UTRECHT UNIVERSITY

MASTER THESIS

Arboreal Singularities in Smooth Manifolds

Author:
Caspar Meijs

Supervisor:
Dr. Álvaro del Pino
Secon Reader:
Dr. Gijs Heuts

Utrecht University

July, 2023

Acknowledgements

First and foremost, I would like to thank Álvaro del Pino Gómez for being my supervisor, suggesting this topic and always being able to find plenty of time for meetings in his busy schedule. I would also like to thank Gijs Heuts for being the Second Reader of this thesis.

I would also like to thank all the people from the Mathematics library. The comradery of studying together on most weekdays, and many weekends and holidays, helped me a great deal. Not only during the writing of this thesis but, but during my entire degree.

Finally, I would like to express my thanks to all my friends and family who endured my incomprehensible ramblings on arboreal singularities during the past year. I would especially like to thank Marieke for being patient with me all those times I was off in another world, thinking about some math problem or another, and for all her support.

Contents

Chapter 1

Introduction

Arboreal singularities were originally introduced by Nadler in [Nad17] as a class of combinatorial singularities of Lagrangian skeleta of symplectic manifolds. Nadler provided explicit models, described by connected non-cyclic graphs, i.e. trees, for these arboreal singularities. Furthermore, it is shown in [Nad17] that certain sheaf theoretical invariants of symplectic manifolds could be combinatorially computed from their Lagrangian skeleton if all singularities were of arboreal type. In [Nad16], Nadler presented a method to deform any Lagrangian skeleton to a skeleton with arboreal singularities such that certain sheaf theoretical invariants are preserved. The combinatorial structure of arboreal singularities, and how this could be used to combinatorially compute sheaf theoretical invariants, was further investigated by Zorn in [Zor18].

In [Sta18] Starston studied arboreal singularities in Weinstein manifolds, which are open exact symplectic manifolds compatible with Morse theory. The central idea of Morse theory is that (smooth) topological properties of a manifold can be investigated by looking at well-behaving smooth functions, called *Morse functions*, and their gradient. In particular, the critical points and gradient flow lines between those critical points of a single well-behaved function reflect the properties of a manifold. A Weinstein manifold is an open exact symplectic manifold equipped with a Morse function. The critical points and gradient flow lines of this Morse function form a Lagrangian skeleton. Starkston showed that, under certain non-degeneracy conditions that do not generally hold, the skeleton of Weinstein manifolds can be generically perturbed to be arboreal.

Expanding on [Sta18] and [Eli17], where the arboreal models are decorated by signs, Álvarez-Gavela, Eliashberg and Nadler introduced the class of *signed arboreal singularities* in [AEN22a]. Whereas the definition of arboreal singularities in [Nad17] only fixes the homeomorphism type of the singularities, these signed arboreal singularities are determined up to ambient symplectomorphism. In [AEN22a], models for signed arboreal singularities, which are described by trees with a decoration of signs on certain edges, are presented.

In [AEN22b] it is shown that the Lagrangian skeleton of any Weinstein manifold that admits a *polarization*, meaning its tangent bundle admits a global field of Lagrangian planes, can be deformed to be arboreal. Furthermore, it is shown that such a Weinstein manifold can be recovered from its arboreal skeleton.

This work aims to build a foundation for an arborealization program in smooth manifolds. The main goals are: (i) to define arboreal singularities in smooth manifolds, (ii) to determine when and how the smooth analogue of Weinstein manifolds, i.e. open smooth manifolds compatible with Morse theory, can be perturbed to admit an arboreal skeleton, and (iii) when and how such a smooth manifold can be recovered from its arboreal skeleton. The smooth structure is less rigid and more malleable than the symplectic structure, often allowing for a more straightforward and transparent treatment of arboreal singularities in smooth manifolds compared to arboreal singularities in symplectic manifolds.

This thesis is structured in the following way. In Chapter 2 and Chapter 3 we discuss some basic notions and central results of Morse theory. First, in Chapter 2 we focus on Morse functions. We define these Morse functions and prove the *Morse lemma*, which states that Morse functions have a ridged local structure around their critical points, and show that smooth maps are generically Morse. Furthermore, we prove the fundamental theorems of Morse theory, which show that the critical points of a Morse function on a manifold determine the topology of that manifold. Then, in Chapter 3 we shift our focus to the dynamics described by the gradient of Morse functions. We show that the flow lines between critical points of such a gradient system give a cellular-like decomposition of a manifold. In general this decompostion is not truly cellular, the closure of a cell is generally not obtained by adding lower index cells. However, this is the case under a certain transversallity condition, which we dicuss. Furthermore, we introduce *X-convex manifolds*, the smooth analogue of Weinstein manifolds, and define the skeleton of a X-convex manifold.

These two chapters give a brief introduction of some of the key concepts and intuitions in Morse theory, but touch only upon a minor part of the rich theory. In particular we do not discuss how Morse theory gives rise to a chain complex, Morse homology or the h-cobordism Theorem. Interested readers are referred to the wide range of books and articles on Morse theory, such as the classic books [Mil63] on Morse Theory and [Mil65] on the h-cobordism theorem by J.W. Milnor or [Bot88] for an historical overview of Morse theory up to the 1980s by R. Bott. More contemporary treatments of Morse theory can be found, among many other books on the subject, in [Mat02], [Nic11] and [ADE13].

In Chapter 4 we discuss *conical singularities*, which serve as a motivation for our definition of arboreal singularities in smooth manifolds. Following [Lau92], we show that if the gradient of a Morse function satisfies a certain tranversallity condition and admits specific local models around its critical points, this gradient induced a stratification with conical singularities.

In Chapter 5 we define the class of arboreal singularities in smooth manifolds and give explicit local models. We show that to every arboreal singularity we can associate a tree with a decoration of signs on its edges.

In Chapter 6 we show that every manifold that can be decomposed as a compact domain with an infinite cylindrical attached to its boundary, can be given the structure of an X-convex manifold with an arboreal skeleton. We do this by taking a X-convex structure, which we show exists using results from Morse theory, and inductively deforming the manifold. In general the singularities of the resulting skeleton admits multiple representations by different trees, in particular different decoration of signs on the edges can be chosen. However, we show that if the manifold is orientable there is a canonical choice of decoration by signs.

In Chapter 7 we present a procedure to recover an X-convex manifold from its arboreal skeleton if both the manifold and its skeleton are orientable. In this procedure, the orientation plays a similar role to the polarization in [AEN22b].

In Chapter 8 we discuss the directions further research might take.

Appendix A provides some background on symplectic, contact and Weinstein manifolds. In Appendix B we discuss arboreal singularities in symplectic manifolds, in particular we focus on the definitions and some of the key properties of arboreal singularities as defined in [Nad17] and [AEN22a]. We compare the models construced in [Nad17] and the models constructed in [AEN22a], and explicitly show that these constructions produce homeomorphic singularities.

Chapter 2

Morse Functions

The idea at the heart of Morse theory is that the topology of a manifold M can be determined by studying smooth functions $M \to \mathbb{R}$. In particular, the topology of M can be described by looking at the critical points and the gradient field of a single so-called *Morse function* $f : M \to \mathbb{R}$

In this chapter we approach Morse theory as a "critical point theory" in the way originally developed by H.C.M. Morse in the 1920s. We focus on how critical points of Morse functions correspond to points where the topology of the domain changes and how this change is determined by the critical point. In the following chapter we will shift our focus and consider Morse theory through the lens of dynamical systems, a point of view initiated by S. Smale in the 1960s, by studying the gradients of Morse functions in more detail.

We will begin the chapter with an illustrative example in Section 2.1. In Section 2.2 we introduce the notion of *Morse functions*, the main object of study within Morse theory, which are functions satisfying a non-degeneracy conditions at their critical points. We formulate and prove the *Morse Lemma*, which states that we have certain local models for Morse functions in the neighbourhood of their critical points. Then we will discuss *gradient* vector fields of Morse functions. These gradient vector fields will play a crucial role in Section 2.3, where we show that the topology of a manifold can be determined by the critical points of a Morse function on that manifold. In particular we show that any Morse function gives a handlebody decomposition of its domain. In Section 2.4 we discuss the limitations of Morse theory as a "cricital point theory" and motivate the need to study the gradients of Morse functions.

This chapter is largely based on the first four sections of [Mil63].

2.1 Height function on the torus

The main philosophy of Morse theory is that the topology of a manifold M can be determined from the critical points of a well-chosen smooth map $f : M \to \mathbb{R}$. We illustrate this with the classic example of the "height" function on a 2-dimensional torus T. Let V be a plane tangent

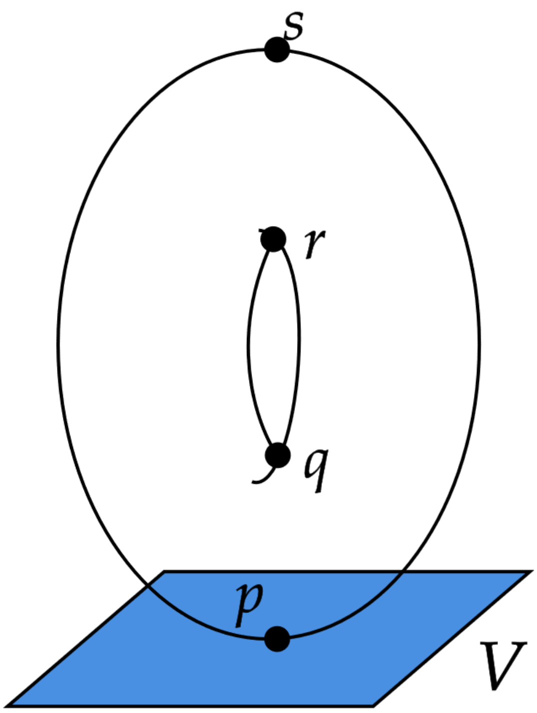


Figure 2.1: The torus T with the points p, q, r, s and the plane V indicated.

to T, as illustrated in Figure 2.1, and define $f : T \to \mathbb{R}$ to be the distance from the V plane. We write

$$T^a = \{x \in T \mid f(x) \leqslant a\}$$

for the sublevel set of $a \in \mathbb{R}$. We make the following observations, as illustrated in Figure 2.2.

1. If $a < f(p)$, then T^a is empty.
2. If $f(p) < a < f(q)$, then T^a is homeomorphic to a 2-cell.
3. If $f(q) < a < f(r)$, then T^a is homeomorphic to a cylinder.
4. If $f(r) < a < f(s)$, then T^a is homeomorphic to a punctured torus having a circle as boundary.
5. If $f(s) < a$, then T^a is the full torus.

We see that the topology of M^a changes as a passes through $f(p), f(q), f(r)$ and $f(s)$. From a homotopy type point of view, the step $1 \to 2$ is given by attaching a 0-cell, since a 0-cell and a 2-cell are homotopic. The step $2 \to 3$ is, in terms of homotopy type, the attaching of a 1-cell. The step $3 \to 4$ is also given by the attaching of a 1-cell. The final step $4 \to 5$ is given by attaching a 2-cell.

The points $p, q, r, s \in T$ are critical points of f, if we choose any coordinate system (x, y) around these points the derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both vanish. Furthermore, as will be shown in the next section, we can pick coordinates (x, y) around p such that $f(x, y) = f(p) + x^2 + y^2$, around q such that $f(x, y) = f(q) + x^2 - y^2$, around r such that $f(x, y) = f(r) + x^2 - y^2$ and around s such that $f(x, y) = f(s) - x^2 - y^2$. We remark that the number of terms with a minus signs in the expression for f correspond to the dimension of the cell that is attached as a passes through the corresponding critical value.

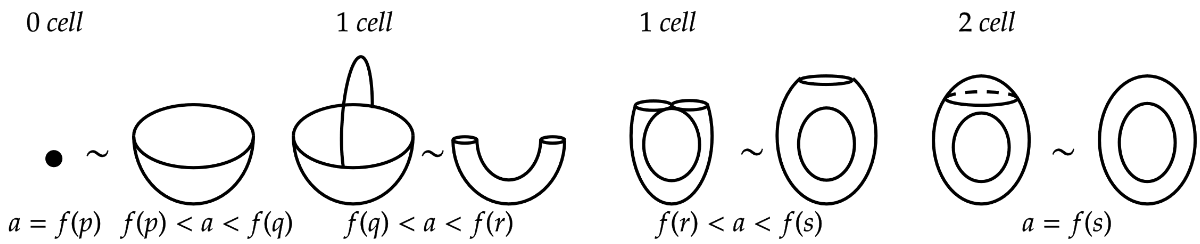


Figure 2.2: Placeholder figure.

2.2 Morse functions

To be able to define *Morse functions*, we first have to introduce some auxiliary notions.

Definition 2.1. Let M be a smooth manifold and $f : M \to \mathbb{R}$ a smooth function. A point $p \in M$ is called a *critical point* of f if $df_p : T_pM \to T_{f(p)}\mathbb{R}$ is zero, we denote the set of critical points of f by $\mathrm{Crit}(f)$.

The real number $f(p) \in \mathbb{R}$ is called a *critical value* of f. △

Note that p is a critical point if and only if, for arbitrary coordinates $(x_1, ..., x_n)$ around p, we have

$$\frac{\partial f}{\partial x_1}(p) = ... = \frac{\partial f}{\partial x_n}(p) = 0.$$

Definition 2.2. If p is a critical point of f we define a bilinear functional $H_f : T_pM \times t_pM \to \mathbb{R}$ of T_pM, called the *Hessian* at p, as

$$H_f(v, w) = \tilde{v}_p(\tilde{w}(f))$$

where $\tilde{v}$ and $\tilde{w}$ are arbitrary extensions of tangent vectors $v, w \in T_pM$ to vector fields. △

Lemma 2.3. *The Hessian is symmetric and well-defined.*

Proof. Begin by noting

$$H_f(v, w) - H_f(w, v) = \tilde{v}_p(\tilde{w}(f)) - \tilde{w}_p(\tilde{v}(f)) = [\tilde{v}, \tilde{w}]_p(f) = 0$$

where $[\tilde{v}, \tilde{w}]$ is the Lie bracket of $\tilde{v}$ and $\tilde{w}$. Because p is a critical point of f all its directional derivatives are zero. Therefore we have that $[\tilde{v}, \tilde{w}]_p(f) = 0$, thus H_f is symmetric.

Now, since $\tilde{v}_p = v$ and $\tilde{w}_p = w$ we see that

$$H_f(v, w) = v(\tilde{w}(f)) = -H_f(w, v) = w(\tilde{v}(f))$$

is independent of both the extension $\tilde{v}$ of v and the extension $\tilde{w}$ of w. □

Definition 2.4. A critical point $p \in M$ of f is called *non-degenerate* if the Hessian at p is non-degenerate, i.e.

$$H_f(v, w) = 0 \text{ for all } w \in T_pM \text{ if and only if } v = 0.$$

A smooth function $f : M \to \mathbb{R}$ is called *Morse* if all the critical points of f are interior, i.e. lie on $M - \partial M$, and are non-degenerate. A Morse function $f : M \to \mathbb{R}$ is called *exhaustive* if for every $c \in \mathbb{R}$ the sublevel set

$$\{x \in M \mid f(x) \leqslant c\}$$

is compact. △

Example 2.5. In Figure 2.3 we have given an example of a Morse function and three examples of non-Morse functions. The function $f(x, y) = x^2 - y^2$ is Morse, it has a single critical point at the origin that is non-degenerate. The function $f(x, y) = x^3 + y^3$ is not Morse, it has a single critical point at the origin that is degenerate. The function $f(x, y) = x^2$ is not Morse, the critical points form the x-axis and are all degenerate. The function $f(x, y) = x^2y^2$ is not Morse, the critical points form the x-axis and y-axis and are all degenerate. △

Note that if we choice local coordinates $(x_1, ..., x_n)$, we can give an explicit representation of H_f with respect to the $\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p$ basis.

We write $v = \sum\limits_i a_i \frac{\partial}{\partial x_i}|_p$ $w = \sum\limits_j b_j \frac{\partial}{\partial x_j}|_p$ and take the extension $\tilde{w} = \sum\limits_j b_j \frac{\partial}{\partial x_j}$ of w. This yields

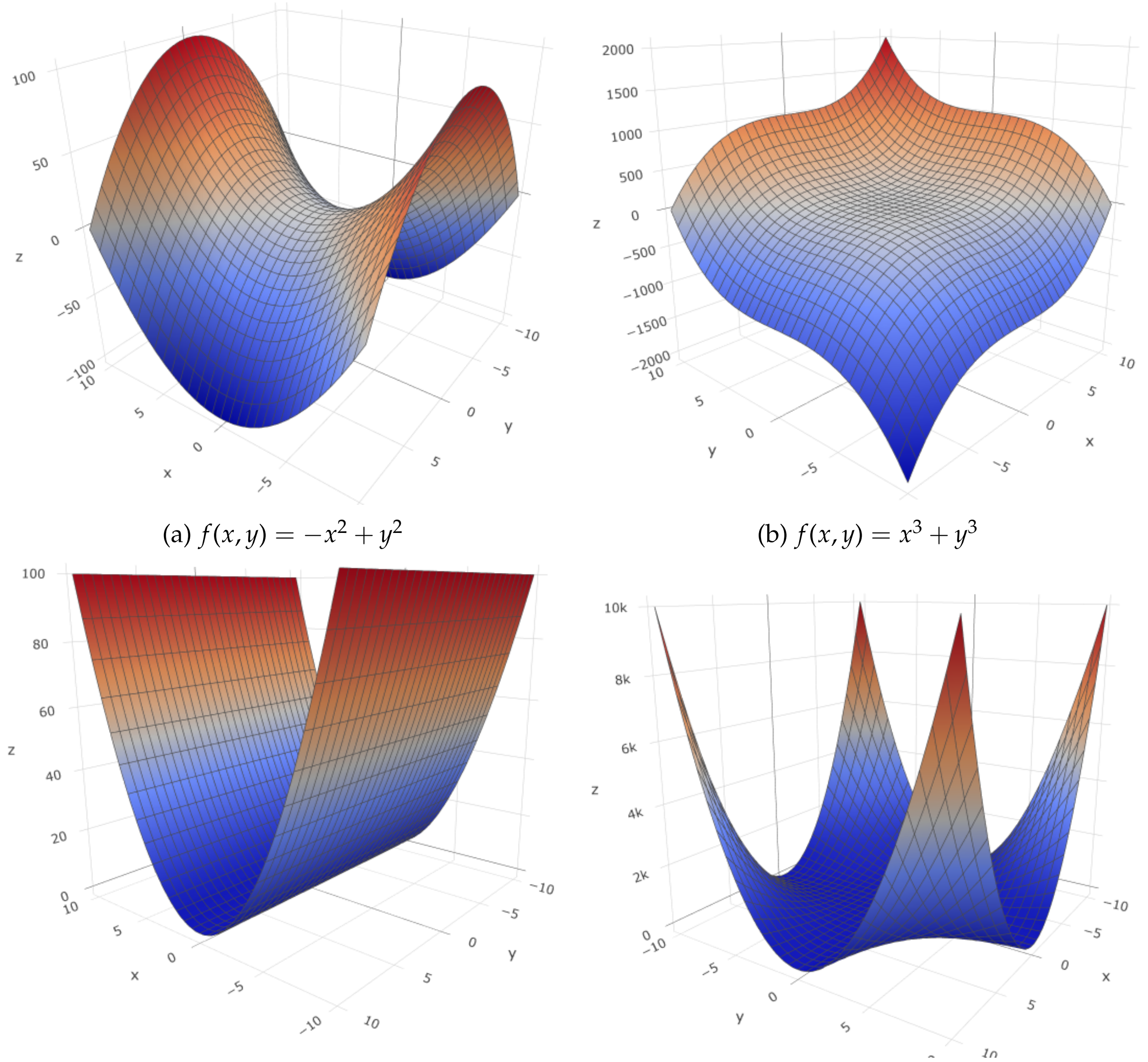
$$H_f(v, w) = v(\tilde{w}(f))(p) = v\left(\sum_j b_j \frac{\partial f}{\partial x_j}\right) = \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}(p).$$

Thus the representation of H_f with respect to the $\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p$ basis is given by the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$.

The Hessian is non-degenarate if and only if the matrix representing the Hessian its respect to the $\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p$ basis is non-singular.

Definition 2.6. The *index* of a bilinear functional H on a vector space V is the maximal dimension of a subspace of V on which H is negative definite.

The index of H_f on T_pM will be called the *index of f at p*. △



(a) $f(x,y) = -x^2 + y^2$ (b) $f(x,y) = x^3 + y^3$

(c) $f(x,y) = x^2$ (d) $f(x,y) = x^2y^2$

Figure 2.3: One Morse function and three functions with degenerate critical points.

We now formulate the Morse Lemma, which will play a central and crucial role in the rest of this thesis. Similar to how Taylor's Theorem states that near a critical point a f is approximated by a quadratic function, the Morse Lemma states that in an appropriate chart a Morse function is precisely a quadratic function around its critical points.

Lemma 2.7 (Morse)**.** *Let p be a non-degenerate critical point for $f : M \to \mathbb{R}$. Then there is a coordinate system $(x_1, ..., x_n)$ on a neighbourhood U around p such that*

$$f(x_1, ..., x_n) = f(p) - x_1^2 - .... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_n^2$$

on U. Furthermore, λ is the index of f at p.

Before we prove the Morse Lemma, we first show the following lemma.

Lemma 2.8. *Let f be a smooth function on convex neighbourhood U of 0 in $\mathbb{R}^n$ with $f(0) = 0$. Then*

$$f(x_1, ..., x_n) = \sum_{i=1}^{n} x_i g_i(x_1, ..., x_n)$$

fore some suitable smooth functions g_i defined on U with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Proof. Using the fundamental theorem of calculus and the chain rule we get

$$f(x_1, ..., x_n) = \int_0^1 \frac{df(tx_1, ..., tx_n)}{dt} dt = \int_0^1 \sum_{i=1}^{n} \frac{df}{dx_i}(tx_1, ..., tx_n) \cdot x_i dt.$$

Thus we can set $g_i(x_1, ..., x_n) = \int_0^1 \sum\limits_{i=1}^{n} \frac{df}{dx_i}(tx_1, ..., tx_n)dt$ giving the desired result. □

We now prove the Morse Lemma.

Proof. We first show f is of the desired form.

Let U be a neighbourhood of p with local coordinates, we can assume that $p = 0$ and $f(p) = f(0) = 0$. Using the previous Lemma we can write

$$f(x_1, ..., x_n) = \sum_{j=1}^{n} x_j g_j(x_1, ..., x_n)$$

for $(x_1, ..., x_n)$ in some neighbourhood of 0. Now, as 0 is a critical point

$$g_j(0) = \frac{\partial f}{\partial x_j}(0) = 0$$

thus we can apply Lemma 2.8 to these g_j to get

$$g_j(x_1, ..., x_n) = \sum_{i=1}^{n} x_i h_{ij}(x_1, ..., x_n)$$

for certain smooth functions h_{ij}. From this we obtain

$$f(x_1, ..., x_n) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x_1, ..., x_n).$$

If we write $\overline{h_{ij}} = \frac{1}{2}(h_{ij} + h_{ji})$ then $\overline{h_{ij}} = \overline{h_{ji}}$ and $f = \sum x_i x_j \overline{h_{ij}}$, thus we assume without loss of generality that $h_{ij} = h_{ji}$. Note that $(h_{ij}(0))$ is precisely Hessian matrix times $1/2$, and thus non-singular by assumption.

We now proceed by induction. Assume there are coordinates $(u_1, ..., u_n)$ in a neighbourhood U of 0 such that

$$f(u) = \pm u_1^2 \pm ... \pm u_{r-1}^2 + \sum_{i,j \geqslant r} u_i u_j H_{ij}(u_1, ..., u_n)$$

for some r and functions H_{ij} such that $(H_{ij}(0))$ is non-singular. We can apply a linear change in the last $n - r + 1$ variables to assume that $H_{rr}(0) \neq 0$, meaning that $H_{rr} \neq 0$ on some open neighbourhood $U' \subset U$ of 0.

We introduce new coordinates $(v_1, ..., v_n)$, where $v_i = u_i$ for $i \neq r$ and

$$v_r(u) = \sqrt{|H_{rr}(u)|}[u_r + \sum_{i>r} u_i H_{ir}(u)/H_{rr}(u)].$$

Using the inverse function theorem we see that $(v_1, ..., v_n)$ are coordinate functions in some sufficiently small neighbourhood of 0.

Without loss of generality we assume $H_{rr} > 0$, then we get

$$v_r^2 = H_{rr}\left(u_r + \sum_{i>r} u_i \frac{H_{ir}}{H_{rr}}\right)^2 = H_{rr}u_r^2 + 2\sum_{i>r} u_r u_i H_{ir} + \frac{\left(\sum_{i>r} u_i H_{ir}\right)^2}{H_{rr}}.$$

Thus

$$\begin{aligned} f &= \pm u_1^2 \pm ... \pm u_{r-1}^2 + \sum_{i,j \geqslant r} u_i u_j H_{ij}(u_1, ..., u_n) \\ &= \pm u_1^2 \pm ... \pm u_{r-1}^2 + u_r^2 H_{rr} + 2\sum_{i>r} u_r u_i H_{ir} + \sum_{i,j>r} u_i u_j H_{ij}(u) \\ &= \pm v_1^2 \pm ... \pm v_{r-1}^2 + v_r^2 + \sum_{i,j>r} u_i u_j \widetilde{H_{ij}}(u) \end{aligned}$$

for new functions $\widetilde{H_{ij}}$ satisfying the induction hypothesis.

Note that under the assumption $H_{rr} < 0$ we get almost the same result, with the difference being that the sign of v_r^2 would be negative.

Using induction we conclude that f has the form

$$f(x_1, ..., x_n) = f(p) - x_1^2 - .... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_n^2.$$

We will now show that λ is the index of f at p. Assume f is of the form $f = f(p) - x_1^2 - ... -$

$x_\lambda^2 + x_{\lambda+1}^2 + ... + x_n^2$, then the Hessian with respect to the basis $\frac{\partial}{\partial x_i}|_p$ is given by

$$\begin{pmatrix} -2 & & & & & 0 \\ & \ddots & & & & \\ & & -2 & & & \\ & & & 2 & & \\ & & & & \ddots & \\ 0 & & & & & 2 \end{pmatrix}.$$

Thus there is a subspace of T_pM of dimension λ where H_f is negative definite and a subspace V of dimension $n - \lambda$ where H_f is positive definite. If there were a subspace of T_pM on which H_f were negative definite of dimension greater than λ, this subspace would have to intersect V. Thus there can not be a subspace of dimension greater than λ on which H_f is negative definite. We conclude that λ is the index of f at p. □

We will call the coordinates $(x_1, ..., x_n)$ on the neighbourhood U around a non-degenerate critical point p on which f has the form

$$f(x_1, ..., x_n) = f(p) - x_1^2 - .... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_n^2$$

the *Morse coordinates* on the *Morse neighbourhood* of p.

Note that every Morse neighbourhood of a non-degenerate critical point p has only one critical point, namely the point p. Thus we have the following direct corollary of the Morse Lemma.

Corollary 2.9. *Non-degenerate critical points are isolated.*

Every manifold admits Morse functions, in fact we have the following well known theorem.

Theorem 2.10. *Let M be a smooth manifold, then the Morse functions form an open dense subset of $C^\infty(M, \mathbb{R})$.*

We give the most important steps of the proof, full details can be found in Section II.6 of [GG74].

Proof. We denote

$$S = \{\sigma = j^1 f(p) \in J^1(M, \mathbb{R}) \mid \text{rank}(df)_p = 0\}$$

and remark that p is a critical point of $f : M \to \mathbb{R}$ if and only if $j^1 f(p) \in S$.

We make the identification $J^1(M, \mathbb{R}) \cong TM \times \mathbb{R}$, we work in local coordinates and assume that $M = \mathbb{R}^n$ and $p = 0$. Now, locally, $j^1 f(x) = (x, df(x), f(x))$ and $S = \mathbb{R}^n \times \{0\} \times \mathbb{R}$, where we have made the identification $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. We see that the critical point p of f is non-degenerate if and only if

$$j^1 f \pitchfork S$$

at p. The assertion now follows directly from the Thom Transversality Theorem. □

2.3 Homotopy type in terms of critical points

Throughout this section we fix a smooth n-dimensional manifold M and a smooth function $f : M \to \mathbb{R}$. We write

$$M^a = f^{-1}((-\infty, a]) = \{p \in M | f(p) \leqslant a\}$$

for the sublevel set.

As seen in Section 2.1 above, the topology of sublevel sets M^a changes when a passes through a critical value. In this section we describe exactly how the topology changes, we will see that passing through a critical value yields a change in topology that corresponds to a λ-handle attachment, where λ is the index of the corresponding critical point. At the end of the section we discuss the challenges of recovering the smooth type of the manifold from a Morse function.

We first show that the topology of M^a only changes when a passes through a critical point, and that it remains stable as a varies over an interval containing only regular values.

Theorem 2.11. *Let $a < b$ and suppose that $f^{-1}([a, b])$ is compact and contains no critical points of f. Then the sublevel sets M^a and M^b are diffeomorphic.*

Furthermore, M^a is a deformation retract of M^b, thus the inclusion $M^a \to M^b$ is a homotopy equivalence.

Before we give the proof we formulate the following definition.

Definition 2.12. Let M be a smooth manifold equipped with a Riemannian metric g and let $f : M \to \mathbb{R}$ be smooth. The *gradient* of f with respect to the Riemannian metric g is the vector field $\text{grad} f \in \mathfrak{X}(M)$ given by

$$g(Y, \text{grad} f) = Y(f) = df(Y)$$

for any vector field Y. △

We now proceed with the proof of Theorem 2.11.

Proof. We fix a vector field X that is gradient for f with respect to some Riemannian metric on M and use the flow to push M^b down to M^a. We define $\rho : M \to \mathbb{R}$ to be a smooth function that is $\frac{1}{X(f)}$ on $f^{-1}([a, b])$ and 0 outside some compact neighbourhood of $f^{-1}([a, b])$.

Now ρX is a vector field that vanishes outside a compact set, meaning it has global flow ψ^s and this flow is a diffeomorphism for every $s \in \mathbb{R}$. Let $x \in M$ and consider the map $s \mapsto f \circ \psi^s(x)$. If $\psi^s(x) \in f^{-1}([a, b])$, then

$$\frac{d}{ds} f \circ \psi^s(x) = \rho X(f) = 1,$$

thus $f \circ \psi^s(x) = f(x) + s$. We see that ψ^{b-a} is a diffeomorphism mapping M^b to M^a.

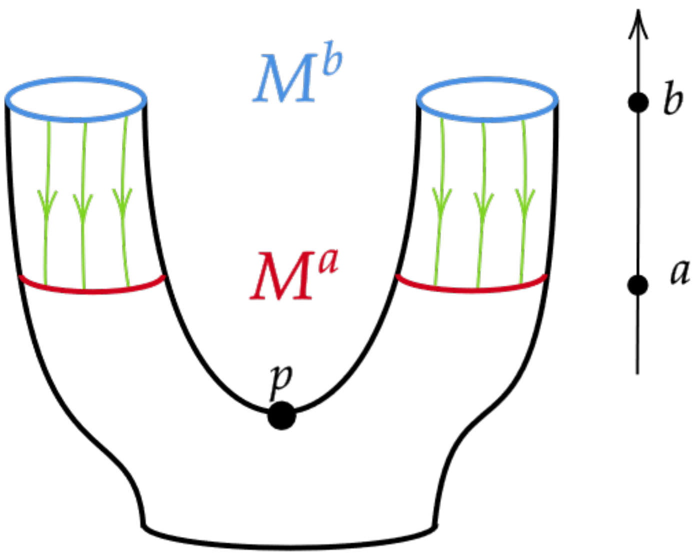


Figure 2.4: Using the flow to push M^b down to M^a.

The map $r_t : M^b \to M^a$ defined as

$$r_t(x) = \begin{cases} x & \text{if } f(x) \leqslant a \\ \psi_{t(a-f(x))}(x) & \text{if } a \leqslant f(x) \leqslant b \end{cases}$$

is a homotopy from $r_0 = id$ to r_1 which is a retraction from M^b to M^a.

We conclude that M^a and M^b are diffeomorphic and that M^a is a deformation retract of M^b. $\square$

The following theorem states how M^a changes as a passes through a critical value.

Theorem 2.13. *Let $f : M \to \mathbb{R}$ be a smooth function, let p be a non-degenerate critical point of index λ and set $c = f(p)$. Suppose that for some sufficiently small $\varepsilon > 0$, $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and contains no critical points other than p. Then the sublevel set $M^{c+\varepsilon}$ is diffeomorphic to $M^{c-\varepsilon}$ with a λ-handle attached. In particular $M^{c+\varepsilon}$ is homotopic to $M^{c-\varepsilon}$ with a λ-cell attached.*

We will not give a complete proof, but only give a sketch the most important steps. A full proof can be found in Section I.3 of [Mil63].

Proof. Consider Morse coordinates $(x, y) \in \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda}$ in a neighbourhood U of p such that

$$f = c - \|x\|^2 + \|y\|^2$$

on U. Let $\varepsilon > 0$ such that both $f^{-1}([c-\varepsilon, c+\varepsilon])$ and the ball of radius 2ε centred at the origin are contained in U.

Let $\chi : M \to \mathbb{R}$ be a smooth bump function compactly supported on some small neighbourhood of p. Then, for some small $\delta > 0$, we can define $F = f - \delta\chi$ such that

- $F(p) < c - \varepsilon$,
- $F^{-1}((-\infty, c + \varepsilon]) = M^{c+\varepsilon}$,
- F and f have the same critical points with the same index.

Since $F^{-1}([c - \varepsilon, c + \varepsilon])$ contains no critical points we can use Theorem 2.11 above to see that $F^{-1}((-\infty, c - \varepsilon])$ is diffeomorphic to, and a deformation retract of, $M^{c+\varepsilon}$.

Now, $F^{-1}((-\infty, c - \varepsilon])$ is the union of $M^{c-\varepsilon}$, drawn in purple in in Figure 2.5, and a small neighbourhood of p, drawn in green.

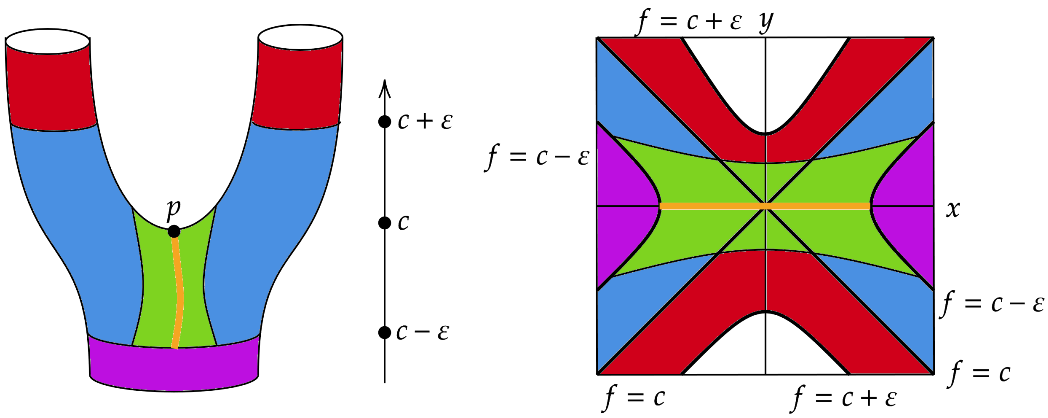


Figure 2.5: The handle attachment in the Morse neighbourhood.

We write

$$H = \overline{F^{-1}((-\infty, c - \varepsilon]) \setminus M^{c-\varepsilon}}$$

for the closure of the neighbourhood of p that is attached. This H is a λ-handle with the core C, drawn in orange, given by

$$C = \{\|x\|^2 \leqslant \varepsilon, \quad y = 0\} \tag{2.1}$$

in the Morse coordinates.

We can construct a deformation retract from $M^{c-\varepsilon} \cup H$ to $M^{c-\varepsilon} \cup C$ as indicated in Figure 2.6. Note that C is a λ-disc with radius ε that is attached along $C \cap M^{c-\varepsilon} = \partial C$, thus $C \cup M^{c-\varepsilon}$ is indeed $M^{c-\varepsilon}$ with a λ-cell attached. $\square$

By inductively applying Theorem 2.13 we get the following theorem, the full details of the proof can be found in Section 2.2 of [Nic11].

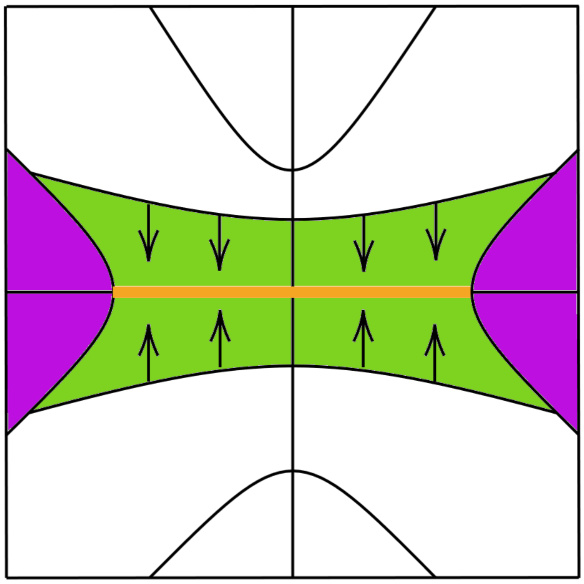


Figure 2.6: Retracting the handle to its core.

Theorem 2.14. *If M is compact and $f : M \to \mathbb{R}$ is Morse, then M has a handlebody structure determined by f, with one handle of index λ_i for every critical point p_i of index λ_i.*

In particular this yields the following corollary.

Corollary 2.15. *If $f : M \to \mathbb{R}$ is an exhaustive Morse function, then M is homotopy equivalent to a CW-complex that has exactly one λ-cell for every critical point of f of index λ.*

Remark 2.16. The handle decomposition of Theorem 2.14 is iterative, it is build up one handle at the time using the previously produced handlebody. There is no combinatorial data describing the construction all at once. One of the goals of this thesis is to address this issue via arborealization, where we are able to give combinatorial data that encodes all handles at once. △

2.4 Limitations to critical point theory

Theorem 2.11 tells us that any exhaustive Morse function on a smooth manifold M gives a handlebody decomposition of M, where every critical point corresponds to a handle with the same index. It is, however, important to stress the attaching maps are induced by the flow of the gradient vector field, meaning the attaching of the handles is also determined by the behaviour of the gradient far away from critical points and dependent on the chosen Riemannian metric. Thus the data of the critical points and their index of a Morse function is not enough to recover M as a handlebody, since this amounts to only giving the index of every handle.

We illustrate this using the following well-known theorem due to Reeb, which can be shown to be a corollary of Theorem 2.11 and the Morse Lemma.

Theorem 2.17 (Reeb)**.** *If M is a compact n-manifold and $f : M \to \mathbb{R}$ a Morse function on M with exactly two critical points, then M is homeomorphic to a sphere.*

Proof. The two critical points $p, q \in M$ must be the minimum and maximum of f, we write $f(p) = a$ and $f(q) = b$, $a < b$. Using the Morse Lemma we must have that $f^{-1}([a, a+\varepsilon])$ and $f^{-1}([b-\varepsilon, b])$ are both n-cells if $\varepsilon > 0$ is small enough. From Theorem 2.11 we know that $M^{a+\varepsilon}$ and $M^{b-\varepsilon}$ are diffeomorphic, thus M is the union of two closed n-cells that are glued along their common boundary under a diffeomorphism.

We conclude that M is homeomorphic to $\mathbb{S}^n$. □

However, this M is not necessarily diffeomorphic to the standard $\mathbb{S}^n$. As famously shown by Milnor in [Mil56], there are manifolds that are homeomorphic but not diffeomorphic to the 7-sphere $\mathbb{S}^7$. In his paper Milnor shows that the manifolds he produces are indeed homeomorphic to $\mathbb{S}^7$ by showing they are compact and constructing a Morse function that has exactly two critical points on them. In fact, some exotic 7-spheres can be constructed by gluing two 7-discs along their boundary via a diffeomorphism that "twists" the boundary, as shown by Milnor in [Mil59].

In general only giving the data of the index of every handle is not even enough to determine the topological type of the resulting handlebody. This can be illustrated by considering a handle attachment to $\mathbb{D}^4$ along a knot $K \cong \mathbb{S}^1 \subset \partial\mathbb{D}^4 = \mathbb{S}^3$ Such a handle attachment corresponds to a so called *Dehn surgery* on its boundary $\mathbb{S}^3$ along K, the details of which can be found in Section 2.1 of [Nic11] and Section 9.G of [Rol76]. These Dehn surgeries can produce vastly different manifolds, depending on the knot K and the framing of its normal bundle.

Any knot $K \subset \mathbb{S}^3$ bounds a smoothly embedded and orientable Riemann surface $\Sigma \subset \mathbb{S}^3$. The interior pointing unit normal along K is a nowhere vanishing section of the normal bundle νK, thus it gives a framing of the bundle which we call the *canonical framing* of the knot K. Note that such a framing gives a diffeomorphism between the tubular neighbourhood T of the knot and the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$. We call $\lambda = \mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times \mathbb{D}^2$ the *longitude* of K and the boundary $\partial\mathbb{D}^2 \times \{1\}$ of a fiber of the normal disc bundle the *meridian* of K.

Any other framing on νK corresponds to a nowhere vanishing section of νK, and thus traces a curve J on $\partial T \cong \mathbb{S}^1 \times \partial\mathbb{D}^2$. In $H_1(\mathbb{S}^1 \times \partial\mathbb{D}^2, \mathbb{Z})$ we can write $[J]$ as

$$[J] = [\lambda] + p[\mu],$$

The integer p is called the *framing coefficient*.

The manifold obtained by attaching a handle to $\mathbb{S}^3$ along a knot K is completely determined by K and the framing coefficient. For instance, attaching a handle along the trivial knot with the canonical framing produces a 4-dimensional manifold with as boundary $\mathbb{S}^2 \times \mathbb{S}^1$, while attaching the handle along the trivial knot with framing coefficient 1 produces $\mathbb{D}^4$ again.

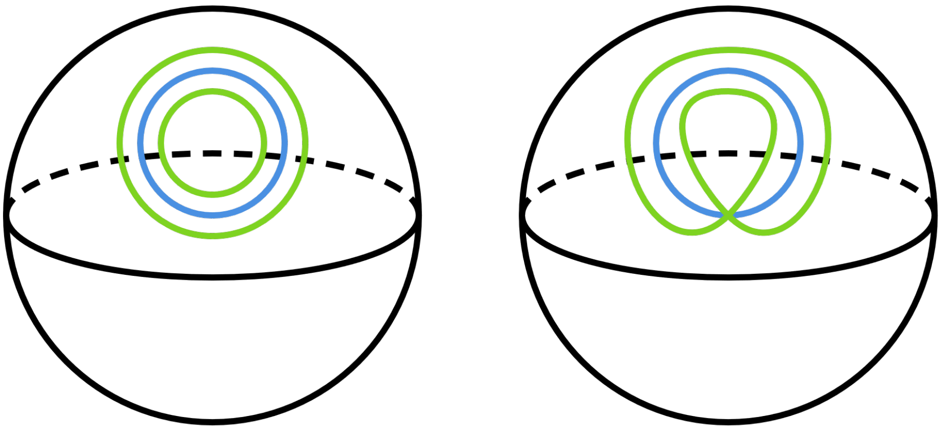


Figure 2.7: Embeddings of the trivial knot into $\mathbb{S}^3$ with different framing coefficients, note that the figure is necessarily inaccurate.

Another example is the attachment of a handle along the trefoil knot with framing coefficient 1, which produces the *Poincaré homology sphere*, which is a 3-manifold with the same homology as the 3-sphere and fundamental group of order 120.

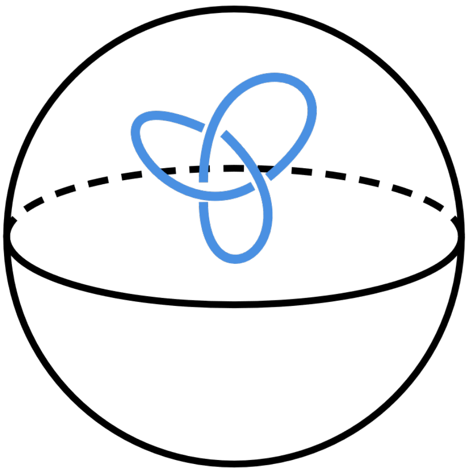


Figure 2.8: Embedding of the trefoil knot into $\mathbb{S}^3$, note that the figure is necessarily inaccurate.

