Utrecht University & Nikhef

Master Thesis

Calculation of Leading Power Jets in Soft-Collinear Effective Theory

Author:
Georgios Smyridis

Supervisor:
prof. dr. Eric Laenen

Second Examiner:
prof. dr. Wilke van der Schee

July 2023

Abstract

The importance of Soft Collinear Effective Theory (SCET) arises from its ability to address the challenges posed by phenomena involving disparate energy scales, where fixed/finite order perturbation theory fails due to the emergence of large logarithms. SCET provides a useful framework for breaking down the calculations into different components that depend on single energy scales, i.e. for derivations of factorisation theorems. The all-order expressions can then be obtained using scale evolution by solving the renormalization group equations (RGEs) for each of the pieces appearing in the factorisation formula. This step is known as resummation. In this master thesis presentation, I delve into the construction of SCET, focusing on the derivation of the leading power Lagrangian and the calculation of leading power jet functions with the SCET Lagrangian as well as regular QCD Lagrangian and compare the results.

To my parents, Chionas and Niki

Contents

Chapter 1

Introduction

The Standard Model (SM) is composed of the electroweak (EW) and strong sectors, with Quantum Chromodynamics (QCD) describing the strong interactions. The Large Hadron Collider (LHC) accelerator complex at CERN collides protons made up of quarks and gluons at very high energies ($\sim$ 13TeV), and thus a large part of the effort to increase the precision of theoretical predictions is devoted to improving the understanding of QCD process which inevitably features in the experiment.

Due to the phenomenon of Asymptotic Freedom, the strong coupling α_s becomes weak for processes involving large momentum transfers. Concretely, α_s becomes perturbative already at a few GeV. Therefore, theoretical calculations can be organised in a perturbative series and systematically improved by computations of higher and higher order corrections in the α_s expansion

$$\mathcal{M}\big(\alpha_s, \{p_i\}, \{p_o\}\big) = \sum_{n=1}^{\infty} \alpha_s^n \mathcal{M}^{(n)}\big(\{p_i\}, \{p_o\}\big). \tag{1.1}$$

Here, $\mathcal{M}$ is the quantity we are interested in calculating, and $\mathcal{M}^{(n)}$ are corresponding n-loop diagrams, both of which depend on the incoming and outgoing particles' momenta.

Naturally, the calculation of the higher order corrections becomes increasingly more complex. However, with the value of the strong coupling at $\alpha_s \sim 0.1$, the first order correction, so-called next-to-leading order (NLO), is around 10% of the leading order value. The next-to-next-to-leading order (NNLO) $\sim$ 1%, and so on. Therefore, the key point is that precise predictions can be obtained through calculation of the first few orders in perturbation theory, as these already describe the physical process well (fixed-order perturbation theory).

Enhanced Logarithms

Despite the clear success of the fixed-order calculations approach, there exist noteworthy drawbacks. It is a well-known fact that fixed-order perturbation theory is unreliable in application to precesses which involve widely separated scales. Example of such processes include production of particles near kinematic threshold. In such regions of the phase space, close to singular limits of the theory, the higher-order corrections are sumpplemented by large logarithms of the scale ratios, say ξ. These large logarithms multiply the small coupling constants which is a priori the expansion parameter of the theory. This leads to a dangerous situation that threatens the convergence of the perturbative series on which the predictive power of the theoretical calculations is based. Namely, the coupling constant α_s is no longer a reliable expansion parameter, since each next order in the perturbative expansion is numerically as important as the previous one. Hence, to make a reliable prediction we have to capture he all-order behaviour of these terms.

Factorisation theorems

The solution to this issue is to divide the problem into pieces which each depend only on one physical scale. This is known as derivation of a *factorisation theorem*. Most factorisation theorems are easy to understand intuitively. For example, the most basic factorisation theorem for the production of lepton pairs in proton-proton collisions (Drell-Yan) has the form

$$\sigma = \sum_{ij} \hat{\sigma}_{ij} \otimes f_{i/P} \otimes f_{j/P} \tag{1.2}$$

Here, the partonic cross section $\hat{\sigma}_{ij}$ describes the production of the two lepton from the two initial state partons i and j, while the parton distribution functions $f_{i/P}$ and $f_{j/P}$ express the probability of finding the partons i and j in each proton with certain momentum fractions in respect to the hadrons. The $\otimes$ denotes the convolution of all these functions. For more complicated processes, such as jet production, the factorisation formula exists and is more complicated. In particular, for the N–jet production in hadronic collisions, the cross section factorises into a partonic cross section $\hat{\sigma}_{ij,k_1,\ldots,k_N}$ (hard function), describing the underlying partonic process to produce N partons, convoluted with N jet functions, J_{k_i}, a soft function, S, and parton distribution functions, $f_{i,j/P}$,

$$\sigma = \sum_{i,j,k_l} \hat{\sigma}_{ij,k_1,\ldots,k_N} \otimes J_{k_1} \otimes \cdots \otimes J_{k_N} \otimes S \otimes f_{i/P} \otimes f_{j/P}. \tag{1.3}$$

The jet functions J_{k_i} are the final-state analog of the parton distribution functions. They describe how the final partons from the hard interactions evolve into the observed jets.

Resummation

The all-order expressions can then be obtained using scale evolution by solving the renormalization group equations (RGEs) for each of the pieces appearing in the factorisation formula. This step is known as *resummation*.

$$\frac{d\sigma}{d\xi} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1} \left[c_{nm}^{(-1)} \left(\frac{\log^m \xi}{\xi}\right)_+ + c_{nm}^{(0)} \log^m \xi + \cdots \right]. \tag{1.4}$$

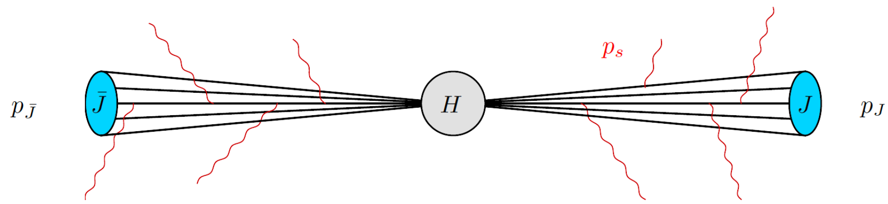


Figure 1.1: Two jet process

We see that the most singular terms in the $\xi \to 0$ are the ones with the coefficients $c_{nm}^{(-1)}$. We refer to these terms as *leading power* (LP) singular terms. Accordingly, the terms with coefficient $c_{nm}^{(0)}$ are known as *next-to-leading power* (NLP) contributions and so on for the terms in the ellipses. For the purposes of the resummation, instead of counting orders in α_s (LO, NLO, NNLO, etc.) as is done in the fixed-order calculations, we count which towers of logarithms are included in the all order result. For instance, the terms with coefficient $c_{n(2n-1)}$ constitute the *leading logarithms* (LL). Terms suppressed by one power in logarithmic counting are reffered to as *next-to leading logarithms* (NLL), and so on.

The current state-of-the-art is the resummation of LP threshold logarithms up to next-to-next-to-next-to leading logarithm (N^{3}LL) accuracy [4][10]. This has been achieved with diagrammatic methods as well as effective theories for QCD. Specifically, the "*Soft Collinear Effective Theory (SCET)*", which is the content of this thesis, is one of the youngest and most successful effective theories constructed specifically for this task. Good resources for further studying on SCET are [3][7].

Chapter 2 of this thesis is dedicated to the step-by-step construction of the leading power SCET Lagrangian. Moving forward to chapter 3, we delve into the computation of the leading power jet functions, which arise from the factorisation of processes such as $B \to X_s\gamma$. This computation is carried out using two different approaches: first, employing the SCET Lagrangian, and second, utilizing the regular QCD Lagrangian. Lastly, in chapter 4, we consolidate the key findings and present a summary, while also offering insights into potential avenues for future research.

Chapter 2

Soft-Collinear Effective Theory (SCET)

This chapter comprises the primary theoretical content of this thesis. In Section 2.1, we provide a concise overview of the rationale behind constructing effective field theories (EFTs), highlighting their advantages over more fundamental theories and emphasizing the essential components of an EFT. Moving forward, Section 2.2 initiates the development of SCET by delineating the kinematic regions it encompasses. Then, in Section 2.3, we outline the construction of the leading power SCET Lagrangian. Specifically, we commence with defining the relevant degrees of freedom and establish a power counting scheme for these fields. Additionally, we briefly explore the Multiple Expansion technique in 2.3.3, keeps the fields with the proper scaling in the soft-collinear interaction terms. The remainder of this section concludes the construction of the leading power Lagrangian. Next, Section 2.4 delves into the symmetries upheld by the leading power SCET Lagrangian, specifically gauge invariance and parametrization invariance. Lastly, Section 2.5 provides an in-depth elucidation of Wilson lines remarking their necessity and highlighting their role in the decoupling transformation.

2.1 Effective Field Theories

A remarkable fact about nature is that interesting phenomena occur over a wide range of energy, length, and time scales. The immense success of physical sciences is that they have managed to describe physical systems and make predictions in this huge range of scales; from down to quarks and leptons at the smallest scales, up to the universe as a whole at the largest, with qualitatively new kinds of structures – nuclei, atoms, molecules, everyday objects, planets, galaxies and so on. Given that all this complexity arises from a set of fundamental laws, it seems weird that one can understand what goes on at one scale without having to understand everything all at once. This fortunate fact, reflects a deep property of Nature called *decoupling*, which states that most (but not all) of the details of very small-distance phenomena tend to be largely irrelevant for the description at much larger scales.

For instance, in quantum mechanics, we are not concerned with the value of the top quark mass when we calculate the energy levels of a hydrogen atom. Of course, given certain precision of an experimental measurement, we might want to be concerned about that. Having this in mind, however, we would still insist that only degrees of freedom relevant to the problem in hand are needed to perform the calculation. In the language of quantum field theory, this implied that operators that are responsible for experimental observables only include fields describing light degrees of freedom. This realisation leads to the concept of effective field theories.

Effective field theories (EFTs) are a model building tool that explicitly implements the strategy outlined above and turns it into a precise, quantitative framework. We usually think of EFTs in terms of energy scales, but EFTs can be built by leveraging hierarchies between all sorts of dimensionful quantities: lengths, time, moments etc.

Main advantages of EFTs

EFTs (Effective Field Theories) serve various purposes in understanding physical systems. They are employed when a more fundamental theory is unavailable, or when extracting precise predictions from a strongly coupled theory is challenging. Additionally, even when not strictly necessary, EFTs offer a convenient framework. The key advantages of an EFT approach can be summarized as follows:

- Simplified Calculations: EFTs greatly simplify calculations by focusing on relevant degrees of freedom and disregarding nonessential aspects for the specific problem at hand.

- Unveiling Symmetries: Isolating relevant degrees of freedom in an EFT can reveal previously obscured symmetries. These symmetries often enable drawing general conclusions without extensive calculations.

- Modular Approach: EFTs facilitate modular calculations by dividing them into two parts. The first part involves the degrees of freedom retained in the EFT calculation, while the second part relates to the neglected physics (matching calculation). This modular approach eliminates unnecessary repetition.

- Resolving Scale-Dependent Issues: Problems involving multiple scales may generate observables dependent on logarithms of their ratios. Perturbative calculations can suffer from decreased accuracy when dealing with large logarithms. EFTs address this by focusing on one scale at a time and employing RG (Renormalization Group) running to sum up the significant logarithms.

Types of EFTs

One can group different EFTs in one of three categories, based on what degrees of freedom they include. The category that will be of interest in this thesis will be the newest and also the most controversial construction. This is an attempt to describe objects that have large energy-momentum transfers, but only in a given, fixed direction.

2.1.1 Main ingredients of EFTs

Now that we have hinted why using an effective theory will be useful in our study, we determine what are the key ingredients for every EFT, and explain their meaning and importance.

1. **Degrees of freedom**. The first step when building an EFT is to figure out what are the degrees of freedom that are relevant to describe the physical system one is interested in. These are the variables that will appear in the effective action. The key word here is "relevant": you can always complicate the description of any phenomenon by adding additional structure, but you will soon reach a point of diminishing returns. Conversely, you can strive for the most economical description but, everything should be made as simple as possible, but not simpler. Sometimes, the degrees of freedom to be used are suggested by symmetry considerations. More often though, the choice of degrees of freedom is an independent input.

2. **Symmetries**. The second step in building an EFT consists in identifying the symmetries that constrain the form of the effective action, and therefore the dynamics of the system. Any term that is compatible with the symmetries of the system should in principle be included in the effective action. As a result, effective actions generally contain an infinite number of terms.

3. **Expansion parameters & Power counting scheme**. The key to handling an action with an infinite number of terms lies in the fact that all EFTs feature one or more expansion parameters. These are small quantities controlling the impact that the physics we choose to neglect could potentially have on the degrees of freedom we choose to keep. For example, in particle physics these expansion parameters are often ratios of energy scales E/Λ, where E is the characteristic energy scale of the process one is interested in, and Λ is the typical energy scale of the UV physics one is neglecting. For this strategy to work, it is crucial to have an explicit power counting scheme, meaning that we should be able to assign a definite order in the expansion parameter to each term in the effective action. This ensures that only a finite number of terms contribute at any given order in perturbation theory, and that we can decide upfront which terms to keep in the action base on the desired level of accuracy.

2.2 Kinematics

As mentioned above, the main interest of this thesis is the Soft Collinear Effective Theory (SCET), which is the newest kind of EFT. In contrast to the other categories, its expansion parameter is not defined as a ratio of energies or masses, but ratios of momenta in certain directions.

Reference Vectors

Before going any further, we need to introduce some notation that is used in SCET. In order to describe the dynamics, it is convenient to choose lightlike reference vectors n_{i-}^{μ} and n_{i+}^{μ} for each collinear direction i. These vectors are given by

$$n_{i-}^{\mu} = (1, \boldsymbol{n}_i) \approx p_i^{\mu}/p^0 \qquad \text{and} \qquad n_{i+}^{\mu} = (1, -\boldsymbol{n}_i) \tag{2.1}$$

The $\boldsymbol{n}_i$ are three-vectors, and the lightlike reference vectors satisfy $n_{i-} \cdot n_{i+} = 2$ and $n_{i+}^2 = n_{i-}^2 = 0$. Using these reference vectors, the metric tensor is decomposed in the following way

$$g^{\mu\nu} = n_{i+}^{\mu}\frac{n_{i-}^{\nu}}{2} + n_{i-}^{\mu}\frac{n_{i+}^{\nu}}{2} + g_{i\perp}^{\mu\nu} \tag{2.2}$$

which defines $g_{i\perp}^{\mu\nu}$. This leads to the realisation that a general four-vector can be written in terms of its light-cone components

$$p^{\mu} = p_{\nu}g^{\mu\nu} = (n_{i+}p)\frac{n_{i-}^{\mu}}{2} + (n_{i-}p)\frac{n_{i+}^{\mu}}{2} + p_{i\perp}^{\mu}, \qquad p^{\mu} = (\underbrace{n_{i+}p}_{-}, \underbrace{n_{i-}p}_{+}, \underbrace{p_{i\perp}}_{\perp}) \tag{2.3}$$

where we have written the four-vector in the component notation. It is immediate to show that the inner product of a four-vector p with itself and another four-vector q take the form

$$p^2 = (n_{i+}p)(n_{i-}p) + p_{i\perp}^2, \qquad p \cdot q = \frac{1}{2}(n_{i+}p)(n_{i-}q) + \frac{1}{2}(n_{i+}q)(n_{i-}p) + p_{i\perp} \cdot q_{i\perp} \tag{2.4}$$

Momentum Regions & Expansion Parameter λ

One of the three main ingredients of EFTs is their expansion parameters. In SCET, the expansion parameter λ, expresses the ratio of momenta magnitudes that flow in different direction. To define it more explicitly, let's consider

$$\lambda^2 \sim \frac{-p_i^2}{Q^2} \sim \frac{-p_j^2}{Q^2} \qquad \text{and} \qquad p_i^2 \sim p_j^2 \sim \lambda^2 Q^2 \tag{2.5}$$

which vanishes in the limit in which we are interested in. Here p_i and p_j are the large momenta flowing in the i and j collinear direction respectively. With the definition of the expansion parameter, we can define the momentum regions that will be of interest to us.

