Master's Thesis

Bulk gravity propagators from boundary Feynman graphs

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Abstract

The AdS/CFT correspondence relates the correlation functions in a d-dimensional conformal field theory (CFT) to the propagators of corresponding bulk fields in an d+1 dimensional Anti-de-Sitter (AdS) space-time on the boundary of which the CFT is placed. This thesis is about generalizing this correspondence to perturbative quantum field theories (QFTs) and deriving the corresponding bulk propagators, in an a priori unknown d+1 dimensional geometry, directly from the Feynman diagrams of the QFT. We show in detail how this can be done for the two-point functions of composite operators of massive scalar fields.

Contents

Chapter 1

Introduction

The goal of this thesis is to explore the connections between gauge theories or quantum field theory in general and string theory. This connection has a long history that goes back to seminal papers of Veneziano [1], [2], [3] and [4]. String theory first emerged as an effective description of the strong force between quarks. It has been known that quarks in QCD confine, that is, the potential energy between a quark and an anti-quark grows as a function of the distance between them. This can be pictured as a gluon field confined in a thin flux tube between them. This led people to think that this flux tube can be modeled as a string.

First, Veneziano [1] suggested a dual resonance model that captured the Regge trajectories experimentally observed in hadrons, which tell us that the mass of hadrons $M^2 \sim J$ for large angular momentum J. This behavior can immediately be seen from a string like object connecting the quarks in the hadrons. 't Hooft realized that high loop Feynman diagrams that involve a test quark and an anti-quark, with the gluons running between them, can be represented as string worldsheets. This will be an important part of our motivation in this thesis.

With this understanding Nambu [4], Polyakov [3] and others wrote down actions for these strings which eventually led to a field of theoretical high energy physics known as "string theory". An important reason for the growth of string theory was the realization of Scherk and Schwarz that the lowest lying excitation of closed strings was a graviton, so then string theory became a candidate for the theory of quantum gravity. At the same time QCD arose as the fundamental theory of strong interactions and the connection between string theory and QCD has been dropped for about two decades.

However this connection between QFTs and strings came back with the seminal papers of Maldacena [5], Gubser, Klebanov and Polyakov [6] and Witten [7] as an exact duality between a special type of QFT, i.e. $\mathcal{N} = 4$ super Yang-Mills in 4D and IIB string theory on the 10D manifold of $AdS_5 \times S_5$. This is qualitatively different than the proposed original connection between gauge theory and string theory in the sense that strings are now propagating in a different space time than the flux tubes. One can Kaluza-Klein reduce the theory on S_5 to obtain string propagation only in AdS_5 which is a 5D manifold that can be visualized as spanned by the 4 Minkowski directions supplemented by a holographic direction z_0.

The original argument behind this connection was based on extended objects in string theory called D-branes, see for example the review [8]. However if this connection is precise then we should, in principle, be able to derive it directly from the QFT generating function. It becomes important then, to understand how this additional holographic dimension can emerge from the Feynman diagrams. To understand this is one of the main motivations of this thesis.

This approach is based on a series of papers by Gopakumar [9–11] where he realized in free field theories that the so-called Schwinger parameters that are used to express

Feynman diagrams (see Chapter 4) give rise to this holographic direction. More precisely the zero-mode of the collection of all Schwinger parameters of a Feynman diagram is directly related to the holographic direction. A manifestation of this connection is the fact that n-point functions in the free QFT can be mapped to Witten diagrams in AdS_5 which describe propagation and interaction of bulk fields that correspond to the QFT operators [7].

In this thesis we will consider generalization of Gopakumar's approach to interacting QFTs in the perturbative limit and derive the corresponding bulk propagators, in an a priori unknown d+1 dimensional geometry. In particular we will focus on the 2-point function of composite operators in a bosonic massive QFT with some polynomial type interaction.

The organization of this thesis is as follows. In Chapter 2 we lay out some of the machinery of AdS/CFT; Chapter 3 contains the expression of multi-point functions in terms of Schwinger parameters; Chapter 4 is devoted to the derivation of the bulk propagators for a free bosonic theory; Chapter 5 contains a generalization to the perturbative case for the two-point function of composite operators of massive scalar fields.