Thus, in general, a manifold is not determined by the handles in its handlebody decomposition, the attaching maps have to be specified. These attaching maps are determined by the gradient. Therefore we will shift our focus from Morse *functions* to Morse *dynamics*, i.e. the dynamics of gradients of Morse functions, in the following chapter.

Chapter 3

Morse Dynamics

In this chapter we study the gradients of Morse functions and how the flow of these gradients determine the manifold. In particular we discuss that, for a compact manifold M equipped with a Riemannian metric and fixed Morse function $f : M \to \mathbb{R}$, the manifold can be decomposed into the stable manifolds of the flow of the gradient of f.

The stable manifolds give a stratification of M by cells, i.e. disjoint open discs of varying dimensions. In this sense they resemble a cellular decomposition, as already remarked in 1949 by Thom in [Tho49]. However, the cells do not generally form a cell-complex in the usual sense because the closure of a cell is generally not obtained by adding lower index cells. By imposing a transversality condition on the flow and compatibility of the vector field with the Morse models, the stratification becomes more well-behaved.

We begin Section 3.1 by defining *gradient-like* vector fields, which can be define without picking a Riemannian metric, and the *(un)stable sets* of gradient-like vector fields. In Subsection 3.1.1 we take a short deviation and consider when and how the theory for compact manifolds can be adapted to non-compact manifolds. In particular we see that a certain class of manifolds, which we call *finite type X-convex*, are completely determined by a compact neighbourhood of their stable manifolds. We conclude this chapter with Section 3.2, where we introduce the *Smale condition*, which is a transversality condition on the stable and unstable manifolds. This condition ensures that the closure of any cell is given by attaching lower dimensional cells. We conclude with the definition of *Morse vector fields*, which are gradient-like vector fields that are compatible with the Morse structure on Morse neighbourhoods. In the following chapter we will show that the stable manifolds of Morse vector fields satisfying the Smale condition form a cell-complex.

3.1 (Un)stable manifolds

We want to develop a theory that is not dependent on the chosen Riemannian metric. A fixed Morse function $f : M \to \mathbb{R}$ has many different gradients, corresponding to different choices of metric on M. It can be difficult to check whether a given vector field X is the gradient of f with respect to *any* metric. Therefore we wish to formulate a necessary condition for X to be a gradient that can be easily checked without knowing the Riemannian metric.

We observe that for any metric g on M have

- $\text{grad}f(f) = g(\text{grad}f, \text{grad}f) > 0$ on $M \setminus \text{Crit}(f)$,
- $\text{grad} = 0$ on $\text{Crit}(f)$.

This motivates the following definition.

Definition 3.1. A vector field $X \in \mathfrak{X}(M)$ is called *gradient-like* for a smooth map $f : M \to \mathbb{R}$ if

- $X(f) > 0$ on $M \setminus \text{Crit}(f)$,
- $X = 0$ on $\text{Crit}(f)$.

△

Note that if X is gradient-like for $f : M \to \mathbb{R}$, the function f must increase along any flow which is not a fixed point and thus gradient-like dynamical systems can not have periodic orbits. In particular, if M is compact and X is gradient-like, any point $x \in M \setminus \text{Crit}(f)$ lies on a unique 1-dimensional manifold "starting" at a critical point p and "ending" at a critical point q.

We capture this behaviour in the following definition.

Definition 3.2. Let X be a gradient-like vector field for a smooth function $f : M \to \mathbb{R}$ and let $p \in \text{Crit}(f)$. The *stable set* of p is defined as

$$\text{Stab}(p) = \{x \in M \mid \lim_{t\to\infty} \varphi^t(x) = p\}$$

and the *unstable set* of p is defined as

$$\text{Unstab}(p) = \{x \in M \mid \lim_{t\to-\infty} \varphi^t(x) = p\}$$

where $\varphi^t : M \to M$ is the flow along X at time t. △

For any smooth vector field $X \in \mathfrak{X}(M)$, at $p \in M$ where X vanishes, the differential d_pX splits the tangent space T_pM as $T_pM = E_p^- \oplus E_p^+ \oplus E_p^0$, where $E^{-/+/0}$ are generalized eigenspaces of T_pM with $-/+/0$ real part. A zero p of the vector field is called *hyperbolic* if $E_p^0 = \{0\}$, a classic result from the study of dynamical systems, see for instance Section 5.10 of [Rob94], is that the (un)stable sets of hyperbolic zeros are manifolds.

Theorem 3.3 (Stable Manifold Theorem)**.** *Let M be an n-dimensional manifold, let $X \in \mathfrak{X}(M)$ be a smooth vector field and let $p \in M$ be a hyperbolic zero of X. Then the stable and unstable sets are the unique smooth invariant manifolds tangent to respectively E_p^- and E_p^+.*

Remark 3.4. Let M be an n-dimensional manifold, let $X \in \mathfrak{X}(M)$ be a gradient vector field for a Morse function $f : M \to \mathbb{R}$ and let $p \in \text{Crit}(f)$ with index λ. Then the linearization D_pX splits the tangent space T_pM at a critical point p of M as $T_pM = E_p^- \oplus E_p^+$, meaning the zeros of X

are all hyperbolic. The stable manifold of p is diffeomorphic to the open disc $\mathbb{D}^\lambda$, the unstable manifold of p is diffeomorphic to the open disc $\mathbb{D}^{n-\lambda}$. △

For a compact M with Morse function $f : M \to \mathbb{R}$ the collection of stable manifolds $\{\mathrm{Stab}(p)\}_{p\in\mathrm{Crit}(f)}$ gives a decomposition of M into disjoint cells, likewise the collection of unstable manifolds $\{\mathrm{Unstab}(p)\}_{p\in\mathrm{Crit}(f)}$ also gives a decomposition of M.

However, in general this decomposition does not have the structure of a cell-complex, since the closure of cells is not necessarily obtained by adding lower dimensional cells. We see this in the example of the height function on the 2-torus in Section 2.1, of which the gradient vector field is drawn in Figure 3.1.

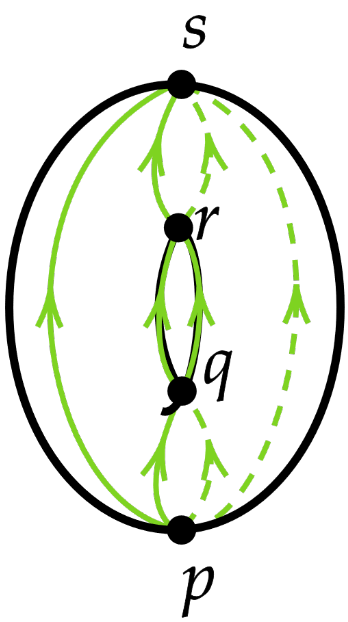


Figure 3.1: A gradient vector field for the Morse function on the 2-torus.

The closure of the 1-cell $\mathrm{Stab}(r)$ is given by adding the point q, which lies on the 1-cell $\mathrm{Stab}(q)$. Thus the closure of a 1-cell is not obtained by adding 0-cells but instead by adding a subset of a 1-cell, meaning the decomposition given by the stable manifolds is not a cell-complex.

3.1.1 Non-compact manifolds

As discussed above, a gradient vector field of a Morse function on a compact manifold M gives a decomposition of M into its stable manifolds. If M is not compact, we do not always have such a decomposition. Not every flow has to settle in a critical point, instead a flow can also go off to infinity. Thus not every point has to be contained in a stable manifold.

However, under certain conditions a non-compact manifold equipped with a vector field can be completely determined, up to diffeomorphism, by a neighbourhood of its stable manifolds.

Definition 3.5. An n-dimensional manifold M is said to be X-*convex* for a complete vector field $X \in \mathfrak{X}(M)$ if there is an exhaustion by compact domains $M = \cup_{k=1}^{\infty} M^k$ such that X is outwardly transverse to the boundary of each M^k.

We call the pair (M, X) a *convex structure*. △

Note that closed manifolds equiped with a complete vector field X are X-convex.

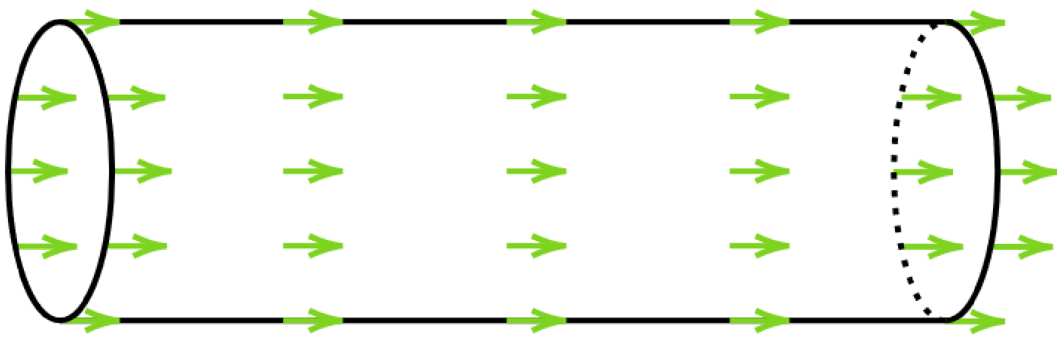


Figure 3.2: This infinite cylinder is not X-convex, since there is no compact domain such that the vector field is outwardly transverse to the boundary.

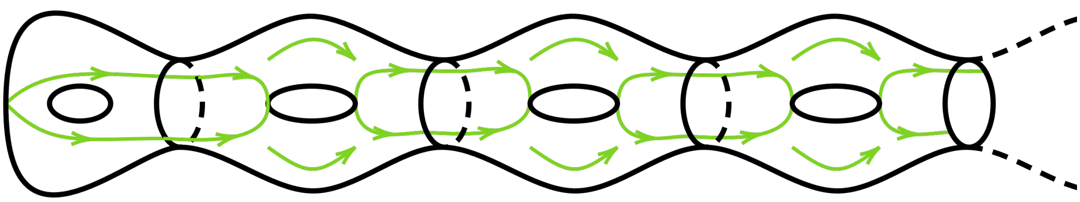


Figure 3.3: This infinite genus surface is X-convex.

Definition 3.6. The *skeleton* of an X-convex manifold M is defined as

$$\text{Skel}(M, X) = \cup_{k=1}^{\infty} \cap_{t>0} \varphi^{-t}(M^k)$$

where $\varphi^t : M \to M$ denotes the flow along X at time t, i.e. the skeleton is the attractor of the backwards flow of X.

We say (M, X) is *finite type* if its skeleton is compact. △

The X-convex infinite genus surface indicated in Figure 3.3 is not finite type, there is no compact domain containing all its critical points and thus its skeleton can not be compact.

The X-convex punctured torus as indicated in Figure 3.4 is finite type

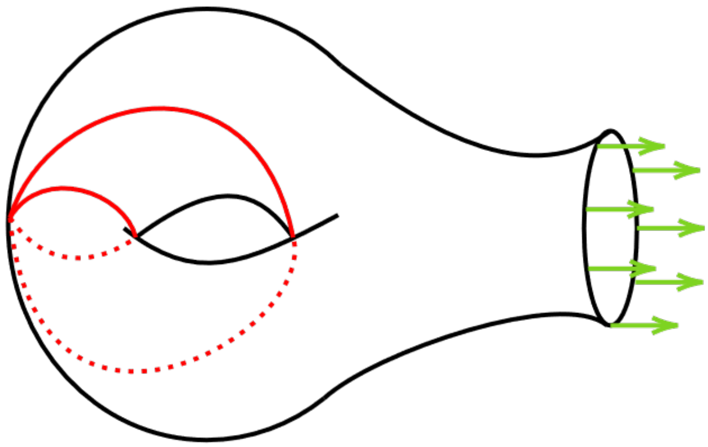


Figure 3.4: An X-convex and finite type surface with its skeleton indicated in red.

Remark 3.7. If X is the gradient of a Morse function $f : M \to \mathbb{R}$, the skeleton is the union of all stable manifolds

$$\text{Skel}(M, X) = \bigcup_{p \in \text{Crit}(f)} \text{Stab}(p).$$

$\triangle$

We always assume that that (M, X) is finite type. Note that, since (M, X) is finite type, there is a compact domain $W \subset M$ such that $\text{Skel}(M, X) \subset W$ and X is outwardly transverse to the boundary ∂W.

Definition 3.8. Let (M, X) be a finite type convex structure, we call any compact $W \subset M$ such that $\text{Skel}(M, X) \subset W$ and X is outwardly transverse to the boundary ∂W a *defining domain* of M. $\triangle$

Informally, one could say that a defining domain $W \subset M$ captures all of the interesting (smooth) topology of M. We make this precise in the following way.

Lemma 3.9. *Let (M, X) be a finite type convex structure and $W \subset M$ a defining domain. Then M is diffeomorphic to the convex manifold*

$$\widehat{M} := W \cup (\partial W \times [0, \infty))$$

obtained by attaching a cylindrical end to ∂W and extending X as $\widehat{X} = e^s X|_{\partial W}$.

Proof. We denote the flow along X at time t by φ^t, and define the map $\Phi : \widehat{M} \to M$ as the identity on W and as

$$\Phi(x, t) = \varphi^t(x)$$

on $\partial W \times [0, \infty)$. Note that Φ is smooth, we show that Φ is a diffeomorphism.

First we show Φ is a bijection. Let $p \in M$, if $p \in W$ then $\Phi(p) = p$. If $x \in M \setminus W$, then the flow through p must pass trough a unique point $x \in \partial W$ at some time $-t < 0$ by the X-convexity condition. Thus, there is a $(x, t) \in \partial W \times [0, \infty)$ such that $\Phi(x, t) = p$. We see that Φ is surjective. The injectivity of Φ follows directedly from the and the uniqueness of flow; if $\Phi(x) = p \in W$ it must be that $x = p$ by the X-convexity condition and if $p \notin W$ there must be a unique $(x, t) \in \partial W \times [0, \infty)$ such that $\Phi(x, t) = p$ by the existence and uniqueness of flow.

Now we show that Φ is an immersion, note that Φ is the identity and thus an immersion on W. For any $t \geqslant 0$, the map $\varphi^t : \partial W \to \varphi^t(\partial W)$ is a diffeomorphism and $\frac{\partial}{\partial t}\varphi^t = X \pitchfork \partial W$. Thus we see that Φ is indeed an immersion, we conclude that Φ is a bijection. □

3.2 Smale vector fields

The following condition assures that the closure of each cell is given by adding lower dimensional cell.

Definition 3.10. A gradient vector field X of a Morse function $f : M \to \mathbb{R}$ is called *Smale* if for every pair of critical points $p, q \in \mathrm{Crit}(f)$ we have

$$\mathrm{Stab}(p) \pitchfork \mathrm{Unstab}(q).$$

△

If M is a closed smooth n-dimensional manifold and X a Smale gradient vector field for a Morse function $f : M \to \mathbb{R}$ with an index λ critical point p, we know that the closure $\overline{\mathrm{Stab}(p)}$ of $\mathrm{Stab}(p)$ is obtained by adding the union of the stable manifolds of smaller index, see for instance Section 3.2 of [ADE13]. Furthermore, as shown in Chapter 4 of [Nic11], the stratification of M by the stable manifolds satisfies the Whitney conditions.

That the closure of every stable manifold is obtained by adding stable manifolds of lower index can be understood in the following way. The transversality condition gives that, if $\mathrm{Stab}(p) \cap \mathrm{Unstab}(q) \neq \emptyset$, we have

$$\dim(\mathrm{Stab}(p) \cap \mathrm{Unstab}(q)) = \dim(\mathrm{Stab}(p)) + \dim(\mathrm{Unstab}(q)) - n.$$

If we write λ_p, λ_q for the index of p and q respectively, we get

$$\dim(\mathrm{Stab}(p) \cap \mathrm{Unstab}(q)) = \lambda_p + n - \lambda_q - n = \lambda_p - \lambda_q.$$

Now, if $x \in \mathrm{Stab}(p) \cap \mathrm{Unstab}(q)$, we must have that the entire integral curve through x must be contained in $\mathrm{Stab}(p) \cap \mathrm{Unstab}(q)$. Thus if $\mathrm{Stab}(p) \cap \mathrm{Unstab}(q) \neq \emptyset$ we must have that $\dim(\mathrm{Stab}(p) \cap \mathrm{Unstab}(q)) \geqslant 1$.

Therefore we see that if $\mathrm{Stab}(p) \cap \mathrm{Unstab}(q) \neq \emptyset$ we must have

$$\lambda_p \geqslant \lambda_1 + 1.$$

Note that the vector field as indicated in Figure 3.1 is not Smale, the intersection

$$\mathrm{Stab}(r) \cap \mathrm{Unstab}(q)$$

is 1-dimensional and thus not transverse. In general, it is difficult to explicitly find a metric such that a gradient is Smale.

However, as shown by Smale, see for instance Theorem A of [Sma61], gradient vector fields of Morse functions are generically Smale. Indeed, a small perturbation yields the Smale vector field indicated in Figure 3.5.

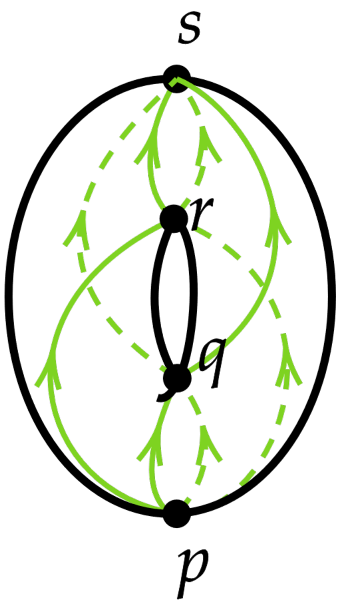


Figure 3.5: A Smale gradient vector field for the Morse function on the 2-torus.

Even though the stable manifolds of a Smale system give a well-behaving cell decomposition, in the sense that the closure $\overline{\mathrm{Stab}(p)}$ of a cell is given by attaching cells of lower dimension and the stratification is Whitney, we can say little about the exact way the stable manifolds of lower index attach to $\mathrm{Stab}(p)$ for general Smale gradients. In particular we de not have specified local models around the singularities of the stratification. However, as we will show in the following chapter, the local structure of $\overline{\mathrm{Stab}(p)}$ is relatively simple if X is Smale and we demand the following compatibility with the Morse model.

Definition 3.11. We say a smooth vector field X on M is *Morse* if it is gradient-like for a Morse function $f : M \to \mathbb{R}$ such that for every critical point there is a Morse neighbourhood on which X corresponds to the gradient of f with respect to the canonical metric on $\mathbb{R}^n$.

We say a smooth vector field X on M is *Morse-Smale* if it is Morse and Smale. △

Remark 3.12. Some texts use the term *gradient-like* to refer to our notion of a *Morse vector field*, i.e. compatibility with the Morse model is imposed in the definition of gradient-like. We make the distinction between gradient-like and Morse vector fields because we later encounter a wider range of gradient-like vector fields, which are compatible with different local models of certain functions satisfying a more general non-degeneracy condition than the Morse condition. △

Note that there are Morse vector fields for every Morse function. One can construct a metric on any manifold M by taking a locally finite cover by charts, pulling back the canonical metric on every chart and gluing these together using a partition of unity. By including the Morse neighbourhoods in this cover and taking an appropriate partition of unity we obtain a metric on M such that the gradient of f is a Morse vector field.

The converse is also true for closed M, Theorem B of [Sma61] asserts that if $X \in \mathfrak{X}(M)$ is Morse, then there is a Morse function $f : M \to \mathbb{R}$ and Riemannian metric g on M such that X is the gradient of f with respect to g. Thus, being Morse is equivalent to being gradient of a Morse function with respect to a certain Riemannian metric.

However, we opt to formulate the definition of Morse vector fields as gradient-like vector fields compatible with the local model because we often work in a set-up where a smooth manifold M (without specified metric) and vector field $X \in \mathfrak{X}(M)$ are given but the Morse function is not specified. Checking the global condition, i.e. checking if for a Morse function f there is any metric such that X is the gradient of f with respect to the metric, is difficult. Comparatively the gradient-like condition is easy to check.

Chapter 4

Submanifolds with Conical Singularities

In this chapter will show, following F. Laudenbach in [Lau92], that the closures of stable manifolds of a Morse-Smale vector field have a relatively simple conical structure.

Consider a closed smooth n-dimensional manifold M and let $X \in \mathfrak{X}(M)$ be a Morse vector field with associated Morse function $f : M \to \mathbb{R}$. Around any critical point $p \in \mathrm{Crit}(f)$ we have a Morse neighbourhood U with Morse coordinates (x, y) such that, on U, we have that

$$f(x, y) = f(p) - \|x\|^2 + \|y\|^2,$$

and X is the gradient of f with respect to the canonical metric on $\mathbb{R}^n$. This situation is shown in Figure 4.1. Then we can see that

$$\mathrm{Stab}(p) \cap U = \{(x, 0) \in U\} \qquad \text{and} \qquad \mathrm{Unstab}(p) \cap U = \{(0, y) \in U\}.$$

Furthermore, as can be seen in Figure 4.2, the vector field is radial in both the stable and unstable manifold on U. In particular, X points inwards, i.e. towards the critical point, in the stable manifold, and X points outwards in the unstable manifold.

This radial structure tells us that if the stable manifold $\mathrm{Stab}(q)$ of another critical point q intersects the unstable manifold of p in its Morse neighbourhood, its closure $\overline{\mathrm{Stab}(q)}$ is locally given by taking the cone of the intersection $\mathrm{Stab}(q) \cap \partial U$ to p. This is illustrated in Figure 4.3.

We begin this chapter by introducing the class of *conical singularities* and stratified spaces whose singularities are all of the conical type, which we call *submanifolds with conical singularities*. Some examples are given and certain important properties of submanifolds with conical singularities are stated and proven. Then, in Section 4.2, we show that stable sets of Morse-Smale vector fields indeed form submanifolds with conical singularities. We conclude this

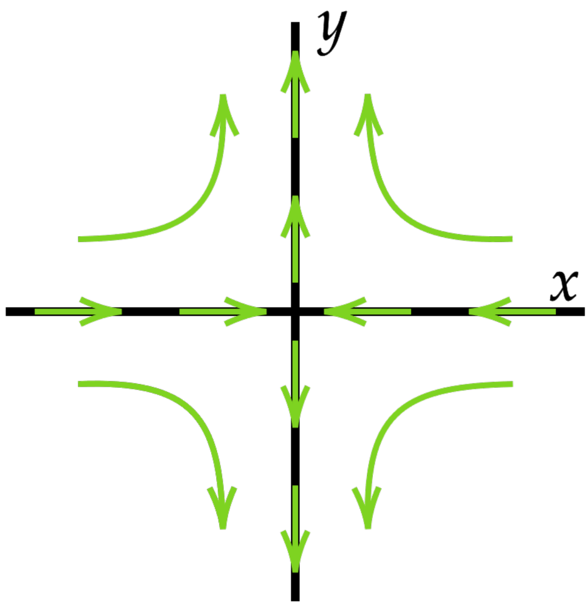


Figure 4.1: A Morse vector field on an appropriate Morse neighbourhood with Morse coordinates (x, y), the stable manifold coincides with the x-axis and the unstable manifold coincides with the y-axis.

chapter with Section 4.3, where we discuss the limitations of this description of stable manifolds as submanifolds with conical singularities if our goal is to recover the smooth type of a manifold from (the neighbourhoods of) its critical points.

Throughout this chapter M is a closed smooth n-dimensional manifold and $X \in \mathfrak{X}(M)$ is a Morse-Smale vector field with associated Morse function $f : M \to \mathbb{R}$ with distinct critical values.

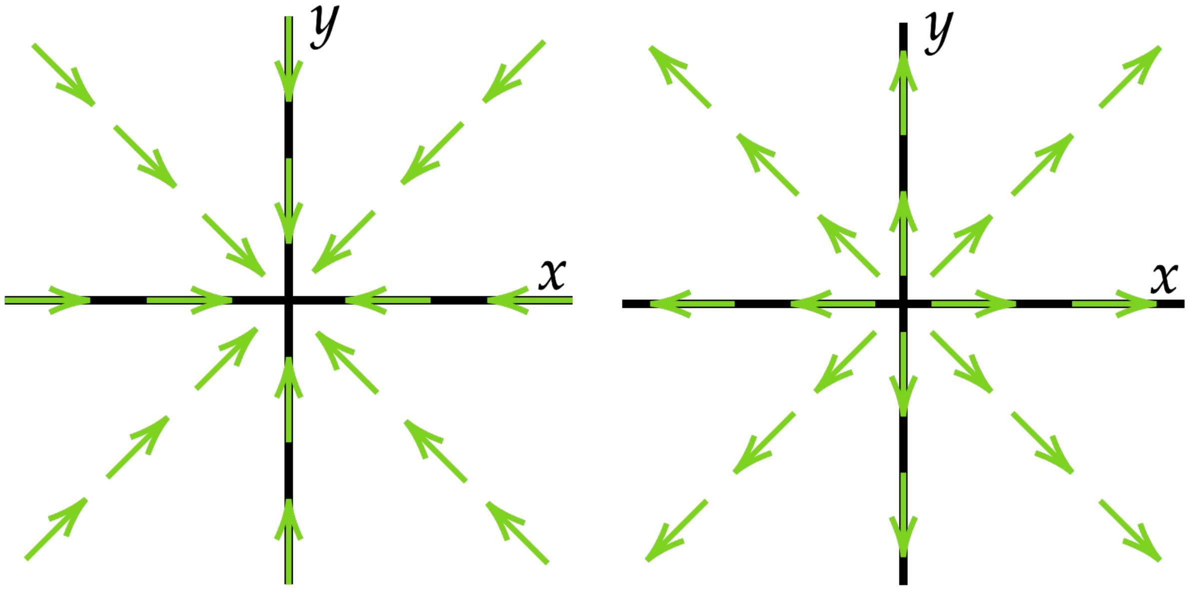
4.1 Submanifolds with conical singularities

Definition 4.1. *Conical singularities* in n-manifolds form the smallest class Con_k^n of germs of closed stratified subsets in n-dimensional smooth manifolds such that the following properties are satisfied:

(i) (Base case) Con_0^n contains $pt = \mathbb{R}^0 \subset \mathbb{R}^n$.

(ii) (Invariance) Con_k^n is invariant with respect to diffeomorphisms.

(iii) (Stabilization) If $\Sigma = (\Sigma_k, ..., \Sigma_0) \subset N$ is in Con_k^n with $k < n$, then the product $\mathbb{D} \times \Sigma = \mathbb{D} \times (\Sigma_k, ..., \Sigma_0, \emptyset)$ is in Con_{k+1}^n.



(a) The radial vector field in the stable manifold. (b) The radial vector field in the unstable manifold.

Figure 4.2: A Morse vector field along to the stable and unstable manifold on the Morse neighbourhood of a critical point.

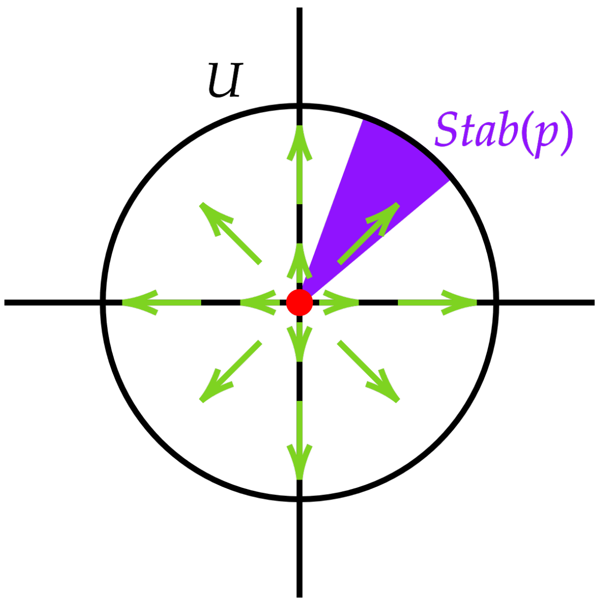


Figure 4.3: The stable manifold Stab(q) is intersects Unstab(p), its closure is locally the cone of its intersection with the boundary of the Morse neighbourhood.

(iv) (Cones) If $\Sigma' = (\Sigma'_{k-1}, ..., \Sigma'_0) \subset \mathbb{S}^{n-1}$ is in Con^{n-1}_{k-1}, then

$$(\mathbb{D}, c\Sigma'_{k-1}, ..., c\Sigma'_0) \subset \mathbb{D}^n$$

is in Con^n_k, here $c\Sigma'_i$ denotes the cone on Σ'_i with respect to the linear structure of $\mathbb{D}^n$.

△

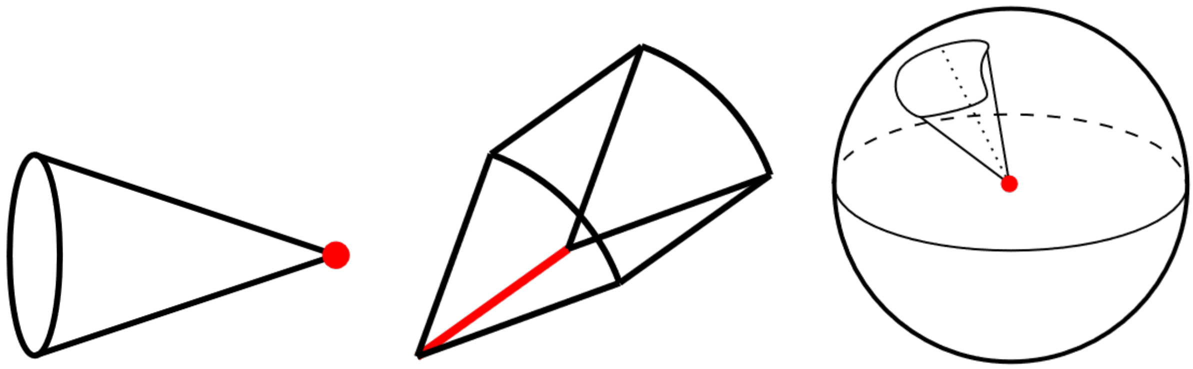


Figure 4.4: Examples of conical singularities.

Definition 4.2. We say a stratified set $\Sigma = (\Sigma_k, ..., \Sigma_0) \subset M$ is a *submanifold with conical singularities* (abridged: *smcs*) of dimension k if all of its singularities are conical singularities. △

Remark 4.3. A smcs of dimension $k = 0$ in M is a set of discrete points in M. A stratified set $\Sigma = (\Sigma_k, ..., \Sigma_0)$ in M is a smcs of dimension k precisely if the following conditions are met:

(i) (Stability) For every $p \in \Sigma_i \setminus \Sigma_{i-1}$ there is a neighbourhood V of p diffeomorphic to a product of discs $\mathbb{D}^i \times \mathbb{D}^{n-i}$ and a smcs $T = (T_{k-i}, ..., T_0)$ such that

$$V \cap \Sigma = \mathbb{D}^i \times (T_{k-i}, ..., T_0, \emptyset, ..., \emptyset).$$

(ii) (Cones) If $p \in \Sigma_0$ there is an closed C^1 n-ball B centred at p such that

$$\Sigma' = \Sigma \cap \partial B = \Sigma \cap \mathbb{S}^{n-1} \text{ is a smcs of dimension } k-1 \text{ in } \mathbb{S}^{n-1}$$

and

$$(B, B \cap \Sigma_k, ..., B \cap \Sigma_0) = (B, c\Sigma'_k, ..., c\Sigma'_1)$$

where $c\Sigma'_i$ denotes the cone on Σ'_i with respect to the linear structure of the parametrized ball B.

△

Example 4.4. Every manifold M with (possibly empty) boundary ∂M is a smcs

$$(M, \partial M, \emptyset, ..., \emptyset).$$

△

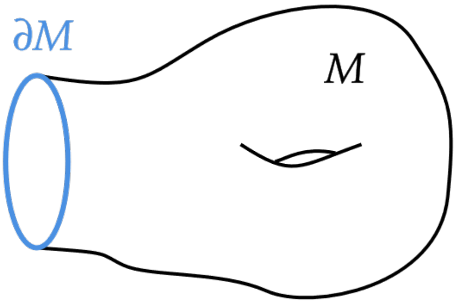


Figure 4.5: A manifold with boundary.

Example 4.5. The stratification of $\mathbb{S}^n$ given by lower dimensional spheres $\mathbb{S}^0 \subset \mathbb{S}^1 \subset ... \subset \mathbb{S}^n$ is a smcs. △

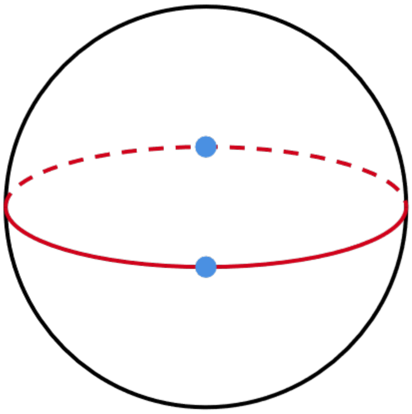


Figure 4.6: The stratification of $\mathbb{S}^2$ given by $\mathbb{S}^0 \subset \mathbb{S}^1 \subset \mathbb{S}^2$.

Nonexample 4.6. The infinite spiral

$$S = \left\{ \left(t \cos\left(\frac{1}{t}\right), t \sin\left(\frac{1}{t}\right) \mid \quad t \in (0, 1] \right) \right\} \cup \{0\} \subset \mathbb{R}^2$$

is a stratified set $(S, \{0\})$ that does not satisfy the cone property at the origin in any chart There is no ambient diffeomorphism from the neighbourhood of the origin to the closed half line in $\mathbb{R}^2$ that is smooth in the origin.

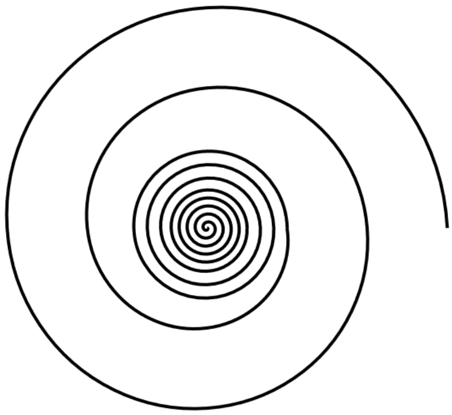


Figure 4.7: The infinite spiral.

It is a well-known fact the transverse intersection of two submanifolds is again a manifold. Likewise the transverse intersection of a submanifold and an smcs is again an smcs, as formulated in the following lemma.

Lemma 4.7. *Let N be a n-dimensional manifold with a smcs $\Sigma = (\Sigma_k, ..., \Sigma_0)$ of dimension k. Suppose $S \subset N$ is a submanifold of codimension q that is transverse to Σ, then*

$$(S \cap \Sigma_k, ..., S \cap \Sigma_q)$$

is a smscs of dimension $k - q$ in S.

If S furthermore has a product neighbourhood $\mathbb{D}^q \times S$ in N with $S = \{0\} \times S$, there is a germ of diffeomorphisms $H : \mathbb{D}^q \times S \to \mathbb{D}^q \times S$ along $\{0\} \times S$ commuting with the projection onto $\mathbb{D}^q$ such that $H(\Sigma) = \mathbb{D}^q \times (\Sigma \cap S)$.

Proof. We begin by proving the first assertion. By transversality we immediately see that $S \cap \Sigma_i = \emptyset$ if $i < q$.

Let $p \in S \cap \Sigma_l$ with $l \geqslant q$, by the stability property there is a chart U around p such that

$$U \cap \Sigma = \mathbb{D}^l \times (T_{k-l}, ..., T_0, \emptyset, ..., \emptyset) = \mathbb{D}^q \times \mathbb{D}^{l-q} \times T$$

where T is a smcs of dimension $k - l$ in $\mathbb{D}^{n-l}$, note that $T_0 = \{p\}$. By transversality there is a projection $\pi_U : \mathbb{D}^q \times \mathbb{D}^{n-q} \to \mathbb{D}^{n-q}$ that induces a diffeomorphism $\varphi_U : U \cap S \to \mathbb{D}^{n-q}$ such that in the corresponding chart on S we have $S \cap \Sigma = \mathbb{D}^{l-q} \times T$. Thus $S \cap \Sigma$ is smcs of dimension $k - q$ in S.

We now assume S has a product neighbourhood $\mathbb{D}^q \times S$ in N with $S = \{0\} \times S$.

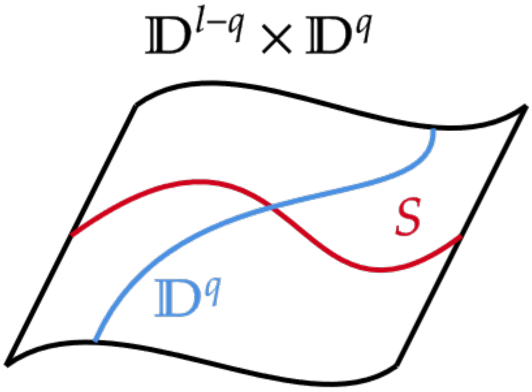


Figure 4.8: The product neighbourhood $\mathbb{D}^q \times S$ can locally be identified with $\mathbb{D}^q \times \mathbb{D}^{l-q}$.

Locally we have projections

$$\varphi_U^{-1} \circ \pi_U : \mathbb{D}^q \times \mathbb{D}^{n-q} \to U \cap S$$

"along the stratification" mapping $U \cap \Sigma$ to $S \cap \Sigma$.

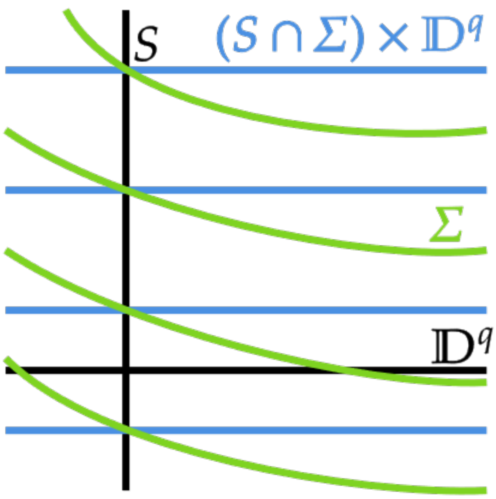


Figure 4.9: The local projection along the stratification.

Note that every local projections $\varphi_U^{-1} \circ \pi_U$ is the identity on $\{0\} \times (S \cap U)$. These local projections can be patched up to a global projection $\pi'' : V \to S$ defined on small tube V around S using a partition of unity.

Furthermore, we have a projection $\pi' : V \to \mathbb{D}^q$, given by the trivialization of the normal bundle.

Now $H = (\pi', \pi'') : V \to \mathbb{D}^q \times S$ is a germ of diffeomorphisms of $\mathbb{D}^q \times S$ along $\{0\} \times S$ and $H(\Sigma \cap V) \subset \mathbb{D}^q \times (\Sigma \cap S)$. □

4.2 Stable manifolds as submanifolds with conical singularities

In this section we show that the closure of a stable manifold of a Morse-Smale system has the structure of a smcs.

Theorem 4.8. *Let X be a Morse-Smale vector field for $f : M \to \mathbb{R}$. If p is a critical point of index λ, then $(\overline{Stab(p)}, \overline{Stab(p)} \setminus Stab(p))$ is a smcs of dimension λ where $\overline{Stab(p)} \setminus Stab(p)$ is stratified by stable manifolds of critical points of index strictly less than λ.*

Proof. Let p be an index λ critical point of f. For $a \in \mathbb{R}$ define

$$S_a(p) = \overline{\mathrm{Stab}(p)} \cap \{f = a\}.$$

Now, if $a < f(p)$ such that $[a, f(p)]$ does not contain any critical values besides $f(p)$, $S_a(p)$ is a sphere. As a decreases $S_a(p)$ remains a sphere until a coincides with a critical value of f at some critical point q of index μ; for $\varepsilon > 0$, $S_{f(q)-\varepsilon}$ is no longer a smooth manifold. Recall that, because X is Morse-Smale, we know $\mu < \lambda$.

Let $(x, y) \in \mathbb{R}^\mu \times \mathbb{R}^{n-\mu}$ be Morse coordinates on a neighbourhood U of q such that on U

$$f(x, y) = f(q) - \|x\|^2 + \|y\|^2$$

and X is the gradient of f with respect to the Euclidean metric on U. We define the level sets

$$V_- = \{f = f(q) - \varepsilon\}, \qquad V_+ = \{f = f(q) + \varepsilon\}$$

and remark that $V_- \cong \mathbb{S}^{\mu-1} \times \mathbb{D}^{n-\mu}$ under the correspondence $(u, \theta v) \leftrightarrow (u \cosh\theta, v \sinh\theta)$ and $V_+ \cong \mathbb{D}^\mu \times \mathbb{S}^{n-\mu-1}$ under the correspondence $(u, \theta v) \leftrightarrow (u \sinh\theta, v \cosh\theta)$. We write

$$S_+ = \mathbb{S}^{\mu-1} \times \{0\} \subset V_- \qquad \text{and} \qquad S_- = \{0\} \times \mathbb{S}^{n-\mu-1} \subset V_+.$$

Note that by the Smale condition $\overline{\mathrm{Stab}(q)} \cap \{f = f(q) + \varepsilon\}$ is transverse to the sphere S_+.

We formulate the following lemma.

Lemma 4.9. *Let Σ_+ be a smcs of dimension k in V_+ that is transverse to S_+ with non-empty intersection. Define Σ_- to be the closure in V_- of the set of points V_- which lie on a gradient line passing through Σ^+, then Σ_- contains S_- and is a smcs of dimension k.*

Proof. On U the flow lines of X are orthogonal trajectories of the surface $-\|x\|^2 + \|y\|^2 =$ constant, thus the gradient lines passing through (x, y) can be parametrized as $(tx, t^{-1}y)$. Therefore, if either x or y is zero the trajectory is a straight line segment trough (x, y) coming from the origin, meaning that no points of S_- lie on a gradient line passing through Σ_+. If both x and y are non-zero the trajectory passing through (x, y) is a hyperbola passing through some point $(u \cosh \theta, u \sinh \theta) \in V_-$ and a corresponding point $(u \sinh \theta, u \cosh \theta) \in V_+$.

Using polar coordinates $(\varphi, \psi, r) \in \mathbb{S}^{i-1} \times \mathbb{S}^{n-i-1} \times [0, 1]$ in $V_- \cong \mathbb{S}^{i-1} \times \mathbb{D}^{n-1}$ and $V_+ \cong \mathbb{D}^i \times \mathbb{S}^{n-i-1}$, we see the the identity map $V_+ \setminus S_+ \to V_- \setminus S_-$ gives the correspondence $(u \cosh \theta, u \sinh \theta) \leftrightarrow (u \sinh \theta, u \cosh \theta)$.

Define $K = \Sigma_+ \cap S_+$, which is a smcs as shown in Lemma 4.7. Note that $\mathbb{S}_+$ has a product neighbourhood $\mathbb{D}^\mu \times \mathbb{S}_+$ in V_+ such that $S_+ = \{0\} \times \mathbb{S}_+$, thus by Lemma 4.7 there is a diffeomorphism $H : D^\mu \times \mathbb{S}^{n-\mu-1} \to D^\mu \times \mathbb{S}^{n-\mu-1}$ along $\{0\} \times \mathbb{S}_+$ such that $H(\Sigma_+) = \mathbb{D}^\mu \times K$ near $\{0\} \times \mathbb{S}_+$.

From the proof of Lemma 4.7 we know, using polar coordinates, that the diffeomorphism H is of the form $H(\varphi, \psi, r) = (\varphi, \overline{\psi}(\varphi, \psi, r), r)$ with $\overline{\psi}(\varphi, \psi, 0) = \psi$. Note that H could also be considered as a map $\widetilde{H} : \mathbb{S}^{\mu-1} \times \mathbb{D}^{n-\mu} \to \mathbb{S}^{\mu-1} \times \mathbb{D}^{n-\mu}$, which is also a diffeomorphism.

Now, we see that

$$H(\Sigma_+ \setminus K) = \{(\varphi, \psi, r) | \varphi \in \mathbb{S}^{\mu-1}, \overline{\psi}(\varphi, \psi, r) \in K, r > 0\},$$

and that in V_-, $\widetilde{H}(\Sigma_- \setminus S_-)$ is given by the same formula. Thus, by taking the closure, we see that

$$\widetilde{H}(\Sigma_-) = \mathbb{S}^{\mu-1} \times cK.$$

We conclude that Σ_- is a smcs containing S_-. □

Using this lemma we see $\overline{\mathrm{Stab}(q)} \cap \{f = f(q) - \varepsilon\}$ is a smcs with a new singular stratum. The theorem now follows by recursion. □

Remark 4.10. Theorem 4.8 implies that, if M is closed, the stable manifolds give the structure of a CW-complex on M. If M is not closed but instead X-convex, the stable manifolds give the structure of a CW-complex on the skeleton Skel(M, X). △

4.3 Limitations

Submanifolds with conical singularities have a high level of regularity and give rise to well-behaving conical models around the singularities, nonetheless, they do have their limitations. Conical models form a continuous family of models, with which we mean a small perturbation of the model gives a new, non-isomorphic, model.

