Elliptic genera in mathematics and physics and a generalization to G-manifolds

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Abstract Elliptic genera are a special type of manifold invariants that only depend on the bordism class of the manifolds. They can be constructed from elliptic curves and take values in modular forms. Important examples include the signature for $4n$-manifolds and the so-called $\widehat{A}$-genus which is related to the Dirac operator. A more general perspective is provided by complex genera which can be studied using formal group laws. This is the right setting for a generalization of elliptic genera to G-manifolds. For this, equivariant bordism and equivariant formal group laws are considered. In physics, elliptic genera arise as partition functions of supersymmetric string theories which has been originally described by Witten. Since then, this perspective has been further considered and gave rise to several developments in the mathematical theory. In this thesis, we will give an introduction to elliptic genera and compare the mathematical and physical perspective. Furthermore, we will explain their relations to formal group laws and outline the generalization of elliptic genera to the equivariant setting.

Contents

1. Introduction

In the study of manifolds, one goal is their classification in terms of *invariants*. Examples include the Euler characteristic and algebraic objects such as homology and cohomology. Invariants of smooth manifolds are unchanged under diffeomorphisms. That is, the invariant only depends on the diffeomorphism class. Another relation between smooth manifolds is *bordism*. The topic of this thesis is a very special type of invariant that depends only on the bordism class of a manifold: *elliptic genera*.

A *genus* φ is a bordism invariant for manifolds that is *additive* and *multiplicative*. That is, a genus satisfies

$$\begin{aligned}\varphi(M \sqcup N) &= \varphi(M) + \varphi(N),\\ \varphi(M \times N) &= \varphi(M)\varphi(N).\end{aligned}$$

We will make this precise by defining a genus as a ring homomorphism out of a bordism ring. An *elliptic genus* is a genus that is constructed from an elliptic curve. In this way, elliptic genera relate the study of manifold invariants with the theory of *elliptic functions* and *modular forms*. Elliptic genera can also be described by *formal group laws*. With this, the theory of elliptic genera gave rise to elliptic cohomology and topological modular forms. In physics, elliptic genera arise as *partition functions* of certain supersymmetric string theories and relate to the *index* of a twisted Dirac operator.

Outline

In Chapter 2, we start by recalling elements of the theory of vector bundles and introduce classifying spaces and characteristic classes for vector bundles with orientation and complex structure. We then turn to defining bordism in Section 2.2. In particular, we will define bordism rings Ω_*^{SO} and Ω_*^{U} of compact manifolds with orientation and with stably almost complex structure. To determine the structure of these rings, we will then introduce *Thom spectra* in Section 2.4. The *Pontryagin-Thom theorem* relates the bordism rings Ω_*^{SO} and Ω_*^{U} to the homotopy groups of the Thom spectra MSO and MU. In Section 2.5, we outline the proof of the Pontryagin-Thom theorem. As a consequence of this, we obtain structural results for the bordism rings Ω_*^{SO} and Ω_*^{U}.

With the structure of the bordism rings, we can define elliptic genera in Chapter 3. Geometrically interesting special cases include the *signature* for $4n$-manifolds and the $\widehat{A}$-*genus* which is related to the index of the Dirac operator. Using elliptic functions, we construct elliptic genera from lattices with a choice of 2-division point in Section 3.2. Equivalently, this yields a construction of elliptic genera from elliptic curves with a 2-torsion point. We then consider the universal elliptic genus that associates modular forms for $\Gamma_0(2)$ to compact oriented manifolds and generalize this to the universal elliptic genus of level N associating modular forms for $\Gamma_1(N)$ to compact almost complex manifolds.

In Chapter 4, we will connect the theory of elliptic genera to formal group laws. These are power series in two variables satisfying axioms that resemble those of an abelian group. Formal group laws can be classified by a universal formal group law over a polynomial ring $L = \mathbb{Z}[x_2, x_4, \dots]$. This is *Lazard's theorem*. We will see that complex genera and formal group laws are in one-to-one correspondence. This is a consequence of *Quillen's theorem* which states that the formal group law over $\pi_*(MU)$ is the universal one. With this, we can describe elliptic genera via their formal group laws. In Section 4.3, we will introduce formal schemes and formal groups providing a more geometric and coordinate-independent perspective on formal group laws.

We will outline the generalization of elliptic genera to G-manifolds in Chapter 5. For this, we first define equivariant bordism rings and describe the equivariant Pontryagin-Thom construction. Then, we define equivariant formal group laws and investigate how the results from the previous chapter carry over to the equivariant setting. Furthermore, we sketch a definition of *equivariant elliptic genera* and discuss further possible directions and applications.

Chapter 6 is devoted to the physics perspective on elliptic genera. We will first give a short introduction to string theory. After this, we discuss Witten's interpretation of the elliptic genus in string theory as a character-valued index of a Dirac-like operator and a partition function of a supersymmetric string theory. Moreover, we will consider the more general *two-variable elliptic genus*, its applications and relation to the universal elliptic genus of level N. Lastly, we discuss the DMVV-formula and its physical background. We also briefly outline how this has led to the development of a mathematical theory of orbifold elliptic genera.

Our approach

The purpose of this thesis is to compare and connect different perspectives on elliptic genera in mathematics and in physics. In Chapter 2, we collect geometric background to construct the bordism rings. For this, we combine different perspectives following [MS74], [Fre13] and [Sto68]. We also fill in some details about complex structures on the tangent and normal bundle of manifolds. The definitions and constructions of elliptic genera in Chapter 3 are largely based on [HBJ92]. We highlight the relation between the elementary constructions and the *universal* elliptic genus which appears in the more recent mathematics and physics literature. In our introduction to formal group laws we mainly follow [Str19]. We then explicitly relate these to the theory of elliptic genera by considering the formal group laws associated to examples of elliptic genera. To later connect to the theory of equivariant formal group laws, we also describe a functorial approach to formal schemes and formal groups following [Str19] and [Str99].

Even if the literature on non-equivariant bordism is relatively complete, the generalizations to the equivariant setting are often not specifically explained and there exist several different approaches. Following [Sin01] and [Han05] in our treatment, we attempt to define all of the necessary notions and outlining more systematically how the theory of bordism generalizes equivariantly, while also referring to original references such as [Was69], [tom70] and [Sto69]. For example, in the literature the difference between stable complex G-structures on the tangent and normal bundle is often not clear. Following [Han05], we point out an important distinction. Using our perspective on formal groups and formal group laws, we then define equivariant formal group laws following [CGK00] and [Gre01]. With this, we can view equivariant complex genera from the point of view of equivariant formal group laws. As a result, we sketch a definition of equivariant elliptic genera using this perspective. We hope that this will give rise to the construction of elliptic genera for G-manifolds and orbifolds through equivariant formal group laws from elliptic curves.

While most of our treatment is mathematical, we also introduce the physical perspective on elliptic genera and compare the approaches. The mathematics and physics literature often differ largely in language and style. We attempt to nevertheless connect them by introducing some of the formalism and motivation in physics to study elliptic genera and explain it in more mathematical terms. As an example, we highlight the connections between the universal elliptic genus of level N and the two-variable elliptic genus in physics following [BL00b] and [Wen15].

Conventions and prerequisites

We assume graduate-level knowledge in pure mathematics and theoretical physics. In particular, the reader should be familiar with differential and algebraic topology. We will also sometimes use the language of category theory. Knowledge of the theory of elliptic curves and modular forms is helpful, but not required. For Chapter 6, we assume knowledge of string theory and quantum field theory.

We use the following conventions. All manifolds are considered to be smooth and to have no boundary if not explicitly mentioned otherwise. Moreover, every ring is assumed to have a unit and to be commutative.

2. Bordism Rings and the Pontryagin-Thom Theorem

Elliptic genera are ring homomorphisms out of *bordism rings*. In this chapter, we define bordism, construct bordism rings and analyse their structure. We start by recalling the theory of vector bundles in Section 2.1 focussing on aspects necessary for the theory of elliptic genera. After this, we will define bordism and construct bordism rings of compact manifolds with orientation and (stably) almost complex structure in Sections 2.2 and 2.3. To obtain the structure of the bordism rings, we will need the Pontryagin-Thom theorem relating them to the homotopy groups of Thom spectra. For this, we introduce elements of stable homotopy theory in Section 2.4. Lastly, in Section 2.5, we will discuss the Pontryagin-Thom theorem and outline its proof and consequences.

2.1. Vector bundles, classifying spaces and characteristic classes

In this section, we will recall aspects of the theory of vector bundles. For this, we assume basic familiarity with the theory and collect statements about classifying spaces and characteristic classes which we need for the construction of bordism rings and elliptic genera. Our main reference for this section is [MS74].

Vector bundles

We start by recalling the definition of a vector bundle.

Definition 2.1. A (real) *vector bundle* is a triple (π, E, B) where $\pi\colon E \to B$ is a continuous surjective map between topological spaces E and B that satisfies the following conditions for every point $b \in B$.

(i) The fibre $\pi^{-1}(b)$ has the structure of a finite dimensional real vector space.

(ii) There exists an open neighbourhood $U \subseteq B$ of b, an integer $k \geq 0$ and a homeomorphism

$$\varphi\colon U \times \mathbb{R}^k \to \pi^{-1}(U)$$

such that $\pi(\varphi(b, x)) = b$ for all $x \in \mathbb{R}^k$, and $x \mapsto \varphi(b, x)$ is an isomorphism of the vector spaces $\mathbb{R}^k$ and $\pi^{-1}(b)$. This condition is called the *local triviality condition.*

The space E is called the *total space* and B is called the *base space.* The vector bundle (π, E, B) can be thought of as a family of vector spaces continuously parametrized by the space B. The integer k is called the dimension and is locally constant. However, we are only interested in the case where k is constant. We then call the vector bundle a *k-dimensional vector bundle* or a *k-plane bundle.* A vector bundle (π, E, B) will often be denoted simply by the map $\pi\colon E \to B$.

Example 2.2. (i) Let B a topological space and define $E := B \times \mathbb{R}^n$. Then the projection $\pi\colon E \to B$ is called the *trivial vector bundle* over B and denoted by $\underline{\mathbb{R}}^n$.

(ii) Let B be a manifold. Then its *tangent bundle* $\tau\colon TM \to M$ is a vector bundle. Choose an embedding $M \hookrightarrow \mathbb{R}^k$ with k sufficiently large. Then the *normal bundle* ν of this embedding is a vector bundle.

(iii) Let $B := \mathbb{R}\mathrm{P}^k$. Recall that $\mathbb{R}\mathrm{P}^k$ is constructed as the space of 1-dimensional linear subspaces of $\mathbb{R}^{k+1}$. Now, define the total space

$$E := \{(v, L) \in \mathbb{R}^{k+1} \times \mathbb{R}\mathrm{P}^k \mid v \in L\}.$$

and projection map by $\pi(v, L) = L$. This defines a 1-dimensional vector bundle called the *tautological bundle* over $\mathbb{R}\mathrm{P}^k$.

In the definition of a vector bundle, the underlying vector spaces are real. Analogously, we can define complex vector bundles by replacing $\mathbb{R}$ by $\mathbb{C}$. Then, there are similar examples as above.

Remark 2.3. If we replace the topological spaces E and B by (smooth) manifolds and the homeomorphism by a diffeomorphism, we obtain the notion of a smooth vector bundle. The tangent bundles of a manifold is an example of a smooth vector bundle.

Definition 2.4. Let $\xi\colon E \to B$ and $\eta\colon E' \to B'$ be vector bundles. A *bundle map* is a map of the total spaces $f\colon E \to E'$ that maps each fibre the E_b for $b \in B$ to a fibre $E'_{b'}$ for some $b' \in B'$ such that this map is an isomorphism of vector spaces. This induces a map on the base spaces $\overline{f}\colon B \to B'$.

When introducing classifying spaces for vector bundles, we will see that vector bundles over a space B can be obtained by *pulling back* a universal bundle along maps from B into a universal classifying space. The pullback bundle is defined as follows.

Definition 2.5. Let $\pi\colon E \to B$ be a vector bundle and let $f\colon B' \to B$ be a continuous map. Consider the pullback square

$$\begin{array}{ccc} f^*E & \xrightarrow{\;g\;} & E \\ {\scriptstyle \pi'}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{\;f\;} & B \end{array}$$

with $f^*E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\} \subseteq B' \times E$. This defines a vector bundle (π', f^*E, B') called the *pullback bundle.*

As an example, consider a vector bundle $\pi\colon E \to B$ and a subspace $B' \subseteq B$. Pulling back along the inclusion $B' \hookrightarrow B$ defines the *restriction* of π to B'. There are several other ways to construct new vector bundles out of existing ones. For instance, by applying constructions for vector spaces fibrewise, we can define the *direct sum* and *tensor product* of vector bundles over the same base space.

Definition 2.6. Let $\pi_1\colon E_1 \to B_1$ and $\pi_2\colon E_2 \to B_2$ be vector bundles. Define the *product vector bundle* by

$$\pi_1 \times \pi_2\colon E_1 \times E_2 \to B_1 \times B_2.$$

The fibre at $(b_1, b_2) \in B_1 \times B_2$ obtains the structure of a vector space as the direct sum of the vectors spaces at b_1 and b_2.

Definition 2.7. Let $\pi_1\colon E_1 \to B$ and $\pi_2\colon E_2 \to B$ be two vector bundles over the same base space B. The *Whitney sum* is defined as the restriction of the bundle

$$\pi_1 \times \pi_2\colon E_1 \times E_2 \to B \times B$$

to the diagonal subspace

$$B \hookrightarrow B \times B, \quad b \mapsto (b, b).$$

Its fibres are given by the direct sum $\pi_1^{-1}(b) \oplus \pi_2^{-1}(b)$ for every $b \in B$.

The tensor product of vector bundles is defined similarly.

Classifying spaces

We will now construct classifying spaces and universal vector bundles over them. Then, a vector bundle over a base space B can be obtained by the pullback of the universal vector bundle along a map from B into the classifying space. We follow Chapter 5 of [MS74].

We start with n-dimensional vector bundles without further structure. Recall the construction of the tautological vector bundle over $\mathbb{R}\mathrm{P}^k$. As a generalization, we define the *Grassmannian* $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ and a universal vector bundle over it as follows. We first define the space $\mathrm{V}_n(\mathbb{R}^{n+k})$ as the set of orthonormal n-frames, that is the set of n-tuples of linearly independent orthonormal vectors in $\mathbb{R}^{n+k}$. The topology on $\mathrm{V}_n(\mathbb{R}^{n+k}) \subseteq (\mathbb{R}^{n+k})^n$ is the subspace topology. There is a free action on $\mathrm{V}_n(\mathbb{R}^{n+k})$ by the group $O(n)$ given by rotating the n-dimensional subspace of $\mathbb{R}^{n+k}$ spanned by the n-frame. The quotient of this action $\mathrm{V}_n(\mathbb{R}^{n+k})/O(n)$ is the set of n-dimensional subspaces of the space $\mathbb{R}^{n+k}$ which we denote by $\mathrm{Gr}_n(\mathbb{R}^{n+k})$. The tautological vector bundle over $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ is defined as follows. Let

$$E = \{(v, V) \in \mathbb{R}^{n+k} \times \mathrm{Gr}_n(\mathbb{R}^{n+k}) \mid v \in V\}$$

and define the tautological bundle $\gamma_{n,k}$ as the projection $E \to \mathrm{Gr}_n(\mathbb{R}^{n+k})$ sending each (v, V) to the n-plane V. This defines a vector bundle over $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ as shown in Lemma 5.2 in [MS74]. Note that the case $n = 1$ recovers the tautological bundle over real projective space $\mathbb{R}\mathrm{P}^k$.

Lemma 2.8. *The Grassmannian* $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ *is a compact smooth manifold.*

Proof. For a proof that the Grassmannian is a topological manifold, we refer to Lemma 5.1 in [MS74] and for the smooth structure we refer to Example 1.36 in [Lee13]. □

Now, consider the inclusions

$$\mathbb{R}^n \subseteq \mathbb{R}^{n+1} \subseteq \ldots \subseteq \mathbb{R}^{n+k} \subseteq \ldots$$

by embedding $(x_1, \ldots, x_n) \in \mathbb{R}^n$ as $(x_1, \ldots, x_n, 0) \in \mathbb{R}^{n+1}$. This induces the chain of inclusions

$$\mathrm{Gr}_n(\mathbb{R}^n) \subseteq \mathrm{Gr}_n(\mathbb{R}^{n+1}) \subseteq \ldots \subseteq \mathrm{Gr}_n(\mathbb{R}^{n+k}) \subseteq \ldots$$

Taking the colimit over k, we obtain

$$\mathrm{Gr}_n(\mathbb{R}^\infty) = \varinjlim_k \mathrm{Gr}_n(\mathbb{R}^{n+k}).$$

We can analogously construct the tautological vector bundle over $\mathrm{Gr}_n(\mathbb{R}^\infty)$ as the bundle with total space consisting of pairs (v, V) where V is an n-plane in $\mathbb{R}^\infty$. The projection then sends (v, V) to the n-plane V. This defines a vector bundle γ_n over $\mathrm{Gr}_n(\mathbb{R}^\infty)$ (see Lemma 5.4 in [MS74]), and is called the *universal* vector bundle. The reason for this is as follows.

Theorem 2.9. *A real vector bundle ξ of real dimension n over a paracompact base space B determines a map $B \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ that is unique up to homotopy.*

Proof. This is Corollary 5.10 in [MS74]. □

Remark 2.10. Here and in the following, we often make the assumption that the base space is paracompact. This is not a restriction since we are mostly interested in vector bundles over compact manifolds. The paracompactness of the base space has several advantages. One of them is that we can choose an inner product on the vector bundle. We will use this in the definition of Thom spaces in Section 2.4. Furthermore, general fibre bundles over paracompact base spaces are fibrations which we implies the existence of lifts. We refer to Section 5.8 of [MS74] for more details.

Taking the pullback of the universal vector bundle γ_n along the map $B \to \mathrm{Gr}_n(\mathbb{R}^\infty)$, we obtain the vector bundle ξ. There is a stronger version of the above theorem which is often referred to as the homotopy classification of vector bundles.

Theorem 2.11. *Let B be a paracompact space. There is a natural bijection between homotopy classes of maps $[B, \mathrm{Gr}_n(\mathbb{R}^\infty)]$ and isomorphism classes of vector bundles over B.*

Proof. See [May99, p.189] for a categorical formulation and proof of this theorem. □

The above discussion carries through upon replacing $\mathbb{R}$ by $\mathbb{C}$. We obtain the complex Grassmannian $\mathrm{Gr}_n(\mathbb{C}^\infty)$ and the universal complex n-plane bundle $\gamma_n^{\mathbb{C}}$ over it.

Theorem 2.12. *A complex vector bundle ξ of complex dimension n over a paracompact base space B determines a map $B \to \mathrm{Gr}_n(\mathbb{C}^\infty)$ that is unique up to homotopy.*

Proof. For the proof, we refer to Theorem 14.6 in [MS74]. □

Complex vector bundles can also be seen as real vector bundle together with a *complex structure*. We will come back to this in Section 2.2. Another important structure on vector bundles is *orientation*.

Definition 2.13. Let V be a real vector space of dimension n. An *orientation* on V is an equivalence class of bases such that two bases $(v_1, \dots, v_n)$ and $(v'_1, \dots, v'_n)$ are equivalent if the matrix A with coefficients a_{ij} defined by $v'_i = \sum_j a_{ij} v_j$ has positive determinant.

The vector space $\mathbb{R}^n$ has a canonical orientation given by the class of the standard basis.

Definition 2.14. Let ξ be a real vector bundle of dimension n. An *orientation* on ξ is a choice of orientation on each of the fibres such that the vector space homeomorphism in the local triviality condition of ξ

$$\varphi\colon U \times \mathbb{R}^n \to \pi^{-1}(U)$$

are fibrewise orientation preserving where $\mathbb{R}^n$ has the canonical orientation.

Analogously to the above, we can construct the oriented Grassmannian $\widetilde{\mathrm{Gr}}_n(\mathbb{R}^\infty)$ as the oriented n-dimensional linear subspaces of $\mathbb{R}^\infty$ and a universal n-dimensional oriented vector bundle $\widetilde{\gamma}_n$ over $\widetilde{\mathrm{Gr}}_n(\mathbb{R}^\infty)$ classifying oriented vector bundles of dimension n. In the following we will denote the Grassmannians by

$$\begin{aligned} BO(n) &= \mathrm{Gr}_n(\mathbb{R}^\infty), \\ BSO(n) &= \widetilde{\mathrm{Gr}}_n(\mathbb{R}^\infty), \\ BU(n) &= \mathrm{Gr}_n(\mathbb{C}^\infty). \end{aligned}$$

This notation has the following reason. Using the language of *principal G-bundles*, one can construct a classifying space BG for every topological group G. This yields a universal principal G-bundle $\pi\colon EG \to BG$ where EG is *weakly contractible*[1]. Then by choosing a representation $G \to O(n)$ and constructing the *associated vector bundle*, one obtains a universal vector bundle over BG that classifies vector bundles with structure group reduced to G. Here we implicitly assume that the base spaces are paracompact such that the vector bundles admit an inner product and the structure group reduces from $GL_n(\mathbb{R})$ to $O(n)$. For oriented vector bundles, we have structure group $G = SO(n)$, and for complex vector bundles of real dimension $2n$, we have $G = U(n)$. Under the identification of principal G-bundles with vector bundles with structure group, the classification theorem for vector bundles (Theorem 2.11) generalizes to the functor represented by BG. For more details, we refer to [Kot12] and Section 23.8 in [May99].

[1] A topological space is called weakly contractible if all of its homotopy groups vanish.

Characteristic classes

To classify unoriented, oriented and complex vector bundles, we have defined the classifying spaces $BO(n)$, $BSO(n)$ and $BU(n)$ respectively. We will now introduce *characteristic classes* which are cohomology classes of the base space that measure the non-triviality of a vector bundle over it. We will see that the cohomology of the classifying spaces are generated by the characteristic classes of their universal bundles.

In the following, we define *Stiefel-Whitney classes*, *Chern classes* and *Pontryagin classes*. There is a fourth type of characteristic classes, the *Euler class* $e(\xi)$, which is defined for oriented vector bundles ξ. Since we will not need this for the description of bordism rings, for its definition, we refer to Chapter 9 in [MS74].

Theorem 2.15 (Stiefel-Whitney classes)**.** *Let $\xi\colon E \to B$ be a real vector bundle. There exist unique cohomology classes $w_i(\xi) \in H^i(B;\mathbb{Z}/2)$ satisfying the following properties.*

(i) The Stiefel-Whitney classes satisfy $w_0(\xi) = 1 \in H^0(B;\mathbb{Z}/2)$, and $w_i(\xi) = 0$ for $i > \operatorname{rank}(\xi)$. The total Stiefel-Whitney class *is $w(\xi) = \sum_{i\geq 0} w_i(\xi) \in H^*(B;\mathbb{Z}/2)$.*

(ii) Let η be another vector bundle over B. Then, the total Stiefel-Whitney class satisfies $w(\xi \oplus \eta) = w(\xi)w(\eta)$.

(iii) Let $f\colon B' \to B$ be map. Then the pullback $f^*\xi$ has Stiefel-Whitney class given by $w_i(f^*\xi) = f^*w_i(\xi) \in H^*(B';\mathbb{Z}/2)$.

(iv) The tautological bundle over the projective space $\mathbb{R}\mathrm{P}^1$ has non-zero Stiefel-Whitney class in $H^*(\mathbb{R}\mathrm{P}^1;\mathbb{Z}/2)$.

Proof. See Theorem 3.1 in [Hat17]. □

Theorem 2.16 (Chern classes)**.** *Let $\xi\colon E \to B$ be a complex vector bundle. There exist unique cohomology classes $c_i(\xi) \in H^{2i}(B;\mathbb{Z})$ called* Chern classes *satisfying the following properties.*

(i) $c_0(\xi) = 1$, and $c_i(\xi) = 0$ for $i > \operatorname{rank}_{\mathbb{C}}(\xi)$. Define $c(\xi) := \sum_{i\geq 0} c_i(\xi) \in H^*(B;\mathbb{Z})$ to be the total Chern class*.*

(ii) Let η be another complex vector bundle over B. The total Chern class satisfies $c(\xi \oplus \eta) = c(\xi)c(\eta)$.

(iii) Let $f\colon B' \to B$ be map. Then the pullback $f^*\xi$ has Chern class given by $c_i(f^*\xi) = f^*c_i(\xi) \in H^*(B';\mathbb{Z})$.

(iv) Let γ be the tautological bundle over $\mathbb{C}\mathrm{P}^n$. Then $c(\gamma) = 1-g$ where $g \in H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is the generating element of the cohomology ring $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$.

Proof. See Theorem 3.2 in [Hat17]. □

The last property ensures that the Chern classes are not trivial and can be seen as a normalization. We will later see that the complex projective spaces are (rational) generators for the complex bordism ring. We write $c_i(M)$ for the i-th Chern classes of the tangent bundle of an (almost) complex manifold.

Example 2.17. The total Chern class of the tangent bundle of $\mathbb{C}\mathrm{P}^n$ is given by

$$c(\mathbb{C}\mathrm{P}^n) = (1+g)^{n+1}.$$

See Theorem 14.10 in [MS74] for a proof.

Chern classes measure the triviality of a complex vector bundle. In particular, adding the trivial bundle does not change the total Chern class.

Proposition 2.18. *Let $\xi\colon E \to B$ be a complex vector bundle and let $\underline{\mathbb{C}}^k$ be the trivial bundle over B. Then, $c(\xi \oplus \underline{\mathbb{C}}^k) = c(\xi)$.*

Proof. This is Lemma 14.3 in [MS74]. □

Next, we define *Pontryagin classes* for a real vector bundle through the Chern classes of its complexification.

Theorem 2.19 (Pontryagin classes)**.** *Let $\eta\colon E \to B$ be a real vector bundle. The* Pontryagin classes *of η defined as*

$$p_i(\eta) := (-1)^i c_{2i}(\eta \otimes \mathbb{C}) \in H^{4i}(B;\mathbb{Z})$$

satisfy the following properties.

(i) $p_0(E) = 1$ and $p_i(E) = 0$ for $2i > \mathrm{rank}_{\mathbb{R}}(\eta)$. Define $p(\eta) := \sum_{i\geq 0} p_i(\eta) \in H^*(B;\mathbb{Z})$ to be the total Pontryagin class*.*

(ii) Let ρ be another real vector bundle over B. The total Pontryagin class satisfies $2[p(\eta \oplus \rho) - p(\eta)p(\rho)] = 0 \in H^*(B;\mathbb{Z})$.

(iii) *Let $f\colon B' \to B$ be a map. Then the pullback $f^*\eta$ has Pontryagin class given by $p_i(f^*\xi) = f^*p_i(\eta) \in H^*(B';\mathbb{Z})$.*

(iv) *Let γ be the tautological bundle over $\mathbb{C}\mathrm{P}^n$, and $\gamma_{\mathbb{R}}$ its underlying real vector bundle. Then $p(\gamma_{\mathbb{R}}) = 1 + g^2$ where $g \in H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is the generator of $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$.*

Proof. The properties of the Pontryagin classes follow from those of the Chern classes. For more details, see [HBJ92, p. 4]. □

As for the Chern classes, we write $p_i(M)$ for the Pontryagin classes of the tangent bundle of a real manifold M. For a complex manifold M, we write $p_i(M)$ for the Pontryagin classes of the underlying real tangent bundle.

Example 2.20. The total Pontryagin class of $\mathbb{C}\mathrm{P}^n$ is given by $p(\mathbb{C}\mathrm{P}^n) = (1+g^2)^{n+1}$. This follows from the total Chern class of $\mathbb{C}\mathrm{P}^n$. Denote by p_i and c_j the Pontryagin and Chern classes of $\mathbb{C}\mathrm{P}^n$. Then by definition of the Pontryagin classes,

$$\begin{aligned}(1 - p_1 + p_2 - \cdots \pm p_n) &= (1 - c_1 + c_2 - \cdots \pm c_n)(1 + c_1 + \cdots + c_n)\\ &= (1-g)^{n+1}(1+g)^{n+1}\\ &= (1-g^2)^{n+1},\end{aligned}$$

and hence, $p(\mathbb{C}\mathrm{P}^n) = (1+g^2)^{n+1}$. This is Example 15.6 in [MS74].

The characteristic classes of the universal vector bundles over the classifying spaces $BO(n)$, $BU(n)$ and $BSO(n)$ are elements of the cohomology of these spaces. Conversely, we have the following theorem.

Theorem 2.21. *(i) The cohomology of the classifying space $BO(n)$ is given by*

$$H^*(BO(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1,\ldots,w_n]$$

where $\deg w_i = i$. The generators w_i are the Stiefel-Whitney classes of the universal vector bundle over $BO(n)$.

(ii) The cohomology of the classifying space $BU(n)$ is given by

$$H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1,\ldots,c_n]$$

where $\deg c_i = 2i$. The generators c_i are the Chern classes of the universal complex vector bundle over $BU(n)$.

(iii) Let R be an integral domain containing $\frac{1}{2}$. The cohomology of the classifying space $BSO(n)$ is given by

$$H^*(BSO(2n+1); R) \cong R[p_1, \ldots, p_n]$$
$$H^*(BSO(2n); R) \cong R[p_1, \ldots, p_{n-1}, e].$$

Here, p_i with degree $4i$ are the Pontryagin classes of the universal vector bundle, and e is the Euler class with degree $2n$ and $e^2 = p_n$.

Proof. For the proof we refer to the following theorems in [MS74]. The first statement is Theorem 7.1, the second statement is Theorem 14.5, and the third statement is Theorem 15.9 in [MS74]. □

Remark 2.22. The reason for the choice of coefficients in (iii) is to eliminate the 2-torsion. Compare with (ii) in Theorem 2.19 which up to 2-torsion is the same property as for the total Chern and Stiefel-Whitney classes. Taking integer coefficients, the structure of the cohomology of $BSO(n)$ would be much more complicated.

To compute the structure of the bordism rings for manifolds with different structures in Section 2.5, the *Thom isomorphism* (Theorem 2.75) is relating the (co)homologies of the base space and the Thom space of a bundle.

For the study of bordism and the definition of elliptic genera, we will need *characteristic numbers*. These are obtained by evaluating characteristic classes of a compact n-manifold M on its *fundamental class*[2] $[M] \in H_n(M)$. Note that from the evaluation of a cohomology class in degree n on a homology class in degree n, we obtain an element in the coefficient ring. The coefficients depend on the type of characteristic class. We make the following definition.

Definition 2.23. Let $i_1, \ldots, i_m \in \mathbb{N}_0$ such that $\sum_{k=1}^m i_k = n$ for some $n \in \mathbb{N}$.

(i) Let M be a compact almost complex $2n$-manifold. The *Chern number* of M associated to $(i_1, \ldots, i_m)$ is defined as

$$\left(\prod_{k=1}^m c_{i_k}(M)\right)[M] \; \in \mathbb{Z}.$$

(ii) Let M be a compact oriented $4n$-manifold. The *Pontryagin number* of M associated to $(i_1, \ldots, i_m)$ is defined as

$$\left(\prod_{k=1}^m p_{i_k}(M)\right)[M] \; \in \mathbb{Z}.$$

[2]For more details, see Section 3.3 on Poincaré duality in [Hat02].

We will see in Theorem 2.77 and Theorem 2.79 that the characteristic numbers of a manifold determine its bordism class. With this, we construct genera as linear combinations of characteristic numbers in Section 3.1. Similarly, one can also construct *Stiefel-Whitney numbers* of compact manifolds. However, since we will be mostly focussing on compact oriented and almost complex manifolds, we refer to Section 4.4 of [MS74].

We end this section by introducing *Chern roots* which we will also need for the construction of elliptic genera. Furthermore, we will use them in Section 6.2 when computing the index of the twisted Dirac operator. Let $\xi = \xi_1 \oplus \cdots \oplus \xi_n$ be the Whitney sum of n complex line bundles ξ_i. Then, the total Chern class factors as

$$c(\xi) = 1 + c_1 + c_2 + \cdots + c_n = \prod_{i=1}^{n}(1 + x_i)$$

where x_i are called *Chern roots* and correspond to the first Chern class of ξ_i. As a result, the Chern classes are the *elementary symmetric functions* in the Chern roots. We will use this point of view in Chapter 3. For an arbitrary complex vector bundle ξ, we can still formally factorize the total Chern class. This is called the *splitting principle*. For more details, see Section 2.3 in [Koc96] and Section 1.4 in [HBJ92].

2.2. Bordism

In this section, we will define bordism and construct bordism rings for manifolds with different structures. The theory of bordism has its roots in the works of Thom [Tho54] and Pontryagin [Pon55], and has seen much development since then. In particular, bordism forms a generalized homology theory as shown by Atiyah [Ati61]. Modern formulations of bordism work with bordism categories. However, we are mainly interested in the construction of bordism rings. For this, our main reference is [Fre13]. Other references include the textbook [Swi02] and the classical [Sto68].

Roughly speaking, a bordism[3] between two n-manifolds M and N is an $(n+1)$-manifold Σ with boundary $\partial\Sigma \cong M \sqcup N$. More precisely, we make the following definition.

Definition 2.24. Let M and N be compact n-manifolds (without boundary). We define a *bordism* from M to N as a tuple $(\Sigma, p, \iota_0, \iota_1)$ where

(i) Σ is a compact $(n+1)$-manifold with boundary,

(ii) p is a partition of its boundary $p\colon \partial\Sigma \to \{0,1\}$. We write $(\partial\Sigma)_i$ for $p^{-1}(i)$,

(iii) ι_0 and ι_1 are (smooth) embeddings

$$\iota_0\colon [0,+1) \times M \to \Sigma$$
$$\iota_1\colon (-1,0] \times N \to \Sigma$$

such that $\iota_0(0,M) = (\partial\Sigma)_0$ and $\iota_1(0,N) = (\partial\Sigma)_1$.

We often only write $\Sigma = (\Sigma, p, \iota_0, \iota_1)$ for a bordism from M to N. Note that in the definition, the maps ι_0 and ι_1 are embeddings of *smooth collars*. These exist by the following theorem.

Theorem 2.25 (Collar neighbourhood theorem)**.** *Let Σ be a manifold with boundary $\partial\Sigma$. Then, there exists a neighbourhood of $\partial\Sigma$ in Σ that is diffeomorphic to $\partial\Sigma \times [0,1)$.*

Proof. For a proof, we refer to Theorem 6.1 of Chapter 4 in [Hir97]. □

Example 2.26. (i) The cylinder $\Sigma := M \times [0,1]$ has boundary $(M \times \{0\}) \sqcup (M \times \{1\})$ and thus provides bordism from M to M. More generally, let $f\colon M' \to M$ be a

[3] Note that in older literature, this is usually referred to as *cobordism*. In modern algebraic topology, the term 'cobordism' is reserved for a cohomology theory that can be defined from bordism.

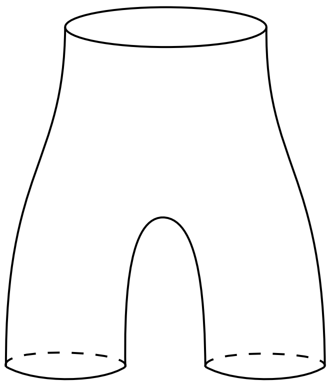
diffeomorphism from a compact n-manifold M' to M. Defining

$$\iota_1\colon (-1,0] \times M' \to \Sigma$$

by using f, we obtain $\partial\Sigma \cong (M \times \{0\}) \sqcup (M' \times \{1\})$. This shows that there is a bordism between diffeomorphic manifolds.

(ii) The 2-disk D^2 is a bordism from S^1 to $\emptyset$, viewing the empty set as a compact 1-manifold. More generally, $\partial D^{n+1} = S^n$, and hence every n-sphere has a bordism to $\emptyset$.

(iii) The so-called *pair of pants* is a bordism between $S^1 \sqcup S^1$ and S^1.



The existence of a bordism between manifolds forms an equivalence relation. For reflexivity, take the cylinder $M \times [0,1]$. For symmetry, we make the following definition of dual bordism.

Definition 2.27. Let M and N be compact n-manifolds and let $(\Sigma, p, \iota_0, \iota_1)$ be a bordism from M to N. The *dual bordism* is defined as $(\Sigma^*, p^*, \iota_0^*, \iota_1^*)$ where $\Sigma^* = \Sigma$, $p^* := 1 - p$, and the embeddings are defined as

$$\iota_0^*\colon [0,+1) \times N \to \Sigma^*, \quad \iota_0^*(t,x) := \iota_1(-t,x), \qquad \text{and} \qquad \iota_1^*\colon (-1,0] \times M \to \Sigma^*, \quad \iota_1^*(t,x) := \iota_0(-t,x).$$

Indeed, the dual bordism provides a bordism from N to M. To show transitivity, we need to glue bordisms together. This is done using the smooth embeddings in the definition of bordism. Let Σ_1 be a bordism from L to M, and let Σ_2 be a bordism from M to N. By gluing Σ_1 and Σ_2 along the manifold M, we can construct a bordism from L to N.

If there exists a bordism between two manifolds, we call them *bordant*. We consider $\emptyset$ to be a compact n-manifold for every $n \in \mathbb{N}_0$. If M is bordant to $\emptyset$, we say that M is *null-bordant* and write $[M] = 0$. We define Ω_n to be the set[4] of bordism equivalence classes of compact n-manifolds. The disjoint union of manifolds $\sqcup$ defines an addition on Ω_n giving it the structure of an abelian group $(\Omega_n, \sqcup)$. The neutral element in Ω_n is the class of the empty n-manifold $\emptyset$.