- **Hard Region** where the components of the momentum scale $k^{\mu} \sim (1, 1, 1)Q$;
- ***i*-th Collinear Region** where k scales as $k^{\mu} \sim (1, \lambda^2, \lambda)Q$; The momenta in this region are close to the collinear direction, i.e. $k^{\mu} \approx Q \cdot n_{i-}^{\mu}/2$
- **Soft Region** where k scales as $k^{\mu} \sim (\lambda^2, \lambda^2, \lambda^2)Q$.

In SCET, each low-energy region listed above is represented by a different field as will become apparent in the next section.

2.3 The SCET Lagrangian to Leading Power

2.3.1 Degrees of Freedom

In this section, we are going to describe the first main ingredient of our effective theory, namely, the relevant degrees of freedom. We will consider the general case where there exist N collinear directions. Then, the quark and gluon fields split into N collinear and one soft fields.

$$\psi(x) \to \underbrace{\psi_1(x) + \cdots + \psi_N(x)}_{\text{N collinear fields}} + \underbrace{\psi_s(x)}_{\text{soft field}} \tag{2.6}$$

$$A^\mu(x) \to \underbrace{A_1^\mu(x) + \cdots A_N^\mu(x)}_{\text{N collinear fields}} + \underbrace{A_s^\mu(x)}_{\text{soft field}}, \tag{2.7}$$

where $\psi_s(x)$ is the soft part of the fermion field and $A_s^\mu(x)$ is the soft part of the gluon field. Note that it is not necessary to introduce fields for other regions, like the hard region, since their contributions will be absorbed into the prefactors of the operators built from soft and collinear fields. These prefactors are called Wilson coefficients and are the coupling constants of the effective theory. By writing down the most general set of operators and by adjusting their Wilson coefficients, as is well illustrated in [3] with exapmles. Here is where SCET differs from traditional effective field theories; here we integrate out a *mode* of the full theory instead of a full heavy field. When constructing the effective Lagrangian, we assume that the momenta of the different fields scale in the proper way. For the construction to make sense, it is important that the external momenta are chosen properly. For example, one must choose the external momentum flowing into a soft field to be soft.

The fermion field in each collinear sector is further split into two components

$$\psi_i(x) = \xi_i(x) + \eta_i(x) \tag{2.8}$$

where the fields $\xi_i(x)$ and $\eta_i(x)$ are defined using projection operators on the full collinear fields:

$$\xi_i(x) = P_{i+}\psi_i(x) \equiv \frac{\not{n}_{i-}\not{n}_{i+}}{4}\psi_i(x) \qquad \text{and} \qquad \eta_i(x) = P_{i-}\psi_i(x) \equiv \frac{\not{n}_{i+}\not{n}_{i-}}{4}\psi_i(x). \tag{2.9}$$

The reason we make this decomposition is that, as we will see below, the two components scale differently in respect to the expansion parameter λ, because the collinear momentum scales differently along the collinear and anti-collinear directions.

2.3.2 Power Counting Scheme

To determine the power of λ with which the different components of each SCET field scales, we look at the two-point correlators.

Collinear Quark Field

We start with the $\xi_i(x)$ component

$$\begin{aligned}\langle 0|T\{\xi_i(x)\bar{\xi}_i(0)|0\rangle &= \langle 0|T\{P_{i+}\psi_i(x)\overline{P_{i+}\psi_i(0)}\}|0\rangle = \langle 0|T\{P_{i+}\psi_i(x)\bar{\psi}_i(0)P_{i-}\}|0\rangle \\ &= P_{i+}\,\langle 0|T\{\psi_i(x)\bar{\psi}_i(0)\}|0\rangle\, P_{i-} = P_{i+}\left(\int \frac{d^4p_i}{(2\pi)^4}\frac{\not{p}_i}{p_i^2+i\epsilon}e^{-ip_i\cdot x}\right)P_{i-} \\ &= \int \frac{d^4p_i}{(2\pi)^4}\frac{1}{p_i^2+i\epsilon}e^{-ip_i\cdot x}\;P_{i+}\not{p}_i P_{i-} = \int \frac{d^4p_i}{(2\pi)^4}\frac{1}{p_i^2+i\epsilon}e^{-ip_i\cdot x}\frac{n_{i+}\cdot p_i}{2}\not{n}_{i-}\end{aligned} \tag{2.10}$$

where in the first line we have used the property ... and the last line the property ... Now, since p_i is a collinear momentum in four dimensions, where the components in the collinear direction scale as λ^0, the component anti-parallel to the collinear direction scales as λ^2, and the two transverse directions scale proportionally to λ. Therefore, the momentum integration measure scales as λ^4. It is immediate to see that $(p_i^2+i\epsilon)^{-1}$ scales as λ^{-2} and $n_{i-}\cdot p_i$ scales proportionally to λ^0. Combining all the scalings together in Eq. we find that

$$\langle 0|T\{\xi_i(x)\bar{\xi}_i(0)\}|0\rangle \sim \lambda^4\frac{1}{\lambda^2} = \lambda^2 \tag{2.11}$$

and therefore, $\xi_i(x)\sim\lambda$.

Similarly, the correlator for the η_i component is

$$\begin{aligned}\langle 0|T\{\eta_i(x)\bar{\eta}_i(0)\}|0\rangle &= \langle 0|T\{P_{i-}\psi_i(x)\overline{P_{i-}\psi_i(0)}\}|0\rangle = \langle 0|T\{P_{i-}\psi_i(x)\bar{\psi}_i(0)P_{i+}\}|0\rangle \\ &= P_{i-}\,\langle 0|T\{\psi_i(x)\bar{\psi}_i(0)\}|0\rangle\, P_{i+} = P_{i-}\left(\int \frac{d^4p_i}{(2\pi)^4}\frac{\not{p}_i}{p_i^2+i\epsilon}e^{-ip_i\cdot x}\right)P_{i+} \\ &= \int \frac{d^4p_i}{(2\pi)^4}\frac{1}{p_i^2+i\epsilon}e^{-ip_i\cdot x}\;P_{i-}\not{p}_i P_{i+} = \int \frac{d^4p_i}{(2\pi)^4}\frac{1}{p_i^2+i\epsilon}e^{-ip_i\cdot x}\frac{n_{i-}\cdot p}{2}\not{n}_{i+}\end{aligned} \tag{2.12}$$

Following the same logic as before, for a collinear momentum p_i,

$$\langle 0|T\{\eta_i(x)\bar{\eta}_i(0)\}|0\rangle \sim \lambda^4\frac{1}{\lambda^2}\lambda^2 = \lambda^4 \tag{2.13}$$

which means that $\eta_i(x)\sim\lambda^2$. Thus, the $\eta_i(x)$ component is suppressed by one power of λ with respect to the component $\xi_i(x)$.

Soft Quark Field

For the soft quark field, since p is soft, it scales as λ^2 in all four directions and $p^2\sim\lambda^4$. Therefore one finds that

$$\langle 0|T\{q(x)\bar{q}(0)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{\not{p}}{p^2+i\epsilon}e^{-ip\cdot x} \sim (\lambda^2)^4\lambda^2\frac{1}{\lambda^4} = \lambda^6 \tag{2.14}$$

and therefore $q(x)\sim\lambda^3$.

Gluon Field

For the gluon field, the two-point function is

$$\langle 0|T\{A^\mu(x)A^\nu(0)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ip\cdot x}\left[g^{\mu\nu}+\xi\frac{p^\mu p^\nu}{p^2}\right] \tag{2.15}$$

If we take the two point-function of the projections of the gluon field along the direction v, we find that

$$\begin{aligned}\langle 0|T\{(v\cdot A(x))(v\cdot A(0))\}|0\rangle &= \langle 0|T\{(v_\mu A^\mu(x))(v_\nu A^\nu(0))\}|0\rangle \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ip\cdot x}\left[v_\mu v_\nu g^{\mu\nu}+\xi\frac{(v_\mu p^\mu)(v_\nu p^\nu)}{p^2}\right] \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ip\cdot x}\left[v^2+\xi\frac{(v\cdot p)(v\cdot p)}{p^2}\right]\end{aligned} \tag{2.16}$$

Let us now consider the case where $v = n_{i-}$ for the collinear gluon field $A_i^\mu(x)$. Since n_{i-} is light-like ($n_{i-}^2 = 0$),

$$\langle 0|T\{(n_{i-}\cdot A_i(x))(n_{i-}\cdot A_i(0))\}|0\rangle = \int \frac{d^4p_i}{(2\pi)^4}\frac{i\xi}{p_i^2+i\epsilon}e^{-ip_i\cdot x}\frac{(n_{i-}\cdot p_i)^2}{p_i^2} \sim \lambda^4\frac{1}{\lambda^2}\frac{\lambda^4}{\lambda^2} \sim \lambda^4 \tag{2.17}$$

which means that $n_{i-}\cdot A_i(x) \sim \lambda^2$. Similarly, one finds that $n_{i+}\cdot A_i(x) \sim \lambda^0$. For any direction v on the $\perp$ plane, $v^2 \sim \lambda^0$ and therefore the quantity inside the bracket of (2.16) scales as λ^0 and thus $A_\perp(x) \sim \lambda$. Finally, for the soft gluon field, since the momentum scales as λ^2 in any direction, and it's norm as λ^4 from (2.15)

$$\langle 0|T\{A_s^\mu(x)A_s^\nu(0)\}|0\rangle \sim (\lambda^2)^4\frac{1}{\lambda^4}\left[\lambda^0+\frac{(\lambda^2)^2}{\lambda^4}\right] \sim \lambda^4 \tag{2.18}$$

Therefore, $A_s^\mu \sim \lambda^2$.

We collect here these results, making also the useful observation that the gluon field scales like its momentum.

$$\xi_i(x) \sim \lambda \qquad \eta_i(x) \sim \lambda^2 \qquad A_s^\mu \sim \lambda^2 \tag{2.19}$$

$$n_{i-}\cdot A_i \sim \lambda^0 \qquad n_{i+}\cdot A_i \sim \lambda^2 \qquad A_{i\perp} \sim \lambda \tag{2.20}$$

We can already make the observation that only the $n_{i+}A_s(x)$ components are not power suppressed with respect to their counterparts in the collinear gluon fields $A_i^\mu(x)$.

2.3.3 Multipole Expansion

In the next section we will present the leading term Lagrangian for strong interactions. It is easy to forsee that because of momentum conservation and how momenta scale, fields belonging to different collinear sectors do not directly interact with each other; only soft exchanges are permitted between the collinear sectors. Therefore, only interaction terms that are a mixture of collinear and soft fields must be considered. An example of such a term is

$$\int d^d x \; \bar{\xi}_c(x) A_s^\mu(x) \xi_c(x) \tag{2.21}$$

Taking the Fourier transform of this interaction we get

$$\int d^d x \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{d^d p_s}{(2\pi)^d} \; e^{-i(p_1+p_2+p_s)\cdot x} \tilde{\bar{\xi}}_c(p_1) \tilde{A}_s^\mu(p_s) \tilde{\xi}_c(p_2) \tag{2.22}$$

where p_1, p_2 are collinear momenta and p_s is soft. Focusing on the exponent, we note that the total momentum scales as $p_1^\mu + p_2^\mu + p_s^\mu \sim (1, \lambda^2, \lambda)Q$, which implies that the position variable x^μ scales as $x^\mu \sim (1/\lambda^2, 1, 1/\lambda)(1/Q)$. By construction, both the collinear terms contribute equally. However, if we consider the scalar product of x^μ with the soft momentum, we find that not all components contribute equally

$$p_s \cdot x = \frac{1}{2} \underbrace{(n_- p_s)(n_+ x)}_{\mathcal{O}(\lambda^0)} + \frac{1}{2} \underbrace{(n_+ p_s)(n_- x)}_{\mathcal{O}(\lambda^2)} + \underbrace{p_{s\perp} \cdot x_\perp}_{\mathcal{O}(\lambda^1)}. \tag{2.23}$$

The dominant term is clearly $(n_- p_s)(n_+ x) \sim \lambda^0$, and the remaining terms are power suppressed in respect to it. If we Taylor expand the soft field arount $x_-^\mu = (n_+ x) n_-^\mu / 2$ we get

$$\begin{aligned} A_s^\mu(x) &= A_s^\mu(x_-) + \underbrace{x_\perp \cdot \partial_\perp}_{\mathcal{O}(\lambda^1)} A_s^\mu(x_-) + \underbrace{x_+ \cdot \partial_-}_{\mathcal{O}(\lambda^2)} A_s^\mu(x_-) + \frac{1}{2} \underbrace{x_{\perp\rho} x_{\perp\sigma} \partial^\rho \partial^\sigma}_{\mathcal{O}(\lambda^2)} A_s^\mu(x_-) + \cdots \\ &= A_s^\mu(x_-) \Big(1 + \mathcal{O}(\lambda) \Big) \end{aligned} \tag{2.24}$$

This is the so-called multipole expansion. The interaction term (2.21) then becomes

$$\int d^d x \; \bar{\xi}_c(x) A_s^\mu(x) \xi_c(x) = \int d^d x \; \bar{\xi}_c(x) A_s^\mu(x_-) \xi_c(x) + \mathcal{O}(\lambda) \tag{2.25}$$

Hence, the soft fields multiplying collinear fields in the SCET Lagrangian are evaluated at a position x_-^μ, rather than the full x^μ position. It is important to stress that the evaluation at x_-^μ is done after the derivatives have acted on the field.

2.3.4 The Lagrangian to Leading Power

Now that we have found the scaling properties of the fields we can move on to determining the effective Lagrangian. The starting point of our discussion is the massless QCD Lagrangian:

$$\mathcal{L}_{\text{QCD}} = \bar{\psi} i \not{D} \psi - \frac{1}{4} (F_{\mu\nu}^a)^2, \tag{2.26}$$

where

$$D_\mu = \partial_\mu - igA^a_\mu t^a. \tag{2.27}$$

The SCET Lagrangian for strong interactions is separated into a soft and N collinear parts as follows

$$\mathcal{L}_{\mathrm{SCET}} = \mathcal{L}_s + \sum_{i=1}^{N} \mathcal{L}_i, \tag{2.28}$$

where each of the collinear sectors is systematically power expanded in the small power counting parameter λ

$$\mathcal{L}_i = \underbrace{\mathcal{L}_i^{(0)}}_{\mathcal{O}(\lambda^0)} + \underbrace{\mathcal{L}_i^{(1)}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathcal{L}_i^{(2)}}_{\mathcal{O}(\lambda^2)} + \cdots . \tag{2.29}$$

The term $\mathcal{L}_i^{(0)}$ in the expansion is called the **leading term contribution**, and the remaining terms denote the power corrections, with the number in the superscript denoting the power suppression in λ with respect to the leading power term.

2.3.5 The soft Lagrangian

Let us first deal with the soft Lagrangian at leading power, which is simply given by

$$\mathcal{L}_s = \bar{\psi}_s i\not{D}_s \psi_s - \frac{1}{4}(F^a_s)_{\mu\nu}(F^a_s)^{\mu\nu}, \tag{2.30}$$

with the soft covariant derivative being

$$iD^\mu_s(x) = i\partial^\mu + g_s A^\mu_s(x) = i\partial^\mu + g_s A^{a,\mu}_s t^a, \tag{2.31}$$

in terms of which we define the soft field strength tensor

$$ig_s F^{\mu\nu}_s = \Big[iD^\mu_s, iD^\nu_s\Big]. \tag{2.32}$$

The reason we have included only soft gluon fields in the covariant derivative of course is because, soft fermionic fields remain soft only if they interact with soft gluons.

