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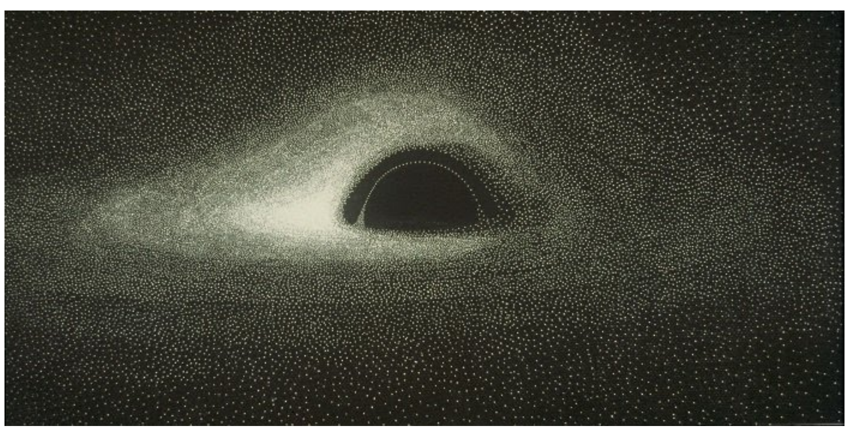
Rotating Black Holes in Dynamical Chern-Simons Gravity and the Four Laws of Black Hole Mechanics



Originally created by J.P. Luminet

Master Thesis
Institute for Theoretical Physics
09-07-2023

Abstract

Although general relativity has proven to be a very successful theory of gravitation, there are strong indications that general relativity requires modification. Dynamical Chern-Simons gravity is one of such modifications of general relativity that has gained popularity in the last fifteen years.

An important prediction of general relativity is the existence of black holes. Although there is a good deal of observational evidence indicating the existence of black holes, it is not yet clear whether the black holes in our universe are well described by general relativity. Therefore, it is critical to study black holes in modified theories of gravity.

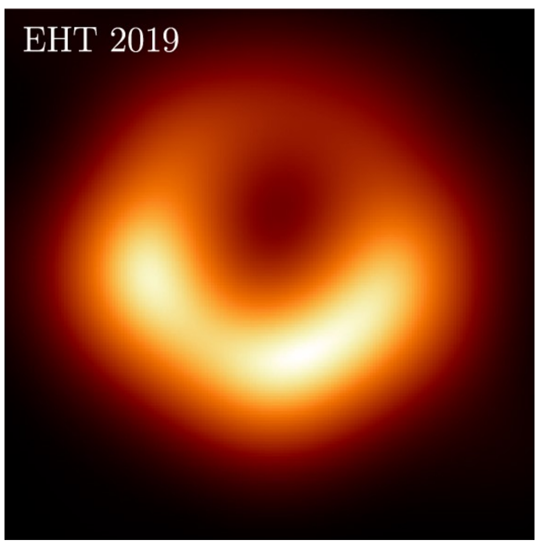
In dynamical Chern-Simons gravity, static black holes do not differ from those in general relativity. However, it is expected that black holes generally are not static, as they are expected to have angular momentum. In other words, they rotate. In that case, the rotating black holes from general relativity, which are described by the Kerr metric, need corrections to satisfy the modified field equations of dynamical Chern-Simons gravity.

In this thesis, we will review the current best description of rotating black holes in dynamical Chern-Simons gravity. Subsequently, we will use this description to test the four laws of black hole mechanics, and draw the comparison with the Kerr black hole from general relativity.

Contents

Introduction

It was in 1979 that French physicist Jean-Pierre Luminet first produced a visual representation of a black hole with accretion disk based on the formulas of general relativity with nothing but an IBM computer, a sheet of paper and ink [1]. At the time, the concept of a black hole was still highly theoretical, hence being able to 'see' such an unfathomable object is nothing short of remarkable.
To produce this 'image' of a black hole, Luminet considered a rotating disk of gas around a static black hole. Although only the accretion disk was taken to be rotating, Luminet argued that a slowly rotating black hole with accretion disk should look similar, and he posed that his image could potentially be a representation of the supermassive black hole M87* at the centre of the galaxy M87.
Almost exactly 40 years later, the EHT consortium captured an image of M87*[2]. Although not quite one to one with Luminet's 'image', many of the defining features were there.
Since 1979 our mathematical understanding of black holes has had a long time to mature, and more and more observational evidence of the existence of black holes has been and is being gathered. However, even though we are almost certain that black holes exist, we are not sure whether the theory of general relativity is sufficient to describe these black holes.



The supermassive black hole M87* at the core of the Messier 87 galaxy, as captured by the EHT consortium.

That general relativity almost certainly does not provide a complete description of gravity is clear, as the theory is not compatible with quantum physics. This realization has led physicists to propose and explore many modified theories of general relativity. Such a theory must provide predictions similar to general relativity in weak gravity environments, such as in our own solar system or in binary pulsar systems, but predictions may differ in environments where gravity is strong, such as black hole environments. Which is why it is interesting to study black holes in modified theories of gravity.

A well-motivated modification of general relativity that has gained popularity in the last fifteen or so years is dynamical Chern-Simons gravity (dCS). This modification has its roots in string theory, loop quantum gravity, and particle physics, but it also comes up naturally when supplementing the Einstein-Hilbert action of general relativity with higher curvature terms. Dynamical Chern-Simons gravity, which was first formulated in [2, 3], extends the Einstein-Hilbert action, by adding a dynamical scalar field coupled to the Chern-Pontryagin scalar, which is quadratic in the curvature.

In dCS gravity, static black holes do not require modifications, however this does not apply to rotating black holes [4]. This has made finding dCS corrections to the Kerr metric a primary point of interest in dCS gravity.

The first dCS corrections to the Kerr metric were found in 2009 by N. Yunes and F. Pretorius [5], however they only allowed for the probing of slowly rotating black holes.

Only recently, in 2019, significantly more accurate dCS corrections to the Kerr metric were found by P. Cano, and A. Ruipérez [6]. These newly found corrections have allowed us to probe moderately fast rotating black holes.

An interesting aspect of black holes in general relativity are the four laws of black hole mechanics. These are a set of four laws, which were discovered in the early seventies [7], connected black hole physics with the four laws of thermodynamics. Since, the thermodynamics of black holes has become a rich, and very well-studied subject. These laws can also be used to put bounds on the energy that can be extracted or released by single or multiple black hole systems.

Whether the laws of black hole mechanics hold in modified theories of gravity is a priori not clear, as many proofs of the laws of black hole mechanics rely on assumptions that cannot be made in many modified theories of gravity, this is for example the case in dCS gravity. This is one of the reasons why studying the laws of black hole mechanics in modified theories of gravity an interesting subject of research.

The structure of this thesis is as follows: In the first chapter, we will briefly go over the main concepts of general relativity and Kerr black holes. Then, in the second chapter, we will discuss the basics of dCS gravity. In the third chapter, we discuss the basic properties of the dCS corrected Kerr black hole, which will include some new work on the local Petrov type. In the final chapter, we will test the four laws of black hole mechanics explicitly using the dCS corrected Kerr metric, and we calculate the upper bound on the amount of energy that can be released from a single or multiple dCS black holes.

1 A Brief Introduction to General Relativity and Black Holes

In this section, we will briefly go over the main concepts of general relativity that we will be using in this thesis. We will also briefly cover the topic of black holes by using the Kerr black hole as an example. This is by no means a complete overview of general relativity or black holes. For that, we will refer to S. Carroll's textbook as an introductory text [8], R. Wald's classic textbook for a more rigorous treatment of general relativity [9], or E. Poisson's textbook for the more advanced topics [10]. The notation and naming conventions used in this thesis are mainly based on those three texts.

1.1 The Einstein Field Equations

The theory of general relativity is built on a four-dimensional pseudo-Riemannian manifold, usually referred to simply as spacetime, which is a manifold equipped with a metric $g_{\mu\nu}$ with a metric signature $(-,+,+,+)$. Such a metric is often referred to as a Lorentzian metric. On top of that, the manifold is equipped with the Levi-Civita connection ∇, which is the unique connection that is metric compatible,

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\sigma{}_{\alpha\mu}\, g_{\sigma\nu} - \Gamma^\sigma{}_{\alpha\nu}\, g_{\sigma\mu} = 0, \tag{1.1}$$

and torsionless,

$$\Gamma^\sigma{}_{\mu\nu} = \Gamma^\sigma{}_{\nu\mu}. \tag{1.2}$$

With these requirements, the connection coefficients are uniquely determined by the metric,

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right). \tag{1.3}$$

Crucial to general relativity is the curvature of spacetime, which is encoded in the Riemann curvature tensor. In terms of the connection coefficients, the Riemann curvature tensor is given by,

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}. \tag{1.4}$$

The dynamics of the metric $g_{\mu\nu}$ are governed by the action,

$$S = \int d^4x \sqrt{-g}\, \left(\kappa R + \mathcal{L}_M\right), \tag{1.5}$$

where g is the determinant of the metric, $\mathcal{L}_M$ is the Lagrangian of matter, $R = R^{\mu\nu}{}_{\mu\nu}$ is the Ricci curvature scalar, and lastly κ is the gravitational coupling constant, which in natural units where $G = c = \hbar = k_b = 1$, is given by $\kappa = 1/16\pi$. We will be employing the natural units system in the rest of this thesis. The gravitational part of Eq. (1.5) is known as Einstein-Hilbert action,

$$S_{\text{EH}} = \int d^4x \sqrt{-g}\, \kappa R. \tag{1.6}$$

By requiring that the metric extremizes the action (1.5), we find the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2\kappa} T_{\mu\nu}, \tag{1.7}$$

where $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$ is the Ricci curvature tensor, and $T_{\mu\nu}$ is the stress-energy tensor of matter, which is defined as,

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}. \tag{1.8}$$

The Einstein field equations are often abbreviated as,

$$G_{\mu\nu} = \frac{1}{2\kappa} T_{\mu\nu}, \tag{1.9}$$

Where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is called the Einstein tensor. The Einstein tensor is automatically conserved, which also implies the conservation of the stress-energy tensor via Eq. (1.9),

$$\nabla_\mu G^{\mu\nu} = 0 \quad \rightarrow \quad \nabla_\mu T^{\mu\nu} = 0. \tag{1.10}$$

By taking the trace of both sides of Eq. (1.7) we get,

$$R = -\frac{1}{2\kappa} T, \tag{1.11}$$

where $T = T^{\mu}{}_{\mu}$ is the trace of the stress-energy tensor. By using Eq. (1.11) in Eq. (1.7) we get the so-called trace-reversed form of the Einstein field equations,

$$R_{\mu\nu} = \frac{1}{2\kappa} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \tag{1.12}$$

where $T = T^{\mu}{}_{\mu}$ is the trace of the stress-energy tensor.
Often we will only be interested in vacuum spacetimes, in which case the Einstein field equations are,

$$R_{\mu\nu} = 0. \tag{1.13}$$

A metric that solves the vacuum Einstein field equations is called a Ricci-flat metric.

1.2 Energy Conditions

General relativity does not tell us what kind of matter is present in the universe. Thus, apart from the vacuum case, in which the stress-energy tensor vanishes, the stress-energy tensor could be anything. If the stress-energy tensor could really be anything, we are left with the situation where any Lorentzian metric can be a solution to the Einstein field equations, as one can simply calculate the Einstein tensor associated with the metric and obtain the stress-energy tensor associated with that metric from Eq. (1.9).
To rule out spacetimes that most physicists deem unphysical, one can put restrictions on the stress-energy tensor. Since we roughly know the properties of the matter that is confirmed to be present in our universe, the restrictions we put on the stress-energy tensor should reflect that. A particular common set of restrictions that are often employed are simply known as *the energy conditions*, which are:

- *Weak Energy Condition:* $T_{\mu\nu}v^\mu v^\nu \geq 0$ for any future-directed *timelike* vector v^μ. Since an observer with four-velocity v^μ measures the local energy density of matter ρ to be $\rho = T_{\mu\nu}v^\mu v^\nu$, the weak energy condition simply states that: locally, an observer should always measure the energy density of matter to be positive.

- *Null Energy Condition:* $T_{\mu\nu}v^\mu v^\nu \geq 0$ for any future-directed *null* vector v^μ. This condition essentially states the same as the weak energy condition but for null observers. This energy condition is actually the weakest energy condition, as it is implied by all other three energy conditions. Conversely, if the null energy condition is violated, then all energy conditions are violated.

- *Strong Energy Condition:* $\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)v^\mu v^\nu \geq 0$ for every future-directed *timelike* or *null* vector v^μ. This energy condition is more of a condition on the Ricci tensor, as through the trace-reversed Einstein field equations (1.12), the strong energy condition states that $R_{\mu\nu}v^\mu v^\nu \geq 0$. This condition ensures that gravity is attractive, or in other words, that matter gravitates towards matter.

- *Dominant Energy Condition:* $-T^\mu_\nu v^\nu$ is a future-directed *timelike* or *null* vector field, for any future-directed *timelike* or *null* vector field v^μ. This condition essentially states that the local four-momentum density measured by a local observer must be timelike or null.

Typically, these energy conditions hold for all known classical matter in the universe, however, many modified theories of gravity violate one or more of these energy conditions. While this is not an immediate cause for concern, theorems in general relativity that assume one or more of these energy conditions, such as the laws of black hole mechanics, need to be reassessed in such theories.

1.3 The Geodesic Equation

In general relativity, test particles, which are free particles moving in curved spacetime, move on geodesics. Geodesics are the generalization of straight lines in curved spacetime. A curve is a geodesic if it extremizes the spacetime distance between any two points on the curve, thus the spacetime distance can be used as the action to describe the motion of test particles in gravitational fields,

$$S_{\text{GD}} = \int d\lambda \sqrt{-\, g_{\alpha\beta}\, u^\alpha u^\beta}. \tag{1.14}$$

Here $u^\alpha = dx^\alpha/d\lambda$ is the four-velocity of the test particle, and λ is an affine parameter[1] that parameterizes the geodesic. By varying the action with respect to the four-velocity, one obtains the geodesic equation,

$$u^\alpha \, \nabla_\alpha u^\beta = 0, \tag{1.15}$$

which is the differential equation that describes the motion of test particles in gravitational fields.

1.4 Killing Tensor Fields

Symmetries are one of the most important concepts in physics, with Noether's theorem giving a one-to-one relation between symmetries and conserved quantities. Symmetries of spacetime are usually represented by Killing vector fields. We say that a vector field ξ^μ is a Killing vector field if the Lie derivative of the metric along said vector field vanishes,

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha \nabla_\alpha \, g_{\mu\nu} + g_{\alpha\mu} \nabla_\nu \, \xi^\alpha + g_{\alpha\nu} \nabla_\mu \, \xi^\alpha = 0. \tag{1.16}$$

Intuitively, this can be understood as the metric being the same along the flow of the vector field ξ^μ. By using the metric compatibility of the Levi-Civita connection (1.3), one can further simplify Eq. (1.16) to,

$$\nabla_{(\nu} \, \xi_{\mu)} = 0, \tag{1.17}$$

here $(.)$ denotes the symmetrization of the indices[2]. Equation (1.17) is often referred to as Killing's equation.

Depending on the coordinates used to express the metric in, finding Killing vectors can be a non-trivial task. However, if in a certain coordinate system the metric does not depend on one or multiple coordinates, then the coordinate vectors associated with these coordinates are Killing vectors. Conversely, if the spacetime admits a Killing vector, then there exists a coordinate system in which the metric does not depend on one of the coordinates.

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[1] When a geodesic is parameterized by a parameter λ that is not affine, the geodesic will satisfy $u^\alpha \, \nabla_\alpha u^\beta = f(\lambda) u^\beta$, for some function $f(\lambda)$.

[2] For a rank two tensor $T_{\mu\nu}$ the symmetrization $(.)$ of the indices is defined as $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$. Furthermore, [.] denotes the *anti*-symmetrization of the indices, which for a rank-two tensor $T_{\mu\nu}$ is defined as $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$.

Similar to Noether's theorem, a Killing vector field implies a conserved quantity. If the four-momentum of a test particle p^α obeys the geodesic equation, and the spacetime admits a Killing vector field ξ^μ, then the quantity,

$$Q_\xi = \xi^\mu p_\mu \quad \rightarrow \quad p^\mu \nabla_\mu Q_\xi = 0, \tag{1.18}$$

is conserved along the geodesic. In some cases, a spacetime has a symmetry that is represented by a higher rank Killing tensor. By a higher rank Killing tensor, we mean a tensor $K^{\mu_1\mu_2\ldots\mu_n}$ of rank $n \geq 2$ that obeys the generalization of Killing's equation to tensors of arbitrary rank,

$$\nabla_{(\nu} K_{\mu_1\ldots\mu_n)} = 0. \tag{1.19}$$

A trivial example of a second rank Killing tensor is the metric, which is a Killing tensor in any spacetime by virtue of the metric compatibility of the Levi-Civita connection (1.3). The conserved quantity associated with higher-rank Killing tensors is given by,

$$Q_K = K^{\mu_1\ldots\mu_n} p_{\mu_1}\ldots p_{\mu_n} \quad \rightarrow \quad p^\mu \nabla_\mu Q_K = 0. \tag{1.20}$$

When a spacetime admits four unique Killing tensors, and thus four unique constants of motion, the geodesic equation, at least in principle, becomes integrable. When the geodesic equation is not integrable, geodesic motion will exhibit chaotic regimes.

1.5 The Kerr Family of Black Holes

One, if not *the* most, interesting predictions of general relativity is the existence of black holes. Black holes, assumed to be the end stage of complete gravitational collapse of a celestial body, simply put, are *regions of no escape*. This means that, once a particle enters a black hole, it is impossible for the particle to exit the black hole again, and escape to infinity. While black holes are one of the best and most studies concepts in physics, precisely answering the question: what is a black hole?, is actually quite difficult [11].

The first black hole solution was found by K. Schwarzschild [12], and independently by J. Droste [13] in 1916. On top of being the first black hole solution, the Schwarzschild metric was actually the first ever solution to the Einstein field equations, apart from the trivial flat spacetime. It seems like there really even was no escape of the concept of black holes.

The black holes described by the Schwarzschild black hole are the simplest example of a black hole, as the Schwarzschild black hole is only characterized by its mass M, and is a vacuum solution of the Einstein field equations.
While the simplicity of the Schwarzschild black hole is a strength, it is also a weakness, as the Schwarzschild black hole does not describe black holes with angular momentum i.e. rotating black holes. If a black hole is formed by the total gravitational collapse of a star, one expects the black hole formed by the gravitational collapse to have angular momentum, since stars typically have angular momentum. It took some time, but in 1963 R. Kerr [14] found an extension to the Schwarzschild metric that also describes black holes with

angular momentum, which are known today as Kerr black holes.

In this section, we will briefly discuss the Kerr black hole and its properties. In particular, we will discuss those properties that are important for the rest of the thesis. For a more detailed review of the Kerr black hole, we refer to the review article by Saul Teukolsky [15].

1.5.1 The Kerr Metric

The Kerr metric is the unique solution to the vacuum Einstein field equations (1.13) that is both stationary and axisymmetric. The Kerr metric is usually represented in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4M^2\chi r\sin^2\theta}{\Sigma}dtd\phi + \Sigma\left(\frac{dr^2}{\Delta}+d\theta^2\right) \\ &+\left(r^2+M^2\chi^2+\frac{2M^3r\chi^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \end{aligned} \tag{1.21}$$

with $\Sigma = r^2 + M^2\chi^2\cos^2\theta$, and $\Delta = r^2 - 2Mr + M^2\chi^2$. Here M represent the mass of the black hole, and $\chi \in [-1, 1]$, represents the spin of the black hole, which is related to the total angular momentum of the black hole by $J = M^2\chi$. The sign of χ denotes the direction of the spin of the black hole. Often, χ is taken to be positive, as the direction of spin is not important for an isolated black hole. Thus, the Kerr metric describes a two-parameter family of black holes.
By setting χ to zero, one obtains the Schwarzschild metric, and by setting M to zero one obtains the flat metric.
When working with a computer algebra system, the trigonometric functions in Eq. (1.21) can drastically slow down calculations. Therefore, it is useful to eliminate them by defining a new coordinate[3] $z = -\cos\theta$. In this new coordinate system, the Kerr metric is given by,

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4M^2\chi r}{\Sigma}dtd\phi + \Sigma\left(\frac{dr^2}{\Delta}+\frac{dz^2}{1-z^2}\right) \\ &+\left(r^2+M^2\chi^2+\frac{2M^3\chi^2 r\left(1-z^2\right)}{\Sigma}\right)\left(1-z^2\right)d\phi^2, \end{aligned} \tag{1.22}$$

with $\Sigma = r^2 + M^2\chi^2 z^2$, and $\Delta = r^2 - 2Mr + M^2\chi^2$. This will be the form of the Kerr metric that we will be using in the rest of this thesis.
It will also be useful to point out that the metric of any stationary and axisymmetric spacetime, can be written in the same form as the Kerr metric,

$$ds^2 = g_{tt}(r,z)dt^2 + 2g_{t\phi}(r,z)dtd\phi + g_{rr}(r,z)dr^2 + g_{zz}(r,z)dz^2 + g_{\phi\phi}(r,z)d\phi^2, \tag{1.23}$$

where the metric components $\{g_{tt}, g_{t\phi}, g_{rr}, g_{zz}, g_{\phi\phi}\}$ are specific to the spacetime.

[3]The minus sign is chosen such that the new coordinate system has the same orientation as the Boyer-Lindquist coordinate system.

