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Finding Generalized Cohomologies from Supersymmetric Field Theories

A Review of the Stolz-Teichner Program

Master Thesis

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Abstract

Generalized Cohomology is a topic in Algebraic Topology. Field Theories are prominent in Theoretical Physics, with connections to the mathematical notions of Topological Field Theories. Supersymmetry can be added to the Field Theories. We will build bridges between these topics. We will introduce supermanifolds and stacks. Using those, we will define suitable bordism categories on which we define Supersymmetric Field Theories. We will see that the Field Theories are a geometric construction of some Generalized Cohomology Theories. We will construct ordinary cohomology from $0|1$-dimensional Field Theories and complexified K-theory and complexified tmf from $1|1$ and $2|1$-dimensional Field Theories respectively. In these constructions, we aim to keep the dimensions general. In particular, we are able to relate even dimensional Field Theories to Siegel Modular forms.

Contents

1 Introduction

Mathematics and Theoretical Physics are tied together with many strings. One such connection is the topic of this thesis. We will investigate how we can obtain Generalized Cohomologies, a topic in Algebraic Topology, from Supersymmetric Field Theories, a topic in Theoretical Physics. Related to the physics of field theories, there is also the mathematical notion of Topological Field Theories or more generally Functorial. These will also play a role in our studies.

A *Generalized Cohomology* is a functor from some category of topological spaces to the category of graded Abelian groups satisfying axioms on homotopy invariance, exactness and excision, see Definition B.1. Compared to ordinary (singular) cohomology, the groups associated to the one point space need not be concentrated in degree zero. Examples of generalized cohomologies include K-theory and elliptic cohomology.

Generalized cohomology theories have vast applications in Theoretical Physics. For example, K-theory can be used to classify D-branes in string theory, [BLT13, Chapter 9.5]. D-branes are objects in string theory where open strings with Dirichlet boundary conditions can end. Conversely, the physical theories can help us find the correct mathematical definitions. We will see this a few times in this thesis too.

The physical field theories we will consider go by the name of *Sigma Models*. These are theories of maps from some domain (commonly called a world line and worldsheet if the dimension is 1 or 2) to some background space X. Supersymmetry comes in when lifting the domain to a supermanifold. The mathematical notion of field theories is in terms of bordism categories. Bordism categories are categories with objects the compact manifolds without boundary and morphisms the manifolds whose boundary corresponds to the source and target object. The composition operation is defined in terms of gluing along the boundaries. The bordisms can be given the extra datum of a map to a background X. Bordism categories are given a symmetric monoidal structure by the disjoint union operation. Functorial field theories are now symmetric monoidal functors from a bordism category to the category of vector spaces. The connection to physical field theories lies precisely in the maps from the bordism to X, which can be interpreted as fields in the relevant Sigma Model.

Sigma Models themselves were first proposed in [GL60] when studying beta decay of pion. Since then, many Sigma Models have been constructed, see e.g., [För77; Jev77; DLD78]. A particular example of a sigma model is String Theory, we embed vibrating strings in a space-time background.

Connecting generalized cohomologies to field theories is an old subject. Already, Edward Witten [Wit87] and Greame Segal [Seg88] tried to connect field theories to elliptic cohomology. The connection between generalized cohomologies and field theories taken here is more modern and goes by the name of the *Stolz-Teichner Program*. This is named after the mathematicians Stephan Stolz and Peter Teichner. Their main objective it to give geometric constructions of generalized cohomology theories in terms of suitable field theories. In their papers [ST04], [HST10], [Hoh+11] and [ST11], they construct suitable bordism categories and obtain singular cohomology for dimension $0|1$ and K-theory in dimension $1|1$. They conjecture that integral elliptic cohomology can be obtained from the $2|1$-dimensional field theories.

The conjecture remains open to date. However, when taking complexified coordinates, it has been shown that tmf, the universal case of elliptic cohomology, can be obtained from $2|1$-dimensional field theories in the style of the Stolz-Teichner program [Ber13a]. In the same paper, complexified K-theory is obtained from $1|1$-dimensional field theories. Studies of $0|2$-dimensional and $1|2$ are done in similar fashion in [Ber13b] and [Ber15] respectively. Equivariant versions are considered in [Ber20].

A different connection between the field theories and cohomologies is due to Kevin Costello [Cos10a; Cos10b], where he constructs the Witten Genus from field theories. One can view Costello's approach as the Hamiltonian equivalent of the Lagrangian based theory studied here.

Considering and applying the methods used in the Stolz-Teichner program is one of the main goals of this thesis. We will present the results of the $0|1$ dimensions and complexified results of the $1|1$ and $2|1$-dimensional case.

We start in Chapters 2 and 3 by introducing the supermanifolds and stacks. The former are the objects which mathematically encode supersymmetry. The latter provide us with a suitable categorical language. Using this framework in Chapter 4, we define supersymmetric bordism categories and field theories in both the mathematical and the physical sense. Already here, we will see that, following [Hoh+11], we can obtain De Rham cohomology from the $0|1$-dimensional field theories. Lastly, in Chapter 5, we consider the field theories in a perturbative sense, by just considering (higher dimensional) tori. This already allows us, using some physical motivation obtained from the physical field theories and following [Ber13a], to construct complexified K-theory and tmf from $1|1$ and $2|1$-dimensional field theories respectively. Moreover, we construct the relevant genera in the given set-up.

Compared to [Hoh+11] and [Ber13a], we aim to keep the story general for as long as possible. Explicitly, we work in arbitrary dimensions and more general structures on the bordisms. The stacks of tori constructed in [Ber13a] in dimensions $1|1$ and $2|1$ with Spin structure group are generalized to higher dimensions and arbitrary struture groups in Chapter 5.2. Moreover, we explicitly show that the constructed objects are differentiable stacks. In [Hoh+11], field theories for $0|q$ are defined, but only the case $q = 1$ is analyzed in detail. We will treat arbitrary q in Theorem 4.32. In [Ber13a], the dimensions $1|1$ and $2|1$ are considered with a specific structure group (the Spin group). We will, using the same methods, generalize to arbitrary dimensions and structures in Theorem 5.29. The original results are recovered with specializing the dimensions and structures. However, the more general view allows us to make extentions. In particular, we obtain a connection to Siegel Modular Forms, Corollary 5.30, in the even bosonic dimensions.

The introduction to supermanifolds, Chapter 2, contains an extensive treatment of the superpoint, its symmetries and their connection to the odd tangent bundles. These turn out to be important objects in the study of field theories. Especially, we consider an arbitrary number of odd dimensions, while in most literature only 1 or 2 dimensions are considered. Most other notions introduced in this section are well-known objects in the realm of supergeometry. We collected them here.

In Chapter 3, we tell the, in a sense folklore, story of stacks in considerable detail. We build up a connection between (differentiable) stacks and (super) Lie groupoids. Such a connection is made in various sources including [BX06; Hei05; Blo07].

The bordism categories on which Functorial Field Theories are defined are build up step by step in Chapter 4.1. We succesively upgrade the notion of bordims by considering Model Geometries, family versions, maps to a background space and supersymmetry. This allows us to see the effect and nuances of each of these additions. The physics of certain Sigma Models is summarized in Chapter 4.3 is mostly due to Witten, [Del+99; Wit87].

This project started with the idea to study the Witten Genus and its properties. To see its relations to Generalized Cohomologies, Bordisms and Sigma Models and see what physical arguments imply for the mathematical definitions and vice versa. During the project, the focus shifted more to studying the relations between Generalized Cohomologies, Bordisms and Sigma Models itself. The Witten Genus is however still constructed in the end of the thesis.