Chapter 2

The Bulk-Boundary correspondence

2.1 General idea of AdS/CFT

The old idea of a connection between gauge theories and strings has been materialised in 1997 by the seminal paper of Maldacena [5] where he conjectured that the $\mathcal{N} = 4$ SYM is in one-to-one correspondence with IIB string theory on $AdS_5 \times S_5$. This is an example of a duality between bulk and boundary in the sense that the string theory lives in the AdS_5 bulk and the QFT lives on the boundary of AdS_5 which is isomorphic to a 4D Minkowski space. This paper was followed by papers of Gubser, Klebanov and Polyakov [6] and Witten [7] who laid down the precise rules as to how to compute n-point functions in this specific QFT from propagation of excitations in the bulk.

The conjecture was based on two distinct descriptions of D3-branes in IIB string theory, which are 3+1 dimensional extended surfaces where the open string boundary conditions are imposed. Just as the open strings are dynamical objects, so are D-branes, fluctuations of which can be described quantum mechanically. It turns out that these quantum fluctuations can be described by a special type of QFT called $\mathcal{N} = 4$ SYM which is a maximally supersymmetric gauge theory in 3+1 dimensions with gauge group $U(N)$. More precisely, this is the QFT description of N D3-branes on top of each other. On the other hand these D-branes are massive due to their tension, hence they curve the background around them, according to the rules of GR. This stems from the fact that these D-branes can also absorb and emit closed strings and the lowest lying excitation of the closed string is the graviton. This fact gives rise to a description of D-branes in terms of the background they curve around them. One can show that this background is described in terms of the so-called Black-brane metric

$$ds^2 = \left(1 + \frac{R^2}{y^4}\right)^{-\frac{1}{2}} \eta_{ij} dx^i dx^j + \left(1 + \frac{R^4}{y^4}\right)^{\frac{1}{2}} (dy^2 + y^2 d\Omega_5^2)$$

that is a 10D background with the isometry of $SO(3,1)$ coming from dimensions transverse to the D3-branes and $SO(6)$ coming from the dimensions perpendicular to the D3-branes.

Another important part of the AdS/CFT dictionary is the relation between composite operators in the gauge theory and the bulk fields on the AdS space. In string theory the bulk fields, which propagate in the target space where the string is embedded (here it is $AdS_5 \times S_5$, as discussed in the next section) is equivalently described as excitations on the worldsheet given by insertions of string vertex operators on the worldsheet, see e.g. Polchinski [12]. For example the stress tensor operator of the gauge theory $T_{\mu\nu}$ corresponds to excitations of the metric in the bulk, the gluon field strength operator $\mathrm{Tr}\, F^2$ corresponds to the dilaton, etc. The full list can be obtained by matching the quantum numbers of the operators that can be classified according to the global symmetry group of the QFT which corresponds to the isometry group of the corresponding $AdS_5 \times S_5$

spacetime. This isometry group turns out to be $SO(4,2) \times SO(6)$ for the AdS_5 and S_5 factors respectively. This precisely corresponds to the global symmetry group of the $\mathcal{N} = 4$ SYM theory where the first factor is the conformal symmetry group in 4D — $\mathcal{N} = 4$ SYM is a conformal theory — and the second factor comes from the so-called R-symmetry group of the theory which rotates the supercharges of $\mathcal{N} = 4$ SYM among themselves. The full list of correspondence between bulk fields and boundary operators can be found in [13].

2.2 Bulk fields

The black brane-metric simplifies when considered in the limit near the D3-branes, or the throat section of the geometry. In this limiting case the geometry becomes $AdS_5 \times S_5$. In Euclidean signature the metric of AdS_{d+1} is given by, with $z_0 \geq 0$,

$$ds^2 = (1/z_0^2)(dz_0^2 + \sum_i dz_i^2). \tag{2.1}$$

This coordinate system is called the Poincare patch for the reason that the translation symmetry on the z_i directions are made manifest. See the review [8] for the AdS metric written in different coordinate systems including the global patch.

Solutions to the classical bulk field equations of motion can be written in terms of the Green's function, or propagator. For example a scalar field with mass m satisfies the equation of motion

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) - m^2\phi = 0,$$

which in terms of the metric (2.1) is given by

$$z_0^{d+1}\frac{\partial}{\partial z_0}\left[z_0^{-d+1}\frac{\partial}{\partial z_0}\phi(z_0, \boldsymbol{z})\right] + z_0^2\frac{\partial}{\partial \boldsymbol{z}^2}\phi(z_0, \boldsymbol{z}) - m^2\phi(z_0, \boldsymbol{z}) = 0.$$