2.3.6 The Collinear Lagrangian

The collinear fermion Lagrangian has a special form since the η_i component of the fermion field is power suppressed with respect to ξ_i, and thus can be integrated out. The covariant derivative in the ith collinear direction is defined as usual by

$$iD_{i\mu}(x) = i\partial_\mu + g_s A_{i\mu}(x) = i\partial_\mu + g_s A^a_{i\mu}(x)t^a \tag{2.33}$$

with t^a being the generators of $SU(3)$ in the adjoint representation. For the time being, we keep both the collinear components of the gluon field and the soft, even though $A_{s\perp}$ and A_{s-} are power suppressed with respect to the collinear gluon field. We will come back to this point when discussing the soft-collinear interactions. Again, starting from the full QCD Lagrangian for the fermionic field in the ith collinear direction we find that the leading power contribution term is

$$\begin{aligned}\mathcal{L}^{(0)}_{i\text{ fermion}} &= \bar{\psi}_i i \not{D}_i \psi_i = (\bar{\xi}_i + \bar{\eta}_i) i \left[\frac{\not{n}_{i+}}{2}(n_{i-} \cdot D_i) + \frac{\not{n}_{i-}}{2}(n_{i+} \cdot D_i) + \not{D}_{i\perp} \right] (\xi_i + \eta_i) \\ &= \bar{\xi}_i \frac{\not{n}_{i+}}{2} i(n_{i-} \cdot D_i)\xi_i + \bar{\xi}_i i\not{D}_{i\perp}\eta_i + \bar{\eta}_i i\not{D}_{i\perp}\xi_i + \bar{\eta}_i \frac{\not{n}_{i-}}{2} i(n_{i+} \cdot D)\eta_i. \end{aligned} \tag{2.34}$$

We use the full covariant derivative where both soft and i-collinear gluon fields are included, because such interactions are allowed. If a collinear fermion absorbs or emits a soft gluon its momentum scales collinearly.

Integrating out η_i

Since the action is quadratic, one can integrate out η_i exactly. A standard and easy way of obtaining the Lagrangian after the η_i field is integrated out consists of employing the equations of motion for the ξ_i field derived from (2.34). The Euler-Lagrange equations of motion for $\bar{\xi}_i$ is

$$0 = \partial_\mu \frac{\partial \mathcal{L}_i}{\partial(\partial_\mu \bar{\xi}_i)} - \frac{\partial \mathcal{L}}{\partial \bar{\xi}_i} = -\frac{\not{n}_{i+}}{2} i n_{i-} \cdot D\xi_i - i\not{D}_\perp \eta_i, \tag{2.35}$$

or equivalently

$$\frac{\not{n}_{i+}}{2} n_{i-} \cdot D\xi_i = -\not{D}_\perp \eta_i. \tag{2.36}$$

Similarly, for $\bar{\eta}_i$ one finds

$$\not{D}_\perp \xi_i = -\frac{\not{n}_{i-}}{2} n_{i+} \cdot D\eta_i. \tag{2.37}$$

Solving for η_i and $\bar{\eta}_i$ one obtains

$$\eta_i = -\frac{\not{n}_{i+}}{2n_{i+} \cdot D} \not{D}_\perp \xi_i \qquad \text{and} \qquad \bar{\eta}_i = -\bar{\xi}_i \not{D}_\perp \frac{\not{n}_{i+}}{2n_{i+} \cdot D}, \tag{2.38}$$

and when replaced in (2.34), one ends up with

$$\mathcal{L}^{(0)}_{\xi,i} = \bar{\xi}_i \frac{\not{n}_{i+}}{2} \left[i n_{i-} D_i + i\not{D}_{i\perp} \frac{1}{i n_{i+} D_i} i\not{D}_{i\perp} \right] \xi_i. \tag{2.39}$$

Irrelevance of the Determinant

The same result is reached when we explicitly integrate out the η_i component in the path integral. When we follow that approach, we get an extra factor, the determinant

$$\det\left(\frac{\not{n}_{i-}}{2} i n_{i+}\cdot D\right) \tag{2.40}$$

We are now going to show that the determinant is irrelevant and therefore can be ignored. First, we demonstrate that the determinant is gauge invariant. If $V \in SU(N)$ so that the quark field transforms according to $\psi \to V\psi$ under gauge transformation, the determinant's covariant derivative will transform as $D \to VDV^\dagger$. Therefore,

$$\begin{aligned}\det\left(\frac{\not{n}_{i-}}{2} i n_{i+}\cdot D\right) \to \det\left(\frac{\not{n}_{i-}}{2} i n_{i+}\cdot VDV^\dagger\right) &= \det(V)\det\left(\frac{\not{n}_{i-}}{2} i n_{i+}\cdot D\right)\det(V^\dagger)\\ &= \det\left(\frac{\not{n}_{i-}}{2} i n_{i+}\cdot D\right)\end{aligned} \tag{2.41}$$

Since the determinant is gauge invariant, it can be computed in any gauge. In the light cone gauge where $n_{i+}\cdot A_i = 0$, $n_{i+}\cdot D_i = n_{i+}\cdot\partial$ and the determinant is independent of the gluon field. Therefore, the determinant remains independent of it in any gauge and is thus an irrelevant multiplicative factor. From the diagramatic point of view, the determinant corresponds to a series of vacuum diagrams of the form

While the collinear quark Lagrangian has somewhat complicated structure, the collinear gluon Lagrangian is simply a copy of the QCD Lagrangian in which the gluon field A^μ is replaced by the collinear gluon field A_i^μ. That is,

$$\mathcal{L}_{g,i} = -\frac{1}{4}(F_i^{\mu\nu})^2 \tag{2.42}$$

where the covariant derivative and the field strength tensor are defined as

$$iD_i^\mu = i\partial^\mu + gA_i^\mu t^a, \tag{2.43}$$

and

$$ig_s F_i^{\mu\nu} = \left[iD_i^\mu, iD_i^\nu\right]. \tag{2.44}$$

2.3.7 Soft-Collinear Interactions

Next, we consider the soft-collinear interaction terms, that is terms that describe the interactions between soft and collinear fields, at leading power. To obtain leading power interactions, we remind ourselves of the scalings of the fields

$$\begin{aligned}\left(n_{i+}\cdot A_i,\ n_{i-}\cdot A_i,\ A_{i\perp}\right) &\sim (1,\lambda^2,\lambda) \qquad \xi_i \sim \lambda\\ \left(n_{i+}\cdot A_s,\ n_{i-}\cdot A_s,\ A_{s,\perp}\right) &\sim (\lambda^2,\lambda^2,\lambda^2) \quad \psi_s \sim \lambda^3\end{aligned}$$

In the SCET Lagrangian for strong interactions, soft-collinear interactions involving soft quarks do not appear at leading order, since ψ_s is power suppressed with respect to ξ_i. Furthermore, only the $n_{i+} \cdot A_s$ component of the soft gluon field is not power suppressed with respect to the corresponding component of the collinear gluon field, so only this component is relevant for the leading soft-collinear interactions. Therefore one can make the replacements

$$A_i^\mu(x) \to \left[n_{i-} \cdot A_i(x) + n_{i-} \cdot A_s(x_-)\right] \frac{n_{i+}^\mu}{2} + n_{i+} \cdot A_i(x) \frac{n_{i-}^\mu}{2} + A_{i\perp}^\mu(x) \Big\} \tag{2.45}$$

in the collinear quark and gluon Lagrangians discussed in the previous subsection. With this substitution one arrives at the final expression of the leading power i-collinear SCET Lagrangian, reading

$$\mathcal{L}_i^{(0)} = \bar{\xi}_i \frac{\not{n}_{i+}}{2} \left[i n_{i-} D_i + g_s n_{i-} A_s(x_{i-}) + i \not{D}_{i\perp} \frac{1}{i n_{i+} D_i} i \not{D}_{i\perp} \right] \xi_i - \frac{1}{4} \big(F_i^{\mu\nu}\big)^2, \tag{2.46}$$

where we made the dependence on $n_{i-} A_s(x_{i-})$ explicit, and we made the decomposition

$$D_i^\mu = (n_{i-} \cdot D_i) \frac{n_{i+}^\mu}{2} + (n_{i+} \cdot D_i) \frac{n_{i-}^\mu}{2} + D_{i\perp}^\mu. \tag{2.47}$$

The field strength tensor is normally defined by

$$i g_s F_i^{\mu\nu} = \left[i D_i^\mu + g_s n_{i-} A_s(x_{i-}) \frac{n_{i+}^\mu}{2}, i D_i^\nu + g_s n_{i-} A_s(x_{i-}) \frac{n_{i+}^\nu}{2} \right] \tag{2.48}$$

With the addition of soft-collinear interactions we have completed the construction of the leading power SCET Lagrangian. Its final form is

$$\mathcal{L} = \bar{\psi}_s i \not{D}_s \psi_s + \sum_{i=1}^{N} \left\{ \bar{\xi}_i \frac{\not{n}_{i+}}{2} \left[i n_{i-} D_i + g_s n_{i-} A_s(x_{i-}) + i \not{D}_{i\perp} \frac{1}{i n_{i+} D_i} i \not{D}_{i\perp} \right] \xi_i - \frac{1}{4} \big(F_i^{\mu\nu}\big)^2 \right\} - \frac{1}{4} (F_s^{\mu\nu})^2. \tag{2.49}$$

2.4 Symmetries

We now discuss two important symmetries of SCET. Both are not symmetries of nature but redundancies in our description. The first one is gauge symmetry which arises because we use four-component fields to describe the two physical polarisations of gauge bosons. The second one is called reparametrisation invariance and arises because we have introduced reference vectors n_{i-}^μ and n_{i+}^μ, in the construction of the effective theory. The choice of these is not unique and physics is independent of their choice.

2.4.1 Reparametrisation Invariance

First, we are briefly discuss reparametrisation invariance. In SCET, one can change the direction of the reference vectors by a small amount, but one can also rescale the light-like reference vectors. The most general infinitesimal transformation is a linear combination of these three types of transformation

$$(\mathbf{I})\begin{cases} n_{i+}^{\mu} \to n_{i+}^{\mu} + \Delta_{\perp}^{\mu} \\ n_{i-}^{\mu} \to n_{i-}^{\mu} \end{cases}, \qquad (\mathbf{II})\begin{cases} n_{i+}^{\mu} \to n_{i+}^{\mu} \\ n_{i-}^{\mu} \to n_{i-}^{\mu} + \epsilon_{\perp}^{\mu} \end{cases} \qquad (\mathbf{III})\begin{cases} n_{i+}^{\mu} \to (1+\alpha) n_{i+}^{\mu} \\ n_{i-}^{\mu} \to (1-\alpha) n_{i-}^{\mu} \end{cases} \tag{2.50}$$

with $\Delta_{\perp} \cdot n_{i+} = \Delta_{\perp} \cdot n_{i-} = \epsilon_{\perp} \cdot n_{i+} = \epsilon_{\perp} \cdot n_{i-} = 0$. For the purpose of this thesis it is not necessary to delve deeper into the details of this symmetry, and we point the interested reader to [9] for more details.

2.4.2 Gauge Symmetries

The original QCD Lagrangian is $SU(3)$ gauge invariant and therefore, the effective Lagrangian has to also be gauge invariant. In this section we will only briefly touch on the gauge transformation properties of the non-abelian gauge Lagrangian; a more involved and detailed discussion can be found in [5]

To accomplish $SU(3)$ gauge invariance, we have to extend the gauge transformations making sure they respect the scaling of the fields. That is to say, after the gauge transformation, each field has to scale in the same fashion as before. For example, transforming a soft field by means of a gauge function $\alpha(x)$ with collinear scaling would turn the soft field into a collinear field. We will consider two types of gauge transformations; the *soft gauge transformation*

$$V_s(x) = \exp\left[i\alpha_s^a(x) t^a \right], \tag{2.51}$$

and the *collinear gauge transformation* along the ith collinear direction

$$V_i(x) = \exp\left[i\alpha_i^a(x) t^a \right]. \tag{2.52}$$

The function $\alpha_s^a(x)$ has soft scaling, i.e. $\partial\alpha_s^a(x) \sim \lambda^2 \alpha_s^a(x)$, while $a_i^a(x)$ has collinear scaling, meaning $\partial\alpha_i(x) \sim (\lambda^2, 1, \lambda)\alpha_i(x)$. We analyse the soft transformations first.

Under a soft gauge transformation the fields transform in the standard way

$$\psi_s(x) \to V_s(x)\psi_s(x) \tag{2.53}$$

$$A^{\mu}(x) \to V_s(x) A_s^{\mu}(x) V_s^{\dagger}(x) + \frac{i}{g} V_s(x) \left[\partial^{\mu}, V_s^{\dagger}(x) \right]. \tag{2.54}$$

The collinear fields transform instead as follows

$$\xi_i(x) \to V_s(x_-)\xi_i(x) \tag{2.55}$$
$$A_i^\mu(x) \to V_s(x_-)A_i^\mu(x)V_s^\dagger(x_-) \tag{2.56}$$

The gauge transformation matrices in Equations (2.55) and (2.56) depend only on x_- from the multipole expansion. The collinear gauge transformations involve a field that scales collinearly, and therefore, the soft fields cannot be transformed under them. That is

$$\psi_s(x) \to \psi_s(x) \qquad\qquad A_s^\mu(x) \to A_s^\mu(x) \tag{2.57}$$

Instead, for a i-collinear gauge transformation, the collinear fields transform as follows

$$\begin{aligned}
&\xi_i(x) \to V_i(x)\xi_i(x)\\
&A_i^\mu(x) \to V_i(x)A_i^\mu(x)V_i^\dagger(x) + \frac{1}{g_s}V_i(x)\left[i\partial + g_s\frac{n_{i+}^\mu}{2}n_{i-}\cdot A_s(x_-), V_i^\dagger(x)\right]\\
&A_j^\mu(x) \to A_j^\mu(x) \qquad i \neq j
\end{aligned} \tag{2.58}$$

It is straightforward to demonstrate that the SCET Lagrangian is manifestly invariant under both collinear and soft gauge transformation (see [5]).

2.5 Wilson Lines

In SCET, as in other EFTs, we encounter non-local operators. In a gauge theory, a product of fields at different space time points is only gauge invariant if the fields are connected by Wilson lines, defined as

$$W(z,y)_A = \mathbb{P}\left\{\exp\left[ig\int_c dx^\mu A_\mu(x)\right]\right\}, \tag{2.59}$$

where $A_\mu(x)$ is the gauge field, and c is the path that connects y with z. The operator $\mathbb{P}$ indicates the path ordering if the colour matrices t^a. The conjugate Wilson line is defined with the opposite ordering prescription. If we parametrise the path c with the parameter s, such that $x(s_y) = y$ and $x(s_z) = z$, then we can write the Wilson line as:

$$W(z,y)_A = \mathbb{P}\left\{\exp\left[ig\int_{s_y}^{s_z} ds\ \frac{dx^\mu}{ds}A_\mu\big(x(s)\big)\right]\right\} \tag{2.60}$$

With this parametrised form, the operator $\mathbb{P}$ indicates the path ordering of the matrix-valued integrals in such a way that an integrand evaluated at a given value s appears to the right of integrands evaluated at larger values of s, while it appears on the left of the integrands evaluated at smaller values of the parameters. We will first write the Wilson lines in a more succinct form by setting the integrands

$$\mathbb{F}(s) = \frac{dx}{ds}\cdot A\big(x(s)\big) = \frac{dx^\mu}{ds}A_\mu^b\big(x(s)\big)t^b \tag{2.61}$$

Then,

$$W(z,y)_A = \mathbb{P}\Big\{ \exp\Big[\int_{s_y}^{s_z} ds\ \mathbb{F}(s)\Big]\Big\}$$
$$= \sum_{n=0}^{\infty} \frac{(ig)^n}{n!} \int_{s_y}^{s_z} ds_1 \int_{s_y}^{s_z} ds_1 \cdots \int_{s_y}^{s_z} ds_n\ \mathbb{P}\big\{\mathbb{F}(s_1)\mathbb{F}(s_2)\cdots\mathbb{F}(s_n)\big\} \tag{2.62}$$

The path ordering prescribes that the non-commuting functions $\mathbb{F}$ should be ordered considering decreasing order of the arguments. Therefore, if $s_1 > s_2 > \cdots > s_n$

$$\mathbb{P}\big\{\mathbb{F}(s_1)\mathbb{F}(s_2)\cdots\mathbb{F}(s_n)\big\} = \mathbb{F}(s_1)\mathbb{F}(s_2)\cdots\mathbb{F}(s_n) \tag{2.63}$$

The integration region is a hypercube. It is possible to subdivide the integration region in $n!$ subregions, which correspond to the $n!$ possible orderings of the elements $\{s_1, s_2, \ldots, s_n\}$. The $n!$ integration regions are simplices, as it easy to see by considering the simple case in which $n = 2$. We can also set $s_y = 0$ and $s_z = 1$ for simplicity.

$$\int_0^1 ds_1 \int_0^1 ds_2\ \mathbb{P}\big\{\mathbb{F}(s_1)\mathbb{F}(s_2)\big\} = \int_0^1 ds_1 \int_0^{s_1} ds_2\ \mathbb{F}(s_1)\mathbb{F}(s_2) + \int_0^1 ds_1 \int_0^{s_1} ds_2\ \mathbb{F}(s_2)\mathbb{F}(s_1)$$
$$= 2\int_0^1 ds_1 \int_0^{s_1} ds_2\ \mathbb{F}(s_1)\mathbb{F}(s_2) \tag{2.64}$$

where in the second line we redefined $s_1 \leftrightarrow s_2$, since they are only dummy variables. This procedure can be generalised to n dimensions

$$\int_0^1 ds_1 \int_0^1 ds_2 \cdots \int_0^1 ds_n\ \mathbb{P}\big\{\mathbb{F}(s_1)\mathbb{F}(s_2)\cdots\mathbb{F}(s_n)\big\} =$$
$$= n! \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\ \mathbb{F}(s_1)\mathbb{F}(s_2) \quad \cdots \mathbb{F}(s_n) \tag{2.65}$$

Therefore,

$$W(z,y)_A = \sum_{n=0}^{\infty} (ig)^n \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\ \mathbb{F}(s_1)\mathbb{F}(s_2)\cdots\mathbb{F}(s_n) \tag{2.66}$$

Now, we will prove a proposition of Wilson lines that will be very important in demonstrating a key feature of the leading power SCET Lagrangian, namely the decoupling transformation.

