Carroll limit of the Dirac Lagrangian

On the dynamics of fermions when taking the speed of light to zero

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Abstract

This thesis delves into the Carroll limit of the Dirac Lagrangian, a concept in theoretical physics that describes physical behavior when the speed of light approaches zero while keeping all other quantities finite. This limit is the opposite of the Galilean limit, where the speed of light is considered infinite. The Carroll limit has been instrumental in the study of null hypersurfaces, such as the event horizons of black holes or null boundaries of asymptotically flat spacetimes. This thesis contributes to a larger effort to find mathematical formulations of the Carroll limit for different particles. This thesis contains a thorough derivation of the Carroll limit of the Dirac Lagrangian and the arising Carroll symmetries. The study identifies that two different Carroll limits of the metric result in two distinct Carroll theories. This divergence arises because two different Carroll Clifford algebras can be defined. The study also identifies a fourth invariant term under Carroll transformations that is not mentioned in earlier work on four dimensional Carroll fermions.

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1 Introduction

The Carroll limit characterizes physical phenomena when the speed of light approaches zero, while all other quantities remain finite. This limit was first introduced by Lévy-Leblond [1] and Sen Gupta [2] in the mid-1960s. The term pays homage to Lewis Carroll, renowned author and mathematician, best known for his novel *Alice's Adventures in Wonderland*, due to the strange and counterintuitive effects of this limit. After approximately half a century of relative obscurity, the Carroll limit has recently gained attention due to its applications in the study of quantum gravity. Aside from that, it forms a part of the broader endeavor to explore Non-Lorentzian physics, a field that also includes the better-known Galilean limit, where the speed of light is considered infinite. A comprehensive overview of non-Lorentzian theories is given in the article by Bergshoeff et al. [3].

The Carroll limit is associated with several interesting phenomena, such as the closure of the lightcone and a degeneracy of the spacetime metric [4]. It plays a role in the study of null hypersurfaces, which are Carroll manifolds. Particularly interesting are the null boundaries of asymptotically flat spacetimes, which serve as the foundation for the emerging field of flat space holography. This new area of research extends the principles of holography, originally developed in the context of black holes and AdS/CFT correspondence, to flat spacetime. The idea is to represent all the information contained in a volume of space by a theory located on the boundary of that space. For research on the relationship between the Carroll limit and asymptotically flat spacetimes, see [5–14].

Another application of the Carroll limit is in the study of black hole event horizons. The emergence of Carrollian symmetries on these horizons has been a subject of recent research. These symmetries provide a new perspective on the structure and dynamics of black holes, and have potential implications for our understanding of quantum gravity. For a breakdown of Carrollian symmetries on black hole event horizons, see [15–18]. Other aspects of Carrollian gravity have also been explored, see works [19–30].

The Carroll limit has also found relevance in various other research areas. Several models with tachyonic aspects that respect Carrollian symmetries have been found [31–33]. The study of Carrollian fluids has also seen a rise of interest [34–37]. Additionally, recent findings have showed connections between the Carroll limit and cosmology, dark energy, and inflation [38]. Lastly, the Carroll limit has been found to intersect with the study of fractons [39].

Carroll field theories, which describe particle behavior on null hypersurfaces, have gained importance. They could potentially shed light on particle behavior on black hole event horizons and the null boundaries of asymptotically flat spacetimes. The dynamics of Carroll particles were first mentioned by Bergshoeff et al. [40]. For additional work on Carroll field theories, including work on scalars, see [38, 41–46].

This thesis aims to contribute to the understanding of particle dynamics in the Carroll limit by examining the Carroll limit of the Dirac Lagrangian in four dimensions. The Dirac Lagrangian [47] is a fundamental component of quantum field theory that describes fermion behavior. It had not been a subject of study yet in the context of the Carroll limit when this thesis started, and was a natural next step in the endeavour of uncovering the particle dynamics on null hypersurfaces. However, during the course of this thesis, four articles discussing Carroll fermions appeared on arxiv [48–51]. This highlights the relevance of the study of Carroll fermions. The results of this thesis have been produced in close but critical consultation of those papers. Specifically the paper by Bagchi et al. [48] has been of importance, because they examine the dynamics of Carroll fermions in four dimensions. We will discuss the findings of this research paper, comparing and contrasting them with the outcomes of this thesis and the reported observations therein.

To set the stage for subsequent discussions, we will start in chapter 2 with a recap of the mathematical structure of the Dirac Lagrangian. Chapter 3 will introduce key concepts in Carrollian physics, by treating the Carroll limit of Lorentz boosts, the algebra of Carroll generators, the ultra-relativistic parametrization of the metric and the Carroll invariant Lagrangians of scalar fields. Building on this foundation, we will delve into Carroll fermions and compare the obtained results to recent literature in chapter 4. Finally, chapter 5 will conclude this thesis and propose potential areas for future research.

2 Relativistic Dirac Lagrangian

We start off by treating the relativistic Dirac Lagrangian to refresh our knowledge on the mathematical structure of fermions. This is the basis from which we start to work our way toward the Carroll Dirac Lagrangian and its dynamics. This chapter closely follows the treatise of fermions from Quantum Field Theory lecture notes by Tong [52].

The Dirac Lagrangian describes the behavior of fermionic particles, the particles that make up all matter. It is given by

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi,$$

where Ψ is the four-component Dirac spinor and $\bar{\Psi}$ the Dirac adjoint, defined as $\bar{\Psi} = \Psi^\dagger\gamma^0$. The gamma matrices form the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu},$$

where in this thesis we focus on 4-dimensional flat spacetimes with metric convention $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. One representation of the gamma matrices that obey the Clifford algebra are

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Pauli matrices have the following properties

$$\sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k \quad \text{and} \quad (\sigma^i)^2 = \mathbb{1},$$

where $\mathbb{1}$ is 2x2 identity matrix. In this representation $\gamma^0 = \gamma^{0\dagger}$ is hermitian and $\gamma^i = -\gamma^{i\dagger}$ is anti-hermitian, and we can see that $(\gamma^0)^2 = \mathbb{1}_4$ and $(\gamma^i)^2 = -\mathbb{1}_4$, where $\mathbb{1}_4$ is the 4x4 identity matrix. We have the following definition for the spinor generators:

$$\Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu].$$

The generators obey identity

$$[\Sigma^{\mu\nu}, \gamma^\rho] = \gamma^\nu\eta^{\mu\rho} - \gamma^\mu\eta^{\nu\rho}, \tag{1}$$

and close the Lorentz algebra:

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = -\Sigma^{\mu\sigma}\eta^{\nu\rho} + \Sigma^{\nu\sigma}\eta^{\mu\rho} - \Sigma^{\nu\rho}\eta^{\mu\sigma} + \Sigma^{\mu\rho}\eta^{\nu\sigma}.$$

Because $\gamma^0 = \gamma^{0\dagger}$ is hermitian and $\gamma^i = -\gamma^{i\dagger}$ is anti-hermitian, we have the following property for all $\mu = 0, 1, 2, 3$:

$$\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}.$$

This allows us to relate the generator to its hermitian conjugate

$$\Sigma^{\mu\nu\dagger} = -\gamma^0 \Sigma^{\mu\nu} \gamma^0.$$

The spinor transformation is given by

$$S = e^{\frac{1}{2}\Theta_{\mu\nu}\Sigma^{\mu\nu}}.$$

The hermitian conjugate of the spinor transformation S can then be expressed as follows:

$$S^\dagger = e^{\frac{1}{2}\Theta_{\mu\nu}\Sigma^{\mu\nu\dagger}} = \gamma^0 e^{-\frac{1}{2}\Theta_{\mu\nu}\Sigma^{\mu\nu}} \gamma^0 = \gamma^0 S^{-1} \gamma^0.$$

The Dirac spinors transform under Lorentz transformations as

$$\begin{aligned} \Psi &\to S(\Sigma)\Psi \\ \bar{\Psi} &\to \bar{\Psi} S^{-1}(\Sigma). \end{aligned}$$

Or, differently stated as

$$\begin{aligned} &\Psi \to e^{\frac{1}{2}\Theta^{\mu\nu}\Sigma_{\mu\nu}}\Psi \\ &\Psi^\dagger \to (e^{\frac{1}{2}\Theta^{\mu\nu}\Sigma_{\mu\nu}}\Psi)^\dagger = \Psi^\dagger e^{\frac{1}{2}\Theta^{\mu\nu}\Sigma^\dagger_{\mu\nu}} \\ &\bar{\Psi} \to \Psi^\dagger e^{\frac{1}{2}\Theta^{\mu\nu}\Sigma^\dagger_{\mu\nu}}\gamma^0 = \bar{\Psi}\gamma^0 e^{-\frac{1}{2}\gamma^0\Theta^{\mu\nu}\Sigma_{\mu\nu}\gamma^0}\gamma^0. \end{aligned}$$

For an infinitesimal transformation we get the following variations for Ψ and $\bar{\Psi}$

$$\begin{aligned} \delta\Psi &= \Theta^{\mu\nu} x_\nu \partial_\mu \Psi + \frac{1}{2}\Theta^{\mu\nu}\Sigma_{\mu\nu}\Psi \\ \delta\bar{\Psi} &= \Theta^{\mu\nu} x_\nu \partial_\mu \bar{\Psi} - \frac{1}{2}\Theta^{\mu\nu}\bar{\Psi}\Sigma_{\mu\nu}. \end{aligned}$$

By using these transformation rules and expression (1), one can show that both terms in the Dirac Lagrangian are Lorentz invariant. The Dirac action is

$$\mathcal{S} = \int d^4x \; i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi.$$

Using the Euler-Lagrange equations for $\bar{\Psi}$ leaves us with the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\Psi = 0,$$

which describes the dynamics of fermionic particles.