The Green's function is defined as a solution with boundary condition

$$K_\Delta(z_0, \boldsymbol{z} - \boldsymbol{x}) \xrightarrow[z_0 \to 0]{} \delta(\boldsymbol{z} - \boldsymbol{x})z_0^{d-\Delta}$$

and is given by the expression

$$K_\Delta(z_0, \boldsymbol{z} - \boldsymbol{x}) = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - \frac{d}{2})}\left(\frac{z_0}{z_0^2 + (\boldsymbol{z} - \boldsymbol{x})^2}\right)^\Delta, \tag{2.2}$$

where Δ will be identified with the scaling dimension of the CFT operator corresponding to ϕ. The general solution for the boundary condition $h(\boldsymbol{x})$ can be written as

$$\phi(z_0, \boldsymbol{z}) = \int d^d\boldsymbol{x} K_\Delta(z_0, \boldsymbol{z} - \boldsymbol{x})h(\boldsymbol{x}).$$

This allows us to represent the propagation of the bulk field as a propagator in the AdS space connected to the boundary. These bulk propagators can be joined using vertices in the bulk which would come from interactions terms of the form ϕ^n with integer $n \geq 2$.

One important fact for us will be the relation between the mass of scalar fields in AdS_5 and the scale dimension Δ of the corresponding composite operator $(mR)^2 = \Delta(\Delta - d)$. This equation follows from expanding the equation of motion above near the

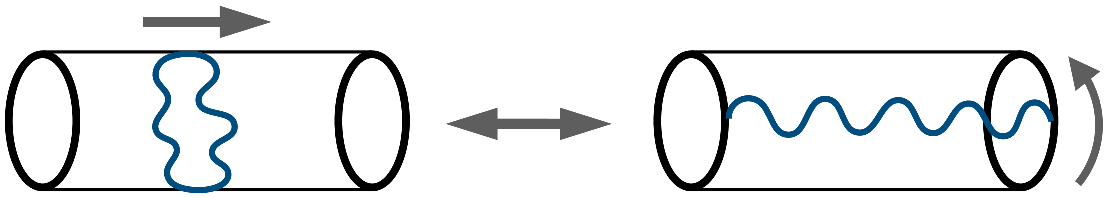

FIGURE 2.1: Equivalence between one-loop diagram of an open string and tree level propagation of a closed string. Taken from Gallegos et al. (2022) [14]

boundary. Near the boundary $z_0 \to 0$ the solution behaves as

$$\phi(z_0, z) \to z_0^{d-\Delta} h(z), \tag{2.3}$$

where Δ will be identified with the scaling dimension of the corresponding CFT operator, see end of section 2.4.

2.3 Large N limit

In the low energy limit [8], one can argue that the bulk and boundary descriptions decouple from each other and they both become equivalent but distinct descriptions of the same quantum mechanical system. On one hand one has the open string description as a QFT, i.e. the $N = 4$ SYM theory in 3+1 dimensions. One the other hand there is the closed string description given by string theory on the black brane background given above, which, in this low energy limit becomes $AdS_5 \times S_5$.

The essence of this correspondence can be understood as an equivalence between the open and closed string descriptions of the same physics processes. As an example think of a closed string that is propagating in d+1 dimensions. Its world-sheet will trace out a cylinder in this d+1 dimensional space time. But the same process can also be described by the partition function of an open string embedded in d-dimensions with one dimension compactified. This equivalence is visualized in Figure 2.1.

It is important to identify the parameters on the two sides of the correspondence. This is usually done in the 't Hooft limit of the gauge theory where one sends the rank of the gauge group, N, to infinity and the Yang-Mills coupling g_{YM} to zero in a correlated manner such that $g_{YM}^2 N = \lambda$ where λ is finite and called the 't Hooft coupling [2]. So there are two independent parameters on the gauge theory side: the 't Hooft coupling λ and the rank of the gauge group N. The first QFT parameter corresponds to the curvature radius of the corresponding $AdS_5 \times S_5$ background in string length unit ℓ_s, that is $\lambda = (R/\ell_s)^4$. In particular when the curvature radius is large, hence the string theory is effectively described by semi-classical gravity, this corresponds to the strong coupling limit of the gauge theory with large λ.

In the free limit in contrast we have $\lambda \to 0$ and can ignore interactions between strings, effectively considering one string with modes. For perturbative QFT we take $\lambda < 1$. Even then, the perturbative expansion is not supposed to converge because, as we know from QFT, the number of Feynman graphs that contribute to any order λ^n in this expansion grows factorially with n. But, we will nevertheless use this perturbative expansion in a formal sense in this thesis, by assuming that it can be defined through e.g. by Borel summation.