1.5.2 Symmetries of the Kerr Geometry

As stated before, the Kerr metric is stationary and axisymmetric. This means that the metric admits a timelike Killing vector t^μ that represents time-translational invariance, and a spacelike Killing vector ϕ^μ that represents the axial symmetry of the spacetime. The coordinates in (1.22) are chosen such that these Killing vectors are the coordinate vectors associated with the timelike coordinate t and the azimuthal coordinate ϕ, hence $t^\mu\partial_\mu = \partial_t$, and $\phi^\mu\partial_\mu = \partial_\phi$. These are the only two independent Killing vectors of the Kerr spacetime.

The Kerr metric also admits two independent second rank Killing tensors, one of which is of course the metric tensor $g_{\mu\nu}$ but the second of which is a bit of a surprise. This Killing tensor is given by,

$$K^{\mu\nu} = 2\Sigma\, l^{(\mu}n^{\nu)} + r^2 g^{\mu\nu}, \tag{1.24}$$

where the vectors l^μ, and n^μ are given by,

$$\begin{aligned} l^\mu\partial_\mu &= \frac{r^2+M^2\chi^2}{\Delta}\,\partial_t + \partial_r + \frac{M\chi}{\Delta}\,\partial_\phi, \\ n^\mu\partial_\mu &= \frac{r^2+M^2\chi^2}{2\Sigma}\,\partial_t - \frac{\Delta}{2\Sigma}\,\partial_r + \frac{M\chi}{2\Sigma}\,\partial_\phi, \end{aligned} \tag{1.25}$$

which are the principle null directions of the Kerr metric. What that exactly means we will discuss in section 3.4.
In geodesic motion, these four Killing tensors generate four constants of motion,

$$\begin{aligned} E &= -t^\mu p_\mu \\ L &= \phi^\mu p_\mu \\ m &= g^{\mu\nu}p_\mu p_\nu \\ C &= K^{\mu\nu}p_\mu p_\nu, \end{aligned} \tag{1.26}$$

where E represents the energy of the test particle, L the angular momentum about the axis of symmetry, m the rest mass of the test particle, and C is called the Carter constant, after B. Carter who discovered this constant of motion in 1968 [16].
Since there are four independent constants of motion, the geodesic equation is fully integrable in the Kerr spacetime.

1.5.3 Event Horizons

A crucial aspect of black holes are their event horizon or event horizons, as there may be multiple. The outer event horizon signifies the boundary between the black hole and the exterior spacetime. In the case of multiple event horizons, there may be another event horizon within the black hole, however, in this thesis we will focus on the outer event horizon only.

First of all, event horizons of stationary black holes are null hypersurfaces. A hypersurface can be defined by an equation,

$$f(x) = 0, \tag{1.27}$$

where $f(x)$ is some function of the spacetime coordinates. From $f(x)$ we can construct the normal vectors n^μ to the hypersurface defined by Eq. (1.27) like,

$$n^\mu \propto g^{\mu\nu}\partial_\mu f(x). \tag{1.28}$$

A hypersurface is null when the normal n^μ to that hypersurface is null $n^\mu n_\mu = 0$.
Apart from being null hypersurfaces event horizons of stationary black holes in general relativity, if not in any theory of gravity, are also Killing horizons, as shown by Hawking [17]. A Killing horizon is a null hypersurface on which the norm of a Killing vector ξ^μ of the spacetime vanishes. We then say that the Killing vector generates the horizon.

In the Kerr spacetime, one can find two event horizons. These event horizons are surfaces where the radial coordinate r is constant, hence the event horizons of the Kerr black hole are defined by the equation,

$$r - r_\pm = 0, \tag{1.29}$$

where r_+ represents the outer horizon and r_- represents the inner horizon. The normal vectors n^μ to these surfaces of constant r, are proportional to the radial coordinate vector,

$$n_\mu \propto \partial_\mu(r - r_\pm) = \delta^r_\mu. \tag{1.30}$$

Thus, the norm of the normal vectors of surfaces of constant r are proportional to the g^{rr} component of the metric,

$$n^\mu n_\mu \propto g^{\mu\nu} n_\mu n_\nu = g^{rr} = \frac{\Delta}{\Sigma}. \tag{1.31}$$

The null surfaces of constant r in the Kerr spacetime can thus be identifies by solving $\Delta = 0$, which gives us,

$$r_\pm = M(1 \pm \sqrt{1-\chi^2}). \tag{1.32}$$

To verify whether these two surfaces are Killing horizons, we take a general Killing vector of the Kerr spacetime,

$$\xi^\mu = t^\mu + A\phi^\mu, \tag{1.33}$$

where it is very important to stress that A is a constant. Then we calculate the norm of this Killing vector on the constant $r_\pm$ surfaces. This gives us,

$$\xi^\mu \xi_\mu = g_{tt}(r_\pm, z) + A\, 2g_{t\phi}(r_\pm, z) + A^2\, g_{\phi\phi}(r_\pm, z). \tag{1.34}$$

The norm of this Killing vector vanishes when,

$$A = \frac{g_{t\phi}}{g_{\phi\phi}}\Big|_{r_\pm} \pm \frac{\sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}^2}\Big|_{r_\pm}. \quad (1.35)$$

One can explicitly check that $g_{t\phi}^2 - g_{tt}g_{\phi\phi} = \Delta(1-z^2)$, which is the negative of the determinant of the $t-\phi$ sector of the metric, and since it is proportional to Δ, this vanishes at $r_\pm$. What we are left with is,

$$A = \frac{g_{t\phi}}{g_{\phi\phi}}\Big|_{r_\pm}, \quad (1.36)$$

which is constant. Hereby we have shown that the inner and outer horizons of the Kerr spacetime are Killing horizons. Since we will be mostly interested in the outer horizon, we will define,

$$\Omega_{\mathrm{H}} = \frac{g_{t\phi}}{g_{\phi\phi}}\Big|_{r_+}, \quad (1.37)$$

such that the Killing vector that generates the outer horizon is written as,

$$\xi^\mu = t^\mu + \Omega_{\mathrm{H}}\,\phi^\mu. \quad (1.38)$$

The quantity Ω_{H} is known as the angular velocity of the outer event horizon, as a test particle falling into the black hole from infinity will have an instantaneous angular velocity of Ω_{H} when it reaches the outer event horizon. For the Kerr black hole, the angular velocity of the event horizon is explicitly given by,

$$\Omega_{\mathrm{H}}^{\mathrm{Kerr}} = \frac{\chi}{2M(1+\sqrt{1-\chi^2})}. \quad (1.39)$$

Henceforth, we will simply refer to the outer event horizon as the event horizon, and refer to the r coordinate that defines the outer event horizon as r_H.

1.5.4 The Ergosphere

A crucial feature of the Kerr spacetime is the existence of an ergosphere. Due to the angular momentum of the black hole, one could say that the spacetime around the black hole is being 'dragged along' with the rotation of the black hole, which is known as frame dragging.

The effect of frame dragging is strongest near the horizon of the black hole and becomes negligible far away from the black hole. At some point, the frame dragging becomes so strong that it is only possible for a null observer to remain static. This is known as the static limit surface, which is the surface on which the time Killing vector t^μ becomes null,

$$t^\mu t_\mu = g_{tt} = -\left(1 - \frac{2Mr}{\Sigma}\right) = 0. \quad (1.40)$$

This implies that the static limit surface of the Kerr spacetime is defined by the equation,

$$r_E(z) = M(1 + \sqrt{1 - \chi^2 z^2}), \tag{1.41}$$

which touches the event horizon at the poles, $r_E(\pm 1) = r_H$. Excluding the interior of the black hole, we have that for $r_H < r < r_E(z)$ the time Killing vector becomes spacelike, which means that not even null observers can remain static. This will be our definition of the ergosphere.

As we will see, the existence of the ergosphere allows for the extraction of energy from the black hole.

2 Introducing Dynamical Chern-Simons Gravity

In this section, we will give a brief introduction to dynamical Chern-Simons gravity (dCS). We will start with a short motivation of the dCS action and a derivation of the modified field equations. Then we will go over known exact solutions to the modified field equations. Lastly, we will derive, and discuss the validity of the modified field equations of dCS gravity in the small coupling limit. This introduction is by no means complete. For a more in depth discussion on dynamical Chern-Simons gravity and its origins, we refer to the review article by S. Alexander and N. Yunes [4].

2.1 The Chern-Simons Action and Equations of Motion

Chern-Simons gravity is a four-dimensional effective extension of general relativity. Chern-Simons gravity extends general relativity by adding a Chern-Simons term to the Einstein-Hilbert action (1.6). This Chern-Simons term is quadratic in the Riemann tensor, making Chern-Simons gravity a quadratic curvature theory.
The Chern-Simons term find its origin in three-dimensional Yang-Mills and gravitational theories [3]. However, one can extend the three-dimensional Chern-Simons term such that it can be introduced in four-dimensional theories of gravity. This was done first by R. Jackiw and S.-Y. Pi in 2003 [2].
The extended four-dimensional Chern-Simons term constructed by Jackiw and Pi is given by,

$$S_{CS} = \kappa \int_{\mathcal{V}} d^4x\sqrt{-g}\,\alpha\,\vartheta\,{}^*RR. \tag{2.1}$$

Here, α is a generic coupling constant, ϑ is a scalar field, and ${}^*RR$ is the Chern-Pontryagin scalar which is given by,

$${}^*RR = {}^*R^{\gamma\delta\alpha\beta}R_{\beta\alpha\gamma\delta} = \frac{1}{2}\epsilon^{\gamma\delta\mu\nu}R^{\alpha\beta}{}_{\mu\nu}R_{\beta\alpha\gamma\delta}, \tag{2.2}$$

here $\epsilon^{\gamma\delta\mu\nu}$ is the Levi-Civita tensor.
The Chern-Pontryagin scalar is parity odd, hence it is actually a pseudo-scalar. That means that under a parity transformation P, the Chern-Pontryagin scalar transforms as,

$$P[{}^*RR] = -{}^*RR. \tag{2.3}$$

The coupling of the scalar field ϑ to the Chern-Pontryagin scalar in Eq. (2.1) is necessary as the Chern-Pontryagin scalar by itself cannot be used to meaningfully extend the Einstein-Hilbert action as it can be written as the divergence of a vector field,

$$\nabla_\mu K^\mu = \frac{1}{2}{}^*RR. \tag{2.4}$$

Here the vector field K^μ, known as the Chern-Simons topological current, is given by,

$$K^\mu = \epsilon^{\mu\alpha\beta\gamma}\Gamma^\nu{}_{\alpha\delta}\left(\partial_\beta\Gamma^\delta{}_{\nu\gamma} + \frac{2}{3}\Gamma^\delta{}_{\beta\lambda}\Gamma^\lambda{}_{\nu\gamma}\right). \tag{2.5}$$

By using Stokes' theorem, we can then write the CS term (2.1) as,

$$S_{CS} = 2\kappa \int_{\partial\mathcal{V}} d^3x \sqrt{|\gamma|}\, \alpha\, \vartheta\, K^\mu n_\mu - 2\kappa \int_{\mathcal{V}} d^4x \sqrt{-g}\, \alpha\, (\partial_\mu \vartheta) K^\mu, \tag{2.6}$$

where $\gamma_{\mu\nu}$ is the induced metric on $\partial\mathcal{V}$ and n^μ is the unit normal to $\partial\mathcal{V}$. The first term is a boundary term and may be discarded from the action. The second term is non-vanishing provided $\partial_\mu\vartheta$ is non-vanishing. This explains why ϑ must be a scalar field, and not just a constant.

The gravitational action, consisting of the Einstein-Hilbert action and the Chern-Simons term, is then given by,

$$S_{CS}^{ND} = S_{EH} + S_{CS} = \kappa \int_{\mathcal{V}} d^4x \sqrt{-g}\, \left(R + \alpha\, \vartheta\, {}^*RR\right), \tag{2.7}$$

Where ND stands for non-dynamical. This action does not dictate the dynamics of the scalar field ϑ, which means that the scalar field can be arbitrarily prescribed. That is why the gravitational theory described by this action is known as non-dynamical Chern-Simons gravity.
By varying the action with respect to metric, we obtain the field equations of non-dynamical CS gravity,

$$G_{\mu\nu} + 4\alpha\, C_{\mu\nu} = 0, \tag{2.8}$$

where $C_{\mu\nu}$ is the so-called Cotton tensor[4], which is given by,

$$C_{\mu\nu} = \nabla^\beta \nabla^\alpha \left[{}^*R_{\alpha(\mu\nu)\beta}\, \vartheta \right]. \tag{2.9}$$

Equivalently, the Cotton tensor can be expressed as,

$$C_{\mu\nu} = \epsilon^{\gamma\beta}{}_{\alpha(\mu|} \nabla_\beta R_{|\nu)\gamma} \nabla^\alpha \vartheta + {}^*R_{\alpha(\mu\nu)\beta}\, \nabla^\beta \nabla^\alpha \vartheta. \tag{2.10}$$

The equivalence of these two expression is proven in Appendix A.

Due to the Einstein tensor automatically satisfying $\nabla_\mu G^{\mu\nu} = 0$, the space of solutions of non-dynamical CS gravity is confined to metrics that satisfy $\nabla_\mu C^{\mu\nu} = 0$. It turns out that this implies that the Chern-Pontryagin scalar of the metric *must* vanish. Thus, the non-dynamical Chern-Simons theory comes with the constraint, ${}^*RR = 0$, known as the Pontryagin constraint. This makes non-dynamical Chern-Simons gravity a constrained theory, for example, ruling out rotating black hole solutions [4]. On top of that, we also have the problem of the scalar field ϑ being completely arbitrary.

[4]Which is related, but not equivalent to the Cotton tensor that appears in three-dimensional Chern-Simons theories. It is also useful to point out that the Cotton tensor is traceless, $C^\mu{}_\mu = 0$.

To cure the Chern-Simons theory of the arbitrariness and the Pontryagin constraint, we can add generic terms to the action to make the scalar field ϑ dynamical. There is of course some arbitrariness to the choice of terms added to the action as well, but we can keep the action for the scalar field quite general. An obvious choice of action is a just a standard kinetic term,

$$S_\vartheta = -\frac{\kappa}{2}\int_{\mathcal{V}} d^4x\sqrt{-g}\, g^{\mu\nu}\partial_\mu\vartheta\partial_\nu\vartheta. \tag{2.11}$$

One could in principle also add a potential term, but since there is no obvious choice of potential, we will leave it out. If we add this term to the non-dynamical CS action, we obtain the dynamical CS action or dCS action, which was first formulated in [18]. The dCS action is then given by,

$$S_{dCS} = S_{EH} + S_{CS} + S_\vartheta = \kappa\int d^4x\sqrt{-g}\left(R + \alpha\,\vartheta\,{}^*RR - \frac{1}{2}\,g^{\mu\nu}\partial_\mu\vartheta\partial_\nu\vartheta\right). \tag{2.12}$$

This action implies the following field equations for the metric[5],

$$G_{\mu\nu} = \frac{1}{2}T^\vartheta_{\mu\nu} - 4\alpha\, C_{\mu\nu}. \tag{2.13}$$

The new term that appears here is the stress-energy associated with S_ϑ, which is given by,

$$T^\vartheta_{\mu\nu} = \partial_\mu\vartheta\partial_\nu\vartheta - \frac{1}{2}g_{\mu\nu}\partial_\sigma\vartheta\partial^\sigma\vartheta. \tag{2.14}$$

This stress-energy tensor is *not* conserved, meaning that $\nabla_\mu T^{\mu\nu}_\vartheta \neq 0$. However, conservation of the Einstein tensor does imply that,

$$\frac{1}{2}\nabla_\mu T^{\mu\nu}_\vartheta = 4\alpha\nabla_\mu C^{\mu\nu}. \tag{2.15}$$

This then in turn implies, instead of the Pontryagin constraint, that the scalar field ϑ must satisfy the sourced wave equation,

$$\Box\vartheta = -\alpha\,{}^*RR, \tag{2.16}$$

which can also be derived by varying the action Eq. (2.12) with respect to ϑ. Since the Chern-Pontryagin scalar is a pseudo-scalar, (2.16) implies that ϑ is also a pseudo-scalar field, which is why it is sometimes called the axion field, which is also the naming convention we will be using in this thesis. The combination $\vartheta\,{}^*RR$ is then a scalar, thus the theory does not break parity at the level of the action.

[5] Of course, one can also add other matter fields to the dCS action, in which case the stress-energy tensor of those fields appears on the right side of the dCS field equations.

Instead of using Eq. (2.13), it is often more convenient to use the trace-reversed form of Eq. (2.13), which is given by

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\vartheta\,\partial_\nu\vartheta - 4\alpha\,\epsilon^{\gamma\beta}{}_{\alpha(\mu|}\nabla_\beta R_{|\nu)\gamma}\nabla^\alpha\vartheta - 4\alpha\;{}^*R_{\alpha(\mu\nu)\beta}\,\nabla^\beta\nabla^\alpha\vartheta, \tag{2.17}$$

where we have used the explicit form of the Cotton tensor given in Eq. (2.10).

Aside from the motivation of the dCS action outlined here, dCS gravity can also be derived from heterotic string theory, loop quantum gravity, and from the standard model, as discussed in [4]. A part from that, dCS gravity also comes up naturally when one considers higher order curvature corrections to the Einstein-Hilbert action, as one can essentially only build four independent quadratic curvature scalars from the Riemann tensor, which are: the Ricci scalar squared R^2, the Ricci tensor squared $R_{\mu\nu}R^{\mu\nu}$, the Kretschmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, and lastly, the Chern-Pontryagin scalar ${}^*RR$. All in all, this makes dCS gravity a very well-motivated extension to the Einstein-Hilbert action that is worth studying.

2.2 Exact Solutions in Dynamical Chern-Simons Gravity

As with most modified theories of gravity, finding exact solutions in dynamical Chern-Simons gravity is not an easy task. However, there are two classes of spacetimes that will trivially solve the dCS equations, provided they are exact solutions to the Einstein field equations. These are all spherically symmetric and all conformally flat solutions to the Einstein field equations. To find out why this is the case, we will treat these two cases separately.

2.2.1 Spherically Symmetric Solutions to the Einstein Field Equations

That any spherically symmetric solution to the Einstein Field equations solve the dCS equations exactly, is purely due to the fact that the Chern-Pontryagin scalar is a pseudo-scalar.
The metric of a spherically symmetric spacetime is parity even, meaning that it does not change under a parity transformation P. Thus, we must have that for spherically symmetric spacetimes $P[{}^*RR] = {}^*RR$, but since the Chern-Pontryagin scalar is inherently parity odd, we also have that $P[{}^*RR] = -{}^*RR$, thus for both of these statements to be true we must have that ${}^*RR = 0$ for spherically symmetric spacetimes.

When a spacetime has a vanishing Chern-Pontryagin scalar, the wave equation for the axion fields given by Eq. (2.16) becomes,

$$\Box\vartheta = 0, \tag{2.18}$$

which usually will not have a well-defined solution that decays to zero at infinity, except for a vanishing field, which in any case is a perfectly fine solution to the wave equation without a source.
If we set the axion field to zero in the dCS equations (2.13), we recover the Einstein field equations (1.7), hence spherically symmetric spacetimes are exact solutions in dCS gravity provided they solve the Einstein field equations.

This specifically means that the Schwarzschild solution is still a solution in dCS, but the Kerr black hole, which is not spherically symmetric, is not a solution in dCS. This is exactly why rotating black holes are a primary area of interest in dCS gravity. The fact that the Schwarzschild solutions persists is, however, still relevant for rotating black holes in dCS. In general relativity, we recover the Schwarzschild solution if we set the spin of the Kerr metric to zero, thus we also expect that rotating black holes in dCS reduce to the Schwarzschild solution when we set their spin to zero. This also automatically implies that the axion field must vanish when we take the spin to zero.

2.2.2 Conformally Flat Solutions to the Einstein Field Equations

A conformally flat spacetime is a spacetime in which the metric can be written as,

$$ds^2 = a(x)\eta_{\mu\nu}dx^\mu dx^\nu, \tag{2.19}$$

where $a(x)$ is some arbitrary function of the spacetime coordinates, and $\eta_{\mu\nu}$ is the flat metric. The curvature of conformally flat spacetimes is completely determined by the Ricci curvature tensor. The Chern-Pontryagin scalar does not contain any information about the Ricci curvature tensor, hence for any conformally flat metric we have that ${}^*RR = 0$[6]. Just like with spherically symmetric spacetimes, we can then set the axion field to zero, and we once again recover that the metric should solve Einstein field equations. Although we will not be dealing with conformally flat spacetimes here, it is important to note that the Friedmann-Lemaître-Robertson-Walker metric, which describes a homogenous, isotropic expanding universe is also still a solution in dCS gravity.

2.3 Dynamical Chern-Simons Gravity in the Small Coupling Limit

In the previous section, we obtained the modified field equations for the metric and the wave equation for the axion field from the dCS action (2.12). In this section, we will expand these equations around any exact vacuum solution of the Einstein field equations (1.13), to leading order in the dCS coupling parameter.

Although trivial solutions to the exact dCS equations are easy to find, like we have seen in the previous section, non-trivial solutions are extremely difficult to find. Furthermore, one could ask whether exact solutions that include all orders in the dCS coupling are even useful. Since general relativity already works well for black hole physics, one should expect the dCS equations to only provide small corrections to observables. On top of that, one expects that the correct effective theory of gravity derived from quantum gravity contains more terms in addition to the dCS term, maybe even infinitely many. In that context, it is not very useful to include beyond leading order dCS corrections to general relativity.