Proposition 2.5.1. The covariant derivative of a Wilson line along the parametrised path vanishes. That is,

$$\frac{dx^\mu}{ds} D_\mu W\big(x(s), x(0)\big)_A = 0 \tag{2.67}$$

Proof First, we start by setting $z = x(s)$ and $y = x(0)$, and then we take the derivative of a Wilson line in respect to s:

$$\begin{aligned}
\frac{d}{ds} W\big(x(s), x(0)\big)_A &= \frac{d}{ds}\left[1 + \sum_{n=1}^{\infty} (ig)^n \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\ \mathbb{F}(s_1)\mathbb{F}(s_2)\cdots\mathbb{F}(s_n)\right] \\
&= \sum_{n=1}^{\infty} (ig)^n \int_0^s ds_2 \int_0^{s_2} ds_3 \cdots \int_0^{s_{n-1}} ds_n\ \mathbb{F}(s)\mathbb{F}(s_2)\mathbb{F}(s_3)\cdots\mathbb{F}(s_n) \\
&= ig\mathbb{F}(s) \sum_{n=0}^{\infty} (ig)^n \int_0^s ds_2 \int_0^{s_2} ds_3 \cdots \int_0^{s_{n-1}} ds_n\ \mathbb{F}(s)\mathbb{F}(s_2)\mathbb{F}(s_3)\cdots\mathbb{F}(s_n) \\
&= ig\mathbb{F}(s) W\big(x(s), x(0)\big)_A = ig\frac{dx^\mu}{ds} A_\mu\big(x(s)\big) W\big(x(s), x(0)\big)_A
\end{aligned} \tag{2.68}$$

Now, we can see that

$$0 = \left(\frac{d}{ds} - ig\frac{dx^\mu}{ds} A_\mu\big(x(s)\big)\right) W\big(x(s), x(0)\big)_A = \frac{dx^\mu}{ds}\left(\partial_\mu - igA_\mu\big(x(s)\big)\right) W\big(x(s), x(0)\big)_A$$

or equivalently,

$$\frac{dx^\mu}{ds} D_\mu W\big(x(s), x(0)\big)_A = 0 \tag{2.69}$$

■

While in principle the path c is arbitrary, in SCET we define the i-th collinear Wilson line along the straight paths $x(s) = x_0 + sn_{i+}$, with the large component of the corresponding gauge field in the exponent by

$$W_i(x) = [x, -\infty n_{i+}] = \mathbb{P}\left\{ \exp\left[ig_s \int_{-\infty}^{0} ds\ n_{i+} \cdot A_i(x + sn_{i+})\right]\right\}, \tag{2.70}$$

and the soft Wilson lines along the straight paths $x(s) = x_0 + sn_{i+}$ given by

$$S_i(x) = \mathbb{P}\left\{ \exp\left[ig_s \int_{-\infty}^{0} ds\ n_{i-} \cdot A_s(x + sn_{i-})\right]\right\}. \tag{2.71}$$

The limits of the integrals in the exponents of Equations (2.70) and (2.71), namely from $-\infty$ to 0, arise from the fact that we are considering *incoming* particles. In Wilson lines that describe *outgoing* particles the limits extend from 0 to ∞.

The usefulness of Wilson lines in the construction of SCET becomes apparent upon consideration of the behaviour of these objects under gauge transformations. Namely, taking as an example the collinear gauge transformations we find that the Wilson lines transform as follows

$$W_i(x) \to V_i(x) W_i(x) V_i^\dagger(-\infty n_{i+}), \tag{2.72}$$

where we consider gauge functions which vanish at infinity $V_i^\dagger(-\infty n_{i+}) = 1$.

Since Wilson lines appear ubiquitously in SCET, we briefly comment on their contri bution to calculation of Feynman diagrams. A Wilson line may source any number of gluons by keeping higher orders in the expansion of the exponential in equations (2.70) and (2.71). The higher number of emissions are suppressed by corresponding powers of. For illustration, we consider the $\mathcal{O}(g_s)$ term. The momentum-space Feynman rule can be obtained Fourier transforming the gauge field and performing the ds integral. By requiring that the Wilson line is well-behaved at infinity, we can fix the $i\delta$ prescription. For example, considering the soft Wilson line in equation (2.70) and expanding to firs order we find the following

$$\begin{aligned} W_i(x) &= \mathbb{P}\Big\{ \exp\Big[i g_s \int_{-\infty}^{0} ds\ n_{i+} \cdot A_i(x + s n_{i+}) \Big] \Big\} \\ &= 1 + i g_s \int_{-\infty}^{0} ds\ n_{i+} \cdot A_i(x + s n_{i+}) + \mathcal{O}(g_s^2) \\ &= 1 + i g_s \int_{-\infty}^{0} ds \int \frac{d^d k}{(2\pi)^d}\ e^{-ik\cdot(x + s n_{i+})} n_{i+}^{\mu} \tilde{A}_{i,\mu,a}(k) T^a + \mathcal{O}(g_s^2) \\ &= 1 + \int \frac{d^d k}{(2\pi)^d}\ e^{-ik\cdot x} \underbrace{\Big[- g_s \frac{n_{i+}^{\mu}}{(n_{i+}\cdot k) - i\delta} T^a \Big]}_{\text{momentum space Feynman rule}} \tilde{A}_{i,\mu,a}(k) T^a + \mathcal{O}(g_s^2) \end{aligned} \tag{2.73}$$

where the termo in the square bracket is the momentum space Feynman rule. This Feynman rule has an eikonal form as expected, namely, it contains a *linear* dependence on the momentum. The same holds for the soft Wilson lines.

2.6 Decoupling Transformation

Here, we introduce one of the core features of SCET that allows it to be so successful. More details on the decoupling transformation can be found in [6]. The interaction between each i-collinear fermionic sector and the soft bosonic sector are contained in the term

$$\mathcal{L}_{i+s} = \bar{\xi}_i(x) \frac{\not{n}_{i+}}{2} i n_{i-} \cdot D_{(i)} \xi_i(x) = \bar{\xi}_i(x) \frac{\not{n}_{i+}}{2} i n_{i-} \cdot \Big[\partial + g_s A_i(x) + g_s A_s(x_-) \Big] \xi_i(x), \tag{2.74}$$

An important feature of the SCET Lagrangian is that the soft-collinear interactions can be completely removed at leading power. This is achieved by the so-called decoupling transformation [71], which is defined by the field redefinitions

$$\xi_i(x) \to S_i(x_-)\xi_i^{(0)}(x), \tag{2.75}$$

$$A_i^\mu(x) \to S_i(x_-)A_i^{(0)\mu}(x)S_i^\dagger(x_-), \tag{2.76}$$

for each collinear direction. As a consequence of these field redefinitions (2.75) and (2.76), one finds that $in_{i-} \cdot D_i\xi(x)$ becomes

$$\begin{aligned}
&\to in_{i-} \cdot \Big[\partial - igS_i(x_-)A_i^{(0)}(x)S_i^\dagger(x_-) - igA_i(x_-)\Big]S_i(x_-)\xi_i^{(0)}(x) \\
&= \Big[in_{i-} \cdot \partial_- S_i(x_-) + in_{i-} \cdot S_i(x_-)\partial + g_s n_{i-} \cdot S_i(x_-)A_i^{(0)}(x) \\
&\qquad\qquad\qquad\qquad + g_s n_{i-} \cdot A_s^{(0)}(x)S_i(x_-)\Big]\xi_i^{(0)}(x) \\
&= \Big[in_{i-} \cdot \big(\partial_- - ig_s A_s(x_-)\big)S_i(x_-) + S_i(x_-)\; in_{i-} \cdot \partial + S_i(x_-)g_s n_{i-} \cdot A_i^{(0)}(x)\Big]\xi_i^{(0)}(x) \\
&= \Big[in_{i-} \cdot \underbrace{D_{s-}S_i(x_-)}_{0} + S_i(x_-)in_{i-} \cdot \partial + S_i(x_-)g_s n_{i-} \cdot A_i^{(0)}(x)\Big]\xi_i^{(0)}(x) \\
&= \Big[S_i(x_-)in_{i-} \cdot \partial + S_i(x_-)g_s n_{i-} \cdot A_i^{(0)}(x)\Big]\xi_i^{(0)}(x) \\
&= S_i(x_-)\Big[in_{i-} \cdot \partial + g_s n_{i-} \cdot A_i^{(0)}(x)\Big]\xi_i^{(0)}(x) \equiv S_i(x_-)in_{i-} \cdot D_i^{(0)}\xi_i^{(0)}(x).
\end{aligned} \tag{2.77}$$

To arrive to this result, in the second line we used the fact that

$$n_{i-} \cdot \partial S_i(x_-) = n_{i-}^a \frac{\partial}{\partial x^a} S_i(x_-) = n_{i-}^a \frac{\partial x_-^\beta}{\partial x^a}\frac{\partial}{\partial x_-^\beta} S_i(x_-) = \frac{n_{i-}^a n_{i+a}}{2} n_{i-}^\beta \frac{\partial}{\partial x_-^\beta} = n_{i-} \cdot \partial_- S_i(x_-), \tag{2.78}$$

since

$$x_-^\beta = \frac{n_{i+} \cdot x}{2} n_{i-}^\beta \Rightarrow \frac{\partial x_-^\beta}{\partial x^a} = \frac{n_{i+a}}{2} n_{i-}^\beta. \tag{2.79}$$

Furthermore, in the fourth equation we made use of the property that the covariant derivative along the Wilson line is zero (Proposition 2.5.1).

Thus, under the field transformations (2.75) and (2.76) the interaction term Lagrangian (2.74) transforms like

$$\mathcal{L}_{i+s} \to \bar{\xi}_i^{(0)} \frac{\not{n}_{i+}}{2} in_{i-} \cdot D_i^{(0)}\xi_i^{(0)}(x), \tag{2.80}$$

so that the soft field no longer appears in the collinear Lagrangian.

The same decoupling happens in the kinetic part of the collinear gluon Lagrangian (see [3]). This kind of transformation is called *decoupling transformation*, since it decouples the soft gluon from the leading power collinear Lagrangian, effectively removing all soft-collinear (at leading power).

The superscript (0) on the decoupled fields is customarily dropped after the field redefinition is performed. This convention is followed in this work, unless for clarity we make this superscript explicit in which case this will be noted.

It is important to stress, however, that at subleading power soft collinear interactions are still present in the Lagrangian. Instead there, one starts from the leading power Lagrangian, where the states can be considered also to be factorised, and considers the sub-leading terms of the Lagrangian as perturbations, or insertions to the the diagrams that are of interest.

Chapter 3

Calculation of Leading Power Jet Functions

In this chapter, our main focus revolves around the assessment of the partonic jet functions denoted as J in the factorisation formula (1.3). To accomplish this, we divide our examination into two sections. Initially, we delve into the evaluation of the leading power SCET jet function, as defined in [2][6], employing the regular QCD Lagrangian. Subsequently, we carry out the same calculations using the SCET Lagrangian in the following section. It is important to note that in both cases, the derivation of the factorisation theorem is performed within the framework of SCET. Once SCET has provided the matrix element definition of the jet functions, we proceed to employ either the QCD or SCET Lagrangian to finalize the calculations.

3.1 Calculation with the QCD Lagrangian

In this section, we employ the standard QCD Feynman rules to compute the SCET jet function, as defined in [6][2]. The key to working in this manner lies in the decoupling transformation, which effectively separates the degrees of freedom in a particular collinear direction from the soft and other collinear components at the leading power. It is worth noting that, in the subsequent analysis, we consider two distinct collinear directions. To maintain consistency with the notation used for light-like reference vectors in Chapter 2, we have the following relationships:

$$n_{1-} = n_{2+} = n \qquad n_{1+} = n_{2-} = \bar{n}. \tag{3.1}$$

[6][2] define the jet function with

$$\frac{\not{n}}{2}(\bar{n} \cdot p)\mathcal{J}(p^2, \mu) = \int d^d x \; e^{-ipx} \left\langle 0 | \mathbb{T}\{\chi(0)\bar{\chi}(x)\} | 0 \right\rangle , \tag{3.2}$$

with $\chi(x)$ being the jet fields defined as $W^\dagger(x)\xi(x)$. These fields are frequently met in SCET because they are used to construct gauge invariant operators because of their transformation properties (gauge invariant building blocks, see [5]). After the decoupling transformation this definition reads

$$\frac{\not{n}}{2}(\bar{n}\cdot p)\mathcal{J}(p^2,\mu) = \int d^dx\; e^{-ipx}\left\langle \frac{\not{n}\not{\bar{n}}}{4} W^\dagger(0)\psi(0)\bar{\psi}(x)W(x)\frac{\not{\bar{n}}\not{n}}{4}\right\rangle$$
$$= \frac{\not{n}\not{\bar{n}}}{4}\left[\int d^dx\; e^{-ipx}\,\langle W^\dagger(0)\psi(0)\bar{\psi}(x)W(x)\rangle\right]\frac{\not{\bar{n}}\not{n}}{4}, \tag{3.3}$$

where the Wilson line is defined by

$$W(x) = \mathbb{P}\left\{\exp\left[ig_s\int_{-\infty}^{0} ds\;\bar{n}\cdot A(x+s\bar{n})\right]\right\} \tag{3.4}$$

The jet function J is the discontinuity of the propagator, i.e.

$$J(p^2,\mu) = \frac{1}{\pi}\mathrm{Im}\left[i\mathcal{J}(p^2,\mu)\right] = \delta(p^2) + \mathcal{O}(\alpha_s). \tag{3.5}$$

By Lorentz invariance and invariance under rescaling of $\bar{n}$, the jet function can only depend on p^2, as ew have written. Physically, as we outlined in the Introduction, the jet function gives something close to the probability of finding a jet with invariant mass p^2; it is not exactly this probability since soft radiation also contributes to the jet masses. This same jet function appears in the factorisation formulas of many processes, like $B \to X_s\gamma$, deep inelastic scattering and direct proton production. Note that the jet function is only useful when evaluated for values of $p^2 \ll Q^2$, for Q some hard scale.

To compute the jet function up to next-to-leading orer, our approach involves inserting the interaction term (in the interaction picture) and subsequently expanding both the Wilson line and the interaction term using a series expansion based on the strong coupling constant. Utilizing Wick's theorem, we identify the various diagrams that are equivalent to this expression. However, we disregard the disconnected graphs as well as those containing closed loops that can be disconnected from the rest of the graph through a cut. The rationale behind this omission is that, in dimensional regularization, these graphs are defined to vanish.