Proposition 2.28. *The bordism group Ω_n is finitely generated and satisfies $2[M] = 0$ for all n-manifolds M. That is, Ω_n is a finite product of $\mathbb{Z}/2$.*

Proof sketch. We will not prove that Ω_n is finitely generated. It follows from (ii) of Theorem 2.31 below which is proved in [Tho54].
By symmetry of the disjoint union, we obtain that the group structure is abelian. For the 2-torsion, observe that $M \times [0, 1]$ can be seen as a bordism from $M \sqcup M$ to $\emptyset$. From this, it follows that $2[M] = [M \sqcup M] = 0$. □

Example 2.29. The bordism groups Ω_n in the first few dimensions are given by

(i) $\Omega_0 \cong \mathbb{Z}/2$. Every 0-dimensional manifold is given by a disjoint union of points. Since $2[\{*\}] = 0$, every 0-manifold is either null-bordant or bordant to the point.

(ii) $\Omega_1 = 0$. Every compact 1-dimensional manifold is given by a disjoint union of circles. But since $[S^1] = 0$, the 1-dimensional bordism group vanishes.

(iii) $\Omega_2 \cong \mathbb{Z}/2$. This follows from the classification of compact surfaces. Compact orientable surfaces are classified by their genus g. All of the genus g surfaces Σ_g are boundaries of handle bodies, and hence $[\Sigma_g] = 0$. Non-orientable compact surfaces are classified by connected sums of $\mathbb{R}\mathrm{P}^2$. The real projective plane $\mathbb{R}\mathrm{P}^2$ is not the boundary of a 3-manifold. Furthermore, one can show that $[M \# N] = [M \sqcup N]$. Hence, $[\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2] = 2[\mathbb{R}\mathrm{P}^2] = 0$, and $[\mathbb{R}\mathrm{P}^2]$ is the only one non-trivial bordism class in Ω_2.

(iv) $\Omega_3 = 0$. One can show that every compact 3-manifold is the boundary of a 4-manifold. This requires some more work.

We can assemble the bordism groups into a *graded ring*. For this, we define

$$\Omega_* := \bigoplus_{n \geq 0} \Omega_n.$$

The Cartesian product of manifolds

$$[M] \times [N] = [M \times N]$$

[4] One might wonder whether we run into set-theoretic problems when defining Ω_n. By considering manifolds as embedded into $\mathbb{R}^\infty$, these problems can be avoided without losing generality.

induces a graded ring structure on Ω_*. By Proposition 2.28, the ring Ω_* consists only of 2-torsion. This is too reductive since we want to construct elliptic genera as ring homomorphisms out of a bordism ring into a ring R. Hence, the image would only be contained in the 2-torsion subgroup of R. In most situations, we take R to be a $\mathbb{Q}$-algebra so R is in particular torsion-free. For the theory of elliptic genera, we are interested in other variants of the bordism ring with bordism classes of manifolds that are endowed with additional structure: oriented and complex bordism. This will give rise to more complicated ring structures and richer invariants.

Remark 2.30. In modern formulations of bordism, one constructs bordism *categories*. In a bordism category Bord_n, the objects are compact $(n-1)$-manifolds and the morphisms are diffeomorphism classes of bordisms between the manifolds. The disjoint union makes Bord_n into a symmetric monoidal category. While the categorical approach captures more information by keeping track of the bordisms (up to diffeomorphism), the Cartesian product does not appear in the formulation. Topological quantum field theories (TQFTs) are symmetric monoidal functors out of these categories and provide a categorical version of bordism invariants. They have applications in low-dimensional topology, representation theory and mathematical physics. See for example [CR18] or [Fre13] for an introductory treatment.

The structure of the ring Ω_* follows from the Pontryagin-Thom theorem which we will see later. Since we will not come back to (unoriented) bordism, we will state the structure theorem of Ω_* here for completeness.

Theorem 2.31. *(i) The bordism ring has the following structure* $\Omega_* \cong \mathbb{Z}/2[x_2, x_4, x_5, \dots]$, *where we have a generator with* $\deg x_i = i$ *for each* i *not of the form* $2^k - 1$.

(ii) Two compact n*-manifolds* M *and* N *are in the same bordism class in* Ω_n *if and only if all of their Stiefel-Whitney numbers are equal.*

This is due to René Thom [Tho54]. Explicit manifolds as generators x_i for Ω_* were first constructed in [Dol56].

Oriented bordism

In the definition of bordism, we have used compact manifolds without any further structure. Endowing the manifolds and bordisms with an orientation or a (stably) almost complex structure, we can construct corresponding bordism rings similarly. These are the bordism rings that we will need for the theory of elliptic genera. We start with bordism for oriented manifolds.

Recall the definition of an orientation on a vector bundle from Definition 2.14. For (smooth) manifolds, there are several different equivalent definitions of orientation. For our purposes, we will make the following definition.

Definition 2.32. An *oriented manifold* is a manifold M together with an orientation on its tangent bundle TM.

For an oriented manifold M, we write $-M$ for the manifold M with opposite orientation. If M has boundary, an orientation on M induces and orientation on ∂M. Consider the short exact sequence

$$0 \to T(\partial M) \to TM|_{\partial M} \to N(\partial M) \to 0$$

where $N(\partial M)$ denotes normal bundle of ∂M. We have that $TM|_{\partial M} \cong T(\partial M) \oplus \underline{\mathbb{R}}$ (with the usual convention of the outward normal vector field). This induces an orientation on $T(\partial M)$.

Let M and N be compact oriented manifolds. Modifying Definition 2.24, for an oriented bordism $(\Sigma, p, \iota_0, \iota_1)$ from M to N, we require Σ to carry an orientation, and the embeddings ι_0 and ι_1 to preserve orientation. Note that we think of $\{0\} \in (-1, 0]$ as embedded by reversing the orientation. The embedding ι_1 is then constructed such that N has to be embedded with opposite orientation in Σ.

Like unoriented bordism, also oriented bordism forms an equivalence relation. Reflexivity and transitivity follow analogously to the unoriented case. For symmetry, we modify the definition of the dual bordism (Definition 2.27) by choosing $\Sigma^* = -\Sigma$. This ensures that the dual embeddings ι_0^* and ι_1^* are orientation-preserving. Let Ω_n^{SO} be the set of oriented bordism classes of n-dimensional compact oriented manifolds. As in the unoriented case, Ω_n^{SO} forms an abelian group with addition given by disjoint union.

Example 2.33. The oriented bordism groups in the first few dimensions are given by

(i) $\Omega_0^{\mathrm{SO}} \cong \mathbb{Z}$. Compact oriented 0-manifolds are disjoint unions of the positively and negatively oriented points $\{+\}$ and $\{-\}$. The closed interval provides a bordism $[\{+\} \sqcup \{-\}] = 0$. Hence, Ω_0^{SO} is freely generated by the bordism class of $\{+\}$ with inverse $\{-\}$.

(ii) $\Omega_1^{\mathrm{SO}} = 0$. Every compact oriented 1-manifold is a disjoint union of circles with orientation. However, since $\partial D^2 = S^1$, the bordism group Ω_1^{SO} is trivial.

(iii) $\Omega_2^{\mathrm{SO}} = 0$. This follows from the classification of surfaces. See Example 2.29.

(iv) $\Omega_3^{\mathrm{SO}} = 0$. The vanishing of the unoriented bordism group Ω_3 generalizes to orientated bordism, but this is a non-trivial fact.

(v) $\Omega_4^{\mathrm{SO}} \cong \mathbb{Z}$. The bordism class of $\mathbb{C}\mathrm{P}^2$ is a generator for Ω_4^{SO}.

(vi) $\Omega_5^{\mathrm{SO}} \cong \mathbb{Z}/2$. This is the first torsion. The *Dold manifold* Y_5 generates Ω_5^{SO} and is also the generator in degree 5 of the unoriented bordism ring. It was constructed by Dold [Dol56] to provide explicit generators of Ω_*.

We can again construct a graded ring Ω_*^{SO} called the *oriented bordism ring*. As becomes clear from Example 2.33, the oriented bordism ring contains both 2-torsion and copies of $\mathbb{Z}$. To determine the structure of the oriented bordism ring, we will construct Thom spectra and use the Pontryagin-Thom theorem in a later section.

Remark 2.34. Note that the oriented bordism only satisfies graded commutativity. That is,

$$[M] \times [N] = (-1)^{\dim M \dim N} [N] \times [M]$$

for compact oriented manifolds M and N. We will see later that all torsion is 2-torsion and that the non-torsion part of Ω_*^{SO} is concentrated in degrees divisible by 4. Hence, the product is commutative in the usual sense.

Complex bordism

The second variant of bordism that will be important for the theory of elliptic genera is *complex bordism* (sometimes called unitary bordism). We will construct the complex bordism ring Ω_*^{U} with elements given by bordism classes of manifolds with *stably almost complex structure*. We follow [Pan11] in our approach. See also [Sto68].

Recall that a linear complex structure on a real vector space V is a linear map $J\colon V \to V$ with $J^2 = -\mathrm{id}_V$. An almost complex structure on a manifold is given by linear complex structures on the tangent spaces that vary continuously.

Definition 2.35. Let M be a (real) manifold. An almost complex structure on M is a vector bundle isomorphism $J\colon TM \to TM$ such that $J^2 = -\mathrm{id}$.

The real vector space $\mathbb{R}^{2n}$ can be endowed with a linear complex structure by identifying it with the complex vector space $\mathbb{C}^n$, sending $(x, y) \in \mathbb{R}^2$ to $x + iy \in \mathbb{C}$ for each of the n copies. Then J essentially corresponds to multiplying by i on every copy of $\mathbb{C}$. Odd dimensional vector spaces $\mathbb{R}^{2n+1}$ do not admit any linear complex structure[5]. This means that we cannot endow manifolds and their bordism both with an almost complex

[5] This can be verified by considering the determinant of the equation $J^2 = -\mathrm{id}$

structure since one of them has odd dimension. Therefore, to define complex bordism, we need to define the notion of *stably* almost complex structures. We say that two vector bundles η and ξ are *stably equivalent* if there exists a vector bundle isomorphism

$$\xi \oplus \underline{\mathbb{R}}^k \to \eta \oplus \underline{\mathbb{R}}^l$$

for some $k, l \geq 0$ where $\underline{\mathbb{R}}^k$ denotes the trivial bundle of dimension k. Similarly, for complex vector bundles, a stable equivalence is an isomorphism between the vector bundles with summands of $\underline{\mathbb{C}}^k$.

Definition 2.36. Let M be an n-manifold. A *stably almost complex structure* on M is given by a bundle isomorphism

$$\rho\colon TM \oplus \underline{\mathbb{R}}^k \to \xi$$

up to stable equivalence. Here, $k \geq 0$ and ξ is a complex vector bundle over M. We denote the stably almost complex structure by $[\xi]$.

Equivalently, we can define a stably almost complex structure on M as a vector bundle isomorphism (up to stable equivalence)

$$J\colon TM \oplus \underline{\mathbb{R}}^k \to TM \oplus \underline{\mathbb{R}}^k$$

for some $k \geq 1$, that satisfies $J^2 = -\mathrm{id}$. Indeed, every complex vector bundle comes equipped with an almost complex structure (when considered as a real vector bundle).

Example 2.37. The sphere S^n for odd n does not admit an almost complex structure. However, with the standard embedding into $\mathbb{R}^{n+1}$, the tangent bundle satisfies

$$TS^n \oplus \underline{\mathbb{R}} \cong TS^n \oplus \nu \cong \underline{\mathbb{R}}^{n+1}$$

where ν is its normal bundle. Since $n+1$ is even, the trivial bundle $\underline{\mathbb{R}}^{n+1}$ carries the canonical complex structure. Hence, S^n admits a stably almost complex structure.

For a stably almost complex manifold M with boundary ∂M, we obtain a stably almost complex structure on ∂M by the identification

$$TM|_{\partial M} \cong T(\partial M) \oplus \underline{\mathbb{R}}.$$

Like in the previous cases, complex bordism defines an equivalence relation. The definition of dual bordism is similar to the oriented case where instead of opposite orientation, we take the opposite complex structure as defined below. We define Ω_n^{U} to be set of bordism equivalence classes of compact stably almost complex n-manifolds. As before, the disjoint union makes Ω_n^{U} into an abelian group, and the Cartesian product induces a graded ring structure on $\Omega_*^{\mathrm{U}} = \oplus_n \Omega_n^{\mathrm{U}}$. This is called the *complex bordism ring*.

In the unoriented bordism group Ω_n, every manifold provides its own inverse, while for oriented bordism Ω_n^{SO}, the inverse of M is the manifold with opposite orientation $-M$. In the complex bordism group Ω_n^{U}, the inverse is given as follows. Let M be a compact n-manifold with stably almost complex structure $[\xi]$. This defines an element $[M,[\xi]] \in \Omega_n^{\mathrm{U}}$. Its inverse is

$$-[M,[\xi]] = [M,[\overline{\xi}]],$$

where $\overline{\xi} := \xi \oplus \underline{\mathbb{C}}^c$ for the trivial conjugate bundle $\underline{\mathbb{C}}^c$. That is, $\overline{\xi}$ is obtained by means of the real vector bundle isomorphism

$$TM \oplus \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^2 \xrightarrow{\omega \oplus \kappa} \xi \oplus \underline{\mathbb{R}}^2$$

with $\kappa(x,y) = (y,-x)$.

To conclude the treatment of complex bordism, we will list the first few complex bordism groups and defer general structural results for Ω_*^{U} to a later section.

Example 2.38. (i) $\Omega_{\mathrm{odd}}^{\mathrm{U}} = 0$. This is a non-trivial fact. Even if an unstable complex structure would suggest this, since we use the stable version, also odd-dimensional manifolds can have such structure. However, all of these manifolds (such as the odd spheres) are null-bordant.

(ii) $\Omega_0^{\mathrm{U}} \cong \mathbb{Z}$. As previously, this is generated by the point (with trivial almost complex structure).

(iii) $\Omega_2^{\mathrm{U}} \cong \mathbb{Z}$. The class of the complex projective line $\mathbb{C}\mathrm{P}^1$ generates Ω_*^{U}.

(iv) $\Omega_4^{\mathrm{U}} \cong \mathbb{Z} \oplus \mathbb{Z}$. This is generated by the classes $[\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1]$ and $[\mathbb{C}\mathrm{P}^2]$.

(v) $\Omega_6^{\mathrm{U}} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. While one could expect this to be generated by the classes $[\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1]$, $[\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^1]$ and $[\mathbb{C}\mathrm{P}^3]$, this does not hold (but it holds rationally as we will see). Instead, one needs to replace the latter by for example $[\mathbb{C}\mathrm{P}^3] + [H_{22}]$ where H_{22} is a so-called *Milnor hypersurface*. We will define those after having introduced the Pontryagin-Thom theorem.

2.3. Stable vector bundles

In the previous section, we have introduced stably almost complex structures on manifolds as a stable complex structure on the tangent bundle. In this section, we will generalize this to define other stable structures. In general, structures on stable vector bundles are called *X-structures*. We will define X-structures on the stable tangent and stable normal bundle of a manifold. This is the right perspective for bordism of manifolds “with more structure” in the sense that the Pontryagin-Thom isomorphism generalizes to bordism with X-structure. As we will see, the Pontryagin-Thom construction relies on the stable normal bundle. Hence, it is more natural to consider stable normal structures on manifolds. This section is mainly based on [Fre13] and [Mil01].

Stable tangential and stable normal bundles

Let M be a manifold. For any n-plane bundle ξ over M there is a map to the classifying space $BO(n)$ which we denote by the same letter

$$\xi\colon M \to BO(n).$$

This map is unique up to homotopy. Adding the trivial line bundle $\underline{\mathbb{R}}$ to ξ yields an $(n+1)$-plane bundle and hence, we obtain a map to $BO(n+1)$. Recall that the classifying space $BO(n)$ is defined as the colimit over the Grassmannians

$$BO(n) = \varinjlim_{k} \mathrm{Gr}_n(\mathbb{R}^{n+k}).$$

Note that for every k, there is an inclusion

$$\mathrm{Gr}_n(\mathbb{R}^{n+k}) \to \mathrm{Gr}_{n+1}(\mathbb{R}^{n+1+k})$$

given by adding a dimension to the $(n+k)$-dimensional surrounding space and to the n-dimensional subspaces. This induces a map on the classifying spaces

$$BO(n) \to BO(n+1).$$

Equivalently, this map can be obtained by applying B to the inclusion $O(n) \hookrightarrow O(n+1)$ since the construction of classifying spaces is functorial. Taking the colimit over n, we obtain

$$BO = \varinjlim_{n} BO(n).$$

One can also show that this is a classifying space for the infinite orthogonal group O. Now, if $M \to BO(n)$ is a map obtained from an n-plane bundle over M, we can think of the composition of this map with $BO(n) \to BO$ as adding arbitrarily many copies of the trivial bundle $\underline{\mathbb{R}}$. We make the following definition.

Definition 2.39. A *stable vector bundle* on a manifold M is a map $\rho\colon M \to BO$ up to homotopy.

Example 2.40. Let M be a manifold of dimension n. Its tangent bundle TM defines a map τ into $BO(n)$. The *stable tangent bundle* is the composite

$$M \xrightarrow{\tau} BO(n) \to BO.$$

Now, choose an embedding $i\colon M \hookrightarrow \mathbb{R}^{n+k}$. The corresponding normal bundle ν_i defines a map into $BO(k)$. The composite

$$M \xrightarrow{\nu_i} BO(k) \to BO.$$

is the *stable normal bundle* of M. This becomes independent of the embedding. This is essentially Theorem 2.70 which we will use in the sketch of the proof of the Pontryagin-Thom theorem.

Next, we want to endow the stable normal and stable tangent bundles with additional structures. We start with orientation. Let M be a manifold of dimension n. Choose an embedding into $\mathbb{R}^{n+k}$ such that

$$TM \oplus \nu \cong \underline{\mathbb{R}}^{n+k}.$$

The trivial bundle $\underline{\mathbb{R}}^{n+k}$ has the canonical orientation. This means an orientation on TM determines and orientation on ν and vice versa. Upon adding trivial bundles $\underline{\mathbb{R}}$, this still holds. Hence, an orientation on the stable tangent bundle is equivalent to an orientation on the stable normal bundle.

Recall that we have defined stably almost complex manifolds by endowing their stable *tangent* bundle with a complex structure. We could also define a complex structure on the stable normal bundle. However, this turns out to be equivalent.

Proposition 2.41. *Let M be a real manifold. There is a natural one-to-one correspondence between stable normal complex structures and stable tangential complex structures on M.*

Proof. Choose an embedding M into $\mathbb{R}^k$ for large enough k, and consider the normal bundle ν. For any real vector bundle ξ, we can define a complex structure on $\xi \oplus \xi$ by $I(v,w) = (-w, v)$. Hence, $\nu \oplus \nu$ has a complex structure. Furthermore, if ξ_1 and ξ_2 have complex structures J_1 and J_2 respectively, the Whitney sum $\xi_1 \oplus \xi_2$ admits the complex structure $J_1 \oplus J_2$. Now, observe that there is an isomorphism of real vector bundles

$$(\nu \oplus \nu) \oplus (TM \oplus \underline{\mathbb{R}}^l) \cong \nu \oplus (\nu \oplus TM) \oplus \underline{\mathbb{R}}^l \cong \nu \oplus \underline{\mathbb{R}}^{k+l}.$$

Hence, a complex structure on $TM \oplus \underline{\mathbb{R}}^l$ determines a complex structure on $\nu \oplus \underline{\mathbb{R}}^{k+l}$. Similarly, by the isomorphism

$$(TM \oplus TM) \oplus (\nu \oplus \underline{\mathbb{R}}^m) \cong TM \oplus \underline{\mathbb{R}}^{k+m},$$

a complex structure on $\nu \oplus \underline{\mathbb{R}}^m$ induces one on $TM \oplus \underline{\mathbb{R}}^{k+m}$. We now show that the constructions are inverse to each other. Let J_{TM} be a complex structure on $TM \oplus \underline{\mathbb{R}}^l$ and let $I_\nu \oplus J_{TM}$ be the complex structure induced on $\nu \oplus \underline{\mathbb{R}}^{k+l}$ where I_ν denotes the canonical complex structure on $\nu \oplus \nu$. Now, this again induces a complex structure on $TM \oplus \underline{\mathbb{R}}^{2k+l}$ which we denote by

$$J'_{TM} \colon (TM \oplus TM) \oplus (\nu \oplus \nu) \oplus (TM \oplus \underline{\mathbb{R}}^l) \to (TM \oplus TM) \oplus (\nu \oplus \nu) \oplus (TM \oplus \underline{\mathbb{R}}^l).$$

By construction, $J'_{TM} = I_{TM} \oplus I_\nu \oplus J_{TM}$ where I_{TM} is the canonical complex structure on $TM \oplus TM$. The complex structure $I_{TM} \oplus I_\nu$ maps

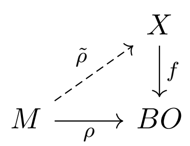
$$\begin{aligned} I_{TM} \oplus I_\nu \colon (TM \oplus TM) \oplus (\nu \oplus \nu) &\to (TM \oplus TM) \oplus (\nu \oplus \nu) \\ (t, u, v, w) &\mapsto (-u, t, -w, v). \end{aligned}$$

Upon identifying this bundle with $\underline{\mathbb{R}}^{2k}$, we obtain the canonical complex structure $I_{\underline{\mathbb{R}}^{2k}}$ on $\underline{\mathbb{R}}^{2k}$. Hence, J'_{TM} is identified with $I_{\underline{\mathbb{R}}^{2k}} \oplus J_{TM}$ and this defines the same class of complex structure as J_{TM}. Since we have only used the orthogonality between TM and ν, by interchanging them, we can use the same argument to show the other inverse property. □

X-structures

Orientation and stably almost complex structures are examples of stable structures on vector bundles which we will call *X-structures*. These were originally defined[6] by Lashof in [Las63]. See also [Sto68] for a detailed account. We make the following definition.

Definition 2.42. Let $f \colon X \to BO$ be a fibration. An *X-structure* on a stable vector bundle $\rho \colon M \to BO$ is a homotopy class of lifts



By definition, ρ is only defined up to homotopy. However, since f is a fibration, the X-structure is independent of the choice of representative in the homotopy class of ρ. Indeed, if H is a homotopy between $\rho_1, \rho_2 \colon M \to BO$, we can lift H along f to obtain a homotopy between $\tilde{\rho}_1$ and $\tilde{\rho}_2$.

[6] In [Las63], these structures are called (B, f)-structures.

Definition 2.43. A *tangential X-structure* on M is an X-structure on its stable tangent bundle. Similarly, a *normal X-structure* on M is an X-structure on its stable normal bundle. We will sometimes just speak of X-structures on manifolds to mean normal X-structures.

Given a stable normal X-structure on M, we can construct a stable tangential structure and vice versa. This is done as follows. Let $Z \in \mathrm{Gr}_n(\mathbb{R}^{n+k})$ be an n-dimensional subspace of $\mathbb{R}^{n+k}$. Using the standard inner product on $\mathbb{R}^{n+k}$, we obtain the space $Z^\perp$ that is orthogonal to Z. This defines map

$$\perp\colon \mathrm{Gr}_n(\mathbb{R}^{n+k}) \to \mathrm{Gr}_k(\mathbb{R}^{n+k}).$$

Taking the double colimit, this induces a map

$$\perp\colon BO \to BO.$$

Composing with $\perp$ interchanges stable tangential and stable normal bundles since they are related by $TM \oplus \nu \oplus \underline{\mathbb{R}}^k \cong \underline{\mathbb{R}}^{n+k}$. Now consider the following diagram

$$\begin{array}{ccccc} & & X^\perp & \longrightarrow & X \\ & \overset{\tilde{\tau}}{\nearrow} & \Big\downarrow{\scriptstyle f^\perp} & & \Big\downarrow{\scriptstyle f} \\ M & \xrightarrow{\tau} & BO & \xrightarrow{\perp} & BO \end{array}$$

where $f^\perp\colon X^\perp \to BO$ is defined[7] as the pullback of f along $\perp$. Let $\tau\colon M \to BO$ be the stable tangent bundle. Then, the composition $\nu = (\perp \circ \tau)$ is the stable normal bundle, and an X-structure on ν induces an $X^\perp$-structure on τ. The same holds the other way around.

We have already seen that an orientation on the stable tangent bundle and an orientation on the stable normal bundle are equivalent. This means, composing with $\perp$ turns one into the other. The same holds for stably almost complex structures. By Proposition 2.41, complex structures on the stable normal bundle correspond to complex structures on the stable tangent bundle. In the language as above, a complex structure on the stable normal bundle is a lift of the map $\nu\colon M \to BO$ to the space BU and similarly for the tangent bundle. Then from the definition of $\perp$ on the Grassmannians taking a subspace to its complement together with the statement of Proposition 2.41 it follows that this the lifts of τ and ν to $BU^\perp$ are equivalent to the lifts to BU.

However, in general, stable tangential X-structures need not correspond to stable normal X-structures. Pin^+-structures and Pin^--structures provide an example for this[8]. Let ξ

[7] To make this construction precise, one should define the pullback level-wise before taking the colimit.

[8] The Lie group $\mathrm{Spin}(n)$ is a double cover of $SO(n)$. Similarly one can define a double cover of $O(n)$. However, there are two such groups $\mathrm{Pin}^\pm(n)$ yielding different structures.

and η be vector bundles with $\xi \oplus \eta \cong \underline{\mathbb{R}}^k$. Then, a Pin^+-structure on ξ is equivalent to a Pin^--structure on η. Hence, also in the stabilized version, the structures on the stable tangent and stable normal bundle are not the same. More on Pin-structures can be found in [KT91].

All of the examples of X-structures we have considered so far come from topological groups. More generally, there are other examples of this form.

Example 2.44. Let $\{G_n\}$ be a sequence of topological groups together with maps $G_n \to G_{n+1}$ and orthogonal representations $G_n \to O(n)$ for each $n \in \mathbb{N}$ such that the following diagram commutes.

$$\begin{array}{ccc} G_n & \longrightarrow & G_{n+1} \\ \downarrow & & \downarrow \\ O(n) & \longrightarrow & O(n+1) \end{array}$$

Here, the bottom map is the inclusion $O(n) \hookrightarrow O(n+1)$. Taking the colimit $n \to \infty$ and applying the classifying space functor B yields the following commutative diagram.

$$\begin{array}{ccccc} BG_n & \longrightarrow & BG_{n+1} & \longrightarrow & BG \\ \downarrow & & \downarrow & & \downarrow \\ BO(n) & \longrightarrow & BO(n+1) & \longrightarrow & BO \end{array}$$

We also call such structures *G-structures*.

Just as the bordism rings Ω_*^{SO} and Ω_*^{U} for orientation and stably almost complex structure, there also exist a bordism rings Ω_*^X of manifolds with general X-structures (on their stable normal bundle). We end this section by listing the most important examples of X-structures and their corresponding bordism rings.

Example 2.45. (i) $X = BO$ with the identity map. A manifold with BO-structure is simply an unoriented manifold and can be seen as a G-structure with $G_n = O(n)$. The bordism ring is $\Omega_*^O = \Omega_*$.

(ii) $X = BSO$ with map induced by $SO(n) \hookrightarrow O(n)$. A BSO-structure on a manifold is an orientation. This gives the oriented bordism ring Ω_*^{SO}. This can be seen as a G-structure with $G_n = SO(n)$.

(iii) $X = BU$ with map induced by $U(n) \hookrightarrow O(2n)$. This is a G-structure with $G_{2n} = U(n)$ and $G_{2n+1} = U(n)$. A manifold with such a structure is a stably almost complex manifold and the corresponding bordism ring is Ω_*^{U}.

(iv) $X = EO$ with the canonical projection map. Such a structure is called *stable framing*. Since EO is weakly contractible, this corresponds to taking trivial groups $G_n = \{1\}$. We denote its bordism ring by Ω_*^{fr}.

Remark 2.46. We have defined bordism as a relation between manifolds. More generally, one can also define bordism a generalized homology theory[9] $\Omega_*^X(-)$ by defining bordism for *singular manifolds* M with X-structure over a space Y, that is for maps $M \to Y$. Choosing $Y = \{*\}$, one recovers the definition above. The homology theory $\Omega_*^X(-)$ is usually referred to as *geometric bordism*.

2.4. Stable homotopy theory

In this section, we will introduce spectra. Most important for us are *Thom spectra* which we will need for the Pontryagin-Thom construction to obtain the structure of the bordism rings Ω_*^{SO} and Ω_*^{U}. We refer to [DK01] and [Mei18].

Spectra

We start by recalling standard notions from homotopy theory. In particular, we will need to introduce several concepts related to topological spaces with basepoints.

A *pointed* or *based space* (X, x_0) is a topological space X together with a chosen basepoint $x_0 \in X$. We often simply write X and keep the basepoint implicit. Note that we can make any (unpointed) topological space X into a pointed space by adding a disjoint basepoint $*$. For this, we write $X_+ = X \sqcup \{*\}$. A *pointed map* between pointed spaces (X, x_0) and (Y, y_0) is a map of spaces $f\colon X \to Y$ such that $f(x_0) = y_0$. We define homotopies between such maps as follows.

Definition 2.47. Let (X, x_0) and (Y, y_0) be pointed spaces and let $f_0\colon X \to Y$ and $f_1\colon X \to Y$ be pointed maps. A *pointed homotopy* from f_0 to f_1 is a map $F\colon [0,1] \times X \to Y$ such that

(i) $F(t, x_0) = y_0$ for all $t \in [0,1]$,

[9] See Chapter 7 of [Swi02] and Section 3.4 of [Koc96] for the axioms of generalized homology theories.

(ii) $F(0,x) = f_0(x)$ and $F(1,x) = f_1(x)$ for all $x \in X$.

We denote by $[X,Y]_*$ the set of pointed homotopy classes of pointed maps $f\colon X \to Y$.

Definition 2.48. Let X and Y be pointed spaces with basepoints x_0 and y_0 respectively.

(i) The *wedge* of X and Y is defined as

$$X \vee Y := (X \sqcup Y)/(\{x_0\} \sqcup \{y_0\})$$

where the basepoint of $X \vee Y$ is given by the point corresponding to $\{x_0\} \sqcup \{y_0\}$.

(ii) The *smash product* of X and Y is the pointed space given by

$$X \wedge Y := (X \times Y)/(X \vee Y)$$

where we think of $X \times \{y_0\} \subseteq X \times Y$ and $\{x_0\} \times Y \subseteq X \times Y$ as subspaces corresponding to X and Y respectively. With this, $X \vee Y$ can be seen as their union in $X \times Y$. The basepoint of the smash product is the point corresponding to $X \vee Y$.

(iii) The *(reduced) suspension* of X is defined as

$$\Sigma X := X \wedge S^1$$

Lemma 2.49. *There is a homeomorphism $S^n \wedge S^k \cong S^{n+k}$.*

Proof. See Lemma 2.27 in [Swi02] for the proof of $S^1 \wedge S^k \cong S^{k+1}$ and apply this inductively. □

As a result, we have $\Sigma S^n \cong S^{n+1}$ and for the iterated reduced suspension of a pointed space X, it follows

$$\Sigma^n X = S^1 \wedge ... \wedge S^1 \wedge X \cong S^n \wedge X.$$

Recall that $\mathrm{Map}(X,Y)$ is the set of maps from (unpointed) spaces X to Y. This set can be topologized with the compact-open[10] topology. Let (X, x_0) and (Y, y_0) be pointed spaces. Then the pointed mapping space $\mathrm{Map}_*(X,Y)$ of pointed maps is topologized as a subspace of the mapping space $\mathrm{Map}(X,Y)$. Then, $\mathrm{Map}_*(X,Y)$ can be regarded as a pointed space with basepoint given by the map $X \mapsto y_0$. With this, we can now define the loop space of a pointed space.

[10]For details, see for example [Hat02, pp. 529-533].

Definition 2.50. The *loop* space of a pointed space X is defined as

$$\Omega X := \mathrm{Map}_*(S^1, X).$$

The loop space and suspension are related in the following way.

Proposition 2.51 (Loop-suspension adjunction)**.** *Let X and Y be pointed spaces. There is a natural bijection*

$$\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}_*(X, \Omega Y).$$

Proof. See for example Theorem 6.37 in [DK01]. □

We are now ready to introduce spectra.

Definition 2.52. A *spectrum* $E = (E_n)_{n \in \mathbb{N}_0}$ is a collection of pointed spaces E_n for $n \in \mathbb{N}_0$ together with pointed maps $\sigma_n \colon \Sigma E_n \to E_{n+1}$ called *structure maps.*

Example 2.53. (i) The *sphere spectrum* $\mathbb{S}$ is the spectrum with n-th space S^n and structure maps $\sigma_n \colon \Sigma S^n \to S^{n+1}$ given by the homeomorphisms from Lemma 2.49.

(ii) The *suspension spectrum* $\Sigma^\infty X$ of a pointed space X has n-th space $\Sigma^n X$. The structure maps are homeomorphisms $\sigma_n \colon \Sigma\Sigma^n X \to \Sigma^{n+1} X$.

Definition 2.54. The *n-th homotopy group* of a spectrum E is defined as the colimit

$$\pi_n(E) := \lim_{k \to \infty} \pi_{n+k}(E_k) = \lim_{k \to \infty} [S^{n+k}, E_k]_*.$$

The sphere spectrum is an important example. We will see how its homotopy groups relate to the homotopy groups of spheres. For this, we recall the following theorem.

Theorem 2.55 (Freudenthal suspension theorem)**.** *The map*

$$\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

induced by the suspension map is an isomorphism for $k \leq n-2$ and a surjection for $k = n-1$.

Proof. For a proof, we refer for example to Corollary 4.24 in [Hat02]. □

A more general statement also holds for finite CW-complexes. The homotopy groups of spheres are very difficult to compute and still remain elusive in many ways. By the Freudenthal suspension theorem, the homotopy groups stabilize after taking iterated suspensions. The homotopy groups of the sphere spectrum are given by the stable homotopy groups of spheres

$$\pi_n \mathbb{S} = \lim_{k\to\infty} \pi_{n+k}(S^k) = \pi_n^{\text{st}}.$$

The first few stable homotopy groups are given by

$$\begin{aligned} \pi_0^{\text{st}} &= \mathbb{Z}, \\ \pi_1^{\text{st}} &= \mathbb{Z}/2, \\ \pi_2^{\text{st}} &= \mathbb{Z}/2, \\ \pi_3^{\text{st}} &= \mathbb{Z}/24, \\ \pi_4^{\text{st}} &= 0. \end{aligned}$$

We finish the general discussion spectra by defining are more well-behaved version of spectra and noting their relations to (co)homology theories.

Definition 2.56. An Ω-*spectrum* is a spectrum E such that the adjoint maps $\widetilde{\sigma}_n \colon E_n \to \Omega E_{n+1}$ are weak homotopy equivalences.

From every Ω-spectrum E, we can construct a *generalized* homology and cohomology theory. For a pointed space X, the reduced homology and cohomology groups are given by

$$\begin{aligned} \widetilde{E}_n(X) &= \varinjlim_k \ \pi_{n+k}(E_k \wedge X) = \varinjlim_k \ [S^{n+k}, E_k \wedge X]_*, \\ \widetilde{E}^n(X) &= \varinjlim_k \ [\Sigma^k X, E_{n+k}]_*. \end{aligned}$$

There is also a converse statement which is the *Brown representability theorem.* For more details and the definition of generalized (co)homology theories, we refer to Chapter 7 of [Swi02] and Section 3.4 of [Koc96].

Example 2.57. *Eilenberg-MacLane spectra* denoted by HA consist of Eilenberg-MacLane spaces $HA_n = K(A, n)$ for an abelian group A. These represent ordinary (co)homology with coefficients in A. There are also spectra KU and KO representing complex and real *topological K-theory.*

Remark 2.58. Some authors refer to the above as *prespectra* and Ω-*prespectra* respectively. With that convention, a *spectrum* is a collection of spaces such that the adjoint maps are homeomorphisms. Then there is a construction called *spectrification* that turns any presprectrum into a spectrum. For our purposes, however, it suffices to consider what we call (Ω-)spectra.

Thom spectra

We now want to construct Thom spectra. For this, we construct the fibre bundles $D(\xi)$, $S(\xi)$, and the Thom space $\mathrm{Th}(\xi)$ for a vector bundle ξ.

Definition 2.59. Let $\xi\colon E \to B$ be a vector bundle over a paracompact space B. Choose an inner product on ξ.

(i) The *disk bundle* $D(\xi)$ is the subbundle of ξ consisting of the vectors with length ≤ 1.

(ii) The *unit sphere bundle* $S(\xi)$ is the subbundle of $D(\xi)$ consisting of the vectors with length $= 1$.

(iii) The *Thom space* $\mathrm{Th}(\xi)$ is defined as the quotient of the total spaces

$$\mathrm{Th}(\xi) = D(\xi)/S(\xi).$$

We consider $\mathrm{Th}(\xi)$ to be a pointed space with basepoint given by the point that corresponds to $S(\xi)$ in the quotient.

Note that the paracompactness is needed for the existence of an inner product. If we drop this assumption, we can still define the Thom space of a vector bundle $\xi\colon E \to B$ as follows. As a set, we have $\mathrm{Th}(\xi) := E \sqcup \{\infty\}$ and topologize it by defining neighbourhood bases $\{\mathcal{B}_x\}$. Fix a neighbourhood base for E. For a point $x \in E$, the neighbourhoods $U_x \in \mathcal{B}_x$ at x are given by those for E. For the point ∞, the neighbourhoods for ∞ are given by U such that the $\xi^{-1}(x) \setminus U \cap \xi^{-1}(x)$ are compact for all $x \in B$. If B is compact, then $\mathrm{Th}(\xi)$ is the one-point compactification of E.

Proposition 2.60. *Let $\xi\colon E \to B$ and $\eta\colon E' \to B'$ be vector bundles over spaces B and B'. Then for their Thom spaces, we have* $\mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta) \cong \mathrm{Th}(\xi \times \eta)$.

