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Spin transport coefficients from Holography

Master's thesis

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Abstract

The hydrodynamic theory of spin current is a useful tool that can be used to describe different phenomena, ranging from the spin in liquid metal to the global spin polarization in heavy-ion collisions. Recently, exhaustive analyses of spin hydrodynamics have been performed and the constitutive relations are obtained. The spin coefficients that appeared in the constitutive relation are vital to the property of the spin current. The heavy-ion collision involves strong interaction so we can't obtain these coefficients from the correlation functions in quantum field theory.

The holography principle, which implies we may use a classical bulk gravity theory to calculate the correlation functions in strongly coupled boundary quantum field theory, could be a method to calculate these spin transport coefficients. In this paper, we choose a simple vector field model as the holography model. This vector field is dual to the trace of contorsion in the hydrodynamic theory. We performed the calculation at an AdS-Schwartzschild background spacetime and we treat the background value of the vector field as a small number. We constructed the counterterms for the vector field action at the probe limit. We obtained Kubo relations for some spin transport coefficients base on the research [1]. We calculated these transport coefficients and found them proportional to the slow falloff mode of the vector field.

Acknowledgement

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Contents

1 Introduction

Hydrodynamics is a low-energy effective theory that can be used to describe classical or quantum many-body systems. It can be used to describe a wide range of phenomena from flow in blood vessels to the plasma emitted by stars. Recently, interest in the hydrodynamic theory in the presence of a conserved angular momentum density has increased. The angular momentum density can be considered as the combination of vorticity and spins. Because of the spin-orbit coupling, vorticity and spin current can influence each other. Takahashi *et al*. [2] reported the first observation of the coupling between vorticity and spin current. Figure 1 shows their equipment to generate the spin current. The left panel shows the Hall effect, which means adding a magnetic field to the electron current causes an electron voltage in the perpendicular direction. The right panel shows that adding a gradient of vorticity causes a spin voltage that can generate the spin current.

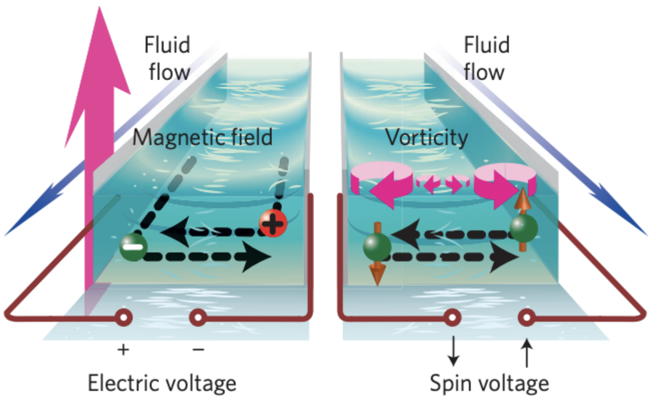

Figure 1: Schemaic illustrations for spin hydrodynamic generation

In the context of ultra-realistic heavy-ion collisions, the fragments on two sides are usually not aligned center-to-center. These collisions create quark-gluon plasma which can be well described by hydrodynamic theory [3–7]. The angular momentum caused by non-central collisions creates a strong vortical structure that couples to the spin current. Extensive research on this subject has been performed [8–13]. Recently, the global spin polarization of Λ and $\bar{\Lambda}$ has been measured at RHIC [14, 15] and hydrodynamic analysis of spin current [16] have shown a good fit with the experimental results.

A comprehensive analysis of spin current hydrodynamics in the presence of torsion has been derived in [1]. The torsion is necessary because it can avoid ambiguity in defining the energy-momentum tensor and the spin current. The constitutive relations for spin current in the second order in derivative have been constructed, which brings tens of transport coefficients. Since heavy-ion collisions involve strong interaction, it's not possible to calculate them from the correlation functions in QFT. These transport coefficients may be determined by using Bayesian analysis on experimental data [17]. The holography principle, which states that a strongly coupled gauge theory is equivalent to a higher dimensional gravity theory, has been used to investigate the hydrodynamic theory in heavy-ion collisions for many years [18–20]. The essential idea is that transport coefficients correspond to the response of current to external sources, using holography we can relate this source to a field in the gravity theory and calculate the response. In fact, the shear viscosity of quark-gluon plasma calculated from holography [21, 22] is similar to the experimental data with hydrodynamic analysis [23]. This result inspired us to investigate the spin current transport coefficients from holography.

In this paper, we choose a simple vector field model as the holography model. This vector field is dual to the trace of contorsion in the hydrodynamic theory. In Chapter 2 we introduced preliminary knowledge about the hydrodynamic theory, spin current, and the basics of holography. In Chapter 3 we summarized some results that we need to use form [1]. In Chapter 4 we investigate our model by solving the equations of motion perturbatively. We obtained some transport coefficients and found them proportional to the slow falloff mode of the vector field.

2 Preliminary knowledge

2.1 Hydrodynamics

Hydrodynamics was invented to describe the dynamics of liquid. From the field theory's point of view, hydrodynamics is an effective field theory at the low momentum limit $\omega \to 0, k \to 0$. It describes the behavior of macroscopic quantities, for example, velocity field and temperature fields.

There are two important ingredients for hydrodynamics: conservation law and constitutive relation. The conservation law comes from the symmetry of a theory and serves as the equation of motion. Complicated microscopic behaviors are simplified as we zoom out the length scale, they become constitutive relations that describe the relations between currents and hydrodynamic variables.

Let's consider relativistic ideal fluid as an example. Translation invariance gives the conservation law of energy-momentum tensor:

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.1}$$

The constitutive relation for ideal fluid is:

$$T^{\mu v} = (\varepsilon + P)u^\mu u^v + P\eta^{\mu v}. \tag{2.2}$$

This relation shows the expectation value of the energy-momentum tensor is only determined by the velocity field $\mu(x)$, energy density field $\epsilon(x)$, and pressure field $P(x)$. Different fluids, though they can obey the same conservation equation, their constitutive relations are different. For example, the constitutive relation for viscous fluid contains an extra term on the right-hand side of (2.1):

$$\tau_{ij} = -\eta \left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_k u^k \right) - \zeta \delta_{ij}\partial_k u^k. \tag{2.3}$$

η is the shear viscosity and ζ is the bulk viscosity.

Bringing the constitutive relation into the conservation law of ideal fluid, we can get the hydrodynamic equations. The time component in conservation law is the relativistic continuity equation:

$$\partial_\mu T^{\mu 0} = 0 \to \frac{d\varepsilon}{d\tau} + (\varepsilon + P)\partial_\mu u^\mu = 0. \tag{2.4}$$

The spatial component in conservation law is:

$$\partial_\mu T^{\mu i} = 0 \to (\varepsilon + P)u^\mu \partial_\mu u^i + \partial^i P = 0. \tag{2.5}$$

This is the relativistic Euler's equation, which is essentially the relativistic version of Newton's second law $F = ma$.

2.2 Spin currents

2.2.1 Currents

Noether's theorem tells us that a continuous global symmetry leads to a conserved current. Assume we have some Lagrangian $\mathcal{L}$ and the variation of a field

ϕ_a under an infinitesimal symmetry transformation is $\delta\phi_a$, then there exists a current:

$$J_a^\mu = -\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\phi_a, \tag{2.6}$$

which is conserved $\partial_\mu J_a^\mu = 0$. For example, the Dirac Lagrangian

$$\mathcal{L}_\psi = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \tag{2.7}$$

is invariant under a global $U(1)$ transformation $\psi \to e^{iq\alpha}\psi$, where q represents the charge of this spinor. Its Noether's current equals to :

$$J^\mu = q\bar{\psi}\gamma^\mu\psi. \tag{2.8}$$

There is another way to obtain this conserved current, which requires promoting the global symmetry to a local gauge symmetry. We can achieve this by promoting the derivative $\partial\mu$ to a covariant derivate ∇_μ. Under a gauge transformation $\phi_a \to U(\alpha(x))$, the covariant derivative should transform as:

$$\nabla_\mu \to U(\alpha(x))\nabla_\mu \tag{2.9}$$

In the Dirac Lagrangian example, the gauge transformation for a spinor is $\psi \to e^{iq\alpha(x)}\psi$. The promotion of derivatives is:

$$\partial_\mu \to \nabla_\mu = \partial_\mu - iqA_\mu. \tag{2.10}$$

A_μ is a gauge field that transforms as $A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x)$. The new Dirac Lagrangian with gauge symmetry is:

$$\mathcal{L}_\psi = i\bar{\psi}\nabla\!\!\!/\psi - m\bar{\psi}\psi. \tag{2.11}$$

We can get the transformation law of action S under this symmetry is:

$$\delta S = \int d^dxJ^\mu\delta A_\mu + E.O.M.\delta\psi, \tag{2.12}$$

where the $E.O.M.$ represent the equation of motion for ψ. Therefore, we can get the electric current by differentiating the on-shell action with respect to the $U(1)$ gauge field. As we will see, the definition of spin current can be achieved in the same way but with a Lorentz gauge field.

2.2.2 Spin current for Dirac spinor

Spin current is the current for Lorentz symmetry. In an N+1 spacetime, the Lorentz symmetry group is $SO(1,N)$, which has N boost generators and $\frac{N(N-1)}{2}$ rotation generators. The Lorentz transformation law for a spinor field ϕ is:

$$\phi \to e^{\frac{i}{2}\lambda_{ab}M^{ab}}\phi, \tag{2.13}$$

where M^{ab} are the generators for Lorentz transformation and λ_{ab} are the parameters to describe the amount of transformation. Spinors are the representations of the Lorentz group, which means two successive Lorentz transformation acting on the spinors should combine to be a Lorentz transformation. Therefore, generators M^{ab} need to satisfy the *Lorentz algebra* relation:

$$\left[M^{ab}, M^{cd}\right] = i\eta^{ac}M^{bd} - i\eta^{bd}M^{ad} + i\eta^{bd}M^{ac} - i\eta^{ad}M^{bc} \tag{2.14}$$

For Dirac spinors, these generators can be represented by gamma matrices γ^{μ}:

$$M^{ab} = \frac{i}{4}\left[\gamma^{a}, \gamma^{b}\right]. \tag{2.15}$$

Bringing the Dirac Lagrangian (2.7) to Noether's theorem, we get the Dirac spin current :

$$J^{\mu ab} = \frac{1}{8}\bar{\psi}\gamma^{\mu}\left[\gamma_{a}, \gamma_{b}\right]\psi. \tag{2.16}$$

Therefore, a full description of spin current $S^{\mu ab}$ needs a spacetime index μ and two group index a, b.