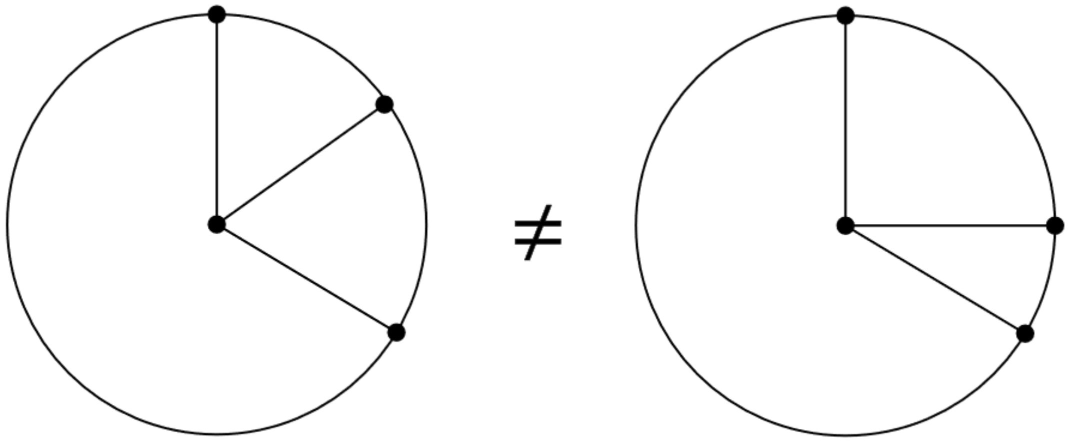


Figure 4.10: Non-isomorphic conical models.

This is illustrated in Figure 4.10, there is no ambient diffeomorphisms between the two models, since such a map could not be smooth in the point of the cone. Therefore the two conical models are not equivalent.

This means that a small perturbation of the Morse-Smale vector field yields vastly different models around every critical point.

To avoid this kind of behaviour we would like to have a discrete family of models, such that a small perturbation yield the same model. Furthermore, our goal is to describe these models combinatorically, allowing for a completely combinatorial description of the complex given by stable manifolds.

Chapter 5

Arboreal Hypersurface Singularities in Smooth Manifolds

In this chapter we define the class of *arboreal hypersurface singularities*, which are an adaptation of the class of *arboreal singularities* as introduced by David Nadler in [Nad17]. Whereas the original arboreal singularities are used to describe Legendrian and Lagrangian singularities in symplectic and contact manifolds, our arboreal hypersurface singularities describe stratified hypersurface singularities in smooth manifolds. Furthermore, we show that each class of germs of arboreal hypersurface singularities is determined by discrete combinatorial data given by a *special signed rooted tree* $\mathbb{T} = (T, \rho, \varepsilon)$, a non-cyclic connected graph T with a condition on the degree of each vertex, a distinct root vertex ρ and signs $\pm$ on each edge.

In the next chapter we discuss how to perturb Morse-Smale systems such that the singularities in their skeleta are arboreal hypersurface singularities. To do so, we "spread out" the critical points into manifolds of critical points.

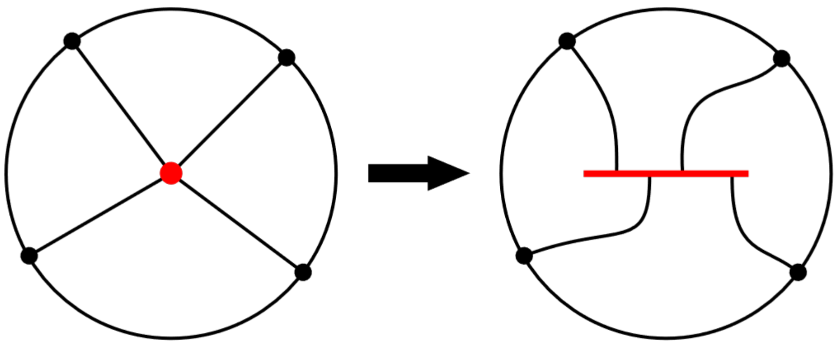


Figure 5.1: Spreading out the critical points.

If we do this in such a matter that the stable manifolds have codimension 1, they are locally separating and we can locally choose an "up" and "down" direction. Recall that in the Morse

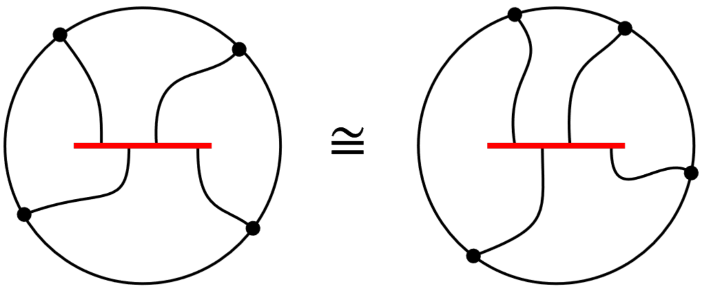


Figure 5.2: Perturbations away from the critical points yield germ diffeomorphic singularities.

case we demanded that, in some specified coordinate system, the vector field is locally radial in the unstable manifold. In the neighbourhood of the spread out critical points we want the vector field to be vertical, i.e. normal to the stable manifold in some coordinate system around the manifold of critical points. Then, close to the critical manifolds, there are coordinates such that the trajectory through a point p in a unstable manifold is locally given as the straight half-line starting at the natural projection of p to the critical manifold and passing through p. In particular, a stable manifold S entering this coordinate neighbourhood of a critical manifold C is locally given as the *vertical cone* of the intersection of S with the boundary of the neighbourhood.

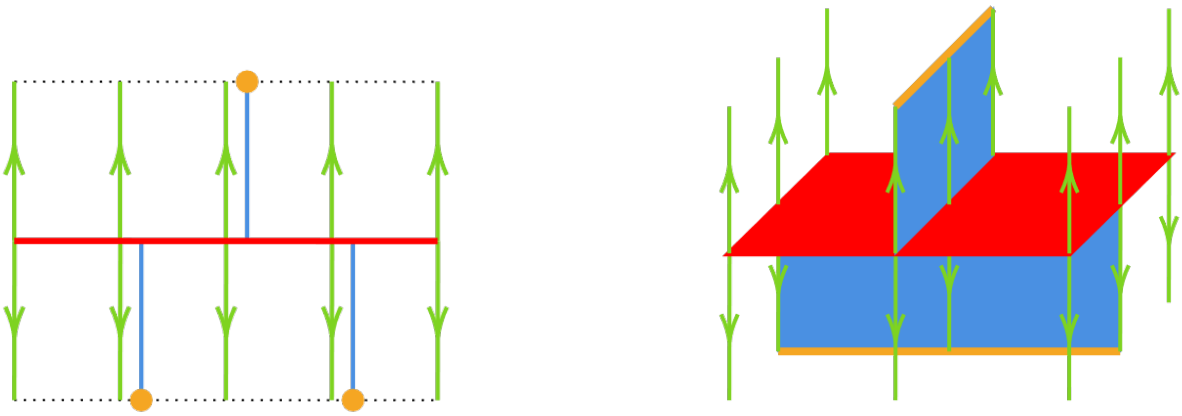


Figure 5.3: Two neighbourhoods of critical manifolds in their unstable manifold. The critical manifolds, indicated in red, have codimension 1. The vector fields, indicated in green, are vertical. Stable manifolds enter the neighbourhood, indicated in orange, and are locally given as a vertical cone, indicated in blue.

We begin this chapter with the definition of arboreal hypersurface singularities in Section 5.1. Then, in Section 5.2, we produce explicit local models for the arboreal hypersurface singularities where each model is associated to a special signed rooted tree. In Section 5.3 we prove the Stability Theorem 5.11 and use it to show that each arboreal hypersurface singularity is germ diffeomorphic to one of the models produced in Section 5.2. Furthermore, we show that this Stability Theorem also holds parametrically. We conclude this chapter with Section 5.4, where we generalize the definition of arboreal hypersurface singularities to allow for strata with boundary. Furthermore, we produce and produce local models associated to special signed rooted trees with certain extra combinatorial data for these generalized arboreal singularities. We prove that a similar Stability Theorem holds for these *generalized arboreal hypersurface* singularities and use this Stability Theorem to conclude that each generalized arboreal hypersurface singularity is germ diffeomorphic to one of the produced models.

5.1 Arboreal hypersurface singularities

Before we can define arboreal hypersurface singularities we first need give a more formal definition of the *vertical cone* operation. Let $H \subset \mathbb{R}^n \times \mathbb{S}^0$ and let $\pi : \mathbb{R}^n \times \mathbb{S}^0 \to \mathbb{R}^n$ be the obvious projection. Given a point $p \in H$ we denote the straight half-line beginning at $\pi(p)$ and passing trough p by $\mathcal{C}(p)$. The *vertical cone* $\mathcal{C}(H) \subset \mathbb{R}^{n+1}$ is the union $\cup_{p \in H} \mathcal{C}(p)$.

Definition 5.1. *Arboreal hypersurface singularities* form the smallest class Arb_n^{hyp} of germs of closed stratified subsets in $n+1$-dimensional smooth manifolds such that the following properties are satisfied:

(i) (Base case) Arb_0^{hyp} contains $pt = \mathbb{R}^0 \subset \mathbb{R}$.

(ii) (Invariance) Arb_n^{hyp} is invariant with respect to diffeomorphisms.

(iii) (Stabilization) If $H \subset M$ is in Arb_n^{hyp}, then the product $H \times \mathbb{R} \subset M \times \mathbb{R}$ is in Arb_{n+1}^{hyp}.

(iv) (Vertical cones) Let $\pi : \mathbb{R}^n \times \mathbb{S}^0 \to \mathbb{R}^n$ be the obvious projection. If $H_1 \subset \mathbb{R}^n \times \mathbb{S}^0$ is an arboreal hypersurface singularity germ from Arb_{n-1}^{hyp} centred at $z_1 \in \mathbb{R}^n \times \mathbb{S}^0$, the union $\mathbb{R}^n \cup \mathcal{C}(H_1)$ of the vertical cone with the zero-section forms an arboreal hypersurface singularity germ from Arb_n^{hyp}.

Furthermore, if $H_1, H_2 \subset \mathbb{R}^n \times \mathbb{S}^0$ are disjoint arboreal hypersurface singularity germs from Arb_{n-1}^{hyp}, such that H_1 and H_2 are centred at $z_1, z_2 \in \mathbb{R}^n \times \mathbb{S}^0$ with $\pi(z_1) = \pi(z_2)$, and the map $\pi|_{H_1 \cup H_2}$ is self-transverse as a map of stratified spaces, then the union $\mathbb{R}^n \cup \mathcal{C}(H_1) \cup \mathcal{C}(H_2)$ of the vertical cones with $\mathbb{R}^n$ forms an arboreal hypersurface singularity germ from Arb_n^{hyp}.

$\triangle$

Note that the condition that $\pi|_{H_1 \cup H_2}$ is self-transverse as a map of stratified spaces is a natural one if we want the models of arboreal hypersurface singularities to be stable, i.e. we want the models to form a finite and discrete family. Transverse intersection is stable while non-transverse intersection is unstable, thus allowing for non-transverse self-intersection would possibly lead to an infinite and continuous family of models.

Remark 5.2. Every arboreal hypersurface singularity admits a canonical stratification, since every arboreal hypersurface singularity can be obtained through a series of stabilizations, vertical cones and diffeomorphisms applied to the base case. This base case, $\mathbb{R}^0 \subset \mathbb{R}$, is a smooth manifold and thus has a canonical stratification. The stabilization $H \times \mathbb{R}$ of a stratified set $H \subset M$ with stratification $(H_n, ..., H_0)$ admits a canonical stratification $(H_n \times \mathbb{R}, H_{n-1} \times \mathbb{R}, ..., H_0 \times \mathbb{R}, \emptyset)$. Similarly, because the transverse intersection of strata yields a manifold, the vertical cone operation applied to stratified $H_1, H_2 \subset M$ gives a set with a canonical stratification. A diffeomorphism $\varphi : (M, H) \to (M', H')$ arboreal hypersurface singularity germs carries a stratification of H to a stratification of H'. △

5.2 Arboreal hypersurface models

We now construct explicit local models for the arboreal hypersurface singularities, to do so we first need to introduce some auxiliary notions.

5.2.1 Fully signed rooted trees

A *graph* G is a set of *vertices* $V(G)$ and a set of *edges* $E(G)$ such that $E(G)$ is a subset of the set of two-elements subsets of $V(G)$. We denote the number of vertices of G by $v(G)$.

Two vertices $\alpha, \beta \in V(G)$ are called *adjacent* if $\{\alpha, \beta\} \in E(G)$, a graph G is called *connected* if any two vertices $\alpha, \beta \in V(G)$ can be linked by a *walk*, i.e. a sequence of edges $\{\alpha, \gamma_1\}, \{\gamma_1, \gamma_2\}, ..., \{\gamma_n, \beta\} \in E(G)$.

A graph G is called *acyclic* if there are no non-empty walks in which all edges are distinct and all vertices except for the first and last are distinct.

Definition 5.3. A *tree* T is a nonempty, finite, connected acyclic graph. The *degree* of a vertex α of a tree T is the number of adjacent vertices, i.e. the number of edges incident on α.

A rooted tree $\mathcal{T} = (T, \rho)$ is a pair of a tree T and a distinguished vertex $\rho \in V(T)$ called the *root*. We denote the set of non-root vertices $N(T) = V(T) - \{\rho\}$ and the number of non-root vertices by $n(T)$.

A *fully signed rooted tree* $\mathbb{T} = (T, \rho, \varepsilon)$ is a rooted tree (T, ρ) and a decoration ε of a sign $\pm$ on each edge of T. △

In a tree T there is always a unique minimal path, meaning a nonrepeating sequence of edges, connecting any two vertices. The vertices $V(T)$ of a rooted tree have a natural poset structure with unique minimum ρ and $\alpha \leqslant \beta$ if the unique minimal path between β and ρ contains α. We call a vertex that is maximal with respect to this partial order a *leaf*.

We introduce the following variation on signed rooted trees.

Definition 5.4. A *special signed rooted tree* $\mathbb{T} = (T, \rho, \varepsilon)$ is a fully signed rooted tree satisfying the following conditions.

- The root ρ has degree at most two.
- Every non-root vertex has degree at most three.
- The edges between any vertex and its (at most two) predecessors do not have the same sign.

△

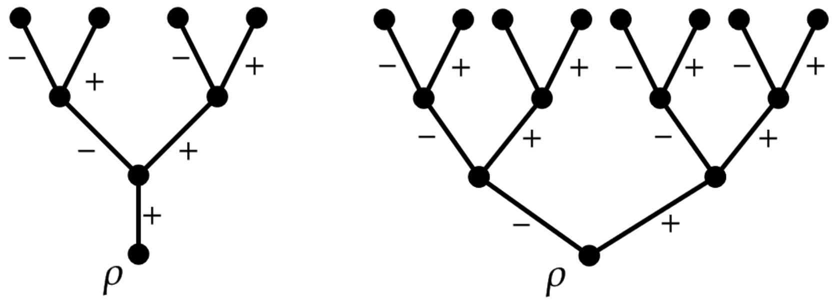


Figure 5.4: Two special signed rooted trees.

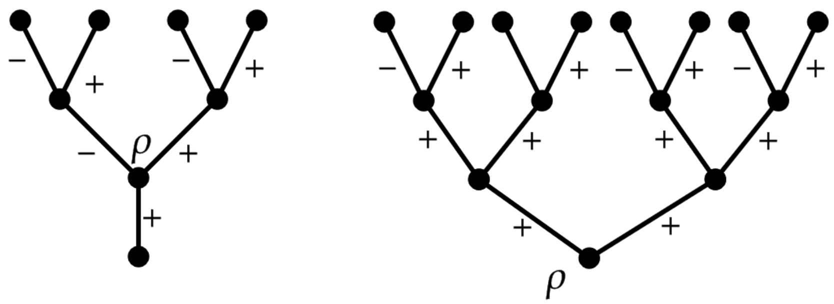


Figure 5.5: Two signed rooted trees that are not special. The tree on the left has a root of degree 3, the tree on the right does not satisfy the condition on the signs.

5.2.2 Arboreal model $\mathbb{T}$-hypersurfaces

Definition 5.5. Let S be a finite set, we write $\mathbb{R}^S$ for the *Euclidean space of S-tuples of real numbers*. △

Remark 5.6. For any finite set S we can fix a bijection $S \to \{1, 2, ..., \|S\|\}$, which induces an isomorphism $\mathbb{R}^S \cong \mathbb{R}^{\|S\|}$. We mostly consider $S = V(\mathbb{T})$, if one wishes to fix a bijection $V(\mathbb{T}) \to \{1, 2, ..., v(\mathbb{T}\}$ it is best to consider an order preserving bijection. △

We now define our models for arboreal hypersurface singularities. These models are a signed variation on Nadler's construction of *rectilinear arboreal hypersurfaces* in [Nad17], similar to Starkston's signed construction in [Sta18].

Definition 5.7. Let $\mathbb{T}$ be a special signed rooted tree. Then the *arboreal model* $\mathbb{T}$*-hypersurface*, $H_{\mathbb{T}}$, associated to $\mathbb{T}$, is the stratified subset of $\mathbb{R}^{V(\mathbb{T})}$ given by the union $H_{\mathbb{T}} = \bigcup_{\alpha \in V(T)} P_\alpha$ were we define

$$P_\alpha := \{x_\alpha = 0,\ \varepsilon_{\beta,\beta'} x_\beta > 0 \text{ for all } \beta < \alpha\} \subset \mathbb{R}^{V(\mathbb{T})}.$$

Here $\varepsilon_{\beta,\beta'}$ denotes the sign of the edge $\{\beta, \beta'\}$ between β and its unique neighbour β' such that $\beta \leqslant \beta' \leqslant \alpha$, i.e. the edge $\{\beta, \beta'\}$ points towards α. △

Each stratum P_α has a co-orientation given by ∂_{x_β}. The geometric meaning of the signs of $\mathbb{T}$ is that, if the vertices $\alpha \leqslant \beta$ share an edge, the sign $\varepsilon_{\alpha,\beta} = \pm 1$ if and only if P_β is on the $\pm$-side of P_α with respect of this co-orientation.

Remark 5.8. Different special signed rooted trees may yield germ diffeomorphic arboreal model hypersurfaces, in particular two different choices of signs on the same underlying tree give germ diffeomorphic arboreal model hypersurfaces.

Let $\mathbb{T}$ and $\mathbb{T}'$ be two special signed rooted trees with the same underlying rooted tree, i.e. $\mathbb{T} = (T, \rho, \varepsilon)$ and $\mathbb{T}' = (T, \rho, \varepsilon')$. We write $\varepsilon_{\alpha,\hat{\alpha}}$ and $\varepsilon'_{\alpha,\hat{\alpha}}$ for the sign in respectively $\mathbb{T}$ and $\mathbb{T}'$ of the edge between a non-root vertex α and its unique predecessor. The map $\mathbb{R}^{V(T)} \to \mathbb{R}^{V(T)}, \quad \{x_\alpha\} \mapsto \varepsilon_{\alpha,\hat{\alpha}} \varepsilon'_{\alpha,\hat{\alpha}} \{x_\alpha\}$ is a diffeomorphism of $(\mathbb{R}^{V(T)}, H_{\mathbb{T}})$ to $(\mathbb{R}^{V(T)}, H_{\mathbb{T}'})$.

These different trees associated to the same (up to ambient diffeomorphism) stratified spaces correspond to different chosen co-orientations of the strata. In the next chapter we discus when and how the strata can be equipped with a canonical co-orientation. △

5.3 Stability of arboreal hypersurface singularities

Definition 5.9. Let $\mathbb{T}$ be a special signed rooted tree with $n = v(\mathbb{T}) \leqslant m$. A closed subset $H \subset M$ of a smooth m-dimensional manifold M is called an *arboreal hypersurface of type* $(\mathbb{T}, m)$ *at* p if the germ of (M, H) at $p \in H$ is diffeomorphic to the germ of the pair $(\mathbb{R}^n \times \mathbb{R}^{m-n}, H_{\mathbb{T}} \times \mathbb{R}^{m-n})$ at the origin.

A closed subset $H \subset M$ of a smooth m-dimensional manifold M is called an *arboreal hypersurface* if it is an arboreal hypersurface of type $(\mathbb{T}, m)$, for varying $\mathbb{T}$, at every $p \in H$. △

The goal of this section is to show that the classes of Arb_m^{hyp} correspond to arboreal hypersurfaces of type $(\mathbb{T}, m)$ for varying special signed rooted trees $\mathbb{T}$ with at most m vertices, as formulated in the following theorem.

Theorem 5.10. *Per dimension m, Definition 5.1 produces only finite many local models up to ambient diffeomorphism. More specifically, every class in the collection* Arb_m^{hyp} *can be assigned a special signed rooted tree* $\mathbb{T}$ *with at most m vertices.*

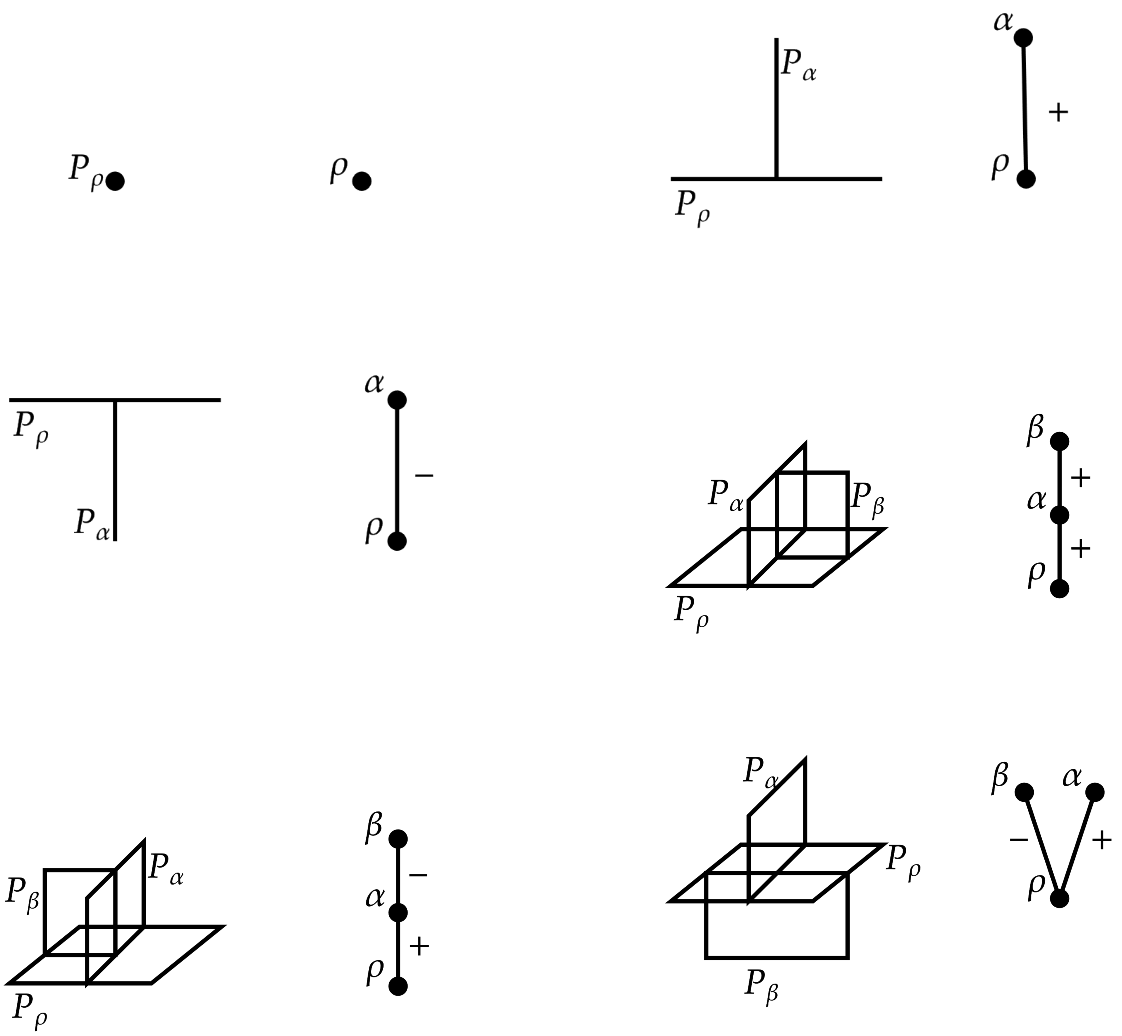


Figure 5.6: Some examples of arboreal model $\mathbb{T}$-hypersurfaces for different special signed rooted trees.

Note that, as a result of Remark 5.8, the tree $\mathbb{T}$ assigned to a class in Arb_m^{hyp} is generally not unique. Any special signed rooted tree with the same underlying rooted tree but different signs can also be assigned to this class.

To be able to prove Theorem 5.10 we first show the following Stability Theorem. This theorem asserts that, under the conditions of Definition 5.1, the vertical cone of two arboreal hypersurface of respective type $(\mathbb{T}, m)$ and $(\mathbb{T}', m)$ union $\mathbb{R}^{m-1}$, is an arboreal hypersurface. This theorem is the analogue of Theorem 3.5 in [AEN22a] for our smooth set-up and is proven using the same technique.

Theorem 5.11 (Stabilty Theorem). *Let $\mathbb{T}$ be a special signed rooted tree, set $m = v(\mathbb{T})$ and let $\varphi_\pm : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{S}^0$ be embeddings mapping 0 to $(0, \pm 1) \in \mathbb{R}^m \times \mathbb{S}^0 = \mathbb{R}^m \times \{-1, 1\}$. We write $\pi : \mathbb{R}^m \times \mathbb{S}^0 \to \mathbb{R}^m$ for the obvious projection.*

Write $\mathbb{T}_+$ and $\mathbb{T}_-$ for the two special signed rooted trees obtained from $\mathbb{T}$ by deleting the root vertex and restricting the signs, where the index $\pm$ of the trees corresponds to the sign of the edge between the root ρ and incident vertex/vertices. We write $H_\pm = \varphi_\pm(H_{\mathbb{T}_\pm} \times \mathbb{R}^{m-n_\pm})$, where $n_\pm = v(\mathbb{T}_\pm)$, and define $H = H_+ \cup H_-$.

Suppose that H projects self-transversely under π. Then $\mathcal{H} = \mathbb{R}^m \times \{0\} \cup \mathcal{C}(H)$, where $\mathcal{C}(H)$ denotes the vertical cone of H, is an arboreal hypersurface of type $(\mathbb{T}, m+1)$ at the origin.

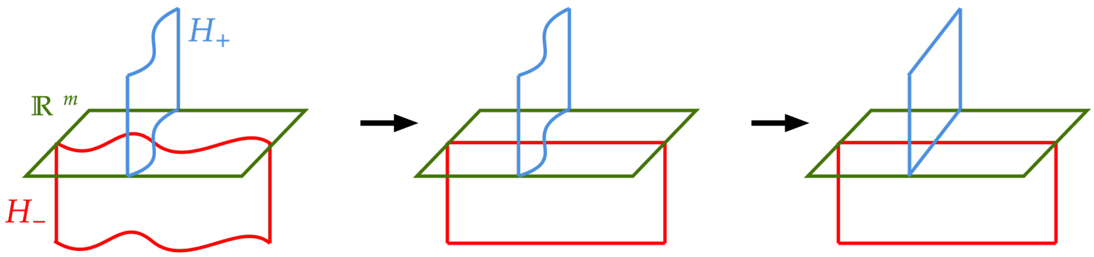


Figure 5.7: The strategy of the proof, we inductively normalize the pieces of $\mathcal{H}$.

Note that one or both of the trees $\mathbb{T}_+$ and $\mathbb{T}_-$ might be empty, which is the case if the root ρ of $\mathbb{T}$ has degree one or zero.

Proof. We need to show that the germ of $\mathcal{H}$ at the origin is diffeomorphic to the germ of the model $(\mathbb{T}, m+1)$-hypersurface singularity, we proceed by induction over the number of vertices of $\mathbb{T}$.

For the base case, assume $\mathbb{T}$ has a single vertex. Then $\mathbb{T}_+ = \mathbb{T}_- = \emptyset$, meaning that $H = \emptyset$ and thus $\mathcal{H} = \mathbb{R}^m \times \{0\}$. Now, the germ of $(\mathbb{R}^{m+1}, \mathbb{R}^m \times \{0\})$ is diffeomorphic to the germ of $(\mathbb{R}^{m+1}, H_\mathbb{T} \times \mathbb{R}^m) = (\mathbb{R}^{m+1}, \{0\} \times \mathbb{R}^m)$.

For the inductive step, assume $\mathbb{T}$ has n vertices and assume the assertion has been established for all special signed rooted trees with $n-1$ vertices. Consider a leaf β of $\mathbb{T}$, without loss of generality $\beta \in V(\mathbb{T}^+)$. We denote the tree obtained from $\mathbb{T}_+$ by deleting β by $\mathbb{T}'_+$ and the tree obtained from $\mathbb{T}$ by deleting β by $\mathbb{T}'$. We first "straighten", or normalize, the pieces of $\mathcal{H}$ corresponding to the vertices of $\mathbb{T}'$, such that they align with the arboreal model hypersurface. Then we normalize the piece of $\mathcal{H}$ corresponding to β without moving the pieces that have already been normalized.

Recall that per definition in $\mathbb{R}^{V(\mathbb{T}^+)}$

$$H_{\mathbb{T}_+} = \bigcup_{\alpha \in V(\mathbb{T}^+)} P_\alpha.$$

By construction the germ of $(\mathbb{R}^m, H_+)$ is diffeomorphic to the germ of $(\mathbb{R}^m, H_{\mathbb{T}_+} \times \mathbb{R}^{m-n_+})$ and thus diffeomorphic to the germ of $(\mathbb{R}^m, (\cup_{\alpha \in V(\mathbb{T}^+)} P_\alpha) \times \mathbb{R}^{m-n_+})$. We write $H_+[\alpha]$ for the piece of H_+ corresponding to P_α under this diffeomorphism.

Note that

$$H_{\mathbb{T}_+} = H_{\mathbb{T}'_+} \times \mathbb{R}^{\{\beta\}} \cup P_\beta.$$

Thus, if we define $H'_+ := H_+ \setminus H_+[\beta]$, we see that H'_+ is an arboreal hypersurface of type $(\mathbb{T}'_+, m)$ at the origin. Because $v(\mathbb{T}') = n - 1$, we have by induction that $\mathcal{H}' := \mathbb{R}^m \times \{0\} \cup \mathcal{C}(H'_+ \cup H_-)$ is an arboreal hypersurface of type $(\mathbb{T}', m)$. Therefore we may assume

$$\mathcal{H}' = H_{\mathbb{T}'} \times \mathbb{R}^{m-n+1}.$$

We write $\mathcal{H}[\alpha] = \mathcal{C}(H_+[\alpha])$ and remark that $\mathcal{H} = \mathcal{H}' \cup \mathcal{C}(H_+[\beta])$.

We now normalize $\mathcal{C}(H_+[\beta])$, to this end let $\mathbb{A}_\beta = (A_\beta, \rho, \varepsilon_\beta)$ be the linear special signed rooted subtree of $\mathbb{T} = (T, \rho, \hat{\varepsilon})$, with vertices $V(A_\beta) = \{\alpha \in V(T) | \alpha \leqslant \beta\}$ and restricted signs. We set $d = v(T) - v(A_\beta)$ to be the number of complementary vertices.

We consider the $(\mathbb{A}_\beta, m)$-hypersurface $\mathcal{K} \subset \mathcal{H}$ given by the union $\mathcal{K} = \cup_{\alpha \in V(A_\beta)} \mathcal{H}[\alpha]$ of the smooth pieces of $\mathcal{H}$ indexed by $\alpha \in V(A_\beta)$. Note that for $\mathbb{A}'_\beta = \mathbb{A}_\beta \cap \mathbb{T}'$ and $\mathcal{K}' = \mathcal{K} \cap \mathcal{H}'$ we have

$$\mathcal{K}' = H_{\mathbb{A}'_\beta} \times \mathbb{R}^{m-n+1+d}.$$

Now we only need to normalize the smooth piece $\mathcal{H}[\beta]$. Observe that $\mathcal{K}$ is an arboreal $(\mathbb{A}_\beta, m)$-

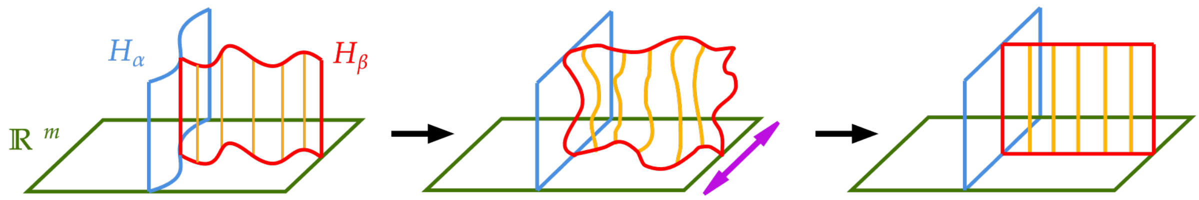


Figure 5.8: First normalizing all but one smooth piece and then treating the complementary directions as parameters.

hypersurface and thus there is a (germ of a) diffeomorphism Φ of the germ of $(\mathbb{R}^m, \mathcal{K})$ to the germ of $(\mathbb{R}^m, H_{\mathbb{A}'_\beta} \times \mathbb{R}^{m-n+d})$. Meanwhile

$$\mathcal{K} = \mathcal{K}' \cup \mathcal{K}[\beta] = H_{\mathbb{A}'_\beta} \times \mathbb{R}^{m-n+1+d} \cup \mathcal{K}[\beta] \qquad \text{and} \qquad H_{\mathbb{A}_\beta} = H_{\mathbb{A}'_\beta} \cup P_\beta.$$

Thus it must be that Φ maps $\mathcal{K}'$ to itself (as a set, not necessarily pointwise), none of the smooth parts of $H_{\mathbb{A}'_\beta}$ are homeomorphic and thus any smooth part of $H_{\mathbb{A}'_\beta}$ must be mapped to itself. Therefore we can normalize $\mathcal{H}[\beta]$ while not moving $\mathcal{K}'$. Using that H projects self-transversely under π we can view the complementary directions $\mathbb{R}^{m-n+1+d}$ as parameters, so we can assure $\mathcal{H}'$ is also preserved. □

We have the following direct corollary.

Corollary 5.12. *Every arboreal hypersurface of type* $(\mathbb{T}, m)$ *is an arboreal hypersurface singularity as in Definition 5.1.*

Proof. Fix $m \geqslant 1$, the claim follows by strong induction over the number of vertices $v(\mathbb{T})$.

For $v(\mathbb{T}) = 1$ we see that $(\mathbb{R}^m, \{0\} \times \mathbb{R}^{m-1})$ is the m-fold stabilization of $(\mathbb{R}, pt)$.

Let $2 \leqslant n \leqslant m$ and assume all arboreal hypersurfaces of type $(\mathbb{T}, m)$ with $v(\mathbb{T}) < n$ are arboreal hypersurface singularities as in Definition 5.1. We fix a special signed rooted tree $\mathbb{T}$ with $v(\mathbb{T}) = n$ and write $\mathbb{T}_\pm$ for the trees obtained from $\mathbb{T}$ by deleting the root vertex as in Theorem 5.11.

We embed $H_{\mathbb{T}_+} \times \mathbb{R}^{m-n_+}$ into $\mathbb{R}^m \times \mathbb{S}^0$ as

$$H_+ = H_{\mathbb{T}_+} \times \mathbb{R}^{m-n_+} \times \{1\}$$

and we embed $H_{\mathbb{T}_-} \times \mathbb{R}^{m-n_-}$ into $\mathbb{R}^m \times \mathbb{S}^0$ as

$$H_- = \mathbb{R}^{m-n_-} \times H_{\mathbb{T}_-} \times \{-1\}.$$

Now we see that

$$H_{\mathbb{T}} \times \mathbb{R}^{m-n} = \mathbb{R}^m \cup \mathcal{C}(H_+) \cup \mathcal{C}(H_-)$$

meaning $H_{\mathbb{T}} \times \mathbb{R}^{m-n} \subset \mathbb{R}^m$ is in Arb^{hyp}_{m-1}, thus the arboreal hypersurface of type $(\mathbb{T}, m)$ is an arboreal hypersurface singularity as in Definition 5.1 by the vertical cone property. □

We are now ready to prove Theorem 5.10.

Proof. We now prove that every class in Arb^{hyp}_m can be assigned a special signed rooted tree $\mathbb{T}$ with at most m vertices by induction over m.

By definition Arb^{hyp}_0 has only a single local model, which is indeed germ diffeomorphic to $(\mathbb{R}, H_\bullet) = (\mathbb{R}, \{0\} \times \mathbb{R})$, where $\bullet$ denotes the special signed rooted tree with one vertex.

Assume every class in Arb^{hyp}_{m-1} can be assigned a special signed rooted tree $\mathbb{T}$ with at most $m-1$ vertices, so if $H \subset M$ is in Arb^{hyp}_{m-1} then (M, H) is germ diffeomorphic to $(\mathbb{R}^m, H_{\mathbb{T}} \times \mathbb{R}^{m-v(\mathbb{T})})$.

Any class in Arb^{hyp}_m with $m \geqslant 1$ can be represented as either a stabilization $H \times \mathbb{R}$ of some $H \subset M$ in Arb^{hyp}_{m-1} or the vertical cone of $H_1, H_2 \subset \mathbb{R}^m \times \mathbb{S}^0$ in Arb^{hyp}_{m-1}.

In the former case the class in Arb^{hyp}_m is represented by the same special signed rooted tree with at most $n-1$ vertices as H, since the germ diffeomorphism $(H, M) \to (\mathbb{R}^m, H_{\mathbb{T}} \times \mathbb{R}^{m-v(\mathbb{T})})$ can be extended to a germ diffeomorphism

$$(M \times \mathbb{R}, H \times \mathbb{R}) \to (\mathbb{R}^{m+1}, H_{\mathbb{T}} \times \mathbb{R}^{m+1-v(\mathbb{T})}).$$

In the latter case the class in Arb_m^{hyp} is represented by the special signed rooted tree $\mathbb{T}$ formed by joining the special signed rooted trees $\mathbb{T}_1$ and $\mathbb{T}_2$ representing H_1 and H_2 at a root. To see that $\mathbb{T}$ has at most m vertices we assume that $v(\mathbb{T}) > m$, meaning that $m_1 + m_2 = v(\mathbb{T}_1) + v(\mathbb{T}_2) \geqslant m$. Then we can assume $H_1 = H_{\mathbb{T}_1} \times \mathbb{R}^{m-m_1}$ and $H_2 = H_{\mathbb{T}_1} \times \mathbb{R}^{m-m_2}$, meaning the strata containing the origin are $\{0\} \times \mathbb{R}^{m-1-m_1}$ and $\{0\} \times \mathbb{R}^{m-1-m_2}$ respectively. Thus it is not possible for these strata to be projected transversely in $\mathbb{R}^m$, which is a condition on the projection in Definition 5.1, thus $v(\mathbb{T}) \leqslant n$.

Now Theorem 5.11 asserts that $\mathbb{R}^m \cup \mathcal{C}(H_1) \cup \mathcal{C}(H_2)$ is an arboreal hypersurface of type $(\mathbb{T}, m+1)$ at the origin. □

5.3.1 Parametric stability

In this section we show that the proof of Theorem 5.11 also holds parametrically, yielding the following parametric stability Theorem.

Theorem 5.13. *Let $\mathbb{T}$ be a special signed rooted tree, let $m \geqslant v(\mathbb{T})$, let $\varphi_\pm^y : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{S}^0$ be families of germs of embeddings mapping 0 to $(0, \pm 1) \in \mathbb{R}^m \times \mathbb{S}^0 = \mathbb{R}^m \times \{-1, 1\}$ and let $\pi : \mathbb{R}^m \times \mathbb{S}^0 \to \mathbb{R}^m$ be the obvious projection.*

Write $\mathbb{T}_+$ and $\mathbb{T}_-$ for the two special signed rooted trees obtained from $\mathbb{T}$ by deleting the root vertex and restricting the signs with the index $\pm$ of the trees corresponds to the sign of the edge between the root ρ and incident vertex/vertices.

We write $H_\pm^y = \varphi_\pm^y(H_{\mathbb{T}_\pm} \times \mathbb{R}^{m-n_\pm})$ where $n_\pm = v(\mathbb{T}_\pm)$, define $H^y = H_+^y \cup H_-^y$ and define $\mathcal{H}^y = \mathbb{R}^m \times \{0\} \cup \mathcal{C}(H^y)$, where $\mathcal{C}(H)$ denotes the vertical cone of H^y.

Suppose that each H^y projects self-transversely under π.

Then there exists a family of (germs of) diffeomorphisms of the germ of $(\mathbb{R}^m, \mathcal{H}^y)$ to the germ of $(\mathbb{R}^m \times H_{\mathbb{T}} \times \mathbb{R}^{m-v(\mathbb{T})})$.

Proof. As in the proof of Theorem 5.11 we proceed by induction.

For the base case assume $\mathbb{T}$ has a single vertex. Then $\mathbb{T}_+ = \mathbb{T}_- = \emptyset$, meaning that $H^y = \emptyset$ and thus $\mathcal{H}^y = \mathbb{R}^m \times \{0\}$. The germ of $(\mathbb{R}^{m+1}, \mathbb{R}^m \times \{0\})$ is diffeomorphic to the germ of $(\mathbb{R}^{m+1}, H_{\mathbb{T}} \times \mathbb{R}^m) = (\mathbb{R}^{m+1}, \{0\} \times \mathbb{R}^m)$.

For the inductive step, assume $\mathbb{T}$ has n vertices and assume the assertion has been established for all special signed rooted trees with $n-1$ vertices. As before we consider a leaf β of $\mathbb{T}$, we assume without loss of generality that $\beta \in V(\mathbb{T}^+)$ and denote the tree obtained from $\mathbb{T}_+$ by deleting β by $\mathbb{T}'_+$ and the tree obtained from $\mathbb{T}$ by deleting β by $\mathbb{T}'$.

We write $H_+^y[\alpha]$ for the smooth piece of H_+ indexed by α and define $H_+^{\prime y} = H_+^y \setminus H_+^y[\beta]$

By induction, $\mathcal{H}^{\prime y} = \mathbb{R}^m \times \{0\} \cup \mathcal{C}(H_+^{\prime y} \cup H_-^y)$ is an arboreal hypersurface of type $(\mathbb{T}', m)$. Therefore we may assume

$$\mathcal{H}^{\prime y} = H_{\mathbb{T}'} \times \mathbb{R}^{m-n+1}.$$

As before we consider $\mathbb{A}_\beta = (A_\beta, \rho, \varepsilon_\beta)$ the linear special signed rooted subtree of $\mathbb{T} = (T, \rho, \hat{\varepsilon})$ with vertices $V(A_\beta) = \{\alpha \in V(T) | \alpha \leqslant \beta\}$ and restricted signs. We set $d = v(T) - v(A_\beta)$ to be the number of complementary vertices.

We look at he family of $(\mathbb{A}_\beta, m)$-hypersurface $\mathcal{K}^y \subset \mathcal{H}^y$ given by the union $\mathcal{K}^y = \cup_{\alpha \in V(A_\beta)} \mathcal{H}^y[\alpha]$ of the smooth pieces of $\mathcal{H}$ indexed by $\alpha \in V(A_\beta)$. For $\mathbb{A}'_\beta = \mathbb{A}_\beta \cap \mathbb{T}'$ and $\mathcal{K}'^y = \mathcal{K}^y \cap \mathcal{H}'^y$, we have

$$\mathcal{K}'^y = H_{\mathbb{A}'_\beta} \times \mathbb{R}^{m-n+1+d}.$$

We now normalize the smooth piece $\mathcal{H}[\beta]$.

