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Phonon-induced spin inertia in ferromagnets

MASTER THESIS

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July 7, 2023

Abstract

Spintronics has been gaining attention due to its possible technical applications. A central topic in spintronics is magnetization dynamics, often described by the Landau-Lifshitz-Gilbert equation. This equation accurately describes the dynamics of many spin systems, and has been experimentally confirmed. However, there have been theoretical predictions that this equation is not complete, and should be expanded with inertial terms. Recently, this prediction has been confirmed by the observation of nutation on top of the precessional magnetization dynamics by various experimental studies, which suggest that more investigation into spin inertia is needed. In this thesis, we demonstrate that spin inertia is an effect due to the environment of the spins. Moreover, we consider an explicit example of a phonon bath as the environment. Our results demonstrate that there is indeed spin inertia, and we provide a calculation of the inertial constant in the case of a ferromagnetic thin film in contact with a bulk phonon bath. To experimentally validate our results we propose investigating an Yttrium Iron Garnet film in contact with a bulk Gadolinium Gallium Garnet substrate.

Acknowledgements

I would like to thank Tim Ludwig, Huaiyang Yuan and Rembert Duine for their supervision and guidance throughout my thesis. The meetings I had with them were not only very instructive, but also very enjoyable. They made me grow as a researcher and made me realise I want to continue in academics, for which I want to express my deepest gratitude.

Also, I would like to thank Lukasz Karwacki, who I have collaborated with during my thesis. His input was very useful, and I have learned a lot from him.

I am extremely lucky, to have made strong friendships during my master. I am grateful to Prajit, Monica, Riley, Giovanni and Audrique for their endless moral support, (too) long coffee breaks, amazing dinners, feedback, and Audrique asking the right questions and looking for minus signs. They have inspired me to work hard, encouraged me to be myself and made me grow as a person. There are no words that can express my gratitude to them.

I am also thankful to Eline, Dónal, Teun, Sjoerd, Robert, Alek, Georgeos, Salma, Artim, Chani, Julius and many others. Their input and support made my university time much more enjoyable.

Finally, I would like to mention my father Vimalathas, brother Irathush and sister Shaumiga. Without their love and support, this endeavor would have been impossible. Also to them, there are no words that can express my appreciation. Thank you!

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1. Introduction

Spintronics is focused on the manipulation of the spin degree of freedom of electrons in addition to their charge degree of freedom and it is becoming more relevant over time. The increased attention is largely due to the possible technical applications for spintronics, such as data storage and information-transferring devices. These applications are promising due to the possibility of increasing the memory and processing capabilities of electronic devices, while reducing the need of power and heat generation. Due to Joule heating, conventional electronic devices based on charge carriers stumble upon a problem, whereas spintronic devices do not face this. Spintronic devices transfer information using spin current carriers. The spin current is carried by quasiparticles called magnons also known as spin waves.

A spin wave is a propagating disturbance of the spins in a magnetic material. Hence, to exhibit spin waves, the material should have regions, often called domains, where the spins are aligned. This is the case for ferromagnetic materials such as Yttrium Iron Garnet (YIG). The collective magnetization dynamics of the collection of spins in such a domain, also known as the macrospin, are often described by the Landau-Lifshitz-Gilbert equation [1],

$$\dot{\mathbf{S}} = \mathbf{S} \times \mathbf{H} - \alpha_0 \mathbf{S} \times \dot{\mathbf{S}}. \tag{1.1}$$

After solving the LLG equation one could see that the term containing the external magnetic field $\mathbf{H}$ leads to precession of the spin as depicted in Fig. 1.1a. Physically, this corresponds to the rotation of the spin around the axis defined by the external magnetic field. The second term on the right hand side describes the damping of the spin toward the external field direction due to its interaction with the environment, with α_0 the Gilbert damping parameter (Fig. 1.1b). Gilbert damping describes the dissipation of the energy of the spin system into its environment, also known as the bath.

The LLG equation is a phenomenological equation, which accurately describes the magnetization dynamics of many spin systems, as confirmed by a number of experimental studies. However, there are numerous theoretical studies [2–4] suggesting an additional inertial term. This prediction was confirmed experimentally by Neeraj *et al.* [5] and Unikandanunni *et al.* [6]. They found evidence of inertial spin dynamics by observing nutation of the magnetization, which confirms the assertion to include inertial terms in the inertial LLG equation. The inertial LLG equation then reads

$$\dot{\mathbf{S}} = \mathbf{S} \times \mathbf{H} - \alpha_0 \mathbf{S} \times \dot{\mathbf{S}} - I\mathbf{S} \times \ddot{\mathbf{S}}, \tag{1.2}$$

where the third term is the inertial term, with I the spin inertia. As a consequence, the spin dynamics change to include nutation as depicted in Fig. 1.1c.

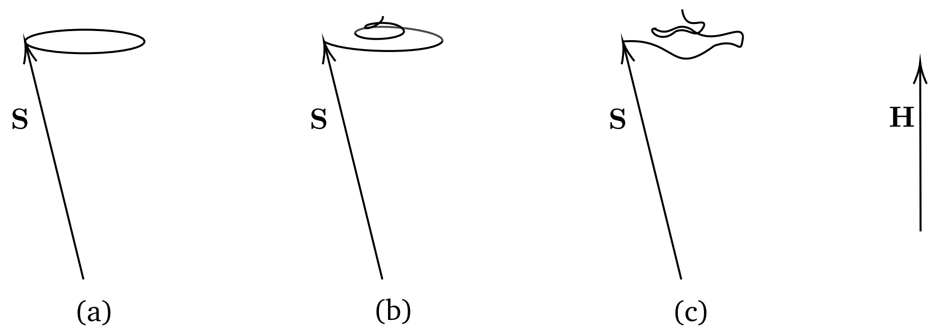

Figure 1.1: *(a) Depiction of a spin precessing around the direction of the external magnetic field* $\mathbf{H}$*. (b) Depiction of a spin precessing including Gilbert damping. (c) Depiction of a sping pressing including both Gilbert damping and spin inertia.*

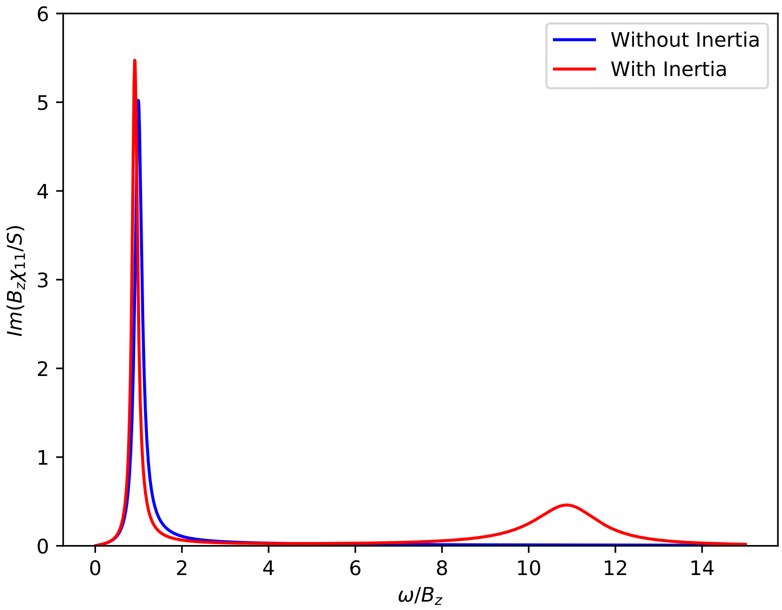

Figure 1.2: *The figure illustrates the dimensionless absorption behavior of the magnetic system as a function of the dimensionless frequency, showcasing two different calculations: one incorporating inertial terms in red and the other excluding them in blue. The absorbtion is calculated from the imaginary part of the susceptibility as defined in Eq.* (1.4).

Nutation results in another resonance spike in the absorption spectrum of a magnetic system in the high frequency domain, in addition to the first resonance peak also known as the ferromagnetic resonance. In experiments, the ferromagnetic resonance is determined by measuring the amplitude and phase of the response of the magnetization as a function of the frequency of the driving field. Neeraj *et al*. [5] observed an additional peak in a higher frequency regime. To understand the ferromagnetic resonance theoretically, we consider a driving field $\mathbf{H}(t)$ constant in the z-direction driving a spin predominantly aligned in the z-direction. Mathematically, the spin and driving field are then given by

$$\mathbf{S}(t) = \begin{pmatrix} S_x(t) \\ S_y(t) \\ S_z \end{pmatrix}, \qquad \mathbf{H}(t) = \begin{pmatrix} H_x(t) \\ H_y(t) \\ H_z \end{pmatrix}, \tag{1.3}$$

where we have linearized the spin in the z-direction. We calculate the absorbtion by rewriting the LLG equation, using Fourier transforms, to

$$\begin{pmatrix} S_x(t) \\ S_y(t) \end{pmatrix} = \begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix} \begin{pmatrix} H_x(t) \\ H_y(t) \end{pmatrix}, \tag{1.4}$$

with χ_{ij} the components of the susceptibility tensor. The χ_{11} component for the ordinary LLG Eq. (1.1), i.e. $I = 0$, is given by

$$\frac{H_z\chi_{11}}{S} = \frac{\left(-i\frac{\omega}{H_z}\right)\alpha_0 S + 1}{-\frac{\omega}{H_z} + \left(-i\left(\frac{\omega}{H_z}\right)\alpha_0 S + 1\right)^2}, \tag{1.5}$$

and for the inertial LLG Eq. (1.2) the χ_{11} component is given by

$$\frac{H_z\chi_{11}}{S} = \frac{\left(-i\frac{\omega}{H_z}\right)\alpha_0 S + 1 - ISH_z\left(\frac{\omega}{H_z}\right)^2}{-\frac{\omega}{H_z} + \left(-i\left(\frac{\omega}{H_z}\right)\alpha_0 S + 1 - ISH_z\left(\frac{\omega}{H_z}\right)^2\right)^2}, \tag{1.6}$$

and we have plotted both in Fig. 1.2. The second peak of the red graph suggests a regime where also nutation effects are relevant, the high-frequency regime. Therefore, further investigation of the underlying theory is necessary. Despite numerous theoretical studies on this topic, the source of the second derivative term in Eq. (1.2) is still unclear.

Only recently, a deeper theoretical understanding of the origin of spin inertia in the form of a second derivative term was provided in Gaspar Quarenta master thesis [7]. They theoretically discovered that spin inertia can arise from the high frequency modes of the environment using the Caldeirra-Leggett approach. In this approach, one assumes that the spin system is linearly coupled to a bath of harmonic oscillators. In turn, one can prove the existence of spin inertia by investigating the so-called bath spectral density function $J(\epsilon)$. This function yields the damping kernel $\alpha(\omega)$ in the generalized LLG equation,

$$\dot{\mathbf{S}}(t) = \mathbf{S}(t) \times \mathbf{H}(t) + \mathbf{S} \times \int \mathrm{d}t' \alpha(t-t')\mathbf{S}(t'). \tag{1.7}$$

While the Caldeirra-Leggett approach has given valuable insights into spin inertia, the physical interpretation of the bath modes in a ferromagnet assumed by Gaspar Quarenta [7] remains to be studied. Hence, in this thesis, we continue their work by providing proof of existence of spin inertia in a more realistic system: a spin system coupled to a phonon bath. Here we consider a realistic spin-phonon coupling in a ferromagnetic system and derive a generalized LLG equation microscopically by eliminating the phonon degree of freedom from the equations of motion of the coupled spin-phonon system. Then, we compare this equation with the one suggested by Gaspar Quarenta [7]. in (1.7) and derive the spectral density function of the phonon bath. Moreover, we provide an explicit calculation of the spin inertia in a specific configuration. This configuration contains a two-dimensional magnetic plate coupled to a three-dimensional phonon bath subject to a perpendicular external magnetic field. In this system, we recover Gilbert damping in the long wavelength limit. Additionally, we find spin inertia in the short wavelength limit and that it is linearly dependent on the size of the bulk phonon bath.

The rest of the thesis is structured as follows. We begin with chapter 2 demonstrating how to obtain the generalized LLG equation as in Verstraten *et al.* [8] and Ref. [7]. Additionaly, we show how to obtain Gilbert damping and spin inertia from the generalized LLG equation. Next, Chapter 3 reveals the main result of the thesis, where we use the approach by Rückriegel and Kopietz [9] to find spin inertia in a spin system coupled to a phonon bath. We conclude this thesis with a discussion, conclusion and outlook section.

2. Spin inertia from high frequency bath modes

In this chapter, we demonstrate how spin inertia arises from high frequency bath modes. To do so, we start with a Hamiltonian describing a spin coupled to a bath. From this Hamiltonian, we derive the generalized Landau-Lifshitz-Gilbert equation using the Keldysh formalism in its path integral form. The Keldysh formalism allows us to derive an action and, in turn, the equations of motion, thereby yielding the generalized LLG equation. Once we have the generalized LLG equation, we will have a discussion on a function which contains all the information related to the spin coupling to its bath; the bath spectral density function. As we will show, the bath spectral density is key in retrieving Gilbert damping and spin inertia [7]. The notion that Gilbert damping is a phenomenon due to the environment is unsurprising, as energy dissipates into the bath. Remarkably, also spin inertia is a phenomenon due to the environment.

2.1 Generalized Landau-Lifshitz-Gilbert equation

We deduce both the Gilbert damping and spin inertia from the generalized LLG equation. Hence, in this section, we rederive the calculations from Ref. [8] obtaining the generalized LLG equation. We obtain the generalized LLG equation by calculating the equations of motion from the action. This action is derived from the Keldysh partition function starting from the Hamiltonian. We consider a system of a macrospin coupled to multiple harmonic oscillators as depicted in Fig. 2.1. In the Caldeira Leggett approach [10–13], one considers a system linearly coupled with a bath of harmonic oscillators. The dynamics are then described by the Hamiltonian,

$$\hat{H} = \hat{H}_{\mathrm{s}} + \hat{H}_{\mathrm{c}} + \hat{H}_{\mathrm{b}}, \tag{2.1}$$

with

$$\hat{H}_{\mathrm{s}} = -\mathbf{H}\cdot\hat{\mathbf{S}}; \tag{2.2}$$

$$\hat{H}_{\mathrm{c}} = \sum_{\alpha} \gamma_\alpha \hat{\mathbf{S}}\cdot\hat{\mathbf{x}}_\alpha; \tag{2.3}$$

$$\hat{H}_{\mathrm{b}} = \sum_{\alpha} \left(\frac{\hat{\mathbf{p}}_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2}\hat{\mathbf{x}}_\alpha^2 \right). \tag{2.4}$$

Here, $\hat{H}_{\mathrm{s}}$ contains the Zeeman term with macrospin $\hat{\mathbf{S}}$ and external magnetic field $\mathbf{H}$. The second term $\hat{H}_{\mathrm{c}}$ describes the linear coupling between the macrospin and the harmonic oscillators of the bath with coupling constant γ_α, the index of the bath oscillators α and the position operator of the harmonic oscillator $\hat{\mathbf{x}}_\alpha$. Finally, $\hat{H}_{\mathrm{b}}$ describes the harmonic oscillators with momentum operator $\hat{\mathbf{p}}_\alpha$, mass m_α and eigenfrequency of the oscillator ω_α.