Similar to what we showed in electric current, we can promote the global Lorentz symmetry to a gauge Lorentz symmetry. The covariant derivative is :

$$\partial_{\mu} \to \nabla_{\mu} = \partial_{\mu} + \frac{1}{2}\omega_{\mu ab}M^{ab}, \tag{2.17}$$

where $\omega_{\mu ab}$ is a gauge field called *spin connection*. From the definition of gauged Dirac Lagrangian 2.11 we can see the variation of action is

$$\delta S = \int d^{d}x J^{\mu ab}\delta\omega_{\mu ab} + \text{ E.O.M. } \delta\psi. \tag{2.18}$$

Readers may wonder about the necessity of having a gauge Lorentz symmetry. In fact, if we want to describe spinors in curved spacetime, then we need to include this gauge symmetry. In curved spacetime, the group index a, b can also be considered as the index in a locally flat frame which we call *Lorentz frame*. To understand the relation between this Lorentz frame and spacetime frame we need to introduce a *Vielbein Formalism*

2.2.3 Vielbein formalism

In the local Lorentz frame, we can think the basis in tangent space is chosen such that the metric is the Minkowski metric. This basis is connected with the spacetime coordinate with the vielbein ${e_{\mu}}^{a}$:

$$g_{\mu\nu} = {e_{\mu}}^{a}{e_{\nu}}^{b}\eta_{ab} \tag{2.19}$$

The inverse of vielbein ${e^{\mu}}_{a}$ is defined by:

$${e^{\mu}}_{a}{e_{\mu}}^{b} = \delta^{a}_{b}. \tag{2.20}$$

With the vielbein and its inverse, we can freely change vectors represented in spacetime coordinate with vectors in the Lorentz frame

$$V^{a} = {e_{\mu}}^{a}V^{\mu}, V_{\mu} = {e_{\mu}}^{a}V_{a}. \tag{2.21}$$

Using the Minkowski metric η_{ab}, spacetime metric $g_{\mu\nu}$, and their inverses, we can freely move the tensor's index up or down. V^a transforms as a vector in local Lorentz transformation and is invariant under spacetime transformation, and vice versa for V^μ. Under a coordinate transformation ξ^μ and a local Lorentz transformation $\theta^a{}_b$, a generic tensor $Q^a{}_b{}^\mu{}_\nu$ transforms as

$$\delta Q^a{}_b{}^\mu{}_\nu = \mathcal{L}_\xi Q^a{}_b{}^\mu{}_\nu - \theta^a{}_c Q^c{}_b{}^\mu{}_\nu - \theta_b{}^c Q^a{}_c{}^\mu{}_\nu, \tag{2.22}$$

Where $\mathcal{L}_\xi$ is the Lie derivative along the ϵ_μ. The Lie derivative describes the difference of a tensor field at the same coordinate under an active coordinate transformation

$$\begin{aligned} x^\mu &\to x'^\mu = x^\mu - \xi^\mu, \\ Q^\mu{}_\nu(x^\mu) &\to Q'^\mu{}_\nu(x'^\mu), \\ \mathcal{L}_\xi Q^\mu{}_\nu(x^\mu) &= Q'^\mu{}_\nu(x^\mu) - Q^\mu{}_\nu(x^\mu). \\ &= \xi^\lambda \partial_\lambda Q^\mu{}_\nu - \partial_\lambda(\xi^\mu) Q^\lambda{}_\nu + \partial_\nu(\xi^\lambda) Q^\mu{}_\lambda. \end{aligned} \tag{2.23}$$

For the Lie derivative with $Q^a{}_b{}^\mu{}_\nu$, just keep in mind the Lorentz index part doesn't change under coordinate transformation, so its Lie derivative takes the same form as in 2.23.

The covariant derivative ∇_μ should make sure that it transforms as a tensor both under local Lorentz transformation and spacetime transformation. For a tensor $X^a{}_b$, the covariant transformation is:

$$\nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega_\mu{}^a{}_c X^c{}_b - \omega_\mu{}^c{}_b X^a_c. \tag{2.24}$$

With the metric compatibility $\nabla_\mu \eta_{ab} = 0$, we find the spin connection is antisymmetric at the last two indexes:

$$\omega_{\mu ab} = \omega_{\mu[ab]} \tag{2.25}$$

Vielbein $e_\mu{}^a$ transforms as a tensor. The transformation rule for $\omega_\mu{}^a{}_b$ is:

$$\delta\omega_\mu{}^a{}_b = \mathcal{L}_\xi \omega_\mu{}^a{}_b + \nabla_\mu \theta^a{}_b \tag{2.26}$$

so it transforms as a one-form under spacetime transformation. Although spin connection doesn't transform as a tensor under local Lorentz transformation, its variation $\delta\omega_\mu{}^a{}_b$ does.

On the other way, we learn from GR that the covariant derivative of V^ν is:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma. \tag{2.27}$$

This equation should be compatible with the case when we view $V^\nu = e^\nu{}_a V^a$ and use the Leibniz rule for covariant derivative. Therefore, we get a relation called the *Vielbein postulate*:

$$0 = \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \Gamma^\sigma_{\mu\nu} e_\sigma{}^a = \nabla_\mu e_\nu{}^a. \tag{2.28}$$

In the absence of torsion, the connection $\Gamma^\sigma_{\mu\nu}$ is symmetric on the two lower indexes, which can be uniquely determined by combing the metric compatibility condition $\nabla_\mu g^{\nu\sigma}$. The result is Christoffel connection $\tilde{\Gamma}^\sigma_{\mu\nu}$, defined as

$$\tilde{\Gamma}^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right). \tag{2.29}$$

A generic connection contains an antisymmetric term in the two lower indexes, which is related to the torsion $T^{\sigma}_{\mu\nu} := 2\Gamma^{\sigma}_{[\mu\nu]}$. Therefore, a generic spin connection can be decomposed as a torsionless part $\tilde{\omega}_{\mu}{}^{a}{}_{b}$ plus a contortion tensor $K_{\mu}{}^{a}{}_{b}$:

$$\omega_{\mu}{}^{a}{}_{b} = \tilde{\omega}_{\mu}{}^{a}{}_{b} + K_{\mu}{}^{a}{}_{b}, \tag{2.30}$$

The contortion tensor is related to the torsion tensor by

$$T^{a}{}_{\mu\nu} = K_{\mu}{}^{a}{}_{\nu} - K_{\nu}{}^{a}{}_{\mu} \tag{2.31}$$

From now on, we denote the covariant derivative with respect to torsionless connection as $\tilde{\nabla}$, the torsionless spin connection can be obtained from the vielbein postulate:

$$\tilde{\omega}_{\mu}{}^{a}{}_{b} = \partial_{\mu} e^{\nu}{}_{b} e_{\nu}{}^{a} - e^{\nu}{}_{b} \tilde{\Gamma}^{\sigma}_{\mu\nu} e_{\sigma}{}^{a} \tag{2.32}$$

2.3 Kubo formula

Kubo formula is the equation that describes the relation between retarded Green function and physical quantity. In order to get the Kubo formula, we need information about constitutive equations and linear response theory. The linear response theory study's the response of an operator's average to an external source up to the linear order. We can study it by using perturbation theory in quantum mechanics. The full Hamiltonian can be written as:

$$H = H_0 + \delta H(t), \tag{2.33}$$

where H_0 is the original Hamiltonian and $\delta H(t)$ is the perturbed Hamiltonian:

$$\delta H(t) = -\int d^3x \phi^{(0)}(t, \boldsymbol{x}) O(\boldsymbol{x}), \tag{2.34}$$

Where $\phi^{(0)}$ is the source of perturbation and O is the operator coupled to it.

The state in a real experiment is usually not a pure state, so we need to introduce a density operator $\rho := \sum_i w_i |\alpha_i\rangle \langle\alpha_i|, \mathrm{tr}(\rho) = 1$ to describe the ensemble. The ensemble average of O is written as

$$\langle O(t, \boldsymbol{x})\rangle_s = \mathrm{tr}[\rho(t) O(\boldsymbol{x})], \tag{2.35}$$

where we use subscript "s" to represent the presence of an external source. In order to see the influence of perturbed Hamiltonian, it's most convenient to work in the interaction picture. The operator $O_I(t, t_0)$ in interaction picture evolves as $O_I = U_0^{-1} O U_0$, where $U_0(t, t_0) = e^{-iH_0(t-t_0)}$ is the time evolution operator of original Hamiltonian. The whole evolution operator $U(t, t_0) = e^{-iH(t-t_0)}$ can also be written as: $U(t, t_0) = U_0(t, t_0) U_1(t, t_0)$, where U_1 should satisfy $i\partial_t U_1 = \delta H_I U_1$. (2.35) is equal to

$$
\begin{aligned}
\langle O(t,\boldsymbol{x})\rangle_s &= \operatorname{tr}\left[\rho\left(t_0\right) U_1^{-1}\left(t,t_0\right) O_I(t,\boldsymbol{x}) U_1\left(t,t_0\right)\right] \\
&= \operatorname{tr}\left[\rho\left(t_0\right)\left(1+i\int_{t_0}^{t} dt'\delta H_I\left(t'\right)+\cdots\right) O_I(t,\boldsymbol{x})\right. \\
&\quad \left.\times\left(1-i\int_{t_0}^{t} dt'\delta H_I\left(t'\right)+\cdots\right)\right] \\
&= \operatorname{tr}\left[\rho\left(t_0\right) O_I(t,\boldsymbol{x})\right] \\
&\quad - i\operatorname{tr}\left[\rho\left(t_0\right)\int_{t_0}^{t} dt'\left[O_I(t,\boldsymbol{x}),\delta H_I\left(t'\right)\right]\right]+\cdots.
\end{aligned} \tag{2.36}
$$

Notice that the $\rho(t_0)$ is just the density matrix at equilibrium ρ_{eq}, which commutes with the H_0. Therefore, we have

$$
\operatorname{tr}\left[\rho\left(t_0\right) O_I(t,x)\right] = \operatorname{tr}\left[\rho_{\text{eq}} O(\boldsymbol{x})\right]. \tag{2.37}
$$