Since $\mathcal{K}^y$ is a family of arboreal $(\mathbb{A}_\beta, m)$-hypersurface, there is a family of diffeomorphism Φ^y of the germs of $(\mathbb{R}^m, \mathcal{K}^y)$ to the germ of $(\mathbb{R}^m, H_{\mathbb{A}'_\beta} \times \mathbb{R}^{m-n+d})$. Meanwhile

$$\mathcal{K}^y = \mathcal{K}'^y \cup \mathcal{K}^y[\beta] = H_{\mathbb{A}'_\beta} \times \mathbb{R}^{m-n+1+d} \cup \mathcal{K}[\beta] \qquad \text{and} \qquad H_{\mathbb{A}_\beta} = H_{\mathbb{A}'_\beta} \cup P_\beta.$$

Thus it must be that Φ^y maps $\mathcal{K}'^y$ to itself (as a set, not necessarily pointwise) meaning we can normalize $\mathcal{H}^y[\beta]$ while not moving $\mathcal{K}'^y$. Using that H^y projects self-transversely under π we can view the complementary directions $\mathbb{R}^{m-n+1+d}$ as parameters so we can also assure $\mathcal{H}'^y$ is also preserved. □

5.4 Generalized arboreal hypersurface singularities

We will need a mild generalization of arboreal hypersurface singularities, similar to the generalization from manifolds to manifolds with boundary, to allow for strata with boundary. These kind of singularities occur when we allow the critical manifolds to have a boundary. This generalization amounts to including one extra base case to Definition 5.1.

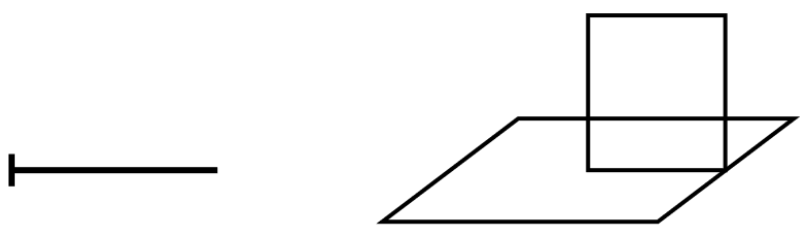


Figure 5.9: Two hypersurface singularities of strata involving boundary.

Definition 5.14. *Generalized arboreal hypersurface singularities* form the smallest class Arb_n^{genhyp} of germs of closed stratified subsets in $n+1$-dimensional smooth manifolds such that the following properties are satisfied:

(i) (Base case) Arb_0^{genhyp} contains $pt = \mathbb{R}^0 \subset \mathbb{R}$ and Arb_1^{genhyp} contains $\{0\} \times [0, \infty) \subset \mathbb{R}^2$.

(ii) (Invariance) Arb_n^{genhyp} is invariant with respect to diffeomorphisms.

(iii) (Stabilization) If $H \subset M$ is in Arb_n^{genhyp}, then the product $H \times \mathbb{R} \subset M \times \mathbb{R}$ is in $\mathrm{Arb}_{n+1}^{genhyp}$.

(iv) (Vertical cones) Let $\pi : \mathbb{R}^n \times \mathbb{S}^0 \to \mathbb{R}^n$ be the obvious projection. If $H_1 \subset \mathbb{R}^n \times \mathbb{S}^0$ is a generalized arboreal hypersurface singularity germ from $\mathrm{Arb}_{n-1}^{genhyp}$ centred at $z_1 \in \mathbb{R}^n \times \mathbb{S}^0$, the union $\mathbb{R}^n \cup \mathcal{C}(H_1)$ of the vertical cone with the zero-section forms a generalized hypersurface singularity germ from Arb_n^{genhyp}.

Furthermore, if $H_1, H_2 \subset \mathbb{R}^n \times \mathbb{S}^0$ are disjoint generalized arboreal hypersurface singularity germs from $\mathrm{Arb}_{n-1}^{genhyp}$, such that H_1 and H_2 are centred at $z_1, z_2 \in \mathbb{R}^n \times \mathbb{S}^0$ with $\pi(z_1) = \pi(z_2)$, and the map $\pi|_{H_1 \cup H_2}$ is self-transverse as a map of stratified spaces, then the union $\mathbb{R}^n \cup \mathcal{C}(H_1) \cup \mathcal{C}(H_2)$ of the vertical cones with $\mathbb{R}^n$ forms a generalized arboreal hypersurface singularity germ from Arb_n^{genhyp}.

△

The rest of this section is dedicated to showing that every class in Arb_n^{genhyp} can be assigned a special signed rooted tree with certain extra data.

Definition 5.15. A *special signed leafy rooted tree* $(\mathbb{T}, \ell_\varepsilon)$ is a special signed rooted tree $\mathbb{T}$ with a collection ℓ of marked leaf vertices, each decorated with a sign $\pm$. △

From a special signed leafy rooted tree $(\mathbb{T}, \ell_\varepsilon)$ we can construct a special signed rooted tree $\mathbb{T}^+$ by adding a vertex above each leaf $\alpha \in \ell$ and giving each edge between a leaf and its added vertex the sign corresponding to the leaf.

Using special signed leafy rooted trees we can define model generalized arboreal hypersurfaces. These models are a signed variation on Nadler's construction of *generalized rectilinear arboreal hypersurface* in [Nad16], similar to the signed construction in [Sta18].

Definition 5.16. Let $(\mathbb{T}, \ell_\varepsilon)$ be a special signed leafy rooted tree, the *generalized arboreal model* $(\mathbb{T}, \ell_\varepsilon)$*-hypersurface*, $H_{(\mathbb{T}, \ell_\varepsilon)}$, associated to $(\mathbb{T}, \ell_\varepsilon)$ is given by the union

$$H_{(\mathbb{T}, \ell_\varepsilon)} = \bigcup_{\alpha \in V(\mathbb{T}^+) \setminus \ell} P_\alpha \subset \mathbb{R}^{v(\mathbb{T}^+)}.$$

△

Example 5.17. The singularity shown at the left in Figure 5.9 is the generalized arboreal model $(\mathbb{T}, \ell_\varepsilon)$-hypersurface associated to the special signed leafy rooted tree with one vertex, which is also a marked leaf which has been given a positive sign.

The righthand singularity depicted in Figure 5.9 is the generalized arboreal model hypersurface associated to the special signed leafy rooted tree with two vertices, $\{\rho, \alpha\}$, with a positive decoration on its one edge and α as marked leaf, decorated with a positive sign. △

Using these models we can generalize Definition 5.9.

Definition 5.18. Let $(\mathbb{T}, \ell_\varepsilon)$ be a special leafy signed rooted tree with $n = v(\mathbb{T}^+) \leqslant m$. A closed subset $H \subset M$ of a smooth m-dimensional manifold M is called a *generalized arboreal hypersurface of type* $(\mathbb{T}, \ell_\varepsilon, m)$ *at* p if the germ of (M, H) at $p \in H$ is diffeomorphic to the germ of the pair $(\mathbb{R}^n \times \mathbb{R}^{m-n}, H_{(\mathbb{T},\ell_\varepsilon)} \times \mathbb{R}^{m-n})$ at the origin.

A closed subset $H \subset M$ of a smooth m-dimensional manifold M is called a *generalized arboreal hypersurface* if it is a generalized arboreal hypersurface of type $(\mathbb{T}, \ell_\varepsilon, m)$, for varying $(\mathbb{T}, \ell_\varepsilon)$, at every $p \in H$. △

Remark 5.19. We note that the base case $pt = \mathbb{R}^0 \subset \mathbb{R}$ in Arb_0^{genhyp} is germ diffeomorphic to $(\mathbb{R}, H_{(\bullet,\emptyset)})$, where $\bullet$ is the rooted tree with one vertex. Furthermore, $\{0\} \times [0, \infty) \subset \mathbb{R}^2$ is germ diffeomorphic to $(\mathbb{R}^2, H_{(\bullet,\bullet)})$, where the root is taken as a leaf and given a positive sign. △

We have a Generalized Stability Theorem for generalized arboreal hypersurface.

Theorem 5.20 (Generalized Stability Theorem)**.** *Let* $(\mathbb{T}, \ell_\varepsilon)$ *be a special signed leafy rooted tree, set* $m = v(\mathbb{T})$, *let* $\varphi_\pm : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{S}^0$ *be embeddings mapping* 0 *to* $(0, \pm 1) \in \mathbb{R}^m \times \mathbb{S}^0 = \mathbb{R}^m \times \{-1, 1\}$ *We write* $\pi : \mathbb{R}^m \times \mathbb{S}^0 \to \mathbb{R}^m$ *for the obvious projection.*

Write $(\mathbb{T}_+, \ell_+)$ *and* $(\mathbb{T}_-, \ell_-)$ *for the two special signed leafy rooted trees obtained from* $(\mathbb{T}, \ell_\varepsilon)$ *by deleting the root vertex and restricting the signs. We write* $H_\pm = \varphi_\pm(H_{\mathbb{T}_\pm} \times \mathbb{R}^{m-n_\pm})$ *where* $n_\pm = v(\mathbb{T}_\pm)$ *and define* $H = H_+ \cup H_-$.

Suppose that H *projects self-transversely under* π. *Then* $\mathcal{H} = \mathbb{R}^m \times \{0\} \cup \mathcal{C}(H)$, *where* $\mathcal{C}(H)$ *denotes the vertical cone of* H, *is an arboreal hypersurface of type* $(\mathbb{T}, \ell, m + 1)$ *at the origin.*

Proof. The proof is very similar to the proof of Theorem 5.11, which is Theorem 5.20 for $\ell = \emptyset$, with some minor variations that we spell out.

We proceed by strong induction over the sum n of the number of vertices of $(\mathbb{T}, \ell)$ and the number leaves in ℓ, note that the base case is already established since a rooted tree with one vertex has no leaves.

Assume that the assertion has been established for all special signed leafy rooted trees with $v(T) + |\ell| < n$. Let $(\mathbb{T}, \ell)$ be a special signed leafy rooted tree with $v(T) + |\ell| = n$, assume $\ell \neq \emptyset$ since this case has already been covered by Theorem 5.11. Let $\beta \in \ell$, without loss of generality $\beta \in V(\mathbb{T}_+)$. Similar to above we write $(\mathbb{T}'_+, \ell'_+)$ for the special signed leafy rooted tree obtained from $(\mathbb{T}_+, \ell_+)$ by removing β. We write $\mathbb{T}_+^+$ for the special signed rooted tree obtained from $(\mathbb{T}_+, \ell_+)$ by adding a vertex above the marked leaves and denote the vertex added above β by $\hat{\beta}$.

The germ of $(\mathbb{R}^m, H_+)$ is diffeomorphic to the germ of $(\mathbb{R}^m, H_{(\mathbb{T}_+,\ell_+)} \times \mathbb{R}^{m-n})$ and per definition

$$H_{(\mathbb{T}_+,\ell_+)} = H_{\mathbb{T}'_+} \times \mathbb{R}^2 \cup P_{\hat{\beta}}.$$

We write $H_+[\hat{\beta}]$ for the smooth piece of H_+ indexed by $\hat{\beta}$ and see that H'_+ is a generalized arboreal hypersurface of type $(\mathbb{T}_+, \ell_+, m)$ at the origin. Thus, by induction, $\mathcal{H}' = \mathbb{R}^m \times \{0\} \cup$

$\mathcal{C}(H'_+ \cup H_-)$ is a generalized arboreal hypersurface of type $(\mathbb{T}'_+, \ell'_+, m)$ and we may assume

$$\mathcal{H}' = H_{(\mathbb{T}'_+,\ell'_+)} \times \mathbb{R}^{m-n+2}.$$

We consider the linear special signed leafy rooted subtree $(\mathbb{A}_{\hat{\beta}}, \ell_{\hat{\beta}})$ and set d to be the number of complementary vertices.

Now we have a $(\mathbb{A}_{\hat{\beta}}, \ell_{\hat{\beta}}, m)$-hypersurface given by the union $\mathcal{K} = \cup_{\alpha \in V(\mathbb{A}_{\hat{\beta}})} \mathcal{H}[\alpha]$ such that for $(\mathbb{A}'_{\hat{\beta}}, \ell'_{\hat{\beta}}) = (\mathbb{A}_{\hat{\beta}}, \ell_{\hat{\beta}}) \cap (\mathbb{T}'_+, \ell'_+)$ and $\mathcal{K}' = \mathcal{K} \cap \mathcal{H}'$ we have

$$\mathcal{K}' = H_{(\mathbb{A}'_{\hat{\beta}},\ell'_{\hat{\beta}})} \times \mathbb{R}^{m-n+2+d}.$$

But $\mathcal{K}$ is an arboreal $(\mathbb{A}_{\hat{\beta}}, \ell_{\hat{\beta}}, m)$-hypersurface and thus there is a diffeomorphism Φ of the germ of $(\mathbb{R}^m, \mathcal{K})$ to the germ of $(\mathbb{R}^m, = H_{(\mathbb{A}'_{\hat{\beta}},\ell'_{\hat{\beta}})} \times \mathbb{R}^{m-n+2+d})$. As before, this diffeomorphism must leave $\mathcal{K}'$ fixed. By viewing the complementary directions as parameters we can assure $\mathcal{H}'$ is also preserved. □

Using the Generalized Stability Theorem we get the following theorem through induction over $v(\mathbb{T}) + |\ell|$ in the same manner as in the proof of Theorem 5.10. Remark 5.19 serves as the induction steps.

Theorem 5.21. *Per dimension m, Definition 5.14 produces only finite many local models up to ambient diffeomorphism. More specifically, every class in the collection Arb_m^{genhyp} can be assigned a special signed leafy rooted tree $(\mathbb{T}, \ell_\varepsilon)$ with $v(\mathbb{T}) + |\ell| \leqslant m$.*

In the same way as for non-generalized arboreal hypersurface singularities, the proof of Theorem 5.22 can also be done parametrically, yielding the following theorem.

Theorem 5.22. *Let $(\mathbb{T}, \ell)$ be a special signed leafy rooted tree, let $m = v(\mathbb{T})$, let $\varphi^y_\pm : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{S}^0$ be families of germs of embeddings mapping 0 to $(0, \pm 1) \in \mathbb{R}^m \times \mathbb{S}^0 = \mathbb{R}^m \times \{-1, 1\}$ and let $\pi : \mathbb{R}^m \times \mathbb{S}^0 \to \mathbb{R}^m$ be the obvious projection.*

Write $(\mathbb{T}_+, \ell_+)$ and $(\mathbb{T}_-, \ell_-)$ for the two special signed leafy rooted trees obtained from $(\mathbb{T}, \ell)$ by deleting the root vertex and restricting the signs. We write $H^y_\pm = \varphi^y_\pm(H_{\mathbb{T}_\pm} \times \mathbb{R}^{m-n_\pm})$ where $n_\pm = v(\mathbb{T}_\pm)$, define $H^y = H^y_+ \cup H^y_-$ and $\mathcal{H}^y := \mathbb{R}^m \times \{0\} \cup \mathcal{C}(H^y)$.

Then there exists a family of (germs of) diffeomorphisms of the germ of $(\mathbb{R}^m, \mathcal{H}^y)$ to the germ of $(\mathbb{R}^m \times H_{(\mathbb{T},\ell,m+1)} \times \mathbb{R}^{m-v(\mathbb{T})+|\ell|})$.

Chapter 6

Producing Arboreal Skeleta

In this chapter we prove that any manifold M that can be decomposed as a compact domain with an infinite cylindrical attached to its boundary, can be equipped with an X-convex structure such that its skeleton is an arboreal hypersurface. We do this by starting with an appropriate Morse function on M and "spreading out" its critical points to manifolds of critical points. This procedure increases the dimension of the stable manifolds, thus it can not produce a hypersurface skeleton if one of the bones has codimension 0. In particular this means that closed manifolds do not admit arboreal skeleta.

In Section 6.1 we define *Morse-Bott functions*, which are smooth functions with a relaxed non-degeneracy condition at their critical points, and *Morse-Bott vector fields*, which are gradient vector fields of Morse-Bott functions that are compatible with their local structure around critical points. Then, in Section 6.2 we discuss properties of finite type X-convex manifolds with Morse-Bott vector fields. We explain how every manifold that can be decomposed as a compact domain with an infinite cylindrical attached to its boundary, can be given a finite type X-convex Morse-Bott structure such that all the stable manifolds have positive codimension. Section 6.3 is dedicated to proving that, if M is a manifold with a finite type X-convex Morse-Bott structure whose skeleton has positive co-dimension, then M admits a finite type X-convex Morse-Bott structure with an arboreal skeleton. We conclude this chapter with Section 6.4, where we discuss how to make consistent choices for the signs on the edges of the trees describing the arboreal singularities.

Throughout this chapter M is an n-dimensional manifold.

6.1 Morse-Bott vector fields

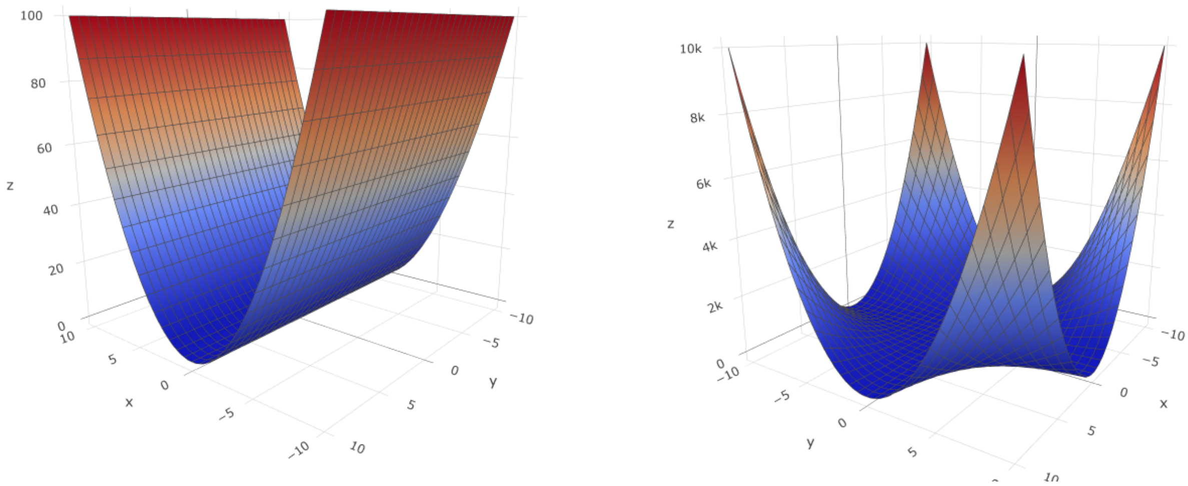
Recall that a Morse function f on M is a smooth map $f : M \to \mathbb{R}$ whose Hessian is non-degenerate at every critical point. A corollary of the Morse Lemma is that the critical points of a Morse function must be isolated, while we want to be able to "spread out" the singularities occurring at critical points. We slightly relax our notion of non-degeneracy to allow for smooth manifolds of critical points. The following notion on non-degeneracy is due to R. Bott, it is slightly to restrictive for our purposes, but serves as a starting point for our even more general definition.

Definition 6.1. A *classic Morse-Bott* function on M is a smooth map $f : M \to \mathbb{R}$ such that each connected component C of $\mathrm{Crit}(f)$ is an embedded submanifold of M and the Hessian of f defines a fiberwise non-degenerate pairing on the normal bundle of C. △

Remark 6.2. If we fix a Riemannian metric on M, the Hessian H_f splits the tangent space T_pM as $E_p^+ \oplus E_p^0 \oplus E_p^-$ into generalized eigenspaces with eigenvalues having positive, zero and negative real part at any critical point $p \in C \subset \mathrm{Crit}(f)$. The non-degeneracy condition on the Hessian of f in the normal bundle of a connected component $C \subset \mathrm{Crit}(f)$ is equivalent to demanding that $TC = E^0|_C$. △

Example 6.3. Recall that the function $f(x, y) = x^2$ is not Morse, since the critical points form the x-axis and are all degenerate. It is, however, classic Morse-Bott.

The function $f(x, y) = x^2y^2$ is not classic Morse-Bott, the critical points form the x-axis and y-axis and thus a connected component of $\mathrm{Crit}(f)$ is not a smooth manifold. △



(a) $f(x, y) = x^2$ (b) $f(x, y) = x^2y^2$

Figure 6.1: A classic Morse-Bott function and a function that is not classic Morse-Bott.

Just like Morse functions, classic Morse-Bott functions admit canonical local models around critical points. This property is captured in the Morse-Bott Lemma, a proof of which can for instance be found in [BH04].

Lemma 6.4 (Morse-Bott Lemma). *Let $f : M \to \mathbb{R}$ be a Morse-Bott function and let $C \subset \mathrm{Crit}(f)$ be a connected component of dimension d. Let $p \in C$, then there exists an open neighbourhood U of p and a smooth chart $\varphi : U \to \mathbb{R}^d \times \mathbb{R}^{m-d}$, such that*

- $\varphi(p) = 0$,

- $\varphi(U \cap C) = \{(x, y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-d-k} \times \mathbb{R}^d | x = y = 0\}$,
- $(f \circ \varphi^{-1})(x, y, z) = f(C) - \|x\|^2 + \|y\|^2$

where $k \leqslant n - d$ is the index of the Hessian at p, and $f(C)$ is the common value of f on C.

A direct corollary of the Morse-Bott Lemma is that the index k of the Hessian is constant on the connected component $C \subset \mathrm{Crit}(f)$, we say C is a *critical manifold* of *index* k.

We generalize the definition of Morse-Bott to allow the critical manifolds to have boundary, but do control the function close to the boundary of the critical manifolds such that it is compatible with the local form of classic Morse-Bott functions.

Definition 6.5. A smooth function $f : M \to \mathbb{R}$ is *Morse-Bott* if for each connected component C of critical points of f we have the following non-degeneracy condition

- C is an embedded submanifold with (possibly empty) boundary in M,
- $TC = E^0|_C$.

Moreover, for any critical manifold C we demand the following local models.

- If $p \notin \partial C$, there is an open neighbourhood U of p and a smooth chart $\varphi : U \to \mathbb{R}^m$ with $\varphi(p) = 0$ such that

  $$\varphi(U \cap C) = \{(x, y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-d-k} \times \mathbb{R}^d | x = y = 0$$

  and

  $$\left(f \circ \varphi^{-1}\right)(x, y, z) = f(C) - \|x\|^2 + \|y\|^2$$

  with $k \leqslant n - d$.

- If $p \in \partial C$, there is an open neighbourhood U of p and a smooth chart $\varphi : U \to \mathbb{R}^m$ with $\varphi(p) = 0$ such that

  $$\varphi(U \cap C) = \{(x, y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-d-k} \times \mathbb{R}^d | x = y = 0, \quad z_1 \leqslant 0$$

  and

  $$\left(f \circ \varphi^{-1}\right)(x, y, z) = f(C) - \|x\|^2 + \|y\|^2 \pm \rho(z_1) z_1^2$$

  with $\rho : \mathbb{R} \to [0, 1]$ a smooth bump function satisfying

  - $\rho = 0$ on $(-\infty, 0]$,
  - $\rho = 1$ on $[\varepsilon, \infty)$ for some $\varepsilon > 0$,
  - $\rho < 0$ and $\rho' > 0$ on $(0, \varepsilon)$ for the same ε.

In the same way as for Morse functions, we say (x, y, z) are the *Morse-Bott coordinates* on the *Morse-Bott neighbourhood U*. △

Remark 6.6. The space of allowed ρ is convex, thus the local model is well defined up to a contractible choice. △

Note that the required local models only impose extra conditions on critical manifolds C with $\partial C \neq \emptyset$, if C has no boundary the Morse-Bott Lemma shows that the non-degeneracy condition implies the existence of these local models.

As before, we consider vector fields compatible with these local models.

Definition 6.7. We say a smooth vector field X on M is *Morse-Bott* if it is gradient-like for a Morse-Bott with boundary function $f : M \to \mathbb{R}$ such that for every critical point there is a Morse-Bott neighbourhood on which X corresponds to the gradient of f with respect to the canonical metric on $\mathbb{R}^n$ △

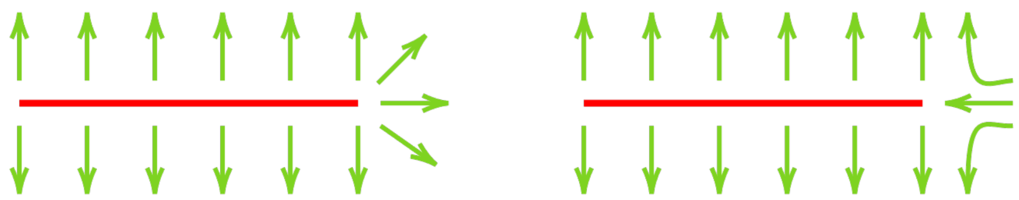


Figure 6.2: A critical manifold of a Morse-Bott vector field with a repelling boundary point and a critical manifold with an attracting boundary point.

Remark 6.8. From the local models we see that the stable and unstable set of a zero p of a Morse-Bott vector field are the unique invariant manifolds that are tangent to respectively E_p^- and E_p^+. △

6.2 Morse-Bott X-convex manifolds

Recall that a convex structure is a pair (M, X) of an n-dimensional manifold M and a complete vector field $X \in \mathfrak{X}(M)$ such that there is an exhaustion by compact domains $M = \cup_{k=1}^{\infty} M^k$ such that X is outwardly transverse to the boundary of each M^k. Furthermore, recall that the skeleton of (M, X) is given by

$$\mathrm{Skel}(M, X) = \cup_{k=1}^{\infty} \cap_{t>0} \varphi^{-t}(M^k)$$

Definition 6.9. A convex structure (M, X) is said to be *Morse (resp. Morse-Smale/Morse-Bott)* if X is Morse (resp. Morse-Smale/Morse-Bott). △

Remark 6.10. The skeleton of a Morse-Bott convex structure is the union of the stable manifolds of all the critical sets. We only consider finite type (M, X), i.e. (M, X) with compact skeleton. The stable manifolds give a disjoint cover of the skeleton, thus every Morse-Bott vector field considered has a finite number of critical manifolds. △

The specific Morse-Bott function $f : M \to \mathbb{R}$ associated to a Morse-Bott convex structure (M, X) is not part of the data, we only require its existence. The function f can be considered to be the primitive of X. In fact, there is never a unique associated Morse-Bott function; any addition of a non-zero constant yields a new function that is again Morse-Bott.

It is, however, often convenient to pick a fixed associated Morse-Bott function for a Morse-Bott vector field.

Definition 6.11. A *Morse-Bott (resp. convex structure* (abridged: *MBc-structure*) (M, X, f) is a Morse-Bott convex structure (M, X) with a choice of $f : M \to \mathbb{R}$ associated to X. Likewise, a *Morse-Smale convex structure* (M, X, f) is a Morse-Smale convex structure (M, X) with a choice of $f : M \to \mathbb{R}$ associated to X. △

Definition 6.12. A *homotopy of MBc-structures* is a 1-parameter family (X_t, f_t) of MBc-structures such that there is an exhaustion by compact domains $M = \cup_{k=1}^{\infty} M^k$ such that for each t, X_t is outwardly transverse to the boundary of every M^k. If there is a homotopy of MBc-structures between two MBc-structures (X_0, f_0) and (X_1, f_1), we call them *MBc-homotopic.* △

We use the following terminology of L. Starkston in [Sta18].

Definition 6.13. If C is a connected component of zeros of a Morse-Bott vector field X, we say $\Delta = \mathrm{Stab}(C)$ is a *bone* of the skeleton and C is the *marrow* of the bone.

We say a bone Δ_1 has a *joint* H on Δ_2 if $H = \overline{\Delta}_2 \cap \Delta_2 t$.

The *index* of a bone is the dimension of the stable manifold of any point in the marrow, we denote the index of δ by $i(\delta)$. △

We record the following properties of joints, which were proven for joints of symplectic arboreal skeleta in Lemma 3.10 of [Sta18].

Lemma 6.14.
1. *If Δ_1 has a non-empty joint on Δ_2, then Δ_2 can not have a non-empty joint on Δ_1.*
2. *If Δ_1 has a non-empty joint on Δ_2, and Δ_2 has index $k > 0$, then there is a bone Δ_3 such that both Δ_1 and Δ_2 have non-empty joint on Δ_3.*

Proof. We write C_1, C_2 for the marrow of Δ_1, Δ_2 respectively. If Δ_1 has a non-empty joint on Δ_2, it must be that the value of Morse-Bott function f on C_1, $f(C_1)$, must be strictly greater than $f(C_2)$. This proves the first assertion.

For the second assertion, consider a point $p \in \overline{\Delta_1} \cap \Delta_2$. There must be a point $q \in C_2$ whose unstable manifold contains p. Because the index of Δ_2 is greater than zero, the stable manifold of q must also contain some point $r \neq p$. This r must be in the unstable manifold of some zero of the Morse-Bott vector field, which lies on the marrow of some bone Δ_3. By the first assertion, we know that $\Delta_1 \neq \Delta_3$.

Now, there is a broken flow-line from Δ_3 through Δ_2 to Δ_1, which by gluing has a nearby unbroken flowline from Δ_3 to Δ_1. □

If a bone has codimension 1 we call it a *hypersurface bone*, if all bones in a skeleton are hypersurface bones we call the skeleton a *hypersurface skeleton*.

Our goal for the rest of this chapter is to find a procedure to homotope a given MBc-structure to a MBc-structure with a hypersurface skeleton such that all its singularities are of (generalized) arboreal type, we call this procedure *arborealization*.

We do this by spreading out the critical points, obtaining higher dimensional critical manifolds and thus higher dimensional bones. This procedure can only increase the dimension of bones, thus it can not produce a hypersurface skeleton if one of the bones has codimension 0.

Definition 6.15. A MBc-structure (X, f) is called *nice* if every bone of the skeleton of (M, X) has positive codimension. △

Note that closed manifolds can not admit a nice MBc-structure, a smooth map on a closed manifold must obtain a maximum at some critical manifold C. Then we see from the local models that the bone Stab(C) must have the same dimension of the manifold.

However, any manifold M that can be decomposed as a compact domain D with boundary and an infinite cylindrical end $\partial D \times [0, \infty)$ attached to this boundary ∂D, i.e.

$$M = D \bigcup_{\partial D \times \{0\}} \partial D \times [0, \infty),$$

admits a nice Morse structure.

Lemma 6.16. *Let M be a manifold with a compact domain with non-empty boundary $D \subset M$ such that $M = D \cup_{\partial D \times \{0\}} \partial D \times [0, \infty)$, then M admits a nice Morse structure.*

Proof. Theorem 8.1 from [Mil65] asserts that D can be equipped with a nice Morse function $f : M \top \mathbb{R}$ whose critical points are all interior. We extend this function as $e^s f$ on the cylindrical end. □

The following theorem shows that it is sufficient to construct an arborealization procedure for nice Morse-Smale convex structures, since every nice MBc-structure is MBc-homotopic to a nice Morse-Smale convex structure.

Theorem 6.17. *Any MBc-structure (X, f) is MBc-homotopic to a Morse-Smale convex structure (Y, g), in particular any nice MBc-structure is MBc-homotopic to a nice Morse-Smale convex structure.*

We give a sketch of the proof, the details can be found in Section 5 of [BH08].

Proof. Let C_i, with $i = 1, ..., k$, be the connected components of critical points of f. For every i let T_i be a small tubular neighbourhood around C_i such that T_i is covered by opens on which X is gradient with respect to the canonical metric. Pick a positive Morse function f_i on C_i and extend it to T_i by making f_i constant in the direction normal to C_i. Let ρ_i be a smooth bump function that is equal to 1 on C_i and compactly supported on T_i, we define

$$g_t = f + t\varepsilon \left(\sum_{i=1}^{k} \rho_i f_i \right).$$

For appropriate ρ_i and ε sufficiently small, g_t is a Morse function for every $t \in (0,1]$ while $g_0 = f$. For every $t \in [0,1]$ we define Y_t to be the gradient of g_t, with respect to the canonical metric, on $T = \cup_{i=1,\dots,k} T_i$ and X outside T, by construction this gives a smooth vector field that is gradient like for g_t. Since the Smale condition is generic, each ρ_i can be chosen such that (Y_1, g_1) is Morse-Smale.

The pair (Y_t, g_t) is a MBc-homotopy between (X, f) and $(Y, g) = (Y_1, g_1)$. □

Remark 6.18. If λ_i is the index of C_i and $p \in C_i$ is a critical point of f_i of index λ_f then p is a critical point of index $\lambda_i + \lambda_f$ of g as produced by the lemma above. △

6.3 The arborealization procedure

This section is dedicated to proving the following theorem.

Theorem 6.19. *Let (X, f) be a nice Morse-Smale convex structure on M. Then there is a homotopy (X_t, f_t) of MBc-structures such that $(X_0, f_0) = (X, f)$ and (X_1, f_1) has an arboreal skeleton.*

The proof consists of two steps: first we produce a MBc-homotopy from an arbitrary Morse-Smale convex structure (X, f) to a MBc-structure with a hypersurface skeleton by spreading out the critical points to critical manifolds. Then we perturb this MBc-structure such that all singularities are arboreal.

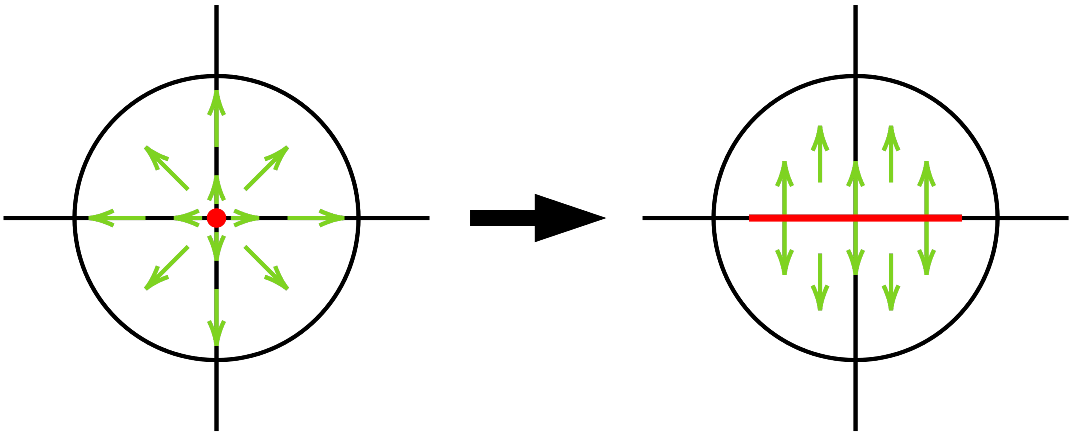


Figure 6.3: Spreading out the critical point of a Morse-Smale vector field to a critical manifold of a Morse-Bott vector field.

We do not want to thicken a critical point to a critical manifold in such a way that stable manifolds of other critical points attach to the boundary of the critical manifold, i.e. we want to avoid the situation where a bone Δ_1 has a joint on the boundary of another bone Δ_2. We say that a skeleton has *interior joints* if for any pair of bones Δ_1, Δ_2 the (possible empty) joint $\overline{\Delta_1} \cap \Delta_2$ is disjoint from the boundary $\partial\Delta_2$.

The following lemma assures that we can move the stable manifolds "out of the way" before we spread out the critical points to critical manifolds.

Lemma 6.20. *Let (X, f) be a nice Morse-Smale convex structure on M and let p be a critical point of f of index k. Then there is a homotopy (X_t, f_t) of nice Morse-Smale convex structures from $(X, f) = (X_0, f_0)$ to (X_1, f_1) such that:*

- *There is a neighbourhood V of p on which we have coordinates $(x, y) = (x_1, ..., x_k, y_1, ..., y_{n-k})$ centred at p such that $f_1(x, y) = f_1(p) - \|x\|^2 + \|y\|^2$ and X_t is the gradient of f_t on V.*
- *The Morse-Smale convex structures (X, f) and (X_t, f_t) agree outside some neighbourhood U of p.*
- *For any other critical point $x \neq p$ of f, the intersection of V and the stable manifold of s is contained in $\{y_1 < 0\}$.*

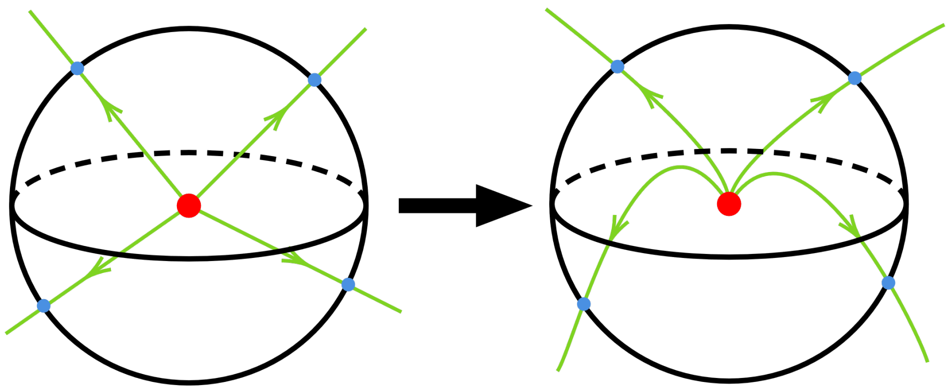


Figure 6.4: Moving the stable manifolds.

Proof. Let U be a Morse-Smale neighbourhood of p with associated coordinates $(x, y) = (x_1, ..., x_k, y_1, ..., y_{n-k})$ centred at p such that $f(x, y) = f(p) - \|x\|^2 + \|y\|^2$ and X is the gradient of f on U.

Fix $\delta > 0$ such that the ball centred at p with radius 2δ with respect to the standard Euclidean metric associated to the coordinates (x, y) is contained in U, we denote this ball by $B_{2\delta}$. We write $S'_{2\delta} := \text{Unstab}(p) \cap S_{2\delta}$ for the intersection of the boundary $S_{2\delta}$ of this ball with the unstable manifold Unstab(p).

Note that on U we have $\text{Unstab}(p) = \{x_1 = ... = x_k = 0\}$, meaning the vector field X is precisely the outward normal of $S'_{2\delta}$. Therefore Unstab(p) and $S_{2\delta}$ have transverse intersection, thus their intersection is a manifold.

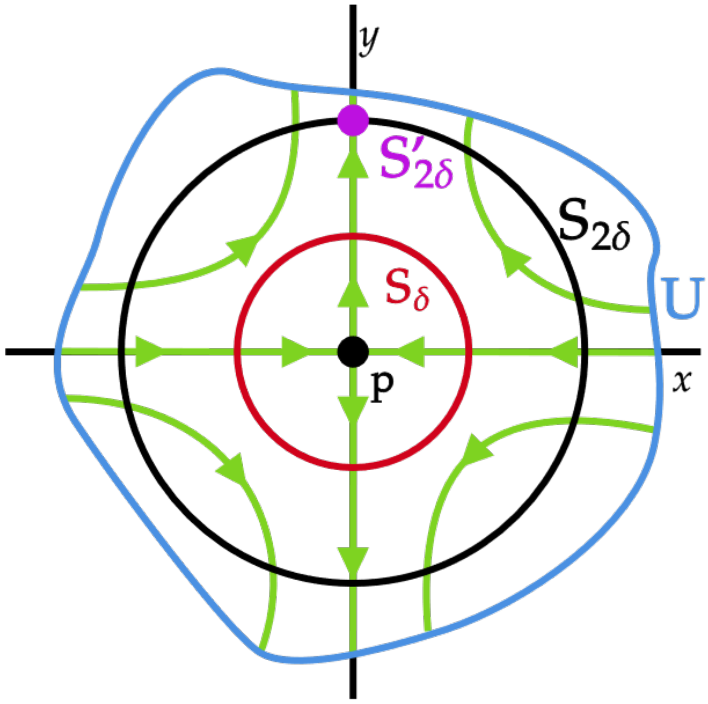


Figure 6.5: The Morse neighbourhood of p.

The Smale condition now gives that for any critical point s of f we must have either

$$\text{Unstab}(p) \cap \text{Stab}(s) = \emptyset \qquad \text{or} \qquad \dim(S'_{2\delta} \cap \text{Stab}(s)) < \dim(S'_{2\delta}).$$

This means that for any critical point s of f the intersection $S'_{2\delta} \cap \text{Stab}(s)$ has measure zero in $S'_{2\delta}$. Since f has a finite number of critical points, the union

$$S = \bigcup_{s \text{ critical poin of } f} S'_{2\delta} \cap \text{Stab}(s)$$

has measure zero in $S'_{2\delta}$ and thus its complement in $S'_{2\delta}$ is dense. Therefore it must be possible to embed a small disc D into $S'_{2\delta}$ that does not intersect any stable manifold. We rotate our coordinate axis in the y hyperplane such that the y_{n-k} axis passes trough the centre of D.

Now, using generalized polar coordinates $(r, \theta_1, ..., \theta_{n-k-1})$ on $U' = U \cap \text{Unstab}(p)$, we can parametrize $S'_{2\delta}$ as

$$\begin{aligned} y_1 &= 2\delta \sin\theta_1 \sin\theta_{k2} ... \sin\theta_{n-k-1} \\ y_1 &= 2\delta \cos\theta_1 \sin\theta_{k2} ... \sin\theta_{n-k-1} \\ &\vdots \\ y_{n-k} &= 2\delta \cos\theta_{n-k-1} \end{aligned}$$

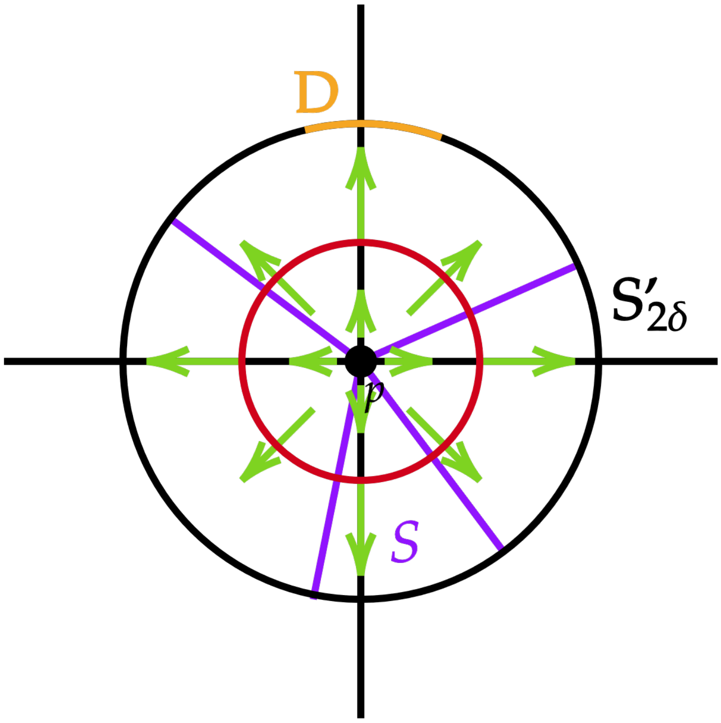


Figure 6.6: The embedded disc D, indicated in yellow, and the stable manifolds in that enter Unstab(p), indicated in purple.

such that D is give by $\theta_{n-1} \leqslant \varepsilon$.

Let $\chi : [0,\infty) \to [0,1]$ be a smooth monotonically decreasing step function which is 1 on a neighbourhood of $[0,\delta^2]$ and vanishes on a neighbourhood of $[4\delta^2,\infty)$. We define

$$\Phi_t : [0,\infty) \times [0,\pi]^{n-1} \to [0,\infty) \times [0,\pi]^{n-k-1}, \quad (r,\theta_1,...,\theta_{n-1}) \mapsto (r,\theta_1,...,H_t(r,\theta_{n-k-1}))$$

where $H_t : [0,\infty) \times [0,\pi] \to [0,\pi]$ is defined as

$$H_t(r,\theta) = \begin{cases} \left(1 + t\chi(r^2)\left(\frac{\pi}{2\varepsilon} - 1\right)\right)\theta & \text{for } 0 \leqslant \theta \leqslant \varepsilon \\ (\theta - \pi)\left(1 - t\chi(r^2)\left(\frac{\pi}{2(\varepsilon-\pi)} + 1\right)\right) + \pi & \text{for } \varepsilon \leqslant \theta \leqslant \pi. \end{cases}$$

We remark that $H_0(r,\theta) = \theta$, $H_t(r,0) = 0$ and $H_t(r,\pi) = \pi$. Furthermore, for all $r \geqslant 2\delta$ we have $H_t(r,\theta) = \theta$ and for all $r \leqslant \delta$ we have $H_1(r,\varepsilon) = \frac{\pi}{2}$.