Proof. Let ξ be of dimension m, and η be of dimension n. The homeomorphism $D^m \times D^n \to D^{m+n}$ yields a homeomorphism

$$D(\xi)_x \times D(\eta)_y \to D(\xi \times \eta)_{(x,y)}$$

for basepoints $x \in B$ and $y \in B'$. This gives rise to a homeomorphism

$$D(\xi) \times D(\eta) \cong D(\xi \times \eta)$$

which maps $D(\xi) \times S(\eta) \cup S(\xi) \times D(\eta)$ onto $S(\xi \times \eta)$. Hence, we get a homeomorphism

$$\begin{aligned} \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta) &= D(\xi) \times D(\eta)/(D(\xi) \times S(\eta) \cup S(\xi) \times D(\eta)) \\ &\cong D(\xi \times \eta)/S(\xi \times \eta) = \mathrm{Th}(\xi \times \eta). \end{aligned}$$ □

This shows that Thom spaces relate the two notions of stability that we have encountered: the stability of vector bundles under adding trivial bundles and the stability of spaces under taking suspensions. Let $\xi \colon E \to B$ be a vector bundle and let $\underline{\mathbb{R}}$ denote the trivial line bundle over B. Note that we can view $\underline{\mathbb{R}} \oplus \xi \cong \underline{\mathbb{R}}_* \times \xi$ where now $\underline{\mathbb{R}}_*$ is the trivial line bundle over a point. Then, indeed

$$\mathrm{Th}(\underline{\mathbb{R}} \oplus \xi) \cong \mathrm{Th}(\underline{\mathbb{R}}_* \times \xi) \cong \mathrm{Th}(\underline{\mathbb{R}}_*) \wedge \mathrm{Th}(\xi) \cong S^1 \wedge \mathrm{Th}(\xi) = \Sigma \,\mathrm{Th}(\xi).$$

Iterating this, we obtain

$$\mathrm{Th}(\underline{\mathbb{R}}^n \oplus \xi) \cong \Sigma^n \,\mathrm{Th}(\xi).$$

In the following, let γ_n be the universal bundle over $BO(n)$. Consider the bundle $\underline{\mathbb{R}} \oplus \gamma_n$ over $BO(n)$. Since this an $(n+1)$-dimensional bundle, we obtain a bundle map to the universal vector bundle γ_{n+1} over $BO(n+1)$. This induces a map on the Thom spaces

$$\sigma_n \colon \Sigma \,\mathrm{Th}(\gamma_n) \cong \mathrm{Th}(\underline{\mathbb{R}} \oplus \gamma_n) \to \mathrm{Th}(\gamma_{n+1}).$$

Now we can define the Thom spectrum MO.

Definition 2.61. The Thom spectrum MO has n-th space $MO_n := \mathrm{Th}(\gamma_n)$ where γ_n is the universal vector bundle over BO. The structure maps of MO are given by σ_n above.

We are mostly interested in Thom spectra MSO and MU that are related to the bordism rings Ω_*^{SO} and Ω_*^{U}. The oriented Thom spectrum is defined analogously to MO.

Definition 2.62. Let $\widetilde{\gamma}_n$ be the universal oriented n-plane bundle over $BSO(n)$. The Thom spectrum MSO has n-th space $MSO_n := \mathrm{Th}(\widetilde{\gamma}_n)$ and structure maps analogous to MO since $\underline{\mathbb{R}} \oplus \widetilde{\gamma}_n$ inherits an orientation from $\underline{\mathbb{R}}$ and $\widetilde{\gamma}_n$.

To define the unitary Thom spectrum, we consider the universal complex n-plane bundle $\gamma_n^{\mathbb{C}}$ over $BU(n)$. Consider the bundle $\underline{\mathbb{C}} \oplus \gamma_n^{\mathbb{C}}$ over the space $BU(n)$. As in the real case, we obtain a bundle map to $\gamma_{n+1}^{\mathbb{C}}$ over $BU(n+1)$ inducing a map on the Thom spaces

$$\Sigma^2 \,\mathrm{Th}(\gamma_n^{\mathbb{C}}) \cong \mathrm{Th}(\underline{\mathbb{C}} \oplus \gamma_n^{\mathbb{C}}) \to \mathrm{Th}(\gamma_{n+1}^{\mathbb{C}}).$$

Note that we now have a double suspension since $\mathbb{C}$ has dimension 2 over $\mathbb{R}$.

Definition 2.63. The unitary Thom spectrum MU has spaces $MU_{2n} := \mathrm{Th}(\gamma_n^{\mathbb{C}})$ and $MU_{2n+1} := \Sigma MU_{2n}$. The structure map

$$\sigma_{2n+1} \colon \Sigma MU_{2n+1} \cong \Sigma^2 MU_{2n} \to MU_{2n+2},$$

is given by the map above, and σ_{2n} is just the identity.

Remark 2.64. We can also construct Thom spectra MX for general X-structures. For example, for the stable framing $X = EO$, we obtain the sphere spectrum $MX \cong \mathbb{S}$. For our purposes, it will suffice to consider MSO and MU.

2.5. The Pontryagin-Thom theorem

In this section, to obtain the structure of the bordism rings Ω_*^{SO} and Ω_*^{U}, we will introduce the Pontryagin-Thom theorem and allude to the proof. The Pontryagin-Thom theorem relates bordism groups to the homotopy groups of the corresponding Thom spectra introduced in the previous section.

There are several versions of the theorem. Historically, the first version considers framed manifolds and is due to Pontryagin [Pon38]. (See also the account [Pon55]). In 1954, Thom [Tho54] constructed the isomorphism for unoriented and oriented bordism determining the structure of the bordism ring Ω_* and $\Omega^{\mathrm{SO}} \otimes \mathbb{Q}$. Note that both Pontryagin and Thom formulated an unstable version. Generalizing their results, Lashof proved the Pontryagin-Thom theorem for general stable X-structures [Las63].

Theorem 2.65 (Pontryagin-Thom for X-structures)**.** *There is an isomorphism of abelian groups*

$$\Omega_n^X \cong \pi_n(MX).$$

In particular, for $X = BSO$ and $X = BU$, we obtain the Pontryagin-Thom isomorphism for oriented and complex bordism. For $X = EO$, that is, for stable framing, we obtain $\Omega_*^{\mathrm{fr}} \cong \pi_*(\mathbb{S}) = \pi_*^{\mathrm{st}}$.

Remark 2.66. Recall that more generally, we can construct a homology theory $\Omega_*^X(-)$ from singular bordism. Similarly, the Thom spectrum MX represents a homology theory often called *homotopical bordism*. Then, the Pontryagin-Thom isomorphism provides a natural isomorphism between the generalized homology theories of geometric and homotopical bordism.

Remark 2.67. One can also define a ring-like structure on the Thom spectra MX using the smash product making them into *ring spectra*. Then, the homotopy groups $\pi_*(MX)$ obtain the structure of a graded ring, and the Pontryagin-Thom isomorphism becomes an isomorphism of graded rings. However, for our purposes, it is enough to consider the isomorphism of abelian groups since $\pi_*(MX)$ inherits the ring structure from Ω_*^X.

The Pontryagin-Thom construction

We now want to give some ideas of the proof of the theorem for unoriented bordism. This can then be generalized to the case of X-structures. For the construction, we need to define the map

$$P_n \colon \Omega_n \to \pi_n(MO),$$

sometimes called *Pontryagin-Thom collapse map*, and show that it is well-defined, a homomorphism, and a bijection. In the following, we will sketch those four steps. Our approach is based on [MS74] and [Mei18]. See also [Swi02].

Step 1: Definition of P_n. Let M be an n-dimensional compact manifold. We want to associate an element of $\pi_n(MO)$ to the bordism class of M. The elements of $\pi_n(MO)$ are pointed homotopy classes of maps $f \colon S^{n+k} \to \mathrm{Th}(\gamma_k)$ for large enough k, where γ_k is the universal vector bundle over $BO(k)$. We need to embed M into a sphere S^{n+k} and obtain a map into $\mathrm{Th}(\gamma_k)$ such that this construction yields an element in the pointed homotopy class of f and is independent of the bordism class of M. The right way to embed M into S^{n+k} is by using tubular neighbourhoods.

Definition 2.68. Let $\iota \colon M \hookrightarrow \mathbb{R}^{n+k}$ be an embedding of an n-manifold with associated normal bundle ν. A *tubular neighbourhood* of M in $\mathbb{R}^{n+k}$ is a lift of embedding to the total space of the normal bundle $\phi \colon E(\nu) \hookrightarrow \mathbb{R}^{n+k}$.

Theorem 2.69. *Tubular neighbourhoods exist.*

This is Theorem 11.1 in [MS74]. Let ϕ be a tubular neighbourhood of M with normal bundle ν. Now extend the embedding to $\mathbb{R}^{n+k} \sqcup \{\infty\} \cong S^{n+k}$, and define a map $g \colon S^{n+k} \to \mathrm{Th}(\nu)$ as follows.

- For points $y = \phi(x) \in \mathbb{R}^{n+k} \subseteq S^{n+k}$, define $g(y) = \phi^{-1}(x) \in E(\nu) \subseteq \mathrm{Th}(\nu)$.
- For points outside the image of ϕ, define $g(y) = \infty \in \mathrm{Th}(\nu)$.

This is continuous precisely by definition of the topology on $\mathrm{Th}(\nu)$ and motivates the construction of the Thom spaces. Now, compose this with $\mathrm{Th}(\nu) \to \mathrm{Th}(\gamma_k) = MO_k$, induced by the map $\nu\colon M \to BO(k)$. Note that this is only defined up to homotopy. Now, define $P_n(M) := ([f], \iota, \phi)$ where f is the composition

$$S^{n+k} \xrightarrow{g} \mathrm{Th}(\nu) \to \mathrm{Th}(\gamma_k)$$

and $[f]$ is its pointed homotopy class, and where we kept track that our construction depends on ι and ϕ.

Step 2: P_n is well-defined. To show that P_n is well-defined, we need to show that $P_n(M) = ([f], \iota, \phi)$ is independent of ι and ϕ, and that P_n is independent of the representative of the bordism class $P_n(M) = P_n([M])$.

Indeed, $([f], \iota, \phi)$ is independent of the choice of embedding ι and choice of tubular neighbourhood ϕ by the following two theorems.

Theorem 2.70. *Any two embeddings $\iota_1, \iota_2\colon M \hookrightarrow \mathbb{R}^{n+k}$ are isotopic for $k \geq n+2$.*

This is the isotopy version of the weak Whitney embedding theorem. See for example Theorem 3.5 of Chapter 1 in [Hir97].

Theorem 2.71. *For a chosen embedding $\iota\colon M \hookrightarrow \mathbb{R}^{n+k}$, any two tubular neighbourhoods $\phi, \phi'\colon E(\nu) \hookrightarrow \mathbb{R}^{n+k}$ are isotopic.*

Proof. See Theorem 5.3 of Chapter 4 in [Hir97]. □

To see that P_n is independent of the choice of M in its bordism class $[M] \in \Omega_n$, let Σ be a bordism between M and N. There is an embedding of Σ into $S^{n+k} \times [0,1]$ such that the manifolds M and N are embedded into $S^{n+k} \times \{0\}$ and $S^{n+k} \times \{1\}$ respectively. This yields a homotopy, and hence $P_n(M) = P_n(N)$.

Step 3: P_n is a homomorphism. Let M and N be compact n-manifolds. We can embed their disjoint union $M \sqcup N$ into S^{n+k} such that M and N are in contained in different hemispheres. In this way, we can view $[f]_M$ and $[f]_N$ separately and add them.

Step 4: P_n is a bijection. For the definition of the inverse map, we need the notion of *transversality.*

Definition 2.72. Let $f\colon M \to N$ be a smooth map between manifolds, and let $Y \subseteq N$ be a compact submanifold. We call f *transverse* to Y, if for every $p \in f^{-1}(Y)$ the following composition is surjective.

$$T_pM \longrightarrow T_{f(p)}N \longrightarrow T_{f(p)}N/T_{f(p)}Y$$

For the inverse map, we need to obtain the bordism class of a manifold. In general, the inverse image $f^{-1}(Y)$ need not be a manifold. Transversality ensures that the inverse construction yields a manifold. That is, if a smooth map $f\colon M \to N$ is transverse to $Y \subseteq N$, then $f^{-1}(Y)$ is a manifold.

Theorem 2.73. *Let $\xi\colon E \to B$ be a smooth vector bundle of dimension k. Every map $f\colon S^m \to \mathrm{Th}(\xi)$ is homotopic to a map $\widetilde{f}$ that is smooth throughout the manifold $\mathrm{Th}(\xi) \setminus \{\infty\}$, and transverse to the zero-section B.*

See Theorem 12.14 in [Swi02]. Now, apply this to the universal vector bundle γ_k over $BO(k)$ for large k. Assign to a class $[\widetilde{f}] \in \pi_{n+k}(\mathrm{Th}(\gamma_k))$ the manifold $\widetilde{f}^{-1}(BO(k))$. This defines a map

$$T_n\colon \pi_{n+k}(\mathrm{Th}(\gamma_k)) \to \Omega_n.$$

One can show that T_n is well-defined and that P_n and T_n are inverse to each other. Note that we fixed a large k and used $\pi_{n+k}(\mathrm{Th}(\gamma_k))$ in place of $\pi_n(MO)$ following the original approach by Thom. One can show that this construction extends to the isomorphism $\Omega_n \cong \pi_n(MO)$. This completes our discussion of the proof.

Structure of the bordism rings

As a consequence of the Pontryagin-Thom theorem, we obtain several structural theorems for the bordism rings. Rationally, that is, after tensoring with $\mathbb{Q}$, the bordism rings are isomorphic to a polynomial ring.

Theorem 2.74 (Thom [Tho54]). *(i) The rational structure of Ω_*^{SO} is given by*

$$\Omega^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}\mathrm{P}^2], [\mathbb{C}\mathrm{P}^4], [\mathbb{C}\mathrm{P}^6], ...]$$

(ii) Two oriented compact n-manifolds M and N represent the same element in $\Omega^{\mathrm{SO}} \otimes \mathbb{Q}$ if and only if their Pontryagin numbers coincide.

The second statement is a consequence of the first one since that Pontryagin numbers are bordism invariants, and the spaces $\mathbb{C}\mathrm{P}^{i_1} \times \cdots \times \mathbb{C}\mathrm{P}^{i_k}$ for different partitions $(i_1, \ldots, i_k)$ of n have linearly independent Pontryagin numbers. The first statement can be shown by using the Pontryagin-Thom theorem, the Hurewicz theorem to relate homotopy groups to homology, the Thom isomorphism theorem applied to the universal oriented bundle $\widetilde{\gamma}_n$, and finally using the rational (co)homology of the classifying space $BSO(k)$.

Theorem 2.75 (Thom isomorphism)**.** *Let $\xi \colon E \to B$ be an oriented vector bundle of dimension n. Then, there is an isomorphism $\widetilde{H}_{n+k}(\mathrm{Th}(\xi); \mathbb{Z}) \cong H_k(B; \mathbb{Z})$.*

This was proved by Thom [Tho52]. See also Chapter 18 in [MS74].

Similarly, we have the following theorem about the rational structure of the complex bordism ring.

Theorem 2.76. *The rational structure of the bordism ring Ω_*^{U} is given by*

$$\Omega_*^{\mathrm{U}} \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}\mathrm{P}^1], [\mathbb{C}\mathrm{P}^2], [\mathbb{C}\mathrm{P}^3], \ldots].$$

The non-rational structures of Ω_*^{SO} and Ω_*^{U} are more complicated. However, for the theory of elliptic genera, we will consider ring homomorphisms into $\mathbb{Q}$-algebras. In this way, we will only rely on the rational structures. Nevertheless, to complete our discussion of bordism, we list refined statements of the structure of the bordism rings without proof.

Theorem 2.77. *(i) All torsion of Ω_*^{SO} is concentrated in degrees $\neq 4k$. Up to torsion, the oriented bordism ring is given by*

$$\Omega_*^{\mathrm{SO}}/\mathrm{torsion} \cong \mathbb{Z}[a_i, i \geq 1]$$

where a_i has degree $4i$.

(ii) All torsion in Ω_*^{SO} is of order 2.

(iii) Two oriented compact n-manifolds M and N are in the same bordism class in Ω_*^{SO} if and only if all of their Stiefel-Whitney and Pontryagin numbers are equal. The Pontryagin numbers determine the bordism class in degrees $4k$, and the Stiefel-Whitney numbers the class in degrees not divisible by 4.

The first statement was shown by Novikov [Nov60] The second and third statement were shown by Wall [Wal60].

The (even) complex projective spaces do not form a set of non-rational generators. However, a set of generators for Ω_*^{U} and for Ω_*^{SO}/torsion is given by the Milnor hypersurfaces.

Definition 2.78. Let $1 \leq i \leq j$. Define the *Milnor hypersurface* H_{ij} as the manifold $H_{ij} \subseteq \mathbb{C}\mathrm{P}^i \times \mathbb{C}\mathrm{P}^j$ defined by the equation $x_0y_0 + x_1y_1 + \ldots x_iy_i = 0$ where x_k and y_k are homogeneous coordinates for $\mathbb{C}\mathrm{P}^i$ and $\mathbb{C}\mathrm{P}^j$, respectively.

Theorem 2.79. *(i) The complex bordism ring Ω_*^{U} is torsion-free.*

(ii) The complex bordism ring Ω_*^{U} is isomorphic to

$$\Omega_*^{\mathrm{U}} \cong \mathbb{Z}[b_i, i \geq 1]$$

where b_i has degree $2i$. Without rationalizing, the generators are not given by complex projective spaces. However, the Milnor hypersurfaces generate Ω_*^{U} (with redundancy).

(iii) Two almost complex compact n-manifolds M and N are in the same bordism class in Ω_*^{U} if and only if all of their Chern numbers are equal.

The statements are due to Novikov and Milnor. See [Nov60] and [Mil60].

3. Elliptic Genera

The theory of elliptic genera has been developed in the 1980s to understand certain properties of known bordism invariants. Using previous vanishing results of the *signature* and the $\widehat{A}$*-genus*, Landweber-Stong [LS88] conjectured a more general vanishing of related invariants. Proving their conjecture, Ochanine [Och87] defined elliptic genera.

The physical interpretation of elliptic genera by Witten [Wit87] also played an important role in the development of the theory. He conjectured elliptic genera to satisfy a certain *rigidity* property which was later shown by Bott-Taubes [BT89]. In Chapter 6, we will allude to Witten's considerations and see how elliptic genera can be interpreted as partition functions of certain string theories. For a more detailed account of the early developments, we refer to [Lan88].

Elliptic genera are relating the study of bordism invariants to the theory of elliptic functions and modular forms. The connections to modular forms have been investigated in [CC88] and [Zag88]. Furthermore, in 1995, Landweber-Ravenel-Stong [LRS95] constructed cohomology theories from elliptic genera. This marks the beginning of the theory of *elliptic cohomology*. We will not cover this, but instead refer to [Mei22] and [DFHH14] for the theory of elliptic cohomology and topological modular forms.

In this chapter, we will first introduce genera as ring homomorphisms out of the oriented bordism ring Ω_*^{SO} and relate them to power series by considering Pontryagin numbers. After defining elliptic genera, we will consider a few geometric examples such as the L-genus and the $\widehat{A}$-genus. In Section 3.2, we will construct elliptic genera from elliptic curves. We will also see that there is a universal elliptic genus that assigns modular forms for a subgroup of $SL_2(\mathbb{Z})$ to compact oriented manifolds.

Finally, in Section 3.3, we will consider complex genera. These are bordism invariants of compact, stably almost complex manifolds. The constructions for oriented manifolds generalize to the complex case and we define complex elliptic genera of level N. Our main reference for this chapter is [HBJ92].

3.1. Definition and geometric examples

The goal of this section is to define elliptic genera as bordism invariants for compact oriented manifolds and consider geometrically relevant examples and properties of elliptic genera. We start by defining a genus.

Definition 3.1. Let R be a $\mathbb{Q}$-algebra. A *genus* with values in R is a ring homomorphism $\varphi\colon \Omega^{\mathrm{SO}} \to R$.

Remark 3.2. In older literature, such a ring homomorphism is called *multiplicative* genus to highlight the multiplicative property $\varphi([M] \times [N]) = \varphi([M])\varphi([N])$.

In Chapter 2, we have seen that the ring structure of Ω_*^{SO} is rather complicated, but that the rational structure of Ω_*^{SO} is given by a polynomial ring generated by the classes of even-dimensional complex projective spaces

$$\Omega^{\mathrm{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}\left[[\mathbb{C}\mathrm{P}^2], [\mathbb{C}\mathrm{P}^4], [\mathbb{C}\mathrm{P}^6], \dots\right]$$

where the degree is the real dimension $\deg[\mathbb{C}\mathrm{P}^{2n}] = 4n$. For a $\mathbb{Q}$-algebra R, a genus $\varphi\colon \Omega^{\mathrm{SO}} \to R$ is fully determined by its rationalization

$$\overline{\varphi}\colon \Omega^{\mathrm{SO}} \otimes \mathbb{Q} \to R.$$

Hence, we will start by considering ring homomorphisms out of $\Omega^{\mathrm{SO}} \otimes \mathbb{Q}$. Hirzebruch described a geometric construction of (rationalized) genera in terms of power series using the theory of characteristic classes. We follow Chapter 1 of [HBJ92] and Section 1.1 of [HSB95] in our treatment. Consider an even normalized power series

$$Q(x) = 1 + a_2x^2 + a_4x^4 + a_6x^6 + \dots \in R[\![x]\!].$$

Let x_i for $i = 1, \dots, n$ be indeterminates each of degree 2. The product

$$Q(x_1)\dots Q(x_n) = 1 + a_2 \sum_{i=1}^{n} x_i^2 + \dots$$

is symmetric in x_i^2 and hence, we can express it in terms of the elementary symmetric polynomials in the variables x_i^2. We recall the following definition.

Definition 3.3. Let $y_1, \dots, y_n$ be indeterminates. The *elementary symmetric polyno-*

mials in y_i are defined as

$$\begin{aligned}
\sigma_1(y_1,\dots,y_n) &:= \sum_{i=1}^{n} y_i,\\
\sigma_2(y_1,\dots,y_n) &:= \sum_{1\le i<j\le n} y_i y_j,\\
\sigma_3(y_1,\dots,y_n) &:= \sum_{1\le i<j<k\le n} y_i y_j y_k,\\
&\vdots\\
\sigma_n(y_1,\dots,y_n) &:= y_1 y_2 \cdots y_n.
\end{aligned}$$

For short-hand notation, we write $\sigma_i = \sigma_i(y_1,\dots,y_n)$.

By collecting the terms with the same weight, the product of the $Q(x_i)$ can be written as

$$\begin{aligned}
Q(x_1)\dots Q(x_n) = {} & 1 + K_1(\sigma_1) + K_2(\sigma_1,\sigma_2) + \ldots + K_n(\sigma_1,\dots,\sigma_n)\\
& + K_{n+1}(\sigma_1,\dots\sigma_n,0) + \ldots\\
=: {} & K(\sigma_1,\sigma_2,\dots,\sigma_n,0,0,\dots),
\end{aligned}$$

where $K_j = K_j(\sigma_1,\dots,\sigma_j)$ are homogeneous polynomials of weight $4j$ in the elementary symmetric polynomials. Indeed, since each x_i has weight 2, and σ_k has degree k in the variables x_i^2, each of the K_j has weight $4j$. The polynomials $\{K_j\}$ from a *multiplicative sequence*. This means for a sequence of indeterminates y_i, y_i', y_i'' such that

$$(1 + y_1 + y_2 + \dots) = (1 + y_1' + y_2' + \dots)(1 + y_1'' + y_2'' + \dots),$$

the polynomials K_j satisfy

$$\begin{aligned}
\sum_{j\ge 0} K_j(y_1,\dots,y_j) &= \sum_{k\ge 0} K_k(y_1',\dots,y_k') \sum_{l\ge 0} K_l(y_1'',\dots,y_l'')\\
\iff \qquad K(y_1,y_2,\dots) &= K(y_1',y_2',\dots)K(y_1'',y_2'',\dots)
\end{aligned}$$

For further details on the theory of multiplicative sequences, we refer to [HSB95]. We now construct genera from the multiplicative sequence $\{K_j\}$. Let M be an oriented compact manifold of dimension $4n$. Associate a genus φ_Q to $Q(x)$ by defining

$$\varphi_Q([M]) := K_n(p_1,\dots,p_n)[M]$$

where x_i are now the Chern roots of the complexified tangent bundle $TM \otimes \mathbb{C}$ and $p_1,\dots,p_n$ are the Pontryagin classes with $p_i \in H^{4i}(M;\mathbb{Z})$, and we evaluate them on the class $[M]$ of the manifold M. Hence, the genus is a linear combination of Pontryagin numbers of M with coefficients in R. Furthermore, we define $\varphi_Q([M]) = 0$ if $4 \nmid \dim(M)$.

Proposition 3.4. *An even normalized power series $Q(x) = 1 + a_2 x^2 + a_4 x^4 + \dots \in R[\![x]\!]$ defines a genus $\varphi_Q \colon \Omega^{\mathrm{SO}} \otimes \mathbb{Q} \to R$.*

Proof. We need to show that the above construction yields a well-defined ring homomorphism φ_Q. Since by construction, φ_Q is a R-linear combination of Pontryagin numbers of the manifolds, and Pontryagin numbers are invariants of oriented bordism, we obtain that it is well-defined. The construction is additive under disjoint union since cohomology becomes a direct sum. For multiplicativity, let $L = M \times N$ be the Cartesian product of two compact oriented manifolds. The total Pontryagin class of L is given by the product of the total Pontryagin classes of M and N (up to 2-torsion). The multiplicative property of K_n then precisely shows that $\varphi_Q([L]) = \varphi_Q([M])\varphi_Q([N])$. □

The power series Q is called *characteristic series* or *Hirzebruch series* of the genus φ_Q.

Example 3.5. The $\widehat{A}$-*genus* is the genus associated to the characteristic series

$$Q(x) = \frac{x/2}{\sinh(x/2)} = 1 - \tfrac{1}{24}x^2 + \tfrac{7}{5760}x^4 - \tfrac{31}{967680}x^6 + \dots$$

It is related to the Dirac operator (provided that the manifolds have spin structure). We will come back to the $\widehat{A}$-genus later, and it will also play a role in the string theory interpretation.

Example 3.6. The L-*genus* is the genus associated to the characteristic series

$$Q(x) = \frac{x}{\tanh(x)} = 1 + \tfrac{1}{3}x^2 - \tfrac{1}{45}x^4 + \tfrac{2}{945}x^6 + \dots$$

We will see later that it is given the *signature* which is an important invariant in the study of $4n$-manifolds.

Since the constant term of $Q(x)$ is 1, the formal power series $1/Q(x)$ also has constant term 1. Since $Q(x)$ is even, the series $1/Q(x)$ is again even. Define an odd power series

$$f(x) := \frac{x}{Q(x)}.$$

Since $f(x)$ has no constant term and $f'(0) = 1$, there exists[1] a unique inverse power series. That is, a power series $g(x) \in R[[x]]$ such that $g(f(x)) = x$.

[1] This can be shown by working mod x^k. See for example Lemma 2.8 of [Str19].

Definition 3.7. Let $Q(x)$ be the characteristic power series associated to a genus $\varphi\colon \Omega^{\mathrm{SO}} \otimes \mathbb{Q} \to R$ where R is a $\mathbb{Q}$-algebra. The *logarithm* of φ is a power series $g(x) \in R[\![x]\!]$ defined as the formal inverse of $f(x) = \frac{x}{Q(x)}$. We often write

$$\log_\varphi(x) := g(x) \quad \text{and} \quad \exp_\varphi(x) := f(x).$$

Remark 3.8. In the next chapter, we will come across another notion of logarithm: the logarithm of a *formal group law*. We will then see how those two notions are related.

Proposition 3.9. *Let $\varphi\colon \Omega^{\mathrm{SO}} \otimes \mathbb{Q} \to R$ be a genus with characteristic power series $Q(x)$. Its logarithm satisfies*

$$\begin{aligned}\log_\varphi(x) = g(x) &= x + \frac{\varphi([\mathbb{C}\mathrm{P}^2])}{3}x^3 + \frac{\varphi([\mathbb{C}\mathrm{P}^4])}{5}x^5 + \frac{\varphi([\mathbb{C}\mathrm{P}^6])}{7}x^7 + \dots \in R[\![x]\!] \\ &= \sum_{n=0}^{\infty} \frac{\varphi([\mathbb{C}\mathrm{P}^{2n}])}{2n+1}x^{2n+1}.\end{aligned}$$

Proof. Recall from Example 2.20 that the total Pontryagin class of $\mathbb{C}\mathrm{P}^n$ is $p(\mathbb{C}\mathrm{P}^k) = (1+g^2)^{n+1}$ where $g \in H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. We also have that $g^2 = p_1$. Therefore, by using the multiplicativity of $\{K_j\}$ iteratively on the sequence

$$1 + p_1 + p_2 + \cdots + p_n = (1+p_1)^{n+1},$$

we obtain

$$K(p_1,\dots,p_n) = \sum_i K_i(p_1,\dots,p_n) = \Big(\sum_j K_j(p_1)\Big)^{n+1} = K(p_1)^{n+1}$$

Now observe that $Q(x) = K(p_1)$. With the above, this yields that

$$\varphi([\mathbb{C}\mathrm{P}^n]) = K_n(p_1,\dots,p_n)[\mathbb{C}\mathrm{P}^n] = \text{coefficient of } x^n \text{ in } Q(x)^{n+1}.$$

Interpreting the power series as Laurent series, we can use the residue theorem to obtain

$$\text{coefficient of } x^n \text{ in } \frac{x^{n+1}}{f(x)^{n+1}} = \mathrm{res}_0\left[\frac{1}{f(x)^{n+1}}\right] = \frac{1}{2\pi i}\oint_\gamma \frac{1}{f(x)^{n+1}}dx,$$

where γ is a contour around the origin with winding number 1. With a transformation of variables $u = f(x)$ and using that g is the inverse to f, the above is equal to

$$\frac{1}{2\pi i}\oint_{f(\gamma)} \frac{g'(u)}{u^{n+1}}du = \mathrm{res}_0\left[\frac{g'(u)}{u^{n+1}}\right] = \text{coefficient of } u^n \text{ in } g'(u).$$

Here, since $f(x) = x + \dots$, we have that $f(\gamma)$ is also a contour around the origin with winding number 1. We have shown that the coefficient of u^n in $g'(u)$ is $\varphi([\mathbb{C}\mathrm{P}^n])$. Formally integrating g' and using that $\mathbb{C}\mathrm{P}^{2n+1}$ is null-bordant, we find that the logarithm indeed has the desired form. $\square$

Since the logarithm is encoding the values of φ on the generators of $\Omega^{\mathrm{SO}} \otimes \mathbb{Q}$, we see that genera φ and odd power series of the form $g(x) = x + \dots$ are in one-to-one correspondence. Since from every $g(x)$, we can construct $f(x)$ and $Q(x)$, there is a one-to-one correspondence

$$\left\{ \substack{\text{Even normalized power series} \\ Q(x)=1+a_2x^2+a_4x^4+\dots \,\in R[\![x]\!]} \right\} \longleftrightarrow \left\{ \text{Genera } \varphi \colon \Omega^{\mathrm{SO}} \otimes \mathbb{Q} \to R \right\}.$$

For a $\mathbb{Q}$-algebra R, this correspondence also holds for genera $\varphi \colon \Omega^{\mathrm{SO}} \to R$ since φ is fully determined by its rationalization. We now come to the definition of *elliptic* genera.

Definition 3.10. An *elliptic genus* is a genus $\varphi \colon \Omega^{\mathrm{SO}} \to R$ with associated logarithm of the form

$$\log_\varphi(x) = \int_0^x \frac{dt}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}}$$

for some $\delta, \varepsilon \in R$.

Note that for general $\mathbb{Q}$-algebras R, we can evaluate the integral formally as a power series over R. For now, we consider the case $R = \mathbb{C}$. If the polynomial $r(t) := 1 - 2\delta t^2 + \varepsilon t^4$ has four distinct roots, the integral occurring in the definition is an *elliptic integral* hence the modifier *elliptic*. This is measured by the discriminant of the polynomial which (up to a factor of 256) is given by

$$\Delta = \varepsilon(\delta^2 - \varepsilon)^2.$$

The polynomial $r(t)$ has four distinct roots if and only if $\Delta \neq 0$. A genus is still called elliptic in the degenerate cases $\Delta = 0$, and those provide interesting geometric examples.

First, let $\varepsilon = 0$. To exclude the trivial[2] genus, we assume $\delta \neq 0$. Now, we can solve the integral using elementary methods and obtain

$$\log_\varphi(x) = \int_0^x \frac{dt}{\sqrt{1 - 2\delta t^2}} = \begin{cases} \sin^{-1}\left(\sqrt{2\delta}x\right)/\sqrt{2\delta} & \text{if } \delta > 0, \\ \sinh^{-1}\left(\sqrt{-2\delta}x\right)/\sqrt{-2\delta} & \text{if } \delta < 0. \end{cases}$$

Choosing $\delta = -\frac{1}{8}$, this recovers the $\widehat{A}$-genus since

$$\log_\varphi(x) = 2\sinh^{-1}(x/2) \quad \Longleftrightarrow \quad Q(x) = \frac{x/2}{\sinh(x/2)}$$

We denote its value on $[M]$ by $\widehat{A}(M)[M]$. Other values of δ simply scale the $\widehat{A}$-genus. The normalization is chosen such that $\widehat{A}(M)[M]$ has integer values provided that M has

[2]If $\delta, \varepsilon = 0$, we simply obtain $\log_\varphi(x) = x$ which corresponds to the trivial genus that assigns $\varphi([\mathbb{C}\mathrm{P}^{2n}]) = 0$ for all $n \geq 1$. This amounts to the power series $Q(x) = 1$.

spin structure. Furthermore, if there is a non-trivial action of S^1 on a spin manifold M, then $\widehat{A}(M)[M] = 0$. This has been shown by Atiyah-Hirzebruch [AH70], and was one of the results inspiring the development of elliptic genera.

The other degenerate case is given by $\delta^2 = \varepsilon$. In this case, the square root can be evaluated. Considering $\delta = \varepsilon = 1$, we obtain

$$\log_\varphi = \int_0^x \frac{dt}{1-t^2} = \tanh^{-1}(x) \quad \Longleftrightarrow \quad Q(x) = \frac{x}{\tanh(x)}$$

This is the L-genus. We denote its value on $[M]$ by $L(M)[M]$. We have the following theorem by Hirzebruch.

Theorem 3.11 (Hirzebruch signature theorem)**.** *The L-genus is the signature. That is, let M be a compact oriented $4n$-manifold. Then*

$$L(M)[M] = \operatorname{sign}(M).$$

Proof. For a proof, we refer to Theorem 8.2.2 in [HSB95]. □

The L-genus assigns 1 to all even complex projective spaces $\mathbb{C}\mathrm{P}^{2n}$. In general, we have the following for values of an elliptic genus.

Proposition 3.12. *Let R be a $\mathbb{Q}$-algebra and $\varphi\colon \Omega^{\mathrm{SO}} \to R$ be an elliptic genus with $\delta, \varepsilon \in R$. Then its values on the complex and quaternionic projective plane are given by*

$$\varphi([\mathbb{C}\mathrm{P}^2]) = \delta \qquad \text{and} \qquad \varphi([\mathbb{H}\mathrm{P}^2]) = \varepsilon.$$

Proof. Consider the derivative of the logarithm of φ. Expanding up to quadratic order yields

$$g'(x) = \frac{1}{\sqrt{1-2\delta x^2 + \varepsilon x^4}} = (1-2\delta x^2 + \varepsilon x^4)^{-\frac{1}{2}} = 1 + \delta x^2 + \dots,$$

and hence, by Proposition 3.9, we have $\varphi(\mathbb{C}\mathrm{P}^2) = \delta$. For the value of φ on the quaternionic projective plane $\mathbb{H}\mathrm{P}^2$, we refer to the computations in Section 1.7 of [HBJ92]. □

While we defined elliptic genera in terms of their logarithm, we have the following equivalent characterization via their power series $f(x)$. We will need this to construct elliptic genera from elliptic functions in Section 3.2.

Proposition 3.13. *Let $\varphi\colon \Omega^{\mathrm{SO}} \to R$ be a genus with values in a $\mathbb{Q}$-algebra R and consider its characteristic power series $Q(x) = x/f(x)$. The genus φ is elliptic if and only if $f(x)$ satisfies the differential equation*

$$(f')^2 = 1 - 2\delta f^2 + \varepsilon f^4.$$

Proof. The genus φ is elliptic if and only if

$$g'(u) = \frac{1}{\sqrt{1 - 2\delta u^2 + \varepsilon u^4}}.$$

By definition, $g(u)$ and $f(x)$ are inverse to each other. Replace $u = f(x)$ and obtain

$$\begin{aligned} \frac{1}{\sqrt{1 - 2\delta f(x)^2 + \varepsilon f(x)^4}} &= g'(f(x)), \\ \iff \qquad 1 - 2\delta f(x)^2 + \varepsilon f(x)^4 &= \frac{1}{g'(f(x))^2} = f(x)^2, \end{aligned}$$

where we used the rule for the derivative of the inverse function $g^{-1} = f$ in the last step. □

Strict multiplicativity

We end this section by collecting geometric properties satisfied by elliptic genera. These properties have played a big role in the historical development of the theory.