[6]In short, this is because the Chern-Pontraygin scalar can also be written as ${}^*CC$. Where $C_{\mu\nu\alpha\beta}$ is the Weyl curvature tensor. Since the Weyl tensor is invariant under conformal transformations of the metric, the Chern-Pontryagin scalar is as well. That means that the Chern-Pontryagin scalar of conformally flat spacetimes are conformally related to the Chern-Pontryagin scalar of the flat metric, which vanishes.

To expand the dCS equations to leading order in the coupling parameter, we first make the ansatz that the metric can be written as,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \alpha\, h^{(1)}_{\mu\nu} + \alpha^2\, h^{(2)}_{\mu\nu} + \mathcal{O}(\alpha^3), \tag{2.20}$$

with $|\alpha^{(n)} h^{(n)}_{\mu\nu}| \ll 1$. Here $g^{(0)}_{\mu\nu}$ is the background spacetime, which is an exact solution to the vacuum Einstein field equations (1.13), which means that,

$$R^{(0)}_{\mu\nu} = 0. \tag{2.21}$$

The ansatz for the axion field also needs to be expanded in the coupling, hence the ansatz for the axion field is,

$$\vartheta = \vartheta^{(0)} + \alpha\, \vartheta^{(1)} + \alpha^2\, \vartheta^{(2)} + \mathcal{O}(\alpha^3). \tag{2.22}$$

Now we insert the ansatz for metric (2.20) and the axion field (2.22) into Eq. (2.16), and then we expand both sides in the coupling parameter. We then find,

$$\Box^{(0)}\vartheta^{(0)} + \alpha\, \left(\Box^{(1)}\vartheta^{(0)} + \Box^{(0)}\vartheta^{(1)}\right) + \mathcal{O}(\alpha^2) = -\alpha\, {}^*RR^{(0)} + +\mathcal{O}(\alpha^2), \tag{2.23}$$

where ${}^*RR^{(0)}$ is the Chern-Pontryagin scalar associated with the background spacetime. The zeroth order part of this equation, $\Box^{(0)}\vartheta^{(0)} = 0$ implies that $\vartheta^{(0)} = 0$, if we require the axion field to vanish at infinity. If we use that again in Eq. 2.23, the wave equation for the axion to leading order in the coupling then becomes,

$$\alpha\, \Box^{(0)}\vartheta^{(1)} = -\alpha\, {}^*RR^{(0)}. \tag{2.24}$$

The axion field thus is at least of order $\mathcal{O}(\alpha)$.

Now we can do the same thing for Eq. (2.17) by using the ansatz for the metric (2.20) and the ansatz for the axion (2.22). Using our newly found knowledge that the axion is at least of order $\mathcal{O}(\alpha)$, Eq. (2.17) to leading order in the coupling becomes,

$$\alpha^2\, R^{(2)}_{\mu\nu} + \mathcal{O}(\alpha^3) = \alpha^2\, \left(\frac{1}{2}\partial_\mu \vartheta^{(1)}\, \partial_\nu \vartheta^{(1)} - 4\, {}^*R^{(0)}_{\alpha(\mu\nu)\beta}\, \nabla^\beta \nabla^\alpha \vartheta^{(1)}\right) + \mathcal{O}(\alpha^3). \tag{2.25}$$

Here the covariant derivatives are those of the background metric, and have used Eq. (2.21), to set the Ricci tensor of the background metric to zero. We have also already set $h^{(1)}_{\mu\nu} = 0$, as there is no source for those terms on the right-hand side of Eq. (2.17). Furthermore, $R^{(2)}_{\mu\nu}$ is given by,

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\left(\nabla^\alpha \nabla_\mu\, h^{(2)}_{\alpha\nu} + \nabla^\alpha \nabla_\nu\, h^{(2)}_{\alpha\mu} - \nabla_\mu \nabla_\nu\, h^{(2)} - \Box\, h^{(2)}_{\mu\nu}\right), \tag{2.26}$$

here the covariant derivatives are the covariant derivates of the background metric and $h = g^{\mu\nu}_{(0)} h^{(2)}_{\mu\nu}$. From this we conclude that the corrections of the metric are, to leading order $\mathcal{O}(\alpha^2)$. That means that to leading order we have,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \alpha^2\, h_{\mu\nu}, \tag{2.27}$$

where $h_{\mu\nu}$ satisfies the differential equation,

$$\alpha^2 \left(\nabla_\alpha \nabla_\mu h^\alpha{}_\nu + \nabla_\alpha \nabla_\nu h^\alpha{}_\mu - \nabla_\mu \nabla_\nu h - \Box h_{\mu\nu}\right) = \partial_\mu \vartheta \, \partial_\nu \vartheta - 8\alpha \, {}^*R_{\alpha(\mu\nu)\beta} \, \nabla^\beta \nabla^\alpha \vartheta, \quad (2.28)$$

where the covariant derivatives, and the dual Riemann tensor are evaluated on the background spacetime. In addition, we have the following equation for the axion field,

$$\Box\vartheta = -\alpha \, {}^*RR, \quad (2.29)$$

with the Chern-Pontryagin scalar evaluated on the background spacetime.

By expanding the dCS equations to leading order in the coupling we have separated the equations for the metric and the equation for the axion, in the sense that we can first solve the equation for the axion on the background spacetime, and subsequently use this solution to solve for the metric corrections. Furthermore, we have also eliminated the third order derivatives, thus the dCS equations have become a set of coupled second order linear partial differential equations for the metric corrections $h_{\mu\nu}$ and a second order linear partial differential equation for the axion ϑ.

2.4 The Coupling Constant, and the Validity of the Small Coupling Limit

When making an approximation, it is of paramount importance to know the limitations of the approximation, and whether the use of the approximation is justified.
The dCS equations in the small coupling limit are only valid when $|\alpha^2 h_{\mu\nu}| \ll 1$, whether this is the case depends on the value of the coupling constant α.
The coupling constant is not a dimensionless quantity. Its dimensions can easily be determined by counting the number of derivatives in each term in the action (2.12).
The Ricci scalar R contains second order derivatives and thus has dimensions $[R] = m^{-2}$. All other terms in the action then should have these dimensions as well. The action of the axion contains a product of two first order derivatives, thus the axion itself is dimensionless. The Chern-Pontryagin scalar ${}^*RR$ is essentially the product of two Riemann tensors, thus has dimensions $[{}^*RR] = m^{-4}$. This implies that the coupling constant, α, must have the dimensions, $[\alpha] = m^2$.

Since the coupling constant α is not dimensionless, it makes no sense to talk about it being small, as that depends entirely on the choice of units. To be able to talk about the coupling constant being small, we must define a dimensionless coupling constant. If we can determine the characteristic length scale l of the system we are interested in, we can define a dimensionless coupling constant,

$$\zeta = \frac{\alpha^2}{l^4}. \quad (2.30)$$

If we are considering dCS corrections to black hole solutions from GR, then this characteristic length scale is the mass of the black hole[7] M hence we will from now on use the dimensionless coupling constant,

$$\zeta = \frac{\alpha^2}{M^4}. \tag{2.31}$$

The added benefit of the introduction of this dimensionless coupling is that, in the small coupling limit any observable, O can be written as,

$$O = O^{\rm GR} + \zeta\delta O + \mathcal{O}(\zeta^2), \tag{2.32}$$

where $O^{\rm GR}$ is the observable evaluated on the background spacetime, and $\delta\mathcal{O}$ is the first order dCS correction to the observable, and we neglect the terms of order $\mathcal{O}(\zeta^2)$.
Coming back to the validity of the leading order expansion of the dCS equations, we can now recast the criterion for validity as,

$$|\zeta h_{\mu\nu}| \ll 1. \tag{2.33}$$

The strictest bounds on the coupling constant α comes from results obtained by the *Neutron Star Interior Composition Explorer* (NICER), combined with the measurement of the neutron star inspiral GW170817 [19]. This measurement puts a bound[8] on α of,

$$\hat{\alpha}^{1/2} \leq 60.3\,\text{km}, \tag{2.34}$$

with a credibility of 90%.
If we take the upper bound, then $\zeta \approx 1$, when $M \approx 5.3 M_\odot$. Thus, it seems that the leading order expansion in the coupling constant of dCS gravity should suffice for astrophysical black holes.

[7] In SI units, $\frac{GM}{c^2}$ is the characteristic length scale of a black hole, which is half of the Schwarzschild radius.
[8] The reported bound on α from is $\alpha^{1/2} \leq 8.5\,\text{km}$, but due to our choice of action we have to include an extra factor of $\kappa^{1/2}$.

3 The Chern-Simons Corrected Kerr Black Hole

With the dCS equations in the small coupling limit in hand, it is now possible to obtain leading order corrections to the Kerr metric. To this end, one must first solve the equations of motion for the axion sourced by the Chern-Pontryagin scalar of the Kerr spacetime, and then solve equation (2.28).
This has turned out to be quite the monumental task, as the first hurdle, obtaining the axion on the Kerr background, has not yet been cleared. However, by expanding in the spin of the Kerr background, it is possible to calculate solutions to arbitrary order in the spin algorithmically.
The downside to this approach is that these solutions need corrections to very high orders in the spin to be valid for fast rotating black holes. Solutions to high order in spin get very lengthy very quickly, thus requiring the aid of computer software to be able to work with these solutions.

In this section, we will sketch how to solve the dCS equations in the small coupling limit to obtain the dCS corrections to the Kerr metric. We will discuss the corrections that have been found so far, and we discuss some basic properties of the dCS corrected Kerr metric. Most attention will be given to the Petrov type of the dCS corrected Kerr metric, as this section will include new results.

3.1 The Axion on a Kerr Background

To find the dCS corrections to the Kerr metric, we first have to solve the equation of motion of the axion (2.29), on a Kerr background (1.22). The Chern-Pontryagin scalar of the Kerr metric is given by,

$$^*RR = -96\frac{M^3\chi\, r}{\Sigma^6} z\left(r^2 - 3M^2\chi^2 z^2\right)\left(3r^2 - M^2\chi^2 z^2\right), \tag{3.1}$$

which is showcased in Fig. 1 in a small region around the event horizon of the Kerr black hole, which is the region of spacetime we will be mostly interested in.
Since the background spacetime on which we solve Eq. (2.29) is stationary and axisymmetric, the axion depends only on the r and z coordinates, $\vartheta = \vartheta(r,z)$. With the explicit expression of the Chern-Pontryagin scalar, the equation of motion for the axion is given by,

$$\Box\vartheta = 96\frac{\alpha M^3\chi\, r}{\Sigma^6} z\left(r^2 - 3M^2\chi^2 z^2\right)\left(3r^2 - M^2\chi^2 z^2\right). \tag{3.2}$$

As stated before, no solution to this complicated differential equation has been found so far. However, if we expand Eq. (3.2) in the spin of the black hole, it is possible to solve for the axion as a power series in the spin i.e. we assume the axion field can be written as,

$$\vartheta = \sum_{n=0}^{\infty} \vartheta^{(n)}(r,z)\,\chi^n. \tag{3.3}$$

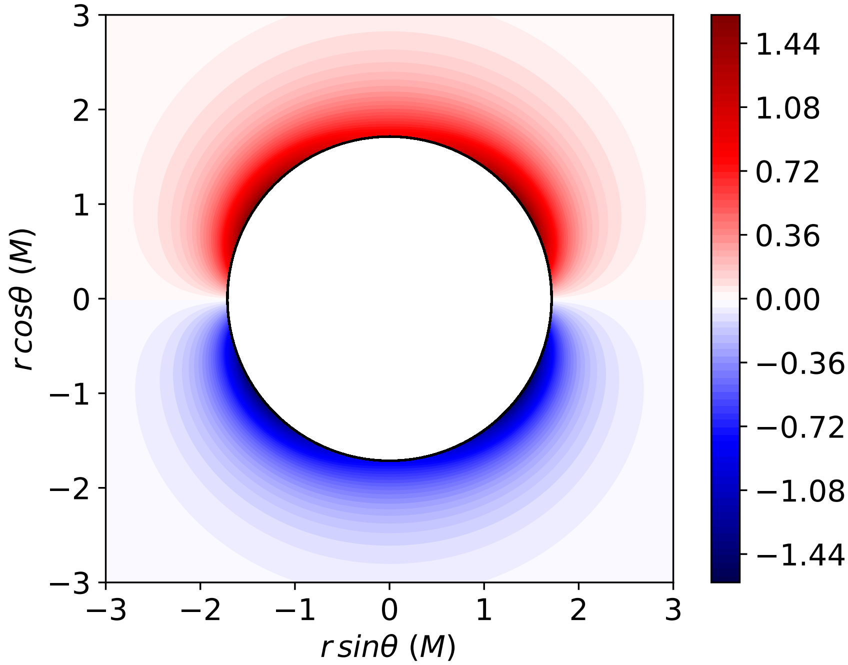


Figure 1: The Chern-Pontryagin scalar of the Kerr spacetime outside the event horizon, with the spin taken to be moderately high, $\chi = 0.7$. The black circle indicates the event horizon.

Where the functions $\vartheta^{(n)}(r,z)$ are to be determined by solving Eq. (3.2) order by order. As an example, to linear order in the spin, the wave equation for the axion becomes,

$$(1-\frac{2M}{r})\partial_r^2\vartheta^{(1)} + \frac{2}{r}(1-\frac{M}{r})\partial_r\vartheta^{(1)} + \frac{1-z^2}{r^2}\partial_z^2\vartheta^{(1)} - \frac{2z}{r^2}\partial_z\vartheta^{(1)} = \alpha M^3\frac{288z}{r^7}, \tag{3.4}$$

The solution to this equation is,

$$\vartheta = -\alpha\chi\frac{5}{2}\frac{z}{r^2}\left(1+\frac{2M}{r}+\frac{18M^2}{5r^2}\right), \tag{3.5}$$

which was first obtained by Yunes and Pretorius in 2009 [5]. However, it is possible to do better. It turns out that, the coefficients $\vartheta^{(n)}(r,z)$ in Eq. (3.3) can always be written as a polynomial in z and $1/r$,

$$\vartheta^{(n)}(r,z) = \sum_{p=0}^{n}\sum_{k=0}^{k_{\max}}\vartheta^{(n,p,k)}z^p r^{-k}, \tag{3.6}$$

where $\vartheta^{(n,p,k)}$ are constant coefficients, and the value of $k_{\max}$ depends on the value of n, and p. By inserting this ansatz in Eq. (3.2) and expanding the wave operator and Chern-Pontryagin scalar in the spin, Eq. (3.2) reduces to a system of algebraic equations, which can be solved order by order. This means that the coefficients $\vartheta^{(n,p,k)}$ only depend on the coefficients $\vartheta^{(m<n,p,k)}$ of the lower order spin terms.

Solving these algebraic equations quickly becomes cumbersome to do by hand, however by using a computer algebra system, solutions to practically arbitrary order can be found, provided one has a sufficient amount of time and processing power available. This approach was first discovered and used by Cano and Ruipérez in 2020 [6]. In this work, a solution up to fourteenth order in the spin was provided, however since then calculations have been done up to twenty-eighth order in the spin [20].
The axion field up to fourteenth order in the spin is shown in Fig. 2 in a small region outside the horizon. The axion field has a strongly resembles the electric potential of an electric dipole. We say the axion has a dipolar character, further emphasized by the absence of a $1/r$ term[9].

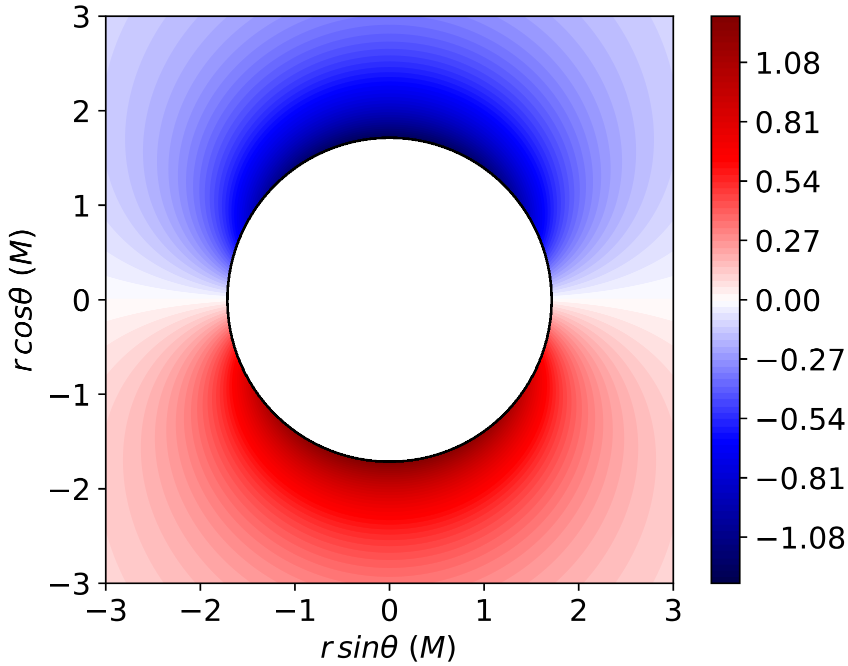


Figure 2: The Axion field on a Kerr background outside the event horizon up to fourteenth order in the spin, with the spin taken to be moderately high, $\chi = 0.7$. The outer event horizon is indicated with the black circle.

[9]We'd like to note that this dipolar character is already captured by the z/r^2 term in the lowest order solution given by Eq. (3.5).

3.2 The Corrected Metric

Once we have obtained the axion field to the desired order in spin, we can determine the right-hand side of (2.28), and solve for the dCS metric corrections to the Kerr spacetime. This was first done by Yunes and Pretorius in 2009 [4], to linear order in the spin. They found that the dCS corrected Kerr metric is given by,

$$ds^2 = ds^2_{\text{Kerr}} + \zeta\chi(1-z^2)\frac{M^4}{r^4}\left(10 + \frac{120M}{7r} + \frac{27M^2}{r^2}\right)d\phi dt, \tag{3.7}$$

where it is understood that ds^2_{Kerr} is also expanded to linear order in the spin. This solution already indicates that the effect of frame dragging is reduced by the dCS corrections to the Kerr metric.

In the following years higher order solutions were found [21], but the door to arbitrary high solutions was really opened by Cano and Ruipére in 2020 [6], who managed to find the corrected metric up to fourteenth order in the spin. To find this solution, Cano and Ruipére used the ansatz,

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma} - \zeta H_1\right)dt^2 - (1+\zeta H_2)\frac{4M^2\chi r}{\Sigma}dtd\phi + (1+\zeta H_3)\,\Sigma\left(\frac{dr^2}{\Delta} + \frac{dz^2}{1-z^2}\right) \\ &+ (1+\zeta H_4)\left(r^2 + M^2\chi^2 + \frac{2M^3\chi^2 r\,(1-z^2)}{\Sigma}\right)\left(1-z^2\right)d\phi^2, \end{aligned} \tag{3.8}$$

where the functions $\{H_1(r,z), H_2(r,z), H_3(r,z), H_4(r,z)\}$ are written as a power series in the spin parameter,

$$H_i = \sum_{n=0}^{\infty} H_i^{(n)}\chi^n, \tag{3.9}$$

and the coefficients $H_i^{(n)}$ are polynomials in z and $1/r$,

$$H_i^{(n)} = \sum_{p=0}^{n}\sum_{k=0}^{k_{\max}} H_i^{(n,p,k)} z^p r^{-k}, \tag{3.10}$$

where $H_i^{(n,p,k)}$ are constant coefficients, and the value of and $k_{\max}$ depends on the value of n, and p. This ansatz reduces the differential equations for $H_i^{(n)}$ to algebraic equations for the coefficients $H_i^{(n,p,k)}$ that can be systematically solved by a computer algebra system. In principle, one can generate solutions to arbitrary order in the spin with this ansatz.

In this thesis, we will be using the dCS corrections to the Kerr metric up to fourteenth order in spin.
We have tried to generate higher order solutions, but it seems that fourteenth order is about the highest order solution one can reasonably find and use, using a standard consumer grade computer.

Recently, the accuracy of the spin expansion has been analysed in [20]. In short, the relevant conclusions we draw from this work are:

- Requiring that metric perturbations, $|\zeta H_i| < 0.5$ on the event horizon for a spin of, $\chi = 0.9$, allows the dimensionless coupling constant to be $\zeta = 0.15$ at most.
  In the rest of this thesis, we will set $\zeta = 0.15$.

- The dCS corrections to the Kerr metric up to order $\mathcal{O}(\zeta\chi^{14})$ are reasonably accurate up to $\chi = 0.8$. If we take the spin to be lower than that, the accuracy dCS corrections improve dramatically.
  In the rest of this thesis, we will set the maximal allowed spin value to be 0.8.

Due to the dCS corrected metric having very length expressions, we will be using the *xAct* Mathematica package to do all of our calculations. This means that most calculations will be done 'under the hood'. The notebook that we have used to do these calculations, as well as more information on the *xAct* package, can be found in Appendix C. Even though we are using a computer, these are still analytical calculations.
Just to give some insight into these dCS corrections to the Kerr metric, the explicit form of the axion field to order $\mathcal{O}(\chi^3)$, as well as metric corrections to order $\mathcal{O}(\chi^2)$, can be found in Appendix B.

3.3 Basic Properties of the dCS Corrected Kerr Black Hole

In this section, we will go over some basic properties of the dCS corrected Kerr metric that will be used in the rest of this thesis. For more details, we refer to [6].