Taking $t_0 \to -\infty$, we can get the response of an operator's expectation value $\delta\langle O(t,\boldsymbol{x})\rangle := \langle O(t,\boldsymbol{x})\rangle_s - \langle O\rangle$ is

$$
\delta\langle O(t,\boldsymbol{x})\rangle = i\int_{-\infty}^{\infty} d^4x'\theta\left(t-t'\right)\left\langle\left[O(t,\boldsymbol{x}),O\left(t',\boldsymbol{x}'\right)\right]\right\rangle \phi^{(0)}\left(t',\boldsymbol{x}'\right) \tag{2.38}
$$

where $\theta(t-t')$ is the Heaviside step function. This expression is related to the *retarded Green's function* $G_R^{OO}(t-t',\boldsymbol{x}-\boldsymbol{x}')$, which is defined as

$$
G_R^{OO}\left(t-t',\boldsymbol{x}-\boldsymbol{x}'\right) := i\theta\left(t-t'\right)\left\langle\left[O(t,\boldsymbol{x}),O\left(t',\boldsymbol{x}'\right)\right]\right\rangle, \tag{2.39}
$$

The OO is used to represent the Green's function between the same operator O. For a generic case, we can consider retarded Green's function between two different operators O_i, O_j. The response 2.38 can be written as

$$
\delta\langle O(t,\boldsymbol{x})\rangle = \int_{-\infty}^{\infty} d^4x' G_R^{OO}\left(t-t',\boldsymbol{x}-\boldsymbol{x}'\right)\phi^{(0)}\left(t',\boldsymbol{x}'\right). \tag{2.40}
$$

In Fourier space, it becomes a simple relation:

$$
\delta\langle O(k)\rangle = G_R^{OO}(k)\phi^{(0)}(k). \tag{2.41}
$$

The retarded Green's function is related to transport coefficients. Here I show the example of Ohm's law:

$$
\langle J^x\rangle = \sigma E_x^{(0)}, \tag{2.42}
$$

where $\langle J^x\rangle$ is the electric current, $E_x^{(0)}$ is the electric field strength and σ is the electric conductivity. Ohm's law is a constitutive relation that describes the relation between electric current and electric field in a conductor, so σ is a transport coefficient. On the other hand, we can think of this current as a response to the perturbation of the electric field. In the gauge $A_0^{(0)} = 0$, use the relation $E_x^{(0)} = -\partial_t A_x^{(0)}$, we can get the relation between electric conductivity and retarded Green's function in Fourier space:

$$
\sigma = \lim_{\omega\to 0,k\to 0} -\frac{G_R^{J_xJ_x}(\omega,k)}{i\omega}. \tag{2.43}
$$

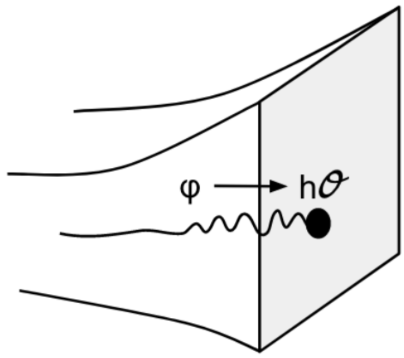

Figure 2: Holographic dictionary. Adapted from [25]

We need to impose the limit condition because Ohm's law is only a low-energy effective relation.

Equations like (2.43) that describe the relation between transport coefficients and Green's function are called the *Kubo formula*. It provides a bridge from the correlation function in microscopic theory to the transport coefficients in the macroscopic theory. For spin current in heavy ion collision, this process involves strong interaction, which means our usual perturbative expansion method in calculating the correlation function doesn't work. However, the holography conjecture states the strongly coupled gauge QFT is equivalent to a classical gravitational theory in a higher spacetime. Therefore, we may calculate Green's function by using holography methods.

2.4 Basics of holography principle

The story of AdS/CFT duality originated from the idea of Maldacena's famous paper [24], which implied that at the t'Hooft 's limit $N \to \infty$ a 4-dimensional $\mathcal{N} = 4$ super Yang-Mills theory corresponds to a $\mathrm{AdS}_5 \times S_5$ supergravity theory. After that, more conjectures about the duality between CFTs and higher dimensional theory on AdS spacetime were imposed. Nowadays, the discussion about this duality is not only limited to CFTs. Inspired by the fact that a 2-dimensional hologram can encode optical information of 3-dimensional objects, people also call this duality as *holography principle*. The lower/higher dimensional theory is referred as boundary/bulk theory respectively.

Roughly speaking, the holography principle claims:

Strongly coupled d-dimensional gauge QFT
= Gravitational theory on a AdS_{d+1} spacetime.

This equivalence relation can be expressed via the GKPW formula(Gubser, Klebanov, Polyakov [26] and Witten [27])

$$Z_{\mathrm{QFT}}\left[\{h_i(x)\}\right] = Z_{\mathrm{Grav}}\left[\{h_i(x)\}\right].$$

On the QFT side Z_{QFT} is the generating functional

$$Z_{\text{QFT}}\left[\{h_i(x)\}\right] \equiv \left\langle e^{i\sum_i \int dx h_i(x)\mathcal{O}_i(x)} \right\rangle_{\text{QFT}}, \tag{2.44}$$

where $h_i(x)$ is the source and $\mathcal{O}_i(x)$ is an operator.

On the gravity side, we need to consider spacetime with a boundary and let the boundary value of a bulk field ϕ_i equal a source h_i on QFT:

$$Z_{\text{Grav.}}\left[\{h_i(x)\}\right] \equiv \int^{\phi_i \to h_i} \left(\prod_i \mathcal{D}\phi_i\right) e^{iS[\{\phi_i\}]}. \tag{2.45}$$

We can also describe the relation $\phi_i \to h_i$ as bulk field ϕ is dual to the operator $\mathcal{O}_i$. The $\mathcal{N} \to \infty$ limit is a classical limit, so we can calculate the gravity theory on a saddle point:

$$Z_{\text{Grav.}}\left[\{h_i(x)\}\right] = e^{iS_{\text{onshell}}\left[\{\phi_i^\star \to h_i\}\right]}. \tag{2.46}$$

It means we set the boundary value of classical bulk fields ϕ_i^* equal to the sources h_i and calculate the action with respect to equations of motion. In practice, we usually found the bulk field diverges when approaching the boundary, so rigorously speaking we are setting the leading divergent modes equal to the sources. We will discuss this in the massive scalar example.

Readers may wonder how can we know which bulk field is dual to which operator. It's worth stressing that the holography principle is still a conjecture while there is much indirect evidence [28–30]. Therefore, instead of asking which bulk field is dual to which operator, it actually works like we assign a bulk field to a source and then add terms to the bulk Lagrangian. For example, the variation of boundary metric δg_{ij} is the source for the energy-momentum T^{ij}. The most natural choice of its bulk dual is still the metric. Therefore, we may choose the Einstein-Hilbert action term to be the bulk action:

$$S[g] = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}\left(R + \frac{d(d-1)}{L^2}\right). \tag{2.47}$$

The cosmological constant term $\frac{d(d+1)}{L^2}$ is included to generate an AdS spacetime.

The reason that people are interested in holography is that we may calculate the correlation functions in strongly coupled boundary QFT by studying classical gravity theory. The n-point function in QFT can be written as

$$\langle O_{i_1}(x_1)\cdots O_{i_n}(x_n)\rangle = \frac{1}{Z_{\text{QFT}}}\frac{\delta^n Z_{\text{QFT}}}{\delta h_{i_1}(x_1)\cdots\delta h_{i_n}(x_n)} \tag{2.48}$$

2.4.1 AdS spacetime

AdS (Anti de Sitter) is a spacetime with constant negative curvature. An n-dimensional AdS spacetime is noted as AdS_n, it can be defined as a hyperbolic-like n-dimensional spacetime embedded in n+1-dimensional flat spacetime with

2 timelike directions:

$$
\begin{aligned}
ds^2 = & - dt^2 - dr^2 + \sum_{i=1}^{d-1} dx_i^2, \\
& - t^2 - r^2 + \sum_{i=1}^{d-1} x_i^2 = -L^2.
\end{aligned}
\tag{2.49}
$$

The curvature of AdS_n is $R = -\frac{n(n-1)}{L^2}$. From this definition, we can see its isometry group is $SO(2, d-1)$. It coincides with the conformal group in d-1 spacetime, which gives us some inspiration about why we need asymptotic AdS bulk spacetime in holography.

In the Hilbert-Einstein action, AdS_n spacetime can be created by adding a cosmological constant term:

$$
S[g] = \frac{1}{2\kappa^2} \int \mathrm{d}^d x \sqrt{-g} \left(R + \frac{(d-1)(d-2)}{L^2} \right). \tag{2.50}
$$

There are many conventions in expressing the metric of AdS_n, the coordinate we choose is a Poincaré-like coordinate:

$$
\mathrm{d}s^2 = \frac{L^2}{r^2} \left(-\mathrm{d}t^2 + \sum_{i=1}^{d-1} dx_i^2 + \mathrm{d}r^2 \right). \tag{2.51}
$$

r here works as a radial direction whose range is $[0, \infty]$. The boundary of AdS_n sits at $r = 0$. To understand this, we can imagine a ball where $r = 0$ represents the sphere and we go into the ball's interior as r increases. The central point of the ball is at $r = \infty$

2.4.2 Gibbons-Hawking term

Einstein's equation can be obtained by requiring the variation of Einstein-Hilbert action to be zero. However, most textbooks only treated the variation of Ricci tensor $R_{\mu\nu}$ as a total covariant derivative and consider it vanishes after we integrate it over the whole manifold and use the Stokes' theorem [31]. However, this is not correct when we consider the spacetime with a boundary. Because the Riemann tensor contains the second derivative of metric, there will be $\nabla \delta g_{\mu\nu}$ term appearing at the boundary after we use Stokes' theorem. This term doesn't have to vanish even when we take the variation of metric vanishes at the boundary. This makes the Lagrangian formalism ill-defined. To eliminate this problem, we can add a Gibbons-Hawking term [32] to the action to eliminate the $\nabla \delta g_{\mu\nu}$ at the boundary:

$$
S = \frac{1}{2\kappa^2} (\int \mathrm{d}^d x \sqrt{-g} R - \int_\partial d^d x \sqrt{\gamma} 2K), \tag{2.52}
$$

where ∂ represent the intergra is performed at the boundary spacetime, γ is the determinant of the induced metric γ_{ij}, K is the trace of extrinsic curvature $K_{\mu\nu}$. The extrinsic curvature is defined as the Lie derivative of induced metric $\gamma_{\mu\nu}$ along the normal vector n^μ perpendicular to the boundary manifold

$$
K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu} \tag{2.53}
$$

The only contribution from the Gibbons-Hawking term in variation cancels with the $\nabla\delta g_{\mu\nu}$ part from the variation of Ricci tensor, so the equation of motion for the metric is the usual Einstein's equation. However, it will contribute to the total action. For example, suppose we are working on a vacuum, then $R = 0$ and the only contribution to the action comes from the Gibbons-Hawking term.