By definition, a genus is multiplicative $\varphi(M \times N) = \varphi(M)\varphi(N)$. The signature satisfies an even stronger multiplicativity property. Let G be a connected compact Lie group and let $E \to B$ be a principal G-bundle where B is a compact oriented manifold. Let G act on a compact oriented manifold M and form the associated bundle $E \times_G M$ over B. With the additional assumption that the orientation on $E \times_G M$ is compatible with the ones on B and M, the signature satisfies

$$\operatorname{sign}(E \times_G M) = \operatorname{sign}(B) \operatorname{sign}(M).$$

This is called *strict multiplicativity* of the signature and has been shown by Chern-Hirzebruch-Serre [CHS57]. Now, let $\xi\colon E \to B$ be a complex vector bundle of complex dimension k. The projectivization $\mathbb{C}\mathrm{P}(\xi)$ is defined as the fibrewise projectivization of the complex vector spaces. This provides an example of the above by taking the Lie group $G = U(k)$. Hence, the signature satisfies

$$\operatorname{sign}(\mathbb{C}\mathrm{P}(\xi)) = \operatorname{sign}(B) \operatorname{sign}(\mathbb{C}\mathrm{P}^{k-1}).$$

If k even, then $\mathrm{sign}(\mathbb{C}\mathrm{P}(\xi))$ vanishes since $\mathrm{sign}(\mathbb{C}\mathrm{P}^{k-1}) = 0$. This follows from the fact that the signature is a bordism invariant and odd-dimensional complex projective spaces are null-bordant. This need not hold for a general genus. However, we have the following defining property of elliptic genera.

Theorem 3.14 (Ochanine [Och87]). *A genus $\varphi\colon \Omega^{\mathrm{SO}} \to R$ is elliptic if and only if it vanishes on $\mathbb{C}\mathrm{P}(\xi)$ for every even-dimensional complex vector bundle ξ.*

Elliptic genera satisfy another multiplicativity property. This has been shown by Bott and Taubes in [BT89] proving the conjectured rigidity by Witten [Wit87].

Theorem 3.15 (Bott-Taubes). *Let φ be an elliptic genus and let $M \to B$ be a fibre bundle with a compact spin manifold F as fibre that has a compact connected Lie group as structure group. Then, $\varphi(M) = \varphi(B)\varphi(F)$.*

This implies one direction of Theorem 3.14 as follows. Let k be even and consider the fibre bundle given by the projectivization of a k-dimensional complex vector bundle with fibre $F = \mathbb{C}\mathrm{P}^{k-1}$ and take $G = U(k)$ as structure group. Odd-dimensional complex projective spaces admit a spin structure. Hence, by applying Theorem 3.15, we obtain that an elliptic genus vanishes on the projectivization of a k-dimensional complex vector bundle with since $\mathbb{C}\mathrm{P}^{k-1}$ is null-bordant.

3.2. Elliptic genera and modular forms

In this section, we will show how elliptic genera can be constructed from elliptic curves together with a chosen 2-torsion point. For this, we will use the theory of elliptic functions, elliptic curves and modular forms. Our approach is based on Chapter 2 of [HBJ92]. This will provide an interpretation for the *universal elliptic genus* taking values in a ring of modular forms.

Definition 3.16. An elliptic genus $\varphi\colon \Omega_*^{\mathrm{SO}} \to \mathbb{Q}[\delta, \varepsilon]$, where δ and ε are the parameters chosen in its logarithm, is called *universal elliptic genus.*

Such a genus is called *universal* because of the following property. Let R be a $\mathbb{Q}$-algebra. Any elliptic genus with values in R factors as

$$\Omega_*^{\mathrm{SO}} \xrightarrow{\varphi} \mathbb{Q}[\delta, \varepsilon] \xrightarrow{\mathrm{eval}} R$$

where the first map is "the" universal elliptic genus and the second map is given by choosing values $\delta, \varepsilon \in R$. Indeed, any compact oriented manifold has only a finite number of non-zero Pontryagin classes. Hence, an elliptic genus associated to $[M]$ according to its characteristic series $Q(x)$ will be a polynomial in δ and ϵ. Furthermore, any element in Ω_*^{SO} is given by a finite linear combination of classes of manifolds. We will interpret $\mathbb{Q}[\delta, \varepsilon]$ as the ring of modular forms and discuss the uniqueness of the universal elliptic genus.

We refer to Appendix A.1 for more details on the background of modular forms. See also Appendix I of [HBJ92].

Elliptic functions and lattices

We want to construct elliptic genera via elliptic functions. For this, we first need the notion of a lattice. A *lattice* is a discrete subgroup L of $\mathbb{C}$ of the form

$$L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$$

with complex numbers $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ that are linearly independent over $\mathbb{R}$, that is, $\mathrm{Im}(\omega_2/\omega_1) \neq 0$. Elliptic functions are functions that are periodic with respect to a lattice. More precisely, we have the following definition.

Definition 3.17. An *elliptic function* is a meromorphic function $f\colon \mathbb{C} \to \overline{\mathbb{C}}$ that is doubly periodic. That is, there exist $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$, such that

$$f(z + \omega_1) = f(z) \qquad \text{and} \qquad f(z + \omega_2) = f(z).$$

In other words, f is periodic with respect to lattice points of the lattice $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. This means we can view f as a meromorphic function $\mathbb{C}/L \to \overline{\mathbb{C}}$.

An important example of an elliptic function is the so-called Weierstraß $\wp$-function.

Definition 3.18. The *Weierstraß $\wp$-function* associated to a lattice L is the function $\wp\colon \mathbb{C} \to \overline{\mathbb{C}}$ defined by

$$\wp(z) := \wp(z, L) := \frac{1}{z^2} + \sum_{\omega \in L \setminus \{0\}} \left[\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right].$$

The following statements are proved in Section 2.1 of [HBJ92]. For more details on the theory of elliptic functions, see [Lan87]. The Weierstraß $\wp$-function is an even function, that is, $\wp(z) = \wp(-z)$ for all $z \in \mathbb{C}$. Its derivative is given by

$$\wp'(z) = \sum_{\omega \in L} \frac{-2}{(z-\omega)^3}.$$

Furthermore, $\wp$ satisfies the following differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2 \cdot \wp(z) - g_3$$

where $g_2 = g_2(L)$ and $g_3 = g_3(L)$ are complex numbers depending on the lattice L. Note that the differential equation resembles the Weierstraß equation for elliptic curves. Indeed, choosing

$$(\wp(z) : \wp'(z) : 1) \mapsto (X_0 : X_1 : 1) \in \mathbb{C}\mathrm{P}^2,$$

we obtain a parametrization of an elliptic curve over $\mathbb{C}$. Elliptic curves over the complex numbers are precisely given by quotients $\mathbb{C}/L$ where L is a lattice.

The complex numbers g_2 and g_3 above are examples of *lattice invariants*. Define

$$s_k(L) := \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^k}.$$

Note that the sum converges for $k > 2$. This defines lattice invariants of weight k. That is, by scaling the lattice, we obtain

$$s_k(\lambda L) = \lambda^{-k} s_k(L), \quad \text{for } \lambda \in \mathbb{C}^\times.$$

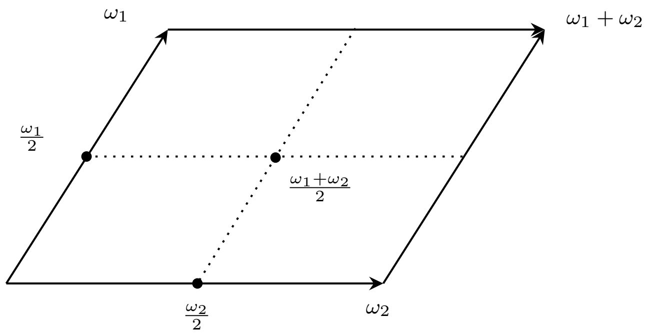
With this definition, g_2 and g_3 are given by

$$g_2(L) = 60 s_4(L),$$
$$g_3(L) = 140 s_6(L).$$

These lattice invariants are related to modular forms called the Eisenstein series. See Appendix A.1.

Elliptic genera from elliptic functions

We now turn to the construction of elliptic genera. Let $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a lattice. The 2-*division points* of L are points $\omega \in \mathbb{C} \setminus L$ such that $2\omega \in L$. Equivalently, $\omega \in \mathbb{C}/L$ with $\omega \neq 0$ such that $2\omega = 0$. In the *fundamental parallelogram*, there are precisely three such points. The fundamental parallelogram of a lattice $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is the parallelogram spanned by $\{0, \omega_1, \omega_2, \omega_1 + \omega_2\}$. Note that this depends on the choice of the basis $\{\omega_1, \omega_2\}$.



One can show that the 2-division points are precisely the zeroes for $\wp'$. Again, see Section 2.1 of [HBJ92] for the proof. Hence, $\wp$ does not have a pole at those points. We make the following definition.

$$e_1 := \wp\left(\frac{\omega_1}{2}\right), \qquad e_2 := \wp\left(\frac{\omega_2}{2}\right), \qquad e_3 := \wp\left(\frac{\omega_1+\omega_2}{2}\right).$$

With this identification, the differential equation for $\wp$ becomes

$$\wp'(z)^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$$

such that we obtain the following relations for the e_i

$$\begin{aligned} e_1+e_2+e_3 &= 0, \\ 4(e_1e_2+e_1e_3+e_2e_3) &= -g_2, \\ 4e_1e_2e_3 &= g_3. \end{aligned}$$

By construction, $\wp(z)-e_1$ has a double zero in $\omega_1/2$. Now, consider the meromorphic function $f(z) := 1/\sqrt{\wp(z)-e_1}$ (with choice of sign such that $f(z)$ has Laurent expansion $f(z) = \frac{1}{z} + \dots$).

Theorem 3.19. *The genus φ_Q defined by the power series $Q(x) = x/f(x)$ with $f(x) = 1/\sqrt{\wp(x)-e_1}$ is an elliptic genus. The parameters are given by $\delta = -\frac{3}{2}e_1$ and $\varepsilon = (e_1-e_2)(e_1-e_3)$.*

Proof. We will show that f satisfies the differential equation. Define

$$(u,v) = (\wp(x), \wp'(x)) \quad \text{and} \quad (w,z) = (f(x), f'(x)).$$

We obtain the following relations

$$w = \frac{1}{\sqrt{u-e_1}} \iff u = w^{-2}+e_1,$$
$$z = \tfrac{d}{dx}(\wp(x)-e_1)^{-\frac{1}{2}} = -\frac{\wp'(x)}{2}(\wp(x)-e_1)^{-\frac{3}{2}} = -\frac{vw^3}{2} \iff v = -2zw^{-3}.$$

Substituting the above into the differential equation for $\wp$ yields

$$\begin{aligned} v^2 &= 4(u-e_1)(u-e_2)(u-e_3), \\ 4z^2w^{-6} &= 4w^{-2}(w^{-2}+e_1-e_2)(w^{-2}+e_1-e_3). \end{aligned}$$

Multiplying both sides by $w^6/4$, and using the relations for e_i, we compute

$$\begin{aligned} z^2 &= (1+(e_1-e_2)w^2)(1+(e_1-e_3)w^2) \\ &= 1+(3e_1-e_1-e_2-e_3)w^2+(e_1-e_2)(e_1-e_3)w^4 \\ &= 1+3e_1w^2+(e_1-e_2)(e_1-e_3)w^4 \\ &= 1-2\delta w^2+\varepsilon w^4 \end{aligned}$$

which gives the differential equation for f by choosing $\delta = -\frac{3}{2}e_1$ and $\varepsilon = (e_1-e_2)(e_1-e_3)$. Hence, φ_Q is an elliptic genus. □

Remark 3.20. Note that the elliptic genera that we obtain in this way do not incorporate the degenerate cases that give us the L-genus and the $\widehat{A}$-genus. However, if we let the lattice degenerate, we can obtain them as limits. First, if $\omega_1 \to 0$, the lattice degenerates to $\mathbb{Z}\omega_2$ and consequently, $e_2 = e_3$. With $e_1 + 2e_2 = 0$, we obtain

$$\varepsilon = (e_1-e_2)^2 = (e_1+\tfrac{1}{2}e_2)^2 = (\tfrac{3}{2}e_1)^2 = \delta^2$$

recovering the case for the L-genus by choosing $\delta = 1$. Second, if $\omega_2 \to 0$ or $\omega_1 \to \omega_2$, we obtain $e_1 = e_2$ or $e_1 = e_3$, respectively. In either case $\varepsilon = 0$, and by choosing $\delta = -\frac{1}{8}$, we obtain the $\widehat{A}$-genus.

Elliptic genera as modular forms

Next, we want to show how δ and ε can be interpreted as modular forms for the congruence subgroup $\Gamma_0(2) \subseteq SL_2(\mathbb{Z})$ defined as

$$\Gamma_0(2) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mod 2 \right\}.$$

In the construction of an elliptic genus in Theorem 3.19, we have implicitly fixed a choice of lattice. If, instead, we do not fix a choice of lattice, but consider the set of lattices $\mathcal{L}$, then $\delta = \delta(L)$ and $\varepsilon = \varepsilon(L)$ can be regarded as lattice invariants of weights 2 and 4, respectively. For $\lambda \in \mathbb{C}^\times$, we have

$$\delta(\lambda L) = \lambda^{-2}\delta(L), \qquad \varepsilon(\lambda L) = \lambda^{-4}\varepsilon(L).$$

Indeed, we have already seen the lattice invariants g_2 and g_3. Similarly, $e_i = e_i(L)$ depend on the lattice, and so do δ and ε. Using the transformation property of $\wp$ under changing from L to λL, we obtain their weights. Note that in the construction for Theorem 3.19, we have also chosen the 2-division point $\omega_1/2$. Since δ and ε depend on this choice, we have constructed them as invariants of lattices with a choice of a 2-division point.

By scaling a lattice $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with a suitable $\lambda \in \mathbb{C}^\times$, we obtain a lattice of the form $\mathbb{Z} \oplus \mathbb{Z}\tau$ with τ in the *upper half plane*

$$\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

In this way, the upper half plane parametrizes lattices of the form $\mathbb{Z} \oplus \mathbb{Z}\tau$. The modular group $SL_2(\mathbb{Z})$ acts on the upper half plane by

$$\gamma \cdot z = \tfrac{az+b}{cz+d}, \quad \text{with } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$$

and any subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$ acts on $\mathcal{H}$ by restriction. *Modular forms* for $SL_2(\mathbb{Z})$ are holomorphic functions f on $\mathcal{H}$ that satisfy the following transformation property for the group action by $SL_2(\mathbb{Z})$

$$f\left(\tfrac{az+b}{cz+d}\right) = (cz+d)^k f(z),$$

where the integer $k \geq 0$ is the *weight* of the modular form. Furthermore, modular forms are required to be *holomorphic at* ∞. This means that a modular form does not have a pole at ∞, and by defining $q = \exp(2\pi i\tau)$, we can expand it in a power series in q around the point ∞. For modular forms for congruence subgroups $\Gamma \subseteq SL_2(\mathbb{Z})$, the holomorphicity at ∞ has to be extended. For $SL_2(\mathbb{Z})$, the point ∞ is a called a *cusp*. While $SL_2(\mathbb{Z})$ only has a cusp at ∞, congruence subgroups can have several cusps. In particular, the cusps of $\Gamma_0(2)$ are given by 0 and ∞. Modular forms for $\Gamma_0(2)$ have to be holomorphic at both 0 and ∞. See Appendix A.1 for more details.

The passage from general lattices to lattices parametrized by $\tau \in \mathcal{H}$ turns lattice invariants into modular forms. Under this correspondence, invariants for lattices with a chosen 2-division point correspond to modular forms for the congruence subgroup $\Gamma_0(2) \subseteq SL_2(\mathbb{Z})$. This can be seen by means of the bijection (see Proposition A.15)

$$\Gamma_0(2)\backslash\mathcal{H} \longleftrightarrow \{(\mathbb{C}/L, P) \mid P \text{ is a 2-division point of } L\}/\text{iso}.$$

induced by sending $\tau \in \mathcal{H}$ to the lattice $L = \mathbb{Z}\oplus\mathbb{Z}\tau$ with 2-division point $P = \frac{1}{2}$. Here, by isomorphism, we mean an isomorphism of elliptic curves $\mathbb{C}/L$ together with a 2-torsion point (which precisely corresponds to a 2-division point of the lattice). Note that $P = \frac{1}{2}$ is the choice that precisely corresponds to choosing $\omega_1/2$ in the lattice $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$.

The upshot[3] of the above discussion is that we can view δ and ε as modular forms of weights 2 and 4, respectively, for the congruence subgroup $\Gamma_0(2)$. The ring $M^{\mathbb{Q}}(\Gamma_0(2))$ of

[3]To be more precise, one also has to show the holomorphicity of the modular forms.

modular forms for $\Gamma_0(2)$ with rational coefficients in their q-expansion is precisely given by

$$M^{\mathbb{Q}}(\Gamma_0(2)) \cong \mathbb{Q}[\delta, \varepsilon].$$

Hence, the universal elliptic genus $\varphi \colon \Omega_*^{\mathrm{SO}} \to \mathbb{Q}[\delta, \varepsilon]$ takes values in modular forms for $\Gamma_0(2)$.

Proposition 3.21. *Let M be a compact oriented manifold of dimension $4n$. The universal elliptic genus φ assigns to M a modular form of weight $2n$ for the congruence subgroup $\Gamma_0(2)$.*

Proof. We have seen that indeed φ takes values in the ring $\mathbb{Q}[\delta, \varepsilon]$ of modular forms for $\Gamma_0(2)$. We need to show that $\varphi([M])$ is of weight $2n$. Consider the differential equation from Proposition 3.13

$$(f'(x))^2 = 1 - 2\delta f(x)^2 + \varepsilon f(x)^4$$

and write the odd power series $f(x) = x + \dots$ as

$$f(x) = \sum_{k \geq 1} c_k x^k \qquad \text{and} \qquad f'(x) = \sum_{k \geq 1} k c_k x^{k-1}$$

Since δ and ε have weights 2 and 4 respectively, it follows inductively that the k-th coefficient c_k has weight $k-1$. Now, since $Q(x)f(x) = x$, we have that

$$(1 + a_2 x^2 + a_4 x^4 + \dots)(1 + c_3 x^2 + c_5 x^4 + c_7 x^6 + \dots) = 1.$$

Hence, by considering the coefficients of x^{2k} on the left-hand side, we see that a_{2k} must be of the same weight as c_{2k+1} which is $2k$. Now, recall the construction of a genus via the Pontryagin classes. The polynomial $K_n(p_1, \dots, p_n)$ is a homogeneous of degree $2n$ in the indeterminates x_i^2. Hence, all the coefficients occurring in $K_n(p_1, \dots, p_n)$ must be of weight $2n$. Therefore, $\varphi([M])$ has weight $2n$ as a modular form. $\square$

The modular forms δ and ε have q-expansions[4]

$$\delta(\tau) = \frac{1}{4} + 6 \sum_{n=1}^{\infty} \sigma_1^{\mathrm{odd}}(n) q^n = \frac{1}{4} + 6 \sum_{n=1}^{\infty} \sum_{\substack{d \mid n \\ d \text{ odd}}} d q^n,$$

$$\varepsilon(\tau) = \frac{1}{16} + \sum_{n=1}^{\infty} \sum_{d \mid n} (-1)^d d^3 q^n,$$

[4] The expansions can be found in Remark 3.7 and Theorem 4.7 in the Appendix I of [HBJ92], and in [Zag88].

with $q = \exp(2\pi i\tau)$. Composing the universal elliptic genus with the map to $\mathbb{Q}[\![q]\!]$ that sends every modular form to its q-expansion, we obtain

$$\Omega^{\mathrm{SO}} \to \mathbb{Q}[\delta, \varepsilon] \to \mathbb{Q}[\![q]\!]$$

assigning to every oriented manifold the q-expansion of the modular form for $\Gamma_0(2)$. The characteristic series for the universal elliptic genus is given by

$$Q(x) = \frac{x}{f(x)} = \frac{x/2}{\tanh x/2} \prod_{n=1}^{\infty} \frac{(1+q^n e^x)(1+q^n e^{-x})(1-q^n)^2}{(1-q^n e^x)(1-q^n e^{-x})(1+q^n)^2}.$$

See for example Theorem 5.6 in Appendix I of [HBJ92]. This theorem also proves $\rho(x,\tau) = 1/f(x,\tau)$ satisfies the properties of a *Jacobi form*, when considering f with its dependence on both x and τ. We refer to Appendix A.1 for the definition of a Jacobi form, and to [EZ85] for more details on the theory of Jacobi forms.

Remark 3.22. In the next chapter, we will see that the above can be refined as follows. The universal genus has image contained in $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$ and the coefficients occurring in its q-expansion are contained in $\mathbb{Z}[\frac{1}{2}]$.

Recall that we chose e_1 as the 2-division point. If, instead, we choose e_2 or e_3, we obtain modular forms for different subgroups, conjugated to $\Gamma_0(2)$. Since $[SL_2(\mathbb{Z}) : \Gamma_0(2)] = 3$ and $\Gamma_0(2)$ is not a normal subgroup, there are precisely three conjugate subgroups, corresponding to the three 2-division points. Now choosing e_2 as the 2-division point, one obtains modular forms for the congruence subgroup

$$\Gamma^0(2) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) \mod 2 \right\}.$$

Switching between modular forms for $\Gamma_0(2)$ and $\Gamma^0(2)$ corresponds to switching the cusps 0 and ∞. Hence, this yields modular forms expanded around the cusp 0. Now, after a suitable transformation of variables, one again obtains modular forms for $\Gamma_0(2)$ from the construction of $f_2(x) = (\wp(x) - e_2)^{-1/2}$, and the characteristic series of this elliptic genus is given by

$$Q_2(x) = \frac{x/2}{\sinh x/2} \prod_{n=1}^{\infty} \left[\frac{(1-q^n)^2}{(1-q^n e^x)\,(1-q^n e^{-x})} \right]^{(-1)^n}.$$

See Section 6.2 of [HBJ92] for more details. One then obtains modular forms $\widetilde{\delta}$ and $\widetilde{\varepsilon}$ different form δ and ε. This raises the question of the uniqueness of the universal genus. One has the following relations

$$\widetilde{\delta} = -\frac{1}{2}\delta \qquad \text{and} \qquad \widetilde{\varepsilon} = \frac{1}{4}(\delta^2 - \varepsilon),$$

which shows that the choice of 2-division point provides an isomorphism of the ring of modular forms preserving the condition $\varepsilon(\delta^2 - \varepsilon)^2 \neq 0$. For further details on the expansions of the modular forms $\widetilde{\delta}$ and $\widetilde{\varepsilon}$, we refer to Appendix I of [HBJ92]. See also [Zag88] for proofs of different formulas for the characteristic series $Q(x)$ and $Q_2(x)$ (denoted there as $P_S(u)$ and $P(u)$, respectively) relating the elliptic genera to Eisenstein series for $\Gamma_0(2)$. In this sense, there are three universal elliptic genera, each one for the choice of e_1, e_2, and e_3 corresponding to the 2-division points. For expressions of the three elliptic genera and the corresponding modular forms in terms of *θ-functions*, highlighting the properties as a Jacobi form, we refer to [Liu96].

3.3. Complex elliptic genera of level N

In the previous section, we have shown the universal elliptic genus assigns modular forms for the congruence subgroup $\Gamma_0(2)$ to compact oriented manifolds. The group $\Gamma_0(2)$ corresponds to a choice of a 2-torsion point of the elliptic curve. More generally, we can choose a primitive[5] N-torsion point and construct *elliptic genera of level N* by assigning modular forms for the congruence subgroup $\Gamma_1(N)$ to manifolds. For this, we require stably almost complex structures on the manifolds. This generalization has been independently considered by Hirzebruch [Hir88] and Witten [Wit88].

In this section, we will briefly allude to this generalization and consider a few relevant examples. For more details, we refer to Chapter 7 of [HBJ92].

We start by defining complex genera. Recall that the bordism ring Ω_*^{U} of manifolds with stably almost complex structure is rationally given by a polynomial ring in the generators $[\mathbb{C}\mathrm{P}^n]$ with degree $2n$ for $n \in \mathbb{N}$.

Definition 3.23. A *complex genus* with values in a $\mathbb{Q}$-algebra R is a ring homomorphism $\varphi\colon \Omega^{\mathrm{U}} \to R$.

As in the oriented case, we can construct such genera from characteristic power series. However, instead of considering a formal factorization of the total Pontryagin class into x_i^2, we now take the factorization of the total Chern class into Chern roots x_i. That is, we take the elementary symmetric functions in the variables x_i rather than x_i^2. Hence, a characteristic power series $Q(x)$ for a complex genus is given by a normalized power series of the form

$$Q(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \in R[\![x]\!].$$

The rest of the construction is completely analogous. In other words, a complex genus is given by a R-linear combination of Chern numbers.

[5] By a *primitive* N-torsion point, we mean that its order in $\mathbb{C}/L$ is precisely N and does not just divide N. For composite numbers N, we have to make this distinction while we did not call the 2-torsion points primitive since 2 is prime.

Example 3.24. For an almost complex compact manifold M of dimension $2n$, the top Chern class $c_n(M) \in H^{2n}(M;\mathbb{Z})$ evaluated on $[M]$ is equal to the Euler characteristic. This defines a complex genus corresponding to the power series $Q(x) = 1 + x$. Indeed, the coefficient of degree n in the product $\prod_{i=1}^{n} Q(x_i)$ is precisely the n-th elementary symmetric function in the x_i.

Example 3.25. The *Todd genus* is the complex genus with power series

$$Q(x) = \frac{x}{1-e^{-x}} = 1 + \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \tfrac{1}{30240}x^6 + \dots$$

The coefficients are related to the *Bernoulli numbers.* We denote its value on a compact stably almost complex manifold M by $\mathrm{Td}(M)[M]$ or simply $\mathrm{Td}[M]$. We will show in Corollary 4.22 that for the complex projective spaces, we have $\mathrm{Td}[\mathbb{C}\mathrm{P}^n] = 1$ for all $n \geq 0$.

Example 3.26. Let R be a $\mathbb{Q}$-algebra. Any (oriented) genus $\varphi\colon \Omega_*^{\mathrm{SO}} \to R$ gives rise to a complex genus by precomposing with map $\Omega_*^{\mathrm{U}} \to \Omega_*^{\mathrm{SO}}$ induced by forgetting the complex structure. The characteristic series is then an even normalized power series.

Example 3.27. Let $R = \mathbb{Q}[y]$. The Hirzebruch χ_y-genus is the genus associated to the power series

$$\widetilde{Q}(x) = \frac{x(1+ye^{-x})}{1-e^{-x}}.$$

This interpolates between some of the above examples. For $y = -1$, we obtain the Euler characteristic. For $y = 0$, we recover the Todd genus, and $y = 1$ yields the signature viewed as a complex genus. However, $\widetilde{Q}(x)$ is not normalized. This means we need to multiply by $1/(1+y)$ such that $Q(x) = \widetilde{Q}(x)/(1+y)$ is a normalized power series. Then, we do not have the Euler characteristic for $y = -1$ any longer.

Instead of defining the notion of a complex *elliptic* genus by means of a logarithm, we outline the construction of complex genera with values in the ring of modular forms for the congruence subgroup $\Gamma_1(N)$ defined as

$$\Gamma_1(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mod N \right\}.$$

Note that $\Gamma_0(2) = \Gamma_1(2)$. Then, the construction works analogously by generalizing to primitive N-division points of a lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$. Just as in the case $N = 2$, we have the following bijection (Proposition A.15).

$$\Gamma_1(N)\backslash\mathcal{H} \longleftrightarrow \{(\mathbb{C}/L, P) \mid P \text{ is a primitive } N\text{-division point of } L\}/\text{iso}.$$

Also for elliptic genera of level N, there is a universal genus.

Definition 3.28. Let $N > 1$ and write $\zeta_N = \exp(2\pi i/N)$. The complex genus with characteristic series

$$Q_N(x) = x\frac{(1-e^{-x}\zeta_N)}{(1-e^{-x})(1-\zeta_N)}\prod_{n=1}^{\infty}\frac{(1-e^{-x}\zeta_N q^n)\left(1-e^{x}\zeta_N^{-1}q^n\right)(1-q^n)^2}{(1-e^{-x}q^n)(1-e^{x}q^n)(1-\zeta_N q^n)\left(1-\zeta_N^{-1}q^n\right)}$$

is called *universal elliptic genus of level* N and is denoted by φ_N.

In the case $N = 2$, the characteristic series $Q_N(x)$ reduces to the characteristic series we obtained in the previous section for the elliptic genus. Since $\Gamma_0(2) = \Gamma_1(2)$, this generalizes the construction. By following the steps for the case of general N, and by using Chern classes in place of Pontryagin classes, we obtain the following generalization of Proposition 3.21.

Proposition 3.29. *Let M be a an almost complex compact manifold of (real) dimension $2n$. Then, $\varphi_N(M)$ is a modular form of weight n for the congruence subgroup $\Gamma_1(N)$.*

Recall that to define modular forms for a congruence subgroup, we need holomorphicity at the cusps. While $\Gamma_0(2)$ only has the cusps 0 and ∞, for $N > 2$, the congruence subgroup $\Gamma_1(N)$ has more cusps. In the case $N = 2$, the two degenerate cases are geometrically important since they correspond to the L-genus and the $\widehat{A}$-genus, respectively. These degenerate cases correspond to the cusps of $\Gamma_0(2)$. For general N, one expects that the genera corresponding to the cusps of $\Gamma_1(N)$, also have geometric relevance. This is indeed the case as some values on the cusps are given by the normalized χ_y-genus for a suitable choice of y. For more details, we refer to Section 7.2 in [HBJ92].

4. Complex Genera and Formal Group Laws

In the previous chapter, we have defined elliptic genera as ring homomorphisms out of the bordism rings Ω_*^{SO} and Ω_*^{U} that come from elliptic curves with torsion points. Considering stably almost complex manifolds, we can construct elliptic genera of level N, that is, assigning modular forms for the congruence subgroup $\Gamma_1(N)$ to stably almost complex manifolds. In the special case $N = 2$, this reduces to the construction of elliptic genera for oriented manifolds.

In this chapter, we will take another perspective on complex genera relating them to *formal group laws*. For a ring R, a formal group law over R is a power series in two variables with coefficients in R that satisfies properties that resemble those of a group. While a priori they seem unrelated to complex genera, we will see that there is in fact in one-to-one correspondence. This is a consequence of Quillen's theorem which states that $\pi_*(MU)$ has a universal property for formal group laws.

After having established this correspondence, we will consider previous examples of elliptic and complex genera from the perspective of formal group laws. Finally, we introduce formal schemes and formal groups and describe how formal group laws can be thought of choosing coordinates and describing the structure of formal groups locally. An important example for this are provided by elliptic curves.

4.1. Formal group laws and the Lazard ring

Formal group laws

We start by introducing formal group laws and some of their properties. We follow [Str19] in our approach.

Definition 4.1. A *formal group law* (FGL) over a ring R is a power series $F(x, y) \in R[\![x, y]\!]$ satisfying the following properties

(i) $F(x, 0) = x \in R[\![x]\!]$,

(ii) $F(x,y) = F(y,x) \in R[\![x,y]\!]$,

(iii) $F(x,F(y,z)) = F(F(x,y),z) \in R[\![x,y,z]\!]$.

Example 4.2. The first examples for formal group laws are given by

(i) The additive formal group law: $F_a(x,y) = x + y$.

(ii) The multiplicative formal group law: $F_m(x,y) = x + y + xy$. More generally, $F_r(x,y) = x + y + rxy$ with $r \in R$ defines a formal group law.

(iii) The hyperbolic tangent formal group law:

$$F_{\tanh}(x,y) = \frac{x+y}{1+xy} = x + y - (x+y)xy + (x+y)x^2y^2 \mp \dots$$

This corresponds to adding parallel velocities in special relativity (when working in units with speed of light $c = 1$). It satisfies $F(\tanh(u), \tanh(v)) = \tanh(u+v)$.

All three examples are of the form $F(x,y) = x + y +$ (higher terms). Indeed, it follows immediately from the first two properties of a formal group law that every formal group law is of this form.

If, for a moment, we assume that x and y are elements of a set, then the properties defining a formal group law $F(x,y)$ can be interpreted as a neutrality, commutativity and an associativity relation respectively. If another property exists that represents the existence of inverses, then formal group laws can be thought of as defining an abelian group law without specifying the elements of the group. Indeed, we have the following inverse property.

Proposition 4.3. *Let F be a formal group law over R. Then there exists a unique power series $i(x) \in R[\![x]\!]$ with $i(0) = 0$ such that $F(x, i(x)) = 0$. The power series $i(x)$ is called the formal inverse.*

Proof. Write $F(x,y) = x + y + \sum_{i,j>0} a_{ij} x^i y^j$. We need to construct the power series $i(x) = \sum_{k>0} b_k x^k$. Let $b_1 = -1$ and define polynomials $i_k(x)$ of degree k inductively to approximate $i(x)$. Let $i_1(x) = b_1 x = -x$. Then $F(x, i_1(x)) = 0 \mod (x^2)$. Now, assume that for $i_k(x)$, we have $F(x, i_k(x)) = 0 \mod (x^{k+1})$. Then, there exists a unique element $b_{k+1} \in R$ such that $F(x, i_k(x)) = -b_{k+1}x^{k+1} \mod (x^{k+2})$. Define

$$i_{k+1}(x) = i_k(x) + b_{k+1}x^{k+1}.$$

We need to show that $F(x, i_{k+1}(x)) = 0 \mod (x^{k+2})$. Observe that for $i, j > 0$, we have

$$x^i i_{k+1}(x)^j = x^i \left[i_k(x) + b_{k+1}x^{k+1} \right]^j = x^i i_k(x)^j \mod (x^{k+2}).$$

With this, we obtain

$$
\begin{aligned}
F(x, i_{k+1}(x)) &= x + i_{k+1}(x) + \sum_{i,j>0} a_{ij} x^i i_{k+1}(x)^j && \mod (x^{k+2}) \\
&= -i_k(x) + i_{k+1}(x) + x + i_k(x) + \sum_{i,j>0} a_{ij} x^i i_k(x)^j && \mod (x^{k+2}) \\
&= b_{k+1} x^{k+1} + F(x, i_k(x)) && \mod (x^{k+2}) \\
&= 0 && \mod (x^{k+2}).
\end{aligned}
$$

Now let $i(x) = \sum_{k>0} b_k x^k$ such that $i(x) = i_k(x) \mod (x^k)$ for every k. Then, we obtain $F(x, i(x)) = 0$ since $F(x, i_k(x)) = 0 \mod (x^{k+1})$ for all k. □

Remark 4.4. A more elaborate way to think about formal group laws is as encoding the group law of a formal group in a neighbourhood around the identity. In this sense, a formal group law really encodes a group structure. We will come back to this in a later section.

Example 4.5. The previous examples have the following inverses.

(i) The additive formal group law has inverse $i_a(x) = -x$.

(ii) The multiplicative formal group law has inverse $i_m(x) = \frac{-x}{1+x} = -x + x^2 - x^3 \pm \dots$ and its generalization has inverse $i_r(x) = \frac{-x}{1+rx}$.

(iii) The hyperbolic tangent formal group law has inverse $i_{\tanh}(x) = -x$.

We now define homomorphisms and isomorphisms of formal group laws.

Definition 4.6. (i) An *homomorphism* of formal group laws $F_1(x, y)$ and $F_2(x, y)$ is a power series $g(x) \in R[\![x]\!]$ with $g(0) = 0$ such that

$$g(F_1(x, y)) = F_2(g(x), g(y)) \in R[\![x, y]\!].$$

If $g'(0) \in R$ is invertible, we say that g is an *isomorphism.* If further $g'(0) = 1$, it is called a *strict isomorphism.*

(ii) A strict isomorphism $g(x)$ between a formal group law $F(x, y)$ and the additive formal group law $F_a(x, y)$ is called *logarithm.* That is,

$$g(F(x, y)) = g(x) + g(y).$$

The logarithm $\log_\varphi(x)$ we defined for an elliptic genus φ is a special case of the logarithm for formal group laws defined above. To see this, we define the *Euler formal group law* $F_E(x,y)$ implicitly by

$$\int_0^{F_E(x,y)} \frac{dt}{\sqrt{r(t)}} = \int_0^x \frac{dt}{\sqrt{r(t)}} + \int_0^y \frac{dt}{\sqrt{r(t)}}$$

where $r(t) = 1 - 2\delta t^2 + \varepsilon t^4$ with $\delta, \varepsilon \in R$. It is easily verified that $F_E(x,y)$ is in fact a formal group law. With this definition, the logarithm $\log_\varphi$ of an elliptic genus φ satisfies

$$\log_\varphi(F_E(x,y)) = \log_\varphi(x) + \log_\varphi(y)$$

which shows that $\log_\varphi(x)$ is indeed a logarithm for the Euler formal group law $F_E(x,y)$. Elliptic integrals of this form were studied by Euler. Before him, Fagnano studied the special case $r(t) = 1 - x^4$ for which the integral computes the arc length of the so-called lemniscate. While these elliptic integrals cannot be computed using elementary functions, Euler derived a closed form for $F_E(x,y)$.

Theorem 4.7 (Euler 1761)**.** *The Euler formal group law F_E has the form*

$$F_E(x,y) = \frac{x\sqrt{r(y)} + y\sqrt{r(x)}}{1 - \varepsilon x^2 y^2}.$$

When defining the Euler formal group law, we are working over a $\mathbb{Q}$-algebra R to formally compute the integral. In particular, it is defined over $\mathbb{Q}[\delta,\varepsilon]$. However, by expanding the square-root, we obtain

$$\sqrt{1 - 2\delta x^2 + \varepsilon x^4} = \sum_{n=0}^{\infty} \frac{1}{2^{2n}(2n-1)} \binom{2n}{n} (2\delta x^2 - \varepsilon x^4)^n.$$

Since $2n-1$ divides the binomial coefficient $\binom{2n}{n}$, all coefficients of $F_E(x,y)$ are contained in

$$\mathbb{Z}[\tfrac{1}{2}][\delta,\varepsilon] \subseteq \mathbb{Q}[\delta,\varepsilon].$$

We will use this fact later when relating formal group laws with complex and elliptic genera.