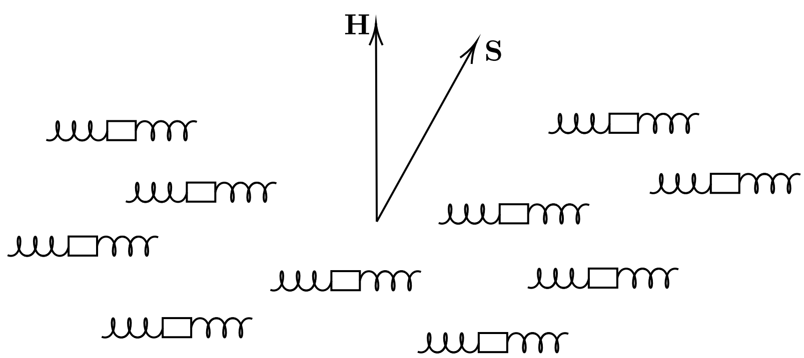

Figure 2.1: *Schematic depiction of the system we consider. Here* $\mathbf{H}$ *is the external magnetic field and* $\mathbf{S}$ *is the macrospin*

We find the action by reading it off from the Keldysh partition function. Before writing down the Keldysh partition function, we discuss some background on the Keldysh formalism as explained in [15]. In the Keldysh formalism, to describe the quantum state of the system, one starts from the equilibrium density matrix in the distant past $\hat{\rho}(-\infty)$. The current density matrix $\hat{\rho}(t)$ is then described by the equilibirum density matrix in the distant past evolved with the time evolution operator,

$$\hat{\rho}(t) = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\hat{\mathcal{U}}_{-\infty,t}, \tag{2.5}$$

where the evolution operator is defined by

$$\hat{\mathcal{U}}_{t,t'} = \mathbb{T}\exp\bigg(-i\int_{t'}^{t}\mathrm{d}t\hat{H}(t)\bigg). \tag{2.6}$$

To calculate the expectation value of an observable $\hat{\mathcal{O}}$ at a time t we use

$$\begin{aligned}\langle\hat{\mathcal{O}}\rangle(t) &\equiv \frac{\mathrm{Tr}\{\hat{\mathcal{O}}\hat{\rho}(t)\}}{\mathrm{Tr}\{\hat{\rho}(t)\}} \\ &= \frac{1}{\mathrm{Tr}\{\hat{\rho}(t)\}}\mathrm{Tr}\{\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\} \\ &= \frac{1}{\mathrm{Tr}\{\hat{\rho}(-\infty)\}}\mathrm{Tr}\{\hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\},\end{aligned} \tag{2.7}$$

where we have used the cyclicity of the trace and used the trace of the density matrix at a distant past for the denominator [15]. The Keldysh partition function is defined by

$$Z = \frac{\mathrm{Tr}\{\hat{U}_{\mathcal{C}}\hat{\rho}(-\infty)\}}{\mathrm{Tr}\{\hat{\rho}(-\infty)\}}, \tag{2.8}$$

where the evolution operator $U_{\mathcal{C}} = \hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,-\infty}$ goes over the Keldysh contour $\mathcal{C}$ as depicted in Fig. 2.2. Basically, we have defined the Keldysh partition function to be the expectation value of the identity operator as could be seen from (2.7).

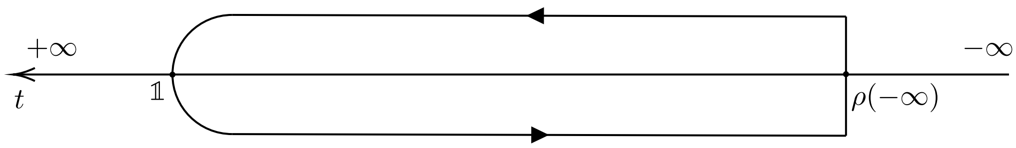

Figure 2.2: *Figure containing the Keldysh closed time contour $\mathcal{C}$, adapted from [15, 16]. The Keldysh contour describes the evolution of the density matrix from a distant past, to time $t = \infty$ and back to the distant past.*

To calculate the Keldysh partition function, we use the following expression of the trace

$$\mathrm{Tr}\{\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}\} = \int \mathrm{d}g\prod_{\alpha}\int \mathrm{d}x_{\alpha}\int \mathrm{d}p_{\alpha}\, e^{ip_{\alpha}x_{\alpha}}\langle p_{\alpha}, g|\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}|x_{\alpha}, g\rangle. \tag{2.9}$$

where we have introduced the spin coherent states $|g\rangle$ [16], and the position x_{α} and momentum p_{α} fields. Following the standard path-integral construction, we split the time-evolution operator into many steps and insert a complete set of eigen states, to obtain the following expression of the Keldysh partition function in path integral form,

$$\begin{aligned}\mathcal{Z} = \int Dg\prod_{\alpha}\int Dx_{\alpha}\int Dp_{\alpha}\exp\bigg[i\oint_{K}\mathrm{d}t\Big(&-i\langle\dot{g}|g\rangle + \langle g|\mathbf{H}\cdot\mathbf{S}|g\rangle - \gamma_{\alpha}\langle g|\mathbf{S}\cdot\dot{\mathbf{x}}|g\rangle \\ &+ p_{\alpha}\dot{x}_{\alpha} - \frac{p_{\alpha}^{2}}{2m} - \frac{m\omega^{2}}{2}x_{\alpha}^{2}\Big)\bigg],\end{aligned} \tag{2.10}$$

referring to appendix A and appendix B for more detailed calculations. The K indexed time integral in the exponent integrates over the Keldysh closed time contour as shown in Fig. 2.2. We can rewrite the Keldysh integral as a regular time integral over the forward and backward contour. Then, the Keldysh partition function can be written as

$$\begin{aligned}
\mathcal{Z} = \int Dg \prod_\alpha \int Dx_\alpha \int Dp_\alpha \exp\Bigg[i \int \mathrm{d}t \Big(& -i\langle \dot{g}^+ | g^+\rangle + \langle g^+ | \mathbf{H}\cdot\hat{\mathbf{S}}^+ | g^+\rangle - \gamma_\alpha \langle g^+ | \mathbf{S}^+ \cdot \dot{\mathbf{x}}^+ | g^+\rangle \\
& + p_\alpha^+ \dot{x}_\alpha^+ - \frac{(p_\alpha^+)^2}{2m} - \frac{m\omega^2}{2}(x_\alpha^+)^2 \\
& + i\langle \dot{g}^- | g^-\rangle - \langle g^- | \mathbf{H}\cdot\hat{\mathbf{S}}^- | g^-\rangle + \gamma_\alpha \langle g^- | \mathbf{S}^- \cdot \dot{\mathbf{x}}^- | g^-\rangle \\
& - p_\alpha^- \dot{x}_\alpha^- + \frac{(p_\alpha^-)^2}{2m} + \frac{m\omega^2}{2}(x_\alpha^-)^2 \Big) \Bigg] \\
= \int Dg \prod_\alpha \int Dx_\alpha \int Dp_\alpha \exp\Bigg[i \int \mathrm{d}t \Bigg(& [-i\langle \dot{g}|g\rangle]^q + [\langle g | \mathbf{H}\cdot\hat{\mathbf{S}} | g\rangle]^q - \Big[\gamma_\alpha \langle g | \hat{\mathbf{S}}\cdot\dot{\mathbf{x}} | g\rangle\Big]^q \\
& + \left[p_\alpha \dot{x}_\alpha - \frac{p_\alpha^2}{2m} - \frac{m\omega^2}{2} x_\alpha^2 \right]^q \Bigg) \Bigg],
\end{aligned} \tag{2.11}$$

where the upper index $+$ and $-$ denote the forward and backward contour respectively. The backward contour goes in the opposite direction as the forward contour, leading to an additional minus sign because of the integration direction. Also, we have introduced notation such that, for example,

$$[-i\langle \dot{g}|g\rangle]^q = -i\langle \dot{g}^+|g^+\rangle + i\langle \dot{g}^-|g^-\rangle. \tag{2.12}$$

Splitting the Keldysh integral into a forward and backward contour is somewhat misleading. It gives the impression that the fields in the forward and backward contour are uncorrelated. However, they must be correlated, intuitively depicted by the points where the forward and backward contour meet in Fig. 2.2. Before solving this problem, we will integrate out the position and momentum fields of the Keldysh partition function to find

$$\mathcal{Z} = \int Dg \exp\left[i \int \mathrm{d}t \Big([-i\langle \dot{g}|g\rangle]^q + [\langle g|\mathbf{H}\cdot\hat{\mathbf{S}}|g\rangle]^q - \frac{1}{2}\mathbf{S}^T(t) \int \mathrm{d}t'\, \alpha(t-t')\mathbf{S}(t') \Big) \right], \tag{2.13}$$

where the tensor $\alpha(t-t')$ is the kernel function containing information about the bath and the vector $\mathbf{S}(t) \equiv \langle g|\hat{\mathbf{S}}(t)|g\rangle$ equals

$$\mathbf{S}(t) = \begin{pmatrix} \mathbf{S}^+(t) \\ \mathbf{S}^-(t) \end{pmatrix}. \tag{2.14}$$

Now we apply a coordinate transformation defined by the Keldysh rotation matrix,

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.15}$$

on (2.14) to obtain

$$\mathbf{S}(t) = \sqrt{2} \begin{pmatrix} \mathbf{S}^c \\ \frac{\mathbf{S}^q}{2} \end{pmatrix}. \tag{2.16}$$

We have defined

$$\mathbf{S}^+ = \mathbf{S}^c + \frac{1}{2}\mathbf{S}^q \tag{2.17}$$

$$\mathbf{S}^- = \mathbf{S}^c - \frac{1}{2}\mathbf{S}^q. \tag{2.18}$$

where $\mathbf{S}^c$ and $\mathbf{S}^q$ represent the classical and quantum components of the spin respectively. In the basis of $\mathbf{S}^c$ and $\mathbf{S}^q$, the kernel function $\alpha(t-t')$ is given by

$$\begin{aligned}\alpha(t-t') &= \begin{pmatrix} 0 & \alpha^A \\ \alpha^R & \alpha^K \end{pmatrix}_{(t-t')} \\ &= \sum_\alpha \frac{\gamma_\alpha^2}{4} \begin{pmatrix} 0 & G_\alpha^A \\ G_\alpha^R & G_\alpha^K \end{pmatrix}_{(t-t')},\end{aligned} \tag{2.19}$$

dependent on the advanced G_α^A, retarded G_α^R and Keldysh G_α^K Green's functions of the bath modes (see appendix C for expressions). The entry of the matrix containing the Keldysh Green's function takes into account the correlations between the forward and backward part of the contour.

We carry out the matrix product in Eq. (2.13) and rewrite the advanced and retarded part of the kernel function in the action such that,

$$\begin{aligned}\mathcal{S} = &\int \mathrm{d}t \ \ -i\langle \dot{g}|g\rangle]^q + [\langle g|\mathbf{H}\cdot\hat{\mathbf{S}}|g\rangle]^q \\ &+ \int \mathrm{d}t \left[\mathbf{S}^q(t)\int \mathrm{d}t\,'\ \alpha_{\text{diss}}(t-t')\mathbf{S}^c(t') + \mathbf{S}^q(t)\alpha^K(t-t')\mathbf{S}^q(t')\right],\end{aligned} \tag{2.20}$$

where we have essentially factored out the $\mathbf{S}^q$ and defined.

$$\alpha_{\text{diss}} = \alpha^A(t'-t) + \alpha^R(t-t'). \tag{2.21}$$

To obtain the equations of motion, we vary the action with respect to the quantum components. Therefore, the quadratic term in quantum components will not matter, because it will vanish in the equation of motions[1]. We then are allowed to consider the following

$$\mathcal{S} = \int \mathrm{d}t \left([-i\langle \dot{g}|g\rangle]^q + [\langle g|\mathbf{H}\cdot\hat{\mathbf{S}}|g\rangle]^q + \hat{\mathbf{S}}^q(t)\int \mathrm{d}t'\ \alpha_{\text{diss}}(t-t')\hat{\mathbf{S}}^c(t')\right). \tag{2.22}$$

To proceed, we have to understand the relation between $|g\rangle$ and $\mathbf{S} = \langle g|\hat{\mathbf{S}}|g\rangle$. The spin coherent states $|g\rangle$ can be represented by the Euler angles [16] in the following way,

$$\begin{aligned}|g\rangle &= g|\uparrow\rangle \\ &= e^{-i\phi S_z}e^{-i\theta S_y}e^{-i\psi S_z}|\uparrow\rangle,\end{aligned} \tag{2.23}$$

where $|\uparrow\rangle$ is an eigenstate of $\hat{\mathbf{S}}$ with maximum eigenvalue S. As a result, we write

$$|g\rangle = e^{-i\phi S_z}e^{-i\theta S_y}|\uparrow\rangle e^{-i\psi S}, \tag{2.24}$$

from which we see that ψ just acts as a phase factor. The remaining two angles ϕ and θ are true rotations depicted in figure 2.3.

[1] If one wants to include noise effects, one needs to be more careful [8, 17]

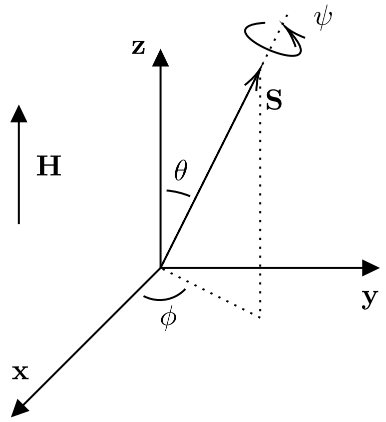

Figure 2.3: *The two remaining Euler angles ϕ and θ are true rotation angles. The third Euler angle ψ is just a phase factor. Figure adapted from [16].*

Then, for $\mathbf{S} = \langle g|\hat{\mathbf{S}}|g\rangle$ it holds that

$$\mathbf{S} = S \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}. \tag{2.25}$$

As a result, we rewrite the action (see appendix D) such that we can read off the equations of motion for ϕ_c and θ_c,

$$\mathcal{S}_{\theta_q} = \int_{-\infty}^{\infty} \mathrm{d}t\theta_q \Big[-\dot{\phi}_c \sin\theta_c + H'_x \cos\theta_c \cos\phi_c + H'_y \cos\theta_c \sin\phi_c - H'_z \sin\theta_c \Big], \tag{2.26}$$

and

$$\mathcal{S}_{\phi_q} = \int_{-\infty}^{\infty} \mathrm{d}t\phi_q \Big[\dot{\theta}_c \sin\theta_c - H'_x \sin\theta_c \sin\phi_c + H'_y \sin\theta_c \cos\phi_c \Big], \tag{2.27}$$

where we have defined an effective magnetic field $\mathbf{H}'$,

$$\begin{pmatrix} H'_x \\ H'_y \\ H'_z \end{pmatrix} = \begin{pmatrix} H_x + \int_{-\infty}^{\infty} \mathrm{d}t'\ \alpha_{\mathrm{diss}}(t-t') S^c_x(t') \\ H_y + \int_{-\infty}^{\infty} \mathrm{d}t'\ \alpha_{\mathrm{diss}}(t-t') S^c_y(t') \\ H_z + \int_{-\infty}^{\infty} \mathrm{d}t'\ \alpha_{\mathrm{diss}}(t-t') S^c_z(t') \end{pmatrix}. \tag{2.28}$$

From this we get the following equations of motion

$$\dot{\phi}_c \sin\theta_c = H'_x \cos\theta_c \cos\phi_c + H'_y \cos\theta_c \sin\phi_c - H'_z \sin\theta_c; \tag{2.29}$$

$$\dot{\theta}_c = H'_x \sin\phi_c - H'_y \cos\phi_c. \tag{2.30}$$

Filling these equations of motion into Eq. (2.25), we get for $\dot{\mathbf{S}}$,

$$\begin{aligned} \dot{\mathbf{S}} &= \mathbf{S} \times \mathbf{H}' \\ &= \mathbf{S} \times \mathbf{H} + \mathbf{S} \times \int_{-\infty}^{\infty} \mathrm{d}t'\ \alpha_{\mathrm{diss}}(t-t')\mathbf{S}(t'), \end{aligned} \tag{2.31}$$

which is the generalized Landau-Lifshitz-Gilbert equation, describing the spin dynamics of the system. The kernel function α_{diss} contains information about the bath, which we will use to retrieve Gilbert damping and spin inertia in the following two sections.