The second parameter, N corresponds to the string coupling constant on the closed string side. More precisely the relation is $g_s = g_{YM}^2$. Rewriting this as above one has $g_s = \lambda/N$. That is for finite λ small string coupling corresponds to large N. Put different, the 't Hooft large N limit of gauge theory maps to perturbative string regime where the interactions between closed strings can be treated perturbatively. This means that the string amplitudes can be described in the genus expansion and this corresponds to the genus expansion of the gauge theory Feynman diagrams in the 't Hooft limit [2].

2.4 The GKPW formula

In the duality the QFT is characterized by the n-point functions

$$G(\{\boldsymbol{x}_n\}) = \langle \prod_{i=1}^{n} O_i(\boldsymbol{x}_i) \rangle_{conn},$$

where the O_i are general products of fields. The bulk theory is characterized by the wave amplitudes of bulk fields ϕ_i. It is natural to formulate the correspondence in terms of the generating functions of those characteristic observables. For the boundary theory

$$\frac{\delta}{\delta h(\boldsymbol{x}_i)} Z_{QFT}[h]\Big|_{h=0} \equiv \langle O_i(\boldsymbol{x}_i) \rangle,$$

where h is the source for the operator O_i. For the bulk Z_{Grav} generates expectation values of wave amplitudes. The generator is a functional of the boundary value h_i of the bulk field ϕ_i, in the sense that near the boundary $\phi(z) \to z_0^{d-\Delta} h(\boldsymbol{x})$. Then the field boundary values h_i are coupled to wave amplitudes in the bulk action in the bulk generator

$$Z_{Grav}[h] \equiv \int^{\phi_i \to h_i} \left(\prod_i D\phi_i \right) e^{iS[\phi]}.$$

When taking the large N limit the bulk fields are classical and wave amplitudes are generated by

$$Z_{Grav}[h] = e^{iS[\{\phi_i^* \to h_i\}]}, \tag{2.4}$$

where ϕ^* is a saddle point of the bulk action S, under the condition that $\phi_i^* \to h_i$ on the boundary. The boundary condition fully determines the bulk field, but we will not prove this here. The conjecture relating the theories is

$$Z_{QFT}[h] = Z_{Grav}[h], \tag{2.5}$$

as first formulated by Gubser-Klebanov-Polyakov and Witten[6][7]. What boundary operators the generators h_i couple to in Z_{QFT} is determined by matching the symmetries on both sides, see e.g. [15].

We can connect the scaling dimension of QFT operators to the mass of bulk fields, by considering the dilatation symmetry of AdS spacetime. Perturbations defined on such a background must transform under this symmetry group. In order for the solution (2.3) from the perturbation to be invariant, the source term h must scale as $h \to \lambda^{\Delta-d} h$. An external operator coupled to the source term h therefore has a scaling dimension Δ, that is $O(x) \to \lambda^{-\Delta} O(\lambda x)$.

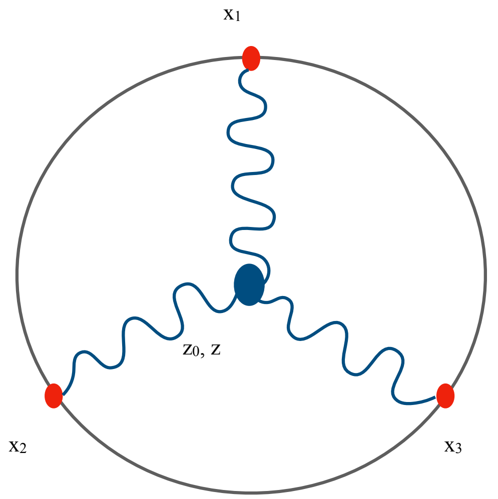

FIGURE 2.2: Witten diagram that corresponds to the three-point function on the boundary theory. Red dots represent insertion of boundary sources which correspond to the boundary values of the bulk fields. The vertex in the bulk comes from a cubic interaction term in the gravity action. Wavy lines represent bulk-to-boundary propagators described above.

