Calabi-Yau manifolds

The Calabi-Yau theorem, explicit constructions, string theory and generalisations to Lie algebroids

Master's thesis

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Abstract

Calabi-Yau manifolds are Ricci-flat Kähler manifolds that are vital to string theory and have many applications in math. The existence of Calabi-Yau metrics on a large class of Kähler manifolds is guaranteed by the Calabi-Yau theorem, due to E. Calabi and S.T. Yau, whose proof is by heavy analysis, which will be the first part of this thesis. Explicit examples of these metrics are very difficult to find, we will present a known construction due to E. Calabi in the second part of this thesis. The third part will consist of an overview of how Calabi-Yau manifolds appear in string theory.

The definition of Calabi-Yau manifolds can be generalised to a space known as a *Lie algebroid*, which is a vector bundle equipped with a structure to make it look like the tangent bundle. In the final part of this thesis, we look at this generalisation and present some rather basic results. Moreover, we will look at Lie algebroids in string theory and present a possible application of Calabi-Yau Lie algebroids in string theory.

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Contents

1 Introduction

One of the biggest open problems in modern day theoretical physics is how to describe gravity as a quantum theory. Quantum field theory describes matter as consisting of fields fluctuating on some geometric background [Sch13]. General relativity, on the other hand, describes gravity as fluctuations in the geometry of the universe itself [Car19]. Therefore, quantising gravity will require turning the geometry of the universe into a quantum field. As it turns out, the standard tools of quantum field theory do not work when applying them to gravity [Sch13], so this problem requires a specialised new theory to solve, it requires a *theory of everything*.

One framework in which gravity can be quantised is *superstring theory* [BBS07; BLT13; GSW87a; GSW87b]. This particular theory assumes that all particles in the universe are vibrating modes of very tiny strings that fill up the universe. This allows for a theory of everything with few free parameters, since really, the only thing we need to describe are the strings themselves, instead of a long list of all particles that make up the current standard model.

As it turns out, this rather simple *ansatz* of a theory of everything has very restrictive consequences. It is only consistent in ten dimensions and it tells us precisely which particles are allowed in the theory [BBS07; BLT13; GSW87a; GSW87b]. One big apparent issue with the theory is that our universe is not ten dimensional, we only have four dimensions: three spatial dimensions and one temporal dimension. The extra six dimensions of superstring theory get "rolled up" into a six dimensional space in a process known as *compactification*. Moreover, superstring theory has a natural symmetry relating bosons and fermions which is quite desirable in this context. Making sure that superstring theory still has this symmetry after compactification requires compactifying on spaces known as *Calabi-Yau manifolds*, as was realised in 1985 by P. Candelas *et al.* [Can+85]. Calabi-Yau manifolds are Kähler manifolds with vanishing Ricci curvature, which means they also solve Einstein's equations in a vacuum, making them a good candidate for a vacuum configuration of the universe.

The process of compactification could also explain a seeming discrepancy: since superstring theory has very few free parameters and the standard model has a lot, we need a mechanism for the appearance of extra free parameters in the effective four dimensional theory of superstring theory. These extra parameters come from the structure of the Calabi-Yau manifold we compactify on. So the hope was for a long time that the only remaining problem was finding "the right" Calabi-Yau manifold. Due to these observations, many people turned to studying Calabi-Yau manifolds and the field blew up.

However, the study of Calabi-Yau manifolds already started in the 1950s, about thirty years earlier, in the works of E. Calabi, e.g. [Cal54], motivated by the study of Ricci curvature combined with the study of Kähler manifolds. They also appeared in the study of holonomy, as they are precisely those manifolds that have (a subgroup of) $SU(n)$ as holonomy group, while not necessarily being symmetric spaces. This means they are one of the seven spaces on M. Berger's list that classifies different holonomy groups that can appear on Riemannian manifolds, as was published in 1955 in [Ber55].

Thus the study of Calabi-Yau manifolds is important in both mathematics and physics. Since these manifolds are, in particular, Riemannian manifolds, the hope is that we can easily find many examples of Calabi-Yau metrics on certain manifolds. Unfortunately, the story is not as simple as that, finding Calabi-Yau metrics on compact spaces requires solving a particularly nasty non-linear partial differential equation known as a *Monge-Ampère equation* [Aub82; Cal54; Joy00; Yau77; Yau78]. The only known explicit example of a Calabi-Yau metric with $SU(n)$ holonomy is due to E. Calabi in 1979 in [Cal79], where he produced such a metric on the canonical bundle of $\mathbb{C}P^{n-1}$, which is a non-compact space. Independently, T. Eguchi and A.J. Hanson found the same metric in the case of $\mathbb{C}P^1$, also in 1979 [EH79].

Even though there are not a lot of explicit examples, solutions to the Monge-Ampère equation do exist. In 1954, in [Cal54], E. Calabi conjectured the existence of a solution to this differential equation under a certain topological condition. He noticed that Ricci-flat Kähler manifolds necessarily have vanishing first Chern class, so he conjectured the converse was also true: compact Kähler manifolds with vanishing first Chern class admit a unique Ricci-flat Kähler metric in every cohomology class carrying a Kähler form. Over 20 years later, in 1977, S.T. Yau finally proved this famous theorem [Yau77; Yau78]. This theorem gives us a large variety of compact manifolds that admit Ricci-flat Kähler metrics, so it provides many examples of Calabi-Yau manifolds, though it doesn't explicitly give us the desired Riemannian metric. Moreover, this Calabi-Yau theorem even gives us a step towards the classification of compact Calabi-Yau manifolds. They are precisely those compact Kähler manifolds with vanishing first Chern class, and the space of Calabi-Yau metrics on such a Kähler manifold is parameterised by the associated Kähler cone.

This kind of classification in the non-compact case is still an open problem, even in the case where we work on the complement of a divisor in a compact Kähler manifold, for a review of results in this setting, see [Küh11]. In this thesis, we propose an alternative approach to finding Ricci-flat Kähler metrics on these open subsets of Kähler manifolds, where we study geometric structures on Lie algebroids. Lie algebroids form a natural setting to do geometry on, in particular, they are vector bundles, so we can put metrics on them. Moreover, they have a natural theory of connections, see e.g. [CFM21], and a metric on a Lie algebroid gives us a Levi-Civita connection, so we can rather easily generalise a lot of constructions in Riemannian geometry to this setting.

A particular type of Lie algebroid is the *elliptic tangent bundle* [CG15], which provides a natural setting to study Kähler metrics with logarithmic divergence towards a submanifold. This particular type of divergence makes them fit into the theory of *log geometry*, see e.g. [Ogu06] for an introduction. In this setting, there is a notion of *log Calabi-Yau manifolds*, which appears in both mathematics and physics literature, see e.g. [DKW13; Ish00; GHK15], but that framework does not approach the problem from a Riemannian viewpoint. In fact, they approach Calabi-Yau theory from an angle that does not require an underlying Kähler metric, which clashes with the classical theory of Calabi-Yau manifolds. Thus, studying Calabi-Yau structures on Lie algebroids might give new insights on Ricci-flat Kähler metrics on the complement of a submanifold of a Kähler manifold and, hopefully, also some new tools for studying log Calabi-Yau spaces.

Moreover, Lie algebroids have been proposed in physics, already in the 2000s, in e.g. [Str04; BZ07], but more recently, as possible non-geometric backgrounds for superstring theory, as can be seen in [Blu+13; Pla19], i.e. a theory of superstrings where the extra six dimensions are not interpreted as physical dimensions, but rather as extra fields on the strings that do not necessarily describe extra dimensions. In this thesis, we will propose a possible application of Calabi-Yau structures on elliptic tangent bundles to string theory, but that suggestion is most likely not a very solid one.

In Chapter 2, we discuss some background material needed for the later chapters, moreover, we give a few well-known examples of Calabi-Yau manifolds that are rather easy to construct using tools from complex geometry. Chapters 3, 4 and 5 form the heart of the thesis. In Chapter 3, we discuss the Calabi-Yau theorem in depth and we provide a proof based on the one given by D.D. Joyce in [Joy00]. In Chapter 4, we provide E. Calabi's construction of an explicit Calabi-Yau metric on the canonical bundle of $\mathbb{C}P^n$, which requires some brute-force. In Chapter 5, we briefly discuss the rich theory of Calabi-Yau manifolds in string theory. Lastly, in Chapter 6, we discuss Lie algebroids and how the theory of Calabi-Yau manifolds can possibly be generalised to that setting, and how it might connect to string theory. This last chapter is by no means a complete description and should be viewed as an outlook for future research.

2 Preliminaries

Throughout this thesis, we will need some preliminary results from analysis and (complex) geometry. This chapter is included to provide these preliminary results, where most proofs/details are omitted for brevity. References will be given where all details can be found and learned.

In Section 2.1, we will introduce the theory of global analysis on Riemannian manifolds and vector bundles thereover, and we shall state some results regarding the solutions of differential equations in these settings, which will mostly be important in Chapter 3. The goal of Section 2.2 is to introduce some basic tools from complex geometry and Kähler geometry that we shall need almost everywhere in this thesis, including the rest of this chapter. In Section 2.3, we will briefly go over the theory of complex line bundles, which will be a helpful tool in Chapter 4, where we will use line bundles to construct an explicit Calabi-Yau metric. Lastly, in Section 2.4, we will turn to stating some basic geometric results for Calabi-Yau manifolds and we will provide some examples of manifolds admitting Calabi-Yau metrics using the famous Calabi-Yau theorem that we will prove in Chapter 3.

2.1 Analysis on manifolds

We start of with a bit of a stand-alone section, introducing some analysis on Riemannian manifolds. This section is particularly important in Chapter 3, where a lot of analysis will be required to prove the Calabi-Yau theorem. The later sections in this chapter do not require this section as prerequisite knowledge, and will, instead, introduce some more geometric tools for the study of complex manifolds and, in particular, Kähler and Calabi-Yau manifolds.

In this section, we will introduce some tools from functional analysis to study PDEs on Riemannian manifolds. In particular, we will introduce Hölder, Lebesgue and Sobolev spaces on Riemannian manifolds and on vector bundles over Riemannian manifolds, we will briefly introduce the theory of differential operators and ellipticity and we will state some results regarding the existence of solutions to differential equations. Along the way, we shall need to relax smoothness conditions, on the one hand by letting smooth differential operators act on Hölder or Sobolev functions, on the other hand, by also considering non-smooth differential operators. We shall also give some results on regularity of solutions to elliptic differential equations, which will provide a helpful tool in studying the differentiability of solutions to such equations.

We will assume the reader is familiar with basic functional analysis and a little bit of Riemannian geometry. Throughout this section, (M^m, g) will be a Riemannian manifold, $d_g : M^{\times 2} \to \mathbb{R}$ will be the associated distance function, $(-,-)_g$ will be the induced fibrewise inner product on $T^*M^{\otimes p} \otimes TM^{\otimes q}$, $|-|_g$ is the associated fibrewise norm, $\nabla : \Gamma(T^*M^{\otimes p} \otimes TM^{\otimes q}) \to \Gamma(T^*M^{\otimes p+1} \otimes TM^{\otimes q})$ will be the Levi-Civita connection associated to g, and vol_g will be the volume form induced by g. Also let $E \to M$ be a real vector bundle over M and equip it with a fibrewise metric h. By slight abuse of notation, we shall also let ∇ denote the metric connection of h on every tensor bundle of E. Moreover, we will assume that all functions are measurable.

2.1.1 Basic theory

There are a few spaces we will need to introduce that will give us a nice framework to work in later down the road. We will not give an in depth discussion of these spaces and their properties, for that we refer the reader to, for instance, [Heb96; Nic22].

We will start with the space of k-times differentiable functions for $k \geq 0$, with $k = 0$ being the space of continuous functions, which we shall denote by $C^k(M)$. This space consists of continuous functions $f : M \to \mathbb{R}$ such that $\nabla^k f$ exists and is continuous. We equip this space with a topology induced by the norm

$$\|f\|_{C^k} := \sum_{i=0}^{k} \sup_M |\nabla^k f|_g \,. \tag{2.1.1}$$

In fact, this turns $C^k(M)$ into a Banach space [Nic22]. Moreover, if a function f is $k+1$ times differentiable, it is also k times differentiable, and by definition, we have $\|f\|_{C^k} \leq \|f\|_{C^{k+1}}$, therefore the inclusion $\iota : C^{k+1}(M) \to C^k(M)$ is continuous, and we can equip $C^\infty(M) := \bigcap_{k=0}^\infty C^k(M)$ with the inverse limit topology, i.e. we take the categorical limit of the sequence $C^1(M) \supset C^2(M) \supset \dots$ in the category of topological vector spaces. Equivalently, we equip it with the topology induced by the seminorms $\|-\|_{C^k}$. One important thing to note is that $C^\infty(M)$ need not be Banach, for this reason, it is often convenient to treat smooth functions as if they were just C^k functions, and reintroduce smoothness later down the road. This trick will become important once we turn to proving the Calabi-Yau theorem in Chapter 3. Using only minor cosmetic changes, we can adapt the above definitions to the setting of vector bundles, and we can define $\Gamma^k(E)$, which consists of k-times differentiable sections, which becomes Banach, and we can define $\Gamma^\infty(E)$ and equip it with the inverse limit topology.

Now we turn to defining α-Hölder spaces, where $\alpha \in (0,1)$. To do this, we define the *α-Hölder coefficient* by

$$[f]_\alpha := \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d_g(x,y)^\alpha} \,. \tag{2.1.2}$$

We define the α-Hölder space by $C^{(0,\alpha)}(M)$, which consists of C^0-functions f such that $[f]_\alpha$ is finite. The norm we put on this is the *α-Hölder norm*, defined by

$$\|f\|_{C^{(0,\alpha)}} := \|f\|_{C^0} + [f]_\alpha \,, \tag{2.1.3}$$

which turns $C^{(0,\alpha)}(M)$ into a Banach space.

Now we wish to generalise this to $C^{(k,\alpha)}(M)$, i.e. the space of k-times differentiable functions such that the k'th derivative is α-Hölder continuous. But now we run into a problem, as we defined the k'th derivative as a tensor field, rather than a function. This can be resolved by working in coordinates and taking a sum over multiindices in the definition of the Hölder coefficient, but since we want to generalise to vector bundles eventually, this is not sufficient for our purposes. Instead, we note that the Levi-Civita connection gives a canonical way of identifying "nearby" tangent spaces by parallel transport along geodesics. To make precise what we mean by this, we define the *injectivity radius* of g. For a point $x \in M$, the *injectivity radius of g at x* is

$$\delta_{g,x} := \sup\{R > 0 \,|\, \exp : B_R(0) \to M \text{ is a diffeomorphism}\}\,, \tag{2.1.4}$$

where $B_R(0)$ is the open ball of radius R around 0 in the fibre $(T_xM, (-,-)_g)$. In fact, the condition we have on R is a closed condition (as we're taking open balls of radius R), and M is compact, so the supremum is finite, and a basic result in Riemannian geometry tells us that the set on the RHS is nonempty, so the supremum is actually achieved in the set. Moreover, we can define the *injectivity radius of g* by

$$\delta_g := \inf_{x \in M} \delta_{g,x}\,. \tag{2.1.5}$$

In fact, we can relatively easily see that $\delta_{g,-} : M \to \mathbb{R}$ is a continuous function, so we can use compactness of M to conclude $\delta_g > 0$. So around every point $x \in M$, we can define the geodesic ball $B_{\delta_g}(x) \subseteq M$, such that for any $y \in B_{\delta_g}(x)$, there is a unique geodesic lying completely in $B_{\delta_g}(x)$ that connects x and y, and we note that this is the same geodesic as the one connecting y and x in $B_{\delta_g}(y)$. Using parallel transport along this geodesic, we can identify T_xM with T_yM canonically, and also E_x and E_y. In particular, given a section $s \in \Gamma(E)$, we can make sense of $|s(x) - s(y)|_h \in \mathbb{R}$, if $d_g(x,y) < \delta_g$. So we can extend the definition of the α-Hölder coefficients to sections of arbitrary vector bundles with metric by

$$[s]_\alpha := \sup_{\substack{x \neq y \in M \\ d_g(x,y) < \delta_g}} \frac{|s(x) - s(y)|}{d_g(x,y)^\alpha}\,. \tag{2.1.6}$$

Using this, we can define $C^{(k,\alpha)}(M)$:

Definition 2.1.1 ((k,α)-Hölder space)**.** The (k,α)-*Hölder space*, denoted by $C^{(k,\alpha)}(M)$, consists of k-times differentiable functions such that the k'th derivative is α-Hölder continuous. The (k,α)-*Hölder norm* is defined by

$$\|f\|_{C^{(k,\alpha)}} := \|f\|_{C^k} + [\nabla^k f]_\alpha\,. \tag{2.1.7}$$

Likewise, we define $\Gamma^{(k,\alpha)}(E)$.

In fact, these spaces are Banach again. One thing to note is that for a C^0 function f, $[f]_\alpha$ as defined in (2.1.2) does not necessarily agree with $[f]_\alpha$ as defined in (2.1.6), but it can be shown that the induced norms on $C^{(0,\alpha)}(M)$ induce the same topology, essentially by equipping M with an open cover induced by the balls $B_{\delta_g}(x)$, extracting a finite subcover using compactness, and then applying the triangle inequality a few times. These Hölder spaces embed nicely into spaces of differentiable functions, which is a fact we will make precise later on in this section.

Next we define some tools needed for analysis, starting with *Lebesgue spaces*. Let $q \geq 1$ be a real number. We shall define the naive q-Lebesgue space $\widetilde{L}^q(M)$ as the space of functions f, such that the *q-Lebesgue norm*

$$\|f\|_{L^q} := \left(\int_M |f|^q \mathrm{vol}_g\right)^{1/q}\,, \tag{2.1.8}$$

is finite. This is not a norm on $\widetilde{L}_q(M)$, as $\|f-g\|_{L^q}=0$ if and only if $f=g$ a.e., so $\|f\|_{L^q}=0$ does not imply $f=0$. To resolve this, we define the *q-Lebesgue space* $L^q(M)$ as the space of equivalence classes of almost everywhere equivalent functions

Definition 2.1.2 (q-Lebesgue space)**.** The *q-Lebesgue space* $L^q(M)$ consists of equivalence classes of functions $[f] := \{g : M \to \mathbb{R} \,|\, f = g \,\text{a.e.}\}$, such that $\|f\|_{L^q} < \infty$. We equip this space with the norm induced by the q-Lebesgue norm $\|-\|_{L^q}$ defined in (2.1.8).

In fact, $L^q(M)$ is a Banach space. We will often pretend like elements of $L^q(M)$ are just functions, keeping in the back of our mind that two things are equal if they are equal almost everywhere. Likewise, if we have a section s of E, we can define the q-Lebesgue norm by

$$\|s\|_{L^q} := \left(\int_M |s|_h^q \mathrm{vol}_g\right)^{1/q}, \tag{2.1.9}$$

and define $L^q(E)$ analogously. Lastly, we define $L^\infty(M)$ and $L^\infty(E)$ as the space of essentially bounded sections up to equivalence almost everywhere, and equip it with the supremum norm, which will be defined as

$$\|[f]\|_{L^\infty} := \inf_{g\in[f]} \sup_{x\in M} |g(x)|_h \,. \tag{2.1.10}$$

In fact, this gives us continuous inclusions $C^0(M) \to L^\infty(M)$ and $\Gamma^0(E) \to L^\infty(E)$, that send continuous functions to their respective equivalence classes.

The relation between the various L^q norms will be very helpful, so we shall give some basic well-known results, named after Hölder.

Theorem 2.1.3 (Hölder inequality)**.** *Let $E, F \to M$ be vector bundles equipped with a metric. Suppose $f \in L^p(E)$, $g \in L^q(F)$ for $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $\|f \otimes g\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^q}$.*

In fact, using one more technical result, we can generalize this inequality to a slightly more general setting

Lemma 2.1.4. *Let $f \in L^{p_1}(E)$, $g \in L^{p_2}(F)$ with $1 \le p_1, p_2 \le \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, with $1 \le p \le \infty$, i.e.*

$$\frac{1}{p_1/p} + \frac{1}{p_2/p} = 1\,,$$

then $\|f \otimes g\|_{L^p} \le \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$.

Proof. By Hölder,

$$\|f\otimes g\|_{L^p} = \left\||f\otimes g|_{h\otimes h'}^p\right\|_{L^1}^{1/p} \le \left\||f|_h^p\right\|_{L^{p_1/p}}^{1/p} \left\||g|_{h'}^p\right\|_{L^{p_2/p}}^{1/p} = \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

□

Theorem 2.1.5 (Generalized Hölder)**.** *Let $E_i \to M$ be vector bundles with metric and let $f_i \in L^{p_i}(E_i)$ with $i \in \{0, \dots, n\}$, with $1 \le p_i \le \infty$, $\frac{1}{p} := \sum_i \frac{1}{p_i}$, with $1 \le p \le \infty$. Then*

$$\left\|\bigotimes_i f_i\right\|_{L^p} \le \prod_i \|f_i\|_{L^{p_i}}\,.$$

Proof. By induction. $n=1$ is trivial, $n=2$ follows from the previous lemma. Suppose it holds for $n-1>2$. Define $\frac{1}{q} := \sum_{i\le n-1} \frac{1}{p_i} \le \frac{1}{p} \le 1$. So we can apply the lemma for $n-1$ to conclude

$$\left\| \bigotimes_{i\le n-1} f_i \right\|_{L^q} \le \prod_{i\le n-1} \|f_i\|_{L^{p_i}} .$$

Now applying the case $n=2$, we see (note $\frac{1}{q}+\frac{1}{p_n}=\frac{1}{p}$ by definition)

$$\left\| f_n \otimes \bigotimes_{i\le n-1} f_i \right\|_{L^p} \le \left\| \bigotimes_{i\le n-1} f_i \right\|_{L^q} \|f_n\|_{L^{p_n}} \le \prod_i \|f_i\|_{L^{p_i}} ,$$

which completes the proof. □

We will use this to prove an interpolation result, which we will call the *Hölder interpolation theorem*, this theorem will be used when discussing the proof for the Calabi-Yau theorem.

Theorem 2.1.6 (Hölder interpolation)**.** *Let $1\le p\le q\le\infty$, suppose $f\in L^p(E)\cap L^q(E)$. Suppose $r\in[p,q]$, i.e. $\frac{1}{r}=\frac{\alpha}{p}+\frac{1-\alpha}{q}$ for some $\alpha\in[0,1]$, then*

$$\|f\|_{L^r} \le \|f\|_{L^p}^{\alpha} \|f\|_{L^q}^{1-\alpha} .$$

Proof. We see

$$\frac{1}{r} = \frac{1}{p/\alpha} + \frac{1}{q/(1-\alpha)} \le 1 ,$$

thus we can apply the Hölder inequality to get

$$\|f\|_{L^r} = \left\| |f|_h^{\alpha} |f|_h^{1-\alpha} \right\|_{L^r} \le \| |f|_h^{\alpha} \|_{L^{p/\alpha}} \left\| |f|_h^{1-\alpha} \right\|_{L^{q/(1-\alpha)}} = \|f\|_{L^p}^{\alpha} \|f\|_{L^q}^{1-\alpha} ,$$

which completes the proof. □

One particularly interesting result following from this is the following

Corollary 2.1.7. *Let $1\le p\le q\le\infty$, if $\|f\|_{L^p}$ and $\|f\|_{L^q}$ are both less than or equal to some C, then so is $\|f\|_{L^r}$ for every $r\in[p,q]$.*

Returning back to the functional analysis, the Hölder inequality tells us that if $p>q\ge 1$, we have

$$\|f\|_{L^q} \le \|f\|_{L^p} \operatorname{vol}_g(M)^{(p-q)/pq} . \tag{2.1.11}$$

In particular, we have an inclusion $\iota : L^p(E)\to L^q(E)$ that is continuous. Therefore, we have a system of Banach spaces again. One thing we note is that for any essentially bounded section f, we have

$$\|f\|_{L^p} = \left(\int_M |f|_h^p \mathrm{vol}_g \right)^{1/p} \le (\|f\|_{L^\infty}^p \operatorname{vol}_g(M))^{1/p} = \|f\|_{L^\infty} \operatorname{vol}_g(M)^{1/p} .$$

In particular, $L^\infty(E)$ embeds continuously into the inverse limit of this system, so in particular, so does $\Gamma^0(E)$.

Lastly, we will discuss *Sobolev spaces*. Defining these spaces requires defining the weak derivative. To do that, we need to define the formal adjoint of the operator $\nabla^k : \Gamma(E)\to\Gamma(E\otimes T^*M^{\otimes k})$. First, consider $\Gamma^\infty(E)$ together with the L^2 inner product, where we note that M is compact, so any smooth function is in particular L^2. Now we can define the adjoint of ∇^k as the unique linear operator $(\nabla^k)^* : \Gamma^\infty(E \otimes T^*M^{\otimes k}) \to \Gamma^\infty(E)$ such that for any $\varphi \in \Gamma^\infty(E)$, $\psi \in \Gamma^\infty(E \otimes T^*M^{\otimes k})$ we have

$$(\nabla^k \varphi, \psi)_{h\otimes g} = (\varphi, (\nabla^k)^*\psi)_h \,. \tag{2.1.12}$$

Now, we can define what it means for an $L^p(E)$ function to be *weakly differentiable*

Definition 2.1.8 (Weak derivative)**.** Let $u \in L^1(E)$, $v \in L^1(E \otimes T^*M^{\otimes k})$. We say $\nabla^k u = v$ *weakly* if for any smooth section $\varphi \in \Gamma^\infty(E \otimes T^*M^{\otimes k})$, we have

$$\int_M (u, (\nabla^k)^*\varphi)_h \mathrm{vol}_g = \int_M (v, \varphi)\mathrm{vol}_g \,. \tag{2.1.13}$$

In this case, we say that v is a *k'th order weak derivative* of u, and that u is *k times weakly differentiable.*

Note that the above definition implies that any function almost everywhere equivalent to a weak derivative is itself a weak derivative, so it is indeed well defined that the weak derivative is a class in L^1. In fact, it can be shown that if a function is k times weakly differentiable, then it is also $k-1$ times weakly differentiable, and its k'th weak derivative is unique up to equivalence almost everywhere. For details, we refer the reader to [Nic22].

A natural next step would be to put some conditions on the k'th order weak derivative. In particular, we will ask it to lie in the p-Lebesgue space again. This leads us to the notion of *Sobolev space.*

Definition 2.1.9 (Sobolev spaces)**.** The *Sobolev space* $L^p_k(E)$ consists of L^p classes u that are k times weakly differentiable such that all derivatives up to order k are in L^p. The (k,p)-*Sobolev norm* is defined to be

$$\|u\|_{L^p_k} = \left(\sum_{j=0}^{k} \left\| \nabla^k u \right\|_{L^p}^p \right)^{1/p} \,. \tag{2.1.14}$$

In fact, this space is Banach again.

It is now clear that we have continuous inclusions $L^p_k(E) \hookrightarrow L^q_k(E)$ whenever $p > q \geq 1$, and we have continuous inclusions $L^p_k(E) \hookrightarrow L^p_l(E)$ whenever $k > l \geq 0$, in particular, we can make sense of $L^p_\infty(E)$ and equip it with the inductive limit topology. However, there is a much stronger result about these spaces, known as the Sobolev embedding theorem

Theorem 2.1.10 (Sobolev embedding)**.** *Let $k \geq l \geq 0$ be integers, $p, q \geq 1$ real numbers, and $\alpha \in (0,1)$. If*

$$\frac{1}{q} \leq \frac{1}{r} + \frac{k-l}{m} \,,$$

then there is a continuous inclusion $L^q_k(E) \hookrightarrow L^r_l(E)$. Moreover, if

$$\frac{1}{q} \leq \frac{k-l-\alpha}{m} \,,$$

then there is a continuous inclusion $L^q_k(E) \hookrightarrow \Gamma^{(l,\alpha)}(E)$.

See [Nic22] for the idea behind the proof. Moreover, we have a result on the compactness of these embeddings, which will be vital for the proof of the Calabi-Yau theorem later on. This result is known as the Rellich-Kondrachov theorem.

Theorem 2.1.11 (Rellich-Kondrachov)**.** *If the inequalities in the previous theorems are strict, the associated embeddings are compact. Moreover, the embedding* $\Gamma^{(k,\alpha)}(E) \hookrightarrow \Gamma^k(E)$ *is compact.*

Again, we refer the reader to [Nic22] for the proof.

2.1.2 Differential operators and ellipticity

Now we will turn to defining differential operators and discussing some results on elliptic differential operators. We will not go very in depth into the theory, see [CB17; KN22; Nic22; Wel80] for a more detailed description. In particular, we will consider the example of the Laplacian. In this part, all vector bundles will be complex and all functions will be assumed to be complex valued, unless stated otherwise.

We start with a definition.

Definition 2.1.12 (Linear differential operator)**.** Let $E, F \to M^m$ be vector bundles. A *linear differential operator* of order at most k from E to F, is a local operator[1] $P : \Gamma^\infty(E) \to \Gamma^\infty(F)$, such that in any simultaneously trivialising chart (U, φ), we have that P_φ is a matrix of linear differential operators of order at most k on $\varphi(U) \subseteq \mathbb{R}^m$. The space of such operators will be denoted by $\mathrm{Dif}_k(E, F)$.

We note that $\mathrm{Dif}_{k-1}(E, F)$ is a linear subspace of $\mathrm{Dif}_k(E, F)$, since a differential operator of order at most $k-1$ is in particular a differential operator of order at most k. Defining differential operators between vector bundles that are of order exactly k is not possible, since this notion is not coordinate invariant, and therefore doesn't lift to manifolds. As an example, the Laplacian on $\mathbb{R}^2$ is of order two: $\Delta = \partial_x^2 + \partial_y^2$, but if we go to polar coordinates, we see $\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2$, which has an order one term. However, we do see that the coefficient of the order two part of the Laplacian transforms in a nice way. This is generally true, we can define an invariant associated to a differential operator of order at most k that gives us the behaviour of the order k part of the differential operator.

To make this precise, note that the order k part of a differential operator $P : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is an order k polynomial in $\partial_1, \dots, \partial_n$ and transforms accordingly between coordinates. We note that we can interpret ∂_i as a fibrewise linear function on the cotangent bundle $T^*\mathbb{R}^n$. So the order k part of P can be interpreted as a fibrewise homogeneous polynomial on $T^*\mathbb{R}^n$ of order k. However, this is not yet how we wish to define this invariant. Instead, we will interpret ∂_i as a fibrewise linear complex-valued function defined by $\partial_i(dx^j) = i\delta_{ij}$, which is an odd looking convention at first sight, but has its roots in the theory of Fourier transforms. The complex-valued fibrewise polynomial on $T^*\mathbb{R}^n$ obtained in this way is called the *principal symbol* of P and we denote it by $\sigma_k(P)$. Because it transforms in a nice way under coordinate changes, this definition immediately generalises to manifolds by defining it in charts. In fact, this definition also carries over to differential operators between vector bundles

Definition 2.1.13 (Principal symbol for differential operators)**.** Let P be a differential operator of order at most k between vector bundles E and F over M, and let $\pi : T^*M \to M$ be the cotangent bundle. The *principal symbol* of P is the section $\sigma_k(P) \in \Gamma(\underline{\mathrm{Hom}}(\pi^*E, \pi^*F))$, such that in any trivialising chart (U, φ), $\sigma_k(P)_\varphi$ is a matrix whose elements are the principal symbols of the elements of the matrix P_φ.

For a proof that this is well defined and an explicit formula, we refer the reader to [Wel80]. The appearance of the cotangent bundle might seem a bit mysterious at face value, but it is precisely the

[1] Meaning $\mathrm{supp}(Ps) \subseteq \mathrm{supp}(s)$ for any $s \in \Gamma^\infty(E)$.

setting we need to work in for this to be well defined. We will now state some properties of the principal symbol, for a proof, see [Wel80], though the proofs are not particularly difficult:

(i) For a differential operator P of order at most k between E and F, and a differential operator Q of order at most l between F and H, the principal symbol of the composition satisfies

$$\sigma_{l+k}(Q \circ P) = \sigma_l(Q) \circ \sigma_k(P)\,. \tag{2.1.15}$$

(ii) σ_k descends to an injective map

$$\sigma_k : \mathrm{Dif}_k(E,F)/\mathrm{Dif}_{k-1}(E,F) \to \Gamma(\underline{\mathrm{Hom}}(\pi^*E, \pi^*F))\,.$$

(iii) For $(x,v) \in T^*M$ and $\lambda \in \mathbb{R}$, we have $\sigma_k(P)(x,\lambda v) = \lambda^k \sigma_k(P)(x,v)$, since it is fibrewise a matrix of homogeneous polynomials of order k. In particular, $\sigma_k(P)$ always vanishes along the zero section $0_{T^*M} \subseteq T^*M$.

One can ask the question what happens away from the zero section. A nice class of operators are the ones where the principal symbol is invertible away from the zero section[1].

Definition 2.1.14 (Elliptic differential operator)**.** A differential operator P of order at most k between equidimensional vector bundles E and F, is called *elliptic* if $\sigma_k(P)(x,v)$ is invertible for any $(x,v) \in T^*M \setminus 0_{T^*M}$.

The reason we are interested in these kinds of operators is that they are invertible in some weak sense, which we shall not try to make precise in this thesis. This will require defining pseudo-differential operators, which are, in general, non-local operators that extend the definition of differential operators to allow more general principal symbols than just homogeneous polynomials. Good references are [CB17] and [Wel80]. It is precisely this invertibility that makes elliptic differential operators nice to work with. In particular, it gives us many results on the existence of solutions to elliptic differential equations, which we shall explore in a bit, after introducing some more theory.

If $E, F \to M$ are hermitian vector bundles with metrics $(-,-)_E$ and $(-,-)_F$, and M is equipped with a density μ, we can define the inner product $\langle -,- \rangle_E : \Gamma(E) \times \Gamma(E) \to \mathbb{C}$ by

$$\langle s, s' \rangle_E := \int_M (s, s')_E \mu\,, \tag{2.1.16}$$

and similarly for F. Then we can do the following:

Definition 2.1.15 (Formal adjoint)**.** Let $k \in \mathbb{N}_0$, let $E, F \to M$ be vector bundles and let $P \in \mathrm{Dif}_k(E,F)$. Then we define the *formal adjoint* P^* of P by the defining property that for any $s \in \Gamma(F)$ and $s' \in \Gamma(E)$,

$$\langle s, Ps' \rangle_F = \langle P^* s, s' \rangle_F\,. \tag{2.1.17}$$

This has the following properties, whose proofs can be found in [CB17] or [Wel80]

(i) P^* is well-defined and unique.

[1] Note that this requires that E and F have the same rank.

(ii) $P^* \in \mathrm{Dif}_k(F, E)$.

(iii) $\sigma_k(P^*) = \sigma_k(P)^*$, in particular, if P is elliptic, so is P^*.

We end this part by discussing an important example, which is a differential operator that we all know and love, the de Rham derivative.

Example 2.1.16 (De Rham derivative)**.** Let M^m be any manifold and let $E = \Lambda^k T^*M$, $F = \Lambda^{k+1}T^*M$ and $P = d$. Then P is a differential operator of degree 1. For its principal symbol, we note that we have $(d\omega)_{i_0\ldots i_k} = \partial_{[i_0}\omega_{i_1\ldots i_k]}$. Writing down the principal symbol is now a straightforward, we see $\sigma_1(d)(\xi) = i\xi \wedge -$. This will rarely be invertible, as we already have that $\Lambda^k T^*M$ and $\Lambda^{k+1}T^*M$ are generically of different rank. However, if m is odd and $k = (m-1)/2$, we see that $\mathrm{rank}(\Lambda^k T^*M) = \mathrm{rank}(\Lambda^{k+1}T^*M)$, the $m = 1$ case is rather easy, so let's take $m = 3$. Given coordinates $\{x^1, x^2, x^3\}$, take the ordered frames $\{dx^1, dx^2, dx^3\}$ and $\{dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2\}$ for T^*M and $\Lambda^2 T^*M$ respectively. We see

$$\sigma^1(d)(\xi) = i \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} . \tag{2.1.18}$$

Unfortunately, we see that this is not invertible away from 0, hence it is not elliptic.

The above gives us a non-example of ellipticity of a differential operator, so here we cannot use the previous theorem. However, the de Rham differential still fits into the theory by considering the following:

Definition 2.1.17 (Elliptic complex)**.** Let $k \in \mathbb{N}_0$. Let (E, P) be a complex of differential operators of order at most k, i.e. a complex

$$0 \to \Gamma(E_0) \xrightarrow{P_0} \Gamma(E_1) \xrightarrow{P_1} \ldots \xrightarrow{P_{n-1}} \Gamma(E_n) \to 0\,,$$

where $E_i \to M$ are complex vector bundles and P_i are a differential operators of order at most k, such that $P_i \circ P_{i-1} = 0$ for every i. Then (E, P) is called *elliptic* if the associated symbol complex

$$0 \to \pi^* E_0 \xrightarrow{\sigma^d(P_0)} \pi^* E_1 \xrightarrow{\sigma^d(P_1)} \ldots \xrightarrow{\sigma^d(P_{n-1})} \pi^* E_n \to 0$$

is exact outside of the zero section.

In fact, since we computed the symbol of the de Rham complex is $\sigma^1(d)(\xi) = \xi \wedge -$, we see that the de Rham complex is, in fact, elliptic.

If the E_i were hermitian vector bundles with metric $(-,-)_i$, we could also define the formal adjoints P_i^*.

Definition 2.1.18 (Laplacian)**.** Let M be a manifold equipped with a density, and let $(E, (-,-), P)$ be a hermitian complex of differential operators of order at most k, i.e. a complex of differential operators of order at most k such that all vector bundles are equipped with a hermitian metric. Then the *Laplacian* Δ of E is a collection of differential operators of order at most $2k$, consisting of $\Delta_i : \Gamma(E_i) \to \Gamma(E_i)$ given by

$$\Delta_i = P_{i-1} \circ P_i^* + P_{i+1}^* \circ P_i\,. \tag{2.1.19}$$

Now we have an important theorem, whose proof can be found in [CB17], though the proof is not particularly difficult.

Theorem 2.1.19. *Let μ be a density on M. A hermitian complex of differential operators $(E, (-,-), P)$ is elliptic if and only if the associated Laplacian Δ consists of elliptic operators.*

In particular, for the de Rham complex over a Riemannian manifold, we have a family of elliptic operators $\Delta_i : \Omega^i(M) \to \Omega^i(M)$.

2.1.3 Solving elliptic equations: the non-smooth setting

As noted in 1984 by H.F. Adu, smooth operators are no place to be ending, but somewhere to start [Sad84]. In order to solve differential equations, it is often helpful to move away from the smooth setting and only return to it later. This is because $C^\infty(M)$ is not a Banach space, so it is usually not particularly nice to work with, moreover, the non smooth setting places us in the regime of the Sobolev embedding theorems 2.1.10, which are quite strong results. Moreover, in Chapter 3, we will need to solve a nonlinear partial differential equation, for which we shall need some linear differential operators with non-smooth coefficients, which can be defined analogously to the smooth setting.

Extending the definition of differential operators to the (k, α)-Hölder setting is rather easy, as applying differential operators to these objects is simply done by taking derivatives. However, we also want to extend the theory to the setting of Sobolev spaces, so it is pleasant that we have the following theorem.

Theorem 2.1.20. *Let $k \in \mathbb{N}_0$ and let $E, F \to M$ be vector bundles. Then any $P \in \mathrm{Dif}_k(E, F)$ uniquely extends for every natural number $l \geq k$ to a map*

$$P : L^2_l(E) \to L^2_{l-k}(F).$$

The proof for this rather important theorem can be found in [Wel80], or in [KN22], where they also treat the case of differential operators with non-smooth coefficients. One should not be surprised by this result, a differential operator of order at most k will take k derivatives of a section that it eats, so one should expect that this also does that when we move to the setting of Sobolev spaces and weak derivatives.

In the particular case of the Laplacian, we have the following useful result, whose proof can be found in [Ros97]:

Theorem 2.1.21. *let $k \in \mathbb{N}_0$. Then $L^2(\Lambda^k T^* M)$ admits an orthonormal decomposition into eigenvalues of Δ_k. Moreover, the spectrum of Δ_k is discrete, nonnegative and all eigenvalues have finite multiplicity.*

We shall now give some theorems regarding regularity of elliptic differential equations. The idea is the following, elliptic operators are invertible in some weak sense, so if P is a differential operator of order at most k and s is a section such that Ps has l derivatives, then we can invert P to show that s must have $l + k$ derivatives. This gives us the following theorem, whose proof can be found in [CB17]:

Theorem 2.1.22 (Elliptic regularity)**.** *Let $k \in \mathbb{N}_0$, $l \in \mathbb{N}$, let $E, F \to M$ be vector bundles and let $P \in \mathrm{Dif}_k(E, F)$. Suppose $f \in L^2_k(E)$ such that $Pf \in L^2_l(F)$, then we have $f \in L^2_{k+l}(E)$. In particular, if $Pf \in L^2_\infty(F)$, then $f \in L^2_\infty(E)$.*

We would also like some results on Hölder spaces, which can be proved using *Schauder estimates*. The nice thing about these estimates is that they carry over to the case of elliptic differential operators with non-smooth coefficients, i.e. if we allowed the matrix of differential operators in Definition 2.1.12 to have (l,α)-Hölder coefficients.

Theorem 2.1.23 (Schauder estimates)**.** *Let $k,l \in \mathbb{N}_0$, $\alpha \in (0,1)$ and let $E,F \to M$ be vector bundles and let P be an elliptic differential operator of order k from E to F with $C^{(l,\alpha)}(M)$-coefficients. Suppose $f \in \Gamma^{(k,\alpha)}(E)$ such that $P(f) \in \Gamma^{(l,\alpha)}(F)$, then there is a constant C that is independent of f, such that*

$$\|f\|_{C^{(k+l,\alpha)}} \leq C(\|P(f)\|_{C^{(l,\alpha)}} + \|f\|_{C^0}) \tag{2.1.20}$$

The proof can be found in [Mor66]. In particular, the above theorem tells us that $f \in C^{(k+l,\alpha)}(E)$, stated in terms of an upper bound on its norm. Usually, many results like this are stated in terms of estimates, this is because estimates are easy to combine into new estimates, which can be very useful, as we will see in Chapter 3.

There are also some useful results regarding solutions to elliptic differential equations, see [Joy00] for a discussion on these results and for the proofs of these theorems.

Theorem 2.1.24 (Solutions to elliptic differential equations I)**.** *Let $k,l \in \mathbb{N}_0$, $\alpha \in (0,1)$, $p > 1$, let $E,F \to M$ be vector bundles and $P \in \mathrm{Dif}_k(E,F)$ be elliptic. Then the images of the maps*

$$P : \Gamma^{(k+l,\alpha)}(E) \to \Gamma^{(l,\alpha)}(F)\,, \quad \textit{and} \quad P : L^p_{k+l}(E) \to L^p_l(F)\,,$$

are closed. Moreover, if $w \in C^{(l,\alpha)}(F)$, there is a $v \in C^{(k+l,\alpha)}(E)$ such that $Pv = w$ if and only if $w \perp \ker P^*$. Moreover, v is unique up to $\ker P$. Likewise, if $w \in L^2_l(F)$ then there is a $v \in L^2_{k+l}(E)$ if and only if $w \perp \ker P^*$, where v is unique up to $\ker P$.

In the non-smooth case, this theorem also works. Here, note that the formal adjoint is still well defined, analogously to the formal adjoint of a smooth operator, see [Joy00] for details. We have the following

Theorem 2.1.25 (Solutions to elliptic differential equations II)**.** *Let $k,l \in \mathbb{N}_0$ such that $k > 0$, let $\alpha \in (0,1)$ and let $E,F \to M$ be vector bundles. Suppose P is a linear elliptic differential operator of order at most k from E to F with $C^{(l,\alpha)}$-coefficients. Then P^* is an elliptic differential operator of order at most k from F to E with $C^{(l-k,\alpha)}$-coefficients. If $w \in \Gamma^{(l,\alpha)}(F)$ then there is a $v \in \Gamma^{(k+l,\alpha)}(E)$ such that $Pv = w$ if and only if $w \perp \ker P^*$, where v is unique up to $\ker P$.*

This completes our discussion of analysis on manifolds.

2.2 Kähler manifolds and some complex geometry

Now that we have introduced some tools from analysis, we will turn to a bit of complex geometry. The tools from the previous section will not appear here, nor in the rest of this chapter, though we will study the Lagrangian on Kähler manifolds a bit. The analysis will return in Chapter 3. The next three sections of this chapter are more intertwined, this section will introduce complex manifolds, the next section will introduce line bundles where we will also study how they relate to complex manifolds, and in the last section, we will use both theories to produce some examples of Calabi-Yau manifolds and to give some basic results about them.

This section will be devoted to introducing tools from (almost-)complex geometry, in particular, we shall study some structure on Kähler manifolds, which are a particular kind of complex manifold. In Subsection 2.2.1, we will recap the theory of almost complex structures and complex manifolds and we will state the Newlander-Nirenberg theorem that relates the two. We shall also spend some time on complex tensor calculus in coordinates. In Subsection 2.2.2, we shall introduce hermitian manifolds, which are simultaneously Riemannian and complex, and we shall introduce Kähler manifolds, which are hermitian manifolds that are also symplectic. We shall also introduce some tools for studying these Kähler manifolds and we shall state some fundamental results about analysis on these spaces and what these results tell us about the geometry of Kähler manifolds. Lastly, we will also do some tensor calculus in coordinates on Kähler manifolds, where we will derive some identities that will be important all throughout this thesis.

Some results of this section will be generalised to the setting of Lie algebroids in Chapter 6, which is why we sometimes spend a bit more time on deriving certain results.

We will assume the reader is familiar with the definition of complex manifolds and has some understanding of differential geometry.

2.2.1 Recap: complex geometry

In this subsection, we will briefly recap some basic (almost-)complex geometry, with the goal to define the $\partial, \bar{\partial}$ and d^c operators, to say a bit about Dolbeault cohomology, and introduce some complex tensor calculus in coordinates, which we will a lot in this thesis. We will be rather quick about many things in this beautiful field of study, for more details, we refer the reader to [GH94; Huy05; KN69; Wel80].

An *almost complex manifold* is a pair (M, J) where M is a smooth manifold and $J : TM \to TM$ is a tangent bundle automorphism such that $J^2 = -\mathrm{id}$.

A *complex manifold* is a manifold that is locally modelled on $\mathbb{C}^m$, such that transition functions are holomorphic maps. These manifolds naturally come equipped with a tangent bundle automorphism $J : TM \to TM$ such that $J^2 = -\mathrm{id}$, making them also almost complex manifolds. Whenever an almost complex structure J comes from a complex structure, we call it *integrable*.

On any almost complex manifold, J decomposes $TM \otimes \mathbb{C}$ in $+i$ and $-i$ eigenbundles, respectively denoted by $TM^{(1,0)}$ and $TM^{(0,1)}$. Given a frame $\{\partial_1, J\partial_1, \partial_2, J\partial_2, \dots, \partial_m, J\partial_m\}$ for TM, we then see that $\partial_{z^i} := \frac{1}{2}(\partial_i - iJ\partial_i)$ define a frame for $TM^{(1,0)}$. Likewise, $\partial_{\bar{z}^i} := \frac{1}{2}(\partial_i + iJ\partial_i)$ define a frame for $TM^{(0,1)}$. Note that often, we will denote $TM \otimes \mathbb{C}$ by just TM, if there's risk of confusion, we will try to be explicit about whether we mean TM or its complexification.

Moreover, J also induces a splitting of $T^*M \otimes \mathbb{C}$ into $+i$ and $-i$ eigenbundles, respectively denoted by $T^*M^{(1,0)}$ and $T^*M^{(0,1)}$, such that we also get a decomposition

$$\Omega^k(M) \cong \bigoplus_{p+q=k} \Omega^{(p,q)}(M),$$

where $\Omega^{(p,q)}(M) := \Gamma^\infty(\Lambda^p T^*M^{(1,0)} \otimes \Lambda^q T^*M^{(0,1)})$. Defining $\{dz^i\}$ to be the dual frame to $\{\partial_{z^i}\}$ and $\{d\bar{z}^i\}$ to be the dual frame to $\{\partial_{\bar{z}^i}\}$, we see that $\Omega^{(p,q)}(M)$ is locally generated by sections of the form $dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge dz^{j_q}$, with $i_1 < \dots < i_p$ and $j_1 < \dots < j_q$ are in $\{1, \dots, m\}$.

There is a rather famous theorem that tells us when an almost complex structure is integrable. This theorem is due to Newlander and Nirenberg in 1957 [NN57]. It states that J is integrable if and only if $TM^{(1,0)}$ is involutive. We will extend this theorem slightly with a few short calculations and turn it into the following:

Theorem 2.2.1 (Newlander-Nirenberg)**.** *Let (M,J) be an almost complex manifold, then the following are equivalent:*

(i) *J is integrable;*

(ii) *$TM^{(1,0)}$ is an involutive distribution in $TM\otimes\mathbb{C}$;*

(iii) *the* Nijenhuis tensor *$N\in\Gamma(T^*M^{\otimes 2}\otimes TM)$, defined for real $X,Y\in\mathfrak{X}(M)$ by*

$$N_J(X,Y)=[X,Y]+J([JX,Y]+[X,JY])-[JX,JY]\,, \tag{2.2.1}$$

vanishes identically;

(iv) *The de Rham operator acts on (p,q)-forms by*

$$d:\Omega^{(p,q)}(M)\to\Omega^{(p+1,q)}(M)\oplus\Omega^{(p,q+1)}(M)\,.$$

Proof. The equivalence of (i) and the others is the main difficulty in this theorem and can be found in [NN57], where they actually prove the equivalence of (i) and (iv).

We start with proving the equivalence of (ii) and (iv). To do this, note that any $\alpha\in\Omega^k(M;\mathbb{C})$ can be written as a finite sum $\alpha=\sum_i\alpha_i$, where $\alpha_i=\alpha_{i1}\wedge\cdots\wedge\alpha_{ik}$ for some $\alpha_{ij}\in\Omega^1(M)$. Therefore, we only need to prove that (ii) is equivalent to $d:\Omega^{(0,1)}(M)\to\Omega^{(1,1)}(M)\oplus\Omega^{(0,2)}(M)$, as the statement for $\Omega^{(1,0)}(M)$ follows from complex conjugation.