3.3.1 Symmetries of the dCS Corrected Kerr Black Hole

Like the Kerr metric, the dCS corrected Kerr metric is stationary and axisymmetric, thus the corrected metric has two independent Killing vectors. The Killing vector associated with stationarity is the coordinate vector of the time coordinate t, and the Killing vector associated with the azimuthal symmetry is the coordinate vector associated with the ϕ coordinate.
Of course, the metric is a Killing tensor, however the Killing tensor in Eq. (1.24) is *not* a Killing tensor of the dCS corrected Kerr metric, hence we lose the Carter constant. This is due to that l^μ and n^μ as defined in (1.25) are not so-called principle null directions of the dCS corrected Kerr metric[10]. Without a fourth constant of motion, the geodesic equation will not be fully integrable.

[10] This will be discussed more in depth in the section on the Petrov type of the dCS corrected Kerr metric.

However, we need not rely on the Carter constant, one could in principle just solve the Killing equation (1.19) by brute force to find a Killing tensor that would generate a fourth independent constant of motion. This was first done by Owen, Vitek, and Yunes in 2021 [22]. In this paper, the authors have shown that up to order $\mathcal{O}(\zeta\chi^2)$ no unique Killing tensor exists of rank 2 up to 6. This also rules out the existence of Killing tensors of this rank, even when including higher order terms in the spin. Thus, it seems likely that the rotating dCS black hole does not have a fourth constant of motion.

3.3.2 The Event Horizon

When it comes to the event horizon of the dCS corrected Kerr black hole, the ansatz (3.8) really shines. The coordinates (r, z) are chosen in such a way that the $r-z$ sector of the metric is simply a conformal rescaling of the $r-z$ sector of the Kerr metric. This means that, in particular, the component g^{rr} is still proportional to Δ, thus the null hypersurfaces of constant r coordinate are still identified by Δ being zero. That means that the r coordinate of the event horizon of the dCS corrected Kerr black hole is still defined by[11],

$$r_H = M(1+\sqrt{1-\chi^2}). \tag{3.11}$$

One can check that this hypersurface is also a Killing horizon, further confirming that this is indeed the event horizon of the dCS corrected Kerr black hole. Like in the Kerr spacetime, the event horizon is generated by the Killing vector,

$$\xi^\mu = t^\mu + \Omega_{\rm H}^{\rm dCS}\phi^\mu, \tag{3.12}$$

with $\Omega_{\rm H}^{\rm dCS}$ still defined by (1.37). Which can explicitly be written as,

$$\Omega_{\rm H}^{\rm dCS} = \Omega_{\rm H}^{\rm Kerr} + \zeta\delta\Omega_{\rm H}, \tag{3.13}$$

where $\Omega_{\rm H}^{\rm Kerr}$ is the horizon angular velocity of the Kerr metric as given by Eq. (1.39), and $\delta\Omega_{\rm H}$ is the dCS correction, which is given by,

$$\begin{aligned}\delta\Omega_{\rm H} = &-\frac{1}{M}\left(\frac{709}{1792}\chi + \frac{169}{1536}\chi^3 + \frac{254929}{2365440}\chi^5 + \frac{613099}{5271552}\chi^7 + \frac{1684631453}{13776322560}\chi^9\right.\\ &\left.+\frac{35249720647}{281036980224}\chi^{11} + \frac{67579939563817}{533970262425600}\chi^{13}\right).\end{aligned} \tag{3.14}$$

This correction is always negative, implying that the angular velocity of the event horizon is smaller in dCS corrected Kerr metric than it is in the Kerr metric, as also seen in Fig. 3.

[11] The coordinate location of the event horizon is physically meaningless, thus this does not mean that the event horizons of Kerr and dCS corrected Kerr black hole are the same.

3.3.3 The Ergosphere

While the analysis of the horizon is very simple due to the choice of coordinates used to express the dCS corrected Kerr metric in, it is a different story for the ergosphere. The static limit surface of the dCS corrected Kerr metric i.e. where the g_{tt} component of the metric vanishes, is now defined by,

$$1 - \frac{2Mr}{\Sigma(r,z)} = \zeta H_1(r,z). \tag{3.15}$$

Solving this equation to obtain a relation $r_E(z)$ between the r and z coordinates is a difficult task. What we can easily do, however, is show that the event horizon and ergosphere touch at the poles, just like in GR,

$$1 - \frac{2Mr_H}{\Sigma(r_H,\pm 1)} = \zeta H_1(r_H,\pm 1). \tag{3.16}$$

That finding $r_E(z)$ is difficult is for the rest of the thesis not important, we will be merely interested in the existence of the ergosphere. To do that, we can simply visualize the g_{tt} component of the metric and determine where it is negative and positive. This is shown in Fig. 4 for a spin of $\chi = 0.7$. Here we see that the ergosphere extends out the furthest at the equator, and touches the horizon at the poles. Thereby making it similar to the ergosphere of the Kerr black hole. We must however note that this visualization is coordinate dependent.

We have indicated the static-limit surface of the Kerr black hole in this figure, however the (t,r,z,ϕ) coordinate system in which we express the Kerr metric is not identical to the (t,r,z,ϕ) coordinate system in which we express the dCS corrected Kerr metric.

A better quantity that can serve as a comparison between the ergospheres would for instance be surface area of the static limit surface.

3.3.4 Energy Conditions

A crucial difference between the Kerr metric and the dCS corrected Kerr metric is that the former is a vacuum spacetime, whereas the latter, is not. It is clear that the dCS corrected black hole supports the axion field, which begs the question: What is the stress-energy tensor associated with the dCS corrected Kerr metric?

Instead of considering dCS gravity as a modified theory of gravity, it can be useful to take the viewpoint that dCS is simply general relativity with some exotic stress-energy tensor. In that case, we should actually interpret the right-hand side of (2.13) as being the stress-energy tensor of dCS gravity,

$$T^{\text{dCS}}_{\mu\nu} = \frac{1}{2}\partial_\mu\vartheta\,\partial_\nu\vartheta - \frac{1}{4}g_{\mu\nu}\,\partial_\sigma\vartheta\,\partial^\sigma\vartheta - 4\alpha\,\epsilon^{\gamma\beta}{}_{\alpha(\mu|}\nabla_\beta R_{|\nu)\gamma}\nabla^\alpha\vartheta - 4\alpha\,{}^*R_{\alpha(\mu\nu)\beta}\,\nabla^\beta\nabla^\alpha\vartheta. \tag{3.17}$$

To see if this is a reasonable stress-energy tensor, we can evaluate the four energy conditions. Since the null energy condition is the weakest energy condition, it makes sense to evaluate this condition first. One can then show that by contracting the stress-energy tensor twice with the null vector field[12],

$$v^\mu \partial_\mu = \partial_t + \left(-\frac{g_{t\phi}}{g_{\phi\phi}} + \frac{\sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}}{g_{\phi\phi}} \right) \partial_\phi, \tag{3.18}$$

we have that everywhere in the exterior of the black hole,

$$T^{\text{dCS}}_{\mu\nu} v^\mu v^\nu \leq 0. \tag{3.19}$$

This means that the null energy condition is clearly violated, implying that all four energy conditions are violated by the dCS corrected Kerr metric [23].

[12]This vector field becomes the Killing vector ξ^μ on the event horizon.

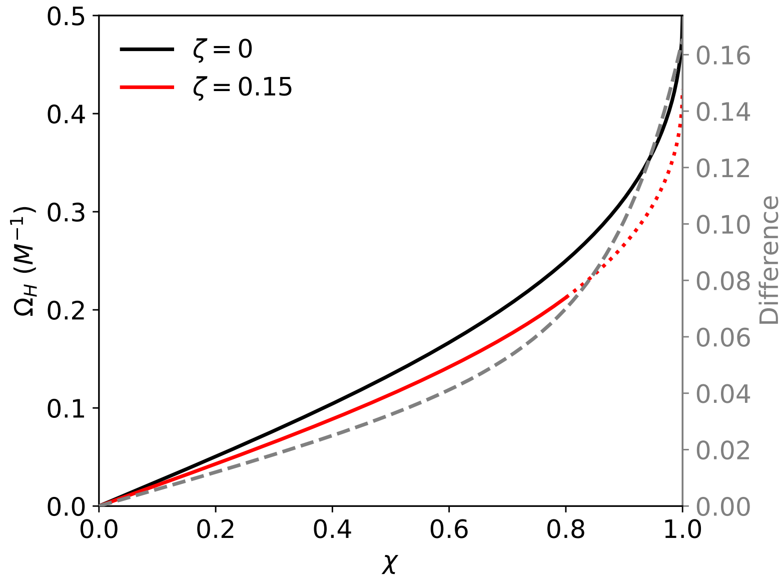


Figure 3: The angular velocity of the event horizon of the Kerr black hole and the dCS corrected Kerr black hole. The grey dashed line indicates the difference between the two.

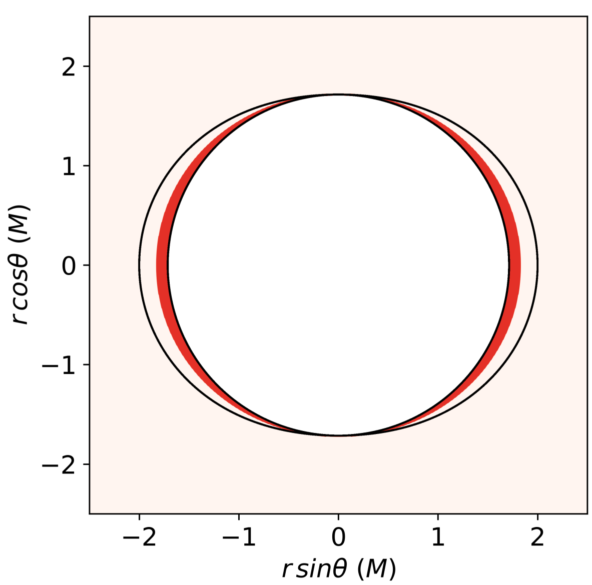


Figure 4: The metric component g_{tt} for a dCS black hole with spin parameter $\chi = 0.7$. In the dark red region, g_{tt} is positive, thus the red region defines the ergosphere of the dCS corrected Kerr black hole. In the light red region g_{tt} is negative. The inner black ring is the outer event horizon and the outer black ring is the boundary of the ergosphere of a Kerr black hole with spin $\chi = 0.7$.

3.4 Petrov Type of the dCS Corrected Kerr Black Hole

In 1954, A. Petrov [24] devised a scheme to classify spacetimes by the algebraic properties of their Weyl curvature. In this classification scheme, a generic spacetime falls into one of six types, known today as the Petrov types. Spacetimes of the same Petrov often have commonalities. For example, the Kerr family of black holes are all of Petrov type D, and whenever we have a vacuum spacetime that is of Petrov Type D, there is a theorem that tells us that there exists a non-trivial rank-two Killing tensor other than the metric [25][13]. In the case of the Kerr black hole, this theorem implies the conservation of the Carter constant, which is generated by (1.24).

Mathematical tools used to study gravitational fields sometimes may only be applicable to spacetimes of a certain Petrov type, an important example is the Teukolsky equation which is used to study perturbations of rotating black holes, which also requires a type D spacetime [26].

The dCS corrected Kerr spacetime being of a different Petrov type than the black holes of GR, thus could imply a drastic difference in properties. Therefore, it is important to study and determine the Petrov type of the dCS corrected Kerr black hole. Before we do that, however, we will have to cover some preliminaries.

3.4.1 Introducing Orthonormal Tetrads

Consider a spacetime with metric, ds^2, and a coordinate system, (x_0, x_1, x_2, x_3). With this coordinate system comes a convenient basis in which we can expand vectors, $(\partial_0, \partial_1, \partial_2, \partial_3)$, and covectors (dx^0, dx^1, dx^2, dx^3). In this 'coordinate basis', the metric is written as,

$$ds^2 = ds^2(\partial_\mu, \partial_\nu)\, dx^\mu \otimes dx^\nu = g_{\mu\nu}\, dx^\mu \otimes dx^\nu, \tag{3.20}$$

where $g_{\mu\nu} = ds^2(\partial_\mu, \partial_\nu)$ are the components of the metric in the coordinate basis. We are so used to using this basis that we often refer to $g_{\mu\nu}$ itself as the metric. However, using the coordinate basis is a choice and not a necessity. We may just as well use another set of vector fields, $(e_{(0)}, e_{(1)}, e_{(2)}, e_{(3)})$, and the associated covector fields, $(e^{(0)}, e^{(1)}, e^{(2)}, e^{(3)})$, as a basis. Such a set of vector fields that we use as a basis we call a tetrad basis. Essentially, that is all there is to it.

The (co)vector fields that constitute the tetrad basis can in turn be expanded in terms of the coordinate basis,

$$e_{(a)} = e_{(a)}{}^{\mu}\, \partial_\mu, \quad e^{(a)} = e^{(a)}{}_{\mu}\, dx^\mu, \quad a \in \{0, 1, 2, 3\}. \tag{3.21}$$

In such a tetrad basis, the metric can be expanded as,

$$ds^2 = ds^2(e_{(a)}, e_{(b)})\, e^{(a)} \otimes e^{(b)} = g_{(a)(b)}\, e^{(a)} \otimes e^{(b)}, \tag{3.22}$$

where, $g_{(a)(b)}$, are the components of the metric in the tetrad basis. It is straightforward to check that the components of the metric in tetrad basis and the components of the metric

[13] The spacetime does not necessarily have to be a vacuum, as the theorem also holds if one adds a cosmological constant, and it also holds in the case of the Kerr-Newman family of black holes, which is not vacuum solution if the black hole is electrically charged.

in the coordinate basis are related by,

$$g_{(a)(b)} = e_{(a)}{}^{\mu} e_{(b)}{}^{\nu} g_{\mu\nu}. \tag{3.23}$$

A particularly useful class of tetrads are 'orthonormal tetrads'. These are tetrad bases in which the components of the metric reduce to those of the Minkowski metric,

$$ds^2 = \eta_{(a)(b)}\, e^{(a)} \otimes e^{(b)} = -e^{(0)} \otimes e^{(0)} + e^{(1)} \otimes e^{(1)} + e^{(2)} \otimes e^{(2)} + e^{(3)} \otimes e^{(3)}. \tag{3.24}$$

Such a choice of tetrad basis is not unique. Consider for example a different tetrad basis, $\{\bar{e}_{(a)}\}$, that is locally related to the orthonormal tetrad basis $\{e_{(a)}\}$ by,

$$\bar{e}_{(a)} = \Lambda_{(a)}{}^{(b)}\, e_{(b)}, \tag{3.25}$$

where $\Lambda_{(a)}{}^{(b)}$ is some transformation matrix. The components of the metric in our newly defined tetrad basis are,

$$\bar{g}_{(a)(b)} = ds^2(\bar{e}_{(a)}, \bar{e}_{(b)}) = \Lambda_{(a)}{}^{(c)}\, \Lambda_{(b)}{}^{(d)}\, ds^2(e_{(c)}, e_{(d)}) = \Lambda_{(a)}{}^{(c)}\, \Lambda_{(b)}{}^{(d)}\, \eta_{(c)(d)}. \tag{3.26}$$

If we require our new tetrad basis to be orthonormal as well, we have must have that,

$$\eta_{(a)(b)} = \Lambda_{(a)}{}^{(c)}\, \Lambda_{(b)}{}^{(d)}\, \eta_{(c)(d)}. \tag{3.27}$$

This is exactly the relation that defines Lorentz transformations. Thus, orthonormal tetrads are locally related by a Lorentz transformations[14].
The tetrad formalism can be a really effective tool when doing calculations, and general relativity can be entirely formulated in terms of tetrads, however the strongest argument for using tetrads as opposed to the usual coordinate bases, is the fact that one has to use tetrads to define fermionic fields on curved spacetime. However, we will be covering that in this thesis.

3.4.2 The Weyl Curvature Tensor and Petrov classification

The curvature of spacetime is expressed by the Riemann curvature tensor. By taking the trace of the Riemann tensor we obtain the Ricci curvature tensor, which plays a central role in the Einstein field equations (1.7), which tell us that the Ricci curvature of spacetime is directly related to the local stress-energy of matter. However, often we are interested in vacuum spacetimes where the local stress-energy of matter vanishes, or we are dealing with compact objects, in which case the local stress-energy is only non-vanishing in a small patch of space. In such cases, where (locally) the Ricci curvature vanishes, the Weyl tensor, which is defined as[15],

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \left(g_{\alpha[\gamma} R_{\delta]\beta} - g_{\beta[\gamma} R_{\delta]\alpha}\right) + \frac{1}{3} R g_{\alpha[\gamma} g_{\delta]\beta}, \tag{3.28}$$

[14]Note that this is not a Lorentz transformation of the coordinates, but simply a Lorentz rotation of the basis vectors that make up the tetrad basis.
[15]In a coordinate basis, we might now like to add.

expresses the curvature of spacetime. Together, the Ricci curvature tensor and the Weyl tensor contain the information provided by the Riemann tensor. We could say that, in general relativity, the Weyl curvature is the curvature of free space. Here we will not be interested in the all the ins and outs of the Weyl tensor, but we will specifically be interested in its algebraic properties, as these algebraic properties give rise to the Petrov classification of spacetimes. The Petrov classification of spacetimes, essentially counts the number of distinct null vector fields k^α which solve the algebraic equation,

$$k^\alpha k^\beta k_{[\mu} C_{\nu]\alpha\beta[\rho} k_{\sigma]} = 0. \tag{3.29}$$

If k^α solves the aforementioned equation, we say that k^α is a principle null direction (PND). For a general spacetime, there can be at most four of these PNDs. To determine the number of distinct PNDs in a spacetime, we use the tetrad formalism. We start with a generic orthonormal tetrad $\{e_{(a)}\}$ with one timelike basis vector $e_{(0)}$ and the other three basis vectors spacelike $\{e_{(1)}, e_{(2)}, e_{(3)}\}$. With this orthonormal tetrad we can construct a new type of tetrad, a complex null tetrad $\{l, n, m, \bar{m}\}$ defined as,

$$\begin{aligned}
l^\alpha &= \frac{1}{\sqrt{2}} \left({e_{(0)}}^\alpha + {e_{(1)}}^\alpha \right), \\
n^\alpha &= \frac{1}{\sqrt{2}} \left({e_{(0)}}^\alpha - {e_{(1)}}^\alpha \right), \\
m^\alpha &= \frac{1}{\sqrt{2}} \left({e_{(2)}}^\alpha + i\, {e_{(3)}}^\alpha \right), \\
\bar{m}^\alpha &= \frac{1}{\sqrt{2}} \left({e_{(2)}}^\alpha - i\, {e_{(3)}}^\alpha \right).
\end{aligned} \tag{3.30}$$

A complex null tetrad, thus consists of two pairs of real valued vectors $\{l, n\}$, and a pair of complex valued vectors $\{m, \bar{m}\}$. As the name suggests, the vectors that form the complex null tetrad are null, and with the only nonvanishing scalar products being, $l^\alpha n_\alpha = -1$, and, $m^\alpha \bar{m}_\alpha = 1$. In this complex null tetrad basis, the metric takes the form,

$$ds^2 = -l \otimes n - n \otimes l + m \otimes \bar{m} + m \otimes \bar{m}. \tag{3.31}$$

In such a basis, the ten independent components of the Weyl tensor are now expressed in five complex scalars, the so-called Weyl scalars, which are defined as,

$$\begin{aligned}
\Psi_0 &= C_{a\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta, \\
\Psi_1 &= C_{a\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^{\bar{b}}, \\
\Psi_2 &= C_{a\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta, \\
\Psi_3 &= C_{a\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta, \\
\Psi_4 &= C_{a\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.
\end{aligned} \tag{3.32}$$

Since the choice of orthonormal tetrad basis is certainly not unique, the complex null tetrad basis is not either. Therefore, the Weyl scalars depend on the choice of basis. It turns out, that if the complex null tetrad is constructed in such a way that the Weyl scalar Ψ_0 vanishes, then l^α is a PND. Thus, finding the distinct PNDs of a spacetime comes down to finding the

Lorentz transformations that transform a general complex null tetrad in such a way that Ψ_0 vanishes. We will not derive exactly how to do this here, but such a transformation can be found by solving the quartic [27],

$$\Psi_0 + 4B\Psi_1 + 6B^2\Psi_2 + 4B^3\Psi_3 + B^4\Psi_4 = 0, \tag{3.33}$$

for the complex scalar B, where the Weyl scalars are computed from general complex null tetrad where Ψ_0 does not vanish.
Having found these complex scalars, the PNDs of a spacetime are then given by,

$$k^\alpha = l^\alpha + \bar{B}m^\alpha + B\bar{m}^\alpha + B\bar{B}n^\alpha. \tag{3.34}$$

To summarize, by using the tetrad formalism, one can transform the algebraic equation (3.29) into the task of finding the roots of the quartic (3.33). The Petrov classification of a spacetime is then assigned into one of six types according to the structure of the roots of this quartic according to the following scheme:

- Petrov type I: Four distinct roots,
- Petrov type II: One double degenerate root and two distinct roots,
- Petrov type III: One triple degenerate root and one distinct root,
- Petrov type D: Two doubly degenerate roots,
- Petrov type N: One quadruple degenerate root,
- Petrov type O: The Weyl curvature vanishes.

A type I spacetime is often called algebraically general, and when a spacetime is not type I it is often called algebraically special. It is important to stress that, even though the Weyl scalars depend on the choice of complex null tetrad, the Petrov classification is an invariant property of the spacetime.