The Euler formal group law comes from an elliptic curve (with 2-torsion point). More generally, any elliptic curve gives rise to formal group laws. Recall that elliptic curves have a group structure with neutral element given by the point at infinity which we denote by O. Then, we obtain a formal group law from the addition on the curve in a neighbourhood around O.

Example 4.8. Let E be an elliptic curve in general Weierstraß form

$$E\colon y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Choosing coordinates $z = -x/y$ and $w = -1/y$, the point O now lies at $(0,0)$ in the affine (z,w)-plane. The equation becomes

$$w = z^3 + a_1zw + a_2z^2w + a_3w^2 + a_4zw^2 + a_6w^3.$$

Then, one can derive from the addition of two points (z_1,w_1) and (z_2,w_2) on the elliptic curve, the formal group law

$$\begin{aligned} F(z_1,z_2) = z_1 + z_2 - a_1z_1z_2 - a_2(z_1^2z_2 + z_1z_2^2) \\ + (2a_3z_1^3z_2 + (a_1a_2 - 3a_3)z_1^2z_2^2 + 2a_3z_1z_2^3) + \dots \end{aligned}$$

over $\mathbb{Z}[a_1,a_2,a_3,a_4,a_6][\![z_1,z_2]\!]$. For more details see Section IV.1 in [Sil09]. See also [BB10] for formal group laws form elliptic curves and relations to elliptic genera.

We resume our discussion of general properties of formal group laws. Note that in general, logarithms for formal group laws over R need not exist. Over a $\mathbb{Q}$-algebra R, however, we have the following.

Proposition 4.9. *Let R be a $\mathbb{Q}$-algebra. Every formal group law over R is uniquely isomorphic to the additive formal group law. That is, every formal group law over R has a unique logarithm.*

Proof. Let $F(x,y)$ be a formal group law over a $\mathbb{Q}$-algebra R and write $F_2(x,y)$ for its formal partial derivative with respect to y. Then,

$$F_2(x,y) = 1 + x + \sum_{i,j>0} ja_{ij}x^iy^{j-1}$$

and hence, $F_2(t,0) = 1 + \dots \in R[\![t]\!]$ has an inverse element $1/F_2(t,0)$. Using formal integration over R, we can define a power series

$$g(x) = \int_0^x \frac{dt}{F_2(t,0)}$$

that satisfies $g(x) = x \mod (x^2)$. Next, we make use of the associativity property

$$F(F(x,y),z) = F(x,F(y,z)).$$

Taking the formal partial derivative with respect to the variable z on both sides and setting $z = 0$, we obtain

$$
\begin{aligned}
&& F_2(F(x,y),0) &= F_2(x,y)F_2(y,0) \\
\iff && \frac{1}{g'(F(x,y))} &= F_2(x,y)\frac{1}{g'(y)} \\
\iff && g'(y) &= F_2(x,y)g'(F(x,y)),
\end{aligned}
$$

where in the first line, we used the chain rule on the right-hand side, and applied the definition of g for the second line. Now, define a power series $h(x,y) = g(F(x,y)) - g(x) - g(y)$. By the above, we have that

$$h_2(x,y) = g'(F(x,y))F_2(x,y) - g'(y) = 0.$$

This means that $h(x,y) = \sum_{i\geq 0} d_i x^i$. However, $h(x,0) = g(x) - g(x) - g(0) = 0$ since $g(0) = 0$. We obtain that $g(F(x,y)) = g(x) + g(y)$ and g is a logarithm.
For uniqueness, consider another logarithm $\widetilde{g}$ and its inverse power series $\widetilde{g}^{-1}$. Then, define a power series $k(x) := g(\widetilde{g}^{-1}(x))$. We need to show that $k(x) = x$. Since $\widetilde{g}$ is a logarithm, its inverse satisfies

$$\widetilde{g}^{-1}(x+y) = F(\widetilde{g}^{-1}(x), \widetilde{g}^{-1}(y)),$$

and we have that $k(x+y) = k(x) + k(y)$. Write $k(x) = \sum_m a_m x^m$. Since we are working over a $\mathbb{Q}$-algebra, we can expand the terms $a_m(x+y)^m$ on the left-hand side with the binomial formula. Comparing with the right-hand side, we obtain that $k(x) = a_1 x$ for some $a_1 \in R^\times$. But since g and $\widetilde{g}$ are strict isomorphisms, we have that $a_1 = 1$ and hence, $k(x) = x$ and we obtain $g = \widetilde{g}$. □

Example 4.10. Let R be a $\mathbb{Q}$-algebra. As formal group laws over R, the previous examples have the following logarithms.

(i) The additive formal group law has logarithm $\log_a(x) = x$.

(ii) The multiplicative formal group law has logarithm $\log_m(x) = \sum_{n\geq 0} \frac{(-1)^n x^{n+1}}{n+1}$. Its generalization $F_r(x,y) = x + y + rxy$ has logarithm

$$\log_r(x) = r^{-1}\log(1+rx) = \sum_{n\geq 0} \frac{(-r)^n x^{n+1}}{n+1}.$$

(iii) The hyperbolic tangent formal group law has logarithm

$$\log_{\tanh}(x) = \tanh^{-1}(x) = \tfrac{1}{2}\log\left(\tfrac{1+x}{1-x}\right) = \sum_{n\geq 0} \frac{x^{2n+1}}{2n+1}.$$

Denote by $\mathrm{FGL}(R)$ be the set of formal group laws over R. Furthermore, let $\mathrm{IPS}(R)$ be the set of invertible power series over R, that is, power series $f(x) = rx + \dots$ with $r \in R^\times$. Consider the subset $\mathrm{IPS}_1(R) \subseteq \mathrm{IPS}(R)$ of power series with $f'(0) = 1$. Examples of such power series are logarithms. As a consequence of Proposition 4.9, we obtain that over a $\mathbb{Q}$-algebra R, every power series in IPS_1 is a logarithm.

Corollary 4.11. *Let R be a $\mathbb{Q}$-algebra. Taking the logarithm of a formal group law over R defines a bijection*

$$\phi\colon \mathrm{FGL}(R) \to \mathrm{IPS}_1(R).$$

Proof. Let F be a formal group law over R. Then by construction in Proposition 4.9,

$$\phi(F)(x) := \log_F(x) = \int_0^x \frac{dt}{F_2(t,0)}.$$

Define an inverse by

$$\phi^{-1}(g)(x,y) := g^{-1}(g(x) + g(y)).$$

By the construction in the Proposition, we know that

$$\phi(F)(F(x,y)) = \log_F(F(x,y)) = \log_F(x) + \log_F(y) = \phi(F)(x) + \phi(F)(y),$$

and hence, $\phi^{-1}\phi(F) = F \in \mathrm{FGL}(R)$. Finally, by uniqueness of the logarithm, we have $\phi(\phi^{-1}(g)) = g \in \mathrm{IPS}_1(R)$. ☐

Remark 4.12. Despite the fact that formal group laws over $\mathbb{Q}$-algebras can be identified with invertible power series with linear term x, this does not mean that such formal group laws are uninteresting. In fact, as the example of the Euler formal group law shows, it is the logarithm which we used in the previous chapter to define an elliptic genus with values in a $\mathbb{Q}$-algebra. In the next section, we will see how the two notions of logarithm are related.

The Lazard ring

We will now see that there is a universal formal group law over a universal ring from which every formal group law over a given ring R arises. We start with the following observation.

Proposition 4.13. *Let $f\colon R \to S$ be a ring homomorphism and let*

$$F_R(x,y) = x + y + \sum_{i,j>0} a_{ij}x^iy^j \in R[\![x,y]\!].$$

be a formal group law over R. Then, we obtain a formal group law over S by pushing forward the coefficients

$$F_S(x,y) = x + y + \sum_{i,j>0} f(a_{ij})x^iy^j \in S[\![x,y]\!].$$

Proof. The power series $F_S(x,y)$ inherits all properties from the formal group law $F_R(x,y)$. □

Proposition 4.14. *There exists a ring R_{univ} and a formal group law over R_{univ} such that there is a one-to-one correspondence*

$$\{\textit{Ring homomorphisms } f\colon R_{\text{univ}} \to R\} \longleftrightarrow \mathrm{FGL}(R)$$

given by pushing forward the universal formal group law over L with respect to f.

Proof. We give a construction of R_{univ}. Let $\mathbb{Z}[\{a_{ij}\}]$ be the ring generated by a_{ij} with $i,j \in \mathbb{N}_0$. Consider a power series

$$F(x,y) = \sum_{i,j\geq 0} a_{ij}x^iy^j \in \mathbb{Z}[\{a_{ij}\}][\![x,y]\!].$$

Making $F(x,y)$ into a formal group law gives equations for the a_{ij}. For instance, $a_{ij} = a_{ji}$ by symmetry, and $a_{00} = 0$, $a_{01} = a_{10} = 1$ by neutrality. Also associativity yields equations for a_{ij}. Now define I to be the ideal in $\mathbb{Z}[\{a_{ij}\}]$ generated by all the equations, and let $R_{\text{univ}} = \mathbb{Z}[\{a_{ij}\}]/I$. By construction, $F(x,y)$ is a formal group law over R_{univ}. Now let R be a ring and consider a formal group law $G(x,y) = \sum_{i,j\geq 0} b_{ij}x^iy^j$ over R. Define a ring homomorphism $f'\colon \mathbb{Z}[\{a_{ij}\}] \to R$ by $f'(a_{ij}) = b_{ij}$. Since b_{ij} satisfy the equations generating I, we obtain $I \subseteq \ker f'$, and hence we obtain a ring homomorphism $f\colon R_{\text{univ}} \to R$ sending $F(x,y)$ to $G(x,y)$. Hence, we have shown surjectivity. Injectivity follows since the set $\pi(\{a_{ij}\})$ generates R_{univ}, where $\pi\colon \mathbb{Z}[\{a_{ij}\}] \to R_{\text{univ}}$ is the projection map. □

The construction of the ring R_{univ} is not very explicit. Lazard's theorem states that the universal ring has the structure of a polynomial ring over $\mathbb{Z}$ in countably many generators.

Theorem 4.15 (Lazard)**.** *There is an isomorphism*

$$R_{\text{univ}} \cong L := \mathbb{Z}[b_2, b_4, \dots]$$

with $\deg(b_i) = i$*. The ring* L *is called the* Lazard ring*.*

Proof. For a proof, we refer to [Ada74]. □

Remark 4.16. The Lazard ring can be naturally equipped with a grading $\deg b_i = i$. Then, L has the universal property for graded formal group laws.

4.2. MU and formal group laws

Aside from the example of the Euler formal group law, our discussion so far has not been related to elliptic and complex genera. In this section, we will see that formal group laws and complex genera are in one-to-one correspondence. This provides us with an alternative perspective on complex genera and enables us to use the theory of formal group laws to study bordism invariants.

Quillen's theorem

We start by sketching a formal group law over $\pi_*(MU)$. A *ring spectrum* is a spectrum E equipped with a multiplication map $E \wedge E \to E$ and a unit map $\mathbb{S} \to E$ that satisfy further properties corresponding to ring axioms up to homotopy. Here, $\mathbb{S}$ is the sphere spectrum. The Thom spectra MO, MSO and MU are examples for such ring spectra with multiplication induced by the smash product. A ring spectrum E with a choice of map of ring spectra $MU \to E$ is called *complex-oriented.* Recall that spectra represent cohomology theories. We call cohomology theories complex-oriented if they are represented by complex-oriented ring spectra. Now, let E^* be a complex-oriented cohomology theory. Then one can compute the cohomology of the infinite complex projective space to be

$$E^*(\mathbb{C}\mathrm{P}^\infty) \cong E^*[\![x]\!],$$

where E^* is the coefficient ring $E^* = E^*(\{*\})$. More generally,

$$E^*((\mathbb{C}\mathrm{P}^\infty)^{\times n}) \cong E^*[\![x_1, \dots x_n]\!].$$

Recall that $\mathbb{C}\mathrm{P}^\infty$ classifies complex line bundles. That is, a line bundle over a space X, yields a map $X \to \mathbb{C}\mathrm{P}^\infty$. The tensor product of complex line bundles induces a multiplication map $m\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ making $\mathbb{C}\mathrm{P}^\infty$ into a homotopy associative and homotopy commutative H-space[1]. On a complex-oriented cohomology theory, this induces a map

$$\mu\colon E^*[\![x]\!] \cong E^*(\mathbb{C}\mathrm{P}^\infty) \to E^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong E^*[\![x_1, x_2]\!].$$

Define a power series $F_\mu(x_1, x_2) := \mu(x)$ where x is the generator of $E^*[\![x]\!]$. By the properties of $\mathbb{C}\mathrm{P}^\infty$ as an H-space, the power series $F_\mu(x_1, x_2)$ defines a formal group law over E^*. As graded rings $\pi_*(E) \cong E^{-*}$, and hence by reversing the grading, we obtain a formal group law over $\pi_*(E)$. Now, taking the complex-oriented cohomology theory represented by the spectrum MU, we obtain a formal group law over $\pi_*(MU)$. It is astonishing that this formal group law is the universal one. This is Quillen's theorem.

Theorem 4.17 (Quillen [Qui69])**.** *The formal group law over $\pi_*(MU)$ is the universal formal group law. That is, the ring homomorphism $L \to \pi_*(MU)$ classifying the formal group law over $\pi_*(MU)$ is an isomorphism.*

Proof. We refer to Theorem 8.2 in Part II of [Ada74]. □

Remark 4.18. Note that $\pi_*(MU)$ and L are graded rings. The above isomorphism is actually an isomorphism of graded rings.

We obtain the following important consequence of Quillen's theorem.

Corollary 4.19. *There is a one-to-one correspondence*

$$\big\{\textit{Complex genera } \varphi\colon \Omega^{\mathrm{U}}_* \to R\big\} \longleftrightarrow \mathrm{FGL}(R).$$

Proof. Recall that by the Pontryagin-Thom theorem, we have $\Omega^{\mathrm{U}}_* \cong \pi_*(MU)$. By Quillen's theorem, we have $\pi_*(MU) \cong L$. Finally, using Lazard's theorem, we obtain the one-to-one correspondence. □

Theorem 4.20 (Mischenko)**.** *The logarithm of the (universal) formal group law on $\pi_*(MU)$ is given by*

$$\log_{MU}(x) = \sum_{n=0}^{\infty} \frac{[\mathbb{C}\mathrm{P}^n]}{n+1} x^{n+1}.$$

[1] An H-space consists of a topological space X with basepoint e, and a multiplication map $m\colon X \times X \to X$ such that $m(e,e) = e$ and the maps $x \mapsto m(x,e)$ and $x \mapsto m(e,x)$ are homotopic to the identity. See for example Section 3.C of [Hat02] for more details.

Proof. For a proof, we refer to Corollary 9.2 in Part II of [Ada74] and Theorem 2.54 in [Mei22]. □

Recall that we have defined the logarithm of a formal group law and the logarithm of a complex genus. For the Euler formal group law, we have already seen that the two notions of logarithm coincide. More generally, we have the following consequence of Mischenko's theorem.

Proposition 4.21. *Let R be a $\mathbb{Q}$-algebra and let $\varphi\colon \Omega_*^{\mathrm{U}} \to R$ be a complex genus. The logarithm of the formal group law associated to φ and the logarithm obtained from its characteristic series are the same power series. It satisfies*

$$\log_\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi([\mathbb{C}\mathrm{P}^n])}{n+1} x^{n+1}.$$

Proof. Let R and S be $\mathbb{Q}$-algebras and let $\psi\colon R \to S$ be a ring homomorphism. Consider the following diagram.

$$\begin{array}{ccc} \mathrm{FGL}(R) & \longrightarrow & \mathrm{FGL}(S) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \mathrm{IPS}_1(R) & \longrightarrow & \mathrm{IPS}_1(S) \end{array}$$

The vertical maps are the bijections from Corollary 4.11 given by taking the logarithm. The top horizontal map is induced by ψ by pushing forward the coefficients (Proposition 4.13). The bottom horizontal map is also obtained from ψ. Since ϕ is defined in terms of the coefficients of the formal group law, the diagram commutes. This shows that we can push forward coefficients of the logarithm. Applying this to the logarithm of the universal formal group law over $\pi_*(MU)$, we obtain the above formula for the logarithm of the formal group law associated to φ.
Recall that for oriented genera, we have seen in Proposition 3.9 that the logarithm $\log_\varphi$ has the above form, but with $\varphi[\mathbb{C}\mathrm{P}^{2n+1}] = 0$. The proof generalizes to complex genera when starting with its characteristic power series

$$Q(x) = \frac{x}{f(x)} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

and using Chern classes of $\mathbb{C}\mathrm{P}^n$ rather than Pontryagin classes. Let $g(x)$ be the inverse power series to $f(x)$. Then,

$$g(x) = \sum_{n=0}^{\infty} \frac{\varphi([\mathbb{C}\mathrm{P}^n])}{n+1} x^{n+1}.$$

Hence, we obtain that the logarithms are indeed equal. □

Complex genera from formal group laws

We now consider examples of complex genera and their corresponding formal group laws. We start with the Todd genus.

Corollary 4.22. *The genus corresponding to* $F(x, y) = x + y - xy$ *is the Todd genus. It has values* $\mathrm{Td}[\mathbb{C}\mathrm{P}^n] = 1$ *for all* $n \geq 0$*.*

Proof. Recall from Example 3.25 that the Todd genus is the complex genus defined by the characteristic series $Q(x) = x/f(x) = \frac{x}{1-e^{-x}}$. Inverting $f(x)$, we obtain the power series

$$g(x) = -\log(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

using the Taylor series expansion of the natural logarithm. Hence, $g(x)$ coincides with the logarithm of the generalized multiplicative formal group law with $r = -1$ as considered in Example 4.10. Furthermore, by Proposition 4.21, we obtain for the values of the Todd genus $\mathrm{Td}[\mathbb{C}\mathrm{P}^n] = 1$ for all $n \geq 0$. □

Example 4.23. The complex genus φ_a corresponding to the additive formal group law has values $\varphi_a([\mathbb{C}\mathrm{P}^n]) = 0$ for all $n \geq 1$ since its logarithm is given by $\log_a(x) = x$.

Recall that any oriented genus gives rise to a complex genus by precomposition with the homomorphism $\Omega_*^{\mathrm{U}} \to \Omega_*^{\mathrm{SO}}$ induced by forgetting the complex structure.

Example 4.24. The L-genus when viewed as a complex genus $\Omega_*^{\mathrm{U}} \to R$ corresponds to the hyperbolic tangent formal group law. Recall that the L-genus has characteristic power series $Q(x) = x/\tanh(x)$. Its logarithm is therefore given by

$$\log_L(x) = \tanh^{-1}(x) = \log_{\tanh}(x).$$

Example 4.25. The Euler formal group law

$$F_E(x, y) = \frac{x\sqrt{r(y)} + y\sqrt{r(x)}}{1 - \varepsilon x^2 y^2}$$

corresponds to the universal elliptic genus $\varphi\colon \Omega_*^{\mathrm{SO}} \to \mathbb{Q}[\delta, \varepsilon]$ when viewed as a complex genus. Indeed, we have seen in the previous section that the logarithms coincide. Choosing values of δ and ε, we obtain the formal group laws for the corresponding special cases. More generally, explicit calculations of the formal group law for the universal elliptic genus of level N can be found in [Bun19].

Using the theory of formal group laws, we can now refine the statement about the universal elliptic genus taking values in modular forms for $\Gamma_0(2)$. For this, we first need the following lemma.

Lemma 4.26. *The forgetful homomorphism modulo torsion* $\Omega_*^{\mathrm{U}} \to \Omega_*^{\mathrm{SO}}/\mathrm{torsion}$ *is surjective.*

Proof. We refer to [Sto68, p.120]. □

Proposition 4.27. *Let R be a $\mathbb{Q}$-algebra. Every elliptic genus $\Omega_*^{\mathrm{SO}} \to R$ factors as*

$$\Omega_*^{\mathrm{SO}} \to \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon] \to R$$

where the first map is the universal elliptic genus and the second map is given by choosing values for δ and ε.

Proof. We have seen that every elliptic genus factors over $\mathbb{Q}[\delta, \varepsilon]$. We will show that the image of the universal elliptic genus is contained in $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$. Precomposed with $\Omega_*^{\mathrm{U}} \to \Omega_*^{\mathrm{SO}}$, the universal elliptic genus corresponds precisely to the Euler formal group law. Recall that the Euler formal group law is defined over the ring $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$. Hence, the image of Ω_*^{U} is generated by the coefficients of the Euler formal group law and lies in the subring $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$. By Lemma 4.26 and since $\mathbb{Q}[\delta, \varepsilon]$ is torsion-free, the image of Ω_*^{U} and Ω_*^{SO} in $\mathbb{Q}[\delta, \varepsilon]$ coincide. Hence, the image of the universal elliptic genus $\varphi\colon \Omega_*^{\mathrm{SO}} \to \mathbb{Q}[\delta, \varepsilon]$ is contained in the ring $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$. □

Remark 4.28. The ring $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$ is isomorphic to the ring of modular forms for $\Gamma_0(2)$ with coefficients of their q-expansion in $\mathbb{Z}[\frac{1}{2}]$. Hence, Proposition 4.27 shows that modular forms assigned to compact oriented manifolds by the universal elliptic genus have coefficients in $\mathbb{Z}[\frac{1}{2}]$.

4.3. Formal schemes and formal groups

In this section, we will introduce formal schemes and formal groups to understand formal group laws from a more geometric and coordinate-free perspective. That is, by choosing coordinates for a formal group, we obtain a formal group law. For instance, we will see how elliptic curves can be made into a formal group and give rise to formal group laws. Furthermore, this perspective will be necessary to generalize the notion of formal group laws to the equivariant setting in the next chapter.

Formal schemes

In algebraic geometry, schemes are ringed spaces that are locally isomorphic to an affine scheme. An affine scheme is a space X that is isomorphic to

$$\mathrm{Spec}(R) = \{\text{prime ideals of } R\}$$

for some ring R equipped with the Zariski topology, together with its sheaf of regular functions $\mathcal{O}_X$.

Our approach to schemes is different from this. We will consider functors

$$X\colon \mathsf{Ring} \to \mathsf{Set}.$$

This is known as the functor-of-points approach in algebraic geometry. See for example Chapter VI in [EH00]. Our treatment mainly follows [Str19] and [Str99].

Definition 4.29. An *affine scheme* is a functor $X\colon \mathsf{Ring} \to \mathsf{Set}$ that is *covariantly representable*. That is, there exists a ring R such that $X(S)$ is isomorphic to $\mathrm{Hom}_{\mathsf{Ring}}(R, S)$ for every ring S. We write $\mathrm{Spec}(R)$ for the functor $S \mapsto \mathrm{Hom}_{\mathsf{Ring}}(R, S)$. A *morphism of schemes* is a natural transformation.

Remark 4.30. It turns out that our category of affine schemes is equivalent to the category of affine schemes in algebraic geometry. Hence, it is appropriate to denote the functor $S \mapsto \mathrm{Hom}_{\mathsf{Ring}}(R, S)$ by $\mathrm{Spec}(R)$.

Remark 4.31. Strictly speaking, we are interested in general schemes since elliptic curves are not affine. General schemes can also be constructed as functors using *Zariski sheaves*. We refer to Section VI.2 of [EH00] and Section 3 of [Str99] for more details. For simplicity, we will restrict ourselves to affine schemes. When speaking about elliptic curves, we will work with an affine patch that contains the point $\mathcal{O}$. This suffices since we will be choosing coordinates around $\mathcal{O}$.

Example 4.32.

(i) Denote by $\mathbb{A}^1$ the forgetful functor $\mathsf{Ring} \to \mathsf{Set}$. This is a scheme since it is isomorphic to $\mathrm{Spec}(\mathbb{Z}[t])$. Indeed, ring homomorphisms $\varphi\colon \mathbb{Z}[t] \to S$ are fully determined by their image on t: by the homomorphism property of φ, the image of each element $a_0 + a_1 t + a_2 t^2 + \cdots \in \mathbb{Z}[t]$ is determined by the element $s = \varphi(t) \in S$. This defines a bijection $\mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Z}[t], S)$ with S as a set.

(ii) Consider functor $\mathbb{A}^n\colon \mathsf{Ring} \to \mathsf{Set}$ sending a ring S to $\mathbb{A}^n(S) := S^n$, the n-fold Cartesian product as a set. By iterating the argument above, this is a scheme as it is isomorphic to the functor $\mathrm{Spec}(\mathbb{Z}[t_1, \ldots, t_n])$.

(iii) The functor $\mathbb{G}_m^1\colon \mathsf{Ring} \to \mathsf{Set}$ defined by $\mathbb{G}_m^1(S) := S^\times$ is a scheme. It is represented by $\mathbb{Z}[t^{\pm 1}]$. Indeed, ring homomorphisms $\varphi\colon \mathbb{Z}[t^{\pm 1}] \to S$ are determined by the image of t. Since

$$1 = \varphi(1) = \varphi(t \cdot t^{-1}) = \varphi(t)\varphi(t^{-1}),$$

we have that $\varphi(t^{-1})$ is the inverse of $\varphi(t)$ in S. Hence, the only possible values for $\varphi(t)$ are in $S^\times$ and these are precisely all.

(iv) Similarly, we can also define $\mathbb{G}_m^n$ to be the functor $\mathsf{Ring} \to \mathsf{Set}$ sending S to the n-fold copy of $S^\times$. It is represented by $\mathbb{Z}[t_1^{\pm 1}, \dots t_n^{\pm 1}]$.

(v) Define the functor $\mathrm{FGL}\colon \mathsf{Ring} \to \mathsf{Set}$ by $\mathrm{FGL}(R) = \{\text{Formal group laws over } R\}$. This is a scheme since it is represented by the Lazard ring L. Recall that L is the universal ring for formal group laws and that for any ring R, any formal group law over R can be obtained by a map $L \to R$. This is precisely the statement that $\mathrm{Hom}_{\mathsf{Ring}}(L, R) = \mathrm{FGL}(R)$ for every ring R.

Definition 4.33. Let $X\colon \mathsf{Ring} \to \mathsf{Set}$ be a functor. Denote by $\mathcal{O}_X$ the set[2] of natural transformations $X \to \mathbb{A}^1$. That is, an element $f \in \mathcal{O}_X$ defines a map $f_R\colon X(R) \to R$ for each ring such that for a ring homomorphism $\phi\colon R \to S$ the following diagram commutes.

$$\begin{array}{ccc} X(R) & \xrightarrow{f_R} & R \\ {\scriptstyle X(\phi)}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ X(S) & \xrightarrow[f_S]{} & S \end{array}$$

This can be made into a ring as follows. Let $f, g \in \mathcal{O}_X$ and define $(f+g)(x) = f(x)+g(x)$ and $(fg)(x) = f(x)g(x)$ for $x \in X(R)$. We call $\mathcal{O}_X$ the *ring of functions* on X.

Note that by the Yoneda lemma, there is an isomorphism $\mathcal{O}_{\mathrm{Spec}(R)} \cong R$. Hence, an affine scheme X is isomorphic to $\mathrm{Spec}(\mathcal{O}_X)$.

Example 4.34. The ring of functions of $\mathbb{A}^1$ is $\mathbb{Z}[x]$.

Remark 4.35. The category of affine schemes Sch is equivalent to the opposite category of rings $\mathsf{Ring}^{\mathrm{op}}$.

We would like to consider functors over a base scheme as follows. Let $X = \mathrm{Spec}(R)$ be an affine scheme. Let $Y\colon \mathsf{Ring} \to \mathsf{Set}$ be a functor together with a map $\pi\colon Y \to X$. Then, X is called the *base scheme* to Y. The ring $\mathbb{Z}$ is an initial object in the category

[2] In general, this need not be a set, but a class. However, in our situations, $\mathcal{O}_X$ turns out to always be a set.

of rings. That is, for every ring S there is a canonical ring homomorphism $\mathbb{Z} \to S$, or equivalently, every ring is a $\mathbb{Z}$-algebra. Since the category Sch is equivalent to $\mathsf{Ring}^{\mathrm{op}}$, the affine scheme $\mathrm{Spec}(\mathbb{Z})$ is a final object. Hence, we can consider any affine scheme, that is representable functor $\mathsf{Ring} \to \mathsf{Set}$ as a functor over $\mathrm{Spec}(\mathbb{Z})$.

For the ring of functions, we obtain the following. Let Y be an affine scheme over another affine scheme X. The ring of functions $\mathcal{O}_Y$ is a $\mathcal{O}_X$ algebra. Indeed, the map $Y \to X$ yields a ring homomorphism $\mathcal{O}_X \to \mathcal{O}_Y$.

Example 4.36. The ring of functions of $\mathbb{A}^1$ over X is $\mathcal{O}_X[x]$.

Next, we introduce *formal schemes*. For us, a formal scheme is a functor $Y\colon \mathsf{Ring} \to \mathsf{Set}$ that is isomorphic to the *formal spectrum* $\mathrm{Spf}(R)$ of a ring R with *linear topology*.

Definition 4.37. Let R be a ring. A *linear topology* on R is defined by a basis consisting of open ideals and their translates, that is, the cosets under addition. This makes R into a topological ring. Morphisms between linearly topologized rings are continuous ring homomorphisms.

Note that the open ideals forms a directed set, that is, a set with a preorder such that every pair of elements of the set has an upper bound. We are mainly interested in the following special case.

Example 4.38. Let R be a ring and let $I \subseteq R$ be an ideal. A basis consisting of the ideals I^n for $n \in \mathbb{N}$ and its translates defines a linear topology on R. This is called the *I-adic topology*.

Definition 4.39. Let R be a linearly topologized ring. Define $\mathrm{Spf}(R)$ as the functor

$$\begin{aligned} \mathrm{Spf}(R)\colon \mathsf{Ring} &\to \mathsf{Set} \\ S &\mapsto \mathrm{Spf}(R)(S) = \mathrm{Hom}_{\mathrm{cont.}}(R, S) \end{aligned}$$

where $\mathrm{Hom}_{\mathrm{cont.}}(R, S)$ is the set of continuous ring homomorphisms, and we consider S to have the discrete topology.

Definition 4.40. A *formal scheme* is a functor $Y\colon \mathsf{Ring} \to \mathsf{Set}$ that is isomorphic to $\mathrm{Spf}(R)$ for a linearly topologized ring R.

Remark 4.41. Note that there are many different conventions for the definition of a formal scheme. For a more general categorical definition of formal schemes as small filtered colimits of affine schemes, see Definition 4.1 in [Str99]. Our definition of formal scheme is called *solid formal scheme* in [Str99].

The first and most important example of a formal scheme is given by the following. Define a functor $\widehat{\mathbb{A}}^1 \colon \mathsf{Ring} \to \mathsf{Set}$ by $\widehat{\mathbb{A}}^1(R) := \mathrm{Nil}(R)$, where

$$\mathrm{Nil}(R) = \{r \in R \mid r^n = 0 \text{ for some } n \in \mathbb{N}\} \subseteq R$$

is the set of the nilpotent elements of the ring R. More generally, we define a functor by $\widehat{\mathbb{A}}^n(R) := \mathrm{Nil}(R)^n$.

Proposition 4.42. *The functor* $\widehat{\mathbb{A}}^n$ *is a formal scheme isomorphic to* $\mathrm{Spf}(\mathbb{Z}[\![x_1 \ldots, x_n]\!])$.

Before we prove this, we will need the *completion* of a ring.

Definition 4.43. Let R be a linearly topologized ring. Taking the inverse limit

$$\widehat{R} = \lim_{\leftarrow I} R/I$$

defines a ring, where I runs over the set of open ideals. This is called the *completion* of the ring R.

Example 4.44. Let $R = \mathbb{Z}[x]$ and consider the ideal $I = (x)$ defining the linear topology called the *I-adic topology.* The completion of R is

$$\widehat{R} = \lim_{\leftarrow n} \mathbb{Z}[x]/(x^n) = \mathbb{Z}[\![x]\!].$$

Proposition 4.45. *Let R be a linearly topologized ring. Then the following holds.*

(i) $\widehat{R} \cong \widehat{\widehat{R}}$. *If* $R \cong \widehat{R}$, *we call* R complete.

(ii) $\mathrm{Spf}(R) \cong \mathrm{Spf}(\widehat{R})$.

Proof. For (i), consider the composite of the completion map and the map obtained from the inverse limit for an open ideal $I \subseteq R$.

$$R \to \widehat{R} \to R/I$$

This is surjective, and hence also $\widehat{R} \to R/I$ is. That means, there exists an ideal $J \subseteq \widehat{R}$ such that $R/I \cong \widehat{R}/J$. The ideals J with their translates form a linear topology on $\widehat{R}$. Taking the inverse limit, we obtain $\widehat{R} \cong \widehat{\widehat{R}}$.

For (ii), let $\varphi \in \mathrm{Hom}_{\mathrm{cont.}}(\widehat{R}, S)$. By precomposition of φ with the completion map $R \to \widehat{R}$, we obtain a continuous homomorphism $R \to S$. Now, let $\varphi \in \mathrm{Hom}_{\mathrm{cont.}}(R, S)$. Consider $\ker \varphi = \varphi^{-1}(\{0\})$. Since S has the discrete topology, we obtain that $I = \ker(\varphi)$ is an open ideal. Hence, φ factors as the continuous homomorphisms $R \to R/I \to S$. Precomposing $R/I \to S$ with the map $\widehat{R} \to R/I$, we obtain a continuous homomorphism in $\mathrm{Hom}_{\mathrm{cont.}}(\widehat{R}, S)$. Hence $\mathrm{Spf}(R)(S) \cong \mathrm{Spf}(\widehat{R})(S)$ for all rings S. □

Proof of Proposition 4.42. We show that $\widehat{\mathbb{A}}^1$ is isomorphic to $\mathrm{Spf}(\mathbb{Z}[x])$ where $\mathbb{Z}[x]$ has the I-adic topology with $I = (x)$. Let S be a ring and consider the set $\mathrm{Hom}_{\mathrm{cont.}}(\mathbb{Z}[x], S)$. This is a subset of $\mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Z}[x], S)$ which is in bijection with S as we have seen before. Hence, an element $\varphi \in \mathrm{Hom}_{\mathrm{cont.}}(\mathbb{Z}[x], S)$ corresponds to $s = \varphi(x) \in S$. Now, since S is endowed with the discrete topology, in particular, the preimage of $0 \in S$ is an open ideal $\ker \varphi$. However the open ideals are simply given by $(x)^n$. This means that $\varphi(x)^n = s^n = 0$ for some n, and hence, $\mathrm{Spf}(\mathbb{Z}[x])(S) \cong \mathrm{Nil}(S) = \widehat{\mathbb{A}}^1(S)$. Now, by Proposition 4.45, $\widehat{\mathbb{A}}^1 \cong \mathrm{Spf}(\mathbb{Z}[x]) \cong \mathrm{Spf}(\mathbb{Z}[\![x]\!])$. The argument generalizes to $n \geq 1$ and we obtain that $\widehat{\mathbb{A}}^n$ is a formal scheme isomorphic to the functor $\mathrm{Spf}(\mathbb{Z}[\![x_1, \ldots, x_n]\!])$. □

Recall that $\mathbb{A}^1 \cong \mathrm{Spec}(\mathbb{Z}[x])$. In this sense,

$$\widehat{\mathbb{A}}^1 \cong \mathrm{Spf}(\mathbb{Z}[\![x]\!]) \cong \mathrm{Spf}(\mathbb{Z}[x])$$

is the formal completion of $\mathbb{A}^1$. Similarly, we can view the formal scheme $\widehat{\mathbb{A}}^n$ as the completion of $\mathbb{A}^n$. A more geometric interpretation is as follows. The ideal $I = (x_1, \ldots, x_n)$ corresponds to the origin of the affine space $\mathbb{A}^n$. Considering its completion at I describes the space in a *formal neighbourhood* around the origin.

Remark 4.46. All of the above also can be defined relatively for functors Y over an affine scheme X. For this, we keep track of a map of functors $Y \to X$.

To define formal groups and ultimately relate them to formal group laws, we define *smooth* formal schemes.

Definition 4.47. Let X be an affine scheme.

(i) Let $Y \colon \mathsf{Ring} \to \mathsf{Set}$ be a functor with a map $\pi \colon Y \to X$. A *formal system of coordinates* on Y is given by maps $x_1, \ldots, x_n \colon Y \to \widehat{\mathbb{A}}^1$ such that the induced map

$$\begin{aligned} Y &\to \widehat{\mathbb{A}}^n \times X \\ y &\mapsto (x_1(y), \ldots, x_n(y), \pi(y)) \end{aligned}$$

is an isomorphism of schemes.

(ii) A formal scheme over X that admits a system of formal coordinates is called an *n-dimensional smooth formal scheme* over X.

Example 4.48. The functor $\widehat{\mathbb{A}}^n$ with coordinate maps given by the projections is an n-dimensional smooth formal scheme over $\operatorname{Spec}(\mathbb{Z})$.

The next example resembles the local description of a smooth curve in a formal neighbourhood. Let $X = \operatorname{Spec}(R)$ be an affine scheme. An element $u \in X(S)$ defines a ring homomorphism $u\colon R \to S$. For power series f over R, we let $u(f)$ be the power series over S obtained by pushforward of the coefficients.