I would like to acknowledge my supervisors Dr. F.L.M Meier and Dr. T.W. Grimm for their suggestions which put me on the right track on various occasions, but also for allowing me the freedom to head in directions I was interested in. I wish to express my gratitude to my fellow students Leon Goertz, Luuk Lagendijk, Jaime Pastor, Tom Vredenbregt and Bas Wensink for taking part in a self-hosted seminar giving the opportunity to study the relevant physics in this thesis together.

2 The Super Side of Things

Supersymmetry will play an important role throughout this thesis. Therefore, we have this chapter devoted to introducing supersymmetry. The study of supersymmetry dates back to Hironari Miyazawa with [Miy66], where he studies symmetry transformation between baryons and mesons in particle physics. After the papers of Miyazawa initially being ignored, supersymmetry has become an important tool in theoretical physics. The symmetry manifest itself by introducing two types of variables: Regular, commuting, ones and anti-commuting ones (Grassman variables). Physicist will, as physicist do, use these notations and compute with them. While this viewpoint works in most cases, it is not completely rigorous in the mathematical sense. We will give a fully rigorous mathematical definition of superspaces.

Throughout this chapter, we will lift notions well-known for ordinary (non-super) mathematics to the case with supersymmetry. We will assume the reader is familiar with standard differential geometry. Prior knowledge of supersymmetry is not required.

The structure of this chapter is as follows. We start by the basics of Super Linear Algebra following [Var04, Chapter 3] and [DM99, Section 1]. Using this setup, we are able to generalize ordinary manifolds to supermanifolds following [DM99, Section 2]. Here, we also see how the physical picture on supersymmetry of (anti)-commuting variables is incorporated in the mathematical definitions, Theorem 2.36. We continue with generalizing the common notions of Lie groups and vector bundles to the super world. For the latter, we follow [DM99, Section 3]. In particular, we will extensively study the odd partner of the super tangent bundle. This will be an important object in the latter study of field theories. Lastly, we will add extra structure to supermanifold in terms of Super Model Geometries.

2.1 Super Linear Algebra

Linear algebra is lifted to the super world, by assuming the vector spaces have a $\mathbb{Z}_2$ grading. We follow [Var04, Chapter 3] and [DM99, Section 1].

Definition 2.1. A **Super vector space** over a field $\mathbb{K}$ is a usual $\mathbb{K}$-vector space V together with a $\mathbb{Z}_2$ grading (representation). In other words, V has a chosen decomposition $V \cong V_0 \oplus V_1$. Here, we identify V_0 with the elements invariant under the representation and V_1 consists of exactly those elements by which the (nontrivial part of the) representation induces a change of sign. If d_i is the dimension of V_i for $i = 0, 1$, then $d_0|d_1$ is the dimension of the super vector space V. ▲

Remark 2.2. Instead of insisting on a field $\mathbb{K}$ and obtaining vector spaces, we could relax to rings and obtain **super monoids**. Most of this section will apply mutatis mutantis to monoids too. ▽

The straightforward, and up to isomorphism only, class of examples of super vector spaces is constructed by taking powers of the ground field $\mathbb{K}$ with appropriate grading. The fact that this yields, up to isomorphism, all super vector spaces can be seen by choosing a basis of homogeneous elements.

Example 2.3. Take $V = \mathbb{K}^{p+q}$ with standard basis e_i for $1 \leq i \leq p+q$. Let $V_0 = \oplus_{i=1}^{p} \mathbb{K}e_i$ and $V_1 = \oplus_{i=p+1}^{p+q} \mathbb{K}e_i$. Equivalently, take the $\mathbb{Z}_2$ representation on V induced by $e_i \mapsto e_i$ for $1 \leq i \leq p$ and $e_i \mapsto -e_i$ for $p+1 \leq p+q$. We denote the resulting super vector space by $\mathbb{K}^{p|q}$. △

In physics or when doing any kind of differential geometry, we will obviously be mostly interested in the case that $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

For our convenience, we give some names to the various elements of a supermanifold:

Definition 2.4. Given a super vector space $V \cong V_0 \oplus V_1$. The elements of $V_0 \cup V_1 \subseteq V$ are called **homogeneous**. Moreover, we associate to each of these homogeneous elements a **parity**: The elements of V_0 are called **even** and have parity $0 \in \mathbb{Z}_2$. The nonzero elements of V_1 are called **odd** and have parity $1 \in \mathbb{Z}_2$. We define the **parity** function $p\colon V_0 \cup V_1 \to \mathbb{Z}_2$ by assigning to every homogeneous element its parity in $\mathbb{Z}_2$. ▲

Remark 2.5. Formulae involving the parity function are, a priori, only defined on the homogeneous part. The parity function is, after all, only defined on those elements. However, in what follows, we will impose that any (multi)linear expression involving the parity function extends linearly to the non-homogeneous elements. ▽

Super vector spaces, just like many mathematical structures, can be organized in a category. We will denote the category of super vector spaces over $\mathbb{K}$ by $sVect_{\mathbb{K}}$. The morphisms in this category are the linear maps of vector spaces which preserve the parity of the homogeneous elements. So for any two super vector space V, W we have the **categorical Hom**

$$\mathrm{Hom}(V, W) = \{\text{Linear maps } V \to W \text{ preserving parity}\}. \tag{2.1.1}$$

Apart from just considering the parity conserving maps, we can instead also look at the **inner hom**, denoted by $\mathbf{Hom}(V, W)$, consisting of all linear maps $V \to W$. The space $\mathbf{Hom}(V, W)$ itself admits the structure of a super vector space, making it an interesting part of the theory. Here, the even part $\mathbf{Hom}(V, W)_0$ consists precisely of the parity preserving maps $\mathrm{Hom}(V, W)$ and the odd component consists of all maps reversing the parity, i.e., sending even elements of V to odd ones in W and vice versa.

We will write $\mathbf{End}(V) = \mathbf{Hom}(V, V)$ for the super vector spaces of endomorphisms on V. Furthermore, we can take the dual V^* as $\mathbf{Hom}(V, \mathbb{K})$. Here, we see $\mathbb{K}$ as a completely even super vector space, i.e., the odd part is trivial. In other words, the space $(V^*)_i$ consists precisely of the linear maps $V \to \mathbb{K}$ which vanish on V_{1-i}.

Given super vector space $V = V_0 \otimes V_1$. We can form a new super vector space by switching the even and odd parts. We simply define $W = W_0 \otimes W_1$, where $W_0 = V_1$ and $W_1 = V_0$. This construction is called the *change of parity* for vector spaces.