2.4.3 Massive scalar example

Let's consider a massive scalar model as the bulk theory. The action for matter part is written as

$$S[\phi] = -\int \mathrm{d}^{d+1}x\sqrt{-g}\left(\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2\right). \tag{2.54}$$

We should keep in mind there is always an Einstein-Hilbert term in the total action that describes the dynamic of metric. However, we may take the value of the matter field to be small enough such that its energy-momentum tensor can be ignored in Einstein's equation. This is the *probe limit*, i.e., the perturbation of the matter field doesn't influence the metric.

The scalar field $\phi(x)$ is dual to an operator in the boundary QFT. In order to see how the source at the boundary influences the scalar field, we need to solve the equation of motion:

$$\nabla^2\phi - m^2\phi = 0. \tag{2.55}$$

The near-boundary solutions are :

$$\phi \sim r^{\Delta_\pm}, \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}. \tag{2.56}$$

The r^{Δ_-} and r^{Δ_+} solutions is called slow falloff and fast falloff respectively. We can write the general form of near boundary solution as:

$$\phi(x,r) = \phi_{(0)}(x)r^{\Delta_-}(1 + \phi_{(1)}(x)r + \cdots) + \tilde{\phi}_{(0)}(x)r^{\Delta_+}(1 + \tilde{\phi}(x)_{(1)}r + \cdots) \tag{2.57}$$

$\Delta_- < 0$, so the slow falloff term diverges as we go to the horizon $r \to 0$. In the holographic dictionary, we need to set the slow falloff mode equal to the source $\phi_{(0)}(x) \to h(x)$. Using the equation of motion and integrating by part, the on-shell action for this bulk theory can be written as an action at a boundary spacetime:

$$S_{\text{onshell}}[\phi] = \int_{r=\epsilon} \mathrm{d}^d x\sqrt{-\gamma}\left(n^i\nabla_i\phi\right), \tag{2.58}$$

where ϵ is the cutoff radius. Because of the slow falloff mode, this on-shell action diverges as the cutoff $\epsilon \to 0$. We need to introduce a counterterm action S_{ct} that consists of covariant local operators to cancel the divergence. The renormalized action is defined as:

$$S_{\text{Ren}} = \lim_{\epsilon\to 0}\left(S_{\text{onshell}} + S_{\text{ct}}\right) \tag{2.59}$$

For massive scalar, the counter terms can be chosen as [33]:

$$S_{\text{ct}} = \int_{r=\epsilon} d^d x\sqrt{-\gamma}\left(\frac{\Delta_-}{2}\phi^2 + \frac{1}{2(\Delta_+ - \Delta_- - 2)}\phi\Box_\gamma\phi\right) + \cdots \tag{2.60}$$

We can get the expectation value of the operator on QFT by variating the renormalized on-shell bulk action with respect to the slow falloff mode:

$$\langle O(x) \rangle = \frac{\delta S_{\text{Ren}}}{\delta \phi_{(0)}} \sim \tilde{\phi}_0(x). \tag{2.61}$$

We find the expectation value of the operator corresponds to the fast falloff mode of its dual bulk field. This pattern is typical even in other bulk models because the kinetic term is quadratic in the derivative. To conclude, the holography principle tells us:

$$\begin{aligned} &\text{slow falloff of bulk field} && \rightarrow \text{source,} \\ &\text{fast falloff of bulk field} && \rightarrow \text{expectation value of operator.} \end{aligned} \tag{2.62}$$

With the holography principle, we can compute the retarded Green's function (2.41) by taking the second functional derivative of the classical bulk action:

$$G_R^{OO}(k) = \frac{\delta S_{\text{Ren}}}{\delta \phi_{(0)}(k) \delta \phi_{(0)}(k)}. \tag{2.63}$$

Then, using the Kubo Fomulas we can get the transport coefficients from holography.

3 Spin current hydrodynamics

In this section, we summarized the necessary results from [16] [1] that we need to use in calculating transport coefficients from the holography.

Suppose the effective action for a spin hydrodynamic system is S. We define the energy-momentum tensor $T^{\mu}{}_{a}$ and spin current $S^{\mu}{}_{ab}$ from the variation of action:

$$\delta S = \int \mathrm{d}^d x |e| \left(T^{\mu}{}_{a} \delta e_{\mu}{}^{a} + \frac{1}{2} S^{\mu}{}_{ab} \delta \omega_{\mu}{}^{ab} \right). \tag{3.1}$$

Recall the relation between the vielbein and metric from (2.19), this definition of energy-momentum tensor is equivalent to the usual definition that we learned from general relativity [31] up to an overall minus sign:

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \tag{3.2}$$

When writing the relation in (3.1), we have implicitly considered the vielbein and spin connection to be independent. However, in the torsionless case, as we have seen in (2.32), the spin connection is determined by the vielbein. This causes ambiguity in defining the energy-momentum tensor and spin current. If we consider that we are variating the action at a torsionless spacetime, then the contribution of $\delta\omega_{\mu}{}^{a}{}_{b}$ should be totally distributed to the variation of vielbein $\delta e_{\mu}{}^{a}$. Therefore, the spin current should be zero. Or, we can consider the variation is performed at a torsionful background, then we set the torsion to zero. In this case, the spin connection is independent of vielbein and the spin current is S^{μ}_{ab}. Therefore, the ambiguity is we can always shuffle the dependence of $\delta\omega_{\mu}{}^{a}{}_{b}$ to a dependence of $\delta e_{\mu}{}^{a}$. To solve this ambiguity, we need to consider spacetime with torsion. Now the spin connection can be considered as a vielbein dependent part $\tilde{\omega}_{\mu}{}^{a}{}_{b}$ plus a contorsion tensor $K\mu^{a}{}_{b}$ as in (2.30). Therefore, we can define the energy-momentum tensor and spin connection as

$$\begin{aligned} T^{\mu}{}_{a} &= \frac{1}{|e|} \left.\frac{\delta S}{\delta e_{\mu}{}^{a}}\right|_{\omega_{\mu}{}^{ab}} = \frac{1}{|e|} \left(\left.\frac{\delta S}{\delta e_{\mu}{}^{a}}\right|_{K_{\mu}{}^{ab}} - \frac{\delta \tilde{\omega}_{\mu}{}^{cd}}{\delta e_{\mu}{}^{a}} \left.\frac{\delta S}{\delta K_{\nu}{}^{cd}}\right|_{e_{\mu}{}^{a}} \right), \\ S^{\mu}{}_{ab} &= \frac{2}{|e|} \left.\frac{\delta S}{\delta \omega_{\mu}{}^{ab}}\right|_{e_{\mu}{}^{a}} = \frac{2}{|e|} \left.\frac{\delta S}{\delta K_{\mu}{}^{ab}}\right|_{e_{\mu}{}^{a}}, \end{aligned} \tag{3.3}$$

where the $|_{\text{parameter}}$ denotes the differential is taken with respect to the fixed parameter. From now on, we consider spin connection and vielbein to be independent parameters, and we will omit the condition $|_{\text{parameter}}$ when writing the functional derivative.

3.1 Conservation law

There are two gauge symmetries in this spin hydrodynamic system: diffeomorphism and local Lorentz symmetry. The conditions that $\delta S = 0$ under these two symmetry transformations can give us two conservation laws.

Under these two transformations,the change in vielbein and spin connection are

$$\begin{aligned}&\text{local Lorentz transformation:}\\&\delta e_\mu{}^a = \mathcal{L}_\xi e_\mu{}^a, \quad \delta\omega_\mu{}^{ab} = \mathcal{L}_{\tilde{\xi}}\omega_\mu{}^{ab}\\&\text{local Lorentz transformation:}\\&\delta e_\mu{}^a = -\theta^a{}_b e^b_\mu, \quad \delta\omega_\mu{}^{ab} = \nabla_\mu\theta^{ab}.\end{aligned} \tag{3.4}$$

Therefore, we have

$$\begin{aligned}0 &= \int \mathrm{d}^d x|e| \left(-\lambda^{ab} e_{\mu b} T^\mu_a + \frac{1}{2}\left(D_\mu\lambda^{ab}\right) S^\mu_{abab}\right.\\0 &= \int \mathrm{d}^d x|e| \left[T^\mu{}_a \left(D_\mu\left(\xi^v e_v{}^a\right) + \xi^v\left(2K_{[v^a\mu]} - \omega_v{}^a{}_b e_\mu{}^b\right)\right)\right.\\&\quad \left.+\frac{1}{2}S^\mu{}_{ab}\left(\xi^v R^{ab}{}_{v\mu} + D_\mu\left(\xi^v\omega_v{}^{ab}\right)\right)\right]\end{aligned} \tag{3.5}$$

After some tedious calculation, we can get the two conservation laws:

$$\begin{aligned}\tilde{\nabla}_\mu T^{\mu v} &= \frac{1}{2}R^{\rho\sigma v\lambda}S_{\lambda\rho\sigma} - T_{\rho\sigma}K^{v\rho\sigma}\\\tilde{\nabla}_\lambda S^\lambda{}_{\mu v} &= 2T_{[\mu v]} + 2S^\lambda_{\rho[\mu}e_{v]}{}^a e^{\rho b}K_{\lambda ab}.\end{aligned} \tag{3.6}$$

Although we describe these two equations as "conservation law" by following the convention of fluid dynamics. We can see the energy-momentum tensor and spin current themself are not conserved. There are some complicated interactions between them that make them transform into each other. There are many tensors in (3.6), in order to further investigate the details of this theory, we need to decompose every tensorial part with respect to velocity u^μ. The part perpendicular to velocity belongs to the subgroup $SO(d-1) \subset SO(1, d-1)$, and we can further decompose them into scalars, vectors, and tensors.

3.2 Constitutive relations

The method that we used to build constitutive relations is bottom-up, i.e., we need to conclude all the possible terms that are allowed by symmetry. Then we can variate the action as in (3.3) to get the constitutive relations for energy-momentum tensor and spin current.