Note that for any $t \in [0,1]$ and $r \in [0,\infty)$ the map $H_t(r,-) : [0,\pi] \to [0,\pi]$ is a diffeomorphism leaving the boundary fixed, from which we see that Φ_t is a diffeomorphism that leaves the boundary fixed for every $t \in [0,1]$. Thus, using generalized polar coordinates, every Φ_t gives a diffeomorphism $\Phi_t : B'_{2\delta} \to B'_{2\delta}$. By using generalized cylindrical coordinates on $B_{2\delta}$ this diffeomorphism can be extended to a diffeomorphism

$$\widehat{\Phi}_t : B_{2\delta} \to B_{2\delta}, \quad (x,r,\theta_1,...,\theta_{n-k-1}) \mapsto (x,r,\theta_1,...,H_t(\sqrt{r^2 + \|x\|^2},\theta_{n-k-1})).$$

This map leaves a neighbourhood of the boundary fixed, since χ vanishes on a neighbourhood of $[4\delta^2, \infty)$ and thus can be extended to a diffeomorphism $\Psi_t : M \to M$ that is the identity outside of $B_{2\delta}$.

The homotopy of nice Morse-Smale (X_t, f_t) is now given by pushing (X, f) forward along Ψ_t. Outside $B_{2\delta}$, and thus also outside U, we have constructed Ψ_t to be the identity. Therefore (X, f) and (X_t, f_t) agree outside of U.

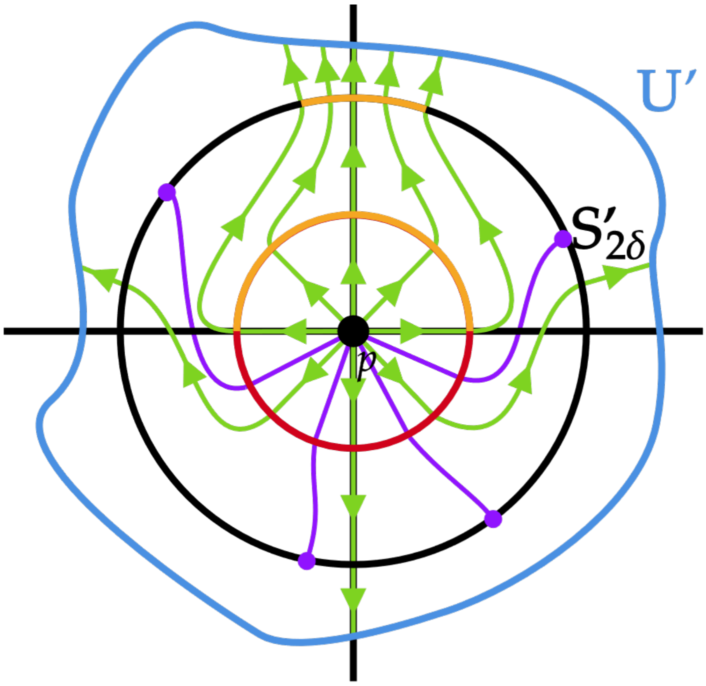


Figure 6.7: The resulting Morse-Smale convex structure (X_1, f_1).

Note that on B_δ

$$f_t(x, y) = (f \circ \Psi_t)(x, y) = f_t(p) - \|x\|^2 + \|y\|^2$$

and X_t is the gradient of f_t with respect to the canonical metric.

To see that the intersection of B_δ and the stable manifold of any critical point $s \neq p$ is contained in $\{y_{n-k} < 0\}$ we first remark that on U' the vector field X is radial. Thus the integral curve passing trough a point $(0, 2\delta, \theta_1, ..., \theta_{n-k-1}) \in U'$ is parametrized by $\{(r, \theta_1, ..., \theta_{n-k-1}) \mid r \in (0, \infty]\}$. Now, let $q = (0, 2\delta, \theta_1, ..., \theta_{n-k-1}) \in \mathrm{Stab}(s)$ for some critical point $s \neq p$. Note that $\Psi_1(q) = q$, thus from the existence and uniqueness of solutions to differential equations and the definition of X_1 as the pushforward of X along a diffeomorphism, we know the integral curve of X passing through q must be taken to the integral curve of X_1 passing through q. In particular this means that the integral curve of X_1 passing through q must pass through $\Psi_1(0, \delta, \theta_1, ..., \theta_{n-k-1}) \in \{y_{n-k} < 0\}$, and by construction $\Psi_1(0, \delta, \theta_1, ..., \theta_{n-k-1}) \in \{y_{n-k} < 0\}$ since $D \cap \mathrm{Stab}(s) = \emptyset$.

Since X_1 is radial on B_δ we see that the intersection of B_δ and Stab(s) is contained in the $\{y_{n-k} < 0\}$ hypersurface. $\square$

The following lemma asserts that we can spread out the critical point into a critical manifold such that its stable manifold had codimension 1.

Lemma 6.21. *There is a homotopy (X_t, f_t) of nice MBc-structures from the standard radial structure $(X, f) = \left(-2\sum_{i=1}^{k} x_i\partial_{x_i} + 2\sum_{j=1}^{n-k} y_j\partial_{y_j}, -\|x\|^2 + \|y\|^2\right)$ on $\mathbb{R}^k \times \mathbb{R}^{n-k}$ with a single critical point of index $k < n$ at the origin to a nice MBc-structure (X_1, f_1) such that*

- *The critical manifold $C = Crit(f_1)$ is a $n-k-1$-dimensional disc contained in*

$$\{(x, y, z) \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{n-k-1} | \quad x = y = 0\}$$

  .

- *The bone $Stab(C)$ has codimension 1.*
- *The MBc-structure (X_1, f_1) agrees with the standard radial structure of index k outside some neighbourhood U of C.*

Proof. Let $\chi : [0, \infty) \to [0, 1]$ be a smooth function such that for small fixed $\delta, \varepsilon > 0$ with $\varepsilon < \delta$

- $\chi = 1$ on $[0, \delta + \varepsilon]$,
- $\chi = 0$ on $[2\delta - \varepsilon, \infty)$,
- $\chi < 1$ and $\chi' < 0$ on $(\delta + \varepsilon, 2\delta - \varepsilon)$.

For $t \in [0, 1]$ we define the family of functions

$$\psi_t : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{n-k-1} \to \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{n-k-1}, \quad p = (x, y, z) \mapsto \left(x, y, \sqrt{1 - t\chi\left(\|p\|^2\right)}z\right).$$

We define $f_t = \psi_t^* f = f \circ \psi_t$, i.e. the pullback of f by ψ_t. Explicitly, for $p = (x, y, z) \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{n-k-1}$,

$$f_t(x, y, z) = -\|x\|^2 + y^2 + \left[1 - t\chi\left(\|p\|^2\right)\right]\|z\|^2.$$

The gradient of f_t using the standard metric gives the family

$$\begin{aligned} X_t = 2\Bigg[&-\sum_{i=1}^{k} x_i\partial_{x_i} + y\partial_y + \left[1 - t\chi(\|p\|^2)\right]\sum_{j=1}^{n-k-1} z_j\partial_{z_j} \\ &- t\chi'(|p|^2)\,\|z\|^2\Big(\sum_{i=1}^{k} x_i\partial_{x_i} + y\partial_y + \sum_{j=1}^{n-k-1} z_j\partial_{z_j}\Big)\Bigg] \end{aligned}$$

of vector fields.

Now, $X_t = 0$ if and only if either $p = 0$ or $t = \chi(\,\|p\|^2\,) = 1$ and $x = y = 0$. We show this by case distinction. If $z = 0$ we see $X_t = 0$ if and only if $x = y = 0$. If $z \neq 0$ and $X_t = 0$ it must be that

$$1 - t\chi(\,\|p\|^2\,) - t\chi'(\,\|p\|^2\,)\,\|z\|^2 = 0.$$

Note that $-t\chi(\,\|p\|^2\,) \geqslant -1$ and $-t\chi'(\,\|p\|^2\,)\,\|z\|^2 \geqslant 0$, thus we must have

$$t\chi(\,\|p\|^2\,) = 1 \qquad \text{and} \qquad t\chi'(\,\|p\|^2\,)\,\|z\|^2 = 0$$

for $X_t = 0$ to hold. We see that $t\chi(\,\|p\|^2\,) = 1$ if and only if $t = \chi(\,\|p\|^2\,) = 1$, which also implies that $t\chi'(\,\|p\|^2\,)\,\|z\|^2 = 0$. In this case $X_t = 0$ if and only if $x = y = 0$.

We observe that indeed $(X_0, f_0) = (X, f)$, that $C = \mathrm{Crit}(f_1)$ is an $n - k - 1$-dimensional disc contained in $\{x = y = 0\}$, that $\mathrm{Stab}(C)$ has codimension 1 and (X_1, f_1) agrees with the radial structure when $\|p\|^2 \geqslant 2\delta$. □

We combine the above two lemmas to obtain the following corollary.

Corollary 6.22. *Let (X, f) be a nice Morse-Smale convex structure on M. Then there is a homotopy (X_t, f_t) of MBc-structures such that $(X_0, f_0) = (X, f)$ and (X_1, f_1) has a hypersurface skeleton with interior joints.*

Proof. The assertion follows by first inductively applying Lemma 6.20 on all critical points of f. We then inductively apply Lemma 6.21 on a Morse neighbourhood of every point, in order of increasing index. □

We now show that a MBc-structure with such a hypersurface skeleton can be perturbed to a MBc-structure with an arboreal skeleton

Lemma 6.23. *Let (X, f) be a MBc-structure on M with a hypersurface skeleton with interior joints, then there is a generic perturbation $\widetilde{X}$ of X such that $(\widetilde{X}, f)$ is a MBc-structures with an arboreal skeleton.*

Note that this generic perturbation of X constitutes a MBc-homotopy.

Proof. On every Morse-Bott neighbourhood there is, using Morse-Bott coordinates (x, y, z), a canonical projection $(x, y, z) \mapsto (x, 0, z)$. With a generic perturbation $\widetilde{X}$ of X, away from the critical manifolds and such that $\widetilde{X}$ is still gradient-like for f, we can make all the hypersurfaces in the skeleton project transversely under the projection $(x, y, z) \mapsto (x, 0, z)$.

Any $s \in \mathrm{Skel}(M, X)$ is contained in a unique bone Δ_1 and lies in the closure of a (possibly empty) collection of bones $\Delta_2, ..., \Delta_m$ having a joint on Δ containing s.

We show by strong induction over m, the number of bones whose closure contains s, that s is an arboreal singularity. If $m = 1$ the point s does not lie on any joint, thus it is either a smooth point of the skeleton or lies on the boundary of Δ_1. In the former case the skeleton is a generalized arboreal hypersurface of type $((\bullet,\emptyset),n)$ at s, in the latter case the skeleton is a generalized arboreal hypersurface of type $((\bullet,\bullet),n)$ at s.

Now let $m > 1$ and suppose that for every point $s' \in \mathrm{Skel}(M, \widetilde{X})$ that lies in the closure of less than m bones, the skeleton has a generalized arboreal singularity at s'.

Note that, by assumption, s is an interior point of Δ_1 if $m > 1$. Furthermore, we may assume that s is a critical point, i.e. we may assume that s lies on the marrow C_1 of Δ_1, by the product symmetry.

Using Morse-Bott coordinates (x, y, z) centred at s we consider the lifts $s_\pm := (0, \pm\delta, 0)$ for sufficiently small $\delta > 0$. It must be that there is a subset

$$\{\Delta_{i_1^-}, ..., \Delta_{i_{m_-}^-}\} \subset \{\Delta_2, ..., \Delta_m\}$$

of bones containing s_- in their closure. Likewise there is a subset

$$\{\Delta_{i_1^+}, ..., \Delta_{i_{m_+}^+}\} \subset \{\Delta_2, ..., \Delta_m\}$$

of bones having s_+ in their closure. Note that these two subsets are disjoint and together contain all $\Delta_2, ..., \Delta_m$.

Since $m_\pm \leqslant m - 1 < m$, the inductive hypothesis yields that the skeleton has a (generalized) arboreal singularity of type $(\mathbb{T}_\pm, \ell_\pm)$ at the points $s_\pm$. Now, by the vertical cone property of arboreal singularities, we know the skeleton has a generalized arboreal singularity of type $(\mathbb{T}, \ell)$ at s. Here $(\mathbb{T}, \ell)$ is the tree obtain by joining $(\mathbb{T}_-, \ell_-)$ and $(\mathbb{T}_+, \ell_+)$ at a root and decorating the edges incident to the root with the signs corresponding to the lift. □

These three lemmas prove Theorem 6.19, together with Theorem 6.17 we conclude that every nice MBc-structure is MBc-homotopic to a MBc-structure with an arboreal skeleton.

6.4 Co-oriented arboreal skeleta

The signs on the vertices of the trees describing arboreal singularities correspond to "up" or "down" in the Morse-Bott neighbourhood. But, in general the Morse-Bott neighbourhoods do not have a canonical choice of up and down, since the bones of a skeleton do not have a canonical co-orientation. Different (local) co-orientations of the bones yield different choices of signs for the trees describing the singularities, as already discussed in Remark 5.8. Thus, if no extra conditions are imposed on M and the skeleton, a skeleton can have many different combinatorial descriptions.

However, if M and all bones of the skeleton are orientable, an orientation of M induces a co-orientation of every bone. We call such a skeleton with an induced co-orientation of every bone a *co-oriented arboreal skeleton*. A co-oriented arboreal skeleton has a unique combinatorial

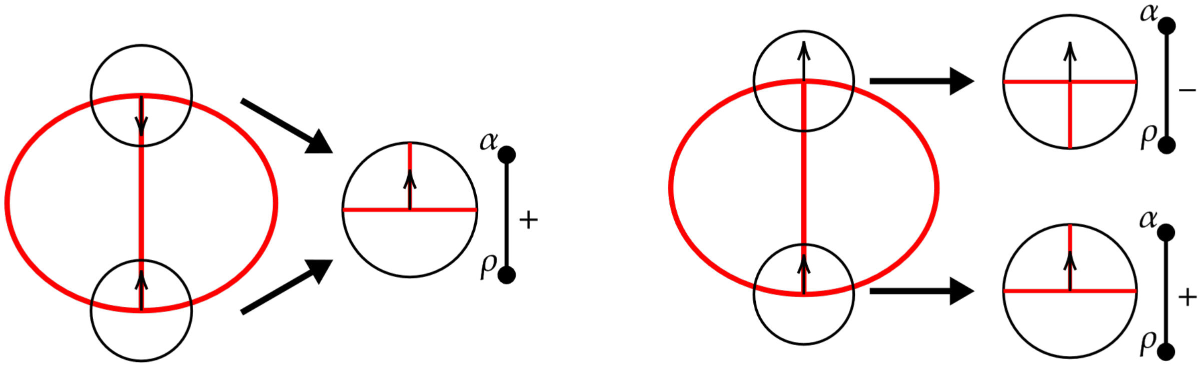


Figure 6.8: Different choices of local co-orientation lead to different combinatorial descriptions of the same arboreal skeleton.

description that is compatible with the co-orientation of each bone. Note that the arborealization procedure described in this chapter produces a skeleton with discs as bones, thus if M is orientable the procedure can be used to obtain a co-oriented arboreal skeleton of M.

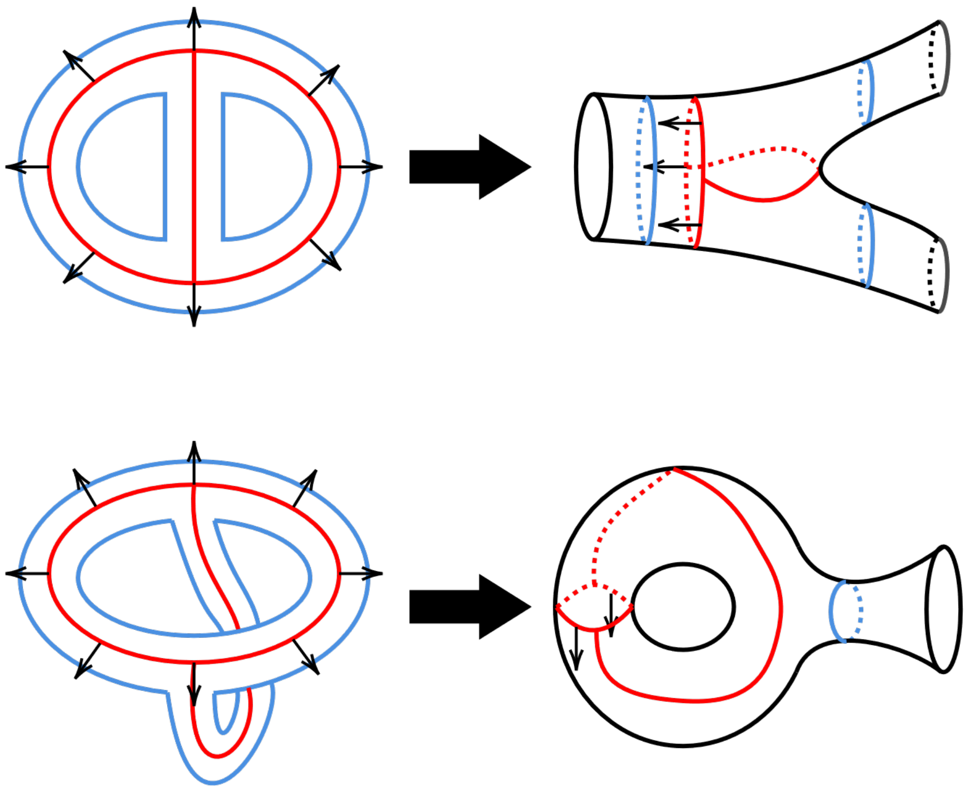


Figure 6.9: Two spaces with diffeomorphic arboreal skeleta, but non-diffeomorphic co-oriented skeleta.

Chapter 7

Thickening Arboreal Skeleta

We wish to be able to recover an X-convex manifold M, as defined in Definition 3.5, with an arboreal skeleton from the combinatorial data of this skeleton. In this chapter we describe how M can be recovered from its skeleton if this skeleton is co-oriented, i.e. M and all bones are orientable as defined in Section 6.4. The orientability condition on M is necessary to produce a thickening of the skeleton that is unique up to diffeomorphism. A given skeleton may have multiple, non-equivalent, thickenings if we allow non-orientable thickenings, as illustrated in Figure 7.1.

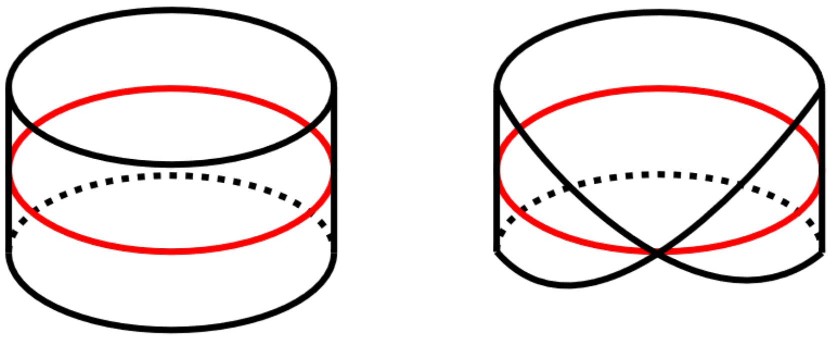


Figure 7.1: Arboreal skeleta can have multiple non-equivalent thickenings if we allow for non-orientable thickenings.

We will recover M by thickening every bone Δ as its trivial line bundle $\Delta \times \mathbb{R}$; we give this line bundle an MBc-structure such that Δ is its skeleton. Then we inductively glue these line bundles using the combinatorial data from the skeleton in such a way that we recover a manifold with the same skeleton as M.

To be able to inductively build up the manifold we need to impose some extra conditions on our skeleton, namely that the skeleton can also be inductively build up via a gluing construction. In particular, we do not allow for situations as pictured in Figure 7.3, where a bone of an

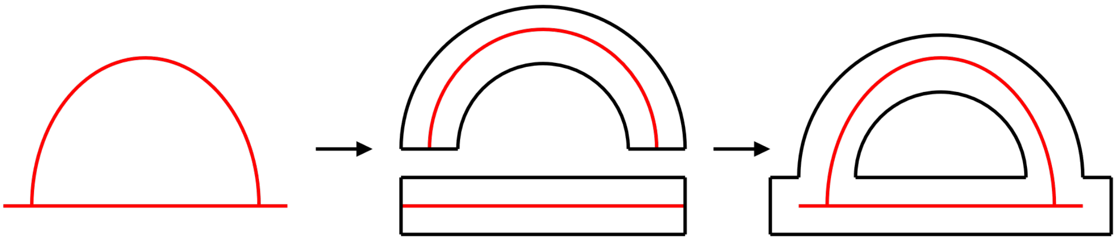


Figure 7.2: Thickening the bones and gluing them using the data from the skeleton.

arboreal skeleton self-intersects. The arboreal skeleta obtained from the arborealization procedure as described in the previous chapter satisfy these additional conditions.

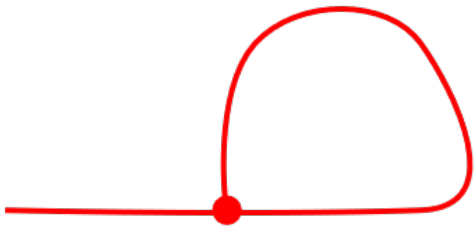


Figure 7.3: An arboreal skeleton where a bone attaches to itself, our procedure will avoid this type of complicated behaviour.

In Section 7.1, we will generalize the definition of X-convex manifolds and MBc-manifolds to allow for manifolds with corners, as we will produce manifolds with corners as intermediate steps during our inductive gluing procedure. This section is an adaptation of Section 2.3 of [AEN22b] to our smooth set-up. Then, in Section 7.2, we will discuss how to glue X-convex manifolds with corners in a way such that we can control what the skeleton of the resulting manifold will be. The techniques developed in this section are the smooth set-up analogue of the techniques developed for Weinstein manifolds with corners in Section 2.5 of [AEN22b]. In Section 7.3 we give an invariant description of an arboreal skeleton, i.e. without needing an ambient space, as we have defined arboreal skeleta as stratified hypersurfaces that are locally given by arboreal models up to *ambient* diffeomorphism. This description is based on the definition of *arboreal spaces* in Section 3.1 of [AEN22b]. The main result of the chapter is Section 7.4, where we give a thickening procedure for co-oriented arboreal skeleta and show that the germ of this thickening at the skeleton is unique up to diffeomorphism.

7.1 Morse-Bott X-convex manifolds with corners

Let $\mathbb{R}^n_+$ denote the subset of $\mathbb{R}^n$ where all of the coordinates are nonnegative, i.e.

$$\mathbb{R}^n_+ = \{(x_1, ..., x_n) \in \mathbb{R}^n | \quad x_1 \geqslant 0, ..., x_n \geqslant 0\}.$$

Definition 7.1. Let M be a topological n-dimensional manifold with boundary. A *chart with corners* for M is a pair (U, φ) of an open set $U \subset M$ and a diffeomorphism $\varphi : U \to \widetilde{U}$, where $\widetilde{U} \subset \mathbb{R}^n_+$ is an open subset. Two charts with corners $(U, \varphi), (V, \psi)$ are smoothly compatible if the composition $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ admits a smooth extension in an open neighbourhood of each point.

A *smooth structure with corners* on a topological manifold M with boundary is a maximal collection of smoothly compatible chart with corners whose domain cover M. A topological manifold with boundary together with a smooth structure with corners is called a *smooth manifold with corners*.

The n-dimensional smooth manifold with corners M has a corner of *order* $k \leqslant n$ at $p \in M$ if there is a neighbourhood of p in M diffeomorphic to a neighbourhood of the origin in $[0, 1)^k \times \mathbb{R}^{n-k}$. We denote the set of order k corners of M by $\partial_k M$. The interior in $\partial_k M$ of a connected component P of $\partial_k M$ is called a *boundary k-face*, we call a boundary 1-face a *boundary face*. △

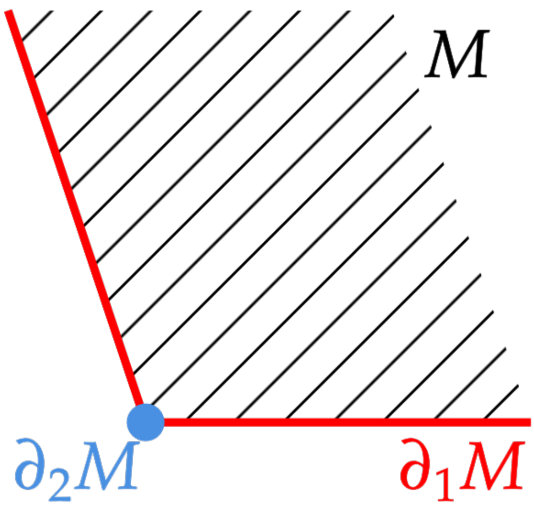


Figure 7.4: A manifold with corners.

We assume that every k-face P has an embedded collar neighbourhood $P \times \mathcal{I}^k \subset M$, where $\mathcal{I}$ denotes the germ of $[0, 1)$ at 0. Near each point $p \in \partial_k M$ we have canonical collar coordinates $p = (x, t)$, where $x \in \partial_k M$ and $t = (t_1, ..., t_k) \in \mathcal{I}^k$. In a neighbourhood of p the k-face $\partial_k M$ is given by $t_1 = ... = t_k = 0$ and for all $j \leqslant k$ the components of $\partial_j M$ whose closure contain p are given by setting j of the t coordinates equal to zero. We assume these collars are compatible in the sense that the remaining $n - j$ coordinates t_i parametrize the collar structure for $\partial_j M \times \mathcal{I}^{n-j}$ near p.

Definition 7.2. Let M be a manifold with corners, $W \subset M$ is an *(embedded) submanifold with corners* if it is a manifold with corners and the inclusion $W \to M$ is a smooth embedding.

A submanifold with corners $W \subset M$ has a *vertical boundary* $\partial_v W = \partial W \cap \mathring{M}$ and a *horizontal* *boundary* $\partial_h W = \partial W \cap \partial M$. △

Definition 7.3. An m-dimensional manifold with corners M is said to be X-*convex* for a complete vector field $X \in \mathfrak{X}(M)$, if there is a compact submanifold with corners $W \subset M$, called a *defining domain*, such that the following properties hold.

- The vector field X is outwards transverse to the vertical boundary $\partial_v W$ and the union of forward trajectories of X starting at $\partial_v W$ is equal to $\partial_v W \cup (M \setminus W)$.
- Each k-face P of the horizontal boundary $\partial_h W$ is an $X|_P$-convex manifold with corners containing a defining domain $N \subset P$, called the *nucleus* of P. Furthermore, on the canonical collar neighbourhood $P \times \mathcal{I}^k$ of P with coordinates (x,t) we have $X_{(x,t)} = (X|_P)_x$.

If manifold with corners M is X-convex we say the pair (M, X) is X-*convex with corners*. △

Note that the defining domain is not part of the data, only its existence is required. An X-convex manifold with corners can have many different defining domains. However, it will often be convenient to consider a certain fixed defining domain. Recall Lemma 3.9, which states that M can be recovered up to diffeomorphism from any defining domain $W \subset M$ by adding cylindrical ends.

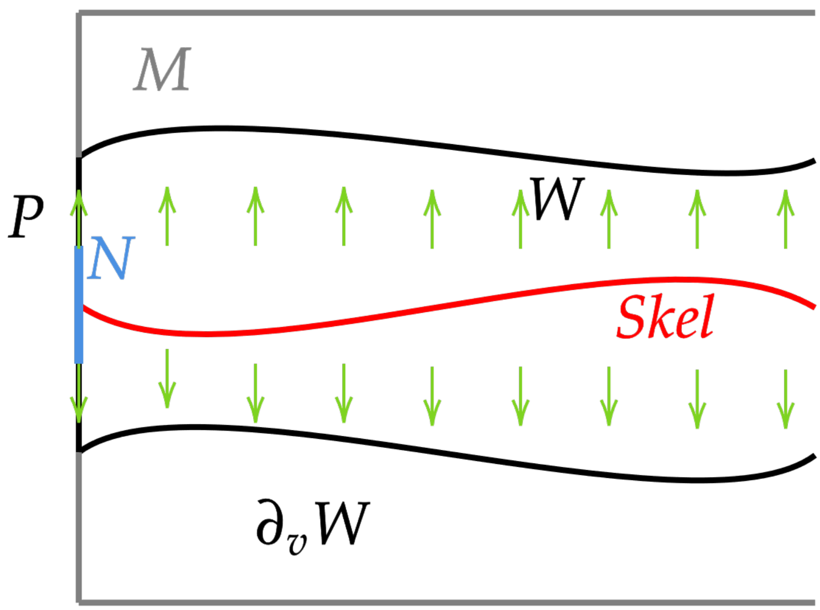


Figure 7.5: An X-convex manifold with corners.

Definition 7.4. The *skeleton* Skel(M, X) of an X-convex manifold with corners with defining domain W is given by

$$\text{Skel}(M, X) = \cap_{t>0} \varphi^{-t}(W)$$

where φ^t is the flow at time t along X. △

Thus the skeleton is defined as the attractor of the negative flow as before for MBc-manifolds without corners.

Remark 7.5. On the canonical collar neighbourhood of a k-face P of the horizontal boundary $\partial_h W$ with nucleus $N \subset P$, the skeleton of (M, X) is given by $\mathrm{Skel}(N, X_N) \times \mathcal{I}^k$. △

Definition 7.6. An X-convex with corners (M, X) is said to be *Morse-Bott* if there is a smooth function $f : M \to \mathbb{R}$ such that

- X is Morse-Bott for f on the interior $\mathring{M} \subset M$,
- $X|_P$ is Morse-Bott for $f|_P$ on every k-face P,
- using coordinates (x, t) on the collar neighbourhood $P \times \mathcal{I}^k$ we have $f(x, t) = f|_P(x)$.

As before, we call the pair (X, f) the *MBc-structure*. △

We stress that the Morse-Bott function f is not part of the data of an MBc-manifold with corners, it is only required to exist, and there is no canonical choice of Morse-Bott function. It is, however, often convenient to pick a Morse-Bott function on an MBc-manifold with corners. We will denote an MBc-manifold with corners (M, X) with a specified choice of Morse-Bott function $f : M \to \mathbb{R}$ by the triple (M, X, f).

Remark 7.7. Without loss of generality we may always assume that a defining domain W of an MBc-manifold with corners (M, X, f) is given by a sublevel set $\{f \leqslant a\}$ such that $\partial_v W$ is a regular level set $\{f = a\}$ and no critical values of f are greater than a. △

In the same way as for MBc-manifolds without corners, we define the notion of homotopy and isomorphism for MBc-manifolds with corners.

Definition 7.8. A *homotopy of MBc-structures* on a manifold with corners M is a 1-parameter family (X_t, f_t) of pairs of MBc-structures.

A diffeomorphism $\Phi : M \to M'$ between MBc-manifolds with corners (M, X) and (M', X') is called a *MBc-manifold with corners isomorphism* if $\Phi^* X' = X$. △

7.2 MBc-buildings

In this section we describe how to build up an MBc-manifold by gluing together MBc-manifolds with corners in such a way that we can control what the skeleton will be. Ultimately, the goal is to start with an arboreal skeleton S and use this gluing procedure to inductively build up a MBc-manifold with skeleton S.

First we describe how to modify an MBc-manifold with corners such that a boundary nucleus on the horizontal boundary becomes an interior set. We call this procedure the *horizontal-to-vertical nucleus conversion*; it is the smooth set-up analogue of the *nucleus-to-hypersurface conversion* for Weinstein manifolds with corners as defined in Section 2.5.1 of [AEN22b]. Then we will describe how to modify an MBc-manifold with corners such that a certain specified defining domain embedded into the vertical boundary becomes a nucleus on the horizontal boundary. This we call the *vertical-to-horizontal nucleus conversion*; it is the smooth set-up analogue *hypersurface-to-nucleus conversion* as defined in Section 2.5.3 of [AEN22b].

7.2.1 Conversion between horizontal and vertical nuclei

Horizontal-to-vertical nucleus conversion

Let (M, X, f) be an MBc-manifold with corners and let P be a boundary face of M with nucleus $N \subset P$. Let $W \subset M$ be a defining domain for (M, X, f), following Remark 7.7 we assume W is a sublevel set $\{f \leqslant a\}$.

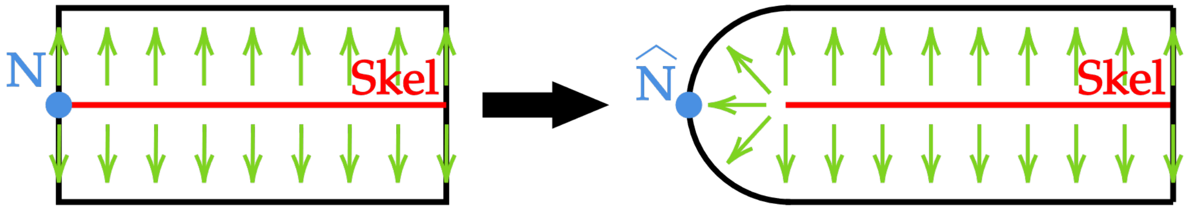


Figure 7.6: Horizontal-to-vertical nucleus conversion.

Consider the canonical collar neighbourhood $U = P \times \mathcal{I}$ of $P \subset M$ with coordinates (x, t). We attach the half-infinite collar $P \times (-\infty, 0]$ to $P \times \mathcal{I}$ along their common intersection $P \times \{0\}$ and write

$$\widehat{U} = U \bigcup_{P\times\{0\}} P \times (-\infty, 0] \qquad \text{and} \qquad \widehat{M} \qquad = M \bigcup_{P\times\{0\}} P \times (-\infty, 0].$$

Recall that on U we have $f(x,t) = f|_P(x)$, we extend the coordinates (x, t) to $\widehat{U}$ and define $\tilde{f} : \widehat{M} \to \mathbb{R}$ to be the extension of f to $\widehat{M}$ given by $\tilde{f} = f$ on M and

$$\tilde{f}(x,t) = f|_P(x) + \rho(t)t^2$$

on $\widehat{U}$. Here $\rho : \mathbb{R} \to [0, 1]$ is an appropriate monotone smooth bump function that is non-negative on $(-\infty, 0)$ and 0 on $[0, \infty)$.

We extend the vector field $X \in \mathfrak{X}(M)$ to $\widehat{M}$ as the gradient of $\hat{f}$ on $P \times (-\infty, 0]$

Recall from Definition 7.6 that, using the coordinates (x, t) on U, we have $f(x, t) = f|_P(x)$. Thus we have that $W \cap U$ is given by $\{f|_P \leqslant a\}$. Fix $\varepsilon > 0$ small, by attaching

$$\{(x,t) \in P \times (-\infty, 0] \mid \tilde{f} \leqslant a + \varepsilon\}$$

to W along their common intersection, we obtain a defining domain $\widehat{W}$ for $(\widehat{M}, \widehat{X}, \hat{f})$. Thus we see that $\widehat{M}$ is $\widehat{X}$-convex with corners, moreover, by construction it is an MBc-manifold with corners.

Definition 7.9. The MBc-manifold with corners $(\widehat{M}, \widehat{X})$ is called the *horizontal-to-vertical nucleus conversion* of MBc-manifold with corners (M, X) and nucleus N. △

Note that the operation of horizontal-to-vertical nucleus conversion only depends on the contractible choices of a collar neighbourhood and smoothing ρ, thus it produces an MBc-structure unique up to MBc-homotopy.

Remark 7.10. There is a canonical lift $\widehat{N}$ of the nucleus $N \subset P$ to the vertical boundary $\partial_v \widehat{W}$. Explicitly, we can parametrize $\widehat{N} \subset \partial_v \widehat{W}$ as

$$\{(x,t) \mid x \in N \quad \rho^2(t)t^2 = \varepsilon\}$$

in the bicollar neighbourhood of N. △

Remark 7.11. By construction $\mathrm{Skel}(\widehat{M}, \widehat{X}) = \mathrm{Skel}(M, X)$. △

Vertical-to-horizontal nucleus conversion

Let (M, X, f) be an n-dimensional MBc-manifold with corners with defining domain W, we assume W is a sublevel set $\{f \leqslant a\}$ of f such that $\partial_v W$ is a regular level set $\{f = a\}$. Let (A, Y, g) be an $(n-1)$-dimensional MBc-manifold with corners with defining domain $B = \{g \leqslant b\}$, we will modify (M, X, f) to obtain an MBc-manifold whose horizontal boundary has a face with nucleus B. From Lemma 3.9 we know this boundary face with nucleus B is diffeomorphic to A.

Fix $\varepsilon > 0$ small, we define

$$B_\varepsilon := \{g \leqslant b + \varepsilon\} \subset A,$$

which we can consider to be the space obtained from B by attaching small cylindrical ends to $\partial_v B$.

Consider an embedding $\varphi : B \hookrightarrow \partial_v W$ that extends to an embedding $\Phi : B_\varepsilon \hookrightarrow \partial_v W$.

Let $B_\varepsilon \times \mathcal{I}$ be the collar neighbourhood of B_ε in W and consider the manifold with corners $\overline{W}$ obtained by the gluing

$$\overline{W} = W \bigcup_{B_\varepsilon \times \{0\}} B_\varepsilon \times [-1, 0].$$

We extend the coordinates (x, t) on $B_\varepsilon \times \mathcal{I}$ to $B_\varepsilon \times \mathcal{I} \cup_{B_\varepsilon \times \{0\}} B_\varepsilon \times [-1, 0]$ and extend $f|_W$ to $\overline{W}$ as

$$\overline{f}(x,t) = (1 - \chi(s))f(x,t) + \chi(s)g(x).$$

Here $\chi : [-1, 1) \to [0, 1]$ is a smooth monotone bump function that is 1 on $[-1, -1/2]$ and 0 on $[0, 1)$.

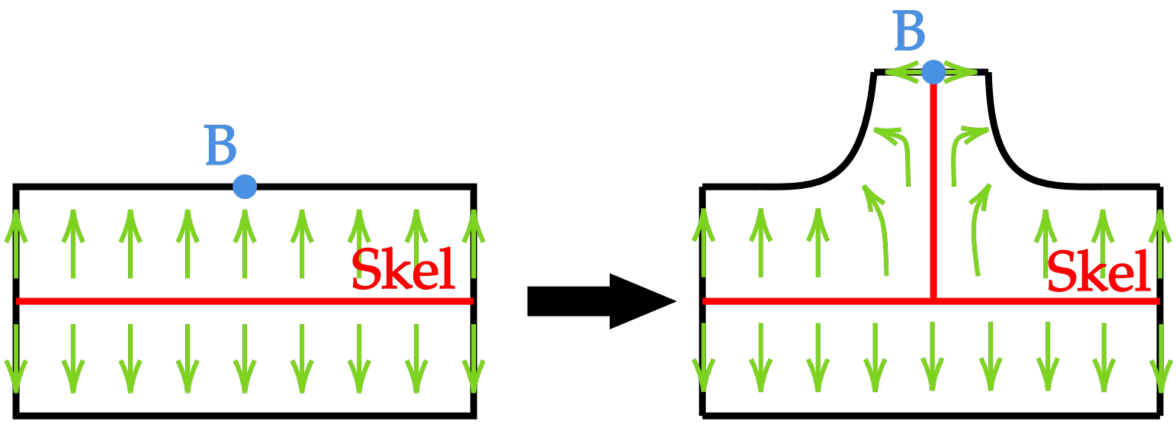


Figure 7.7: Vertical-to-horizontal nucleus conversion.

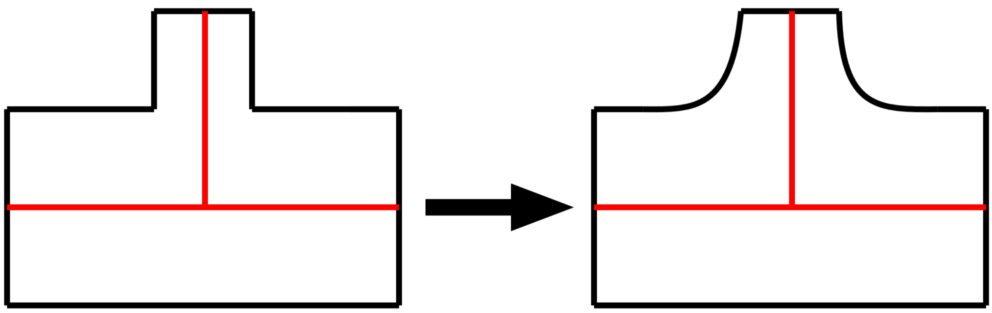


Figure 7.8: Smoothening the construction.

Now we smoothen this construction by replacing $B_\varepsilon \times [-1,0]$ with

$$\{(x,t) \in A \times [-1,0] \quad | \quad g(x) \leqslant b + (1-\chi(t)/2)\varepsilon\}.$$

From this smoothened $\overline{W}$ we can obtain an MBc-manifold by attaching a cylindrical end

$$\overline{M} = \overline{W} \bigcup_{\partial_v W} \partial_v W \times [0,\infty),$$

extending $\overline{f}$ to $\overline{M}$ as $e^s\overline{f}$ (here s denotes the cylindrical coordinate), and extending $X|_W$ to $\overline{M}$ as the gradient of $\overline{f}$ outside W. Note that $\overline{W}$ is a defining domain for $\overline{M}$ and $B_{\varepsilon/2}$ is a boundary face of the horizontal boundary $\partial_h\overline{W}$ with nucleus B.

Definition 7.12. The MBc-manifold with corners $(\overline{M},\overline{X})$ is called the *vertical-to-horizontal nucleus conversion* of the MBc-manifold with corners (M,X) and nucleus B along φ. △

Note that the vertical-to-horizontal nucleus conversion depends only on φ and the contractible choices of collar neighbourhoods and smoothing function χ, thus the result of the vertical-to-horizontal nucleus conversion along φ is unique up to MBc-manifold with corners homotopy.

Remark 7.13. By construction we have that $\mathrm{Skel}(\overline{M}, \overline{X})$ is the union of $\mathrm{Skel}(M, X)$ and the saturation of the skeleton $\mathrm{Skel}(A, Y) \subset B \subset \partial_h \overline{W}$ by the backwards flow, i.e.

$$\mathrm{Skel}(\overline{M}, \overline{X}) = \mathrm{Skel}(M, X) \bigcup \cup_{t \geqslant 0} \overline{\varphi}^{-t} \mathrm{Skel}(A, Y)$$

where $\overline{\varphi}^t$ denotes the flow along $\overline{X}$ at time t.

Thus, under the transversality conditions as specified in Definition 5.1, the skeleton of $\overline{M}$ is arboreal if the skeleton of M is a smooth manifold and the skeleton of A is arboreal. △

Lemma 7.14. *Let $(\overline{M}, \overline{X})$ be the vertical-to-horizontal nucleus conversion of the n-dimensional MBc-manifold (M, X) with nucleus $B \subset A$ along $\varphi : B \to W$, where W is a defining domain of M and B is a defining domain of an $(n-1)$-dimensional MBc-manifold (A, Y). Let the skeleton of M be smooth and the skeleton of A be arboreal, and assume the projection*

$$\pi : W \to Skel(M, X)$$

given by the backward flow of X restricts to a self-transverse map $\pi|_{Skel(A,Y)}$ of stratified spaces. Then $Skel(\overline{M}, \overline{X})$ is arboreal.

Proof. This follows directly from the vertical cone property of arboreal hypersurface singularities as given in Definition 5.1. □

7.2.2 Gluing

We distinguish two different types of gluing of MBc-manifolds with corners. The first is *horizontal gluing*, where two MBc-manifolds with corners are glued along horizontal boundary faces with isomorphic nuclei. The second type of gluing is *vertical gluing*, where a horizontal boundary face of one MBc-manifold with corners is glued into the vertical boundary of another MBc-manifold with corners. This section is an adaptation of Section 2.6 of [AEN22b].

Horizontal gluing

Let $(M, X), (M', X')$ be two MBc-manifolds with corners, let $P_0, ..., P_l$ be non-adjacent boundary faces of M and $P'_0, ..., P'_l$ be non-adjacent boundary faces of M'. Denote by N_j and N'_j the nuclei of P_j and P'_j and suppose that for each $1 \leqslant j \leqslant l$ there is an isomorphism $\psi_j : N_j \to N'_j$ of MBc-manifolds with corners.