Super vector spaces admit direct sums in the obvious manner. The even resp. odd component of the direct sum super vector space is simply the direct sum of the even resp. odd components, i.e., $(V \oplus W)_i = V_i \oplus W_i$. For tensor products, we take the usual formula for graded vector spaces. However, the indices are taken modulo 2:

Definition 2.6. For super vector spaces V, W define the homogeneous components of the tensor product $V \otimes W$ as

$$(V \otimes W)_i = \bigoplus_{j+m=i} V_j \otimes W_m. \tag{2.1.2}$$

Here the indices i, j, m are taken in $\mathbb{Z}_2$. More explicitly, we set $(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$ and $(V \otimes W)_1 = (V_1 \otimes W_0) \oplus (V_0 \otimes W_1)$. ▲

The tensor product is both associative and symmetric. This means that there exists natural isomorphisms $a_{U,V,W} \colon (U \otimes V) \otimes W \cong U \otimes V(\otimes W)$ and $c_{V,W} V \otimes W \cong W \otimes V$. However, even though the isomorphisms are natural, there is *not* a unique choice of such natural isomorphisms. In the usual setting of standard linear algebra, one takes the associativity isomorphism $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$ and symmetry isomorphism $v \otimes w \mapsto w \otimes v$. In our setting of super linear algebra, we take the associativity isomorphism $a_{U,V,W}$ the same, but change the symmetry isomorphism to

$$c_{V,W} \colon V \otimes W \to W \otimes V, \quad v \otimes w \mapsto (-1)^{p(v)p(w)} w \otimes v. \tag{2.1.3}$$

Notice that, a priori, this is only defined on homogeneous elements. As remarked, we can extend linearly to all non-homogeneous elements. The motivation for this particular choice of commutativity isomorphism stems from quantum physics. In quantum mechanics, there are two kinds of indistinguishable particles: bosons and fermions. Following physical arguments, whenever we have two (or more) bosons the wave functions need to be symmetrized, while for fermions the wave functions need to be antisymmetrized (that is, taking the (anti)-symmetric sum of the products of the wave functions of the individual particles). Notice that in the case of fermions, whenever we interchange two particles, the resulting antisymmetrized wave function will obtain a sign change. While if we swap two bosons, the symmetrized wave function will not change at all. In the case, we interchange a boson with a fermion, we obviously have distinguishable particles. Hence, the resulting wave function (which is just the simple product of the wave function for the individual particles) will not change either.

2.1.1 The Rule of Signs

Returning to our super linear algebra setting, suppose that the even part correspond to some bosonic object, while the odd part correspond to some fermionic object, then we obtain the sign change precisely when two odd elements are interchanged. This is exactly what Eq. (2.1.3) tells us: Only in the piece $V_1 \otimes W_1 \to W_1 \otimes V_1$ there is a sign change. The principle of changing the sign whenever two odd/fermionic elements are swapped is called **the rule of signs**.

Upon taking this different sign convention in the symmetry isomorphism, we have wiped a subtle point under the carpet: its consistency. To illustrate this potential problem, consider some tensor product of an arbitrary number of super vector spaces and we permute the components in some way. For consistency, the sign change of the permutation must be the same, no matter how we write the permutation. As a well known fact, we know that any permutation can be decomposed in a sequence of adjacent exchanges, but the decomposition is not unique. A particular decomposition together with the rule of signs induces a sign convention. What is required to show is that the sign does not depend on the decomposition, but only the permutation.

In the relatively simple example of the sign, one can show the consistency by a direct argument, see [Var04, Proposition 3.1.1 and Corollary 3.1.2]. More generally, we can attack the problem by Mac Lane's *Coherence Theorem* for symmetric monoidal categories. This theorem essentially says that whenever some particular diagrams involving $a_{U,V,W}$ and $c_{V,W}$ commute, all such diagrams commute. See [Mac71, Chapter XI.1].

Proposition 2.7. *Let U, V, W, X be $\mathbb{K}$ super vector spaces. The following diagrams commute.*

$$\begin{tikzcd}
 & & (U \otimes V) \otimes (W \times X) \arrow[drr, "a_{U,V,W\otimes X}"] & & \\
((U \otimes V) \otimes W) \otimes X \arrow[urr, "a_{U\otimes V,W,X}"] \arrow[dr, "a_{U,V,W}\otimes id_X"'] & & & & U \otimes (V \otimes (W \otimes X)) \\
 & (U \otimes (V \otimes W)) \otimes X \arrow[rr, "a_{U,V\otimes W,X}"] & & U \otimes ((V \otimes W) \otimes X) \arrow[ur, "id_U \otimes a_{V,W,X}"'] &
\end{tikzcd} , \tag{2.1.4}$$

$$\begin{tikzcd}
(V \otimes \mathbb{K}) \otimes W \arrow[r, "a_{V,\mathbb{K},W}"] \arrow[dr, "\rho_V \otimes id_W"'] & V \otimes (\mathbb{K} \otimes W) \arrow[d, "id_V \otimes \lambda_W"] \\
 & V \otimes W
\end{tikzcd} , \tag{2.1.5}$$

$$\begin{tikzcd}
(U \otimes V) \otimes W \arrow[r, "a_{U,V,W}"] \arrow[d, "c_{U,V}\otimes id_W"] & U \otimes (V \otimes W) \arrow[r, "c_{U,V\otimes W}"] & (V \otimes W) \otimes U \arrow[d, "a_{V,W,U}"] \\
(V \otimes U) \otimes W \arrow[r, "a_{V,U,W}"] & V \otimes (U \otimes W) \arrow[r, "id_V \otimes c_{U,W}"] & V \otimes (W \otimes U)
\end{tikzcd} . \tag{2.1.6}$$

Here $\rho_V : V \otimes \mathbb{K} \to V$ and $\lambda_W : \mathbb{K} \otimes W \to W$ are the canonical isomorphisms. Moreover, there holds

$$c_{V,W} \circ c_{W,V} = id_{V\otimes W}. \tag{2.1.7}$$

Proof. Commutativity of the diagrams (2.1.4) and (2.1.5) is unchanged compared to standard linear algebra. Chasing a homogeneous element $u \otimes v \otimes w \in U \otimes V \otimes W$ in diagram (2.1.6) yields

$$\begin{tikzcd}
(u\otimes v)\otimes w \arrow[r, "a_{U,V,W}"] \arrow[d, "c_{U,V}\otimes \mathrm{id}_W"] & u\otimes(v\otimes w) \arrow[r, "c_{U,V\otimes W}"] & (-1)^{p(u)p(v\otimes w)}(v\otimes w)\otimes u \arrow[d, "a_{V,W,U}"] \\
(-1)^{p(u)p(v)}(u\otimes v)\otimes w \arrow[r, "a_{V,U,W}"] & (-1)^{p(u)p(v)}v\otimes(u\otimes w) \arrow[r, "\mathrm{id}_V \otimes c_{U,W}"] & (-1)^{p(u)(p(v)+p(w))}v\otimes(w\otimes u)
\end{tikzcd} . \tag{2.1.8}$$

By definition of the tensor product for super vector spaces, we have that $p(v \otimes w) = p(v) + p(w)$. Therefore, the diagram above commutes for homogeneous elements. Extending linearly gives the result for non-homogeneous elements.

The identity $c_{V,W} \circ c_{W,V} = \mathrm{id}_{V\otimes W}$ is easily verified in the same fashion. □

Corollary 2.8. *Any diagram involving just $a_{U,V,W}$, $c_{V,W}$ and ρ_V and λ_W from the proposition commutes. In particular, any decomposition in adjacent exchanges of a permutation of some tensor product of super vector spaces gives the same resulting sign.*

Proof. The first statement is the Coherence Theorem by Mac Lane. See [Mac71, Chapter XI.1]. For the second statement, suppose that we have two decompositions of the same permutation

σ. We can form the following diagram

$$\begin{tikzcd} V_1\otimes\cdots\otimes V_n \arrow[r, "\mathrm{id}_{V_1}\otimes\cdots\otimes c_{V_i,V_{i+1}}\otimes\cdots\otimes \mathrm{id}_{V_n}"] \arrow[d, equal] & V_1\otimes\cdots\otimes V_{i+1}\otimes V_i\otimes\cdots\otimes V_n \arrow[r] & \cdots \arrow[r] & V_{\sigma(1)}\otimes\cdots\otimes V_{\sigma(n)} \arrow[d, equal] \\ V_1\otimes\cdots\otimes V_n \arrow[r, "\mathrm{id}_{V_1}\otimes\cdots\otimes c_{V_j,V_{j+1}}\otimes\cdots\otimes \mathrm{id}_{V_n}"] & V_1\otimes\cdots\otimes V_{j+1}\otimes V_j\otimes\cdots\otimes V_n \arrow[r] & \cdots \arrow[r] & V_{\sigma(1)}\otimes\cdots\otimes V_{\sigma(n)} \end{tikzcd} . \tag{2.1.9}$$

Here, the two rows execute the decompositions step by step. Since the diagram commutes by the Coherence Theorem, we must obtain equal signs from the two different decompositions. □

2.1.2 Superalgebras and Rings

We will now lift the notions of algebras and rings, well known in ordinary linear algebra, to the super world.