So take any $\alpha\in\Omega^{(0,1)}(M)$. Then (iv) is equivalent to saying that for any $X,Y\in\Gamma(TM^{(1,0)})$, we have $d\alpha(X,Y)=0$. We see $d\alpha(X,Y)=X(\alpha(Y))-Y(\alpha(X))-\alpha([X,Y])$. Now, since $\alpha\in\Omega^{(0,1)}(M)$, we have $\alpha(X)=\alpha(Y)=0$, hence we see that (iv) is equivalent to saying that $\alpha([X,Y])=0$ for any $(0,1)$-form α and any $(1,0)$-vector fields X,Y. Since this must hold for any α, (iv) is equivalent to saying $[X,Y]\in TM^{(1,0)}$ for any $X,Y\in\Gamma(TM^{(1,0)})$, i.e. $TM^{(1,0)}$ is involutive, proving (ii) $\iff$ (iv).

So it remains to prove (ii) $\iff$ (iii). To do this, we let v,w be real vector fields and consider $X=v-iJv$ and $Y=w-iJw$, where we note that any $(1,0)$-vector field can be written like this. We see

$$[X,Y]=[v,w]-[Jv,Jw]-i([v,Jw]+[Jv,w])\,.$$

Therefore, involutivity of $TM^{(1,0)}$ is equivalent to having $[v,Jw]+[Jv,w]=J([v,w]-[Jv,Jw])$ for any v,w real vector fields, i.e. $N_J(X,Y)=0$ for any real vector fields X,Y, showing the equivalence of (iii) and (ii). One last remark is that this N_J is indeed a tensor, as we have for any $f\in C^\infty(M)$,

$$N_J(X,fY)=fN_J(X,Y)+X(f)Y+J((JX)(f)Y+X(f)JY)-(JX)(f)JY=fN_J(X,Y)\,,$$

completing the proof. □

In particular, we see that the de Rham operator on a complex manifold splits as $d=\partial+\bar\partial$, defined in holomorphic coordinates (z^α) by

$$\partial:\Omega^{(p,q)}(M)\to\Omega^{(p+1,q)}(M)\,;\qquad \partial(\omega_{\gamma\bar\delta}dz^\gamma\wedge d\bar z^{\bar\delta})=\partial_{z^\alpha}\omega_{\gamma\bar\delta}dz^\alpha\wedge dz^\gamma\wedge d\bar z^{\bar\delta}\,, \tag{2.2.2}$$

where γ is a multi-index of order p, and δ is a multi-index of order q. Likewise,

$$\bar{\partial} : \Omega^{(p,q)}(M) \to \Omega^{(p,q+1)}(M) ; \qquad \bar{\partial}(\omega_{\gamma\bar{\delta}} dz^{\gamma} \wedge d\bar{z}^{\bar{\delta}}) = \partial_{\bar{z}^{\bar{\alpha}}} \omega_{\gamma\bar{\delta}} d\bar{z}^{\bar{\alpha}} \wedge dz^{\gamma} \wedge d\bar{z}^{\bar{\delta}} . \tag{2.2.3}$$

These operators square to 0, so they define chain complexes. In particular, on any complex manifold M, for every $p \in \mathbb{N}_0$, we have a chain complex

$$\Omega^{(p,0)}(M) \xrightarrow{\bar{\partial}} \Omega^{(p,1)}(M) \xrightarrow{\bar{\partial}} \dots$$

The associated cohomology is called the *Dolbeault cohomology* of M, and will be denoted by

$$H^{(p,q)}(M) := \frac{\ker(\bar{\partial} : \Omega^{(p,q)}(M) \to \Omega^{(p,q+1)}(M))}{\operatorname{im}(\bar{\partial} : \Omega^{(p,q-1)}(M) \to \Omega^{(p,q)}(M))} . \tag{2.2.4}$$

The numbers $h^{p,q} := \dim(H^{(p,q)}(M))$ are known as the *Hodge numbers* of M.

We also define the conjugate d^c of the de Rham operator as $d^c := i(\bar{\partial} - \partial)$, such that $dd^c = 2i\partial\bar{\partial}$. The advantage of d^c is that it is a real operator, i.e. it maps real forms to real forms, which is a property that ∂ and $\bar{\partial}$ do not have.

Lastly, later on, we will work in coordinates and it might be a bit confusing how real things correspond to imaginary things, which is what we will now quickly consider.

Given a real frame $\{\partial_1, J\partial_1, \dots, \partial_m, J\partial_m\}$ for TM, we can form the associated frames $\{\partial_{z^\alpha}\}$ and $\{\partial_{\bar{z}^{\bar{\alpha}}}\}$ for $TM^{(1,0)}$ and $TM^{(0,1)}$. We will use Latin indices $\{a, b, \dots\}$ to run over $\{\partial_1, \partial_2, \dots, \partial_m\}$, we will use barred Latin indices $\{\bar{a}, \bar{b}, \dots\}$ to run over $\{J\partial_1, J\partial_2, \dots, J\partial_m\}$ and we use Latin indices $\{i, j, \dots\}$ to run over $\{\partial_1, J\partial_1, \dots, \partial_m, J\partial_m\}$. We use Greek indices $\{\alpha, \beta, \dots\}$ to run over $\{\partial_{z^1}, \dots, \partial_{z^m}\}$ and barred Greek indices $\{\bar{\alpha}, \bar{\beta}, \dots\}$ to run over $\{\partial_{\bar{z}^1}, \dots, \partial_{\bar{z}^m}\}$.

Given a tangent vector $u = u^i\partial_i$, we see that the indices are related by

$$u^i\partial_i = u^\alpha\partial_\alpha + u^{\bar{\alpha}}\partial_{\bar{\alpha}} , \tag{2.2.5}$$

so we get

$$u^a = \tfrac{1}{2}(u^\alpha + u^{\bar{\alpha}}) ; \tag{2.2.6}$$

$$u^{\bar{a}} = \tfrac{1}{2i}(u^\alpha - u^{\bar{\alpha}}) . \tag{2.2.7}$$

We can invert this to get

$$u^\alpha = u^a + iu^{\bar{a}} ; \tag{2.2.8}$$

$$u^{\bar{\alpha}} = u^a - iu^{\bar{a}} . \tag{2.2.9}$$

Likewise,

$$u_a = u_\alpha + u_{\bar{\alpha}} ; \tag{2.2.10}$$

$$u_{\bar{a}} = i(u_\alpha - u_{\bar{\alpha}}) ; \tag{2.2.11}$$

$$u_\alpha = \tfrac{1}{2}(u_a - iu_{\bar{a}}) ; \tag{2.2.12}$$

$$u_{\bar{\alpha}} = \tfrac{1}{2}(u_a + iu_{\bar{a}}) . \tag{2.2.13}$$

We see that all these relations are linear. Therefore, things like the first Bianchi identity for the Riemann tensor will carry over from Latin indices to Greek indices. Moreover, $u^iv_i = u^\alpha v_\alpha + u^{\bar{\alpha}}v_{\bar{\alpha}}$ as is easily calculated. Therefore, these new Greek indices behave very much like the Latin indices, so we can translate tensor calculus to these indices with only minor changes.

2.2.2 Kähler geometry

Kähler manifolds are a particular kind of complex manifold with a Riemannian structure that is particularly well behaved. Let M^m is a complex manifold of complex dimension m, i.e. real dimension $2m$, with complex structure J. Suppose we are given a Riemannian metric g on M. We say that g is a *hermitian metric* if for any $x \in M$, and any pair of (real) tangent vectors $u, v \in T_xM$, we have $g_x(J_xu, J_xv) = g_x(u, v)$, i.e. that J_x is g_x-orthogonal for every x. A complex manifold equipped with a hermitian metric will be called a *hermitian manifold*.

For $u, v \in T_xM^{(1,0)}$, we have the following identity

$$g(u, v) = g(Ju, Jv) = i^2 g(u, v) \implies g(u, v) = 0\,. \tag{2.2.14}$$

Likewise, g vanishes on pairs of $(0,1)$-vectors. Instead, we have that if u is a $(1,0)$ tangent vector, we can decompose into real and imaginary part $u = u_{\mathbb{R}} + iu_{\mathbb{C}}$, such that $u_{\mathbb{R}}$, $u_{\mathbb{C}}$ are real, and we see

$$g(u, \overline{u}) = g(u_{\mathbb{R}}, u_{\mathbb{R}}) + g(u_{\mathbb{C}}, u_{\mathbb{C}})\,. \tag{2.2.15}$$

So whenever $u \neq 0$, this will be strictly positive. Moreover, for two $(1,0)$-vectors u, v, we have

$$g(u, \overline{v}) = g(u_{\mathbb{R}}, v_{\mathbb{R}}) + g(u_{\mathbb{C}}, v_{\mathbb{C}}) + i(g(u_{\mathbb{C}}, v_{\mathbb{R}}) - g(u_{\mathbb{R}}, v_{\mathbb{C}})) = \overline{g(v, \overline{u})}\,. \tag{2.2.16}$$

This naturally leads us to the following observation

Lemma 2.2.2. *There is a one-to-one correspondence between hermitian metrics on M and tensors $g \in \Gamma(T^*M^{(1,0)} \otimes T^*M^{(0,1)})$ such that for any $x \in M$ and $u, v \in T_xM^{(1,0)}$,*

1. $g(u, \overline{u}) \geq 0$, *with equality if and only if* $u = 0$.
2. $g(u, \overline{v}) = \overline{g(v, \overline{u})}$.

Proof. One injection was discussed before the theorem. So now we have to find an inverse. So suppose g is a complex tensor with values in $T^*M^{(1,0)} \otimes T^*M^{(0,1)}$ satisfying the assumptions of the theorem. We wish to construct a hermitian metric $\widetilde{g}$ that equals g when restricted to $TM^{(1,0)} \otimes TM^{(0,1)}$ and show that it is unique. Since any vector $u \in T_xM$ decomposes as $u = u^{(1,0)} + u^{(0,1)}$, where $u^{(1,0)} \in T_xM^{(1,0)}$ and $u^{(0,1)} \in T_xM^{(0,1)}$. So using the conditions, we see that for any pair $u, v \in T_xM$, $\widetilde{g}$ must satisfy

$$\widetilde{g}(u, v) = \widetilde{g}(u^{(1,0)}, v^{(1,0)}) + \widetilde{g}(u^{(1,0)}, v^{(0,1)}) + \widetilde{g}(u^{(0,1)}, v^{(1,0)}) + \widetilde{g}(u^{(0,1)}, v^{(0,1)})\,.$$

Since we want $\widetilde{g}$ to be a hermitian metric, (2.2.14) tells us $\widetilde{g}(u^{(1,0)}, v^{(1,0)}) = \widetilde{g}(u^{(0,1)}, v^{(0,1)}) = 0$. Moreover, $\widetilde{g}$ must equal g when restricting to $TM^{(1,0)} \otimes TM^{(0,1)}$, so the above reduces to

$$\widetilde{g}(u, v) = g(u^{(1,0)}, v^{(0,1)}) + g(v^{(1,0)}, u^{(0,1)})\,.$$

One immediately sees that $\widetilde{g}(Ju, Jv) = \widetilde{g}(u, v)$. Moreover, it is a quick check that this is a pointwise real inner product when plugging in real tangent vectors, as those are precisely the tangent vectors u such that $u^{(0,1)} = \overline{u^{(1,0)}}$. Therefore, we see that the injection described before the theorem is invertible, therefore, the theorem is proven. □

The upshot is that we can now define a hermitian metric as a section of $T^*M^{(1,0)} \otimes T^*M^{(0,1)}$ such that $g(u,\overline{u}) \geq 0$ with equality if and only if $u = 0$, and $g(u,\overline{v}) = \overline{g(v,\overline{u})}$, which is a definition that is better adapted to the setting of complex manifolds.

Our next observation is that, for two real tangent vectors $u, v \in T_xM$, we have a real, nondegenerate two form defined by $\omega(u,v) := g(Ju, v)$. This is called the *hermitian form associated to* g. Again, looking at complex vectors, we see

$$\omega(u,v) = g(J(u^{(1,0)} + u^{(0,1)}), v^{(1,0)} + v^{(0,1)}) = ig(u^{(1,0)}, v^{(0,1)}) - ig(v^{(1,0)}, u^{(0,1)})\,. \tag{2.2.17}$$

So it follows that ω is a $(1,1)$-form. In particular, for $u, v \in T_xM^{(1,0)}$,

$$\omega(u,\overline{v}) = ig(u,\overline{v}) = -\overline{\omega(v,\overline{u})}\,. \tag{2.2.18}$$

Moreover, one sees that g also follows from ω by $g(u,v) = \omega(u, Jv)$, and that $\omega(Ju, Jv) = \omega(u,v)$, i.e. that ω is J-invariant. This gives us another equivalent definition of a hermitian manifold, namely a complex manifold (M,J) equipped with a real J-invariant nondegenerate two-form ω, such that $\omega(u, Ju) > 0$ whenever $u \neq 0$ is real. Real two-forms η that satisfy the condition $\eta(u, Ju) > 0$ when $u \neq 0$ is real, are called *positive two-forms*.

Now we have all structure needed to define Kähler manifolds. They are a particular kind of hermitian manifold, namely those where the hermitian form is not only nondegenerate, but is symplectic.

Definition 2.2.3 (Kähler manifold)**.** A *Kähler manifold* is a hermitian manifold (M,g) such that the associated hermitian form ω satisfies $d\omega = 0$. In this case, we call g a *Kähler metric* and ω the associated *Kähler form*.

Kähler manifolds are very interesting, as they lie in the intersection of complex, symplectic and Riemannian geometry. The study of Kähler manifolds goes very deep and their geometry is very rich. In this thesis, we will derive a few results, and state some others without proof. A full discussion is outside the scope of this thesis, for more details, see, for instance, [GH94; Huy05; KN69; Wel80].

The particular definition of a Kähler manifold is a rather strong one, yet it might not be entirely obvious why this is an interesting definition when one approaches Kähler manifold theory from a Riemannian and complex viewpoint, and not from a symplectic viewpoint. The following might give a bit more motivation

Proposition 2.2.4 (Alternative Kähler characterisations)**.** *Let* (M,g,J) *be a hermitian manifold with Levi-Civita connection* ∇ *and hermitian two-form* ω*, then the following conditions are equivalent*

(i) $d\omega = 0$;
(ii) $\nabla J = 0$;
(iii) $\nabla\omega = 0$.
(iv) *The holonomy group of* (M,g) *lies inside* $U(m)$.

Proof. Our proof is based on the one in [KN69]. The equivalence of (ii) and (iii) follows from the fact that g is covariantly conserved and nondegenerate. The equivalence of (ii) and (iv) follows from the holonomy principle: holonomy is inside $U(m)$ if and only if the almost complex structure is preserved. Lastly, $\nabla\omega = 0$ implies $d\omega = 0$, as $d\omega$ is the antisymmetrisation of $\nabla\omega$. The remaining implication to show is that (i) implies (ii), which we will now show.

We see that for any vector fields X, Y, Z, we have

$$g((\nabla_X J)Y, Z) = g(\nabla_X(JY), Z) + g(\nabla_X Y, JZ)\,.$$

Moreover, the respective Koszul formulas tell us

$$\begin{aligned} g(\nabla_X Y, Z) &= \tfrac{1}{2}(X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\qquad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X))\,; \\ d\omega(X,Y,Z) &= X(g(JY,Z)) - Y(g(JX,Z)) + Z(g(JX,Y)) \\ &\qquad - g(J[X,Y],Z) + g(J[X,Z],Y) - g(J[Y,Z],X)\,. \end{aligned}$$

So we see

$$\begin{aligned} g((\nabla_X J)Y, Z) &= \tfrac{1}{2}(Y(g(X,JZ)) - Z(g(X,JY)) + JY(g(X,Z)) - JZ(g(X,Y)) \\ &\quad + g([X,JY],Z) - g([X,Z],JY) - g([JY,Z],X) \\ &\quad + g([X,Y],JZ) - g([X,JZ],Y) - g([Y,JZ],X)) \\ &= \tfrac{1}{2}(d\omega(X,Y,Z) - d\omega(X,JY,JZ) \\ &\quad - g([Y,Z],JX) - g(J[JY,Z],JX) - g(J[Y,JZ],JX) + g([JY,JZ],JX)) \\ &= \tfrac{1}{2}(d\omega(X,Y,Z) - d\omega(X,JY,JZ) - g([Y,Z] + J([JY,Z] + [Y,JZ]) - [JY,JZ],JX))\,. \end{aligned}$$

Now, we see the Nijenhuis tensor appearing in the last term on the RHS, so since the complex structure is integrable, we see

$$g((\nabla_X J)Y, Z) = \tfrac{1}{2}(d\omega(X,Y,Z) - d\omega(X,JY,JZ))\,. \tag{2.2.19}$$

So $d\omega = 0$ implies $\nabla J = 0$, meaning (i) implies (ii), which finishes the proof. □

One sees that Kähler manifolds are precisely those manifolds where the complex structure is covariantly conserved.

On a given complex manifold (M, J), there might be certain cohomology classes that contain real positive J-invariant $(1,1)$-forms. Before, we showed that these forms induce a hermitian structure on M, and because we assumed they lie in a cohomology class, they are closed. Therefore, these are precisely the objects that induce Kähler structures. A cohomology class carrying a Kähler form is called a *Kähler class.*

Now, as on any Riemannian manifold, we have the Hodge star operator $\star : \Omega^p(M) \to \Omega^{2m-p}(M)$ defined by $\alpha \wedge \star\beta = g(\alpha,\beta)\mathrm{vol}$, for every $\alpha, \beta \in \Omega^p(M)$. Here, vol is the volume form induced by g. If we then assume our manifold is compact, we also get an inner product

$$\langle \alpha, \beta \rangle = \int_M g(\alpha,\beta) \star 1\,, \tag{2.2.20}$$

so we can define adjoints of ∂ and $\bar\partial$. We find

$$\partial^* = -\star\bar\partial\star; \qquad \bar\partial^* = -\star\partial\star\,. \tag{2.2.21}$$

From this, similar equations can be derived for d^* and $(d^c)^*$. In fact, one can show $\partial\bar\partial^* = -\bar\partial^*\partial$ and $\bar\partial\partial^* = -\partial^*\bar\partial$ [Huy05]. Moreover, we can now define the Laplacians as $\Delta_D = DD^* + D^*D$ for $D = \partial, \bar\partial, d, d^c$.

A property of compact Kähler manifolds is that, while ∂, $\bar\partial$ and d are all different operators, their Laplacians are actually the same up to a constant prefactor.

Lemma 2.2.5. $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta_d$.

The proof requires quite a long calculation, see for instance [Huy05]. The above condition tells us that the space of harmonic forms of ∂ , $\bar\partial$ and d are all the same. To get a nice consequence, we need the following theorem as well:

Theorem 2.2.6 (Hodge decomposition)**.** *Let (M, g) be a compact hermitian manifold, then we have the following orthogonal decompositions:*

$$\Omega^{(p,q)}(M) = \mathrm{im}(\partial) \oplus \ker(\Delta_\partial) \oplus \mathrm{im}(\partial^*) \tag{2.2.22}$$

$$\Omega^{(p,q)}(M) = \mathrm{im}(\bar\partial) \oplus \ker(\Delta_{\bar\partial}) \oplus \mathrm{im}(\bar\partial^*) \,, \tag{2.2.23}$$

where p, q are nonnegative integers, and all maps are chosen at the level where they make sense.

For the proof, we refer the reader to [Wel80]. Because the Laplacians all agree for Kähler manifolds, we define the space of (p, q)-*harmonic forms* as

$$\mathcal{H}^{(p,q)}(M, g) := \ker(\Delta_D : \Omega^{(p,q)}(M) \to \Omega^{(p,q)}(M)) \,, \tag{2.2.24}$$

for $D = \partial$, $\bar\partial$, d . In particular, we see that any d-cohomology class contains a unique harmonic representative, moreover, this class satisfies that all (p, q)-components are harmonics as well. Moreover, every Dolbeault cohomology class also carries a unique harmonic representative. As a consequence of the Hodge decomposition theorem, we get the following important result, called the *dd^c-lemma*, or sometimes the $\partial\bar\partial$-lemma if one is mostly interested in complex forms.

Lemma 2.2.7 (dd^c-lemma)**.** *Let (M, g) be a compact Kähler manifold, then for any closed (p, q)-form α , the following are equivalent:*

(i) *α is d-exact;*

(ii) *α is ∂-exact;*

(iii) *α is $\bar\partial$-exact;*

(iv) *α is $\partial\bar\partial$-exact;*

(v) *α is orthogonal to $\mathcal{H}^{(p,q)}(M, g)$.*

Moreover, if $\alpha \in \Omega^k(M)$ is real and d-exact, then α is dd^c-exact as a real form.

The proof can be found in [Huy05], it follows relatively easily from the Hodge decomposition theorem.

The dd^c-lemma might seem rather technical at first, but it is a very effective tool to describe Kähler manifolds. It tells us that every cohomology class, whether it be a de Rham cohomology class or a Dolbeault cohomology class, carries a unique harmonic representative. In particular, because complex conjugation induces isomorphisms $\mathcal{H}^{(p,q)} \cong \mathcal{H}^{(q,p)}$, we get the following relations between the Hodge numbers:

Theorem 2.2.8. *On a compact Kähler manifold (M^m, g) ,*

$$h^{p,q} = h^{q,p} = h^{m-p,m-q} = h^{m-q,m-p} \,.$$

Proof. $h^{p,q} = h^{q,p}$ was established before the theorem, so we just have to show $h^{p,q} = h^{m-q,m-p}$, which is known as *Hodge duality*. From [Huy05], we know that Δ commutes with $\star$, therefore we see that $\alpha \in \Omega^{(p,q)}(M)$ is harmonic if and only if $\star\alpha \in \Omega^{(m-q,m-p)}(M)$ is harmonic. This completes the proof. □

Secondly, we have

Theorem 2.2.9. *Let (M^m, g) be a compact Kähler manifold. Then we have the following*

$$H^i_{\mathrm{dR}}(M) \cong \bigoplus_{p+q=i} H^{(p,q)}_{\bar\partial}(M)\,.$$

In particular, we see that the i'th Betti number of M is related to the Hodge numbers of M by $b_i(M) = \sum_{p+q=i} h^{p,q}$.

Proof. This is a direct consequence of the observation below Equation (2.2.24), that any d-cohomology class carries a unique harmonic representative, whose (p,q)-components are also harmonic. □

These two results can be combined to give a topological obstruction for manifolds to be Kähler, we see that any compact Kähler manifold needs to have even odd-degree Betti numbers, i.e. $b_{2i+1} \in 2\mathbb{Z}$ for any $i \in \mathbb{N}_0$.

As another tool, we have the following lemma, which guarantees the existence of certain local functions called *Kähler potentials*, which will play a key role in Chapter 4.

Lemma 2.2.10 (Existence of Kähler potentials)**.** *Let (M, g) be a Kähler manifold with Kähler form ω. Let U be a contractible open. Then there is a real function φ on U such that*

$$\omega|_U = dd^c\varphi\,. \tag{2.2.25}$$

This result follows from the Poincaré lemma [Huy05]. Moreover, if (U, z^α) is a holomorphic chart, and φ is a Kähler potential on U, then we see

$$g_{\alpha\bar\beta} = 2\partial_\alpha\partial_{\bar\beta}\varphi\,. \tag{2.2.26}$$

Here, note that our convention dictates $dz^\alpha \wedge d\bar z^\beta(\partial_\alpha, \partial_{\bar\beta}) = 1$.

The volume form of (M, g) relates to ω in a rather nice way. To see this, let $x \in M$, and pick a frame for the tangent bundle that is orthonormal at x, given by $(\partial_{x^1}, J\partial_{x^1}, \ldots, \partial_{x^m}, J\partial_{x^m})$. The reason that we can pick an frame like this is that J is an orthogonal transformation. We also define $\partial_{y^i} := J\partial_{x^i}$.

Now we will calculate the volume form. To do this, first note that we have a preferred orientation of our manifold induced by the complex structure. This has orientation form $dx^1 \wedge dy^1 \wedge \cdots \wedge dx^m \wedge dy^m$. Since this frame is orthonormal, the previous top form is also the volume form vol_g of our Kähler manifold at the point x. By direct computation, we see

$$\omega^m_x = m!\frac{i^m}{2^m} dz^1 \wedge d\bar z^1 \wedge \cdots \wedge dz^m \wedge d\bar z^m\,. \tag{2.2.27}$$

Since we have $dz^k \wedge d\bar z^k = -2i\, dx^k \wedge dy^k$, and $x \in M$ was arbitrary, we get

$$\omega^m = m!\,\mathrm{vol}_g\,. \tag{2.2.28}$$

Moreover, we know $\omega = ig_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^{\bar\beta}$, so we see

$$\omega^m = i^m m! \det(g_{\alpha\bar\beta})\, dz^1 \wedge d\bar z^1 \wedge \cdots \wedge dz^m \wedge d\bar z^m\,. \tag{2.2.29}$$

The next thing we consider is the Levi-Civita connection associated to g. As it turns out, when applying the Levi-Civita connection to $TM^{(1,0)}$, it agrees with the *Chern connection* on that bundle [Huy05]. This particular connection is the unique holomorphic connection that commutes with g. To make the discussion slightly more general, we will jump to the setting of a holomorphic vector bundle $E \to M$ equipped with a hermitian connection h, defined in the smooth sense. Recall that there is a well defined $\bar{\partial}_E : \Omega^{(p,q)}(M; E) \to \Omega^{(p,q+1)}(M; E)$ operator on such vector bundles, which can be defined locally in holomorphic trivializations. Note that transition functions are then holomorphic, so they are $\bar{\partial}$-closed, thus this definition is coordinate invariant.

Lemma 2.2.11 (Existence of Chern connection)**.** *Let (E, h) be a hermitian, holomorphic vector bundle over a complex manifold M. Then there is a unique connection ∇ on E such that for any $u, v \in \Gamma(E)$, $d(h(u, v)) = h(\nabla u, v) + h(u, \nabla v)$, and $\nabla u \in \Omega^{(1,0)}(M; E)$ for any (local) holomorphic section u.*

The proof is straightforward and can be found in [Huy05]. If we want to write down what this looks like in coordinates, we obtain

$$A^{\lambda}{}_{\mu\alpha} = h^{\lambda\bar{\nu}} \partial_\alpha h_{\mu\bar{\nu}} \, , \tag{2.2.30}$$

where $h^{\lambda\bar{\mu}}$ is the transpose inverse matrix of $h_{\lambda\bar{\mu}}$, i.e. $h^{\lambda\bar{\mu}} h_{\nu\bar{\mu}} = \delta^{\lambda}_{\nu}$. Note that this is how the connection acts on E, we equip $\overline{E}$ with the structure induced by complex conjugating everything, i.e. simply put a bar over every symbol, e.g. the connection is determined by $\bar{A}^{\bar{\lambda}}{}_{\bar{\mu}\bar{\alpha}} := \overline{A^{\bar{\lambda}}{}_{\bar{\mu}\bar{\alpha}}}$. In fact, we can now explicitly calculate the associated curvature tensor in coordinates to find

$$K^{\lambda}{}_{\mu\alpha\bar{\beta}} = -\partial_{\bar{\beta}} A^{\lambda}{}_{\mu\alpha} \, , \tag{2.2.31}$$

with all other components vanishing. See [KN69] for the calculation. Here, we use $\alpha, \beta, \ldots$ to run over a $TM^{(1,0)}$-frame, and we use $\lambda, \mu, \ldots$ to run over the E-trivialization. Note that some authors (e.g. Calabi [Cal79]) choose to define the curvature tensor with an extra minus sign to get a more natural way of lowering the λ index. In our convention (which is shared by Kobayashi and Nomizu [KN69]),

$$K_{\lambda\bar{\mu}\alpha\bar{\beta}} = -h_{\nu\bar{\mu}} K^{\nu}{}_{\lambda\alpha\bar{\beta}} \, , \tag{2.2.32}$$

in the other convention, the RHS would not have this minus sign.

One thing we can see from the above calculation is that the curvature tensor is in $\Omega^{(1,1)}(M; \mathrm{End}(E))$, which is an interesting fact on its own.

Returning to Kähler manifolds, we will now want to calculate the associated Ricci curvature tensor, which we will denote by Ric if we're working independent of coordinates, and by R_{ij} when working in coordinates. Moreover, we will denote the Riemann tensor in coordinates by $R^{i}{}_{jkl}$, to indicate that we're in the special case of the curvature tensor of the tangent bundle. To calculate the Ricci tensor, we note that the first Bianchi identity tells us

$$R_{\alpha\bar{\beta}} = R^{\gamma}{}_{\alpha\gamma\bar{\beta}} = R^{\gamma}{}_{\gamma\alpha\bar{\beta}} \, , \tag{2.2.33}$$

where we used that the curvature form is $(1,1)$. Moreover, we have $R_{\bar{\alpha}\beta} = \overline{R_{\alpha\bar{\beta}}}$, and all other components vanish. Translating back to Latin indices then simply gives us

$$R_{\bar{a}\bar{b}} = R_{ab} = R_{\alpha\bar{\beta}} + R_{\bar{\alpha}\beta} \, ; \quad R_{a\bar{b}} = -R_{\bar{a}b} = i(-R_{\alpha\bar{\beta}} + R_{\bar{\alpha}\beta}) \, . \tag{2.2.34}$$

In particular, it is J-invariant and as a sanity check, it is real. Moreover, we see that $\rho := \mathrm{Ric}(J-,-)$ defines a J-invariant real two-form with coordinates

$$\rho_{ab} = \rho_{\bar{a}\bar{b}} = i(R_{\alpha\bar{\beta}} - R_{\bar{\alpha}\beta})\,; \quad \rho_{a\bar{b}} = -\rho_{\bar{a}b} = R_{\alpha\bar{\beta}} + R_{\bar{\alpha}\beta}\,. \tag{2.2.35}$$

If we want to explicitly calculate this, we note that the curvature on $\Lambda^{(m,0)}TM := \Lambda^m(TM^{(1,0)})$ is given by

$$\begin{aligned}[\nabla_\alpha, \nabla_{\bar{\beta}}](\partial_1 \wedge \cdots \wedge \partial_m) &= R^1{}_{\alpha 1\bar{\beta}}\partial_1 \wedge \partial_2 \wedge \cdots \wedge \partial_m + \partial_1 \wedge R^2{}_{\alpha 2\bar{\beta}}\partial_2 \wedge \cdots \wedge \partial_m + \partial_1 \wedge \cdots \wedge R^m{}_{\alpha m\bar{\beta}}\partial_m \\ &= R_{\alpha\bar{\beta}}\partial_1 \wedge \cdots \wedge \partial_m\,.\end{aligned}$$

So $R_{\alpha\bar{\beta}}$ is the curvature of $\Lambda^{(m,0)}TM$. However, the metric on this bundle is given by a scalar, namely $\det(g_{\alpha\bar{\beta}})$. Note that $\det(g_{\alpha\bar{\beta}})$ is a real function, as $g_{\alpha\bar{\beta}}$ is a hermitian matrix. Therefore, we see

$$R_{\alpha\bar{\beta}} = -\partial_{\bar{\beta}}((\det(g_{\gamma\bar{\delta}}))^{-1}\partial_\alpha \det(g_{\gamma\bar{\delta}})) = -\partial_\alpha\partial_{\bar{\beta}} \log\det(g_{\gamma\bar{\delta}})\,. \tag{2.2.36}$$

This means we get Ricci form

$$\rho = -i\partial\bar{\partial}\log\det(g_{\gamma\bar{\delta}}) = -\tfrac{1}{2}dd^c \log\det(g_{\gamma\bar{\delta}})\,. \tag{2.2.37}$$

In particular, this is a closed real $(1,1)$-form, and it therefore defines a real cohomology class. In fact, since Ric is the trace of the curvature tensor on $\Lambda^{(m,0)}TM$, we see the cohomology class $[\rho]$ is in fact $2\pi c_1(M)$, with $c_1(M)$ being the first (real) Chern class of M, i.e. the first Chern class of $TM^{(1,0)}$. Here, the construction of the first Chern class is done using invariant polynomials, see e.g. [KN69; Tu17], in Section 2.3, we will construct the first Chern class as an integral class.

Lastly, there is a nice formula of the Laplacian acting on a $(p,0)$-form on a Kähler manifold, which we will now state

Lemma 2.2.12. *Let (M,g) be a compact Kähler manifold and let $\xi \in \Omega^{(p,0)}(M)$, then we have*

$$(\Delta\xi)_{\alpha_1\ldots\alpha_p} = -\nabla^i\nabla_i\xi_{\alpha_1\ldots\alpha_p} + pg^{\beta\bar{\gamma}}\xi_{\beta[\alpha_1\ldots\alpha_{p-1}}R_{\alpha_p]\bar{\gamma}}\,, \tag{2.2.38}$$

where $[-]$ denotes antisymmetrisation of the indices.

This result follows from a straightforward computation, see e.g. [BLT13] for details.

Corollary 2.2.13. *If (M,g) is a compact Kähler manifold such that* $\mathrm{Ric} = 0$, *then a $(p,0)$-form ξ is homolorphic if and only if it is parallel.*

Proof. Firstly, we note that $(p,0)$-forms are holomorphic if and only if they are harmonic, which follows from the fact that $H^{(p,0)}_{\bar{\partial}}(M)$-classes have a unique representative, since $\mathrm{im}(\bar{\partial}) = 0$, moreover, this representative is holomorphic by definition, so because every class carries a unique harmonic representative, we know that a $(p,0)$-form ξ is holomorphic if and only if it is harmonic.

By Lemma 2.2.12, if $\Delta\xi = 0$, then $\nabla^i\nabla_i\xi = 0$, but then $\langle\nabla^i\nabla_i\xi, \xi\rangle = 0$, which means $\langle\nabla\xi, \nabla\xi\rangle = 0$, so $\nabla\xi = 0$, i.e. ξ is parallel.

Conversely, if $\nabla\xi = 0$, Lemma 2.2.12 tells us $\Delta\xi = 0$. ☐

2.3 Line bundles on complex manifolds

Now that we did a bit of work on studying complex manifolds and Kähler manifolds, we will turn to studying line bundles. These objects appear in many places, as we shall see. Moreover, they are a rather important tool in this thesis, in the next section, we will use them to study some properties of Calabi-Yau manifolds and we will use them to produce some examples of Calabi-Yau manifolds. Moreover, they will play a vital role in Chapter 4, where we will construct an explicit example of a Calabi-Yau metric on the total space of a line bundle over $\mathbb{C}P^n$.

We will start the section by briefly sketching the classification of smooth complex line bundles by their first integral Chern class, then we will introduce the canonical line bundle over any complex manifold, we will introduce the theory of divisors, which will play a minor role in this thesis, and we will spend some time going in depth on holomorphic line bundles over complex projective space, which will play a central role in Chapter 4.

We will assume the reader is familiar with certain characteristic classes defined for vector bundles, has some familiarity with complex manifolds, and has some understanding of differential geometry.

2.3.1 Classifying line bundles

Line bundle are defined as rank one complex vector bundles over some manifold M. They are often easy to work with, yet just nontrivial enough to give us some useful results. Complex line bundles over a manifold M define a group $(\mathrm{Vb}^1(M), \otimes)$ consisting of isomorphism classes of line bundles with the tensor product as a group operation.

An important result in the theory of vector bundles is that isomorphism classes of smooth (or even continuous) vector bundles are classified by suitable equivalence classes of their transition functions, which can be proved using clutching constructions, see e.g. [Ati89; GH94; Kar78].

To properly describe vector bundles in this way, we have to specify what happens to the transition functions on triple overlaps of trivialisation domains, i.e. when there's multiple transition functions defined at once. So suppose $\mathcal{U}$ is an open cover for the manifold M, and let $E \to M$ be a complex vector bundle of rank k admitting trivialisations over $\mathcal{U}$. We denote the trivialisation over $U \in \mathcal{U}$ by $\varphi_U : E|_U \to U \times \mathbb{C}^k$. Then set of transition functions

$$\check{g} := \{g_{UV} := \varphi_U \cap \varphi_V^{-1} : U \cap V \to GL(k;\mathbb{C}) | U, V \in \mathcal{U}\} \tag{2.3.1}$$

satisfies the following identities:

$$g_{UU} = 1\,, \quad \forall U \in \mathcal{U}\,. \tag{2.3.2}$$

$$g_{U_1U_2} \cdot g_{U_2U_3} = g_{U_1U_3}\,, \quad \forall U_1, U_2, U_3 \in \mathcal{U}\,. \tag{2.3.3}$$

In fact, using the first condition, the second condition can be rewritten as

$$g_{U_1U_2} \cdot g_{U_2U_3} \cdot g_{U3U1} = 1\,, \quad \forall U_1, U_2, U_3 \in \mathcal{U}\,. \tag{2.3.4}$$

Condition (2.3.2) is known as the *skew-symmetry* condition, while Condition (2.3.4) is known as the *cocycle condition.* By the clutching constructions found in e.g. [Ati89; GH94; Kar78], we see that the converse is also true, if $\mathcal{U}$ is a cover consisting of open discs, i.e. contractible opens, then any such collection of functions satisfying (2.3.2) and (2.3.4) defines a vector bundle. However, two such collections of functions can define the same vector bundle, if we have a collection

$$\check{f} := \{f_U : U \to GL(k;\mathbb{C}) | U \in \mathcal{U}\}\,, \tag{2.3.5}$$

then

$$\check{g}' := \{g'_{UV} := f_U \cdot g_{UV} \cdot f_V^{-1} | U, V \in \mathcal{U}\}\,,$$

defines the same vector bundle. As it turns out, this is the only anomaly appearing, we have the following theorem

Theorem 2.3.1. *Let $\mathcal{U}$ be an open cover of discs and $k \in \mathbb{N}_0$. Then the set of isomorphism classes of rank k vector bundles is isomorphic to the set of collections $\check{g}$ as in (2.3.1), satisfying the skew-symmetry (2.3.2) and cocycle condition (2.3.4), modulo equivalence by collections $\check{f}$ as in (2.3.5).*

Proof. One direction was discussed before the theorem. For the other direction, suppose E and F are isomorphic vector bundles, and let $\Phi : E \to F$ be an explicit basepoint preserving isomorphism. Then E and F admit trivialisations over $\mathcal{U}$ given by, respectively, $\{\varphi_U : E|_U \to U \times \mathbb{C}^k | U \in \mathcal{U}\}$ and $\{\varphi'_U : F|_U \to U \times \mathbb{C}^k | U \in \mathcal{U}\}$. We define $\check{f}$ as the collection of functions $f_U := \varphi'_U \circ \Phi \circ \varphi_U^{-1}$. Letting $\check{g}$ be the transition functions of E and $\check{g}'$ be the transition functions of F, we see that we have

$$\begin{aligned}
g'_{UV} &= \varphi'_U \circ (\varphi'_V)^{-1} \\
&= \varphi'_U \circ \Phi \circ \varphi_U^{-1} \circ \varphi_U \circ \Phi^{-1} \circ \Phi \circ \varphi_V^{-1} \circ \varphi_V \circ \Phi^{-1} \circ (\varphi'_V)^{-1} \\
&= f_U \circ g_{UV} \circ f_V^{-1}\,.
\end{aligned}$$

Thus completing the proof. □

In the particular case $k = 1$, i.e. the case of line bundles, we see that $GL(1;\mathbb{C}) \cong \mathbb{C}^*$ is abelian, therefore, things become a bit simpler. In fact, the picture fits into the theory of Čech cohomology. For brevity, we will not develop this theory here, an excellent introduction can be found in [GH94].

The main result is that there is a group isomorphism $c_1 : \mathrm{Vb}^1(M) \to \check{H}^1(M; C^\infty(-;\mathbb{C}^*))$, i.e. the first Čech cohomology group with values in the sheaf $C^\infty(-;\mathbb{C}^*)$. Furthermore, using the theory of sheaf cohomology and the exact sequence of sheaves given by

$$0 \to \mathbb{Z} \hookrightarrow C^\infty(-;\mathbb{C}) \xrightarrow{\exp(2\pi i -)} C^\infty(-;\mathbb{C}^*) \to 0\,, \tag{2.3.6}$$

it can even be shown that $\check{H}^1(M; C^\infty(-;\mathbb{C}^*))$ is isomorphic to $H^2(M;\mathbb{Z})$. For details, we refer the reader to [God64; GH94; Ten75]. This finally tells us that $(\mathrm{Vb}^1(M), \otimes) \cong H^2(M;\mathbb{Z})$, with the isomorphism given by the first Chern class. In particular, under Poincaré duality, we get that line bundles correspond to integral homology cycles of real codimension 2.

To show that this particular definition coincides with the definition of the first Chern class in terms of invariant polynomials, we refer the reader to [Huy05].

2.3.2 Examples of line bundles and divisors

Now that we have discussed some abstract subtleties and we have classified complex line bundles, it is time to give some examples.

In this thesis, we will be interested in Calabi-Yau manifolds, which are defined as Ricci flat Kähler manifolds. In the previous section we showed that the Ricci curvature on a Kähler manifold M^m is precisely the curvature of the *anticanonical bundle* $-K_M := \Lambda^{(m,0)}TM$, i.e. the top exterior power of the $(1,0)$-part of the tangent bundle. In particular, Ricci-flatness requires this line bundle to have vanishing real first Chern class, i.e. the first Chern class $-[K_M]$ of the anticanonical bundle[1] needs to be a torsion class in $H_{n-2}(M;\mathbb{Z})$. Therefore we have found a cohomological obstruction to the existence of Ricci-flat Kähler metrics, namely the canonical class $[K_M]$, i.e. the first Chern class of the canonical bundle $K_M := \Lambda^{(m,0)}T^*M$. Rather interestingly, a deep result by E. Calabi [Cal54] and S.T. Yau [Yau77; Yau78] shows that the converse is also true, if M is a compact Kähler manifold such that $[K_M]$ is a torsion class, then M admits a Ricci-flat Kähler metric, we shall go more in depth in Chapter 3.

Computing the canonical bundle of a complex submanifold can be done with the help of the following theorem:

Theorem 2.3.2 (Adjunction formula)**.** *Let M be a complex manifold, let N be an embedded complex submanifold and let $\mathcal{N}$ denote the normal bundle to N. Then we have the following isomorphism*

$$K_N \cong K_M|_N \otimes \det(\mathcal{N})\,, \tag{2.3.7}$$

called the adjunction formula*.*

In fact, all involved bundles are holomorphic, so this is even an isomorphism of holomorphic line bundles, see [Huy05] for details and for a proof.

An important manifold that is often quite easy to produce examples from is complex projective space $\mathbb{C}P^n$. The (co-)homology groups of this space are centered in even degree [Hat01]:

$$H^i(\mathbb{C}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\,, & i \text{ even, } 0 \le i \le 2n\,; \\ 0\,, & \text{else.} \end{cases} \tag{2.3.8}$$

Moreover, we know $H_{2n-2}(\mathbb{C}P^n;\mathbb{Z})$ is generated by $[\mathbb{C}P^{n-1}]$. In particular, we see that (isomorphism classes of) line bundles over $\mathbb{C}P^n$ are classified by the integers. We denote by $\mathcal{O}(k)$ the line bundle corresponding to $k \in \mathbb{Z}$.

To study these line bundles, we first turn to a nice description of $\mathcal{O}(-1)$ based on the approach taken by [Huy05]. First note that we can describe $\mathbb{C}P^n$ as the set of all lines in $\mathbb{C}^{n+1}$. Consider $L \subseteq \mathbb{C}P^n \otimes \mathbb{C}^{n+1}$ consisting of pairs (ℓ, v) , where $v \in \ell$. We see that this defines a line bundle over $\mathbb{C}P^n$, whose total space is a complex manifold and whose projection map is holomorphic, therefore it is even a holomorphic line bundle. This line bundle has local trivialisations over the fundamental opens $U_i \subseteq \mathbb{C}P^n$ for $i = 0, \dots, n$ defined by $U_i = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n : z_i \neq 0\}$, where $[z_0 : \cdots : z_n]$ denotes the line through $(z_0, \dots, z_n)$. The frame for this local trivialisation is precisely the vector $(z_0, \dots, 1, \dots, z_n)$, with 1 in the i'th slot.

[1] As a slight bit of abuse of notation, $-K_M$ will be used both for the class of the anticanonical bundle as well as for the anticanonical bundle itself.

When changing coordinates between U_i and U_j, we see that we multiply our frame vector by z_i/z_j, therefore the change of coordinates function is z_j/z_i. This has a simple pole at the copy of $\mathbb{C}P^{n-1}$ defined by $z_i = 0$. If we had carefully followed all the isomorphisms discussed above, we see that this means the first Chern class in fact corresponds to $-[\mathbb{C}P^{n-1}] \in H_{2n-2}(\mathbb{C}P^n;\mathbb{Z})$, and therefore this is indeed the line bundle $\mathcal{O}(-1)$, which we have now equipped with a canonical holomorphic structure. For details, see [Huy05].

Using this holomorphic structure, we can also put holomorphic structures on $\mathcal{O}(k)$ for any $k \in \mathbb{Z}$, because this is precisely $\mathcal{O}(-1)^{\otimes(-k)}$, where $\mathcal{O}(-1)^{\otimes(-1)} := \mathcal{O}(1)$. For positive $k \in \mathbb{N} \setminus \{0\}$, we can find a rather nice description of holomorphic sections of $\mathcal{O}(k)$. Let $p \in \mathbb{C}[z_0, \dots, z_n]_k$ be a homogeneous polynomial of degree k on $\mathbb{C}^{n+1}$. We note that on the fundamental opens U_i discussed before, we can define holomorphic functions $p_i([z_0 : \dots : z_n]) := p(z_0/z_i, \dots, 1, \dots, z_n/z_i)$. On overlaps $U_i \cap U_j$, we see that $p_j = (z_i/z_j)^k p_i$, therefore, these p_i together define a section of $\mathcal{O}(k)$. Thus homogeneous polynomials of degree k on $\mathbb{C}^{n+1}$ define holomorphic sections of $\mathcal{O}(k) \to \mathbb{C}P^n$. In fact, as is shown in [Huy05], all holomorphic sections of $\mathcal{O}(k)$ arise in this way.

We can also explicitly compute the canonical class of this manifold. On U_i, the canonical bundle is generated by $\Lambda_{k\neq i} d(z_k/z_i) := d(z_0/dz_i) \wedge \dots \wedge \widehat{d(z_i/z_i)} \wedge \dots \wedge d(z_n/dz_i)$, and on $U_i \cap U_j$, we see

$$\Lambda_{k\neq i} d(z_k/dz_i) = -\frac{z_j^2}{z_i^2}\left(\prod_{n\neq i,j} \frac{z_j}{z_i}\right) d(z_0/z_j) \wedge \dots \wedge d(z_i/z_j) \wedge \dots d(z_n/z_j)\,,$$

here, $d(z_i/z_j)$ on the right hand side is in the same position as $d(z_j/z_i)$ is on the left hand side. We see that the change of coordinate function has a pole of order $n+1$ at $z_i = 0$, thus we have for the canonical class $K_{\mathbb{C}P^n} = (-n-1)[\mathbb{C}P^{n-1}]$, i.e.

$$K_{\mathbb{C}P^n} = \mathcal{O}(-n-1)\,. \tag{2.3.9}$$

In particular, we see that no complex projective space of positive dimension admits a Ricci flat metric. In the case $n = 1$, this is just saying that the sphere S^2 doesn't admit a metric with vanishing curvature, which is a well-known result.

Now we wish to study holomorphic line bundles a bit. As a bit of motivation as though why these are important in this thesis, in Chapter 4, we will use holomorphic line bundles to find explicit examples of Calabi-Yau manifolds. The group of holomorphic line bundles over a complex manifold M will be denoted by $\mathrm{Pic}(M)$, the group operation is given by $\otimes$.

In the setting of holomorphic line bundles, Hartogs theorem in complex geometry [Huy05] tells us that nonzero meromorphic sections need to have poles and zeroes on embedded complex codimension one submanifolds. Since such submanifolds also define homology classes, one could wonder if there's a relation between the first Chern class of a line bundle and potential zeroes and poles of sections of this bundle. This correspondence can be formulated more precisely using *divisors*.

Definition 2.3.3 (Divisor)**.** A *divisor* D on a complex manifold M is a locally finite formal sum of embedded codimension one complex submanifolds $\{X_i\}_{i\in I}$ given by

$$D = \sum_{i\in I} a_i X_i\,, \tag{2.3.10}$$

where $a_i \in \mathbb{Z}$ are some integers, we also define $I_>$ as the set of $i \in I$ such that $a_i > 0$ and $I_<$ as the set of $i \in I$ such that $a_i < 0$. The set of all divisors forms a group with the obvious group structure, which we denote by $\mathrm{Div}(M)$.

For an in depth discussion on divisors and their relation to holomorphic line bundles, see [Huy05] or [GH94]. As a bit of terminology, we shall say that a function f has order of vanishing a_i on X_i if it has a pole of order $-a_i$ if $i \in I_<$ or a zero of order a_i if $i \in I_>$ along X_i.

For a divisor $D = \sum_{i\in I} a_i X_i$, we can define a line bundle $\mathcal{O}(D)$ by requiring it to have a meromorphic section σ that has precisely order of vanishing a_i at X_i for all $i \in I$. Note that the map $\mathcal{O} : \mathrm{Div}(M) \to \mathrm{Pic}(M)$ is not injective, if D and D' are divisors, and M has a meromorphic function f such that $\mathrm{div}(f) = D' - D$, where $\mathrm{div}(f)$ is the divisor $\sum_{i\in I} a_i X_i$, where f has order of vanishing a_i on X_i, then $\mathcal{O}(D) = \mathcal{O}(D')$. If such an f exists, we say that D and D' are *linearly equivalent*. [Huy05] shows that $\mathcal{O} : \mathrm{Div}(M)/(\text{linear equivalence}) \to \mathrm{Pic}(M)$ is an injection whose image is generated by those line bundles that admit a global meromorphic section.

This gives us a tool for studying holomorphic line bundles when the sheaf of meromorphic sections over a complex manifold is known. In particular, we see that on $\mathbb{C}P^n$, the only holomorphic line bundles that admit global meromorphic sections are the $\mathcal{O}(k)$'s that we defined previously, see [Huy05] for details.

We can also use these results to study algebraic submanifolds of $\mathbb{C}P^n$ as follows. Let $M \subseteq \mathbb{C}P^n$ be the zero set of some homogeneous polynomial $p \in \mathbb{C}[x_0, \dots, x_n]_k$, where k is some positive integer, and assume M is smooth. By the above, p defines a section of $\mathcal{O}(k)$, hence the normal bundle to M is $\mathcal{O}(k)|_M$. Therefore, the adjunction formula (2.3.7) tells us

$$K_M \cong \mathcal{O}(k)|_M \otimes K_{\mathbb{C}P^n}|_M \cong \mathcal{O}(-n-1+k)|_M . \tag{2.3.11}$$

In fact, we can get even more from this. Let $\{X_i\}_{i=1,\dots,m}$ be a family of projective hypersurfaces in $\mathbb{C}P^n$ defined by polynomials $p_i \in \mathbb{C}[x_0, \dots, x_n]_{a_i}$, where a_i are positive integers. Assume that X_i intersect transversally at $M := \bigcap_{i=1}^m X_i$ and that M is smooth. We see that M has normal bundle $\mathcal{N}_M = \bigoplus_{i=1}^m \mathcal{O}(a_i)|_M$, such that $\det(\mathcal{N}_M) = \mathcal{O}(\sum_{i=1}^m a_i)|_M$. In particular, by the adjunction formula,

$$K_M \cong \mathcal{O}\left(-n-1+\sum_{i=1}^m a_i\right)\Bigg|_M . \tag{2.3.12}$$

Manifolds appearing in this fashion are called *complete intersections* in $\mathbb{C}P^n$. In fact, if one of the a_i's is equal to 1, then M can be regarded as a complete intersection in $\mathbb{C}P^{n-1}$ by simply restricting to $Z(p_i) \cong \mathbb{C}P^{n-1}$. So complete intersections arise as the intersection of transverse hypersurfaces of degree ≥ 2.