3.4.3 The Petrov Classification of the dCS Corrected Kerr Spacetime

To determine the Petrov type of the dCS corrected Kerr spacetime, we will take a general approach that works for any stationary and axisymmetric spacetime. In such a spacetime, the metric can always be written in the form given by Eq. (1.23).
Then, the following set of vectors will constitute an orthonormal tetrad,

$$\begin{aligned}
t_\alpha dx^\alpha &= \sqrt{-g_{tt}}dt - \frac{g_{t\phi}}{\sqrt{-g_{tt}}}d\phi,\\
r_\alpha dx^\alpha &= \sqrt{g_{rr}}dr,\\
z_\alpha dx^\alpha &= \sqrt{g_{zz}}dz,\\
\phi_\alpha dx^\alpha &= -\frac{\sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{\sqrt{-g_{tt}}}d\phi.
\end{aligned} \tag{3.35}$$

This can be checked by using Eq. (3.24). Subsequently, we construct the complex null tetrad,

$$\begin{aligned} l^\alpha &= \frac{1}{\sqrt{2}}\left(t^\alpha + \phi^\alpha\right), \\ n^\alpha &= \frac{1}{\sqrt{2}}\left(t^\alpha - \phi^\alpha\right), \\ m^\alpha &= \frac{1}{\sqrt{2}}\left(r^\alpha + iz^\alpha\right), \\ \bar{m}^\alpha &= \frac{1}{\sqrt{2}}\left(r^\alpha - iz^\alpha\right). \end{aligned} \tag{3.36}$$

We will use this complex null tetrad to calculate the Weyl scalars as defined in (3.32). One can explicitly check that in this complex null tetrad basis, the Weyl scalars, Ψ_1 and Ψ_3 vanish for a generic stationary axisymmetric spacetime. The quartic (3.33) then reduces to,

$$\Psi_0 + 6B^2\Psi_2 + B^4\Psi_4 = 0. \tag{3.37}$$

Besides still being a quartic in B, this polynomial can also be viewed as quadratic in B^2, which either has two distinct roots or one double root, which corresponds to either four distinct roots or two double roots in terms of B. Thus, a generic stationary axisymmetric spacetime can only be of Petrov type I or D[16]. Whether a quadratic has two distinct or a double root can be determined by the discriminant D of the quadratic, which in this case is,

$$D = 36\Psi_2^2 - 4\Psi_0\Psi_4. \tag{3.38}$$

If the discriminant vanishes, then we have a double root and the spacetime is Petrov type D, if the discriminant is nonvanishing, then the spacetime is Petrov type I. A thorough analysis of the Petrov type of the dCS corrected spacetime was first carried out by Yani, Yunes, and Tanaka in 2017 where they used dCS correct Kerr spacetime up to second order in the black hole spin [22]. The trio concluded that up to linear order in the black hole spin, the dCS corrected Kerr spacetime is, like the Kerr spacetime, Petrov type D. However, if one includes the second order spin corrections, the spacetime is Petrov Type I. Even though this is analysis was carried out with a corrected metric up to quadratic order in the spin, including higher order corrections in the spin is not necessary. This can be easily seen by considering (3.38).

Since we are working with a metric expand in the spin, we must also expand the discriminant in the spin,

$$D(r,z) = \sum_{n=0}^{\infty} D^{(n)}(r,z)\chi^n. \tag{3.39}$$

[16] Or Type O if the Weyl curvature vanishes, but this case we will disregard.

Since the spacetime is only type D if the discriminant vanishes, the discriminant must vanish order by order in the spin.

Nonetheless, we will still carry out the analysis of the Petrov type with the dCS corrected metric up to fourteenth order in spin. In particular, we will consider the local Petrov type of the dCS corrected Kerr spacetime. This has not been considered before[17].

Since the Weyl scalars, and thus the discriminant are locally defined, we can also consider the Petrov type of the spacetime locally. If we use the metric (3.8), and calculate the discriminant (3.38), we can make a few interesting observations.
We will not show the explicit form of the discriminant here, but we can present the discriminant as a contour plot in a small region around the black hole. As the discriminant is generally a complex quantity, we have shown the real and complex parts of the discriminant separately in Fig. 5 for a dCS black hole with a very small spin of, $\chi = 0.01$. From these figures we can indeed gather that the Petrov type is indeed almost everywhere type I, however there are two important surfaces where the Petrov type is locally of type D. One of these surfaces is the event horizon, clearly visible in both figures as a white arc around $r = 2$, which is approximately where the event horizon is located for a very slow rotating black hole. One can also explicitly check that,

$$D(r_H, z) = 0, \tag{3.40}$$

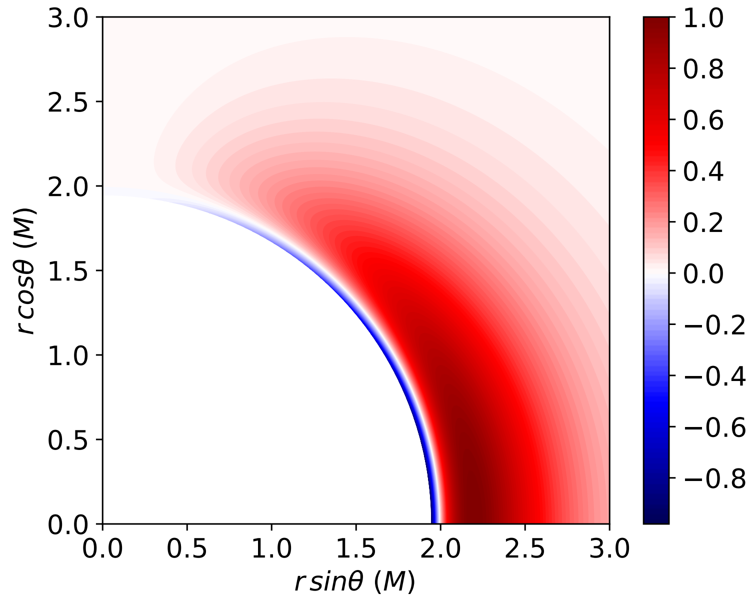
when expanded to order, $\mathcal{O}(\zeta\chi^{14})$. One can also explicitly check that the axis of symmetry is locally of type D,

$$D(r, \pm 1) = 0. \tag{3.41}$$

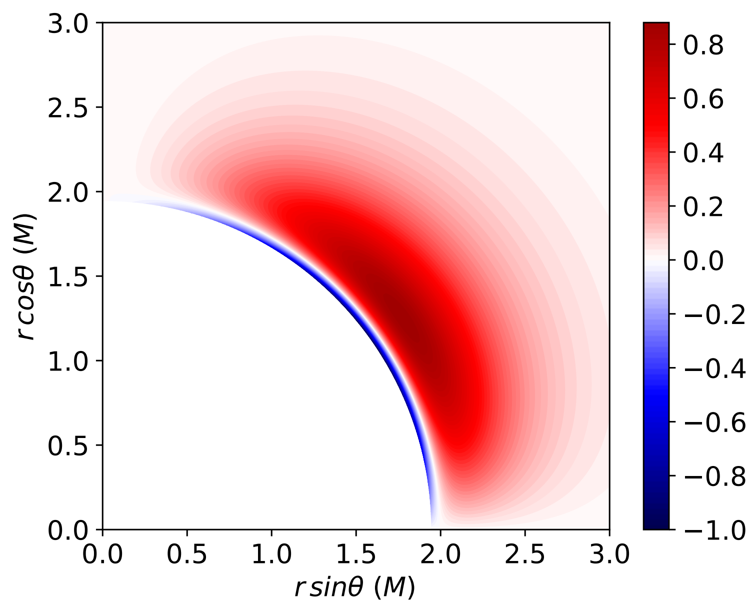
The fact that the event horizon and the axis of symmetry of a black hole are algebraically special surfaces has been noted before, for instance by D. Papadopoulos and B. Xanthopoulos in 1984 [28], for general static and axisymmetric local black holes[18]. However, this does not directly apply to the dCS corrected Kerr black hole. That the event horizon is an algebraically special surface, is easy to forget in general relativity, as stationary axisymmetric black holes in general relativity are everywhere of type D.
In a 2013 paper by I. Tanatarov and O. Zaslavksii [29], on the Petrov classification of event horizon of axisymmetric black holes, the duo argue that due to the event horizon being of a different Petrov type than the of the exterior of the black, an observer could locally distinguish the event horizon from the rest of the spacetime, contrary to the popular notion that event horizons are locally not detectable. We must however add that not a lot of literature exists on this topic, and the deeper mathematical meanings as to why the these algebraically special surfaces appear as well as the implications of the event horizon and axis of symmetry being algebraically special surfaces is unclear. Therefore, although an interesting mathematical observation, we decided not to pursue this topic further.

[17] Initially our analysis of the Petrov type was purely to check the results obtained by Yani, Yunes, and Tanaka, however by doing this we noticed the interesting properties of the local Petrov type.

[18] That is, non-rotating black holes, around which, for a finite distance, there are no non-gravitational fields present.



(a) $Re(D)$



(b) Im(D)

Figure 5: The real and imaginary parts of the discriminant that determines the local Petrov type, with the spin taken to be very small, $\chi = 0.01$. Where this discriminant is non-zero, the local Petrov type is I, where the discriminant vanishes, the local Petrov Type is D. Only a small region inside the horizon is presented, thus no data is provided in the region bounded by the blue arc. The event horizon is clearly visible as a type D surface.

4 The Four Laws of Black Hole Mechanics

In 1969, during a conference held in Florence, Italy, R. Penrose showed that it is possible to extract energy from a rotating black hole in general relativity [30, 31]. In a nutshell, Penrose showed that one could arrange a situation where a composite particle on a geodesic enters the ergosphere of a rotating black hole. Subsequently, this composite particle decays into two particles, after which the particles will travel along separate geodesics. One of the particles eventually falls into the black hole, whereas the other will again exit the ergosphere having gained energy and angular momentum as compared to the energy and angular momentum of the initial composite particle. Since for every winner, there is a loser, the black hole must have lost an amount of energy and angular momentum equal to the energy and angular momentum that the particle emerging from the ergosphere has been enhanced with.

A year after Penrose introduced the Penrose process, D. Christodoulou considered the loss of mass of a black hole due to the Penrose process [32]. Christodoulou showed that this loss of mass is not indefinite, but that the black hole reaches its minimal mass when all of its angular momentum has been extracted. Christodoulou coined this mass the irreducible mass, and he argued that the only transformations of the black hole's mass and angular momentum that can be reversed are those that leave the irreducible mass unchanged. Any other transformation can only increase the irreducible mass. Christodoulou also showed that up to 29% of the mass energy of an extremal Kerr black hole can be extracted via the Penrose process, a fact that we will rediscover shortly. At the time, it became clear that Christodoulou's work on the Penrose process, with its reversible and irreversible processes, had a lot in common with the second law of thermodynamics. This ultimately begged the question: is there a connection between the four laws of thermodynamics and black hole physics? It turned out that there indeed is such a connection.

A little over two years after Christodoulou's work was published, J. Bardeen, B. Carter, and S. Hawking published a set of four laws, which they called the four laws of black hole mechanics, each of which having a clear analogue with the four laws of thermodynamics [7]. Despite not co-authoring this specific paper, a name we certainly must not forget to mention is J. Bekenstein, who in 1972 published an important paper on black holes and the second law of thermodynamics [33].

Since then, a lot of work has been done to deepen our understanding of these laws in the context of general relativity. However, when we consider theories that supplement the Einstein-Hilbert action with additional terms, it is not a priori clear that these four laws of black hole mechanics still hold. This is because proofs of the four laws hinge on assumptions that often cannot be made in modified theories of gravity, for example: the energy conditions, which are not satisfied in dCS gravity. Therefore, we will revisit the four laws of black hole mechanics in the context of dCS gravity.

In this section, we will go over each law one by one. First, we will simply state the law that is under consideration, and then we will give some general background information. After that, we will evaluate the law using the dCS corrected Kerr metric and study the implications.

4.1 The Zeroth Law of Black Hole Mechanics

The zeroth law of black mechanics states:

> *The surface gravity κ of a stationary black hole is constant over the event horizon.*

This statement is analogous to the zeroth law of thermodynamics, which states that the temperature of a thermal system is uniform in equilibrium.
The surface gravity is defined by,

$$\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu, \tag{4.1}$$

where ξ^μ is the Killing vector that generates the event horizon of the black hole, and κ is the surface gravity. But how is this related to temperature?
In their 1972 paper on the laws of black hole mechanics, J. Bardeen, B. Carter, and S. Hawking explicitly stated that the surface gravity and thermodynamic temperature are only *analogously* related to each other, and that the surface gravity is not related to the actual temperature of a black hole. They argued that the effective temperature of a black hole should be absolute zero. They argued that this is because a black hole can never be in equilibrium with a black body of finite non-zero temperature, as some radiation of the black body will always be absorbed by the black hole, but by definition the black hole itself cannot emit any radiation.
Later, sometime around 1975, Hawking, by considering the effects of Killing horizons on quantum fields, showed that black holes in fact do emit radiation [34]. It then became clear that the surface gravity is in fact directly related to the temperature of this radiation in any theory of gravity by,

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi}, \tag{4.2}$$

which is now known as the Hawking temperature. By considering this relation between surface gravity and temperature, we may also reformulate the zeroth law as,

> *The temperature T_{H} of a stationary black hole is constant over the event horizon.*

The original proof of the zeroth law of black hole mechanics assumes the dominant energy condition [7], which does not hold in dCS, thereby rendering the proof inapplicable. However, the constancy of the surface gravity of the event horizon can still be proven under a different set of reasonable assumptions without making an appeal to the energy conditions or the field equations [35]. Here we will take an even simpler approach. Since we know the metric (approximately), we can directly calculate the surface gravity, and check if it is constant on the event horizon.

Recall that, the event horizon of a rotating black hole in dCS is a Killing horizon that is generated by the Killing vector ξ^μ as given by Eq. (1.38). It turns out that evaluating Eq. (4.1) directly is usually not a very convenient way to calculate the surface gravity, however, we can express Eq. (4.1) equivalently as,

$$-\partial_\mu \xi^2 = 2\kappa \xi_\mu. \tag{4.3}$$

This expression is more convenient as there are no covariant derivatives involved, hence we do not need to evaluate the connection coefficients (1.3). There is a problem, however. The coordinates in which we have expressed the metric of dCS corrected Kerr black hole are not regular on the event horizon. Preferably, we change to a different set of coordinates that *are* regular on the event horizon, but, if we are careful, we can still obtain the correct result with our ill-behaving coordinates.
First, consider the left-hand side of equation (4.3). By inserting (1.38) into (4.3) we obtain,

$$-\partial_\mu \xi^2 = -\partial_\mu (g_{tt} + 2\Omega_{\rm H}\, g_{t\phi} + \Omega_{\rm H}^2\, g_{\phi\phi}). \tag{4.4}$$

This expression is completely regular when evaluated on the event horizon, as only the g_{rr} component of the metric diverges there. Furthermore, one can show by explicit calculation that only the partial derivative with respect to the radial coordinate is non-zero, leaving us with,

$$-\partial_\mu \xi^2|_{r_H} = -\partial_r \xi^2|_{r_H} \delta^r{}_\mu. \tag{4.5}$$

Now we tackle the right-hand side of (4.3). This turns out to be a more subtle. First of all, we will use that the Killing vector ξ^μ is both normal and tangent to the event horizon. Then, by using Eq. (1.30) we should have that ξ_μ must be proportional to $\delta^r{}_\mu$, thus we may write,

$$\xi_\mu = \xi_r\, \delta^r{}_\mu, \tag{4.6}$$

where ξ_r is to be determined. Note that this is only true *on* the event horizon. We can then determine ξ_r by considering the square of ξ^μ,

$$\xi^2 = \xi^\mu \xi_\mu = \xi_r^2\, g^{rr}. \tag{4.7}$$

Then, we can determine what ξ_r should be by taking the on-horizon limit of the following expression,

$$\xi_r = \lim_{r \to r_H} \sqrt{\frac{\xi^2}{g^{rr}}}. \tag{4.8}$$

Since both ξ^2 and g^{rr} go to zero on the horizon, we may use l'Hopital's rule to evaluate the limit as,

$$\xi_r = \lim_{r \to r_H} \sqrt{\frac{\partial_r \xi^2}{\partial_r g^{rr}}}. \tag{4.9}$$

Altogether, this gives us the following expression for the surface gravity of the event horizon,

$$\kappa = -\frac{\partial_r \xi^2|_{r_H}}{2\xi_r}. \tag{4.10}$$

If we then evaluate this expression for the rotating dCS black hole, we find a surface gravity,

$$\kappa_{\text{dCS}} = \kappa_{\text{Kerr}} + \zeta\delta\kappa, \tag{4.11}$$

where κ_{Kerr} is the surface gravity of the Kerr black hole,

$$\kappa_{\text{Kerr}} = \frac{\sqrt{1-\chi^2}}{2M(1+\sqrt{1-\chi^2})}, \tag{4.12}$$

and $\delta\kappa$ is the leading order dCS correction to the surface gravity,

$$\begin{aligned}\delta\kappa = \frac{1}{M}\bigg(&\frac{2127}{7168}\chi^2 + \frac{14423}{86016}\chi^4 + \frac{429437}{3153920}\chi^6 + \frac{125018653}{984023040}\chi^8 + \frac{20524079857}{165315870720}\chi^{10} \\ &+ \frac{276424697191}{2248295841792}\chi^{12} + \frac{2482747013891}{20341724282880}\chi^{14}\bigg).\end{aligned} \tag{4.13}$$

Thus, we find that the surface gravity of the dCS corrected Kerr black hole is constant on the horizon, hence the zeroth law of black hole mechanics holds for the dCS corrected Kerr black hole.

Besides this, it is interesting to consider the features of the surface gravity. In general relativity, a black hole with a fixed mass M has a maximal surface gravity when it is non-rotating, and it vanishes when the black hole is extremal, as can be seen in Fig. 6. The dCS correction, however, given by Eq. (4.13), increases with the spin of the black hole. Thus, a dCS black hole has a higher surface gravity than a Kerr black hole. Via the Hawking temperature (4.2), we can then also say that the horizon of a dCS black hole is at a higher temperature than the horizon of a Kerr black hole.
This is ultimately related to the fact that a Kerr black hole of fixed mass and spin has a higher horizon angular velocity than a dCS corrected Kerr black hole of the same mass and spin.

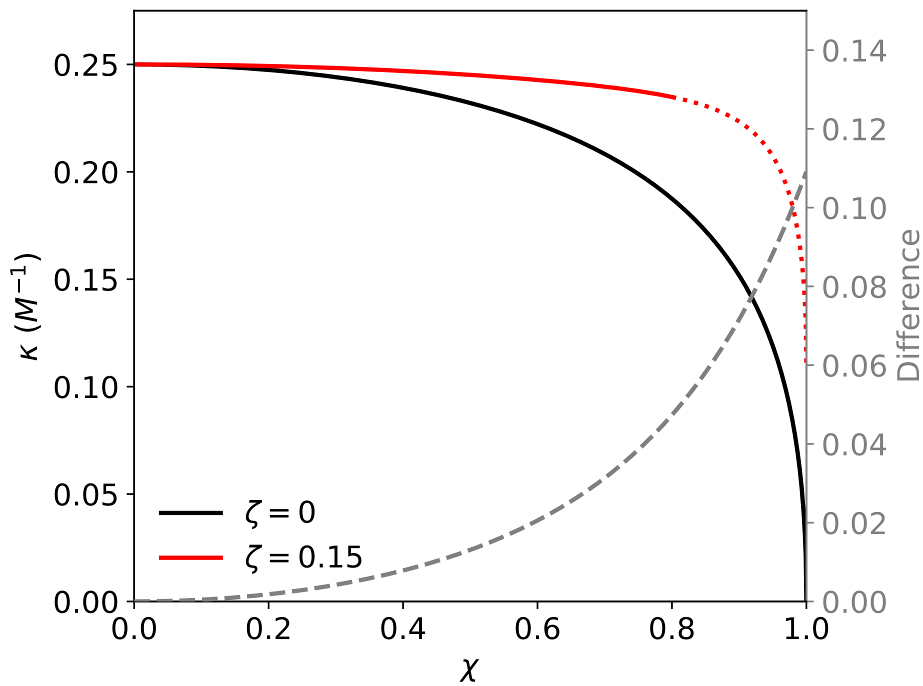


Figure 6: The surface gravity of the event horizon in GR and in dCS. The grey dashed line indicates the difference between the two.

4.2 The First Law of Black Hole Mechanics

The first law of black hole mechanics states:

> *Nearby stationary black holes solutions are related by:* $TdS = dM - \Omega_{\rm H} dJ$.

This law specifically applies to uncharged black holes, however, it can be naturally extended to include charged black holes as well. When this law was first put forth by Bardeen, Carter and Hawking, the first law was stated as:

> *Nearby stationary black holes solutions are related by:* $\frac{\kappa}{8\pi} d\mathcal{A} = dM - \Omega_{\rm H} dJ$.