Definition 2.9. A $\mathbb{K}$**-superalgebra** is a $\mathbb{K}$ super vector space A with an algebra multiplication $\mu\colon A\otimes A\to A$ which preserves parity. I.e., the map μ is a morphism of super vector spaces. The **supercenter** $Z(A)$ of a superalgebra A is given by the span of the homogeneous elements a such that for any $b\in A_0\cup A_1$ there holds $ab=(-1)^{p(a)p(b)}ba$. A superalgebra is **supercommutative** when it is equal to its supercenter. ▲

We give some basic properties of superalgebras:

Proposition 2.10. *(a) A super vector space A with an associative map $\mu\colon A\otimes A\to A$ is a superalgebra if and only if $p(\mu(a,b))=p(a)+p(b)$ for all homogeneous $a,b\in A_0\cup A_1$.*

(b) If A has a unit 1, then 1 has even parity.

(c) The space A_0 is a purely even subalgebra of A. Moreover, there holds $A_0A_1\subseteq A_1$ and $A_1A_1\subseteq A_0$.

Proof. (a) The statement that μ preserves parity, precisely tells us that $p(\mu(a\otimes b))=p(a\otimes b)=p(a)+p(b)$ for all homogeneous $a,b\in A_0\cup A_1$.

(b) From part a) we have that $p(1)=p(\mu(1\otimes 1))=p(1)+p(1)$. Hence, $p(1)=0$ and thus 1 is even.

(c) Suppose $a,b\in A_0$. Then $p(\mu(a\otimes b))=p(a)+p(b)=0$ and thus $\mu(a\otimes b)\in A_0$. The other cases follow similarly.

□

Example 2.11. For any super vector space V, the endomorphisms $\mathbf{End}(V)$ are a superalgebra under multiplication. Its supercenter is given by the scalar multiplications $\mathbb{K}\cdot\mathrm{id}_V$. The only supercommutative cases are $\mathbb{K}^{1|0}$ and $\mathbb{K}^{0|1}$. △

Example 2.12. Let $A=\mathbb{K}[t]$ with t odd and $t^2=1$. This is canonically a superalgebra. Notice that since $t^2\neq 0$, we have that $t\notin Z(A)$. Hence, $Z(A)=\mathbb{K}$ and thus A is not supercommutative. On the other hand, if we consider A just as an algebra, then it is commutative. This example illustrates that the superness can significantly change the objects under investigation. △

Example 2.13. An important example of a supercommuting superalgebra is the exterior algebra $\Lambda^\bullet(V)$ of a (totally even) vector space V. In finite dimension, we obtain, up to isomorphism, $\mathbb{K}[\theta_1,\cdots,\theta_q]$ satisfying the relation $\theta_i\theta_j+\theta_j\theta_i=0$ for all $0<i,j\leq q$. △

Example 2.14. Following [Mei13, Defintion 2.1], we define for a $\mathbb{K}$ vector space V and symmetric bilinear form $B\colon V \times V \to \mathbb{K}$ the **Clifford Algebra** to be the quotient

$$\mathrm{Cl}(V, B) = T(V)/\mathcal{T}(V, B). \tag{2.1.10}$$

Here, we have denoted $T(V)$ for the tensor algebra and $\mathcal{T}(V, B)$ for the ideal generated by all elements of the form

$$v \otimes w + w \otimes v - 2B(v, w), \qquad v, w \in V. \tag{2.1.11}$$

The tensor algebra has a natural $\mathbb{Z}$ grading. It collapsed to a $\mathbb{Z}/2$ grading upon taking the quotient by the ideal $\mathcal{T}(V, B)$. △

As a special case, we consider the $\mathbb{Z}$-superalgebras. Notice that we are working over a ground ring now, see Remark 2.2. In this case, we obtain the notion of a super ring.

Definition 2.15. A **superring** is a $\mathbb{Z}$-superalgebra. ▲

Example 2.16. For a topological space, the cohomology ring $H^\bullet$ is a superring under the cup-product. The odd degrees of cohomology give the odd components of the algebra. △

2.2 Supermanifolds

A smooth structure on ordinary (real or complex) manifolds is usually given by a smooth atlas of charts. These give us local coordinates on the manifold, allowing us to do analysis. In particular, we can define for a smooth manifold M its algebra $C^\infty(M)$ of real (or complex) valued functions. This algebra turns out to be as fundamental as the smooth atlas. More precisely, we can equivalently define a smooth structure on a manifold by its function algebra.

Definition 2.17. Let X be a topological space. A **presheaf** of sets is an assignment $\mathcal{F}$ which assign for every open set $U \subseteq X$ a set $\mathcal{F}U$ and for every inclusion of opens $V \subseteq U$ a map $\mathrm{res}_{V,U} \colon \mathcal{F}U \to \mathcal{F}V$ called the **restriction morphism**. We require the functorial properties that $\mathrm{res}_{U,U}$ is the identity morphism and that for opens $W \subseteq V \subseteq U$ there holds for the composition $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$.

A presheaf is called a **sheaf** if additionally for any open cover $(U_i)_{i\in I}$ of an open $U \subseteq X$ the following two properties hold

- (**Locality**) Let $s, t \in \mathcal{F}U$. If $\mathrm{res}_{U_i,U}(s) = \mathrm{res}_{U_i,U}(t)$ for all $i \in I$, then $s = t$.

- (**Gluing**) Suppose that we have $s_i \in \mathcal{F}U_i$ for all $i \in I$ such that $\mathrm{res}_{U_i\cap U_j,U_i}(s_i) = \mathrm{res}_{U_i\cap U_j,U_j}(s_j)$ for all $i, j \in I$. Then there is an $s \in \mathcal{F}U$ such that $\mathrm{res}_{U_i,U}(s) = s_i$ for all $i \in I$.

▲

Remark 2.18. The notion of a presheaf above coincides with the general notion of a presheaf in category theory, being contravariant functors. For this, consider the poset category with objects the opens of X and morphisms the inclusions of opens $V \subseteq U$. Contravariant functors on this category exactly satisfy the properties in the definition above. ▽

Remark 2.19. Instead of taking values in the category of sets, we can land anywhere. Examples include rings, algebras, vector spaces, and of course their super counterparts. ▽

Example 2.20. A trivial example of a sheaf is given by sending every open U of a space X to the empty set. △

Example 2.21. For every topological space X, we can assign to an open U the set of continuous functions $C^0(U)$, the restriction morphisms are given by the restriction of functions (hence the name). Notice that using the ring structure of $C^0(U)$, we can see this as a *sheaf of rings* too. We will denote this sheaf by C^0. Moreover, passing to smooth manifolds (defined by some atlas), we can also consider the set (or ring) $C^\infty(U)$ of smooth functions. This gives us a sheaf C^∞ on the manifold. △

Continuing on this last example of function spaces, a map $f\colon X \to Y$ of topological spaces induces a map $f^*\colon C^0(Y) \to C^0(X)$ by taking the pullback, i.e., by precomposing. Notice that this pullback commutes with the restrictions. Hence, we obtain a good candidate for morphisms between sheaves. However, for a general sheaf $\mathcal{F}$, the set $\mathcal{F}X$ has a priori no relation to X (apart from $\mathcal{F}$). Therefore, a map $f\colon X \to Y$ of topological spaces need not induce a canonical map on a sheaf. The solution is to simply remember the map on the sheaf too.