Now we see that if we have $\sum_i a_i = n+1$, then the resulting complete intersection had trivial anticanonical bundle, hence it admits a Ricci-flat Kähler metric.

2.4 Calabi-Yau manifolds

The previous sections have introduced some tools from complex geometry and the theory of line bundles. Now we will finally get to the main subject of this thesis: Calabi-Yau manifolds. We will study their structure, keeping in mind that this is useful for string theory, as their geometric structure influences the four dimensional effective theory, as we shall see in Chapter 5.

We will start by stating the Calabi-Yau theorem, which will be proven in Chapter 3 and discuss some consequences it has and we will study some basic results about the geometry of Calabi-Yau manifolds. Then we will give a rather short list of examples of spaces that admit Calabi-Yau metrics. The main example of a Calabi-Yau manifold will be postponed to Chapter 4, where we will provide an explicit example of a nontrivial Calabi-Yau metric on the canonical bundle of complex projective space due to E. Calabi [Cal79].

We assume the reader is familiar with some complex geometry and Riemannian geometry, as well as the results discussed in Sections 2.2 and 2.3.

2.4.1 Structure of Calabi-Yau manifolds

The definition of Calabi-Yau manifolds may vary from author to author. In many string theory contexts, for instance, the requirement is that global holonomy is equal to $SU(n)$, see e.g. [BBS07; BLT13; GSW87b], this is to fit them nicely into the theory of string compactifications, which is a setting where it is required for the lower dimensional theory to have just enough supersymmetry. In other settings, e.g. [Huy05], Calabi-Yau manifolds are defined as Kähler manifolds admitting a global nonvanishing holomorphic top form, which is equivalent to saying the global holonomy is insude $SU(n)$. In this thesis, we will be mostly interested in Kähler manifolds admitting a Ricci-flat Kähler metric, i.e. with reduced holonomy in $SU(n)$, which, for compact manifolds, is equivalent to saying that $[K_M]$ is a torsion class. This is a consequence of the *Calabi-Yau theorem*, which we already hinted at before and will be the main star of Chapter 3.

Theorem 2.4.1 (Calabi-Yau)**.** *Let (M^m, g) be a compact Kähler manifold with Ricci form ρ and Kähler form ω. Let ρ' be a real, closed $(1,1)$-form cohomologous to ρ. Then there is a unique Kähler metric g' on M with Kähler form ω' such that ω' is cohomologous to ω, and ρ' is the Ricci form of g'.*

In particular, we have the following corollary, which is mainly how we will apply this theorem in this section:

Corollary 2.4.2. *Let M be a complex manifold admitting Kähler metrics such that $c_1(M) := c_1(-K_M) = 0$ as a real cohomology class, then M has a Ricci-flat Kähler metric.*

Proof. Let g be any Kähler metric on M. As discussed in Section 2.2, the Ricci form ρ of g is a representative of $2\pi c_1(M)$, thus $[\rho] = 0$, i.e. there is a Kähler metric g' with Kähler form cohomologous to ω such that g' is Ricci-flat. □

We start with a few theorems to give a bit of a feeling for what kind of structure the different (inequivalent) definitions of Calabi-Yau manifolds imply.

Firstly, we consider manifolds whose anticanonical bundle is holomorphically trivial

Theorem 2.4.3. *Let M^m be a compact complex manifold admitting Kähler metrics. Then M admits a Kähler metric g with global holonomy inside $SU(m)$ if and only if the canonical bundle K_M is holomorphically trivial.*

Proof. First, assume that M has a Kähler metric with global holonomy inside $SU(n)$. Then the global holonomy of the Chern connection on K_M is trivial. Thus, let $x \in M$ and let Ω_x be a generator for

$K_{M,x}$, then extend Ω_x to a global form Ω by parallel transport, which is well defined as global holonomy of K_M is trivial. Then we see that Ω satisfies the differential equation $\bar{\partial}_{K_M}\Omega = 0$, as it is parallel, hence it is also holomorphic. Therefore, K_M admits a global holomorphic section and therefore it is trivial as a holomorphic line bundle.

Conversely, suppose K_M is holomorphically trivial. Then we see $[K_M] = 0$, in particular, $c_1(M) = 0$, hence by the Calabi-Yau theorem, M admits a Ricci-flat Kähler metric g. Because g is Kähler, $\mathrm{Hol}(g) \subseteq U(m)$. Now, let Ω be any global nonvanishing holomorphic top form. Then in particular, $\Delta\Omega = 0$, so Corollary 2.2.13, Ω is parallel with respect to g, thus the global holonomy group of g fixes Ω, such that $\mathrm{Hol}(g) \subseteq SU(m)$. □

Corollary 2.4.4. *Let M^m be a Kähler manifold with global holonomy inside $SU(m)$, then M admits a spin structure.*

Proof. In [BH59], it is shown that M admits a spin structure if and only if it is orientable and $w_2(M) = c_1(M) \mod 2 = 0$. Complex manifolds are always orientable, and $[K_M] = 0$ means $c_1(M) = 0$, completing the proof. □

Moreover, we see that K_M being holomorphically trivial implies that there is a unique (up to a constant) global holomorphic $(m,0)$-form. Also, we see that any closed $(m,0)$-form Ω satisfies $d\Omega = \bar{\partial}\Omega = 0$, in particular, closed $(m,0)$-forms are precisely the holomorphic $(m,0)$-forms. Thus, we get

Proposition 2.4.5. *Let M^m be a compact complex manifold admitting Kähler metrics, such that K_M is holomorphically trivial. Then*

$$h^{m,0} := \dim(H^{(m,0)}(M)) = 1\,. \tag{2.4.1}$$

Now consider the special case where M^m admits a Kähler metric g such that $\mathrm{Hol}(g) = SU(m)$. Then we even get the following

Proposition 2.4.6. *Let (M,g) be a compact Kähler manifold with* $\mathrm{Hol}(g) = SU(m)$. *Then we have*

$$h^{p,0} = 0\,, \qquad p = 1, \ldots, m-1\,. \tag{2.4.2}$$

Proof. By Corollary 2.2.13, holomorphic p-forms are parallel, and since $\mathrm{Hol}(g) = SU(m)$, we know that parallel forms must transform under the scalar representation of $SU(m)$, so for $p \neq 0, m$, the only parallel $(p,0)$-form is 0. See e.g. [BLT13] for details. □

In particular, using Theorem 2.2.8, we see that the for $m = 3$, we have only two independent Hodge numbers, namely $h^{1,1}$ and $h^{2,1}$, which is an important fact in superstring theory, as we will discuss in Chapter 5.

Lastly, we discuss some results on Ricci-flat Kähler manifolds.

Lemma 2.4.7. *Let M^m be a compact complex manifold admitting Kähler metrics. Then the following are equivalent*

(i) *M admits a Ricci-flat Kähler metric;*
(ii) *$c_1(M) = 0$ as a real class;*
(iii) *M admits a metric with restricted holonomy inside $SU(m)$;*

(iv) *a positive power of K_M is trivial as a smooth vector bundle.*

Proof. By the Calabi-Yau theorem and the discussion at the end of Section 2.2, we see that M admits a Ricci-flat Kähler metric if and only if $c_1(M) = 0$, thus estabilishing (i) $\iff$ (ii). Moreover, any Ricci-flat Kähler metric has restricted holonomy inside $SU(m)$, as being Ricci-flat and Kähler implies that the induced connection on K_M is flat, thus we have (i) $\implies$ (iii).

Now, let g be a metric on M that has restricted holonomy inside $SU(m)$. By Proposition 2.2.4, g is Kähler. Moreover, we see that the induced connection on K_M has trivial holonomy and is therefore flat, hence g is Ricci-flat, thus establishing (iii) $\implies$ (i).

Lastly, we see that $c_1(M) = 0$ as a real class if and only if $[K_M]$ is a torsion class. This means that there is some natural number $n \in \mathbb{N}_0$ such that $n[K_M] = 0$, which means $[K_M^{\otimes n}] = 0$, i.e. $K_M^{\otimes n}$ is trivial as a smooth vector bundle, thus establishing (ii) $\iff$ (iv). ☐

2.4.2 Examples of Calabi-Yau manifolds

Now that we have some basic understanding of the structure of Calabi-Yau manifolds, we turn to some examples. In general, finding explicit Kähler metrics g such that $\mathrm{Hol}(g) = SU(m)$ is very difficult and the only known example is a noncompact one: the *Eguchi-Hanson space* [BLT13; Cal79; EH79]. We will discuss this example quite in depth in Chapter 4. However, there is an algorithm by S.K. Donaldson that can be employed to find numerical approximations for Ricci-flat Kähler metrics [Don01].

From here on out, we shall define *Calabi-Yau manifolds* as manifolds that admit a Ricci-flat Kähler metric, which we shall call a *Calabi-Yau metric*.

Since finding explicit Calabi-Yau metrics is so difficult outside of trivial examples, the nontrivial examples that we shall give will be examples of complex manifolds with vanishing real first Chern class. The Calabi-Yau theorem then tells us that there exists a Ricci-flat Kähler metric on these spaces.

Example 2.4.8 (Trivial examples)**.** $\mathbb{C}^n$ is a Calabi-Yau manifold, where the Calabi-Yau metric is the standard one. Likewise, $\mathbb{T}^{2n} := \mathbb{C}^n/\mathbb{Z}^{2n}$ is a Calabi-Yau manifold, where the Calabi-Yau metric is the standard one.

Example 2.4.9 (Calabi-Yau curve)**.** Let M be a compact Riemann surface, then M is Calabi-Yau if and only if it is diffeomorphic to the torus. This is because the first Chern class of a compact Riemann surface satisfies $c_1(M) = \chi(M) = 2 - 2g$, where g is the genus of M [GH94].

Example 2.4.10 (Calabi-Yau surface)**.** The four torus $\mathbb{C}^2/\mathbb{Z}^4$ is a trivial example of a Calabi-Yau surface, however, there are also some nontrivial examples. There are many known results on compact complex surfaces, see [Bar+04] for an excellent review. Specifically, two well known nontrivial examples of two dimensional Calabi-Yau's are the *K3 surfaces*, which are Calabi-Yau manifolds with $SU(2)$ holonomy, and the *Enriques surfaces*, which are Calabi-Yau manifolds with nonvanishing canonical class K_M, but do have $2K_M = 0$.

Example 2.4.11. [CICY's] By Equation (2.3.12), we see that a smooth intersection M of m transversally intersecting hypersurfaces $\{X_i\}_{i=0}^m$ in $\mathbb{C}P^n$ of degree $\{a_i \geq 2\}_{i=0}^m$[1] has canonical bundle $\mathcal{O}(-n-1+$

[1] Recall that if some $a_i = 1$, then we can write M as a CICY in $\mathbb{C}P^{n-1}$ by restricting to $X_i \cong \mathbb{C}P^{n-1}$.

$\sum_i a_i)|_M$. Therefore, if $\sum_i a_i = n+1$, we see that M is Calabi-Yau. This gives us a large family of examples of Calabi-Yau manifolds known as *CICY*'s, which is short for *complete intersection Calabi-Yau manifolds*.

The particular case $n - m = 3$, i.e. when M is a 3 dimensional CICY, then admits the following description. If, following [Joy00], we employ the notation $[M] = (n|a_1, \ldots, a_m)$ for the class of complete intersections arising from hypersurfaces $\{X_i\}_{i=1}^m$ of degree $\{a_i\}_{i=1}^m$, we see that the only classes of 3 dimensional CICY's are

$$(4|5)\,,\ (5|4,2)\,,\ (5|3,3)\,,\ (6|3,2,2)\,,\ (7|2,2,2,2)\,.$$

Calabi-Yau's of complex dimension 3 are the most important ones in the context of superstring theory, which is a theory that works in 10 real dimensions but needs to be "compactified" on a 6 (real) dimensional manifold to describe general relativity in 4 dimensions, as we shall describe in Chapter 5.

3 Calabi-Yau theorem

The previous chapter introduced some analysis on Riemannian manifolds, some complex geometry, some line bundle theory and some basic results about Calabi-Yau manifolds, where we also stated the Calabi-Yau theorem.

This Calabi-Yau theorem is a big theorem on the existence of compact Calabi-Yau manifolds, that was conjectured by E. Calabi in 1954 [Cal54] and finally proven in 1978 by S.T. Yau [Yau77; Yau78]. For this reason, literature often refers to this theorem by the *Calabi conjecture*, or by *Yau's theorem*. In this work, we will merge the two and refer to it as the *Calabi-Yau theorem*. Note that this theorem only tells us something about the compact case, there are also non-compact Calabi-Yau manifolds and we will provide an explicit example in Chapter 4.

In this chapter, we will start by studying the Calabi-Yau theorem a bit and provide a proof, based on the approach taken by D.D. Joyce [Joy00], which is based on the original proof by Yau, relying on the continuity method, noting that there is an alternative proof from 1985 by H.D Cao [Cao85], based on the method of Ricci flow, which we will not discuss here. Note that the proof by Cao still requires some of the results from Yau, so it's not a completely independent proof.

The proof we provide will be by analysis, where we will need most of the results from Section 2.1, and some results from Section 2.2.

We will assume all manifolds are connected, the disconnected case can be extracted relatively easily from the following discussion, but requires some bookkeeping that is quite tedious and does not grant any useful insight.

3.1 Statement of the theorem

The Calabi-Yau theorem concerns the existence of Kähler metrics with prescribed Ricci form on compact Kähler manifolds. So let M^m be a compact Kähler manifold with Kähler metric g . Since we have seen that the associated Ricci form ρ is a real, closed $(1,1)$-form lying in $2\pi c_1(M)$, a natural question would be to ask which real $(1,1)$-forms in this cohomology class are also Ricci-forms of certain Kähler metrics. The Calabi-Yau theorem guarantees that *any* such $(1,1)$-form satisfies this.

Theorem 3.1.1 (Calabi-Yau)**.** *Let (M^m, g) be a compact Kähler manifold with Ricci form ρ and Kähler form ω. Let ρ' be a real, closed $(1,1)$-form cohomologous to ρ. Then there is a unique Kähler metric g' on M with Kähler form ω' such that ω' is cohomologous to ω, and ρ' is the Ricci form of g'.*

In particular, we see that this immediately gives a seemingly stronger result.

Corollary 3.1.2. *Let M^m be a compact complex manifold admitting Kähler metrics. Let ρ be a real, closed $(1,1)$-form lying in $2\pi c_1(M)$. Then there is a unique Kähler form ω in every Kähler class of M, such that the Ricci form of ω is ρ.*

To prove this famous theorem, we shall need quite some brutal analysis. To see how this analysis pops up, we notice that the Calabi-Yau theorem can be rephrased in terms of a nonlinear partial differential equation, as we will now show.

Since ρ and ρ', as defined in the above theorem, are cohomologous, they differ by an exact form. So by the dd^c lemma, we conclude that there is a function $f \in C^\infty(M;\mathbb{R})$ (unique up to a constant, remember that all manifolds are assumed to be connected), such that $\rho' = \rho - \frac{1}{2}dd^c f$. The factor $-\frac{1}{2}$ is completely arbitrary and is chosen for later convenience. Moreover, since ω^m is a volume form, we see that $(\omega')^m = F \cdot \omega^m$ for some strictly positive $F \in C^\infty(M;\mathbb{R})$. In particular, using Equation (2.2.29), if we jump to coordinates, we see

$$\det(g'_{\alpha\bar\beta}) = F\det(g_{\alpha\bar\beta})\,.$$

Therefore, Equation (2.2.37) tells us

$$\tfrac{1}{2}dd^c\log(F) = \rho - \rho' = \tfrac{1}{2}dd^c f\,.$$

In particular, we see that $\log(F) - f$ is dd^c closed, and therefore constant, thus we can find a constant $A > 0$ such that $\log(A) = \log(F) - f$, meaning $F = Ae^f$, i.e.

$$(\omega')^m = Ae^f\omega^m\,. \tag{3.1.1}$$

Since ω and ω' are cohomologous, they induce the same volume on M, so we see

$$A = \frac{m!\mathrm{vol}_g(M)}{\int_M e^f\omega^m}\,, \tag{3.1.2}$$

where $\mathrm{vol}_g(M)$ is the volume of g. Now we are ready to reformulate the Calabi-Yau theorem.

Theorem 3.1.3 (Calabi-Yau II)**.** *Let (M, g) be a compact Kähler manifold with Kähler form ω. Let $f \in C^\infty(M;\mathbb{R})$ and let $A := \mathrm{vol}_g(M)/\int_M e^f\mathrm{vol}_g$. Then there is a unique Kähler metric g' on M such that its associated Kähler form ω' is cohomologous to ω and $(\omega')^m = Ae^f\omega^m$.*

Now we have rewritten the Calabi-Yau to get an equation for the condition on ρ', we see that the Calabi-Yau theorem can be interpreted as a result about cohomologous Kähler metrics with a prescribed volume form.

There is one more condition that we can rewrite, namely the condition that ω and ω' are cohomologous. This condition tells us that there is a real function φ on M such that $\omega' = \omega + dd^c\varphi$. Moreover, this φ is unique up to a constant, so we can choose normalisation condition $\int_M \varphi\,\mathrm{vol}_g = 0$. We note that any form of the type $\omega + dd^c\varphi$ is both real and $(1,1)$, however it need not be positive[1], which is the last condition we need for $\omega + dd^c\varphi$ to be a Kähler form. However, we can, in fact, show that if $\omega + dd^c\varphi$ satisfies the equation in the previous theorem, it is necessarily positive.

Lemma 3.1.4. *Let (M,g) be a compact Kähler manifold with Kähler form ω. Let $f \in C^0(M;\mathbb{R})$ and define $A := \mathrm{vol}_g(M)/\int_M e^f \mathrm{vol}_g$. Suppose that $\varphi \in C^2(M;\mathbb{R})$ satisfies $(\omega + dd^c\varphi)^m = Ae^f\omega^m$, then $\omega' := \omega + dd^c\varphi$ is positive.*

Proof. We note that ω' is positive if and only if $\omega'(-, J-)$ is a hermitian metric, which is a problem we will solve in coordinates. Note that ω' is positive if and only if the following matrix is positive definite:

$$g'_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \frac{\partial\varphi}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}},$$

as then we know g' is a hermitian metric. Rewriting the condition $(\omega + dd^c\varphi)^m = Ae^f\omega^m$, we see

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\varphi}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}}\right) = Ae^f \det(g_{\alpha\bar{\beta}}). \tag{3.1.3}$$

So we see that $g'_{\alpha\bar{\beta}}$ has positive determinant, so no nonzero eigenvalues. Since all manifolds are assumed to be connected, it suffices to show that there is some point in M where all eigenvalues of $g'_{\alpha\bar{\beta}}$ are strictly positive. We're in luck, as M is compact, we know that φ must have a minimum, at which we know that the Hessian $\partial_i\partial_j\varphi$ is positive semidefinite (here, i and j are real indices, so they run over the ∂_x and ∂_y's). We define $m \times m$ matrices

$$A_{ab} := \frac{\partial\varphi}{\partial x^a \partial x^b};$$
$$B_{ab} := \frac{\partial\varphi}{\partial x^a \partial y^b};$$
$$C_{ab} := \frac{\partial\varphi}{\partial y^a \partial y^b},$$

such that the complex Hessian becomes

$$\frac{\partial^2\varphi}{\partial z \partial \bar{z}} = \tfrac{1}{4}\left(A + C + i(B - B^T)\right).$$

For a complex vector $u = v + iw$, with v and w real, we then see

$$u^* \frac{\partial^2\varphi}{\partial z \partial \bar{z}} u = \tfrac{1}{4}((v^TAv + w^TCw - v^TBw - w^TB^Tv) + (w^TAw + v^TCv + w^TBv + v^TB^Tw)). \tag{3.1.4}$$

Since $\partial_i\partial_j\varphi$ is positive semi-definite, we know that for any real vector $(u_1, u_2)^T$, we have

$$u_1^TAu_1 + u_2^TCu_2 + u_1^TBu_2 + u_2^TB^Tu_1 \geq 0.$$

In particular, if we pick $u_1 = v$ and $u_2 = -w$, we see that the first term on the RHS of (3.1.4) is non-negative, and if we pick $u_1 = w$ and $u_2 = v$, we see that the second term on the RHS of (3.1.4) is non-negative. So we see that the complex Hessian at a minimum is a positive semi-definite hermitian matrix, in particular it has non-negative eigenvalues. Since we also know $g_{\alpha\bar{\beta}}$ has positive eigenvalues everywhere, as it is a hermitian metric, we see that $g'_{\alpha\bar{\beta}}$ has positive eigenvalues at this minimum, thus completing the proof. □

[1] Recall that a positive $(1,1)$-form is a real form η such that for any real vector $u \neq 0$, $\eta(u, Ju) > 0$.

Thus we get the final version of the Calabi-Yau theorem, which is now phrased in terms of a differential equation:

Theorem 3.1.5 (Calabi-Yau III)**.** *Let (M, g) be a compact Kähler manifold with Kähler form ω. Let f be a smooth, real function on M and define $A := \mathrm{vol}_g(M)/\int_M e^f \mathrm{vol}_g$. Then there is a unique smooth real function φ such that*

(i) $(\omega + dd^c\varphi)^m = Ae^f\omega^m$.

(ii) $\int_M \varphi \,\mathrm{vol}_g = 0$.

The following is equivalent to part (i)*, when written out in local holomorphic coordinates:*

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\varphi}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}}\right) = Ae^f \det(g_{\alpha\bar{\beta}}) . \tag{3.1.5}$$

This equation is a particular kind of *complex Monge-Ampère equation*. Solving it is quite tricky, as it is highly nonlinear. For an overview of results on this type of equation, see [Aub82], in this thesis we will only consider this specific form of Monge-Ampère equation.

3.2 Proof of theorem

The approach we will take is called the *continuity method*. The idea is that we write down a continuous one-parameter family of differential equations $P_t(\varphi) = 0$, to be specified later, such that at time 0, the equation is easy to solve, or even trivial, and at time 1, we have the equation we want to solve. Then we define

$$S := \{t \in [0, 1] : P_t(\varphi) = 0 \text{ has solution given suitable initial conditions}\} ,$$

and we shall prove that S is both open and closed in $[0, 1]$. Since $[0, 1]$ is connected, this will imply that S is either empty, or $[0, 1]$. Since we assumed $P_0(\varphi) = 0$ is easy to solve and has a solution given suitable initial conditions, we know that $0 \in S$, and therefore $1 \in S$, i.e. $P_1(\varphi) = 0$ has a solution, which is the problem we're trying to solve.

To make this more precise, we give the following definition, which is the setup for the proof of the Calabi-Yau theorem. We will not assume smoothness of f and φ below, we will assume $f \in C^{(3,\alpha)}(M)$, and we shall ask the question when there is a solution φ to the Monge-Ampère equation such that $\varphi \in C^{(5,\alpha)}(M)$. This is because Hölder spaces often have nice topological properties, e.g. we have the Sobolev embedding theorem 2.1.10.

Definition 3.2.1. Let (M, g) be a Kähler manifold with Kähler form ω. Let $\alpha \in (0, 1)$ and let $f \in C^{(3,\alpha)}(M; \mathbb{R})$. Let S be the subset of $[0, 1]$ such that there exists a $\varphi \in C^{(5,\alpha)}(M; \mathbb{R})$ and an $A > 0$, such that $(\omega + dd^c\varphi)^m = Ae^{tf}\omega^m$.

At time 0, the above differential equation is satisfied by $\varphi = 0$ and $A = 1$, so we know $0 \in S$. Also note that this A is uniquely determined by φ, if it exists. So if we can show that S is clopen, we know then also $1 \in S$, so then we have at least one solution $\varphi \in C^{(5,\alpha)}(M; \mathbb{R})$ to the Monge-Ampère equation we want to solve. What remains to show is then that this solution is unique, and that this solution is smooth if f is smooth. Now we have cut up our problem in several smaller components:

(i) Show that S is closed;

(ii) Show that S is open;
(iii) Show that φ is smooth whenever f is smooth;
(iv) Show that φ is unique.

Showing that S is closed is significantly harder than showing the other three items on the list. This is why it took 24 years after the theorem was conjectured for Yau finally formulate a proof using the continuity method, for which he received a Fields medal in 1982.

We will state four theorems (numbered CY 1-4), which together imply the Calabi-Yau theorem, the proofs of these theorems will be given later down the road, in Sections 3.4-3.7. The first two will be sufficient to prove that S is closed, the second one will also give us smoothness, the third one will give us that S is open, and the last one gives us uniqueness. There will be some functional analysis involved with different norms all over the place, see Section 2.1 for the respective definitions.

Theorem CY1. Let (M,g) be a compact Kähler manifold with Kähler form ω. Let $Q_1 \geq 0$. Then there exist $Q_2, Q_3, Q_4 \geq 0$ depending on only M, g and Q_1, such that the following holds:

Suppose $f \in C^3(M)$, $\varphi \in C^5(M)$ and $A > 0$ satisfy

$$\|f\|_{C^3} \leq Q_1\,, \quad \int_M \varphi \operatorname{vol}_g = 0\,, \quad (\omega + dd^c\varphi)^m = Ae^f\omega^m\,.$$

Then $\|\varphi\|_{C^0} \leq Q_2$, $\|dd^c\varphi\|_{C^0} \leq Q_3$ and $\|\nabla dd^c\varphi\|_{C^0} \leq Q_4$.

Where, like before, we view ∇ as a map $\nabla : \Gamma(T^*M^{\otimes k}) \to \Gamma(T^*M^{\otimes k+1})$, so no antisymmetrisation. This theorem is about *a priori estimates*, it gives us bounds that any solution to the Monge-Ampère equation must satisfy, without assuming we put in a specific function f. Theorem CY1 is Yau's contribution to the proof of the Calabi-Yau theorem. Together with the following theorem, we can prove that S is closed

Theorem CY2. Let (M,g) be a compact Kähler manifold with Kähler form ω and complex structure J. Let $Q_1, Q_2, Q_3, Q_4 \geq 0$ and $\alpha \in (0,1)$. Then there is a $Q_5 \geq 0$ depending only on $M, g, J, Q_1, Q_2, Q_3, Q_4$ and α such that the following holds.

Suppose $f \in C^{(3,\alpha)}(M)$, $\varphi \in C^5(M)$ and $A > 0$ satisfy $(\omega + dd^c\varphi)^m = Ae^f\omega^m$ and the inequalities

$$\|f\|_{C^{(3,\alpha)}} \leq Q_1\,, \quad \|\varphi\|_{C^0} \leq Q_2\,, \quad \|dd^c\varphi\|_{C^0} \leq Q_3\,, \quad \|\nabla dd^c\varphi\|_{C^0} \leq Q_4\,.$$

Then $\varphi \in C^{(5,\alpha)}(M)$ and $\|\varphi\|_{C^{(5,\alpha)}} \leq Q_5$. Moreover, if $f \in C^{(k,\alpha)}(M)$ for some $k \geq 3$, then $\varphi \in C^{(k+2,\alpha)}(M)$, and if f is smooth, so is φ.

Note that if f is $C^{(3,\alpha)}$, it is in particular C^3, and we have $\|f\|_{C^3} \leq \|f\|_{C^{(3,\alpha)}}$. So if we have $\|f\|_{C^{(3,\alpha)}} \leq Q_1$, CY1 gives us Q_2, Q_3 and Q_4 to put into CY2, and then CY2 gives us a Q_5 such that φ is $C^{(5,\alpha)}(M)$ with norm $\|\varphi\|_{C^{(5,\alpha)}} \leq Q_5$. It is clear that CY2 gives us smoothness of φ, so that's one of the four ingredients of the proof of the Calabi-Yau theorem complete.

Corollary 3.2.2. *Let $f \in C^\infty(M)$. If $A > 0$ and $\varphi \in C^{(5,\alpha)}(M)$ satisfy $\int_M \varphi \operatorname{vol}_g = 0$ and $(\omega + dd^c\varphi)^m = Ae^f\omega^m$, then φ is smooth.*

Moreover, CY1 and CY2 together imply that S is closed:

Corollary 3.2.3. *S is closed.*

Proof. We shall show that S has all limit points. As $[0,1]$ is Hausdorff, this will imply closedness. Let $\{t_i\}$ be a sequence in S converging to t'. We shall show that t' is in S using CY1 and CY2. Since $t_i \in S$ for every i by assumption, we can find $\varphi_i \in C^{(5,\alpha)}(M)$ and $A_i > 0$ for every i, such that

$$\int_M \varphi_i = 0\,; \quad (\omega + dd^c\varphi_i)^m = A_i e^{t_i f}\omega^m\,.$$

Let $Q_1 := \|f\|_{C^{(3,\alpha)}}$, and use CY1 to find Q_2, Q_3, Q_4 as described there, and use CY2 to find Q_5 as described there. Since we have $\|t_i f\|_{C^{(3,\alpha)}} = t_i Q_1 \le Q_1$, CY1 tells us $\|\varphi_i\|_{C^0} \le Q_2$, $\|dd^c\varphi_i\|_{C^0} \le Q_2$ and $\|\nabla dd^c\varphi_i\|_{C^0} \le Q_4$ for every i. Therefore, CY2 tells us that φ_i is $C^{(5,\alpha)}$, with $\|\varphi_i\|_{C^{(5,\alpha)}} \le Q_5$, for every i.

So we see that $\{\varphi_i\}$ is a bounded sequence in $C^{(5,\alpha)}(M)$. However, the Rellich-Kondrachov theorem 2.1.11 tells us $C^{(5,\alpha)}(M) \hookrightarrow C^5(M)$ is compact, therefore, $\{\varphi_i\}$ has a convergent subsequence in $C^5(M)$. Let $\{\varphi_{i_j}\}$ be such a convergent subsequence that converges to φ'. Define $A' := \mathrm{vol}_g(M)/\int_M e^{t'f}\mathrm{vol}_g$. Because M is compact, we have that $\{A_{i_j}\}$ converges to A', because $\{t_{i_j}\}$ converges to t'. Now, since $\{\varphi_{i_j}\}$ converges in $C^5(M)$, it converges in $C^2(M)$, so we can take the limit of the Monge-Ampère equation to get

$$\int_M \varphi' = 0\,; \quad (\omega + dd^c\varphi')^m = A' e^{t'f}\omega^m\,.$$

However, now we can apply CY1 and CY2 again to conclude that φ' is in fact $C^{(5,\alpha)}(M)$, so we see that $t' \in S$, i.e. S is closed. □

So now we have two out of four. What remains to show is that S is open, which is given by CY3, and uniqueness, which will be given by CY4

Theorem CY3. Let (M,g) be a Kähler manifold with Kähler form ω. Let $\alpha \in (0,1)$ and suppose $f' \in C^{(3,\alpha)}(M)$, $\varphi' \in C^{(5,\alpha)}(M)$ and $A' > 0$ satisfy

$$\int_M \varphi' \,\mathrm{vol}_g = 0\,; \quad (\omega + dd^c\varphi')^m = A' e^{f'}\omega^m\,.$$

Then whenever $f \in C^{(3,\alpha)}(M)$ is such that $\|f - f'\|_{C^{(3,\alpha)}}$ is sufficiently small, then there is a $\varphi \in C^{(5,\alpha)}(M)$ and an $A > 0$ such that

$$\int_M \varphi \,\mathrm{vol}_g = 0\,; \quad (\omega + dd^c\varphi)^m = A e^{f}\omega^m\,.$$

In the above theorem, "sufficiently small" can be more precisely formulated using a standard analysis definition: for every $f' \in C^{(3,\alpha)}(M)$, there is an $\varepsilon > 0$ such that $\|f - f'\|_{C^{(3,\alpha)}} < \varepsilon$ implies that such a φ exists. Since $\|tf - t'f\|_{C^{(3,\alpha)}} = |t - t'|\|f\|_{C^{(3,\alpha)}}$, the following is an immediate consequence of CY3:

Corollary 3.2.4. *S is open.*

Finally, uniqueness of the solution to the Monge-Ampère equation is guaranteed by the following result, which was already known to Calabi in the '50s [Cal54]:

Theorem CY4. Let (M,g) be a Kähler manifold with Kähler form ω. Let $f \in C^1(M)$, and let $A > 0$ be some real number, then there is at most one $\varphi \in C^3(M)$ such that $\int_M \varphi \,\mathrm{vol}_g = 0$ and $(\omega + dd^c\varphi)^m = A e^f \omega^m$.

Since the assumption $\int_M \varphi \, \mathrm{vol}_g = 0$ is just a normalisation condition, we see that we could have also removed this and obtained that φ is unique up to addition of a constant. This concludes the proof of the Calabi-Yau theorem.

3.3 Some intermediate calculations

Before jumping into the proof of CY1, we will do some computations that appear in multiple places or might be interesting results on their own.

Lemma 3.3.1. *Let η and σ be real, positive $(1,1)$-forms on a Kähler manifold (M,g). Then the fibrewise inner product $(\eta,\sigma)_g$ is strictly positive.*

Proof. Since J is orthogonal, we have $(\eta,\sigma)_g = (\eta(-,J-),\sigma(-,J-))_g$. In an orthonormal basis, this gives us $(\eta,\sigma)_g = \mathrm{Tr}(\eta(-,J-)\sigma(-,J-))$. Since η and σ are positive, $\eta(-,J-)\sigma(-,J-)$ has strictly positive eigenvalues, thus $\mathrm{Tr}(\eta(-,J-)\sigma(-,J-)) > 0$, proving the lemma. ☐

Corollary 3.3.2. *Let η and σ be real, positive $(1,1)$-forms on a Kähler manifold (M,g). Then $\eta \wedge \star\sigma$ is a positive top form with respect to vol_g.*

This leads us to the following rather useful lemma, which we will require a lot in the rest of the proof:

Lemma 3.3.3. *Let (M,g) be a compact complex Kähler manifold with Kähler form ω. Let φ be a real function on M and let ω' be a positive real $(1,1)$-form. Then*

$$d\varphi \wedge d^c\varphi \wedge \omega^{m-1} = \tfrac{1}{m}|\nabla\varphi|_g^2\omega^m .$$

Moreover, there exist nonnegative functions F_j for $j = 1,\dots,m-1$ such that

$$d\varphi \wedge d^c\varphi \wedge \omega^{m-1-j} \wedge (\omega')^j = F_j\omega^m .$$

Proof. We start by calculating $\star\omega$:

$$\omega \wedge \star\omega = (\omega,\omega)_g\mathrm{vol}_g = (\omega(-,J-),\omega(-,J-))_g\mathrm{vol}_g = (g,g)_g\mathrm{vol}_g = 2m\,\mathrm{vol}_g = \frac{2m}{m!}\omega^m .$$

Therefore, $\star\omega = \frac{2}{(m-1)!}\omega^{m-1}$. We see

$$d\varphi \wedge d^c\varphi \wedge \omega^{m-1} = \frac{(m-1)!}{2} d\varphi \wedge d^c\varphi \wedge \star\omega = \frac{1}{2m}(d\varphi \wedge d^c\varphi,\omega)_g\,\omega^m = \frac{1}{2m}((d\varphi)^2 + (d^c\varphi)^2, g)_g\,\omega^m .$$

We note that, in coordinates, $|\nabla\varphi|_g^2 = |d\varphi|_g^2 = |d^c\varphi|_g^2 = g^{ij}d\varphi_i d\varphi_j$, and that $g^{ij}d\varphi_i d\varphi_j = ((d\varphi)^2, g)_g$. Therefore, we see

$$d\varphi \wedge d^c\varphi \wedge \omega^{m-1} = \tfrac{1}{m}|\nabla\varphi|^2\omega^m .$$

This concludes the proof of the first part of the lemma. For the second half, note that $d\varphi \wedge d^c\varphi$ is a nonnegative real $(1,1)$-form. So if we can show that $\star(\omega^{m-1-j} \wedge (\omega')^j)$ is positive, we can use the same reasoning as above to conclude that $d\varphi \wedge d^c\varphi \wedge \omega^{m-1-j} \wedge (\omega')^j$ is a nonnegative top form, which then implies the second part of the lemma.

To show this, we will work in coordinates. Around some point $x \in M$, we pick a unitary frame, i.e. a frame where $g_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$ at x. Since we know that ω' is a positive $(1,1)$-form, it has an associated hermitian metric $g'_{\alpha\bar{\beta}}$, which then commutes with $g_{\alpha\bar{\beta}}$ at x. So we can diagonalise $g'_{\alpha\bar{\beta}}$ using unitary transformations, such that $g'_{\alpha\bar{\beta}} = \operatorname{diag}(a_1, \dots, a_m)$ at x, where $a_1, \dots, a_m$ are positive real numbers. Then we see $\omega' = i(a_1 dz^1 \wedge d\bar{z}^1 + \cdots + a_m dz^m \wedge d\bar{z}^m)$ at x, so by direct computation we conclude that $\star(\omega^{m-1-j} \wedge (\omega')^j)$ is a positive $(1,1)$-form at x, which was an arbitrary point, so it is a positive $(1,1)$-form globally, hence the lemma is proved. □

As briefly discussed in the proof of the previous theorem, given a C^2 solution to the Monge-Ampère equation, i.e. a function $f \in C^0(M)$, $\varphi \in C^2(M)$ and $A > 0$ such that $(\omega + dd^c\varphi)^m = Ae^f\omega^m$, we can always find a frame around a point $x \in M$ such that, at x, we have that g and the C^0 metric g', associated to $\omega' := \omega + dd^c\varphi$, are diagonal, this is stated in the following lemma:

Lemma 3.3.4. *In the situation above, we can always find a frame such that*

$$g_{\alpha\bar{\beta}}(x) = \delta_{\alpha\bar{\beta}} \quad \textit{and} \quad g'_{\alpha\bar{\beta}}(x) = \operatorname{diag}(a_1, \dots, a_m)$$

for real numbers $a_1, \dots, a_m$. Moreover, in this frame,

$$\omega(x) = i\sum_j dz^j \wedge d\bar{z}^j \quad \textit{and} \quad \omega'(x) = i\sum_j a_j dz^j \wedge d\bar{z}^j\,.$$

The proof is straightforward and just requires simultaneously diagonalising g' and g at p, which can always be done if we pick a nice enough frame such that g is the identity matrix. Now we define the ∂-Laplacian by $\Delta\psi = -g^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}\psi$ for $\psi \in C^\infty(M)$, and we extend the definition for $\psi \in C^2(M)$. From now on, we will do calculations in the above mentioned frame and we will do calculations at the point x. In this frame, we see that the complex Hessian $\partial_\alpha\partial_{\bar{\beta}}\varphi$ is diagonal at x, as we have $g'_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \partial_\alpha\partial_{\bar{\beta}}\varphi$, so in particular we have the following result, which follows from this discussion and from Equation (3.1.5):

Lemma 3.3.5. *In the situation described above, we have*

$$\prod_j a_j = Ae^{f(x)}\,; \quad \partial_\alpha\partial_{\bar{\alpha}}\varphi(x) = a_\alpha - 1 \quad \textit{and} \quad \Delta\varphi(x) = -\operatorname{Tr}(\partial_\alpha\partial_{\bar{\beta}}\varphi(x)) = m - \sum_j a_j\,. \tag{3.3.1}$$

Now we do a few calculations

$$\begin{aligned}
|dd^c\varphi|_g^2 &= -4(\partial\bar{\partial}\varphi, \partial\bar{\partial}\varphi)_g \\
&= 8(g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\partial_\alpha\partial_{\bar{\delta}}\varphi\partial_\gamma\partial_{\bar{\beta}}\varphi) \\
&= 8\sum_j (a_j - 1)^2\,.
\end{aligned}$$

Likwise, one can find $|g'_{ij}|_g^2$ and $|g'^{\,ij}|_g^2$. We summarise these results in the following lemma:

Lemma 3.3.6. *In the situation above, at the point x, we have*

$$|dd^c\varphi|_g^2 = 8\sum_j (a_j - 1)^2\,; \quad |g'_{ij}|_g^2 = 2\sum_j a_j^2 \quad \textit{and} \quad |g'^{\,ij}|_g^2 = 2\sum_j a_j^{-2} \tag{3.3.2}$$

Now we will use some of the above results to get some bounds the C^0 norms of g', $(g')^{-1}$ and $dd^c\varphi$ in terms of f and $\Delta\varphi$. These results will help us in the following sections, as it allows us to estimate certain things by just finding estimates for the Laplacian.

Proposition 3.3.7. *Let (M, g) be a compact Kähler manifold with Kähler form ω. Let $f \in C^0(M)$, $\varphi \in C^2(M)$ and $A > 0$ such that they solve the Monge-Ampère equation. Let $\omega' := \omega + dd^c\varphi$ and let g' be the associated Kähler metric. Then*

$$\Delta\varphi \leq m - mA^{1/m}e^{f/m} < m\,, \tag{3.3.3}$$

and there are constants c_1, c_2 and c_3 depending only on m and upper bounds for $\|f\|_{C^0}$ and $\|\Delta\varphi\|_{C^0}$, such that

$$\left\|g'_{ij}\right\|_{C^0} \leq c_1\,; \quad \left\|g'^{\,ij}\right\|_{C^0} \leq c_2 \quad \textit{and} \quad \|dd^c\varphi\|_{C^0} \leq c_3\,, \tag{3.3.4}$$

where all Banach norms are defined with respect to the metric g.

Proof. We will do some calculations at the point x. The first inequality follows from Lemma 3.3.5 and the fact that the arithmetic mean $(a_1 \dots a_m)^{1/m}$ is less than or equal to the geometric mean $\frac{1}{m}\sum_j a_j$, which is a known fact, but can be proven e.g. by

$$\log\left(\frac{\sum_j a_j}{m}\right) \geq \sum_j \frac{1}{m}\log(a_j) = \log((a_1 \dots a_m)^{1/m})\,,$$

where the inequality, called *Jensen's inequality*, essentially follows from the fact that the log is concave, such that the log of the mean is always greater or equal than the mean of the logs. Moreover, we note

$$\left(\sum_j a_j\right)^2 \geq \sum_j a_j^2\,,$$

as all a_j are positive. Moreover, $(a_j - 1)^2 \leq a_j^2 + 1$, so putting these together, and using $m - \Delta\varphi = \sum_j a_j$ and $|dd^c\varphi|_g^2 = 8\sum_j (a_j - 1)^2$, we get

$$|dd^c\varphi|_g^2 \leq 8m + 8(m - \Delta\varphi)^2\,.$$

Moreover, as $|g'_{ij}|_g^2 = 2\sum_j a_j^2$, we also have

$$|g'_{ij}|_g^2 \leq 2(m - \Delta\varphi)^2\,.$$

These calculations were done at a point, but the given formula is not dependent on the explicit frame anymore, so they hold globally. Setting $c_3 = 2\|\Delta\varphi\|_{C^0}^2 + 4m\|\Delta\varphi\|_{C^0} + 2m^2$ then suffices, as $2(m - \Delta\varphi)^2 \leq c_3$. So then also $c_1 = 4c_3 + 8m$ suffices. Moreover, these estimates only depend on m and $\|\Delta\varphi\|_{C^0}$, so not even explicitly[1] on $\|f\|_{C^0}$.

Now we will work more globally, so not specifically at x, if we need to work at x we will make it explicit. For the remaining estimate, we note that ω' and ω induce the same volume, as M is compact and they are cohomologous symplectic forms, so we need $\int_M e^{\log A + f}\,\mathrm{vol}_g = \int_M \mathrm{vol}_g$, in particular, $-\sup f \leq \log A \leq -\inf f$, as the equality cannot be satisfied if $\log A + f$ is strictly greater than 0 everywhere, or if $\log A + f$ is strictly smaller than 0 everywhere. Therefore,

$$|\log A| \leq \max(|\sup f|, |\inf f|) = \sup|f| = \|f\|_{C^0}\,.$$

[1] Eventually we will have to estimate $\|\Delta\varphi\|_{C^0}$ in terms of certain bounds on norms of f, so there is still an implicit dependence here.

We see

$$e^{-2\|f\|_{C^0}} \leq Ae^f \leq e^{2\|f\|_{C^0}}\,.$$

Now, Lemma 3.3.5 tells us $\prod_j a_j = Ae^{f(x)}$, so we see

$$a_j^{-1} = A^{-1}e^{-f(x)}\prod_{k\neq j} a_k\,.$$

Now, we have also seen $a_k \leq \sum_j a_j = m - \Delta\varphi(x)$, so putting things together, we see

$$a_j^{-2} \leq e^{4\|f\|_{C^0}}(m-\Delta\varphi(x))^{2m-2}\,.$$

So we see

$$|g'^{\,ij}|_g^2 = 2\sum_j a_j^{-2} \leq 2me^{4\|f\|_{C^0}}\sum_{k=0}^{2m-2}\binom{2m-2}{k} m^k \,\|\Delta\varphi\|_{C^0}^{2m-2-k} =: c_2\,,$$

therefore this choice of c_2 is sufficient. □

3.4 Proof of CY1

In this section, we discuss the proof of CY1, which was Yau's main contribution to the proof of the Calabi-Yau theorem. The proof will be quite involved an will use a lot of estimation of functions, and estimating norms in terms of other norms.

Theorem CY1. Let (M,g) be a compact Kähler manifold with Kähler form ω . Let $Q_1 \geq 0$. Then there exist $Q_2, Q_3, Q_4 \geq 0$ depending on only M , g and Q_1 , such that the following holds:

Suppose $f \in C^3(M)$, $\varphi \in C^5(M)$ and $A > 0$ satisfy

$$\|f\|_{C^3} \leq Q_1\,, \quad \int_M \varphi\,\mathrm{vol}_g = 0\,, \quad (\omega + dd^c\varphi)^m = Ae^f\omega^m\,.$$

Then $\|\varphi\|_{C^0} \leq Q_2$, $\|dd^c\varphi\|_{C^0} \leq Q_3$ and $\|\nabla dd^c\varphi\|_{C^0} \leq Q_4$.

In this section, we assume without loss of generality that $A = 1$. We can do this by modifying the function f by addition of a constant. In the entire section, we assume we are working in the setting as described in the theorem above. We see that there are 3 steps involved: estimates of order 0, i.e. $\|\varphi\|_{C^0}$, estimates of order 2, i.e. $\|dd^c\varphi\|_{C^0}$ and estimates of order 3, i.e. $\|\nabla dd^c\varphi\|_{C^0}$.

3.4.1 Estimates of order zero

Firstly, we discuss the zeroth order estimates, i.e. the estimates of φ itself, not its derivatives.

We start with a technical result, which will follow from a calculation

Proposition 3.4.1. *Let $p > 1$ be a real number, then we have the following inequality*

$$\int_M \left|\nabla|\varphi|^{p/2}\right|_g^2 \mathrm{vol}_g \leq \frac{mp^2}{4(p-1)}\int_M (1-e^f)\varphi|\varphi|^{p-2}\,\mathrm{vol}_g\,. \tag{3.4.1}$$

Proof. The proof is by direct computation. We start from $(\omega')^m = e^f\omega^m$ and $\omega - \omega' = -dd^c\varphi$. This tells us

$$(1-e^f)\omega^m = \omega^m - (\omega')^m = -dd^c\varphi \wedge (\omega^{m-1} + \omega^{m-2}\wedge\omega' + \cdots + (\omega')^{m-1}) \,.$$

By Stokes' theorem,

$$\int_M d\left(\varphi|\varphi|^{p-2} d^c\varphi \wedge (\omega^{m-1}\wedge\cdots\wedge(\omega')^{m-1})\right) \mathrm{vol}_g = 0\,.$$

Combining the two, using $d\omega = d\omega' = 0$ and using

$$d(\varphi|\varphi|^{p-2}) = \mathrm{sign}(\varphi) d|\varphi|^{p-1} = (p-1)|\varphi|^{p-2} d(\mathrm{sign}(\varphi)|\varphi|) = (p-1)|\varphi|^{p-2}d\varphi\,,$$

we get

$$\int_M \varphi|\varphi|^{p-2}(1-e^f)\omega^m = (p-1)\int_M |\varphi|^{p-2} d\varphi\wedge d^c\varphi\wedge(\omega^{m-1}\wedge\cdots\wedge(\omega')^{m-1})\,.$$

Applying Lemma 3.3.3, we find nonnegative functions F_j for $j = 1,\dots,m-1$, such that we get

$$\int_M \varphi|\varphi|^{p-2}(1-e^f)\omega^m = \frac{p-1}{m}\int_M |\varphi|^{p-2}(|\nabla\varphi|_g^2 + F_1 + \cdots + F_{m-1})\omega^m\,.$$

Since $\omega^m = m!\,\mathrm{vol}_g$, we see

$$\int_M \varphi|\varphi|^{p-2}(1-e^f)\mathrm{vol}_g = \frac{p-1}{m}\int_M |\varphi|^{p-2}(|\nabla\varphi|_g^2 + F_1 + \cdots + F_{m-1})\mathrm{vol}_g\,.$$

Since φ is C^5, $|\varphi|$ is C^5 almost everywhere, hence the following computation can be done in the weak sense

$$|\nabla|\varphi|^{p/2}|_g^2 = \left|\frac{p}{2}|\varphi|^{(p-2)/2}\nabla|\varphi|\right|_g^2 = \frac{p^2}{4}|\varphi|^{p-2}|\nabla\varphi|_g^2\,.$$

This finally gives us the inequality

$$\int_M |\nabla|\varphi|^{p/2}|_g^2\,\mathrm{vol}_g \le \frac{mp^2}{4(p-1)}\int_M (1-e^f)\varphi|\varphi|^{p-2}\,\mathrm{vol}_g\,,$$

proving the proposition. □

Now we define $\varepsilon := \frac{m}{m-1}$.