2.2 Origin of Gilbert damping

In this section, we derive the ordinary LLG Eq. (1.1) from the generalised Landau-Lifshitz-Gilbert equation,

$$\dot{\mathbf{S}}(t) = \mathbf{S}(t) \times \mathbf{H} + \mathbf{S}(t) \times \int \mathrm{d}t' \; \alpha_{\mathrm{diss}}(t-t')\mathbf{S}(t'), \tag{2.32}$$

for a macrospin $\mathbf{S}$ subject to an external magnetic field $\mathbf{H}$ and coupled to some bath. By defining the bath spectral density function $J(\epsilon)$ as

$$J(\epsilon) = \pi \sum_{\alpha} \frac{\gamma_\alpha^2}{2m_\alpha \omega_\alpha} \delta(\epsilon - \omega_\alpha), \tag{2.33}$$

we rewrite the kernel function such that,

$$\alpha_{\mathrm{diss}}(\omega) = -\frac{2}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\epsilon \frac{\epsilon J(\epsilon)}{(\omega + i0)^2 - \epsilon^2}. \tag{2.34}$$

The bath spectral density function $J(\epsilon)$ contains all the information related to the coupling of the spin system to its bath and it is closely related to the density of stats of the phonons. In general, this function can take any form. However, we assume the bath spectral density function to be linear for now; the bath is assumed to be Ohmic. This approximation is often realistic for lower frequencies and will lead us to the ordinary LLG equation. To see this, we apply contour integration and find that the kernel function is linear in ω,

$$\begin{aligned}\alpha_{\mathrm{diss}}(\omega) &= -\frac{2}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\epsilon \frac{\alpha_0 \epsilon^2}{(\omega + i0)^2 - \epsilon^2} \\ &= i\alpha_0 \omega. \end{aligned} \tag{2.35}$$

The kernel function in the generalized LLG Eq. (2.32) is in time space. Hence, we Fourier transform Eq. (2.35), and obtain

$$\alpha_{\mathrm{diss}}(t-t') = \int \frac{\mathrm{d}\omega}{2\pi} i\alpha_0 \omega e^{-i\omega(t-t')} \tag{2.36}$$

$$= \alpha_0 \partial_{t'} \delta(t-t'), \tag{2.37}$$

to plug it in Eq. (2.32), which returns the ordinary LLG equation,

$$\dot{\mathbf{S}} = \mathbf{S} \times \mathbf{H} - \alpha_0 \mathbf{S} \times \dot{\mathbf{S}}. \tag{2.38}$$

To arrive at the inertial LLG Eq. (1.2), we will assume that the bath spectral density function is linear for low frequencies, but will deviate from the linear approximation for higher frequencies. This is justified, because the bath spectral density function should go to zero for infinite frequency; the bath should not be able to absorb an infinite amount of energy.

2.3 Origin of spin inertia

In the previous section, we considered the case that the bath spectral density function is linear. In this section we relax this assumption and demonstrate that spin inertia arises from the non-linear contributions of $J(\epsilon)$. We start our analysis with substracting the zero frequency contributions from the kernel function

$$\begin{aligned}\tilde{\alpha}(\omega) &= \alpha_{\mathrm{diss}}(\omega) - \alpha_{\mathrm{diss}}(0) \\ &= -\frac{2}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\epsilon \frac{\epsilon J(\epsilon)}{(\omega + i0)^2 - \epsilon^2} - \frac{2}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\epsilon \frac{J(\epsilon)}{\epsilon} \\ &= -\frac{2}{\pi} \int_{-\infty}^{\infty} \mathrm{d}\epsilon \frac{\omega^2}{\epsilon} \frac{J(\epsilon)}{[(\omega + i0)^2 - \epsilon^2]}. \end{aligned} \tag{2.39}$$

We are allowed to use this kernel function, because the additional term does not contribute to the generalised LLG Eq. (2.32); it has no effect on the physics. This can be seen by recalling that the Fourier transform of a constant function in ω equals a delta function $\delta(t-t')$, which in turn gives us $\hat{\mathbf{S}}(t) \times \hat{\mathbf{S}}(t) = \mathbf{0}$. An important assumption that we make now, which will be justified in Sec. 3.3, is that the bath spectral density function is linear for just low frequencies and not for high frequencies. An example is given in Fig. 2.4.

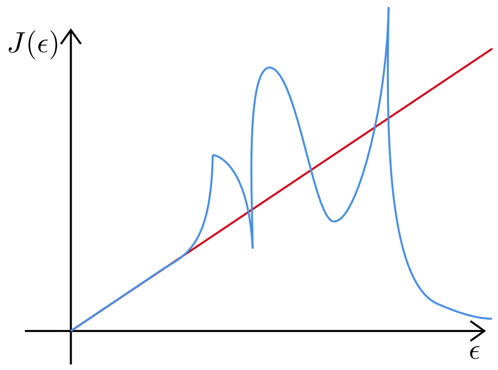

Figure 2.4: *Depicted are the Ohmic bath spectral density in red and the general bath spectral density in blue. The general bath spectral density function has the only requirement, that it is approximately linear for low frequencies. If the bath spectral density function is non-Ohmic for higher frequencies, spin inertia would arise and roughly corresponds to the area between the red and blue graphs apart from the factor of* $\frac{1}{\epsilon^3}$.

Due to this linear low frequency behaviour, we introduce a linear function $J_{\text{lf}}(\omega)$ and split the kernel function into a low and high frequency part,

$$\begin{aligned}\tilde{\alpha}(\omega) &= -\frac{2}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\epsilon \frac{\omega^2\left[J_{\text{lf}}(\epsilon) + J(\epsilon) - J_{\text{lf}}(\epsilon)\right]}{\epsilon\left[(\omega+i0)^2-\epsilon^2\right]} \\ &= \underbrace{-\frac{2}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\epsilon \frac{\omega^2 J_{\text{lf}}(\epsilon)}{\epsilon\left[(\omega+i0)^2-\epsilon^2\right]}}_{\alpha_{\text{lf}}} \underbrace{-\frac{2}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\epsilon \frac{\omega^2\left[J(\epsilon) - J_{\text{lf}}(\epsilon)\right]}{\epsilon\left[(\omega+i0)^2-\epsilon^2\right]}}_{\alpha_{\text{hf}}}. \end{aligned} \tag{2.40}$$

The low frequency term, α_{lf}, gives us Gilbert damping back, as J_{lf} is Ohmic. For the high frequency term, α_{hf}, we define ϵ_c such that $J(\epsilon) \approx J_{\text{lf}}(\epsilon)$ for $\epsilon < \epsilon_c$. For $\epsilon > \epsilon_c$, we assume that $\omega \ll \epsilon_c$. Exploiting these approximations in their respective regimes gives us

$$\begin{aligned}\alpha_{\text{hf}}(\omega) &= -\frac{2}{\pi}\int_{0}^{\epsilon_c} \mathrm{d}\epsilon \frac{\omega^2\left[J(\epsilon) - J_{\text{lf}}(\epsilon)\right]}{\epsilon\left[(\omega\pm i0)^2-\epsilon^2\right]} - \frac{2}{\pi}\int_{\epsilon_c}^{\infty} \mathrm{d}\epsilon \frac{\omega^2\left[J(\epsilon) - J_{\text{lf}}(\epsilon)\right]}{\epsilon\left[(\omega\pm i0)^2-\epsilon^2\right]} \\ &\approx 0 + \frac{2\omega^2}{\pi}\int_{\epsilon_c}^{\infty} \mathrm{d}\epsilon \frac{J(\epsilon) - J_{\text{lf}}(\epsilon)}{\epsilon^3} \\ &= I\omega^2, \end{aligned} \tag{2.41}$$

where

$$I \equiv \frac{2}{\pi}\int_{0}^{\infty} \mathrm{d}\epsilon \frac{J(\epsilon) - J_{\text{lf}}(\epsilon)}{\epsilon^3}, \tag{2.42}$$

is the spin inertia constant, independent of ω. Returning to time space, the Fourier transform of the high frequeny part of the kernel function in (2.40) yields

$$\begin{aligned}\alpha_{\text{hf}}(t-t') &= \int \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega(t-t')} I\omega^2 \\ &= -I\partial_{t'}^2 \delta(t-t'), \end{aligned} \tag{2.43}$$

and by filling this in Eq. (2.32), we obtain

$$\dot{\mathbf{S}} = \mathbf{S} \times \mathbf{H} - \alpha_0 \mathbf{S} \times \dot{\mathbf{S}} - I\mathbf{S} \times \ddot{\mathbf{S}}. \tag{2.44}$$

So, unless the bath is Ohmic up to arbitrarily high frequencies, we find spin inertia I given by Eq. (2.42). This constant roughly corresponds to the area between the low and high frequency part of the bath spectral density function, apart from the factor of $\frac{1}{\epsilon^3}$. In the next chapter, we explicitly consider a phonon bath and provide an expression for the spin inertia.

3. Phonon bath

In this chapter, we show our main results; spin inertia in ferromagnets in contact with a phonon bath. For this, we use the results by Rückriegel and Kopietz [9] who did not consider spin inertia, but wrote down the appropriate coupling between spins and phonons that we use as a starting point. Rückriegel and Kopietz found, among other things, that the properties of dissipation are determined by the phonon dynamics captured by the generalized LLG equation. We adapt their generalized LLG equation to align with our own generalized LLG Eq. (2.31), enabling us to obtain information about the coupling between the ferromagnet and the phonon bath. This information is contained in the bath spectral density function $J(\epsilon)$, which we need to identify from the generalized LLG equation. From $J(\epsilon)$ we are able to show spin inertia explicitly in example configurations containing a ferromagnetic thin film in contact with a bulk phonon bath subject to an external magnetic field.

3.1 Rederivation of the generalized Landau-Lifshitz-Gilbert equation

In this section, we rederive the results by Rückriegel and Kopietz [9], with the generalized LLG equation as the destination. To arrive at the generalized LLG equation, we will calculate the spin equations of motion from the Hamiltonian. This Hamiltonian models the spin dynamics in YIG using a spin lattice coupled to a phonon bath subject to an external magnetic field, which can be described by,

$$H(t) = H_{\mathrm{s}}(t) + H_{\mathrm{c}} + H_{\mathrm{p}}. \tag{3.1}$$

The magnetic part $H_{\mathrm{s}}(t)$ is described by an exchange interaction and Zeeman term,

$$H_{\mathrm{s}}(t) = -\frac{1}{2}\sum_{ij}\sum_{\alpha\beta}\delta^{\alpha\beta}J_{ij}\hat{S}_i^{\alpha}\hat{S}_j^{\beta} - \sum_i \mathbf{H}(t)\cdot\hat{\mathbf{S}}_i, \tag{3.2}$$

where α, β are indices for the x-,y- or z-direction, J_{ij} is the ferromagnetic exchange coupling, $\mathbf{H}(t)$ is the time-dependent external magnetic field and $\hat{\mathbf{S}}_i$ is the operator for the spin localized at site $\mathbf{R}_i$ of a cubic lattice. Note that, unlike Rückriegel and Kopietz, we disregard dipolar interactions, because we want to focus on the effects of the environment, and eventually develop a theory for a macrospin. The elastic (phonon) part is described by

$$H_{\mathrm{p}}(t) = \frac{1}{N}\sum_{\mathbf{k}\lambda}\left[\frac{P_{-\mathbf{k}\lambda}P_{\mathbf{k}\lambda}}{2M} + \frac{M}{2}\omega_{\mathbf{k}\lambda}^2 X_{-\mathbf{k}\lambda}X_{\mathbf{k}\lambda}\right], \tag{3.3}$$

where N is the number of sites, M is the effective mass of a unit cell, $\omega_{\mathbf{k}\lambda}$ is the dispersion of the acoustic phonons with wavevector $\mathbf{k}$ and polarization λ, and $P_{\mathbf{k}\lambda}$ and $X_{\mathbf{k}\lambda}$ are the canonical momentum and position operators associated with the bath modes. Finally, the magnetoelastic coupling term is desribed by

$$H_{\mathrm{c}} = \frac{1}{S^2}\sum_i\sum_{\alpha\beta}B_{\alpha\beta}\hat{S}_i^{\alpha}\hat{S}_i^{\beta}X_i^{\alpha\beta}, \tag{3.4}$$

where $B_{\alpha\beta}$ is the e_α, e_β-directional magnetoelastic coupling constant and $X_i^{\alpha\beta}$ is the e_α-directional strain tensor of the phonon displacements at site $\mathbf{R}_i$,

$$X_i^{\alpha\beta} = \frac{1}{2}\left[\frac{\partial X_\alpha(\mathbf{r})}{\partial r_\beta} + \frac{\partial X_\beta(\mathbf{r})}{\partial r_\alpha}\right]_{\mathbf{r}=\mathbf{R}_i}. \tag{3.5}$$

The magnetoelastic coupling describes the coupling between the spin and the lattice degrees of freedom. The coupling leads to interactions between magnons and phonons and it can be described by a scattering process (see appendix E).

Now that we have the Hamiltonian of the system, we can calculate the equations of motion by using the Heisenberg equations of motion. The Heisenberg equation for the spin operator $\hat{\mathbf{S}}_i$ reads

$$\frac{d}{\mathrm{d}t}\hat{S}_i^\gamma(t) = \frac{i}{\hbar}\left[H(t), \hat{S}_i^\gamma(t)\right], \tag{3.6}$$

for each component. Using the commutation relations for the spin operators, $\left[\hat{S}^j, \hat{S}^k\right] = i\hbar\epsilon_{jkl}\hat{S}^l$, we obtain the following for the commutator on the right hand side,

$$\left[H(t), \hat{S}_i^\gamma\right] = -i\hbar\left(\hat{\mathbf{S}}_i(t) \times \mathbf{H}(t)\right)^\gamma - i\hbar\left(\hat{\mathbf{S}}_i(t) \times \sum_j J_{ij}\hat{\mathbf{S}}_i(t)\right)^\gamma \tag{3.7}$$

$$+\frac{i\hbar}{S^2}\sum_{\alpha\beta} B_{\alpha\beta}X^{\alpha\beta}\left(\hat{S}_i^\alpha(t)\sum_k \epsilon_{\beta\gamma k}\hat{S}_i^k(t) + \sum_j \epsilon_{\alpha\gamma j}\hat{S}_i^j(t)\hat{S}_i^\beta(t)\right). \tag{3.8}$$

The third term can be rewritten using

$$-\frac{2}{S^2}\sum_{\alpha\beta} B_{\alpha\beta}X_i^{\alpha\beta}\sum_j \epsilon_{\alpha\gamma j}S_i^j S_i^\beta = \left(\hat{\mathbf{S}}_i \times \hat{\mathbf{F}}_i\right)_\gamma; \tag{3.9}$$

$$-\frac{2}{S^2}\sum_{\alpha\beta} B_{\alpha\beta}X_i^{\alpha\beta}\sum_k \epsilon_{\beta\gamma k}\hat{S}_i^\alpha\hat{S}_i^k = \left(\hat{\mathbf{F}}_i \times \hat{\mathbf{S}}_i\right)_\gamma, \tag{3.10}$$

where we have defined the magnetoelastic field,

$$\hat{\mathbf{F}}_i(t) \equiv -\frac{2}{S^2}\sum_{\alpha\beta} B_{\alpha\beta}\mathbf{e}_\alpha X_i^{\alpha\beta}(t)\hat{S}^\beta, \tag{3.11}$$

which describes the impact of the magnetoelastic coupling on the spin-dynamics. Plugging these expressions into the Heisenberg Eq. (3.6) for the spin yields

$$\dot{\hat{\mathbf{S}}}_i(t) = \hat{\mathbf{S}}_i(t) \times \left[\mathbf{H}(t) + \sum_j J_{ij}\mathbf{S}_j\right] + \frac{1}{2}\left[\hat{\mathbf{S}}_i(t) \times \hat{\mathbf{F}}_i(t) - \hat{\mathbf{F}}_i(t) \times \hat{\mathbf{S}}_i(t)\right]. \tag{3.12}$$

Similarly, we calculate the equations of motion for the canonical momentum and position operator of the phonons. Next, we plug the solution into these equations of motion into the spin equations of motion (3.12) to arrive at the generalized LLG equation of motion. The Heisenberg equations for the canonical momentum and position operator respectively read

$$\dot{X}_{\mathbf{k}\lambda} = i\left[H(t), X_{\mathbf{k}\lambda}\right] \tag{3.13}$$

$$\dot{P}_{\mathbf{k}\lambda} = i\left[H(t), P_{\mathbf{k}\lambda}\right]. \tag{3.14}$$