Definition 2.22. Let X and Y be topological spaces and sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$ on X and Y respectively. A map from the pair $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ consists of

- a continuous map $f\colon X \to Y$,
- a map $f_V^*\colon \mathcal{O}_Y V \to \mathcal{O}_X f^{-1}(V)$ for every open $V \subseteq Y$ which commutes with restrictions. I.e., there holds $\mathrm{res}_{f^{-1}(V'),f^{-1}(V)} \circ f_V^* = f_{V'}^* \circ \mathrm{res}_{V',V}$ for all open $V', V \subseteq Y$.

▲

Remark 2.23. The pullback maps f_V^* should be seen as morphisms in the target category, see Remark 2.19. So in the case of rings, we should require the pullback maps to be ring homomorphisms and similarly for any other target category. ▽

For us, the most important case will be the sheaves of rings. We will call a pair $(X, \mathcal{O}_X)$ of a topological space with a sheaf of rings a **ringed space**. The sheaf $\mathcal{O}_X$ is called the structure sheaf. Together with the maps of Definition 2.22, we have formed the category of ringed spaces. The canonical case is a topological space, with the sheaf of continuous functions. In this case, a map on the topological spaces induce a map on the structure sheaf by precomposition.

Ringed spaces allow us to do geometry. In fact, we can define smoothness of manifolds just in terms of their structure sheaf.

Theorem 2.24. *Let $(X, \mathcal{O}_X)$ be a ringed space which is locally isomorphic to $(\mathbb{R}^n, C^\infty)$. Then there exists a smooth atlas A on X such that (X, C^∞) w.r.t the atlas A is isomorphic to $(X, \mathcal{O}_X)$.*

Proof. For a proof, see [Nes03, Chapter 7]. □

Remark 2.25. Completely analogously, we can look at the complex valued smooth function instead. Since a complex valued function is not nothing more than two real valued ones, we obtain similar properties. ▽

Remark 2.26. Related to the theorem above, there is the Gelfand-Naimark Theorem, which states that any commutative C^*-algebra is isomorphic to $C(X)$ for some locally compact Hausdorff space X. For a precise statement and proof of the Gelfand-Naimark Theorem, see for example [Fol16, Theorem 1.20 and 1.31]. ▽

This theorem allows us to view smooth structures more algebraically as sheaf of rings. This point of view will allow us to define the notion of a *supermanifold*. The idea is to change the sheaves of (necessarily commutative) rings by sheaves of commutative superrings. For ordinary manifolds, these sheaves need to be locally isomorphic to $(\mathbb{R}^n, C^\infty)$. For supermanifolds, we should find some suitable local model giving a superstructure.

Definition 2.27. A **superdomain**, denoted by $U^{p|q}$ is the super ringed space $(U, C^{\infty p|q})$ where $U \subseteq \mathbb{R}^p$ and $C^{\infty p|q}$ is the sheaf given by $C^{\infty p|q}V = C^\infty(V) \otimes_{\mathbb{R}} \Lambda^\bullet(\mathbb{R}^q)$ for open $V \subseteq U$. Here, $\Lambda^\bullet(\mathbb{R}^q)$ denotes the exterior algebra. As seen in Example 2.13, we can identify $C^{\infty p|q}V$ with $C^\infty(V)[\theta_1, \cdots, \theta_q]$ where $\theta_i\theta_j + \theta_j\theta_i = 0$ for $1 \leq i, j \leq q$. ▲

In particular, we have the superdomain $\mathbb{R}^{p|q} = (\mathbb{R}^p, C^{\infty p|q})$ of super real space. This gives us a suitable local model to define supermanifolds.

Definition 2.28. A **supermanifold** is a super ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to $\mathbb{R}^{p|q}$. The local isomorphisms are the **supercharts** of the supermanifold. ▲

Remark 2.29. In the case we want to consider real analytic, holomorphic or similar supermanifolds, we just have to change our superdomains into the relevant local model. E.g., for the holomorphic setting, we take our superdomains to be $(U, H^{p|q})$ where $U \subseteq C^n$ open and $H^{p|q} = H(U) \otimes_{\mathbb{C}} \Lambda^\bullet(\mathbb{C}^q)$. Here, $H(U)$ denotes the holomorphic maps on U. ▽

Remark 2.30. We will denote to a supermanifold $(X, \mathcal{O}_X)$ usually by just X. If we want to make explicit reference to the underlying ordinary manifold, then we use the notation $|X|$. The underlying ordinary manifold is also called the **reduced** manifold. ▽

Remark 2.31. At this point, there might seem to be an apparent notational mismatch: We have the supermanifold $\mathbb{R}^{p|q}$ and the super vector space $\mathbb{R}^{p|q}$. The manifold has reduced manifold $\mathbb{R}^p$ while the underlying vector space of the super vector space $\mathbb{R}^{p|q}$ is $\mathbb{R}^{p+q}$. We will resolve this mismatch at the end of this section, Lemma 2.41. Moreover, it will be clear from the context whether we are looking at supermanifolds or super vector spaces. ▽

Example 2.32. Any superdomain trivially is a supermanifold. In particular, $\mathbb{R}^{p|q}$ is. △

Example 2.33. Any ordinary d-dimensional manifold X is a supermanifold of dimension $d|0$. △

Example 2.34. For an ordinary manifold X, the sheaf of differential forms $\Omega^\bullet(X)$ is locally free. Hence, the pair $(X, \Omega^\bullet(X))$ defines a supermanifold. This goes by the name of the *odd* or *shifted tangent bundle*. It turns this manifold can be identified with the mapping space $\mathrm{SMfld}(\mathbb{R}^{0|1}, X)$, which will be important to use when we consider (vacuum) field theories over some manifold, see Definition 5.20. We will consider this manifold in more detail and also motivate its name in Chapter 2.4.2. △

Example 2.35. Let $E \to M$ be an ordinary real vector bundle over M. The space of section $\Gamma(\Lambda^\bullet E)$ of the exterior bundle $\Lambda^\bullet E$ forms a sheaf of super rings over M. We obtain a supermanifold $(M, \Gamma(\Lambda^\bullet E)$ of dimension $\dim(M)|\mathrm{rank}(E)$. Marjorie Bachelor showed in [Bat79] that every supermanifolds, up to isomorphism, can be obtained from this construction. △

The intuitive picture of a supermanifold is that of a real manifold with a *cloud of odd stuff* around it. The cloud cannot be seen when mapping the structure sheaf into any completely even ring, as per definition of maps of superrings. Nevertheless, the odd cloud does contribute to the geometry and the even and odd parts do interact.