2.5 Multi-point functions

QFT multi-point functions can be expressed in terms of the bulk fields and their boundary conditions. From the generating functional

$$Z_{QFT}[\{h_i(x)\}] = \langle e^{i\sum_i \int dx h_i(x) O_i(x)} \rangle_{QFT},$$

and using (2.5) and (2.4) it is easy to see n-point functions are given by

$$\langle \prod_i O_i(\boldsymbol{x}_i) \rangle = i \left[\prod_i \frac{\delta}{\delta h_i(x)} \right] S_{Grav}[\{\phi_i^* \to h_i\}].$$

In terms of the Green's function (2.2), the n-point function is given by

$$\langle \prod_i O_i(\boldsymbol{x}_i) \rangle = g_3 \int d^d \boldsymbol{z} \int_0^\infty \frac{dz_0}{z_0^{d+1}} \prod_{i=1}^{3} K(\boldsymbol{z} - \boldsymbol{x}_i, z_0), \tag{2.6}$$

where g_3 is the cubic interaction strength of the bulk action. This can be represented in terms of Witten diagrams which are essentially Feynman diagrams for propagation and interaction of the bulk fields inside the AdS space. For example the three-point function on the boundary corresponds to the Witten diagram shown in Figure 2.2.

Chapter 3

Schwinger parametrization of QFT correlation functions

3.1 Perturbative expansion

We will mainly be interested in the combinatorics of fields whose symmetry transformations are given by one of the classical groups. For the sake of simplicity this group is taken to be $O(N)$, but these results are more generally applicable. Let Φ be a $N \times N$ real matrix and let the theory be invariant under $O(N)$ transformations of Φ. A general gauge invariant theory is given by the action

$$S = N \int d^d \boldsymbol{x} \, \mathrm{Tr} \left(-\frac{1}{2}(\partial\Phi)^2 - \frac{M^2}{2}\Phi^2 + \sum_h \frac{\lambda_h}{h!}\Phi^h \right),$$

which was chosen such that we will get a non-trivial 1/N expansion. Vertices are proportional to N, propagators to 1/N, see e.g. [16]. We will be calculating the gauge-invariant, field index independent quantities

$$\Omega(\{\boldsymbol{x}_n\}) = \langle \prod_{i=1}^{n} \mathrm{Tr}\, \Phi^J(\boldsymbol{x}_i) \rangle_{conn}.$$

In a diagrammatic expansion of $\Omega(\{\boldsymbol{x}_n\})$, one can use a double line notation with an index label for each line and such that the sum over the index of each line is factored out. Each diagram is proportional to N^{L-E+V}, where L, E and V are the number of closed index-lines, edges, and vertices respectively. It is given by

$$\Omega(k) = \sum_{diagrams} N^{L-E+V} \frac{1}{\epsilon_{sym}} \Omega_F(k),$$

where the sum is performed over all diagrams F in the double line notation, including non-planar graphs, but $\Omega_F(k)$ does not depend on the double line structure of the diagram F.

3.2 Schwinger parametrization

Let the numbers $r \in 1, \cdots, I$ and $b \in 1, \cdots, n_e$ be associated with respectively the edges and external vertices of F. Let a and k denote $\{a_r\}$ and $\{k_b\}$ respectively. The trees $\{T_1\}$ associated with the diagram F are defined as those diagrams with as many propagators removed from F, such that the diagram contains no loops, but is still connected. The 2-trees $\{T_2\}$ are defined as the diagrams constructed from a tree with one extra propagator removed. In terms of the Schwinger parameters a, Ω_F is given by

$$\Omega_F(k) = \delta^{(d)}\left(\sum_b k_b\right)\int_0^\infty \left(\prod_r da_r\right) U(a)^{-d/2} \exp\left[-(\sum_r a_r m_r^2 + P(a;k))\right] S(a;k). \tag{3.1}$$

Here m_r are mass terms generically defined on each edge of the graph, but in the rest of the thesis we will set $m_r = m$ which then corresponds to the bare mass of the scalar in the QFT. The Symanzik polynomials U and A are given by

$$U(a) = \sum_{T_1} \prod_{r \notin T_1} a_r,$$

$$A(a;k) = U(a)P(a;k) = \sum_{T_2}\left(\prod_{r\notin T_2} a_r\right)\left(\sum_{b\in \mathcal{I}_{T_2}} k_b\right)^2.$$

Here $\mathcal{I}$ denotes either of the connected parts of a 2-tree. We have $S(a;k) \propto \lambda_h^{2(l+1-J)/(h-2)}$, where l is the number of loops of the diagram. The factor $S(a,k)$ in general involves other factors for theories that involve fermions and/or vector fields [17][14]. In this thesis, we will assume a bosonic QFT.