Here κ is the surface gravity, M and J are the mass and angular momentum of the black hole, $\Omega_{\rm H}$ is the angular frequency of the black hole event horizon, and $\mathcal{A}$ is the surface area of the black hole event horizon, which is defined as,

$$\mathcal{A} = \oint_{\rm H} d^2x \sqrt{|\gamma|}, \tag{4.14}$$

where γ is the determinant of the induced metric, and the integral is over the event horizon at a constant time.
This is quite a remarkable result, as it relates the geometry of the event horizon, $\mathcal{A}$, and quantities locally defined on the event horizon, κ and $\Omega_{\rm H}$, to quantities that are measured in the asymptotically flat region of the spacetime, M and J. Besides that, it seems to almost

exactly mimic the first law of thermodynamics,

$$TdS = dU - dW, \tag{4.15}$$

if we identify the change in black hole mass dM with the change in energy dU, the change in angular momentum $\Omega_{\rm H} dJ$ with the work done on the system dW, and the change in the event horizon area $\frac{\kappa}{8\pi} d\mathcal{A}$ with the change in entropy TdS.
As we have already related the temperature of a black hole to the surface gravity via Eq. (4.2), it seems that the event horizon surface area should play the part of entropy[19],

$$S_{\rm BH} = \frac{\mathcal{A}}{4}. \tag{4.16}$$

We now call this the Bekenstein-Hawking entropy. With these definitions, we may write the first law of black hole mechanics as,

$$TdS_{\rm BH} = dM - \Omega_{\rm H} dJ. \tag{4.17}$$

If one tries to apply this law to the dCS corrected Kerr black hole, they will come to the conclusion that this relation does not hold. On the surface, this seems like a problem, but really, this is not a cause for concern. The first law can be fixed.
Although the relation between entropy and surface area of a black hole proposed by Bekenstein and Hawking is quite elegant, there is a priori no reason to assume that this relation holds in modified theories of gravity.
Should we have to modify the definition of entropy as well? This modified definition should reduce to Eq. (4.16) when applied to general relativity, and it should probably depend on the event horizon, as this is the only part of the black hole that is accessible to an outside observer. But what should this definition be? Luckily, we are not the first ones to have asked this question. In fact, our question has already been answered thirty years ago: we should use the Wald entropy.

4.2.1 The Wald Entropy

In 1993 R. Wald proposed a new definition of a black hole's entropy with which the first law of black hole mechanics holds for a wide class of theories of gravity, if in such a theory the zeroth law holds [36], and if the black hole's mass and angular momentum are well-defined. With the knowledge that we have so far, it seems that this should be a good definition of entropy in dCS gravity. The definition of entropy that Wald proposed is,

$$S = -8\pi \oint_{\rm H} d^2x \sqrt{|\gamma|} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\alpha\beta}} n_{[\mu}\sigma_{\nu]} n_{[\alpha}\sigma_{\beta]}. \tag{4.18}$$

[19] This is also what Jacob Bekenstein figured out in 1972 when he proposed that black holes should have an entropy that is proportional to surface area of their event horizon. Bekenstein proposed that the proportionality constant between entropy and the surface area should be $\frac{\log 2}{8\pi}$, or at least very close to that. However, the first law together with the definition of the Hawking temperature implied that the proportionality constant should simply be $\frac{1}{4}$.

Here $\mathcal{L}$ is the Lagrangian of the theory, n_μ is the unit normal to a constant time hypersurface, ($n^\mu n_\mu = -1$), and σ_μ is the unit normal to the event horizon within this constant time hypersurface, ($\sigma^\mu \sigma_\mu = 1$). The anti-symmetrized product of these two normal vectors has the property,

$$n^{[\mu}\sigma^{\nu]}n_{[\mu}\sigma_{\nu]} = -\frac{1}{2}. \tag{4.19}$$

Note that, like the Bekenstein-Hawking entropy, the Wald entropy is still an integral over the event horizon on, and as we shall see, the Wald entropy reduces to the Bekenstein-Hawking entropy when we take the Lagrangian to be the Einstein-Hilbert Lagrangian.

Let's now apply Wald's entropy formula to dCS gravity. Evidently the kinetic term of the axion field in the dCS Lagrangian is not relevant here, as it does not contain the Riemann tensor, thus the Wald entropy of a dCS black hole is determined by the Lagrangian,

$$\mathcal{L} = \kappa\big(R + \alpha\,\vartheta\,{}^*RR\big). \tag{4.20}$$

The derivative of this Lagrangian with respect to the Riemann tensor can then be calculated in a straightforward manner by using,

$$\frac{\partial R_{abcd}}{\partial R_{\mu\nu\alpha\beta}} = \delta^\mu{}_a \delta^\nu{}_b \delta^\alpha{}_c \delta^\beta{}_d. \tag{4.21}$$

We begin by considering the entropy contribution of the Einstein-Hilbert term, for this we have to calculate the derivative of the Ricci scalar with respect to the Riemann tensor,

$$\begin{aligned} \frac{\partial R}{\partial R_{\mu\nu\alpha\beta}} &= g^{ac}g^{bd}\frac{\partial R_{abcd}}{\partial R_{\mu\nu\alpha\beta}} \\ &= g^{ac}g^{bd}\delta^\mu{}_a \delta^\nu{}_b \delta^\alpha{}_c \delta^\beta{}_d \\ &= g^{\mu\alpha}g^{\nu\beta}. \end{aligned} \tag{4.22}$$

Then we contract with the normal vectors n^μ, and σ^μ to obtain the integrand of the Wald entropy,

$$g^{\mu\alpha}g^{\nu\beta}n_{[\mu}\sigma_{\nu]}n_{[\alpha}\sigma_{\beta]} = n^{[\mu}\sigma^{\nu]}n_{[\mu}\sigma_{\nu]} = -\frac{1}{2}, \tag{4.23}$$

where we have used Eq. (4.19). The entropy contribution of the Einstein-Hilbert term then reduces to a quarter of the event horizon surface area, which is simply the Bekenstein-Hawking entropy of the black hole,

$$S^{\mathrm{EH}} = \frac{1}{4}\oint_{\mathrm{H}} d^2x\sqrt{|\gamma|} = \frac{\mathcal{A}}{4}. \tag{4.24}$$

To calculate the contribution from the dCS part of the Lagrangian, we calculate the derivative of the Chern-Pontryagin scalar with respect to the Riemann tensor,

$$\begin{aligned}\frac{\partial\, {}^*RR}{\partial R_{\mu\nu\alpha\beta}} &= \frac{1}{2}\epsilon^{abcd} g^{kn} g^{lm} \frac{\partial}{\partial R_{\mu\nu\alpha\beta}} R_{abkl} R_{cdmn} \\ &= \epsilon^{ab\mu\nu} R_{ab}{}^{\beta\alpha} \\ &= 2\, {}^*R^{\mu\nu\beta\alpha}.\end{aligned} \tag{4.25}$$

This results in an entropy contribution,

$$\delta S^{\text{dCS}} = \alpha \oint_{\text{H}} d^2x \sqrt{|\gamma|}\, \vartheta\, {}^*R^{\mu\nu\alpha\beta} n_{[\mu}\sigma_{\nu]} n_{[\alpha}\sigma_{\beta]}. \tag{4.26}$$

Since the dual of the Riemann tensor is antisymmetric on the first and second pair of indices, we may drop the antisymmetrisation brackets on the normal vectors. The total Wald entropy of a dCS black hole then is given by,

$$S^{\text{dCS}} = \frac{\mathcal{A}}{4} + \alpha \oint_{\text{H}} d^2x \sqrt{|\gamma|}\, \vartheta\, {}^*R^{\mu\nu\alpha\beta} n_\mu \sigma_\nu n_\alpha \sigma_\beta. \tag{4.27}$$

When specifically applied to the dCS corrected Kerr black hole, the Wald entropy can then be divided into three parts,

$$S^{\text{dCS}} = \frac{1}{4}\mathcal{A}^{\text{Kerr}} + \zeta \delta S^{\text{dCS}} = \frac{1}{4}\mathcal{A}^{\text{Kerr}} + \zeta \left(\frac{1}{4}\delta\mathcal{A}^{\text{dCS}} + \delta S^{\vartheta}\right). \tag{4.28}$$

This consists of the background part, which is the Bekenstein-Hawking entropy of the Kerr black hole (4.16), where the surface area of the event horizon is given by,

$$\mathcal{A}^{\text{Kerr}} = 8\pi M^2 \left(1 + \sqrt{1-\chi^2}\right). \tag{4.29}$$

Then we have a correction to the Bekenstein-Hawking entropy due to the dCS corrections to the Kerr metric, which changes the horizon surface area by,

$$\begin{aligned}\delta\mathcal{A}^{\text{dCS}} = \pi M^2 \,\Bigg(&-\frac{915}{112}\chi^2 - \frac{25063}{6720}\chi^4 - \frac{528793}{295680}\chi^6 - \frac{39114883}{53813760}\chi^8 - \frac{618487273}{7749181440}\chi^{10} \\ &+ \frac{1975280860769}{5796387717120}\chi^{12} + \frac{17922709822092007}{28634155322572800}\chi^{14}\Bigg).\end{aligned} \tag{4.30}$$

This contribution is negative, meaning that the surface area of a dCS corrected Kerr black hole is smaller than the surface area of a Kerr black hole. Note that the last two terms in the expansion are actually positive, but the total contribution is still negative if we restrict the spin to be within the accepted range allowed by the approximate metric.
Lastly, we have an explicit contribution from the quadratic curvature term in the dCS action. To leading order in the coupling, this contribution is completely determined by the axion field on a Kerr background. The dCS corrections to the metric do not come into play

here. Therefore, we call this the axion contribution to the entropy, which is given by,

$$\begin{aligned}\delta S^{\vartheta} = \pi M^2 \Bigg(&\frac{29}{8}\chi^2 + \frac{311}{160}\chi^4 + \frac{33511}{26880}\chi^6 + \frac{1971919}{2257920}\chi^8 + \frac{96205849}{149022720}\chi^{10} \\ &+ \frac{84400129507}{170481991680}\chi^{12} + \frac{34572926824993}{88650635673600}\chi^{14} \Bigg).\end{aligned} \tag{4.31}$$

Altogether, this gives us a total contribution to the Wald entropy due to the dCS corrections of the Kerr metric of,

$$\begin{aligned}\delta S^{\mathrm{dCS}} = \pi M^2 \Bigg(&\frac{709}{448}\chi^2 + \frac{5437}{5376}\chi^4 + \frac{945691}{1182720}\chi^6 + \frac{12760691}{18450432}\chi^8 + \frac{587646343}{939294720}\chi^{10} \\ &+ \frac{135895944179}{234197483520}\chi^{12} + \frac{16211067412583}{29665014579200}\chi^{14} \Bigg),\end{aligned} \tag{4.32}$$

which is an increase in the total entropy as compared to the Bekenstein entropy of a Kerr black hole. These corrections are shown in Fig. 7 and Fig. 8 as a function of the black hole spin, keeping the mass of the black hole constant. Even though the dCS corrections to the entropy are an increasing function of spin, the total entropy of the dCS corrected Kerr black hole is still a decreasing function of spin.

In principle, the first law must hold with Wald's definition of entropy. Nonetheless, we can still explicitly check that this is true. By explicitly using Eq. (4.28), the first law implies that we should have,

$$T dS^{\mathrm{dCS}} = T\frac{\partial S^{\mathrm{dCS}}}{\partial M} dM + T\frac{\partial S^{\mathrm{dCS}}}{\partial J} dJ = dM - \Omega_{\mathrm{H}} dJ. \tag{4.33}$$

Since we have expressed the Wald entropy in terms of mass and spin, it is more convenient to rephrase this as,

$$T dS^{\mathrm{dCS}} = T\frac{\partial S^{\mathrm{dCS}}}{\partial M} dM + T\frac{\partial S^{\mathrm{dCS}}}{\partial \chi} d\chi = \left(1 - 2M\chi\Omega_{\mathrm{H}}\right) dM - M^2 \Omega_{\mathrm{H}} d\chi. \tag{4.34}$$

Here we used that $dJ = 2M\chi dM + M^2 d\chi$. Then the first law holds if,

$$\begin{aligned} 1 - 2M\chi\Omega_{\mathrm{H}} - T\frac{\partial S^{\mathrm{dCS}}}{\partial M} &= 0, \\ T\frac{\partial S^{\mathrm{dCS}}}{\partial \chi} + \Omega_{\mathrm{H}} &= 0. \end{aligned} \tag{4.35}$$

We have confirmed that this does indeed hold up to order $\mathcal{O}(\zeta\chi^{14})$. Thus, the first law of black hole mechanics holds for the dCS corrected Kerr black hole provided we use Wald's definition of entropy given by Eq. (4.27).

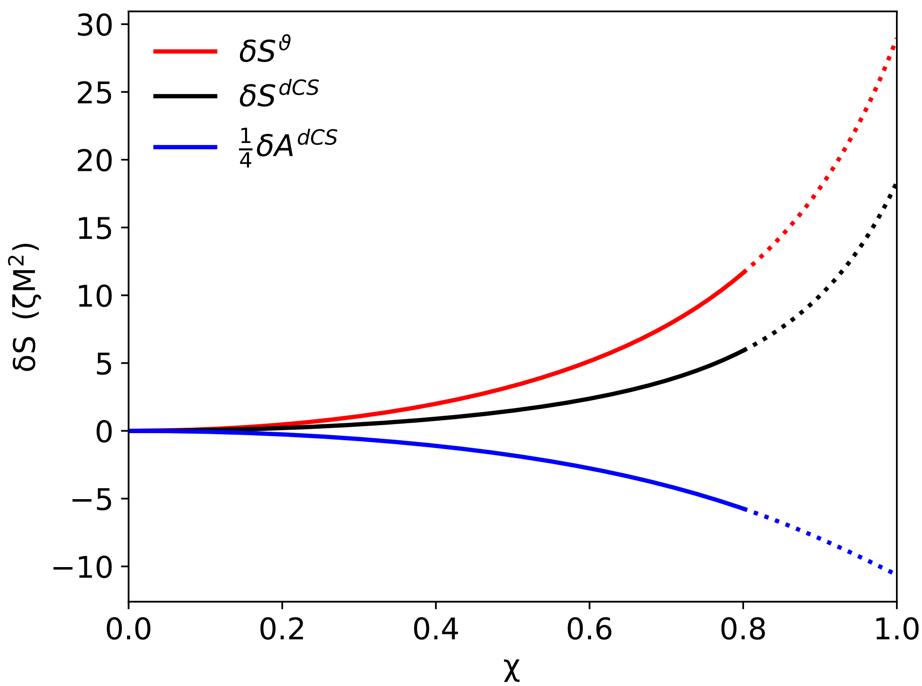


Figure 7: The dCS corrections to the Wald entropy as a function of the spin. The contribution from the axion on the Kerr background increases the entropy, whereas the contribution from the changed the horizon surface area decreases the entropy. However, overall, their sum results in a positive contribution to the Wald entropy that grows with the spin.

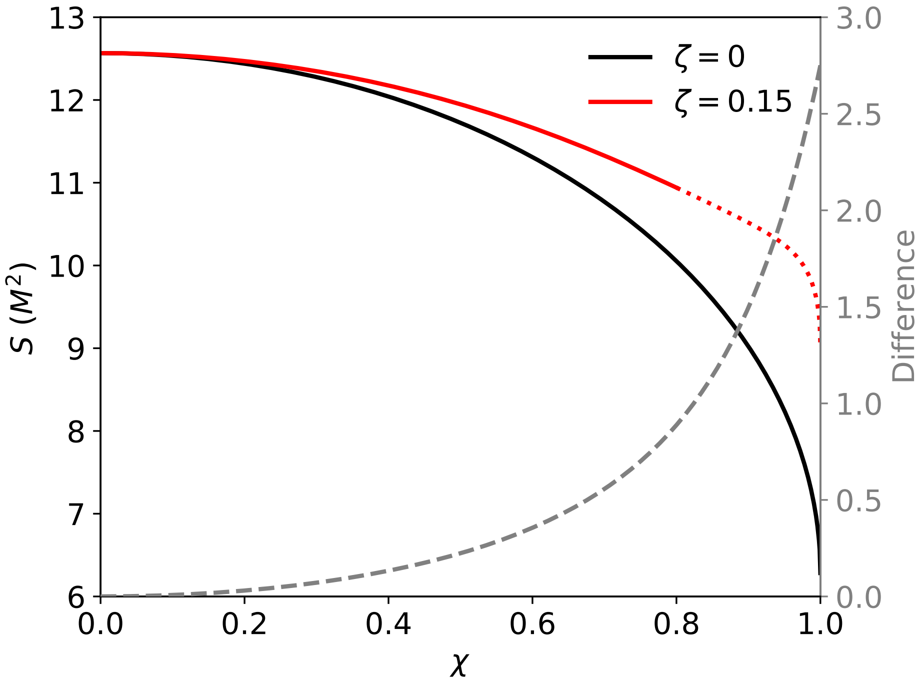


Figure 8: The total entropy of a rotating black hole in GR and in dCS as given by the Wald entropy. Even though the dCS entropy corrections increase with spin, the total entropy is still a decreasing function of spin.

4.3 The Second Law of Black Hole Mechanics

The second law of black hole mechanics states:

> *The entropy of a black hole cannot be decreased by any classical process,* $dS \geq 0$.

This statement is actually stronger than the second law of thermodynamics, which states that the entropy of a closed system cannot decrease, but entropy may still be exchanged between two systems. In general relativity, two black holes cannot exchange entropy, thus the entropy of each black hole is separately non-decreasing by means of any classical process. This does not require a quasi-static process, indeed in any classical process the entropy of a black hole cannot be decreased. A key ingredient in the proof of the second law is the null energy condition [37], which, as discussed, does not hold in dCS. That makes this proof inapplicable as well. We will not be able to present a formal proof of the second law in dCS gravity here, however, what we can do is consider what happens to a black hole's entropy due to the effect of the Penrose process. In general relativity, the Penrose process serves as an informal proof of the second law. This will also be our strategy here. We will consider the effects of the Penrose process on the dCS corrected Kerr black hole. As we will see, one cannot decrease the entropy of a dCS black hole with the Penrose process, suggesting that the second law holds.

4.3.1 The Penrose Process

In the introduction to this chapter, we have already briefly discussed the Penrose process qualitatively, so let us now discuss it quantitively. We will do this in a general setting without initially making any appeal to a specific gravitational theory.

Consider a stationary axisymmetric spacetime with a coordinate system, (t, r, θ, ϕ), that is adapted to the spacetime symmetries. This means that we have two Killing vectors t^μ and ϕ^μ, corresponding with the time-translational and axial symmetry, that are the coordinate vectors of the t and ϕ coordinates respectively. We will assume this spacetime describes a rotating black hole with well-defined mass M and angular momentum J, and we will assume that the outer event horizon is a Killing horizon, generated by the Killing vector,

$$\xi^\mu = t^\mu + \Omega_{\rm H}\, \phi^\mu, \tag{4.36}$$

with $\Omega_{\rm H}$ the event horizon angular velocity, which must be a constant. We will assume that the event horizon is a surface of constant r. Lastly, we will assume there is an ergosphere in the exterior of the black hole i.e. the region where the Killing vector t^μ is spacelike.

Now consider a particle travelling on a geodesic inside the ergosphere with four-momentum p^μ. As the particle travels on a geodesic trajectory, the quantities,

$$\begin{aligned} dM &= -t^\mu p_\mu, \\ dJ &= \phi^\mu p_\mu, \end{aligned} \tag{4.37}$$

which are the energy and angular momentum of the particle respectively, are constants of motion. We will assume that these quantities are much smaller than the mass energy M and angular momentum J of the black hole. Since t^μ is spacelike in the ergosphere, the energy of the particle can be of either sign. The sign of the angular momentum of the particle depends on whether the particle moves with or against the rotation of the black hole. In the latter case, the angular momentum is negative. Thus, we can arrange for both the energy and angular momentum of the particle to be negative, then, when the particle falls into the black hole, the mass and angular momentum of the black hole decreases by dM and dJ respectively.

One may believe that this is the end of the story, however, it is a priori not clear that a particle with negative mass and angular momentum will eventually cross the event horizon in the future. Therefore, to make sure it does cross the event horizon, it might be a good idea to have an observer keep an eye on our particle.

A particularly useful set of observers are observers that co-rotate with the black hole at constant r and θ, while also having zero angular momentum, so-called zero angular momentum observers (ZAMOs). The four-velocity of such an observer is,

$$u^\mu = t^\mu + \Omega(r, \theta)\, \phi^\mu. \tag{4.38}$$

This four-velocity should really be normalized such that $u^\mu u_\mu = -1$, but that is not important here. The quantity $\Omega(r, \theta)$ is the angular velocity of the ZAMO, which is defined by,

$$\Omega(r,\theta) = -\frac{g_{t\phi}}{g_{\phi\phi}}, \tag{4.39}$$

which is largest near the horizon and vanishes at infinity.
Even though the conserved energy of the infalling particle is negative, any observer will always locally measure the energy of the particle to be positive. In particular, If we take this observer to be a ZAMO, we get the condition,

$$u^\mu p_\mu = t^\mu p_\mu + \Omega(r,\theta)\, \phi^\mu p_\mu = -dM + \Omega(r,\theta)\, dJ \geq 0, \tag{4.40}$$

which can also be written as,

$$dM \geq \Omega(r,\theta)\, dJ. \tag{4.41}$$

Hence, when the particle crosses the event horizon, a ZAMO near the horizon should measure the particle's energy to be positive. Near, or on the horizon we have,

$$\lim_{r\to r_H} \Omega(r,\theta) = \Omega_{\mathrm{H}}. \tag{4.42}$$

Thus, the condition for the particle to be able to cross the horizon is,

$$dM \geq \Omega_{\mathrm{H}}\, dJ. \tag{4.43}$$

When the particle crosses the event horizon, we assume that its mass and angular momentum adds to the mass and angular momentum of the black hole, which leaves us with a stationary black hole of mass $M + dM$ and angular momentum $J + dJ$.
If in this spacetime, the first law holds, we see, by using Eq. (4.43) in combination with the first law, that the entropy of the black hole must increase by the effects of the Penrose process,

$$dS = \frac{1}{T}\left(dM - \Omega_{\mathrm{H}}\, dJ\right) \geq 0. \tag{4.44}$$

As stated before, this is not a formal proof, but it does hint towards the second law being true in dCS gravity.