By using the canonical commution relations $[X_{\mathbf{k}\lambda}, P_{\mathbf{k}'\lambda'}] = iN\delta_{\mathbf{k},-\mathbf{k}}\delta_{\lambda\lambda'}$, $[X_{\mathbf{k}\lambda}, X_{\mathbf{k}'\lambda'}] = 0$ and $[P_{\mathbf{k}\lambda}, P_{\mathbf{k}'\lambda'}] = 0$, we obtain the following for the right hand side of Eq. (3.13),

$$i\left[H(t), X_{\mathbf{k}\lambda}\right] = \frac{P_{\mathbf{k}\lambda}}{M}, \tag{3.15}$$

from which the equation of motion for the canonical position follows,

$$\dot{X}_{\mathbf{k}\lambda} = \frac{P_{\mathbf{k}\lambda}}{M}. \tag{3.16}$$

For the right hand side of (3.14) we get

$$i\left[H(t), P_{\mathbf{k}\lambda}\right] = -M\omega_{\mathbf{k}\lambda}^2 X_{\mathbf{k}\lambda} + MA_{\mathbf{k}\lambda}, \tag{3.17}$$

where

$$A_{\mathbf{k}\lambda}(t) = \frac{i}{2MS^2}\sum_i \sum_{\alpha\beta} e^{-i\mathbf{k}\cdot\mathbf{R}_i} B_{\alpha\beta}(\mathbf{k}_{\alpha\beta}\cdot e_{-\mathbf{k}\lambda})\hat{S}_i^\alpha(t)\hat{S}_i^\beta(t), \tag{3.18}$$

and

$$\mathbf{k}_{\alpha\beta} = k_\alpha e_\beta + k_\beta e_\alpha. \tag{3.19}$$

The equation of motion for the canonical momentum directly follows,

$$\dot{P}_{\mathbf{k}\lambda} = -M\omega_{\mathbf{k}\lambda}^2 X_{\mathbf{k}\lambda} + MA_{\mathbf{k}\lambda}(t). \tag{3.20}$$

By taking an additional time derivative of the equations of motions (3.16) and (3.20), we arrive at the following second order differential equation

$$\left(\partial_t^2 + \omega_{\mathbf{k}\lambda}^2\right) X_{\mathbf{k}\lambda} = A_{\mathbf{k}\lambda}(t). \tag{3.21}$$

For this differential equation, the general solution reads

$$X_{\mathbf{k}\lambda}(t) = X_{\mathbf{k}\lambda}(0)\cos(\omega_{\mathbf{k}\lambda}t) + \frac{P_{\mathbf{k}\lambda}(0)}{M\omega_{\mathbf{k}\lambda}}\cos(\omega_{\mathbf{k}\lambda}t) + \int_0^t \mathrm{d}t' \, \frac{\sin\left[\omega_{\mathbf{k}\lambda}(t-t')\right]}{\omega_{\mathbf{k}\lambda}} \frac{dA_{\mathbf{k}\lambda}(t')}{\mathrm{d}t'}, \tag{3.22}$$

with the initial conditions of the phonon coordinates and momenta $X_{\mathbf{k}\lambda}(0)$ and $P_{\mathbf{k}\lambda}(0)$. We rewrite the general solution to

$$X_{\mathbf{k}\lambda}(t) = \tilde{X}_{\mathbf{k}\lambda}(0)\cos(\omega_{\mathbf{k}\lambda}t) + \frac{P_{\mathbf{k}\lambda}(0)}{M\omega_{\mathbf{k}\lambda}}\cos(\omega_{\mathbf{k}\lambda}t) + \frac{A_{\mathbf{k}\lambda}(t)}{\omega_{\mathbf{k}\lambda}^2} - \int_0^t \mathrm{d}t' \, \frac{\cos\left[\omega_{\mathbf{k}\lambda}(t-t')\right]}{\omega_{\mathbf{k}\lambda}^2} A_{\mathbf{k}\lambda}(t'), \tag{3.23}$$

defining $\tilde{X}_{\mathbf{k}\lambda}(0)$ through

$$\tilde{X}_{\mathbf{k}\lambda}(0) = X_{\mathbf{k}\lambda}(0) - \frac{A_{\mathbf{k}\lambda}(0)}{\omega_{\mathbf{k}\lambda}^2}. \tag{3.24}$$

Before plugging this solution into (3.12), we rewrite Eq. (3.12). We do this by approximating the quantum mechanical spin operators $\hat{\mathbf{S}}_i(t)$ by the classical vectors $\mathbf{S}_i(t)$. This is justified, because the saturation magnetization of YIG is quite high; the length S of the $\mathbf{S}_i$ vectors is approximately 14. As a result, we may say

$$\mathbf{S}_i(t) \times \mathbf{F}_i(t) = -\mathbf{F}_i(t) \times \mathbf{S}_i(t). \tag{3.25}$$

from which the following spin equation of motion follows,

$$\dot{\mathbf{S}}_i(t) = \hat{\mathbf{S}}_i(t) \times \left[\mathbf{H}(t) + \sum_j J_{ij}\mathbf{S}_j\right] + \mathbf{S}_i(t) \times \mathbf{F}_i(t). \tag{3.26}$$

Comparing this equation to Eq. (2.31), we see that the magnetoelastic field strength $\mathbf{F}_i(t)$ must contain information about the coupling of the spin to the bath. By plugging the solution (3.23) into Eq. (3.11) we obtain,

$$\mathbf{F}_i(t) = \delta\mathbf{h}_i(t) + \bar{\mathbf{h}}_i(t) - \int_0^t \mathrm{d}t' \, \sum_j \mathbb{G}_{ij}(t,t')\dot{\mathbf{S}}_j(t'), \tag{3.27}$$

where the first term is defined by

$$\delta\mathbf{h}_i(t) = -\frac{i}{NS^2}\sum_{\alpha\beta} B_{\alpha\beta}\mathbf{e}_\alpha \sum_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{R}_i}(\mathbf{k}_{\alpha\beta}\cdot\mathbf{e}_{\mathbf{k}\lambda})\left[\tilde{X}_{\mathbf{k}\lambda}(0)\cos(\omega_{\mathbf{k}\lambda}t) + \frac{P_{\mathbf{k}\lambda}(0)}{M\omega_{\mathbf{k}\lambda}}\sin(\omega_{\mathbf{k}\lambda}t)\right]S_i^\beta(t). \tag{3.28}$$

The second term, the induced magnetic field $\bar{\mathbf{h}}(t)$, is defined by

$$\bar{\mathbf{h}}(t) = \frac{1}{2S^4}\sum_{\alpha\beta}\sum_{\mu\nu}\mathbf{e}_\alpha B_{\alpha\beta}B_{\mu\nu}S_i^\beta(t)\sum_j S_j^\mu(t)S_j^\nu(t)\frac{1}{N}\sum_{\mathbf{k}\lambda}\frac{(\mathbf{k}_{\alpha\beta}\cdot\mathbf{e}_{\mathbf{k}\lambda})(\mathbf{k}_{\mu\nu}\cdot e_{-\mathbf{k}\lambda})}{M\omega_{\mathbf{k}\lambda}^2} \tag{3.29}$$

and the third term contains the damping kernel function $\mathbb{G}$ with its components defined by

$$\mathbb{G}_{\alpha\beta}(\mathbf{R}_i-\mathbf{R}_j, t-t') = \frac{1}{NS^2}B_{\alpha z}B_{\beta z}\sum_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot(\mathbf{R}_i-\mathbf{R}_j)}(\mathbf{k}_{\alpha z}\cdot\mathbf{e}_{\mathbf{k}\lambda})(\mathbf{k}_{\beta z}\cdot\mathbf{e}_{\mathbf{k}\lambda})\frac{\cos\left[\omega_{\mathbf{k}\lambda}(t-t')\right]}{M\omega_{\mathbf{k}\lambda}^2}. \tag{3.30}$$

In the end we find the following generalized LLG equation

$$\dot{\mathbf{S}}(\mathbf{R}_i,t) = \mathbf{S}(\mathbf{R}_i,t)\times\left[\mathbf{H}(t)+\mathbf{h}(\mathbf{R}_i,t)+\sum_j J_{ij}\mathbf{S}(\mathbf{R}_j,t)\right] \tag{3.31}$$

$$-\,\mathbf{S}(\mathbf{R}_i,t)\times\int_0^t \mathrm{d}t'\sum_j \mathbb{G}(\mathbf{R}_i-\mathbf{R}_j,t-t')\dot{\mathbf{S}}(\mathbf{R}_j,t'), \tag{3.32}$$

by plugging Eq. (3.27) into Eq. (3.12), where we have defined $\mathbf{h}_i(t) = \bar{\mathbf{h}}_i(t) + \delta\mathbf{h}_i(t)$. We disregard $\mathbf{h}_i(t)$ for the rest of the thesis, as it does not have an effect on the spin inertia.

3.2 Extracting bath spectral density function from gLLG

To get information about spin inertia, extracting the bath spectral density function would be the next step, because we have found the generalized Landau-Lifshitz-Gilbert Eq. (3.31) for the phonon bath. However, this equation is not of the same form as (2.32). The latter is formulated for a macrospin while (3.31) is formulated for a lattice of spins. We wish to approximate this system with a macrospin. This is realised by Fourier transforming (3.31), which will result in an expression containing $\mathbf{S}(\mathbf{k})$. Here $\mathbf{k}$ is the wavevector of the precessing magnetization. We only allow for homogeneous precession, i.e., $\mathbf{k} = 0$, in the macrospin approximation.

The Fourier transform of the left hand side of (3.31) is

$$\sum_i e^{-i\mathbf{R}_i\cdot\mathbf{k}}\dot{\mathbf{S}}(\mathbf{R}_i,t) \equiv \dot{\mathbf{S}}(\mathbf{k},t). \tag{3.33}$$

Assuming that J_{ij} only depends on the distance between spins $\mathbf{R}_i - \mathbf{R}_j$, the third term of the right hand side of (3.31) becomes

$$\sum_{ij} e^{-i\mathbf{R}_i\cdot\mathbf{k}}\mathbf{S}(\mathbf{R}_i,t)\times J(\mathbf{R}_i-\mathbf{R}_j)\mathbf{S}(\mathbf{R}_j,t) = \sum_{\mathbf{k}'}\frac{1}{N}\mathbf{S}(\mathbf{k}',t)\times J(\mathbf{k}-\mathbf{k}')\mathbf{S}(\mathbf{k}-\mathbf{k}',t). \tag{3.34}$$

In the macro spin approximation, we are only interested in zero wavevectors, i.e., $\mathbf{k} = \mathbf{k}' = \mathbf{0}$. However, there is no exchange interaction between different directions; $J(\mathbf{R}_i - \mathbf{R}_j)$ is proportional to the identity matrix. As a result, we end up with a term $\mathbf{S}(t)\times\mathbf{S}(t) = 0$. This is expected, because the exchange interaction takes place among different spins that are misaligned, while in the macrospin approximation the entire spin system can be regarded as just one (macro)spin.

Finally, the fourth term of (3.31) becomes

$$-\sum_i e^{-i\mathbf{R}_i\cdot\mathbf{k}}\mathbf{S}(\mathbf{R}_i,t)\times\int_0^t \mathrm{d}t'\ \sum_j \mathbb{G}(\mathbf{R}_i-\mathbf{R}_j,t-t')\dot{\mathbf{S}}(\mathbf{R}_j,t')$$
$$\approx -\int_0^t \mathrm{d}t'\ \frac{1}{N}\mathbf{S}(\mathbf{k}=\mathbf{0},t)\times\mathbb{G}(\mathbf{k}=\mathbf{0},t-t')\dot{\mathbf{S}}(\mathbf{k}=\mathbf{0},t'), \tag{3.35}$$

in the macrospin limit. We calculate the damping kernel with

$$\mathbb{G}(\mathbf{k}=\mathbf{0},t-t')=\sum_i \mathbb{G}(\mathbf{R}_i,t-t'), \tag{3.36}$$

from which we find the following expression for the generalized LLG equation for phonon baths,

$$\dot{\mathbf{S}}(\mathbf{k}=0,t)=\mathbf{S}(\mathbf{k}=0,t)\times\mathbf{H}(t)-\mathbf{S}(\mathbf{k}=\mathbf{0},t)\times\int_0^t \mathrm{d}t'\ \frac{1}{N}\sum_i \mathbb{G}(\mathbf{R}_i,t-t')\dot{\mathbf{S}}(\mathbf{k}=\mathbf{0},t'). \tag{3.37}$$

As all wavevectors are zero, we will drop the $\mathbf{k}$ dependence in our notation. The final step, before we can compare equations (3.31) and (2.32), is to apply partial integration on the second term of Eq. (3.31),

$$-\int_0^t \mathrm{d}t'\ \sum_i \frac{1}{N}\mathbb{G}(\mathbf{R}_i,t-t')\dot{\mathbf{S}}(t')$$
$$=\int_{-\infty}^{\infty}\mathrm{d}t'\ \frac{1}{N}\left(\delta(t-t')\sum_i \mathbb{G}(\mathbf{R}_i,t-t')+\Theta(t-t')\sum_i \dot{\mathbb{G}}(\mathbf{R}_i,t-t')\right)\mathbf{S}(t') \tag{3.38}$$

from which it follows, by comparison, that $\alpha(t-t')$ equals

$$\alpha(t-t')=-\frac{1}{N}\delta(t-t')\sum_i \mathbb{G}(\mathbf{R}_i,t-t')-\frac{1}{N}\Theta(t-t')\sum_i \dot{\mathbb{G}}(\mathbf{R}_i,t-t'). \tag{3.39}$$

To extract the bath spectral density function, which contains information about Gilbert damping and spin inertia, we write this kernel function in the form of Eq. (2.34) using Fourier transformations,

$$\alpha^{\alpha\beta}(\omega)=-\frac{1}{N}G(t=0)-\frac{1}{N}\sum_i\sum_{\mathbf{k}\lambda}\frac{B_{\alpha z}B_{\beta z}}{MNS^2}e^{i\mathbf{k}\cdot\mathbf{R}_i}(\mathbf{k}_{\alpha z}\cdot\mathbf{e}_{\mathbf{k}\lambda})(\mathbf{k}_{\beta z}\cdot\mathbf{e}_{-\mathbf{k}\lambda})\frac{1}{(\omega+i0)^2-\omega_{\mathbf{k}\lambda}^2} \tag{3.40}$$

where we have used that

$$\begin{aligned}\int\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega t}\frac{1}{(\omega+i0)^2-\epsilon^2}&=\int\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega t}\frac{1}{\omega+i0+\epsilon}\frac{1}{\omega+i0-\epsilon}\\&=\begin{cases}-\frac{e^{i\epsilon t}}{2i\epsilon}+\frac{e^{-i\epsilon t}}{2i\epsilon} & t>0\\ 0 & t<0\end{cases}\\&=-\Theta(t)\frac{\sin(\epsilon t)}{\epsilon}.\end{aligned} \tag{3.41}$$

The result of (3.41) can be proven using contour integration, with the concerning poles depicted in Fig. 3.1. Also, we may disregard the $-\frac{1}{N}\mathbb{G}(t=0)$ term in (3.40), as it is constant in ω and hence will not contribute to Eq. (2.32); a similar argument has been used in Eq. (2.39).

Now, to establish the connection between Eq. (3.40) and (2.34), we define the bath spectral density function $J(\epsilon)$ to be

$$J^{\alpha\beta}(\epsilon)=\frac{\pi}{N}\sum_i\sum_{\mathbf{k}\lambda}\frac{2B_{\alpha z}B_{\beta z}}{MNS^2\omega_{\mathbf{k}\lambda}}e^{i\mathbf{k}\cdot\mathbf{R}_i}(\mathbf{k}_{\alpha z}\cdot\mathbf{e}_{\mathbf{k}\lambda})(\mathbf{k}_{\beta z}\cdot\mathbf{e}_{-\mathbf{k}\lambda})\delta(\epsilon-\omega_{\mathbf{k}\lambda}). \tag{3.42}$$

Pluggin this back into Eq. (2.34) returns Eq. (3.40),

$$\begin{aligned}\alpha^{\alpha\beta}(\omega) &= -\int \frac{\mathrm{d}\epsilon}{\pi}\frac{\epsilon}{(\omega+i0)^2-\epsilon^2}\frac{\pi}{N}\sum_i\sum_{\mathbf{k}\lambda}\frac{B_{\alpha z}B_{\beta z}}{MNS^2\omega_{\mathbf{k}\lambda}}e^{i\mathbf{k}\cdot\mathbf{R}_i}(\mathbf{k}_{\alpha z}\cdot\mathbf{e}_{\mathbf{k}\lambda})(\mathbf{k}_{\beta z}\cdot\mathbf{e}_{-\mathbf{k}\lambda})\delta(\epsilon-\omega_{\mathbf{k}\lambda})\\ &= -\frac{1}{N}\sum_i\sum_{\mathbf{k}\lambda}\frac{B_{\alpha z}B_{\beta z}}{MNS^2}e^{i\mathbf{k}\cdot\mathbf{R}_i}(\mathbf{k}_{\alpha z}\cdot\mathbf{e}_{\mathbf{k}\lambda})(\mathbf{k}_{\beta z}\cdot\mathbf{e}_{-\mathbf{k}\lambda})\frac{1}{(\omega+i0)^2-\omega_{\mathbf{k}\lambda}^2}. \end{aligned} \tag{3.43}$$

The bath spectral density function allows us to calculate the Gilbert damping and spin inertia. The spin inertia constant I will have a tensorial form, as the Gilbert damping $\alpha^{\alpha\beta}(\omega)$ has a tensorial form. In the next section, we discuss a case where the spin inertia vanishes and two more cases where it does not.