2.2.1 Morphisms of Super Manifolds

Maps of supermanifolds can be defined completely analogously as we did for ringed spaces in Definition 2.22. Notice that for the (super) Euclidean case of a map $f\colon \mathbb{R}^{p|q} \to \mathbb{R}^{p|q}$, it consists of a map $|f|\colon \mathbb{R}^p \to \mathbb{R}^{p'}$ on the base manifold and a homomorphism of sheaves

$$f^*\colon C^\infty(\mathbb{R}^{p'})[\theta_1,\dots,\theta_{q'}] \to C^\infty(\mathbb{R}^p)[\theta_1,\dots,\theta_q]. \tag{2.2.1}$$

The function $|f|$ can be recovered from f^* by taking the odd parameters to zero. In turn, the homomorphism f^* is completely determined by its action on $C^\infty(\mathbb{R}^{p'})$ and the generators $\theta_1,\dots,\theta_{q'}$. Therefore, we can equivalently view a morphism as a map $f\colon \mathbb{R}^{p|q} \to \mathbb{R}^{p|q}$ as a map on the even and odd coordinate functions x_i and θ_j. Here, the map $x_i\colon \mathbb{R}^p \to \mathbb{R}$ is the projection on the ith coordinate and $\theta_j \in C^\infty(\mathbb{R}^p)[\theta_1,\dots,\theta_q]$ is the constant map with image θ_j. More explicitly, the map f becomes an assignment

$$(\vec{x},\vec{\theta}) = (x_1,\dots,x_p,\theta_1,\dots,\theta_q) \mapsto (f_1^{even}(\vec{x},\vec{\theta}),\dots,f_{p'}^{even}(\vec{x},\vec{\theta}), f_1^{odd}(\vec{x},\vec{\theta}),\dots,f_{q'}^{odd}(\vec{x},\vec{\theta})). \tag{2.2.2}$$

This coordinate approach is commonly used in physics and makes doing computations much easier. From such an assignment on coordinates we can recover the sheaf homomorphism $C^\infty(\mathbb{R}^{p'})[\theta_1,\dots,\theta_{q'}] \to C^\infty(\mathbb{R}^p)[\theta_1,\dots,\theta_q]$, by viewing the assignment as a coordinate transformation acting by precomposition. We will switch freely between the different descriptions.

The view of reducing a map of supermanifolds to an assignment on the even and odd components can be formalized in the *Chart Theorem*.

Theorem 2.36 (Chart Theorem)**.** *Let $U^{p|q}$ be a superdomain and S a supermanifold. There is a natural bijection, explicitly given for $U^{p|q} = \mathbb{R}^{p'|q'}$ in the text above, between maps $S \to U$ of supermanifolds and collections of p even functions $f_i^{even} \in \mathcal{O}(S)$ and q odd functions $f_j^{odd} \in \mathcal{O}(S)$ such that for the reduced map $|f|$ obtained by taking the odd variables to zero there holds $|f|(m) \in |U|$ for all $m \in |S|$.*

Proof. For the case that S is a superdomain, a detailed proof can be found in [Lei80, Theorem 2.1.7]. The general case follows by a similar proof, see [CF07, Theorem 3.1.2]. ☐

Example 2.37. Consider the map $f\colon \mathbb{R}^{1|2} \to \mathbb{R}^{1|0}$ given by the assignment

$$(x,\theta_,\theta_2) \mapsto (x + \theta_1\theta_2). \tag{2.2.3}$$

Under the associated homomorphism $f^*\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})[\theta_1,\theta_2]$, the map $(x \mapsto x^2) \in C^\infty(\mathbb{R})$ is mapped to

$$(x,\theta_1,\theta_2) \mapsto (x+\theta_1\theta_2)^2 = x^2 + 2x\theta_1\theta_2. \tag{2.2.4}$$

More generally, a map $g \in C^\infty(\mathbb{R})$ is mapped under the homomorphism f^* to

$$(x,\theta_1,\theta_2) \mapsto g(x) + g'(x)\theta_1\theta_2, \tag{2.2.5}$$

where g' denotes the derivative of g. △

Do notice, however, that not all assignments as in Eq. (2.2.2) give a well-defined morphism of supermanifolds: The even components f_i^{even} should really have even values, while the odd components f_j^{odd} must be odd.

Example 2.38. The assignment $f\colon \mathbb{R}^{1|1} \to \mathbb{R}^{1|0}$ given by

$$(x, \theta_1) \mapsto x + \theta_1 \tag{2.2.6}$$

does not give a map of supermanifolds. Indeed, the structure sheaf of $\mathbb{R}^{1|0}$ is completely even, while the identity function on $\mathbb{R}$ under the associated homomorphism f^* would be mapped to $(x \mapsto x + \theta_1)$ which mixes even and odd variables. △

The function sheaf $C^\infty(X)$ of a supermanifold X has a decomposition in the even and odd function. This decomposition can be recovered from functions using the Chart Theorem.

Corollary 2.39. *For a supermanifold X, we can identify $C^\infty(X)^{ev} = SMfld(X, \mathbb{R}^1)$ and $C^\infty(X)^{odd} = SMfld(X, \mathbb{R}^{0|1})$.*

2.2.2 Functor of Points Viewpoint

With the morphisms of supermanifolds, we have constructed the **category of supermanifolds** SMfld. As mentioned in Example 2.35 every isomorphism class of supermanifolds can be obtained from vector bundles over ordinary manifolds. However, the required isomorphism cannot be chosen canonically. In fact, morphisms in the category of supermanifolds differ from morphisms of ordinary vector bundles. [DM99, Section 2.1.5]

Associated to SMfld we have the category of presheaves $\mathrm{SMfld}^{op} \to \mathrm{Set}$ on it.

Definition 2.40. We call a presheaf $\mathrm{SMfld}^{op} \to \mathrm{Set}$ a **generalized supermanifold**. A generalized supermanifold is **representable** by a supermanifold X if it is naturally isomorphic to $S \mapsto \mathrm{SMfld}(S, X)$. Here, we write $\mathrm{SMfld}(S, X)$ for the set of morphisms $S \to X$. ▲

There are two points to make about this notion of generalized supermanifolds. Firstly, we have the Yoneda Lemma which tells us that the maps between representables are cannonically identified with the natural transformations between the generalized supermanifolds. Secondly, set of morphism $\mathrm{SMfld}(S, X)$ can canonically be identified with the set of algebra functions $\mathrm{Alg}(\mathcal{O}(X), \mathcal{O}(S))$ between the structure sheaves. This turns geometry problem into algebra, which gives us a wealth of tools available to analyze them. The approach of considering the presheaf represented by a (super) manifold is referred to as *the functor of points* approach.

A particular important case mapping space is the mappings out of $\mathbb{R}^{0|q}$, the super point with q odd dimensions. It will play a crucial role in field theories, see Definition 5.20. The foundation will lie in the odd tangent bundles, which we already have seen for ordinary manifolds in Example 2.34. At this point, we can consider one further interesting case:

Lemma 2.41 ([DM99, Lemma 3.1.1])**.** *The mapping space $SMfld(\mathbb{R}^{0|1}, \mathbb{R}^{p|q})$ is a super vector space isomorphic to $\mathbb{R}^{p|q}$.*

Proof. We can decompose, using the group structure of $\mathbb{R}^p$, a morphism $f\colon \mathbb{R}^{0|1} \to \mathbb{R}^{p|q}$ in an even morphism $\mathbb{R}^{0|0} \to \mathbb{R}^p$ on the reduced manifolds and an odd morphism $\mathbb{R}^{0|1} \to \mathbb{R}^{p|q}$ for which the reduced manifold is mapped to the origin. This gives us a decomposition of $\mathrm{SMfld}(\mathbb{R}^{0|1}, \mathbb{R}^{p|q})$ into the even component $\mathrm{SMfld}(\mathbb{R}^{0|0}, \mathbb{R}^{p|q})$ of maps factoring through $\mathbb{R}^{0|0}$ and maps $\mathrm{SMfld}(\mathbb{R}^{0|1}, \mathbb{R}^{p|q})$ with reduced image the origin. This decomposition gives easily the isomorphism with $\mathbb{R}^{p|q}$. □

2.2.3 Supermanifolds with Boundary

Ordinary manifolds can be generalized to allow them having *boundaries*. Manifolds with boundary are a useful concept when considering bordisms. So for the purpose of generalizing bordisms to the super setting, which we do in Chapter 4.1.3, we will now define supermanifolds with boundary. Supermanifolds with boundary were first defined in [VZ87] precisely for the purpose of considering bordisms and homotopical properties of supermanifolds.