Chirality in quantum field theory refers to a certain property of the solutions of the Dirac equation. We can define two projection operators, called chirality operators, by

$$\begin{aligned} P_L &= \frac{1}{2}(\mathbb{1}_4 - \gamma^5) \\ P_R &= \frac{1}{2}(\mathbb{1}_4 + \gamma^5) \end{aligned}$$

where γ^5 is another gamma matrix. This γ^5 defined by

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3,$$

where in our representation

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

Here we have $(\gamma^5)^2 = \mathbb{1}_4$ and $\gamma^5 = \gamma^{5\dagger}$ is hermitian. The γ^5 matrix also has the property that it anticommutes with the other gamma matrices:

$$\{\gamma^5, \gamma^\mu\} = 0.$$

The projection operators have the following properties

$$\begin{aligned} &P_L + P_R = \mathbb{1}_4 \\ &P_L P_R = P_R P_L = 0 \\ &P_L^2 = P_L \\ &P_L^\dagger = P_L \\ &P_R^2 = P_R \\ &P_R^\dagger = P_R. \end{aligned}$$

The operators project out independent components of the Dirac spinor. If we decompose the Dirac spinor in Weyl spinors

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

and then use the projection operators on Ψ, we obtain

$$P_L\Psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \quad \text{and} \quad P_R\Psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}.$$

We label the states projected out by P_L as left-handed and the states projected out by P_R as right-handed. The Weyl spinors transform in the same way as the Dirac spinor, because γ^5 anticommutes with γ^μ. We can see that therefore it commutes with the generator. The variations of these components are

$$\begin{aligned} \delta P_R\Psi &= P_R(\Theta^{\mu\nu} x_\nu \partial_\mu \Psi) + \frac{1}{2} P_R(\Theta^{\mu\nu}\Sigma_{\mu\nu}\Psi) \\ &= \Theta^{\mu\nu} x_\nu \partial_\mu (P_R\Psi) + \frac{1}{2}\Theta^{\mu\nu}\Sigma_{\mu\nu} P_R\Psi = \delta\psi_R \\ \delta P_L\Psi &= P_L(\Theta^{\mu\nu} x_\nu \partial_\mu \Psi) + \frac{1}{2} P_L(\Theta^{\mu\nu}\Sigma_{\mu\nu}\Psi) \\ &= \Theta^{\mu\nu} x_\nu \partial_\mu (P_L\Psi) + \frac{1}{2}\Theta^{\mu\nu}\Sigma_{\mu\nu} P_L\Psi = \delta\psi_L, \end{aligned}$$

which are the same transformations as for the Dirac spinor.

In chapter 4 we will look at which of these different components and properties of the fermionic algebra change under the Carroll limit. All of the above described properties will be used to compare our Carroll fermions to. This is to illuminate the differences that appear in these two theories. We will now head over to an introduction on Carrollian physics, describing some of the fundamentals we will need in later discussions.

3 Introduction to Carrollian physics

This chapter will treat some of the important preliminaries of Carrollian physics. We first shortly describe the algebra of Carroll boosts, by taking the limit of Lorentz boost transformations. Then we proceed by explaining the Carroll algebra of boosts and rotations, and how they differ from their Lorentzian counterparts. Next, we discuss the 'pre-ultra-local' parametrization of the metric, which is the small speed of light expansion of the Lorentzian geometry. Lastly, we treat the algebra of Carroll scalar fields and the two Carroll theories that arise from taking the limit.

3.1 Carroll boost transformations

In this paragraph we treat the Carroll boost transformations, following [38]. We start by reviewing the Lorentz boosts and taking the limit of the speed of light going to zero. We denote the Lorentz boost transformation parameter by β. Under a Lorentz boost in x-direction, we have

$$ct' = \frac{1}{\sqrt{1-\beta^2}}\left(ct - \beta x\right), \quad x' = \frac{1}{\sqrt{1-\beta^2}}\left(x - \beta ct\right), \quad y' = y, \quad z' = z,$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

is the Lorentz boost factor. To get the Carroll transformations, we first use the following scaling

$$\beta \to cb,$$

where the parameter b becomes the Carroll boost parameter. Keeping it fixed in the limit $c \to 0$ results in the Carroll boosts:

$$t' = t - bx, \quad x' = x, \quad y' = y, \quad z' = z.$$

This can be contrasted with taking the non-relativistic Galilei limit, which can be found by setting $\beta = c^{-1}b$ and then letting $c \to \infty$. This results in the Galilei transformations:

$$t' = t, \quad x' = x - bt, \quad y' = y, \quad z' = z.$$

In a universe governed by Galilei transformations, time is an absolute, while in a Carroll universe, space is absolute. As the Lorentz boost factor $\gamma \to 1$ in the Carroll limit, Lorentz length contraction does not occur. While a Carroll spacetime doesn't have time dilation, it does exhibit a time shift for events that occur at different spatial locations. The transformations show that time is relative in a Carroll universe, whereas space is a relative in a Galilean universe.

As the speed of light approaches zero, spacetime distances transition to spatial distances. This results in a simplification of the Minkowski metric, where if $c \to 0$ we obtain $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \to dx^2 + dy^2 + dz^2$. The Carroll group maintains the invariance of these spatial distances. The Lorentz invariant expression $-c^2t^2 + x^2 + y^2 + z^2$ transforms to the Carroll invariant expression $x^2 + y^2 + z^2$ in the Carroll limit. Light rays, characterized by this invariant being zero, adopt a different behavior in Carroll spacetime, where they are expressed as $\vec{x} = 0$ and any given time, t. This indicates that the time coordinate t parametrises the light cone. When $\vec{x} = 0$, light does not move in space and the light cone closes up. This stands in contrast to what happens in the Galilean limit, where the speed of light is assumed infinite and the light cone opens up towards the x-axis.

3.2 Carrollian boost and rotation generators

In this paragraph we discuss how to get the Carroll boost and rotation generators from taking the Carroll limit of the Lorentzian counterparts. We start off by considering the Lorentz boost an rotation generators. The Lorentz boost generators are given by

$$L_i = \frac{1}{c}x_i\partial_t + ct\partial_i$$

and the rotation generators by

$$J_{ij} = x_i\partial_j - x_j\partial_i.$$

The Lorentz algebra commutation relations of the boost L_i and rotation generators J_{ij} are

$$\begin{aligned}
&[L_i, L_j] = -J_{ij} \\
&[J_{ij}, L_k] = -\delta_{jk}L_i + \delta_{ik}L_j \\
&[J_{ij}, J_{kl}] = -\delta_{jk}J_{il} + \delta_{ik}J_{jl} - \delta_{il}J_{jk} + \delta_{jl}J_{ik}.
\end{aligned}$$

To get to the Carroll boost generators, we first define new generators $C_i \equiv cL_i$. Upon taking the limit of $c \to 0$ of these generators, we find expression

$$C_i = x_i\partial_t.$$

While the Carroll boosts change compared to Lorentz boosts, Carroll rotation generators remain the same as their Lorentzian counterparts. The Carroll algebra commutation relations are given by

$$\begin{aligned}
&[C_i, C_j] = 0 \\
&[J_{ij}, C_k] = -\delta_{jk}C_i + \delta_{ik}C_j \\
&[J_{ij}, J_{kl}] = -\delta_{jk}J_{il} + \delta_{ik}J_{jl} - \delta_{il}J_{jk} + \delta_{jl}J_{ik}.
\end{aligned}$$

We see that the Carroll boost commutation relation becomes zero, due to the spatial derivative term dropping out of the boost generator definition. These relations become important in our discussion of the Carroll spinor generator algebra in chapter 4.