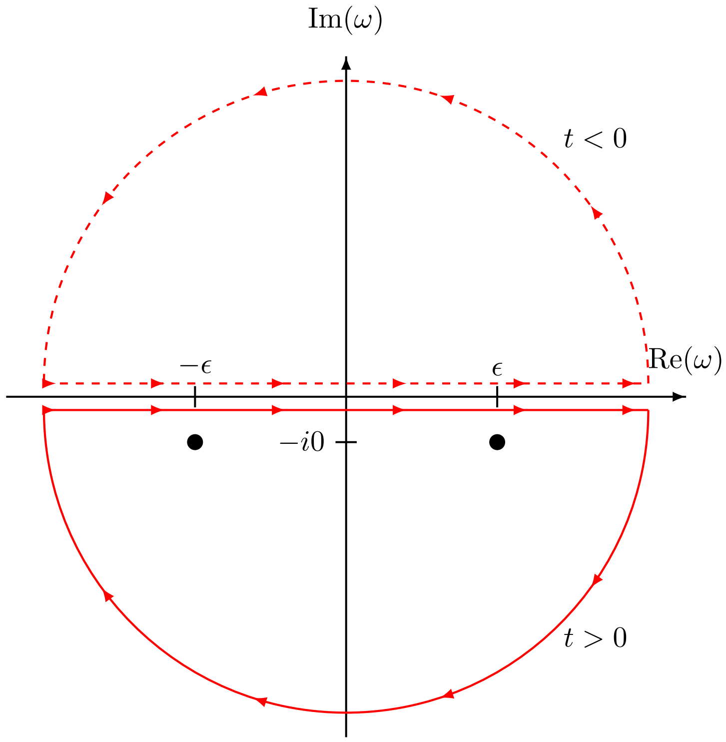

Figure 3.1: *The red, solid contour is for $t > 0$ and the poles of the integral are then situated at $\omega = \epsilon - i0$ and $\omega = -\epsilon - i0$. The red, dashed contour is for $t < 0$ and then the integral has no poles.*

3.3 Existence spin inertia

In this section, we show a special case leading to vanishing spin inertia, and two cases showcasing spin inertia. When considering a 3D spin system in contact with a 3D phonon bath, we will demonstrate that there is no spin inertia from the bulk. When considering a 2D spin system in contact with a 3D phonon bath, on the other hand, will will show that there is, indeed, spin inertia. Finally, for two different realisations of this system, we provide an explicit calculation of spin inertia appearing in the inertial LLG equation, which is the main result of this thesis.

3.3.1 3D spin system in contact with 3D phonon bath

In previous section 3.2, we found that in a 3D spin system in contact with a 3D phonon bath results in a bath spectral density function as in Eq. (3.42). The sum over i runs over all the lattice sites in the 3D lattice. Note the following identity

$$\begin{aligned}\sum_i e^{i\mathbf{k}\cdot\mathbf{R}_i} &= \sum_{m,n} e^{ik_x n_x a + ik_y n_y a + ik_z n_z a}\\ &= N_x\delta_{k_x,0}N_y\delta_{k_y,0}N_z\delta_{k_z,0}, \end{aligned} \tag{3.44}$$

where m, n are integers, and N_x, N_y and N_z denote the number of lattice sites in the x, y and z direction respectively. Filling this identity into (3.42) results in $J(\epsilon) = 0$. This result can also be obtained from a physical argument. The bath spectral density function scales with the phonon momentum $\mathbf{k}$. We find that this momentum should be the sum of the magnon momenta in the scattering process due to momentum conservation (see Fig. E.1 in appendix E). However, the magnon momenta should be zero, because of the macrospin approximation, and thus the phonon momenta should also be zero. As a result, $J(\epsilon)$ vanishes. Then, we must conclude that there is no Gilbert damping and spin inertia due to a phonon bath from the bulk in the macrospin approximation, which follows directly from Eq. (2.39) when filling in the vanishing bath spectral density function.

3.3.2 Spin inertia for a magnetic plate perpendicular to the magnetic field

We saw that, in the macrospin approximation, the bath spectral density vanishes from the bulk. The bath spectral density function does not vanish when reducing the spin system to just a 2D plane perpendicular to the external magnetic fields. The new situation is depicted in Fig. 3.2.

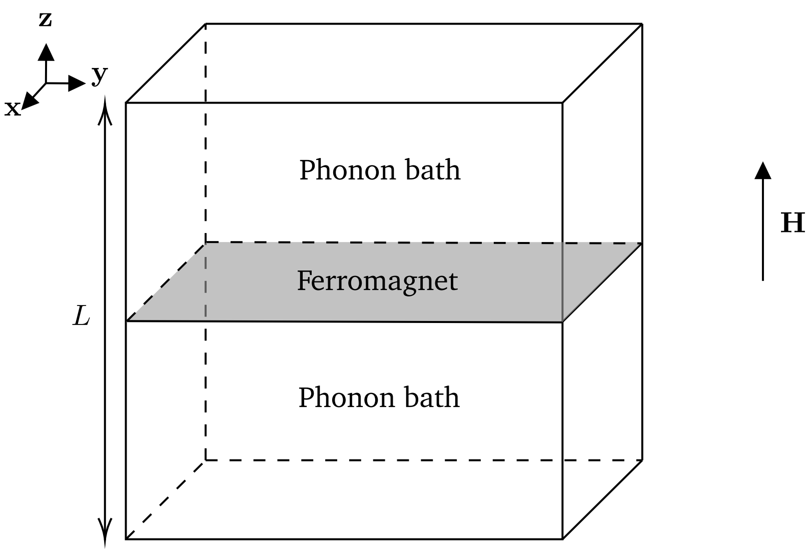

Figure 3.2: *Schematic visualisation of the system we consider. The ferromagnet is situated on the xy-plane and is in contact with the bulk phonon bath subject to an external magnetic field in the z-direction.*

Considering a 2D spin system instead of a 3D one, has consequences for the bath spetral density function. The sum in the bath spectral density function, now, runs over just the lattice sites of the spin; the xy-plane. As a result, we use

$$\sum_{n_x,n_y} e^{ik_x n_x a + ik_y n_y a} = N_x \delta_{k_x,0} N_y \delta_{k_y,0} \, . \tag{3.45}$$

Here N_x and N_y denote the number of spins in the x and y direction respectively. For the bath spectral density function in Eq. (3.42) we obtain

$$J^{\alpha\beta} = \pi \sum_{k_z \lambda} \frac{2B_{\alpha z} B_{\beta z} N_x N_y}{M S^2 N^2 \omega_{\mathbf{k}\lambda}} (\mathbf{k}_{\alpha z} \cdot \mathbf{e}_{\mathbf{k}\lambda})(\mathbf{k}_{\beta z} \cdot \mathbf{e}_{-\mathbf{k}\lambda}) \delta(\epsilon - \omega_{\mathbf{k}\lambda}), \tag{3.46}$$

where we sum over the phonon momenta k_z. This could have been predicted from physical arguments as well. When considering a magnetic plate in the xy direction, the confinement in the z direction is strict. As a result of Heisenberg's uncertainty principle, the phonons can take all the momenta in the z direction.

Now, we will calculate the bath spectral density function (3.46). In a simple cubic lattice considering nearest and next to nearest neighbour coupling (see Fig. 3.3), the dispersion in the z direction is known [18],

$$\omega_{\mathbf{k}1,2} = \sqrt{\frac{4K_2}{M}} \left|\sin\left(\frac{1}{2}k_z a\right)\right|, \tag{3.47}$$

$$\omega_{\mathbf{k}3} = \sqrt{\frac{4K_1 + 8K_2}{M}} \left|\sin\left(\frac{1}{2}k_z a\right)\right|, \tag{3.48}$$

where M is the effective mass of each lattice site, K_1 is the force constant for first neighboring interactions, and K_2 is the force constant for second neighboring interactions. The third dispersion corresponds to the longitudinal polarization and the other two the transversal polarizations.

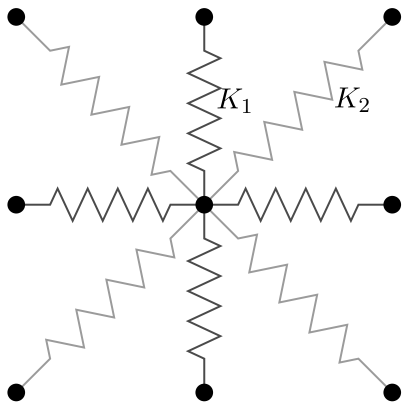

Figure 3.3: *Depiction of nearest-neighbour (force constant K_1) and next-to-nearest-neighbour (force constant K_2) interactions in a simple cubic lattice.*

Using the fact that $k_x = k_y = 0$, we obtain

$$\begin{aligned}\mathbf{k}_{\alpha z} &= \begin{cases} k_z \mathbf{e}_\alpha & \alpha \neq z \\ 2k_z \mathbf{e}_z & \alpha = z \end{cases} \\ &= (1 + \delta_{\alpha,z})\, k_z \mathbf{e}_\alpha. \end{aligned} \tag{3.49}$$

The polarization vectors are given by [19]

$$\begin{aligned} \mathbf{e}_{\mathbf{k}1} &= (1,0,0) & \mathbf{e}_{-\mathbf{k}1} &= (1,0,0) \\ \mathbf{e}_{\mathbf{k}2} &= (0,i,0) & \mathbf{e}_{-\mathbf{k}2} &= (0,-i,0) \\ \mathbf{e}_{\mathbf{k}3} &= (0,0,i) & \mathbf{e}_{-\mathbf{k}3} &= (0,0,-i), \end{aligned} \tag{3.50}$$

where the third one corresponds to the longitudinal polarization and the other to the transversal ones. Explicitly calculating the bath spectral density in (3.46), by filling in the dispersion and polarization vectors, gives

$$J^{xx}(\epsilon) = \pi \sum_{k_z} \frac{B_\perp^2 N_x N_y}{MS^2N^2\sqrt{\frac{4K_2}{M}}\left|\sin\left(\frac{1}{2}k_z a\right)\right|} k_z^2 \delta\left(\epsilon - \sqrt{\frac{4K_2}{M}}\left|\sin\left(\frac{1}{2}k_z a\right)\right|\right); \tag{3.51}$$

$$J^{yy}(\epsilon) = \pi \sum_{k_z} \frac{B_\perp^2 N_x N_y}{MS^2N^2\sqrt{\frac{4K_2}{M}}\left|\sin\left(\frac{1}{2}k_z a\right)\right|} k_z^2 \delta\left(\epsilon - \sqrt{\frac{4K_2}{M}}\left|\sin\left(\frac{1}{2}k_z a\right)\right|\right); \tag{3.52}$$

$$J^{zz}(\epsilon) = 4\pi \sum_{k_z} \frac{B_\parallel^2 N_x N_y}{MS^2N^2\sqrt{\frac{4K_1+8K_2}{M}}\left|\sin\left(\frac{1}{2}k_z a\right)\right|} k_z^2 \delta\left(\epsilon - \sqrt{\frac{4K_1+8K_2}{M}}\left|\sin\left(\frac{1}{2}k_z a\right)\right|\right), \tag{3.53}$$

for the diagonal terms. The off diagonal terms vanish. Note that there are no phonon states with frequency larger than the Debye frequency $\omega_D = 2\sqrt{\frac{K}{M}}$, and no phonon states with negative frequency. As a result, $J^{\alpha\alpha}(\epsilon)$ vanishes in those frequency regions. Defining the continuously differentiable function $h(k) = \epsilon - 2\left(\frac{K}{M}\right)^{\frac{1}{2}} \sin\left(\frac{1}{2}ka\right)$, we see that

$$\delta(h(k)) = \sum_n \frac{\delta(k - k_n)}{\left|\frac{dh}{dk}(k_n)\right|}, \tag{3.54}$$

with k_n such that $h(k_n) = 0$. For $0 < \epsilon < 2\sqrt{\frac{K}{M}}$ the only solutions to $h(k_n) = 0$ are

$$k_0 = \pm\frac{2}{a}\sin^{-1}\left[\frac{1}{2}\left(\frac{M}{K}\right)^{\frac{1}{2}}\epsilon\right]. \tag{3.55}$$

The derivative term in (3.54) equals,

$$\left|\frac{dh}{dk}(k_0)\right| = a\left(\frac{K}{M}\right)^{\frac{1}{2}}\cos\left(\frac{1}{2}k_0 a\right). \tag{3.56}$$

Using this, and after replacing the sum with an integral in the continuum limit, the bath spectral density function becomes

$$J^{xx}(\epsilon) = J^{yy}(\epsilon) = \frac{LB_\perp^2 N_x N_y}{MS^2N^2\epsilon}\left\{\frac{2}{a}\sin^{-1}\left[\frac{1}{2}\left(\frac{M}{K_2}\right)^{\frac{1}{2}}\epsilon\right]\right\}^2 \frac{1}{a\sqrt{\frac{K_2}{M} - \frac{1}{4}\epsilon^2}} \tag{3.57}$$

$$J^{zz}(\epsilon) = \frac{4LB_\parallel^2 N_x N_y}{MS^2N^2\epsilon}\left\{\frac{2}{a}\sin^{-1}\left[\frac{1}{2}\left(\frac{M}{K_1+2K_2}\right)^{\frac{1}{2}}\epsilon\right]\right\}^2 \frac{1}{a\sqrt{\frac{K_1+2K_2}{M} - \frac{1}{4}\epsilon^2}}. \tag{3.58}$$

To know the low frequency behaviour, we expand the bath spectral density and retain the lowest order terms

$$\begin{aligned} J^{xx}(\epsilon) = J^{yy}(\epsilon) &= \frac{LB_\perp^2 N_x N_y}{MS^2N^2\epsilon}\left[\frac{1}{a}\sqrt{\frac{M}{K_2}}\epsilon + \mathcal{O}(\epsilon^3)\right]^2\left[\frac{1}{a\sqrt{\frac{K_2}{M}}} + \mathcal{O}(\epsilon^2)\right] \\ &\approx \frac{LB_\perp^2 N_x N_y}{S^2a^3N^2}\sqrt{\frac{M}{K_2^3}}\,\epsilon; \end{aligned} \tag{3.59}$$

$$\begin{aligned} J^{zz}(\epsilon) &= \frac{LB_\parallel^2 N_x N_y}{MS^2N^2\epsilon}\left[\frac{1}{a}\sqrt{\frac{M}{K_1+2K_2}}\epsilon + \mathcal{O}(\epsilon^3)\right]^2\left[\frac{1}{a\sqrt{\frac{K_1+2K_2}{M}}} + \mathcal{O}(\epsilon^2)\right] \\ &\approx \frac{LB_\parallel^2 N_x N_y}{S^2a^3N^2}\sqrt{\frac{M}{(K_1+2K_2)^3}}\,\epsilon \end{aligned} \tag{3.60}$$

and see that the bath spectral densities are linear for low frequencies which indicates that we recover Gilbert damping. The bath spectral density functions for each direction behave similarly. To see how exactly they behave we have plotted the functions

$$J(x) = C\frac{\left[\sin^{-1}(x)\right]^2}{x\sqrt{1-x^2}}, \tag{3.61}$$

$$J_{\text{lf}}(x) = Cx, \tag{3.62}$$

with constant $C = \frac{2B^2LN_xN_y}{a^3KS^2N^2}$ and dimensionless energy $x = \frac{\epsilon}{2\sqrt{\frac{K}{M}}}$ in Fig. 3.4. We have dropped the tensorial indices, but C and $J(x)$ are still tensors. For J^{xx} and J^{yy} we have that $K = K_2$ and for J^{zz} we have $K = K_1 + 2K_2$. As we can see in Fig. 3.4, the bath spectral density function has a Van Hove singularity [20] at the Debye frequency and vanishes above the Debye frequency. Due to