Chapter 4

Gopakumar's computation for free QFTs

The QFT 3-point function of single trace operators $\mathrm{Tr}\,\Phi^2$ of a free massless theory can be written in the form (2.6), the derivation of which was given by Gopakumar [9, 10]. This thesis will follow the same method, for which we start with the Schwinger parametrization

$$\Omega_F(k_1,k_2,k_3) = \Omega(k_1,k_2,k_3) = \delta^{(d)}\left(\sum_b^3 k_b\right)\int_0^\infty (\prod_r^3 da_r)U(a)^{-d/2}\exp\{-P(a,k)\},$$

where irrelevant prefactors are not shown, as is the case in the remainder of this section. We make the reparametrization $\sigma_r = 1/a_r$ along with

$$\hat{U}(\sigma) = (\prod_r \sigma_r)U(\{1/\sigma_r\}),$$

$$\hat{P}(\sigma;k) = P(\{1/\sigma_r\};k),$$

such that

$$\Omega_F(k_1,k_2,k_3) = \delta^{(d)}\left(\sum_b^3 k_b\right)\int_0^\infty (\prod_r^3 d\sigma_r\sigma_r^{\frac{d}{2}-2})\hat{U}(\sigma)^{-d/2}\exp\{-\hat{P}(\sigma;k)\}. \tag{4.1}$$

The polynomials $\hat{U}$ and $\hat{P}$ for the diagram under consideration are given by

$$\hat{U}(\sigma) = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1,$$

$$\hat{P}(\sigma;k) = \frac{1}{\hat{U}(\sigma)}(\sigma_1 k_1^2 + \sigma_2 k_2^2 + \sigma_3 k_3^2).$$

The correspondence between electrical circuits and the Schwinger parametrization provides motivation to redefine the integration variables σ with a trick called the "star-triangle duality". In this correspondence the new variables ρ would be given by the inductances along the three prongs of a star-shaped circuit, replacing the resistances in the triangle. The new variables ρ_r are defined by

$$\frac{1}{\rho_i} \equiv \frac{\sigma_i}{\hat{U}(\sigma)},$$

such that in terms of ρ

$$\sigma_i = S_i(\rho) \equiv \frac{\prod_r \rho_r}{(\sum_r \rho_r)\rho_i}; \qquad \det\left(\frac{\partial S}{\partial \rho}\right) = \frac{\prod_r \rho_r}{(\sum_r \rho_r)^3};$$
$$\hat{U}(S(\rho)) = \frac{\prod_r \rho_r}{\sum_r \rho_r}; \qquad \hat{P}(S(\rho)) = \sum_r \frac{k_r^2}{\rho_r}.$$

With this reparametrization, for (4.1) we find

$$\Omega_F(k_1,k_2,k_3) = \delta^{(d)}\left(\sum_b^3 k_b\right)\int_0^\infty \left[\prod_r^3 d\rho_r \rho_r^{d/2-3} e^{-\frac{k_r^2}{\rho_r}}\right](\sum_r \rho_r)^{-d+3}.$$

The prongs of the diagram can be rewritten into Bulk-Boundary propagators. We then use the identity

$$\frac{1}{\alpha^\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty dt t^{\beta-1}e^{-\alpha t}$$

to add what will correspond to the radial coordinate of AdS space:

$$\Omega_F(k_1,k_2,k_3) = \delta^{(d)}\left(\sum_b^3 k_b\right)\int_0^\infty \frac{dt}{t^{d/2+1}}\left[\prod_r^3 \int_0^\infty d\rho_r \rho_r^{d/2-3} t^{d/2-1} e^{-t\rho_r} e^{-\frac{k_r^2}{\rho_r}}\right].$$

Fourier transform yields

$$\begin{aligned}\Omega_F(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) &= \int_0^\infty \frac{dt}{t^{d/2+1}}\int d^d\boldsymbol{z}\left[\prod_r^3 \int_0^\infty d\rho_r \rho_r^{-3} t^{d/2-1} e^{-\rho_r(t+(\boldsymbol{x}_r-\boldsymbol{z})^2}\right] \\ &= \int_0^\infty \frac{dt}{t^{d/2+1}}\int d^d\boldsymbol{z}\prod_r^3 K_2(\boldsymbol{x}_r,\boldsymbol{z};t)\end{aligned}$$

where

$$K_\Delta(\boldsymbol{x},\boldsymbol{z};t) = \frac{t^{\Delta/2}}{(t+(\boldsymbol{x}-\boldsymbol{z})^2)^\Delta}.$$

We recognise this expression as the product of three bulk to boundary propagators K inside an integral over the bulk.