3.3 'Pre-ultra-local' parametrization

We will proceed with showing how to obtain an expression for the Carroll flat space metric. This paragraph follows the geometry conventions presented by Hansen et al. [53] In this paper a 'pre-ultra-local' metric is introduced, which is the small speed of light expansion of the Lorentzian geometry. This PUL parametrization corresponds to a split of the tangent bundle in 'temporal' and 'spatial' components. In leading order terms the following metric is considered

$$\begin{aligned}
&\eta_{\mu\nu} = -c^2\tau_\mu\tau_\nu + h_{\mu\nu}\\
&\text{where}\quad \tau_\mu = (-1,0,0,0)\\
&\text{and}\quad h_{\mu\nu} = \text{diag}(0,1,1,1)\\
&\eta^{\mu\nu} = -\frac{1}{c^2}v^\mu v^\nu + h^{\mu\nu}\\
&\text{where}\quad v^\mu = (1,0,0,0)\\
&\text{and}\quad h^{\mu\nu} = \text{diag}(0,1,1,1).
\end{aligned}$$

Where we have

$$\tau_\mu v^\mu = -1, \qquad \tau_\mu h^{\mu\nu} = 0, \qquad h_{\mu\nu}v^\nu = 0, \qquad \delta^\mu_\nu = -v^\mu\tau_\nu + h^{\mu\rho}h_{\rho\nu}.$$

The last identity can be shown using

$$\begin{aligned}
&\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu\\
&(-\frac{1}{c^2}v^\mu v^\rho + h^{\mu\rho})(-c^2\tau_\rho\tau_\nu + h_{\rho\nu}) = \delta^\mu_\nu\\
&-\frac{1}{c^2}v^\mu v^\rho h_{\rho\nu} + v^\mu v^\rho\tau_\rho\tau_\nu + h^{\mu\rho}h_{\rho\nu} - c^2h^{\mu\rho}\tau_\rho\tau_\nu = \delta^\mu_\nu\\
&-v^\mu\tau_\nu + h^{\mu\rho}h_{\rho\nu} = \delta^\mu_\nu.
\end{aligned}$$

We will use this metric in chapter 4 to take the Carroll limit, where we find two degenerate metrics which we can use to define our Carroll Clifford algebras with.

3.4 Carroll scalar fields

To illuminate the behavior of fields in the Carroll limit, and give an insight into earlier work done on Carroll fields, we will discuss the Carroll invariant Lagrangians of scalar fields in this paragraph. We will closely follow the derivation by de Boer et al. [38]. We can start off by considering the Lagrangian of relativistic real scalar field ϕ:

$$\mathcal{L} = \frac{1}{2c^2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_i\phi)^2.$$

To take the Carroll limit, we first rescale $\phi \to c\phi$ and then take $c \to 0$. This leaves us with Carroll invariant Lagrangian

$$\mathcal{L}_c = \frac{1}{2}\dot{\phi}^2.$$

We can see that in this theory the spatial derivative terms drop out of the equation. However, we can find a second Carroll theory. This is done by first defining an auxiliary field χ as

$$\chi = -\frac{1}{c^2}\partial_t\phi.$$

Using this auxiliary field in our relativistic Lagrangian gives

$$\mathcal{L} = -\frac{c^2}{2}\chi^2 + \chi\partial_t\phi - \frac{1}{2}(\partial_i\phi)^2.$$

We can now take the Carroll limit $c \to 0$ to obtain

$$\mathcal{L}_c = \chi\partial_t\phi - \frac{1}{2}(\partial_i\phi)^2.$$

We can see that there are two types of Carroll theories. The authors note in the paper that these two types of Carroll limits generally seem to exist, where the first Lagrangian arises due to the electric limit and the second Lagrangian to the magnetic limit. The terminology originates from the Galilean analogues, particularly from the non-relativistic theory of electromagnetism where two unique limits exist: the electric and magnetic limits. In these limits, either the electric effects dominate over the magnetic ones, or the magnetic effects dominate over the electric effects.

4 Carroll Fermions

Now we will head over to treating the algebra of Carroll fermions. We start by taking the Carroll limit of the metric, which leaves us with two distinct metric terms. One will form the basis of our theory of lower fermions, which will be discussed in paragraph 4.1. The other will give us the theory for upper fermions, which will be discussed in paragraph 4.2. We dub these different theories as lower and upper, following nomenclature used in current literature [46]. Then lastly we will discuss recent developments in the literature on Carroll fermions and compare our results to the findings demonstrated there.

4.0.1 Taking the limit $c \to 0$ of the metric tensor

First we want to find the Carroll metric tensors to define our new Clifford algebras with. We start with the PUL parametrization described in paragraph 3.3, and take the $c \to 0$ limits of this metric. For the lower indices we have

$$\eta_{\mu\nu} = -c^2 \tau_\mu \tau_\nu + h_{\mu\nu},$$

where taking the limit of $c \to 0$ gives us $\eta_{\mu\nu} \to h_{\mu\nu}$, with $h^{\mu\nu} = \mathrm{diag}(0, 1, 1, 1)$. We are thus left with the spatial component of the metric. For the upper indices we have

$$\eta^{\mu\nu} = -\frac{1}{c^2} v^\mu v^\nu + h^{\mu\nu}.$$

First we rescale the metric by using $\eta_{\mu\nu} \to -c^2 \eta_{\mu\nu}$. Then when taking the limit of $c \to 0$, we obtain $-c^2 \eta_{\mu\nu} \to v^\mu v^\nu$, with $v^\mu v^\nu = \mathrm{diag}(1, 0, 0, 0)$. Now just the time component remains. For these two metrics we get two different Carroll theories, one for lower fermions, and one for upper fermions.

4.1 Lower fermions

Due to the degeneracy of the metric, we proceed with working out two different Carroll theories. We start by treating the fermion algebra for the metric with lower indices.

4.1.1 Carroll Clifford algebra

For the lower Carroll fermions the Carroll Clifford algebra can be defined using

$$\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = -2h_{\mu\nu},$$

where $h_{\mu\nu} = (0, 1, 1, 1)$. We put forward two representations that satisfy these constraints, but note that these two are not the only possible representations. The first one, from how on dubbed representation A, is given by

$$\tilde{\gamma}_0^A = \begin{pmatrix} 0 & 0 \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \tilde{\gamma}_i^A = \begin{pmatrix} i\sigma_i & 0 \\ 0 & -i\sigma_i \end{pmatrix}.$$

The second representation, dubbed representation B, is given by

$$\tilde{\gamma}_0^B = \frac{1}{2}\begin{pmatrix} i\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -i\mathbb{1} \end{pmatrix}, \qquad \tilde{\gamma}_i^B = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.$$

For both representations we have that $\tilde{\gamma}_0$ is non-hermitian and $\tilde{\gamma}_i^\dagger = -\tilde{\gamma}_i$ is anti-hermitian. The transformation between these two representations is given by

$$\gamma_\mu^A U = U \gamma_\mu^B$$

where

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} i\mathbb{1} & \mathbb{1} \\ \mathbb{1} & i\mathbb{1} \end{pmatrix}.$$

Now we want to move forward by finding the Carroll spinor generator algebra. We define

$$\tilde{\Sigma}_{\mu\nu} \equiv \frac{1}{4}[\tilde{\gamma}_\mu, \tilde{\gamma}_\nu],$$

where we want to find the commutation relations of the generators

$$[\tilde{\Sigma}_{\mu\nu}, \tilde{\Sigma}_{\rho\sigma}].$$

We will make use of the following identities

$$\tilde{\Sigma}_{\mu\nu} = \frac{1}{2}(\tilde{\gamma}_\mu\tilde{\gamma}_\nu + h_{\mu\nu}) \tag{2}$$

$$[\tilde{\gamma}_\mu, \tilde{\gamma}_\nu] = 2(\tilde{\gamma}_\mu\tilde{\gamma}_\nu + h_{\mu\nu}) \tag{3}$$

$$\tilde{\gamma}_\nu\tilde{\gamma}_\mu = (-2h_{\mu\nu} - \tilde{\gamma}_\mu\tilde{\gamma}_\nu) \tag{4}$$

$$2\tilde{\Sigma}_{\mu\nu} - h_{\mu\nu} = \tilde{\gamma}_\mu\tilde{\gamma}_\nu. \tag{5}$$