Lemma 7.15. *We can extend each ψ_j to an MBc-manifold with corners isomorphism $\Psi_j : P_j \to P'_j$.*

Proof. By definition P_j is the union of N_j and all the forward trajectories of X starting at $\partial_v N_j$, meaning that if $x \in P_j$ either $x \in N_j$ or there is a unique pair $y \in \partial_v N_j$ and $t \in (0, \infty)$ such that $x = \varphi^t(y)$, where φ^t denotes the flow along X at time t. Likewise P'_j is the union of N'_j and all the forward trajectories of X' starting at $\partial_v N'_j$, for every $x' \in P'_j$ either $x' \in N'_j$ or $x' = \varphi'^t(y')$

for some $y' \in \partial_v N_j'$ and $t \in (0, \infty)$. We extend Ψ_j to P_j as mapping $x = \varphi^t(y)$ to $\varphi'^t(\psi_j(y))$. By construction

$$\Psi_j^* X' = X.$$

□

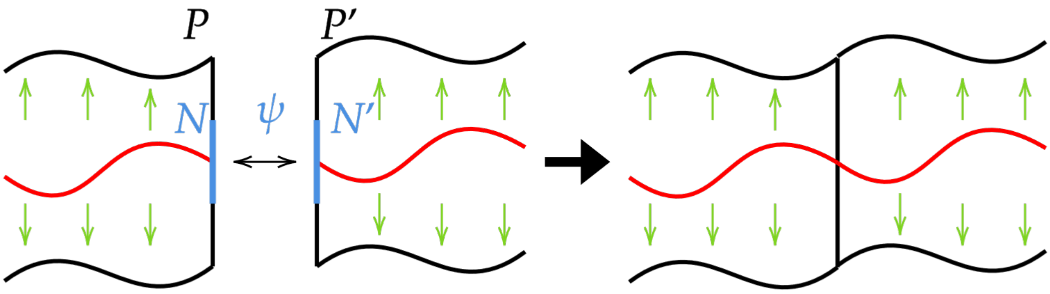


Figure 7.9: Horizontal gluing.

Definition 7.16. The *horizontal gluing* of (M, X) and (M', X') along $\{\psi_j\}_j$ is the MBc-manifold $(M \cup M', X \cup X')$ where we identify $P_j \sim P_j'$ via Ψ_j. △

Remark 7.17. The skeleton of the horizontal gluing of M and M' is the union of the skeleta of M and M',

$$\mathrm{Skel}(M \cup M', X \cup X') = \mathrm{Skel}(M, X) \cup \mathrm{Skel}(M', X'),$$

where the skeleta are glued along $\{\psi_j\}_j$ restricted to the skeleta of N_j. △

Vertical Gluing

Definition 7.18. Consider a pair of n-dimensional MBc-manifold with corners (M, X) and (M', X') with defining domains W and W' respectively.

Let (A, Y) be an $(n-1)$-dimensional MBc-manifold with corners with defining domain B, suppose we have an embedding $B \hookrightarrow \partial_v W$ that extends to an embedding $B_\varepsilon \hookrightarrow \partial_v W$, with B_ε as above. Let N be a nucleus of a horizontal boundary face of W' and let $\varphi : N \to A$ be an MBc-manifold with corners isomorphism.

Then the *vertical gluing* of (M, X) and (M', X') along φ is the composition of the vertical-to-horizontal nucleus conversion of (M, X) with nucleus $\varphi : B \hookrightarrow \partial_v W$ as in Definition 7.12 and the horizontal gluing along φ as in Definition 7.2.2. △

Note that the resulting MBc-manifold with corners is unique up to isomorphism and the skeleton of the vertical gluing is the union

$$\mathrm{Skel}(\overline{M}, \overline{X}) \cup \mathrm{Skel}(M, X)$$

where $(\overline{M}, \overline{X})$ denotes the vertical-to-horizontal nucleus conversion of (M, X) given by $B \hookrightarrow \partial_v W$.

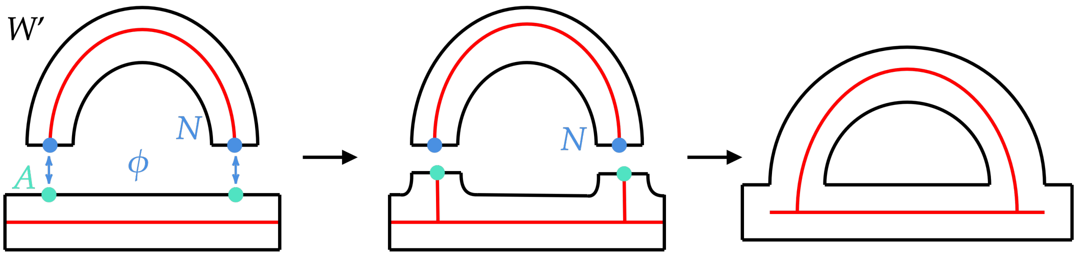


Figure 7.10: Vertical gluing.

7.2.3 MBc-buildings

MBc-buildings are the iterated vertical gluing of MBc-manifolds with corners, they are the smooth set-up analogue of Wc-buildings as defined in Section 2.7 of [AEN22b].

Definition 7.19. Let $(M_1, X_1), ..., (M_k, X_k)$ be MBc-manifolds with corners with defining domains $W_1, ..., W_k$. We inductively define a *k-level MBc-building*

$$(M_k \xrightarrow{\varphi_{k-1}} M_{k-1} \xrightarrow{\varphi_{k-2}} ... \xrightarrow{\varphi_1} M_1).$$

A 1-*level MBc-building* is (M_1, X_1), suppose the $k-1$-level MBc-building

$$Q = (M_{k-1} \xrightarrow{\varphi_{k-2}} ... \xrightarrow{\varphi_1} M_1)$$

is defined.

Consider a collection of nuclei $\{N_i\}_i$ of boundary faces $\{P_i\}_i$ of Q. Write $N = \cup_i N_i$ and let $\varphi_{k-1} : N \hookrightarrow \partial_v W_k$ be an embedding that extends to an embedding $N_\varepsilon \hookrightarrow \partial_v W_k$.

Then the *k-level MBc-building*

$$(M_k \xrightarrow{\varphi_{k-1}} M_{k-1} \xrightarrow{\varphi_{k-2}} ... \xrightarrow{\varphi_1} M_1) = M_k \xrightarrow{\varphi_{k-1}} Q$$

is defined as the vertical gluing of Q to M_k along φ_{k-1}. △

7.3 Arboreal Spaces

Recall that we assigned a generalized arboreal model hypersurface $H_{(\mathbb{T},\ell)} \subset \mathbb{R}^{n(\mathbb{T})}$ to every special signed leafy rooted tree $(\mathbb{T}, \ell)$. Whether a stratified hypersurface $H \subset M$ is arboreal is dependent on the way H lies in M, for every $p \in H$ the germ of (M, H) at p must be diffeomorphic to the germ of some $(\mathbb{R}^m, H_{(\mathbb{T},\ell)} \times \mathbb{R}^{m-n})$ at the origin. Thus we can not yet talk about an arboreal space without its ambient space, which we want to be able to do to recover the smooth type of an MBc manifold from the combinatorial data of its skeleton.

For $c \geqslant 0$ and $m \geqslant n(\mathbb{T}) + c$ we set $d = m - n(\mathbb{T}) - c$ and define

$$H(\mathbb{T}, \ell, m, c) := H_{(\mathbb{T},\ell)} \times \mathbb{R}^d \times \mathbb{R}^c_{\geqslant 0} \subset \mathbb{R}^{m-c} \times \mathbb{R}^c_{\geqslant 0}.$$

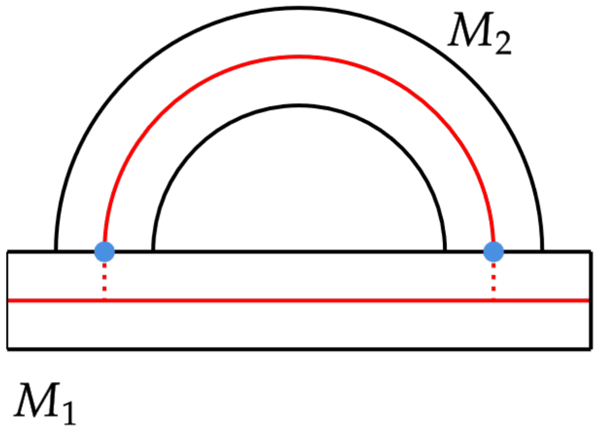


Figure 7.11: An MBc-building.

We denote by $\mathcal{O}(\mathbb{T}, m, c)$ the structure sheaf of functions restricted from smooth functions on $\mathbb{R}^{m-c} \times \mathbb{R}^c_{\geqslant 0}$ by the inclusion $H(\mathbb{T}, \ell, m, c) \hookrightarrow \mathbb{R}^{m-c} \times \mathbb{R}^c_{\geqslant 0}$.

Definition 7.20. An m-dimensional *arboreal space with corners* $(A, \mathcal{O})$ is a locally ringed compact Hausdorff space locally modelled on

$$(H(\mathbb{T}, \ell, m, c), \mathcal{O}(\mathbb{T}, \ell, m, c))$$

for varying special signed leafy rooted trees $(\mathbb{T}, \ell)$ and $c \leqslant m - n(\mathbb{T})$.

If $(A, \mathcal{O})$ is locally modelled on $(H(\mathbb{T}, \ell, m, c), \mathcal{O}(\mathbb{T}, \ell, m, c))$ for varying special signed leafy rooted trees $(\mathbb{T}, \ell)$ and $c \leqslant 1$, we call $(A, \mathcal{O})$ simply an *arboreal space with boundary*. If $(A, \mathcal{O})$ is locally modelled on $(H(\mathbb{T}, \ell, m, 0), \mathcal{O}(\mathbb{T}, \ell, m, 0))$ for varying special signed leafy rooted trees $(\mathbb{T}, \ell)$, we call $(A, \mathcal{O})$ an *arboreal space*. △

Note that the boundary of an arboreal space with boundary is again an arboreal space (without boundary).

Remark 7.21. Recall that the model generalized arboreal hypersurface of the special signed leafy rooted tree $(\mathbb{T}, \ell)$ is a union of disjoint pieces, $H_{(\mathbb{T},\ell)} = \bigcup_{\alpha \in V(\mathbb{T}) \setminus \ell} P_\alpha$, meaning we also have the disjoint smooth hypersurfaces

$$H(\mathbb{T}, \ell, m, c)_\alpha := P_\alpha \times \mathbb{R}^d \times \mathbb{R}^c_{\geqslant 0}.$$

Thus an arboreal space with boundary A is the union $A = \cup_j A_j$ of disjoint smooth pieces A_j, which are maximal connected subsets that restrict to a smooth piece $H(\mathbb{T}, \ell, m, c)_\alpha$ in every model. Furthermore, since A is compact, A must be the union of a finite collection of smooth pieces (possibly with boundary). △

7.4 Thickening co-oriented arboreal skeleta

Consider an arboreal space A.

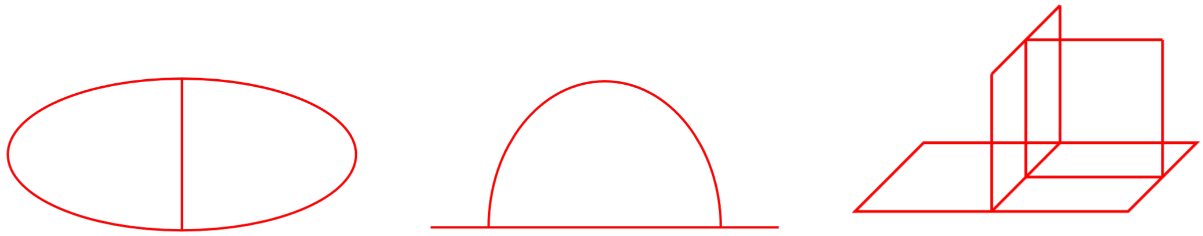


Figure 7.12: One arboreal space and two arboreal spaces with boundary

Definition 7.22. An *orientable arboreal space* is an arboreal space whose smooth pieces are all orientable. △

Remark 7.23. The smooth pieces of the skeleta of the MBc-manifolds obtained from the arborealization procedure as described in Chapter 6 are discs. Thus the arborealization procedure produces skeleta which are orientable arboreal spaces. △

As explained in Remark 7.21, the arboreal space A can be written as the disjoint union $A = \cup_j A_j$ of smooth pieces. Equivalently, A is the union $A = \cup_j \mathbf{A}_j$ of closed smooth manifolds with corners such that each point of A lies in the interior of precisely one $\mathbf{A}_j$, where the interior is taken with respect to the subspace topology in A.

Definition 7.24. An *arboreal n-building* is an orientable arboreal n-space with corners, A, whose closed smooth manifold with corners pieces can be ordered $\mathbf{A}_1, ..., \mathbf{A}_l$ such that:

- each $\mathbf{A}_{<j} = \cup_{i<j} \mathbf{A}_i$ is an arboreal space with boundary;
- $\mathbf{A}_{<j+1}$ is obtained from $\mathbf{A}_{<j}$ by gluing a collection $S^j = \cup_i S^j_i$ of boundary components $S^j_i \subset \partial \mathbf{A}_{<j}$ to $\mathbf{A}_j$ along a map $\psi_j : S^j \to \mathbf{A}_j$.

We call the pieces $\mathbf{A}_j$, that are closed smooth manifolds with corners, the *building blocks*. △

Remark 7.25. By construction the boundary of each $\mathbf{A}_{<j}$ is an arboreal $(n-1)$-building with the building blocks given by the boundaries $\partial \mathbf{A}_i$. △

Note that the arborealization procedure described in Chapter 6 produces an arboreal skeleton that has the structure of an arboreal building.

The blocks of an arboreal building admit a thickening to MBc-manifolds as trivial line bundles, these thickenings form the building blocks of an MBc-building that produces M.

Definition 7.26. Let $A = \cup_{1 \leqslant j \leqslant l} \mathbf{A}_j$ be an arboreal n-building, the *canonical MBc-thickening* of a block $\mathbf{A}_j$ is the n-dimensional MBc-manifold with corners

$$M(\mathbf{A}_j) = \mathbf{A}_j \times \mathbb{R}$$

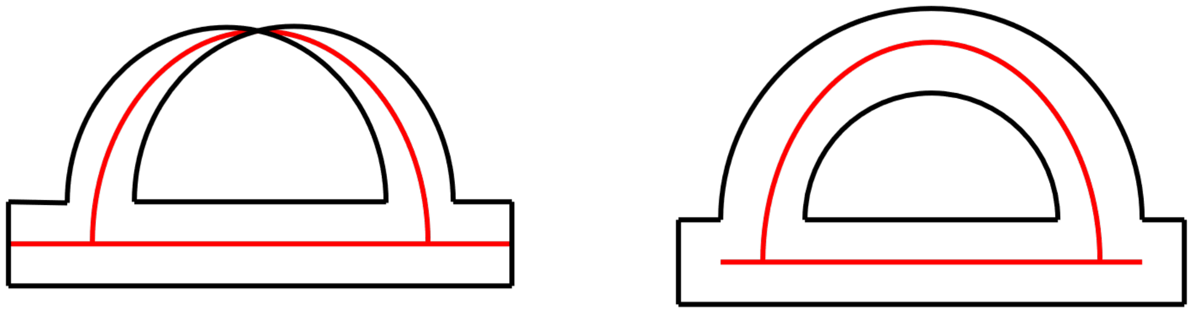
with the Morse-Bott vector field $X = 2t\partial_t$, the gradient of the Morse-Bott function $f : A \times \mathbb{R} \to \mathbb{R}, \quad (x,t) \mapsto t^2$. A defining domain is given by $W(\mathbf{A}_j) = \mathbf{A}_j \times [-1,1]$, which has vertical boundary $\partial_v W(\mathbf{A}_j) = \mathbf{A}_j \times S^0$.

We give $M(\mathbf{A}_j)$ the product orientation induced by the orientation of $\mathbf{A}_j$ and the canonical orientation of $\mathbb{R}$. $\triangle$

The rest of this chapter will be devoted to proving the following theorem, which is the main result of this chapter.

Theorem 7.27. *Let $A = \cup_{1 \leqslant j \leqslant k} \mathbf{A}_j$ be an arboreal building, then there is a thickening of A to an orientable n-dimensional MBc-manifold with corners (M, X) that has A as its skeleton such that $\partial A \subset \partial M$.*

Furthermore, the germ at A of such a thickening is unique up to diffeomorphism leaving A pointwise fixed.



(a) A thickening we do not allow, since it is not orientable.

(b) The orientable thickening.

We first show the following lemma.

Lemma 7.28. *Let (M, X) be an MBc-manifold thickening of $A = \cup_{1 \leqslant j \leqslant k} \mathbf{A}_j$, then the germ of M at A has the structure of a k-level MBc-building.*

Proof. We proceed by induction over the number k of building blocks of A.

Let $k = 1$, then (M, X) is per definition a 1-level MBc-building.

Let $k > 1$ and assume the assertion has been established for $k - 1$. We take a small tubular neighbourhood N of $\mathbf{A}_k$ such that $N \pitchfork A$ and $\mathrm{Skel}(M, X) \cap N$ retracts to $\mathbf{A}_k$, as indicated in Figure 7.14. Note that this tubular neighbourhood is the trivial line bundle over $\mathbf{A}_k$, thus it is diffeomorphic to the canonical thickening of $\mathbf{A}_k$. Under the identification of N and the canonical thickening $M(\mathbf{A}_k)$, we have

$$\mathrm{Skel}(M, X) \cap N = \mathbf{A}_k \cup \mathcal{C}(N \cap A),$$

where $\mathcal{C}(N \cap A)$ is the vertical cone of $N \cap A$, i.e. the saturation of $N \cap A$ under the backward flow of the Morse-Bott vector field on $M(\mathbf{A}_k)$.

This tubular neighbourhood N "splits" the germ of M at A into a piece contained in N, which we denote M_k, and a piece whose skeleton is isomorphic to $A_{<k}$, which we denote $M_{<k}$. The germ of M is obtained through a horizontal gluing of M_k and $M_{<k}$ along $i : N \cap A \hookrightarrow N$. Note

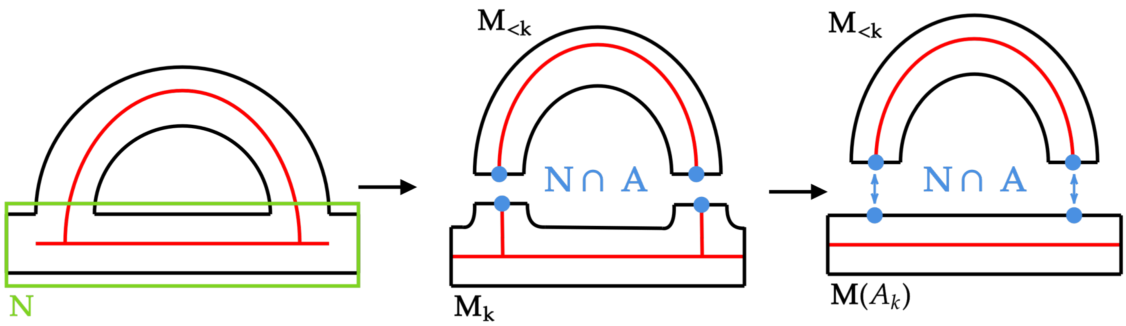


Figure 7.14: Splitting the germ of M at A to obtain an MBc-building.

that M_k is the vertical-to-horizontal nucleus conversion of the canonical thickening of $\mathbf{A}_k$ and nucleus $N \cap A$ along $i : N \cap A \hookrightarrow N$. Thus, equivalently, the germ of M is obtained through a vertical gluing of $M_{<k}$ to the canonical thickening of $\mathbf{A}_k$ along $i : N \cap A \hookrightarrow N$.

By out induction hypothesis, the germ of $M_{<k}$ at $A_{<k}$ is a $(k-1)$-level MBc-building. We conclude that, per Definition 7.19, the germ of M at A is a k-level MBc-building. □

We now prove Theorem 7.27.

Proof. We proceed by induction over the number k of building blocks of A. For $k = 1$ the canonical MBc-thickening of the single block is an MBc-manifold thickening. Any orientable MBc-manifold thickening of $A = \mathbf{A}_1$ is an orientable line bundle over $\mathbf{A}_1$, and since $\mathbf{A}_1$ is an orientable, connected, closed manifold with corners it admits only one orientable line bundle up to diffeomorphism, being the trivial line bundle. So any MBc-manifold thickening of $A = \mathbf{A}_1$ is diffeomorphic to the canonical MBc-thickening of $\mathbf{A}_1$.

Let $k > 1$, assume the assertions have been established for all orientable arboreal buildings with $k-1$ blocks and let A be an arboreal building with k blocks. By definition, $A_{<k}$ is an arboreal building with $k-1$ blocks, meaning it has an MBc-manifold thickening $M(A_{<k})$ with defining domain $W(A_{<k})$ such that the germ of $M(A_{<k})$ at $A_{<k}$ is unique up to diffeomorphism leaving $A_{<k}$ fixed. Recall that by the definition of arboreal n-buildings, $\mathbf{A}_{<k+1}$ is obtained from $\mathbf{A}_{<k}$ by gluing a collection $S^k = \cup_i S_i^k$ of boundary components $S_i^k \subset \partial \mathbf{A}_{<k}$ to $\mathbf{A}_k$ along a map $\psi_k : S^k \to \mathbf{A}_k$. We write P_i^k for the boundary face of the horizontal boundary $\partial_h W(A_{<k})$ containing S_i^k and denote $P^k = \cup_i P_i^k$.

We formulate the following lemma.

Lemma 7.29. *The gluing map $\psi_k : S^k \to \mathbf{A}_k$ can be extended to an orientation preserving embedding (or, equivalently, an orientation preserving injective immersion since we consider germs) Ψ_k of the germ of $\partial \mathbf{A}_{<k}$ at S^k into $\mathbf{A}_k$ that leaves S^k pointwise fixed. The map Ψ_k can be canonically lifted to an orientation preserving embedding $\widetilde{\Psi}_k$ of the germ of $\partial \mathbf{A}_{<k}$ at S^k into $\mathbf{A}_k \times \mathbb{S}^0$.*

Furthermore, the space of orientation preserving embeddings of the germ of $\partial \mathbf{A}_{<k}$ at S^k into $\mathbf{A}_k$ leaving S^k pointwise fixed is contractible, meaning the embedding Ψ_k is unique up to isotopy.

Proof. As stated in Remark 7.25, the boundary of $A_{<k}$ is an arboreal building with $k-1$ blocks. This means that S^k is an arboreal space with an orientable thickening T, and the germ of T at S^k is unique up to diffeomorphism. Any neighbourhood of $\psi_k(S^k)$ in $\mathbf{A}_k$ constitutes an oriented thickening of $\psi_k(S^k)$. Because ψ_k is an embedding and the germ of any thickening of S^k at S^k is unique up to diffeomorphism, we see that the germ of T at S^k is diffeomorphic to the germ of $\mathbf{A}_k$ at $\psi_k(S^k)$. Thus, the gluing map $\psi_k : S^k \to \mathbf{A}_k$ can be extended to an orientation preserving embedding of the germ of $\partial \mathbf{A}_{<k}$ at S^k into $\mathbf{A}_k$

The existence of the lift follows from the definition of arboreal singularities and how a special signed leafy rooted tree determines an arboreal model.

To put it more precisely, every point $x \in S^k$ has a neighbourhood $U \subset S^k$, a collar neighbourhood $U \times \mathcal{I} \subset A_{<k}$ and a neighbourhood $V \subset A_{<k+1}$, such that $U \subset U \times \mathcal{I} \subset V$. Now, V is modelled on $(H(\mathbb{T}, \ell, m, c), \mathcal{O}(\mathbb{T}, \ell, m, c))$, and $U \times \mathcal{I} \subset V$ corresponds to one of the at most two subtrees obtained from $\mathbb{T}$ by deleting the root vertex. The sign of the edge between the root of $\mathbb{T}$ and the root of the subtree corresponding to $U \times \mathcal{I} \subset V$ gives the lift of $\psi_k(x)$; if the sign is positive $\Psi_k(x) = \psi_k(x) \times \{1\}$ and if the sign is negative $\Psi_k(x) = \psi_k(x) \times \{-1\}$.

Let Ψ'_k be another orientation preserving embedding of the germ of $\partial \mathbf{A}_{<k}$ at S^k into $\mathbf{A}_k \times \mathbb{S}^0$ leaving S^k pointwise fixed. Fix a Riemannian metric on $\mathbf{A}_k \times \mathbb{S}^0$. For all $x \in \partial \mathbf{A}_{<k}$ sufficiently close to S^k we must have that $\Psi_k(x)$ and $\Psi'_k(x)$ are in a geodesically convex neighbourhood. Following these geodesics gives an isotop0y H_t between $H_0 = \Psi_k$ and $H_1 = \Psi'_k$ that leaves S^k pointwise fixed. Consider a point $p \in S^k$, we denote the maximal connected smooth piece of S^k containing p by S. By assumption $\Psi_k|_{S^k} = \Psi'_k|_{S^k} = H_t|_{S^k}$ for all $t \in [0,1]$. Thus we see that

$$d(\Psi_k)_p|_{T_pS} = d(\Psi'_k)_p|_{T_pS} = d(H_t)_p|_{T_pS}$$

for all $t \in [0,1]$. We decompose T_pM as $T_pM = T_pS \oplus \nu_pS$, where νS is the normal bundle of S. Note that S is a hypersurface and thus ν_pS is 1-dimensional, we pick a basis vector of ν_pS and denote it by v. Since both Ψ_k and Ψ'_k are orientation preserving and their differentials restricted to the tangent bundle of S agree, it must be that $d(\Psi_k)_pv$ and $d(\Psi'_k)_pv$ lie on the same side of the hypersurface T_pS.

Therefore any convex combination of $d(\Psi_k)_pv$ and $d(\Psi'_k)_pv$ must lie on the same side of the hypersurface $T_{\Psi_k(p)}\Psi_k(S) = T_{\Psi'_k(p)}\Psi'_k(S)$, from which we see that $d(H_t)_pv$ always lies on the same side of the hypersurface $T_{\Psi_k(p)}\Psi_k(S)$. Thus the determinant of $d(H_t)_p$ never changes sign, in particular, this means the determinant of $d(H_t)_p$ never vanishes. Thus H_t is an isotopy of injective immersions close to S^k.

We remark that this construction of the isotopy can be done with any number of parameters, from which we conclude that the space orientation preserving embeddings leaving S^k pointwise fixed is contractible. □

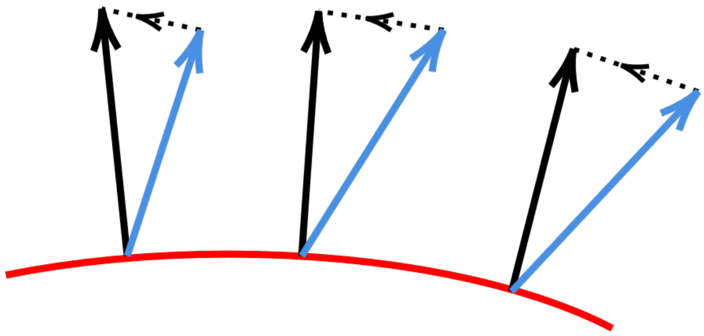


Figure 7.15: The homotopy between the νS component of the differentials $d\Psi_k$ and $d\Psi'_k$.

We now obtain a an MBc-manifold thickening $M(A_{<k+1})$ of A through a vertical gluing of $M(A_{<k+1})$ to $M(\mathbf{A}_k)$ via $\widetilde{\Psi_k}$.

Let M be another MBc-manifold thickening of A. We split the germ of M at A, as in Lemma 7.28, into M_k and $M_{<k}$ such that the germ of M is obtained through a vertical gluing of $M_{<k}$ to the canonical thickening of $\mathbf{A}_k$. By our induction hypothesis, the germ of $M_{<k}$ at $A_{<k}$ is unique up to diffeomorphism. By Lemma 7.29 the space of admissible vertical gluing maps is contractible, thus the germ at A of the result of the vertical gluing of $M_{<k}$ to M_k must be diffeomorphic to the germ at A of the thickening $M(A_{<k+1})$ produced described above. □

Remark 7.30. The construction of the thickening as an MBc-building is inductive, but in contrast to the handlebody given by a Morse function as discussed in Chapter 2 all the data of this building is given at once by the skeleton. △

Chapter 8

Outlook

In this chapter we discuss some open questions that further research might aim to answer.

One question is how the results of Chapter 7 can be generalized to non-orientable manifolds and skeleta with non-orientable smooth pieces. To be able to do this, more data should be provided with the arboreal skeleton. One expects this data to amount to giving a collection of Čech-cocycles.

Another question is how the arborealization program can be used to compute invariants of manifolds. Following the backwards flow gives a retraction of an X-convex manifold to its skeleton, thus the manifold and its skeleton have the same homotopy type. Therefore it should be straightforward to recover homotopy invariants from the skeleton. Since it is possible to recover an orientable manifold from its co-oriented arboreal skeleton, one would expect that all smooth invariants of the manifolds can be computed from its skeleton.

Furthermore, one wonders under which conditions a cobordism between two manifolds can be recovered from the intersection of the skeleton of the cobordism with its boundary. To be able to recover the cobordism, more data should be provided. For instance, in the situation as drawn in Figure 8.1, it should be indicated that all the points on the intersection with the boundary are connected via the skeleton. The question is what extra data should be given and how this data can be presented.

Another question is how different skeleta of the same manifold can be related and how this could be used to develop a homotopy theory. Can the results of Chapter 6 and Chapter 7 be generalized to a 1-parametric set-up, and how do the phenomena of 1-parameter families of functions, such as birth-death pairs and handle slides, translate to the arboreal setting?

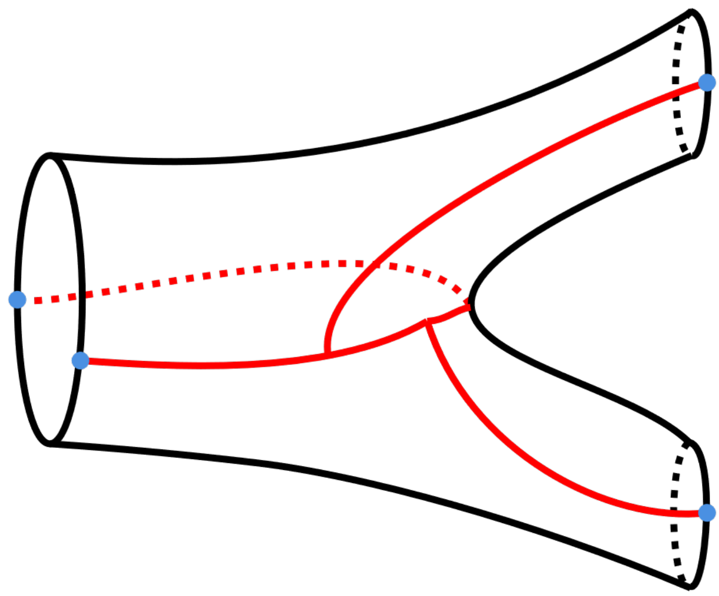


Figure 8.1: A cobordism with an arboreal skeleton and its intersection with the boundary.

Appendix A

Symplectic Geometry, Contact Geometry and Weinstein Manifolds

In this chapter we give the definition and some examples of symplectic, contact and Weinstein manifolds, and record some notions and results from the study of these manifolds crucial for our understanding of arboreal Lagrangian and Legendrian singularities. This chapter offers only a minimal introduction to the vast field of study of symplectic and contact geometry, discussing only what is needed to be able to define Weinstein manifolds and arboreal singularities.

In Section A.1 we define *symplectic vector spaces*. Then, in Section A.2, we discuss *symplectic manifolds*, which are manifolds with a symplectic structure on their tangent bundle. We give examples and formulate the well-known *Darboux's Theorem* for symplectic manifolds. In Section A.3 we discuss the closely related notion of contact manifolds. We give examples, explain the connection between symplectic and contact manifolds and formulate Darboux's Theorem for contact manifold. We conclude with the definition of *Weistein manifolds*, which are symplectic manifolds compatible with Morse theory, in Section A.4.

The first two sections are based on Chapters 2 and 3 of [MS17] and Chapters 1, 2 and 3 of [Sil01]. The third section is based on Chapter 1 and 2 [Han08] and the fourth section is based on Chapter 11 of [CE12].

A.1 Symplectic vector spaces

Definition A.1. A *symplectic vector space* is a pair (V, ω) of a finite dimensional real vector space V with a bilinear form $\omega : V \times V \to \mathbb{R}$ such that

- The form ω is *skew-symmetric*, i.e.

$$\omega(v, w) = -\omega(w, v)$$

for all $v, w \in V$.

- The form ω is non-degenerate, i.e.

$$\ker \omega = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in V\}$$

is trivial.

△

A skew-symmetric version of the Gram-Schmidt process yields the following theorem.

Theorem A.2. *Let (V, ω) be a symplectic vector space. Then there is a basis $e_1, ..., e_n, f_1, ..., f_n$ of V such that*

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \qquad \text{for all } i \text{ and } j$$
$$\omega(e_i, f_j) = \delta_{ij} \qquad \text{for all } i \text{ and } j.$$

We call such a basis $e_1, ..., e_n, f_1, ..., f_n$ of V a *symplectic basis*.

Note that a direct corollary of this theorem is that every symplectic vector space is even dimensional.

Definition A.3. Let $(V, \omega), (V', \omega')$ be symplectic vector spaces. A linear map $\varphi : V \to V'$ is called *symplectic* if $\varphi^* \omega' = \omega$. A linear symplectic bijection is called a *linear symplectomorphism*. △

Definition A.4. Let (V, ω) be a symplectic vector space. The *symplectic complement* of a linear subspace $W \subset V$ is defined as the subspace

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}.$$

A subspace $W \subset V$ is called

- *isotropic* if $W \subset W^\omega$;
- *coisotropic* if $W^\omega \subset W$;
- *symplectic* if $W \cap W^\omega = \{0\}$;
- *Lagrangian* if $W = W^\omega$.

△

Remark A.5. From Theorem A.2 we see that an isotropic subspace $W \subset V$ of a $2n$-dimensional symplectic vector space has at most dimension n, a Lagrangian subspace is an isotropic subspace of maximal dimension. △

A.2 Symplectic manifolds

Definition A.6. Let M be a manifold. A 2-form $\omega \in \Omega^2(M)$ is say to be *symplectic* if ω is closed, i.e. $d\omega = 0$, and $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is symplectic for all $p \in M$.

A *symplectic manifold* is a pair (M, ω) of a manifold M equipped with a symplectic form ω.

A *symplectic submanifold* of M is a submanifold $S \subset M$ such that $\omega|_S$ is a symplectic form on S.

An *isotropic submanifold* of M is a submanifold on which the symplectic form restricts to zero. A closed subset of a symplectic manifold is called an *isotropic subset* if it is stratified by isotropic submanifolds. If an isotropic submanifold has maximal dimension, i.e. half the dimension of the ambient symplectic manifold, we say it is a *Lagrangian submanifold*. △

Example A.7. Let $M = \mathbb{R}^{2n}$ with coordinates $x_1, ..., x_n, p_1, ..., p_n$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dp_i$$

is a symplectic form on M. This form is called the *standard form* on $\mathbb{R}^{2n}$. △

Example A.8. Let M be an n-dimensional manifold and consider its cotangent bundle T^*M. Let U be a coordinate neighbourhood for M with coordinates $x_1, ..., x_n$ and associated coordinate neighbourhood T^*U with coordinates $x_1, ..., x_n, \xi_1, ..., \xi_n$ for T^*M. Then

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i$$

is a symplectic form on T^*M. △

Using the symplectic basis of the tangent space from Theorem A.2 at every point $p \in M$, we get the following result.

Proposition A.9. *Let M be an $2n$-dimensional manifold, a 2-form $\omega \in \Omega^2(M)$ is symplectic if and only if $\omega^n \neq 0$, i.e. ω is a volume form.*

Proof. First let $\omega \in \Omega^2(M)$ be symplectic. For any $p \in M$ we have a symplectic basis as in Theorem A.2. Now, using the notation of Theorem A.2, we see that $\omega_p(e_1, ..., e_n, f_1, ..., f_n) \neq 0$.

Now let $\omega \in \Omega^2(M)$ be a 2-form such that ω^n is a volume form and assume ω is not symplectic. Then there must be a $p \in M$ such that ω_p is not symplectic, thus there must be a $0 \neq v \in T_pM$ such that $\omega(v, w) = 0$ for all $w \in T_pM$. Now, we see that for all $\lambda \in \mathbb{R}$ it must hold that $\omega(\lambda v, w) = 0$ for all $w \in T_pM$. Therefore the linear subspace $< v >$ must be contained in the kernel of ω. From this we directedly see that it must be that $\omega_p^n = 0$, which is in contradiction with the assumption that ω is a volume form.

We conclude that $\omega \in \Omega^2(M)$ is symplectic if and only if $\omega^n \neq 0$. □

Definition A.10. Let $(M, \omega), (M', \omega')$ be $2n$-dimensional symplectic manifolds. A diffeomorphism $\varphi : M \to M'$ is called a *symplectomorphism* if $\varphi^*\omega' = \omega$. △

A central result in symplectic geometry is Darboux's Theorem, which states that any $2n$-dimensional symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Theorem A.11 (Darboux). *Let (M, ω) be a symplectic manifold of dimension $2n$. Then every $p \in M$ has a coordinate neighbourhood U with a coordinate map $\varphi : U \to U' \subset \mathbb{R}^{2n}$ such that $\varphi^*\omega_0 = \omega$.*

As a result of Darboux's Theorem all symplectic manifolds of the same dimension are locally equivalent. In particular, the only local invariant of symplectic manifolds up to symplectomorphism is the dimension of the manifold.

A.3 Contact manifolds

Definition A.12. A *hyperplane field* ξ on a manifold M is a codimension 1 subbundle of the tangent bundle TM. △

We note that, locally, any hyperplane field ξ is the kernel of some 1-form $\alpha \in \Omega^1(M)$.

Definition A.13. Let M be an $(2n+1)$-dimensional manifold, a hyperplane field ξ is said to be *maximally non-integrable* if for every 1-form α such that $\xi = \ker \alpha$, we have

$$\alpha \wedge (d\alpha)^n \neq 0.$$

We call such a hyperplane field ξ a *contact structure*, we call a 1-form α such that $\xi = \ker \alpha$ a *contact form*. △

Remark A.14. Equivalently, one could say that a hyperplane field $\xi \subset TM$ in the tangent bundle of a $(2n+1)$-dimensional manifold is maximally non-integrable if for very 1-form α such that $\xi = \ker \alpha$, we have $(d\alpha)^n|_\xi \neq 0$. Following Proposition A.9 this implies that $(\xi_p, d\alpha|_{\xi_p})$ is a symplectic vector space and that $d\alpha|_\xi$ is a symplectic form on ξ. △

Example A.15. On $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x_1, y_1, ..., x_{n+1}, y_{n+1})$, the 1-form

$$\alpha_0 = dz + \sum_{i=1}^{n} x_i dy_i$$

is a contact form. The contact structure $\xi = \ker \alpha_0$ is called the *standard contact structure on* $\mathbb{R}^{2n+1}$. △

Example A.16. Let $(x_1, y_1, ..., x_{n+1}, y_{n+1})$ be Cartesian coordinates on $\mathbb{R}^{2n+n}$ and consider the unit sphere $\mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2}$. Then the 1-form

$$\alpha = \sum_{i=1}^{n+1} x_i dy_i - y_i dx_i$$

is an contact form on $\mathbb{S}^{2n+1}$. The contact structure $\xi = \ker \alpha$ is called the *standard contact structure on the unit sphere* $\mathbb{S}^{2n+1}$. △

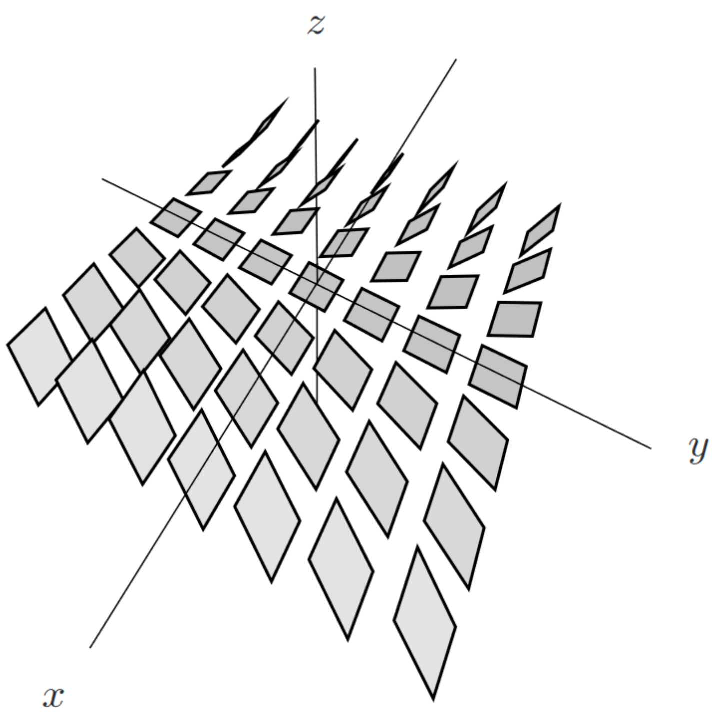


Figure A.1: The contact structure $\ker(dz + xdy)$ ([Han08], p. 4)

Example A.17. Let M be an n-dimensional manifold. Note that the space J^1M of 1-jets can be canonically identified with $T^*M \times \mathbb{R}$. A point in J^1M is a triple (x, p, z), where $x \in M$, p is a linear form on T_xM and $z \in \mathbb{R}$. Then the 1-form $\alpha = dz - \sum p_i dx_i$ is a contact form, we call

$$\xi_{st} := \ker \alpha$$

the *standard contact structure on the* 1*-jet bundle*. △

Definition A.18. Let (M, ξ) be an $(2n + 1)$-dimensional contact manifold. A submanifold $L \subset M$ is called *isotropic* if $T_pL \subset \xi_p$ for all $p \in L$. A closed subset of M stratified by isotropic submanifolds of M is called an *isotropic subset* of M.

If an isotropic submanifold $L \subset M$ is of dimension n, we say L is a *Legendrian submanifold*. △

Remark A.19. Let $L \subset M$ be an isotropic submanifold of an $(2n + 1)$-dimensional contact manifold (M, ξ) and let α be the 1-form defining ξ.

Then $\alpha|_L = 0$, thus $d\alpha|_L = 0$, meaning that for every $p \in L$ we have that $T_pL \subset \xi_p$ is an isotropic subspace of the $2n$-dimensional symplectic vector space $(\xi_p, d\alpha|_{\xi_p})$. Thus a Legendrian submanifold is an isotropic submanifold of maximal dimension. △

Example A.20. Let M be an n-dimensional manifold, consider the space $J^1M = T^*M \times \mathbb{R}$ of

1-jets with the standard contact structure. Any smooth map $f : M \to J^1M$ defines a section

$$x \mapsto j^1f(x) = \big(x, df(x), f(x)\big)$$

of the bundle $J^1M \to M$. Note that f is a embedding and $f^*(dz - pdx) = 0$, from which we see that $j^1(M)$ is a Legendrian submanifold of J^1M. △

Lemma A.21. *The germ at the origin of a isotropic submanifold $L \subset T^*\mathbb{R}^n$ of the cotangent bundle with the standard Liouville structure $\lambda = pdx$ admits a unique lift to an isotropic germ at the origin of the 1-jet bundle $J^1\mathbb{R}^n$ with the standard contact structure $\xi = ker\ \alpha$ where $\alpha = dz - pdx$.*

Proof. We write $i : L \to T^*\mathbb{R}^n$ for the inclusion. Note that $i^*\lambda$ is closed, meaning, locally, $i^*\lambda$ is exact.