Lemma 3.4.2. *There are constants $C_1, C_2 > 0$ depending only on M and g such that for every $\psi \in L_1^2(M)$, $\|\psi\|_{L^{2\varepsilon}} \le C_1(\|\nabla\psi\|_{L^2}^2 + \|\psi\|_{L^2}^2)$. Moreover, if ψ satisfies $\int_M \psi\,\mathrm{vol}_g = 0$, we have $\|\psi\|_{L^2} \le C_2\|\nabla\psi\|_{L^2}$.*

Proof. By the Sobolev Embedding Theorem 2.1.10, we have that $L^{2\varepsilon}(M)$ is continuously embedded in $L_1^2(M)$, so we can find a constant C_1 depending only on M and g such that for every $\psi \in L^{2\varepsilon}$, $\|\psi\|_{L^{2\varepsilon}}^2 \le C_1\,\|\psi\|_{L_1^2}^2$. Since we have

$$\|\psi\|_{L_1^2}^2 = \|\psi\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2\ ,$$

the first part of the lemma is proved. For the second part, we consider the operator $d^*d : C^\infty(M) \to C^\infty(M)$. We note that $\mathrm{Ker}(d^*d)$ are precisely the constant functions, as $\chi \in \mathrm{Ker}(d^*d)$ means

$$0 = (d^*d\chi, \chi)_g = (d\chi, d\chi)_g = |d\chi|_g^2\,,$$

and therefore $d\chi = 0$, i.e. χ is constant. Moreover, Theorem 2.1.21 tells us that d^*d has a positive, discrete spectrum of eigenvalues, as M is compact. In particular, d^*d has a minimal nonzero eigenvalue $\lambda_1 > 0$. Therefore, assuming ψ is smooth, $\int_M \psi \,\mathrm{vol}_g = 0$ tells us that ψ is L^2 orthogonal to $\mathrm{Ker}(d^*d)$, i.e. ψ is a sum of eigenvectors of d^*d with eigenvalues greater or equal than λ_1. Therefore, $\|d\psi\|_{L^2}^2 = (\psi, d^*d\psi)_g \geq \lambda_1 \|\psi\|_{L^2}^2$. Now, since $C^\infty(M)$ is dense in $L^2_1(M)$, this inequality still holds if ψ is not assumed to be smooth. Therefore, picking $C_2 := \lambda_1^{-1/2}$ gives us the second part of the lemma. □

Now we will turn to a priori estimation of $\|\varphi\|_{L^p}$. This will be done by induction on p in some sense.

Lemma 3.4.3. *There is a constant $C_3 > 0$ depending only on M, g and Q_1 such that if $p \in [2, 2\varepsilon]$, we have $\|\varphi\|_{L^p} \leq C_3$.*

Proof. Using Proposition 3.4.1 for $p = 2$, and using the inequality $|1 - e^f| \leq e^{Q_1}$, which follows from $\|f\|_{C^0} \leq \|f\|_{C^3} \leq Q_1$, we get $\|\nabla\varphi\|_{L^2}^2 \leq me^{Q_1}\|\varphi\|_{L^1}$. Since we assumed normalisation condition $\int_M \varphi \,\mathrm{vol}_g = 0$, Lemma 3.4.2 tells us $\|\varphi\|_{L^2} \leq C_2 \|\nabla\varphi\|_{L^2}$. Moreover, Hölders inequality tells us $\|\varphi\|_{L^1} \leq \|\varphi\|_{L^2}\,\mathrm{vol}_g(M)^{1/2}$. Combining these results gives us the inequality

$$\|\nabla\varphi\|_{L^2}^2 \leq mC_2 e^{Q_1}\mathrm{vol}_g(M)^{1/2}\|\nabla\varphi\|_{L^2}\,.$$

This gives $\|\nabla\varphi\|_{L^2}^2 \leq mC_2e^{Q_1}\mathrm{vol}_g(M)^{1/2} =: c$. Hence $\|\varphi\|_{L^2} \leq cC_2$, so Lemma 3.4.2 tells us

$$\|\varphi\|_{L^{2\varepsilon}}^2 \leq C_1(c^2 + c^2C_2^2)\,.$$

We define $C_3 := \max(cC_2, cC_1^{1/2}(1+C_2)^{1/2})$, which now only depends on M, g and Q_1. We see $\|\varphi\|_{L^2}, \|\varphi\|_{L^{2\varepsilon}} \leq C_3$, so the Hölder Interpolation Theorem 2.1.6 gives us $\|\varphi\|_{L^p} \leq C_3$ for every $p \in [2, 2\varepsilon]$, thus proving the lemma. □

Using this as a base case, we will extend to the general case $p \geq 2$ using induction. The statement is as follows:

Proposition 3.4.4. *There are constants Q_2 and C_4 depending only on M, g and Q_1 such that for every $p \geq 2$, we have $\|\varphi\|_{L^p} \leq Q_2(C_4p)^{-m/p}$.*

Proof. Define $C_4 > 0$ by $C_4 = C_1\varepsilon^{m-1}(me^{Q_1} + \frac{1}{2})$ and choose $Q_2 > 0$ such that

$$Q_2 \geq \max(C_3\mathrm{vol}_g(M)^{1/2}, C_3, 1)(C_4p)^{m/p}\,, \quad \text{for } p \geq 2\,.$$

Note that $\lim_{p\to\infty}(C_4p)^{m/p} = 0$, so such a constant exists. In fact, $(C_4p)^{m/p}$ attains its maximum at $p = eC_4^{-1}$, so picking $Q_2 = \max(C_3\mathrm{vol}_g(M)^{1/2}, C_3, 1)e^{C_4m/e}$ suffices. Now, if $p \in [2, 2\varepsilon]$, then $\|\varphi\|_{L^p} \leq C_3$ by Lemma 3.4.3, so by the way we chose Q_2, $\|\varphi\|_{L^p} \leq Q_2(C_4p)^{-m/p}$, as desired. For the inductive step, assume there is some $k \geq 2\varepsilon$ such that $\|\varphi\|_{L^p} \leq Q_2(C_4p)^{-m/p}$ for every $p \in [2, k]$. We will show that the inequality holds for every $p \in [2, \varepsilon k]$, such that it holds for every $p \geq 2$.

Let $p \in [2, k]$. Then we have $p^2/4(p-1) \leq p$. Moreover, we have $|1 - e^f| \leq e^{Q_1}$, so Proposition 3.4.1 tells us

$$\left\|\nabla|\varphi|^{p/2}\right\|_{L^2}^2 \leq mpe^{Q_1}\|\varphi\|_{L^{p-1}}^{p-1}\,.$$

Next, we apply Lemma 3.4.2 to $\psi := |\varphi|^{p/2}$ to get

$$\|\varphi\|_{L^{\varepsilon p}}^{p} \leq C_1 \left(\left\| \nabla |\varphi|^{p/2} \right\|_{L^2}^{2} + \|\varphi\|_{L^p}^{p} \right) .$$

Combining these gives us

$$\|\varphi\|_{L^{\varepsilon p}}^{p} \leq mpC_1 e^{Q_1} \|\varphi\|_{L^{p-1}}^{p-1} + C_1 \|\varphi\|_{L^p}^{p} .$$

Define $q := \varepsilon p$. Since $p \in [2,k]$, the induction hypothesis tells us $\|\varphi\|_{L^p} \leq Q_2(C_4 p)^{-m/p}$, and by the way we picked Q_2, $Q_2(C_4 p)^{-m/p} \geq 1$, thus $(Q_2(C_4 p)^{-m/p})^{p-1} \leq (Q_2(C_4 p)^{-m/p})^{p}$. Moreover, since $\|\varphi\|_{L^1} \leq \|\varphi\|_{L^2} \operatorname{vol}_g(M)^{1/2} \leq C_3 \operatorname{vol}_g(M)^{1/2}$ by the Hölder inequality, and since $1 \leq p-1 \leq p$, Hölder interpolation tells us that $\|\varphi\|_{L^{p-1}} \leq Q_2(C_4 p)^{-m/p}$. Putting everything together again, we get

$$\|\varphi\|_{L^q}^{p} \leq Q_2^{p}(C_4 p)^{-m}(mpC_1 e^{Q_1} + C_1) .$$

Moreover, by definition, $(Q_2(C_4 q)^{-m/q})^p = Q_2^p (C_4 p\varepsilon)^{1-m}$. Since $p \geq 2$, the way we defined C_4 tells us $mpC_1 e^{Q_1} + C_1 \leq C_4 p \varepsilon^{1-m}$. This then tells us

$$\|\varphi\|_{L^q}^{p} \leq Q_2^{p}(C_4 p\varepsilon)^{1-m} = (Q_2(C_4 q)^{-m/q})^p .$$

So finally, we obtain $\|\varphi\|_{L^q} \leq Q_2(C_4 q)^{-m/q}$ for every $q \in [2\varepsilon, \varepsilon k]$, thus proving the proposition. □

We finally arrive at the a priori estimate of order zero:

Corollary 3.4.5. *φ satisfies $\|\varphi\|_{C^0} \leq Q_2$.*

Proof. We note that the C^0-norm is the limit of the L^p-norms as p goes to ∞, where it might be helpful to note that all continuous functions are L^p for every p on a compact manifold. Since $\|\varphi\|_{L^p} \leq Q_2(C_4 p)^{-m/p}$ and $(C_4 p)^{-m/p}$ goes to 1 as p goes to ∞, we see that $\|\varphi\|_{C^0} \leq Q_2$, thus giving us the a priori estimate of order zero. □

3.4.2 Estimates of order two

We again work in the setting we described earlier, so (M, g) is a Kähler manifold with Kähler form ω, $f \in C^3(M)$, $\varphi \in C^5(M)$ are such that the Monge-Ampère equation is satisfied, assuming $A = 1$, $\omega' = \omega + dd^c\varphi$ and g' is the Kähler metric associated to ω'. ∇ is the Levi-Civita connection associated to g, with connection matrix Γ, which we interpret as a map

$$\nabla : \Gamma(T^*M^{\otimes k} \otimes TM^{\otimes l}) \to \Gamma(T^*M^{\otimes k+1} \otimes TM^{\otimes l}) ,$$

and we introduce the shorthand notation (in coordinates)

$$\nabla_{i_1 \dots i_k} T = \nabla_{i_1} \nabla_{i_2} \dots \nabla_{i_k} T ,$$

where T is some tensor. The Riemann tensor of g will be denoted by $R^i{}_{jkl}$. We note that for a function ψ, we have $\nabla_{\alpha\bar{\beta}}\psi = \partial_\alpha \partial_{\bar{\beta}} \psi$, so we can define the Laplacians $\Delta, \Delta' : C^\infty(M) \to C^\infty(M)$ by $\Delta\psi := -g^{\alpha\bar{\beta}} \nabla_{\alpha\bar{\beta}}\psi$ and $\Delta'\psi := -g'^{\,\alpha\bar{\beta}} \nabla_{\alpha\bar{\beta}}\psi$ both in terms of the Levi-Civita connection of g. Note that these Laplacians are the $\bar{\partial}$-Laplacians, so half of the usual d-Laplacians. Then we extend the definitions of these Laplacians to $C^k(M)$ for $k \geq 2$. Now we state a proposition which reduces some fourth order derivative to a third order derivative.

Proposition 3.4.6. *In the situation above, we have*

$$\Delta'\Delta\varphi = -\Delta f + g^{\alpha\bar\beta} g'^{\,\gamma\bar\delta} g'^{\,\epsilon\bar\zeta} \nabla_{\alpha\bar\delta\epsilon}\varphi \nabla_{\bar\beta\gamma\bar\zeta}\varphi + g'^{\,\alpha\bar\beta} g^{\gamma\bar\delta}(R^{\bar\epsilon}{}_{\bar\beta\gamma\bar\delta}\nabla_{\alpha\bar\epsilon}\varphi - R^{\bar\epsilon}{}_{\bar\beta\alpha\bar\delta}\nabla_{\gamma\bar\epsilon}\varphi)\,. \tag{3.4.2}$$

Proof. We take the log of the Monge-Ampère equation to get $\log\det(g_{\alpha\bar\beta} + \partial_\alpha\partial_{\bar\beta}\varphi) = f + \log\det g_{\alpha\bar\beta}$. Then we apply $\nabla_{\bar\gamma}$ and use that it is the Levi-Civita connection, i.e. $\nabla g = 0$, to get

$$\nabla_{\bar\gamma} f = \nabla_{\bar\gamma}(\log\det(g_{\alpha\bar\beta} + \partial_\alpha\partial_{\bar\beta}\varphi)) = \mathrm{Tr}_{\alpha\bar\beta}\nabla_{\bar\gamma}(\log(g_{\alpha\bar\beta} + \partial_\alpha\partial_{\bar\beta}\varphi))\,,$$

where we used the well known identity $\mathrm{Tr}\log = \log\det$, and that g commutes with ∇. Now using the formula we know for the derivative of the logarithm of a matrix (and that the trace is cyclic), we get

$$\nabla_{\bar\gamma} f = \mathrm{Tr}((g'_{\alpha\bar\beta})^{-1}\nabla_{\bar\gamma}(g_{\alpha\bar\beta} + \partial_\alpha\partial_{\bar\beta}\varphi)) = g'^{\,\alpha\bar\beta}\nabla_{\bar\gamma}\partial_\alpha\partial_{\bar\beta}\varphi = g'^{\,\alpha\bar\beta}\nabla_{\bar\gamma\alpha\bar\beta}\varphi\,.$$

Therefore,

$$\Delta f = -g^{\gamma\bar\delta}\nabla_\gamma(g'^{\,\alpha\bar\beta})\nabla_{\bar\gamma\alpha\bar\beta}\varphi - g^{\gamma\bar\delta} g'^{\,\alpha\bar\beta}\nabla_{\gamma\bar\delta\alpha\bar\beta}\varphi\,.$$

Now, $g'^{\,\alpha\bar\beta} g'_{\gamma\bar\beta} = \delta^\alpha_\gamma$, so we see

$$0 = \nabla_\gamma \delta^\alpha_\beta = g'_{\beta\bar\delta}\nabla_\gamma g'^{\,\alpha\bar\delta} + g'^{\,\alpha\bar\delta}\nabla_\gamma g'_{\beta\bar\delta} = g'_{\beta\bar\delta}\nabla_\gamma g'^{\,\alpha\bar\delta} + g'^{\,\alpha\bar\delta}\nabla_{\gamma\beta\bar\delta}\varphi\,.$$

So we see $\nabla_\gamma g'^{\,\alpha\bar\beta} = -g'^{\,\delta\bar\beta} g'^{\,\alpha\bar\epsilon}\nabla_{\gamma\bar\epsilon\delta}\varphi$. So combining some things, we see

$$g'^{\,\alpha\bar\beta} g^{\gamma\bar\delta}\nabla_{\gamma\bar\delta\alpha\bar\beta}\varphi = -\Delta f + g^{\alpha\bar\beta} g'^{\,\gamma\bar\delta} g'^{\,\epsilon\bar\zeta}\nabla_{\alpha\bar\delta\epsilon}\varphi\nabla_{\bar\beta\gamma\bar\zeta}\varphi\,.$$

Now we note that we have

$$\nabla_{\alpha\bar\beta\gamma\bar\delta}\varphi = \nabla_\alpha(\partial_{\bar\beta\gamma\bar\delta}\varphi - \Gamma^{\bar\epsilon}{}_{\bar\delta\bar\beta}\partial_{\gamma\bar\epsilon}\varphi) = \partial_{\alpha\bar\beta\gamma\bar\delta}\varphi + \Gamma^\epsilon{}_{\gamma\alpha}\partial_{\bar\beta\epsilon\bar\delta}\varphi - \partial_\alpha(\Gamma^{\bar\epsilon}{}_{\bar\delta\bar\beta}\partial_{\gamma\bar\epsilon}\varphi) - \Gamma^\zeta{}_{\gamma\alpha}\Gamma^{\bar\epsilon}{}_{\bar\delta\bar\beta}\partial_{\zeta\bar\epsilon}\varphi\,.$$

Now, using $R^\alpha{}_{\beta\gamma\bar\delta} = -\partial_{\bar\delta}\Gamma^\alpha{}_{\beta\gamma}$, we see

$$\nabla_{\alpha\bar\beta\gamma\bar\delta}\varphi - \nabla_{\gamma\bar\delta\alpha\bar\beta}\varphi = (R^{\bar\epsilon}{}_{\bar\beta\gamma\bar\delta}\partial_{\alpha\bar\epsilon}\varphi - R^{\bar\epsilon}{}_{\bar\beta\alpha\bar\delta}\partial_{\gamma\bar\epsilon}\varphi)\,.$$

Lastly, we have $\Delta'\Delta\varphi = g'^{\,\alpha\bar\beta} g^{\gamma\bar\delta}\nabla_{\alpha\bar\beta\gamma\bar\delta}\varphi$, so if we put everything together, we see

$$\Delta'\Delta\varphi = -\Delta f + g^{\alpha\bar\beta} g'^{\,\gamma\bar\delta} g'^{\,\epsilon\bar\zeta}\nabla_{\alpha\bar\delta\epsilon}\varphi\nabla_{\bar\beta\gamma\bar\zeta}\varphi + g'^{\,\alpha\bar\beta} g^{\gamma\bar\delta}(R^{\bar\epsilon}{}_{\bar\beta\gamma\bar\delta}\partial_{\alpha\bar\epsilon}\varphi - R^{\bar\epsilon}{}_{\bar\beta\alpha\bar\delta}\partial_{\gamma\bar\epsilon}\varphi)\,,$$

thus proving the proposition. ☐

Now we will turn to finding an estimate for $\|\Delta\varphi\|_{C^0}$ so that we can use Proposition 3.3.7 to find the second order estimate we need.

Proposition 3.4.7. *There is a constant $C_{\Delta\varphi}$ depending only on M, J, g and Q_1 such that $\|\Delta\varphi\|_{C^0} \leq C_{\Delta\varphi}$.*

Proof. We define a function F on M by $F := \log(2m - \Delta\varphi) - \kappa\varphi$ for some real number κ to be determined later. By Proposition 3.3.7, we have that $m - \Delta\varphi > 0$, so this is well defined. We calculate

$$\begin{aligned}\Delta' F &= -g'^{\,\alpha\bar\beta}\nabla_\alpha\nabla_{\bar\beta}\log(2m - \Delta\varphi) - \kappa\Delta'\varphi \\ &= g'^{\,\alpha\bar\beta}\nabla_\alpha((2m - \Delta\varphi)^{-1}\nabla_{\bar\beta}\Delta\varphi) - \kappa\Delta'\varphi \\ &= -(2m - \Delta\varphi)^{-1}\Delta'\Delta\varphi + (2m - \Delta\varphi)^{-2} g'^{\,\alpha\bar\beta}\nabla_\alpha\Delta\varphi\,\nabla_{\bar\beta}\Delta\varphi - \kappa\Delta'\varphi\end{aligned}$$

We note that g and g' define pointwise hermitian metrics on the complex cotangent spaces. Therefore, taking tensor products, we see that $g \otimes g' \otimes g'$ defines a hermitian metric on $T^*M^{\otimes 3}$. Thus, we have the following inequality

$$g^{\alpha\bar{\beta}} g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} [(2m - \Delta\varphi)\nabla_{\alpha\bar{\delta}\epsilon}\varphi - g'_{\alpha\bar{\delta}}\nabla_\epsilon \Delta\varphi][(2m - \Delta\varphi)\nabla_{\bar{\beta}\gamma\bar{\zeta}}\varphi - g'_{\bar{\beta}\gamma}\nabla_{\bar{\zeta}}\Delta\varphi] \geq 0\,,$$

where we note that the Laplacian of a real function is real. Now we note that $g^{\alpha\bar{\beta}} g'_{\alpha\bar{\beta}} = m - \Delta\varphi$, and that $\nabla_{\alpha\beta} = \nabla_{\beta\alpha}$, as the curvature is $(1,1)$. So we see that the above inequality becomes

$$(2m - \Delta\varphi)^2 g^{\alpha\bar{\beta}} g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} \nabla_{\alpha\bar{\delta}\epsilon}\varphi \nabla_{\bar{\beta}\gamma\bar{\zeta}}\varphi - (3m - \Delta\varphi) g'^{\,\alpha\bar{\beta}} \nabla_\alpha \Delta\varphi\, \nabla_{\bar{\beta}}\Delta\varphi \geq 0\,.$$

We note that $g'^{\,\alpha\bar{\beta}}\nabla_\alpha\Delta\varphi\nabla_{\bar{\beta}}\Delta\varphi \geq 0$, so we get

$$-(2m - \Delta\varphi)^{-1} g^{\alpha\bar{\beta}} g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} \nabla_{\alpha\bar{\delta}\epsilon}\varphi \nabla_{\bar{\beta}\gamma\bar{\zeta}}\varphi + (2m - \Delta\varphi)^{-2} g'^{\,\alpha\bar{\beta}} \nabla_\alpha \Delta\varphi\, \nabla_{\bar{\beta}}\Delta\varphi \leq 0\,.$$

Now, $|g'^{\,\alpha\bar{\beta}}|_g \leq g'^{\,\alpha\bar{\beta}} g_{\alpha\bar{\beta}}$ and we see $|\nabla_{\alpha\bar{\beta}}\varphi|_g = |g'_{\alpha\bar{\beta}} - g_{\alpha\bar{\beta}}|_g \leq 2m - \Delta\varphi$. Therefore, there exists a constant C_5 depending only on M, g and m, such that

$$|g'^{\,\alpha\bar{\beta}} g^{\gamma\bar{\delta}} (R^{\bar{\epsilon}}{}_{\bar{\beta}\gamma\bar{\delta}} \nabla_{\alpha\bar{\epsilon}}\varphi - R^{\bar{\epsilon}}{}_{\bar{\beta}\alpha\bar{\delta}} \nabla_{\gamma\bar{\epsilon}}\varphi)| \leq C_5 (2m - \Delta\varphi) g'^{\,\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\,.$$

We also note $|\Delta f| \leq |g^{\alpha\bar{\beta}}|_g |\nabla_{\alpha\bar{\beta}} f|_g \leq mQ_1$. Thus, we see that

$$\Delta' F \leq (2m - \Delta\varphi)^{-1} mQ_1 + \kappa(m - g'^{\,\alpha\bar{\beta}} g_{\alpha\bar{\beta}}) + C_5 g'^{\,\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\,.$$

Since M is compact, there must be a point $x \in M$ where F attains a maximal value, such that $\Delta' F \geq 0$. This means that, at x,

$$(\kappa - C_5) g'^{\,\alpha\bar{\beta}} g_{\alpha\bar{\beta}} \leq (2m - \Delta\varphi)^{-1} mQ_1 + \kappa m\,.$$

Moreover, by Proposition 3.3.7, we have $m - \Delta\varphi \geq m e^{f/m} \geq m e^{-Q_1/m}$. This then gives $(2m - \Delta\varphi)^{-1} \leq \frac{1}{m} e^{Q_1/m}$. Choosing $\kappa = C_5 + 1$ and defining $C_6 := Q_1 e^{Q_1/m} + \kappa m$, we then see $g'^{\,\alpha\bar{\beta}} g_{\alpha\bar{\beta}} \leq C_6$ at x.

Now we pick a frame such that $g_{\alpha\bar{\beta}} = \mathrm{diag}(1, \ldots, 1)$ and $g' = \mathrm{diag}(a_1, \ldots, a_m)$ at x. Lemma 3.3.5 then gives us

$$m - \Delta\varphi = \sum_j a_j\,; \quad g'^{\,\alpha\bar{\beta}} g_{\alpha\bar{\beta}} = \sum_j a_j^{-1} \quad \text{and} \quad \prod_j a_j = e^{f(x)}\,.$$

We obtain

$$2m - \Delta\varphi = m + e^{f(x)} (g'^{\,\alpha\bar{\beta}} g_{\alpha\bar{\beta}})^{m-1} \leq m + e^{Q_1} C_6^{m-1}\,.$$

So at x, we have

$$F(x) \leq \log(m + e^{Q_1} C_6^{m-1}) - \kappa\varphi(x) \leq \log(m + e^{Q_1} C_6^{m-1}) - \kappa \inf\varphi\,.$$

Using the zeroth order estimate $\|\varphi\|_{C^0} \leq Q_2$, we see $\inf\varphi \geq -Q_2$, so we see $F(x) \leq \log(m + e^{Q_1} C_6^{m-1}) + \kappa Q_2$. Since x was the maximum of F, we see that this inequality holds globally. This gives us

$$0 < 2m - \Delta\varphi \leq e^{\log(m + e^{Q_1} C_6^{m-1}) + \kappa Q_2 + \kappa\varphi} \leq (m + C_6^{m-1} e^{Q_1}) e^{2\kappa Q_2}\,.$$

Therefore, choosing $C_{\Delta\varphi} = \max(2m, |2m - (m + C_6^{m-1} e^{Q_1}) e^{2\kappa Q_2}|)$ gives us $\|\Delta\varphi\|_{C^0} \leq C_{\Delta\varphi}$. $\square$

Combining this result with Proposition 3.3.7 gives us the following:

Corollary 3.4.8. *There are constants c_1, c_2, Q_3 depending only on M, g, J and Q_1 such that*

$$\left\|g'_{\alpha\bar{\beta}}\right\|_{C^0} \leq c_1\,; \quad \left\|g'^{\,\alpha\bar{\beta}}\right\|_{C^0} \leq c_2 \quad \textit{and} \quad \|dd^c\varphi\|_{C^0} \leq Q_3\,.$$

Note that the c_1 and the c_2 are ever so slightly different than those in Proposition 3.3.7, as we are only considering $g'^{\,\alpha\bar{\beta}}$, and not $g'^{\,ij}$. This result concludes the section on second order estimates. Now we only have to do the third order estimates.

3.4.3 Estimates of order three

We define a function $S := \frac{1}{4}|\nabla dd^c\varphi|_{g'}$, such that, in local coordinates,

$$S^2 = g'^{\,\alpha\bar{\beta}} g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} \nabla_{\alpha\bar{\delta}\epsilon}\varphi \nabla_{\bar{\beta}\gamma\bar{\zeta}}\varphi\,.$$

We will calculate the Laplacian of this monster, which will be quite the calculation. We introduce shorthand notation $T_{;ij\ldots k} := \nabla_{ij\ldots k}T$ for any tensor T. So $-\Delta' S^2 = g'^{\,\alpha\bar{\beta}} S^2_{;\alpha\bar{\beta}}$. Let's begin

$$\begin{aligned}
-\Delta' S^2 &= g'^{\,\alpha\bar{\beta}} S^2_{;\alpha\bar{\beta}} \\
&= g'^{\,\alpha\bar{\beta}}(g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}} \varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}})_{;\alpha\bar{\beta}} \\
&= g'^{\,\alpha\bar{\beta}}((g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}})_{;\bar{\beta}} \varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}} (\varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}})_{;\bar{\beta}})_{;\alpha}\,.
\end{aligned}$$

We note

$$g'^{\,\alpha\bar{\beta}}{}_{;\gamma} = -g'^{\,\alpha\bar{\delta}} g'^{\,\epsilon\bar{\beta}} g'_{\epsilon\bar{\delta};\gamma} = -g'^{\,\alpha\bar{\delta}} g'^{\,\epsilon\bar{\beta}} \varphi_{;\gamma\bar{\delta}\epsilon}\,.$$

So we see

$$\begin{aligned}
(g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}})_{;\bar{\beta}} &= g'^{\,\gamma\bar{\delta}}{}_{;\bar{\beta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}}{}_{;\bar{\beta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}}{}_{;\bar{\beta}} \\
&= -(g'^{\,\gamma\bar{\iota}} g'^{\,\kappa\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\iota}} g'^{\,\kappa\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\iota}} g'^{\,\kappa\bar{\theta}})\varphi_{;\bar{\beta}\bar{\iota}\kappa}\,.
\end{aligned}$$

Moreover,

$$(\varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}})_{;\bar{\beta}} = \varphi_{;\bar{\beta}\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}} + \varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\bar{\beta}\bar{\delta}\epsilon\bar{\theta}}\,.$$

So we see

$$\begin{aligned}
-\Delta' S^2 = g'^{\,\alpha\bar{\beta}}(&g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}}(\varphi_{;\bar{\beta}\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}} + \varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\bar{\beta}\bar{\delta}\epsilon\bar{\theta}}) \\
&- (g'^{\,\gamma\bar{\iota}} g'^{\,\kappa\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\iota}} g'^{\,\kappa\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\iota}} g'^{\,\kappa\bar{\theta}})\varphi_{;\bar{\beta}\bar{\iota}\kappa}\varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}})_{;\alpha}\,.
\end{aligned}$$

Only one more derivative to take. Using the same reasoning as above, we get

$$\begin{aligned}
-\Delta' S^2 = g'^{\alpha\bar{\beta}}(&-(g'^{\,\gamma\bar{\iota}} g'^{\,\kappa\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\iota}} g'^{\,\kappa\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\iota}} g'^{\,\kappa\bar{\theta}})\varphi_{;\alpha\bar{\iota}\kappa}(\varphi_{;\bar{\beta}\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}} + \varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\bar{\beta}\bar{\delta}\epsilon\bar{\theta}}) \\
&+ g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}}(\varphi_{;\alpha\bar{\beta}\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}} + \varphi_{;\bar{\beta}\gamma\bar{\zeta}\eta}\varphi_{;\alpha\bar{\delta}\epsilon\bar{\theta}} + \varphi_{;\alpha\gamma\bar{\zeta}\eta}\varphi_{;\bar{\beta}\bar{\delta}\epsilon\bar{\theta}} + \varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\alpha\bar{\beta}\bar{\delta}\epsilon\bar{\theta}}) \\
&+ (g'^{\,\gamma\bar{\lambda}} g'^{\,\mu\bar{\iota}} g'^{\,\kappa\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\iota}} g'^{\,\kappa\bar{\lambda}} g'^{\,\mu\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\iota}} g'^{\,\kappa\bar{\delta}} g'^{\,\epsilon\bar{\lambda}} g'^{\,\mu\bar{\zeta}} g'^{\,\eta\bar{\theta}} \\
&+ g'^{\,\gamma\bar{\iota}} g'^{\,\kappa\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\lambda}} g'^{\,\mu\bar{\theta}} + g'^{\,\gamma\bar{\lambda}} g'^{\,\mu\bar{\delta}} g'^{\,\epsilon\bar{\iota}} g'^{\,\kappa\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\lambda}} g'^{\,\mu\bar{\iota}} g'^{\,\kappa\bar{\zeta}} g'^{\,\eta\bar{\theta}} \\
&+ g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\iota}} g'^{\,\kappa\bar{\lambda}} g'^{\,\mu\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\iota}} g'^{\,\kappa\bar{\zeta}} g'^{\,\eta\bar{\lambda}} g'^{\,\mu\bar{\theta}} + g'^{\,\gamma\bar{\lambda}} g'^{\,\mu\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\iota}} g'^{\,\kappa\bar{\theta}} \\
&+ g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\lambda}} g'^{\,\mu\bar{\zeta}} g'^{\,\eta\bar{\iota}} g'^{\,\kappa\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\lambda}} g'^{\,\mu\bar{\iota}} g'^{\,\kappa\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\iota}} g'^{\,\kappa\bar{\lambda}} g'^{\,\mu\bar{\theta}})\varphi_{;\alpha\bar{\lambda}\mu}\varphi_{;\bar{\beta}\bar{\iota}\kappa}\varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}} \\
&- (g'^{\,\gamma\bar{\iota}} g'^{\,\kappa\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\iota}} g'^{\,\kappa\bar{\zeta}} g'^{\,\eta\bar{\theta}} + g'^{\,\gamma\bar{\delta}} g'^{\,\epsilon\bar{\zeta}} g'^{\,\eta\bar{\iota}} g'^{\,\kappa\bar{\theta}})(\varphi_{;\alpha\bar{\beta}\bar{\iota}\kappa}\varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}} \\
&+ \varphi_{;\bar{\beta}\bar{\iota}\kappa}\varphi_{;\alpha\gamma\bar{\zeta}\eta}\varphi_{;\bar{\delta}\epsilon\bar{\theta}} + \varphi_{;\bar{\beta}\bar{\iota}\kappa}\varphi_{;\gamma\bar{\zeta}\eta}\varphi_{;\alpha\bar{\delta}\epsilon\bar{\theta}}))\,.
\end{aligned}$$

After some more work it can be shown that the above equation "simplifies" to (see e.g. [Aub70; Yau78])

$$
\begin{aligned}
-\Delta' S^2 = {} & g'^{\,\alpha\bar\beta} g'^{\,\gamma\bar\delta} g'^{\,\epsilon\bar\zeta} g'^{\,\eta\bar\theta} ([\varphi_{;\bar\beta\gamma\bar\zeta\eta} - g'^{\,\iota\bar\kappa}\varphi_{;\bar\beta\iota\bar\zeta}\varphi_{;\gamma\bar\kappa\eta}][\varphi_{;\alpha\bar\delta\epsilon\bar\theta} - g'^{\,\iota\bar\kappa}\varphi_{;\alpha\bar\kappa\eta}\varphi_{;\bar\delta\iota\bar\theta}] \\
& + [\varphi_{;\alpha\gamma\bar\zeta\eta} - g'^{\,\iota\bar\kappa}\varphi_{;\alpha\bar\zeta\iota}\varphi_{;\gamma\bar\kappa\eta} - g'^{\,\iota\bar\kappa}\varphi_{;\alpha\bar\kappa\eta}\varphi_{;\iota\bar\zeta\gamma}][\varphi_{;\bar\beta\bar\delta\epsilon\bar\theta} - g'^{\,\iota\bar\kappa}\varphi_{;\bar\beta\epsilon\bar\kappa}\varphi_{;\bar\delta\iota\bar\theta} - g'^{\,\iota\bar\kappa}\varphi_{;\bar\beta\iota\bar\theta}\varphi_{;\bar\kappa\epsilon\bar\delta}]) \\
& - g'^{\,\alpha\bar\beta}(2g'^{\,\gamma\bar\delta} g'^{\,\epsilon\bar\zeta} g'^{\,\eta\bar\theta} + g'^{\,\gamma\bar\zeta} g'^{\,\eta\bar\delta} g'^{\,\epsilon\bar\theta})\varphi_{;\gamma\bar\theta\alpha}\varphi_{;\bar\zeta\eta\bar\beta}[f_{;\epsilon\bar\delta} - R_{\epsilon\bar\delta}] \\
& + g'^{\,\alpha\bar\beta} g'^{\,\gamma\bar\delta} g'^{\,\epsilon\bar\zeta}[\varphi_{;\bar\beta\alpha\bar\zeta} f_{;\gamma\bar\delta\epsilon} + \varphi_{;\gamma\bar\delta\epsilon} f_{;\bar\beta\alpha\bar\zeta}] \\
& + g'^{\,\alpha\bar\beta} g'^{\,\gamma\bar\delta} g'^{\,\epsilon\bar\zeta} g'^{\,\eta\bar\theta}[\varphi_{;\bar\delta\gamma\bar\theta}(R^{\iota}{}_{\eta\alpha\bar\zeta}\varphi_{;\gamma\bar\beta\iota} + R^{\bar\kappa}{}_{\bar\zeta\bar\beta\gamma}\varphi_{;\alpha\bar\kappa\eta} + R^{\iota}{}_{\eta\bar\beta\gamma}\varphi_{;\alpha\bar\zeta\iota}) \\
& + \varphi_{;\gamma\bar\delta\eta}(R^{\bar\kappa}{}_{\bar\theta\bar\beta\epsilon}\varphi_{;\bar\delta\alpha\bar\kappa} + R^{\iota}{}_{\epsilon\alpha\bar\delta}\varphi_{;\bar\beta\iota\bar\theta} + R^{\bar\kappa}{}_{\bar\theta\alpha\bar\delta}\varphi_{;\bar\beta\epsilon\bar\kappa})] \\
& + g'^{\,\alpha\bar\beta} g'^{\,\gamma\bar\delta}[\varphi_{;\bar\beta\epsilon\bar\delta}(g'^{\,\eta\bar\zeta} R^{\epsilon}{}_{\gamma\bar\zeta\alpha;\eta} - g'^{\,\epsilon\bar\zeta} R_{\gamma\bar\zeta;\alpha}) + \varphi_{;\alpha\bar\zeta\gamma}(g'^{\epsilon\bar\theta} R^{\bar\zeta}{}_{\bar\delta\epsilon\bar\beta;\bar\theta} - g'^{\,\epsilon\bar\zeta} R_{\bar\delta\epsilon;\bar\beta})]\,.
\end{aligned}
$$

Inspecting the above, we see that the terms carrying a fourth derivative in φ are both in the first two lines of the right hand side. However, these lines are both nonnegative, as they are the g' norm of fourth order tensors. Moreover, we already have bounds in terms of Q_1, c_2 and $\left\|R^i{}_{jkl}\right\|_{C^1}$ for the g norm of everything else on the right hand side, except for $|\varphi_{;\alpha\bar\beta\gamma}|_g$ and $|\varphi_{;\alpha\bar\beta\gamma}|_g^2$. So we see that there is a constant $C_{\Delta' S^2}$ depending only on Q_1, c_2 and $\left\|R^i{}_{jkl}\right\|_{C^1}$, such that

$$\Delta' S^2 \leq C_{\Delta' S^2}(|\varphi_{;\alpha\bar\beta\gamma}|_g + |\varphi_{;\alpha\bar\beta\gamma}|_g^2)\,.$$

We note

$$|\varphi_{;\alpha\bar\beta\gamma}|_g^2 \leq |g^{\alpha\bar\beta}|_{g'}^6 S^2 \leq c_1^6 S^2\,. \tag{3.4.3}$$

Therefore, we have the following

Lemma 3.4.9. *There exists a constant C_7 depending only on Q_1, c_1, c_2 and $\left\|R^i{}_{jkl}\right\|_{C^1}$ such that*

$$\Delta' S^2 \leq C_7(S^2 + S)\,.$$

Moreover, (3.4.3) tells us that finding an a priori estimate for S^2 is enough to give us an a priori estimate for $\|\nabla dd^c\varphi\|_{C^0}$. Therefore, we just need to prove the following:

Proposition 3.4.10. *There is a constant C_{S^2} depending only on M, g, J and Q_1 such that*

$$\left\|S^2\right\|_{C^0} \leq C_{S^2}\,.$$

Proof. The first thing we note is that

$$S^2 = |\varphi_{;\alpha\bar\beta\gamma}|_{g'}^2 \leq |g'^{\,\alpha\bar\beta} g'^{\,\gamma\bar\delta} g'^{\,\epsilon\bar\zeta}|_{g\otimes g'\otimes g'}\, g^{\alpha\bar\beta} g'^{\,\gamma\bar\delta} g'^{\,\epsilon\bar\zeta} \varphi_{;\alpha\bar\delta\epsilon}\varphi_{;\bar\beta\gamma\bar\zeta} \leq mc_2\, g^{\alpha\bar\beta} g'^{\,\gamma\bar\delta} g'^{\,\epsilon\bar\zeta} \varphi_{;\alpha\bar\delta\epsilon}\varphi_{;\bar\beta\gamma\bar\zeta}\,.$$

So by Proposition 3.4.6, we see that there exists a constant C_8 depending only on M, g, J and Q_1 such that

$$\Delta'\Delta\varphi \geq c_2^{-1} m^{-1} S^2 - C_8\,.$$

This means that

$$
\begin{aligned}
\Delta'(S^2 - 2c_2 m C_7 \Delta\varphi) &\leq C_7(S^2 + S) - 2c_2 m C_7(c_2^{-1} m^{-1} S^2 - C_8) \\
&= -C_7(S - \tfrac{1}{2})^2 + 2c_2 m C_7 C_8 + \tfrac{1}{4} C_7\,.
\end{aligned}
$$

Again, at a maximum $x \in M$ of $S^2 - 2c_2 m C_7 \Delta\varphi$, we see that the LHS is strictly positive, so at x,

$$(S - \tfrac{1}{2})^2 \leq 2c_2 m C_8 + \tfrac{1}{4}\,.$$

So there is a constant C_9 depending only on c_2, C_8 and the a priori estimate for $\|\Delta\varphi\|_{C^0}$ we found earlier, such that $S^2 - 2c_2 m C_7 \Delta\varphi \leq C_9$. Because x was a maximum, this inequality holds globally. Moreover, S^2 is nonnegative. So we can apply the estimate for $\|\Delta\varphi\|_{C^0}$ again to obtain the a priori estimate C_{S^2} for $\|S^2\|_{C^0}$ that depends only on C_9, C_7, c_2 and $\|\Delta\varphi\|_{C^0}$, therefore, it depends only on M, g, J and Q_1, which concludes the proof of the proposition. □

Corollary 3.4.11. *There is a constant Q_4 depending only on M, g, J and Q_1 such that*

$$\|\nabla dd^c\varphi\|_{C^0} \leq Q_4\,.$$

This concludes the proof of Theorem CY1.

3.5 Proof of CY2

In this section, we will prove Theorem CY2, which is restated below. In the rest of this section, we will always assume we are in the setting of this theorem.

Theorem CY2. Let (M, g) be a compact Kähler manifold with Kähler form ω and complex structure J. Let $Q_1, Q_2, Q_3, Q_4 \geq 0$ and $\alpha \in (0,1)$. Then there is a $Q_5 \geq 0$ depending only on $M, g, J, Q_1, Q_2, Q_3, Q_4$ and α such that the following holds.

Suppose $f \in C^{(3,\alpha)}(M)$, $\varphi \in C^5(M)$ and $A > 0$ satisfy $(\omega + dd^c\varphi)^m = Ae^f\omega^m$ and the inequalities

$$\|f\|_{C^{(3,\alpha)}} \leq Q_1\,, \quad \|\varphi\|_{C^0} \leq Q_2\,, \quad \|dd^c\varphi\|_{C^0} \leq Q_3\,, \quad \|\nabla dd^c\varphi\|_{C^0} \leq Q_4\,.$$

Then $\varphi \in C^{(5,\alpha)}(M)$ and $\|\varphi\|_{C^{(5,\alpha)}} \leq Q_5$. Moreover, if $f \in C^{(k,\alpha)}(M)$ for some $k \geq 3$, then $\varphi \in C^{(k+2,\alpha)}(M)$, and if f is smooth, so is φ.

The proof of this theorem will be done using a method called *bootstrapping* [Joy00], the idea is that, if $f \in C^{(k,\alpha)}(M)$, we produce a method to find estimates for $\|\varphi\|_{C^{(l,\alpha)}}$ depending on $\|f\|_{C^{(l-2,\alpha)}}$, $\|\varphi\|_{C^{(l-1,\alpha)}}$ and $l, \alpha, M, g, J, Q_1, \ldots, Q_4$, which we can inductively apply to find a bound for $\|\varphi\|_{C^{(k+2,\alpha)}}$, which will then only depend on $k, \alpha, M, g, j, Q_1, \ldots, Q_4$ and $\|f\|_{C^{(k,\alpha)}}$.

For the proof, we will need the following three lemmas from functional analysis. The first and the third will follow from Schauder estimates, see Theorem 2.1.23, the proof for the second one can be found in [Joy00].

Lemma 3.5.1. *Let $k \geq 0$ be an integer and let $\alpha \in (0,1)$. Then there exists a constant $E_{k,\alpha}$ depending only on M, g, k and α such that if $\psi \in C^2(M)$ and $\xi \in C^{(k,\alpha)}(M)$ such that $\Delta\psi = \xi$, then $\psi \in C^{(k+2,\alpha)}(M)$ and $\|\psi\|_{C^{(k+2,\alpha)}} \leq E_{k,\alpha}(\|\xi\|_{C^{(k,\alpha)}} + \|\psi\|_{C^0})$.*

Lemma 3.5.2. *Let $\alpha \in (0,1)$, then there is a constant E'_α depending only on M, g, α, $\left\|g'_{ij}\right\|_{C^0}$ and $\left\|g'^{\,ij}\right\|_{C^{(0,\alpha)}}$ such that if $\psi \in C^2(M)$ and $\xi \in C^0(M)$ such that $\Delta'\psi = \xi$, then $\psi \in C^{(1,\alpha)}(M)$ and $\|\psi\|_{C^{(1,\alpha)}} \leq E'_\alpha(\|\xi\|_{C^0} + \|\psi\|_{C^0})$.*

Lemma 3.5.3. *Let $k \geq 0$ be an integer and let $\alpha \in (0,1)$. Then there is a constant $E'_{k,\alpha}$ depending only on $M, g, k, \alpha, \left\|g'_{ij}\right\|_{C^0}$ and $\|g'^{\,ij}\|_{C^{(k,\alpha)}}$ such that for any $\psi \in C^2(M)$ and $\xi \in C^0(M)$ such that $\Delta'\psi = \xi$, we have $\psi \in C^{(k+2,\alpha)}(M)$ and $\|\psi\|_{C^{(k+2,\alpha)}} \leq E'_{k,\alpha}(\|\xi\|_{C^{(k,\alpha)}} + \|\psi\|_{C^0})$.*

As explained before, the problem will be solved inductively, so we start with a base case

Proposition 3.5.4. *There is a constant D_1 depending on $M, g, J, Q_1, \dots, Q_4$ and α such that $\|\varphi\|_{C^{(3,\alpha)}} \leq D_1$.*

Proof. In this proof, all estimates will be only depending on $M, g, J, Q_1, \dots, Q_4$ and α. By Proposition 3.3.7, we can find estimates c_1 and c_2 for $\left\|g'_{ij}\right\|_{C^0}$ and $\|g'^{\,ij}\|_{C^0}$, respectively. Moreover, we see $\nabla g'^{\,ij} = -g'^{\,ik}g'^{\,lj}\nabla g'_{kl} = -g'^{\,ik}g'^{\,lj}\nabla(dd^c\varphi(-,J-))_{lk}$, i.e. we can find an estimate for $\|\nabla g'^{\,ij}\|_{C^0}$ using J and the estimates for $\|g'^{\,ij}\|_{C^0}$ and $\|\nabla dd^c\varphi\|_{C^0}$. Using the estimates for $\|g'^{\,ij}\|_{C^0}$ and $\|\nabla g'^{\,ij}\|_{C^0}$, we can then find an estimate for $\|g'^{\,ij}\|_{C^{(0,\alpha)}}$.

This puts us in the setting of Lemma 3.5.2, i.e. we have an estimate E'_α such that $\|\Delta\varphi\|_{C^{(1,\alpha)}} \leq E'_\alpha(\|\Delta'\Delta\varphi\|_{C^0} + \|\Delta\varphi\|_{C^0})$. Moreover, we can apply Proposition 3.4.6 to find an estimate D_2 for $\|\Delta\Delta'\varphi\|_{C^0}$. Thus, we see $\|\Delta\varphi\|_{C^{(1,\alpha)}} \leq E'_\alpha(D_2 + \frac{m}{2}Q_3)$, where we used $\|\Delta\varphi\|_{C^0} = \left\|g^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}\varphi\right\|_{C^0} \leq \frac{m}{2}\|dd^c\varphi\|_{C^0}$.

Now we are in the setting of Lemma 3.5.1, so we have an estimate $E_{1,\alpha}$ such that

$$\|\varphi\|_{C^{(3,\alpha)}} \leq E_{1,\alpha}(\|\Delta\varphi\|_{C^{(1,\alpha)}} + \|\varphi\|_{C^0}) \leq E_{1,\alpha}(E'_\alpha(D_2 + \tfrac{m}{2}Q_3) + Q_2) =: D_1\,,$$

completing the proof. □

Now we proceed to the induction step

Proposition 3.5.5. *Let $k \geq 2$ and suppose $f \in C^{(k,\alpha)}(M)$, $\varphi \in C^{(k+1,\alpha)}(M)$, such that we have an a priori estimate D for $\|\varphi\|_{C^{(k+1,\alpha)}}$ depending only on $M, g, J, Q_1, \dots, Q_4, k, \alpha, \|f\|_{C^{(k-1,\alpha)}}$ and $\|\varphi\|_{C^{(k,\alpha)}}$. Then $\varphi \in C^{(k+2,\alpha)}(M)$ and we can find an a priori estimate D' for $\|\varphi\|_{C^{(k+2,\alpha)}}$ depending only on $M, g, J, Q_1, \dots, Q_4, k, \alpha, \|f\|_{C^{(k,\alpha)}}$ and D.*

Proof. We will start by estimating $\|\Delta'\Delta\varphi\|_{C^{(k-2,\alpha)}}$ using Proposition 3.4.6, which tells us

$$\Delta'\Delta\varphi = -\Delta f + g^{\alpha\bar{\beta}}g'^{\,\gamma\bar{\delta}}g'^{\,\epsilon\bar{\zeta}}\nabla_{\alpha\bar{\delta}\epsilon}\varphi\nabla_{\bar{\beta}\gamma\bar{\zeta}}\varphi + g'^{\,\alpha\bar{\beta}}g^{\gamma\bar{\delta}}(R^{\bar{\epsilon}}{}_{\bar{\beta}\gamma\bar{\delta}}\nabla_{\alpha\bar{\epsilon}}\varphi - R^{\bar{\epsilon}}{}_{\bar{\beta}\alpha\bar{\delta}}\nabla_{\gamma\bar{\epsilon}}\varphi)\,. \tag{3.5.1}$$

Note that $\|\Delta f\|_{C^{(k-2,\alpha)}}$ can be estimated by $m\|f\|_{C^{(k,\alpha)}}$. Moreover, since derivatives of $g'^{\,\alpha\bar{\beta}}$ can be computed using derivatives of φ, we see that the $C^{(k-2,\alpha)}$-norm of all other terms on the right hand side can be estimated in terms of $\|g'^{\,ij}\|_{C^0}$, $\|\varphi\|_{C^{(k+1,\alpha)}}$ and M, g, J. Therefore, we can estimate $\|\Delta'\Delta\varphi\|_{C^{(k-2,\alpha)}}$ in terms of $\|f\|_{C^{(k,\alpha)}}$, $\|g'^{\,ij}\|_{C^0}$, $\|\varphi\|_{C^{(k+1,\alpha)}}$ and M, g, J. Moreover, using Proposition 3.3.7, we can find an estimate for $\|g'^{\,ij}\|_{C^0}$ in terms of $\|\varphi\|_{C^{(2,\alpha)}}$, so we can find an estimate $F_{k,\alpha}$ for $\|\Delta'\Delta\varphi\|_{C^{(k-2,\alpha)}}$ in terms of $\|f\|_{C^{(k,\alpha)}}$, $\|\varphi\|_{C^{(k+1,\alpha)}}$ and M, g, J, as desired.