Proposition 4.49. *Let $f(x,y) \in R[\![x,y]\!]$ be a power series over a ring R. Define a functor $Y\colon \mathsf{Ring} \to \mathsf{Set}$ by*

$$Y(S) = \{(x,y,u) \in \widehat{\mathbb{A}}^2(S) \times X(S) \mid (u(f))(x,y) = 0\}.$$

If $f(0,0) = 0$ and $(\partial_2 f)(0,0) \in S^\times$, then

$$\begin{aligned} Y &\to \widehat{\mathbb{A}}^1 \times X \\ (x,y,u) &\mapsto (x,u) \end{aligned}$$

is an isomorphism. Hence, Y is a 1-dimensional formal scheme over X.

Proof. For the proof, we refer to Proposition 5.6 in [Str19]. □

Example 4.50. Formal completions of elliptic curves are examples of formal schemes. Consider the polynomial

$$g(x,y,z) = y^2z + a_1xyz + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2 - a_6z^3$$

with $a_i \in R$. This defines an elliptic curve E over $\mathbb{C}$ (if the discriminant is non-vanishing). Define the functor $\widehat{E}$ given by

$$\widehat{E}(S) = \{(x,z,u) \in \widehat{\mathbb{A}}^2(S) \times X(S) \mid (u(g))(x,1,z) = 0\}.$$

Recall from Example 4.8 that an elliptic curve gives rise to a formal group law. In particular, we chose coordinates such that the point $\mathcal{O}$ lies at the origin. This is precisely what we have done here with taking $f(x,z) = g(x,1,z)$. We will see that this is an example of how formal groups with coordinates around zero give rise to formal group laws.

Formal groups

Definition 4.51. Let X be an affine scheme. A *(1-dimensional commutative) formal group* over X is a 1-dimensional smooth formal scheme $\mathbb{G}$ over X such that $\pi^{-1}(x)$ has the structure of an abelian group for every $x \in X(R)$ and every ring R, and this structure depends naturally on R.

In the following, we will refer to a 1-dimensional commutative formal group simply as formal group. Morphisms of formal groups are given by morphisms of formal schemes that are compatible with the group structure.

Example 4.52. (i) The *additive formal group* $\widehat{\mathbb{G}}_a$ is the formal scheme $\widehat{\mathbb{A}}^1$ with the usual addition on $\widehat{\mathbb{A}}^1(R) = \mathrm{Nil}(R)$ for each ring R. This defines a formal group over $\mathrm{Spec}(\mathbb{Z})$.

(ii) Define a functor $\widehat{\mathbb{G}}_m \colon \mathsf{Ring} \to \mathsf{Set}$ by

$$\widehat{\mathbb{G}}_m(R) = \{a \in R \mid a = 1 \mod \mathrm{Nil}(R)\}.$$

This is a formal group over $\mathrm{Spec}(\mathbb{Z})$ called the *multiplicative formal group*. Indeed, defining a coordinate by $x(a) = 1 - a$, we obtain an isomorphism $\widehat{\mathbb{G}}_m \cong \widehat{\mathbb{A}}^1$.

The above examples generalize to formal groups over a base scheme X.

More precisely, we can define a (commutative) formal group over X as an abelian group object in the category of formal schemes over X. That is, a formal scheme $\mathbb{G}$ together with maps

$$\mu \colon \mathbb{G} \times_X \mathbb{G} \to \mathbb{G}, \qquad e \colon X \to \mathbb{G}, \qquad i \colon \mathbb{G} \to \mathbb{G}$$

representing multiplication, identity element and inverse map. These maps are required to satisfy compatibility conditions representing associativity, commutativity and inverse property. In terms of rings of functions, we obtain maps dual to the above. In particular, the multiplication map induces a comultiplication map on the ring of functions S

$$\Delta \colon S \to S \mathbin{\widehat{\otimes}_R} S$$

where $X = \mathrm{Spec}(R)$ and $\widehat{\otimes}_R$ is the *completed tensor product* over the base ring R. The completed tensor product of completed topological rings $\widehat{S}$ and $\widehat{T}$ over a ring R is given by

$$\widehat{S} \mathbin{\widehat{\otimes}_R} \widehat{T} = \lim_{\leftarrow I,J} S/I \otimes_R T/J$$

where I and J denote the open ideals of S and T respectively. Since all formal groups $\mathbb{G}$ over X are isomorphic to $\widehat{\mathbb{A}}^1 \times X$ as formal schemes, it suffices to consider its ring of functions. This is given by $\mathcal{O}_X[\![x]\!]$ and the comultiplication map is of the form

$$\Delta\colon \mathcal{O}_X[\![x]\!] \to \mathcal{O}_X[\![x]\!] \mathbin{\widehat{\otimes}} \mathcal{O}_X[\![x]\!] \cong \mathcal{O}_X[\![1 \otimes x, x \otimes 1]\!].$$

Now, the image of x under Δ is determined by the multiplication map of the formal group $\mathbb{G}$. From every formal group, we obtain a formal group law by $\Delta(x)$.

We first need to choose an isomorphism, that is, an appropriate coordinate x. By definition of a formal group $\mathbb{G}$ over X, there is a map $e\colon X \to \mathbb{G}$ that provides the "identity element" which we denote by 0. A coordinate on $\mathbb{G}$ satisfying $x(0) = 0$ is called *normalized coordinate*. One can show that every formal group $\mathbb{G}$ over X admits a normalized coordinate x. Using the normalized coordinate, we define

$$\begin{aligned} \mathbb{G} \times_X \mathbb{G} &\to \widehat{\mathbb{A}}^1 \\ (u, v) &\mapsto x(u + v). \end{aligned}$$

Since $\mathbb{G} \times_X \mathbb{G} \cong \widehat{\mathbb{A}}^2 \times X$, we obtain

$$x(u + v) = \sum_{i,j \geq 0} a_{ij} x(u)^i x(v)^j =: F(x(u), x(v))$$

where $F(s, t)$ is a power series in $\mathcal{O}_X[\![s, t]\!]$. By the properties of a formal group, this defines a formal group law. In terms of the ring of functions, we have

$$\Delta(x) = F(1 \otimes x, x \otimes 1).$$

The interpretation is as follows. A formal group can be thought of as the formal neighbourhood around the identity of a scheme with a group structure. Choosing a coordinate around zero, we obtain a formal group law. In this way, formal group laws encode the local behaviour of the group structure of a formal group.

Example 4.53. (i) For the additive formal group $\widehat{\mathbb{G}}_a$, we have the comultiplication on the ring of functions

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

(ii) For the multiplicative formal group $\widehat{\mathbb{G}}_m$, we have

$$\Delta(x) = 1 \otimes x + x \otimes 1 - (1 \otimes x)(x \otimes 1).$$

Conversely, let $F(x, y) \in R[\![x, y]\!]$ be a formal group law over a ring R. We can define a formal group $\mathbb{G}_F$ over $X = \mathrm{Spec}(R)$ by setting $\mathbb{G}_F = \widehat{\mathbb{A}}^1 \times X$ as a formal scheme and defining the group law as follows. Let $x \in X(S) = \mathrm{Hom}_{\mathsf{Ring}}(R, S)$. By $x\colon R \to S$, we can regard S as an R-algebra and push forward the coefficients of F to S. Define the addition of $u, v \in \widehat{\mathbb{A}}^1(S) = \mathrm{Nil}(S) \cong \pi^{-1}\{x\}$ by

$$u +_F v = F(u, v).$$

By the properties of the formal group law, this turns $\mathrm{Nil}(S)$ into an abelian group. Hence, $\mathbb{G}_F$ is a formal group.

Remark 4.54. In our approach to formal groups, we have first defined formal schemes and then considered abelian group objects. Alternatively, one can look at group schemes, that is, group objects in the category of formal schemes and complete them to become formal groups. Those two approaches are equivalent, but our approach is more suited for our situation. One advantage of the group scheme approach is that the above consideration of the ring of functions can be made for group schemes already. Then, one can show that an affine group scheme is equivalent to a *Hopf algebra*. See for example [Moo11, p. 34]. Completion of an affine group scheme and the ring of functions turns the Hopf algebra into a *complete topological Hopf algebra* where the multiplication and comultiplication maps are defined using the completed tensor product $\widehat{\otimes}$ instead of the ordinary tensor product similar to what we have seen above to obtain formal group laws. We will come back to this in Chapter 5 when defining equivariant formal group laws.

Example 4.55. In Example 4.50, we have seen how an elliptic curve can be made into formal scheme $\widehat{E}$. The group structure on an elliptic curve gives rise to a group structure on $\widehat{E}$. Then, the formal group law arising from $\widehat{E}$ is precisely the formal group law that we have seen in Example 4.8.

We now consider the logarithm of a formal group law in the perspective of formal groups. Recall that the logarithm of a formal group law F is a (strict) isomorphism from F to the additive formal group law. For formal groups, additive coordinates are the counter-part of logarithms.

Definition 4.56. Let $\mathbb{G}$ be a formal group over X. An *additive coordinate* on $\mathbb{G}$ is a formal coordinate such that $x(u + v) = x(u) + x(v)$ for every $(u, v) \in \mathbb{G} \times_X \mathbb{G}$. That is, the map

$$\begin{aligned} \mathbb{G} &\to \widehat{\mathbb{A}}^1 \times X \\ u &\mapsto (x(u), \pi(u)) \end{aligned}$$

is an isomorphism of formal groups over X.

Every formal group law over a $\mathbb{Q}$-algebra has a logarithm (Proposition 4.9). In the language of formal groups, we have the following statement.

Proposition 4.57. *Let $\mathbb{G}$ be a formal group over $X = \operatorname{Spec}(A)$ and let A be a $\mathbb{Q}$-algebra. Then, $\mathbb{G}$ has an additive coordinate.*

Proof. See Proposition 5.20 in [Str19]. □

We end this section by connecting to the theory of elliptic genera. For this, we follow Sections 3.8 and 3.9 of [Mei22]. Recall from Section 3.2 that the universal elliptic genus is constructed from the affine Jacobi quartic

$$C\colon y^2 = 1 - 2\delta x^2 + \varepsilon x^4.$$

This describes the affine part of an elliptic curve with identity element $(0, 1)$. To obtain an elliptic curve, one needs to appropriately glue this to the curve

$$C'\colon y^2 = x^4 - 2\delta x^2 + \varepsilon.$$

The resulting curve $\overline{C}$ indeed defines an elliptic curve over an affine scheme $\operatorname{Spec} R$. See Section 3.8 of [Mei22] for the argument. This is called a *Jacobi quartic*. There is a universal elliptic curve of this form with $R = \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon][(\delta^2 - \varepsilon)^{-1}\varepsilon^{-1}]$. This is universal in the sense that it is isomorphic to the *universal elliptic curve* with choice of 2-torsion point and a choice of *invariant differential*[3].

Now, recall from Section 4.2 that the Euler formal group law correspond to the universal elliptic genus. The universal Jacobi quartic can be completed at the identity element given by $(0, 1)$. This yields a formal group $\widehat{C}$ which gives rise to a formal group law upon choosing a coordinate around the identity. One can show that this precisely gives the Euler formal group law. For more details, we refer to Section 3.9 in [Mei22].

[3] See for example Section III.5 of [Sil09].

5. Equivariant Complex Genera

In the previous chapter, we have seen that complex genera with values in R are in one-to-one correspondence to formal group laws over R. Using the Pontryagin-Thom isomorphism for complex bordism and the Quillen isomorphism, we have the following composition.

$$\Omega_*^U \xrightarrow{\substack{\text{Pontryagin-}\\ \text{Thom}}} \pi_*(MU) \xrightarrow{\text{Quillen}} L \xrightarrow{\text{FGL}} R$$

We would like to generalize this chain of maps to the *equivariant* setting to define complex and elliptic genera for G-manifolds and relate them to G-equivariant versions of formal group laws. A manifold together with an action by a suitable group G is called a G-manifold. While first definitions can be made for any topological group G, equivariant homotopy theory is only well-understood for compact Lie groups G. In our treatment, for simplicity, we will restrict to finite groups G. For more details on equivariant homotopy theory, we refer to [Blu17], [MPC96] and [LMS86].

In this chapter, we will construct the G-equivariant analogues of complex bordism Ω_*^{U}, the homotopy groups of the unitary Thom spectrum $\pi_*(MU)$, and the Lazard ring L classifying equivariant formal group laws. Furthermore, we will see how the maps between these objects generalize to the equivariant setting. It turns out that the equivariant Pontryagin-Thom map will not be an isomorphism, and that for equivariant formal group laws, we will have to restrict to abelian groups. However, we can still look at equivariant complex genera arising from equivariant formal groups.

5.1. Equivariant bordism and the equivariant Pontryagin-Thom theorem

To transport the theory of elliptic genera to the equivariant setting, in this section, we will introduce equivariant bordism and an equivariant version of the Pontryagin-Thom construction. We follow [Sin01] and [Han05] in our approach. See also [tom70].

Equivariant geometric bordism

We start with G-spaces and G-manifolds. In the following, by G we denote a finite group.

Definition 5.1. (i) A *G-space* is a topological space X together with a continuous G-action.

(ii) Let X and Y be G-spaces. A *G-equivariant map* from X to Y is a map $f\colon X \to Y$ such that $gf(x) = f(gx)$ for all $g \in G$.

(iii) A *G-manifold* is a manifold M together with a smooth G-action.

For pointed versions of the above, we require that G fixes the chosen basepoint of the spaces.

Definition 5.2. A *G-vector bundle* is a vector bundle (π, E, B) where E and B are equipped with G-actions and $\pi\colon E \to B$ is a G-equivariant map such that for all $g \in G$, the map on the fibres $\pi^{-1}(b) \to \pi^{-1}(gb)$ is a vector space isomorphism for all $b \in B$.

This definition works for both real and complex vector bundles. Recall that we can think of a complex vector bundle of dimension n as a real vector bundle of dimension $2n$ with a complex structure J. From this perspective, a complex G-vector bundle is a real G-vector bundle where J is compatible with the action of G.

Example 5.3. The tangent bundle of a G-manifold is a G-vector bundle.

Definition 5.4. Let $\pi\colon E \to B$ and $\pi'\colon E' \to B'$ be G-vector bundles. A *morphism* from π to π' a morphism of vector bundles that is G-equivariant.

Definition 5.5. Let M be a compact manifold with G-action. A *tangentially stably almost complex G-structure* on M is a real G-vector bundle isomorphism

$$TM \oplus \underline{\mathbb{R}}^k \cong \xi$$

for some $k \geq 0$, where G acts trivially on $\underline{\mathbb{R}}^k$, and ξ is a complex G-bundle. Two such structures on M are identified if upon adding G-trivial summands of $\underline{\mathbb{C}}$, there is an isomorphism of complex vector bundles.

Recall that in the non-equivariant setting, stably almost complex structures on the tangent bundle and on the normal bundle are equivalent. This is not the case equivariantly. Consider the following example as given in [Han05].

Example 5.6. The antipodal action makes the sphere S^2 into a $\mathbb{Z}/2$-manifold. It can be endowed with a complex $\mathbb{Z}/2$-structure on its stable normal bundle a follows. Let $\mathbb{Z}/2$ act by multiplication with -1 on $\mathbb{R}^3$ and consider the embedding of S^2 into $\mathbb{R}^3$. This has equivariant normal bundle $\nu = \underline{\mathbb{R}}$ with trivial $\mathbb{Z}/2$-action and hence, one obtains a normally stably almost complex $\mathbb{Z}/2$-structure on M. However, S^2 does not admit a tangentially stably almost complex $\mathbb{Z}/2$-structure. This is a consequence of the fact that the quotient space $S^2/(\mathbb{Z}/2) \cong \mathbb{R}\mathrm{P}^2$ is not orientable and can therefore not carry a stably almost complex structure. Indeed, any $\mathbb{Z}/2$-equivariant stable complex structure on the tangent bundle of S^2 would give rise to a stably almost complex structure on $\mathbb{R}\mathrm{P}^2$ and hence an orientation.

We now define a bordism between two compact n-manifolds M and N that are endowed with tangentially stably almost complex G-structure. Let Σ be a compact $(n+1)$-manifold with boundary that is endowed with a tangentially stably almost complex G-structure. By $\overline{N}$, we denote the G-manifold N with *opposite* stably almost complex structure. This is defined similarly to the non-equivariant setting. For a stably almost complex G-structure on N given by ξ, its opposite is defined as the equivalence class of $\xi \oplus \underline{\mathbb{C}}^c$ where G acts trivially on the conjugate bundle $\underline{\mathbb{C}}^c$. Let $\partial\Sigma \cong M \sqcup \overline{N}$ be a partition of the boundary into G-manifolds M and $\overline{N}$ such that the induced tangentially stably almost complex structures on M and $\overline{N}$ are compatible with the G-action. Such a Σ is called a complex G-bordism for M and N. This again forms an equivalence relation. For reflexivity, we take $M \times [0,1]$, but let G act trivially on the interval $[0,1]$. Symmetry works as in the non-equivariant case, and for transitivity, the necessary equivariant version of the collar theorem exist. See for example Theorem 20.3 in [CF64].

As before, we can construct an abelian group from the equivalence classes by using the disjoint union. We denote this group by $\Omega_n^{U,G}$. The Cartesian product induces a multiplication making

$$\Omega_*^{U,G} = \bigoplus_{n \geq 0} \Omega_n^{U,G}$$

into a graded ring called the *G-equivariant complex bordism ring.*

Remark 5.7. For a more precise definition of G-equivariant complex bordism and the associated equivariant homology theory using families of subgroups, we refer to [Sto69].

With the construction of the equivariant complex bordism ring, we can now generalize the definition of genera to the equivariant setting.

Definition 5.8. A *G-equivariant complex genus* is a ring homomorphism $\varphi\colon \Omega_*^{U,G} \to R$ for a $\mathbb{Q}$-algebra R.

Equivariant homotopical bordism

Next, we would like to define the G-equivariant homotopical complex bordism ring, that is, the equivariant analogue of $\pi_*(MU)$. Note that we will not define G-spectra and their homotopy groups separately, but make an ad hoc definition of $\pi_*^G(MU_G)$. There are several different approaches in stable equivariant homotopy theory to define equivariant spectra and their corresponding equivariant (co)homology theories. For our purposes, it suffices to define $\pi_*^G(MU_G)$ directly. Let γ_k be the universal bundle over $BU(k)$. The homotopy groups of the Thom spectrum MU are given by the following.

$$\pi_n(MU) = \varinjlim_k [S^{n+2k}, \mathrm{Th}(\gamma_k)]_*$$

Note that at first sight there is a slight difference to the definitions in Section 2.4. Recall that we defined $MU_{2k} = \mathrm{Th}(\gamma_k)$, and MU_{2k+1} as its suspension. Furthermore, in the definition of the homotopy groups of spectra, we index over all natural numbers. Here, we essentially index over the even numbers. However, the definition is equivalent since taking the colimit, we obtain the same.

For our definition of $\pi_*^G(MU_G)$, we will need the following notions.

(i) Classes of pointed G-homotopies.

(ii) Representation spheres S^V for representations V of G.

(iii) The classifying space $BU(n,G)$ for complex G-vector bundle and the universal bundle over it.

(i) A *G-homotopy* between G-equivariant maps $f, g\colon X \to Y$ is a G-equivariant map

$$H\colon X \times [0,1] \to Y$$

where G acts trivially on $[0,1]$ such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$. If f and g are pointed maps, we require H to be pointed and G acts trivially on the base point. We write $[-,-]_*^G$ for the pointed homotopy classes of G-equivariant maps.

(ii) Let V be a finite dimensional representation of G. Denote the one-point compactification of V by S^V called the *representation sphere*. This is a pointed space with a G-action. For the n-dimensional trivial real representation $V = \mathbb{R}^n$, we obtain $S^{\mathbb{R}^n}$ which is S^n with trivial G-action. In this sense representation spheres generalize the usual spheres. For more details, see Example 1.1.5 in [Blu17].

(iii) Let $\mathcal{U}$ be an infinite dimensional complex representation of G containing a countably infinite direct sum of each of the irreducible finite dimensional complex representations of G. This is called a *complete complex G-universe*. The Grassmannian $\mathrm{Gr}_n(\mathcal{U})$ is the space of n-dimensional subspaces of $\mathcal{U}$. This is a classifying space for complex G-vector bundles. Hence, we write $BU(n, G) = \mathrm{Gr}_n(\mathcal{U})$. As in the non-equivariant setting, there exists a universal complex G-vector bundle over γ_n^G over $BU(n, G)$. For more details on equivariant vector bundles and their classification, see for example [Was69].

We are now ready to define the equivariant homotopical complex bordism ring.

Definition 5.9. Let γ_n^G be the universal bundle over $BU(n, G)$, and let $\mathrm{Th}(\gamma_n^G)$ be its Thom space. We define

$$\pi_n^G(MU_G) := \varinjlim_V \left[S^{V \oplus \underline{n}}, \mathrm{Th}(\gamma_{|V|}^G)\right]_*^G$$

where the colimit runs over finite dimensional complex representations V and where $|V| := \dim_{\mathbb{C}}(V)$.

Note that we take the colimit over representations, but only index the universal bundles by their dimension. One could also use the Thom spaces of bundles over $BU(n, G)$ indexed by a representations V. However, this yields the same equivariant homotopical bordism groups. For more details, see [Sin01].

If G is the trivial group, the above definition reduces to $\pi_*(MU)$. For $G = \mathbb{Z}/2$, computations have been carried out by Strickland [Str01] to obtain a presentation as a module over the ring $\pi_*(MU)$. Further methods have been developed to compute $\pi_*^G(MU_G)$ as a module over $\pi_*(MU)$. See for example the PhD thesis of Abram [Abr13]. In the case of abelian G, the following structural result has been observed by Comezaña.

Theorem 5.10 (Comezaña [MPC96])**.** *Let G be an abelian finite group. Then $\pi_*^G(MU_G)$ is a free $\pi_*(MU)$-module concentrated in even degrees.*

It has been conjectured that this holds for all groups G. This is called the *evenness conjecture*. For more details, see [Uri18]. However, the evenness conjecture does not hold for all groups G. See [ASSU23] for a counter-example.

The equivariant Pontryagin-Thom construction

Recall the Pontryagin-Thom construction in Section 2.5. We now sketch the generalization to the equivariant setting.

Above, we defined the equivariant unitary bordism ring using manifolds with *tangentially* stable almost complex G-structures. The non-equivariant Pontryagin-Thom construction is defined using the Thom space of the normal bundle of a manifold. In the non-equivariant setting, this is not problematic since by Proposition 2.41, complex structures on the stable normal and stable tangent bundle are in one-to-one correspondence. However, as we have seen, in the equivariant setting, complex G-structures on the stable tangent bundle do not correspond to those on the stable normal bundle. Hence, to construct the Pontryagin-Thom map, we need a complex G-structure on the normal bundle of an embedding of a manifold induced by the complex G-structure on its tangent bundle. This is done as follows. Consider a tangentially almost complex G-manifold and an embedding $M \hookrightarrow W$ into a complex representation W of G. Similar to the proof of Proposition 2.41, we have the following isomorphism.

$$(\nu \oplus \nu) \oplus (TM \oplus \mathbb{R}^k) \cong \nu \oplus \underline{W} \oplus \underline{\mathbb{R}}^k$$

We obtain a complex G-structure on $\nu \oplus \underline{W} \oplus \underline{\mathbb{R}}^k$. The difference is that we now have a non-trivial G-action on W. We use the following trick. Consider the embedding $W \hookrightarrow W \oplus W$ into the first summand and compose with the embedding of M into W to obtain an embedding of M into $W \oplus W$. The normal bundle of this embedding is $\nu \oplus \underline{W}$ where ν is the normal bundle of the embedding into W. Using the above isomorphism, we have constructed a stable complex G-structure on $\nu \oplus \underline{W}$. Hence, by taking $V = W \oplus W$ in the following, we obtain an induced complex G-structure on the embedding of M into V.

Now, we are ready to describe the equivariant Pontryagin-Thom construction. Let M be a manifold with tangentially stably almost complex G-structure, and let V be a complex representation of G such that M embeds into V such that the normal bundle ν of the embedding obtains a complex G-structure as described above. Now extend this embedding to the representation sphere S^V, and identify the normal bundle ν with a tubular neighbourhood of M in S^V. By collapsing every outside ν to the basepoint, we obtain a map $g\colon S^V \to \mathrm{Th}(\nu)$ as in the non-equivariant construction. Consider the composition

$$S^V \xrightarrow{g} \mathrm{Th}(\nu) \to \mathrm{Th}(\gamma^G_{|\nu|})$$

where the second map is induced by the classifying map of the complex G-bundle ν. We define this to be the equivariant Pontryagin-Thom construction $P^G_n([M])$.

Theorem 5.11. *The equivariant Pontryagin-Thom construction*

$$P^G_n\colon \Omega^{U,G}_n \to \pi^G_n(MU_G)$$

is a well-defined homomorphism.

This is Theorem 3.7 in [Sin01], but no proof is given there. See Proposition 3.1 in [Was69] for a proof of the equivariant Pontryagin-Thom construction in the non-complex case.

Unlike in the non-equivariant setting, the equivariant Pontryagin-Thom construction is *not* an isomorphism. We can see that the Pontryagin-Thom homomorphism viewed as a graded ring homomorphism is not surjective since there exist non-zero elements in negative degree called *Euler classes*. A complex representation V can be viewed as a complex G-vector bundle over a point with dimension $|V|$. The classifying map $V \to BU(n, G)$ induces a map on the Thom spaces $S^V \to \mathrm{Th}(\gamma^G_{|V|})$. By precomposition with the inclusion $S^0 \to S^V$ induced by the map $0 \to V$, this yields an element

$$e_V \in \pi^G_{-|V|}(MU_G),$$

the Euler class, which is non-zero if and only if G and V are non-trivial. On the other hand, equivariant geometric bordism obviously does not have negative degrees. However, even on non-negative degrees, P^G_n is not an isomorphism. This has the following reason.

Recall that to define the inverse map of the Pontryagin-Thom construction, we need *transversality* (see Section 2.5). While for many theorems in equivariant differential topology, there are equivariant analogues, the transversality theorem does not hold equivariantly. Recall that the transversality theorem ensures that we can approximate maps from spheres to the Thom space of a smooth vector bundle by a map that transverse to the base space. In the equivariant setting, the existence of such equivariant homotopy fails in general. For an example of this failure and more details on the equivariant Pontryagin-Thom construction, see [Was69] and [tom70]. See also [BH72] for a stabilized version of equivariant geometric bordism for which the equivariant Pontryagin-Thom construction is an isomorphism. This gives $\pi^G_*(MU_G)$ a more geometric interpretation, but does not provide a description of G-equivariant genera.

5.2. Equivariant formal group laws and the equivariant Quillen isomorphism

In this section, we will define equivariant formal group laws and relate them to equivariant bordism. We refer to [CGK00] and [Gre01].

Equivariant formal group laws

Recall from Section 4.3 that non-equivariant formal group laws arise from formal groups by choosing coordinates. Formal groups $\mathbb{G}$ are group objects in the category of (smooth) formal schemes, and formal schemes encode the behaviour of a scheme locally in a formal neighbourhood. This can be thought of as a neighbourhood around the identity and is by definition isomorphic to the formal neighbourhood $\widehat{\mathbb{A}}^n$ of the origin in the affine space $\mathbb{A}^n$. For a 1-dimensional formal group, the choice of normalized coordinate for $\mathbb{G}$ is the choice of isomorphism to $\widehat{\mathbb{A}}^1$ and provides a description of the group law. As we have seen, this gives rise to an element in the ring of functions $\mathcal{O}_X[\![x, y]\!]$ which is a formal group law over $\mathcal{O}_X$. Furthermore, recall that for the formal group $\widehat{\mathbb{A}}^1$ over an affine scheme $X = \mathrm{Spec}(R)$, the ring of functions of $\widehat{\mathbb{A}}^1$ is $R[\![x]\!]$, where $R = \mathcal{O}_X$. Here, x is identified with the coordinate of the formal group $\mathbb{G}$. Then, the group law $\mu\colon \mathbb{G} \times_X \mathbb{G} \to \mathbb{G}$ is encoded in coordinates in terms of the ring of functions by the map

$$\begin{aligned} \Delta\colon R[\![x]\!] &\to R[\![x]\!]\widehat{\otimes}R[\![x]\!] \cong R[\![x \otimes 1, 1 \otimes x]\!] \\ x &\mapsto F(x \otimes 1, 1 \otimes x) \end{aligned}$$

where $\widehat{\otimes}$ is the completed tensor product of the complete rings $R[\![x]\!]$. The map $\Delta = \mu^*$ encodes the formal group law and provides a coproduct for the *complete topological Hopf algebra* $R[\![x]\!]$. A complete topological Hopf algebra is a modified version of an ordinary Hopf algebra[1]. In particular, the underlying ring R is endowed with a linear topology and complete with respect to this topology, and the tensor product in the definition of the (co)multiplication maps is replaced with the completed tensor product $\widehat{\otimes}_R$.

Definition 5.12. A *complete topological Hopf R-algebra* is a tuple $(H, m, \Delta, \eta, \varepsilon, S)$ where H is a topological R-algebra, the multiplication and comultiplication

$$\begin{aligned} m\colon H \,\widehat{\otimes}_R\, H &\to H \\ \Delta\colon H &\to H \,\widehat{\otimes}_R\, H \end{aligned}$$

and the unit η, counit ε, and antipode S are continuous maps satisfying the usual relations for Hopf algebras.

Now, choosing a complete topological Hopf algebra H that is isomorphic to $R[\![x]\!]$ amounts to choosing a (smooth) formal group $\mathbb{G}$ over $\mathrm{Spec}(R)$. The choice of isomorphism for $H \cong R[\![x]\!]$ corresponds to the choice of coordinate and determines the formal group law coming from $\mathbb{G}$. This the right setting to generalize to equivariant formal groups and equivariant formal group laws. The definition is due to Cole-Greenlees-Kriz [CGK00]. See also the more introductory account [Gre01].

First, we define $G^* := \mathrm{Hom}(G, S^1)$ to be the Pontryagin dual group to G. We will describe G-equivariant formal groups in terms of the representations G^*. Since S^1 is abelian any homomorphism $G \to S^1$ factors through the abelianization of G. Therefore, using G^* to define equivariant formal group laws, we can only capture equivariance for

[1]For the theory of Hopf algebras also with a view towards group schemes, see [Und11].

abelian groups. Hence, in the following, we will restrict ourselves to *abelian* finite groups G. Following [CGK00], we denote the trivial representation by $\epsilon \in G^*$, that is, the group homomorphism $G \to S^1$ sending every element to the identity in S^1. We also introduce the ring of R-valued functions on G^* defined as

$$R^{G^*} := \mathrm{Map}(G^*, R)$$

with ring structure inherited from R. This will be used to define the topology on the complete topological Hopf algebra.

The definition in [CGK00] of equivariant formal group laws is given for general abelian compact Lie groups. However, for simplicity we restrict to finite abelian groups.

Definition 5.13. Let G be a finite abelian group. A *G-equivariant formal group law* over a commutative ring R consists of a tuple $(H, \theta, y(\epsilon))$ where

(i) H is a complete topological Hopf R-algebra,

(ii) $\theta \colon H \to R^{G^*}$ is a homomorphism of Hopf R-algebras whose kernel is the ideal defining the topology on H,

(iii) $y(\epsilon) \in H$ is a regular element generating the kernel of the ϵ-th component, θ_ϵ of θ.

Remark 5.14. In the more general case of G being an abelian compact Lie group, condition (ii) is slightly more complicated. One then requires θ to be a homomorphism of *topological* Hopf R-algebras and the topology on H is defined by finite intersections of of kernels of $\theta_\alpha \colon H \to R$ for $\alpha \in G^*$.

As in the non-equivariant setting, there is a G-equivariant formal group law over the ring $\pi_*^G(MU_G)$ arising from the tensor product of G-equivariant complex line bundles. For more details, see [CGK00]. See also [CGK02] for the equivariant complex-oriented cohomology theories represented by the equivariant version of the spectrum MU. Recall that the Lazard ring L classifies formal group laws over a ring R by ring homomorphisms $L \to R$. Similarly, one can construct a universal ring L_G classifying G-equivariant formal group laws.

Theorem 5.15 ([CGK00])**.** *Let G be a finite group. There exists a ring L_G that classifies G-equivariant formal group laws over R by means of the bijection*

$$\{\textit{Ring homomorphisms } f \colon L_G \to R\} \longleftrightarrow \{G\textit{-equivariant formal group laws over } R\}.$$

The difference to the non-equivariant case is that equivariant formal group laws over R are not simply power series over R, but more complicated Hopf-algebra structures.

However, it can be shown that from a ring homomorphism $R \to S$, an equivariant formal group law over R is pushed forward to S.

By construction of L_G, there is a ring homomorphism $L_G \to \pi_*^G(MU_G)$ that determines the formal group law over $\pi_*^G(MU_G)$. Recall that in the non-equivariant setting, Quillen's theorem states that this is an isomorphism. The equivariant version of Quillen's theorem has been conjectured by Greenlees in [Gre00] and [Gre01]. It is shown by Hanke-Wiemeler [HW18] that the conjecture is true for $G = \mathbb{Z}/2$ by using calculations of the equivariant homotopical bordism ring $\pi_*^{\mathbb{Z}/2}(MU_{\mathbb{Z}/2})$ carried out by Strickland [Str01]. The conjecture was proved by Hausmann in 2022 for all abelian compact Lie groups.

Theorem 5.16 (Hausmann [Hau22])**.** *The map $L_G \to \pi_*^G(MU_G)$ is an isomorphism for an abelian compact Lie group G.*

His proof is using a *global* approach. Global homotopy theory studies homotopical objects with simultaneous action by *all* compact Lie groups. See [Sch18] for a reference on global homotopy theory.

5.3. Applications of equivariant genera

After having introduced equivariant versions of bordism and formal group laws, we return to genera. Recall that an equivariant complex genus with values in a $\mathbb{Q}$-algebra R is a ring homomorphism

$$\varphi\colon \Omega_*^{U,G} \to R.$$

With the constructions in the previous sections, we have the following composition of maps.

$$\Omega_*^{U,G} \xrightarrow{\text{equiv. PT}} \pi_*^G(MU_G) \xrightarrow{\text{equiv. Quillen}} L_G \xrightarrow{\text{equiv. FGL}} R$$

While the middle map coming from the equivariant version of Quillen's theorem is an isomorphism, we have seen that the equivariant Pontryagin-Thom construction is not an isomorphism. Hence, equivariant complex genera are not in one-to-one correspondence with equivariant formal group laws. However, from every equivariant formal group law, we obtain a map $\pi_*^G(MU_G) \cong L_G \to R$, and precomposing with the equivariant Pontryagin-Thom construction, yields a G-equivariant complex genus. Not every equivariant complex genus will arise in this way, and two G-equivariant formal group laws can give rise to the same G-equivariant genus.

We propose to investigate which equivariant complex genera indeed come from an equivariant formal group law and which equivariant formal group laws give rise to the same equivariant elliptic genus.

Equivariant elliptic genera

Recall that we initially defined elliptic genera as ring homomorphisms $\varphi\colon \Omega_*^{\mathrm{SO}} \to R$ that are defined via its logarithm from an elliptic curve in Jacobi quartic form. Extending the perspective to complex genera, we have seen that this generalizes to elliptic genera of level N. With the theory of formal groups and formal group laws, we can reformulate this as follows. An elliptic genus is a ring homomorphism $\varphi\colon \Omega_*^{\mathrm{U}} \to R$ such that its associated formal group law is obtained from the completion of an elliptic curve. Having introduced equivariant versions of bordism and formal group laws, we propose the following sketch of a definition.

Definition sketch. Let G be an abelian finite group. A *G-equivariant elliptic genus* is a G-equivariant complex genus

$$\varphi\colon \Omega_*^{U,G} \to R$$

that is given by a G-equivariant formal group law *coming from* an elliptic curve.

To make this more precise, we would first need to understand how elliptic curves can give rise to equivariant formal group laws. The starting point for this investigation would be the simplest non-trivial case $G = \mathbb{Z}/2$. Note that calculations in $\mathbb{Z}/2$-equivariant bordism and formal group laws are very complicated. For instance, a presentation of $\pi_*^{\mathbb{Z}/2}(MU_{\mathbb{Z}/2})$ in terms of $\pi_*(MU)$ is given by Strickland in [Str01]. Furthermore, Example 3.1 in [Gre01] shows the underlying ring of the complete topological Hopf algebra for a $\mathbb{Z}/2$-equivariant formal group law.

Applications as orbifold elliptic genera

Let G be a finite abelian group and M be a G-manifold. If the smooth action of G on M is free, then the orbit space M/G is again a smooth manifold[2]. If, however, the action has fixed points, the quotient becomes singular at those points.

An *orbifold* is a generalization of a manifold that incorporates such singular points modelled as fixed point of a local action. Roughly, speaking an orbifold is a space that

[2]For a proof of this in the more general case where G is a Lie group, see Theorem 21.10 in [Lee13]. Since we take G to be finite, the action is automatically proper.

locally looks like $\mathbb{R}^n/\Gamma$ where Γ is a finite group acting on $\mathbb{R}^n$. Note that in general, Γ is not the same for all neighbourhoods. Classically, this has been defined by Satake [Sat56] as V-manifolds, and later used by Thurston in the theory of 3-manifolds who coined the term orbifold (see Chapter 13 of [Thu02]). A more modern approach to the theory of orbifolds is through the language of *Lie groupoids*. See for example [Moe02].

One particularly important class of orbifolds are *global quotient orbifolds* given by quotients of G-manifold M denoted by $M /\!\!/ G$. Homotopical constructions for global quotient orbifolds are usually done in terms of the G-manifold presenting them. For example, the orbifold cohomology of the global quotient orbifold $M /\!\!/ G$ is equal to the G-equivariant cohomology of the manifold M. See Example 2.11 in [ALR07]. Furthermore, the Euler characteristic of a quotient orbifold is defined via the Euler characteristic of the underlying manifold. See for example [HH90] or [Tam01]. Hence, we expect that using our definition of G-equivariant genera, suitable definitions for complex and elliptic genera for global quotient orbifolds can be made. This could then be compared with existing notions of elliptic genera for orbifolds.