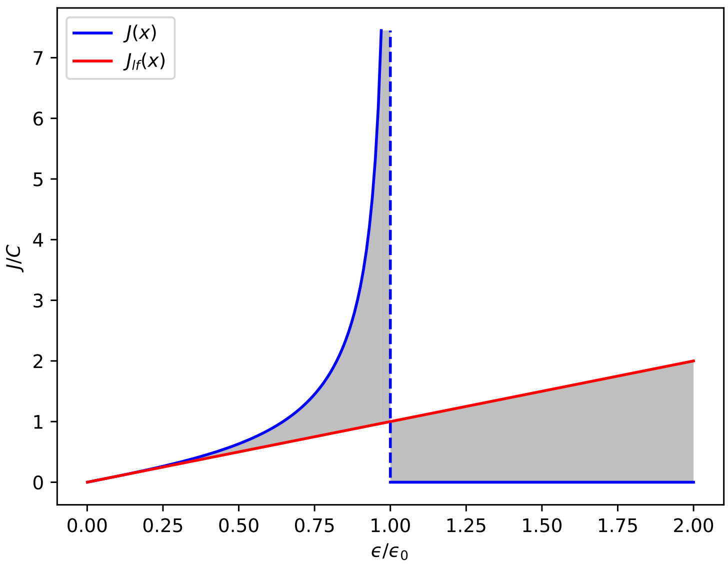

Figure 3.4: *In blue the bath spectral density J/C as a function of the dimensionless frequency ϵ/ϵ_0, with $\epsilon_0 = 2\sqrt{\frac{K}{M}}$ the Debye frequency and $C = \frac{2B^2 L N_x N_y}{a^3 K S^2 N^2}$ and in red the low frequency approximation.*

this non-linear high frequency behaviour, there must be spin inertia. In order to calculate it we use Eq. (2.42) and see that

$$\begin{aligned} I &\equiv \frac{2}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\epsilon \frac{J(\epsilon) - J_{\mathrm{lf}}(\epsilon)}{\epsilon^3} \\ &= \frac{1}{2\pi} C \left(\frac{K}{M}\right)^{-1} \left[\int_0^1 \frac{\frac{[\sin^{-1}(x)]^2}{x\sqrt{1-x^2}} - x}{x^3} dx + \int_1^{\infty} \frac{-1}{x^2} dx\right] \\ &= \frac{1}{2\pi} C \left(\frac{K}{M}\right)^{-1} \left[\int_0^{\pi/2} \frac{x^2 - \sin^2(y)\cos(y)}{\sin^4(x)} dx - 1\right], \end{aligned} \tag{3.63}$$

where we have used trigonometric identities to rewrite the integral. We calculate this integral numerically,

$$\begin{aligned} I &= \frac{1}{2\pi} C \left(\frac{K}{M}\right)^{-1} [1.928... - 1] \\ &\approx \frac{C}{2\pi}\left(\frac{K}{M}\right)^{-1}. \end{aligned} \tag{3.64}$$

From Eq. (2.38) we see that the Gilbert damping α_0, a tensor as well, equals $C/2\sqrt{\frac{K}{M}}$, as we have

$$J_{\mathrm{lf}}(\epsilon) = \frac{C}{2\sqrt{\frac{K}{M}}}\epsilon, \tag{3.65}$$

for the low frequency part of the bath spectral density function. For the spin inertia we then obtain the main result of our thesis,

$$I^{\alpha\beta} \approx \frac{\alpha_0^{\alpha\beta}}{\pi}\sqrt{\frac{M}{K}}, \tag{3.66}$$

showing spin inertia for a ferromagnetic thin film in contact with a bulk phonon bath. Here, $\sqrt{\frac{M}{K}}$ has the dimension of time and is often called the angular momentum relaxation time in literature [2, 5, 6].

Regarding spin inertia, the setup of the system play a crucial role. Consider, for example, the length of the lattice L (see Fig. 3.2). The spin inertia tensor I scales, just as the Gilbert damping tensor, with L. This could be interesting for experimental purposes. A typical experimental setup, which would mirror our theory in this subsection, consists of a thin Yttrium Iron Garnet (YIG) film in contact with a bulk Gadolinium Gallium Garnet (GGG) substrate subject to an external magnetic field. The YIG film would correspond to the 2D ferromagnet and the GGG substrate to the 3D phonon bath. Increasing the length of the GGG substrate should lead to an increase in spin inertia and Gilbert damping.

In the next subsection, we give another example why the specific setup of the systems is important.

3.3.3 Spin inertia for a magnetic plate in plane of the magnetic field

In this subsection, we consider the magnetic plate to be in plane to the external magnetic field (see Fig. 3.5), which gives slightly different results.

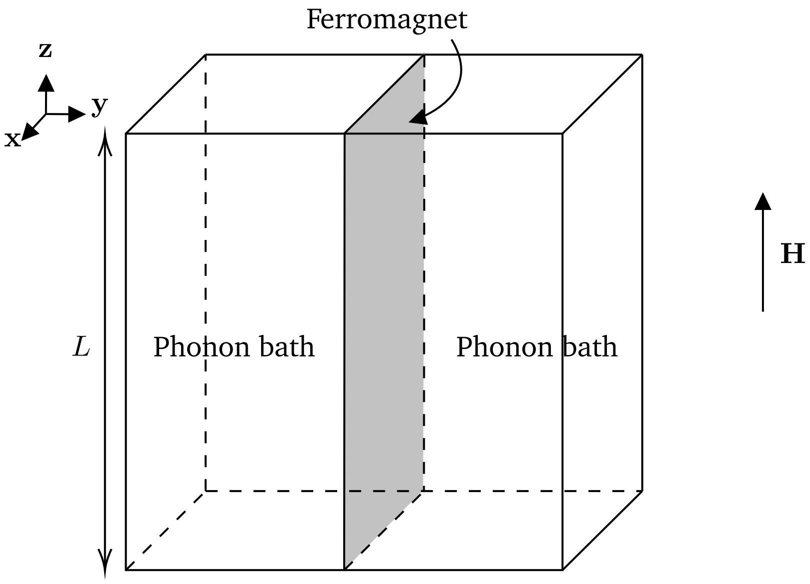

Figure 3.5: *Schematic visualisation of the system we consider. The ferromagnet is situated on the xz-plane and is in contact with the bulk phonon bath with the external magnetic field in the z-direction.*

Now, we use the following identity,

$$\sum_{m,n} e^{ik_x ma + ik_y na} = N_x \delta_{k_x,0} N_z \delta_{k_z,0} \ , \tag{3.67}$$

because we consider the magnetic plate to be on the xz-plane. Here N_x and N_z denote the number of spins in the x and z direction respectively. This results in the bath spectral density function,

$$J^{\alpha\beta} = \pi \sum_{\mathbf{k}\lambda} \frac{B_{\alpha z} B_{\beta z} N_x N_z}{M S^2 N^2 \omega_{k_z \lambda}} (\mathbf{k}_{\alpha z} \cdot \mathbf{e}_{\mathbf{k}\lambda})(\mathbf{k}_{\beta z} \cdot \mathbf{e}_{-\mathbf{k}\lambda}) \delta(\epsilon - \omega_{\mathbf{k}\lambda}), \tag{3.68}$$

where we sum over the phonon momenta k_z. The dispersion relation in this direction is given by [18],

$$\omega_{\mathbf{k}2} = \sqrt{\frac{4K_1 + 8K_2}{M}} \left| \sin\left(\frac{1}{2} k_z a\right) \right| , \tag{3.69}$$

$$\omega_{\mathbf{k}1,3} = \sqrt{\frac{4K_2}{M}} \left| \sin\left(\frac{1}{2} k_z a\right) \right| , \tag{3.70}$$

where the first one corresponds to the longitudinal polarization and the other two the transversal polarizations. Using the fact that $k_x = k_z = 0$, we obtain

$$\mathbf{k}_{\alpha z} = \delta_{\alpha y} k_\alpha \mathbf{e}_z. \tag{3.71}$$

For the polarization vectors we know [19]

$$\begin{aligned} \mathbf{e}_{\mathbf{k}1} &= (0,0,-1) & \qquad \mathbf{e}_{-\mathbf{k}1} &= (0,0,-1) \\ \mathbf{e}_{\mathbf{k}2} &= (0,i,0) & \mathbf{e}_{-\mathbf{k}2} &= (0,-i,0) \\ \mathbf{e}_{\mathbf{k}3} &= (i,0,0) & \mathbf{e}_{-\mathbf{k}3} &= (-i,0,0), \end{aligned} \tag{3.72}$$

where the second one corresponds to the longitudinal polarization and the other to the transversal ones. Explicitly calculating bath spectral density gives

$$J^{xx}(\epsilon) = 0 \tag{3.73}$$

$$J^{yy}(\epsilon) = \pi \sum_{k_y} \frac{B_\perp^2 N_x N_z}{MS^2N^2\sqrt{\frac{4K_2}{M}}\left|\sin\left(\frac{1}{2}k_z a\right)\right|} k_y^2 \delta\left(\epsilon - \sqrt{\frac{4K_2}{M}}\left|\sin\left(\frac{1}{2}k_z a\right)\right|\right) \tag{3.74}$$

$$J^{zz}(\epsilon) = 0, \tag{3.75}$$

for the diagonal terms. The off diagonal terms vanish. Using similar calculations as for the xy plane we obtain

$$J^{xx}(\epsilon) = 0 \tag{3.76}$$

$$J^{yy}(\epsilon) = \frac{LB_\perp^2 N_x N_z}{MS^2N^2\epsilon}\left\{\frac{2}{a}\sin^{-1}\left[\frac{1}{2}\left(\frac{M}{K_2}\right)^{\frac{1}{2}}\epsilon\right]\right\}^2 \frac{1}{a\sqrt{\frac{K_2}{M} - \frac{1}{4}\epsilon^2}} \tag{3.77}$$

$$J^{zz}(\epsilon) = 0, \tag{3.78}$$

which lead to the following expression for the Gilbert damping tensor

$$\alpha_0^{\alpha\beta} = \begin{cases} \frac{B^2 L N_x N_z M^{1/2}}{a^3 K^{3/2} S^2 N^2} & \text{if } \alpha = y, \beta = y \\ 0 & \text{else} \end{cases}, \tag{3.79}$$

and for the spin inertia tensor

$$I^{\alpha\beta} \approx \frac{\alpha_0^{\alpha\beta}}{\pi}\sqrt{\frac{M}{K}}. \tag{3.80}$$

The difference regarding the spin inertia (and Gilbert damping) with respect to the setup of 3.3.2 is that there is no spin inertia in the x and the z direction, confirming that the specific setup of the system is important. Also, this setup could be interesting for experiments, as only the direction of the external magnetic field has to be changed with respect to the setup in 3.3.2.

4. Discussion, conclusion and outlook

In this thesis, we calculated the spin inertia for the case of a ferromagnetic thin film in contact with a bulk phonon bath. Our approach started by rederiving the result from Gaspar Quarenta's master's thesis [7]; the high-frequency bath modes of the environment can lead to spin inertia, which showcases itself in the inertial LLG equation. The rederivation involved the Keldysh formalism to formulate the generalized Landau-Lifshitz-Gilbert equation containing the kernel function $\alpha(t-t')$. The kernel function is dependent on the bath spectral density function $J(\epsilon)$, which contains all the information regarding the coupling of the spin system to its environment. From this function, we found an expression for the spin inertia I. Our findings indicate the presence of spin inertia for any non-Ohmic expression of the bath spectral density function.

To arrive at the main result of this thesis, we considered the phonon bath to get an explicit expression for the bath spectral density function. To obtain this expression, we calculated the kernel function $\alpha(t-t')$ using the approach from Rückriegel and Kopietz [9]. We have predicted that there is spin inertia in magnetic systems in contact with a phonon bath. Furthermore, we explicitly calculated the spin inertia tensor in a two dimensional ferromagnetic system coupled to a three dimensional phonon bath perpendicular to the external magnetic field, and found that the inertial constant scales linearly with the Gilbert damping. We also found spin inertia in a similar system, but with an in-plane external magnetic field, leading to slightly different results; there is only spin inertia perpendicular to the magnetic plate.

Despite the promising results, we have some matters to discuss. For the first point, we have to recall that spin inertia is an effect originating from the higher frequency modes in the environment. We have explicitly seen that a linear dispersion will only lead to Gilbert damping. Spin inertia arises from the non-linear behaviour of the phonon dispersion for higher frequencies. This non-linear phonon dispersion is adapted from Ref. [18], where the dispersion is obtained from a lattice model (high-frequency or short-wavelength phonons) in a simple cubic lattice. However, our theory is built upon the work by Rückriegel and Kopietz [9]. Although they worked with a lattice model as well, they assume the long-wavelength limit for the phonon system. Therefore, it is unclear if the magnetoelastic coupling and the phonon dispersion we used is compatible. We present two ways of proceeding forward. The first way is to determine the phonon dispersion for higher frequencies in the continuum model. The second way is to rederive the results by Rückriegel and Kopietz [9] for a full lattice model for the spins and the lattice vibrations including a lattice version of the magnetoelastic coupling.

The second topic that needs to be discussed involves one of the approximations we made. It is the macrospin approximation and it simplified our calculations substantially for the case of a ferromagnetic thin film in contact with a bulk phonon bath. If the external magnetic field is homogeneous, the macrospin approximation is valid. However, it might be possible to consider a larger thin film by regarding multiple spins. One does this by starting from the non-local generalized Landau Lifshitz Gilbert equation [21].

The final topic regards the possibility to pursue experiments as a result of the theoretical developments made in this thesis. In this thesis, we have emphasized the role of the specific setup of the systems. Recall that we provided explicit calculations of the spin inertia in the example of a two dimensional ferromagnet in contact with a three dimensional phonon bath in section 3.3. This can be realised by placing a thin Yttrium Iron Garnet (YIG) film in a bulk Gadolinium Gallium Garnet substrate (GGG) similar to the setup in Ref. [22]. Here YIG contains the magnetic system and the GGG acts as the phonon bath. One could verify if changing the external magnetic field from perpendicular to an in-plane magnetic field has consequences for the spin inertia, as we have predicted theoretically. Also, one could verify if spin inertia scales linearly with the thickness of the phonon baths.

A slightly different configuration would consist of a ferromagnetic thin film in contact with a bulk phonon bath only on top of the film. Such a configuration is better feasable experimentwise. Theoretically, however, one needs to be more careful. Defining the phonons in Fourier space is more complicated, since the phonon lattice is only defined on the upper half volume. If one, nevertheless, still manages to define the phonons, one can follow the calculations in Ch. 3 to obtain an expression for the spin inertia. We would expect the difference to be a factor $\frac{1}{2}$ with respect to the full phonon bath case. On the thin film only phonons with positive wavevector would have been created and implementing this into our calculations would lead to an inertial constant half as big as the inertial constant for the case of a bulk phonon bath both on top and below of the ferromagnetic thin film.

Finally, a natural question to ask is if spin inertia could also arise from an electron bath as well. The electron bath would originate from the electrons in the ferromagnet. Developments in this area could help to gain a more deeper understanding of spin inertia as was observed in [5,6]. Moreover, transferring the underlying theory behind spin inertia to antiferromagnets would be interesting as well.