Now we are in the setting of Lemma 3.5.3, we have $\Delta\varphi \in C^{(k,\alpha)}(M)$ and we have an estimate $E'_{k-2,\alpha}$ depending only on M, g, k and α such that

$$\|\Delta\varphi\|_{C^{(k,\alpha)}} \leq E'_{k-2,\alpha}(\left\|\Delta'\Delta\varphi\right\|_{C^{(k-2,\alpha)}} + \|\Delta\varphi\|_{C^0}) \leq E'_{k-2,\alpha}(F_{k,\alpha} + \tfrac{m}{2}Q_3)\,.$$

Now we are in the setting of Lemma 3.5.1, so $\varphi \in C^{(k+2,\alpha)}(M)$ and we have an estimate $E_{k,\alpha}$ depending only on M, g, k and α such that

$$\|\varphi\|_{C^{(k+2,\alpha)}} \leq E_{k,\alpha}(\|\Delta\varphi\|_{C^{(k,\alpha)}} + \|\varphi\|_{C^0}) \leq E_{k,\alpha}(E'_{k-2,\alpha}(F_{k,\alpha} + Q_3) + Q_2) =: D'\,,$$

completing the proof. $\square$

Thus, combining the base case and the induction step, we see that if $f \in C^{(k,\alpha)}(M)$, we have $\varphi \in C^{(k+2,\alpha)}(M)$, and we can find an a priori estimate for $\|\varphi\|_{C^{(k+2,\alpha)}}$ in terms of $M, g, J, Q_1, \dots, Q_4, k, \alpha$ and $\|f\|_{C^{(k,\alpha)}}$. *A fortiori*, if $f \in C^{(3,\alpha)}$, we see that $\varphi \in C^{(5,\alpha)}(M)$, and we have an a priori estimate Q_5 depending only on $M, g, J, Q_1, \dots, Q_4$ and α such that $\|\varphi\|_{C^{(5,\alpha)}} \leq Q_5$. Moreover, if f is smooth, it is $C^{(k,\alpha)}$ for any k, so that φ is $C^{(k,\alpha)}$ for any k, and we see that φ is also smooth, completing the proof of CY2.

3.6 Proof of CY3

In this section, we discuss the proof of CY3, which is restated below. The proof requires some tools from functional analysis on Banach spaces.

Theorem CY3. Let (M, g) be a Kähler manifold with Kähler form ω. Let $\alpha \in (0,1)$ and suppose $f' \in C^{(3,\alpha)}(M)$, $\varphi' \in C^{(5,\alpha)}(M)$ and $A' > 0$ satisfy

$$\int_M \varphi' \operatorname{vol}_g = 0\,; \quad (\omega + dd^c\varphi')^m = A' e^{f'} \omega^m\,.$$

Then whenever $f \in C^{(3,\alpha)}(M)$ is such that $\|f - f'\|_{C^{(3,\alpha)}}$ is sufficiently small, then there is a $\varphi \in C^{(5,\alpha)}(M)$ and an $A > 0$ such that

$$\int_M \varphi \operatorname{vol}_g = 0\,; \quad (\omega + dd^c\varphi)^m = A e^{f} \omega^m\,.$$

Proof. Define $X := \{\varphi \in C^{(5,\alpha)}(M) : \int_M \varphi \operatorname{vol}_g = 0\}$, this is a closed linear subspace of $C^{(5,\alpha)}(M)$ and we equip it with the subspace topology. Now define $U \subseteq X$ by $U = \{\varphi \in X : \omega + dd^c\varphi \text{ is positive}\}$. Since positivity is an open condition, we see that U is open in X. Now suppose $\varphi \in U$ and $a \in \mathbb{R}$, then because $\omega + dd^c\varphi$ is positive, we see that $(\omega + dd^c\varphi)^m$ is a positive multiple of ω^m at every point, so there is a unique real valued function f such that $(\omega + dd^c\varphi)^m = e^{a+f}\omega^m$. Since $\varphi \in C^{(5,\alpha)}(M)$, we see $f \in C^{(3,\alpha)}(M)$.

Using this, we can define a continuous map of Banach manifolds $F : U \times \mathbb{R} \to C^{(3,\alpha)}(M)$, by $F(\varphi, a) = f$, such that $(\omega + dd^c\varphi)^m = e^{a+f}\omega^m$. We will show that F is differentiable. To do this, let ε be a real number. We see that for any $\psi \in X$, $(\varphi, a) \in U \times \mathbb{R}$,

$$(\omega + dd^c(\varphi + \varepsilon\psi))^m = (\omega + dd^c\varphi)^{m-1} \wedge (\omega + dd^c\varphi + m\varepsilon dd^c\psi + O(\varepsilon^2))\,.$$

Now, defining $\omega' := \omega + dd^c\varphi$ and letting g' be the associated Kähler metric with Hodge star $\star'$, we compute

$$dd^c\psi \wedge (\omega')^{m-1} = \frac{(m-1)!}{2} dd^c\psi \wedge \star'\omega' = \frac{1}{2m}(dd^c\psi, \omega')_{g'}(\omega')^m = -\frac{1}{m}\Delta'\psi(\omega')^m\,,$$

where $\Delta' = -g'^{\alpha\bar{\beta}}\nabla_{\alpha\bar{\beta}}$. Thus,

$$(\omega + dd^c(\varphi + \varepsilon\psi))^m = (\omega + dd^c\varphi)^m(1 - \varepsilon\Delta'\psi + O(\varepsilon^2)) = e^{a+f-\varepsilon\Delta'\psi+O(\varepsilon^2)}\omega^m\,,$$

where $f := F(\varphi, a)$. So now looking at the linear term in ε, we find the derivative of F

$$dF_{(\varphi,a)} : X \times \mathbb{R} \to C^{(3,\alpha)}(M)\,, \quad dF_{(\varphi,a)}(\psi, b) = -b - \Delta'\psi\,. \tag{3.6.1}$$

We see that Δ' is a linear elliptic differential operator with $C^{(3,\alpha)}$-coefficients. Moreover, M is connected, so $\ker \Delta'$ are the constant functions. We compute the dual of Δ' with respect to g, so let $\psi_1 \in C^\infty(M)$ and $\psi_2 \in C^\infty(M)$. We see

$$\begin{aligned}
\langle \Delta'\psi_1, \psi_2\rangle &= \int (\Delta'\psi_1, \psi_2) \operatorname{vol}_g \\
&= \int (\Delta'\psi_1, \psi_2)(e^{a+f})^{-1} \operatorname{vol}_{g'} \\
&= \int (\psi_1, \Delta'(e^{-f}\psi_2))e^{-a} \operatorname{vol}_{g'} \\
&= \int (\psi_1, e^f \Delta'(e^{-f})\psi_2) \operatorname{vol}_g \\
&= \langle \psi_1, e^f \Delta'(e^{-f}\psi_2)\rangle ,
\end{aligned}$$

where we used self-adjointness of the Laplacian. Thus we see $(\Delta')^* = e^f \Delta' e^{-f}$, i.e. $\ker(\Delta')^*$ consists of scalar multiples of e^f. In particular, $\psi \perp \ker \Delta'$ if and only if $\int_M \psi \operatorname{vol}_g = 0$, and $\psi \perp \ker(\Delta')^*$ if and only if $\langle \psi, e^{-f}\rangle_g = 0$.

Thus, by Theorem 2.1.25, for $\chi \in C^{(3,\alpha)}(M)$, there is a $\psi \in C^{(5,\alpha)}(M)$ such that $\Delta'\psi = \chi$ if and only if $\langle \chi, e^{-f}\rangle_g = 0$. By the same theorem, ψ is unique up to a constant, so there is a unique on such that $\int_M \psi \operatorname{vol}_g = 0$. Moreover, for any $\chi \in C^{(3,\alpha)}(M)$, there is a unique $b \in \mathbb{R}$ such that $\langle \chi + b, e^{-f}\rangle = 0$, as $\int_M e^{-f} \operatorname{vol}_g$ is nonzero.

Putting everything together, we see that for any $\chi \in C^{(3,\alpha)}(M)$, there is a unique $b \in \mathbb{R}$ and a unique $\psi \in C^{(5,\alpha)}(M)$ such that $\Delta'\psi = -\chi - b$. Therefore, there is a unique $(\psi, b) \in X \times \mathbb{R}$ such that $dF_{(\varphi,a)}(\psi, b) = \chi$, which means $dF_{(\varphi,a)}$ is invertible at any point in $U \times \mathbb{R}$. Therefore, by the Inverse Mapping Theorem for Banach spaces [Joy00], there is an open neighborhood $U' \subseteq U \times \mathbb{R}$ of (φ', a') and an open neighborhood $V' \subseteq C^{(3,\alpha)}(M)$ of f' such that $F|_{U'}$ is a homeomorphism onto V'.

Because the topology on $C^{(3,\alpha)}(M)$ is induced by $\|-\|_{C^{(3,\alpha)}}$, we see that this means that there is an $\varepsilon > 0$ such that $\|f - f'\|_{C^{(3,\alpha)}} < \varepsilon$ implies that there is a $\varphi \in C^{(5,\alpha)}(M)$ and an $A > 0$ such that

$$\int_M \varphi \operatorname{vol}_g = 0 ; \quad (\omega + dd^c\varphi)^m = Ae^f\omega^m ,$$

thus completing the proof. $\square$

3.7 Proof of CY4

Finally, we prove CY4. This proof is by Calabi in 1954 [Cal54]. The proof is the shortest of the four, but still requires a bit of work and a few tricks. We start by stating the theorem again.

Theorem CY4. Let (M, g) be a Kähler manifold with Kähler form ω. Let $f \in C^1(M)$, and let $A > 0$ be some real number, then there is at most one $\varphi \in C^3(M)$ such that $\int_M \varphi \operatorname{vol}_g = 0$ and $(\omega + dd^c\varphi)^m = Ae^f\omega^m$.

Proof. Suppose φ_1 and φ_2 are C^3 solutions. We shall prove that their difference is d-closed, because then we know that their difference must be constant (as we assumed M is connected), so then the normalisation condition will tell us that they're equal.

We define $\omega_i := \omega + dd^c\varphi_i$ for $i = 1, 2$, by Lemma 3.1.4, these are both positive $(1,1)$-forms. Thus they define Kähler metrics g_1 and g_2, respectively. We have $\omega_1^m = \omega_2^m$, and $\omega_1 - \omega_2 = dd^c(\varphi_1 - \varphi_2)$. We can relate the two by noting that

$$\begin{aligned} 0 &= \omega_1^m - \omega_2^m \\ &= (\omega_1 - \omega_2) \wedge (\omega_1^{m-1} + \omega_1^{m-2} \wedge \omega_2 + \cdots + \omega_2^{m-1}) \\ &= dd^c(\varphi_1 - \varphi_2) \wedge (\omega_1^{m-1} + \omega_1^{m-2} \wedge \omega_2 + \cdots + \omega_2^{m-1}), \end{aligned}$$

where it might be helpful to note that these are all two-forms, so the wedge product is commutative, so there's no antisymmetry involved. Because M is compact, Stokes' tells us

$$\int_M d\left((\varphi_1 - \varphi_2) d^c(\varphi_1 - \varphi_2) \wedge (\omega_1^{m-1} + \omega_1^{m-2} \wedge \omega_2 + \cdots + \omega_2^{m-1})\right) = 0.$$

Combining the two and using $d\omega_1 = d\omega_2 = 0$, we see

$$\int_M d(\varphi_1 - \varphi_2) \wedge d^c(\varphi_1 - \varphi_2) \wedge (\omega_1^{m-1} + \omega_1^{m-2} \wedge \omega_2 + \cdots + \omega_2^{m-1}) = 0.$$

By Lemma 3.3.3, we see that all terms in the integrand must vanish separately, as they are all nonnegative top forms. In particular, the first term gives us

$$|d(\varphi_1 - \varphi_2)|_{g_1}^2 = 0,$$

i.e. $d(\varphi_1 - \varphi_2) = 0$, so $\varphi_1 - \varphi_2$ is constant. Using the normalisation condition $\int_M \varphi_i \operatorname{vol}_g = 0$, we see $\varphi_1 = \varphi_2$, thus proving the theorem. $\square$

4 Calabi's construction

So far, we have proved the Calabi-Yau theorem, which does not offer a lot of insight into how to actually find explicit examples of Calabi-Yau metrics on compact manifolds, which requires us to solve the differential equation popping up in Theorem 3.1.5. Unfortunately, explicitly solving this equation turns out to be very difficult. In fact, there are very few known explicit nontrivial examples of Calabi-Yau metrics [Joy00]. As it turns out, going to a non-compact setting makes it slightly easier to find examples, as we shall see in this chapter.

We will discuss a known explicit example of a Ricci-flat Kähler metric on the canonical bundle of $\mathbb{C}P^n$. The approach will be quite brute-force and was invented by Calabi [Cal79]. The idea is to start with a Kähler-Einstein manifold M, e.g. $\mathbb{C}P^n$ with the Fubini-Study metric, and take a holomorphic line bundle $L \to M$ with a constant curvature hermitian metric h that has curvature opposite to the curvature of M, e.g. $K_{\mathbb{C}P^n} \to \mathbb{C}P^n$. Then cancel both curvatures by "pushing it to ∞" to obtain a complete Ricci-flat Kähler metric on the total space of L. This is done by modifying the Kähler potential of a canonical Kähler metric we introduce on L.

This method allows us to find explicit Calabi-Yau metrics on the total space of $K_{\mathbb{C}P^n}$. The Calabi-Yau metric on $K_{\mathbb{C}P^1} \cong T^*\mathbb{C}P^1$ that we produce in this way was found independently from Calabi by T. Eguchi and A.J. Hanson in 1979 [EH79].

This construction will mostly require the tools introduced in Sections 2.2 and 2.3.

4.1 Kähler-Einstein metrics on the total space of line bundles

Kähler potentials will play a central role in this chapter, the method we shall describe will be one to find Kähler potentials belonging to Kähler-Einstein metrics. However, while it is true that any Kähler form can be written as $dd^c\varphi$ for some real valued function φ, it is not true that $dd^c\varphi$ necessarily defines a Kähler form, thus after solving these differential equations, we have to take extra care that the resulting form is, in fact, positive. To make the distinction more precise, we define forms that are of *Kähler type*.

Definition 4.1.1 (Kähler type form)**.** A $(1,1)$-form ω is of *Kähler type* if and only if it is real and closed.

Now we see that Kähler forms are precisely the Kähler type forms that are positive. Moreover, $dd^c\varphi$ is always of Kähler type if φ is real. Conversely, by the Poincaré lemma, any Kähler type form can be written as $dd^c\varphi$ for some real function φ.

Now let $\pi : E \to M$ be a holomorphic hermitian vector bundle of rank k with hermitian metric h, over a (not necessarily compact) Kähler manifold (M,g) of dimension m. Let ∇ denote the Chern connection of h. Pick a holomorphic trivialising chart $(z^1,\dots,z^m;e_1,\dots,e_k)$, where e_λ are linear independent local holomorphic sections, let $h_{\lambda\bar\mu} := h(e_\lambda,\bar e_{\bar\mu})$ and let $h^{\lambda\bar\mu}$ denote the coordinates of the inverse matrix, such that $h^{\lambda\bar\mu}h_{\lambda\bar\nu} = \delta^{\bar\mu}_{\bar\nu}$. Let $A^\lambda{}_{\bar\mu\alpha}$ denote the corresponding connection matrix and let $K^\lambda{}_{\mu\alpha\bar\beta}$ be the curvature form. From Section 2.2, we have the following formulas

$$A^\lambda{}_{\mu\alpha} = h^{\lambda\bar\nu}\partial_\alpha h_{\mu\bar\nu}\,; \tag{4.1.1}$$

$$K^\lambda{}_{\mu\alpha\bar\beta} = -\partial_{\bar\beta}A^\lambda{}_{\mu\alpha} = h^{\lambda\bar\nu}h^{\rho\bar\sigma}\partial_{\bar\beta}h_{\rho\bar\nu}\partial_\alpha h_{\mu\bar\sigma} - h^{\lambda\bar\nu}\partial_\alpha\partial_{\bar\beta}h_{\mu\bar\nu}\,. \tag{4.1.2}$$

The idea is to find Kähler forms on the total space of E, so we need to work on T^*E eventually. Before we jump to that, we will define a slightly more convenient frame for TE than just $(\partial_{z^\alpha};\partial_{\zeta^\lambda})$, where ζ^λ are the coordinate functions corresponding to the frame $(e_1,\dots,e_k)$. Instead, we define a frame consisting of horizontal lifts of tangent vectors to M. So let σ be a local holomorphic section of E such that $\nabla\sigma = 0$ at $x \in M$. Then we can take $d_x\sigma : T_xM \to T_{(x,\sigma(x))}E$. In fact, because σ is flat at x, its components σ^λ satisfy the differential equation $\partial_\alpha\sigma^\lambda = -A^\lambda{}_{\mu\alpha}\sigma^\mu$, at x. In particular, $d_x\sigma(v^\alpha\partial_{z^\alpha})$ only depends on the value of σ at x. So at the point $(x,\zeta) \in E$, we define the vector $\nabla_{z^\alpha} := \partial_{z^\alpha} - A^\lambda{}_{\mu\alpha}\zeta^\mu\partial_{\zeta^\lambda}$, which is the horizontal lift of ∂_{z^α} to (x,ζ). The holomorphic frame we pick on TE is then $(\nabla_{z^\alpha};\partial_{\zeta^\lambda})$. The corresponding dual frame is then $(dz^\alpha;\nabla\zeta^\lambda)$, with $\nabla\zeta^\lambda := A^\lambda{}_{\mu\alpha}\zeta^\mu dz^\alpha + d\zeta^\lambda$ at the point (x,ζ).

We also define the square distance to 0_M as follows: at a point $(x,\zeta) \in E$, let $t(x,\zeta) := h_{\lambda\bar\mu}\zeta^\lambda\bar\zeta^{\bar\mu}$. We wish to put a natural Kähler type form ω_E on the total space of E. Natural here means a few things:

(i) $\omega_E|_{0_M}$ agrees with ω_M, the Kähler form on M.

(ii) $t/2$ must be a potential for vertical vectors, i.e. $\omega_E(v,w) = i\partial\bar\partial t(v,w)$, where v and w are vertical tangent vectors.

(iii) Vertical vectors must be ω_E orthogonal to horizontal vectors.

These three conditions together imply that ω_E must be of the form

$$\omega_E = i(g_{\alpha\bar\beta} + \widetilde g_{\alpha\bar\beta})dz^\alpha \wedge d\bar z^{\bar\beta} + ih_{\lambda\bar\mu}\nabla\zeta^\lambda \wedge \nabla\bar\zeta^{\bar\mu}\,, \tag{4.1.3}$$

where $\widetilde{g}_{\alpha\bar{\beta}}$ are functions on E that vanish on 0_M. Since we want ω_E to be of Kähler type, it needs to be closed. Looking at the $d\bar{\zeta}^{\bar{\mu}} \wedge dz^\alpha \wedge d\bar{z}^{\bar{\beta}}$ part of $d\omega_E$, we see

$$\partial_{\bar{\mu}}\widetilde{g}_{\alpha\bar{\beta}} - \partial_{\bar{\beta}}(h_{\lambda\bar{\mu}}A^\lambda{}_{\nu\alpha}\zeta^\nu) + h_{\lambda\bar{\rho}}A^\lambda{}_{\nu\alpha}A^{\bar{\rho}}{}_{\bar{\mu}\bar{\beta}}\zeta^\nu = 0\,. \tag{4.1.4}$$

Therefore,

$$\partial_{\bar{\mu}}\widetilde{g}_{\alpha\bar{\beta}} = K_{\lambda\bar{\mu}\alpha\bar{\beta}}\zeta^\lambda\,. \tag{4.1.5}$$

Note that the right hand side is independent of $\bar{\zeta}^{\bar{\mu}}$, thus we see

$$\widetilde{g}_{\alpha\bar{\beta}} = K_{\lambda\bar{\mu}\alpha\bar{\beta}}\zeta^\lambda\bar{\zeta}^{\bar{\mu}}\,, \quad \text{and} \quad \omega_E = i\left(g_{\alpha\bar{\beta}} + K_{\lambda\bar{\mu}\alpha\bar{\beta}}\zeta^\lambda\bar{\zeta}^{\bar{\mu}}\right)dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + ih_{\lambda\bar{\mu}}\nabla\zeta^\lambda \wedge \nabla\bar{\zeta}^{\bar{\mu}}\,. \tag{4.1.6}$$

To see that this indeed defines a Kähler type form, we let Φ be a potential for g, i.e. such that $\omega_M = dd^c\Phi$, we define $\Psi := \pi^*\Phi + t/2$, and we see

$$\begin{aligned}
dd^c\Psi &= \pi^*dd^c\Phi + \tfrac{1}{2}dd^c(h_{\lambda\bar{\mu}}\zeta^\lambda\bar{\zeta}^{\bar{\mu}}) \\
&= ig_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + i\frac{\partial^2 h_{\lambda\bar{\mu}}}{\partial z^\alpha\partial\bar{z}^{\bar{\beta}}}\zeta^\lambda\bar{\zeta}^{\bar{\mu}}dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + i\frac{\partial h_{\lambda\bar{\mu}}}{\partial\bar{z}^{\bar{\beta}}}\bar{\zeta}^{\bar{\mu}}d\zeta^\lambda \wedge d\bar{z}^{\bar{\beta}} + i\frac{\partial h_{\lambda\bar{\mu}}}{\partial z^\alpha}\zeta^\lambda dz^\alpha \wedge d\bar{\zeta}^{\bar{\mu}} + ih_{\lambda\bar{\mu}}d\zeta^\lambda \wedge d\bar{\zeta}^{\bar{\mu}} \\
&= ig_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + i\frac{\partial^2 h_{\lambda\bar{\mu}}}{\partial z^\alpha\partial\bar{z}^{\bar{\beta}}}\zeta^\lambda\bar{\zeta}^{\bar{\mu}}dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + i\frac{\partial h_{\lambda\bar{\mu}}}{\partial\bar{z}^{\bar{\beta}}}\bar{\zeta}^{\bar{\mu}}\left(\nabla\zeta^\lambda - ih^{\lambda\bar{\nu}}\frac{\partial h_{\rho\bar{\nu}}}{\partial z^\alpha}\zeta^\rho dz^\alpha\right) \wedge d\bar{z}^{\bar{\beta}} + ih_{\lambda\bar{\mu}}\nabla\zeta^\lambda \wedge d\bar{\zeta}^{\bar{\mu}} \\
&= ig_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + i\left(\frac{\partial^2 h_{\lambda\bar{\mu}}}{\partial z^\alpha\partial\bar{z}^{\bar{\beta}}}\zeta^\lambda\bar{\zeta}^{\bar{\mu}} - h^{\lambda\bar{\nu}}\frac{\partial h_{\rho\bar{\nu}}}{\partial z^\alpha}\frac{\partial h_{\lambda\bar{\mu}}}{\partial\bar{z}^{\bar{\beta}}}\zeta^\rho\bar{\zeta}^{\bar{\mu}}\right)dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + ih_{\lambda\bar{\mu}}\nabla\zeta^\lambda \wedge \nabla\bar{\zeta}^{\bar{\mu}} \\
&= i\left(g_{\alpha\bar{\beta}} - h_{\lambda\bar{\mu}}K^\lambda{}_{\nu\alpha\bar{\beta}}\zeta^\nu\bar{\zeta}^{\bar{\mu}}\right)dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + ih_{\lambda\bar{\mu}}\nabla\zeta^\lambda \wedge \nabla\bar{\zeta}^{\bar{\mu}} \\
&= i\left(g_{\alpha\bar{\beta}} + K_{\lambda\bar{\mu}\alpha\bar{\beta}}\zeta^\lambda\bar{\zeta}^{\bar{\mu}}\right)dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + ih_{\lambda\bar{\mu}}\nabla\zeta^\lambda \wedge \nabla\bar{\zeta}^{\bar{\mu}} \\
&= \omega_E
\end{aligned}$$

So ω_E indeed defines a Kähler type form with potential $\Psi = \pi^*\Phi + t/2$. This is a Kähler form if and only if it is positive. Note that positivity for vertical vectors follows from the fact that h is a hermitian metric, so we just need positivity for horizontal vectors.

If Z is a horizontal vector in $T_{(x,\zeta)}E^{(1,0)}$ such that $K_{\lambda\bar{\mu}\alpha\bar{\beta}}\zeta^\lambda\bar{\zeta}^{\bar{\mu}}Z^\alpha\bar{Z}^{\bar{\beta}} < 0$, then we can always find a real number r such that $(g_{\alpha\bar{\beta}} + r^2K_{\lambda\bar{\mu}\alpha\bar{\beta}}\zeta^\lambda\bar{\zeta}^{\bar{\mu}})Z^\alpha\bar{Z}^{\bar{\beta}} < 0$, i.e. the horizontal lift of Z to $(x, r\zeta)$ has negative length. Thus positivity for horizontal vectors requires nonnegativity of $K_{\lambda\bar{\mu}\alpha\bar{\beta}}\zeta^\lambda\bar{\zeta}^{\bar{\mu}}Z^\alpha\bar{Z}^{\bar{\beta}}$ for any $(x,\zeta) \in E$ and any $Z \in T_xM^{(1,0)}$. This leads us to the following definition

Definition 4.1.2. The curvature $K_{\lambda\bar{\mu}\alpha\bar{\beta}}$ is called *positive*, respectively *nonnegative*, *negative* and *nonpositive*, if for any $(x,\zeta) \in E \setminus 0_M$ and any $Z \in T_xM^{(1,0)} \setminus \{0\}$, we have

$$K_{\lambda\bar{\mu}\alpha\bar{\beta}}\zeta^\lambda\bar{\zeta}^{\bar{\mu}}Z^\alpha\bar{Z}^{\bar{\beta}} > 0\,,$$

respectively ≥ 0, < 0 and ≤ 0.

Thus, ω_E is a Kähler form globally if and only if the curvature of h is nonnegative.

Now, if the curvature was not nonnegative, we note that if ω_E is still positive at some $(x,\zeta) \in E$, then it is positive at all $(x, r\zeta) \in E$ for $r \in [0,1]$. Likewise, if ω_E is not positive at some $(x,\zeta) \in E$, then it is not positive at any $(x, r\zeta) \in E$ for $r \geq 1$. Thus for any $x \in M$, the set $U_x \subseteq E_x$ of points where ω_E

is positive, is connected. Therefore, if M is connected, the set $U \subseteq E$ of points where ω_E is positive, is connected. Moreover, it is clear that $0_M \subseteq U$ and that U is open, so this procedure always provides a Kähler form ω_E on a connected open submanifold of E, even if the curvature of h is not nonnegative.

Since we eventually want to find Kähler forms with specific curvature, we change the potential Ψ a little bit to get a Kähler type form on some $U(k)$-invariant subbundle $F \subseteq E$, defined by some open subset[1] $I \subseteq \mathbb{R}^+$, by $F = \{(x,\zeta) \in E : t(x,\zeta) \in I\}$. We let $u : I \to \mathbb{R}$ be any function, and we consider the potential $\Psi = \pi^*\Phi + u \circ t/2$, such that $\omega_F := dd^c\Psi$ is a Kähler type form on F. We compute

$$\begin{aligned}
\omega_F &= \pi^* dd^c\Phi + i\partial\bar{\partial} u \circ t \\
&= i g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + iu'(t)\partial\bar{\partial}t + iu''(t)\partial t \wedge \bar{\partial} t \\
&= i\left(g_{\alpha\bar{\beta}} + u'(t) K_{\lambda\bar{\mu}\alpha\bar{\beta}} \zeta^\lambda \bar{\zeta}^{\bar{\mu}}\right) dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + iu'(t) h_{\lambda\bar{\mu}} \nabla\zeta^\lambda \wedge \nabla\bar{\zeta}^{\bar{\mu}} \\
&\quad + iu''(t)\left(\frac{\partial h_{\lambda\bar{\mu}}}{\partial z^\alpha}\zeta^\lambda\bar{\zeta}^{\bar{\mu}} dz^\alpha + h_{\lambda\bar{\mu}}\bar{\zeta}^{\bar{\mu}} d\zeta^\lambda\right) \wedge \left(\frac{\partial h_{\rho\bar{\sigma}}}{\partial \bar{z}^{\bar{\beta}}}\zeta^\rho\bar{\zeta}^{\bar{\sigma}} d\bar{z}^{\bar{\beta}} + h_{\rho\bar{\sigma}}\zeta^\rho d\bar{\zeta}^{\bar{\sigma}}\right) \\
&= i\left(g_{\alpha\bar{\beta}} + u'(t) K_{\lambda\bar{\mu}\alpha\bar{\beta}} \zeta^\lambda \bar{\zeta}^{\bar{\mu}}\right) dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + i\left(u'(t) h_{\lambda\bar{\mu}} + u''(t) h_{\lambda\bar{\nu}} h_{\rho\bar{\mu}} \zeta^\rho \bar{\zeta}^{\bar{\nu}}\right) \nabla\zeta^\lambda \wedge \nabla\bar{\zeta}^{\bar{\mu}}\,.
\end{aligned}$$

We summarise this result in the following lemma:

Lemma 4.1.3. *Let $E \to M$ be a holomorphic hermitian vector bundle of rank k, with hermitian metric h, square distance to the origin t and curvature K over a Kähler manifold (M,g) with local Kähler potential $\pi^*\Phi$. Then for an open subset $I \subseteq \mathbb{R}^+$ and function $u : I \to \mathbb{R}$, the form*

$$\omega_F := i\left(g_{\alpha\bar{\beta}} + u'(t) K_{\lambda\bar{\mu}\alpha\bar{\beta}} \zeta^\lambda \bar{\zeta}^{\bar{\mu}}\right) dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + i\left(u'(t) h_{\lambda\bar{\mu}} + u''(t) h_{\lambda\bar{\nu}} h_{\rho\bar{\mu}} \zeta^\rho \bar{\zeta}^{\bar{\nu}}\right) \nabla\zeta^\lambda \wedge \nabla\bar{\zeta}^{\bar{\mu}}\,, \tag{4.1.7}$$

is a $U(k)$-invariant Kähler type form on $F := \{(x,\zeta) \in E : t(\zeta) \in I\}$, with local potential $\Psi = \pi^*\Phi + u \circ t/2$.

Remark 4.1.4. Earlier, we chose the potential $\Psi = \pi^*\Phi + t/2$ such that it agrees with ω_M at the zero section, such that ω_E acts on vertical vectors according to the potential $t/2$, and such that vertical vectors are orthogonal to horizontal vectors. Even though it appears we are only changing the second condition by adding this extra function u, we are sneakily also possibly removing the zero section from E by picking F defined by an interval $I \subseteq \mathbb{R}^+$ that does not contain 0. In that case we do not have a proper justification for picking such a potential, as the first condition no longer makes sense then. Nevertheless, we are eventually interested in potentials that can be extended to the zero section, hence the choice of this potential is still a natural one in that setting.

We will be interested in the case when ω_F is a Kähler form, i.e. when it is positive. Since horizontal and vertical directions are ω_F-orthogonal, positivity follows from checking positivity in horizontal and vertical directions separately. We start with the vertical directions, as those don't carry a curvature term.

Proposition 4.1.5. *The Kähler type form ω_F restricted to vertical directions, is positive definite if and only if u satisfies the differential inequalities*

$$u'(x) > 0\,; \qquad u'(x) + xu''(x) > 0\,.$$

If E is a line bundle such that the zero section is not part of F, then the first condition is no longer necessary.

[1] Meaning it is open in $\mathbb{R}^+$, i.e. it is allowed to be of the form $[0,a)$.

Proof. Let $g_F := \omega_F(-, J-)$ denote the corresponding bilinear form. The necessity of the second condition follows from $g_F\left(\zeta\partial_\zeta, \bar\zeta\partial_{\bar\zeta}\right) > 0$, for $\zeta \in F \setminus \{0\}$. To prove the necessity of the first inequality whenever $k \geq 2$, note that, at a point $\zeta \neq 0$, we can always find a vector $(\zeta')^\lambda\partial_{\zeta^\lambda}$ such that

$$h_{\mu\bar\nu}\bar\zeta^\nu\nabla\zeta^\mu\left((\zeta')^\lambda\partial_{\zeta^\lambda}\right) = h_{\mu\bar\nu}(\zeta')^\mu\bar\zeta^\nu = 0\,,$$

as we are in two or more dimensions, so we can always find a vector that is h-orthogonal to ζ. Plugging in this vector and its hermitian conjugate into g_F tells us that $u'(t)$ has to be strictly positive for g_F to be positive definite in the vertical directions. Likewise, if $k = 1$ and the zero section is part of F, we see that positive definiteness on the zero section requires $u'(0) > 0$. The second inequality tells us $xu'(x)$ is strictly increasing, therefore we have that $u'(x) > 0$ everywhere by connectedness. When the zero section is not part of F, we see that $m = 1$ implies that the second condition is enough, i.e. $u'(x) > 0$ is no longer necessary.

To prove that these conditions are sufficient, we see that

$$g_F(X^\lambda\partial_{\zeta^\lambda}, \overline{X}^\mu\partial_{\bar\zeta^\mu}) = u'(t)|X|^2 + u''(t)|h(X,\bar\zeta)|^2$$

If $u'(t) > 0$ and $u''(t) \geq 0$, we see that this is strictly positive if $X \neq 0$. If $u''(t) < 0$, Cauchy-Schwarz tells us

$$u'(t)|X|^2 + u''(t)|h(X,\bar\zeta)|^2 \geq (u'(t) + u''(t)t)|X|^2\,,$$

so the second inequality tells us this is strictly positive if $X \neq 0$. If $k = 1$ and the zero section is not part of F, we see that $X = \lambda\zeta$ for some $\lambda \in \mathbb{C}$. Therefore,

$$u'(t)|X|^2 + u''(t)|a(X,\bar\zeta)|^2 = (u'(t) + u''(t)t)|\lambda|^2 t\,,$$

so if $X \neq 0$, i.e. $|\lambda| > 0$, we see that it is sufficient to just assume the second inequality to prove positive definiteness whenever $m = 1$ and F does not contain the zero section. □

This proposition takes care of the vertical vectors, so now we're left with the horizontal ones. Since we do not care about global vectors as of yet, we define a slightly weaker notion of nonnegativity for curvature, namely being bounded below

Definition 4.1.6 (Lower bounds for curvature)**.** The curvature $K_{\lambda\bar\mu\alpha\bar\beta}$ is said to be *bounded below* by $c \in \mathbb{R}$ at $x \in M$ if and only if for every $\zeta \in E_x$ and every $Z \in T_xM^{(1,0)}$, we have

$$K_{\lambda\bar\mu\alpha\bar\beta}\zeta^\lambda\bar\zeta^{\bar\mu}Z^\alpha\bar Z^{\bar\beta} > c\, h_{\lambda\bar\mu}\zeta^\lambda\bar\zeta^{\bar\mu}g_{\alpha\bar\beta}Z^\alpha\bar Z^{\bar\beta}\,.$$

c is called a *lower bound* for K at x. Moreover, if K is bounded below by some c at every $x \in M$, we say K is *uniformly bounded below* by c and c is called a *uniform lower bound* for K.

Likewise, we can define upper bounds for curvature. Note that if M is compact, then K is always uniformly bounded below. This notion gives us the following theorem:

Theorem 4.1.7. *Let $E \xrightarrow{\pi} M$ be a holomorphic vector bundle with hermitian metric h and associated Chern connection ∇, where M is a Kähler manifold with Kähler metric g. If the curvature of ∇ is uniformly bounded below, then there is a complete Kähler metric g_y on the fibre $\mathbb{C}^k$ that is $U(k)$-invariant, such that the induced Kähler type form on the total space of E, determined by the hermitian metric, by g_y and by g, is positive definite on E. Moreover, if g is complete, then so is the induced Kähler structure on E.*

Proof. The strategy will be to find an appropriate function u and use the methods we have just developed. Firstly, if the curvature is uniformly bounded below by some $c \geq 0$, we note that we can just use $u(x) = x$. The metric that is induced on the fibres then is then the standard metric on $\mathbb{C}^k$, which is complete. If M is complete, the Hopf-Rinow theorem tells us that the metric induced on the total space is also complete.

So now assume the curvature is uniformly bounded below by some $-b$ with $b > 0$. We claim that the choice

$$u(x) = \frac{2}{3b}\log(c+x) - \frac{1}{3}\log(\log(c+x)), \qquad c := e^b$$

suffices. We will first show that this choice of metric will be positive definite in the vertical direction, i.e. that the conditions of the previous proposition are satisfied. We see

$$u'(x) = \frac{2}{3b(c+x)} - \frac{1}{3\log(c+x)(c+x)} = \frac{1}{3(c+x)}\left(\frac{2}{b} - \frac{1}{\log(c+x)}\right).$$

One thing we immediately note is that $\log(c+x) \geq \log(c) = b$, therefore

$$u'(x) \geq \frac{1}{3b(c+x)} > 0.$$

Secondly, we see

$$u''(x) = -\frac{1}{3(c+x)^2}\left(\frac{2}{b} - \frac{1}{\log(c+x)}\right) + \frac{1}{3(c+x)^2(\log(c+x))^2}.$$

So we see

$$xu''(x) + u'(x) = \frac{1}{3(c+x)^2}\left(\frac{2c}{b} - \frac{c}{\log(c+x)} + \frac{x}{(\log(c+x))^2}\right).$$

We see

$$xu''(x) + u'(x) \geq \frac{1}{3(c+x)^2}\left(\frac{c}{b} + \frac{x}{(\log(c+x))^2}\right) > 0.$$

Thus we have proven that this is positive definite in the vertical direction. Now we want to show that $i\partial\bar{\partial}(u \circ t)$ defines a complete Kähler metric on the fibre. By the Hopf-Rinow theorem, it suffices to show that length from the origin to infinity diverges. We see that this length is given by

$$\int_0^\infty dx\sqrt{xu''(x) + u'(x)} \geq \int_0^\infty dx\sqrt{\frac{c}{3b(c+x)^2}} = \sqrt{\frac{c}{3b}}\int_c^\infty \frac{dx}{x}.$$

Since the rightmost side diverges logarithmically, we see that the Kähler metric induced on the fibre is indeed complete.

It remains to show positive definiteness in the horizontal direction. To do this, we note that by assumption,

$$(g_{\alpha\bar{\beta}} + u'(t)K_{\lambda\bar{\mu}\alpha\bar{\beta}}\zeta^\lambda\bar{\zeta}^\mu)dz^\alpha d\bar{z}^\beta \geq (1 - btu'(t))g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta.$$

Here, the inequality means that if a conjugate pair $(Z, \overline{Z})$ of horizontal tangent vectors is plugged in, then the inequality holds. We note that for every $x \geq 0$,

$$1 - bxu'(x) = 1 - \frac{bx}{3(c+x)}\left(\frac{2}{b} - \frac{1}{\log(c+x)}\right) = \frac{1}{3} + \frac{1}{3(c+x)}\left(2c + \frac{bx}{\log(x+c)}\right) \geq \frac{1}{3}.$$

Thus we see that the Kähler metric in the horizontal direction is bounded below by $\frac{1}{3}g$. Thus the Kähler metric is indeed positive definite in the horizontal direction. Again, if g is complete, we see by Hopf-Rinow that the Kähler metric on the total space is complete. $\square$

Corollary 4.1.8. *The total space of a holomorphic hermitian vector bundle over a compact Kähler manifold admits a complete Kähler metric.*

In particular, $\mathbb{C}P^n \setminus \{*\}$, i.e. $\mathbb{C}P^n$ with a point removed, always admits a complete Kähler metric, as it can be realised as a holomorphic line bundle over $\mathbb{C}P^{n-1}$.

Thus we have a method of producing complete Kähler metrics on certain open manifolds. The next step is to precisely control the curvature of these metrics to produce examples of open Calabi-Yau's. To make our life easier, we will work over Kähler manifolds with an exceptional amount of symmetry, we will work over *Kähler-Einstein manifolds*.

Definition 4.1.9 (Kähler-Einstein manifold)**.** A Kähler manifold (M, g) is called *Kähler-Einstein* if there is a constant $k_0 \in \mathbb{R}$ such that $\mathrm{Ric} = k_0 g$. Equivalently, if $\rho = k_0\omega$, where ρ is the Ricci form and ω is the Kähler form. In this case, we will say that the Ricci curvature is constant and that g is a *Kähler-Einstein metric.*

In particular, Ricci-flat Kähler manifolds are precisely those Kähler-Einstein manifolds with $k_0 = 0$.

Since the Ricci tensor satisfies Equation (2.2.36), i.e.

$$R_{\alpha\bar{\beta}} = -\partial_{\bar{\beta}}((\det(g_{\gamma\bar{\delta}}))^{-1}\partial_\alpha \det(g_{\gamma\bar{\delta}})) = -\partial_\alpha\partial_{\bar{\beta}} \log\det(g_{\gamma\bar{\delta}}) , \tag{4.1.8}$$

we see that the Kähler-Einstein condition translates to

$$\det(g_{\alpha\bar{\beta}}) = |f|^2 e^{-2k_0\Phi} , \tag{4.1.9}$$

where f is an some nonvanishing holomorphic function. If $k_0 \neq 0$, we can redefine $\Phi \to \Phi + \frac{1}{2k_0}\log(|f|^2)$, and we can redefine $\Phi \to \Phi/2$, i.e. $\omega = i\partial\bar{\partial}\Phi$, such that

$$\det(g_{\alpha\bar{\beta}}) = e^{-k_0\Phi} . \tag{4.1.10}$$

Now, if $k_0 = 0$, we have vanishing Ricci curvature. Since Ricci curvature is precisely the curvature of $TM^{(1,0)}$, we see that vanishing Ricci curvature means that we can locally pick a flat frame and normalise it, i.e. a frame such that $\det(g_{\alpha\bar{\beta}}) = 1$.

All in all, on a Kähler-Einstein manifold, we can always pick our setting such that Equation (4.1.10) holds locally.

We now want to find Kähler-Einstein metrics on the total space of holomorphic vector bundles $E \to M$, however, this is a very difficult task. To make things easier, we restrict ourselves to the setting of holomorphic line bundles $L \to M$. We assume L can be equipped with a hermitian metric h of constant curvature, i.e. such that $K_{\alpha\bar{\beta}} = -\partial_\alpha\partial_{\bar{\beta}}\log(h) = -l g_{\alpha\bar{\beta}}$ for some $l \in \mathbb{R}$. This is then the condition

$$h = |f|^2 e^{l\Phi} , \tag{4.1.11}$$

for some holomorphic function f, where we note that we redefined Φ such that $g_{\alpha\bar{\beta}} = \partial_\alpha\partial_{\bar{\beta}}\Phi$. Again, by picking an appropriate frame for L, this can be simplified to the condition

$$h = e^{l\Phi} . \tag{4.1.12}$$

Now, we note that first real Chern class of a line bundle must always be the image of an integral class, so we see that this constant curvature condition must mean that $\frac{1}{2\pi}l\omega$ defines an integral class. Moreover, the converse is also true, if L is a holomorphic line bundle with first Chern class $\frac{1}{2\pi}l\omega$, then it admits a hermitian metric with constant curvature l [Cal54].

Since $\rho \in 2\pi c_1(M)$, $\frac{1}{2\pi}k_0\omega$ is an integral cohomology class. So if $k_0 \neq 0$, we see that there is a nonzero lattice $\Lambda \subseteq \mathbb{R}$ such that $l \in \Lambda$ means $\frac{1}{2\pi}l\omega$ is an integral cohomology class. However, if $k_0 = 0$, things are a bit more difficult, as $\frac{1}{2\pi}k_0\omega = 0$, so integrality of this class tells us nothing about ω. We will ignore this difficulty and from now on, we will assume $[\omega]$ is an integral class, i.e. g is a *Hodge metric*, where we note that if $k_0 \neq 0$, this can always be achieved by rescaling g.

Theorem 4.1.10. *Let M be an $(n-1)$-dimensional complex manifold with Hodge metric g of constant Ricci curvature k_0. Let $L \xrightarrow{\pi} M$ be a hermitian line bundle of constant curvature $-l$. For every $x_0 > 0$, define the real hypersurface*

$$H_{x_0} := \{q \in L : t(q) = x_0\}.$$

Let $u(x)$ be a function of one real variable defined on an interval $I \subseteq \mathbb{R}^>$ around x_0. Then the Kähler type form $\omega_L := i\partial\bar{\partial}\Psi = i\partial\bar{\partial}(\pi^*\Phi + u \circ t)$, is positive definite if and only if u satisfies the differential inequalities

$$1 + lxu'(x) > 0; \qquad u'(x) + xu''(x) > 0. \tag{4.1.13}$$

If these conditions are satisfied, and if furthermore $l \neq 0$, then the metric induced by $i\partial\bar{\partial}\Psi$ has constant Ricci curvature k if and only if u satisfies the equation

$$(1 + lxu'(x))^{n-1}(u'(x) + xu''(x)) = cx^{l^{-1}(k_0-k-l)}e^{-ku(x)}, \tag{4.1.14}$$

where c is some positive constant. If $l = 0$, one forces $k = k_0$, and the condition becomes

$$\begin{cases} u(x) = 2k^{-1}\log(1 + c_0kx^c) + c_1\log(x) + c_2, & (l = 0, k = k_0 \neq 0); \\ u(x) = c_0^2x^c + c_1\log(x) + c_2, & (l = 0, k = k_0 = 0), \end{cases} \tag{4.1.15}$$

where c, c_0, c_1, c_2 are arbitrary real constants such that $c_0 \neq 0$.

Proof. We have

$$i\partial\bar{\partial}\Psi = i(1 + lxu'(x))g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta + ie^{l\Phi}(u'(x) + xu''(x))|\nabla\zeta|^2. \tag{4.1.16}$$

From this, it directly follows that the conditions

$$1 + lxu'(x) > 0; \qquad u'(x) + xu''(x) > 0,$$

are necessary and sufficient for this form to be positive definite.

The volume form of this form is given by

$$\mathrm{vol}_L = (1 + lxu'(x))^{n-1}e^{-k_0\Phi}e^{l\Phi}(u'(x) + xu''(x)), \tag{4.1.17}$$

as we assumed $\det(g_{\alpha\bar{\beta}}) = e^{-k_0\Phi}$. We want the curvature of the metric induced on the total space to be constant constant and equal to k, we see that this condition is

$$\det(\partial_\alpha\partial_{\bar{\beta}}\Psi) = |f|^2e^{-k\Psi}, \tag{4.1.18}$$

where f is some nonzero holomorphic function. So the condition becomes

$$(1 + lxu'(x))^{n-1}(u'(x) + xu''(x)) = |f|^2\exp((k_0 - l - k)\Phi - ku(x)). \tag{4.1.19}$$

We now see that, if $l \neq 0$, the condition $h = e^{l\Phi}$ means that $e^{\Phi} = |g|^2 x^{l^{-1}}$, where g is some holomorphic function (note that we assume $x \neq 0$ in the theorem), which we can incoorporate into the definition of f, therefore, we get the equation

$$x^{-l^{-1}(k_0-k-l)}(1+lxu'(x))^{n-1}(u'(x)+xu''(x))e^{ku(x)} = |f|^2\,. \tag{4.1.20}$$

We will now reason that $|f|^2$ is constant. Suppose that it's not, then we note that we must be able to find a point $p \in M$ where $d_p|f|^2 \neq 0$, and then we can locally find a hypersurface Σ of constant $|f|^2$. We note that this hypersurface must agree with the hypersurface of constant t, as we have Equation (4.1.20) relating the two. This means that ∂_{z^i} and $\partial_{\bar{z}^i}$ must lie in $T_p\Sigma \otimes \mathbb{C}$, which means these vectors must be horizontal. This means the curvature of h must be zero at p, which is a contradiction, as we assumed $l \neq 0$. Therefore, $|f|^2$ must be constant. Thus we have derived the equation corresponding to $l \neq 0$.

On the other hand, suppose $l = 0$. Then Equation (4.1.16) tells us that the induced Kähler metric locally splits into a Kähler metric on M and a Kähler metric on the fibre of L. For this to have constant curvature, we therefore need to have $k = k_0$. Then Equation (4.1.19) becomes

$$u'(x) + xu''(x) = |f|^2 \exp(-ku(x))\,. \tag{4.1.21}$$

Moreover, the induced metric on the fibre must have circular symmetry, therefore $|f|^2(e^{i\theta}x) = |f|^2(x)$ for any $\theta \in \mathbb{R}$, so we see that $f = a\zeta^b$ for some $a, b \in \mathbb{R}$, with a nonzero. Thus we see

$$u'(x) + xu''(x) = ax^b \exp(-ku(x))\,. \tag{4.1.22}$$

Now, this is a second order linear ODE, so we see that Equation (4.1.15) are indeed the possible solutions. □

Now, in the previous theorem we assumed $x \neq 0$. The next step is to extend to the zero section

Proposition 4.1.11. *If the induced Kähler metric metric g_L from the previous theorem is defined around 0, it can be regularly extended to the zero section if and only if, in the general case, $k = k_0 - l$ and $u'(0) > 0$. In the particular case $l = 0$, this condition becomes*

$$u(x) = 2k^{-1}\log(1 + c_0^2 kx) + c_2\,; \qquad (k = k_0 \neq 0, c_0 \neq 0)\,,$$

or

$$u(x) = c_0^2 x + c_2\,; \qquad (k = k_0 = 0, c_0 \neq 0)\,.$$

Proof. When $l = 0$, we want u to extend to the zero section, and we want $u'(x) + xu''(x)$ to extend to the zero section and be positive. Thus we see that the given formulas are the only possibilities. Therefore, we will focus on the case $l \neq 0$. Multiplying both sides of Equation (4.1.14) by $l^{-1}(k_0 - k) - kxu'(x)$ and integrating, we get

$$\frac{k_0}{nl^2}[(1+lxu'(x))^n - 1] - \frac{k}{(n+1)l^2}[(1+lxu'(x))^{n+1} - 1] = cx^{l^{-1}(k_0-k)}e^{-ku(x)} + c_2\,. \tag{4.1.23}$$

We can describe the left hand side as $P(xu'(x))$, where $P(w)$ is the polynomial

$$P(w) = \frac{k_0}{nl^2}((1+lw)^n - 1) - \frac{k}{(n+1)l^2}((1+lw)^{n+1} - 1) = \sum_{j=1}^{n+1} \frac{n(k_0-k) - k_0(j-1)}{n(n+1)} \binom{n+1}{j} l^{j-2} w^j\,. \tag{4.1.24}$$

Note that $P(xu'(x))$ admits a Taylor expansion around 0, therefore, the right hand side of Equation (4.1.23) must also. Thus we see that $l^{-1}(k_0 - k)$ must be a nonnegative integer. Furthermore, the linear term of $P(xu'(x))$ is

$$\frac{k_0 - k}{l} x u'(0) \, ,$$

while it does not have a constant term. Therefore, we must either have $k_0 = k$ and

$$c_2 = -ce^{-ku(0)} \, ,$$

or $l^{-1}(k_0 - k) = 1$ and $c_2 = 0$, to make the linear terms on both sides of (4.1.23) match. In the first scenario, we see that the linear term on the left hand side vanishes, therefore so should the linear term on the right hand side. However, this linear term is

$$-cku'(0)e^{-ku(0)}x \, ,$$

so if $k \neq 0$, the condition $u'(0) > 0$ tells us that this is impossible. If $k = 0$, however, the right hand side of (4.1.23) becomes constant. The equation we then obtain is of the form $u'(x) = c/x$, which does not have solutions for $x = 0$. Therefore, the first condition cannot happen if we can extend $u(x)$ regularly to the zero section.