The first occurrence of *orbifold elliptic genera* in the mathematical literature is due to Borisov-Libgober [BL03]. Instead of adopting a topological point of view, their approach uses connections between complex differential geometry and complex algebraic geometry and define an elliptic genus for a special type of singular varieties. This has been developed to prove the so-called DMVV-formula which relates the elliptic genus of a manifold and its *symmetric products* which is a special type of global quotient orbifold. After having introduced the string theoretic perspective on elliptic genera, we will discuss the DMVV-formula in Section 6.3 and outline the theory of Borisov-Libgober.

A theory of orbifold elliptic genera that is closer to our approach is that of Ando-French [AF07]. In their paper, they construct the so-called *orbifold sigma genus* from the perspective of equivariant elliptic cohomology defined by a supersingular elliptic curve and compare it to the orbifold elliptic genus defined in [BL03]. See also [Gan04] for further homotopy theoretic generalization of orbifold elliptic genera. This is also shown to reproduce an analogue of the DMVV-formula.

6. Elliptic Genera in String Theory

In this chapter, we will explain how elliptic genera arise in physics. We start by giving a review of string theory focussing on the most important aspects to understand the physical perspective on elliptic genera in the subsequent section. There, we will see how considering a specific type of superstring theory, the universal elliptic genus arises as its partition function. This has been described by Witten in [Wit87]. Further developments have led to the two-variable elliptic genus (see [Wit94] and [KYY94]) which has found further applications in superstring theory.

As an example, we will discuss the DMVV-formula [DMVV97] that formally relates the two-variable elliptic genus of a manifold and its symmetric products. The formula was first derived with physical arguments and implied an extension of the definition of elliptic genera to orbifolds. A mathematical proof of this has been provided by Borisov-Libgober [BL03] constructing a mathematical theory of *orbifold elliptic genera*. We will briefly outline their approach in the end of this chapter.

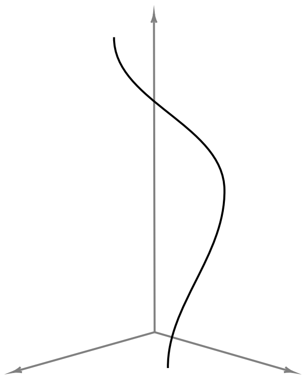
6.1. An overview of string theory

The following introduction is based on the first sections of [Ton09] and [BBS06].

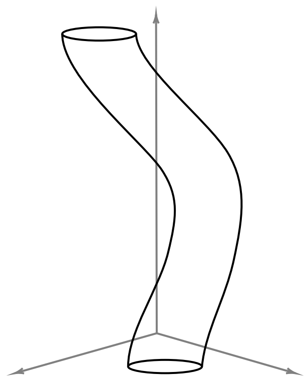
The standard model in particle physics successfully describes three of the four fundamental interactions: electromagnetism, the weak and the strong interaction. Many predictions of the standard model have been in agreement with experiments to great numerical accuracy. Nevertheless, there are still open problems such as the explanation of dark matter that lie beyond the standard model. The fourth fundamental interaction, gravity, is described by Einstein's theory of general relativity. The underlying principle is that the geometry of spacetime is dynamical and interacts with matter. While predictions of general relativity are in good agreement with observations such as the detection of gravitational waves, its fundamental assumptions seem incompatible with quantum mechanics. That is, the theory of gravity cannot be quantized using standard methods.

The hope is that there is a general theory combining both particle physics at the microscopic limit and general relativity at the macroscopic limit. This is what is called *quantum gravity.* There have been several attempts at describing a theory of quantum gravity. One of which is string theory.

The fundamental idea of string theory is that elementary point particles are replaced by 1-dimensional *strings* that can either be *open* or *closed.* Oscillations of those strings then give rise to the particles we observe. While particles trace out worldlines through spacetime, the time-evolution of a string is described by a 2-dimensional *worldsheet.*



Worldline



Worldsheet

The worldsheet Σ of a string is embedded in the *target space* M that ultimately describes spacetime. The target space M usually has dimension $D = 26$ or $D = 10$ depending on the type of theory. Most commonly, 10-dimensional supersymmetric theories are considered which have a symmetry between bosonic and fermionic degrees of freedom. To effectively obtain a 4-dimensional target space that could describe the spacetime of our universe, the extra six dimensions are "curled up" and become small with respect to the four spacetime dimensions. This process is called *compactification.* Through compactification, the geometry of the compactified space influences the physics of the 4-dimensional low-energy effective theory.

The target space M is a manifold if the theory admits the description of a so-called *non-linear sigma model.* In supersymmetric theories, the target manifolds are locally of the form

$$\mathbb{R}^{1,3} \times \mathrm{CY}_3$$

where $\mathbb{R}^{1,3}$ is 4-dimensional Minkowski space, and CY_3 is a *Calabi-Yau* manifold of complex dimension 3. Through compactifications via symmetries acting on the target space, the target space can also obtain the structure of an orbifold rather than a manifold. This identification then requires adding so-called *twisted sectors* from which a new theory is obtained.

String theory automatically includes gravity and gives rise to gauge theories like the standard model. This makes it a good candidate for a theory of quantum gravity. However, there is a very large number of possible choices for the geometry of the compactified space that give rise to many different possible configurations called *vacuum states*. Even if one of those vacua could describe our universe, the number of vacua is just too large and the choice of such a specific vacuum is not built into string theory. Furthermore, many aspects of string theory have not yet been fully understood.

Nevertheless, there are several reasons why it can still be useful to study string theory. The first one is that conceptually, string theory can help us understand the constraints of *the* theory of quantum gravity since many of its necessary features and relevant aspects are already part of string theory. Furthermore, the techniques developed in string theory can find applications in other fields of physics. One example is the AdS/CFT-correspondence which describes the relationship between a theory in an Anti-de Sitter spacetime and a conformal field theory on its boundary. This has been applied to nuclear physics and condensed matter physics.

Another reason is that string theory provides new mathematical insights. Since a variety of different mathematical objects appear in string theory, this has given rise to conjectures and uncovered connections between different areas of mathematics. One famous example is *mirror symmetry* which relates geometrically different Calabi-Yau manifolds through dualities of the corresponding string theories. Also the development of elliptic genera has been influenced by the string theoretic perspective which we will explain in the next section.

In the following, we will use the following conventions. The Minkowski metric in D dimensions has signature $(-,+,\dots,+)$, and we work in natural units.

Bosonic string theory

The simplest string theory is bosonic string theory. We will now give a short description of the most important aspects of the theory that are also needed in the supersymmetric generalization. For this, we follow [Ton09] and [BLT13].

The starting point for bosonic string theory is the Nambu-Goto action

$$S_{NG} = -T \int_{\Sigma} dA$$

which is a generalization of the action of a relativistic point particle to an integral over the two-dimensional worldsheet. Here, the constant T is the *string tension* and is a free parameter of the theory. The worldsheet Σ is usually parametrized by coordinates $\sigma^\alpha = (\sigma, \tau)$, where σ represents the spacelike direction, and τ represents the timelike direction. In its most basic form, bosonic string theory is considered in D-dimensional Minkowsi spacetime. There are D scalar fields which are labelled by $X^\mu(\sigma,\tau)$ where $\mu = 0,\dots,D-1$ and can be viewed as coordinates on spacetime. We then have

$$dA = \sqrt{-\det \gamma} \quad \text{where} \quad \gamma_{\alpha\beta} = \tfrac{\partial X^\mu}{\partial \sigma^\alpha}\tfrac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu}$$

is the pullback of the metric $\eta_{\mu\nu}$ of the Minkowski spacetime to the worldsheet. The square root in the action provides difficulties when quantizing the theory. Therefore, instead of working with the Nambu-Goto action, an auxiliary field $h^{\alpha\beta}$ is introduced. Again generalizing from the one-dimensional case, this yields the *Polyakov action*

$$S_P = -\frac{T}{2}\int_\Sigma d\sigma d\tau \sqrt{-\det h}\; h^{\alpha\beta}\partial_\alpha X^\mu(\sigma,\tau)\partial_\beta X^\nu(\sigma,\tau)\eta_{\mu\nu}$$

which is classically equivalent to the Nambu-Goto action. Here, $h^{\alpha\beta}$ is the worldsheet metric which is now a dynamical object. More generally, for possibly curved target manifolds, the spacetime metric $\eta_{\mu\nu}$ can be replaced by a general metric $g_{\mu\nu}$.

The Polyakov action has the following symmetries: *Poincaré invariance*, *diffeomorphism invariance* and *Weyl invariance*. While Poincaré invariance is a global symmetry, the other two are local symmetries. Diffeomorphism invariance is the invariance under change of coordinates. Weyl invariance, on the other hand, is not a symmetry under coordinate transformations, but a symmetry under changing the metric locally by a positive factor.

Using the symmetries of the action, a suitable choice of gauge can be made in which the worldsheet metric is flat. With this, the Polyakov action simplifies and it becomes easier to deal with the equations of motion. Then, the so-called *lightcone coordinates*

$$\sigma^+(\sigma,\tau) = \sigma + \tau, \qquad \sigma^-(\sigma,\tau) = \sigma - \tau$$

are introduced on the worldsheet. In these coordinates, the equations of motion are given by

$$\partial_+\partial_- X^\mu(\sigma^+,\sigma^-) = 0$$

where $\partial_\pm$ is the partial derivative with respect to $\sigma^\pm$. Solutions are given by fields X^μ which split as

$$X^\mu(\sigma,\tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-).$$

Here, X_L and X_R are called *left-* and *right-moving sectors* and can be expressed in Fourier expansion using fixed boundary conditions. The boundary conditions depend on whether the string is *open* or *closed*. The Fourier modes are denoted by $\overline{\alpha}^\mu_m$ and α^μ_m and for left- and right-moving modes respectively.

The theory satisfies additional constraints which can be expressed in terms of the Fourier modes. For this, we define

$$\begin{aligned} L_n &= \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m, \\ \overline{L}_n &= \frac{1}{2}\sum_m \overline{\alpha}_{n-m}\cdot\overline{\alpha}_m. \end{aligned}$$

After quantizing, the constraints of the modes are given by

$$\begin{aligned} L_n\,|\text{phys}\rangle &= 0 \quad \text{if } n\neq 0, \\ (L_0-a)\,|\text{phys}\rangle &= 0, \end{aligned}$$

and similarly for $\overline{L}_n$. Here, $|\text{phys}\rangle$ are physical states, and a is the *normal ordering constant* that is introduced when quantizing. In the quantum theory, operators do not commute in general. Hence, to quantize the classical theory, a choice of ordering of the operators has to be made. This introduces an ambiguity captured by the normal ordering constant a. For bosonic string theory to be consistent, the *critical dimension* is $D = 26$. This is required by Lorentz invariance and the consistent choice of the normal ordering constant $a = 1$ upon quantizing.

One way to derive the critical dimension is through the perspective of *conformal field theory.* The L_n can be viewed as the Fourier modes of the *stress-energy tensor* $T_{\alpha\beta}$ of the theory and span the *Virasoro algebra* satisfying

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

which is the central extension of the Witt algebra. Here, c is called the *central charge* or *conformal anomaly.* The reason it is called conformal anomaly is that computing the expectation value of the trace of the stress-energy tensor yields

$$\langle T^{\alpha}_{\alpha}\rangle = -\frac{c}{12}R$$

where R is the Ricci scalar of the worldsheet. For a conformal field theory, the expectation values needs to vanish. This is the case for $R = 0$, but to obtain a theory for general dynamical backgrounds, we need that $c = 0$. Using the path integral formalism, the gauge fixing using diffeomorphism and Weyl invariance introduces the so-called system of *bc ghosts* that has central charge $c_{bc} = -26$. Now, each bosonic scalar contributes $c_{\text{boson}} = 1$ to the total central charge. This means that there need to be 26 bosonic degrees of freedom and therefore, $D = 26$.

When calculating the states of the bosonic string, the lowest state has negative mass squared. This means that the bosonic string contains a tachyon which is problematic. One solution is to introduce fermions and supersymmetry to remove tachyons from the spectrum.

Superstring theory

Bosonic string theory only includes bosonic degrees of freedom. The idea of superstring theory is to introduce fermionic degrees of freedom on the worldsheet using supersymmetry. These are labelled by $\psi^\mu(\sigma,\tau)$ and quantized using *anti-commutators* instead of commutators. This adds further terms to the action (considered in superconformal gauge)

$$S_{\text{ferm}} \propto \int d\sigma d\tau \, \overline{\psi}^\mu(\sigma,\tau)\rho^\alpha \partial_\alpha \psi_\mu(\sigma,\tau).$$

Here, ρ^α for $\alpha = 0, 1$ are the 2-dimensional Dirac matrices acting on the spinors. The imposed supersymmetry relates bosonic and fermionic degrees of freedom. Recall that symmetries in physics are associated to charges via the Noether theorem. For supersymmetry, these charges are called *supercharges* and are of fermionic nature.

Adding fermionic degrees of freedom to the theory, one can carry out similar calculations as for the bosonic case. When choosing boundary conditions for the fermions, there are two possibilities: Periodic boundary conditions are called *Ramond boundary conditions* while anti-periodic boundary conditions are referred to as *Neveu-Schwarz boundary conditions.*

The algebra describing the corresponding superconformal theory then has additional generators arising from the additional terms in the action. Furthermore, the fermionic degrees of freedom contribute an additional ghost system called the *$\beta\gamma$ system.* This has $c_{\beta\gamma} = 11$, and hence, the supersymmetric string theory now has to have $c = 15$. Since every fermionic degree of freedom contributes $c_{\text{ferm.}} = 1/2$, and by supersymmetry, there are the same number of bosons and fermions we find that the critical dimension is $D = 10$ such that $c = 15 = 10(1 + 1/2)$.

There are five types of superstring theories in $D = 10$ dimensions:

type I, type IIA, type IIB, heterotic $SO(32)$, heterotic $E_8 \times E_8$.

We are mostly interested in type II and heterotic string theory. Type II string theory has supersymmetry in both the left- and right-moving modes. While type IIA is not chiral, that is, there is a symmetry between left- and right-moving modes, type IIB is a chiral theory. Furthermore, Type II string theory also includes *D-branes* which are higher dimensional dynamical objects that describe the end-points of open strings with *Dirichlet boundary conditions.* For instance, D-branes can be used to describe higher-dimensional black holes, and D-brane state-counting has been found to reproduce the entropy of such a black hole. We will discuss this more in detail in Section 6.3.

In heterotic string theory, the left- and right-moving sector are treated independently. The left-moving sector corresponds to a bosonic string in $D = 26$ dimensions, and the right-moving sector to a superstring in $D = 10$ dimensions. To make it a 10-dimensional theory, the bosonic string is compactified on an even, self-dual lattice $\Gamma^{16} \subseteq \mathbb{R}^{16}$. In 16 dimensions there are only two such lattices. The resulting gauge groups for the 10-dimensional theory are $E_8 \times E_8$ and $SO(32)$ which distinguish the two heterotic string theories.

The different superstring theories are related by several dualities such as *S-duality* and *T-duality*. This suggests that these theories are in fact part of a larger theory. Modern formulations are in terms of an 11-dimensional universal theory called *M-theory* that has the supersymmetric string theories as its limits.

6.2. Elliptic genera as partition functions

In this section, we will allude to the interpretation of elliptic genera in physics due to Witten [Wit87]. His work connected the mathematical theory of elliptic genera with ideas in physics that built on the previous work by Pilch and Schellekens-Warner (see [PSW87] and [SW87]) on *anomaly cancellation* in string theory. In his construction, Witten related elliptic genera to partition functions of type II supersymmetric string theory and formally defined an operator whose index is the elliptic genus. Witten also constructed the so-called *Witten genus* which assigns modular forms for the full modular group $SL_2(\mathbb{Z})$ to compact manifolds with *string* structure. We mainly follow his original paper [Wit87] and explain further background provided in [Wit88], the introductory account in [Lan88], and Section 6 of [HBJ92]. A mathematical proof for certain *rigidity* properties conjectured in [Wit87] can be found in [BT89].

We start by considering the $\widehat{A}$-genus. Let M be an even dimensional manifold with spin structure. Then there are two irreducible representations of the spin group and the *spinor bundle*[1] splits as $S = S^+ \oplus S^-$. The Dirac operator is an *elliptic operator*[2]

$$\mathcal{D}\colon \Gamma(S^+) \to \Gamma(S^-)$$

acting on spinors. Here $\Gamma(S^{\pm})$ denotes the sections of the $S^{\pm}$-bundle. Note that the Dirac operator is often denoted by $i\not{D}$. The kernel and cokernel of an elliptic operator O are finite dimensional. The *index* of O then is defined as

$$\mathrm{Ind}(O) = \dim(\ker O) - \dim(\operatorname{coker} O).$$

[1]The spinor bundle is the associated bundle to the principal bundle of spin frames. A spinor field is a smooth section in this bundle. In the physics literature, a spinor field is usually referred to simply as spinor. For more details, we refer Section 9.4.2 in [Nak18].

[2]For the theory of elliptic operators and the Atiyah-Singer Index theorem, we refer to [AS68], and [Fre21]. Also compare the account in [BT89].

A version of the *Atiyah-Singer index theorem* states that for a M compact spin manifold of dimension $4n$, the following holds

$$\operatorname{Ind}\mathcal{D} = \widehat{A}(M)[M].$$

The index formula can be extended to the *twisted* Dirac operator. Consider a representation of the spin group and take the associated bundle V over M. The twisted Dirac operator

$$\mathcal{D}_V \colon \Gamma(S^+ \otimes V) \to \Gamma(S^- \otimes V)$$

is acting on spinors with values in V. Then, the following generalization of the index theorem holds

$$\operatorname{Ind}\mathcal{D}_V = (\widehat{A}(M)\operatorname{ch}(V))[M]$$

where $\operatorname{ch}(V)$ is the Chern character of the complexification of V.

The $\widehat{A}$-genus vanishes on spin manifolds with an action by S^1 that lifts to the spin bundle. In terms of index theory, the S^1-action gives rise to an *equivariant* or *character-valued* index

$$\operatorname{Ind}^{S^1}(\mathcal{D})$$

which indeed vanishes as shown by Atiyah-Hirzebruch [AH70]. This is called *rigidity* of the $\widehat{A}$-genus. The question whether this generalizes to the twisted Dirac operator was one of the motivations in the early developments of elliptic genera. Landweber-Stong [LS88] considered a series of representations R_k of $\mathrm{Spin}(n)$ showing the rigidity of twisted versions of the $\widehat{A}$-genus.

Witten realized that the representations R_k are corresponding to the mass levels of the Neveu-Schwarz sector of open superstrings. In the following, let $|\Omega\rangle$ denote the ground state transforming as a $\mathrm{Spin}(n)$ singlet. Furthermore, let 1 be the trivial representation, and T be the *fundamental representation*[3] of $\mathrm{Spin}(n)$. Then there is the following correspondence.

Representation R_k	States at k-th mass level in the NS-sector
$R_0 = 1$	$\lvert\Omega\rangle$
$R_1 = T$	$\psi^i_{-\frac{1}{2}}\lvert\Omega\rangle$
$R_2 = \Lambda^2 T \oplus T$	$\psi^i_{-\frac{1}{2}}\psi^j_{-\frac{1}{2}}\lvert\Omega\rangle$ and $\alpha^i_{-1}\lvert\Omega\rangle$
$R_3 = \Lambda^3 T \oplus (T \otimes T) \oplus T$	$\psi^i_{-\frac{1}{2}}\psi^j_{-\frac{1}{2}}\psi^l_{-\frac{1}{2}}\lvert\Omega\rangle$, $\psi^i_{-\frac{1}{2}}\alpha^j_{-1}\lvert\Omega\rangle$, and $\psi^j_{-\frac{3}{2}}\lvert\Omega\rangle$
$\vdots$	$\vdots$

[3]Here, by fundamental representation of a Lie group, we mean its smallest faithful representation. In the case of matrix Lie groups, this is simply given by the matrices themselves. For more on representation theory of Lie groups and Lie algebras, we refer to [Hal15]

Here, ψ^i_{-l} is a fermionic creation operator for a fermion with spin $l = \frac{1}{2}, \frac{3}{2}, \ldots$ and the superscript i indexes the representation. Similarly, α^i_m is a bosonic creation operator for a boson with spin $m = 1, 2, 3, \ldots$ and representation indexed by i. The fermionic states $\psi^i_{-l} |\Omega\rangle$ transform as T and similarly for the bosonic states $\alpha^i_{-m} |\Omega\rangle$. If two types of states are transforming as S and T, together they transform as $S \oplus T$. Furthermore, a state having several fermionic creation operators on $|\Omega\rangle$ transforms as the exterior powers of T. Lastly, states with bosonic and fermionic creation operator transform as the tensor product of the individual representations. Hence, the states on the right-hand side transform precisely as given by the representations on the left-hand side. Now, let E be a vector bundle over M. Define the following formal series of bundles

$$S_t E := \bigoplus_{k=0}^{\infty} (S^k E) t^k \qquad \text{and} \qquad \Lambda_t E := \bigoplus_{k=0}^{\infty} (\Lambda^k E) t^k.$$

where $S^k E$ and $\Lambda^k E$ denote the symmetric and alternating bundles, respectively. With his interpretation, Witten argued that the representations R_k can be assembled into a formal series

$$\bigoplus_{k \geq 0} R_k q^{k/2} = \bigotimes_{l \in \{\frac{1}{2}, \frac{3}{2}, \ldots\}} \Lambda_{q^l} T \bigotimes_{m \in \{1,2, \ldots\}} S_{q^m} T.$$

Now, when evaluated, the index of the Dirac operator twisted with the above representation yields a power series in q.

$$\widehat{A}(M) \operatorname{ch}\Big[\bigotimes_{l \in \{\frac{1}{2}, \frac{3}{2}, \ldots\}} \Lambda_{q^l} T \bigotimes_{m \in \{1,2, \ldots\}} S_{q^m} T \Big][M].$$

Here $\operatorname{ch}(-)$ denotes the Chern character of the complexified bundles which is defined as follows. Let $x_1, \ldots, x_n$ be the Chern roots of a complex vector bundle ξ of dimension n. We define the *Chern character* to be

$$\operatorname{ch}(\xi) := \sum_{i=1}^{n} e^{x_i}.$$

For complex vector bundles ξ and η, the Chern character satisfies

$$\begin{aligned} \operatorname{ch}(\eta \oplus \xi) &= \operatorname{ch}(\eta) + \operatorname{ch}(\xi) \\ \operatorname{ch}(\eta \otimes \xi) &= \operatorname{ch}(\eta) \operatorname{ch}(\xi). \end{aligned}$$

Furthermore, the Chern character evaluated on the bundles $S_t E$ and $\Lambda_t E$ yields the following.

$$\operatorname{ch}(S_t E) = \prod_{i=1}^{n} \frac{1}{1 - t e^{x_i}} \qquad \text{and} \qquad \operatorname{ch}(\Lambda_t E) = \prod_{i=1}^{n} 1 + t e^{x_i}$$

where n is the dimension and x_i are the Chern roots of E. We refer to Section 1.5 of [HBJ92] for more details. Now, evaluating the Chern characters of the vector bundles above, yields an expression for the universal elliptic genus. However, this is not the same q-expansion as in Section 3.2, but corresponds to the choice of the 2-division point $e_3 = \wp(\frac{\omega_1+\omega_2}{2})$. One obtains up to a factor the q-expansion $\varphi_3(M)(q)$ of the universal elliptic genus constructed from e_3. There are two other bundles that yield the universal elliptic genera constructed from e_1 and e_2, that is with characteristic power series $Q(x)$ and $Q_2(x)$ as discussed in Section 3.2. For more details, we refer to [Wit88], and to Sections 6.1 and 6.2 of [HBJ92] for a more mathematical account on this. See also [Liu96] for an expression of the three expansions for the universal elliptic genus in terms of θ-functions.

We have seen that the $\widehat{A}$-genus arises as the index of the Dirac operator. Along the lines of [Wit82], Witten interprets the Dirac operator as the supercharge of supersymmetric quantum mechanics. In supersymmetric quantum field theory, Witten proposed that the elliptic genus should also be the index of a suitable operator, the Dirac-Ramond operator (see [Wit85]). This can be interpreted as a formal extension of the Dirac operator to the *free loop space*

$$\mathcal{L}M = C^\infty(S^1, M)$$

of a manifold M. However, the free loop space is an infinite dimensional manifold and there are no index theorems in infinite dimensions.

Witten's physical description is as follows. Consider a closed string type II string theory that has Ramond boundary condition for the right-movers and Neveu-Schwarz boundary conditions for the left-movers. The supercharge of the right-moving degrees of freedom is defined as

$$\mathcal{Q} = \int_0^{2\pi} d\sigma\, g_{ij}(X^k(\sigma))\partial_+ X^i(\sigma)\psi_+^j(\sigma).$$

Here, g_{ij} is the metric of the target space manifold M, and X^k its coordinates with superpartners ψ^k. The operator satisfies $\mathcal{Q}^2 = L_0$ which is the Hamiltonian of the right-movers. Let F_R be the number operator for right-moving fermions. The supercharge operator satisfies

$$(-1)^{F_R}\mathcal{Q} + \mathcal{Q}(-1)^{F_R} = 0$$

and commutes with the momentum operator $P = \overline{L}_0 - L_0$. Hence, $\mathcal{Q}$ and P have the same eigenbasis. The character-valued index of $\mathcal{Q}$ is then formally defined as

$$F(q) = \sum_\lambda b_\lambda q^\lambda$$

where λ are the eigenvalues of the momentum operator P, and b_λ is the index of $\mathcal{Q}$ upon restriction to the eigenspace of the eigenvalue corresponding to λ. By considering the path integral formulation, Witten argues that the eigenvalues of P are of the form

$$\lambda = -\frac{n}{16} + \frac{k}{2}$$

where n is the dimension of the target space, and k is a natural number. This means

$$F(q) = q^{-n/16} \sum_{k\in\mathbb{Z}} b_k q^{k/2}.$$

To make the eigenvalues of $\mathcal{Q}$ accessible to calculation, the *large-volume limit* is considered by taking the metric of M to be $g = \lim_{t\to\infty} t g_0$, where g_0 is a fixed metric. From this, the eigenvalues of $\mathcal{Q}$ are given by the eigenvalues of the twisted Dirac operator. Since the eigenvalues are topological invariants of the theory, deforming the space does not change them. Hence, the b_k are equal to the index of the twisted Dirac operator $\mathrm{Ind}(\mathcal{D}_{R_k})$ and

$$\begin{aligned} F(q) &= q^{-n/16} \sum_{k\in\mathbb{Z}} \mathrm{Ind}(\mathcal{D}_{R_k}) q^{k/2} \\ &= q^{-n/16} \Big(\widehat{A}(M)\,\mathrm{ch}\Big[\bigotimes_{l\in\{\frac{1}{2},\frac{3}{2},\,\dots\}} \Lambda_{q^l} T \bigotimes_{m\in\{1,2,\,\dots\}} S_{q^m} T \Big]\Big)[M] \\ &= \left(\frac{\eta\left(-q^{1/2}\right)}{\eta(q)\eta(-q)} \right)^d \varphi_3(M)(q) \end{aligned}$$

hence yielding the universal elliptic genus up to a factor. Here d is the dimension of M. See [Wit87] for more details. Recall that for a spin manifold M with an S^1-action that lifts to the spin bundle, the $\widehat{A}$-genus is rigid. Witten conjectured that if the action S^1 commutes with the twisted Dirac operator, the character-valued index of the twisted Dirac operator is constant. This has been proved by Bott-Taubes in [BT89].

As a result of the correspondence of states and representations and the definition of $\mathcal{Q}$, the character-valued index of $\mathcal{Q}$ can be expressed in terms of the *partition function* of the type II superstring that is considered. That is, we have the following expression.

$$F(q) = \mathrm{Tr}\left[(-1)^{F_R} q^{\overline{L}_0} \overline{q}^{L_0} \right]$$

Here, F_R is the fermion number of right-moving degrees of freedom. Recall that L_0 and $\overline{L}_0$ are the Virasoro generators for the right- and left-moving degrees of freedom respectively. Furthermore, recall that $\mathcal{Q}^2 = L_0$. States in the right-moving sector with non-zero eigenvalues L_0 are related by supersymmetry and cancel out of the trace because of the $(-1)^{F_R}$. Hence, $F(q)$ only depends on q and not on $\overline{q}$. Note this is the original equation (13) in [Wit87]. In the later physics literature, the conventions for right- and left-moving degrees of freedom are interchanged.

6.3. The DMVV-formula

In this section, we will consider further developments and applications of the string theoretic version of the elliptic genus. First, we will discuss the *two-variable elliptic genus* as introduced in physics in [Wit94] and [KYY94] for superconformal field theories, and reviewed mathematically in [BL00b]. See also [Wen15] for a more detailed comparison of the two perspectives.

After this, we will introduce the DMVV-formula [DMVV97] relating the two-variable elliptic genus of a manifold to the two-variable elliptic genus of its symmetric products. We will discuss the physical setting of a counting formula for dyons in which the DMVV-fomula has been conjectured in [DVV97].

Finally, we briefly allude to the work of Borisov-Libgober [BL03] and [BL00a] developing a theory of orbifold elliptic genera for singular varieties and orbifolds and proving the DMVV-formula.

The two-variable elliptic genus

The string theoretic version of elliptic genus has been further extended by Witten in [Wit94] considering minimal models and Landau-Ginzburg models. Building on Witten's work, Kawai-Yamada-Yang [KYY94] considered the elliptic genus for $N = 2$ superconformal field theories (SCFTs). These are conformal field theories that have supersymmetry where N denotes the type of supersymmetry. For an introduction to SCFTs and their description in terms of superconformal algebras, we refer to [BP09]. The $N = 2$ superconformal algebra contains a $U(1)$-current. Thus, the definition of the elliptic genus depends on an additional parameter corresponding to this current. This version of the elliptic genus is therefore sometimes referred to as the *two-variable elliptic genus*.

Let J_0 be the $U(1)$-charge operator for the left-moving degrees of freedom, that is, the zero modes of the $U(1)$ current $J(z)$. Furthermore, let L_0 and $\overline{L}_0$ be Virasoro generators for the left- and right-moving[4] degrees of freedom respectively. The elliptic genus for the superconformal field theory is then defined as

$$Z(\tau, z) = \mathrm{Tr}\left[(-1)^F q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24} y^{J_0}\right].$$

Here, $q = \exp(2\pi i\tau)$, $y = \exp(2\pi i z)$, and $F = F_L - F_R$ is the fermion number operator. The constants c and $\overline{c}$ represent the central charge of the theory and the trace is taken over the Hilbert space of the Ramond sector. The expression on the right-hand side does

[4]Note that now the convention for notation has been switched between left and right for L_0 and $\overline{L}_0$.

not depend on $\overline{\tau}$ since the states cancel out in trace due to supersymmetry. However, it is customary to include the term $\overline{q}^{\overline{L}_0-\overline{c}/24}$ for completeness.

Note that this is the usual form of the elliptic genus in the physics literature while it differs slightly from Witten's original conventions. Note that in the previous section, the central charge was not written out explicitly.

If the theory admits the description of a sigma model, the two-variable elliptic genus can be mathematically defined for compact complex manifolds. Let M be a compact complex manifold and let T be the holomorphic tangent bundle of M, and T^* the cotangent bundle. Then, the two-variable elliptic genus for M is defined as

$$\chi(M;\tau,z) = \mathrm{Td}(M)\,\mathrm{ch}(\mathcal{E}_{q,y})[M]$$

where $q = \exp(2\pi i\tau)$, $y = \exp(2\pi i z)$ and $\mathcal{E}_{q,y}$ is the bundle

$$\mathcal{E}_{q,y} = y^{-\frac{\dim M}{2}}\left(\bigotimes_{n\in\mathbb{N}}\Lambda_{-yq^{n-1}}T^*\otimes\Lambda_{-y^{-1}q^n}T\bigotimes_{n\in\mathbb{N}}S_{q^n}T^*\otimes S_{q^n}T\right).$$

The expression in this form is taken from [BL00b, eq. 3] and was originally proposed in terms of θ-functions in Section 5.5 of [KYY94]. The target manifolds considered in string theory are usually Calabi-Yau manifolds. If M is a Calabi-Yau manifold of complex dimension d, then the two-variable elliptic genus $\chi(M;\tau,z)$ is a *weak Jacobi form* of weight 0 and index $d/2$. This is shown in Theorem 2.2 in [BL00b]. See Appendix A.1 for a definition of weak Jacobi forms.

Note that the above expression in terms of bundles on M has a similar form as the elliptic genus considered by Witten in [Wit87]. The difference is that here, the Todd-genus is used in place of the $\widehat{A}$-genus since complex manifolds are considered. The two-variable elliptic genus is related to the universal elliptic genus of level N that we have seen in Section 3.3 at least when Calabi-Yau manifolds are considered. This relationship is as follows.

Let M be a compact Calabi-Yau manifold of complex dimension d, then

$$\varphi_N(M)(\tau) = \frac{\eta(\tau)^{3d}}{\theta(-\frac{1}{N},\tau)^d}\,\chi(M;\tau,\tfrac{1}{N})$$

where $\varphi_N(M)(\tau)$ is the universal elliptic genus of level N evaluated on M as function of τ (see Example 3.28). Furthermore, the η-function and the θ-function are given by

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

$$\theta(z,\tau) = q^{\frac{1}{8}}(2\sin\pi z)\prod_{n=1}^{\infty}(1-q^n)\prod_{n=1}^{\infty}(1-q^n y)(1-q^n y^{-1})$$

with $q = \exp(2\pi i\tau)$ and $y = \exp(2\pi i z)$. For the proof, we refer to Proposition 2.4 in [BL00b][5]. Furthermore, for more details on the relationship between conformal field theoretic elliptic genera and the mathematical counterpart, we refer to [Wen15]. There, also a relationship of the elliptic genus of the $K3$-surface with *Mathieu Moonshine* is explained. This has been originally discovered in [EOT10]. See also [GHV10].

The DMVV-formula

Let M be a Kähler manifold. The *N-th symmetric product* of M

$$\Sigma^N M = M^{\times N} /\!\!/ \Sigma_N$$

is the orbifold quotient of the symmetric group Σ_N acting on the N-fold Cartesian product $M^{\times N}$. With this, the *DMVV-formula* [DMVV97] reads

$$\sum_{N=0}^{\infty} p^N \chi(\Sigma^N M; \tau, z) = \prod_{\substack{n>0,\ m\geq 0 \\ l\in\mathbb{Z}}} \frac{1}{(1 - p^n q^m y^l)^{c(nm,l)}}$$

where $c(k,l)$ are the coefficients of the two-variable elliptic genus of M

$$\chi(M; \tau, z) = \sum_{k\geq 0} \sum_{l\in\mathbb{Z}} c(k,l) q^k y^l$$

with $q = \exp(2\pi i\tau)$ and $y = \exp(2\pi i z)$. The coefficients can be obtained by expanding the two-variable elliptic genus of M in powers of q and y. If M is a Calabi-Yau manifold and its elliptic genus has the property of a weak Jacobi form, the coefficients $c(k,l)$ only depend[6] on the quantity $4k - l^2$.

In [DMVV97], Dijkgraaf-Moore-Verlinde-Verlinde provide physical arguments to derive this formula by relating partition functions of different string theories. Every summand on the left-hand side of the DMVV-formula can be regarded as a partition function of a string on the target spacetime $\Sigma^N M \times S^1 \times \mathbb{R}$ where the string winds around the S^1 with winding number 1. The right-hand side is described by the partition function of a second-quantized strings on $M \times S^1$ where the coefficients $c(nm,l)$ correspond to the number of bosonic and fermionic states of the string wound around S^1 with winding number n and momentum m, and where $F_L = l$.

It is shown that the Hilbert space of the theory on the left-hand side decomposes into Hilbert spaces that correspond to string states wound n times around S^1. This decomposition is depending on conjugacy classes of the symmetric group Σ^N. Then, the identity

[5] In terms of the convention in Proposition 2.4 in [BL00b], we have chosen $\alpha = 0$ and $\beta = 1$ for our primitive N-division point. The characteristic series for the universal elliptic genus of level N used there is given on pp. 456-457. Our convention for the N-torsion point correspond to $k = 0$, $l = 1$.

[6] For a proof of this, see Theorem 2.2 in [EZ85] and the discussion on weak Jacobi forms on pp. 104-105.

for the elliptic genera is derived by considering the partition functions and using the additive and multiplicative properties of the elliptic genus.

The physical origins of the DMVV-formula

The origins of the DMVV-formula lie in the earlier work by Dijkgraaf-Verlinde-Verlinde [DVV97] where this formula is conjectured for the $K3$-surface. We will now briefly outline their considerations.

One of their aims was to provide a formula that counts certain states of a string theory to obtain a microscopic explanation for the macroscopic *Bekenstein-Hawking entropy*

$$S_{BH} = \frac{A}{4}$$

which is a well-known result in the theory of black holes. Here, A is the area of the event-horizon and depends on parameters such as the charge according to the specific configuration of the black hole. This builds on work by Strominger-Vafa [SV96] where 5-dimensional extremal black holes are considered. For details on the theory of black holes in string theory, we refer to Chapter 11 of [BBS06].

In [DVV97], a degeneracy formula is derived that is counting microscopic states of *dyons* in a 4-dimensional string theory with $N = 4$ supersymmetry, and it is shown to also reproduce the Bekenstein-Hawking entropy formula for 4-dimensional extremal black holes. Dyons are (hypothetical) particles that carry both magnetic and electric charge. To consider the dyonic spectrum of a $N = 4$ string theory in 4 dimensions, two perturbative theories are described that are dual to each other via the so-called *string-string duality*: type II string theory compactified on $K3 \times T^2$ where T^2 is the 2-torus, and heterotic string theory compactified on the 6-torus T^6.