Chapter 5

Weakly interacting QFTs

5.1 Effective mass term

For massless QFTs, the bulk-boundary propagators can be decoupled from one another, except from an integral over their dependence on the bulk coordinates. We will see how a mass term prevents this decoupling in the two-point function of composite operators $\mathrm{Tr}\,\Phi^J$ for massive fields. For non-composite operators this was shown using the same method by Gallegos, Gürsoy and Zinnato [14]. First, we separate the Schwinger parameter dependence in the expression, with exception of an effective mass term. In the Schwinger parametrization, a diagram F of the two-point function with composite external vertices is given by (3.1)

$$\Omega_F(k) = \delta^{(d)}(k_1+k_2)\int_0^\infty (\prod_r da_r) U(a)^{-d/2}\exp[-(\sum_r a_r m_r^2 + P(a;k))].$$

Using the identity

$$\int_0^\infty d\tau\delta(\tau^l - U(a)) = \frac{1}{lU(a)^{\frac{l-1}{l}}}$$

and performing a suitable coordinate transformation of a and τ [14], one finds

$$\Omega_F(k) = \delta(k)\int (\prod db_r) l\delta(1-U(b))A^{\frac{ld}{2}-I}\int d\tau\tau^{I-1-\frac{ld}{2}}\exp[-\tau(A(b)^{-1}\sum_r b_r m_r^2 + k_1^2)].$$

Let us write our result explicitly for composite operators. The only dependence on the Feynman graphs is contained in the Symanzik polynomials and in the power

$$\Delta \equiv \frac{d}{2}(1+l) - I - 2,$$

which can be understood as the scaling dimension of the Feynman graph. Let h be the coordination number of the internal vertices, J of the external vertices. Writing Δ in terms of J and h instead of I one finds

$$\Delta = \left(\frac{d}{2} - \frac{h}{h-2}\right) l + \frac{2J-h}{h-2} + \frac{d}{2} - 2.$$

Furthermore the effective mass terms is defined as

$$M^2(b) \equiv A(b)^{-1}\sum_r b_r m_r^2.$$

With these definitions, we have

$$\Omega_F(k) = l\delta(k)\int (\prod db_r)\delta(1-U(b))A^{\Delta+2-d/2}\int d\tau\tau^{-3-\Delta+d/2}\exp[-\tau(M^2(b)+k_1^2)].$$

5.2 Integral over the bulk

In this section we rewrite the τ integral into the product of two bulk-to-boundary propagators. Our intermediate result can be written as

$$\Omega_F(k) = \delta(k_1 + k_2) \int d\mu(b) \int_0^\infty d\tau \tau^{-3-\nu} e^{-\tau(k_1^2 + M(b)^2)},$$

where we define $\nu \equiv \Delta - d/2$ and

$$\int d\mu(b) \equiv l \int_0^\infty [\prod_r db_r] \delta(1 - U(b)) A^{\nu+2}.$$

By using the identity

$$\int_0^\infty d\tau \tau^\beta e^{-\tau k^2} = \frac{\Gamma(2+2c)}{\Gamma(1+c)^2} \int_0^\infty d\alpha_1 d\alpha_2 (\alpha_1\alpha_2)^c (\alpha_1 + \alpha_2)^{\beta-1-2c} e^{-(\alpha_1+\alpha_2)k^2}$$

with $\beta = -3 - \nu$, the 2-point function becomes

$$\Omega_F(k) = \delta(k_1 + k_2) \frac{\Gamma(2+2c)}{\Gamma(1+c)^2} \int_0^\infty d\alpha_1 d\alpha_2 \int d\mu(b) (\alpha_1\alpha_2)^c (\alpha_1 + \alpha_2)^{-4-\nu-2c} e^{-(\alpha_1+\alpha_2)(k_1^2+M^2)}.$$

Then t is introduced using the definition of the Γ function

$$\Gamma(\nu + 4 + 2c) = (\alpha_1 + \alpha_2)^{\nu+4+2c} \int_0^\infty dt t^{\nu+3+2c} e^{-(\alpha_1+\alpha_2)t}.$$