We will focus on the second condition, i.e. $l = k_0 - k$ and $c_2 = 0$. Then we get the equation

$$P(xu'(x)) = cxe^{-ku(x)} \, , \tag{4.1.25}$$

where $k = k_0 - l$. Note that the derivative of P at 0 is 1, therefore, the inverse function theorem tells us we can find a unique analytic inverse $F(v)$ defined around 0 such that $F(P(w)) = w$. We introduce the parameter

$$\xi := F(cxe^{-ku(x)}) \, . \tag{4.1.26}$$

Note that $x = 0$ means $\xi = 0$, as $P(0) = 0$. Thus, we see that we have $xu'(x) = \xi$. Moreover, if we differentiate both sides of $P(\xi) = cxe^{-ku(x)}$, we get

$$\begin{aligned} P'(\xi)d\xi &= c(e^{-ku(x)} - kxu'(x)e^{-ku(x)})dx \\ &= P(\xi)(1 - k\xi)\frac{dx}{x} \, . \end{aligned}$$

We see

$$P'(w) = \frac{k_0}{l}(1 + lw)^{n-1} - \frac{k}{l}(1 + lw)^{n-1}(1 + lw) = (1 + lw)^{n-1}(1 - kw) \, . \tag{4.1.27}$$

So we see

$$x = x_0 \exp\left(\int_{\xi_0}^{\xi} \frac{(1 + lw)^{n-1}}{P(w)} dw\right) \, ; \qquad u = u_0 + \int_{\xi_0}^{\xi} \frac{w(1 + lw)^{n-1}}{P(w)} dw \, . \tag{4.1.28}$$

Here, $x_0 > 0$ is such that $cx_0e^{-ku_0}$ lies in the interval where F is defined, and these formulas hold for every other ξ in this interval. This formula can be used to extend u to the 0 section regularly. This completes the proof. □

Note that ξ has a geometric interpretation, if we take the disc $D := \{t \leq r\}$ in some fibre, we see that the area is precisely $2\pi h\xi(r)$, so ξ indicates the factor by how much the area of discs change due to the introduction of u in the Kähler potential.

We have three interesting cases, the last of which is the most important one for us. Before we state them we need a technical result about roots of unity, which we shall quickly state and prove to save the reader some time in trying to prove it themselves.

Lemma 4.1.12. *If $\omega = e^{2\pi i/n}$, for some $n \in \mathbb{N} \setminus \{0\}$, we have*

$$\prod_{j=1}^{n-1}(z-\omega^j) = 1 + z + z^2 + ... + z^{n-1}\,,$$

for every $z \in \mathbb{C}$.

Proof. We note that

$$\prod_{j=0}^{n-1}(z-\omega^j) = z^n - 1\,,$$

as the roots of unity are precisely the roots of $z^n - 1$. We also note that

$$z^n - 1 = (z-1)(1 + z + z^2 + ... + z^{n-1})\,.$$

When $z \neq 1$, dividing both sides by $z - 1$ gives the desired result. Since both sides are continuous functions, we see that this equality extends to $z = 1$. $\square$

In particular, we have

$$\prod_{j=1}^{n-1}(1-\omega^j) = n\,.$$

We will now give the special cases we talked about earlier:

1. If $k_0 = 0$, i.e. if M is Ricci flat, and $k = -l$, then we have $P(w) = (n+1)^{-1}l^{-1}[(1+lw)^{n+1} - 1]$. Then, applying partial fraction expansion, we get

$$\begin{aligned} x &= x_0 \exp\left(\int_{\xi_0}^{\xi} \frac{(n+1)l(1+lw)^{n-1}}{(1+lw)^{n+1}-1} dw\right) \\ &= x_0 \exp\left(\sum_{j=0}^{n} \frac{(n+1)\omega^{-j}}{\prod_{p=1}^{n}(1-\omega^p)} \log\left(\frac{1-\omega^j + l\xi}{1-\omega^j + l\xi_0}\right)\right) \\ &= x_0 \exp\left(\sum_{j=0}^{n} \omega^{-j} \log\left(\frac{1-\omega^j + l\xi}{1-\omega^j + l\xi_0}\right)\right) \end{aligned}$$

where $\omega := e^{2\pi i/(n+1)}$. The last equality follows from the lemma. Likewise, we get

$$\begin{aligned} u(x) &= u_0 + \int_{\xi_0}^{\xi} \frac{(n+1)lw(1+lw)^{n-1}}{(1+lw)^{n+1}-1} dw \\ &= u_0 + \frac{1}{l}\log\left(\frac{x_0 P(\xi)}{x P(\xi_0)}\right) \\ &= \frac{1}{l}\log\left(\frac{P(\xi)}{cx}\right)\,. \end{aligned}$$

Summarizing, we have

$$x = x_0 \exp\left(\sum_{j=0}^{n} \omega^{-j} \log\left(\frac{1-\omega^j + l\xi}{1-\omega^j + l\xi_0}\right)\right)\,; \qquad u(x) = \frac{1}{l}\log\left(\frac{P(\xi)}{cx}\right)\,. \tag{4.1.29}$$

2. If $l \neq 0$, $k_0 = -nl$, $k = -(n+1)l$. Then we see $P(w) = w(1+lw)^n$. The integrals become elementary,
$$x = x_0 \frac{\xi(1+l\xi_0)}{\xi_0(1+l\xi)}; \qquad u(x) = u_0 + \frac{1}{l}\log\left(\frac{1+l\xi}{1+l\xi_0}\right). \tag{4.1.30}$$

3. Lastly, the case that's most interesting for our purposes, $l = k_0$ and $k = 0$. We see $P(w) = (nl)^{-1}((1+lw)^n - 1)$. Therefore,
$$x = x_0 \frac{P(\xi)}{P(\xi_0)}. \tag{4.1.31}$$
Likewise, we see
$$u = u_0 + \int_{\xi_0}^{\xi} \frac{nlw(1+lw)^{n-1}}{(1+lw)^n - 1} dw = u_0 + n(\xi - \xi_0) + n\int_{\xi_0}^{\xi} \frac{1-(1+lw)^{n-1}}{(1+lw)^n - 1} dw.$$
Applying partial fraction expansion, we can calculate the remaining integral,
$$\begin{aligned} n\int_{\xi_0}^{\xi} \frac{1-(1+lw)^{n-1}}{(1+lw)^n - 1} dw &= \frac{1}{l}\sum_{j=0}^{n-1} \frac{n(\omega^j - 1)}{\prod_{p=1}^{n-1}(1-\omega^p)} \log\left(\frac{1-\omega^j + l\xi}{1-\omega^j + l\xi_0}\right) \\ &= \frac{1}{l}\sum_{j=1}^{n-1}(\omega^j - 1)\log\left(\frac{1-\omega^j + l\xi}{1-\omega^j + l\xi_0}\right), \end{aligned}$$
where $\omega = e^{2\pi i/n}$. In the last line, we applied the lemma and we noted that the $j = 0$ term vanishes, as $1 - 1 = 0$. In total, we get
$$u(x) = u_0 + n(\xi - \xi_0) - \frac{1}{l}\sum_{j=1}^{n-1}(1-\omega^j)\log\left(\frac{1-\omega^j + l\xi}{1-\omega^j + l\xi_0}\right). \tag{4.1.32}$$

Now we have defined a Kähler-Einstein metric on some maximal domain $E \subseteq L$ and we have tackled the question whether this can be extended to the zero section. The remaining question is completeness.

Theorem 4.1.13. *Let M be an $(n-1)$-dimensional Kähler manifold with constant Ricci curvature k_0, and let $L \xrightarrow{\pi} M$ be a holomorphic line bundle with hermitian metric of constant curvature $-l$. Then the Einstein-Kähler metric, adapted to these data on a neighborhood of 0 on total space of L, and positive definite on a maximal domain $E \subseteq L$, defines a complete Einstein-Kähler structure on E if and only if M is complete, $k = k_0 - l$, $l \geq 0$ and $k \leq 0$.*

Proof. We will begin by proving the necessity of the conditions. The necessity that M be complete is immediate, as it can be realised as the zero section of L, $k = k_0 - l$ follows from the previous proposition and the condition $k \leq 0$ follows from Myers' theorem, as the total space of L is not compact.

The remaining condition is $l \geq 0$. The proof will be by contradiction, so suppose $l < 0$ and that L is complete with the induced Kähler metric. We see that $u(x)$ satisfies the condition $xu'(x) = \xi$, where ξ lies in the interval $(0, \alpha_1)$, where α_1 is the smallest positive root of $P(w)$, or $(0, \infty)$ if $P(w)$ has no positive roots. This is because there is a $P(\xi)$ in the denominator of the integral that $u(x)$ amounts to, so if P has a root at some $\xi > 0$, $u'(x)$ cannot be defined there. The induced Kähler type metric is positive definite when ξ lies in the interval $(0, \min(-l^{-1}, \alpha_1))$, by the condition $1 + lxu'(x) > 0$. The other condition is satisfied then as well, as
$$u'(x) + xu''(x) = \frac{d}{dx}(xu'(x)) = \frac{d\xi}{dx} = x^{-1}P(\xi)(1+l\xi)^{1-n} > 0.$$

We study the asymptotic behaviour as $\xi \to \min(-l^{-1}, \alpha_1)$. To do this, first note that we have the identity

$$P(w) = w(1+lw)^n - \frac{k_0 + nl}{n(n+1)} w^2 \sum_{j=1}^{n} j(1+lw)^{j-1}\,, \tag{4.1.33}$$

this fact can be proven using the identity

$$1 + nx(1+x)^n - (1+x)^n = x^2 \sum_{j=1}^{n} j(1+x)^{j-1}\,,$$

which can be easily proven by induction on $n \geq 1$.

Using this expression for $P(w)$, we see that it has a root in $(0, -l^{-1})$ if and only if $k_0 > -nl$. To get a contradiction, we will prove that infinity is not infinitely far away, then we know that the metric cannot be complete. Therefore, take a path $\zeta = \tau$ for $\tau > 0$. We see

$$\left(\frac{ds}{d\tau}\right)^2 = 2(u'(x) + xu''(x))h = 2h\frac{d\xi}{dx}\,,$$

where $x = a\tau^2$. Therefore,

$$\frac{d\tau}{d\xi} = \frac{1}{2\sqrt{hx}}\frac{dx}{d\xi}\,.$$

Thus,

$$\left(\frac{ds}{d\xi}\right)^2 = \frac{1}{2x}\frac{dx}{d\xi} = \frac{(1+l\xi)^{n-1}}{2P(\xi)}\,.$$

Hence,

$$s = \int_0^{\xi} \left(\frac{(1+lw)^{n-1}}{2P(w)}\right)^{1/2} dw\,. \tag{4.1.34}$$

From Equation (4.1.27), we know that the roots of $P'(w)$ are at $-l^{-1}$ and at k^{-1}. Firstly, we see that if $k_0 > -nl$, then $k = k_0 - l > -(n+1)l$, i.e. $k^{-1} < l^{-1}$. Now we note that $P(w)$ has a root at 0, so it cannot have a root in $(0, k^{-1}]$, as it is either strictly negative or strictly positive in that interval. Now, we noted earlier that $P(w)$ has a root α_1 in $(0, -l^{-1})$, so by this reasoning, it must be a simple root. Therefore, we see that s converges as $\xi \to \alpha_1$. Secondly, if $k_0 = -nl$, we see that $P(w) = w(1+lw)^n$, which has a root at $w = -l^{-1}$, such that we get $s \to \sqrt{2}(\sqrt{-l^{-1}} + 1)$ as $\xi \to -l^{-1}$. Thirdly, if $k_0 < -nl$, the integral converges trivially as $\xi \to -l^{-1}$, as $P(w)$ has no roots in $(0, -l^{-1}]$.

So we see that $l \geq 0$ is necessary for completeness. What remains is to show sufficiency. So suppose M is complete, $l \geq 0$ and $k = k_0 - l \leq 0$. Then $P(\xi) = \int_0^{\xi}(1-kw)(1+lw)^{n-1}dw$, see Equation (4.1.27), is strictly positive for $\xi > 0$. Thus, we see that Equation (4.1.28) is well defined for any $\xi \geq 0$. If $k < 0$, we see that $P(w)$ is a polynomial of order $n+1$, so we see that the integral in Equation (4.1.34) diverges as $\xi \to \infty$, so $U(1)$ invariance in the fibre tells us that E is geodesically complete, so by the Hopf-Rinow theorem, E is complete. Likewise, if $k = 0$, $P(w)$ is a polynomial of order n, so the integral still diverges as $\xi \to \infty$, i.e. E is also complete. This shows sufficiency, thus the theorem is proven. $\square$

4.2 The Eguchi-Hanson space

Now that we have this construction, we will finally apply it to construct explicit Ricci-flat Kähler manifolds. To do this, we first note that in the case where k_0 and l are nonzero, taking the limit as $\xi_0 \to 0$ in

Equation (4.1.28) gives us

$$x = \frac{1}{cnl}((1+l\xi)^n - 1) \implies \xi = \frac{(1+nlcx)^{1/n} - 1}{l}\,,$$

where c is an arbitrary constant, so we can pull the nl into its value. Plugging this into u, we then see

$$u(x) = u(0) + \frac{n}{l}((1+cx)^{1/n} - 1) - \sum_{j=1}^{n-1} \frac{1-\omega^j}{l} \log\left(\frac{(1+cx)^{1/n} - \omega^j}{1-\omega^j}\right). \tag{4.2.1}$$

In particular, if M is a Kähler-Einstein manifold with Ricci curvature k_0, we see that K_M with the induced metric has curvature $-k_0$, so in this setting, there is a Ricci-Flat Kähler metric on the total space of K_M induced by the $u(x)$ of this form.

So let $M = \mathbb{C}P^{n-1}$ and equip it with the Fubini-Study form. We note that $\mathcal{O}(-1)$ is the blow-up of $\mathbb{C}^n$ at the origin, so we can equip it with the metric induced by the standard metric on $\mathbb{C}^n$ to turn it into a hermitian vector bundle. Since the Fubini-Study form is defined by the Kähler potential $\Phi = \log(|\sigma|^2)$, where σ is any local holomorphic section of $\mathcal{O}(-1)$, i.e. $\omega^{\mathrm{FS}} = i\partial\bar{\partial}\log(|\sigma|^2)$. We then see that $\mathcal{O}(-1)$ has constant curvature -1, by definition. Thus, this means that $K_{\mathbb{C}P^{n-1}} = \mathcal{O}(-n)$ has constant curvature $-n$. I.e. ω^{FS} is a Kähler-Einstein metric of constant curvature n. Thus the above procedure gives us a Ricci-flat Kähler metric on the total space of $K_{\mathbb{C}P^{n-1}}$.

To make things a bit more explicit, we note that $K_{\mathbb{C}P^{n-1}} \cong \tau^{\otimes n}$, and that the Kähler potential pulled back to $\tau^{\otimes n}$ has explicit form $\pi^*\Phi = \log(r^2)$, by definition. So, considering the map $(-)^n : \tau \setminus \{0\} \to K_{\mathbb{C}P^1} \setminus \{0\}$, we can pull back everything to $\tau \setminus \{0\}$ to get potential

$$\Psi = \log(r^2) + u(0) + (1+cr^{2n})^{1/n} - 1 - \sum_{j=1}^{n-1} \frac{1-\omega^j}{n} \log\left(\frac{(1+cr^{2n})^{1/n} - \omega^j}{1-\omega^j}.\right)$$

Now, we note that

$$\log(r^2) = \tfrac{1}{n}\log(((1+r^{2n})^{1/n})^n - 1) = \tfrac{1}{n}\sum_{j=0}^{n-1}\log((1+r^{2n})^{1/n} - \omega^j)\,.$$

Thus, picking appropriate values for c and $u(0)$, which were *a priori* left undetermined, we see

$$\Psi = (1+r^{2n})^{1/n} + \tfrac{1}{n}\sum_{j=0}^{n-1}\omega^j\log((1+r^{2n})^{1/n} - \omega^j)\,. \tag{4.2.2}$$

In the particular case $n = 2$, i.e. on the complex manifold $(T^*\mathbb{C}P^1)^{(1,0)}$, we have

$$\Psi = (1+r^4)^{1/2} + \tfrac{1}{2}(\log(1+r^4)^{1/2} - 1) - \log((1+r^4)^{1/2} + 1))\,,$$

so we see

$$g_{\alpha\bar{\beta}} = \partial_\alpha\partial_{\bar{\beta}}\Psi = \partial_\alpha\partial_{\bar{\beta}} r^2 \frac{\partial\Psi}{\partial r^2} + \partial_\alpha r^2 \partial_{\bar{\beta}} r^2 \frac{\partial^2\Psi}{(\partial r^2)^2}\,.$$

We see $\partial_\alpha r^2 = \bar{z}^\alpha$ and $\partial_{\bar{\beta}} r^2 = z^{\bar{\beta}}$. So we compute

$$\partial_{r^2}\Psi = \frac{r^2}{\sqrt{1+r^4}} + \frac{1}{r^2\sqrt{1+r^4}} = r^{-2}\sqrt{1+r^4}\,,$$

hence

$$\partial_{r^2}\partial_{r^2}\Psi = \frac{1}{\sqrt{1+r^4}} - \frac{\sqrt{1+r^4}}{r^4} = -\frac{1}{r^4\sqrt{1+r^4}}\,.$$

So we see that the metric becomes

$$g_{\alpha\bar{\beta}} = \frac{\sqrt{1+r^4}}{r^2}\,\delta_{\alpha\bar{\beta}} - \frac{1}{r^4\sqrt{1+r^4}}\,\bar{z}^\alpha z^{\bar{\beta}}\,. \tag{4.2.3}$$

This is known as the *Eguchi-Hanson* metric on $(T^*\mathbb{C}P^1)^{(1,0)}$, which was discovered independently from Calabi by T. Eguchi and A.J. Hanson in 1979 [EH79]. Note that $(T^*\mathbb{C}P^1)^{(1,0)}$ is diffeomorphic to $T^*\mathbb{C}P^1$, so this construction can be used to put a Ricci-flat Kähler structure on $T^*\mathbb{C}P^1$.

One property of this metric is that it is *asymptotically locally Euclidean* (ALE), which is to say that it converges to a locally Euclidean metric as $r \to \infty$ sufficiently quickly [Joy00]. This property makes it so that it can be used to locally approximate certain K3 surfaces.

To see this, we briefly sketch Kummer's construction of the K3 surface [Kum75]. We start with a four torus $T^4 = \mathbb{C}^2/\mathbb{Z}^4$. We define an involution $\iota : T^4 \to T^4$ by letting it act on $\mathbb{C}^2$ by $\iota(z_1, z_2) = (-z_1, -z_2)$. We see that this involution has 16 fixed points on the torus, so the quotient $T^4/\mathbb{Z}_2$ has 16 singularities that can be modelled on $\mathbb{C}^2/\mathbb{Z}_2$. Blowing up $\mathbb{C}^2/\mathbb{Z}_2$ at the fixed point gives us $\mathcal{O}(-2)$, which is $T^*\mathbb{C}P^1$. Moreover, it is known that blowing up $T^4/\mathbb{Z}_2$ at the fixed points gives a K3 surface [Joy00], so this particular construction of a K3 surface looks like $T^*\mathbb{C}P^1$ around a blown up point. Now, there is no known explicit formula for the Ricci-flat Kähler metric on any K3 surface [Joy00], however, the Ricci-flat Kähler metric on Kummer's construction of the K3 surface can be approximated by the Eguchi-Hanson metric around the blown up points, as this metric is asymptotically locally Euclidean, meaning the Eguchi-Hanson metrics around different blown up points are approximately the same far away from the blown up points. For details, we refer the reader to [Joy00].

5 Calabi-Yau manifolds in string theory

So far, we have studied and proved the Calabi-Yau theorem and we have given a nontrivial example of an explicit Calabi-Yau metric on $K_{\mathbb{C}P^n}$. In this chapter, we will give a huge application of the theory of Calabi-Yau manifolds, as Calabi-Yau manifolds are a cornerstone of superstring theory.

Superstring theory is a theory of quantum gravity. The idea is that particles are not described by worldlines, but rather by two-dimensional *worldsheets*. These are described by embeddings of some two dimensional surface Σ into some target manifold M^m, which is a Lorentzian manifold with metric G. We can describe this as a nonlinear sigma model on Σ. If the embedding is the only field we put on the string, then we have a *bosonic string theory*, which always has a tachyon in its ground state [GSW87a; BLT13; BBS07]. This is a rather unfortunate fact, we want to get rid of this tachyon. One way to do so is by introducing fermions on the worldsheet and using spacetime supersymmetry, i.e. supersymmetry on the target space M^m.

There are two different formalisms one can use to get supersymmetry. On the one hand, one can upgrade the target space M to a *supermanifold*, which has both bosonic and fermionic coordinates. Then one can write down an action for the superstring that has manifest spacetime supersymmetry. This is known as the *GS formalism*, named after M.B. Green and J.H. Schwarz [GS81; GS82a; GS82b].

Equivalently, one could introduce fermions on the worldsheet and impose supersymmetry there. This is known as the *RNS formalism*, after P. Ramond, A. Neveau and J.H. Schwarz [Ram71; NS71]. We will follow the RNS formalism and our main reference will be [BLT13]. This approach gives us manifest worldsheet supersymmetry, but not spacetime supersymmetry, which will have to be imposed later down the road.

5.1 Superstrings in RNS formalism

For the moment, we assume the target space is flat, i.e. $G_{\mu\nu} = \eta_{\mu\nu}$. We start with the Polyakov action for the bosonic string, i.e.

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu\,, \tag{5.1.1}$$

where h is the worldsheet metric, $\alpha, \beta, \ldots$ run over the worldsheet coordinates, $\mu, \nu, \ldots$ run over the spacetime coordinates $\sigma^0 = \tau$, $\sigma^1 = \sigma$, and spacetime indices are lowered and raised according to $\eta_{\mu\nu}$.

To get $N = 1$ supersymmetry on the worldsheet, one has to add two kinds of particles. Firstly, every X^μ needs a fermionic superpartner ψ^μ . Then, the zweibein $e_\alpha{}^a$ must have a superpartner χ_α , known as the *gravitino* on the worldsheet. Here, we let $a, b, \ldots$ run over Minkowski space. Since we want these to be superpartners, we want the fermions to be real, so we impose a Majorana condition on them[1]. Denoting the two dimensional Dirac matrices by ρ (which define Dirac matrices on the worldsheet by applying the zweibein), we get the following action

$$S = -\frac{1}{8\pi}\int_\Sigma d^2\sigma\sqrt{-h}\left(\frac{2}{\alpha'}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + 2i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu - i\bar{\chi}_\alpha\rho^\alpha\rho^\beta\psi^\mu\left(\sqrt{\frac{2}{\alpha'}}\partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta\psi^\mu\right)\right). \tag{5.1.2}$$

Using symmetries of the action, one can further simplify this action to *superconformal gauge*, in which this action takes the form

$$S = -\frac{1}{4\pi}\int_\Sigma d^2\sigma(\tfrac{1}{\alpha'}\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu)\,, \tag{5.1.3}$$

where η is the Minkowski metric. See [BLT13] for details. Note that this differs from the one in [GSW87a]. On the one hand, the factor in front of the action is different, which is due to [GSW87a] choosing units where $\alpha' = \frac{1}{2}$. On the other hand, the second term in this action carries an extra sign. This is due to chosen conventions for the Dirac algebra in two dimensions. In our convention, which is used in [BLT13], we have

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ ; \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,$$

which gives us $\{\rho^a, \rho^b\} = 2\eta^{ab}$, where $\{-, -\}$ denotes the anti-commutator.

In superconformal gauge, we see that we have global worldsheet supersymmetry, which is generated by Majorana spinors ϵ such that $\partial_\alpha\epsilon = 0$, and is explicitly given by

$$\delta_\epsilon X^\mu = i\sqrt{\tfrac{\alpha'}{2}}\,\bar{\epsilon}\psi^\mu\,; \tag{5.1.4}$$

$$\delta_\epsilon\psi^\mu = \sqrt{\tfrac{1}{2\alpha'}}\,\rho^\alpha\partial_\alpha X^\mu\epsilon\,. \tag{5.1.5}$$

[1] This means that there are more fermionic degrees of freedom in the theory than bosonic degrees of freedom. To get the theory properly, one would have to introduce auxiliary scalar fields F^μ for the multiplet (X^μ, ψ^μ) and a scalar field A for the multiplet $(e_\alpha{}^a, \chi_\alpha)$, but these fields decouple, so we ignore them. See [BLT13] for details.

Jumping to lightcone gauge, (5.1.3) becomes

$$S = \frac{1}{2\pi}\int_\Sigma d\sigma^+ d\sigma^- (\tfrac{2}{\alpha'}\partial_+ X^\mu \partial_- X_\mu + i(\psi^\mu_+ \partial_- \psi_{\mu+} + \psi^\mu_- \partial_+ \psi_{\mu-}))\,. \tag{5.1.6}$$

Here, we defined $\psi = (\psi_+, \psi_-)^T$. Note that ψ_+ and ψ_- are both real Weyl spinors, which makes sense as Majorana-Weyl spinors exist in $1+1$ dimensions [BLT13].

We obtain equations of motion given by

$$\partial_+\partial_- X^\mu = 0\,; \tag{5.1.7}$$

$$\partial_+\psi^\mu_- = \partial_-\psi^\mu_+ = 0\,. \tag{5.1.8}$$

Before we do mode expansions to find the solutions to these equations, we note quickly that ψ^μ do not have the same geometric interpretation that the X^μ do. Therefore, if we consider a closed string, it is not obvious that they should obey periodic boundary conditions like the X^μ do. In fact, for a string of length ℓ, and picking a variation $\delta\psi^\mu(\tau_0)\delta\psi^\mu(\tau_1) = 0$, we see that we get

$$\delta S = \int_{\tau_0}^{\tau_1} (\psi^\mu_+ \delta\psi_{\mu+} - \psi^\mu_- \delta\psi_{\mu-})\big|_{\sigma=0}^{\sigma=\ell} = 0\,.$$

For the closed string, we see that we get the requirement

$$(\psi^\mu_+ \delta\psi_{\mu+} - \psi^\mu_- \delta\psi_{\mu-})(\sigma+\ell) = (\psi^\mu_+ \delta\psi_{\mu+} - \psi^\mu_- \delta\psi_{\mu-})(\sigma)\,,$$

for which we see

$$\psi^\mu_+(\sigma+\ell) = \pm\psi^\mu_+(\sigma)\,; \tag{5.1.9}$$

$$\psi^\mu_-(\sigma+\ell) = \pm\psi^\mu_-(\sigma)\,, \tag{5.1.10}$$

where we note that ψ is Majorana, so other phase factors are not allowed. We see that we can either have periodic (Ramond, R) or anti-periodic (Neveau-Schwarz, NS) boundary conditions. Spacetime Poincaré invariance requires that all $\psi^0_+, \psi^1_+, \ldots, \psi^m_+$ get the same boundary conditions, and likewise for ψ^μ_-, however, ψ^μ_+ and ψ^μ_- could still have different boundary conditions. So we have four total options for boundary conditions of, respectively, ψ^μ_+ and ψ^μ_-, namely R-R, R-NS, NS-R and NS-NS. These four options define the four different sectors of our theory.

For the open string, we need that $\psi^\mu_+\delta\psi_{\mu+} - \psi^\mu_-\delta\psi_{\mu-}$ vanishes at 0 and ℓ independently. Therefore, we need to impose

$$\psi^\mu_+(0) = \pm\psi^\mu_-(0)\,;$$
$$\psi^\mu_+(\ell) = \pm\psi^\mu_-(\ell)\,.$$

We can freely change the sign of ψ^μ_-, so the only boundary condition that is physical is the one at ℓ. We will not consider branes in this part of the thesis, so for now we assume we can only have Neumann boundary conditions for ψ^μ_+ and ψ^μ_- at the boundaries. This means that spacetime Poincaré invariance must be preserved at both boundaries of the open string, so we see that we must pick the same sign for every μ at ℓ. Thus, we see that we get

$$\psi^\mu_+(0) = \psi^\mu_-(0)\,; \tag{5.1.11}$$

$$\psi^\mu_+(\ell) = \pm\psi^\mu_-(\ell)\,, \tag{5.1.12}$$

where we again refer to the positive boundary condition at ℓ as Ramond, or R, and to the negative boundary condition at ℓ as Neveau-Schwarz, or NS. The open NN superstring has only two sectors, the NS sector and the R sector, depending on the boundary condition.

We can do mode expansions for the theory. The bosonic part gets the same result as in bosonic string theory, so we focus on the fermionic part. We see that for the closed string, with R boundary conditions for the respective chiralities, we get

$$\psi^\mu_+(\tau,\sigma) = \sqrt{\frac{2\pi}{\ell}} \sum_{n\in\mathbb{Z}} \bar{d}^\mu_n \, e^{-2\pi i n(\sigma+\tau)/\ell} \, ; \tag{5.1.13}$$

$$\psi^\mu_-(\tau,\sigma) = \sqrt{\frac{2\pi}{\ell}} \sum_{r\in\mathbb{Z}} d^\mu_n \, e^{-2\pi i n(\sigma-\tau)/\ell} \, . \tag{5.1.14}$$

Likewise, for NS boundary conditions in the respective chiralities, we get

$$\psi^\mu_+(\tau,\sigma) = \sqrt{\frac{2\pi}{\ell}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \bar{b}^\mu_r \, e^{-2\pi i r(\sigma+\tau)/\ell} \, ; \tag{5.1.15}$$

$$\psi^\mu_-(\tau,\sigma) = \sqrt{\frac{2\pi}{\ell}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r \, e^{-2\pi i r(\sigma-\tau)/\ell} \, . \tag{5.1.16}$$

Here, the normalisations are arbitrary, but have been chosen for later convenience.

For the NN open superstring with R boundary conditions, we have

$$\psi^\mu_\pm(\tau,\sigma) = \sqrt{\frac{\pi}{\ell}} \sum_{n\in\mathbb{Z}} d^\mu_n \, e^{-i\pi n(\sigma\pm\tau)/\ell} \, , \tag{5.1.17}$$

and for NS boundary conditions, we get

$$\psi^\mu_\pm(\tau,\sigma) = \sqrt{\frac{\pi}{\ell}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r \, e^{-i\pi r(\sigma\pm\tau)/\ell} \, . \tag{5.1.18}$$

The Majorana condition tells us $(d^\mu_n)^* = d^\mu_{-n}$, with analogous relations for all other Fourier coefficients. This completes our discussion of the classical superstring, now we turn to quantising.

Using the canonical anticommutation relations $\{\psi^\mu_A(\sigma), \psi^\nu_B(\sigma)\} = 2\pi\, \eta^{\mu\nu}\delta(\sigma-\sigma')\delta_{AB}$, with $A, B, \ldots$ spinor indices, we can derive the anticommutation relations

$$\{d^\mu_n, d^\nu_{n'}\} = \eta^{\mu\nu}\delta_{nn'} \, , \tag{5.1.19}$$

and analogous relations for the other Fourier modes, mixed anticommutation relations vanish. In particular, the d^μ_0 modes (up to a constant factor) satisfy the Dirac algebra, and are therefore Dirac matrices.

Letting α^μ_n denote the Fourier modes of the bosonic sector of the theory, we can now define the ground state of the R sector by $\alpha^\mu_n|0\rangle = d^\mu_n|0\rangle = 0$ for any $n > 0$, and likewise we can define the ground state for the NS sector by $\alpha^\mu_n|0\rangle = b^\mu_r|0\rangle = 0$ for $r, n > 0$. [BLT13] then shows that for the closed string, b^μ_r, d^μ_n and α^μ_n increase $\alpha' m^2$ by $-2r$, $-2n$ and $-2n$, respectively for $r, n < 0$, and for the open string, by $-r$, $-n$

and $-n$. In particular, the d_0^μ do not change the mass and therefore leave the ground state invariant. This means that the R sector ground state becomes an $SO(1, m-1)$ spinor.

When going to lightcone gauge, using the residual supersymmetry, one can gauge away $\psi^+ := \frac{1}{\sqrt{2}}(\psi^0 + \psi^1)$, where the superscript is now a spacetime index, not a spinor index. Moreover, as [BLT13] shows, the d_n^-, b_r^- (and their barred counterparts in case of a closed superstring) can be expressed, respectively, in terms of d_n^i and α_n^i and in terms of b_r^i and α_n^i, with i running over the transverse directions. In particular, we see that $b^\mu_{-1/2}|0\rangle$ in the NS sector transforms as a vector under $SO(m-2)$. The only way this is possible is if $b^i_{-1/2}|0\rangle$ is a massless vector under $SO(1, m-1)$. However, this means that the ground state of the NS sector has $\alpha' m^2 = -1/2$, i.e. it is a tachyon. Moreover, using ζ-function renormalisation, [BLT13] also shows that the normal order ambiguity in the NS sector satisfies $a_{\mathrm{NS}} = -\frac{1}{16}(m-2)$. So using $\alpha' m^2 = a_{\mathrm{NS}} = -\frac{1}{2}$ for the ground state, we obtain the critical dimension $m = 10$ for the superstring.

In fact, after imposing the equations of motion for the R sector ground state in $m = 10$, we only have 8 independent components, transforming as a Majorana-Weyl spinor under $\mathrm{Spin}(8)$ in the standard, or the conjugate representation. See [BLT13].

Moreover, we see that the R sector contains the spacetime fermions of the theory, while the NS sector contains the spacetime bosons. Thus, in the case of a closed superstring, the R-R and the NS-NS sector are bosonic, while the R-NS and NS-R sectors describe the fermions in the spectrum. In particular, in the NS sector, we get anticommuting operators mapping bosons to bosons, which is rather odd.

Finally, we see that the NS sector spectrum masses are half integer spaced, while the R sector masses are integer spaced, in particular, this means we cannot have spacetime supersymmetry, which requires massive bosons to have fermionic counterparts of equal mass.

All in all, there are quite some issues with the spectrum as of right now. Luckily, we are not forced to consider the full spectrum of the superstring. Rather, we can project out many states to get a theory that doesn't have these issues. This process is known as *GSO projection*, named after F. Gliozzi, J. Scherk and D.I. Olive [GSO77], see also [BLT13]. The basic idea is to define an operator $(-1)^F$, which counts the fermions. Then we associate $(-1)^F|0\rangle_{\mathrm{NS}} := -|0\rangle_{\mathrm{NS}}$, and multiply by -1 every time we add a fermion into the theory. Projecting the spectrum on $(-1)^F = 1$ for the NS sector then projects out the half integer states, including the tachyon. Likewise, in the R sector, we define $(-1)^F$ in a slightly more involved way, such that it is $+1$ on the vacuum in the standard representation of $\mathrm{Spin}(8)$, and -1 on the vacuum in the conjugate representation of $\mathrm{Spin}(8)$. Then GSO projection requires choosing a projection onto $(-1)^F = +1$ or $(-1)^F = -1$ for the R sector.

For the closed string, GSO projecting can be done in two inequivalent ways, one where the $(-1)^F$ in the R sector agrees with $(-1)^{\bar{F}}$, and one where the signs are changed. $(-1)^F = (-1)^{\bar{F}} = +1$ leads to *Type IIB superstring theory*. Its massless spectrum consists of the bosons

$$[(1) + (28) + (35)_v]_{(\mathrm{NS},\mathrm{NS})} + [(1) + (28) + (35)_s]_{(\mathrm{R},\mathrm{R})} \,,$$

where $(35)_v$ are the degrees of freedom of a (massless) graviton, $(35)_s$ are the degrees of freedom of a four-form with self dual exterior derivative, (28) are the degrees of freedom of an on-shell two-form, and (1) represents a scalar. The fermions in the massless spectrum are

$$[(8)_c + (56)_c]_{(\mathrm{R},\mathrm{NS})} + [(8)_c + (56)_c]_{(\mathrm{NS},\mathrm{R})} \,,$$

the $(8)_c$ are on-shell spin-1/2 spinors transforming in the conjugate representation of $\mathrm{Spin}(8)$, known as *dilatinos*, while the $(56)_c$ are spin-3/2 particles known as *gravitinos*. We see that the fermions and the bosons in the theory have equal number of degrees of freedom. In fact, it can be shown that this admits $N = 2$ spacetime supersymmetry, i.e. supersymmetry with two generators, at least when $M = \mathbb{R}^{1,9}$.

Choosing $(-1)^F = -(-1)^{\bar{F}}$ gives *Type IIA superstring theory*, which has particles

$$[(1) + (28) + (35)_v]_{(\mathrm{NS,NS})} + [(8)_v + (56)_v]_{(\mathrm{R,R})} + [(8)_c + (56)_c]_{(\mathrm{R,NS})} + [(8)_s + (56)_s]_{(\mathrm{NS,R})} ,$$

where the (NS,R)-sector now transforms under the standard representation of $\mathrm{Spin}(8)$, i.e. the (R,NS)- and (NS,R)-sectors are anti-chiral. Here, $(56)_v$ are the degrees of freedom of an on shell three-form field. We again obtain $N = 2$ spacetime supersymmetry.

Lastly, one can get a theory of unoriented open strings by taking Type IIB superstring theory and projecting onto the diagonal, i.e. making sure left and right movers are equal. The way we do this is by defining the *worldsheet parity operator* $P_\Sigma(\tau, \sigma) = (\tau, \ell - \sigma)$, where ℓ is the string length, and projecting onto the part of the spectrum that is symmetric under P_Σ . This gives *Type I superstring theory*, which has spectrum

$$[(1) + (35)_v]_{(\mathrm{NS,NS})} + [(28)]_{(\mathrm{R,R})} + [(8)_c + (56)_c]_{(\mathrm{NS,R})+(\mathrm{R,NS})} .$$

Again, see [BLT13] for details. This has $N = 1$ spacetime supersymmetry.

This completes our brief recap of superstring theory. The next step is to consider nontrivial target spaces, in order to obtain an effective theory in four dimensions.

5.2 Nontrivial target spaces: Calabi-Yau manifolds

Now we will consider compactifications to reduce $m = 10$ to a four dimensional theory, the idea is to make the extra six dimensions "small enough" such that they are no longer visible at low enough energies. If we assume no background fields, this requires the extra six dimensions to form a Calabi-Yau manifold, as was realised in 1985 by P. Candelas, G.T. Horowitz, A. Strominger and E. Witten [Can+85]. Our main reference will be [BLT13] again.

We shall start by showing why these Calabi-Yau manifolds pop up here, and then we shall use their properties to derive the effective four dimensional bosonic massless spectrum of the theory, also combining the bosons into supermultiplets, i.e. multiplets invariant under supersymmetry transformations From there, the massless fermionic spectrum could be derived, which we will not do here.

We let (M^{10}, G) be a ten dimensional Lorentzian manifold, such that it "looks like" a four dimensional manifold at low enough energies. The way that we do it is by assuming it decomposes like $(M^{10}, G) \cong (M^4, \widetilde{g}) \times (K^6, g)$, where M^4 is our four dimensional universe, and K^6 is some compact manifold[1], where we allow the metric g to vary smoothly over the M^4.

The first thing we see is that (K^6, g) should be a Riemannian manifold, as then $(M^4, \widetilde{g})$ inherits the time direction from M^{10} , i.e. $\widetilde{g}$ is a Lorentz metric, such that the effective four dimensional theory becomes a theory of general relativity. Moreover, we need (M^{10}, G) to have spacetime supersymmetry, so the particular (K^6, g) we choose should respect that.

[1]One would be inclined to assume the seemingly more general case where M^{10} is a K^6-fibre bundle over M^4 , however, such a generalisation isn't needed, as a fibre bundle is locally trivial, and we're interested in the local theory on M^4 anyway.

So suppose we start with $N = 1$ supersymmetry in ten dimensions. We let Q denote the generator of the supersymmetry, i.e. Q is a Majorana-Weyl spinor of $SO(1,9)$. If we let a spinor ϵ parameterise a supersymmetry transformation, we require $\bar{\epsilon}Q|0\rangle = 0$, i.e. supersymmetry leaves the vacuum invariant. In particular, since Q is an infinitesimal supersymmetry transformation, we see that for any field φ, we get $\delta_\epsilon\varphi = [\bar{\epsilon}Q, \varphi]$, which is just the adjoint action. In particular, we see that $\langle\delta_\epsilon\varphi\rangle = \langle 0|[\bar{\epsilon}Q,\varphi]|0\rangle = 0$ is a necessary condition for ϵ to parameterise a supersymmetry transformation. The first thing to note is that the vaccuum expectation value of any fermion must vanish, as fermions necessarily transform nontrivially under $\mathrm{Spin}(1,9)$, such that they break Poincaré invariance on Minkowski space. Thus any bosonic field has $\langle\delta_\epsilon\varphi_{\text{boson}}\rangle \propto \langle\varphi_{\text{fermion}}\rangle = 0$ irregardless of ϵ.

Thus whether ϵ parameterises a supersymmetry transformation is dependent on $\langle\delta_\epsilon\varphi_{\text{fermion}}\rangle = 0$. In particular, if ψ_μ denotes the graviton in $m = 10$, one can show

$$\delta_\epsilon\psi_\mu = \nabla_\mu\epsilon + \varphi\,,$$

where φ satisfies $\langle\varphi\rangle = 0$, and ∇ denotes the spin connection on M^{10}. Now, we see that the condition becomes $\langle\nabla_\mu\epsilon\rangle = 0$, so in particular, ϵ must be covariantly constant on K^6.

Thus, we see that spacetime supersymmetry requires (K^6, g) to be a Riemannian manifold that admits a spin structure such that it also has a global parallel spinor. We suppose for the moment that K^6 admits a spin structure, which requires K^6 to be orientable and the Stiefel-Whitney class $w_2(K^6)$ to vanish [BH59]. We will show the existence of a parallel spinor is related to the holonomy group $\mathrm{Hol}(g) \subseteq SO(6)$.

First, note the exceptional isomorphism $\mathrm{Spin}(6) \cong SU(4)$, since these are both simply connected integrations of $\mathfrak{so}(6)$ [Hum72]. So spinors on K^6 transform under $SU(4)$ or its conjugate representation, hence they can be described by four tuples of complex numbers. The holonomy principle tells us that a spinor is parallel if and only if it is parallelly transported along any piecewise smooth $\gamma : [0,1] \to K^6$. Such spinors therefore only exist if the action of $\mathrm{Hol}(g)$ has a fixed spinor. Since the spinors transform under $SU(4)$, having a spinor fixed by parallel transport means that $\mathrm{Hol}(g)$ actually lies inside $SU(3)$. Thus K^6 admits a parallel spinor if and only if $\mathrm{Hol}(g) \subseteq SU(3)$.

Now note that by Corollary 2.4.4, we see that any K^6 with holonomy inside $SU(3)$ admits a spin structure. Thus we see that the manifolds we are looking for are precisely the (K^6, g) with $\mathrm{Hol}(g) \subseteq SU(3)$.

Moreover, we see that if $\mathrm{Hol}(g) \subseteq SU(2)$, then $\mathrm{Hol}(g)$ even fixes (at least) two chiral spinors, so K^6 has (at least) two fibrewise linearly independent parallel spinors of the same chirality. This means that the $N = 1$ supersymmetry gets upgraded to an $N = 2$ supersymmetry on M^4, where the different supersymmetries come from the $N = 1$ supersymmetry on M^{10} with two independent spinors on K^6 giving two inequivalent ways of applying supersymmetry transformations.

Now, as [BLT13] mentions, supersymmetric extensions of the standard model require $N = 1$ supersymmetry, otherwise the theory does not allow for chiral matter. Therefore, in string theory, the most interesting manifolds are the manifolds that have $\mathrm{Hol}(g) = SU(3)$.

So now suppose we take $M^{10} \cong M^4 \times CY^3$, where CY^3 is a three complex dimensional (i.e. it has six real dimensions) Calabi-Yau manifold with global holonomy $\mathrm{Hol}(g) = SU(3)$. We want to know how the effective four dimensional spectrum looks like. We will only consider the Type II superstring, the Type I theory can be obtained by projecting the spectrum of the Type IIB superstring like before. To make things easier, we take a low energy limit, such that massive modes of the string get heavily suppressed.

Moreover, we take a *supergravity approximation*, which is to say we assume our CY^3 is much larger than our string, such that stringy effects do not change the theory. This assumption might seem a bit weird at first, given that the whole point is that the CY^3 should be small, such that it only appears at very high energies. However, the string itself lives at the Planck scale, so this assumption is just telling us that the CY^3 is slightly larger than the Planck scale.

To see how the geometry of the CY^3 then influences the low energy effective theory, we first look at the equations of motion for the various fields in the theory to find the possible modes. We will look at just the bosonic sector, as the fermionic sector modes can always be derived from there using supersymmetry. The bosonic sector contains a graviton, and various k-form fields depending on whether we consider the IIA or IIB superstring (note that a scalar field is a 0-form). The equations of motion of a k-form field A are governed by its associated field strength $F = dA$, in particular, there is some gauge symmetry left, namely $A \mapsto A + d\Lambda$, where Λ is some $(k-1)$-form. Depending on which directions we chose to lie on the CY^3, A induces l-forms on the M^4 for $l = 0, \ldots, k$, depending on the mode of the associated $(k-l)$-form on CY^3. E.g. if $A_{\mu\nu}$ is a two-form on M^{10}, it decomposes into $A_{mn} \oplus A_{mi} \oplus A_{ij}$, where $\mu, \ldots$ runs over the M^{10}, $m, \ldots$ runs over the M^4 and $i, \ldots$ runs over the CY^3, so we get a two-form on M^4 governed by a scalar on CY^3, a one-form governed by a one-form and a scalar governed by a two-form.

If A denotes a massless k-form on CY^3, its action is given by

$$S_A \propto \int_{CY^3} F \wedge \star F\,. \tag{5.2.1}$$

Now, by the Hodge decomposition theorem, $A = d\alpha + d^*\beta + \gamma$ for some harmonic γ, such that $d^*A = d^*d\alpha$. Thus we can fix the gauge by choosing $A \mapsto A - d\alpha$, i.e. $d^*A = 0$. Then we see that the equations of motion of A become

$$\Delta A = 0\,, \tag{5.2.2}$$

i.e. $d^*\beta = 0$. Thus we see that massless on-shell k-form instantons on CY^3 are precisely the harmonics.

Since harmonic forms represent cohomology, Proposition 2.4.6 tells us that scalars and six-forms have 1 mode, one- and five-forms have only the trivial mode, two- and four-forms have $h^{1,1}$ modes, and three-forms have $2h^{2,1} + 2$ modes.

Now we also have a graviton in the theory. This splits as $h_{\mu\nu} = h_{mn} + h_{mi} + h_{ij}$, so we get a graviton on M^4 governed by a scalar, a one-form governed by a one-form (which has no modes and therefore does not contribute) and some scalars on M^4 governed by h_{ij}, which is not a two-form! Now, in order to preserve supersymmetry, we note that the h_{ij} must preserve the Calabi-Yau structure, i.e. $g_{ij} + h_{ij}$ must still be a Ricci-flat Kähler metric. However, this metric need not be Kähler with respect to *the same* complex structure, i.e. h_{ij} can also change the complex structure of the underlying manifold. Thus, the h_{ij} split into two parts. Firstly parameters that fix the complex structure but change the metric to a different Ricci-flat Kähler metric, which means, by the Calabi-Yau theorem, that they generate translations in the Kähler cone. Secondly, we have h_{ij} that change the complex structure. [BLT13] shows that these *complex structure moduli* are generated by $H^1_{\bar\partial}(M; TM^{(1,0)})$, i.e. the Dolbeault cohomology of $TM^{(1,0)}$ at level 1, which could also be shown using e.g. the Kodaira-Spencer map and its properties [KS58].

Now, we note that our $M := CY^3$ has a holomorphic volume form Ω that is unique up to scaling. Thus, if we have a form α representing a class in $H^1_{\bar\partial}(M; TM^{(1,0)})$, we see that $\Omega \wedge \alpha$ defines a $\bar\partial$-closed $(2,1)$-form, where it is understood that the $TM^{(1,0)}$-part of α is plugged into Ω by interior multiplication.

Moreover, since Ω is holomorphic, we see that

$$\Omega \wedge \bar{\partial}\alpha = \bar{\partial}(\Omega \wedge \alpha)\,, \tag{5.2.3}$$

such that this process gives us a well defined map $\varphi : H^1_{\bar{\partial}}(M; TM^{(1,0)}) \to H^{(2,1)}(M)$. Now, if $\alpha \in \Omega^{(2,1)}(M)$, we see $\alpha = \sum_i \beta_i \wedge \gamma_i$, for some $\beta_i \in \Omega^{(2,0)}(M)$ and $\gamma_i \in \Omega^{(0,1)}(M)$. The β_i define global $(2,0)$-forms, thus nondegeneracy of Ω tells us they can be uniquely written as $\iota_{X_i}\Omega$ for some $X_i \in \Gamma(TM^{(1,0)})$, such that $\alpha = \Omega \wedge \sum_i X_i \otimes \gamma_i$. Therefore, any $(2,1)$-form α can be uniquely written as $\Omega \wedge \beta$ for some $\beta \in \Omega^{(0,1)}(M; TM^{(1,0)})$, where β is $\bar{\partial}$-closed if and only if α is. This shows surjectivity of φ. Now, if $\Omega \wedge \alpha = \Omega \wedge \beta + \bar{\partial}\gamma$ for some $\gamma \in \Omega^{(2,0)}(M)$, we know that there exists a unique $X \in \Gamma(TM^{(1,0)})$, such that $\gamma = \iota_X \Omega$, such that $\alpha = \beta + \bar{\partial}(X)$, i.e. they differ by an exact form, showing injectivity of φ. Thus, φ defines an isomorphism between $H^1_{\bar{\partial}}(M; TM^{(1,0)})$ and $H^{(2,1)}(M)$. In particular, we see that the complex structure deformations of CY^3 are classified by $H^{(2,1)}(M)$.

Now we have classified the different modes our fields can have on the CY^3, so it is time to combine the bosons on M^4 into supermultiplets, i.e. minimal supersymmetry invariant collections of particles. One thing to note is that we started with $N = 2$ supersymmetry on M^{10}, as we're considering Type II superstring theories. This means we get $N = 2$ supersymmetry on our M^4. According to [BLT13], the allowed massless supermultiplets we can consider are, firstly,

$$(2, 2 \times \tfrac{3}{2}, 1)_{\pm}\,,$$

where the $\pm$ indicates that the states with both helicities should be present in the multiplet. This multiplet contains the graviton of spin 2 together with two gravitino's of spin-3/2 and a *graviphoton*, which is a massless vector particle in this supermultiplet. We appropriately call this the *gravity multiplet*. The second multiplet we can use is

$$(1, 2 \times \tfrac{1}{2}, 0)_{\pm}\,,$$

consisting of a vector, two spin-1/2 particles and a scalar. We call this a *vector multiplet*. Lastly, we have a supermultiplet consisting of

$$(\tfrac{1}{2}, 2 \times 0, -\tfrac{1}{2}) + \text{h.c.}\,,$$

consisting of a spin-1/2 particle, its antichiral partner, and two spin-0 bosons, which we call a *hypermultiplet*. Note that we do not have Majorana-Weyl spinors in 6 Euclidean dimensions, so we have to make sure the supermultiplets are real, i.e. we need to add hermitian conjugates to the supermultiplets. By collecting them in this way, we are thus making sure the multiplets are invariant under CPT transformations.