If the second law holds, it is possible to put constraints on the amount of energy that can be extracted from a single black hole, and it can also be used to put constraints on the amount of energy that can be converted into gravitational waves in a binary black hole merger event. This is what we will do now.

4.3.2 Extracting Energy from an Isolated Black Hole

An interesting question to ask is: how much energy can one extract from a single black hole by Penrose processes? The second law provides an answer. The most efficient way to extract energy from a black hole is by a process where the entropy of the black hole remains constant,

$$dS = 0 \to dM = \Omega_{\rm H}\, dJ. \tag{4.45}$$

We say that this is an ideal Penrose process. From this differential relation, we can calculate how the mass of the black hole changes as a function of its angular momentum. To make further calculations easier, we will work with the spin, χ instead of the angular momentum J. In terms of the spin, Eq. (4.45) becomes,

$$dM = \frac{M^2\Omega_{\rm H}}{1 - 2M\chi\Omega_{\rm H}} d\chi, \tag{4.46}$$

where we have used that $dJ = 2M\chi dM + M^2 d\chi$. At first glance, this does not seem to help us, but let us now consider what this differential relation looks like for the Kerr black hole. By using the explicit form of the event horizon angular velocity of the Kerr black hole as given by Eq. (1.39) we obtain,

$$\frac{M^2\Omega_{\rm H}^{\rm Kerr}}{1 - 2M\chi\Omega_{\rm H}^{\rm Kerr}} = \frac{M}{2}\cdot\frac{\chi}{1-\chi^2+\sqrt{1-\chi^2}}, \tag{4.47}$$

which gives the following differential relation between the mass and the spin of the black hole,

$$\frac{dM}{M} = \frac{1}{2}\cdot\frac{\chi\, d\chi}{1-\chi^2+\sqrt{1-\chi^2}}. \tag{4.48}$$

This differential relation is separable, hence solving for the mass of the black hole in terms of the spin is straightforward. We integrate both sides of Eq. (4.48) and take the exponential to obtain,

$$M(\chi) = M_0 \exp\left\{\frac{1}{2}\int_{\chi_0}^{\chi} d\chi' \frac{\chi'}{1-\chi'^2+\sqrt{1-\chi'^2}}\right\}, \tag{4.49}$$

here M_0 and χ_0 represent the initial mass and spin of the black hole.
Now all that is left to do is to perform the integration on the right-hand side of Eq. (4.49), which is straightforward as well. The final result then is,

$$M(\chi) = M_0 \left(\frac{1+\sqrt{1-\chi_0^2}}{1+\sqrt{1-\chi^2}}\right)^{1/2}. \tag{4.50}$$

This expression represents a curve of constant entropy, in the parameter space (M, χ) of the Kerr black hole. If we follow this curve, starting from the point (M_0, χ_0), we eventually reach the point $(M_{\rm irr}, 0)$, where the black hole has lost all of its spin, and we are left with a Schwarzschild black hole of mass,

$$M_{\rm irr} = M_0 \left(\frac{1+\sqrt{1-\chi_0^2}}{2}\right)^{1/2}, \tag{4.51}$$

which is known as the irreducible mass. Thus, by means of any process where $dS \geq 0$, a Kerr black hole with an initial mass and spin of (M_0, χ_0) can never have a mass that is less than $M_{\rm irr}$. Equivalently, we can say that a Kerr black hole with an initial mass and spin of (M_0, χ_0) can at most lose a fraction,

$$\Delta\mathcal{M}_{\rm BH} = 1 - \frac{M_{\rm irr}}{M_0} = 1 - \left(\frac{1+\sqrt{1-\chi_0^2}}{2}\right)^{1/2}, \tag{4.52}$$

of its mass. If we start with an extremal Kerr black hole, $\chi_0 = 1$, the maximal amount of mass that can be extracted is $\Delta\mathcal{M}_{\rm BH} = 1 - \frac{1}{\sqrt{2}}$, or about 29% of its initial mass.

In the case of the dCS corrected Kerr black hole, the same relation given by Eq. (4.46) holds between the mass and spin of the black hole. If we insert the explicit form of the horizon angular velocity of dCS corrected Kerr black hole given by Eq. (3.13), into Eq. (4.46), we obtain the differential equation,

$$\dot{M} + F(\chi, M, \alpha^2) = 0. \tag{4.53}$$

Here the dot denotes a derivative with respect to the spin parameter χ, and the function F is given by,

$$F(\chi, M, \alpha^2) = \frac{M^2(\Omega_{\rm H}^{\rm Kerr} + \frac{\alpha^2}{M^4}\delta\Omega_{\rm H})}{1 - 2M\chi(\Omega_{\rm H}^{\rm Kerr} + \frac{\alpha^2}{M^4}\delta\Omega_{\rm H})}. \tag{4.54}$$

Here we have used Eq. (3.13), and we have replaced ζ with α^2/M^4 to make the mass dependence of F clearer. Since we only know the metric of the dCS corrected Kerr black hole up to order $\mathcal{O}(\alpha^2)$, the differential equation (4.53) is only valid up to order $\mathcal{O}(\alpha^2)$, in which case, the leading order solution to (4.53) can be written as,

$$M_{\rm dCS}(\chi) = M_{\rm Kerr}(\chi) + \alpha^2\,\delta M(\chi). \tag{4.55}$$

We then insert this ansatz into Eq. (4.53), and expand to order $\mathcal{O}(\alpha^2)$. Equation (4.53) then becomes,

$$\dot{M}_{\rm Kerr} + F(\chi, M_{\rm Kerr}, 0) + \alpha^2\Big(\delta\dot{M} + \frac{\partial F(\chi, M_{\rm Kerr}, 0)}{\partial\alpha^2} + \frac{\partial F(\chi, M_{\rm Kerr}, 0)}{\partial M}\delta M\Big) = 0, \tag{4.56}$$

where F is evaluated at $(\chi, M_{\rm Kerr}, 0)$ after differentiating.

To solve Eq. (4.56) we separately have to solve the zeroth order $\mathcal{O}(\alpha^0)$ part and the leading order in coupling part $\mathcal{O}(\alpha^2)$. The zeroth order part of Eq. (4.56), is just the ideal Penrose process in the Kerr spacetime, which means that $M_{\rm Kerr}$ is given by Eq. (4.50). Then we have the leading order in coupling part, which will gives us the dCS correction to the ideal Penrose process,

$$\delta\dot{M} - \frac{M_{\rm Kerr}(\chi)^2\Omega_{\rm H}^{\rm Kerr}}{1 - 2M_{\rm Kerr}(\chi)\chi\Omega_{\rm H}^{\rm Kerr}}\delta M - \frac{M_{\rm Kerr}(\chi)^2\delta\Omega_{\rm H}}{(1 - 2M_{\rm Kerr}(\chi)\chi\Omega_{\rm H}^{\rm Kerr})^2} = 0, \tag{4.57}$$

where $\Omega_{\rm H}^{\rm Kerr}$ and $\delta\Omega_{\rm H}$ are evaluated at $M_{\rm Kerr}(\chi)$. This differential equation comes with the initial condition, $\delta M(\chi_0) = 0$, as the initial mass and spin of the black hole do not depend on the coupling parameter.
In addition to only being interested in the leading order in coupling solution, we are also limited by the fact that the metric we are using is only valid up to fourteenth order in spin. Thus, it only makes sense to look for a power series solution up to fourteenth order in spin. Obtaining a general solution is then straightforward, however a generic choice of initial spin χ_0 will lead us to having to solve for the roots of a fourteenth order polynomial to satisfy the initial condition. The only choice that avoids this problem is choosing $\chi_0 = 0$, which will give us the following solution,

$$\delta M(\chi) = -\frac{1}{M_0^3}\left(\frac{709}{3584}\chi^2 + \frac{8747}{86016}\chi^4 + \frac{5784463}{75694080}\chi^6 + \frac{1534220603}{23616552960}\chi^8 + \frac{9347435299}{160306298880}\chi^{10} + \frac{38736160452331}{719454669373440}\chi^{12} + \frac{55277078724807509}{1093571097447628800}\chi^{14}\right). \tag{4.58}$$

However, it now seems our initial black hole already has zero spin, so how can we extract any information from this? The solution to Eq. (4.56) represents a curve of constant entropy[20]. The initial conditions (M_0, χ_0) merely serve to pick out the exact value of the entropy of that curve, which means it does not really matter which initial conditions we pick. We can later just pick any point on that curve and choose that as the initial state of the black hole. The only downside to choosing $\chi_0 = 0$, instead of keeping it generic, is that we now no longer can find an explicit form of the irreducible mass as a function of the initial spin.

Equipped with the solutions to equation (4.58), we can now see how these curves of constant entropy look like for the Kerr case and the dCS corrected Kerr case. This is showcased in Fig. 9.
The point where the red and black curves cross represents the initial states of the black hole, thus we are considering a black hole that initially has moderately high spin of, $\chi_{\rm i} = 0.8$. The other points on the curves then represent the states that the initial black hole can reach in an idealized Penrose process. It is also important to note that these curves separate the parameter space of the black holes into two regions. From our initial black hole state, it is possible to reach any state above the curves. This means that starting from our initial Kerr or dCS black hole, we can pick any point on or above their respective constant entropy curve and there then exists a path in parameter space that connects these two initial and final state by a path along which $dS \geq 0$. The points below the curves cannot be reached, as they require that $dS < 0$.
From these constant entropy curves, it is also clear that the ideal Penrose process in dCS gravity is less efficient than the ideal Penrose process in GR, in the sense that you have to decrease the spin of the black hole by a greater amount in dCS gravity to achieve the same decrease in mass as compared to the Kerr case. This is, of course, directly related to the fact that the horizon angular velocity of the dCS corrected Kerr black hole is lower than

[20] As a sanity check, we have checked that $dS(M_{\rm dCS}(\chi), \chi)/d\chi = 0$ up to order $\mathcal{O}(\zeta\chi^{14})$.

the horizon angular velocity of the Kerr black hole, for a black hole with the same mass and spin. This can also be seen from Eq. (4.46) since this equation tells us that the slope of $M(\chi)$ decreases if one decreases Ω_{H}.

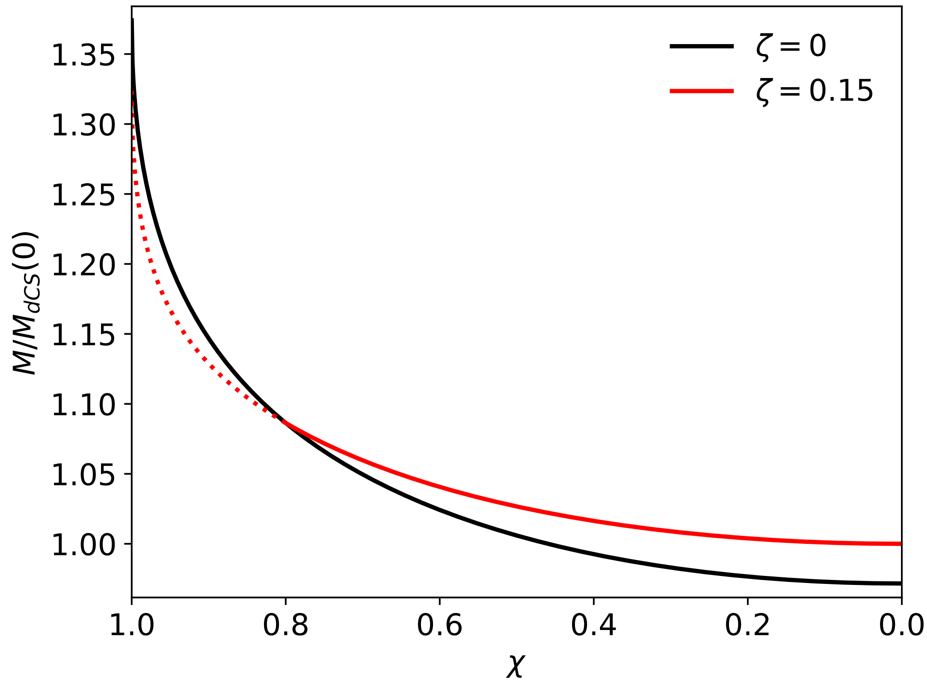


Figure 9: The ideal Penrose process, or the constant entropy curves starting from a black hole with a black hole of initial spin $\chi_i = 0.8$ and mass $M(\chi_i)$ in dCS gravity and GR.

With these constant entropy curves, we can now also determine what the upper bound is on the amount of mass that can be extracted from a single black hole in GR and in dCS. This upper bound is shown in Fig. 10.
Let us consider, in specific, a black hole of initial spin $\chi_i = 0.8$ again. In GR, the second law puts a bound on the extraction of mass of about 10.6% whereas in dCS this is only 7.9%, which is a difference of about 2.7% – quite significant. Of course, this difference very much depends on the value of the coupling constant.

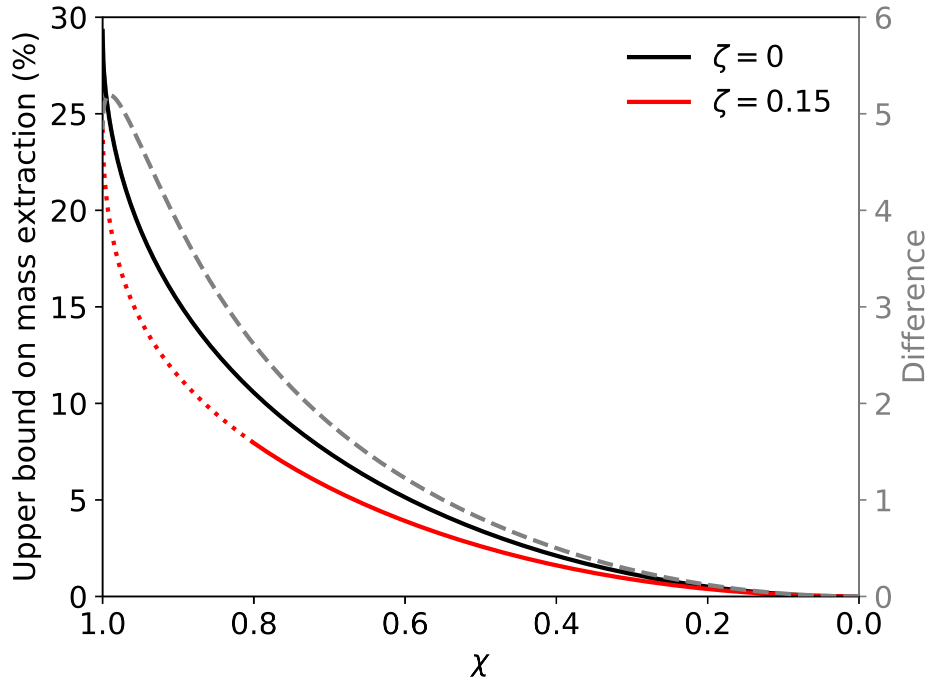


Figure 10: The upper bound on the fraction of the initial mass that can be extracted from a single black hole with a spin of χ, in dCS and in GR. The dashed grey line indicates the difference between the two.

4.3.3 The Second Law and Binary Black Hole Mergers

Another interesting setting to discuss is binary black hole mergers. If the second law holds in a general setting, then the remnant black hole of a binary black hole merger event should have an entropy of at least the sum of the entropies of the two black holes prior to merging,

$$S_{\text{rem}} \geq S_1 + S_2. \tag{4.59}$$

It would then be interesting to consider how much mass energy could be extracted from the merger event. This energy would then be converted into gravitational waves. The maximal amount of energy will be extracted when the entropy of the remnant is equal to the sum of the entropies of the merging black holes,

$$S_{\text{rem}} = S_1 + S_2. \tag{4.60}$$

The number of parameters in this situation is six, as we now have three black holes. However, if we wish to calculate the true upper bound on the amount of energy extracted, we can reduce the number of free parameters to two.

First of all, we may assume that the remnant is a Schwarzschild black hole, in which case the entropy of the remnant black hole is,

$$S_{\text{rem}} = 4\pi M_{\text{rem}}^2. \tag{4.61}$$

If the remnant black hole had non-zero spin it could still lose mass via the Penrose process, so fixing the spin of the remnant to be zero gives us the largest upper bound.
Then, the smallest possible mass of the remnant in terms of the entropy of the merging black holes would be,

$$\begin{aligned} 4\pi M_{\text{rem}}^2 &= S_1(M_1,\chi_1) + S_2(M_2,\chi_2), \\ &= S_1^{\text{Kerr}} + S_2^{\text{Kerr}} + \frac{\alpha^2}{M_1^4}\delta S_1^{\text{dCS}} + \frac{\alpha^2}{M_2^4}\delta S_2^{\text{dCS}}. \end{aligned} \tag{4.62}$$

If we choose the spin of the merging black holes to be as large as possible, we maximize the mass difference, $M_1 + M_2 - M_{\text{rem}}$, thus we will choose the merging black holes to have the same spin, and as high as possible. Keeping the restriction that we will not be considering spins higher than 0.8 for the dCS corrected Kerr black hole, we will set $\chi_1 = \chi_2 = 0.8$. We will also define $\bar{M} = \frac{M_2}{M_1} \geq 1$, and $\zeta_1 = \frac{\alpha^2}{M_1^4}$. We will also strip all dimensionful parameters from the entropies of the merging black holes by defining,

$$\begin{aligned} S_i^{\text{Kerr}} &= M_i^2 \bar{S}_i^{\text{Kerr}}, \\ \delta S_i^{\text{dCS}} &= M_i^2 \delta\bar{S}_i^{\text{dCS}}. \end{aligned} \tag{4.63}$$

After some algebra, we can then rewrite Eq. (4.62) as,

$$4\pi\left(\frac{M_{\text{rem}}}{M_1}\right)^2 = \left((1+\bar{M}^2)\bar{S}^{\text{GR}}(\chi) + \zeta_1(1+\frac{1}{\bar{M}^2})\delta\bar{S}^{\text{dCS}}(\chi)\right). \tag{4.64}$$

This equation still has three masses, however the fractional difference between the total mass before the merging and the remnant mass is only a function of the ratio of the masses of the merging black holes,

$$\Delta\mathcal{M}_{\text{BH-BH}} = 1 - \frac{M_{\text{rem}}}{M_1+M_2} = 1 - \left(\frac{M_{\text{rem}}}{M_1}\right)\frac{1}{(1+\bar{M})}, \tag{4.65}$$

where M_{rem}/M_1 is obtained from Eq. (4.64). This fractional difference is shown in Fig. 11. We see a clear difference between the bound set by the GR case and the dCS case when the masses of the black holes are similar. In this mass ratio regime, the upper bound differs by a few percent. Another noticeable difference is that the upper bound is highest in GR when the black holes have the same mass, but in dCS the upper bound is highest when one of the black holes is slightly more massive than the other.
As the mass ratio increases, the upper bound on the change in mass becomes smaller and smaller, as does the difference between GR and dCS. This is due to the fact that the Kerr part of the entropy of the merging black holes scales as, $\bar{M}^2$ whereas the dCS part of the entropy scales as, $\bar{M}^{-2}$. Hence, ultimately, the second law does not put a significantly different upper bound on the mass extraction unless the merging black holes have a similar mass. When the mass ratio is large, there is almost no difference between GR and dCS, even though we have taken the dCS coupling constant to be quite large.

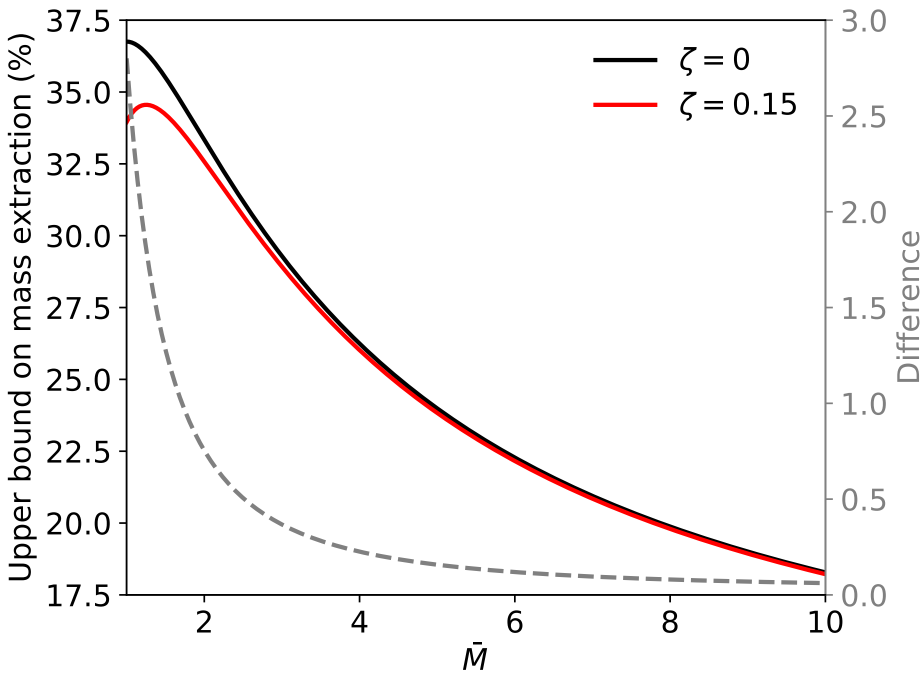


Figure 11: The upper bound percentage of total mass energy that can be converted to gravitational waves in a BH-BH merger event in dCS gravity and GR. The dashed grey line indicates the difference between the two.