A. Derivation spin dependent part Keldysh partition function

In this appendix, we will rederive the spin dependent part apparent following Keldysh partition function

$$\mathcal{Z} = \int Dg \prod_{\alpha} \int Dx_{\alpha} \int Dp_{\alpha} \exp\Big[i \oint_K \mathrm{d}t \Big(-i\langle \dot{g}|g\rangle + \langle g|\mathbf{H}\cdot\hat{\mathbf{S}}|g\rangle - \gamma_{\alpha}\langle g|\hat{\mathbf{S}}\cdot\dot{\mathbf{x}}|g\rangle \\ + p_{\alpha}\dot{x}_{\alpha} - \frac{p_{\alpha}^2}{2m} - \frac{m\omega^2}{2}x_{\alpha}^2\Big)\Big], \quad \text{(A.1)}$$

from the Hamiltonian (2.1) and for notational reasons we define the effective field

$$\mathbf{H}' = \mathbf{H} - \gamma_{\alpha}\dot{\mathbf{x}}. \quad \text{(A.2)}$$

We rederive the spin part of Eq. (A.1) and do this analogously to Ref. [16], but in the Keldysh formalism in its path integral form. The definition of the Keldysh path integral is

$$Z = \frac{\mathrm{Tr}\{\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}(-\infty)\}}{\mathrm{Tr}\{\hat{\rho}(-\infty)\}}. \quad \text{(A.3)}$$

For the trace we have

$$\mathrm{Tr}\{\hat{O}\} = C\int \mathrm{d}g\ \langle g|\hat{O}|g\rangle, \quad \text{(A.4)}$$

and note that

$$\hat{1}_S = C\int \mathrm{d}g_j\ |g_j\rangle\langle g_j|, \quad \text{(A.5)}$$

where $\hat{1}_S$ is the unit operator and C comes from the Haar measure. Following the standard path-integral construction, we split the time evolution operator in (A.3) into N steps given by

$$\begin{aligned}\langle g_j|\hat{\mathcal{U}}_{\pm\delta_t}|g_{j-1}\rangle &= \langle g_j|\exp\big(\mp i\hat{H}\delta_t\big)|g_{j-1}\rangle \\ &\approx \langle g_j|1 \mp i\hat{H}\delta_t|g_{j-1}\rangle,\end{aligned} \quad \text{(A.6)}$$

where δ_t is the length of the step. Working it out and re-exponentiating it gives us

$$\begin{aligned}\langle g_j|1 \mp i\hat{H}_{\mathrm{spin}}\delta_t|g_{j-1}\rangle &= \langle g_j|1 \mp i\mathbf{H}'\cdot\hat{\mathbf{S}}\delta_t|g_{j-1}\rangle \\ &= \langle g_j|g_{j-1}\rangle \mp \langle g_j|i\mathbf{H}'\cdot\hat{\mathbf{S}}\delta_t|g_{j-1}\rangle \\ &= 1 - \langle g_j|g_j\rangle + \langle g_j|g_{j-1}\rangle \mp i\delta_t\langle g_j|\mathbf{H}'\cdot\hat{\mathbf{S}}|g_{j-1}\rangle \\ &\approx \exp\big(\langle g_j|g_{j-1}\rangle - \langle g_j|g_j\rangle \mp i\delta_t\langle g_j|\mathbf{H}'\cdot\hat{\mathbf{S}}|g_{j-1}\rangle\big).\end{aligned} \quad \text{(A.7)}$$

For the forward contour, the matrix elements then read

$$\begin{aligned}\langle g_{i+1}|e^{+i\mathbf{H}'\cdot\hat{\mathbf{S}}\delta_t}|g_i\rangle &\approx \exp\left(\langle g_{i+1}|g_i\rangle - \langle g_i|g_i\rangle + i\delta_t\langle g_{i+1}|\mathbf{H}'\cdot\hat{\mathbf{S}}|g_i\rangle\right) \\ &= \exp\!\left(i\delta_t\left(-i\frac{\langle g_{i+1}|g_i\rangle - \langle g_i|g_i\rangle}{\delta_t} + \langle g_{i+1}|\mathbf{H}'\cdot\hat{\mathbf{S}}|g_i\rangle\right)\right),\end{aligned} \quad \text{(A.8)}$$

and for the backward contour, the matrix elements read

$$\begin{aligned}\langle g_i|e^{-i\mathbf{H}'\cdot\hat{\mathbf{S}}\delta_t}|g_{i+1}\rangle &\approx \exp\left(\langle g_i|g_{i+1}\rangle - \langle g_{i+1}|g_{i+1x}\rangle - i\delta_t\langle g_{i+1}|\mathbf{H}'\cdot\hat{\mathbf{S}}|g_i\rangle\right) \\ &= \exp\!\left(i\delta_t\left(-i\frac{\langle g_i|g_{i+1}\rangle - \langle g_{i+1}|g_{i+1}\rangle}{\delta_t} + \langle g_i|\mathbf{H}'\cdot\hat{\mathbf{S}}|g_{i+1}\rangle\right)\right).\end{aligned} \quad \text{(A.9)}$$

In the limit of $N \to \infty$, we then obtain the spin part apparent in the following Keldysh partition function,

$$\mathcal{Z} = \int Dg \prod_{\alpha} \int Dx_{\alpha} \int Dp_{\alpha} \exp\Big[i \oint_K \mathrm{d}t \Big(-i\langle \dot{g}|g\rangle + \langle g|\mathbf{H}\cdot\hat{\mathbf{S}}|g\rangle - \gamma_{\alpha}\langle g|\hat{\mathbf{S}}\cdot\dot{\mathbf{x}}|g\rangle + p_{\alpha}\dot{x}_{\alpha} - \frac{p_{\alpha}^2}{2m} - \frac{m\omega^2}{2}x_{\alpha}^2 \Big)\Big], \quad \text{(A.10)}$$

which is the desired result.

B. Derivation harmonic oscillator part Keldysh partition function

Similar to appendix A we rederive the harmonic oscillator part of Eq. (A.1). The one operator we should consider for the path integral is

$$\hat{1} = \int dx_{j+1} dp_j |x_{j+1}\rangle\langle p_j| e^{ip_j x_{j+1}}, \tag{B.1}$$

and for the trace we use

$$\mathrm{Tr}\{\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}\} = \int \mathrm{d}x\mathrm{d}p \; e^{ipx}\langle p|\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}|x\rangle. \tag{B.2}$$

We plug in Eq. (B.1) N times into the evolution operator. For the matrix elements on the forward contour, we have

$$\langle p_j|\hat{\mathcal{U}_{\delta_t}}|x_j\rangle \approx e^{-ip_j x_j} \exp\left(-i\delta_t\left(\frac{p^2}{2m} + \frac{m\omega^2}{2}x^2\right)\right), \tag{B.3}$$

and for the matrix elements on the backward contour, we have

$$\langle p_j|\hat{\mathcal{U}_{-\delta_t}}|x_j\rangle \approx e^{-ip_j x_j} \exp\left(i\delta_t\left(\frac{p^2}{2m} + \frac{m\omega^2}{2}x^2\right)\right). \tag{B.4}$$

The harmonic oscillator part apparent in Eq. (A.1) then becomes

$$\begin{aligned}\mathcal{Z} = \lim_{N\to\infty}\int \prod_{i=0}^{N} dx_i dp_i dp_i' dx_i' \exp\Bigg[& i\delta_t \sum_{j=0}^{N-1}\left(\frac{p_j x_{j+1} - p_j x_j}{\delta_t} - \frac{p_j^2}{2m} - \frac{m\omega^2}{2}x_j^2\right) \\ & + i\delta_t \sum_{i=0}^{N-1}\left(\frac{p_j x_{j-1} - p_j x_j}{\delta_t} + \frac{p_j^2}{2m} + \frac{m\omega^2}{2}x_j^2\right)\Bigg] \\ = \int DxDp \exp\left[i\oint_K dt\left(p\dot{x} - \frac{p^2}{2m} - \frac{m\omega^2}{2}x^2\right)\right], \end{aligned} \tag{B.5}$$

which is the desired result.

C. Integrating out the momentum and position fields

Now, we integrate out the momentum field in the harmonic oscillator part of Keldysh partition function (Ref. [15] Eq. (2.22))

$$\begin{aligned}\mathcal{Z}_{\rm HO} &= \int DxDp\exp\left[i\oint_K \mathrm{d}t\left(p\dot{x}-\frac{p^2}{2m}\right)\right]\exp\left[-i\oint_K \mathrm{d}t\left(\frac{m\omega^2}{2}x^2\right)\right]\\ &= \int Dx\exp\left[i\oint_K \mathrm{d}t\left(\frac{m\dot{x}^2}{2}\right)\right]\exp\left[-i\oint_K \mathrm{d}t\left(\frac{m\omega^2}{2}x^2\right)\right]\\ &= \int Dx\exp\left[i\oint_K \mathrm{d}t\left(\frac{m\dot{x}^2}{2}-\frac{m\omega^2}{2}x^2\right)\right],\end{aligned} \tag{C.1}$$

from which we read

$$\mathcal{S} = \oint_K \mathrm{d}t\left[-i\langle\dot{g}|g\rangle + BS - \gamma\sum_\alpha Sx_\alpha + \sum_\alpha\frac{m_\alpha}{2}(\dot{x}_\alpha^2-\omega_\alpha^2 x_\alpha^2)\right]. \tag{C.2}$$

Subsequently, we split the fields x_α and S into two components, $x_\alpha^+(t)$ and $x_\alpha^-(t)$, defined on the forward and backward branches of the contour respectively. For convenience, we define

$$\mathcal{S}_{\rm HO} \equiv \oint_K \mathrm{d}t\left[-\gamma\sum_\alpha Sx_\alpha + \sum_\alpha\frac{m_\alpha}{2}(\dot{x}_\alpha^2-\omega_\alpha^2 x_\alpha^2)\right], \tag{C.3}$$

which then becomes

$$\mathcal{S}_{\rm HO} = \int_{-\infty}^{\infty}\mathrm{d}t\left[-\gamma\sum_\alpha(\mathbf{S}^+x_\alpha^+-\mathbf{S}^-x_\alpha^-)+\sum_\alpha\frac{m_\alpha}{2}((\dot{x}_\alpha^+)^2-(\dot{x}_\alpha^-)^2-\omega_\alpha^2(x_\alpha^+)^2+\omega_\alpha^2(x_\alpha^-)^2)\right]. \tag{C.4}$$

The Keldysh rotation is defined as

$$x^{\rm cl}(t) = \frac{1}{2}\left[x^+(t)+x^-(t)\right]; \tag{C.5}$$

$$x^q(t) = \frac{1}{2}\left[x^+(t)-x^-(t)\right], \tag{C.6}$$

and applying this to the action in Eq. (C.4) yields

$$\begin{aligned}\mathcal{S}_{\rm HO} &= \int_{-\infty}^{\infty}\mathrm{d}t\left[-\gamma\sum_\alpha\mathbf{S}\hat{\sigma}_1\mathbf{x}_\alpha+\sum_\alpha 2m_\alpha\left(\dot{x}^{\rm cl}\dot{x}^q-\omega_\alpha^2x^{\rm cl}x^q\right)\right]\\ &= \int_{-\infty}^{\infty}\mathrm{d}t\left[-\gamma\sum_\alpha\mathbf{S}^T\hat{\sigma}_1\mathbf{x}_\alpha+\sum_\alpha 2m_\alpha\left(-x^{\rm cl}\ddot{x}^q-\omega_\alpha^2x^{\rm cl}x^q\right)\right]\\ &\equiv \mathcal{S}_{\rm coupl}+\mathcal{S}_{\rm HO1},\end{aligned} \tag{C.7}$$

where we have used partial integration in the second line. From Ref. [15], we see that

$$\mathcal{S}_{HO1} = \sum_\alpha 2m_\alpha\mathbf{x}^TG_\alpha^{-1}\mathbf{x}, \tag{C.8}$$

where

$$\mathbf{x}\equiv\begin{pmatrix}x^{\rm cl}\\ x^q\end{pmatrix}; \qquad G_\alpha^{-1}=\begin{pmatrix}0 & [G^{-1}]^A\\ [G^{-1}]^R & [G^{-1}]^K\end{pmatrix}. \tag{C.9}$$

The retarded and advanced matrix components are given by

$$\frac{1}{2}[G_\alpha^{-1}]^{R/A} = m_\alpha(i\partial_t \pm i0)^2 - m_\alpha\omega_\alpha^2. \tag{C.10}$$

and in the Fourier representation by

$$G_\alpha^{-1}(\omega) = \frac{m_\alpha}{2}(\omega^2 - \omega_\alpha^2). \tag{C.11}$$

The inverse of (C.9) reads

$$\hat{G}^{\alpha\beta}(t,t') = \begin{pmatrix} G^K(t,t') & G^R(t,t') \\ G^A(t,t') & 0 \end{pmatrix}, \tag{C.12}$$

and in the Fourier representation the advanced and retarded part yield

$$G^{R/A}(\epsilon) = \frac{1}{2}\frac{1}{(\epsilon \pm i0)^2 - \omega_\alpha^2}. \tag{C.13}$$

The fluctuation dissipation theorem [15] implies

$$G^K(\epsilon) = \coth\frac{\epsilon}{2T}\left[G^R(\epsilon) - G^A(\epsilon)\right], \tag{C.14}$$

for the Keldysh part of the Greens function. For the total action, we then write

$$\mathcal{S} = \oint_K \mathrm{d}t \left[-i\langle\dot{g}|g\rangle + BS - \gamma\sum_\alpha \mathbf{S}^T\hat{\sigma}_1\mathbf{x}_\alpha + \sum_\alpha 2m_\alpha \mathbf{x}^T G_\alpha^{-1}\mathbf{x}\right]. \tag{C.15}$$

Integrating out the position fields gives us

$$\begin{aligned} \mathcal{S} &= \oint_K \mathrm{d}t \left[-i\langle\dot{g}|g\rangle + BS + \sum_\alpha \mathbf{S}_i^T G_{ij}^{-1}\mathbf{S}_j\right] \\ &= \oint_K \mathrm{d}t \left[-i\langle\dot{g}|g\rangle + BS + \oint_K \mathrm{d}t'\ \mathbf{S}(t)^T\alpha(t-t')\mathbf{S}(t')\right], \end{aligned} \tag{C.16}$$

with

$$\begin{aligned} \alpha(t-t') &= \begin{pmatrix} 0 & \alpha^A \\ \alpha^R & \alpha^K \end{pmatrix}_{(t-t')} \\ &= \sum_\alpha \frac{\gamma_\alpha^2}{4}\begin{pmatrix} 0 & G_\alpha^A \\ G_\alpha^R & G_\alpha^K \end{pmatrix}_{(t-t')}, \end{aligned} \tag{C.17}$$

which is the desired result.

D. Euler angles

The spin coherent state dependent part of the action reads

$$\mathcal{S}_g = \oint_K \mathrm{d}t[-i\langle\dot{g}|g\rangle + \langle g|\mathbf{H}\cdot\hat{\mathbf{S}}|g\rangle], \tag{D.1}$$

and we want to write this down in the Euler angle representation. We have

$$\begin{aligned}|g(\phi,\theta,\psi)\rangle &\equiv e^{-i\phi\hat{S}_3}e^{-i\theta\hat{S}_2}e^{-i\psi\hat{S}_3}|\uparrow\rangle \\ &= e^{-i\phi\hat{S}_3}e^{-i\theta\hat{S}_2}|\uparrow\rangle e^{-i\psi S},\end{aligned} \tag{D.2}$$

which means that we can ignore the $e^{-i\psi\hat{S}_3}$ in the spin coherent state. For the time derivative of the spin coherent state, we get

$$\langle\dot{g}| = \langle\uparrow|\left[e^{i\theta\hat{S}_2}(i\dot{\phi}\hat{S}_3)e^{i\phi\hat{S}_3} + (i\dot{\theta}\hat{S}_2)e^{i\theta\hat{S}_2}e^{i\phi\hat{S}_3}\right]. \tag{D.3}$$

Carrying out the inner product yields

$$\begin{aligned}\langle\dot{g}|g\rangle &= \langle\uparrow|\left[e^{i\theta\hat{S}_2}(i\dot{\phi}\hat{S}_3)e^{i\phi\hat{S}_3} + (i\dot{\theta}\hat{S}_2)e^{i\theta\hat{S}_2}e^{i\phi\hat{S}_3}\right]\left[e^{-i\phi\hat{S}_3}e^{-i\theta\hat{S}_2}\right]|\uparrow\rangle e^{-i\psi S} \\ &= \langle\uparrow|\left[e^{i\theta\hat{S}_2}(i\dot{\phi}\hat{S}_3)e^{-i\theta\hat{S}_2}\right]|\uparrow\rangle + (i\dot{\theta})\langle\uparrow|\hat{S}_2|\uparrow\rangle \\ &= (i\dot{\phi})\langle\uparrow|\hat{S}_3\cos(-\theta) + \hat{S}_2\sin(-\theta)|\uparrow\rangle \\ &= i\dot{\phi}\cos(\theta)S.\end{aligned} \tag{D.4}$$

Now, we choose a different gauge (see Refs. [23, 24]),

$$\int \mathrm{d}t\left[S\dot{\phi}\cos(\theta)\right]^q = -\int \mathrm{d}t\left[S\dot{\phi}(1-\cos(\theta))\right]^q, \tag{D.5}$$

and this choice has no physical impact on the theory. We obtain the following action

$$\mathcal{S} = \int \mathrm{d}t\left[-S\dot{\phi}(1-\cos\theta) + \mathbf{H}\cdot\mathbf{S}\right]^q - \int \mathrm{d}t\int \mathrm{d}t' S^T(t)\alpha(t-t')S(t'). \tag{D.6}$$