If we carefully follow the compactifications, and if we let $\alpha, \beta, \ldots$ and $\bar{\alpha}, \bar{\beta}, \ldots$ denote complex indices, we get the following supermultiplets in our effective four dimensional theory for Type IIA, where we only give the bosons, the fermions can be obtained by applying supersymmetry transformations:

$$\begin{aligned}
\text{1 Gravity multiplet:} &\qquad \{h_{mn}, (C_1)_m\} \\
\text{1 Hypermultiplet:} &\qquad \{\Phi, B_{mn}, (C_3)_{\alpha\beta\gamma}, (C_3)_{\bar{\alpha}\bar{\beta}\bar{\gamma}}\} \\
h^{2,1}\ \text{Hypermultiplets:} &\qquad \{h_{\alpha\beta}, h_{\bar{\alpha}\bar{\beta}}, (C_3)_{\alpha\bar{\beta}\bar{\gamma}}, (C_3)_{\bar{\alpha}\beta\gamma}\} \\
h^{1,1}\ \text{Vectormultiplets:} &\qquad \{(C_3)_{m\alpha\bar{\beta}}, g_{\alpha\bar{\beta}}, B_{\alpha\bar{\beta}}\}\,.
\end{aligned}$$

Here h is the graviton, $B_{\mu\nu}$ is the $(\mathrm{NS},\mathrm{NS})$-two-form, i.e. the two-form in the $(\mathrm{NS},\mathrm{NS})$-sector, Φ is the dilaton, i.e. the scalar in the $(\mathrm{NS},\mathrm{NS})$-sector, and C_1 and C_3 are, respectively, the one-form and the three-form in the (R,R)-sector.

Likewise, for the Type IIB superstring, we get the following massless bosons in the effective four dimensional theory:

$$\begin{aligned} 1\text{ Gravity multiplet:}\quad & \{h_{mn}, (C_4)_{m\alpha\beta\gamma}\} \\ 1\text{ Hypermultiplet:}\quad & \{\Phi, B_{mn}, C_0, (C_2)_{mn}\} \\ h^{1,1}\text{ Hypermultiplets:}\quad & \{h_{\alpha\bar{\beta}}, B_{\alpha\bar{\beta}}, (C_2)_{\alpha\bar{\beta}}, (C_4)_{mn\alpha\bar{\beta}}\} \\ h^{2,1}\text{ Vectormultiplets:}\quad & \{(C_4)_{m\alpha\bar{\beta}\bar{\gamma}}, h_{\alpha\beta}, h_{\bar{\alpha}\bar{\beta}}\}\,. \end{aligned}$$

Here, C_0, C_2 and C_4 denote the various k-forms in the (R,R)-sector. Note that C_4 has self dual field strength, so a few of its degrees of freedom are removed.

We see a peculiar feature of the theory: compactifying a Type IIA superstring on a CY^3 gives the same massless spectrum as compactifying a Type IIB superstring on a $\widehat{CY^3}$ whose Hodge diamond is flipped, i.e. $\hat{h}^{1,1} = h^{2,1}$ and $\hat{h}^{2,1} = h^{1,1}$, where h are the Hodge numbers of CY^3 and $\hat{h}$ are the Hodge numbers of $\widehat{CY^3}$. This particular symmetry is known as *mirror symmetry*.

This gives a rather interesting geometric question: given a three-dimensional Kähler manifold (M,g) with $\mathrm{Hol}(g) = SU(3)$, does there exist another three-dimensional Kähler manifold (M',g') with $\mathrm{Hol}(g') = SU(3)$, such that the Hodge diamond of M' is the flipped Hodge diamond of M?

As of yet, this question remains unanswered, it is part of a field of study known as *mirror symmetry*. This field of study is rather big, for an overview, see e.g. [Hor+03]. In [CLS90], a lot of evidence for this conjecture has been shown using computer analysis on Calabi-Yau's that arise as submanifolds of weighted $\mathbb{C}P^4$'s, which is a generalisation of the quintic in $\mathbb{C}P^4$ from Example 2.4.11.

5.3 Breaking supersymmetry

Now we have compactified Type II string theory on a CY^3 to get an effective four dimensional theory with $N = 2$ supersymmetry. This is still not sufficient, we said earlier we want $N = 1$ spacetime supersymmetry in the effective four dimensional theory. Thus, we have to break a supersymmetry. One way to do this is by doing an *orientifold projection*. Recall that we constructed the spectrum of the Type I superstring in Minkowski space by taking a Type IIB superstring and identifying left- and right-moving modes using the worldsheet partiy operator P_Σ. Here, we do something similar, but to keep a theory of closed strings, we first assume that we can find a holomorphic isometric involution $\theta : CY^3 \to CY^3$, i.e. a holomorphic map such that $\theta^2 = \mathrm{id}$. This leaves two possibilities, since θ is a holomorphic isometry, either $\theta^*\Omega = \Omega$, or $\theta^*\Omega = -\Omega$. Now, we want our string spectrum to be invariant under simultaneous application of P_Σ on the worldsheet, and θ on the CY^3. However, this does mean we want $(P_\Sigma, \theta)^2 = \mathrm{id}$, so [BLT13] notes that if $\theta^*\Omega = -\Omega$, we need to include a factor of $(-1)^{F_L}$ on the P_Σ, where F_L is the left moving fermion number. Thus, we define an operator S such that $S = P_\Sigma$ if θ preserves Ω, and $S = (-1)^{F_L} P_\Sigma$ if θ changes the sign of Ω.

Now, we can apply this to the full string theory by extending θ to an isometric involution on $M^{10} \cong M^4 \times CY^3$ by letting it act trivially on the M^4 factor. Then we project onto the states that are invariant under (S,θ).

Now, we can do a similar trick for the Type IIA superstring, only now we need θ to also change the chiralities. Thus, we pick θ to be an anti-holomorphic involution on CY^3. We see $\theta^*\Omega = e^{2\pi i\phi}\,\overline{\Omega}$, where ϕ is some constant phase factor. However, we can get rid of ϕ by multiplying Ω by a phase factor $e^{2\pi i\phi}$, so without loss of generality, $\theta^*\Omega = \overline{\Omega}$. The associated involution of the theory is then $((-1)^{F_L}P_\Sigma, \theta)$, and we project onto the states that are invariant under this operator.

Another way to break the $N = 2$ supersymmetry would be to introduce extra background fields on the CY^3 in order to get rid of some extra degrees of freedom. For instance, one could turn on a flux of the (NS, NS)-three-form $H := dB$. The way to do this would be to let H be an exact three-form that defines a nonzero cohomology class in $H^3(CY^3;\mathbb{R})$, which is at least two-dimensional. These kind of fluxes generally back-react on the metric to break Ricci-flatness, but if we're lucky, we could still find parallel spinors if we introduce torsion into the connection. This goes beyond the scope of this thesis and will not go into details, we refer the reader to [BLT13] of [DK07].

This completes our (rather brief) discussion of superstring theory and the role Calabi-Yau manifolds play in the theory. This is by no means meant as a complete overview, as superstring theory is a very deep subject with many interesting results. Some things we didn't mention were the *heterotic string*, introduced in 1995 by D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm [Gro+95], which is another way of getting a string theory with fermions in $m = 10$, with $N = 1$ supersymmetry. Here, the idea is to have a left-moving sector of a bosonic string theory combines with a right-moving sector of a superstring theory.

Moreover, we have not mentioned *branes*, which can be viewed as higher dimensional analogues of strings. Introducing branes into the theory can be used to e.g. get a theory of open superstrings with Dirichlett boundary conditions, whose endpoints end on a brane. These branes will have to be charged under the various (R, R)-forms in order to be able to couple to these strings. The upshot is then also that these branes become stable objects. These objects were introduced in the context of superstring theory by J. Polchinski in 1995 [Pol95].

Then there is also M-theory, where the fundamental objects are not strings, but rather branes of different dimensions. This is a famous theory due to E. Witten. This is a theory with critical dimension $m = 11$, and to get minimal supersymmetry in $m = 4$, it would have to be compactified on a G_2-manifold, i.e. a Riemannian manifold with G_2 holonomy. For an introduction, see [BBS07].

Moreover, string theory has many dualities associated to it, which intertwine various types of string theories and M-theory. One example we saw above was mirror symmetry, which is a symmetry between Type IIA and IIB superstrings. Mirror symmetry can be thought of as a type of T-duality (T stands for "target space"), which also appears as a duality between different string theories compactified on tori. For an introduction to dualities in string theory, see [BBS07].

Applying T-duality transformations to certain settings with (NS, NS)-flux present requires introducing new fluxes that lead to a *non-geometric* theory, i.e. the theory itself is no longer a theory of strings compactified on some manifold, but rather the extra six dimensions in the theory are assumed to be various fields on the string that do not necessarily have a strict geometric interpretation. One model of such strings is a model of *symplectic gravity*, which is described in [Blu+13], and uses the theory of Lie algebroids, to which we will get back in Section 6.4.

Lastly, there is a notion of F-theory, which is a theory in 12 dimensions. When we assume the

12-dimensional manifold is a T^2 fibration over some 10-dimensional manifold, one obtains Type IIB superstring theory. This is a rather deep theory due to C. Vafa [Vaf96], and goes well beyond the scope of this thesis, for a brief introduction, see [BBS07].

6 Outlook: Lie algebroids

So far, we have studied the classical theory of Calabi-Yau manifolds, by studying their existence, using the Calabi-Yau theorem. Moreover, we provided an explicit example of a non-compact Calabi-Yau manifold and we studied a particularly big application of the theory, namely the theory of superstrings. Now we will venture into a bit more uncharted territory and we will propose and study a possible generalisation of Calabi-Yau manifolds to the setting of Lie algebroids.

As a bit of motivation, in literature, there is a notion of *log Calabi-Yau manifolds*, see e.g. [DKW13; Ish00; GHK15], which are defined by (existence of) a global holomorphic top forms with particular divergence behaviour towards a divisor. The downside of this definition of log Calabi-Yau manifolds is that it doesn't ask whether this holomorphic top form is induced by a Ricci-flat Kähler metric. Moreover, since Ricci-flat Kähler metrics can also have a torsion canonical class, they do not necessarily admit a global holomorphic top form, so this definition of log Calabi-Yau also does not allow for those kinds of settings. By using Lie algebroids, we can define a setting where Riemannian metrics with logarithmic divergences are allowed, so we can formulate an alternative definition of log Calabi-Yau manifolds where we do take into account the underlying Riemannian metric.

Lie algebroids form a natural setting to do geometry on, as we will discuss in Sections 6.1 and 6.2, so it is natural to try to ask the same questions as in the Riemannian setting. Thus, in Section 6.3, we will start with developing the theory of Riemannian Lie algebroids, in particular Kähler and Calabi-Yau Lie algebroids.

Finally, in Section 6.4 we discuss a bit how Lie algebroids pop up in modern day string theory and in Section 6.5 we shall suggest a place where Calabi-Yau Lie algebroids might pop up in physics, though that suggestion is very underdeveloped as of yet and might not be of interest after all.

Note that much of this chapter, namely Sections 6.2, 6.3 and 6.5, is work-in-progress research and is far from a completely finished story, so many of the results are only intermediate and a final conclusion is lacking as of yet.

6.1 Lie algebroids: basic results and examples

We will start by introducing Lie algebroids and giving a few examples. In the next two sections, we will put geometric structures on these Lie algebroids and we will see how classical geometry can quite easily be lifted to Lie algebroids. In Section 6.4 we will also give an application of Lie algebroids in physics, using a few of the geometric structures that we introduce on Lie algebroids.

In this section, we assume the reader is familiar with differential geometry, we will not develop the full theory of Lie algebroids as it is not needed for this thesis. A good resource to learn about Lie algebroids is [CFM21].

Lie algebroids are vector bundles equipped with a particular structure that makes them act like the tangent bundle in many ways. They were introduced in 1967 by J. Pradines as a tool to study Lie groupoids infinitesimally [Pra67]. The correspondence between Lie algebroids and Lie groupoids has since been extensively studied, with a famous result by M. Crainic and R.L. Fernandes giving necessary and sufficient conditions for Lie algebroids to be integrated to Lie groupoids [CF03]. In this thesis, however, we will not discuss any Lie groupoids. Rather, we will focus on the Lie algebroids themselves, as many geometric constructions can be quite easily generalised to this setting, as we shall see in Chapter 6.

Let's begin with a definition

Definition 6.1.1 (Anchored vector bundle)**.** An *anchored vector bundle* is a pair $(\mathcal{A}, \rho)$, where $\mathcal{A} \to M$ is a real vector bundle, and $\rho : \mathcal{A} \to TM$ is a morphism of vector bundles.

The idea behind this definition is that it gives vectors in the vector bundle a sense of direction. In particular, given a function $f \in C^\infty(M)$ and a vector $v \in \mathcal{A}$, we can make sense of the derivative of f in the direction of v because we can compute $\rho(v)(f)$.

However, the tangent bundle also carries an internal differentiation, namely the Lie bracket. Imitating this on an anchored vector bundle, we get the notion of a Lie algebroid

Definition 6.1.2 (Lie algebroid)**.** A *Lie algebroid* is a triple $(\mathcal{A}, \rho, [-,-]_\mathcal{A})$, where $(\mathcal{A}, \rho)$ is an anchored vector bundle over M, and $[-,-]_\mathcal{A}$ is a Lie bracket on the sheaf of sections $\Gamma^\infty(\mathcal{A})$, such that the Leibniz rule is satisfied:

$$[v, fw]_\mathcal{A} = \rho(v)(f)w + f[v,w]_\mathcal{A}\,, \qquad \forall f \in C^\infty(M)\,,\, v, w \in \Gamma(\mathcal{A})\,. \tag{6.1.1}$$

Note that often we will refer to $\mathcal{A}$ as the Lie algebroid and leave the structure maps implicit.

In particular, this definition gives us the following:

Proposition 6.1.3. *Let $(\mathcal{A}, \rho, [-,-]_\mathcal{A})$ be a Lie algebroid, then the induced map $\rho : \Gamma(\mathcal{A}) \to \mathfrak{X}(M)$ is a morphism of Lie algebras.*

Proof. Let $f \in C^\infty(M)$, $v, w, u \in \Gamma(\mathcal{A})$. The Jacobi identity tells us

$$[[v,w]_\mathcal{A}, fu]_\mathcal{A} + [[w, fu]_\mathcal{A}, v]_\mathcal{A} + [[fu, v]_\mathcal{A}, w]_\mathcal{A} = 0\,,$$

and computing some stuff tells us

$$\begin{aligned}
&[[v,w]_\mathcal{A}, fu]_\mathcal{A} = f[[v,w]_\mathcal{A}, u]_\mathcal{A} + \rho([v,w]_\mathcal{A})(f)u\,;\\
&[[w,fu]_\mathcal{A}, v]_\mathcal{A} = f[[w,u]_\mathcal{A}, v]_\mathcal{A} + \rho(w)(f)[u,v]_\mathcal{A} - \rho(v)\circ\rho(w)(f)u - \rho(v)(f)[w,u]_\mathcal{A}\,;\\
&[[fu,v]_\mathcal{A}, w]_\mathcal{A} = f[[u,v]_\mathcal{A}, w]_\mathcal{A} - \rho(v)(f)[u,w]_\mathcal{A} + \rho(w)\circ\rho(v)(f)u - \rho(w)(f)[u,v]_\mathcal{A}\,.
\end{aligned}$$

Putting everything together and applying the Jacobi identity gives us

$$0 = (\rho([v,w]_\mathcal{A}) - (\rho(v)\circ\rho(w) - \rho(w)\circ\rho(v)))(f)u = (\rho([v,w]_\mathcal{A}) - [\rho(v), \rho(w)])(f)u\,.$$

Since f and u were arbitrary, we finally obtain

$$(\rho([v,w]_\mathcal{A}) = [\rho(v), \rho(w)]\,, \tag{6.1.2}$$

as desired. ∎

In particular, $\rho(\mathcal{A}) \subseteq TM$ defines a singular involutive distribution on M. We have the following theorem that tells us this singular distribution is in fact integrable into a partition of M into immersed submanifolds.

Theorem 6.1.4. *Let $\mathcal{A}$ be a Lie algebroid. Then there is a partition of M into connected immersed submanifolds $\mathcal{S} = \{L \subseteq M\}$ such that $M = \coprod_{L\in\mathcal{S}} L$ and $T_xL = \rho(\mathcal{A}_x)$, for every $L \in \mathcal{S}$ and $x \in L$.*

See [CFM21] for a proof. Note that in the regular case, i.e. when $\rho(\mathcal{A})$ is a regular distribution, this is just the Frobenius theorem. Analogously to the regular setting, we will call $\mathcal{S}$ the (singular) *foliation* of $\mathcal{A}$ and $L \in \mathcal{S}$ will be called a *leaf* of $\mathcal{A}$.

Moreover, we see that ρ is not necessarily injective, therefore it can have a kernel. In fact, we see

Lemma 6.1.5. *Let $\mathcal{A}$ be a Lie algebroid, $x \in M$. Then $\ker(\rho_x)$ admits a natural Lie bracket inherited from $[-,-]_{\mathcal{A}}$.*

Proof. Let $v, w \in \ker(\rho_x)$. Suppose $\tilde{v}$ and $\tilde{w}$ are local extensions of v and w, respectively. We claim that $[\tilde{v}, \tilde{w}]_{\mathcal{A}}(x)$ is independent of the chosen extension. To prove this, pick a frame $\{e_1, \dots, e_k\}$ for $\mathcal{A}$ around x and let $\tilde{w}^i$ be the coordinate functions of $\tilde{w}$. Then we see $\rho_x(\tilde{v})(\tilde{w}^i) = 0$, hence

$$[\tilde{v}, \tilde{w}]_{\mathcal{A}}(x) = \sum_i [\tilde{v}, e_i]_{\mathcal{A}}(x),$$

which is clearly independent of the chosen extension $\tilde{w}$, so by skew-symmetry, it is also independent of the chosen extension $\tilde{v}$. Then we define $[v, w]_x := [\tilde{v}, \tilde{w}]_{\mathcal{A}}(x)$ as the Lie bracket on $\ker(\rho_x)$. ☐

We call $\mathfrak{g}_x := (\ker(\rho_x), [-,-]_x)$ the *isotropy Lie algebra* of $\mathcal{A}$ at x. This turns out to be a leafwise invariant.

Lemma 6.1.6. *Let $\mathcal{A}$ be a Lie algebroid. Let L be a leaf of $\mathcal{A}$, then for any $x, y \in L$, $\mathfrak{g}_x \cong \mathfrak{g}_y$.*

Proof. Note that $E_L := \ker(\rho|_L)$ defines a vector bundle over L. Now there's a neat trick, we see by involutivity of $\rho(\mathcal{A})$ that we can restrict $[-,-]_{\mathcal{A}}$ to $\Gamma(\mathcal{A}|_L)$. Now we have a short exact sequence of vector bundles

$$0 \to E_L \hookrightarrow \mathcal{A}|_L \xrightarrow{\rho} TL \to 0.$$

Pick a splitting $\tau : TL \to \mathcal{A}|_L$. Now we can define a connection $\nabla : E_L \to \Omega^1(E_L)$ by $\nabla_X v = [\tau(X), v]_{\mathcal{A}|_L}$, where $X \in \mathfrak{X}(L)$ and $v \in \Gamma(E_L)$. This is a connection because

$$[\tau(fX), v]_{\mathcal{A}|_L} = f[\tau(X), v]_{\mathcal{A}|_L} + \rho(v)(f)\tau(X) = f[\tau(X), v]_{\mathcal{A}|_L},$$

and

$$[\tau(X), fv]_{\mathcal{A}|_L} = f[\tau(X), v]_{\mathcal{A}|_L} + \rho(\tau(X))(f)v = f[\tau(X), v]_{\mathcal{A}|_L} + X(f)v.$$

Moreover, we see that the bracket $[-,-]_x$ of $\mathfrak{g}_x$ varies smoothly over E_L, as for sections $v, w \in \Gamma(E_L)$, we have $[v, w]_x = [v, w]_{\mathcal{A}|_L}(x)$. Thus $[-,-]_x$ defines a tensor on E_L. We also have that ∇ is compatible with $[-,-]_x$, as we have for sections $v, w \in \Gamma(E_L)$ and $X \in \mathfrak{X}(L)$,

$$\nabla_X [v, w]_{\mathcal{A}|_L} = [\tau(X), [v, w]_{\mathcal{A}|_L}]_{\mathcal{A}|_L} = [[\tau(X), v]_{\mathcal{A}|_L}, w]_{\mathcal{A}|_L} + [v, [\tau(X), w]_{\mathcal{A}|_L}]_{\mathcal{A}|_L} = [\nabla_X v, w]_{\mathcal{A}|_L} + [v, \nabla_X w]_{\mathcal{A}|_L}.$$

In particular, parallel transport preserves $[-,-]_x$, so parallel transport along a path $\gamma : x \Rightarrow y$ in L defines an isomorphism of Lie algebras $\mathfrak{g}_x \cong \mathfrak{g}_y$. ☐

We end this section with some examples of Lie algebroids

Example 6.1.7 (Foliation)**.** By the Frobenius theorem, foliations of M, i.e. a partitions of M into equidimensional immersed submanifolds $M = \coprod_{L\in\mathcal{S}} L$, correspond to regular involutive distributions, i.e. subbundles $D \subseteq TM$ such that $[v,w] \in \Gamma(D)$ for any $v, w \in \Gamma(D)$. This gives D the structure of a Lie algebroid with anchor map $\iota : D \hookrightarrow TM$ and bracket $[-,-]_D := [-,-]|_{\Gamma(D)}$.

Example 6.1.8 (Lie algebra)**.** Let $\mathfrak{g}$ be a Lie algebra. Then the trivial map $\mathfrak{g} \to *$ defines a vector bundle over $*$. This is a Lie algebroid with anchor $\rho : \mathfrak{g} \to 0 = T*$ and the Lie bracket of $\mathfrak{g}$ as its bracket. It has a single leaf, namely $*$, and the isotropy Lie algebra is just $\mathfrak{g}$.

Example 6.1.9 (Poisson manifold)**.** We will discuss this example rather briefly and we will not prove too much. The theory of Poisson manifolds is very rich with many deep results, see e.g. [CFM21] for an introduction, where this example will be discussed in much, much more detail.

A Poisson manifold is a pair (M,π) of a smooth manifold M and a bivectorfield $\pi \in \mathfrak{X}^2(M)$ that commutes with itself, i.e. the Schouten-Nijenhuis bracket $[\pi,\pi]_{\mathrm{SN}} = 0$. Such bivectors define a map $\pi^\sharp : T^*M \to TM$ by $\pi^\sharp(\alpha) = \pi(\alpha, -)$, and it defines a Lie bracket on $\Gamma(T^*M)$ by

$$[\alpha,\beta]_\pi := \pi^\sharp(\alpha)(\beta) - \pi^\sharp(\beta)(\alpha) - d(\pi(\alpha,\beta)).$$

This turns T^*M into a Lie algebroid called the *cotangent Lie algebroid* of (M,π). The associated foliation inherits a symplectic structure ω_L on every leaf $L \in \mathcal{S}$, defined by $\omega_L(X,Y) = -\pi(\alpha,\beta)$, where $X, Y \in T_xL$ and $\alpha, \beta \in T^*_xM$ are covectors such that $\pi^\sharp(\alpha) = X$ and $\pi^\sharp(\beta) = Y$.

In particular, symplectic manifolds have a natural Lie algebroid structure on their cotangent bundle.

Example 6.1.10 (Log-tangent bundle)**.** Let M be a manifold of dimension $m+1$ and take an embedded codimension 1 submanifold $N \hookrightarrow M$. Let $\mathfrak{X}_N$ be the sheaf of vector fields on M that are tangent to M. We can find slice charts (U,φ) with coordinates $\{x_0,\ldots,x_m\}$ around N such that $\varphi(U\cap N) = \{x_0 = 0\}$, see Figure 6.1.1. We see that in this chart, $\mathfrak{X}_N(U)$ is generated by the vector fields $\{x_0\partial_{x_0}, \partial_{x_1}, \ldots, \partial_{x_m}\}$.

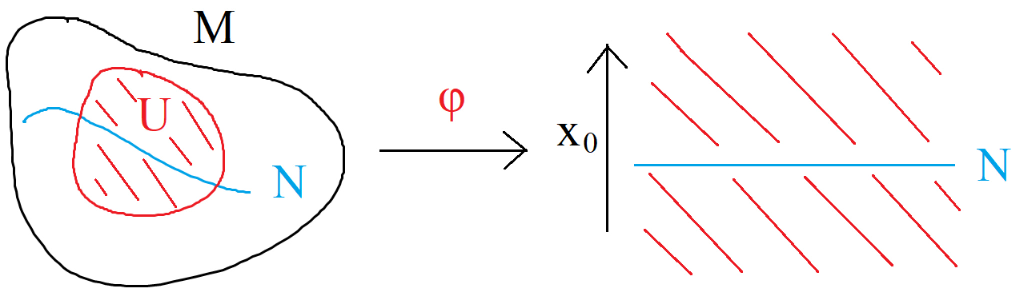

Figure 6.1.1: Slice chart for N.

In charts (U,φ) such that $U\cap N = \emptyset$, we have that $\mathfrak{X}_N(U)$ is generated by $\{\partial_{x_0}, \partial_{x_1}, \ldots, \partial_{x_m}\}$, i.e. $\mathfrak{X}_N$ is a locally free sheaf and therefore defines a vector bundle $TM(-\log(N))$. The terminology comes from the idea that the dual bundle $TM(\log(N)) := (TM(-\log(N)))^*$ is now generated by $\{dx_0/x_0, dx_1, \ldots, dx_m\}$, i.e. we have made sense of the *logarithmic form* $d\log(x_0) := dx_0/x_0$. Note that $\mathfrak{X}_N$ is closed under $[-,-]_{TM}$, therefore $TM(-\log(N))$ defines a Lie algebroid over M with bracket $[-,-]_{TM}|_{\mathfrak{X}_N}$ and anchor induced by the inclusion $\mathfrak{X}_N \hookrightarrow \mathfrak{X}$. This Lie algebroid is called the *log-tangent bundle*, or sometimes the *b-tangent bundle*, which is how it was introduced by R.B. Melrose [Mel93].

The leaves of this Lie algebroid are the connected components of $M \setminus N$ and the connected components of N. The isotropy Lie algebras are all abelian.

This example generalises to the setting of complex manifolds in a rather nice way

Example 6.1.11 (Elliptic tangent bundle)**.** Let M be a complex manifold of (complex) dimension $m+1$ with sheaf of holomorphic functions $\mathcal{O}_M$, and let $N \hookrightarrow M$ be an embedded complex submanifold of (complex) codimension 1. Just like in the previous example, we define $\mathfrak{X}_N^{\mathrm{hol}}$ as the sheaf of holomorphic vector fields tangent to N, and we can take holomorphic slice charts, i.e. holomorphic charts (U, φ) with coordinates $\{z_0, \dots, z_m\}$ such that $\varphi(U \cap N) = \{z_0 = 0\}$. We see that $\mathfrak{X}_N^{\mathrm{hol}}(U)$ in this chart is generated by $\{z_0 \partial_{z_0}, \partial_{z_1}, \dots, \partial_{z_m}\}$. in charts (U, φ) with $U \cap N = \emptyset$, $\mathfrak{X}_N^{\mathrm{hol}}(U)$ is generated by $\{\partial_{z_0}, \partial_{z_1}, \dots, \partial_{z_n}\}$, so $\mathfrak{X}_N^{\mathrm{hol}}$ is a locally free sheaf of $\mathcal{O}_M$-modules, so it defines a holomorphic vector bundle $TM(-\log(N))$. Likewise, we can define the anti-holomorphic vector bundle $TM(-\log(\overline{N}))$ that, in slice charts, is generated by $\{\bar{z}_0 \partial_{\bar{z}_0}, \partial_{\bar{z}_1}, \dots, \partial_{\bar{z}_m}\}$. Thus we can make sense of holomorphic logarithmic forms $d\log(z_0)$ and anti-holomorphic logarithmic forms $d\log(\bar{z}_0)$. This idea of a *holomorphic tangent bundle* is the complex analogue of the real log tangent bundle, and it has existed already since the 1970s [Del70]. By studying this in the broader context of smooth vector bundles, and not just holomorphic vector bundles, we can extract a real vector bundle from this, as was done in [CG15], which is what we show now.

We consider the smooth vector bundle $TM(-\log(N)) \oplus TM(-\log(\overline{N}))$, whose sheaf of sections is closed under $[-,-]_{TM}$, and we note that it has a canonical involution $\iota(v) = \bar{v}$ that defines a real structure. So we can take the underlying real vector bundle $TM(-\log|N|)$, whose sheaf of sections $\Gamma(TM(-\log|N|))$ is a subsheaf of $\mathfrak{X}$, of smooth vector fields. Moreover, $\Gamma(TM(-\log|N|))$ is closed under $[-,-]_{TM}$, therefore $TM(-\log|N|)$ is a Lie algebroid with bracket $[-,-]_{TM}|_{\Gamma(TM(-\log|N|))}$ and anchor induced by the inclusion $\Gamma(TM(-\log|N|)) \hookrightarrow \mathfrak{X}$.

In a slice chart $\{z_0 = re^{i\theta}, z_1 = x_1 + iy_1, \dots, z_m = x_m + iy_m\}$, we have that $\Gamma(TM(-\log|N|))$ is generated by

$$\{r\partial_r = z_0 \partial_{z_0} + \bar{z}_0 \partial_{\bar{z}_0}, \partial_\theta = i(z_0 \partial_{z_0} - \bar{z}_0 \partial_{\bar{z}_0}), \partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_m}, \partial_{y_m}\}.$$

So the dual $TM(\log|N|)$ is generated by $\{d\log r, d\theta, dx_1, dy_1, \dots, dx_m dy_m\}$.

The leaves of this Lie algebroid are the connected components of $M \setminus N$ and the connected components of N. The isotropy Lie algebras are all abelian.

6.2 Basic geometric constructions

Now that we have introduced these Lie algebroids along with claims that we can put geometric structure on them, it is finally time to show what we mean. Since Lie algebroids have the same algebraic structure as the tangent bundle, i.e. a way to differentiate functions and a Lie bracket, most things that are defined on TM using a formula carry over to the Lie algebroid, using the same formula, which means the "mimicked" structure on the algebroid has a lot of the same properties that the classical structure on the tangent bundle would have. We will show we can define a cohomology theory analogously to de Rham cohomology, and we will introduce a theory of $\mathcal{A}$-connections, curvature and even parallel transport. For this, we shall need a notion of morphism between Lie algebroid over different bases, as will become apparent later. We will also give some examples. In the next section, we will put more structure on $\mathcal{A}$, namely something analogous to a Riemannian metric, and we will see that the classical theory of the Levi-Civita connection also carries over.

The first thing we can do is generalise the de Rham cohomology to this setting, as we have all ingredients needed for a Koszul-type formula for the de Rham derivative.

Definition 6.2.1 ($\mathcal{A}$-differential forms)**.** Let $\mathcal{A} \to M$ be a Lie algebroid. Then for any $k \in \mathbb{N}_0$, we can define the space of *$\mathcal{A}$-differential k-forms* by $\Omega^k(\mathcal{A}) = \Gamma^\infty(\Lambda^k \mathcal{A}^*)$. Likewise, for a vector bundle $E \to M$, one has $\mathcal{A}$-differential k-forms with values in E, which forms the space $\Omega^k(\mathcal{A}; E) = \Gamma^\infty(\Lambda^k \mathcal{A}^* \otimes E)$.

Moreover, we have a de Rham type derivative $d_{\mathcal{A}} : \Omega^k(\mathcal{A}) \to \Omega^{k+1}(\mathcal{A})$ called the *$\mathcal{A}$-differential*, defined as follows: let $v_0, \ldots, v_k \in \Gamma^\infty(\mathcal{A})$ and let $\omega \in \Omega^k(\mathcal{A})$, then we define

$$d_{\mathcal{A}}\omega(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i \rho(v_i)(\omega(v_0, ..., \widehat{v_i}, ..., v_k)) + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([v_i, v_j]_{\mathcal{A}}, v_0, ..., \widehat{v_i}, ..., \widehat{v_j}, ..., v_k)\,. \tag{6.2.1}$$

Because the formula for the Lie algebroid differential agrees with the formula for the de Rham differential, many properties with a purely algebraic proof directly carry over to this setting. In particular, we have the following:

Lemma 6.2.2. *The $\mathcal{A}$-differential satisfies*

(i) $d_{\mathcal{A}}^2 = 0$;

(ii) *for every $\omega \in \Omega^k(\mathcal{A})$ and $\eta \in \Omega^{k+l}(\mathcal{A})$, $d_{\mathcal{A}}(\omega \wedge \eta) = d_{\mathcal{A}}\omega \wedge \eta + (-1)^k \omega \wedge d_{\mathcal{A}}\eta$.*

Therefore, $(\Omega^\bullet(\mathcal{A}), d_{\mathcal{A}})$ defines a chain complex, such that $d_{\mathcal{A}}$ is a graded differential. Therefore, we get an associated cohomology theory $H^\bullet(\mathcal{A}) := H^\bullet(\Omega^\bullet(\mathcal{A}))$ that is compatible with $\wedge$. This cohomology is known as the *Lie algebroid cohomology* of $\mathcal{A}$.

Generically, the Lie algebroid cohomology is difficult to compute, as it has no Poincaré lemma.

Example 6.2.3 (Foliations)**.** Let D be a rank k foliation of M^m viewed as a Lie algebroid, i.e. an involutive distribution $D \subseteq M$ of rank k. Let (U, φ) be a foliation chart with codomain $\mathbb{R}^m$, i.e. $\varphi_*(D) = \{x_{k+1}, \ldots, x_m = 0\}$. On U, $\Omega^l(D)$ is $C^\infty(\mathbb{R}^m)$-generated by $\{dx^{i_1} \wedge \cdots \wedge dx^{i_l} | i_1 < \cdots < i_l \le k\}$. If $\omega \in \Omega^l(D)$, we see that $d_{\mathcal{A}}\omega = 0$ if and only if $\omega|_L$ is closed, where L is any leaf in $\varphi(U)$, i.e. $L = \mathbb{R}^k \times \{\lambda\}$ for some $\lambda \in \mathbb{R}^{m-k}$. Thus, a closed D-l-form on U is smooth $(m-k)$-parameter family of closed forms $\{\omega_\lambda\}_{\lambda \in \mathbb{R}^{m-k}}$. Likewise, an exact D-l-form on U is a smooth $(m-k)$-parameter family of exact forms. Thus foliation Lie algebroids do have a Poincaré lemma, i.e. $H^0(D|_U) = C^\infty(\mathbb{R}^{m-k})$ and $H^i(D|_U) = \{0\}$ for $i > 0$.

However, global cohomology can be very wild also outside of level 0, for instance, take $S^n \times \mathbb{R}$ with distribution $D = TS^n$. Then we see $H^n(D) = C^\infty(\mathbb{R})$, which is infinite dimensional.

Example 6.2.4 (Lie algebras)**.** Let $\mathfrak{g}$ be a Lie algebra viewed as a Lie algebroid over a point, then $H^\bullet(\mathfrak{g})$ is isomorphic to the Lie algebra cohomology of $\mathfrak{g}$. See [CE48] for details and examples of Lie algebra cohomology. In particular, this Lie algebroid has pointwise cohomology, so no Poincaré lemma.

Example 6.2.5 (Log-tangent bundle)**.** Let $TM(-\log(N)) \to M$ be a log-tangent bundle. Then we have $H^k(TM(-\log(N)) \cong H^k(M) \oplus H^{k-1}(N)$, see [Mel93].

Example 6.2.6 (Elliptic tangent bundle)**.** Let $TM(-\log|N|)$ be an elliptic tangent bundle, then we have $H^k(-\log|N|) \cong H^k(M \setminus N) \oplus H^{k-1}(S(NN))$, where $S(NN) \to N$ is the circle bundle of the normal bundle to N, see [CG15].

Moreover, since we now have differential $\mathcal{A}$-forms and an exterior derivative, we could study symplectic Lie algebroids, i.e. Lie algebroids equipped with a nondegenerate closed 2-$\mathcal{A}$-form. We will not do that in this thesis, we refer the reader to e.g. [Kla17].

Instead, we will focus more on the Riemannian geometry one could define on Lie algebroids. So we want appropriate generalisations of the Riemannian theory to this setting. Many of these generalisations are canonical, they have been done before in e.g. [Blu+13]. One thing we will need for that is a notion of $\mathcal{A}$-connection, which can be thought of as a way of covariantly differentiating sections of a vector bundle in $\mathcal{A}$ directions. It is defined as follows:

Definition 6.2.7 ($\mathcal{A}$-connection)**.** Let $E \to M$ be a vector bundle and $\mathcal{A} \to M$ be a Lie algebroid. Then an $\mathcal{A}$-*connection* on E is a map $\nabla : \Gamma(E) \to \Omega^1(\mathcal{A}; E)$, satisfying the Leibniz rule, i.e. for any $f \in C^\infty(M)$ and any $s \in \Gamma^\infty(E)$, we have

$$\nabla(fs) = d_{\mathcal{A}} f \otimes s + f\nabla s\,. \tag{6.2.2}$$

Like in the case of honest connections, i.e. TM-connections, any $\mathcal{A}$-connection induces a map $\nabla : \Omega^k(\mathcal{A}; E) \to \Omega^{k+1}(\mathcal{A}; E)$, which is defined on pure tensors in $\Gamma^\infty(\Lambda^k \mathcal{A}^* \otimes E)$ by

$$\nabla(\omega \otimes s) = d_{\mathcal{A}}\omega \otimes s + (-1)^{|\omega|}\omega \wedge \nabla s\,; \qquad \omega \in \Omega^k(M)\,,\ s \in \Gamma^\infty(E)\,. \tag{6.2.3}$$

We see

$$\nabla^2(\alpha \otimes s) = \alpha \wedge \nabla^2 s\,. \tag{6.2.4}$$

Moreover, like in the case of TM-connections, one sees that for $\alpha \in \Omega^k(\mathcal{A}; E)$, ∇ is given by the Koszul formula

$$\nabla\alpha(v_0, \dots, v_k) = \sum_{i=0}^{k} (-1)^i \nabla_{v_i} \alpha(v_0, \dots, \widehat{v_i}, \dots, v_k) + \sum_{i \le j} (-1)^{i+j} \alpha([v_i, v_j]_{\mathcal{A}}, v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_k)\,. \tag{6.2.5}$$

In particular, for $s \in \Gamma^\infty(E)$, we see

$$\nabla^2 s(v, w) = \nabla_v \nabla_w s - \nabla_w \nabla_v s - \nabla_{[v,w]_{\mathcal{A}}} s\,. \tag{6.2.6}$$

One sees that the above equation is tensorial, hence we can write $\nabla^2 s = F_\nabla(s)$ for some $F_\nabla \in \Omega^2(\mathcal{A}, \mathrm{End}(E))$. This endomorphism valued $\mathcal{A}$-2-form is appropriately called the *curvature* of ∇. Combining this with Equation (6.2.4), we see that for any $\alpha \in \Omega^k(\mathcal{A}, E)$, we have

$$\nabla^2 \alpha = F_\nabla \wedge \alpha\,, \tag{6.2.7}$$

where it is understood that the $\mathrm{End}(E)$ part of F_∇ acts on the E part of α. So we see that F_∇ measures two things, Equation (6.2.6) tells us it measures the failure of ∇ to be a Lie algebra morphism, and Equation (6.2.7) tells us it measures the failure of ∇ to be a cochain differential.

For these reasons, often flat $\mathcal{A}$-connections, i.e. $\mathcal{A}$-connections ∇ on some vector bundle $E \to M$ such that $F_\nabla = 0$, are called $\mathcal{A}$-*representations.* See [CFM21] for details.

Note that, like in the case of TM-connections, we can also find a local model for $\mathcal{A}$-connections, i.e. if $E \to M$ is a rank k vector bundle, and $\{e_1, \ldots, e_k\}$ is a frame, ∇ decomposes as $\nabla = d_{\mathcal{A}} + A$, where A is a matrix of one forms, i.e. for coordinate functions $\{s^i\}_{i=1,\ldots,k}$, we have $\nabla(s^i e_i) = d_{\mathcal{A}}(s^i)e_i + s^j A_j{}^i e_i$. Thus, we can derive formulas like

$$F_\nabla = d_{\mathcal{A}}A + A \wedge A\,, \tag{6.2.8}$$

like in the case of TM-connections.

Because we have this local form, we see that we are not too far of defining parallel transport along $\mathcal{A}$-connections, as in the TM case, this would follow from the existence and uniqueness theorem of solutions to linear first order differential equations, which is also what we start to see appearing here. But there's a problem, to define parallel transport along a curve $\gamma : [0,1] \to M$, we first have to be able to take directional derivatives along the tangent vector to the curve, but we see that on the one hand, we can only take directional derivatives in $\mathrm{im}(\rho)$-directions, while on the other hand, taking a directional derivative in an $\mathrm{im}(\rho)$-direction requires lifting it to a vector in $\mathcal{A}$, which cannot be done canonically in the generic case, as we might have isotropy.

The solutions to both these issues is to define the notion of $\mathcal{A}$-path.

Definition 6.2.8 ($\mathcal{A}$-path)**.** Let $\pi : \mathcal{A} \to M$ be a Lie algebroid. Then an $\mathcal{A}$-*path* is a smooth map $\gamma : [0,1] \to \mathcal{A}$ such that $\rho \circ \gamma = \frac{d}{dt}(\pi \circ \gamma)$.

Remark 6.2.9. One can define *piecewise smooth* $\mathcal{A}$-*paths* as paths $\gamma : [0,1] \to \mathcal{A}$ that are compositions of finitely many $\mathcal{A}$-paths. We say that $\gamma_0, \gamma_1 : [0,1] \to \mathcal{A}$ are *composable* if $\pi \circ \gamma_1(0) = \pi \circ \gamma_0(1)$, i.e. we do not assume $\gamma_1 \# \gamma_0$ is continuous inside $\mathcal{A}$. This is reminiscent of the fact that the composition of honest smooth paths $\gamma_0, \gamma_1 : [0,1] \to M$ need not be continuously differentiable everywhere and can have "kinks".

We see that the notion of $\mathcal{A}$-path is precisely what we need to get past both issues described above the definition. On the one hand, $\mathcal{A}$-paths project down to paths on the base manifold whose velocities lie inside $\mathrm{im}(\rho)$, while on the other hand, their velocities naturally lie inside $\mathcal{A}$, as that is how we defined them. Lastly, to define parallel transport, we define the pull back connections of Lie algebroids, as we need to pull back the problem to $(\pi \circ \gamma)^* E \to [0,1]$.

If $\mathcal{A} \to M$ and $\mathcal{B} \to N$ are Lie algebroids, and $\Phi : \mathcal{B} \to \mathcal{A}$ is a vector bundle map covering $\varphi : N \to M$ such that

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\Phi"] \arrow[d, "\rho_{\mathcal{B}}"'] & \mathcal{A} \arrow[d, "\rho_{\mathcal{A}}"] \\ N \arrow[r, "\varphi"'] & M \end{tikzcd}\,,$$

commutes, we can pull back $\mathcal{A}$-connections on vector bundles $E \to M$ to $\mathcal{B}$-connections on $\varphi^* E \to N$, using the formula

$$(\Phi^* \nabla)_v \varphi^* s = \varphi^*(\nabla_{\Phi(v)} s)\,. \tag{6.2.9}$$

Moreover, if the following diagram commutes

$$\begin{tikzcd} \Omega^\bullet(\mathcal{A}) \arrow[r, "\Phi^*"] \arrow[d, "d_{\mathcal{A}}"'] & \Omega^\bullet(\mathcal{B}) \arrow[d, "d_{\mathcal{B}}"'] \\ \Omega^\bullet(\mathcal{A}) \arrow[r, "\Phi^*"] & \Omega^\bullet(\mathcal{B}) \end{tikzcd}, \tag{6.2.10}$$

then, following the formulas, we see $\Phi^* F_\nabla = F_{\Phi^*\nabla}$. The attentive reader might have noticed we haven't yet defined morphisms between Lie algebroids, but now that we have the above result, we see that the following definition comes quite naturally

Definition 6.2.10 (Lie algebroid morphisms)**.** Let $\mathcal{A} \to M$ and $\mathcal{B} \to N$ be Lie algebroids. Then a *Lie algebroid morphism* from $\mathcal{B}$ to $\mathcal{A}$ is a vector bundle map $\Phi : \mathcal{B} \to \mathcal{A}$ such that Diagram (6.2.10) commutes.

In particular, if $\mathcal{A}, \mathcal{B} \to M$ are Lie algebroids and $\Phi : \mathcal{B} \to \mathcal{A}$ is a vector bundle map covering the identity, the Koszul formula (6.2.1) tells us that Φ is a Lie algebroid morphism if and only if it is a Lie algebra morphism from $\Gamma^\infty(\mathcal{B})$ to $\Gamma^\infty(\mathcal{A})$.

We see that $\mathcal{A}$-paths $\gamma : [0,1] \to \mathcal{A}$ can be lifted to a Lie algebroid morphism $\gamma : T[0,1] \to \mathcal{A}$ by $\gamma(d/dt|_t) := \gamma(t)$, where the condition $\rho \circ \gamma = d/dt(\pi \circ \gamma)$ is precisely what tells us $\gamma^* \circ d_{\mathcal{A}} = d_{[0,1]} \circ \gamma^*$, where we note that $\Omega^i([0,1]) = 0$ for $i \geq 2$. Therefore, an $\mathcal{A}$-connection ∇ on a vector bundle $E \to M$ can be pulled back to an honest $T[0,1]$-connection $\gamma^*\nabla$ on $\gamma^* E \to [0,1]$ [1]. We can define parallel transport along γ using parallel transport along this $T[0,1]$-connection.

Definition 6.2.11 (Parallel transport along $\mathcal{A}$-connections)**.** Let $\mathcal{A} \to M$ be a Lie algebroid, $E \to M$ a vector bundle, ∇ an $\mathcal{A}$-connection on E and $\gamma : [0,1] \to \mathcal{A}$ a smooth $\mathcal{A}$-path. Then *parallel transport* along γ, denoted by $T_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$, is defined as the parallel transport of the induced $T[0,1]$-connection $\gamma^*\nabla$ on γ^*E. In the piecewise smooth case, it is defined as the composition of parallel transport along the smooth pieces of γ.

We see that T_γ is always an invertible linear map, which directly follows from the result in the classical case, the inverse is $T_{-\gamma}$, where $-\gamma(t) := \gamma(1-t)$. In particular, if γ is a (piecewise) smooth $\mathcal{A}$-*loop* at some $x \in M$, i.e. an $\mathcal{A}$-path γ such that $\pi(\gamma(0)) = \pi(\gamma(1)) = x$, we have $T_\gamma \in GL(E_x)$.

Since we now have a well defined notion of parallel transport, we can use this to study holonomy of $\mathcal{A}$-connections, if $E \to M$ is a rank k vector bundle equipped with an $\mathcal{A}$-connection, for any point $x \in M$, we define $\mathrm{Hol}_x(\nabla) \subseteq GL(E_x)$ by

$$\mathrm{Hol}_x(\nabla) = \{T_\gamma : E_x \to E_x | \gamma : [0,1] \to \mathcal{A} \text{ is an } \mathcal{A}\text{-loop at } x\}. \tag{6.2.11}$$

Lie algebroid holonomy has been studied by e.g. R.L. Fernandes [Fer02], with a few interesting results.

One thing to note is that $\mathcal{A}$-paths can never leave a leaf, which follows from the following theorem, taken from [AS09]:

Theorem 6.2.12. *Let $\mathcal{A} \to M^m$ be a Lie algebroid, then for every $x \in M$ such that the leaf L through x is k-dimensional, then there is an open neighborhood U of x and a submersion $\varphi : U \to (N, \mathcal{F})$, where $(N, \mathcal{F})$ is a manifold of dimension $n-k$ equipped with a singular foliation (in an appropriate sense), such that $\varphi^*\mathcal{F}$ is the singular foliation of $\mathcal{A}|_U$.*

[1] Note the slight abuse of notation, $\gamma^*E := (\pi \circ \gamma)^*E$.

In particular, the $\mathcal{F}$-leaf through $\varphi(x)$ is 0-dimensional, hence a point. Then for an $\mathcal{A}$-path γ, we see that $\gamma_\varphi := \varphi(\pi \circ \gamma)$ defines an $\mathcal{F}$-path on N, i.e. a path whose velocity lies in $\mathcal{F}$. In particular, around $x \in \operatorname{im}\gamma$, it stays inside the leaf $L|_U = \varphi^{-1}[\varphi(x)]$ through x, as γ_φ is the constant path at $\varphi(x)$, since it has 0 velocity.

Thus locally, $\mathcal{A}$-paths cannot jump between leaves, so they cannot jump between leaves in finite time, so $\mathcal{A}$-paths always lie in a single leaf.

In particular, we see that this means holonomy of $\mathcal{A}$-connections is no longer the same for any $x \in M$, but rather it's only a leafwise invariant. Here, note that any two points x, y in a leaf L always can be connected by an $\mathcal{A}$-path, since the short exact sequence

$$0 \to \ker(\rho|_L) \hookrightarrow \mathcal{A}|_L \xrightarrow{\rho} TL \to 0$$

splits, hence we can embed TL into $\mathcal{A}|_L$, so any L-path can be lifted to an $\mathcal{A}$-path. Generically, Lie algebroid holonomy really is a leafwise invariant, the behaviour of the holonomy can jump quite irregularly between leaves, as illustrated by the following example, communicated to me by J. Pedregal Pastor, to appear in [Ped23].

Example 6.2.13 (Action Lie algebroid)**.** Let G be Lie group and $G \circlearrowright M$ be an action of G on M. This then induces a map $(-)_M : \mathfrak{g} \to \mathfrak{X}(M)$ defined by $\xi \mapsto \xi_M$, where ξ_M is the infinitesimal vector field of ξ, i.e.

$$\xi_M(x) = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot x\,.$$

Then we can give $\mathcal{A} := M \times \mathfrak{g}$ the structure of a Lie algebroid, with $\rho(x, \xi) = \xi_M(x)$ and $[-,-]_{\mathcal{A}}$ such that it agrees with $[-,-]_{\mathfrak{g}}$ on infinitesimal vector fields, i.e.

$$[\xi, \eta]_{\mathcal{A}}(x) = [\xi(x), \eta(x)]_{\mathfrak{g}} + \rho(\xi(x))(\eta) - \rho(\eta(x))(\xi)\,,$$

where ξ and η are interpreted as functions $\xi, \eta : M \to \mathfrak{g}$. We look at $\mathcal{A}$-connections on $\mathcal{A}$. Since $\mathcal{A}$ is a trivial vector bundle, these are defined by $d_{\mathcal{A}} + A$, where $\mathcal{A}$ is an $\operatorname{End}(\mathfrak{g})$-valued $\mathcal{A}$-1-form.