We start with, using relation (2),

$$\begin{aligned} [\tilde{\Sigma}_{\mu\nu}, \tilde{\Sigma}_{\rho\sigma}] =& \frac{1}{2}[\tilde{\Sigma}_{\mu\nu}, \tilde{\gamma}_\rho\tilde{\gamma}_\sigma] + \frac{1}{2}[\tilde{\Sigma}_{\mu\nu}, h_{\rho\sigma}] \\ =& \frac{1}{2}([\tilde{\Sigma}_{\mu\nu}, \tilde{\gamma}_\rho]\tilde{\gamma}_\sigma + \tilde{\gamma}_\rho[\tilde{\Sigma}_{\mu\nu}, \tilde{\gamma}_\sigma]). \end{aligned}$$

Where we see that, using relations (3) and (4),

$$\begin{aligned} [\tilde{\Sigma}_{\mu\nu}, \tilde{\gamma}_\rho] =& \frac{1}{2}[\tilde{\gamma}_\mu\tilde{\gamma}_\nu, \tilde{\gamma}_\rho] + \frac{1}{2}[h_{\mu\nu}, \tilde{\gamma}_\rho] \\ =& \frac{1}{2}(\tilde{\gamma}_\mu[\tilde{\gamma}_\nu, \tilde{\gamma}_\rho] + [\tilde{\gamma}_\mu, \tilde{\gamma}_\rho]\tilde{\gamma}_\nu) \\ =& (\tilde{\gamma}_\mu(\tilde{\gamma}_\nu\tilde{\gamma}_\rho + h_{\nu\rho}) + (\tilde{\gamma}_\mu\tilde{\gamma}_\rho + h_{\mu\rho})\tilde{\gamma}_\nu) \\ =& (\tilde{\gamma}_\mu\tilde{\gamma}_\nu\tilde{\gamma}_\rho + \tilde{\gamma}_\mu h_{\nu\rho} + \tilde{\gamma}_\mu\tilde{\gamma}_\rho\tilde{\gamma}_\nu + h_{\mu\rho}\tilde{\gamma}_\nu) \\ =& (\tilde{\gamma}_\mu\tilde{\gamma}_\nu\tilde{\gamma}_\rho + \tilde{\gamma}_\mu h_{\nu\rho} + \tilde{\gamma}_\mu(-2h_{\nu\rho} - \tilde{\gamma}_\nu\tilde{\gamma}_\rho) + h_{\mu\rho}\tilde{\gamma}_\nu) \\ =& \tilde{\gamma}_\nu h_{\mu\rho} - \tilde{\gamma}_\mu h_{\nu\rho}. \end{aligned}$$

Using this and relation (5), we see that

$$\begin{aligned}
[\tilde{\Sigma}_{\mu\nu},\tilde{\Sigma}_{\rho\sigma}] =& \frac{1}{2}((-\tilde{\gamma}_\mu h_{\nu\rho}+\tilde{\gamma}_\nu h_{\mu\rho})\tilde{\gamma}_\sigma+\tilde{\gamma}_\rho(-\tilde{\gamma}_\mu h_{\nu\sigma}+\tilde{\gamma}_\nu h_{\mu\sigma}))\\
=& \frac{1}{2}(-\tilde{\gamma}_\mu\tilde{\gamma}_\sigma h_{\nu\rho}+\tilde{\gamma}_\nu\tilde{\gamma}_\sigma h_{\mu\rho}-\tilde{\gamma}_\rho\tilde{\gamma}_\mu h_{\nu\sigma}+\tilde{\gamma}_\rho\tilde{\gamma}_\nu h_{\mu\sigma})\\
=& \frac{1}{2}(-(2\tilde{\Sigma}_{\mu\sigma}-h_{\mu\sigma})h_{\nu\rho}+(2\tilde{\Sigma}_{\nu\sigma}-h_{\nu\sigma})h_{\mu\rho}\\
&-(2\tilde{\Sigma}_{\rho\mu}-h_{\rho\mu})h_{\nu\sigma}+(2\tilde{\Sigma}_{\rho\nu}-h_{\rho\nu})h_{\mu\sigma})\\
=& -\tilde{\Sigma}_{\mu\sigma}h_{\nu\rho}+\tilde{\Sigma}_{\nu\sigma}h_{\mu\rho}-\tilde{\Sigma}_{\nu\rho}h_{\mu\sigma}+\tilde{\Sigma}_{\mu\rho}h_{\nu\sigma}.
\end{aligned}$$

Looking at metric $h_{\mu\nu}=\mathrm{diag}(0,1,1,1)$ we have

$$h_{00}=0,\quad h_{0i}=0,\quad h_{ij}=\delta_{ij}.$$

We can use these identities to get the following Carroll spinor generator commutation relations:

$$\begin{aligned}
&[\tilde{\Sigma}_{0i},\tilde{\Sigma}_{0j}]=0\\
&[\tilde{\Sigma}_{0i},\tilde{\Sigma}_{jk}]=-\delta_{ij}\tilde{\Sigma}_{0k}+\delta_{ik}\tilde{\Sigma}_{0j}\\
&[\tilde{\Sigma}_{ij},\tilde{\Sigma}_{kl}]=-\delta_{jk}\tilde{\Sigma}_{il}+\delta_{ik}\tilde{\Sigma}_{jl}-\delta_{il}\tilde{\Sigma}_{jk}+\delta_{jl}\tilde{\Sigma}_{ik}.
\end{aligned}$$

These close the same algebra as the generators stated in paragraph 3.2, where we see that the commutator between the boost generators becomes zero. We can therefore see that the used definition renders a faithful representation of the Carroll spinor generators.

4.1.2 Carroll invariant Lagrangians

Using this definition of the Carroll generators, we proceed by defining the Carroll invariant Lagrangians for lower fermions. In the Lorentz invariant case, Dirac adjoints are defined as $\bar{\Psi}=\Psi^\dagger\gamma^0$. This definition is needed such that under Lorentz transformations the Dirac spinor and its adjoint change as

$$\begin{aligned}
&\Psi\to S(\Sigma)\Psi\\
&\bar{\Psi}\to\bar{\Psi}S^{-1}(\Sigma).
\end{aligned}$$

In the Carroll theory we need to define a new Dirac adjoint that serves the same purpose. Therefore we define new Dirac adjoint $\bar{\Psi}=\Psi^\dagger\Lambda$. This matrix Λ has to obey the following relation:

$$\tilde{\gamma}_\mu^\dagger=\Lambda\tilde{\gamma}_\mu\Lambda.$$

A suitable representation for Λ for both $\tilde{\gamma}^\mu$ representations A and B is

$$\Lambda=\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix},$$

which is the same as the representation of the γ^0 matrix in the Lorentzian Clifford algebra stated in chapter 2. We can see that $\Lambda = \Lambda^{-1}$ and $\Lambda^2 = \mathbb{1}_4$. Because of the relation we found between $\tilde{\gamma}_\mu$ and $\tilde{\gamma}_\mu^\dagger$, we can find a relation between $\tilde{\Sigma}_{\mu\nu}^\dagger$ and $\tilde{\Sigma}_{\mu\nu}$, using

$$\begin{aligned}\tilde{\Sigma}_{\mu\nu}^\dagger =&(\frac{1}{4}[\tilde{\gamma}_\mu, \tilde{\gamma}_\nu])^\dagger \\ =&\frac{1}{4}[\tilde{\gamma}_\nu^\dagger, \tilde{\gamma}_\mu^\dagger] \\ =&\frac{1}{4}(\Lambda\tilde{\gamma}_\nu\tilde{\gamma}_\mu\Lambda - \Lambda\tilde{\gamma}_\mu\tilde{\gamma}_\nu\Lambda) \\ =&-\Lambda\tilde{\Sigma}_{\mu\nu}\Lambda.\end{aligned}$$

Knowing this, we can find the Carroll transformations of the Dirac spinor and its adjoint

$$\begin{aligned}&\Psi \to e^{\frac{1}{2}\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}}\Psi \\ &\Psi^\dagger \to (e^{\frac{1}{2}\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}}\Psi)^\dagger = \Psi^\dagger e^{\frac{1}{2}\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}^\dagger} \\ &\bar{\Psi} \to \Psi^\dagger e^{\frac{1}{2}\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}^\dagger}\Lambda = \bar{\Psi}\Lambda e^{-\frac{1}{2}\Lambda\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}\Lambda}\Lambda.\end{aligned}$$

