = [D, D]$ is naturally oriented.*

Proof. Let X and Y be two vector fields such that locally $D = \langle X, Y \rangle$ around $p \in M$. Then $[X(p), Y(p), [X, Y](p)]$ is an orientation of $\mathcal{E}$. Changing X or Y, or their signs, leaves this orientation the same, so it is independent of the choice of X and Y. □

Finally, we see that an oriented manifold with an oriented Engel structure is in fact parallelizable.

Proposition C.17. *Let (M, D) be an Engel manifold with D and M oriented. Then $TM \cong \mathbb{R}^4$.*

Proof. Recall that an Engel structure comes with a flag $\mathcal{W} \subset D \subset \mathcal{E} \subset TM$. We note that

$$TM = \mathcal{W} \oplus \frac{D}{\mathcal{W}} \oplus \frac{\mathcal{E}}{D} \oplus \frac{TM}{\mathcal{E}}.$$

Using Proposition C.15 and Proposition C.16 and the fact that D and M are oriented, we note that this is the sum of four oriented line bundles. Thus indeed $TM \cong \mathbb{R}^4$. □

C.4 $(2, 3, 5)-$structures

To conclude this appendix, we will look at $(2, 3, 5)-$structures and orientations. The next result shows that an orientation of a $(2, 3, 5)-$structure induces an orientation on the manifold and vice versa.

Proposition C.18. *Let M be a manifold which admits a $(2, 3, 5)-$structure D. Then M is orientable if and only if D is orientable.*

Proof. Recall that a formal $(2, 3, 5)-$structure comes with a flag $D \subset \mathcal{E} \subset TM$, and the isomorphisms

$$D \wedge D \cong \det(D) \cong \mathcal{E}/D \quad \text{and} \quad D \wedge \mathcal{E}/D \cong TM/\mathcal{E}.$$

We note that $TM/\mathcal{E} \cong D$ by the isomorphism induced by taking the tensor product with $\mathcal{E}/D$. This follows from the second isomorphism above. Using this, we get

$$TM \cong D \oplus \mathcal{E}/D \oplus TM/\mathcal{E} \cong D \oplus \det(D) \oplus D.$$

Therefore,

$$\det(TM) \cong \det(D)^3 \cong \det(D).$$

From Lemma C.13 it now follows that indeed M is orientable if and only if D is orientable. □

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