More specifically, below we calculate the contributions to $\mathcal{J}(p^2,\mu)$, regularising the divergences with dimensional regularisation, with dimension $d = 4 - 2\epsilon$.

$$\mathcal{A} = \int d^dx\; e^{-ipx}\left\langle W^\dagger(0)\psi(0)\bar{\psi}(x)W(x)\exp i\int d^4y\;\mathcal{L}_{int}(y)\right\rangle. \tag{3.6}$$

The Wilson lines expanded read

$$W(x) = 1 + ig_s\int_{-\infty}^{0} ds\;\bar{n}\cdot A(x+s\bar{n}) - g_s^2\int_{-\infty}^{0} ds\int_{-\infty}^{s} d\lambda\;\left[\bar{n}\cdot A(x+s\bar{n})\right]\left[\bar{n}\cdot A(x+\lambda\bar{n})\right], \tag{3.7}$$

and the conjugate

$$W^\dagger(0) = 1 - ig_s \int_{-\infty}^{0} ds\ \bar{n}\cdot A(s\bar{n}) - g_s^2 \int_{-\infty}^{0} ds \int_{s}^{0} d\lambda\ \big[\bar{n}\cdot A(s_2\bar{n})\big]\big[\bar{n}\cdot A(s_1\bar{n})\big]. \qquad (3.8)$$

In a similar fashion, the interaction term

$$\exp\left[i\int dy^4\ g_s\bar{\psi}(y)\gamma^\mu A_\mu(y)\psi(y)\right] = 1 + ig_sT^a\int d^4y\ \bar{\psi}(y)\gamma^\mu A^a_\mu(y)\psi(y)$$
$$- \frac{g_s^2}{2}T^aT^b\int d^4y\int d^4z\ \bar{\psi}(y)\gamma^\mu A^a_\mu(y)\psi(y)\bar{\psi}(z)\gamma^\mu A^b_\mu(z)\psi(z). \qquad (3.9)$$

With this expansion we find that

$$\mathcal{A} = \int d^dx\ e^{-ipx}\Bigg[g_s^2\int_{-\infty}^{0} ds\int_{-\infty}^{0} d\lambda\,\langle[\bar{n}\cdot A(s\bar{n})][\bar{n}\cdot A(x+\lambda\bar{n})]\psi(0)\bar{\psi}(x)\rangle$$
$$+ g_s\int_{-\infty}^{0} ds\int d^dy\,\langle[\bar{n}\cdot A(s\bar{n})]\mathcal{L}_{int}(y)\psi(0)\bar{\psi}(x)\rangle$$
$$- g_s\int_{-\infty}^{0} ds\int d^dy\,\langle[\bar{n}\cdot A(x+s\bar{n})]\mathcal{L}_{int}(y)\psi(0)\bar{\psi}(x)\rangle$$
$$- g_s^2\int_{-\infty}^{0} ds\int_{s}^{0} d\lambda\,\langle[\bar{n}\cdot A(s\bar{n})][\bar{n}\cdot A(\lambda\bar{n})]\psi(0)\bar{\psi}(x)\rangle$$
$$- g_s^2\int_{-\infty}^{0} ds\int_{-\infty}^{s} d\lambda\,\langle[\bar{n}\cdot A(x+s\bar{n})][\bar{n}\cdot A(x+\lambda\bar{n})]\psi(0)\bar{\psi}(x)\rangle$$
$$- \frac{1}{2}\int d^dy\int d^dz\ \langle\mathcal{L}_{int}(y)\mathcal{L}_{int}(z)\psi(0)\bar{\psi}(x)\rangle\Bigg]. \qquad (3.10)$$

By following this procedure, we obtain four distinct contributions. The first contribution corresponds to the leading order and can be attributed to the quark propagator. Moving to the next order, the first contribution is associated with the exchange of a gluon among the Wilson lines, which we refer to as the NLO (next-to-leading order) contribution. The second and third corrections, which we will now refer to as "vertex" terms, involve the exchange of a gluon between a Wilson line and the fermion. Moving further, we encounter the "eikonal self-energy" corrections as the next two terms, where a gluon is emitted and absorbed by the same Wilson line. Lastly, we consider the quark self-energy correction, completing the set of contributions.

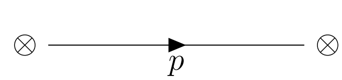


Figure 3.1: Leading order jet diagram

For the leading order contribution we set the Wilson lines to 1. Therefore, the leading order $\mathcal{J}^{(0)}$ function can is calculated

$$
\begin{aligned}
\frac{\not{n}}{2}(\bar{n}\cdot p)\mathcal{J}^{(0)}(p^2,\mu) &= \int d^dx\; e^{-ipx}\frac{\not{n}\not{\bar{n}}}{4}\,\langle\psi(0)\bar{\psi}(x)\rangle\,\frac{\not{\bar{n}}\not{n}}{4} = \int d^dx\int\frac{d^dk}{(2\pi)^d}\;e^{i(k-p)x}\frac{\not{n}\not{\bar{n}}\not{k}\not{\bar{n}}\not{n}}{16k^2} \\
&= \int\frac{d^dk}{(2\pi)^d}\;\frac{(k\cdot\bar{n})}{k^2}\frac{\not{n}}{2}\int d^dx\;e^{ix(k-p)} = \int d^dk\;\frac{(k\cdot\bar{n})}{k^2}\frac{\not{n}}{2}\delta^d(k-p) \\
&= (p\cdot\bar{n})\frac{\not{n}}{2}\frac{1}{p^2}, \qquad (3.11)
\end{aligned}
$$

or

$$
\boxed{\mathcal{J}^{(0)}(p^2,\mu) = \frac{1}{p^2}.} \qquad (3.12)
$$

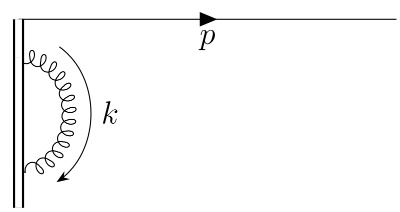


Figure 3.2: "Eikonal self-energy" diagram

For the NLO corrections, first, we calculate the "eikonal self energy" correction diagram depicted in Figure 3.2, which is given by

$$
\begin{aligned}
\mathcal{A}^{(1)}_{E1} &= -g_s^2\bar{n}^\mu\bar{n}^\nu T^aT^b\int d^dx\;e^{-ipx}\int_{-\infty}^{0}ds\int_{-\infty}^{s}d\lambda\,\langle\psi(0)\bar{\psi}(x)A^a_\mu(x+s\bar{n})A^b_\nu(x+\lambda\bar{n})\rangle \\
&= -g_s^2\bar{n}^\mu\bar{n}^\nu T^aT^b\int d^dx\;e^{-ipx}\int_{-\infty}^{0}ds\int_{-\infty}^{s}d\lambda\,\langle\psi(0)\bar{\psi}(x)\rangle\,\langle A^a_\mu(x+s\bar{n})A^b_\nu(x+\lambda\bar{n})\rangle \\
&= -g_s^2\bar{n}^\mu\bar{n}^\nu T^aT^b\int d^dx\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dq}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}\int_{-\infty}^{0}ds\int_{-\infty}^{s}d\lambda \\
&\qquad\qquad\times\frac{i\not{k}}{k^2}\,\langle\tilde{A}^a_\mu(q)\tilde{A}^b_\nu(l)\rangle\,e^{-ix(p-k+q+l)e^{-iq\bar{n}s}}e^{-il\bar{n}\lambda}
\end{aligned}
\qquad (3.13)
$$

The gluon field propagator gives the metric tensor which contracts $\bar{n}$ with itself. For light-like $\bar{n}$, $\bar{n}^2 = 0$, and therefore $\mathcal{A}^{(1)}_{E1} = 0$. The same holds true for the second eikonal self energy correction, that is, $\mathcal{A}^{(1)}_{E2} = 0$, and hence, these terms don't contribute to the jet function.

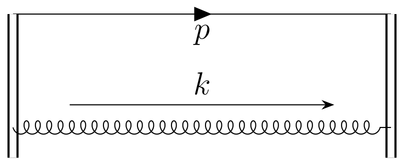


Figure 3.3: Wilson line - Wilson line interaction diagram

The next contribution we calculate is the interaction between the two Wilson Lines, depicted in Figure 3.3, and given by

$$\begin{aligned}\mathcal{A}^{(1)}_{ww} &= g_s^2 T^a T^b \bar{n}^\mu \bar{n}^\nu \int d^dx\; e^{-ipx} \int_{-\infty}^0 ds \int_{-\infty}^0 d\lambda\, \langle A^a_\mu(s\bar{n})\psi(0)\bar{\psi}(x) A^b_\nu(x+\lambda\bar{n})\rangle \\ &= g_s^2 T^a T^b \bar{n}^\mu \bar{n}^\nu \int d^dx\; e^{-ipx}\, \langle\psi(0)\bar{\psi}(x)\rangle \int_{-\infty}^0 ds \int_{-\infty}^0 d\lambda\, \langle A^a_\mu(s\bar{n}) A^b_\nu(x+\lambda\bar{n})\rangle \\ &= g_s^2 T^a T^b \bar{n}^\mu \bar{n}^\nu \int d^dx\; e^{-ipx}\, \langle\psi(0)\bar{\psi}(x)\rangle \int \frac{d^dq}{(2\pi)^d} \int \frac{d^dk}{(2\pi)^d} \langle \tilde{A}^a_\mu(q)\tilde{A}^b_\nu(k)\rangle \int_{-\infty}^0 ds \int_{-\infty}^0 d\lambda\; e^{i(sq+\lambda k+x)\bar{n}} \\ &= ig_s^2 C_F \int d^dx\; e^{-ipx}\, \langle\psi(0)\psi(x)\rangle \int \frac{d^dk}{(2\pi)^d}\, \frac{\bar{n}^2}{k^2(\bar{n}\cdot k)^2}. \end{aligned} \tag{3.14}$$

Again, because $\bar{n}$ is lightlike, $\mathcal{A}_{ww}$ vanishes, and it does not contribute to the jet function.

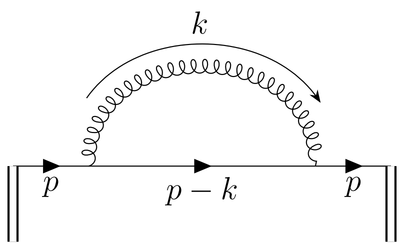


Figure 3.4: Quark self energy diagram

Now, we calculate the fermion self-energy contribution, shown in Figure 3.4

$$\begin{aligned}\mathcal{A}^{(1)}_{P} &= -g_s^2 \int d^dx\; e^{-ipx} \int d^dy \int d^dz\;\; \langle\psi(0)\bar{\psi}(y)\rangle\,\gamma^\nu\,\langle\psi(y)\bar{\psi}(z)\rangle\,\gamma^\mu\,\langle\psi(z)\bar{\psi}(x)\rangle\,\langle A_\nu(y)A_\mu(z)\rangle \\ &= -g_s^2 \int d^dx \int d^dy \int d^dz \int \frac{d^dk}{(2\pi)^d} \int \frac{d^dq}{(2\pi)^d} \int \frac{d^dl}{(2\pi)^d} \int \frac{d^dj}{(2\pi)^d} \frac{\not{j}\gamma^\mu \not{q}\gamma_\mu \not{l}}{k^2q^2l^2j^2} e^{ix(l-p)} e^{iy(j-q-k)} e^{iz(q-l-k)}. \end{aligned}$$

After we carry out the space integrations and the momentum integrations over q, l and j, we find that

$$\mathcal{A}^{(1)}_{P}(p^2,\mu) = g_s^2 \mu^{2\epsilon} C_F \frac{\not{p}}{p^2} \underbrace{\left[\int \tilde{dk}\; \frac{\gamma^\mu(\not{p}-\not{k})\gamma_\mu}{k^2(k-p)^2}\right]}_{I_k} \frac{\not{p}}{p^2}. \tag{3.15}$$

Figure 3.5: "Vertex" diagrams.

The momentum integral, gives the quark self energy, and it evaluates as

$$\Sigma(p) = \int \frac{d^dk}{(2\pi)^d}\frac{\gamma^\mu(\not{p}-\not{k})\gamma_\mu}{k^2(p-k)^2} = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{\gamma^\mu(\not{p}-\not{k})\gamma_\mu}{\Big[(1-x)k^2 + x(k^2+p^2-2k\cdot p)\Big]^2}$$
$$= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{\gamma^\mu(\not{p}-\not{k})\gamma_\mu}{\Big[(k-xp)^2-(-p^2)x(1-x)\Big]^2}. \quad (3.16)$$

We first focus on the denominator, which can be rewritten as

$$(k-xp)^2 + p^2x(1-x) \to k^2 + p^2x(1-x), \quad (3.17)$$

where the loop momentum has been shifted $k^\mu \to k^\mu + xp^\mu$. The numerator in turn is shifted to

$$\gamma^\mu\big[(1-x)\not{p}-\not{k}\big]\gamma_\mu = -(d-2)\big[(1-x)\not{p}-\not{k}\big] \to -2(1-\epsilon)(1-x)\not{p}, \quad (3.18)$$

where we have ignored the term proportional to $\not{k}$, since the integral remains invariant under the transformation $k^\mu \to -k^\mu$. Combining the two calculations, we find that

$$\mathcal{A}_P^{(1)}(p^2,\mu) = \frac{-\alpha_s C_F}{4\pi}\frac{\not{p}}{p^2}\left(\frac{4\pi\mu^2}{-p^2}\right)^\epsilon (1-\epsilon)\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}. \quad (3.19)$$

From (3.3) and the fact that $P_+\not{p}P_- = (\bar{n}\cdot p)\not{n}/2$, we arrive at

$$\boxed{\mathcal{J}_P^{(1)}(p^2,\mu) = \frac{-\alpha_s C_F}{4\pi p^2}\left(\frac{e^{\gamma_E}\mu^2}{-p^2}\right)^\epsilon (1-\epsilon)\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} = -\frac{\alpha_s C_F}{4\pi}\left[\frac{1}{\epsilon} - \log\frac{-p^2}{\mu^2} + 1\right].} \quad (3.20)$$

in $\overline{\text{MS}}$.