However, there are an infinite number of such terms. Luckily, we are working on a low momentum limit $k_\mu \to 0$, so we can consider the action as an expansion in terms of derivative ∇_μ. If the action doesn't contain any explicit derivative, then it describes a perfect fluid. In this paper, we only include the leading order non-ideal contribution to the action.

We should be careful in counting the order of derivative expansion. The conservation (3.6) implies that the first-order term in the antisymmetric part of energy-momentum tensor $T_{[\mu\nu]}$ should be second-order in spin current and the symmetric part in energy-momentum tensor. To avoid confusion, we will refer to the order as the derivative order in spin current and the symmetric part in energy-momentum tensor. Therefore, for the leading-order non-ideal part, we need to include second-order terms in the action because of the presence of $T_{[\mu\nu]}$. The action can be written as

$$S = \int d^d x \left(P + S_{(1)} + S_{(2)}\right) \tag{3.7}$$

Where P represents the action for ideal fluid, $S_{(1)}, S_{(2)}$ represent the first and second-order derivative terms respectively. The complete analysis can be very long, here we only give an example of constructing constitutive relations from the hydrostatic ideal fluid and summarize the rest results.

3.3 Hydrostatic ideal fluid

If the fluid is influenced by a time-independent source, then it will eventually decay to an equilibrium state which we describe as hydrostatic. The hydrostatic state is a solution to the equations of motion, so it put a constraint on the hydrodynamic solution. We can first construct the constitutive relations for hydrostatic fluid and then add hydrodynamic terms which vanish on the hydrostatic limit. Here we take the ideal fluid as an example to show how can we construct the constitutive relation in the hydrostatic case.

The time independence of hydrostatic fluid implies we can find a coordinate in which all variables of this fluid don't change with time. In a general coordinate, this corresponds to a timelike Killing vector V^μ along all variables are invariant. However, in our case, we also have the gauge Lorentz symmetry, so any two fields that can be connected by a local Lorentz symmetry should describe the same state. Therefore, the right condition for hydrostatic is that there exists a timelike Killing vector V^μ and a local Lorentz transformation field ${\theta_V}^a{}_b$ that makes the vielbein and spin connection invariant:

$$\begin{aligned} 0 &= \mathcal{L}_V e^a{}_\mu - {\theta_V}^a{}_b e^b{}_\mu, \\ 0 &= \mathcal{L}_V {\omega_\mu}^a{}_b + \nabla_\mu {\theta_V}^a{}_b. \end{aligned} \tag{3.8}$$

It would be convenient to choose a gauge such that $V^\mu = (1, 0, \cdots, 0)$, ${\theta_V}^a{}_b = 0$. But this complete form is important to keep variables covariant.

To construct the constitutive relations, we first need to choose the hydrodynamic variables. These variables should be: ${e_\mu}^a, {\omega_\mu}^a{}_b, V^\mu, {\theta_V}^a{}_b$. The ${e_\mu}^a$ and V^μ transforms as tensor, but ${\omega_\mu}^a{}_b$ and ${\theta_V}^a{}_b$ don't. In order to make our final expression covariant, we need to combine them to be a covariant form:

$$\mu^{ab} = \frac{V^\mu {\omega_\mu}^{ab} + {\theta_V}^{ab}}{\sqrt{-V^2}} \tag{3.9}$$

We can also define

$$T^{-1} = \sqrt{-V^2} \quad u^\mu = \frac{V^\mu}{\sqrt{-V^2}}, \tag{3.10}$$

which we can show that they can be interpreted as the temperature and velocity field of the fluid respectively. Now, we have the hydrodynamic variables

$T, u^\mu, \mu^{ab}, e^a{}_\mu$, then we can use them to construct all possible scalars and add them into the action. The action for hydrostatic ideal fluid can be expressed as

$$S = \int d^d x |e| P \left(T, u^\mu, \mu^{ab}, e^a{}_\mu \right). \tag{3.11}$$

We can decompose μ^{ab} with respect to the velocity to extract its unique components:

$$\mu^{ab} = u^a m^b - u^b m^a + M^{ab}, \tag{3.12}$$

where $u_a M^{ab} = 0$ and $u_a m^a = 0$. After this decomposition, the scalars formed by contracting and μ^{ab} and other variables can be expressed in terms of contraction between m^a and M^{ab}. We can construct tensors $\mathcal{M}_{(n)}{}^a{}_b$

$$\begin{aligned} &\mathcal{M}_{(0)}{}^a{}_b = \delta^a{}_b, \quad \mathcal{M}_{(1)}{}^a{}_b = M^a{}_b, \\ &\mathcal{M}_{(n)}{}^a{}_b = M^a{}_{c_2} M^{c_2}{}_{c_3} \dots M^{c_n}{}_b, \quad n \geq 2. \end{aligned} \tag{3.13}$$

Besides scalar T, the rest of available scalars are given by the contraction of a pair of m^a with these tensors or by taking the trace of them:

$$\begin{aligned} m_{(n)} &= m_c \mathcal{M}_{(2n)}{}^{cd} m_d, \\ M_{(n)} &= \mathcal{M}_{(2n)}{}^c{}_c. \end{aligned} \tag{3.14}$$

There is only a finite degree of freedom in m^a, M^{ab}, so the number of independent scalars is also finite. The action becomes a function of these scalars $P(m_{(n)}, M_{(n)}, T)$, and we can get the energy-momentum tensor $T^{\mu\nu}_{id}$ and spin current $S_{id}{}^{\mu\nu\rho}$ by taking variation

$$T_{id}{}^\mu{}_a = \frac{1}{|e|} \frac{\delta}{\delta e^a{}_\mu} \int d^d x |e| P, \quad S_{id}{}^\mu{}_{ab} = \frac{2}{|e|} \frac{\delta}{\delta \omega_\mu{}^{ab}} \int d^d x |e| P. \tag{3.15}$$

The results are

$$\begin{aligned} T^{\mu\nu}_{id} &= \epsilon u^\mu u^\nu + P \Delta^{\mu\nu} + u^\mu \Delta^{\nu\alpha} P_\alpha, \\ S_{id}{}^\lambda \mu\nu &= u^\lambda \rho_{\mu\nu}, \end{aligned} \tag{3.16}$$

where $\Delta^{\mu\nu} := g^{\mu\nu} + u^\mu u^\nu$ is a projection orthogonal to the velocity field. Comparing this constitutive relation with the energy-momentum tensor of a conventional ideal fluid (2.2), we find the P should be identified as the pressure.

ϵ is given by

$$\epsilon = -P + sT + \frac{1}{2} \rho_{\alpha\beta} \mu^{\alpha\beta}, \tag{3.17}$$

and s, ρ_{ab} and P_a are given by the variation of pressure with respect to T, μ^{ab} and u^a:

$$s = \frac{\partial P}{\partial T}, \quad \frac{1}{2} \rho_{ab} = \frac{\partial P}{\partial \mu^{ab}}, \quad P_a = \frac{\partial P}{\partial u^a}. \tag{3.18}$$

Therefore, by comparing the expression of energy density ϵ, we can identify $s, T, \rho_{\alpha\beta}, \mu_{\alpha\beta}$ as entropy density, temperature, spin density, and spin chemical potential respectively.

Is (3.16) our final results for ideal fluid? Actually, there is a subtlety. So far, we have treated the μ^{ab} as zeroth order in derivative since there is no explicit derivative. However, the static conditions (3.8) actually give constraints between these hydrodynamic variables:

$$\begin{aligned} u^\mu K_\mu{}^{ab} &= \mu^{ab} + e^a{}_\mu e^b{}_\nu \left(\Omega^{\mu\nu} - 2u^{[\mu}a^{\nu]}\right) \\ T e^\rho{}_a e^\sigma{}_b \dot{\nabla}_\lambda \frac{\mu^{ab}}{T} &= R^{\rho\sigma}{}_{\lambda\alpha}u^\alpha - 2e^\rho{}_a e^\sigma{}_b K_{\lambda c}{}^{[a}\mu^{b]c} \end{aligned} \tag{3.19}$$

where $\Omega^{\mu\nu}, a^\nu$ are components from the derivative of the velocity field. Therefore, for consistency, we should treat $T, e_\mu{}^a, u^\mu$ as order zero, and $\mu^{ab}, K_\mu{}^{ab}$ as order one. This implies that the $m_{(n)}, M_{(n)}$ are also not zeroth order scalar. If we count carefully, we can find that $m_{(0)}, M_1$ should be considered as order two. Therefore, our previous constitutive relations (3.16) for ideal fluids are not really "ideal", it contains higher derivative terms. Since we want to explore the leading order of non-ideal fluid behavior, we need to expand the pressure P to quadratic order

$$P\left(m_{(n)}, M_{(n)}, T\right) = P_0(T) + \rho_m(T)m_{(0)} + \rho_M(T)M_{(1)} + \mathcal{O}\left(\nabla^4\right). \tag{3.20}$$

And we can find the energy-momentum tensor and spin current that comes from this pressure term are

$$\begin{aligned} T^{\mu\nu}_{id} =& \left(\epsilon_0 + \left(\rho_m + T\rho_m'\right)m_\alpha m^\alpha + \left(\rho_M + T\rho_M'\right)M_{\alpha\beta}M^{\alpha\beta}\right)u^\mu u^\nu \\ &+ \left(P_0 + \rho_m m_\alpha m^\alpha + \rho_M M^{\alpha\beta}M_{\alpha\beta}\right)\Delta^{\mu\nu} \\ &+ u^\mu m_\alpha M^{\alpha\nu}\left(2\rho_m - 4\rho_M\right) + \mathcal{O}\left(\nabla^4\right), \\ S^{\lambda\mu\nu}_{id} =& u^\lambda\left(4\rho_m m^{[\mu}u^{\nu]} - 4\rho_M M^{\mu\nu}\right) + \mathcal{O}\left(\nabla^4\right), \end{aligned} \tag{3.21}$$