Therefore, locally, there is a primitive f of $i^*\lambda$. Now, a lift to an isotropic germ is given by the map (i, f). Such a lift is unique up to a translation along the z-axis, thus there is a unique lift to an isotropic germ at the origin. □

Definition A.22. Let $(M, \xi), (M', \xi')$ be $(2n + 1)$-dimensional contact manifolds. A diffeomorphism $\varphi : M \to M'$ is called a *contactomorphism* if $df(\xi) = \xi'$. △

Theorem A.23. *Let (M, ξ) be an $(2n + 1)$-dimensional contact manifold with contact form α and let $p \in M$ be a point. Then p has a coordinate neighbourhood U with coordinates $x_1, ..., x_n, y_1, ..., y_n, z$ such that $p = (0, ..., 0)$ and*

$$\alpha|_U = dz + \sum_{i=1}^{n} x_i dy_i.$$

A.4 Weinstein manifolds

Definition A.24. Let (M, ω) be a $2n$-dimensional symplectic manifold equipped with an exact symplectic form ω. A *Liouville form* is a choice of primitive λ, i.e. λ is a 1-form such that $\omega = d\lambda$. The vector field Z that is ω-dual to λ, i.e. such that $\iota_Z\omega = \lambda$, is called the *Liouville field* of λ. △

Remark A.25. If Z is a Liouville vector field for a Liouville form λ, Cartan's formula yields

$$L_Z\omega = \iota_Z d\omega + d\iota_Z\omega = d\lambda = \omega.$$

This means if Z integrates to a flow $Z^t : M \to M$, it satisfies $(Z^t)^*\omega = e^t\omega$. This means that symplectic form expands as one flows along Z. Similarly

$$L_Z\lambda = \iota_Z\omega + d\iota_Z\lambda = \lambda$$

meaning the flow of Z expands the Liouville form, i.e. $(Z^t)^*\lambda = e^t\lambda$. △

Definition A.26. A *Liouville domain* is a compact exact symplectic manifold (M, λ) with boundary such that the Liouville field Z is outwardly transverse to the boundary ∂M. △

Note that

$$\lambda \wedge (d\lambda)^{n-1} = (\iota_Z d\lambda) \wedge (d\lambda)^{n-1} = \frac{1}{n}\iota_Z (d\lambda)^n = \frac{1}{n}\iota_Z \omega^n.$$

Thus positive transversality of Z at the boundary is equivalent to requiring $\lambda_{\partial M}$ to be a contact form such that the orientation defined on ∂M by the volume form $\lambda \wedge (d\lambda)^{n-1}_{\partial M}$ coincides with the orientation of ∂M as the boundary of M.

Definition A.27. The *skeleton* of a Liouville domain (M, λ) is the attractor of the negative flow of the Liouville field, i.e. the subset

$$\text{Skel}(M, \lambda) = \bigcap_{t>0} Z^{-t}(M) \subset M.$$

△

Note that the skeleton of a Liouville domain is a closed subset of a compact set and thus compact.

Definition A.28. A *Liouville manifold* (M, λ) is an exact symplectic manifold M such that the Liouville field Z is complete, i.e. has global flow, and there is an exhaustion $M = \bigcup\limits_{k=1}^{\infty} M^k$ by compact domains with smooth boundaries along which the Liouville field Z is outwardly transverse. △

Remark A.29. Any Liouville domain (M, λ) can be completed to a Liouville manifold $(\widehat{M}, \widehat{\lambda})$ by attaching the forward trajectories of Z starting at ∂M. Explicitly,

$$\widehat{M} = M \bigcup_{\partial M \sim \partial M \times \{0\}} (\partial M \times [0, \infty))$$

and λ is extended to $\widehat{M}$ as $e^s(\lambda|_{\partial M})$ on the attached cylinder. We will call a Liouville manifold obtained by completing a Liouville domain a *finite type Liouville manifold*, the Liouville domain giving the Liouville manifold is called *defining*. △

Definition A.30. The *skeleton* of a Liouville manifold (M, λ) is given by

$$\text{Skel}(M, \lambda) = \bigcup_{k=1}^{\infty} \bigcap_{t>0} Z^{-t}(M^k),$$

i.e. it is the attractor of the negative flow of the Liouville vector field. △

Lemma A.31. *A Liouville manifold $(\widehat{M}, \widehat{\lambda})$ is of finite type if and only if its skeleton is compact.*

Proof. Consider a finite type Liouville manifold $(\widehat{M}, \widehat{\lambda})$ with defining domain M. Since all points in $\widehat{M} - M$ lie on the forward trajectory of Z starting at a point ∂M and the Liouville field is outwardly transverse we see that $\text{Skel}(\widehat{M}, \widehat{\lambda}) = \text{Skel}(M, \lambda)$. Thus the skeleton of a finite type Liouville manifold is compact.

Now consider a Liouville manifold $(\widehat{M}, \widehat{\lambda})$ with compact skeleton. Let k be sufficiently large such that a neighbourhood of the skeleton of $\widehat{M}$ is contained in $\widehat{M}^k$, we write $\Sigma = \partial \widehat{M}^k$. The forward flow of Z starting from Σ gives a diffeomorphism $M - \mathrm{Int}(\widehat{M}^k) \simeq \Sigma \times [0, \infty)$ because for every point $p \in \widehat{M}$ the negative flow $Z^{-t}(p)$ gets arbitrarily close to the skeleton as $t \to \infty$ and thus is contained in $\widehat{M}^k$ for sufficiently large t. Under this diffeomorphism we see that the Liouville form $\lambda = \iota_Z \omega$ corresponds to $e^t \alpha$ where $t \in \mathbb{R}$ is the parameter of the flow and $\alpha = \lambda_\Sigma$. $\square$

The Weinstein condition amounts to the additional demand that the Liouville field is Morse-Bott.

Definition A.32. A *Weinstein domain (resp. Weinstein manifold)* (M, λ, φ) is a Liouville domain(resp. Liouville manifold) (M, λ) with a Morse-Bott with boundary Liouville field Z. $\triangle$

As shown in Section 2.2 of [Sta18], the skeleton of a Weinstein domain is an isotropic set.

Proposition A.33. *Let M be a Weinstein domain with Liouville vector field Z. For any connected component C of zeros of Z, the stable manifold $Stab(C)$ is an isotropic submanifold (possibly with boundary). Thus, the skeleton of a Weinstein domain is an isotropic subset.*

Appendix B

Arboreal Lagrangian and Legendrian Singularities in Symplectic and Contact Manifolds

In this chapter we define the classes of *arboreal Lagrangian singularities* and *arboreal Legendrian singularities*, which are stratified Lagrangian singularities in symplectic manifolds (resp. Legendrian singularities in contact manifolds). Furthermore, we give local models for these singularities and show that each class of germs of arboreal Lagrangian singularities (resp. arboreal Legendrian singularities) is determined by discrete combinatorial data given by a non-cyclic connected graph, i.e. a *tree*, with some extra combinatorial data.

We begin this chapter with the definition of arboreal Lagrangian and Legendrian singularities in Section B.1 as originally formulated in [AEN22a]. Then, in Section B.2, we produce arboreal models via a gluing construction as first defined by Nadler in Section 2.1 of [Nad17]. We can associate such a gluing to every tree. This gluing construction fixes only the homeomorphism type of the singularity, not the smooth, symplectic or contact type. In Section B.3 we give another construction of arboreal models, following Section 2 of [AEN22a], where arboreal singularities are produced as positive conormal bundles of certain hypersurfaces. These models are associated to trees with extra data on the vertices and edges, and the construction fixes the diffeomorphism type of the model. These two constructions are equivalent, in the sense that the singularities they produce are homeomorphic, as we show in Section B.4.

We will then, in Section B.5, discuss a Stability Theorem from Section 3 of [AEN22a] which shows that each of the arboreal Lagrangian singularities is germ diffeomorphic to one of the arboreal models. We conclude with Section B.6, where we generalize the arboreal models to allow for strata with boundary. We prove that a similar Stability Theorem holds for these *generalized arboreal singularities*.

B.1 Arboreal Lagrangian and Legendrian singularities

Before we can define arboreal Lagrangian and Legendrian singularities we need to discuss some auxiliary notions.

Recall from Lemma A.21 that the germ of an isotropic submanifold of $T^*\mathbb{R}^n$ at the origin lifts to an isotropic germ at the origin of the 1-jet bundle $J^1\mathbb{R}^n$. As a corollary, the germ at the origin of any simply-connected isotropic subset $L \subset T^*\mathbb{R}^n$ admits a unique lift to the germ at the origin of an isotropic subset $\widehat{L} \subset J^1\mathbb{R}^n$. We want an arboreal Lagrangian singularity in $T^*\mathbb{R}^n$ to lift to an arboreal Legendrian singularity in $J^1\mathbb{R}^n$. We wish for our arboreal Lagrangian and Legendrian singularities to be compatible with this lift.

Furthermore, we want to have a coning operation analogous to the vertical cone in the definition of arboreal hypersurface singularities. From Darboux's Theorem we know that all contact manifolds of the same dimension are locally contactomorphic, so we define this coning operation for a specific class of contact manifolds, namely unit spheres with the standard contact structure. For an isotropic subset $\Lambda \subset S^*\mathbb{R}^n \cong \mathbb{S}^{2n-1}$ of the cosphere bundle we define its *Liouville cone* $C(\Lambda) \subset T^*\mathbb{R}^n$ to be the closure of the trajectories of the Liouville vector field $p\frac{\partial}{\partial p}$ going trough Λ. We note that this Liouville cone is isotropic.

We are now ready to define arboreal Lagrangian and Legendrian singularities, the following definition is Definition 1.1 from [AEN22a].

Definition B.1. *Arboreal Lagrangian (resp. Legendrian) singularities* form the smallest class Arb_n^{symp} ($resp.\mathrm{Arb}_n^{cont}$) of germs of closed stratified isotropic subsets in $2n$-dimensional symplectic (res0p. $(2n+1)$-dimensional contact) manifolds such that

1. (Invariance) Arb_n^{symp} is invariant with respect to symplectomorphisms and Arb_n^{cont} is invariant with respect to contactomorphisms.
2. (Base case) Arb_0^{symp} contains $pr = \mathbb{R}^0 = T^*\mathbb{R}^0 = pt$.
3. (Stabilizations) If $L \subset (X, \omega)$ is in Arb_n^{symp}, then the product $L \times \mathbb{R} \subset (X \times T^*\mathbb{R}, \omega + dp \wedge dq)$ is in $\mathrm{Arb}_{n+1}^{symp}$.
4. (Legendrian lifts) If $L \subset T^*\mathbb{R}^n$ is in Arb_n^{symp}, then the Legendrian lift $\widehat{L} \subset J^1\mathbb{R}^n$ is in Arb_n^{cont}.
5. (Liouville cones) Let $\Lambda_1, ..., \Lambda_k \subset S^*\mathbb{R}^n$ be a finite disjoint union of arboreal Legendrian germs from $\mathrm{Arb}_{n-1}^{cont}$ centred at points $z_1, ..., z_k \in S^*\mathbb{R}^n$. Let $\pi : S^*\mathbb{R}^n \to \mathbb{R}^n$ be the front projection and suppose
   - $\pi(z_1) = ... = \pi(z_k)$;
   - For any i and stratum $Y \subset \Lambda_i$, the restriction $\pi|_Y : Y \to \mathbb{R}^n$ is an embedding;
   - For any distinct $i_1, ..., i_l$, and any set of strata $Y_{i_1} \subset \Lambda_{i_1}, ..., Y_{i_l} \subset \Lambda_{i_l}$, the restriction $\pi|_{Y_{i_1} \cup ... \cup Y_{i_l}}$ is self-transverse.

Then the union $\mathbb{R}^n \cup C(\Lambda_1) \cup ... \cup C(\Lambda_k)$ forms an arboreal Lagrangian germ from Arb_n^{symp}.

△

B.2 Gluing construction

We begin by giving the construction of arboreal singularities as originally introduced by Nadler in [Nad17]. These arboreal singularities are constructed by gluing Euclidean spaces, this produces topological stratified spaces whose strata have no canonical smooth structure. Thus this original construction yields homeomorphism classes of singularities, rather than symplectomorphism or contactomorphism classes.

Before we give the gluing construction of arboreal sinuglarities, we first recall some notions from Chapter 5. A *graph* G is a set of *vertices* $V(G)$ and a set of *edges* $E(G)$ such that $E(G)$ is a subset of the set of two-elements subsets of $V(G)$. Two vertices $\alpha, \beta \in V(G)$ are called *adjacent* if $\{\alpha, \beta\} \in E(G)$, a vertex $\tau \in V(G)$ is caled a *terminal vertex* if it has a single adjacent vertex. A graph G is called *connected* if any two vertices $\alpha, \beta \in V(G)$ can be linked by a *walk*, i.e. a sequence of edges $\{\alpha, \gamma_1\}, \{\gamma_1, \gamma_2\}, ..., \{\gamma_n, \beta\} \in E(G)$. A graph G is called *acyclic* if there are no non-empty walks in which all edges are distinct and all vertices except for the first and last are distinct. A *tree* T is a nonempty, finite, connected acyclic graph.

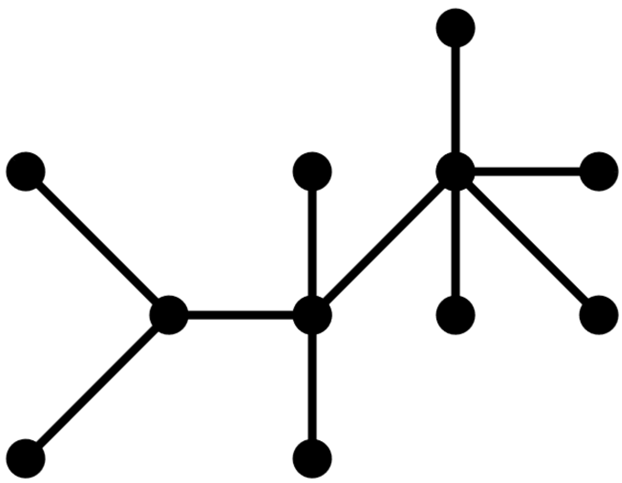


Figure B.1: A tree.

For any finite set S we write $\mathbb{R}^S$ for the Euclidean space of S-tuples of real numbers, note that $\mathbb{R}^S \cong \mathbb{R}^{\|S\|}$. In particular, for a tree T, we introduce for each vertex $\alpha \in V(T)$ the Euclidean space $\mathfrak{L}_T(\alpha) \simeq \mathbb{R}^{V(T)\setminus\{\alpha\}}$ of tuples $\{x_\gamma(\alpha)\}_{\gamma \in V(T)\setminus\{\alpha\}}$ of real numbers.

Definition B.2. For an edge $\{\alpha, \beta\} \in E(T)$ we define the $\{\alpha, \beta\}$*-edge gluing* as the quotient of the disjoint union of Euclidean spaces

$$\left(\mathfrak{L}_T(\alpha)\coprod\mathfrak{L}_T(\beta)\right)/\sim$$

with $\{x_\gamma(\alpha)\}\sim\{x_\gamma(\beta)\}$ whenever

$$x_\beta(\alpha)=x_\alpha(\beta)\geqslant 0 \qquad \text{and} \qquad x_\gamma(\alpha)=x_\gamma(\beta) \text{ for all } \gamma\neq\alpha,\beta\in V(T).$$

△

Definition B.3. The *arboreal singularity* $\mathfrak{L}_T$ associated to a tree T is the quotient of the disjoint union of Euclidean spaces

$$\mathfrak{L}_T=\Big(\coprod_{\alpha\in V(T)}\mathfrak{L}_T(\alpha)\Big)/\sim$$

where $\sim$ is the equivalence relation generated by the edge gluing for all edges $\{\alpha,\beta\}\in E(T)$. △

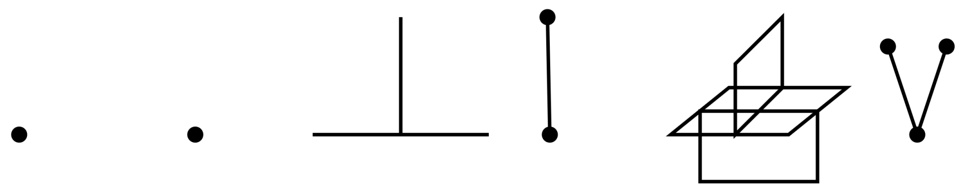


Figure B.2: Some arboreal singularities obtained from the gluing construction with the associated trees.

The following Lemma explains how the Euclidean spaces $\mathfrak{L}_T(\alpha)$ and $\mathfrak{L}_T(\beta)$ are glued in $\mathfrak{L}_T$ for non-adjacent vertices $\alpha,\beta\in V(T)$.

Lemma B.4. *Let $\alpha,\beta\in V(T)$ be two vertices and suppose the shortest path between them consists of k edges with the successive adjacent vertices $\alpha=\gamma_0,\gamma_1,...,\gamma_k=\beta$.*

Then, inside of $\mathfrak{L}_T$, the Euclidean spaces $\mathfrak{L}_T(\alpha)$ and $\mathfrak{L}_T(\beta)$ are glued under the identification $\{x_\gamma(\alpha)\}\sim\{x_\gamma(\beta)\}$ whenever

$$x_{\gamma_i}(\alpha)=x_{\gamma_{i-1}}(\beta)\geqslant 0 \textit{ for all } 1\leqslant i\leqslant k$$
$$x_\gamma(\alpha)=x_\gamma(\beta) \textit{ for all } \gamma\neq\gamma_0,...,\gamma_k\in V(T).$$

Proof. Because T is acyclic only the edges on the shortest path between α and β play a role.

We proceed by induction over k, the number of edges in the shortest path between α and β. Note that for $k=1$ the statement of the Lemma is exactly the $\{\alpha,\beta\}$-edge gluing.

Assume the statement holds for $k-1$, meaning that $\mathfrak{L}_T(\alpha)$ and $\mathfrak{L}_T(\gamma_{k-1})$ are glued in $\mathfrak{L}_T$ by making the identification $\{x_\gamma(\alpha)\}\sim\{x_\gamma(\gamma_{k-1})\}$ whenever

$$x_{\gamma_i}(\alpha)=x_{\gamma_{i-1}}(\gamma_{k-1})\geqslant 0 \text{ for all } 1\leqslant i\leqslant k-1$$
$$x_\gamma(\alpha)=x_\gamma(\gamma_{k-1}) \text{ for all } \gamma\neq\gamma_0,...,\gamma_{k-1}\in V(T).$$

Note that the $\{\gamma_{k-1}, \beta\}$-edge gluing makes the identification $\{x_\gamma(\gamma_{k-1})\} \sim \{x_\gamma(\beta)\}$ whenever

$$x_\beta(\gamma_{k-1}) = x_{\gamma_{k-1}}(\beta) \geqslant 0$$
$$x_\gamma(\gamma_{k-1}) = x_\gamma(\beta) \text{ for all } \gamma \neq \gamma_{k-1}, \beta \in V(T).$$

Thus we see that $\{x_\gamma(\alpha)\} \sim \{x_\gamma(\beta)\}$ whenever

$$x_{\gamma_i}(\alpha) = x_{\gamma_{i-1}}(\beta) \geqslant 0 \text{ for all } 1 \leqslant i \leqslant k$$
$$x_\gamma(\alpha) = x_\gamma(\beta) \text{ for all } \gamma \neq \gamma_0, ..., \gamma_k \in V(T).$$

□

The following Lemma gives an alternative description of $\mathfrak{L}_T$ that will be useful when comparing this gluing construction to the conormal construction in Section B.3.

Lemma B.5. *Let T be a tree and suppose $\tau \in V(T)$ is a terminal vertex of T. Define the tree T_τ with vertices $V(T) \setminus \{\tau\}$ and edges $E(T) \setminus \{\tau, \alpha\}$.*

Then there is a canonical homeomorphism

$$\mathfrak{L}_T \simeq \left(\mathfrak{L}_{T_\tau} \times \mathbb{R}^{\{\tau\}}\right) \coprod_{\mathfrak{L}_{T_\tau}(\alpha) \times \{0\}} \left(\mathfrak{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}_{\leqslant 0}\right)$$

Proof. Note that for any $\gamma \in V(T_\tau)$ we have

$$\mathfrak{L}_T(\gamma) = \mathbb{R}^{V(T) \setminus \{\gamma\}} = \mathbb{R}^{V(T_\tau) \setminus \{\gamma\} \cup \{\tau\}} = \mathfrak{L}_{T_\tau}(\gamma) \times \mathbb{R}^{\{\tau\}}$$

and

$$\mathfrak{L}_T(\tau) = \mathbb{R}^{V(T) \setminus \{\tau\}} = \mathfrak{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}.$$

Thus we see that

$$\begin{aligned}
\mathfrak{L}_T &= \left(\coprod_{\gamma \in V(T)} \mathfrak{L}_T(\gamma)\right) / \sim \\
&= \left(\mathfrak{L}_T(\tau) \coprod_{\gamma \in V(T_\tau)} \mathfrak{L}_{T_\tau}(\gamma) \times \mathbb{R}^{\{\tau\}}\right) / \sim \\
&= \left(\mathfrak{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}} \coprod \mathfrak{L}_{T_\tau} \times \mathbb{R}^{\{\tau\}}\right) / \sim
\end{aligned}$$

where $\mathfrak{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}$ is attached to $\mathfrak{L}_{T_\tau} \times \mathbb{R}^{\{\tau\}}$ by the $\{\tau, \alpha\}$-edge gluing. This means that $\mathfrak{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}$ gets attached along

$$\mathfrak{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}_{\geqslant 0} \subset \mathfrak{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}$$

which can be less redundantly described as attaching

$$\mathfrak{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}_{\leqslant 0} \subset \mathfrak{L}_{T_\tau}(\alpha) \times \mathbb{R}^{\{\alpha\}}$$

along $\mathfrak{L}_{T_\tau}(\alpha) \times \{0\}$. □

B.3 Conormal construction

We now give the construction of arboreal models as originally introduced in [AEN22a]. To every *signed rooted tree*, which is a tree with certain extra data on vertices and edges, we will associate a cooriented graphical hypersurface. The *positive conormal*, which we define below, of these graphical hypersurfaces will be our arboreal singularities. As we will see, this construction yields stratified singularities with symplectic (resp. contact) structures on each strata. This construction is motivated by the following notions.

Let $T^*\mathbb{R}^n$ denote the cotangent bundle with canonical 1-form $pdx = \sum_{i=1}^{n} p_i dx_i$ where $p = (p_1, ..., p_n)$ are the dual coordinates to $x = (x_1, ..., x_n)$. Let $J^1\mathbb{R}^n = \mathbb{R} \times \mathbb{R}T^*\mathbb{R}^n$ denote the 1-jet bundle with contact form $dx_0 + pdx = dx_0 + \sum_{i=1}^{n} p_i dx_i$.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ with graph $\Gamma_f = \{x_0 = f(x)\} \subset \mathbb{R} \times \mathbb{R}^n$ we have the *conormal Lagrangian of the graph*

$$L_{\Gamma_f} = \left\{x_0 = f(x), p_i = -p_0 \frac{\partial f}{\partial x_i}\right\} \subset T^*\mathbb{R}^{n+1}$$

and the *conormal Legendrian of the graph*

$$\Lambda_{\Gamma_f} = \left\{x_0 = f(x), p_0 = 1, p_i = -\frac{\partial f}{\partial x_i}\right\} \subset J^1\mathbb{R}^n.$$

B.3.1 Quadratic fronts

We first define the maps whose graphs we will consider.

Definition B.6. For $i \geqslant 0$ we define functions $h_i : \mathbb{R}^i \to \mathbb{R}$ by the inductive formula

$$h_0 = 0, \qquad h_i = h_i(x_1, ..., x_i) = x_1 - h_{i-1}(x_2, ..., x_i)^2.$$

For $0 \leqslant j < i \leqslant n$, set

$$h_{i,j} = h_{i-j}(x_{j+1}, ..., x_i).$$

△

Note that $h_{i,0} = h_i(x_1, ..., x_i)$ and $h_{i,i-1} = h_1(x_i) = x_i$.

Example B.7. For small i we have

$$h_1(x_1) = x_1, \qquad h_2(x_1, x_2) = x_1 - x_2^2,$$
$$h_3(x_1, x_2, x_3) = x_1 - (x_2 - x_3^2)^2, \qquad h_4(x_1, x_2, x_3, x_4) = x_1 - (x_1 - (x_2 - x_3^2)^2)^2.$$

△

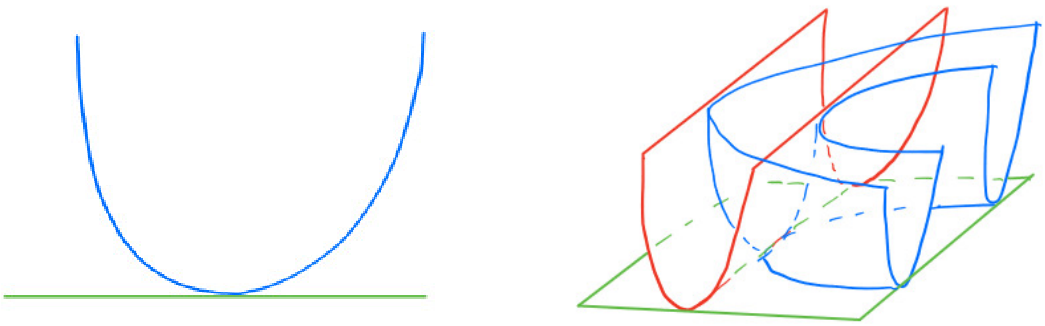
Definition B.8. For $n \geqslant 0$ and $i = 0, ..., n$ we define the smooth graphical hypersurfaces

$$^n\Gamma_i = \{x_0 = h_i^2\} \subset \mathbb{R}^{n+1}$$

and the union

$$^n\Gamma = \bigcup_{i=0}^{n} {}^n\Gamma_i.$$

△



(a) The hypersurfaces $^1\Gamma_0$, in green, and $^1\Gamma_1$, in blue.

(b) The hypersurfaces $^2\Gamma_0$, in green, $^2\Gamma_1$, in red, and $^2\Gamma_2$, in blue.

Figure B.3: Quadratic fronts, ([AEN22a], p. 5, p. 6).

Remark B.9. Note that we have the identities

$$^n\Gamma_i =^i \Gamma_i \times \mathbb{R}^{n-i} \qquad \text{for all } i = 0, ..., n$$
$$^n\Gamma_i \cap^n \Gamma_0 =^{n-1} \Gamma_{i-1} \qquad \text{for all } i = 1, ..., n.$$

△

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ we can equip the graphical hypersurface $\Gamma_f = \{x_0 = f(x)\} \subset \mathbb{R} \times \mathbb{R}^n$ with the a coorientation given by the conormal that is positive in the ∂_{x_0} direction. We call this coorientation the *graphical coorientation*, we equip every $^n\Gamma_i$ with the graphical coorientation.

B.3.2 Conormal Lagrangians and Legendrians

We now look at the conormal Lagrangians and Legendrians of these quadratic fronts.

For $i = 0$, let $^nL_0 = \mathbb{R}^n \subset T^*\mathbb{R}^n$ denote the zero-section. For $i = 1, ..., n$ we denote the conormal Lagrangian

$$^nL_i = L_{^{n-1}\Gamma_{i-1}} \subset T^*\mathbb{R}^n$$

of the graph ${}^{n-1}\Gamma_{i-1} \subset \mathbb{R}^n$ and consider their union

$$^nL = \bigcup_{i=0}^{n} {}^nL_i.$$

Similarly, for $i = 0, ..., n$, introduce the conormal Legendrian

$$^n\Lambda_i = \Lambda_{^n\Gamma_i} \subset J^1\mathbb{R}^n$$

of the graph ${}^n\Gamma_i \subset \mathbb{R}^{n+1}$ and consider their union

$$^n\Lambda = \bigcup_{i=0}^{n} {}^n\Lambda_i.$$

Remark B.10. Observe that ${}^nL_i \subset T^*\mathbb{R}^n$ is given by the equations

$$x_1 = h_{i,1}^2, \qquad \sum_{j=2}^{n} p_j dx_j = -\sum_{j=2}^{n} p_1 \frac{\partial h_{i,1}^2}{\partial x_j} dx_j = -p_1 dh_{i,1}^2 = -p_1 dx_1$$

Thus the Liouville form pdx vanishes on ${}^nL_i \subset T^*\mathbb{R}^n$, meaning that $T({}^nL_i) \subset \ker(dx_0 + pdx)$. Therefore the lift of ${}^nL_i$ to $J^1\mathbb{R}^n = \mathbb{R} \times T^*\mathbb{R}^n$ zero primitive, i.e. $\{0\} \times^n L_i$ is Legendrian. △

We have the following Lemma on compatibility between this lift and the conormal Legendrian.

Lemma B.11. *The contactomorphism*

$$S : J^1\mathbb{R}^n \to J^1\mathbb{R}^n, \quad (x_0, x, p) \mapsto (x_0 - p_1^2/4, x_1 + p_1/2, x_2, ..., x_n, p_1, ..., p_n)$$

takes the Legendrian ${}^n\Lambda_i$ isomorphically to the Legendrian $\{0\} \times^n L_i$ and thus the union ${}^n\Lambda$ isomorphically to the union $\{0\} \times^n L$.

Proof. We note that ${}^n\Lambda_i \subset J^1\mathbb{R}^n$ is defined by the equations

$$x_0 = h_i^2 \qquad pdx = -dh_i^2 = -2h_i dh_i = -2h_i(dx_1 - 2h_{i,1}dh_{i,1})$$

and thus in particular $p_1 = -2h_i$ and $\sum_{j=2}^n p_j dx_j = 4h_i h_{i,1} dh_{i,1} = -2p_1 h_{i,1} dh_{i,1}$ and ${}^nL_i \subset T^*\mathbb{R}^n$ is given by the equations

$$x_1 = h_{i,1}^2 \qquad \sum_{j=2}^{n} p_j dx_j = -p_1 dh_{i,1}^2 = -2p_1 h_{i,1} dh_{i,1}.$$

We write $(\widehat{x_0}, \widehat{x}, \widehat{p}) = S(x_0, x, p)$ for $(x_0, x, p) \in^n \Lambda_i$ and see

$$\widehat{x_0} = x_0 - p_1^2/4 = \pm(x_0 - h_i^2) = 0 \qquad \widehat{x_1} = x_1 + p_1/2 = x_1 - h_i = x_1 - (x_1 - h_{i,1}^2) = h_{i,1}^2.$$

Thus we conclude that ${}^n\Lambda_i$ is taken isomorphically to the Legendrian $\{0\} \times^n L_i$ and thus the union ${}^n\Lambda$ is taken isomorphically to the union $\{0\} \times^n L$. □

B.3.3 Distinguished quadrants

To construct the arboreal models we need to specify distinguished quadrants of the ${}^n\Gamma$ and introduce signs, the geometric meaning of which will become clear below.

For a list of signs $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$ we define the involution $\sigma_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\sigma_\varepsilon(x_1, ..., x_n) = (\varepsilon_1 x_1, ..., \varepsilon_n x_n).$$

Define the *signed domain quadrant* ${}^n R_i^\varepsilon \subset \mathbb{R}^n$ to be the closed subspace

$${}^n R_i^\varepsilon = \{\varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leqslant 0, ..., \varepsilon_{i-1}\varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leqslant 0\}.$$

Define the cooriented hypersurface ${}^n\Gamma_i^\varepsilon \subset \mathbb{R}^{n+1}$ to be the restricted signed graph

$${}^n\Gamma_i^\varepsilon = \{x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon\}|_{{}^n R_i^\varepsilon}$$

with the graphical coorientation. More explicitly ${}^n\Gamma_i^\varepsilon$ is given by the equations

$$x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon, \qquad \varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leqslant 0, ..., \varepsilon_{i-1}\varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leqslant 0.$$

Define the union

$${}^n\Gamma^\varepsilon = \bigcup_{i=0}^{n} {}^n\Gamma_i^\varepsilon.$$

Remark B.12. Note that $h_{i,i-1} = x_i$ and thus $\varepsilon_{i-1}\varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon = x_i$, therefor every ${}^n\Gamma_i^\varepsilon$ only depends on the first i signs $\varepsilon_0, ..., \varepsilon_{i-1}$ but not on ε_i. In particular the union ${}^n\Gamma^\varepsilon$ is independent of ε_n. △

Lemma B.13. *Fix a list of signs $\varepsilon = (\varepsilon_0, ..., \varepsilon_n)$ and set $\varepsilon' = (\varepsilon_1, ..., \varepsilon_n)$. The homeomorphism*

$$s : \varepsilon_0\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \varepsilon_0\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n, \quad (x_0, x_1, x_2, ..., x_n) \mapsto (x_0, x_1 + \varepsilon_0\sqrt{\varepsilon_0 x_0}, x_2, ..., x_n)$$

gives an identification

$$s({}^n\Gamma_i^\varepsilon) = \varepsilon_0\mathbb{R}_{\geqslant 0} \times {}^{n-1}\Gamma_{i-1}^{\varepsilon'}$$

for every $0 < i \leqslant n$.

Proof. Recall ${}^n\Gamma_i^\varepsilon$ is defined by

$$x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon, \qquad \varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leqslant 0, ..., \varepsilon_{i-1}\varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leqslant 0.$$

and thus in particular

$$x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon, \qquad \varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon = \varepsilon_0\varepsilon_1 h_i \circ \sigma_\varepsilon \leqslant 0.$$

When $\varepsilon_0 x_0 \geqslant 0$ and $\varepsilon_0\varepsilon_1 h_i \circ \sigma_\varepsilon \leqslant 0$ the equation $x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon$ is equivalent to $\sqrt{\varepsilon_0 x_0} = -\varepsilon_0\varepsilon_1 h_i \circ \sigma_\varepsilon$. This can be rewritten in the form

$$x_1 + \varepsilon_0\sqrt{\varepsilon_0 x_0} = \varepsilon_1 h_{i-1,1}^2 \circ \sigma_{\varepsilon'}.$$

Since $\varepsilon x_0 = h_i^2 \circ \sigma_\varepsilon \geqslant 0$ we see that s maps ${}^n\Gamma_i^\varepsilon$ into $\varepsilon_0\mathbb{R}_{\geqslant 0} \times \{x_1 = \varepsilon_1 h_{i-1,1}^2 \circ \sigma_{\varepsilon'}\}$. Now note that the functions $h_{i,1}, ..., h_{i,i-1}$ cutting out ${}^n\Gamma_i^\varepsilon$ are independent of the coordinates x_0, x_1 and thus push forward to the same functions $h_{i,1}, ..., h_{i,i-1}$ that cut out

$${}^{n-1}\Gamma_{i-1}^{\varepsilon'} \subset \{x_1 = \varepsilon_1 h_{i-1,1}^2 \circ \sigma_{\varepsilon'}\}.$$

To see that s gives a cooriented identification note that the coorientations of ${}^n\Gamma_i^\varepsilon$ and ${}^{n-1}\Gamma_{i-1}^{\varepsilon'}$ are positive in respectively the ∂_{x_0} and ∂_{x_1} direction. The ∂_{x_1} component of $s_*\partial_{x_1}$ is in the positive direction and thus s gives a cooriented identification. □

The following corollary explains the geometric meaning of the signs ε.

Corollary B.14. *Fix signs $\varepsilon = (\varepsilon_0, ..., \varepsilon_n)$. For $i = 0, ..., n-1$, we have $\varepsilon_i = \pm 1$ if and only if ${}^n\Gamma_{i+1}$ is on the $\pm$-side of ${}^n\Gamma_i$ with respect to the graphical dx_0-coorientation.*

Moreover, for $i = 1, ..., n-1$, we have $\varepsilon_i = \pm 1$ if and only if ${}^n\Gamma_{i+1} \cap^n \Gamma_0$ is on the $\pm$-side of ${}^n\Gamma_i \cap^n \Gamma_0$ with respect to the graphical dx_1-coorientation.

Proof. We proof the assertions by induction. For $i = 0$ we have ${}^n\Gamma_0 = \{x_0 = 0\}$ and ${}^n\Gamma_1 = \{x_0 = \varepsilon_0(\varepsilon_1 x_1)^2 = \varepsilon_0 x_1^2,\ \varepsilon_0\varepsilon_1(\varepsilon_1 x_1) = \varepsilon_0 x_1 \leqslant 0\}$ and the first assertion holds.

For $i > 0$ both assertions follow immediately by inductively applying Lemma B.13. □

We record the following direct corollary of Lemma B.13 that will prove useful in comparing the different constructions of arboreal singularities.

Corollary B.15. *The map $s : \varepsilon_0\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n \to \varepsilon_0\mathbb{R}_{\geqslant 0} \times \mathbb{R}^n$ restricts to a homeomorphism*

$$s|_{{}^n\Gamma^\varepsilon} :^n \Gamma^\varepsilon \xrightarrow{\simeq} \{0\} \times \mathbb{R}^n \cup \varepsilon_0\mathbb{R}^{\geqslant 0} \times^{n-1} \Gamma^{\varepsilon'}.$$

By recursively applying the map of Lemma B.13 we get homeomorphisms

$${}^n\Gamma_i^\varepsilon \simeq \mathbb{R}_{\geqslant 0}^i \times \{0\} \times \mathbb{R}^{n-i}$$

$${}^n\Gamma^\varepsilon \simeq \bigcup_{i=0}^{n} \mathbb{R}_{\geqslant 0}^i \times \{0\} \times \mathbb{R}^{n-i}.$$

Fix signs $\varepsilon = (\varepsilon_0, ., , , .\varepsilon_{n-1})$. For $i = 0$ let ${}^nL_0^\varepsilon = \mathbb{R}^n \subset T^*\mathbb{R}^n$ denote the zero-section. For $i = 1, ..., n$ we introduce the *positive conormal bundle*

$${}^nL_i^\varepsilon = \nu_{\mathbb{R}^n}^+\ {}^{n-1}\Gamma_{i-1}^\varepsilon \subset T^*\mathbb{R}^n$$

determined by the graphical orientation and define the union

$$^{n}L^{\varepsilon} = \bigcup_{i=0}^{n} {}^{n}L_i^{\varepsilon}.$$

Note that $^{n}L_i^{\varepsilon}$ is Lagrangian.

Fix signs $\varepsilon = (\varepsilon_0, ., , , .\varepsilon_n)$ and for $i = 0, ..., n$ define the Legendrian

$$^{n}\Lambda_i^{\varepsilon} \subset J^1\mathbb{R}^n$$

that is the lift of the front $^{n}\Gamma_i^{\varepsilon} \subset \mathbb{R}^{n+1}$ and their union

$$^{n}\Lambda^{\varepsilon} = \bigcup_{i=0}^{n} {}^{n}\Lambda_i^{\varepsilon}.$$

We have the following Lemma similar to Lemma B.11

Lemma B.16. *Fix signs $\varepsilon = (\varepsilon_0, ..., \varepsilon_n)$ and set $\varepsilon' = (\varepsilon_1, ..., \varepsilon_n)$. The contactomorphism*

$$S_{\varepsilon_0} : J^1\mathbb{R}^n \to J^1\mathbb{R}^n, \quad (x_0, x, p) \mapsto (x_0 - \varepsilon_0 p_1^2/4, x_1 + \varepsilon_0 p_1/2, x_2, ..., x_n, p_1, ..., p_n)$$

takes the Legendrian $^{n}\Lambda_i^{\varepsilon}$ isomorphically to the Legendrian $\{0\} \times {}^{n}L_i^{\varepsilon'}$ and thus the union $^{n}\Lambda^{\varepsilon}$ isomorphically to the union $\{0\} \times^{n} L^{\varepsilon'}$.

Proof. The assertion follows from the proof of Lemma B.11 with the extra conditions

$$\varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leqslant 0, ..., \varepsilon_{i-1}\varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leqslant 0.$$

Observe that if $\varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leqslant 0$ and $p_1 = -2\varepsilon_0\varepsilon_1 h_{i,0} \circ \sigma_\varepsilon$, then $p_1 \geqslant 0$, meaning that we indeed map to the positive conormal. As in the proof of Lemma B.11 the remaining functions are independent of x_0 and x_1. We conclude that S_{ε_0} takes $^{n}\Lambda_i^{\varepsilon}$ to $\{0\} \times^{n} L_i^{\varepsilon'}$. □

B.3.4 Conormal arboreal models

Before we construct the models for arboreal singularities, we first fix some terminology. Recall a *rooted tree* $\mathcal{T} = (T, \rho)$ is a pair of a tree T and a distinguished vertex $\rho \in V(T)$ called the *root*. We denote the set of edges of $\mathcal{T}$ by $E(\mathcal{T}) = E(T)$ and introduce the set of non-root vertices $N(\mathcal{T}) = V(\mathcal{T}) \setminus \{\rho\}$, we write $n(\mathcal{T}) = |N(\mathcal{T})|$.

Recall that the vertices $V(\mathcal{T}) = V(T)$ of a rooted tree have a natural poset structure with unique minimum ρ and $\alpha \leqslant \beta$ if the unique shortest path between β and ρ contains α. For every non-root vertex α of $\mathcal{T}$ there is a unique *parent vertex* $\widehat{\alpha}$ such that $\widehat{\alpha} \leqslant \alpha$ and there is no $\beta \neq \alpha, \widehat{\alpha} \in V(\mathcal{T})$ with $\widehat{\alpha} \leqslant \beta \leqslant \alpha$. We call a non-root vertex α that is adjacent to exactly one vertex a *leaf* and denote the set of all leaves by $L(T)$.

Definition B.17. A *signed rooted tree* $\widehat{\mathcal{T}} = (T, \rho, \varepsilon)$ is a rooted tree (T, ρ) such that each edge of T not adjacent to the root ρ has a decoration of a sign ± 1. △

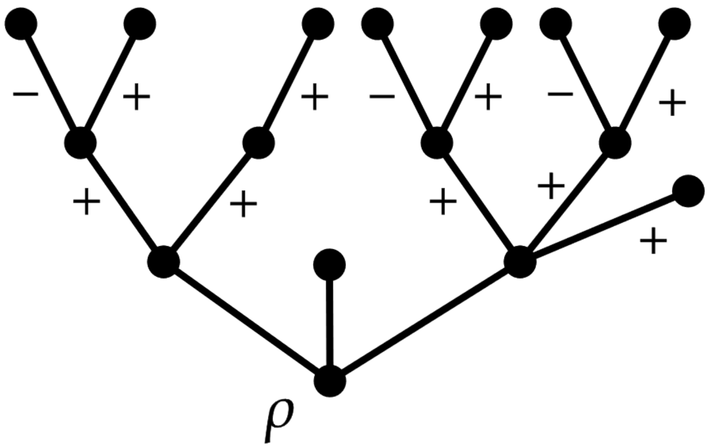


Figure B.4: A signed rooted tree.

Note that signed rooted trees are different from *fully signed rooted trees*, in the latter case all edges have a sign while in the former case the edges adjecent to the root do not have a sign.

Given a signed rooted tree $\widehat{\mathcal{T}} = (T, \rho, \varepsilon)$ we write $V(\widehat{\mathcal{T}}) = V(T)$ for the set of vertices, $E(\widehat{\mathcal{T}}) = E(T)$ for the set of edges, $N(\widehat{\mathcal{T}}) = N(\mathcal{T}) = V(\widehat{\mathcal{T}}) \setminus \{\rho\}$ for the set of non-root vertices and $v(\widehat{\mathcal{T}}) = \left|V(\widehat{\mathcal{T}})\right|$, $n(\widehat{\mathcal{T}}) = \left|N(\widehat{\mathcal{T}})\right|$.

We first produce conormal arboreal models for a subclass of signed rooted trees.

Conormal arboreal models for linear trees

Definition B.18. For $n \geqslant 0$, the signed rooted tree with vertices $V(A_n) = \{0, 1, ..., n\}$, edges $E(A_n) = \{\{i, i+1\} | i = 0, ..., n-1\}$ and root $\rho = 0$ is called the *linear signed rooted n-tree*. △

By definition, the sign a of a linear signed rooted n-tree is a length $n-1$ list of signs $(a_{\{1,2\}}, ..., a_{\{n-1,n\}})$. We set $\varepsilon = (a_{\{1,2\}}, ..., a_{\{n-1,n\}}, 1)$, the length n list of signs obtained by padding a by adding a single 1 at the end.

Definition B.19. Let $\mathcal{A}_n = (A_n, \rho, a)$ be a linear signed rooted tree and set $\varepsilon = (a, 1)$.

1. The *arboreal $\mathcal{A}_0$-front* is the empty set $H_{\mathcal{A}_0} = \emptyset$ inside the point $\mathbb{R}^0$. For $n \geqslant 1$, the arboreal $\mathcal{A}_n$-front $H_{\mathcal{A}_n}$ is the cooriented hypersurface

$$H_{\mathcal{A}_n} =^{n-1} \Gamma^{\varepsilon} \subset \mathbb{R}^n.$$

2. For $n \geqslant 0$, the *arboreal $\mathcal{A}_n$-Lagrangian* $L_{\mathcal{A}_n}$ is the union of the zero-section and positive

conormal

$$L_{\mathcal{A}_n} = \mathbb{R}^n \cup \nu^+_{\mathbb{R}^n} H_{\mathcal{A}_n} \subset T^*\mathbb{R}^n.$$

3. For $n \geqslant 0$, the *arboreal $\mathcal{A}_n$-Legendrian*, $\Lambda_{\mathcal{A}_n}$, is the lift

$$\Lambda_{\mathcal{A}_{n+1}} = \{0\} \times L_{\mathcal{A}_n}.$$

△

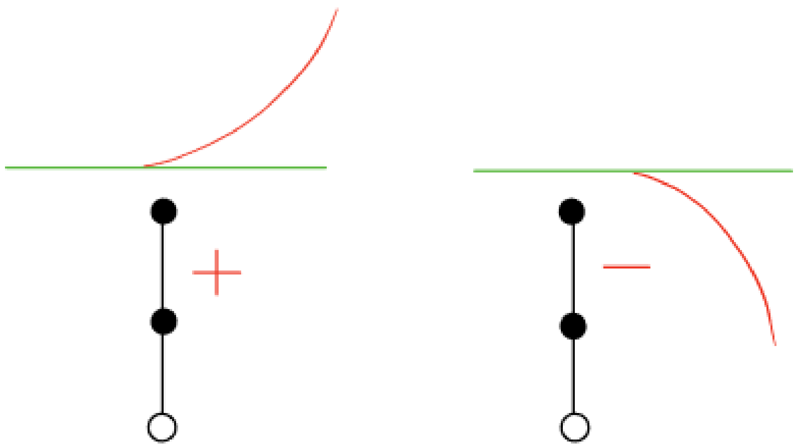


Figure B.5: Two A_2 fronts with positive and negative sign, ([AEN22a], p. 13)

We can also give a more invariant description of these arboreal models by viewing the ambient space $\mathbb{R}^n$ as $\mathbb{R}^{N(\mathcal{A}_n)}$ where the ordering of the coordinates matches that of $N(\mathcal{A}_n)$.