This gives us as variations for Ψ and $\bar{\Psi}$

$$\begin{aligned}\delta\Psi &= \Theta^{\mu\nu}x_\nu\partial_\mu\Psi + \frac{1}{2}\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}\Psi \\ \delta\bar{\Psi} &= \Theta^{\mu\nu}x_\nu\partial_\mu\bar{\Psi} - \frac{1}{2}\Theta^{\mu\nu}\bar{\Psi}\tilde{\Sigma}_{\mu\nu}.\end{aligned}$$

We can then find four Carroll invariant terms

$$\begin{aligned}\delta(\bar{\Psi}\Psi) =&\Theta^{\mu\nu}x_\nu(\partial_\mu\bar{\Psi})\Psi - \frac{1}{2}\bar{\Psi}\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}\Psi + \Theta^{\mu\nu}x_\nu\bar{\Psi}(\partial_\mu\Psi) + \frac{1}{2}\bar{\Psi}\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}\Psi \\ =&\partial_\mu[\Theta^{\mu\nu}x_\nu\bar{\Psi}\Psi]\end{aligned}$$

$$\begin{aligned}\delta(\bar{\Psi}v^\rho\tilde{\gamma}_\rho\Psi) =&\Theta^{\mu\nu}x_\nu(\partial_\mu\bar{\Psi})v^\rho\tilde{\gamma}_\rho\Psi - \frac{1}{2}\bar{\Psi}\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}v^\rho\tilde{\gamma}_\rho\Psi \\ &+ \Theta^{\mu\nu}x_\nu\bar{\Psi}v^\rho\tilde{\gamma}_\rho(\partial_\mu\Psi) + \frac{1}{2}\bar{\Psi}v^\rho\tilde{\gamma}_\rho\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}\Psi \\ =&-\frac{1}{2}\bar{\Psi}\Theta^{\mu\nu}[\tilde{\Sigma}_{\mu\nu}, \tilde{\gamma}_\rho]v^\rho\Psi + \partial_\mu[\Theta^{\mu\nu}x_\nu\bar{\Psi}v^\rho\tilde{\gamma}_\rho\Psi] \\ =&-\frac{1}{2}\bar{\Psi}\Theta^{\mu\nu}(\tilde{\gamma}_\nu h_{\mu\rho} - \tilde{\gamma}_\mu h_{\nu\rho})v^\mu\Psi + \partial_\mu[\Theta^{\mu\nu}x_\nu\bar{\Psi}v^\rho\tilde{\gamma}_\rho\Psi] \\ =&\partial_\mu[\Theta^{\mu\nu}x_\nu\bar{\Psi}v^\rho\tilde{\gamma}_\rho\Psi]\end{aligned}$$

$$\begin{aligned}\delta(i\bar{\Psi}v^\rho\partial_\rho\Psi) =&i\Theta^{\mu\nu}x_\nu(\partial_\mu\bar{\Psi})v^\rho\partial_\rho\Psi - \frac{i}{2}\bar{\Psi}\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}v^\rho\partial_\rho\Psi \\ &+ i\Theta^{\mu\nu}x_\nu\bar{\Psi}v^\rho\partial_\rho(\partial_\mu\Psi) + \frac{i}{2}\bar{\Psi}\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}v^\rho\partial_\rho\Psi \\ =&\partial_\mu[\Theta^{\mu\nu}x_\nu i\bar{\Psi}v^\rho\partial_\rho\Psi]\end{aligned}$$

$$
\begin{aligned}
\delta(i\bar{\Psi}v^\rho\tilde{\gamma}_\rho v^\sigma\partial_\sigma\Psi) =& i\Theta^{\mu\nu}x_\nu(\partial_\mu\bar{\Psi})v^\rho\tilde{\gamma}_\rho v^\sigma\partial_\sigma\Psi - \frac{i}{2}\bar{\Psi}\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}v^\rho\tilde{\gamma}_\rho v^\sigma\partial_\sigma\Psi \\
&+ i\Theta^{\mu\nu}x_\nu\bar{\Psi}v^\rho\tilde{\gamma}_\rho v^\sigma\partial_\sigma(\partial_\mu\Psi) + \frac{i}{2}\bar{\Psi}v^\rho\tilde{\gamma}_\rho\Theta^{\mu\nu}\tilde{\Sigma}_{\mu\nu}v^\sigma\partial_\sigma\Psi \\
=& -\frac{i}{2}\bar{\Psi}\Theta^{\mu\nu}[\tilde{\Sigma}_{\mu\nu},\tilde{\gamma}_\rho]v^\rho v^\sigma\partial_\sigma\Psi + \partial_\mu[\Theta^{\mu\nu}x_\nu i\bar{\Psi}v^\rho\tilde{\gamma}_\rho v^\sigma\partial_\sigma\Psi] \\
=& -\frac{i}{2}\bar{\Psi}\Theta^{\mu\nu}(\tilde{\gamma}_\nu h_{\mu\rho} - \tilde{\gamma}_\mu h_{\nu\rho})v^\rho v^\sigma\partial_\sigma\Psi + \partial_\mu[\Theta^{\mu\nu}x_\nu i\bar{\Psi}v^\rho\tilde{\gamma}_\rho v^\sigma\partial_\sigma\Psi] \\
=& \partial_\mu[\Theta^{\mu\nu}x_\nu i\bar{\Psi}v^\rho\tilde{\gamma}_\rho v^\sigma\partial_\sigma\Psi].
\end{aligned}
$$

For $\delta(\bar{\Psi}v^\rho\tilde{\gamma}_\rho\Psi)$ and $\delta(\bar{\Psi}v^\rho\tilde{\gamma}_\rho v^\sigma\partial_\sigma\Psi)$ we used that $h_{\mu\nu}v^\mu = 0$. All these terms leave us with total derivatives, which ensure the Lagrangian to be Carroll invariant. We can see that the first two terms are two mass terms, while the latter two contain time derivatives. For lower fermions we have no invariant terms which contain spatial derivatives.

By using found invariant terms we can make the following Carroll invariant Lagrangians for massless fermions

$$
\begin{aligned}
\mathcal{L}_1 &= i\bar{\Psi}v^\mu\partial_\mu\Psi = i\bar{\Psi}\partial_t\Psi \\
\mathcal{L}_2 &= i\bar{\Psi}v^\mu\tilde{\gamma}_\mu v^\nu\partial_\nu\Psi = i\bar{\Psi}\tilde{\gamma}_0\partial_t\Psi.
\end{aligned}
$$

And for massive fermions we can define the following Lagrangians

$$
\begin{aligned}
\mathcal{L}_3 &= i\bar{\Psi}v^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi = i\bar{\Psi}\partial_t\Psi - m\bar{\Psi}\Psi \\
\mathcal{L}_4 &= i\bar{\Psi}v^\mu\tilde{\gamma}_\mu v^\nu\partial_\nu\Psi - m\bar{\Psi}v^\mu\tilde{\gamma}_\mu\Psi = i\bar{\Psi}\tilde{\gamma}_0\partial_t\Psi - m\bar{\Psi}\tilde{\gamma}_0\Psi \\
\mathcal{L}_5 &= i\bar{\Psi}v^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi = i\bar{\Psi}\partial_t\Psi - m\bar{\Psi}\tilde{\gamma}_0\Psi \\
\mathcal{L}_6 &= i\bar{\Psi}v^\mu\tilde{\gamma}_\mu v^\nu\partial_\nu\Psi - m\bar{\Psi}v^\mu\tilde{\gamma}_\mu\Psi = i\bar{\Psi}\tilde{\gamma}_0\partial_t\Psi - m\bar{\Psi}\Psi.
\end{aligned}
$$

These combinations of terms will later on allow us to find specific field equations for the components of the Carroll-Dirac spinor.

4.1.3 Equations of motion and Hamiltonians

We now want to proceed by finding the equations of motion for the different Lagrangians and find the Hamiltonian densities. We start defining the Carroll-Dirac spinor as

$$\Psi = \begin{pmatrix}\phi \\ \chi\end{pmatrix}.$$

For the equations of motion we use convention A for our representation of $\tilde{\gamma}_0$.

We then have that the equation of motion for $\mathcal{L}_1$ is given by

$$i\partial_t\Psi = 0$$

and the equation of motion for $\mathcal{L}_2$ is given by

$$i\tilde{\gamma}_0\partial_t\Psi = 0.$$

We can see here that the field equations for $\mathcal{L}_1$ are

$$\phi = \phi(\vec{x}) \quad \text{and} \quad \chi = \chi(\vec{x}),$$

and for $\mathcal{L}_2$ are

$$\phi = \phi(\vec{x}) \quad \text{and} \quad \chi = 0.$$

Both Hamiltonian densities $\mathcal{H}_1$ and $\mathcal{H}_2$ give zero.