Adding hydrodynamic terms and adding higher order terms into the action (3.7)

$$\begin{aligned} \mathcal{W}_{(1)} &= \chi_1^{(1)}\kappa, \\ \mathcal{W}_{(2)} &= \sum \chi_i^{(2)} S_{(i)}, \end{aligned} \tag{3.22}$$

where κ comes from the decomposition of the contorsion tensor and is the only order one scalar term, $S_{(i)}$ are the order two scalars. We can get the complete constitutive relations for energy-momentum tensor and spin current. Here we only show the equation for spin current:

$$\begin{aligned} S^{\lambda\mu\nu} =& 2\chi_1^{(1)}\Delta^{\lambda[\mu}u^{\nu]} - 2\chi_1^{(2)}M^{\lambda[\mu}u^{\nu]} + 2\chi_2^{(2)}u^\lambda M^{\mu\nu} - 2\chi_3^{(2)}u^\lambda u^{[\mu}m^{\nu]} + 4\chi_4^{(2)}\Delta^{\lambda[\mu}m^{\nu]} \\ &+ 4\chi_5^{(2)}\kappa\Delta^{\lambda[\mu}u^{\nu]} + 2u^\lambda\left(2\chi_6^{(2)}k^{[\mu}u^{\nu]} + 2\chi_7^{(2)}K^{\mu\nu} + \chi_{10}^{(2)}\kappa_A^{\mu\nu}\right) \\ &+ 2u^\lambda\chi_8^{(2)}\mathcal{K}_V{}^{[\mu}u^{\nu]} + 4\Delta^{\lambda[\mu}\left(\chi_8^{(2)}k^{\nu]} + 2\chi_9^{(2)}\mathcal{K}_V{}^{\nu]}\right) \\ &- 2\chi_{10}^{(2)}K^{\lambda[\mu}u^{\nu]} - 4\chi_{11}^{(2)}\kappa_A^{\lambda[\mu}u^{\nu]} + 4\chi_{12}^{(2)}\kappa_S{}^{\lambda[\mu}u^{\nu]} \\ &+ 4\chi_{13}^{(2)}\mathcal{K}_A{}^{\lambda\mu\nu} + 4\chi_{14}^{(2)}\mathcal{K}_T{}^{\mu\nu\lambda} \\ &+ 2u^\lambda\left(\chi_1^{(2)}\kappa_A^{\mu\nu} + \chi_2^{(2)}K^{\mu\nu} - \chi_3^{(2)}u^{[\mu}k^{\nu]}\right) - 2\chi_4^{(2)}u^\lambda u^{[\mu}\mathcal{K}_V^{\nu]}. \end{aligned} \tag{3.23}$$

Don't get scared of this long equation, $\chi_1^{(1)}, \chi_i^{(2)}$ are the transport coefficients that we want to calculate, and we have already seen $u^\mu, M^{\mu\nu}, m^\mu, \Delta^{\mu\nu}$. The rest part is just some variables that come from the decomposition of the contorsion tensor.

3.4 Kubo formulas

In order to use holography to calculate the transport coefficients, we need to get the Kubo formulas. The retarded Green's functions we use are defined as:

$$\begin{aligned}
G^{\nu\rho,\mu\sigma} &= e^{\sigma a}\frac{\delta}{\delta e_\mu{}^a}|e|T^{\nu\rho}, \\
G^{\nu\rho,\mu\alpha\beta} &= e^{\alpha a}e^{\beta b}\frac{\delta}{\delta\omega_{\mu ab}}|e|T^{\nu\rho}, \\
G^{\nu\rho\sigma,\mu\alpha} &= e^{\alpha a}\frac{\delta}{\delta e_\mu{}^a}|e|S^{\nu\rho\sigma}, \\
G^{\nu\rho\sigma,\mu\alpha\beta} &= e^{\alpha a}e^{\beta b}\frac{\delta}{\delta\omega_{\mu ab}}|e|S^{\nu\rho\sigma}.
\end{aligned} \tag{3.24}$$

Decompose the energy-momentum tensor and spin current into sectors that are orthogonal to each other, we can get the Kubo formulas. Here we summarize the Kubo formulas that we need to use in the following calculation

$$\begin{aligned}
\chi_2^{(2)} + 2\chi_7^{(2)} &= -\lim_{k\to 0}\lim_{\omega\to 0}\frac{1}{(d-2)k^2}\,\mathrm{Im}\left(G^{\lambda\rho\sigma,\mu\nu}\Pi_{(0)\rho\nu}u_{(0)\lambda}k_\sigma u_{(0)\mu}\right), \\
\chi_3^{(2)} + 2\chi_6^{(2)} &= -\lim_{k\to 0}\lim_{\omega\to 0}\frac{1}{k^2}\,\mathrm{Im}\left(G^{\lambda\rho\sigma,\mu\nu}u_{(0)\lambda}u_{(0)\rho}k_\sigma u_{(0)\mu}u_{(0)\nu}\right) \\
\chi_4^{(2)} + \chi_8^{(2)} &= \lim_{k\to 0}\lim_{\omega\to 0}\frac{1}{2(d-2)k^2}\,\mathrm{Im}\left(G^{\lambda\rho\sigma,\mu\nu}\Pi_{(0)\mu\nu}u_{(0)\lambda}u_{(0)\rho}k_\sigma\right), \\
\chi_8^{(2)} &= \lim_{k\to 0}\lim_{\omega\to 0}\frac{1}{2(d-2)k^2}\,\mathrm{Im}\left(G^{\lambda\rho\sigma,\mu\nu}\Pi_{(0)\lambda\rho}k_\sigma u_\mu u_\nu\right), \\
\chi_9^{(2)} &= \lim_{k\to 0}\lim_{\omega\to 0}\frac{1}{8(d-2)^2k^2}\,\mathrm{Im}\left(G^{\lambda\rho\sigma,\mu\nu}k_\rho\Pi_{(0)\lambda\sigma}\Pi_{(0)\mu\nu}\right), \\
\chi_5^{(2)} + \frac{(d-2)\chi_{12}^{(2)}}{d-1} &= -\lim_{k\to 0}\lim_{\omega\to 0}\frac{1}{k^4}\,\mathrm{Im}\left(G^{\lambda\rho\sigma,\mu\nu}k_\lambda u_{(0)\rho}k_\sigma u_{(0)\mu}k_\nu\right), \\
\chi_5^{(2)} - \frac{\chi_{12}^{(2)}}{d-1} &= \lim_{k\to 0}\lim_{\omega\to 0}\frac{(d-1)}{2(d-2)k^2}\,\mathrm{Im}\left(G^{\lambda\rho\sigma,\mu\nu}\Pi_{(0)\lambda\rho}u_{(0)\sigma}u_{(0)\mu}k_\nu\right),
\end{aligned} \tag{3.25}$$

where k^μ is the spatial part for momentum vector and $\Pi_{(0)\mu\nu} := \Delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ is a projection orthogonal to u^μ and k^μ. Note that the above equations are obtained around the Minkowski spacetime, so we also need to consider a Minkowski boundary spacetime in the calculations of holography.

4 Spin transport coefficients from holography

In order to use the holography method to calculate spin transport coefficients, we need to choose a bulk theory that describes the dynamic of the energy-momentum tensor and spin current's dual fields. The energy-momentum tensor $T^{\mu a}$ is dual to the vielbein $e_{\mu a}$, which contains the same information as the metric. Therefore, we can identify the induced metric of the bulk metric at the boundary to the metric of the boundary field. Consider a d-dimensional boundary spacetime and a d+1-dimensional bulk spacetime. The model we use to describe the dynamic of the bulk metric is the Einstein-Hilbert action:

$$S[g] = \kappa \int \mathrm{d}^{d+1}x\sqrt{-g}\,(R - 2\Lambda)\,, \tag{4.1}$$

where $\Lambda = -\frac{d(d-1)}{2L^2}$ and from now on, we choose $d = 4$.

Spin current $S^{\mu ab}$ is dual to the spin connection $\omega_{\mu ab}$. In the study of classical gravity theory, it's usually convenient to work on the *probe limit*, which means we consider other fields are small enough such that their dynamic doesn't influence the metric. This is not a trivial statement, because in general, the coupling between metric and other fields is complicated and we can't guarantee a perturbation in other fields doesn't cause a perturbation in metric. However, as we will show in the following content, this probe limit works fine in our model. Therefore, if we consider the metric is only a background metric, using the relation (2.30), we can find the dynamic part of the spin connection only comes from the contorsion $K_{\mu ab}$. There is a paper [34] states that if we decompose the torsion T_{abc} into irreducible pieces under Lorentz group: a trace part T_a, a totally antisymmetric part $T_{[abc]}$ and the rest part W_{abc}. At a 5-dimensional bulk spacetime, this decomposition can be written as

$$T_{abc} = \frac{1}{4}\left(\eta_{ac}T_b - \eta_{bc}T_a\right) + T_{[abc]} + W_{abc}, \tag{4.2}$$

where

$$T_b := T^a{}_{ab}, \quad T_{[abc]} := \frac{1}{3}\left(T_{abc} + T_{cab} + T_{bca}\right). \tag{4.3}$$

Then the dynamical model for torsion containing no ghosts and tachyons should only have T_a and $T_{[abc]}$. We choose, to further simplify the model, a simple toy massive vector model to describe the T_a.

4.1 Massive vector as a bulk gravity model

The total action we choose for bulk theory is

$$S = \int d^{d+1}x\sqrt{|g|}\left(\kappa(\tilde{R}(g) - 2\Lambda) - \frac{1}{4}\alpha F^{\alpha\beta}F_{\alpha\beta} - \frac{1}{2}m^2 A^\alpha A_\alpha\right) - \kappa\int_\partial \mathrm{d}^d x\sqrt{\gamma}2K, \tag{4.4}$$

where κ, α, m are the coupling constants, and we have also included a Gibbons-Hawking term on the boundary. Although we can redefine the A_α to absorb the α constant, in holography we want A_α at the boundary equal to the trace part of torsion T_α, so we need to keep this α constant. Note that the because the torsion is related to contorsion by (2.31), T_a is equal to the trace of contorsion:

$$K_a := K^b{}_{ba}. \tag{4.5}$$

Therefore, in our model, the only dynamical part of contorsion/torsion is their trace. This put a limit on the transport coefficients: if a transport coefficient is not coupled with the trace of contorsion, then our model is not capable to calculate it. Why don't we further simplify the model by taking the mass in (4.4) to be zero? That's because for a massless vector, an extra $U(1)$ gauge symmetry would appear. This may cause some extra conservation laws that we don't expect to have in our spin hydrodynamics. The equations of motion for action (4.4) are:

$$\alpha\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) - \sqrt{-g}m^2A^\nu = 0 \tag{4.6}$$

$$\kappa R_{\mu\nu} + \frac{2\kappa}{1-d}\Lambda g_{\mu\nu} - \frac{1}{2}\alpha\left(F_\mu{}^\beta F_{\nu\beta} - \frac{1}{2(d-1)}g_{\mu\nu}F_{\mu\nu}F^{\mu\nu}\right) + \frac{m^2}{2}A_\mu A_\nu = 0, \tag{4.7}$$

where $F_\mu\nu := \partial\mu A_\nu - \partial_\nu A_\mu$ is the field strength, $R_{\mu\nu} := R^\lambda{}_{\mu\lambda\nu}$ is the Ricci tensor. In general, it's not easy to solve Einstein's equation (4.7) analytically. Also, since we are only interested in using the linear response theory to determine the transport coefficients, we choose to solve these two equations perturbatively. Let's expand the vector field and metric as

$$g_{\mu\nu}(x) = g^0_{\mu\nu}(x) + \epsilon h_{\mu\nu}(r)e^{-i\omega t+ikz}, \tag{4.8}$$

$$A_\mu(x) = A^0_\mu(x) + \epsilon a_\mu(r)e^{-i\omega t+ikz}, \tag{4.9}$$

where $\epsilon \ll 1$, $g^0_{\mu\nu}, A^0_\mu$ are the background fields. In these equations, we also expressed the expansion in Fourier mode and we choose the spatial momentum only in the z-direction. Notice that taking derivative in (4.6) gives a constraint on the vector field

$$\partial_\nu(\sqrt{-g}A^\nu) = 0, \tag{4.10}$$

To avoid confusion, we use i, j, k, l to represent the non-radial index. We choose $a_r(r)$ to be a non-dynamical component which is determined by $a_i(r)$.