Finally, we solve for the contribution of the two "vertex terms" (Figure 3.5) combined. In explicit, one of the amplitudes that contribute is

$$\begin{aligned}
\mathcal{A}^{(1)}_{V_0} &= -g_s^2\mu^{2\epsilon}\bar{n}^\mu T^a T^b \int d^d x\, e^{-ipx} \int d^d y \int_{-\infty}^{0} ds\, \langle A^a_\mu(x+s\bar{n})\bar{\psi}(y)\gamma^\nu A^b_\nu(y)\psi(y)\psi(0)\bar{\psi}(x)\rangle \\
&= -g_s^2\mu^{2\epsilon}\bar{n}^\mu T^a T^b \int d^d x\, e^{-ipx} \int d^d y \int_{-\infty}^{0} ds\, \langle A^a_\mu(x+\lambda\bar{n})A^b_\nu(y)\rangle\, \langle\psi(0)\bar{\psi}(y)\rangle\, \gamma^\nu\, \langle\psi(y)\bar{\psi}(x)\rangle \\
&= ig_s^2\mu^{2\epsilon}C_F \int d^d x \int d^d y \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \frac{e^{-ix\cdot(p+k-q)}e^{-iy\cdot(l+q-k)}}{l^2 q^2 k^2(\bar{n}\cdot k)} \not{l}\,\not{\bar{n}}\,\not{q} \\
&= \frac{-ig_s^2\mu^{2\epsilon}C_F}{p^2} \int \frac{d^d k}{(2\pi)^d} \frac{\not{p}\,\not{\bar{n}}(\not{p}+\not{k})}{(p+k)^2 k^2(\bar{n}\cdot k)}.
\end{aligned} \tag{3.21}$$

Similarly, we find that

$$\begin{aligned}
\mathcal{A}^{(1)}_{V_x} &= g_s \int_{-\infty}^{0} ds \int d^d y\, \langle[\bar{n}\cdot A(s\bar{n})]\bar{\psi}(y)\gamma^\nu A_\nu(y)\psi(y)\psi(0)\bar{\psi}(x)\rangle \\
&= \frac{-ig_s^2\mu^{2\epsilon}C_F}{p^2} \int \frac{d^d k}{(2\pi)^d} \frac{(\not{p}+\not{k})\not{\bar{n}}\,\not{p}}{(p+k)^2 k^2(\bar{n}\cdot k)}.
\end{aligned} \tag{3.22}$$

We proceed now to calculate the sum of (3.21) and (3.22)

$$\mathcal{A}^{(1)}_{V} \equiv \mathcal{A}^{(1)}_{V_0} + \mathcal{A}^{(1)}_{V_x} = \frac{2ig_s^2 C_F\mu^{2\epsilon}}{-p^2} \int \frac{d^d k}{(2\pi)^d} \frac{(\not{p}+\not{k})\not{\bar{n}}\,\not{p} + \not{p}\,\not{\bar{n}}(\not{p}+\not{k})}{(p+k)^2 k^2[2(\bar{n}\cdot k)]}. \tag{3.23}$$

As always, first we parametrise the denominator, introducing Feynman parameters (A.3). Specifically,

$$\mathcal{A}^{(1)}_{V} = \frac{4ig_s^2\mu^{2\epsilon}C_F}{-p^2} \int_0^1 dx \int_0^1 dy \int \frac{d^d k}{(2\pi)^d} \frac{y^{-2}\left[(\not{p}+\not{k})\not{\bar{n}}\,\not{p} + \not{p}\,\not{\bar{n}}(\not{p}+\not{k})\right]}{\left[\left(k+xp+\frac{1-y}{y}\bar{n}\right)^2 - 2x\frac{1-y}{y}(\bar{n}\cdot p) + x(1-x)p^2\right]^3}. \tag{3.24}$$

The next step is to make the shift in the loop momentum $k^\mu \to k^\mu - xp^\mu - \frac{1-y}{y}\bar{n}^\mu$ keeping in mind that $\bar{n}^2 = 0$.

$$\begin{aligned}
\mathcal{A}_V^{(1)} &= \frac{8ig_s^2\mu^{2\epsilon}C_F}{-p^2}\not{p}\not{\bar{n}}\not{p}\int_0^1 dx\int_0^1 dy\int\frac{d^dk}{(2\pi)^d}\frac{(1-x)y^{-2}}{\Big[k^2-2x\frac{1-y}{y}(\bar{n}\cdot p)+x(1-x)p^2\Big]^3} \\
&= \frac{4g_s\mu^{2\epsilon}C_F}{-p^2(4\pi)^{d/2}}\not{p}\not{\bar{n}}\not{p}\frac{\Gamma(1+\epsilon)}{\epsilon}\int_0^1 dx\int_0^1 dy\ (1-x)y^{-2}\Big[2x\frac{1-y}{y}(\bar{n}\cdot p)-x(1-x)p^2\Big]^{-1-\epsilon} \\
&= \frac{4g_s\mu^{2\epsilon}C_F\Gamma(\epsilon)}{-p^2(4\pi)^{d/2}}\not{p}\not{\bar{n}}\not{p}\int_0^1 dx\int_0^\infty dz\ (1-x)\Big[2xz(\bar{n}\cdot p)-x(1-x)p^2\Big]^{-1-\epsilon} \\
&= \frac{g_s\mu^{2\epsilon}C_F\Gamma(\epsilon)}{-p^2(\bar{n}\cdot p)(4\pi)^{d/2}}\not{p}\not{\bar{n}}\not{p}\int_0^1 dx\ (1-x)x^{-1}\Big[-x(1-x)p^2\Big]^{-\epsilon} \\
&= \frac{\alpha_s C_F\not{p}\not{\bar{n}}\not{p}}{-4\pi p^2(\bar{n}\cdot p)}\left(\frac{4\pi\mu^2}{-p^2}\right)^\epsilon\Gamma(\epsilon)\int_0^1 dx\ (1-x)^{1-\epsilon}x^{-1-\epsilon}
\end{aligned} \tag{3.25}$$

Lastly, we take into account the equation $\not{p}\not{\bar{n}}\not{p} = \not{p}\big[2(\bar{n}\cdot p)-\not{p}\not{\bar{n}}\big] = 2(\bar{n}\cdot p)\not{p} - p^2\not{\bar{n}}$. The term proportional to $\not{\bar{n}}$ can be dropped because it will vanish when we place $\mathcal{A}_V$ between the projection operators. Thus, we find that, in $\overline{\text{MS}}$

$$\boxed{\mathcal{J}_V^{(1)}(p^2,\mu) = \frac{\alpha_s C_F}{-\pi p^2}\left(\frac{e^{\gamma_E}\mu^2}{-p^2}\right)^\epsilon\frac{\Gamma(\epsilon)\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}.} \tag{3.26}$$

If we expand in ϵ

$$\mathcal{J}_V^{(1)}(p^2,\mu) = \frac{\alpha_s C_F}{-\pi p^2}\left[-\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\left(1-\log\frac{-p^2}{\mu^2}\right)-\left(2-\frac{\pi^2}{24}-\log\frac{-p^2}{\mu^2}=\frac{1}{2}\log^2\frac{-p^2}{\mu^2}\right)\right], \tag{3.27}$$

where we see the emergence of double poles that are connected to soft and collinear divergences, as expected.

3.2 Calculation with the SCET Lagrangian

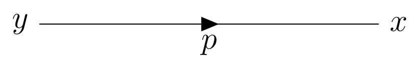
In this section, we perform the calculation of the jet function defined in (3.28)

$$\frac{\not{n}}{2}(\bar{n}\cdot p)\mathcal{J}(p^2,\mu) = \int d^dx\ e^{-ipx}\,\langle W^\dagger(0)\xi(0)\bar{\xi}(x)W(x)\rangle \tag{3.28}$$

with the SCET Lagrangian. Below, in Figure 3.6 we present the corresponding Feynman rules [1], which are evidently more complicated than the ones for regular QCD.

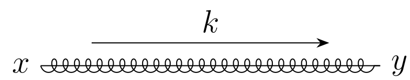
The leading order contribution to our calculations is simply given by the collinear quark propagator over i

$$\frac{\not{n}}{2}(\bar{n}\cdot p)\mathcal{J}^{(0)}(p^2,\mu) = \frac{(\bar{n}\cdot p)}{p^2}\frac{\not{n}}{2}, \tag{3.33}$$



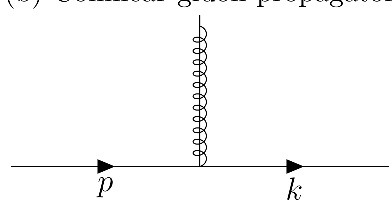
$$\langle 0|\xi(x)\bar{\xi}(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\; e^{ip(x-y)} \frac{i(\bar{n}\cdot p)}{p^2}\frac{\not{n}}{2} \tag{3.29}$$

(a) Collinear fermionic propagator



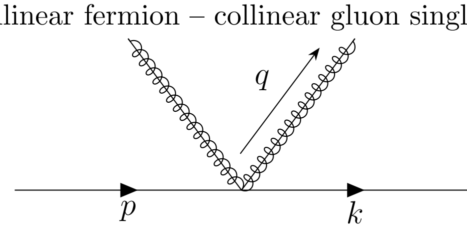
$$\langle 0|A^a_\mu(x)A^b_\nu(y)|0\rangle = \int \frac{d^4p}{(2\pi^4)}\; e^{ip(x-y)} \frac{-ig_{\mu\nu}\delta^{ab}}{p^2} \tag{3.30}$$

(b) Collinear gluon propagator



$$V_1^{\mu,a}(p,k) = ig_sT^a\left[n^\mu + \frac{\gamma_\perp^\mu \not{p}_\perp}{\bar{n}\cdot p} + \frac{\not{k}_\perp\gamma_\perp^\mu}{\bar{n}\cdot k} - \frac{\not{k}_\perp\not{p}_\perp}{(\bar{n}\cdot k)(\bar{n}\cdot p)}\bar{n}^\mu\right]\frac{\not{\bar{n}}}{2} \tag{3.31}$$

(c) Collinear fermion – collinear gluon single vertex



$$\begin{aligned} V_2^{\mu,\nu,a,b}(p,k,q) =& \frac{ig_s^2T^aT^b}{\bar{n}\cdot(p-k)}\left[\gamma_\perp^\mu\gamma_\perp^\nu - \frac{\gamma_\perp^\mu\not{p}_\perp}{\bar{n}\cdot p}\bar{n}^\nu - \frac{\not{k}_\perp\gamma_\perp^\nu}{\bar{n}k}\bar{n}^\mu + \frac{\not{k}_\perp\not{p}_\perp}{(\bar{n}\cdot p)(\bar{n}\cdot k)}\bar{n}^\mu\bar{n}^\nu\right]\frac{\not{\bar{n}}}{2} \\ &+ \frac{ig_s^2T^bT^a}{\bar{n}\cdot(p+k)}\left[\gamma_\perp^\nu\gamma_\perp^\mu - \frac{\gamma_\perp^\nu\not{p}_\perp}{\bar{n}\cdot p}\bar{n}^\nu - \frac{\not{k}_\perp\gamma_\perp^\mu}{\bar{n}\cdot k}\bar{n}^\nu + \frac{\not{k}_\perp\not{p}_\perp}{(\bar{n}\cdot p)(\bar{n}\cdot k)}\bar{n}^\mu\bar{n}^\nu\right]\frac{\not{\bar{n}}}{2} \end{aligned} \tag{3.32}$$

(d) Collinear fermion – collinear gluon double vertex

Figure 3.6: Feynman rules for the SCET Lagrangian

and therefore

$$\boxed{\mathcal{J}^{(0)}(p^2,\mu)=\frac{1}{p^2}.} \tag{3.34}$$

For the next to leading order corrections, first, we calculate the fermionic self energy correction diagrams, given in Figure 3.7. For the one on the left, we find that its expression is

Figure 3.7: Self energy corrections to the jet function

$$i\Sigma_c(p)=\int\frac{d^dk}{(2\pi)^d}\frac{-g_s^2C_F\big[\bar{n}\cdot(p+k)\big]}{8k^2(p+k)^2}\left[n^\mu+\frac{\gamma_\perp^\mu\not{p}_\perp}{\bar{n}\cdot p}+\frac{(\not{p}_\perp+\not{k}_\perp)\gamma_\perp^\mu}{\bar{n}\cdot(p+k)}-\frac{(\not{p}_\perp+\not{k}_\perp)\not{p}_\perp}{[\bar{n}\cdot(p+k)](\bar{n}\cdot p)}\bar{n}^\mu\right]\not{\bar{n}}\not{n}$$
$$\times\left[n_\mu+\frac{\gamma_\mu^\perp(\not{p}_\perp+\not{k}_\perp)}{\bar{n}(p+k)}+\frac{\not{p}_\perp\gamma_\mu^\perp}{\bar{n}\cdot p}-\frac{\not{p}_\perp(\not{p}_\perp+\not{k}_\perp)}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}\bar{n}_\mu\right]\not{\bar{n}}. \tag{3.35}$$

A useful property of $\not{n}$ and $\not{\bar{n}}$ is that they anti-commute with any $\not{v}_\perp$, i.e. $\{\not{n},\not{v}_\perp\}=\{\not{\bar{n}},\not{v}_\perp\}=0$. With this remark, we can drag all $\not{n}$ and $\not{\bar{n}}$ to the left, keeping in mind that $\not{\bar{n}}\not{n}\not{\bar{n}}=4\not{\bar{n}}$, ending up with

$$i\Sigma_c(p)=-g_s^2C_F\frac{\not{\bar{n}}}{2}\int\frac{d^dk}{(2\pi)^d}\left[n^\mu+\frac{\gamma_\perp^\mu\not{p}_\perp}{\bar{n}\cdot p}+\frac{(\not{p}_\perp+\not{k}_\perp)\gamma_\perp^\mu}{\bar{n}\cdot(p+k)}-\frac{(\not{p}_\perp+\not{k}_\perp)\not{p}_\perp}{[\bar{n}\cdot(p+k)](\bar{n}\cdot p)}\bar{n}^\mu\right]$$
$$\times\left[n_\mu+\frac{\gamma_\mu^\perp(\not{p}_\perp+\not{k}_\perp)}{\bar{n}(p+k)}+\frac{\not{p}_\perp\gamma_\mu^\perp}{\bar{n}\cdot p}-\frac{\not{p}_\perp(\not{p}_\perp+\not{k}_\perp)}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}\bar{n}_\mu\right]. \tag{3.36}$$

Now, we distribute the terms in the square brackets keeping in mind that $n^2=\bar{n}^2=n_\mu\gamma_\perp^\mu=\bar{n}\gamma_\perp^\mu=0$. $\Sigma_c(p)$ then becomes

$$i\Sigma_c(p)=-g_s^2C_F\frac{\not{\bar{n}}}{2}\int\frac{d^dk}{(2\pi)^d}\frac{\bar{n}\cdot(p+k)}{(p+k)^2}\Bigg\{-(\bar{n}\cdot n)\frac{\not{p}_\perp(\not{p}_\perp+\not{k}_\perp)}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}+\frac{\gamma_\perp^\mu\not{p}_\perp\gamma_\mu^\perp(\not{p}_\perp+\not{k}_\perp)}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}+\frac{p_\perp^2\gamma_\perp^\mu\gamma_\mu^\perp}{(\bar{n}\cdot p)^2}$$
$$+\frac{(\not{p}_\perp+\not{k}_\perp)\gamma_\perp^\mu\gamma_\mu^\perp(\not{p}_\perp+\not{k}_\perp)}{[\bar{n}\cdot(p+k)]^2}+\frac{(\not{p}_\perp+\not{k}_\perp)\gamma_\perp^\mu\not{p}_\perp\gamma_\mu^\perp}{[\bar{n}\cdot(p+k)](\bar{n}\cdot p)}-(\bar{n}\cdot n)\frac{(\not{p}_\perp+\not{k}_\perp)\not{p}_\perp}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}\Bigg\}. \tag{3.37}$$

The first and last term can be combined to give

$$
\begin{aligned}
-(\bar{n}\cdot n)\frac{\not{p}_\perp(\not{p}_\perp+\not{k}_\perp)}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}-(\bar{n}\cdot n)\frac{(\not{p}_\perp+\not{k}_\perp)\not{p}_\perp}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}=&-(\bar{n}\cdot n)\frac{\{\not{p}_\perp,\not{p}_\perp+\not{k}_\perp\}}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}\\
=&\frac{-4(p_\perp^2+p_\perp\cdot k_\perp)}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}.
\end{aligned}
\tag{3.38}
$$

To simplify the rest of the expression, first we have to calculate the contractions $\gamma_\perp^\mu\gamma_\mu^\perp$ and $\gamma_\perp^\mu\not{p}_\perp\gamma_\mu^\perp$. We start with $\gamma_\perp^\mu\gamma_\mu^\perp$.

$$
\gamma_\perp^\mu\gamma_\mu^\perp=\left(\gamma^\mu-\frac{n^\mu}{2}\not{\bar{n}}-\frac{\bar{n}^\mu}{2}\not{n}\right)\gamma_\mu^\perp=\gamma^\mu\gamma_\mu^\perp=\gamma^\mu\gamma_\mu-\frac{1}{2}\{\not{\bar{n}},\not{n}\}=d-2. \tag{3.39}
$$

As for $\gamma_\perp^\mu\not{p}_\perp\gamma_\mu^\perp$, first we calculate the commutator $\{\not{p}_\perp,\gamma_\perp^\mu\}$

$$
\{\not{p}_\perp,\gamma_\perp^\mu\}=\left\{\not{p}_\perp,\gamma^\mu-\frac{n^\mu}{2}\not{\bar{n}}-\frac{\bar{n}^\mu}{2}\not{n}\right\}=\{\not{p}_\perp,\gamma^\mu\}=2p_\perp^\mu. \tag{3.40}
$$

With this result, it is easy to calculate

$$
\gamma_\perp^\mu\not{p}_\perp\gamma_\mu^\perp=\gamma_\perp^\mu\{\not{p}_\perp,\gamma_\mu^\perp\}-\gamma_\perp^\mu\gamma_\mu^\perp\not{p}_\perp=-(d-4)\not{p}_\perp. \tag{3.41}
$$