Using the heterotic formulation, the electric and magnetic charges are described as elements on the Lorentzian lattice $q_e, q_m \in \Gamma^{22,6}$. For this, a special class of states called *BPS states* is considered. BPS states are unchanged under small changes of the theory and lend themselves to computations in both perturbative formulations. States that are either purely electric or purely magnetic are 1/2 BPS states, that is, these are BPS states that preserve 1/2 of the supersymmetry charges. General dyonic states however, are 1/4 BPS states. The degeneracy formula counting the dyonic states is given as

$$d(q_e, q_m) = \oint d\rho d\sigma d\nu \frac{\exp\big(i\pi(q_m^2\rho + q_e^2\sigma + 2q_e \cdot q_m\nu)\big)}{\Phi(\rho, \sigma, \nu)}.$$

Here, ρ, σ and ν lie in the interval $(0, 1)$ and are parameters for the Narain lattice $\Gamma^{3,2}$ (see Appendix of [DVV97]). They transform under the group $SL_2(\mathbb{Z})$ considered as a subgroup of $\mathrm{SO}(3, 2; \mathbb{Z})$ acting on the lattice, and the denominator $\Phi(\rho, \sigma, \nu)$ is invariant under transformation of $SL_2(\mathbb{Z})$. It is shown that this degeneracy formula matches the asymptotic growth of the Bekenstein-Hawking entropy in the large-charge limit

$$d(q_e, q_m) \sim \exp\Big(\pi\sqrt{q_e^2 q_m^2 - (q_e \cdot q_m)^2}\Big)$$

where the entropy is

$$S = \pi\sqrt{q_e^2 q_m^2 - (q_e \cdot q_m)^2}.$$

In the dual formulation of type II string theory, these states amount to five-branes wrapped around the $K3$-surface. After restricting the supersymmetry, the five-brane states that have precisely the right configuration can be counted with the two-variable elliptic genus for the $K3$-surface

$$\chi(K3; \tau, z) = \mathrm{Tr}(-1)^{F_L - F_R} q^{(L_0 - c/24)} y^{J_0}.$$

As usual, $q = \exp(2\pi i\tau)$ and $y = \exp(2\pi i z)$. Furthermore, F_L and F_R are the fermion numbers and correspond to the zero-modes of the $U(1)$-current. For the $K3$ surface, we have $c = 6$. With this, the DMVV-formula

$$\sum_{n=0}^{\infty} \exp(2\pi i n\sigma)\, \chi(\Sigma^n K3; \sigma, \nu) = \prod_{\substack{n>0,\ m\geq 0 \\ l\in\mathbb{Z}}} \frac{1}{(1 - \exp(2\pi i(n\rho + m\sigma + l\nu)))^{c(nm,l)}}$$

for the $K3$-surface was conjectured in [DVV97] as a generalization of a known generating function for counting 1/2 BPS states that was given in terms of the Euler characteristic of $K3$ and its symmetric powers. Compare the above with the general DMVV-formula replacing the complex exponentials with periodic variables accordingly.

Orbifold elliptic genera and a proof of the DMVV-formula

To provide a mathematical proof of the DMVV-formula, Borisov-Libgober [BL03] (see also the review [BL00a]) developed a mathematical theory of orbifold elliptic genera from the point of view of algebraic geometry.

In their paper [BL03], Borisov-Libgober construct the orbifold elliptic genus of X/G where X is an algebraic variety with faithful action by a finite group G (see Definition 4.1 in [BL03]) using the so-called *holomorphic Lefschetz number*. The DMVV-formula for their definition of the orbifold elliptic genus is then proved for smooth varieties X in Theorem 4.4 in [BL03]. After having introduced the necessary mathematical formalism, the proof the DMVV-formula goes along the original proposed arguments in [DMVV97]. Recall that for Calabi-Yau manifolds, the two-variable elliptic genus is related to the elliptic genus of level N. Calabi-Yau manifolds can be viewed from both an complex differential geometric and an complex algebraic point of view since they can be hypersurfaces of certain complex projective varieties.

In [BL03], another type of elliptic genus introduced: the *singular elliptic genus* (see Definition 3.1 in [BL03]). This defines an invariant of a specific type of singular variety in terms of the Jacobi θ-functions similar to how the two-variable elliptic genus can be defined.

7. Conclusion and Outlook

The theory of elliptic genera relates the study of bordism invariants with elliptic curves, modular forms, formal group laws, and index theory. Furthermore, they have applications in string theory. We provided general background and first definitions as well as an overview over the different aspects of elliptic genera and their perspectives in mathematics and physics.

In Chapter 2, we introduced the geometric and homotopical foundations to describe bordism. For this, we defined stable structures on vector bundles to endow manifolds with orientation and stably almost complex structures and construct the bordism rings Ω_*^{SO} and Ω_*^{U}. Furthermore, we introduced Thom spectra MSO and MU whose homotopy groups compute the structure of the bordism rings. The isomorphism is given by the Pontryagin-Thom construction.

We introduced elliptic genera in Chapter 3 as ring homomorphisms out of Ω_*^{SO} whose logarithm is given by an elliptic integral. Examples are given by geometrically relevant bordism invariants such as the $\widehat{A}$-genus or the signature. We also constructed the universal elliptic genus $\varphi\colon \Omega_*^{\mathrm{SO}} \to \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$ which assigns to oriented manifolds modular forms for the congruence subgroup $\Gamma_0(2)$. More generally, for each N there is a universal elliptic genus of level N assigning modular forms for $\Gamma_1(N)$ to almost complex manifolds.

Complex genera are in one-to-one correspondence to formal group laws. We have defined formal group laws in Chapter 4 and described the connection to complex genera through Quillen's theorem. With this, we gave the description of complex and elliptic genera in terms of their formal group laws. Finally, we provided a more geometric description of formal group laws from the perspective of formal groups.

In Chapter 5, we generalized the theory of bordism and formal group laws to the equivariant setting. For this, we defined the equivariant complex bordism ring $\Omega_*^{U,G}$ and the homotopical version $\pi_*^G(MU_G)$ for finite groups G. The equivariant Pontryagin-Thom construction is well-defined, but not an isomorphism. Furthermore, for G abelian, we defined G-equivariant formal group laws which are classified by the equivariant Lazard ring L_G. Since the equivariant version of Quillen's theorem holds, we can consider equivariant complex genera obtained from equivariant formal group laws. With this, we proposed the definition of equivariant elliptic genera as equivariant genera coming from an equivariant formal group law of an elliptic curve. Further research is needed to construct examples. For this, previous work of the case $G = \mathbb{Z}/2$ in the theory of equivariant bordism and equivariant formal groups can be used. Another application of the generalization to G-equivariant genera would be the construction of genera for global quotient orbifolds. A suitable definition of this can then be compared with existing notions of orbifold elliptic genera as considered in [BL03], [AF07] and [Gan04].

The physics perspective on elliptic genera was discussed in Chapter 6. This description is due to Witten who gave a string theoretic interpretation of elliptic genera as partition functions of supersymmetric string theories. Witten formally described an operator on the free loop space whose index is the universal elliptic genus. This has been further developed from a physical perspective in terms of the two-variable elliptic genus which relates to the universal elliptic genus of level N. The main example for us was the DMVV-formula suggesting a definition of an elliptic genus for the symmetric product of a manifold which ultimately gave rise to the first mathematical theory of orbifold elliptic genera in [BL03]. The elliptic genus of the K3-surface has also played a role in Moonshine phenomena related to the Mathieu groups.

A. Appendix

A.1. Modular forms

In this appendix, we collect elements of the theory of modular forms relevant for our treatment of elliptic genera. We refer to [BD20] and [DS05]. Also compare Appendix I of [HBJ92].

Modular forms for the modular group

The *modular group*

$$SL_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

acts on the upper half plane by

$$\gamma\tau = \tfrac{a\tau+b}{c\tau+d}, \quad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

The *fundamental domain* of the action of $SL_2(\mathbb{Z})$ is the open set

$$\mathcal{D} := \{\tau \in \mathcal{H} \mid -\tfrac{1}{2} < \mathrm{Re}(\tau) < \tfrac{1}{2} \text{ and } |\tau| > 1\} \subseteq \mathcal{H}.$$

Its closure contains a representative for every orbit of the action. Two distinct points $\tau, \tau' \in \overline{\mathcal{D}}$ that lie in the same orbit are on the boundary of $\mathcal{D}$. For more details, we refer to Theorem 1.5 in [BD20]. The modular group $SL_2(\mathbb{Z})$ is generated by the matrices

$$T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \qquad \text{and} \qquad S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

Note that in some of the literature, $-S$ is taken instead of S.

Definition A.1. A meromorphic function $f\colon \mathcal{H} \to \mathbb{C} \cup \{\infty\}$ is called *weakly modular* of weight k if it satisfies the following transformation under the action of $SL_2(\mathbb{Z})$.

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \qquad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

To define modular forms, we also need *holomorphicity at* ∞. The point at ∞ is often called the *cusp*. For this, consider the holomorphic function

$$\begin{aligned} q\colon \mathcal{H} &\to \mathbb{C} \\ \tau &\mapsto \exp(2\pi i\tau). \end{aligned}$$

If f is a meromorphic function that is weakly modular of weight k, then in particular, $f(\tau+1) = f(\tau)$ by taking the matrix T. Hence, we can write f as

$$f(\tau) = \widetilde{f}(\exp(2\pi i\tau)) = \widetilde{f}(q)$$

for a meromorphic function $\widetilde{f}\colon D^* \to \mathbb{C} \cup \{\infty\}$ on the punctured open unit disk

$$D^* = \{q \in \mathbb{C} \mid |q| < 1\} \setminus \{0\}.$$

Definition A.2. A meromorphic function $f\colon \mathcal{H} \to \mathbb{C} \cup \{\infty\}$ that is weakly modular of weight k is called *meromorphic at* ∞ if the associated function $\widetilde{f}$ can be continued to the unpunctured open unit disk $D = \{q \in \mathbb{C} \mid |q| < 1\}$. If $\widetilde{f}$ does not have a pole at $q = 0$, then f is called *holomorphic at* ∞. If f is holomorphic at ∞, the series expansion

$$\widetilde{f}(q) = \sum_{n=0}^{\infty} a_n q^n$$

is called the *q-expansion* of f.

Note that usually, we denote $\widetilde{f}$ simply by f, using that $q = \exp(2\pi i\tau)$. We are now ready to define modular forms.

Definition A.3. A *modular form* of weight k for the group $SL_2(\mathbb{Z})$ is a holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ that satisfies the following conditions.

(i) f is weakly modular of weight k,

(ii) f is holomorphic at ∞.

If, additionally, $f(\infty) = 0$, we say that f is a *cusp form*. This is equivalent to $a_0 = 0$ in the q-expansion of f.

The set M_k of modular forms of weight k forms a vector space over $\mathbb{C}$. The spaces M_k can be assembled into a graded ring since the weight of modular forms multiplies. The so-called *valence formula* computes the dimension of M_k. In particular, there are no modular forms of odd weight k, and the dimensions of the space S_k of cusp forms is related to the space of modular forms M_k. See Section 3.5 of [DS05] or Sections 2.7 and 2.8 of [BD20].

Example A.4. Let $k \geq 4$ and define the *Eisenstein series* G_k as the holomorphic function

$$G_k \colon \mathcal{H} \to \mathbb{C}, \quad \tau \mapsto \sum_{\substack{(m,n)\in\mathbb{Z}\oplus\mathbb{Z} \\ (m,n)\neq 0}} \frac{1}{(m\tau+n)^k}.$$

The Eisenstein series G_k is a modular form for $SL_2(\mathbb{Z})$ of weight k. One typical normalization of the Eisenstein series is by normalizing the coefficient $a_1 = 1$. The *normalized Eisenstein series* of weight k is denoted by E_k. We have the following q-expansion

$$\begin{aligned} E_k(\tau) &= \frac{(k-1)!}{2(2\pi i)^k} G_k(\tau) \\ &= -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \right) q^n \end{aligned}$$

where B_k are the *Bernoulli numbers*. For more details, see Sections 2.2 and 2.3 of [BD20]. Note that the definition of the Eisenstein series G_k is the same as $s_k(\mathbb{Z} \oplus \mathbb{Z}\tau)$ in Section 3.2.

Example A.5. Define the modular form $\Delta \colon \mathcal{H} \to \mathbb{C}$ by

$$\Delta(\tau) = \frac{(240E_4(\tau))^3 - (504E_6(\tau))^2}{1728}.$$

Since E_4 and E_6 are modular forms of weight 4 and 6 respectively, the functions Δ is a modular form of weight 12 for $SL_2(\mathbb{Z})$. We have the following relation $\Delta(\tau) = \eta(\tau)^{24}$ where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

is the Dedekind η-function. For more details, we refer to Sections 2.5 and 2.6 of [BD20].

Example A.6. The *j-function* is defined as

$$j(\tau) = \frac{(240E_4(\tau))^3}{\Delta(\tau)}.$$

This is a *modular function.* That is, j is a meromorphic function on $\mathcal{H}$ that is meromorphic at ∞ and $j(\gamma\tau) = j(\tau)$. It is not holomorphic at ∞ since Δ is a cusp form and E_4 is not. See Section 2.5 of [BD20]. The j-function has q-expansion

$$j(q) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

These coefficients are related to the dimensions of the irreducible representations of monster group. This is the famous unexpected connection between representation theory and modular forms that was titled *monstrous moonshine.*

Modular forms for congruence subgroups

In Sections 3.2 and 3.3, we have seen that elliptic genera take values in modular forms for $\Gamma_0(2)$ and $\Gamma_1(N)$. We will now give the definition of modular forms for those subgroups.

Define a subgroup of $SL_2(\mathbb{Z})$ by

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N \right\}.$$

Definition A.7. A *congruence subgroup* of level N is a subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$ such that $\Gamma(N) \subseteq \Gamma$ for an integer $N \geq 1$. We call $\Gamma(N)$ the *principal congruence subgroup* of level N.

The most important examples of congruence subgroups are given by

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mod N \right\}$$
$$\Gamma_1(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mod N \right\}$$

Note that in general, $\Gamma_1(N) \subseteq \Gamma_0(N)$. In the case of $N = 2$, we have $\Gamma_1(2) = \Gamma_0(2)$ by considering the determinant mod 2. To extend the definition of modular forms to congruence subgroups, we make the following definition.

Definition A.8. Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup. A meromorphic function $f\colon \mathcal{H} \to \mathbb{C} \cup \{\infty\}$ is called *weakly modular* of weight k for Γ if it satisfies

$$f(\gamma z) = (cz + d)^k f(z) \qquad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

Next, we extend the notion of cusps to congruence subgroups. The modular group $SL_2(\mathbb{Z})$ only has a cusp at ∞. Congruence subgroups will have more cusps, but always have a cusp at ∞.

Consider the rational projective line

$$\mathbb{Q}\mathrm{P}^1 = \mathbb{Q} \cup \{\infty\}.$$

Let $t \in \mathbb{Q}\mathrm{P}^1$ and let

$$\gamma t = \tfrac{at+b}{ct+d}, \quad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

This defines a transitive action of $SL_2(\mathbb{Z})$ on $\mathbb{Q}\mathrm{P}^1$ where we set $\gamma t = \infty$ if $ct + d = 0$, and $\gamma\infty = a/c$. For details, we refer to Section 3.2 of [BD20].

Definition A.9. Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup. The *set of cusps* of Γ is defined as

$$\text{Cusps}(\Gamma) = \Gamma \backslash \mathbb{Q}\text{P}^1$$

An equivalent definition is given by the double quotient

$$\text{Cusps}(\Gamma) = \Gamma \backslash SL_2(\mathbb{Z})/SL_2(\mathbb{Z})_\infty$$

since there is a bijection between $SL_2(\mathbb{Z})/SL_2(\mathbb{Z})_\infty$ and $\mathbb{Q}\text{P}^1$. Here, $SL_2(\mathbb{Z})_\infty$ is the stabilizer subgroup of the point ∞.

Example A.10. The congruence subgroup $\Gamma_0(2)$ has cusps at 0 and ∞.

In general, the cusps of congruence subgroups can be *regular* or *irregular*, and differ by their *width*. We will not defines these notions, but again refer to Section 3.2 of [BD20]. For a given cusp c of Γ and weakly modular function f for Γ, one can again define a function $\widetilde{f}_c$ on the punctured open unit disk D^* and ask whether it extends to the entire open unit disk D.

Definition A.11. Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup. A meromorphic function $f\colon \mathcal{H} \to \mathbb{C} \cup \infty$ that is weakly modular of weight k for Γ is called *holomorphic at the cusps of* Γ if for all cusps c, the function $\widetilde{f}_c(q_c)$ can be continued to the open disk and does not have a pole at $q_c = 0$.

With this, we are ready to define modular forms for congruence subgroups.

Definition A.12. Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup. A *modular form for* Γ *of weight* k is a holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ satisfying

(i) f is weakly modular of weight k for Γ,

(ii) f is holomorphic at the cusps of Γ.

If f vanishes at all cusps of Γ, we call f a *cusp form* for Γ.

Again, one can show that the set $M_k(\Gamma)$ of modular forms of weight k for Γ forms a $\mathbb{C}$-vector space and that these can be assembled into a graded ring.

Example A.13. The modular forms $\delta(\tau)$ and $\varepsilon(\tau)$ as defined in the end of Section 3.2 are examples of modular forms for $\Gamma_0(2)$. In particular, δ has weight 2, and ε has weight 4. The ring of modular forms for $\Gamma_0(2)$ is isomorphic to $\mathbb{C}[\delta, \varepsilon]$. This is Theorem 4.3 in Appendix I of [HBJ92] and similar statements hold after replacing $\mathbb{C}$ with $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{2}]$. Then those become the coefficient rings for the q-expansions of the modular forms for $\Gamma_0(2)$.

Example A.14. Another example of modular forms for the group $\Gamma_0(2)$ is provided by the Eisenstein series E_k^* of weight k for $\Gamma_0(2)$ defined as

$$E_k^*(\tau) = E_k(\tau) - 2^{k-1} E_k(2\tau)$$

See [Zag88] for more details[1] and relations to δ and ε.

Recall our discussion of the relation between modular forms and lattices in Section 3.2. Lattices of the form $\mathbb{Z} \oplus \mathbb{Z}\tau$ are parametrized by $\tau \in \mathcal{H}$ such that two lattices L and L' given by τ and τ' are the same if and only if τ and τ' lie in the same orbit. Upon restricting to the subgroup $\Gamma_1(N)$, we have the following proposition that relates modular forms for $\Gamma_1(N)$ with the choice of torsion points of a lattice.

Proposition A.15. *There is a bijection*

$$\Gamma_1(N)\backslash\mathcal{H} \longleftrightarrow \{(\mathbb{C}/L, P) \mid P \text{ is a primitive } N\text{-division point of } L\}/iso.$$

induced by the map $\tau \mapsto (L = \mathbb{Z} \oplus \mathbb{Z}\tau, P = \frac{1}{2})$.

Proof. This is Proposition 7.4 in the Appendix I of [HBJ92]. □

Lastly, we want to define Jacobi forms. We have seen in Section 3.2 that the function $\rho(x, \tau)$ defined from the characteristic series of the universal elliptic genus is a Jacobi form. (More precisely, this is proved in Theorem 5.6 in the Appendix I of [HBJ92]). Moreover, we have seen in Section 6.3 that the two-variable elliptic genus of a Calabi-Yau manifold is a weak Jacobi form. Other examples include the Weierstraß $\wp$-function when considered as a function of two variables.

Definition A.16. A *Jacobi form of weight k and index m* for $SL_2(\mathbb{Z})$ is a holomorphic function $f\colon \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the following equations

$$f\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k \exp\left(\frac{2\pi i m c z^2}{c\tau+d}\right) f(\tau, z), \qquad \text{for } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$$
$$f(\tau, z + \lambda\tau + \mu) = \exp\left(-2\pi i m \left(\lambda^2\tau + 2\lambda z\right)\right) f(\tau, z), \qquad \text{for } \lambda, \mu \in \mathbb{Z}$$

[1] This is equation (13) and there the notation G_k^* is used instead of E_k^*.

and having expansion

$$f(\tau, z) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm}} c(n, r) e^{2\pi i (n\tau + rz)} = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm}} c(n, r) q^n y^r$$

with $q = \exp(2\pi i \tau), y = \exp(2\pi i z)$.

Dropping the condition $r^2 \leq 4nm$ on the summation index r in the expansion of f, we obtain the notion of a *weak Jacobi form.* For more details on the theory of Jacobi forms, we refer to [EZ85].

Bibliography

[Abr13] William C. Abram. *Equivariant Complex Cobordism.* PhD thesis, University of Michigan, 2013.

[Ada74] J. F. Adams. *Stable homotopy and generalised homology.* Chicago Lectures in Mathematics. The University of Chicago Press, 1974.

[AF07] Matthew Ando and Christopher P. French. *Discrete torsion for the supersingular orbifold sigma genus*, volume 342 of *London Mathematical Society Lecture Note Series*, pages 1–25. Cambridge University Press, 2007.

[AH70] Michael Atiyah and Friedrich Hirzebruch. *Spin-Manifolds and Group Actions*, pages 18–28. Springer Berlin Heidelberg, 1970.

[ALR07] Alejandro Adem, Johann Leida, and Yongbin Ruan. *Orbifolds and Stringy Topology.* Cambridge Tracts in Mathematics. Cambridge University Press, 2007.

[AS68] M. F. Atiyah and I. M. Singer. The index of elliptic operators: III. *Annals of Mathematics*, 87(3):546–604, 1968.

[ASSU23] Andrés Angel, Eric Samperton, Carlos Segovia, and Bernardo Uribe. Oriented and unitary equivariant bordism of surfaces. https://arxiv.org/abs/2111.02693, 2023.

[Ati61] Michael F. Atiyah. Bordism and cobordism. *Proceedings of the Cambridge Philosophical Society*, 57:200–208, 1961.

[BB10] Victor Buchstaber and Elena Bunkova. Elliptic formal group laws, integral hirzebruch genera and krichever genera, 2010.

[BBS06] K. Becker, M. Becker, and J.H. Schwarz. *String Theory and M-Theory: A Modern Introduction.* Cambridge University Press, 2006.

[BD20] P. Bruin and S. Dahmen. Modular forms, 2020. http://www.few.vu.nl/~sdn249/modularforms16/notes.pdf.

[BH72] Theodor Bröcker and Edward C. Hook. Stable equivariant bordism. *Mathematische Zeitschrift*, 129(3):269–277, 1972.

[BL00a] Lev A. Borisov and Anatoly Libgober. Elliptic genera of singular varieties, orbifold elliptic genus and chiral deRham complex. https://arxiv.org/abs/math/0007126, 2000.

[BL00b] Lev A. Borisov and Anatoly Libgober. Elliptic genera of toric varieties and applications to mirror symmetry. *Inventiones mathematicae*, 140(2):453–485, 2000.

[BL03] Lev A. Borisov and Anatoly Libgober. Elliptic genera of singular varieties. *Duke Mathematical Journal*, 116(2):319–351, 2003.

[BLT13] R. Blumenhagen, D. Lüst, and S. Theisen. *Basic Concepts of String Theory*. Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 2013.

[Blu17] Andrew Blumberg. Equivariant homotopy theory. https://github.com/adebray/equivariant_homotopy_theory, Lecture notes published by Arun Debray, 2017.

[BP09] Ralph Blumenhagen and Erik Plauschinn. *Introduction to Conformal Field Theory: With Applications to String Theory*, volume 779 of *Lecture Notes in Physics*. Springer Berlin Heidelberg, 2009.

[BT89] Raoul Bott and Clifford Taubes. On the rigidity theorems of witten. *Journal of the American Mathematical Society*, 2(1):137–186, 1989.

[Bun19] E. Yu. Bunkova. Universal formal group for elliptic genus of level n. *Proceedings of the Steklov Institute of Mathematics*, 305:33–52, 2019.

[CC88] D.V. Chudnovsky and G. V. Chudnovsky. Elliptic modular functions and elliptic genera. *Topology*, 27(2):163–170, 1988.

[CF64] P. E. Conner and E. E. Floyd. *Differentiable Periodic Maps*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 33. Springer Berlin Heidelberg, 1964.

[CGK00] Michael Cole, J. P. C. Greenlees, and I. Kriz. Equivariant formal group laws. *Proceedings of the London Mathematical Society*, 81(2):355–386, 2000.

[CGK02] Michael Cole, John Greenlees, and Igor Kríz. The universality of equivariant complex bordism. *Mathematische Zeitschrift*, 239:455–475, 2002.

[CHS57] S. Chern, F. Hirzebruch, and J.-P. Serre. On the index of a fibered manifold. *Proceedings of the American Mathematical Society*, 8:587–596, 1957.

[CR18] N. Carqueville and I. Runkel. Introductory lectures on topological quantum field theory. *Banach Center Publications*, 114:9–47, 2018.

[DFHH14] C.L. Douglas, J. Francis, A.G. Henriques, and M.A. Hill. *Topological Modular Forms*. Mathematical Surveys and Monographs. American Mathematical Society, 2014.

[DK01] James F. Davis and Paul A. Kirk. *Lecture notes in algebraic topology*, volume 35 of *Graduate Studies in Mathematics*. American Mathematical Society, 2001.

[DMVV97] Robbert Dijkgraaf, Gregory Moore, Erik Verlinde, and Herman Verlinde. Elliptic genera of symmetric products and second quantized strings. *Communications in Mathematical Physics*, 185(1):197–209, April 1997.

[Dol56] Albrecht Dold. Erzeugende der thomschen algebra $\mathfrak{N}$. *Mathematische Zeitschrift*, 65:25–35, 1956.

[DS05] Fred Diamond and Jerry Shurman. *A First Course in Modular Forms*, volume 228 of *Graduate Texts in Mathematics*. Springer New York, 2005.

[DVV97] Robbert Dijkgraaf, Erik Verlinde, and Herman Verlinde. Counting dyons in $\mathrm{N} = 4$ string theory. *Nuclear Physics B*, 484(3):543–561, January 1997.

[EH00] D. Eisenbud and J. Harris. *The Geometry of Schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 2000.

[EOT10] Tohru Eguchi, Hirosi Ooguri, and Yuji Tachikawa. Notes on the K3 surface and the mathieu group $\mathbb{M}_{24}$. *Experimental Mathematics*, 20(1):91–96, 2010.

[EZ85] M. Eichler and D. Zagier. *The Theory of Jacobi Forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston, 1985.

[Fre13] Daniel S. Freed. Bordism: Old and New. https://web.ma.utexas.edu/users/dafr/bordism.pdf, 2013.

[Fre21] Daniel S. Freed. The Atiyah-Singer index theorem. https://arxiv.org/abs/2107.03557, 2021.

[Gan04] Nora Ganter. Orbifold genera, product formulas and power operations. *Advances in Mathematics*, 205:84–133, 2004.

[GHV10] Matthias R. Gaberdiel, Stefan Hohenegger, and Roberto Volpato. Mathieu moonshine in the elliptic genus of K3. *Journal of High Energy Physics*, 2010(10), 2010.

[Gre00] J. P. C. Greenlees. The coefficient ring of equivariant homotopical bordism classifies equivariant formal group laws over noetherian rings, 2000. preprint.

[Gre01] J. P. C. Greenlees. Equivariant formal group laws and complex oriented cohomology theories. *Homology, Homotopy and Applications*, 3(2):225 – 263, 2001.

[Hal15] Brian Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, volume 222 of *Graduate Texts in Mathematics*. Springer International Publishing, 2015.

[Han05] Bernhard Hanke. Geometric versus homotopy theoretic equivariant bordism. *Mathematische Annalen*, 332:677 – 696, 2005.

[Hat02] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002.

[Hat17] Allen Hatcher. Vector bundles and K-theory, 2017. https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf.

[Hau22] Markus Hausmann. Global group laws and equivariant bordism rings. *Annals of Mathematics*, 195(3):841 – 910, 2022.

[HBJ92] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung. *Manifolds and Modular Forms*. Vieweg+Teubner Verlag Wiesbaden, 1992.

[HH90] Friedrich Hirzebruch and Thomas Höfer. On the Euler number of an orbifold. *Mathematische Annalen*, 286, 1990.

[Hir88] Friedrich Hirzebruch. Elliptic Genera of Level N for Complex Manifolds. *Differential Geometrical Methods in Theoretical Physics*, pages 37–63, 1988.

[Hir97] M.W. Hirsch. *Differential Topology*. Graduate Texts in Mathematics. Springer New York, 1997. Corrected reprint of the 1976 original.

[HSB95] F. Hirzebruch, R.L.E. Schwarzenberger, and A. Borel. *Topological Methods in Algebraic Geometry: Reprint of the 1978 Edition*. Classics in Mathematics. Springer Berlin-Heidelberg, 1995. Translated version of the 1956 original Neue Topologische Methoden in der Algebraischen Geometrie.

[HW18] Bernhard Hanke and Michael Wiemeler. An equivariant quillen theorem. *Advances in Mathematics*, 340:48–75, 2018.

[Koc96] S.O. Kochman. *Bordism, Stable Homotopy, and Adams Spectral Sequences*. Fields Institute for Research in Mathematical Sciences Toronto: Fields Institute monographs. American Mathematical Society, 1996.

[Kot12] Chris Kottke. Bundles, classifying spaces and characteristic classes, 2012. https://ckottke.ncf.edu/docs/bundles.pdf.

[KT91] R.C. Kirby and L.R. Taylor. *Pin structures on low-dimensional manifolds*, volume 2 of *London Mathematical Society Lecture Note Series*, page 177–242. Cambridge University Press, 1991.

[KYY94] Toshiya Kawai, Yasuhiko Yamada, and Sung-Kil Yang. Elliptic genera and N = 2 superconformal field theory. *Nuclear Physics B*, 414(1-2):191–212, 1994.

[Lan87] Serge Lang. *Elliptic Functions*, volume 112 of *Graduate Texts in Mathematics*. Springer New York, 1987.

[Lan88] P.S. Landweber, editor. *Elliptic Curves and Modular Forms in Algebraic Topology*, Lecture Notes in Mathematics. Springer, 1988.

[Las63] R. Lashof. Poincaré duality and cobordism. *Transactions of the American Mathematical Society*, 109:257–277, 1963.

[Lee13] J.M. Lee. *Introduction to Smooth Manifolds*. Number 218 in Graduate Texts in Mathematics. Springer New York, 2013.

[Liu96] Kefeng Liu. Modular forms and topology. *Contemporary Mathematics*, 193:237–262, 1996.

[LMS86] L. G. Lewis, J. P. May, and M. Steinberger. *Equivariant Stable Homotopy Theory*. Springer Berlin Heidelberg, 1986.

[LRS95] Peter S. Landweber, Douglas C. Ravenel, and R. Stong. Periodic cohomology theories defined by elliptic curves. *Contemporary Mathematics*, 181:317–337, 1995.

[LS88] Peter S. Landweber and Robert E. Stong. Circle actions on spin manifolds and characteristic numbers. *Topology*, 27(2):145–161, 1988.

[May99] J.P. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999.

[Mei18] Lennart Meier. Introduction to stable homotopy theory. https://webspace.science.uu.nl/~meier007/IntroStable.pdf, 2018.

[Mei22] Lennart Meier. From Elliptic Genera to Topological Modular Forms. https://webspace.science.uu.nl/~meier007/TMF-Lecture.pdf, 2022.

[Mil60] J. Milnor. On the cobordism ring Ω^* and a complex analogue, part I. *American Journal of Mathematics*, 82(3):505–521, 1960.

[Mil01] Haynes Miller. Notes on cobordism. https://math.mit.edu/~hrm/papers/cobordism.pdf, 2001.

[Moe02] Ieke Moerdijk. Orbifolds as Groupoids: an Introduction, 2002. https://arxiv.org/abs/math/0203100.

[Moo11] Ben Moonen. Chapter III: Basic theory of group schemes. https://www.math.ru.nl/personal/bmoonen/BookAV/BasGrSch.pdf, 2011.

[MPC96] J.P. May, R.J. Piacenza, and M. Cole. *Equivariant Homotopy and Cohomology Theory: Dedicated to the Memory of Robert J. Piacenza*. Regional conference series in mathematics. American Mathematical Society, 1996.

[MS74] J.W. Milnor and J.D. Stasheff. *Characteristic Classes*. Annals of mathematics studies. Princeton University Press, 1974.

[Nak18] M. Nakahara. *Geometry, Topology and Physics*. Taylor & Francis, 2018.

[Nov60] S. P. Novikov. Some problems in the topology of manifolds connected with the theory of Thom spaces. *Soviet Mathematics. Doklady*, 1:717–719, 1960.

[Och87] Serge Ochanine. Sur les genres multiplicatifs definis par des integrales elliptiques. *Topology*, 26(2):143–151, 1987.

[Pan11] Taras Panov. Complex bordism. http://www.boma.mpim-bonn.mpg.de/data/32print.pdf, 2011.

[Pon38] L. S. Pontrjagin. Classification of continuous maps of a complex into a sphere, Communication I. *Doklady Akademii Nauk SSSR*, 19(3):147–149, 1938. Translated by R. V. Gamkrelidze, 1987.

[Pon55] L. S. Pontrjagin. Smooth manifolds and their applications in homotopy theory. *Doklady Akademii Nauk SSSR*, 45, 1955. Translated in AMS Translation Series 2, Vol. 11, 1959.

[PSW87] K. Pilch, A.N. Schellekens, and N.P. Warner. Path integral calculation of string anomalies. *Nuclear Physics B*, 287:362–380, 1987.

[Qui69] Daniel G. Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bulletin of the American Mathematical Society*, 75:1293–1298, 1969.

[Sat56] I. Satake. On a generalization of the notion of manifold. *Proceedings of the National Academy of Sciences of USA the*, 42(6), 1956.

[Sch18] S. Schwede. *Global Homotopy Theory*. New Mathematical Monographs. Cambridge University Press, 2018.

[Sil09] J.H. Silverman. *The Arithmetic of Elliptic Curves*. Graduate Texts in Mathematics. Springer New York, 2009.

[Sin01] Dev P. Sinha. Computations of complex equivariant bordism rings. *American Journal of Mathematics*, 123(4):577–605, 2001.

[Sto68] R. E. Stong. *Notes on Cobordism Theory*. Mathematical notes. Princeton University Press, 1968.

[Sto69] R. E. Stong. Complex and oriented equivariant bordism. *Topology of Manifolds*, pages 291–316, 1969.

[Str99] Neil P. Strickland. Formal schemes and formal groups. *Contemporary Mathematics*, 239:263–352, 1999.

[Str01] Neil P. Strickland. Complex cobordism of involutions. *Geometry & Topology*, 5(1):335 – 345, 2001.

[Str19] Neil P. Strickland. Formal groups. https://strickland1.org/courses/formalgroups/fg.pdf, 2019.

[SV96] Andrew Strominger and Cumrun Vafa. Microscopic origin of the bekenstein-hawking entropy. *Physics Letters B*, 379(1-4):99–104, 1996.

[SW87] A.N. Schellekens and N.P. Warner. Anomalies, characters and strings. *Nuclear Physics B*, 287:317–361, 1987.

[Swi02] R.M. Switzer. *Algebraic Topology: Homotopy and Homology*. Classics in Mathematics. Springer, 2002.

[Tam01] Hirotaka Tamanoi. Generalized orbifold euler characteristic of symmetric products and equivariant Morava K-theory. *Algebraic & Geometric Topology*, 1(1):115–141, 2001.

[Tho52] René Thom. Espaces fibrés en sphères et carrés de steenrod. *Annales scientifiques de l'École Normale Supérieure*, 69(3):109–182, 1952.

[Tho54] René Thom. Quelques propriétés globales des variétés différentiables. *Commentarii Mathematici Helvetici*, 28:17–86, 1954.

[Thu02] William P. Thurston. The Geometry and Topology of Three-Manifolds. http://www.msri.org/publications/books/gt3m/, 2002.

[tom70] Tammo tom Dieck. Bordism of G-manifolds and integrality theorems. *Topology*, 9(4):345–358, 1970.

[Ton09] D. Tong. String theory, 2009. http://www.damtp.cam.ac.uk/user/tong/string.html, Lecture notes.

[Und11] R.G. Underwood. *An Introduction to Hopf Algebras*. Springer New York, 2011.

[Uri18] Bernardo Uribe. The evenness conjecture in equivariant unitary bordism. *Proceedings of the International Congress of Mathematicians (ICM 2018)*, 2018.

[Wal60] C. T. C. Wall. Determination of the cobordism ring. *Annals of Mathematics*, 72(2):292–311, 1960.

[Was69] Arthur G. Wasserman. Equivariant differential topology. *Topology*, 8(2):127–150, 1969.

[Wen15] Katrin Wendland. Snapshots of conformal field theory. *Mathematical Physics Studies*, pages 89–129, 2015.

[Wit82] Edward Witten. Supersymmetry and Morse theory. *Journal of Differential Geometry*, 17(4):661 – 692, 1982.

[Wit85] Edward Witten. Global anomalies in string theory. *Symposium on Anomalies, Geometry, Topology*, 1985.

[Wit87] Edward Witten. Elliptic genera and quantum field theory. *Communications in Mathematical Physics*, 109(4):525 – 536, 1987.

[Wit88] Edward Witten. The index of the Dirac operator in loop space. *Elliptic Curves and Modular Forms in Algebraic Topology*, pages 161–181, 1988.

[Wit94] Edward Witten. On the Landau-Ginzburg description of $N = 2$ minimal models. *International Journal of Modern Physics A*, 09(27):4783–4800, 1994.

[Zag88] Don Zagier. Note on the Landweber-Stong elliptic genus. *Elliptic Curves and Modular Forms in Algebraic Topology*, pages 216–224, 1988.