The temperature of this theory comes from the black hole. As far as we know, there is no analytical black hole solution for an asymptotic AdS spacetime with a nonvanishing massive vector. Although we can choose a numerical expansion to describe this type of black hole [?, 35], extra difficulties would be involved. Therefore, we choose to first solve the equations at an AdS-Schwartzshild spacetime with $A^0_\mu(x) = 0$, and the background metric is

$$\begin{aligned} ds^2 &= \frac{L^2}{r^2}\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + dx^2 + dy^2 + dz^2\right), \\ f(r) &= 1 - \frac{r^4}{r_h^4}. \end{aligned} \tag{4.11}$$

Near the boundary $r \to 0$, $f(r) \to 0$, so this is an asymptotic AdS spacetime. The event horizon of this black hole sits at $r = r_h$.

Inserting this background into the equations of motion, we find there are some sectors that describe the coupling between components in $a_\mu(r), h_{\mu\nu}(r)$, these relations are shown in Table 1. We find that the metric and vector fields

a_x				
a_y				
a_z	a_t	a_r		
h_{xy}				
h_{ty}	h_{yz}			
h_{tx}	h_{xz}			
h_{tt}	h_{tz}	h_{xx}	h_{yy}	h_{zz}

Table 1: Sectors for coupled variables. Variables at the same line are coupled to each other

are decoupled in the linear order, which validates that we can take the probe limit in the regime of our research. In fact, we can find the decoupling between metric and the vector field in the leading order from the structure of equations of motion (4.6,4.7). Since we are expanding the vector field around zero, the vector field and the field strength are considered first-order terms. Therefore, it is easy to read from the equations of motion that the coupling between the metric and the vector field happens at least in the second order.

With the decoupling of metric and vector field, we can investigate their perturbation equations separately. There are already many excellent papers about the holography treatment for pure gravity model in AdS-Schwartzschild spacetime [33, 36, 37]. In the following context, we will focus on the solutions for the vector field.

4.2 Near boundary analysis

The equation of motion for a_y is

$$a_y(r)\left[-k^2 r - \frac{m^2}{r\alpha} + \frac{r\omega^2}{f(r)}\right] + a_y'(r)\left[-f(r) + rf'(r)\right] + rf(r)ay''(r) = 0. \quad (4.12)$$

Let's investigate it near boundary $r \ll 0$. Using the ansatz

$$a_y = r^\beta(a_{y(0)} + a_{y(1)}r + a_{y(2)}r^2 + \cdots), \quad (4.13)$$

we find at leading order in r, the equation requires

$$\alpha(-2+\beta)\beta - m^2 = 0 \quad (4.14)$$

Therefore, there are two solutions corresponding to

$$\beta_- = \Delta = 1 - \sqrt{M^2+1}, \quad \beta_+ = 2 - \Delta = 1 + \sqrt{M^2+1}, \quad (4.15)$$

where we defined $M^2 := \frac{m^2}{\alpha} > 0$, so $\Delta < 0$. We also find that the odd order coefficient $a_{y(i)}$ should vanish because the metric only contains r to the power of an even number. Repeat this procedure for other components $a_{i(0)}$, we found their near boundary expansion can be expressed as

$$a_i = r^{\Delta}\left(a_{i(0)} + a_{i(2)}r^2 + \cdots\right) + r^{2-\Delta}\left(\tilde{a}_{i(0)} + \cdots\right). \tag{4.16}$$

Since Δ is a negative number, r^{Δ} term diverges as r approached 0. $a_{i(o)}$ is the slow falloff mode and $\tilde{a}_{i(0)}$ is the fast falloff mode. Following the holography principle, we set the slow falloff mode at the boundary as the same value of the contorsion's trace.

The value of Δ is related to the scaling dimensions of contorsion's trace and the spin current coupled with it. Under a scale transformation

$$x^{\mu} \to \Lambda x^{\mu}, \tag{4.17}$$

the AdS-Schwartzschild spacetime is invariant. If a quantity ϕ transforms under this scaling as

$$\phi \to \Lambda^{-a}\phi, \tag{4.18}$$

then we say ϕ has scaling dimension a. A_{μ} is a one-form, so its scaling dimension is 1. Therefore, we can infer that the scaling dimension of $a_{i(0)}, \tilde{a}_{i(0)}$ are $1+\Delta$ and $3-\Delta$ respectively.

To investigate the scaling dimension in the boundary, we can write the variation caused by δK_{μ} as as

$$\delta S = \int d^4x\sqrt{g}\delta K_{\mu} S^{\mu}, \tag{4.19}$$

where we use S^{μ} to represent the component in spin current that is coupled to the K_{μ}. After we identify the slow falloff mode with the source, we get the scaling dimension for controsion's trace K_i is $1+\Delta$. δS is invariant under the scaling transformation, so we can get S^{μ} has scaling dimension $3-\Delta$. If we know the scaling dimension of the spin current in our hydrodynamic theory, then we get the value of Δ and add a constraint to the parameters α, m.

4.3 Near horizon analysis

Expand the equation of motion near the horizon, using the ansatz $a_i = (r-r_h)^{\beta}$, we find the leading order solutions are

$$a_i \sim (r-r_h)^{\frac{\pm i\omega}{|f'(rh)|}} \tag{4.20}$$

Recovering the $e^{-i\omega t+ikx}$ term, we can find these two solutions correspond to propagation into/ away from the black hole, we call them infalling and outgoing solutions respectively. The direction of propagation becomes clear when we use the tortoise coordinate r^*

$$dr^* = \frac{dr}{f(r)}. \tag{4.21}$$

The solutions become

$$a_i e^{-i\omega t+ikx} \sim e^{i\omega(t\pm r^*)}, \tag{4.22}$$

where $t + r^*$ corresponds to the outgoing solution and $t - r^*$ corresponds to the ingoing solution. In order to get the correct retarded Green's function, we need to choose the ingoing solution [38]. This choice is also natural since we are considering perturbations added at the boundary. One should be careful that after choosing only one solution, the fast falloff mode in (4.16) depends on the slow falloff mode. This dependence can be computed by solving the equation of motion in the whole region.

4.4 Counterterms

The slow falloff mode corresponds to the divergence of a_i in the near boundary, which causes a divergence in the action. To remove this divergence, we need to add counterterms that are expressed as local functions in the boundary.

In the probe limit, the on-shell action for the matter field is

$$\begin{aligned} S_{\text{onshell}} &= \int d^4x \partial_\mu \left(-\frac{\alpha}{2} \sqrt{-g} A_\nu F^{\mu\nu} \right) \\ &= \int_{r=\epsilon} d^4x \sqrt{-g} \frac{\alpha}{2} A_\nu F^{r\nu} \\ &= \int_{r=\epsilon} d^4x \frac{\alpha}{2} \epsilon^2 \left[\delta\eta^{ij} a_{i(0)} a_{j(0)} r^{2\Delta-2} - \eta^{ij}\eta^{kl} \left(\frac{(2+2\Delta)\partial_i\partial_j a_{k(0)} a_{l(0)}}{4\Delta} \right.\right. \\ &\qquad \left.\left. + \frac{\partial_i a_{j(0)} \partial_k a_{l(0)}}{2-\Delta} \right) r^{2\Delta} + O(r^{2\Delta+2}) \right], \end{aligned} \tag{4.23}$$

where in the first line we used the equation of motion, in the second line we integrate the total derivative in the radial direction and introduce a cut-off ϵ, and in the third line we brought the near boundary expansion into this equation. Obviously, the number of divergent terms depends on the value of Δ. In our research, we restrict its range to $\Delta > -1$. We found the divergence can be cancelled by these counterterms:

$$S_{\text{CT}} = \int_{r=\epsilon} d^4x \sqrt{-\gamma} \alpha \left[\frac{-\Delta}{2} A_i A^i + \frac{1}{8\Delta} F_{ij} F^{ij} + \left(\frac{1}{2(2-\Delta)} + \frac{1}{4\Delta} \right) \partial_i A^i \partial_j A^j \right] \tag{4.24}$$

The renormalized action S is defined as:

$$S = \lim_{\epsilon\to 0} S_{\text{onshell}} + S_{\text{CT}} \tag{4.25}$$

4.5 Retarded Green's function

With the renormalized bulk action, we can start to calculate the retarded Green's function from holography. There are two things that we should be careful of. The first is that the only dynamical part in contorsion is its trace, so we need to express the $\frac{\delta}{\delta K_{\mu\nu\rho}}$ in terms of $\frac{\delta}{\delta K_\mu}$ by using the chain rule:

$$\frac{\delta}{\delta K_{\mu\nu\rho}} = (g^{\mu\nu}\delta^\rho_\sigma - g^{\mu\rho}\delta^\nu_\sigma) \frac{\delta}{\delta K_\sigma}. \tag{4.26}$$