The first to fifth term in (3.37) then become

$$
\begin{aligned}
&-(d-4)\frac{\not{p}_\perp(\not{p}_\perp+\not{k}_\perp)}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}+(d-2)\frac{p_\perp^2}{(\bar{n}\cdot p)^2}+(d-2)\frac{(p_\perp+k_\perp)^2}{[\bar{n}\cdot(p+k)]^2}-(d-4)\frac{(\not{p}_\perp+\not{k}_\perp)\not{p}_\perp}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}\\
&=-(d-4)\frac{\{\not{p}_\perp,\not{p}_\perp+\not{k}_\perp\}}{(\bar{n}\cdot p)[\bar{n}\cdot(p+k)]}+(d-2)\left[\frac{p_\perp^2}{(\bar{n}\cdot p)^2}+\frac{(p_\perp+k_\perp)^2}{[\bar{n}\cdot(p+k)]^2}\right].
\end{aligned}
\tag{3.42}
$$

Putting (3.38) and (3.42) in (3.37) we get that

$$
\boxed{
\begin{aligned}
i\Sigma_c(p)=g_s^2\mu^{2\epsilon}C_F\frac{\not{\bar{n}}}{2}\int\frac{d^dk}{(2\pi)^d}\Bigg\{&2(d-2)\frac{p_\perp^2+p_\perp\cdot k_\perp}{(\bar{n}\cdot p)(p+k)^2k^2}\\
&-(d-2)\left[\frac{(p_\perp+k_\perp)^2}{[\bar{n}\cdot(p+k)]^2}+\frac{p_\perp^2}{(\bar{n}\cdot p)^2}\right]\frac{\bar{n}\cdot(p+k)}{(p+k)^2k^2}\Bigg\}
\end{aligned}
}
$$

The result can be found readily in [1], in $\overline{\text{MS}}$,

$$
\Sigma_c(p)=\frac{\alpha_sC_F}{4\pi}\frac{p^2}{(\bar{n}\cdot p)}\frac{\not{\bar{n}}}{2}\left(\frac{e^{\gamma_E}\mu^2}{-p^2}\right)^\epsilon(1-\epsilon)\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}. \tag{3.43}
$$

The next step is to put the self-energy between two collinear propagators

$$
I_P(p^2)=\frac{i(\bar{n}\cdot p)}{p^2}\frac{\not{n}}{2}\Sigma_c(p)\frac{i(\bar{n}\cdot p)}{p^2}\frac{\not{n}}{2}=-\frac{-\alpha_sC_F}{4\pi p^2}\left(\frac{e^{\gamma_E}\mu^2}{-p^2}\right)^\epsilon(1-\epsilon)\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}. \tag{3.44}
$$

(a) Diagram for V_1 (b) Diagram for V_2

Therefore, we find that

$$\boxed{\mathcal{J}_P^{(1)}(p^2,\mu) = \frac{-\alpha_s C_F}{4\pi p^2}\left(\frac{e^{\gamma_E}\mu^2}{-p^2}\right)^{\epsilon}(1-\epsilon)\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}.} \tag{3.45}$$

Next, we calculate the vertex diagrams

$$iI_{V_1} = g_s^2\mu^{2\epsilon}C_F\int\frac{d^dk}{(2\pi)^d}\frac{[\bar{n}\cdot(p+k)]\bar{n}_\mu}{k^2(\bar{n}\cdot k)(p+k)^2}\frac{\not{n}}{2}\left[n^\mu + \frac{\gamma_\perp^\mu(\not{p}_\perp+\not{k}_\perp)}{\bar{n}\cdot(p+k)} + \frac{\not{p}_\perp\gamma_\perp^\mu}{\bar{n}\cdot p} - \frac{\not{p}_\perp(\not{p}_\perp+\not{k}_\perp)}{[\bar{n}\cdot(p+k)](\bar{n}\cdot p)}\bar{n}^\mu\right]\frac{\not{\bar{n}}}{2}\frac{i(\bar{n}\cdot p)}{p^2}\frac{\not{n}}{2}$$

$$\begin{aligned} iI_{V_1} &= \frac{ig_s^2\mu^{2\epsilon}C_F(\bar{n}\cdot p)}{p^2}\int\frac{d^dk}{(2\pi)^d}\frac{[\bar{n}\cdot(p+k)]}{k^2(\bar{n}\cdot k)(p+k)^2}\frac{\not{n}\not{\bar{n}}\not{n}}{8} \\ &= \frac{ig_s^2\mu^{2\epsilon}C_F(\bar{n}\cdot p)}{p^2}\frac{\not{n}}{2}\underbrace{\int\frac{d^dk}{(2\pi)^d}\frac{[\bar{n}\cdot(p+k)]}{k^2(\bar{n}\cdot k)(p+k)^2}}_{I_k} \end{aligned} \tag{3.46}$$

First, we are solving the integral over the loop momentum k. Using the parametrisation

(A.3)

$$
\begin{aligned}
I_k &= \int_0^1 dx \int_0^1 dy \int \frac{d^dk}{(2\pi)^d} \frac{4y[\bar{n}\cdot(p+k)]}{\left[yk^2 + 2xy(p\cdot k) + 2(1-y)(\bar{n}\cdot k) + xyp^2\right]^3} \\
&= \int_0^1 dx \int_0^1 dy \int \frac{d^dk}{(2\pi)^d} \frac{4y^{-2}[\bar{n}\cdot(p+k)]}{\left[\left(k + xp + \frac{1-y}{y}\bar{n}\right)^2 - \left(xp + \frac{1-y}{y}\bar{n}\right)^2 + xp^2\right]^3} \\
&= \int_0^1 dx \int_0^1 dy \int \frac{d^dk}{(2\pi)^d} \frac{4y^{-2}\left[(1-x)(\bar{n}\cdot p) + (\bar{n}\cdot k)\right]}{\left[k^2 - 2x\frac{1-y}{y}(\bar{n}\cdot p) + x(1-x)p^2\right]^3} \\
&= \frac{-2i\Gamma(1+\epsilon)(\bar{n}\cdot p)}{(4\pi)^{d/2}} \int_0^1 dx \int_0^1 dy\; y^{-2}(1-x)\left[2x\frac{1-y}{y}(\bar{n}\cdot p) - x(1-x)p^2\right]^{-1-\epsilon} \\
&= \frac{-2i\Gamma(1+\epsilon)(\bar{n}\cdot p)}{(4\pi)^{d/2}} \int_0^1 dx \int_0^\infty dz\; (1-x)x^{-1}\left[2xz(\bar{n}\cdot p) - x(1-x)p^2\right]^{-1-\epsilon} \\
&= \frac{-i(-p^2)^{-\epsilon}}{(4\pi)^d}\Gamma(\epsilon) \int_0^1 dx\; (1-x)^{1-\epsilon}x^{-1-\epsilon} \\
&= \frac{-i}{(4\pi)^2}\left(\frac{4\pi}{-p^2}\right)^{\epsilon}\Gamma(\epsilon)\frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)}
\end{aligned} \tag{3.47}
$$

We can put this result back in (3.46) and get back

$$
I_{V_1} = -\frac{\alpha_s C_F}{2\pi}\frac{(\bar{n}\cdot p)}{p^2}\frac{\not{n}}{2}\left(\frac{e^{\gamma_E}\mu^2}{-p^2}\right)^{\epsilon}\frac{\Gamma(\epsilon)\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)} \tag{3.48}
$$

We find the same contribution for the vertex V_2 and thus, the contribution for both "vertex" corrections to $\mathcal{J}$ is

$$
\boxed{\mathcal{J}_V(p^2,\mu) = -\frac{\alpha_s C_F}{\pi p^2}\left(\frac{e^{\gamma_E}\mu^2}{-p^2}\right)^{\epsilon}\frac{\Gamma(\epsilon)\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}} \tag{3.49}
$$

Finally, we have the diagram in 3.9, which vanishes.

We see that the contributions for $\mathcal{J}(p^2,\mu)$ to leading order are the same when they are calculated with the SCET Feynman rules and the regular QCD Feynman rules. From (3.5) we can find the jet function $J(p^2,\mu)$. More explicitly,

$$
\boxed{J(p^2) = \delta(p^2) - \frac{\alpha_s C_F}{4\pi p^2}\left(\frac{e^{\gamma_E}\mu^2}{-p^2}\right)^{\epsilon}\left[4\frac{\Gamma(\epsilon)\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)} + (1-\epsilon)\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}\right] + (\alpha_s^2)} \tag{3.50}
$$

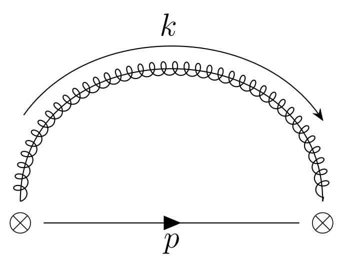


Figure 3.9: Diagram for interaction between Wilson lines.

Chapter 4

Conclusions & Outlook

In this thesis, we have observed that the leading power jet functions, derived through factorisation theorems within SCET, can be computed using either the SCET Lagrangian or the regular QCD Lagrangian. The rationale behind this lies in the decoupling transformation, which effectively separates the soft and collinear sectors. Following this transformation, the collinear fermions can be regarded as fermions within full QCD, but constrained to move in specific directions.

Looking ahead to future work, there is an opportunity to place greater emphasis on the definition and computation of sub-leading jet functions. While these jet functions have received less attention compared to their leading power counterparts, they play a crucial role in achieving high precision in collider physics experiments.

Appendix A

Useful Formulas

A.1 Parametrisations

To evaluate loop integrals in quantum field theory, it is often helpful to introduce Feynman or Schwinger parameters [8]. The Feynman parameters are based on easily verifiable mathematical identities. The simplest is

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A+(B-A)x]^2} = \int_0^1 dx \int_0^1 dy\ \delta(1-x-y)\frac{1}{[xA+yB]^2} \tag{A.1}$$

A parametrisation that is really useful when we have terms linear in momenta in the denominator is derived below.

$$\begin{aligned}
\frac{1}{ABC} &= \int_0^1 dx \int_0^1 dy\ \delta(1-x-y)\frac{1}{(xA+yB)^2}\frac{1}{C} \\
&= \int_0^1 dx \int_0^1 dy\ \delta(1-x-y)\frac{1}{\alpha^2 C} \\
&= -\frac{\partial}{\partial\alpha}\int_0^1 dx \int_0^1 dy\ \delta(1-x-y)\frac{1}{\alpha C} \\
&= -\frac{\partial}{\partial\alpha}\int_0^1 dx \int_0^1 dy \int_0^1 dz\ \frac{\delta(1-x-y)}{[z\alpha+(1-z)C]^2} \\
&= \int_0^1 dx \int_0^1 dy \int_0^1 dz\ \delta(1-x-y)\frac{2z}{[z\alpha+(1-z)C]^2},
\end{aligned} \tag{A.2}$$

where $\alpha = xA+yB$. We now perform the integration over y and change the variable name $z \to y$ to get the final form of the parametrisation

$$\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy\ \frac{2y}{\left[y\Big(x(A-B)+B\Big)+(1-y)C\right]^3}. \tag{A.3}$$

A.2 Loop integration

Two really useful formulas that are used throughout this thesis repeatedly are the results of the following loop momentum integrals [11]

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)} \Delta^{d/2-n} \tag{A.4}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2-\Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(n-d/2-1)}{\Gamma(n)} \Delta^{1+d/2-n} \tag{A.5}$$

Appendix B

Gamma Matrices Properties

B.1 Properties of Gamma Matrices

1. $\gamma^\mu \gamma_\mu = 4\mathbb{1}_4$
2. $\gamma^\nu \gamma^\mu \gamma_\nu = -2\gamma^\mu$
3. $\gamma^\nu \gamma^\mu \gamma^\rho \gamma_\nu = 4\eta^{\mu\rho}\mathbb{1}_4$
4. $\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma \gamma_\nu = -2\gamma^\sigma \gamma^\mu \gamma^\rho$

Proposition B.1.1. The Gamma matrices satisfy the following anti-commutation relations:

1. $\{\gamma^\mu \gamma^\nu, \gamma^\rho\} = 2(\eta^{\nu\rho}\gamma^\mu + \eta^{\mu\rho}\gamma^\nu)$
2. $\{\gamma^\mu \gamma^\nu, \gamma^\rho \gamma^\sigma\} = 2(\eta^{\nu\rho}\gamma^\mu \gamma^\sigma + \eta^{\mu\rho}\gamma^\nu \gamma^\sigma + \eta^{\mu\sigma}\gamma^\rho \gamma^\nu + \eta^{\nu\sigma}\gamma^\rho \gamma^\mu)$
3. $\gamma^\mu \gamma^\nu \gamma^\rho = -\gamma^\rho \gamma^\nu \gamma^\mu + 2(\eta^{\mu\nu}\gamma^\rho - \eta^{\mu\rho}\gamma^\nu + \eta^{\nu\rho}\gamma^\mu)$

Proof

1. We start from the the anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$ and we find that

$$\{\gamma^\mu \gamma^\nu, \gamma^\rho\} = \gamma^\mu \{\gamma^\nu, \gamma^\rho\} + \{\gamma^\mu, \gamma^\rho\}\gamma^\nu = 2(\eta^{\nu\rho}\gamma^\mu + \eta^{\mu\rho}\gamma^\nu) \tag{B.1}$$

2. Using the previous two relations above, we can easily calculate that

$$\begin{aligned} \{\gamma^\mu \gamma^\nu, \gamma^\rho \gamma^\sigma\} &= \{\gamma^\mu \gamma^\nu, \gamma^\rho\}\gamma^\sigma + \gamma^\rho \{\gamma^\mu \gamma^\nu, \gamma^\sigma\} \\ &= 2(\eta^{\nu\rho}\gamma^\mu \gamma^\sigma + \eta^{\mu\rho}\gamma^\nu \gamma^\sigma + \eta^{\mu\sigma}\gamma^\rho \gamma^\nu + \eta^{\nu\sigma}\gamma^\rho \gamma^\mu) \end{aligned} \tag{B.2}$$

3. To prove this result, we use the commutation relations of gamma matrices repeatedly.

■

B.2 Fermionic Projection Operators

In the followin, $n \equiv n_{i-}$ and $\bar{n} \equiv n_{i+}$.

Proposition B.2.1. The operators $P_+ := \not{n}\not{\bar{n}}/4$ and $P_- := \not{\bar{n}}\not{n}/4$ are projection operators.

Proof

For a set of two operators to be projection operators, they have to satisfy the conditions $P_\pm^2 = P_\pm$, $P_+ + P_- = 1$ and $P_+ \cdot P_- = 0$. A necessary result for the proof of these conditions is the anti-commutator $\{\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma\}$. The calculation goes as follows, keeping in mind that .

Then,

Using this result then for P_+

$$\begin{aligned}
16P_+^2 &= \not{n}\not{\bar{n}}\not{n}\not{\bar{n}} = n_\mu\bar{n}_\nu n_\rho\bar{n}_\sigma\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \\
&= -n_\mu\bar{n}_\nu n_\rho\bar{n}_\sigma\gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu + 2n_\mu\bar{n}_\nu n_\rho\bar{n}_\sigma(\eta^{\nu\rho}\gamma^\mu\gamma^\sigma + \eta^{\mu\rho}\gamma^\nu\gamma^\sigma + \eta^{\mu\sigma}\gamma^\rho\gamma^\nu + \eta^{\nu\sigma}\gamma^\rho\gamma^\mu) \\
32P_+^2 &= 2\Big((\bar{n}\cdot n)n_\mu\bar{n}_\sigma\gamma^\nu\gamma^\sigma + (n^2)\bar{n}_\nu\bar{n}_\sigma\gamma^\nu\gamma^\sigma + (n\cdot\bar{n})n_\rho\bar{n}_\nu\gamma^\rho\gamma^\nu + (\bar{n})^2 n_\mu n_\rho\gamma^\rho\gamma^\mu\Big) \\
16P_+^2 &= 4n_\mu\bar{n}_\sigma\gamma^\nu\gamma^\sigma = 16P_+
\end{aligned} \tag{B.3}$$

Therefore, indeed $P_+^2 = P_+$. Similarly one can show easily that $P_-^2 = P_-$. Next, we show that

$$\begin{aligned}
4(P_+ + P_-) &= \not{n}\not{\bar{n}} + \not{\bar{n}}\not{n} = n_\mu\bar{n}_\nu(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \\
&= n_\mu\bar{n}_\nu\{\gamma^\mu, \gamma^\nu\} = 2n_\mu\bar{n}_\nu\eta^{\mu\nu}\mathbb{1} \\
&= 2(n\cdot\bar{n})\mathbb{1} = 4\mathbb{1}
\end{aligned} \tag{B.4}$$

Therefore, $P_+ + P_- = 1$.