After Fourier transforming, introducing Lagrange multipliers z

$$\delta(k_1 + k_2) = \frac{1}{(2\pi)^d} \int d^d z e^{iz(k_1+k_2)},$$

we have

$$\Omega_F(x) = \int d\mu(b) \frac{2^{-d}\Gamma(2+2c)}{\Gamma(\nu+4+2c)\Gamma(1+c)} \int d^d z \int_0^\infty dt \prod_{i=1}^{2} d\alpha_i (\alpha_i)^c t^{\nu+3+2c} e^{-\alpha_i t}$$
$$\times \int_{-\infty}^{\infty} dk_i e^{ik_i(z+x_i) - \alpha_i(k_i^2+M^2)}.$$

Performing the momentum integral yields

$$\Omega_F(x) = \frac{2^{-d}\Gamma(2+2c)}{\Gamma(\nu+4+2c)\Gamma(1+c)} \int d^d z dt \int d\mu(b) t^{\nu+3+2c} \prod_{i=1}^{2} d\alpha_i (\alpha_i)^{c-d/2} e^{-\alpha_i t + \frac{-(z+x_i)^2}{4\alpha_i} - \alpha_i M^2}. \quad (5.1)$$

This can be written as the bulk integral of the product of two bulk-boundary propagators, if one defines propagators that depend on the Schwinger parameters b. Here ν is given by, for composite operators,

$$\nu = \left(\frac{d}{2} - \frac{h}{h-2}\right) l + \frac{2J-h}{h-2} - 2.$$

As a result of this computation, we have seen that the two-point function of composite operators $\mathrm{Tr}\,\Phi^J$ with $J > 1$ an integer, in a massive scalar QFT can also be rewritten as a product of two bulk-to-boundary propagators, but unlike the example of massless

QFT in [10], here we obtain a more complicated Green's function than for AdS in the bulk. This is expected because the mass terms in the boundary QFT break conformal symmetry. Hence the bulk dual cannot be AdS recalling matching of the symmetries that we explained above on the bulk and the boundary sides of the correspondence.

5.3 Conjecture

Regarding the form of the two point function that we expressed above in terms of bulk-to-boundary propagators, one can make the following tempting conjecture without proof. The standard Polyakov action is based on Weyl symmetry on the worldsheet. This restricts the possible terms in the action severely. However, there exists generalizations of string theories with broken world-sheet Weyl invariance. These are called non-critical string theories [12]. The simplest of such Weyl symmetry breaking terms is a "cosmological constant" on the worldsheet,

$$S_\mu = \mu \int \sqrt{g}\, d^2\sigma \,.$$

This term clearly breaks the Weyl symmetry $g_{ab} \to e^{2\varphi} g_{ab}$. In [14] presence of such a term was argued to follow from the mass term in the corresponding boundary QFT. In the absence of Weyl symmetry one expects the gluing of the strings that propagate from the two boundary points — which corresponds to insertions of the boundary composite operators — toward the center of the bulk to be more complicated than critical string theories. This is because in the critical case one can use Weyl symmetry to remove fluctuations on the worldsheet, hence the gluing procedure becomes rigid [14]. Here, in the presence of mass terms, one should separately glue these fluctuations on the two sides of the worldsheets that are being glued together. We conjecture that the integral over $\mu(b)$ above is precisely coming from such ripples on the worldsheet.

Chapter 6

Discussion

In this thesis we explored the connection between perturbative quantum field theories and their holographic representation in terms of an asymptotically AdS space. In particular we focused on a matrix-valued scalar field theory with a power-law interaction and asked the question whether the 2-point function of this theory in the quantum perturbative loop expansion can be represented as Witten diagrams in the bulk space-time. The correspondence is expected to hold for composite operators in the bulk of the form $\mathrm{Tr}\,\Phi^J$. Therefore we considered the three point function $\langle \mathrm{Tr}\,\Phi^J\,\mathrm{Tr}\,\Phi^J\,\mathrm{Tr}\,\Phi^J \rangle_{conn}$. The case of massless boundary fields were studied before in [14]. Here we considered the massive case. Our results are shown in (5.1). We conclude from this that indeed the two-point function can be represented as a Witten diagram that is given by joining two bulk-to-boundary propagators at a bulk vertex whose location is then integrated over the bulk space.

We found that this bulk vertex is more complicated than the case of the massless boundary theory because it involves integration over the additional parameters that characterize the bulk-to-boundary propagators. It is tempting to conjecture that these extra parameters correspond to ripples on the worldsheet — recall that in general the bulk-to-boundary propagator is given by string propagation — which cannot be ironed out as in the case of the usual Weyl invariant string theory because the mass term on the boundary quantum field theory most probably leads to a non-critical string theory with Weyl invariance lost, see [14] for a discussion on this point. It would be important to clarify this issue in future work.

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