Also, we want to give the other parts in terms of Euler angles. Recall the spin in terms of Euler angles (Fig 2.3). For the forward and backward part of the spin this, we have

$$\mathbf{S}^+ = \begin{pmatrix}\sin\theta_+\cos\phi_+ \\ \sin\theta_+\sin\phi_+ \\ \cos\theta_+\end{pmatrix}, \qquad \mathbf{S}^- = \begin{pmatrix}\sin\theta_-\cos\phi_- \\ \sin\theta_-\sin\phi_- \\ \cos\theta_-\end{pmatrix}. \tag{D.7}$$

The classical and quantum components of the Euler angles are, similarly to the spin, defined by

$$\theta_\pm \equiv \theta_c \pm \frac{\theta_q}{2}; \tag{D.8}$$

$$\phi_\pm \equiv \phi_c \pm \frac{\phi_q}{2}, \tag{D.9}$$

and using trigonometric identities we obtain for the quantum components of the spin that

$$S_x^q = 2\cos\theta_c\sin\frac{\theta_q}{2}\cos\phi_c\cos\frac{\phi_q}{2} - 2\sin\theta_c\cos\frac{\theta_q}{2}\sin\phi_c\sin\frac{\phi_q}{2}; \tag{D.10}$$

$$S_y^q = 2\sin\theta_c\cos\frac{\theta_q}{2}\cos\phi_c\sin\frac{\phi_q}{2} + 2\cos\theta_c\sin\frac{\theta_q}{2}\sin\phi_c\cos\frac{\phi_q}{2}; \tag{D.11}$$

$$S_z^q = -2\sin\theta_c\sin\frac{\theta_q}{2}. \tag{D.12}$$

For the classical part of the spin, we obtain

$$S_x^c = \sin\theta_c \cos\frac{\theta_q}{2}\cos\phi_c\cos\frac{\phi_q}{2} - \cos\theta_c\sin\frac{\theta_q}{2}\sin\phi_c\sin\frac{\phi_q}{2}; \tag{D.13}$$
$$S_y^c = \sin\theta_c \cos\frac{\theta_q}{2}\sin\phi_c\cos\frac{\phi_q}{2} + \cos\theta_c\sin\frac{\theta_q}{2}\cos\phi_c\sin\frac{\phi_q}{2}; \tag{D.14}$$
$$S_z^c = \cos\theta_c\cos\frac{\theta_q}{2}. \tag{D.15}$$

Rewriting the first part of the action (D.6) in terms of the classical components ϕ_c, ϕ_q, θ_c and θ_q yields,

$$\int \mathrm{d}t\left[-S\dot{\phi}(1-\cos(\theta))\right]^q = \int_{-\infty}^{\infty}\mathrm{d}tS\left[-2\dot{\phi}_c\sin\theta_c\sin\frac{\theta_q}{2} + \phi_q\sin\theta_c\cos\frac{\theta_q}{2}\dot{\theta}_c + \phi_q\cos\theta_c\sin\frac{\theta_q}{2}\frac{\dot{\theta}_q}{2}\right], \tag{D.16}$$

where we have used trigonometric identities partial integration. We are only interested in terms linear to the quantum components. Hence,

$$\int \mathrm{d}t\left[-S\dot{\phi}(1-\cos(\theta))\right]^q = \int_{-\infty}^{\infty}\mathrm{d}tS\left[-\dot{\phi}_c\sin\theta_c\theta_q + \phi_q\sin\theta_c\dot{\theta}_c\right], \tag{D.17}$$

where we have used $\cos\frac{\theta_q}{2} \approx 1$, $\cos\frac{\phi_q}{2} \approx 1$, $\sin\frac{\theta_q}{2} \approx \frac{\theta_q}{2}$ and $\sin\frac{\phi_q}{2} \approx \frac{\theta_q}{2}$, because we disregard higher order terms. Similarly, the second part of (D.6) becomes

$$\begin{aligned}\int \mathrm{d}t[\mathbf{H}\cdot\mathbf{S}]^q \approx S\int_{-\infty}^{\infty}\mathrm{d}t\Big[&H_x\left\{\theta_q\cos\theta_c\cos\phi_c - \phi_q\sin\theta_c\sin\phi_c\right\}\\ &+ H_y\left\{\phi_q\sin\theta_c\cos\phi_c + \theta_q\cos\theta_c\sin\phi_c\right\}\\ &- H_z\theta_q\sin\theta_c\Big],\end{aligned} \tag{D.18}$$

up to linear order in quantum components. Before moving on to the third term in (D.6), we first note that expanding in terms of quantum components and only retaining linear contributions of those, has the following implications for the quantum part of the spin,

$$S_x^q \approx \theta_q\cos\theta_c\cos\phi_c - \phi_q\sin\theta_c\sin\phi_c; \tag{D.19}$$
$$S_y^q \approx \phi_q\sin\theta_c\cos\phi_c + \theta_q\cos\theta_c\sin\phi_c; \tag{D.20}$$
$$S_z^q \approx -\theta_q\sin\theta_c, \tag{D.21}$$

and for the classical part of the spin,

$$S_x^c = \sin\theta_c\cos\phi_c; \tag{D.22}$$
$$S_y^c = \sin\theta_c\sin\phi_c; \tag{D.23}$$
$$S_z^c = \cos\theta_c. \tag{D.24}$$

Filling this in each term in the inner product (in x, y, z-space) of the third term in (D.6) gives us

$$S_x^q(t)\int_{-\infty}^{\infty}\mathrm{d}t'\;\alpha_{\mathrm{diss}}(t-t')S_x^c(t') = (\theta_q\cos\theta_c\cos\phi_c - \phi_q\sin\theta_c\sin\phi_c)\int_{-\infty}^{\infty}\mathrm{d}t'\;\alpha_{\mathrm{diss}}(t-t')\sin\theta_c'\cos\phi_c'; \tag{D.25}$$

$$S_y^q(t)\int_{-\infty}^{\infty}\mathrm{d}t'\;\alpha_{\mathrm{diss}}(t-t')S_y^c(t') = (\phi_q\sin\theta_c\cos\phi_c + \theta_q\cos\theta_c\sin\phi_c)\int_{-\infty}^{\infty}\mathrm{d}t'\;\alpha_{\mathrm{diss}}(t-t')\sin\theta_c'\sin\phi_c'; \tag{D.26}$$

$$S_z^q(t)\int_{-\infty}^{\infty}\mathrm{d}t'\;\alpha_{\mathrm{diss}}(t-t')S_z^c(t') = (-\theta_q\sin\theta_c)\int_{-\infty}^{\infty}\mathrm{d}t'\;\alpha_{\mathrm{diss}}(t-t')\cos\theta_c'. \tag{D.27}$$

The integral does not contain quantum components. As a result, we can rewrite the action such that we can read off the equations of motion for ϕ_c and θ_c:

$$\begin{aligned}\mathcal{S}_{\theta_q} = \int_{-\infty}^{\infty} \mathrm{d}t\theta_q \Bigg[& -\dot{\phi}_c \sin\theta_c + H_x \cos\theta_c \cos\phi_c + H_y \cos\theta_c \sin\phi_c - H_z \sin\theta_c \\ & + \cos\theta_c \cos\phi_c \int_{-\infty}^{\infty} \mathrm{d}t'\ \alpha_{\mathrm{diss}}(t-t') S_x^c(t') \\ & + \cos\theta_c \sin\phi_c \int_{-\infty}^{\infty} \mathrm{d}t'\ \alpha_{\mathrm{diss}}(t-t') S_y^c(t') \\ & - \sin\theta_c \int_{-\infty}^{\infty} \mathrm{d}t'\ \alpha_{\mathrm{diss}}(t-t') S_z^c(t') \Bigg],\end{aligned} \tag{D.28}$$

and

$$\begin{aligned}\mathcal{S}_{\phi_q} = \int_{-\infty}^{\infty} \mathrm{d}t\phi_q \Bigg[& \dot{\theta}_c \sin\theta_c - H_x \sin\theta_c \sin\phi_c + H_y \sin\theta_c \cos\phi_c \\ & - \sin\theta_c \sin\phi_c \int_{-\infty}^{\infty} \mathrm{d}t'\ \alpha_{\mathrm{diss}}(t-t') S_x^c(t') \\ & + \sin\theta_c \cos\phi_c \int_{-\infty}^{\infty} \mathrm{d}t'\ \alpha_{\mathrm{diss}}(t-t') S_y^c(t') \Bigg],\end{aligned} \tag{D.29}$$

which are the desired results. We present the equation of motion in the main text.

E. Magnetoelastic coupling

The magnetoelastic coupling term in the Hamiltonian reads

$$H_{\mathrm{c}} = \frac{1}{S^2} \sum_i \sum_{\alpha\beta} B_{\alpha\beta} \hat{S}_i^\alpha \hat{S}_i^\beta X_i^{\alpha\beta}, \tag{E.1}$$

with S the length of the spin, i the index of the lattice site, α, β indexing the x, y or z-direction, $B_{\alpha\beta}$ the magnetoelastic coupling constant and $X_i^{\alpha\beta}$ the strain tensor,

$$X_i^{\alpha\beta} = \frac{1}{2} \left[\frac{\partial X_\alpha(\mathbf{r})}{\partial r_\beta} + \frac{\partial X_\beta(\mathbf{r})}{\partial r_\alpha} \right]_{\mathbf{r}=\mathbf{R}_i}. \tag{E.2}$$

After bosonizing the spin operators into magnon annihalation and creation operators ($b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$) using the Holstein-Primakoff transformation and after going to momentum space, we obtain the following Hamiltonian [19],

$$\begin{aligned} H_{\mathrm{c}} = & \sum_{\mathbf{k}\lambda} \left[\Gamma_{\mathbf{k}\lambda} b_{-\mathbf{k}} X_{\mathbf{k}\lambda} + \Gamma^*_{-\mathbf{k}\lambda} b^\dagger_{\mathbf{k}} X_{\mathbf{k}\lambda} \right] + \frac{1}{\sqrt{N}} \sum_{\mathbf{k},\mathbf{q},\mathbf{q}'} \delta_{\mathbf{q}-\mathbf{q}'-q,0} \sum_\lambda \Gamma^{\mathrm{an}}_{\mathbf{q}\mathbf{q}',\lambda} b^\dagger_{\mathbf{q}} b_{\mathbf{q}'} X_{\mathbf{k}\lambda} \\ & + \frac{1}{\sqrt{N}} \sum_{\mathbf{k},\mathbf{q},\mathbf{q}'} \delta_{\mathbf{q}+\mathbf{q}'+q,0} \sum_\lambda \Gamma^{bb}_{\mathbf{q}\mathbf{q}',\lambda} b_{\mathbf{q}} b_{\mathbf{q}'} X_{\mathbf{k}\lambda} + \frac{1}{\sqrt{N}} \sum_{\mathbf{k},\mathbf{q},\mathbf{q}'} \delta_{\mathbf{q}+\mathbf{q}'-q,0} \sum_\lambda \Gamma^{\bar{b}\bar{b}}_{\mathbf{q}\mathbf{q}',\lambda} b^\dagger_{\mathbf{q}} b^\dagger_{\mathbf{q}'} X_{\mathbf{k}\lambda}, \end{aligned} \tag{E.3}$$

where the interaction vertices are given by

$$\Gamma_{\mathbf{k}\lambda} = \frac{B_\perp}{\sqrt{2S}} \left[i k_z e^x_{\mathbf{k}\lambda} + k_z e^y_{\mathbf{k}\lambda} + (i k_x + k_y) e^z_{\mathbf{k}\lambda} \right]; \tag{E.4}$$

$$\Gamma^{\mathrm{an}}_{\mathbf{q}\mathbf{q}',\lambda} = U_{\mathbf{q}-\mathbf{q}',\lambda}; \tag{E.5}$$

$$\Gamma^{bb}_{\mathbf{q}\mathbf{q}',\lambda} = V_{-\mathbf{q}-\mathbf{q}',\lambda}; \tag{E.6}$$

$$\Gamma^{\bar{b}\bar{b}}_{\mathbf{q}\mathbf{q}',\lambda} = V^*_{-\mathbf{q}-\mathbf{q}',\lambda}, \tag{E.7}$$

and

$$U_{\mathbf{k},\lambda} = \frac{i B_\parallel}{S} \left[k_x e^x_{\mathbf{k}\lambda} + k_y e^y_{\mathbf{k}\lambda} - 2 k_z e^z_{\mathbf{k}\lambda} \right]; \tag{E.8}$$

$$V_{\mathbf{k},\lambda} = \frac{i B_\parallel}{S} \left[k_x e^x_{\mathbf{k}\lambda} - k_y e^y_{\mathbf{k}\lambda} \right] + \frac{B_\perp}{S} \left[k_y e^x_{\mathbf{k}\lambda} + k_x e^y_{\mathbf{k}\lambda} \right]. \tag{E.9}$$

The scattering processes described by the Hamiltonian in Eq. (E.3) are depicted in Fig. E.1. Note that the interaction should obey the conservation of momentum.

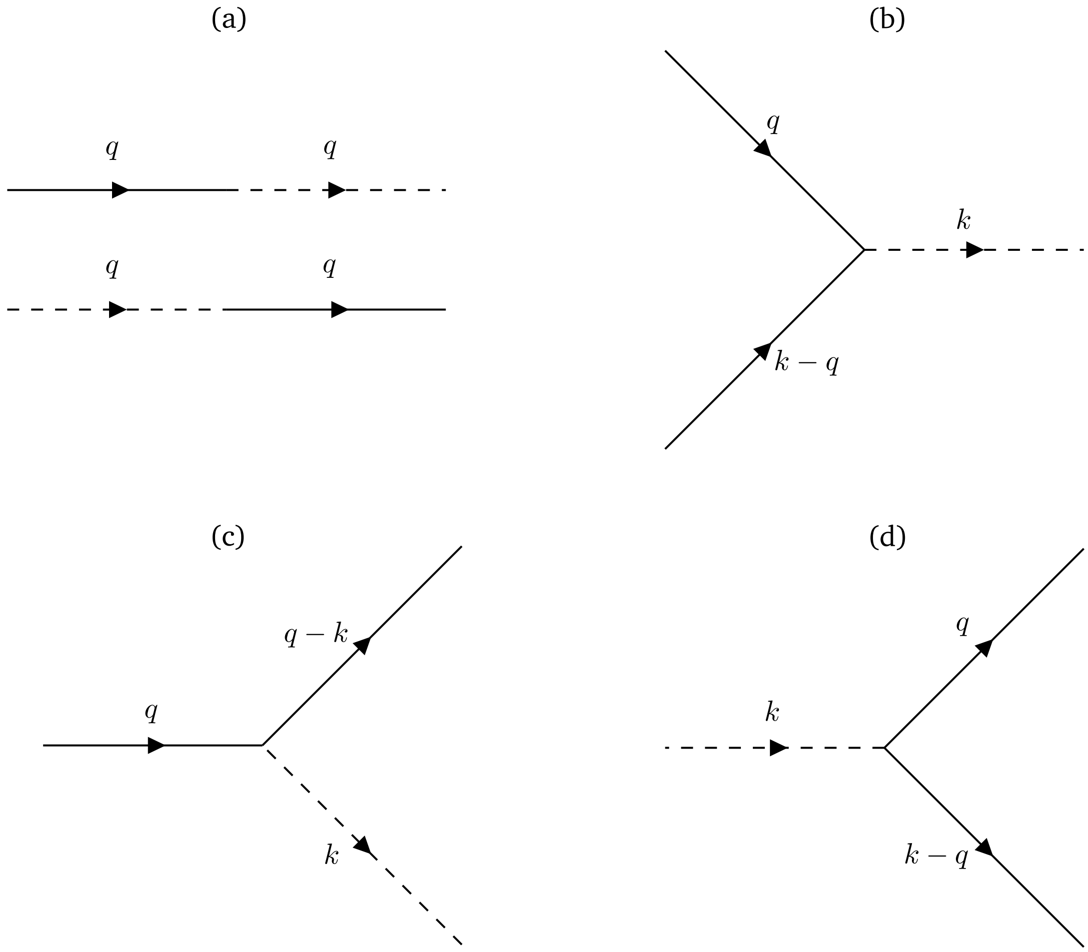

Figure E.1: *Feynman diagrams depicting interactions between magnons (solid lines) and phonons (dashed lines) coming from the Hamiltonian in Eq.* (E.3) *These interactions obey conservation of momentum.*

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