■

Proposition B.2.2. The operators defined in Proposition B.2.1 satisfy the following properties:

1. $\not{n}P_+ = \not{\bar{n}}P_- = 0$
2. $P_+\not{n} = \not{n}P_- = \not{n}$ and $P_-\not{\bar{n}} = \not{\bar{n}}P_+ = \not{\bar{n}}$
3. $\overline{P_+\psi} = \bar{\psi}P_-$ and $\overline{P_-\psi} = \bar{\psi}P_+$

Proof

1. For the first property, we prove it for P_+ and the proof for P_- is similar. It is straightforwad to show that

$$\not{n}P_+ = \frac{1}{4}\not{n}\not{n}\bar{\not{n}} = \frac{1}{4}n_\mu n_\nu \bar{n}_\rho \gamma^\mu\gamma^\nu\gamma^\rho = \frac{1}{4}n_\mu n_\nu \bar{n}_\rho \left(2\eta^{\mu\nu}\gamma^\rho - \gamma^\nu\gamma^\mu\right) \Rightarrow 2\not{n}P_+ = \frac{1}{2}n^2\bar{\not{n}} \tag{B.5}$$

Because $n^2 = 0$, we get that $\not{n}P_+$ vanishes as well.

2. For the second property, we are going to use first property. Namely,

$$\bar{\not{n}} = \bar{\not{n}}(P_+ + P_-) = \bar{\not{n}}P_+ + \bar{\not{n}}P_- = \bar{\not{n}}P_+ \tag{B.6}$$

Similarly for P_-. Also, it is straightforward to show

$$P_+\not{n} = \not{n}\frac{\bar{\not{n}}\not{n}}{4} = \frac{\bar{\not{n}}\not{n}}{4}\not{n} = P_-\not{n} = \not{n} \tag{B.7}$$

The same for $P_-\bar{\not{n}}$.

3. To prove this property, we need another result for the

$$\overline{\gamma^\mu\gamma^\nu\psi} = (\gamma^\mu\gamma^\nu\psi)^\dagger\gamma^0 = \psi^\dagger(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger\gamma^0 \tag{B.8}$$

Since $(\gamma^0)^2 = \mathbb{1}$ we can enter it between terms without changing the result, and we get

$$\overline{\gamma^\mu\gamma^\nu\psi} = \left(\psi^\dagger\gamma^0\right)\left(\gamma^0(\gamma^\nu)^\dagger\gamma^0\right)\left(\gamma^0(\gamma^\mu)^\dagger\gamma^0\right) \tag{B.9}$$

Using the result $\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$, this equation becomes

$$\overline{\gamma^\mu\gamma^\nu\psi} = \bar{\psi}\gamma^\nu\gamma^\mu \tag{B.10}$$

Now, since $n_\mu, \bar{n}_\nu \in \mathbb{R}$, it is straightforward to show that

$$\overline{P_+\psi} = \overline{\left(\frac{\not{n}\bar{\not{n}}}{4}\psi\right)} = \frac{n_\mu\bar{n}_\nu}{4}\,\overline{\gamma^\mu\gamma^\nu\psi} = \frac{n_\mu\bar{n}_\nu}{4}\bar{\psi}\gamma^\nu\gamma^\mu = \bar{\psi}\left(\frac{\bar{\not{n}}\not{n}}{4}\right) = \bar{\psi}P_- \tag{B.11}$$

The proof for the second part is exactly the same.

■

Proposition B.2.3. For the components of the collinear fermion field we have the following properties:

1. $\not{n}\xi = \bar{\xi}\not{n} = 0$
2. $\bar{\not{n}}\eta = \bar{\eta}\bar{\not{n}} = 0$
3. $\bar{\xi}\not{D}_\perp\xi = \bar{\eta}\not{D}_\perp\eta = 0$

Proposition B.2.4. For any vector $v \in \mathbb{R}^4$

$$\not{n}\not{v}\not{n} = 2(n \cdot v)\not{n} \qquad \text{and} \qquad \not{\bar{n}}\not{v}\not{\bar{n}} = 2(\bar{n} \cdot v)\not{\bar{n}} \tag{B.12}$$

Proof By B.1.1(3) we have that

$$\begin{aligned} \not{n}\not{v}\not{n} &= n_\mu v_\nu n_\rho \gamma^\mu \gamma^\nu \gamma^\rho = n_\mu v_\nu n_\rho \big[- \gamma^\rho \gamma^\nu \gamma^\mu + 2(\eta^{\mu\nu}\gamma^\rho - \eta^{\mu\rho}\gamma^\nu + \eta^{\nu\rho}\gamma^\mu)\big] \\ &= -\not{n}\not{v}\not{n} + 2\big[(n \cdot v)\not{n} - (n^2)\not{v} + (n \cdot v)\not{n}\big] \Rightarrow 2\not{n}\not{v}\not{n} = 4(n \cdot v)\not{n} \end{aligned} \tag{B.13}$$

Which gives the desired result. We work similarly for the second one. ■

Proposition B.2.5. For any vector $v \in \mathbb{R}^4$

$$P_+\not{v}P_- = \frac{(v \cdot \bar{n})}{2}\not{n} \qquad \text{and} \qquad P_-\not{v}P_+ = \frac{(v \cdot n)}{2}\not{\bar{n}} \tag{B.14}$$

Proof Using Proposition B.2.4 it is straightforward to calculate

$$P_+\not{v}P_- = \frac{1}{16}\not{n}\not{\bar{n}}\not{v}\not{\bar{n}}\not{n} = \frac{(v \cdot \bar{n})}{8}\not{n}\not{\bar{n}}\not{n} = \frac{(v \cdot \bar{n})(n \cdot \bar{n})}{4}\not{n} = \frac{(v \cdot \bar{n})}{2}\not{n} \tag{B.15}$$

■

Proposition B.2.6. For any $v, u \in \mathbb{R}^d$

$$\{\not{v}_\perp, \not{u}_\perp\} = 2(v \cdot u) - (v \cdot n)(u \cdot \bar{n}) - (v \cdot \bar{n})(u \cdot n) \tag{B.16}$$

Proposition B.2.7. For any $v \in \mathbb{R}^d$

$$\gamma^\mu_\perp \not{\bar{n}}\not{n}\gamma^\perp_\mu = (d-2)\not{\bar{n}}\not{n} \tag{B.17}$$

Proof

We start from writing

$$\gamma^\mu_\perp \not{\bar{n}}\not{n}\gamma^\mu_\perp = \left(\gamma^\mu - \frac{n^\mu}{2}\not{\bar{n}} - \frac{\bar{n}^\mu}{2}\not{n}\right)\not{\bar{n}}\not{n}\gamma^\perp_\mu \tag{B.18}$$

Since, $\bar{n}^\mu \gamma^\perp_\mu = n^\mu \gamma^\perp_\mu = 0$, from the first parenthesis we keep only γ^μ

$$\gamma^\mu_\perp \not{v}\not{\bar{n}}\not{n}\gamma^\perp_\mu = \gamma^\mu \not{\bar{n}}\not{n}\left(\gamma_\mu - \frac{n_\mu}{2}\not{\bar{n}} - \frac{\bar{n}^\mu}{2}\not{n}\right) \tag{B.19}$$

We can observe that we can ignore the term that is proportional to $\not{n}$ inside the parenthesis, since its contribution vanishes immediately. Therefore,

$$\begin{aligned}
\gamma_\perp^\mu \not{v}\not{\bar{n}}\not{n}\gamma_\mu^\perp &= \gamma^\mu \not{\bar{n}}\not{n}\left(\gamma_\mu - \frac{n_\mu}{2}\not{\bar{n}}\right) \\
&= \gamma^\mu \not{\bar{n}}\not{n}\gamma_\mu - \frac{1}{2}\not{n}\not{\bar{n}}\not{n}\not{\bar{n}} \\
&= (d-4)\not{\bar{n}}\not{n} + 8 - 2\not{n}\not{\bar{n}} \qquad \text{(B.20)}
\end{aligned}$$

Remembering also that $\not{n}\not{\bar{n}} + \not{\bar{n}}\not{n} = 4$, we arrive to the desired result.

■

Proposition B.2.8. For any $v \in \mathbb{R}^d$

1. $$\gamma_\perp^\mu \not{v}\not{\bar{n}}\not{n}\gamma_\mu^\perp = (d-4)\not{v}\not{\bar{n}}\not{n} - 4(\bar{n}\cdot v)\not{n} \qquad \text{(B.21)}$$

2. if v has only components in the perpendicular direction, i.e. $v = v_\perp$, then

$$\gamma_\perp^\mu \not{v}_\perp \not{\bar{n}}\not{n}\gamma_\mu^\perp = (d-4)\not{v}_\perp\not{\bar{n}}\not{n} \qquad \text{(B.22)}$$

Proof

1. Similarly to the proof of the previous result, we start from the equation

$$\begin{aligned}
\gamma_\perp^\mu \not{v}\not{\bar{n}}\not{n}\gamma_\mu^\perp &= \gamma^\mu \not{v}\not{\bar{n}}\not{n}\left(\gamma_\mu - \frac{n^\mu}{2}\not{\bar{n}}\right) \\
&= \gamma^\mu \not{v}\not{\bar{n}}\not{n}\gamma_\mu - \frac{1}{2}\not{n}\not{v}\not{\bar{n}}\not{n}\not{\bar{n}} \\
&= (d-4)\not{v}\not{\bar{n}}\not{n} - 2\not{n}\not{\bar{n}}\not{v} - 2\not{n}\not{v}\not{\bar{n}} \\
&= (d-4)\not{v}\not{\bar{n}}\not{n} - 2\not{n}\{\not{\bar{n}}, \not{v}\} \\
&= (d-4)\not{v}\not{\bar{n}}\not{n} - 4(\bar{n}\cdot v)\not{n} \qquad \text{(B.23)}
\end{aligned}$$

2. For the second part the result is immediate from the first part, noticing that $(\bar{n}\cdot v_\perp) = 0$. What remains to be shown is that

$$\gamma_\perp^\mu \not{v}_\perp \not{\bar{n}}\not{n}\gamma_\mu^\perp = -\gamma_\perp^\mu \not{\bar{n}}\not{n}\not{v}_\perp\gamma_\mu^\perp. \qquad \text{(B.24)}$$

   To that end, we calculate first that

$$\{\not{v}_\perp, \not{\bar{n}}\not{n}\} = \{\not{v}_\perp, \not{\bar{n}}\}\not{n} + \not{\bar{n}}\{\not{v}_\perp, \not{n}\} = 2(\bar{n}\cdot v_\perp)\not{n} + 2(n\cdot v_\perp)\not{\bar{n}} = 0, \qquad \text{(B.25)}$$

   which means that $\not{v}_\perp\not{\bar{n}}\not{n} = -\not{\bar{n}}\not{n}\not{v}_\perp$.

■

Proposition B.2.9. For any $v \in \mathbb{R}^d$

1.

$$\gamma^\mu_\perp \not{v}\not{\bar{n}}\not{n}\not{v}\gamma^\perp_\mu = -2(d-4)(n\cdot v)\not{\bar{n}}\not{v} + 2(d-4)(\bar{n}\cdot v)\not{n}\not{v} + v^2(d-2)\not{\bar{n}}\not{n} + 2(\bar{n}\cdot v)(n\cdot v)(\not{n}\not{\bar{n}} - \not{\bar{n}}\not{n}) \tag{B.26}$$

If v has components only in the perpendicular direction, i.e. $v = v_\perp$, then

$$\gamma^\mu_\perp \not{v}_\perp \not{\bar{n}}\not{n}\not{v}_\perp \gamma^\perp_\mu = (d-2)v^2_\perp \not{\bar{n}}\not{n} \tag{B.27}$$

Proof

Our starting point is the equation

$$\begin{aligned}
\gamma^\mu_\perp \not{v}\not{\bar{n}}\not{n}\not{v}\gamma^\perp_\mu &= \gamma^\mu \not{v}\not{\bar{n}}\not{n}\not{v}\left(\gamma_\mu - \frac{n_\mu}{2}\not{\bar{n}} - \frac{\bar{n}_\mu}{2}\not{n}\right) \\
&= \gamma^\mu \not{v}\not{\bar{n}}\not{n}\not{v}\gamma_\mu - \frac{1}{2}\not{n}\not{v}\not{\bar{n}}\not{n}\not{v}\not{\bar{n}} - \frac{1}{2}\not{\bar{n}}\not{v}\not{\bar{n}}\not{n}\not{v}\not{n} \\
&= -2(d-4)(n\cdot v)\not{\bar{n}}\not{v} + 2(d-4)(\bar{n}\cdot v)\not{n}\not{v} + v^2\left[(d-2)\not{\bar{n}}\not{n} + 2\not{n}\not{\bar{n}}\right] \\
&\quad - \frac{1}{2}\not{n}\not{v}\not{\bar{n}}\not{n}\not{v}\not{\bar{n}} - 2(\bar{n}\cdot v)(n\cdot v)\not{\bar{n}}\not{n}
\end{aligned} \tag{B.28}$$

Finally, with repeated commutations we find that

$$-\frac{1}{2}\not{n}\not{v}\not{\bar{n}}\not{n}\not{v}\not{\bar{n}} = -2\left[v^2 - (n\cdot v)(\bar{n}\cdot v)\right]\not{n}\not{\bar{n}} \tag{B.29}$$

Then, the final result becomes

$$\gamma^\mu_\perp \not{v}\not{\bar{n}}\not{n}\not{v}\gamma^\perp_\mu = -2(d-4)(n\cdot v)\not{\bar{n}}\not{v} + 2(d-4)(\bar{n}\cdot v)\not{n}\not{v} + v^2(d-2)\not{\bar{n}}\not{n} + 2(\bar{n}\cdot v)(n\cdot v)(\not{n}\not{\bar{n}} - \not{\bar{n}}\not{n}) \tag{B.30}$$

■

Bibliography

[1] Christian W. Bauer, Sean Fleming, Dan Pirjol, and Iain W. Stewart. An effective field theory for collinear and soft gluons: heavy to light decays. *Physical Review D*, 63(11), may 2001.

[2] Christian W. Bauer, Dan Pirjol, and Iain W. Stewart. Soft-collinear factorization in effective field theory. *Physical Review D*, 65(5), feb 2002.

[3] Thomas Becher, Alessandro Broggio, and Andrea Ferroglia. *Introduction to Soft-Collinear Effective Theory*. Springer International Publishing, 2015.

[4] Thomas Becher, Matthias Neubert, and Gang Xu. Dynamical threshold enhancement and resummation in drell-yan production. *Journal of High Energy Physics*, 2008(07):030–030, jul 2008.

[5] M. Beneke and Th. Feldmann. Multipole-expanded soft-collinear effective theory with non-abelian gauge symmetry. *Physics Letters B*, 553(3-4):267–276, feb 2003.

[6] S.W. Bosch, B.O. Lange, M. Neubert, and G. Paz. Factorization and shape-function effects in inclusive b-meson decays. *Nuclear Physics B*, 699(1-2):335–386, nov 2004.

[7] C. Bauer I.Steward. *Introduction to Soft-Collinear Effective Theory*. MIT Opencourseware, 2014.

[8] U-Rae Kim, Sungwoong Cho, and Jungil Lee. The art of schwinger and feynman parametrizations. *Journal of the Korean Physical Society*, 82(11):1023–1039, 2023.

[9] Aneesh V. Manohar, Thomas Mehen, Dan Pirjol, and Iain W. Stewart. Reparameterization invariance for collinear operators. *Physics Letters B*, 539(1-2):59–66, jul 2002.

[10] S. Moch and A. Vogt. Higher-order soft corrections to lepton pair and higgs boson production. *Physics Letters B*, 631(1-2):48–57, dec 2005.

[11] Matthew D. Schwartz. Quantum Field Theory and the Standard Model. 2013.