We look at the particular case of the circle action $U(1) \circlearrowright \mathbb{C}$. We see that sections of $\mathcal{A}$ are simply functions $\xi : \mathbb{C} \to \mathbb{R}$. Moreover, A is an honest $\mathcal{A}$-1-form, i.e. A is just some function. Lastly, $\mathcal{A}$-paths $\gamma : [0,1] \to M$ must lie in orbits of $U(1)$, i.e. γ has constant norm.

We look at parallel transport equation along a constant rank path $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = re^{i\theta(t)}$, such that $\dot{\gamma}$ is $\dot{\theta} : [0,1] \to \mathcal{A}$ whenever $r \neq 0$, and $\dot{\gamma}$ is an arbitrary function on $[0,1]$ whenever $r = 0$. When $r \neq 0$, the parallel transport equation tells us

$$\frac{d}{dt}\xi = -A\dot{\theta}\xi\,.$$

We see

$$\xi(1) = \exp\left(-\int_0^1 A\dot{\theta}dt\right)\xi(0)\,.$$

We note that we can interpret A as a function on S^1, as γ is a path of constant norm, so we can rewrite $A\dot{\theta}dt = Ad\theta$.

To compute holonomy, we see that γ must be a closed loop, which defines a homology class on S^1, so we see

$$\xi(1) = \exp\left(-\int_\gamma Ad\theta\right)\xi(0)\,.$$

Thus, we see that $\mathrm{Hol}(r) = 1$ if $Ad\theta$ is exact, and $e^{\lambda\mathbb{Z}}$ else, where λ is the Poincaré dual of the cohomology class of $Ad\theta$, i.e. the average value of A on S^1.

If $r = 0$, we still have the above integral, but now we see that A is a constant and that $\dot{\gamma}$ is an arbitrary function on $[0,1]$, so we get

$$\xi(1) = \exp\left(-A(0)\int_0^1 \dot{\gamma}dt\right)\xi(0)$$

Since γ is always a closed loop and $\int_0^1 \dot{\gamma}dt$ is just the average value of $\dot{\gamma}$, which can take on any value, we see that holonomy is 1 if $A(0) = 0$ and $\mathbb{R}_>$ else.

In particular, if $A(0)$ is nonzero, we see that $Ad\theta$ is not exact around 0, so then holonomy at 0 is continuous, but holonomy around 0 is discrete.

6.3 Riemannian Lie algebroids and complex structures

Up to this point, we have considered basic differential geometric constructions that can be defined on Lie algebroids. The next step would be to put some extra geometric structure on the Lie algebroid and see what kind of consequences such structures might have. In particular, we will define Riemannian geometry, (almost-)complex geometry and Kähler/Calabi-Yau geometry on these Lie algebroids, and we will give a few examples.

Analogously to putting Riemannian metrics on manifolds, we will now consider Riemannian metrics on Lie algebroids. They are just honest metrics on the vector bundle $\mathcal{A} \to M$.

Definition 6.3.1 (Riemannian Lie algebroid)**.** Let $\mathcal{A} \to M$ be a Lie algebroid. Then a *Riemannian metric* on $\mathcal{A}$ is a vector bundle metric $g \in \Gamma^\infty(\mathrm{Sym}^2(\mathcal{A}^*))$. Then pair $(\mathcal{A}, g)$ is called a *Riemannian Lie algebroid.*

Note that there are no compatibility assumptions in the definition. This is reminiscent of Riemannian metrics being vector bundle metrics on TM, where there are also no compatibility assumptions.

One of the fundamental results in Riemannian geometry is the existence of a canonical connection, namely the Levi-Civita connection. As it turns out we also have that in this setting. But first we need to define torsion.

Definition 6.3.2. Let $\mathcal{A}$ be a Lie algebroid and let ∇ be an $\mathcal{A}$-connection on $\mathcal{A}$. Then the *torsion* of ∇ is a tensor field $T_\nabla \in \Gamma^\infty(\Lambda^2\mathcal{A}^* \otimes \mathcal{A})$, defined by

$$T_\nabla(v,w) = \nabla_v w - \nabla_w v - [v,w]_{\mathcal{A}}\,. \tag{6.3.1}$$

As promised, we have the following:

Theorem 6.3.3. *Let $(\mathcal{A}, g)$ be a Riemannian Lie algebroid. Then there is a unique torsion free $\mathcal{A}$-connection ∇ on $\mathcal{A}$ that is metric, i.e. for any $v, w \in \Gamma^\infty(\mathcal{A})$,*

$$d_{\mathcal{A}}(g(v,w)) = g(\nabla v, w) + g(v, \nabla w)\,. \tag{6.3.2}$$

Proof. We will start by proving uniqueness, then provide an explicit formula for the connection, shamelessly copying the proof from Riemannian geometry of the tangent bundle. So suppose we have a connection satisfying the above conditions. We see that for sections $v, w, u \in \Gamma^\infty(\mathcal{A})$,

$$
\begin{aligned}
d_{\mathcal{A}}g(w,u)(v) + d_{\mathcal{A}}g(v,u)(w) - d_{\mathcal{A}}g(v,w)(u) \\
&= g(\nabla_v w + \nabla_w v, u) + g(\nabla_v u - \nabla_u v, w) + g(\nabla_w u - \nabla_w u, v) \\
&= g(\nabla_v w + \nabla_w v, u) + g([v,u]_{\mathcal{A}}, w) + g([w,u]_{\mathcal{A}}, v) \\
&= 2g(\nabla_v w, u) - g([v,w]_{\mathcal{A}}, u) + g([v,u]_{\mathcal{A}}, w) + g([w,u]_{\mathcal{A}}, v)\,.
\end{aligned}
$$

Thus we obtain the *Koszul formula* for the Levi-Civita connection of the Riemannian Lie algebroid

$$
\begin{aligned}
g(\nabla_v w, u) = \tfrac{1}{2}\big(&d_{\mathcal{A}}g(w,u)(v) + d_{\mathcal{A}}g(v,u)(w) - d_{\mathcal{A}}g(v,w)(u) \\
&+ g([v,w]_{\mathcal{A}}, u) - g([v,u]_{\mathcal{A}}, w) - g([w,u]_{\mathcal{A}}, v)\big)\,.
\end{aligned}
\tag{6.3.3}
$$

Since u was arbitrary, g is nondegenerate, and the right hand side does not depend on ∇, we see that this formula uniquely characterizes the Levi-Civita connection.

Also note that given $v, w \in \Gamma^\infty(\mathcal{A})$, the Koszul formula defines a section $\nabla_v w \in \Gamma^\infty(\mathcal{A})$, so to prove existence of the Levi-Civita connection, all we have to show is that this assignment indeed satisfies the definition of a connection. This can be done by direct computation and is exactly the same as in Riemannian geometry. $\square$

Note that if $\mathcal{A}$ has non-abelian isotropy, then there is no local frame $\{e_i\}$ for $\mathcal{A}$ such that $[e_i, e_j]_{\mathcal{A}} = 0$, so the usual formula for the Christoffel symbol in Riemannian geometry does not carry over, one needs to also take into account the possible isotropy terms appearing from the Koszul formula (6.3.3). An interesting question would be studying geodesics of the Levi-Civita connection, i.e. $\mathcal{A}$-paths γ such that $(\gamma^*\nabla)_{d/dt}\gamma = 0$. Doing this, one could also define geodesic completeness of a Riemannian Lie algebroid. We conjecture that Riemannian Lie algebroids over a compact manifold are complete, i.e. geodesics exist for any $t \in \mathbb{R}$. However, we have not yet made significant progress in this area and we leave the general case for future research.

An interesting property of the classical Levi-Civita connection is that $d\omega$ is the antisymmetrisation of $\nabla\omega$, with $\omega \in \Omega^k(\mathcal{A})$. This property also carries over to the Lie algebroid setting

Lemma 6.3.4. *Let ∇ be a torsion free $\mathcal{A}$-connection on $\mathcal{A}$, then for any $\omega \in \Omega^k(\mathcal{A})$, $v_0, \ldots, v_k \in \Gamma^\infty(\mathcal{A})$,*

$$
d\omega(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{v_i}\omega)(v_0, \ldots, \widehat{v_i}, \ldots, v_k)\,. \tag{6.3.4}
$$

Proof. The proof follows from the Koszul formula and the fact that ∇ on $C^\infty(M)$ is given by $\nabla_v f = \rho(v)(f)$. The Koszul formula tells us

$$
\begin{aligned}
d\omega(v_0, \ldots, v_k) = &\sum_i (-1)^i \rho(v_i)(\omega(v_0, \ldots, \widehat{v_i}, \ldots, v_k)) \\
&+ \sum_{i<j} (-1)^{i+j} \omega([v_i, v_j]_{\mathcal{A}}, v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_k)\,.
\end{aligned}
$$

Then we have

$$\begin{aligned}\rho(v_i)(\omega(v_0,\dots,\widehat{v_i},\dots,v_k)) = (\nabla_{v_i}\omega)(v_0,\dots,\widehat{v_i},\dots,v_k)&\\ +\sum_{j<i}(-1)^j\omega(\nabla_{v_i}v_j,v_0,\dots,\widehat{v_j},\dots,\widehat{v_i},\dots,v_k)&\\ -\sum_{j>i}(-1)^j\omega(\nabla_{v_i}v_j,v_0,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_k)\,.&\end{aligned}$$

Thus, we see

$$\begin{aligned}\sum_{i=0}^{k}(-1)^i\rho(v_i)(\omega(v_0,\dots,\widehat{v_i},\dots,v_k)) = \sum_i(-1)^i(\nabla_{v_i}\omega)(v_0,\dots,\widehat{v_i},\dots,v_k)&\\ -\sum_{i<j}(-1)^{i+j}\omega(\nabla_{v_i}v_j-\nabla_{v_j}v_i,v_0,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_k)\,.&\end{aligned}$$

The result then follows from the torsion-freeness condition $\nabla_{v_i}v_j-\nabla_{v_j}v_i=[v_i,v_j]_{\mathcal{A}}$. □

We will need this property later on when discussing Kähler Lie algebroids, but it is an interesting property on its own.

The curvature of the Levi-Civita connection will be called the *Riemann curvature* of g and will be denoted by Riem, or by $R^i{}_{jkl}$ in a local frame. Classically, the Riemann curvature tensor has a few well-known symmetries, with purely algebraic proofs, so they immediately carry over to this setting.

Proposition 6.3.5. *The Riemann curvature tensor has the following properties:*

(i) $\mathrm{Riem}(v,w)=-\mathrm{Riem}(w,v)$;
(ii) $g(\mathrm{Riem}(v_1,v_2)v_3,v_4)=-g(\mathrm{Riem}(v_1,v_2)v_4,v_3)$;
(iii) $g(\mathrm{Riem}(v_1,v_2)v_3,v_4)=g(\mathrm{Riem}(v_3,v_4)v_1,v_2)$;
(iv) $\mathrm{Riem}(v,w)u+\mathrm{Riem}(w,u)v+\mathrm{Riem}(u,v)w=0$, *known as the* first Bianchi identity;
(v) $\nabla\mathrm{Riem}=0$, *known as the* second Bianchi identity, *where ∇ is the $\mathcal{A}$-connection on* $\mathrm{End}(\mathcal{A})$, *and* Riem *is interpreted as a two-form in* $\Omega^2(\mathcal{A};\mathrm{End}\mathcal{A})$.

In a local frame, these become

(i) $R_{ijkl}=-R_{ijlk}$;
(ii) $R_{ijkl}=-R_{jikl}$;
(iii) $R_{ijkl}=R_{klij}$;
(iv) $R_{ijkl}+R_{iklj}+R_{iljk}=0$;
(v) $\nabla_iR_{jklm}+\nabla_lR_{jkmi}+\nabla_mR_{jkil}=0$.

Likewise, one can define the *Ricci tensor* Ric, in coordinates

$$R_{ij}=R^k{}_{ikj}\,. \tag{6.3.5}$$

One sees that this is then is then a symmetric second order tensor, i.e. $\mathrm{Ric}\in\mathrm{Sym}^2(\mathcal{A}^*)$.

Examples of Riemannian Lie algebroids are, for instance, foliations with a smoothly varying metric on the leaves. There, the Levi-Civita connections and all curvatures can be computed leafwise. A more interesting case is that of compact semisimple Lie algebras, i.e. Lie algebras whose Killing form is negative definite.

Example 6.3.6 (Compact semisimple Lie algebra)**.** Let $\mathfrak{g}$ be a compact semisimple Lie algebra viewed as a Lie algebroid over a point. Then we see that $\mathfrak{g}$ admits a canonical Riemannian metric, namely $-\kappa$, where κ is the Killing form, i.e. $\kappa(\xi,\eta) = \mathrm{Tr}(\mathrm{ad}_\xi \, \mathrm{ad}_\eta)$.

Since the anchor of $\mathfrak{g}$ is trivial, we see that the Koszul formula (6.3.3) becomes

$$-\kappa(\nabla_\xi \eta, \zeta) = \tfrac{1}{2}(-\kappa([\xi,\eta],\zeta) + \kappa([\xi,\zeta],\eta) + \kappa([\eta,\zeta],\xi)) .$$

The Killing form satisfies $\kappa([\xi,\eta],\zeta) = \kappa(\xi,[\eta,\zeta])$, so we see

$$\kappa(\nabla_\xi \eta, \zeta) = \tfrac{1}{2}\kappa([\xi,\eta],\zeta) .$$

Since ζ was arbitrary and κ is nondegenerate as it is negative definite, we see

$$\nabla_\xi \eta = \tfrac{1}{2}[\xi,\eta] . \tag{6.3.6}$$

We see

$$\mathrm{Riem}(\xi,\eta) = \tfrac{1}{4}\mathrm{ad}_{[\eta,\xi]} . \tag{6.3.7}$$

Therefore, we see

$$R_{ij} = -\tfrac{1}{4}\sum_k \kappa([e_k,e_i],[e_k,e_j]) . \tag{6.3.8}$$

The holonomy group must be a subgroup of $GL(\mathfrak{g})$. Moreover, since parallel transport along a curve $\xi(t)$ is infinitesimally generated by the vector field $-\frac{1}{2}\mathrm{ad}_{\xi(t)}$, which lies tangent to $\mathrm{Ad}(G) \subseteq GL(\mathfrak{g})$, i.e. the integration of $\mathrm{ad}(\mathfrak{g})$. Thus, we get $\mathrm{Hol}(-\kappa) \subseteq \mathrm{Ad}(G)$. Moreover, parallely transporting some vector η along the constant path ξ gives us

$$\frac{d\eta}{dt} = -\nabla_\xi \eta = -\tfrac{1}{2}\mathrm{ad}_\xi \eta ,$$

which then gives us

$$\eta(1) = \exp(-\tfrac{1}{2}\mathrm{ad}_\xi)\eta(0) ,$$

so the group generated by these expressions is a subgroup of the holonomy group of $-\kappa$, i.e. $\mathrm{Ad}(g) = \exp(\mathrm{ad}(\mathfrak{g})) \subseteq \mathrm{Hol}(-\kappa)$. So we see $\mathrm{Hol}(-\kappa) = \mathrm{Ad}(G)$. Moreover, the parallel transport equation also tells us that geodesic are necessarily constant paths, and since all constant paths satisfy the geodesic equation, we see that the geodesics are all constant paths, in particular, this metric is complete.

We look at the particular example $\mathfrak{g} = \mathfrak{so}(3)$. We can realise this Lie algebra as $(\mathbb{R}^3, \times)$, i.e. $\mathbb{R}^3$ equipped with the cross product. This Lie algebra has $-\kappa(v,w) = 2v \cdot w$. The Levi-Civita connection is given by $\nabla_v w = \frac{1}{2} v \times w$, we have

$$\mathrm{Riem}(v,w) = \tfrac{1}{4}\begin{pmatrix} 0 & v_1w_2 - v_2w_1 & v_1w_3 - v_3w_1 \\ v_2w_1 - v_1w_2 & 0 & v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 & v_3w_2 - v_2w_3 & 0 \end{pmatrix} .$$

And for the Ricci tensor, we have

$$R_{ij} = \tfrac{1}{2}\delta_{ij} .$$

In particular, $\mathfrak{so}(3)$ with $-\kappa$ has constant Ricci curvature $\frac{1}{4}$. The holonomy group of $-\kappa$ is $SO(3)$ with the standard action and the geodesics are the constant paths.

This also gives an example of an *Einstein Lie algebroid*, i.e. a Riemannian Lie algebroid with constant Ricci curvature.

The next step would be to define almost complex structures on Lie algebroids. This is also done in the same way as on the tangent bundle, i.e. by a bundle map that squares to -1.

Definition 6.3.7 (Almost complex Lie algebroid)**.** Let $\mathcal{A} \to M$ be a Lie algebroid. Then an *almost complex structure* on $\mathcal{A}$ is a bundle map $J : \mathcal{A} \to \mathcal{A}$ covering the identity on M, such that $J^2 = -\mathrm{id}$. The pair $(\mathcal{A}, J)$ is called an *almost complex Lie algebroid*.

This gives us the usual decompositions $\mathcal{A} \otimes \mathbb{C} \cong \mathcal{A}^{(1,0)} \oplus \mathcal{A}^{(0,1)}$ into, respectively, $+i$ and $-i$ eigenbundles of J, and $\Lambda^k \mathcal{A}^* \cong \bigoplus_{p+q=k} \Lambda^{(p,q)} \mathcal{A}^*$ defined by $\Lambda^{(p,q)} \mathcal{A}^* := \Lambda^p(\mathcal{A}^{(1,0)})^* \otimes \Lambda^q(\mathcal{A}^{(0,1)})^*$. In particular, we get a space of (p,q)-forms $\Omega^{(p,q)}(\mathcal{A}) := \Gamma^\infty(\Lambda^{(p,q)}\mathcal{A}^*)$ and a decomposition $\Omega^k(\mathcal{A}) \cong \bigoplus_{p+q=k} \Omega^{(p,q)}(\mathcal{A})$. Moreover, we have

$$d_{\mathcal{A}} : \Omega^{(p,q)}(\mathcal{A}) \to \Omega^{(p-1,q+2)}(\mathcal{A}) \oplus \Omega^{(p,q+1)}(\mathcal{A}) \oplus \Omega^{(p+1,q)}(\mathcal{A}) \oplus \Omega^{(p+2,q-1)}(\mathcal{A}) ,$$

which can be proved by locally decomposing into $(1,0)$ and $(0,1)$ forms, and then applying the Leibniz rule.

Since Lie algebroids are not actually tangent bundles, we do not have frames that arise from coordinates on the base manifold. Thus the definition of integrability of almost complex structures through holomorphic coordinates does not carry over to this setting. Luckily, we have the Newlander-Nirenberg theorem 2.2.1, and we can just define it through involutivity of $\mathcal{A}^{(1,0)}$.

Definition 6.3.8 (Complex Lie algebroid)**.** Let $(\mathcal{A}, J)$ be an almost complex Lie algebroid. Then we say J is *integrable* if $\mathcal{A}^{(1,0)}$ is involutive, i.e. $[\mathcal{A}^{(1,0)}, \mathcal{A}^{(1,0)}]_{\mathcal{A}} \subseteq \mathcal{A}^{(1,0)}$, where we extended the scalars of the bracket linearly. In this case, we say J is a *complex structure* and that $(\mathcal{A}, J)$ is a *complex Lie algebroid*.

Remark 6.3.9. In literature, the term *complex Lie algebroid* is sometimes used for a complex vector bundle with an anchor that is a bundle map mapping into $TM \otimes \mathbb{C}$ and a Lie bracket satisfying the Leibniz rule. In this thesis, we will not use these objects very much and we think the above definition is a more suitable candidate for the term *complex Lie algebroid*. For the other objects, we propose the name *Lie algebroids over* $TM \otimes \mathbb{C}$. Likewise, we can define Lie algebroids over $TM^{(1,0)}$ and $TM^{(0,1)}$ if M is a complex manifold. In particular, in Example 6.1.11, the bundle $TM(-\log(N))$ is a Lie algebroid over $TM^{(1,0)}$ and $TM(-\log(\overline{N}))$ is a Lie algebroid over $TM^{(0,1)}$.

One large class of examples of complex Lie algebroids comes from foliations consisting of complex manifolds with smoothly varying complex structure. Another quick example is the example of a complex Lie algebra

Example 6.3.10. Let $\mathfrak{g}$ be a complex Lie algebra, viewed as a real Lie algebroid over $*$ with complex structure J induced by multiplication by i. Then J is an integrable complex structure, as $v \in \mathfrak{g}^{(1,0)}$ can be written as $v = v_R - iJv_R$ for some $v_R \in \mathfrak{g}$, so we see for $v, w \in \mathfrak{g}^{(1,0)}$,

$$[v, w] = [v_R - iJv_R, w_R - iJw_R] = [v_R, w_R] - [Jv_R, Jw_R] - i([v_R, Jw_R] + [Jv_R, w_R]) .$$

Since J comes from multiplication by i in the complex vector space $\mathfrak{g}$, it commutes with the Lie bracket, so we see

$$[v, w] = 2[v_R, w_R] - iJ(2[v_R, w_R]) ,$$

so $[v,w] \in \mathfrak{g}^{(1,0)}$, meaning J is integrable.

Perhaps a more interesting case is when J does not come from a complex multiplication of a complex Lie algebra. In this case, we see that $\mathfrak{g}^{(1,0)}$ is involutive if and only if for any $v,w \in \mathfrak{g}$,

$$[v,Jw]+[Jv,w]=J[v,w]-J[Jv,Jw] \iff [v,w]+J([v,Jw]+[Jv,w])-[Jv,Jw]=0\,,$$

so here we see this well-known form of the Nijenhuis tensor appearing again.

One thing to note is that the equivalence of (ii), (iii) and (iv) from Theorem 2.2.1 immediately carries over to this case, with the same proof, i.e. we have:

Theorem 6.3.11 (Different integrability criteria)**.** *Let $(\mathcal{A},J)$ be an almost complex Lie algebroid, then the following are equivalent:*

(i) *J is integrable;*

(ii) *the* Nijenhuis tensor $N \in \Gamma((\mathcal{A}^*)^{\otimes 2} \otimes \mathcal{A})$, *defined for real $v,w \in \Gamma(\mathcal{A})$ by*

$$N_J(v,w)=[v,w]+J([Jv,w]+[v,Jw])-[Jv,Jw]\,, \tag{6.3.9}$$

vanishes identically;

(iii) *The algebroid differential $d_{\mathcal{A}}$ acts on $\mathcal{A}$-(p,q)-forms by*

$$d_{\mathcal{A}}:\Omega^{(p,q)}(\mathcal{A}) \to \Omega^{(p+1,q)}(\mathcal{A}) \oplus \Omega^{(p,q+1)}(\mathcal{A})\,.$$

Proof. See the proof of Theorem 2.2.1, where we note that item (ii) in that theorem is how we defined integrability in the Lie algebroid case. □

In particular, we see that $d_{\mathcal{A}}$ decomposes into $\partial_{\mathcal{A}}:\Omega^{(p,q)}(\mathcal{A}) \to \Omega^{(p+1,q)}(\mathcal{A})$ and $\bar{\partial}_{\mathcal{A}}:\Omega^{(p,q)}(\mathcal{A}) \to \Omega^{(p,q+1)}(\mathcal{A})$, which satisfy $\partial_{\mathcal{A}}^2=\bar{\partial}_{\mathcal{A}}^2=0$ and $\partial_{\mathcal{A}}\bar{\partial}_{\mathcal{A}}=-\bar{\partial}_{\mathcal{A}}\partial_{\mathcal{A}}$. Thus, we can define Dolbeault cohomology for complex Lie algebroids as well.

When we have a complex Lie algebroid over a complex manifold, one could ask for the two complex structures to be compatible in some sense.

Definition 6.3.12 (Compatibility of complex structures)**.** Let $\mathcal{A} \to M$ be an (almost) complex Lie algebroid over an (almost) complex manifold. Then the two (almost) complex structures are said to be *compatible* if the following diagram commutes

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\rho_{\mathcal{A}}"] \arrow[d, "J_{\mathcal{A}}"'] & TM \arrow[d, "J_M"] \\ \mathcal{A} \arrow[r, "\rho_{\mathcal{A}}"'] & TM \end{tikzcd}\,.$$

Compatibility is precisely the condition that makes sure $\rho:\mathcal{A}^{(1,0)} \to TM^{(1,0)}$. In particular, if the complex structures are compatible, we get that ρ commutes with $\bar{\partial}$ and with ∂. Moreover, if $\mathcal{A} \to M$ is an almost complex Lie algebroid over an almost complex manifold with compatible almost complex structures, we see that the Nijenhuis tensors satisfy $N(\rho(v),\rho(w))=\rho(N(v,w))$.

One sees that if a Lie algebroid $\mathcal{A} \to M$ is *transitive*, i.e. if $\rho:\mathcal{A} \to TM$ is a surjection, and we can find a complex structure J such that $\ker\rho$ is invariant under J, then J induces a unique compatible complex structure on the base manifold. We summarise in the following proposition.

Proposition 6.3.13. *Let $(\mathcal{A}, J_\mathcal{A}) \to (M, J_M)$ be an almost complex Lie algebroid over an almost complex manifold. Then for any $x \in M$ and $v, w \in \mathcal{A}_x$, we get $N_{J_M}(\rho(v), \rho(w)) = \rho(N_{J_\mathcal{A}}(v, w))$.*

Moreover, if $(\mathcal{A}, J_\mathcal{A}) \to M$ is a transitive complex Lie algebroid such that $\ker \rho$ *is J-invariant, then there is a unique complex structure J_M on M that is compatible with $J_\mathcal{A}$, which is defined by $J_M(\rho(v)) = \rho(J_\mathcal{A}(v))$.*

An interesting question would be to ask which manifolds admit transitive complex Lie algebroids of sufficiently low rank. In particular, does S^6 admit a transitive complex Lie algebroid of rank 8 ?[1] What about $2n-1$ manifolds and transitive complex Lie algebroids of rank $2n$?

Conversely, one might be interested in complex structures on Lie algebroids that do not come from complex structures on the base. For instance, in [CG15] it is shown that the elliptic tangent bundle of Example 6.1.11 can be defined on real manifolds as well. The upshot would be that then $TM(-\log|N|)$ no longer comes equipped with a canonical complex structure inherited from the base, and one can define an almost complex structure that intertwines logarithmic directions with tangent directions.

We leave these questions for future research and we continue with our quest to define Kähler structures.

Definition 6.3.14 (Hermitian Lie algebroid)**.** A *hermitian Lie algebroid* is a triple $(\mathcal{A}, g, J)$ of a Lie algebroid $\mathcal{A}$, a Riemannian metric g on $\mathcal{A}$ and an almost complex structure J on $\mathcal{A}$, such that $g(J-, J-) = g(-,-)$, i.e. J is g-orthogonal. The two-form $\omega := g(J-,-) \in \Omega^2(\mathcal{A})$ is called the *hermitian two-form* of $(\mathcal{A}, g, J)$. If $d_\mathcal{A}\omega = 0$, we call $(\mathcal{A}, g, J)$ a *Kähler Lie algebroid* and we call ω the *Kähler form* of $(\mathcal{A}, g, J)$.

Likewise, one can define *almost hermitian Lie algebroids* and *almost Kähler Lie algebroids* by dropping the assumption that J is integrable.

The following proposition immediately carries over from the classical case, where we note that we have Lemma 6.3.4, see the proof of Proposition 2.2.4.

Proposition 6.3.15 (Alternative Kähler characterisations)**.** *Let $(\mathcal{A}, g, J)$ be a rank k hermitian Lie algebroid with Levi-Civita connection ∇ and hermitian two-form ω, then the following conditions are equivalent*

(i) $d_\mathcal{A}\omega = 0$;
(ii) $\nabla J = 0$;
(iii) $\nabla \omega = 0$.
(iv) *The holonomy groups of $(\mathcal{A}, g)$ lie inside $U(k)$.*

One unfortunate thing is that Hodge theory does not naturally carry over to this setting. Many results from Hodge theory require the base manifold to be compact, as results require integrating over the entire manifold. In the Lie algebroid setting, a symplectic two-form on a Lie algebroid does induce a volume form by taking the top exterior power, however, it does not induce a volume form on the base. This means we cannot use symplectic forms on Lie algebroids to integrate over the base manifold, hence we have no direct way of translating Hodge theory to this setting.

M. Crainic pointed out to me that there is a notion of Poisson manifold of compact type in Poisson geometry [CFM15; CFM16], where some classical symplectic dualities can be generalised to the setting

[1] By dimensionality, a transitive Lie algebroid over M^m of rank m is necessarily TM, as the anchor must be a fibrewise isomorphism, so a rank 6 complex Lie algebroid on S^6 is equivalent to a complex structure on S^6.

of Poisson manifolds that are not symplectic. Perhaps a Kähler Lie algebroid admitting an integration into a Lie groupoid with compact s-fibres would admit some kind of Hodge theory. We leave this for future research.

Using the setting of Kähler Lie algebroids, one can easily define *Kähler-Einstein Lie algebroids* as Kähler Lie algebroids with constant Ricci curvature, and one can define *Calabi-Yau Lie algebroids* as Kähler Lie algebroids with vanishing Ricci curvature. We note that the following proposition immediately carries over to the setting of Lie algebroids.

Proposition 6.3.16. *Let $\mathcal{A}$ be a rank k complex Lie algebroid carrying a Ricci-flat Kähler metric g, then* $\mathrm{Hol}_0(g) \subseteq SU(k)$ *on every leaf. Moreover, if* $\mathrm{Hol}(g) \subseteq SU(k)$ *on every leaf, then $\mathcal{A}$ has a nonvanishing $(k,0)$-form $\Omega \in \Omega^{(k,0)}(\mathcal{A})$.*

Here, the restricted holonomy group is generated by $\mathcal{A}$-paths whose induced loop on the base manifold is null-homotopic relative to the leaf it lies on, in particular, it is a leafwise invariant. See [Ped23] for details.

Example 6.3.17 (Elliptic tangent bundle)**.** Let $\mathcal{A}$ be the elliptic tangent bundle associated to a codimension one complex submanifold $N \subseteq M$. Since we have that the connected components of $M \setminus N$ are leaves where ρ is an isomorphism, we see that a Kähler metric g on $\mathcal{A}$ induces a Kähler structure[1] on the open manifold $M \setminus N$. Moreover, if g is Kähler-Einstein (or even Ricci-flat), then so is the induced Kähler structure on $M \setminus N$.

An interesting case is when we let $\mathcal{A}$ be the elliptic tangent bundle with respect to $(\{0\} \sqcup \{\infty\}) \subseteq \mathbb{C}P^1$. Let $g = d\log(z)\, d\log(\bar{z})$. We see $d\log(z) = -d\log(1/z)$, so g is indeed a well defined metric on $\mathcal{A}$. Since $\mathcal{A}$ has rank 2, the associated hermitian form is closed, hence g defines a Kähler structure on $\mathcal{A}$. A calculation then shows that the associated Ricci tensor is identically zero. So we see that $\mathcal{A}$ is, in fact, a Ricci-flat Kähler Lie algebroid. In fact, [Küh11] shows that this is actually the only class of examples of a Calabi-Yau elliptic tangent bundle over a compact complex curve, i.e. there is no Calabi-Yau elliptic tangent bundle over a Riemann surface of nonnegative genus. Note that we can construct many other Calabi-Yau elliptic tangent bundles over $\mathbb{C}P^1$ by applying Möbius transformations to this example.

If we now let M be any Calabi-Yau manifold, we see that the elliptic tangent bundle with respect to $(M \times (\{0\} \sqcup \{\infty\})) \subseteq M \times \mathbb{C}P^1$ also inherits a Calabi-Yau Lie algebroid structure.

The interesting part of the above example is that there has been quite some study on Kähler-Einstein metrics on complements of complex submanifolds. See [Küh11] for a review. In particular, the Tian-Yau theorem [TY87; TY90; TY91] tells us that whenever M is a particular type of complex manifold, known as a *Fano manifold*, if we have a smooth normal crossings divisor D, i.e. a family of smooth codimension one complex embedded submanifolds $\{X_i\}$, such that at any point where k of them intersect, let's say $X_{i_1}, \dots, X_{i_k}$, we can find local coordinates such that $X_{i_j} = \{z_j = 0\}$, and moreover if D represents $-K_M$, we have that $M \setminus D$ has a Calabi-Yau metric. However, these metrics usually have rather strange behaviour towards the divisor, where the Kähler potential on $M \setminus D$ is approximated by $(-\log(r^2))^{m+1/m}$ around the divisor [Küh11].

[1]Since any such Kähler metric must have logarithmic divergence towards the divisor, and since for any $a > 0$, $\int_a^0 dr/(r^2)$ diverges, I think this even makes $M \setminus N$ a complete Kähler manifold, but I don't have a rigorous proof as of yet.

Now, the elliptic tangent bundle can also be defined with respect to a normal crossing divisor D like above, see [CG15], where we now let $TM(-\log|D|)$ be locally generated by $\{z^1\partial_{z^1},\ldots,z^k\partial_{z^k},\partial_{z^{k+1}},\ldots,\partial_{z^m}\}$ and their complex conjugates. So we see that a Calabi-Yau metric on $TM(-\log|D|)$ would induce a Calabi-Yau metric on $M\setminus D$ with nice behaviour towards the divisor, namely logarithmic behaviour. Studying Ricci-flat Kähler metrics on elliptic tangent bundles might therefore give some new insights in this problem, though it might very well be that the existence of these is obstructed outside of cases with a lot of symmetry.

Lastly, in literature, there is a notion of *log Calabi-Yau manifold* [DKW13]. This is defined as an elliptic tangent bundle carrying a holomorphic volume form, where we note that $TM(-\log D)$ is a holomorphic vector bundle, so it makes sense to talk about holomorphic $(p,0)$-forms. However, this definition does not force that this volume form also comes from some Kähler metric on the Lie algebroid. In fact, it ignores the Kähler question altogether. Thus, studying Kähler elliptic tangent bundles might tell us when log Calabi-Yau structures actually come from Calabi-Yau metrics on the associated elliptic tangent bundle. This will give a new approach to this problem from a Riemannian perspective.

6.4 Lie algebroids in string theory: symplectic gravity

Up to this point this chapter has consisted of a mathematical discussion on geometry on Lie algebroids and the possible applications to mathematics. However, as mentioned before, Lie algebroids have also appeared in string theory, in this section, we present the main idea given in [Blu+13], which uses Lie algebroids. In the next section, we will speculate a bit about a possibility of also introducing Calabi-Yau Lie algebroids in string theory.

The idea is to take a vector bundle $E\to M$ that is isomorphic to $TM\to M$ as a vector bundle, the pick an explicit isomorphism $\rho: E\to TM$ covering id, and pull back the Lie bracket on TM to E by $[v,w]_E := \rho^{-1}[\rho(v),\rho(w)]_{TM}$. Then $(E,\rho,[-,-])$ becomes a Lie algebroid. The upshot is now that we have an extra degree of freedom, namely we can change the anchor map with some Lie group. Then we can look for theories that are invariant under such transformations.

A particular example of a Lie algebroid admitting such behaviour is a symplectic manifold (M,ω). The Lie algebroid is then T^*M and the anchor map is $\omega^\sharp := (\omega^\flat)^{-1}$, defined by $\omega^\flat(X) := \iota_X\omega$. However, there is no explicit need for this two-form to be closed to obtain a Lie algebroid structure on T^*M, in fact, we only really need it to be nondegenerate.

So assume (M,g) is a Riemannian manifold equipped with a background $(\mathrm{NS},\mathrm{NS})$-three-form flux H, i.e. $dH=0$. Then we can locally find two-forms B such that $H=dB$, which we assume to be nondegenerate. Using these two-forms, we can locally put a Lie algebroid structure on T^*M, however, these two-forms B are only unique up to a gauge transformation $B\mapsto B+d\xi$ for some one-form ξ. Therefore, we need to make sure that whatever theory we put on T^*M is invariant under gauge transformations. Moreover, this is only a local theory, to find a global theory requires patching these theories together, but if the theory is gauge invariant, we know that we can always do this.

Locally, we define the two-vector β as the inverse of B, i.e. $\beta^{ij}B_{jk}=\delta^i_k$, and we see that infinitesimal gauge transformations correspond to

$$\delta_\xi\beta = (\beta^\sharp\otimes\beta^\sharp)(d\xi)\,, \tag{6.4.1}$$

i.e. in coordinates,

$$(\delta_\xi \beta)^{ij} = \beta^{ik}\beta^{jl}(d\xi)_{kl}\,. \tag{6.4.2}$$

To further study this object, we define the R-flux $\Theta := \frac{1}{2}[\beta,\beta]_{\mathrm{SN}}$, i.e. the failure of β to be Poisson[1], and we equip T^*M with the *Koszul bracket*, i.e. $[\xi,\eta]_K := \iota_{\beta^\sharp(\xi)}d\eta - \iota_{\beta^\sharp(\eta)}d\xi + d(\beta(\xi,\eta))$, which, in coordinates, becomes

$$[\xi,\eta]_K = (\beta^{ij}(\xi_i\partial_j\eta_k - \eta_i\partial_j\xi_k) + \partial_k\beta^{ij}\xi_i\eta_j)dx^k\,. \tag{6.4.3}$$

To relate this to the Lie bracket we put on T^*M by pulling back along β, we compute

$$[\beta^{ij}\xi_i\partial_j, \beta^{kl}\eta_k\partial_l]_{TM} = (\beta^{ij}\beta^{kl}(\xi_i\partial_j\eta_k - \eta_i\partial_j\xi_k) + \beta^{ij}(\xi_i\eta_k - \eta_i\xi_k)\partial_j\beta^{kl})\partial_l\,.$$

Now, using $\partial_i(B_{jk}\beta^{kl}) = 0$, we see $\partial_j\beta^{kl} = \beta^{km}\beta^{ln}\partial_j B_{mn}$, such that

$$[\beta^{ij}\xi_i\partial_j, \beta^{kl}\eta_k\partial_l]_{TM} = \beta^{ij}\beta^{kl}(\xi_i\partial_j\eta_k - \eta_i\partial_j\xi_k + \beta^{mn}(\xi_i\eta_m - \eta_i\xi_m)\partial_j B_{kn})\partial_l$$

Noting that we defined the bracket on T^*M by also left multiplying by B, we then see

$$[\xi,\eta] = B_{mn}([\beta^{ij}\xi_i\partial_j, \beta^{kl}\eta_k\partial_j]^m_{TM})dx^n = \beta^{ij}(\xi_i\partial_j\eta_k - \eta_i\partial_j\xi_k + \beta^{mn}(\xi_i\eta_m - \eta_i\xi_m)\partial_j B_{kn})dx^k\,.$$

Plugging in Equation (6.4.3), we see

$$[\xi,\eta] = [\xi,\eta]_K - \beta^{ij}\beta^{kl}((\xi_i\eta_k - \eta_i\xi_k)\partial_j B_{lm} + \partial_m B_{jl}\xi_i\eta_k)dx^m\,,$$

i.e.

$$[\xi,\eta] = [\xi,\eta]_K - \iota_{\beta^\sharp(\eta)}\iota_{\beta^\sharp(\xi)}H\,. \tag{6.4.4}$$

So if β is Poisson, then the Koszul bracket is precisely the bracket of the Lie algebroid T^*M, thus we can also interpret the R-flux as the failure of $\beta^\sharp$ to be an algebra homomorphism. Moreover, note that the Koszul bracket is only a Lie bracket if β is Poisson, as the Jacobiator is precisely given by

$$\mathrm{Jac}(\xi,\eta,\zeta) = (\beta^\sharp([\xi,\eta]_K) - [\beta^\sharp(\xi),\beta^\sharp(\eta)])(\zeta) + \text{c.p.}\,, \tag{6.4.5}$$

which vanishes if and only if β is Poisson. One might wonder what the point of any of this is, since it seems like we're intentionally misdefining things: why use the Koszul bracket when we have the algebroid bracket available? Well, the idea is that applying a gauge transformation will yield a "non-geometric" transformation of the theory, however, part of this transformation is a usual diffeomorphism, the part that is "non-geometric" can be described in terms of the Koszul-bracket. As is shown in [Blu+13], we have

$$\delta_\xi\beta = \beta^\sharp(\xi)(\beta) - \hat{\delta}_\xi\beta\,, \tag{6.4.6}$$

where

$$\hat{\delta}_\xi\beta := \hat{L}_\xi\beta + (\beta^\sharp\otimes\beta^\sharp)(d\xi)\,. \tag{6.4.7}$$

Here, $\hat{L}$ is the "non-geometric" Lie derivative with respect to β, given on vector fields by

$$\hat{L}_\xi X = \iota_\xi d_\beta X + d_\beta\iota_\xi X\,, \tag{6.4.8}$$

[1] Note that this means that a vanishing H-flux implies there is no R-flux. The idea is to translate a setting with H-flux to a setting with R-flux by moving everything to T^*M.

with $d_\beta X := [\beta, X]_{\mathrm{SN}}$, and given on one-forms by

$$\hat{L}_\xi \eta = [\xi, \eta]_K \,. \tag{6.4.9}$$

Moreover, [Blu+13] also shows

$$\delta_\xi \hat{g} = \beta^\sharp(\xi)(\hat{g}) - \hat{\delta}_\xi \hat{g} \,, \tag{6.4.10}$$

where $\hat{g} = (\beta^\sharp \otimes \beta^\sharp)(g)$ and

$$\hat{\delta}_\xi \hat{g} = \hat{L}_\xi \hat{g} \,. \tag{6.4.11}$$

So we see that gauge transformations are generated by usual diffeomorphisms in the $\beta^\sharp(\xi)$-direction, together with a "non-geometric" part given by $\hat{\delta}_\xi$, which we will call an infinitesimal β-*diffeomorphism*, i.e. β-diffeomorphisms are generated by transformations of this form.

Now, for general (r,s)-tensors $\hat{T}$, we define $\hat{\delta}_\xi \hat{T}$, which we can do, because $\hat{T}$ can always be uniquely written as $\hat{T} = ((B^\flat)^{\otimes r} \otimes (\beta^\sharp)^{\otimes s})T$ for some (s,r)-tensor T. Then we do the infinitesimal transformation $B \mapsto B + d\xi$ and define

$$\delta_\xi \hat{T} := (((B + d\xi)^\flat)^{\otimes r} \otimes ((\beta + \delta_\xi \beta)^\sharp)^{\otimes s})T - \hat{T} \,, \tag{6.4.12}$$

and

$$\hat{\delta}_\xi \hat{T} := \beta^\sharp(\xi)(\hat{T}) - \delta_\xi \hat{T} \,. \tag{6.4.13}$$

We say $\hat{T}$ is a β-*tensor* if $\hat{\delta}_\xi \hat{T} = \hat{L}_\xi \hat{T}$.

[Blu+13] shows a few things, which follow from straightforward computations:

(i) Θ is a β-tensor;

(ii) The algebra of β-diffeomorphisms does not close, but it does close up to honest diffeomorphisms, so the algebra of β-diffeomorphisms and honest diffeomorphisms does close;

(iii) For two β-one-forms ξ and η, $[\xi, \eta]$ is a β-one-form;

(iv) The Levi-Civita connection $\hat{\nabla}$ of $\hat{g}$ maps β-tensors to β-tensors.

In particular, we have all the tools we need to define a differential geometric theory of β-tensors.

The next step is to write down a β-invariant action, then we have a dynamical theory with nontrivial R-flux. Now, since our algebroid is the cotangent bundle, and sections of $\Lambda^n T^*M$ can be integrated, we want to find a β-volume form, or rather, a top form that is a β-volume form up to a total derivative. [Blu+13] shows that $\sqrt{-|\hat{g}|}|\beta^{-1}|dx^m$ satisfies these conditions. The Lagrangian that is then considered for a symplectic gravity theory is

$$\mathcal{L} = e^{-2\varphi}(\hat{R} - \tfrac{1}{12}\Theta^{ijk}\Theta_{ijk} + 4\hat{g}_{ij} D^i \varphi D^j \varphi) \,, \tag{6.4.14}$$

where $D^i := \beta^\sharp(dx^i)$, and φ is the dilaton field. The associated action is manifestly invariant under β-diffeomorphisms, so this gives a well defined functional. Moreover, this precise form is given such that it looks like the action for the bosonic string.

The upshot is now that we pulled back everything to the cotangent bundle, such that the canonical frames corresponding to coordinates do not come from the coframes of the coordinate system, but rather to the frames of the coordinate system pulled back via β. Thus, many things become nontrivial for the coframes corresponding to coordinate systems, in particular, $[dx^i, dx^j]$ does not necessarily vanish. This gives a rather peculiar theory, which is nevertheless perfectly well defined due to everything being invariant under β-diffeomorphism.

In [Blu+13], they solve the associated equations of motion in some particular cases, and they indeed find some dynamics with nontrivial R-fluxes.

One thing to note is that in Section 6.3 of [Blu+13], a cohomology theory is defined, which the authors call *quasi-Poisson cohomology*, together with a notion of *co-Calabi-Yau manifold*. This quasi-Poisson cohomology they define is isomorphic to ordinary de Rham cohomology, and the notion of co-Calabi-Yau coincides with the usual notion of Calabi-Yau manifold, transported to the cotangent bundle using β.

This gives us an explicit example of a string theory where the extra six dimensions are "non-geometric". Examples of these kinds of theories also arise naturally from the setting of toroidal compactifications with nontrivial H-flux. Compactifying on tori puts us in the setting where T-duality transformations are a canonical thing to consider. As it turns out, applying two T-duality transformations to a theory with background H-flux gives a nongeometric theory where monodromy around the torus acts by T-duality. These kinds of theories are most naturally formulated in terms of *Courant algebroids*, which are kind of like Lie algebroids, but where the target space is $TM \oplus T^*M$ equipped with a natural bracket. For details on these kinds of constructions, see e.g. the review [Pla19] for the physical picture, or [CG10] for the mathematical picture.

6.5 Calabi-Yau Lie algebroids in string theory?

The previous section gave an application of Lie algebroids to string theory. Ideally, we would also find an application of Calabi-Yau Lie algebroids to physics. The idea we present here is still speculatory in nature and as of right now, just a possible mechanism without a proper framework to couple it to string theory.

The idea is to start with a spacetime $M^{10} \cong M^4 \times M^6$, where M^6 is some six-dimensional complex manifold. On the M^6, we take a four dimensional elliptic tangent bundle corresponding to $N \hookrightarrow M^6$, possibly wrapping a nontrivial homology cycle, see Example 6.1.11. Now, if we could find a Calabi-Yau metric on $TM(-\log|N|)$, such that the holonomy group is $SU(3)$, and such that $TM(-\log|N|)$ admits spinors, then we have a global parallel spinor on $M^6 \setminus N$, and a metric with logarithmic divergence towards N.

One question that appears in this theory is whether or not the bundle $TM(-\log|N|)$ admits spinors. One thing to note is that it does admit a complex structure induced by the complex structure of M, so it is orientable. Therefore, the question reduces to whether the honest first Chern class $c_1(TM(-\log|N|)) \in H^2(M;\mathbb{Z})$ vanishes mod 2, as then we know that the second Stiefel-Whitney class is zero. In particular, does $\mathrm{Hol}(g) \subseteq SU(m)$ leafwise mean that $TM(\log|N|)^{(m,0)}$ is trivial?

Now we turn to describing a local model on these kinds of spaces, giving differential equations that should be satisfied to have a Kähler form on the space.

A hermitian two-form on $TM(-\log|N|)$ would, in a slice chart, be given by

$$\omega = f(\mathbf{z},\overline{\mathbf{z}})d\log z_0 \wedge d\log \bar{z}_0 + \sum_{i=1}^{n} g_i(\mathbf{z},\overline{\mathbf{z}})d\log(z_0) \wedge d\bar{z}_i - \sum_{i=1}^{n} \overline{g_i}(\mathbf{z},\overline{\mathbf{z}})dz_i \wedge d\log(\bar{z}_0) + \sum_{i,j=1}^{n} h_{ij}(\mathbf{z},\overline{\mathbf{z}})dz_i \wedge d\bar{z}_j\,,$$

where $-if$ is a positive real valued smooth function and g_i and h_{ij} are complex valued smooth functions such that $h_{ji} = -\overline{h_{ij}}$, which is imposed by reality, and all eigenvalues of $-ih$ are positive, which is imposed by positive definiteness of the metric.

Now, the Kähler condition says $d_{TM(-\log|N|)}\omega = 0$, which translates to the following differential equations

(i) $\partial_{z_i} f + z_0 \partial_{z_0} \overline{g_i} = 0\,; \quad i = 1, \dots, n\,;$

(ii) $\partial_{z_i} g_j - z_0 \partial_{z_0} h_{ij} = 0\,; \quad i, j = 1, \dots, n\,;$

(iii) $\partial_{z_i} h_{jk} - \partial_{z_j} h_{ik} = 0\,; \quad i, j, k = 1, \dots, n\,.$

An example of a six dimensional space satisfying these properties would be $K3 \times \mathbb{C}P^1$, where $N = K3 \times (\{0\} \sqcup \{\infty\})$, see Example 6.3.17. This example is, in particular, Ricci-flat, so it has some of the desirable properties Calabi-Yau manifolds have.

Such spaces are Calabi-Yau in the bulk, but the Kähler potential degenerates like $\log(z)\log(\bar{z})$ towards the divisor. As far as the author is aware, these kinds of spaces do not appear in string theory as of now.

Perhaps the divisor N in this picture could be interpreted as some kind of brane. Given the setup, this would mean N has to be a spacetime filling seven-brane, as the metric on M^{10} has to be continuous. However, this does mean that the volume of the $M^6 \setminus N$ diverges, given that the volume form is a logarithmic form. Thus, this kind of space might not be well-suited as a compactifying space for string theory. But perhaps the metric with logarithmic divergence need not be literally interpreted as a metric, but perhaps as some second auxiliary metric, like in bigravity theories, cf. e.g. [HR11].

Moreover, in [DKW13], the authors present a theory on certain log-Calabi-Yau spaces (i.e. elliptic tangent bundles with a holomorphic volume form) that appear as degenerations of smooth Calabi-Yaus in the context of F-theory, where they show there is some holographic behaviour, i.e. a part of the theory is localised on the divisor. With a bit of imagination, such behaviour could be studied in the case where the Calabi-Yau structure is induced by a Riemannian metric on the algebroid.

Note again that this suggestion is heavily under-developed as of now. Perhaps there is another, more useful application in physics to be found. We will keep on looking for something as the theory develops, and we encourage the reader to do the same!

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