Boundaries of ordinary manifolds are, roughly speaking, sharp edges where the manifold “ends”. Rigorously, we assert that the charts of a manifold X must be diffeomorphisms between opens in X and opens of the half space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \geq 0\}$. The boundary of X will now be exactly those pieces which are mapped to hyperplane with $x_1 = 0$ of $\mathbb{R}^n$. It can straightforwardly be shown that the boundary point of X do not depend on the chosen atlas. In other words, when passing from ordinary manifolds without boundary to ordinary manifolds with boundary, we have changed our local model of the space from the whole $\mathbb{R}^n$ to the half space $\mathbb{R}^n_+$. This change of local model motivates the following definition.

Definition 2.42 ([VZ87, Definition 1])**.** A **supermanifold with boundary** of dimension $p|q$ is a ringed space locally isomorphic to open subspaces of the model space

$$\mathbb{R}^{p|q}_+ = \left(\mathbb{R}^p_+, C^\infty(\mathbb{R}^p_+) \otimes \Lambda^\bullet(\mathbb{R}^q)\right). \tag{2.2.7}$$

Here, we write $\mathbb{R}^p_+$ for the half space $\mathbb{R}^p_+ = \{(x_1, \ldots, x_p) \in \mathbb{R}^p | x_1 \geq 0\}$. The open subspaces of $\left(\mathbb{R}^p_+, C^\infty(\mathbb{R}^p_+) \otimes \Lambda^\bullet(\mathbb{R}^q)\right)$ are **superdomains with boundary**. ▲

The reduced manifold $|X|$ of a supermanifold with boundary X is an ordinary manifold with boundary. This boundary is an embedded submanifold $\partial|X| \hookrightarrow |X|$ with a canonical embedding. We like to have a notion of boundaries of supermanifolds too. The boundary will then be some $p-1|q$-dimensional submanifold of X, where $p|q$ is the dimension of X. Naturally, the underlying ordinary manifold must be the boundary $\partial|X|$ of the reduced manifold. However, this does not completely describe the boundary supersubmanifold yet

Example 2.43 ([Loz+04])**.** The space $\mathbb{R}^{1|2}_+$ has topological boundary the point 0. However, the embeddings $\iota, \iota' \colon \mathbb{R}^{0|2} \to \mathbb{R}^{1|2}_+$ written in coordinates (t, θ_1, θ_2) of $\mathbb{R}^{1|2}$ as

$$\iota(\theta_1, \theta_2) = (0, \theta_1, \theta_2), \tag{2.2.8}$$
$$\iota'(\theta_1, \theta_2) = (\theta_1\theta_2, \theta_1, \theta_2) \tag{2.2.9}$$

are different embeddings which are equally valid as boundary. In this simple case of a superdomain, one might be tempted to prefer the embedding ι. However, for a general supermanifold with boundary, we cannot make this choice consistently since in different charts the boundary might look different. Do notice that the two embedded supermanifolds here are isomorphic as supermanifolds. △

Definition 2.44 ([VZ87, Definition 2])**.** A **boundary of a supermanifold** X, whose dimension is $p|q$, is a supermanifold Y of dimension $(p-1)|q$ together with a smooth embedding of supermanifolds

$$\iota \colon Y \hookrightarrow X \tag{2.2.10}$$

such that ι reduces to a diffeomorphism $|\iota| \colon |Y| \to \partial|X|$ from the reduced manifold $|Y|$ to the boundary $\partial|X|$ of the reduced manifold $|X|$. ▲

As we have seen in Example 2.43 boundaries of supermanifolds are not unique. However, all boundaries of a supermanifold are isomorphic as supermanifolds.

Proposition 2.45 ([VZ87, Proposition 2])**.** *Any two boundaries of a supermanifold with boundary are isomorphic as supermanifolds.*

2.3 Super Lie Groups

In ordinary differential geometry, a Lie group is a smooth manifold which is also a group such that the multiplication and inversion maps are smooth. This can be formulated completely in categorical terms by demanding a Lie group to be a group object in the category of smooth manifolds. This categorical viewpoint is vital for generalizing the notion of Lie groups to Super Lie groups. The reason for this is that in the super case, we don't have points to turn to. So, a condition like "there exists an $e \in G$ such that $eg = g$ for all $g \in G$" does not make sense. Instead, following [Fio10], we make a categorical definition.

Definition 2.46. A **super Lie group** G is a supermanifold G with maps

$$\mu\colon G \times G \to G, \tag{2.3.1}$$
$$i\colon G \to G, \tag{2.3.2}$$
$$e\colon \mathbb{R}^{0|0} \to G, \tag{2.3.3}$$

called the multiplication, inversion and unit map respectively. These maps are further required to satisfy the usual relation (associativity, unit, etc.). These can be summarized by requiring the following diagrams to commute.

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu\times\mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_G\times\mu"] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd}, \tag{2.3.4}$$

$$\begin{tikzcd} G \times G \arrow[dr, "\mu"'] & G \arrow[l, "\langle \mathrm{Id}_G, \hat{e}\rangle"'] \arrow[r, "\langle \hat{e}, \mathrm{Id}_G\rangle"] \arrow[d, "\mathrm{Id}_G"] & G \times G \arrow[dl, "\mu"] \\ & G & \end{tikzcd} \qquad \begin{tikzcd} G \times G \arrow[dr, "\mu"'] & G \arrow[l, "\langle \mathrm{Id}_G, i\rangle"'] \arrow[r, "\langle i, \mathrm{Id}_G\rangle"] \arrow[d, "\hat{e}"] & G \times G \arrow[dl, "\mu"] \\ & G & \end{tikzcd}. \tag{2.3.5}$$

Here, the map $\hat{e}\colon G \to G$ is the constant map on the unit of G. I.e., it is the composition of the unique map $G \to \mathbb{R}^{0|0}$ and e. ▲

Example 2.47 ([Ber13a, Example 1.11])**.** Let V be a vector space and a symmetric pairing $R\colon \Delta \times \Delta \to V$. Denote $\Pi\Delta$ for the completely odd space of Δ, i.e., for finite dimensional Δ, the superspace $\mathbb{R}^{0|q}$ with $q = \dim(\Delta)$. We can define the supergroup $V \times \Pi\Delta$ with underlying manifold V and supergroup action given by

$$(\vec{v}, \vec{\theta}) \cdot (\vec{v}', \vec{\theta}') = (\vec{v} + \vec{v}' + R(\vec{\theta}, \vec{\theta}'), \vec{\theta} + \vec{\theta}'). \tag{2.3.6}$$

Notice that in this case, we have a short exact sequence of groups

$$0 \to V \to V \times \Pi\Delta \to \Pi\Delta \to 0. \tag{2.3.7}$$

In case $V = \mathbb{R}^p$ and $\Pi\Delta = \mathbb{R}^{0|q}$, we write $\mathbb{E}^{p|q}_R = V \times \Pi\Delta$ and call it the group of **super Euclidean translations** for the pairing R. We are using the notation $\mathbb{E}^{p|q}_R$ to explicitly distinguish between the space $\mathbb{R}^{p|q}$ and the supergroup of translations $\mathbb{E}^{p|q}_R$. △