The second thing is that in the spin hydrodynamic theory, we treat the spin connection and the vielbein as independent variables, but in the bulk theory we would like to treat the contorsion and vielbein as independent variables. We need to replace the differential with respect to the vielbein in the hydrodynamic theory to

$$\left.\frac{\delta}{\delta e_\mu{}^a}\right|_{\omega_\mu{}^{ab}} = \left.\frac{\delta}{\delta e_\mu{}^a}\right|_{K_\mu{}^{ab}} - \frac{\delta \tilde{\omega}_\mu{}^{cd}}{\delta e_\mu{}^a} \left.\frac{\delta}{\delta K_\nu{}^{cd}}\right|_{e_\mu{}^a}. \tag{4.27}$$

However, notice that $\frac{\delta \tilde{\omega}_\mu{}^{cd}}{\delta e_\mu{}^a}$ is in the linear order of momentum, we can treat it as subleading terms in the hydrodynamic limit $\omega, k \to 0$

The Green's functions (3.24) can be expressed in terms of the functional derivative of contorsion's trace and metric:

$$\begin{aligned}
G^{\nu\rho,\mu\sigma} &= 2\frac{\delta}{\delta g_{\mu\sigma}}\left(\frac{\delta S}{\delta g_{\nu\rho}}\right), \\
G^{\nu\rho,\mu\sigma} &= 2\left[g^{\mu\alpha}\frac{\delta}{\delta K_\beta}\left(\frac{\delta S}{\delta g_{\nu\rho}}\right) - g^{\mu\beta}\frac{\delta}{\delta K_\alpha}\left(\frac{\delta S}{\delta g_{\nu\rho}}\right)\right], \\
G^{\lambda\alpha\beta,\mu\rho} &= \frac{1}{2}\frac{\delta}{\delta g_{\mu\rho}}\left(\frac{\delta S}{\delta K_\beta}g^{\lambda\alpha} - \frac{\delta S}{\delta K_\alpha}g^{\lambda\beta}\right), \\
G^{\mu\rho\sigma,\nu\alpha\beta} &= \frac{1}{2}\left(g^{\nu\alpha}\frac{\delta}{\delta K_\beta} - g^{\nu\beta}\frac{\delta}{\delta K_\alpha}\right)\left(\frac{\delta S}{\delta K_\sigma}g^{\mu\rho} - \frac{\delta S}{\delta K_\rho}g^{\mu\sigma}\right).
\end{aligned} \tag{4.28}$$

4.6 Spin transport coefficients

Now we can calculate the transport coefficients in (3.25) by using the retarded Green's function above. We found that if we choose the background value of the vector field to be zero, then these transport coefficients vanish at the leading order. To make the results more interesting, we choose to set the background vector field to be a small value $A_{B\mu}$ that is in the same order as ϵ and satisfy the infalling condition. The near boundary expansion of $A_{B\mu}$ takes the same form as (4.16):

$$A_{Bi} = r^\Delta\left(A_{Bi(0)} + A_{Bi(2)}r^2 + \cdots\right) + r^{2-\Delta}\left(\tilde{A}_{Bi(0)} + \cdots\right). \tag{4.29}$$

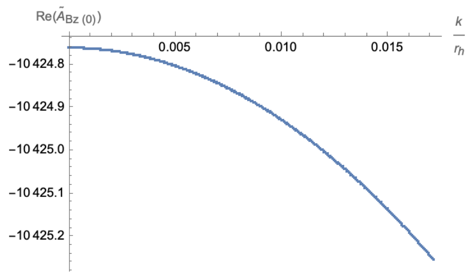

Figure 3: Real part for $\tilde{A}_{Bz(0)}$

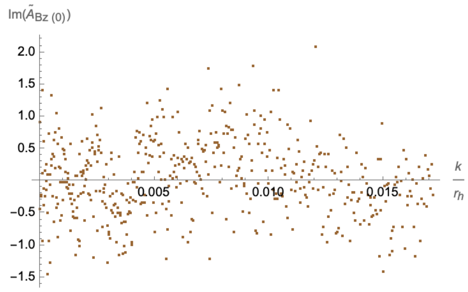

Figure 4: Imaginary part for $\tilde{A}_{Bz(0)}$

The results for transport coefficients in the leading order are:

$$\begin{aligned}
\chi_2^{(2)} + 2\chi_7^{(2)} &= \lim_{k\to 0}\lim_{\omega\to 0} \frac{1-\Delta}{k}\alpha\, \mathrm{Im}\left(\tilde{A}_{Bz(0)}\right),\\
\chi_3^{(2)} + 2\chi_6^{(2)} &= \lim_{k\to 0}\lim_{\omega\to 0} \frac{1-\Delta}{2k}\alpha\, \mathrm{Im}\left(\tilde{A}_{Bz(0)}\right),\\
\chi_4^{(2)} &= 0,\\
\chi_5^{(2)} &= \lim_{k\to 0}\lim_{\omega\to 0} \frac{1-\Delta}{k}\alpha\, \mathrm{Im}\left(\tilde{A}_{Bz(0)}\right),\\
\chi_8^{(2)} &= \lim_{k\to 0}\lim_{\omega\to 0} -\frac{1-\Delta}{4k}\alpha\, \mathrm{Im}\left(\tilde{A}_{Bz(0)}\right),\\
\chi_9^{(2)} &= 0,\\
\chi_{12}^{(2)} &= \lim_{k\to 0}\lim_{\omega\to 0} -\frac{3(1-\Delta)}{2k}\alpha\, \mathrm{Im}\left(\tilde{A}_{Bz(0)}\right).
\end{aligned} \tag{4.30}$$

These results are of the same form. They all depend on the imaginary part of the fast falloff mode for the background vector field. Although we can explain some technical details like the non-vanishing contribution to these transport coefficients comes from $\frac{\delta S}{\delta a_{\mu(0)}}$ and $\frac{\delta^2 S}{\delta g_{\rho\sigma}\delta a_{\mu(0)}}$ and they take a similar form, we haven't found a generic explanation for this phenomenon.

$\tilde{A}_{Bz(0)}$ is a function of the value of contorsion's trace in the hydrodynamic theory. This function depends on the value of $\omega, k, \frac{m^2}{\gamma}, r_h$. In principle, we can solve the dependence numerically. Although we tried to work on the numerical methods, there are still some difficulties that require more time to fix. For example, in the Figure3.4. we showed numerical results for the fast falloff. The algorithm we used to solve differential equations is the Chebyshev method [39] which converts the linear differential equations to a set of linear equations and solves them iteratively. Although the real part solution seems to be fine, the imaginary part in Figure 4 shows our result is still not reliable. A complete analysis to perform the numerics correctly is beyond our research regime, we left it for future work.

5 Conclusion and outlook

5.1 Conclusion

Let's summarize what we have done in this paper. In the preliminary knowledge, we introduced that the spin current is the current of gauge Lorentz symmetry. In order to describe the gauge Lorentz symmetry, we need to introduce spin connection as the gauge field. To imagine this gauge Lorentz symmetry, we can consider a local Lorentz frame in which the metric is the Minkowski metric. The basis in the local Lorentz frame is related to the basis in the spacetime coordinate by the vielbein. Then a local Lorentz frame can be considered as only changing the basis in this local Lorentz frame, which doesn't change any physical quantities. In the hydrodynamic theory, there are two important ingredients: conservation law and constitutive relation. The spin coefficients appear in the constitutive relation and they describe the responses of currents to external sources. In quantum mechanics, we can get the linear response to a perturbation described by retarded Green's function, which is related to transport coefficients by the Kubo formula. The retarded Green's function can be computed by correlation functions in quantum field theory. However, we would like to use the spin current in heavy-ion collisions, which involve strong interaction. Inspired by the fact that the shear viscosity calculated from holography is similar to the experimental result for quark-gluon plasma, we want to use holography to investigate these transport coefficients. The holography principle is a conjecture that states a strongly coupled gauge theory in the boundary spacetime is equivalent to a classical gravity theory in the bulk spacetime. Therefore, we may use it to calculate Green's function from a classical bulk theory.

Then, we summarized the results for the hydrodynamic theory of spin current form [1] The spin connection contains a vielbein-dependent part and a contorsion part. We want to make the spin connection independent of the vielbein to avoid ambiguity in defining the spin current, so we need to include a torsion in our theory. The conservation laws from diffeomorphism and gauge Lorentz symmetry actually give two non-conservation equations for energy-momentum tensor and spin current. In order to obtain the constitutive relations for spin current, we consider an action containing all the possible scalars up to the second order in the derivative. To construct the scalars, we need to decompose all tensors into scalars, vectors, and tensors of the rotation subgroup. Then the constitutive relations are derived by taking the derivative of the action to the vielbein and spin current.

Finally, we came to our holography model for the spin hydrodynamic theory. We considered a toy model which only contains a massive vector field. The massive vector is dual to the trace of the contorsion tensor. Although a simple model gives us some benefits, it also limits the spin transport coefficients that we can calculate. We choose to solve the equations of motion on an AdS-Schwartzschild background spacetime and solve equations in the leading order of perturbative expansion. We found the vector field and gauge field are decoupled at the leading order, so we can work in the probe limit in which we don't need to consider the back reaction of the vector field to the background spacetime. The near-boundary solution diverges for the vector field so we need to add counter terms. We found the counterterms in the probe limit with the restriction $\Delta > -1$. Based on the research [1], we found the Kubo formulas for some transport coefficients which appeared in the second derivative order in the hydrodynamic model. We calculated these transport coefficients and found them proportional to the fast falloff mode of the contorsion's trace.

5.2 Outlook

We need to improve the numerical method substantially to extract the imaginary part of the fast falloff. The imaginary part of fast falloff's dependence on the momentum k is vital because (4.30) suggest that only if the leading order contribution of this fast falloff is linear in k then we can get finite transport coefficients. If the leading order contribution is zeroth order in k, then it suggests our toy model is not suitable for describing spin hydrodynamics. If we find the imaginary part is linear in k, then we can further investigate these transport coefficients' dependence on variables like the value of contorsion's trace, parameters in our bulk model, and the temperature in the bulk spacetime.

We must admit that the model we choose is a toy model that is not likely to describe the real physics of spin current. Even if the holography conjecture is true and our model describes the right dynamics for the contorsion's trace, it would be difficult to extract data from experiments to get the transport coefficients related only to the trace of contorsion. Nevertheless, we could still further improve the model by including the dynamic terms of other components in contorsion. It's definitely a tough task because there are many models/background spacetimes/parameters that we need to choose or fit. If we find a holography model that can predict similar spin transport coefficients as those obtained from Bayesian analysis in experimental data, then it will be a strong evidence for the holography conjecture.

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