4.4 The Third Law of Black Hole Mechanics

The third law of black hole mechanics states:

> *It is impossible to reduce the surface gravity of a black hole to zero by means of any process, no matter how idealized, by finitely many operations.*

The third law of black hole mechanics mirrors the third law of thermodynamics, which states the same for the temperature of a thermodynamic system. Of course, as we have seen, we may interpret the surface gravity as the temperature of the black hole, making the analogy even stronger. But what exactly is meant by finitely many operations? To make sense of what a finite operation is, we must take the dynamics of the black hole into account, in which case, we cannot define the surface gravity any more, so we cannot infer what happens to the surface gravity in a dynamical setting.
The third law is a delicate matter, which was also noted by Bardeen, Carter, and Hawking, as they posed the third law without proof in [7]. We can, however, get an idea of how the third law works by considering some examples.
Take the Schwarzschild black hole, the surface gravity is given by,

$$\kappa_{\text{Schw}} = \frac{1}{4M}. \tag{4.66}$$

To reach zero surface gravity, one must add an infinite amount of mass to the black hole, which is not possible in finitely many operations. For the Kerr black hole, the situation is a bit more complex. The surface gravity is,

$$\kappa_{\text{Kerr}} = \frac{\sqrt{1-\chi^2}}{2M(1+\sqrt{1-\chi^2})}. \tag{4.67}$$

It is clear that in this case we also cannot reduce the surface gravity in finitely many steps by just adding mass, however if we can let the spin of the black hole approach unity, the surface gravity vanishes. Thus, to reduce the surface gravity to zero, we would need to spin up the black hole until it becomes extremal.

The third law thus deals with (near) extremal black holes, which is a setting in which our solutions for the rotating dCS black hole are not valid, thus no concrete statements can be made regarding the third law in this work. There have, however, been attempts to find solutions to the dCS equations for near-extremal black holes, but as of now, no solutions of that type exist [38].

Discussion and Conclusions

In this thesis, we have used the most accurate dCS corrections to the Kerr metric [6] that are currently available to study the properties of rotating black holes in dynamical Chern-Simons gravity. With these corrections, it became possible to probe moderately fast rotating black holes, whereas previously, it was only possible to study slowly rotating black holes.

Our main goal was to study the four laws of black hole mechanics in the context of dynamical Chern-Simons gravity. We verified the zeroth law by explicitly checking that the surface gravity of the event horizon is constant. This also revealed to us that the dCS corrections tend to increase the temperature of the event horizon, which can be explained by the fact that the horizon angular velocity, which is directly related to the surface gravity, is lowered by the dCS corrections.
To study the first law, we calculated the explicit form of the Wald entropy in dCS gravity. Although one would a priori expect that the first law holds in dCS with Wald's definition of entropy, we nonetheless evaluated the Wald entropy with the dCS corrected Kerr metric, which allowed us to verify the first law with a direct calculation. This showed us that the corrections to the Bekenstein-Hawking entropy coming from the Wald entropy tend to increase with the spin of the black hole, which is in contrast with the Bekenstein-Hawking entropy of the black hole, which decreases with spin.
We studied the second law by considering the Penrose process. Although this analysis does not constitute a formal proof, it did hint towards the second law holding true in dCS.
We then calculated the upper bound on the amount of energy that can be extracted from a single rotating black hole under the assumption that the second law holds. We found that due to the dCS corrections, less energy can be extracted from a rotating black hole. This is again directly related to the angular velocity of the event horizon being lowered by the dCS corrections.
We also considered the upper bound on the amount of energy that can be released in a binary black hole merger event, assuming that the second law holds. In this situation, we also found that less energy can be released due to the dCS corrections, however the difference between the dCS and GR upper bound very quickly approaches zero when the mass ratio of the black holes is large.
We were unable to do any meaningful work on the third law, as this requires studying the behaviour of the black hole in the extremal limit, which is currently not possible with the available dCS corrections to the Kerr metric.

While reviewing and checking the already known properties of the dCS corrected Kerr metric, we discovered that event horizon and axis of symmetry are algebraically special surfaces of the dCS corrected Kerr spacetime, in the sense that on these surfaces the local Petrov type is D, whereas it is type I everywhere else in the exterior of the black hole. Since the local Petrov type is an invariant, this means that one might be able to locally detect the event horizon by determining the local Petrov type. This is not possible in GR, where stationary black holes are type D everywhere.

Since the dCS corrected Kerr metric discovered in [6] is only an approximate solution to the dCS equations, all the calculations that are done in this thesis are only valid up to the accuracy of the approximate solution. This is a clear limitation of the results. However, since it is unlikely that there exist exact closed form solutions describing rotating black holes in dCS gravity, this will remain a limitation.

Outlook

For future work, it would be interesting to be able to probe dCS corrected Kerr black holes near the extremal limit. We would then be able to work on the third law of black hole mechanics, but it would also be interesting and useful in general to study the properties of dCS corrected Kerr black holes in the extremal limit. To this end one could for instance extend the corrections from [6], which has already been partly done in [20], or one could start from the ground up and solve the dCS equations on an extremal Kerr background, which also was already partly done in [38].

It would also be interesting to study dynamical perturbations of the dCS corrected Kerr black hole to better study the second law of black hole mechanics.

An obvious extension to work done here would be to repeat the calculations done in this thesis for other modified theories of gravity. The solution found in [6] also includes corrections to the Kerr metric due to the Einstein-dilaton Gauss-Bonnet extension to the Einstein-Hilbert action, as well as corrections due to cubic curvature terms. Extending the calculations done in this thesis to include these additional corrections to the Kerr metric should be straightforward.

References

[1] J-P Luminet. “Image of a spherical black hole with thin accretion disk”. In: *Astronomy and Astrophysics, vol. 75, no. 1-2, May 1979, p. 228-235.* 75 (1979), pp. 228–235.

[2] Kazunori Akiyama et al. “First M87 event horizon telescope results. IV. Imaging the central supermassive black hole”. In: *The Astrophysical Journal Letters* 875.1 (2019), p. L4.

[3] Stanley Deser, Roman Jackiw, and S Templeton. “Three-dimensional massive gauge theories”. In: *Physical Review Letters* 48.15 (1982), p. 975.

[4] Stephon Alexander and Nicolas Yunes. “Chern–Simons modified general relativity”. In: *Physics Reports* 480.1-2 (2009), pp. 1–55.

[5] Nicolas Yunes and Frans Pretorius. “Dynamical Chern-Simons modified gravity: Spinning black holes in the slow-rotation approximation”. In: *Physical Review D* 79.8 (2009), p. 084043.

[6] Pablo A Cano and Alejandro Ruipérez. “Leading higher-derivative corrections to Kerr geometry”. In: *Journal of High Energy Physics* 2019.5 (2019), pp. 1–50.

[7] James M Bardeen, Brandon Carter, and Stephen W Hawking. “The four laws of black hole mechanics”. In: *Communications in mathematical physics* 31 (1973), pp. 161–170.

[8] Sean M Carroll. *Spacetime and geometry*. Cambridge University Press, 2019.

[9] Robert M Wald. *General relativity*. University of Chicago press, 2010.

[10] Eric Poisson. *A relativist’s toolkit: the mathematics of black-hole mechanics*. Cambridge university press, 2004.

[11] Erik Curiel. “The many definitions of a black hole”. In: *Nature Astronomy* 3.1 (2019), pp. 27–34.

[12] Karl Schwarzschild. “Über das gravitationsfeld eines massenpunktes nach der einsteinschen theorie”. In: *Sitzungsberichte der königlich preussischen Akademie der Wissenschaften* (1916), pp. 189–196.

[13] Johannes Droste. “The field of a single centre in Einstein’s theory of gravitation, and the motion of a particle in that field”. In: *Ned. Acad. Wet., SA* 19 (1917), p. 197.

[14] Roy P. Kerr. “Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics”. In: *Phys. Rev. Lett.* 11 (5 Sept. 1963), pp. 237–238.

[15] Saul A Teukolsky. “The kerr metric”. In: *Classical and Quantum Gravity* 32.12 (2015), p. 124006.

[16] Brandon Carter. “Global structure of the Kerr family of gravitational fields”. In: *Physical Review* 174.5 (1968), p. 1559.

[17] Stephen W Hawking and George FR Ellis. *The large scale structure of space-time*. Cambridge university press, 2023.

[18] Tristan L Smith et al. “Effects of Chern-Simons gravity on bodies orbiting the Earth”. In: *Physical Review D* 77.2 (2008), p. 024015.

[19] Hector O Silva et al. “Astrophysical and theoretical physics implications from multimessenger neutron star observations”. In: *Physical review letters* 126.18 (2021), p. 181101.

[20] Pablo A Cano, Alexander Deich, and Nicolás Yunes. “Accuracy of the slow-rotation approximation for black holes in modified gravity in light of astrophysical observables”. In: *arXiv preprint arXiv:2305.15341* (2023).

[21] Kent Yagi, Nicolas Yunes, and Takahiro Tanaka. “Slowly rotating black holes in dynamical Chern-Simons gravity: Deformation quadratic in the spin”. In: *Physical Review D* 86.4 (2012), p. 044037.

[22] Caroline B Owen, Nicolás Yunes, and Helvi Witek. “Petrov type, principal null directions, and Killing tensors of slowly rotating black holes in quadratic gravity”. In: *Physical Review D* 103.12 (2021), p. 124057.

[23] Nikos Chatzifotis et al. “Scalarization of Chern-Simons-Kerr black hole solutions and wormholes”. In: *Physical Review D* 105.8 (2022), p. 084051.

[24] Aleksei Zinovevich Petrov. *Einstein spaces*. Elsevier, 2016.

[25] Martin Walker and Roger Penrose. “On quadratic first integrals of the geodesic equations for type {22} spacetimes”. In: *Communications in Mathematical Physics* 18 (1970), pp. 265–274.

[26] Saul A Teukolsky. “Perturbations of a rotating black hole. I. Fundamental equations for gravitational, electromagnetic, and neutrino-field perturbations”. In: *Astrophysical Journal, Vol. 185, pp. 635-648 (1973)* 185 (1973), pp. 635–648.

[27] Subrahmanyan Chandrasekhar. *The mathematical theory of black holes*. Vol. 69. Oxford university press, 1998.

[28] Dimitrios Papadopoulos and Basilis C Xanthopoulos. “Local black holes are type D on the horizon.” In: *Nuovo Cimento B Serie* 83 (1984), pp. 113–126.

[29] IV Tanatarov and OB Zaslavskii. “What happens to Petrov classification, on horizons of axisymmetric dirty black holes”. In: *Journal of Mathematical Physics* 55.2 (2014).

[30] Carla Rodrigues Almeida. “The thermodynamics of black holes: from Penrose process to Hawking radiation”. In: *The European Physical Journal H* 46.1 (2021), p. 20.

[31] Roger Penrose and RM Floyd. “Extraction of rotational energy from a black hole”. In: *Nature Physical Science* 229.6 (1971), pp. 177–179.

[32] Demetrios Christodoulou. “Reversible and irreversible transformations in black-hole physics”. In: *Physical Review Letters* 25.22 (1970), p. 1596.

[33] Jacob D Bekenstein. “Black holes and entropy”. In: *Physical Review D* 7.8 (1973), p. 2333.

[34] Stephen W Hawking. “Particle creation by black holes”. In: *Communications in mathematical physics* 43.3 (1975), pp. 199–220.

[35] Istvan Râcz and Robert M Wald. “Extensions of spacetimes with Killing horizons”. In: *Classical and Quantum Gravity* 9.12 (1992), p. 2643.

[36] Robert M Wald. “Black hole entropy is the Noether charge”. In: *Physical Review D* 48.8 (1993), R3427.

[37] Ted Jacobson, Gungwon Kang, and Robert C Myers. “Increase of black hole entropy in higher curvature gravity”. In: *Physical Review D* 52.6 (1995), p. 3518.

[38] Robert McNees, Leo C Stein, and Nicolás Yunes. “Extremal black holes in dynamical Chern–Simons gravity”. In: *Classical and Quantum Gravity* 33.23 (2016), p. 235013.

A Equivalent Expressions of the Cotton Tensor

In this appendix, we show that the Cotton tensor can be equivalently written as,

$$C_{\mu\nu} = \nabla^{\beta}\nabla^{\alpha}\Big[\,{}^{*}R_{\alpha(\mu\nu)\beta}\,\vartheta\Big] = \epsilon^{\gamma\beta}{}_{\alpha(\mu|}\nabla_{\beta}R_{|\nu)\gamma}\nabla^{\alpha}\vartheta + {}^{*}R_{\alpha(\mu\nu)\beta}\,\nabla^{\beta}\nabla^{\alpha}\vartheta. \tag{A.1}$$

Starting with the expression,

$$C_{\mu\nu} = \nabla^{\beta}\nabla^{\alpha}\Big[\,{}^{*}R_{\alpha(\mu\nu)\beta}\,\vartheta\Big], \tag{A.2}$$

we first distribute the covariant derivatives to obtain,

$$C_{\mu\nu} = \Big[\nabla^{\alpha}\vartheta\nabla^{\beta} + \big[\nabla^{\beta}\vartheta + \vartheta\nabla^{\beta}\big]\nabla^{\alpha} + \nabla^{\beta}\nabla^{\alpha}\vartheta\Big]\,{}^{*}R_{\alpha(\mu\nu)\beta}. \tag{A.3}$$

The first term can be rewritten as,

$$\nabla^{\beta}\,{}^{*}R_{\alpha\mu\nu\beta} = \frac{1}{2}\epsilon_{\alpha\mu}{}^{\gamma\delta}\nabla_{\beta}R_{\gamma\delta\nu}{}^{\beta} = \frac{1}{2}\epsilon_{\alpha\mu}{}^{\gamma\delta}\Big[\nabla_{\delta}R_{\gamma\nu} - \nabla_{\gamma}R_{\delta\nu}\Big] = \epsilon_{\alpha\mu}{}^{\gamma\delta}\nabla_{\delta}R_{\gamma\nu}. \tag{A.4}$$

Here we used that the covariant derivative of the Levi-Civita tensor vanishes due to metric compatibility of the Levi-Civita connection,

$$\nabla_{\mu}\,\epsilon_{\alpha\beta\gamma\delta} = 0. \tag{A.5}$$

We also used the single contracted differential Bianchi identity,

$$\nabla_{\mu}R_{\alpha\beta\nu}{}^{\mu} = \nabla_{\beta}R_{\alpha\nu} - \nabla_{\alpha}R_{\beta\nu}. \tag{A.6}$$

The second term vanishes due to the fact that the dual Riemann tensor is divergenceless on the left two indices,

$$\nabla^{\alpha}\,{}^{*}R_{\alpha\mu\nu\beta} = \frac{1}{2}\nabla_{\alpha}\,\epsilon^{\alpha}{}_{\mu}{}^{\gamma\delta}R_{\gamma\delta\nu\beta} = -\frac{1}{2}\epsilon_{\mu}{}^{\alpha\gamma\delta}\nabla_{\alpha}R_{\gamma\delta\nu\beta} = -\frac{1}{2}\epsilon_{\mu}{}^{\alpha\gamma\delta}\nabla_{[\alpha}R_{\gamma\delta]\nu\beta} = 0. \tag{A.7}$$

Here we used the differential Bianchi identity in the last step,

$$\nabla_{[\alpha}R_{\beta\gamma]\mu\nu} = 0. \tag{A.8}$$

Although not important here, it can be shown analogously that the dual Riemann tensor is also divergenceless on the second index. Since $\nabla^{\alpha}\,{}^{*}R_{\alpha\mu\nu\beta}$ vanishes we have that,

$$C_{\mu\nu} = \nabla^{\alpha}\vartheta\nabla^{\beta}\,{}^{*}R_{\alpha(\mu\nu)\beta} + {}^{*}R_{\alpha(\mu\nu)\beta}\nabla^{\beta}\nabla^{\alpha}\vartheta. \tag{A.9}$$

Which, using (A.4) can be written as,

$$C_{\mu\nu} = \epsilon^{\gamma\beta}{}_{\alpha(\mu|}\nabla_{\beta}R_{|\nu)\gamma}\nabla^{\alpha}\vartheta + {}^{*}R_{\alpha(\mu\nu)\beta}\,\nabla^{\beta}\nabla^{\alpha}\vartheta, \tag{A.10}$$

Which concludes the proof.

B Explicit Metric Corrections and Axion Field of the dCS Corrected Kerr Metric

In this appendix, we present the solutions of the metric corrections $\{H_1, H_2, H_3, H_4\}$, and the axion field ϑ up to order $\mathcal{O}(\chi^3)$. For convenience, we have set the mass M to unity, however one can reimplement the mass by making the substitutions,

$$\begin{aligned}\vartheta(r,z) &\to \frac{1}{M^2}\vartheta(r/M,z),\\ H_i(r,z) &\to \frac{1}{M^4}H_i(r/M,z).\end{aligned} \tag{B.1}$$

The explicit form of the metric corrections and the axion field are:

$$\begin{aligned}\frac{\vartheta(r,z)}{\sqrt{\zeta}} =&\chi\left(-\frac{9z}{r^4}-\frac{5z}{r^3}-\frac{5z}{2r^2}\right)+\\ &\chi^3\left(\frac{100z^3}{3r^6}+\frac{12z^3}{r^5}+\frac{2z}{5r^5}+\frac{3z^3}{r^4}+\frac{3z}{5r^4}+\frac{z}{2r^3}+\frac{z}{4r^2}\right)+\mathcal{O}(\chi^5),\end{aligned} \tag{B.2}$$

$$\begin{aligned}H_1(r,z) = \chi^2\Bigg(&\frac{342z^2}{r^9}-\frac{9279z^2}{637r^8}-\frac{20268}{637r^8}-\frac{19280z^2}{1001r^7}-\frac{11710}{637r^7}-\frac{1094689z^2}{42042r^6}-\\ &\frac{30707}{3234r^6}+\frac{298393z^2}{84084r^5}+\frac{1074}{7007r^5}+\frac{80291z^2}{24024r^4}-\frac{271}{12012r^4}+\\ &\frac{80291z^2}{24024r^3}-\frac{271}{12012r^3}+\frac{72185}{48048r^2}-\frac{72185}{48048r}\Bigg)+\mathcal{O}(\chi^4),\end{aligned} \tag{B.3}$$

$$\begin{aligned}H_2(r,z) = &\left(-\frac{27}{2r^5}-\frac{60}{7r^4}-\frac{5}{r^3}\right)+\chi^2\Bigg(\frac{171z^2}{r^8}+\frac{81219z^2}{637r^7}-\frac{10134}{637r^7}+\frac{4701743z^2}{126126r^6}-\\ &\frac{447949}{57330r^6}-\frac{32689z^2}{5544r^5}-\frac{564161}{194040r^5}-\frac{1852791z^2}{224224r^4}+\frac{154675}{96096r^4}-\frac{3310225z^2}{1345344r^3}+\\ &\frac{1153277}{1345344r^3}+\frac{462029z^2}{672672r^2}+\frac{457841}{3363360r^2}+\frac{72185}{96096r}-\frac{72185}{96096}\Bigg)+\mathcal{O}(\chi^4),\end{aligned} \tag{B.4}$$

$$\begin{aligned}H_3(r,z) = \chi^2\Bigg(&-\frac{99z^2}{r^8}-\frac{57843z^2}{1274r^7}+\frac{639}{1274r^7}-\frac{455055z^2}{28028r^6}+\frac{2005}{2548r^6}+\\ &\frac{5891z^2}{3822r^5}+\frac{41549}{42042r^5}+\frac{3425z^2}{8008r^4}+\frac{8581}{12012r^4}-\frac{2969z^2}{4004r^3}+\\ &\frac{887}{1716r^3}-\frac{14015z^2}{6864r^2}+\frac{270}{1001r^2}-\frac{72185}{48048r}+\frac{72185}{48048}\Bigg)+\mathcal{O}(\chi^4),\end{aligned} \tag{B.5}$$

$$H_4(r,z) = \chi^2 \left(-\frac{45z^2}{r^8} - \frac{54}{r^8} - \frac{705z^2}{637r^7} - \frac{27897}{637r^7} + \frac{234445z^2}{14014r^6} - \frac{40995}{1274r^6} + \frac{294806z^2}{21021r^5} - \frac{18587}{1617r^5} + \frac{183353z^2}{24024r^4} - \frac{12993}{2002r^4} + \frac{80291z^2}{24024r^3} - \frac{12241}{3432r^3} - \frac{85145}{48048r^2} - \frac{72185}{48048r} + \frac{72185}{48048} \right) + \mathcal{O}(\chi^4). \tag{B.6}$$

C xAct Mathematica Package

xAct is a package for Wolfram Mathematica developed by Jose M. Martin-Garcia that is specifically geared towards doing analytical tensor calculus in general relativity and differential geometry. The package was first released publicly in 2004 and since has been constantly tested and improved. The package can be found on `http://www.xact.es/`. Here one can also find installation instructions, example notebooks, and more. There is also an active public Google Groups specifically for discussing all things related to *xAct*.

Without the *xAct* package, this thesis would not have been possible, as all calculations with the dCS corrected Kerr metric have been done with the *xAct* package. All of these calculations have been collected into a single notebook, which is available on Github.