We move on by looking at the Lagrangians with mass terms. For these Lagrangians we get different field equations. We first start by working out the equation of motion for $\mathcal{L}_3$:

$$i\partial_t\Psi = m\Psi.$$

This gives field solutions

$$\phi = e^{-imt}\phi(\vec{x}) \quad \text{and} \quad \chi = e^{-imt}\chi(\vec{x}).$$

For the Hamiltionian density $\mathcal{H}_3$ we have

$$\begin{aligned}\Pi_\Psi &= \frac{\partial\mathcal{L}_3}{\partial\dot{\Psi}} = i\bar{\Psi}\\ \mathcal{H}_3 &= \Pi_\Psi\dot{\Psi} - \mathcal{L}_3 = m\bar{\Psi}\Psi.\end{aligned}$$

The equation of motion for $\mathcal{L}_4$ with convention A for our representation of $\tilde{\gamma}_0$ is

$$i\tilde{\gamma}_0\partial_t\Psi = m\tilde{\gamma}_0\Psi,$$

which gives field solution

$$\phi = e^{-imt}\phi(\vec{x}).$$

For the Hamiltionian density $\mathcal{H}_4$ we get

$$\begin{aligned}\Pi_\Psi &= \frac{\partial\mathcal{L}_4}{\partial\dot{\Psi}} = i\bar{\Psi}\tilde{\gamma}_0\\ \mathcal{H}_4 &= \Pi_\Psi\dot{\Psi} - \mathcal{L}_4 = m\bar{\Psi}\tilde{\gamma}_0\Psi = m\phi^\dagger\phi.\end{aligned}$$

For massive Carroll fermions the Hamiltonian is nonzero.

The equation of motion for $\mathcal{L}_5$ with convention A for our representation of $\tilde{\gamma}_0$ is

$$i\partial_t\Psi = m\tilde{\gamma}_0\Psi,$$

which gives field solution

$$\chi = -imt\phi(\vec{x}) + \chi(\vec{x}).$$

For the Hamiltionian density $\mathcal{H}_5$ we get

$$\begin{aligned}\Pi_\Psi &= \frac{\partial \mathcal{L}_5}{\partial \dot{\Psi}} = i\bar{\Psi} \\ \mathcal{H}_5 &= \Pi_\Psi \dot{\Psi} - \mathcal{L}_5 = m\bar{\Psi}\tilde{\gamma}_0\Psi = m\phi^\dagger\phi.\end{aligned}$$

The equation of motion for $\mathcal{L}_6$ with convention A for our representation of $\tilde{\gamma}_0$ is

$$i\tilde{\gamma}_0\partial_t\Psi = m\Psi,$$

which gives field solutions

$$\phi = 0 \quad \text{and} \quad \chi = 0.$$

This is therefore not a good physical representation.

4.1.4 Notes on Carroll chirality

In this paragraph we describe some steps taken towards defining Carroll projection operators. We will use the chiral convention for Lorentzian gamma matrices γ_μ and γ_5, which is

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

We then start by using our representation B for $\tilde{\gamma}_\mu$, which is

$$\tilde{\gamma}_0 = \frac{1}{2}\begin{pmatrix} i\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -i\mathbb{1} \end{pmatrix}, \qquad \tilde{\gamma}_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.$$

We can see that for these representations, the following relations hold

$$\begin{aligned}\gamma_0 &= \tilde{\gamma}_0 + \tilde{\gamma}_0^\dagger \\ \gamma_5 &= i(\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger) \\ \tilde{\gamma}_0 &= \frac{1}{2}(\gamma_0 - i\gamma_5) \\ \tilde{\gamma}_0^\dagger &= \frac{1}{2}(\gamma_0 + i\gamma_5) \\ \tilde{\gamma}_i &= \gamma_i.\end{aligned}$$

We first try and start by defining $\tilde{\gamma}_5$ in a similar way as its Lorentzian counterpart. This gives us

$$\begin{aligned}\tilde{\gamma}_5 &\equiv i\tilde{\gamma}_0\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3 \\ &= \frac{i}{2}(\gamma_0 - i\gamma_5)\gamma_1\gamma_2\gamma_3 \\ &= \frac{i}{2}(\gamma_0 - i\gamma_5) \\ &= i\tilde{\gamma}_0.\end{aligned}$$

We see here that $\tilde{\gamma}_5$ can be defined in terms of $\tilde{\gamma}_0$, and it therefore not linearly independent like its Lorentzian counterpart. Defining an projection operator $\tilde{P} = \frac{1}{2}(\mathbb{1}_4 - i\tilde{\gamma}_5)$ does not work, because for this definition $\tilde{P}^2 \neq \tilde{P}$. We therefore need to find different candidates for projection operators. We propose

$$\begin{aligned}\tilde{P}_L &= \frac{1}{2}(\mathbb{1}_4 - i(\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger))\\ \tilde{P}_R &= \frac{1}{2}(\mathbb{1}_4 + i(\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger)).\end{aligned}$$

These new operators obey the same properties as the Lorentzian projection operators:

$$\begin{aligned}&\tilde{P}_L + \tilde{P}_R = \mathbb{1}_4\\ &\tilde{P}_L^2 = \frac{1}{4}(\mathbb{1}_4 - 2i(\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger) - (\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger)(\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger))\\ &= \frac{1}{4}(\mathbb{1}_4 - 2i(\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger) + (\tilde{\gamma}_0\tilde{\gamma}_0^\dagger + \tilde{\gamma}_0^\dagger\tilde{\gamma}_0))\\ &= \frac{1}{2}(\mathbb{1}_4 - i(\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger)) = \tilde{P}_L\\ &\tilde{P}_R^2 = \frac{1}{4}(\mathbb{1}_4 + 2i(\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger) - (\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger)(\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger))\\ &= \frac{1}{4}(\mathbb{1}_4 + 2i(\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger) + (\tilde{\gamma}_0\tilde{\gamma}_0^\dagger + \tilde{\gamma}_0^\dagger\tilde{\gamma}_0))\\ &= \frac{1}{2}(\mathbb{1}_4 + i(\tilde{\gamma}_0 - \tilde{\gamma}_0^\dagger)) = \tilde{P}_R\\ &P_{L,R}^\dagger = P_{L,R}\end{aligned}$$

In our representation B, these operators take the following matrix form:

$$\begin{aligned}\tilde{P}_L &= \begin{pmatrix}\mathbb{1} & 0\\ 0 & 0\end{pmatrix}\\ \tilde{P}_R &= \begin{pmatrix}0 & 0\\ 0 & \mathbb{1}\end{pmatrix}.\end{aligned}$$

These operators however do not commute with our Carroll generator $\tilde{\Sigma}_{\mu\nu}$, and therefore the different components of the Carroll-Dirac spinor do not transform independently under Carroll transformations. Therefore chirality does not seem to be trivially conserved for Carroll fermions.

4.2 Upper fermions

Now that we have treated the theory of lower fermions, we move on to the algebra for the fermions with the upper index metric.

4.2.1 Carroll Clifford algebra

First we define the upper Carroll Clifford algebra using the timelike invariant metric component $v^\mu v^\nu$:

$$\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = -2v^\mu v^\nu,$$

where $v^\mu v^\nu = \mathrm{diag}(1,0,0,0)$. A viable matrix representation that satisfies the constraints is

$$\hat{\gamma}^0 = \begin{pmatrix} i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix}, \qquad \hat{\gamma}^i = \begin{pmatrix} 0 & 0 \\ -\sigma^i & 0 \end{pmatrix}$$

where $(\gamma^0)^2 = -\mathbb{1}_4$ and $(\gamma^i)^2 = 0$. We can use this to define the Carroll generator for spinors as

$$\hat{\Sigma}^{\mu\nu} \equiv \frac{1}{4}[\hat{\gamma}^\mu, \hat{\gamma}^\nu].$$

However, defining this as the generator for the upper gamma matrices, contrary to the lower gamma matrices, renders a problem. In this definition the rotation generators become zero, so $\hat{\Sigma}^{ij} = 0$. In a paper by Bagchi et al. [46] a way to work around this problem is proposed. We will treat this in paragraph 4.3.
We will still proceed with this definition to find a Carroll boost invariant Lagrangian. We can follow very similar steps as for the lower gamma matrices to get an expression for the commutator of the generator and the upper gamma matrix

$$[\hat{\Sigma}^{\mu\nu}, \hat{\gamma}^\rho] = \hat{\gamma}^\nu v^\mu v^\rho - \hat{\gamma}^\mu v^\nu v^\rho.$$