We rename the smooth pieces of the $\mathcal{A}_n$-front, indexing them by the non-root vertices

$$H_i =^{n-1} \Gamma^{\varepsilon}_{i-1} \subset H_{\mathcal{A}_n}, \qquad i \in N(\mathcal{A}_n) = \{1, ..., n\}.$$

Then we can write the $\mathcal{A}_n$-Lagrangian as the union of smooth pieces L_i indexed by the vertices, where $L_0 = \mathbb{R}^n \subset L_{\mathcal{A}_n}$ and

$$L_i = \nu^+_{\mathbb{R}^n} H_i \subset L_{\mathcal{A}_n}, \qquad i \in N(\mathcal{A}_n) = \{1, ..., n\}.$$

Lastly, we can write the $\mathcal{A}_n$-Legendrian as the union of smooth pieces Λ_i indexed by the vertices

$$\Lambda_i = \{0\} \times L_i, \qquad i \in V(\mathcal{A}_n) = \{0, ..., n\}.$$

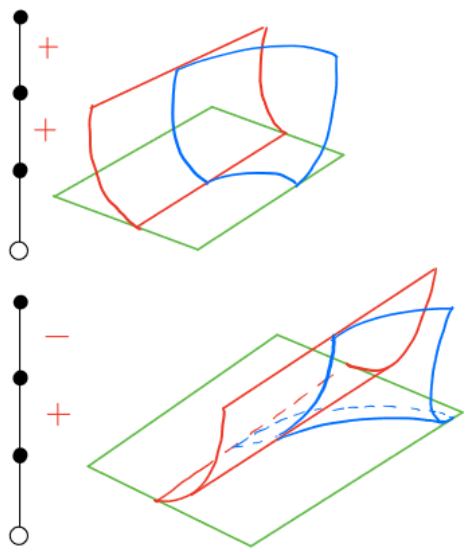


Figure B.6: Two A_3 fronts with different choices of signs, ([AEN22a], p. 14)

Conormal arboreal models for general trees

We now construct conormal arboreal models for general trees, let $\widehat{\mathcal{T}} = (T, \rho, \varepsilon)$ be a signed rooted tree. To each leaf $\beta \in L(T)$ we can associate the linear signed rooted tree $\mathcal{A}_\beta = (A_\beta, \rho, a_\beta)$ where A_β is the full subtree of T with vertices $V(A_\beta) = \{\alpha \leqslant \beta \in V(T)\}$ and restricted sign decoration a_β.

For each leaf $\beta \in L(T)$ the inclusion $N(\mathcal{A}_\beta) \subset N(\widehat{\mathcal{T}})$ induces a natural projection

$$\pi_\beta : \mathbb{R}^{N(\widehat{\mathcal{T}})} \to \mathbb{R}^{N(\mathcal{A}_\beta)}.$$

Definition B.20. Let $\widehat{\mathcal{T}} = (T, \rho, \varepsilon)$ be a signed rooted tree.

1. The *arboreal model* $\widehat{\mathcal{T}}$*-front* is the singular hypersurface given by the union

$$H_{\widehat{\mathcal{T}}} = \bigcup_{\beta \in L(\widehat{\mathcal{T}})} \pi_\beta^{-1}(H_{\mathcal{A}_\beta}) \subset \mathbb{R}^{N(\widehat{\mathcal{T}})}$$

   where $\mathcal{A}_\beta \subset \mathbb{R}^{N(\mathcal{A}_\beta)}$ is the arboreal $\mathcal{A}_\beta$-front.

2. The *arboreal model* $\widehat{\mathcal{T}}$*-Lagrangian* is the union to the zero-section and positive conormal

$$L_{\widehat{\mathcal{T}}} = \mathbb{R}^{N(\widehat{\mathcal{T}})} \cup \nu^+_{\mathbb{R}^{N(\widehat{\mathcal{T}})}} H_{\widehat{\mathcal{T}}} \subset T^*\mathbb{R}^{N(\widehat{\mathcal{T}})}.$$

3. The *arboreal model* $\widehat{\mathcal{T}}$*-Legendrian* is the lift

$$\Lambda_{\widehat{\mathcal{T}}} = \{0\} \times L_{\widehat{\mathcal{T}}} \subset J^1\mathbb{R}^{N(\widehat{\mathcal{T}})}.$$

△

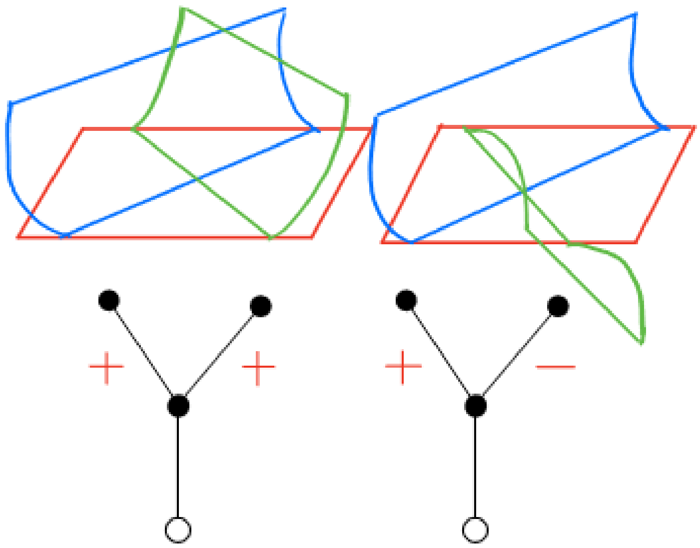


Figure B.7: Two arboreal fronts for non-linear trees with different choices of signs, ([AEN22a], p. 15)

Similar to our approach for the $\mathcal{A}_{n+1}$ case, we can index the smooth pieces of the $\widehat{\mathcal{T}}$-front by non-root vertices

$$H_\alpha = \pi_\beta^{-1}(H_{\mathcal{A}_\beta}) \subset H_{\widehat{\mathcal{T}}}, \qquad \alpha \in N(\widehat{\mathcal{T}})$$

where $\beta \in L(\widehat{\mathcal{T}})$ is any leaf with $\alpha \leqslant \beta$ and $H_{\mathcal{A}_\beta} \subset H_{\mathcal{A}_\beta}$ is the corresponding smooth piece. No that every H_α has a coorientation induced by the coorientation of $H_{\mathcal{A}_\beta}$.

Likewise, we can index the smooth pieces of the $\widehat{\mathcal{T}}$-Lagrangian by vertices where $L_\rho = \mathbb{R}^{N(\widehat{\mathcal{T}})} \subset L_{\widehat{\mathcal{T}}}$ and

$$L_\alpha = \nu^+_{\mathbb{R}^{N(\widehat{\mathcal{T}})}} H_\alpha \subset L_{\widehat{\mathcal{T}}}, \qquad \alpha \in N(\widehat{\mathcal{T}})$$

and we can index the smooth pieces of the $\widehat{\mathcal{T}}$-Legendrian by vertices

$$\Lambda_\alpha = \{0\} \times L_\alpha \subset \Lambda_{\widehat{\mathcal{T}}}, \qquad \alpha \in V(\widehat{\mathcal{T}}).$$

Remark B.21. Let $\widehat{\mathcal{T}}$ be a signed rooted tree and let τ be a leaf of $\widehat{\mathcal{T}}$. Then we have identifications

$$\begin{aligned} H_{\widehat{\mathcal{T}}} &= H_{\widehat{\mathcal{T}}\setminus\{\tau\}} \times \mathbb{R}^{\tau} \cup H_{\tau}, \\ L_{\widehat{\mathcal{T}}} &= L_{\widehat{\mathcal{T}}\setminus\{\tau\}} \times \mathbb{R}^{\tau} \cup L_{\tau}, \\ \Lambda_{\widehat{\mathcal{T}}} &= \Lambda_{\widehat{\mathcal{T}}\setminus\{\tau\}} \times \mathbb{R}^{\tau} \cup \Lambda_{\tau}. \end{aligned}$$

△

B.4 Comparing the gluing and conormal construction

We will now show that the arboreal model $L_{\widehat{\mathcal{T}}}$ Lagrangian corresponding to a signed rooted tree $\widehat{\mathcal{T}} = (T, \rho, \varepsilon)$ is homeomorphic to the arboreal singularity $\mathfrak{L}_T$ of T as defined in Section B.2. We begin by showing this for linear trees.

Lemma B.22. *For any linear signed rooted tree $\mathcal{A}_n = (A_n, \rho, a)$ there is a stratified homeomorphism*

$$\mathfrak{L}_{A_n} \simeq L_{\mathcal{A}_n}.$$

Proof. We use induction, first note $L_{\mathcal{A}_0} = \mathbb{R}^0$ and $\mathfrak{L}_{A_0} = \mathbb{R}^0$ and thus $\mathfrak{L}_{A_0} \simeq L_{\mathcal{A}_0}$. Now assume

$$\mathfrak{L}_{A_n} \simeq L_{\mathcal{A}_n}$$

for some $n \geqslant 1$.

Directly from the definition we see

$$\begin{aligned} L_{\mathcal{A}_{n+1}} &= \mathbb{R}^n \cup \nu^+_{\mathbb{R}^n} H_{\mathcal{A}_{n+1}} \subset T^*\mathbb{R} \\ &= \mathbb{R}^n \cup \nu^+_{\mathbb{R}^n}\, {}^{n-1}\Gamma^{\varepsilon} \\ &= \mathbb{R}^n \cup \nu^+_{\mathbb{R}^n} \left(\bigcup_{i=0}^{n-1} {}^{n-1}\Gamma^{\varepsilon}_i \right) \\ &= \mathbb{R}^n \cup \nu^+_{\mathbb{R}^n} \left(\bigcup_{i=0}^{n-2} {}^{n-1}\Gamma^{\varepsilon}_i \right) \bigcup \nu^+_{\mathbb{R}^n}\, {}^{n-1}\Gamma^{\varepsilon}_{n-1} \end{aligned}$$

We first take a closer look at the first two factors. Recall that for $0 \leqslant i < n$ we have

$${}^{n-1}\Gamma^{\varepsilon}_i \simeq {}^{n-2}\Gamma^{\varepsilon'}_i \times \mathbb{R},$$

where $\varepsilon' = (\varepsilon_1, ..., \varepsilon_n)$, giving

$$\mathbb{R}^n \cup \nu^+_{\mathbb{R}^n} \left(\bigcup_{i=0}^{n-2} {}^{n-1}\Gamma^{\varepsilon}_i \right) = L_{\mathcal{A}_n} \times \mathbb{R}.$$

We now consider the last factor. Up to homeomorphism the positive conormal $\nu^+_{\mathbb{R}^n} H$ of a hyperspace $H \subset \mathbb{R}^n$ is obtained by attaching a half line $\mathbb{R}_{\geqslant 0}$ in the positive conormal direction at every point. Note that the coorientation of ${}^{n-1}\Gamma^{\varepsilon}_{n-1}$ and any ${}^{n-1}\Gamma^{\varepsilon}_{i}$ agree on their intersection, which can be seen inductively by applying Lemma B.13.

Thus $\nu^+_{\mathbb{R}^n}\, {}^{n-1}\Gamma^{\varepsilon}_{n-1} \subset L_{\mathcal{A}_{n+1}}$ can seen as attaching ${}^{n-1}\Gamma^{\varepsilon}_{n-1} \times \mathbb{R}_{\geqslant 0}$ to $\mathbb{R}^n \cup \nu^+_{\mathbb{R}^n}\left(\bigcup_{i=0}^{n-2} {}^{n-1}\Gamma^{\varepsilon}_{i}\right)$ along

$$ {}^{n-1}\Gamma^{\varepsilon}_{n-1} \times \{0\} \cup \left(\bigcup_{i=0}^{n-2} {}^{n-1}\Gamma^{\varepsilon}_{i} \cap^{n-1} \Gamma^{\varepsilon}_{n-1}\right) \times \mathbb{R}_{\geqslant 0}. $$

Recall, as stated in Corollary B.15, that we have the canonical identification

$$ {}^{n-1}\Gamma^{\varepsilon}_{i} \simeq \mathbb{R}^{i}_{\geqslant 0} \times \{0\} \times \mathbb{R}^{n-1-i}. $$

Thus we have canonical identifications

$$ {}^{n-1}\Gamma^{\varepsilon}_{n-1} \simeq \mathbb{R}^{n-1}_{\geqslant 0} \times \{0\}, $$
$$ \bigcup_{i=0}^{n-2} {}^{n-1}\Gamma^{\varepsilon}_{i} \cap^{n-1} \Gamma^{\varepsilon}_{n-1} \simeq \bigcup_{i=0}^{n-2} \mathbb{R}^{i}_{\geqslant 0} \times \{0\} \times \mathbb{R}^{n-2-i}_{\geqslant 0} \times \{0\}. $$

Now, note that

$$ \mathbb{R}^{n-1}_{\geqslant 0} \times \{0\} \times \{0\} \bigcup \left(\bigcup_{i=0}^{n-2} \mathbb{R}^{i}_{\geqslant 0} \times \{0\} \times \mathbb{R}^{n-2-i}_{\geqslant 0} \times \{0\} \times \mathbb{R}_{\geqslant 0}\right) $$

is exactly the boundary of the positive quadrant

$$ Q_n := \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} | x_{n-1} = 0, x_i \geqslant 0 \text{ for all } i \neq n-1\} $$

in the $x_{n-1} = 0$ hyperplane of $\mathbb{R}^{n+1}$. The gluing in of $\nu^+_{\mathbb{R}^n}\, {}^{n-1}\Gamma^{\varepsilon}_{n-1}$ is precisely attaching the interior of the positive quadrant to the boundary. Thus

$$ L_{\mathcal{A}_{n+1}} \simeq \mathbb{R}^n \cup \nu^+_{\mathbb{R}^n}\left(\bigcup_{i=0}^{n-2} {}^{n-1}\Gamma^{\varepsilon}_{i}\right) \coprod_{\partial Q_n} Q_n. $$

Now note that the quadrant Q_n admits the representation $Q_n \simeq \partial Q_n \times \mathbb{R}_{\geqslant 0}$ and that there is a piecewise linear homeomorphism $\partial Q_n \simeq \mathbb{R}^{n-1}$. We conclude that

$$ L_{\mathcal{A}_{n+1}} \simeq L_{\mathcal{A}_n} \times \mathbb{R} \coprod_{\mathbb{R}^{n-1} \times \{0\}} \mathbb{R}^{n-1} \times \mathbb{R}_{\geqslant 0} $$

which is the exact same representation as appears in Lemma B.5. We conclude that

$$ \mathfrak{L}_{A_n} \simeq L_{\mathcal{A}_n}. \tag{B.1} $$

□

Theorem B.23. *For any signed rooted tree $\widehat{\mathcal{T}} = (T, \rho, \varepsilon)$, there is a stratified homeomorphism*

$$\mathfrak{L}_T \simeq L_{\widehat{\mathcal{T}}}.$$

Proof. We proceed by induction over the number n of non-root vertices. Note that the case of $n = 0$ is a direct result of Lemma B.22 above.

Assume the statement holds for all signed rooted trees with $n-1$ non-root vertices for some $n \geqslant 1$. Let $\widehat{\mathcal{T}} = (T, \rho, \varepsilon)$ be a signed rooted tree with n non-root vertices, we assume $\widehat{\mathcal{T}}$ is not a linear signed rooted tree, as this case is covered by Lemma B.22. Consider any leaf $\tau \in L(\widehat{\mathcal{T}})$, denote its associated linear signed rooted tree $\mathcal{A}_\tau \subset \widehat{\mathcal{T}}$, and denote the signed rooted tree with $n-1$ non-root vertices obtained by deleting τ by $\widehat{\mathcal{T}}_\tau$. Recall, from the discussion in Section B.3,

$$\begin{aligned} L_{\widehat{\mathcal{T}}} &= \cup_{\alpha \in V(\widehat{\mathcal{T}})} L_\alpha \\ &= \mathbb{R}^{V(\widehat{\mathcal{T}})} \cup_{\alpha \in N(\widehat{\mathcal{T}})} \nu^+_{\mathbb{R}^{N(\widehat{\mathcal{T}})}} H_\alpha \\ &= L_{\widehat{\mathcal{T}}_\tau} \times \mathbb{R}^{\{\tau\}} \cup \nu^+_{\mathbb{R}^{N(\widehat{\mathcal{T}})}} H_\tau \end{aligned}$$

Now consider a vertex $\alpha \notin \mathcal{A}_\tau$ and a leaf $\beta \in L(\widehat{\mathcal{T}})$ with $\alpha \leqslant \beta$. Note that $N(\mathcal{A}_\beta) \cap N(\mathcal{A}_\tau) = \emptyset$, thus there is an inclusion $N(\mathcal{A}_\beta) \subset N(\widehat{\mathcal{T}}) \setminus N(\mathcal{A}_\tau)$ that induces a natural projection $\tilde{\pi}_\beta : \mathbb{R}^{N(\widehat{\mathcal{T}}) \setminus N(\mathcal{A}_\tau)} \to \mathbb{R}^{N(\mathcal{A}_\beta)}$ which is a restriction of the projection $\pi_\beta : \mathbb{R}^{N(\widehat{\mathcal{T}})} \to \mathbb{R}^{N(\mathcal{A}_\beta)}$. Recall that $H_\alpha = \pi_\beta^{-1}(H_{\mathcal{A}_{\beta,\alpha}})$, meaning that

$$H_\alpha = \tilde{\pi}_\beta^{-1}(H_{\mathcal{A}_{\beta,\alpha}}) \times \mathbb{R}^{N(\mathcal{A}_\tau)}$$

From this we see that H_α and H_τ are not tangent along their intersection and thus must have different conormals along their intersection. In contrast, for any $\gamma \in \mathcal{A}_\tau$ we have that H_τ and H_γ have the same conormal along their intersection as a result of the proof of Lemma B.22.

Thus $\nu^+_{\mathbb{R}^{N(\widehat{\mathcal{T}})}} H_\tau \subset L_{\widehat{\mathcal{T}}}$ can be seen as attaching $H_\tau \times \mathbb{R}_{\geqslant 0}$ to $\mathbb{R}^{V(\widehat{\mathcal{T}})} \cup_{\alpha \in N(\widehat{\mathcal{T}}_\tau)} \nu^+_{\mathbb{R}^{N(\widehat{\mathcal{T}})}} H_\alpha$ along

$$H_\tau \times \{0\} \cup \left(\bigcup_{\gamma \in V(\mathcal{A}_\tau \setminus \{\tau\})} (H_\gamma \cap H_\tau) \times \mathbb{R}_{\geqslant 0} \right).$$

By the discussion above we can conclude that

$$L_{\widehat{\mathcal{T}}} \simeq L_{\widehat{\mathcal{T}}_\tau} \times \mathbb{R} \coprod_{\mathbb{R}^{n-1} \times \{0\}} \mathbb{R}^{n-1} \times \mathbb{R}_{\geqslant 0}$$

and thus

$$\mathfrak{L}_T \simeq L_{\widehat{\mathcal{T}}}.$$

□

B.5 Stability of the conormal construction

Now that we have constructed the models for arboreal fronts, Lagrangians and Legendrians, we are ready to define arboreal fronts, Lagrangians and Legendrians.

Definition B.24. Arboreal fronts, Lagrangians and Legendrians are defined as follows.

1. A closed subset $H \subset M$ of a $(m+1)$-dimensional manifold M is called an *arboreal front* if the germ of (M, H) at any point $m \in M$ is diffeomorphic to the germ of $(\mathbb{R}^n \times \mathbb{R}^{m-n}, H_{\widehat{\mathcal{T}}} \times \mathbb{R}^{m-n})$ at the origin, for a signed rooted tree $\widehat{\mathcal{T}}$ with $n = n(\widehat{\mathcal{T}}) \leqslant m$.
2. A closed subset $L \subset X$ of a $2m$-dimensional symplectic manifold (X, ω) is called an *arboreal Lagrangian* if the germ of (X, L) at any point $\lambda \in L$ is symplectomorphic to the germ of the pair $(T^*\mathbb{R}^n \times T^*\mathbb{R}^{m-n}, L_{\widehat{\mathcal{T}}} \times \mathbb{R}^{m-n})$ at the origin, for a signed rooted tree $\widehat{\mathcal{T}}$ with $n = n(\widehat{\mathcal{T}}) \leqslant m$.
3. A closed subset $\Lambda \subset X$ of a $(2m+1)$-dimensional contact manifold (Y, ξ) is called an *arboreal Legendrian* if the germ of (Y, Λ) at any point $\lambda \in L$ is contactomorphic to the germ of the pair $(J^1(\mathbb{R}^n \times \mathbb{R}^{m-n} = J^1\mathbb{R}^n\mathbb{R}^n \times T^*\mathbb{R}^{m-n}, \Lambda_{\widehat{\mathcal{T}}} \times \mathbb{R}^{m-n})$ at the origin, for a signed rooted tree $\widehat{\mathcal{T}}$ with $n = n(\widehat{\mathcal{T}}) \leqslant m$.

The pair $(\widehat{\mathcal{T}}, m)$ is called the *arboreal type* of the germ of H, L or Λ. △

The goal of this section is to prove the following theorem.

Theorem B.25. *To each class in Arb_n^{symp} we can assign a signed rooted tree $\widehat{\mathcal{T}} = (T, \rho, \varepsilon)$ with at most $n+1$ vertices, thus Definition B.1 produces only finitely many local models up to ambient symplectomorphism per dimension n.*

This theorem is a corollary of the following Stability Theorem, which is Theorem 3.5 from [AEN22a]. Let t^*M denote the germ of T^*M along the zero-section M.

Theorem B.26. *Let $\widehat{\mathcal{T}}_1, ..., \widehat{\mathcal{T}}_k$ be signed rooted trees with roots $\rho_1, ..., \rho_k$. Fix $m > n = \sum_j n(\widehat{\mathcal{T}}_j)$ and let $\varphi_j : t^*\mathbb{R}^m \to J^1\mathbb{R}^m$ be germs of embeddings with disjoint images. Denote $z_j := \varphi_j(0)$, $\Lambda^j = \varphi_j(L_{\widehat{\mathcal{T}}_j} \times \mathbb{R}^{m-n(\widehat{\mathcal{T}}_j)})$, $j = 1, ..., k$. Suppose that*

1. *$\pi(z_j) = 0$ for all $j = 1, ..., k$;*
2. *the arboreal Legendrian $\Lambda := \cup_{j=1}\Lambda^j$ projects transversely under the front projection $\pi : J^1\mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^m$.*

Then there is a signed rooted tree $\widehat{\mathcal{T}}$, obtained by joining all $\widehat{\mathcal{T}}_j$ to a common root ρ along their roots ρ_j and picking appropriate signs for the edges $\{\rho_j, \alpha\}$, such that $\mathbb{R}^m \cup C(\Lambda)$ is an arboreal Lagrangian of type $(\widehat{\mathcal{T}}, m)$. Here C denotes the Liouville cone. Equivalently, there is a diffeomorphism ψ between the germ of $H_{\widehat{\mathcal{T}}} \times \mathbb{R}^{m-n(\widehat{\mathcal{T}})}$ and the germ of the front $\pi(\Lambda)$.

We can now prove Theorem B.25.

Proof. The assertion follows by induction over dimension n. By definition Arb_0^{symp} has only a single local model that is represented by the signed rooted tree consisting of only one vertex.

Now assume the assertion has been established for $n-1 \geqslant 0$. Any class in Arb_n^{symp} with $n \geqslant 1$ can be represented as either a stabilization $L \times \mathbb{R}$ of some L in $\mathrm{Arb}_{n-1}^{symp}$ or the Liouville cone of appropriate $\Lambda_1, ..., \Lambda_k$ from $\mathrm{Arb}_{n-1}^{cont}$.

In the first case the class in Arb_n^{symp} is represented by the same signed rooted tree as L, which has less than $n-1$ by the induction assumption.

In the second case the class is represented by a signed rooted tree $\widehat{\mathcal{T}}$ formed by joining the trees $\widehat{\mathcal{T}_j}$ with root ρ_j assigned to the Legendrians Λ_j at a common root ρ.

From the transversality condition on the projection we know $\widehat{\mathcal{T}}$ has at most n vertices. □

B.6 Generalized arboreal models

To allow for interactions between strata when some strata have boundary, one needs a mild generalization of our arboreal models. Nadler introduced these *generalized arboreal singularities* in [Nad16], we expand on the definition in [Nad16] by adding signs to the construction in a fashion inspired by the construction by Starkston in [Sta18]. Every generalized arboreal singularity can be assigned a signed rooted tree with some extra data.

Definition B.27. A *signed leafy rooted tree* $(\widehat{\mathcal{T}}, \ell)$ is a signed rooted tree $\widehat{\mathcal{T}}$ with a collection ℓ of marked leaf vertices. △

From a signed leafy rooted tree $(\widehat{\mathcal{T}}, \ell)$ we can construct a signed rooted tree $\widehat{\mathcal{T}}^+$ by adding a vertex above each leaf $\tau \in \ell$ and giving each edge between a leaf $\tau \in \ell$ and its added vertex the sign of the unique edge incident on τ.

Recall that we could index the smooth pieces of the arboreal model $\widehat{\mathcal{T}}^+$ front, Lagrangian and Legendrian by the vertices of $\widehat{\mathcal{T}}^+$.

Definition B.28. Let $(\widehat{\mathcal{T}}, \ell)$ be a signed leafy rooted tree.

1. The *generalized arboreal model* $(\widehat{\mathcal{T}}, \ell)$*-front* is the multi-cooriented hypersurface

$$H_{(\widehat{\mathcal{T}},\ell)} = \bigcup_{\alpha \in N(\widehat{\mathcal{T}}^+)\setminus \ell} H_\alpha \subset H_{\widehat{\mathcal{T}}^+} \subset \mathbb{R}^{N(\widehat{\mathcal{T}}^+)}$$

where H_α is the smooth piece of $H_{\widehat{\mathcal{T}}^+}$ indexed by α.

2. The *generalized arboreal model* $(\widehat{\mathcal{T}}, \ell)$*-Lagrangian* is the union

$$L_{(\widehat{\mathcal{T}},\ell)} = \bigcup_{\alpha \in V(\widehat{\mathcal{T}}^+)\setminus \ell} L_\alpha \subset L_{\widehat{\mathcal{T}}^+} \subset T^*\mathbb{R}^{N(\widehat{\mathcal{T}}^+)}$$

where L_α is the smooth piece of $L_{\widehat{\mathcal{T}}^+}$ indexed by α.

3. The *generalized arboreal model* $(\widehat{\mathcal{T}}, \ell)$*-Legendrian* is the union

$$\Lambda_{(\widehat{\mathcal{T}},\ell)} = \bigcup_{\alpha \in V(\widehat{\mathcal{T}}^+)\setminus \ell} \Lambda_\alpha \subset \Lambda_{\widehat{\mathcal{T}}^+} \subset J^1\mathbb{R}^{N(\widehat{\mathcal{T}}^+)}$$

where Λ_α is the smooth piece of $\Lambda_{\widehat{\mathcal{T}}^+}$ indexed by α.

△

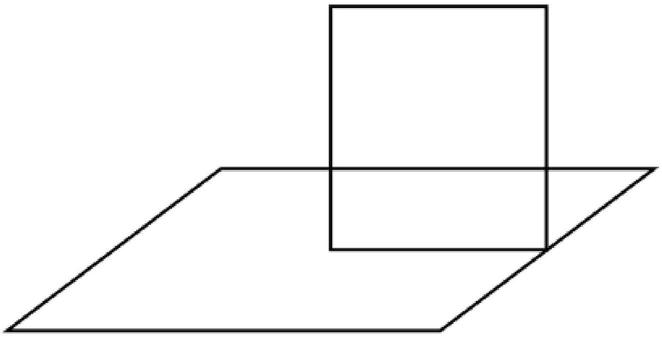


Figure B.8: A generalized arboreal singularity.

The following description of generalized arboreal models will prove useful.

Lemma B.29. *Let $(\widehat{\mathcal{T}}, \ell)$ be a signed leafy rooted tree, denote the signed rooted tree obtained by deleting all leaves in ℓ from $\widehat{\mathcal{T}}$ by $\widehat{\mathcal{T}}^-$ and denote the set of vertices added above the leaves ℓ by ℓ^+. Then we have canonical identities*

$$H_{(\widehat{\mathcal{T}},\ell)} = H_{\widehat{\mathcal{T}}^+} \setminus H_{\widehat{\mathcal{T}}} \times \mathbb{R}^{|\ell|} \cup H_{\widehat{\mathcal{T}}^-} \times \mathbb{R}^{2|\ell|},$$
$$L_{(\widehat{\mathcal{T}},\ell)} = L_{\widehat{\mathcal{T}}^+} \setminus L_{\widehat{\mathcal{T}}} \times \mathbb{R}^{|\ell|} \cup L_{\widehat{\mathcal{T}}^-} \times \mathbb{R}^{2|\ell|}.$$

Proof. Note that

$$H_{(\widehat{\mathcal{T}},\ell)} = H_{\widehat{\mathcal{T}}^+} \setminus \bigcup_{\tau \in \ell} H_\tau,$$
$$L_{(\widehat{\mathcal{T}},\ell)} = L_{\widehat{\mathcal{T}}^+} \setminus \bigcup_{\tau \in \ell} H_\tau.$$

By Remark B.21 we know that for any $\tau \in \ell^+$

$$H_{\widehat{\mathcal{T}}^+} = H_{\widehat{\mathcal{T}}^+\setminus\{\tau\}} \times \mathbb{R}^\tau \cup H_\tau,$$
$$L_{\widehat{\mathcal{T}}^+} = L_{\widehat{\mathcal{T}}^+\setminus\{\tau\}} \times \mathbb{R}^\tau \cup L_\tau.$$

From this we get

$$\begin{aligned} H_{\widehat{\mathcal{T}}^+} &= H_{\widehat{\mathcal{T}}} \times \mathbb{R}^{|\ell|} \cup_{\tau \in \ell^+} H_\tau, \\ L_{\widehat{\mathcal{T}}^+} &= L_{\widehat{\mathcal{T}}} \times \mathbb{R}^{|\ell|} \cup_{\tau \in \ell^+} L_\tau. \end{aligned}$$

Meaning that

$$\begin{aligned} H_{\widehat{\mathcal{T}}^+} \setminus H_{\widehat{\mathcal{T}}} \times \mathbb{R}^{|\ell|} &= \cup_{\tau \in \ell^+} H_\tau, \\ L_{\widehat{\mathcal{T}}^+} \setminus L_{\widehat{\mathcal{T}}} \times \mathbb{R}^{|\ell|} &= \cup_{\tau \in \ell^+} L_\tau. \end{aligned}$$

Similarly we get

$$\begin{aligned} H_{\widehat{\mathcal{T}}} &= H_{\widehat{\mathcal{T}}^-} \times \mathbb{R}^{|\ell|} \cup_{\tau \in \ell} H_\tau, \\ L_{\widehat{\mathcal{T}}} &= L_{\widehat{\mathcal{T}}^-} \times \mathbb{R}^{|\ell|} \cup_{\tau \in \ell} L_\tau. \end{aligned}$$

Meaning that

$$\begin{aligned} H_{\widehat{\mathcal{T}}^+} \setminus H_{\widehat{\mathcal{T}}} \times \mathbb{R}^{|\ell|} \cup H_{\widehat{\mathcal{T}}^-} \times \mathbb{R}^{2|\ell|} &= H_{\widehat{\mathcal{T}}^+} \setminus \bigcup_{\tau \in \ell} H_\tau, \\ L_{\widehat{\mathcal{T}}^+} \setminus L_{\widehat{\mathcal{T}}} \times \mathbb{R}^{|\ell|} \cup L_{\widehat{\mathcal{T}}^-} \times \mathbb{R}^{2|\ell|} &= L_{\widehat{\mathcal{T}}^+} \setminus \bigcup_{\tau \in \ell} H_\tau. \end{aligned}$$

□

Definition B.30. We can now define generalized arboreal singularities.

1. A closed subset $H \subset M$ of a $(m+1)$-dimensional manifold M is called a *generalized arboreal front* if the germ of (M, H) at any point $m \in M$ is diffeomorphic to the germ of $(\mathbb{R}^n \times \mathbb{R}^{m-n}, H_{(\widehat{\mathcal{T}},\ell)} \times \mathbb{R}^{m-n})$ at the origin, for a signed leafy rooted tree $(\widehat{\mathcal{T}}, \ell)$ with $n = n(\widehat{\mathcal{T}}) + |\ell| \leqslant m$.
2. A closed subset $L \subset X$ of a $2m$-dimensional symplectic manifold (X, ω) is called a *generalized arboreal Lagrangian* if the germ of (X, L) at any point $\lambda \in L$ is symplectomorphic to the germ of the pair $(T^*\mathbb{R}^n \times T^*\mathbb{R}^{m-n}, L_{(\widehat{\mathcal{T}},\ell)} \times \mathbb{R}^{m-n})$ at the origin, for a signed leafy rooted tree $(\widehat{\mathcal{T}}, \ell)$ with $n = n(\widehat{\mathcal{T}}) + |\ell| \leqslant m$.
3. A closed subset $\Lambda \subset X$ of a $(2m+1)$-dimensional contact manifold (Y, ξ) is called a *generalized arboreal Legendrian* if the germ of (Y, Λ) at any point $\lambda \in L$ is contactomorphic to the germ of the pair $(J^1(\mathbb{R}^n \times \mathbb{R}^{m-n} = J^1\mathbb{R}^n\mathbb{R}^n \times T^*\mathbb{R}^{m-n}, \Lambda_{(\widehat{\mathcal{T}},\ell)} \times \mathbb{R}^{m-n})$ at the origin, for a signed leafy rooted tree $(\widehat{\mathcal{T}}, \ell)$ with $n = n(\widehat{\mathcal{T}}) + |\ell| \leqslant m$.

The triple $(\widehat{\mathcal{T}}, \ell, m)$ is called the *generalized arboreal type* of the germ of H, L or Λ. △

Using the Stability Theorem B.26 we can show the following Stability Theorem for generalized arboreal Lagrangians.

Theorem B.31. *For $j = 1, ..., k$, let $(\widehat{\mathcal{T}_j}, \ell_j)$ be signed leafy rooted trees with roots ρ_j. Fix $m > n = \sum_j n(\widehat{\mathcal{T}_j}) + |\ell_j|$ and let $\varphi_j : t^*\mathbb{R}^m \to J^1\mathbb{R}^m$ be germs of embeddings with disjoint images.*

Denote $z_j := \varphi_j(0)$, $\Lambda^j = \varphi_j(L_{(\widehat{\mathcal{T}_j},\ell_j)} \times \mathbb{R}^{m-n(\widehat{\mathcal{T}_j})-|\ell|})$, $j = 1, ..., k$. Suppose that

1. *$\pi(z_j) = 0$ for all $j = 1, ..., k$;*
2. *the arboreal Legendrian $\Lambda := \cup_{j=1}^k \Lambda^j$ projects transversely under the front projection $\pi : J^1\mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^m$.*

Then there is a signed leafy rooted tree $(\widehat{\mathcal{T}}, \ell)$, obtained by joining all $(\widehat{\mathcal{T}_j}, \ell_j)$ to a common root ρ along their roots ρ_j and picking appropriate signs for the edges $\{\rho_j, \alpha\}$, such that $\mathbb{R}^m \cup C(\Lambda_y)$ is a generalized arboreal Lagrangian of type $(\widehat{\mathcal{T}}, \ell, m)$. Here C denotes the Liouville cone. Equivalently, there is a diffeomorphism ψ between the germ of $H_{(\widehat{\mathcal{T}},\ell)} \times \mathbb{R}^{m-n(\widehat{\mathcal{T}})-|\ell|}$ and the germ of the front $\pi(\Lambda)$.

Proof. The assertion follows from applying Theorem B.26 to different signed rooted trees obtained from the signed leafy rooted tree $(\widehat{\mathcal{T}}, \ell)$.

Let $\widehat{\mathcal{T}}_j^-$ denote the signed rooted tree obtained from $\widehat{\mathcal{T}_j}$ by deleting the marked leaves ℓ_j and let $\widehat{\mathcal{T}}_j^+$ denote signed rooted tree obtained by adding a vertex above each leaf $\tau \in \ell_j$ as defined above. We write

$$\Lambda^{j,-} = \varphi_j(L_{\widehat{\mathcal{T}}_j^-} \times \mathbb{R}^{m-n(\widehat{\mathcal{T}}^-)}), \qquad \Lambda^- = \cup_{j=0}^k \Lambda^{j,-},$$
$$\Lambda^{j,0} = \varphi_j(L_{\widehat{\mathcal{T}}_j} \times \mathbb{R}^{m-n(\widehat{\mathcal{T}})}), \qquad \Lambda^0 = \cup_{j=0}^k \Lambda^{j,0}$$
$$\Lambda^{j,+} = \varphi_j(L_{\widehat{\mathcal{T}}_j^+} \times \mathbb{R}^{m-n(\widehat{\mathcal{T}}^+)}), \qquad \Lambda^+ = \cup_{j=0}^k \Lambda^{j,+}.$$

Then, using Theorem B.26 we know there are families of diffeomorphisms $\psi^{-/0/+}$ between the germ of $H_{\widehat{\mathcal{T}}^{-/0/+}}$ and the germ of the front $\pi(\Lambda^{-/0/+})$.

Now, by Lemma B.29 we know

$$L_{(\widehat{\mathcal{T}_j},\ell_j)} = L_{\widehat{\mathcal{T}}_j^+} \setminus L_{\widehat{\mathcal{T}}_j^0} \times \mathbb{R}^{|\ell|} \cup L_{\widehat{\mathcal{T}}_j^-} \times \mathbb{R}^{|2\ell|}$$

and since φ_j is an embedding

$$\Lambda^j = \Lambda^{j,+} \setminus \Lambda^{j,0} \cup \Lambda^{j,-}$$

and since all φ_j have disjoint images

$$\Lambda = \Lambda^+ \setminus \Lambda^0 \cup \Lambda^-.$$

Meanwhile

$$H_{(\widehat{\mathcal{T}},\ell)} = H_{\widehat{\mathcal{T}}^+} \setminus H_{\widehat{\mathcal{T}}^0} \times \mathbb{R}^{|\ell|} \cup H_{\widehat{\mathcal{T}}^-} \times \mathbb{R}^{2|\ell|}.$$

Thus the diffeomorphisms $\psi^{-/0/+}$ between the germ of $H_{\widehat{\mathcal{T}}^{-/0/+}}$ and the germ of the front $\pi(\Lambda^{-/0/+})$ give a diffeomorphism between the germ of $H_{(\widehat{\mathcal{T}},\ell)}$ and the germ of the front $\pi(\Lambda)$. $\square$

Bibliography

[ADE13] M. Audin, M. Damian, and R. Erné. *Morse Theory and Floer Homology*. Universitext. Springer London, 2013. ISBN: 9781447154969.

[AEN22a] D. Alvarez-Gavela, Y. Eliashberg, and D. Nadler. *Arboreal models and their stability*. 2022. arXiv: 2101.04272 [math.SG].

[AEN22b] D. Alvarez-Gavela, Y. Eliashberg, and D. Nadler. *Positive arborealization of polarized Weinstein manifolds*. 2022. arXiv: 2011.08962 [math.SG].

[BH04] A. Banyaga and D. Hurtubise. "A proof of the Morse-Bott Lemma". In: *Expositiones Mathematicae* 22.4 (2004), pp. 365–373. ISSN: 0723-0869. DOI: https://doi.org/10.1016/S0723-0869(04)80014-8. URL: https://www.sciencedirect.com/science/article/pii/S0723086904800148.

[BH08] A. Banyaga and D. Hurtubise. *The Morse-Bott inequalities via dynamical systems*. 2008. arXiv: 0709.0959 [math.AT].

[Bot88] R. Bott. "Morse theory indomitable". en. In: *Publications Mathématiques de l'IHÉS* 68 (1988), pp. 99–114. URL: http://www.numdam.org/item/PMIHES_1988__68__99_0/.

[CE12] K. Cieliebak and Y. Eliashberg. *From Stein to Weinstein and Back: Symplectic Geometry of Affine Complex Manifolds*. Vol. 59. American Mathematical Society Colloquium publications. American Mathematical Society, 2012. ISBN: 9780821885338.

[Eli17] Y. Eliashberg. *Weinstein manifolds revisited*. 2017. arXiv: 1707.03442 [math.SG].

[GG74] M. Golubitsky and V. Guillemin. *Stable mappings and their singularities*. Vol. 14. Graduate Texts in Mathematics. Springer New York, 1974. ISBN: 9780387900735.

[Han08] G. Hansjörg. *An Introduction to Contact Topology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2008. DOI: 10.1017/CBO9780511611438.

[Lau92] F. Laudenbach. "Appendix: On the Thom-Smale complex". In: *Astérisque* 205 (1992), pp. 219–233.

[Mat02] Y. Matsumoto. *An Introduction to Morse Theory*. Europe and Central Asia Poverty Reduction and Economic Manag. American Mathematical Society, 2002. ISBN: 9780821810224.

[Mil56] J.W. Milnor. "On Manifolds Homeomorphic to the 7-Sphere". In: *Annals of Mathematics* 64 (1956), p. 399.

[Mil59] J.W. Milnor. "Differentiable Structures on Spheres". In: *American Journal of Mathematics* 81.4 (1959), pp. 962–972. ISSN: 00029327, 10806377.

[Mil63] J.W. Milnor. *Morse Theory*. Vol. 51. Annals of mathematics studies. Princeton University Press, 1963.

[Mil65] J.W. Milnor. *Lectures on the h-cobordism theorem*. Princeton university press, 1965.

[MS17] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*. Oxford University Press, Mar. 2017. ISBN: 9780198794899. DOI: 10.1093/oso/9780198794899.001.0001. URL: https://doi.org/10.1093/oso/9780198794899.001.0001.

[Nad16] D. Nadler. *Non-characteristic expansions of Legendrian singularities*. 2016. arXiv: 1507.01513 [math.SG].

[Nad17] D. Nadler. "Arboreal singularities". In: *Geometry & Topology* 21.2 (2017), pp. 1231–1274.

[Nic11] L. Nicolaescu. *An Invitation to Morse Theory*. Universitext. Springer New York, 2011. ISBN: 9781461411055.

[Rob94] C. Robinson. "Dynamical Systems: Stability, Symbolic Dynamics, and Chaos". In: 1994.

[Rol76] D. Rolfsen. *Knots and Links*. Mathematics lecture series. Publish or Perish, 1976. ISBN: 9780914098164.

[Sil01] A.C. da Silva. *Lectures on Symplectic Geometry*. Lecture Notes in Mathematics nr. 1764. Springer, 2001. ISBN: 9783540421955. DOI: 10.1007/978-3-540-45330-7. URL: https://doi.org/10.1007/978-3-540-45330-7.

[Sma61] S. Smale. "On Gradient Dynamical Systems". In: *Annals of Mathematics* 74 (1961), p. 199.

[Sta18] L. Starkston. *Arboreal singularities in Weinstein skeleta*. Sept. 2018. DOI: 10.1007/s00029-018-0441-z. URL: https://doi.org/10.1007%2Fs00029-018-0441-z.

[Tho49] R. Thom. "Sur une partition en cellules associée à une fonction sur une variété". In: *Comptes Rendus Hebdomadaires des Seances de l Academie des Sciences* 228.12 (1949), pp. 973–975.

[Zor18] A. Zorn. "A combinatorial model of lagrangian skeleta". PhD thesis. UC Berkeley, 2018.