For the boost generators and the upper gamma matrices we find the following two commutation relations

$$[\hat{\Sigma}^{0i}, \hat{\gamma}^0] = \hat{\gamma}^i, \qquad \text{and} \qquad [\hat{\Sigma}^{0i}, \hat{\gamma}^j] = 0. \tag{6}$$

The commutation relations between the generators all become zero

$$[\hat{\Sigma}^{\mu\nu}, \hat{\Sigma}^{\rho\sigma}] = -\hat{\Sigma}^{\mu\sigma} v^\nu v^\rho + \hat{\Sigma}^{\nu\sigma} v^\mu v^\rho - \hat{\Sigma}^{\nu\rho} v^\mu v^\sigma + \hat{\Sigma}^{\mu\rho} v^\nu v^\sigma = 0,$$

Because $v^\mu v^\nu = \mathrm{diag}(1,0,0,0)$ and $\hat{\Sigma}^{ij} = 0$. This therefore does not close the Carroll algebra and does not form a true Carroll generator structure, due to the vanishing rotation operator. However, we can still find a boost invariant Lagrangian for upper fermions. Using the matrix representation stated above for $\hat{\gamma}^0$ and $\hat{\gamma}^i$, we obtain the following expression for the Carroll boost generators

$$\hat{\Sigma}^{0i} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ i\sigma^i & 0 \end{pmatrix}.$$

In this representation, we can again find a Λ matrix that satisfies the following conditions. Given

$$\Lambda = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \text{we have that} \quad \hat{\gamma}^{\dagger\mu} = \Lambda \hat{\gamma}^{\mu} \Lambda \quad \text{such that} \quad \hat{\Sigma}^{\dagger 0i} = -\Lambda \hat{\Sigma}^{0i} \Lambda.$$

We can use this Λ to define the Carroll-Dirac adjoint $\Psi = \bar{\Psi}^\dagger \Lambda$. We proceed by looking at how the spinors change under these Carroll boosts.

4.2.2 Carroll boost invariant Lagrangian

Under the Carroll boosts, the spinor and its adjoint change as

$$\begin{aligned}
\delta\Psi &= \Theta_{0i} x^i \partial_t \Psi + \Theta_{0i} \hat{\Sigma}^{0i} \Psi \\
\delta\bar{\Psi} &= \Theta_{0i} x^i \partial_t \bar{\Psi} - \Theta_{0i} \bar{\Psi} \hat{\Sigma}^{0i}
\end{aligned}$$

We can then show Carroll boost invariance for the term $i\bar{\Psi}\hat{\gamma}^\mu \partial_\mu \Psi$ and $m\bar{\Psi}\Psi$. We will start with the former. Varying this term gives

$$\delta(i\bar{\Psi}\hat{\gamma}^\mu \partial_\mu \Psi) = \delta(i\bar{\Psi}(\hat{\gamma}^0 \partial_t + \gamma^i \partial_i)\Psi),$$

where we will treat the terms separately. The first term gives

$$\begin{aligned}
\delta(i\bar{\Psi}\hat{\gamma}^0 \partial_t \Psi) =& i\Theta_{0i} x^i \partial_t \bar{\Psi} \hat{\gamma}^0 \partial_t \Psi - i\bar{\Psi}\Theta_{0i} \hat{\Sigma}^{0i} \hat{\gamma}^0 \partial_t \Psi \\
& + i\bar{\Psi}\hat{\gamma}^0 \partial_t (\Theta_{0i} x^i \partial_t \Psi) + i\bar{\Psi}\Theta_{0i} \hat{\gamma}^0 \hat{\Sigma}^{0i} \partial_t \Psi \\
=& \partial_t [i\Theta_{0i} x^i \bar{\Psi} \hat{\gamma}^0 \partial_t \Psi] - i\bar{\Psi}\Theta_{0i} [\hat{\Sigma}^{0i}, \hat{\gamma}^0] \partial_t \Psi \\
=& \partial_t [i\Theta_{0i} x^i \bar{\Psi} \hat{\gamma}^0 \partial_t \Psi] - i\bar{\Psi}\Theta_{0i} \hat{\gamma}^i \partial_t \Psi,
\end{aligned}$$

where we use $[\hat{\Sigma}^{0i}, \hat{\gamma}^0] = \hat{\gamma}^i$. The second term gives

$$\begin{aligned}
\delta(i\bar{\Psi}\hat{\gamma}^j \partial_j \Psi) =& i\Theta_{0i} x^i \partial_t \bar{\Psi} \hat{\gamma}^j \partial_j \Psi - i\bar{\Psi}\Theta_{0i} \hat{\Sigma}^{0i} \hat{\gamma}^j \partial_j \Psi \\
& + i\bar{\Psi}\hat{\gamma}^j \partial_j (\Theta_{0i} x^i \partial_t \Psi) + i\bar{\Psi}\Theta_{0i} \hat{\gamma}^j \hat{\Sigma}^{0i} \partial_j \Psi \\
=& \partial_t [i\Theta_{0i} x^i \bar{\Psi} \hat{\gamma}^j \partial_j \Psi] + i\bar{\Psi}\Theta_{0i} \hat{\gamma}^i \partial_t \Psi - i\bar{\Psi}\Theta_{0i} [\hat{\Sigma}^{0i}, \hat{\gamma}^j] \partial_j \Psi \\
=& \partial_t [i\Theta_{0i} x^i \bar{\Psi} \hat{\gamma}^0 \partial_t \Psi] + i\bar{\Psi}\Theta_{0i} \hat{\gamma}^i \partial_t \Psi,
\end{aligned}$$

where we use $[\hat{\Sigma}^{0i}, \hat{\gamma}^j] = 0$. We then can see that the total variation gives

$$\delta(i\bar{\Psi}\hat{\gamma}^\mu \partial_\mu \Psi) = \delta(i\bar{\Psi}(\hat{\gamma}^0 \partial_t + \gamma^i \partial_i)\Psi) = \partial_t [i\Theta_{0i} x^i \bar{\Psi}(\hat{\gamma}^0 \partial_t + \hat{\gamma}^i \partial_i)\Psi].$$

This leave us with total time derivative, which ensures the Lagrangian to be Carroll boost invariant. Now we proceed with $m\bar{\Psi}\Psi$:

$$\begin{aligned}
\delta(\bar{\Psi}\Psi) =& \bar{\Psi}\Theta_{0i} x^i (\partial_t \Psi) + \bar{\Psi}\Theta_{0i} \hat{\Sigma}^{0i} \Psi + \Theta_{0i} x^i (\partial_t \bar{\Psi})\Psi - \Theta_{0i} \bar{\Psi} \hat{\Sigma}^{0i} \Psi \\
=& \partial_t [i\Theta_{0i} x^i \bar{\Psi}\Psi].
\end{aligned}$$

This also leave us with total time derivative. Now we can proceed to define the Carroll boost invariant Lagrangian for massless upper fermions as

$$\mathcal{L}_7 = i\bar{\Psi}\hat{\gamma}^\mu\partial_\mu\Psi = i\bar{\Psi}(\hat{\gamma}^0\partial_t + \hat{\gamma}^i\partial_i)\Psi,$$

and for massive upper fermions as

$$\mathcal{L}_8 = i\bar{\Psi}\hat{\gamma}^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi = i\bar{\Psi}(\hat{\gamma}^0\partial_t + \hat{\gamma}^i\partial_i)\Psi - m\bar{\Psi}\Psi.$$

These retain their spatial derivatives, in contrast with the Lagrangians for lower fermions.

4.2.3 Equations of motion and Hamiltonians

We first start by working out the equation of motion for $\mathcal{L}_7$:

$$i\hat{\gamma}^\mu\partial_\mu\Psi = 0.$$

For the Hamiltionian density $\mathcal{H}_7$ we have

$$\begin{aligned}\Pi_\Psi &= \frac{\partial\mathcal{L}_7}{\partial\dot{\Psi}} = i\bar{\Psi}\hat{\gamma}^0,\\ \mathcal{H}_7 &= \Pi_\Psi\dot{\Psi} - \mathcal{L}_7 = -i\bar{\Psi}\hat{\gamma}^i\partial_i\Psi.\end{aligned}$$

We note here that the Hamiltonian is nonzero, in contrast to the massless lower fermion Hamiltonian densities. Working out the equation of motion for $\mathcal{L}_8$ gives:

$$(i\hat{\gamma}^\mu\partial_\mu - m)\Psi = 0.$$

For the Hamiltionian density $\mathcal{H}_8$ we have

$$\begin{aligned}\Pi_\Psi &= \frac{\partial\mathcal{L}_8}{\partial\dot{\Psi}} = i\bar{\Psi}\hat{\gamma}^0,\\ \mathcal{H}_8 &= \Pi_\Psi\dot{\Psi} - \mathcal{L}_8 = -i\bar{\Psi}\hat{\gamma}^i\partial_i\Psi + m\bar{\Psi}\Psi.\end{aligned}$$

The massive upper fermion Hamiltonian is nonzero.

4.3 Relevant research on Carroll fermions

We now move on to compare our findings to recent literature on Carroll fermions. The important paper to compare our results to is the work done by Bagchi et al. [46], that appeared during the course of this thesis. In this paper four-dimensional Carroll fermions are treated. The research in this thesis overlaps to a large extent with the article.

The paper starts the formulation of a Carroll fermion theory with the same metric conventions we have used. They define the lower and upper matrices in a similar way as our convention. For the lower fermions, the generator is defined equivalent to what we have reported. They also note that it closes the homogeneous part of the Carroll algebra. After that they move on to define a new Dirac spinor adjoint, using a new matrix Λ. They report the following terms to be invariant under Carroll tranformations:

$$\begin{aligned}&\bar{\Psi}\tilde{\gamma}_0\partial_t\Psi\\ &m\bar{\Psi}\Psi\\ &m\bar{\Psi}\tilde{\gamma}_0\Psi.\end{aligned}$$

We can see that these three terms match three of the four invariant terms we have found, however we have also found invariant term

$$i\bar{\Psi}\partial_t\Psi,$$

of which there is no mention in the paper. They proceed by showing that the Hamiltonian for the massless fermion is zero.

Then for the upper index fermion, they report the same problem, that defining the generator as

$$\hat{\Sigma}^{\mu\nu} \equiv \frac{1}{4}[\hat{\gamma}^{\mu}, \hat{\gamma}^{\nu}],$$

results in a vanishing rotation generator $\hat{\Sigma}^{ij} = 0$. They propose a new generator

$$\hat{\Sigma}_c^{ij} \equiv \frac{1}{4}[\hat{\gamma}^i, \hat{\gamma}_c^j] + \frac{1}{4}[\hat{\gamma}_c^i, \hat{\gamma}^j],$$

where $\hat{\gamma}_c^i$ is defined as

$$\hat{\gamma}_c^i = -\mathcal{C}\hat{\gamma}^i\mathcal{C}^{-1},$$

with

$$\mathcal{C} = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}.$$

They then report to find a rotation matrix of the form

$$\hat{\Sigma} = \frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}.$$

We have not been able to recover the same result, due to σ^2 commuting with itself and and anticommuting with σ^1 and σ^3. The explicit matrix representations that they have given however do close the homogeneous part of the Carroll algebra.

If we redefine the rotation matrix as

$$\hat{\Sigma}_t^{ij} \equiv \frac{1}{4}[\hat{\gamma}^i, \hat{\gamma}^{j\dagger}] + \frac{1}{4}[\hat{\gamma}^{i\dagger}, \hat{\gamma}^j],$$

we do find proper explicit matrix representations for our rotation generators. This together with the former boost generator

$$\hat{\Sigma}^{0i} \equiv \frac{1}{4}[\hat{\gamma}^0, \hat{\gamma}^i],$$

does close the homogeneous part of the Carroll algebra

$$\begin{aligned}
&[\hat{\Sigma}^{0i}, \hat{\Sigma}^{0j}] = 0 \\
&[\hat{\Sigma}^{0i}, \hat{\Sigma}_t^{jk}] = -\delta^{ij}\hat{\Sigma}^{0k} + \delta^{ik}\hat{\Sigma}^{0j} \\
&[\hat{\Sigma}_t^{ij}, \hat{\Sigma}_t^{kl}] = -\delta^{jk}\hat{\Sigma}_t^{il} + \delta^{ik}\hat{\Sigma}_t^{jl} - \delta^{il}\hat{\Sigma}_t^{jk} + \delta^{jl}\hat{\Sigma}_t^{ik}.
\end{aligned}$$

The open question remains if the boost invariant Lagrangian that we have found is also invariant under the newly defined rotation operator. Apart from this, it remains to be seen if this is a proper way to redefine the rotation generator, and if another way can be found to solve the problem of the vanishing rotation generator.

The authors do note that under their redefinition of the rotation generator, the found Lagrangian is rotation invariant, and go on to find that the Hamiltonian for upper fermions is non-vanishing. They remark that lower fermions seem to correspond to the electric Carroll theory, and upper fermions to the magnetic Carroll theory.

5 Conclusions and outlook

We have seen in this thesis that the two different Carroll limits of the metric result in two different Carroll theories, one for lower fermions and one for upper fermions. This split arises because two different Carroll Clifford algebras can be defined for the two metrics. We have treated the algebra for lower fermions and upper fermions in four dimensions.

We started with defining a new Clifford algebra for lower fermions, for which we found two different matrix representations of the lower Carroll gamma matrices that satisfy the constraints. Using the newly defined gamma matrices, we constructed a Carroll spinor algebra that closes the algebra for Carroll boosts and rotations. This definition of the Carroll spinor generator has the property that the commutator between the boosts becomes zero, which is in line with the formulation of the Carroll algebra in earlier literature. We then went on to define a new Carroll-Dirac adjoint spinor, that allowed us to find four Carroll invariant components. With these components we formed four Carroll invariant Lagrangians, two describing massless Carroll fermions and four describing massive Carroll fermions. In these invariant terms, all spatial derivatives drop out. We found that the massless Carroll fermions have a zero Hamiltonian density, while the massive Carroll fermions have a nonzero Hamiltonian density. We also discussed that chirality does not seem to be trivially conserved in the Carroll limit.

Then we proceeded with defining the Clifford algebra for upper fermions. We stated a matrix representation that satisfied the constraints, from which we went on to define the Carroll generators in a similar fashion as for the lower gamma indices. However, here we ran into a problem, because using this definition causes the rotation matrices to vanish. This thus is not a valid definition for the Carroll generators, because it does not recover the Carroll algebra. However, it did render a proper expression for the boost generators, which we used to find a Carroll boost invariant term for the upper fermions. We have also found a boost invariant mass term. We went on to define a boost invariant Lagrangian for massless upper fermions and for massive upper fermions. The Hamiltonian densities for massless and massive upper fermions are both nonzero.

We then went on to discuss relevant work on Carroll fermions, and comparing our results earlier results. Our research on Carroll fermions aligns significantly with the work of Bagchi et al., which also explores four-dimensional Carroll fermions. Both studies use the same metric conventions and define the lower and upper Carroll Clifford algebras similarly. Bagchi et al. also define the generator for lower fermions in a manner equivalent to our approach, noting its closure of the homogeneous part of the Carroll algebra. They further define a new Dirac spinor adjoint and identify three invariant terms under Carroll transformations, which match three of the four invariant terms we found. However, they do not mention the fourth term we identified.

For upper fermions, Bagchi et al. encounter the same issue of a vanishing rotation generator. They propose a new generator and find a rotation matrix that closes the Carroll algebra. We were unable to replicate this result due to the commuting and anticommuting properties of the matrices involved. However, by redefining the rotation matrix, we found proper explicit matrix representations for our rotation generators that close the homogeneous part of the Carroll algebra. It remains to be seen whether the boost invariant Lagrangian we found is also invariant under the newly defined rotation operator and whether this redefinition of the rotation generator is appropriate. This is an open question that requires further exploration.

Bagchi et al. find their Lagrangian to be rotation invariant under their redefined rotation generator and note a non-vanishing Hamiltonian for upper fermions. They suggest that lower fermions correspond to the electric Carroll theory and upper fermions to the magnetic Carroll theory. Further research is necessary to increase our understanding of the Carroll limit of fermions and how the structures we have discussed emerge from their relativistic parent theories. This would clarify the origin of the two distinct fermion species we have identified. Determining whether the lower and upper fermions correspond to electric and magnetic theories, as per current nomenclature, would further elucidate the properties of the theories we have uncovered.

It is still unclear how and if there is a form of chirality for Carroll fermions. This remains up for further study. Another next step in research could be finding Carroll invariant interaction terms to add to the Lagrangians. Lastly, the natural path now is to progress towards quantization of the theory of Carroll fermions.

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