Example 2.48 ([CF07, Example 3.4.3])**.** Let $V = V_0 \oplus V_1$ be a super vector space. The space of invertible linear parity preserving transformations $\mathrm{GL}(V)$ can be identified with $\mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$, seen as ordinary vector spaces. If $p = \dim(v_0)$ and $q = \dim(V_1)$, then we can see $\mathrm{GL}(V)$ as an open of $\mathbb{R}^{p^2+q^2}$. We will give $\mathrm{GL}(V)$ the structure sheaf from the embedding in $\mathbb{R}^{p^2+q^2|2pq}$. I.e., for an open $U \subseteq \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \subseteq \mathbb{R}^{p^2+q^2}$, we have that

$$\mathcal{O}_{\mathrm{GL}(V)}(U) = C^\infty(U)[\theta_1, \ldots, \theta_{2pq}] \tag{2.3.8}$$

with the usual anti-commutativity $\theta_i\theta_j + \theta_j\theta_i = 0$. △

Lemma 2.49. *Let $(S, \mathcal{O}_S)$ a supermanifold. The space of maps SMfld$(S, GL(\mathbb{R}^{p|q}))$ can be identified with $Aut_{\mathcal{O}_S}(\mathcal{O}_S^{p|q})$, the automorphisms of $\mathcal{O}_S^{p|q} = \mathcal{O}_S^p \otimes_{\mathbb{R}} \Lambda^\bullet(\mathbb{R}^q)$.*

Proof. By the Chart Theorem, Theorem 2.36, we can identify morphisms of supermanifolds $f\colon S \to \mathrm{GL}(\mathbb{R}^{p|q}) \subseteq \mathbb{R}^{p^2+q^2|2pq}$ as collections of p^2+q^2 even elements and $2pq$ odd elements of the structure sheaf $\mathcal{O}_S$ such that the induced reduced map has image in $\mathrm{GL}(\mathbb{R}^p) \times \mathrm{GL}(\mathbb{R}^q) \subseteq \mathbb{R}^{p^2+q^2}$. In turn, these collections, can be identified with an invertible matrix with coefficients in $\mathcal{O}_S$. The invertibility follows from the fact that the reduced map has image in $\mathrm{GL}(\mathbb{R}^p) \times \mathrm{GL}(\mathbb{R}^q)$ is invertible. This shows that $\mathrm{SMfld}(S, \mathrm{GL}(\mathbb{R}^{p|q}))$ can be identified with the automorphisms of $\mathcal{O}_S^{p|q}$. □

2.3.1 Super Lie Groupoids

In the same spirit of super Lie groups, we define super Lie groupoids. The difference between groups and groupoids is that in groupoids the multiplication is only partially defined. Elements are given a *source* and *target*. Two elements can only be composed if the target of the one element agrees with the source of the other. Without any smooth structure, a groupoid can be identified with a category where all morphisms are isomorphisms. In ordinary smooth geometry a *Lie groupoid* is a pair of two manifolds Γ_0 and Γ_1 with two surjective submersion $s, t\colon \Gamma_1 \to \Gamma_0$, the source and target map. Obviously in this smooth setting, we also assume that the inversion map and (partially defined) multiplication map are smooth.

We now lift the notion to the case for supermanifold, using a similar categorical approach as for super Lie groups.

Definition 2.50 ([Tom10, Definition 2.1])**.** A **Super Lie Groupoid** is a pair of supermanifolds (Γ_1, Γ_0) with five morphisms satisfying some axioms.

- The source and target submersive epimorphisms $s, t\colon \Gamma_1 \to \Gamma_0$.
- The multiplication map $m\colon \Gamma_{1\,t}\!\times_s \Gamma_1 \to \Gamma_1$ defined on the fiber product $\Gamma_{1\,t}\!\times_s \Gamma_1$ in the diagram

$$\begin{tikzcd} \Gamma_{1\,t}\!\times_s \Gamma_1 \arrow[r, "\mathrm{Pr}_2"] \arrow[d, "\mathrm{Pr}_1"] & \Gamma_1 \arrow[d, "t"] \\ \Gamma_1 \arrow[r, "s"] & \Gamma_0 \end{tikzcd} \tag{2.3.9}$$

- The inversion map $i\colon \Gamma_1 \to \Gamma_1$.
- The identity map $e\colon \Gamma_0 \to \Gamma_1$.

These maps are assumed to satisfy $i \circ i = \mathrm{Id}_{\Gamma_1}$ and $s \circ i = t$ and make the following diagrams commute.

$$\begin{tikzcd}
\Gamma_0 & \Gamma_0 \arrow[l, "\mathrm{Id}_{\Gamma_0}"'] \arrow[r, "\mathrm{Id}_{\Gamma_0}"] \arrow[d, "e"] & \Gamma_0 \\
& \Gamma_1 \arrow[ul, "s"] \arrow[ur, "t"'] &
\end{tikzcd} \tag{2.3.10}$$

$$\begin{tikzcd}
\Gamma_{1\,t}\!\times_s \Gamma_1 \arrow[r, "m"] \arrow[d, "\mathrm{Pr}_1"] & \Gamma_1 \arrow[d, "s"] \\
\Gamma_1 \arrow[r, "s"] & \Gamma_0
\end{tikzcd} \qquad
\begin{tikzcd}
\Gamma_{1\,t}\!\times_s \Gamma_1 \arrow[r, "m"] \arrow[d, "\mathrm{Pr}_2"] & \Gamma_1 \arrow[d, "t"] \\
\Gamma_1 \arrow[r, "t"] & \Gamma_0
\end{tikzcd} \tag{2.3.11}$$

$$\begin{tikzcd}
\Gamma_{1\,t}\!\times_s \Gamma_{1\,t}\!\times_s \Gamma_1 \arrow[r, "\mathrm{Id}_{\Gamma_1} \times m"] \arrow[d, "m \times \mathrm{Id}_{\Gamma_1}"] & \Gamma_{1\,t}\!\times_s \Gamma_1 \arrow[d, "m"] \\
\Gamma_{1\,t}\!\times_s \Gamma_1 \arrow[r, "m"] & \Gamma_1
\end{tikzcd} \tag{2.3.12}$$

$$\begin{tikzcd}
\Gamma_1 \arrow[r, "\langle e\circ s, \mathrm{Id}_{\Gamma_1}\rangle"] \arrow[dr, "\mathrm{Id}_{\Gamma_1}"'] & \Gamma_{1\,t}\!\times_s \Gamma_1 \arrow[d, "m"] & \Gamma_1 \arrow[l, "\langle \mathrm{Id}_{\Gamma_1}, e\circ t\rangle"'] \arrow[dl, "\mathrm{Id}_{\Gamma_1}"] \\
& \Gamma_1 &
\end{tikzcd} \tag{2.3.13}$$

$$\begin{tikzcd}
\Gamma_{1\,t}\!\times_s \Gamma_1 \arrow[r, "\langle \mathrm{Id}_{\Gamma_1}, i\rangle"] \arrow[dr, "e\circ s"'] & \Gamma_{1\,t}\!\times_s \Gamma_1 \arrow[d, "m"] & \Gamma_{1\,t}\!\times_s \Gamma_1 \arrow[l, "\langle i, \mathrm{Id}_{\Gamma_1}\circ t\rangle"'] \arrow[dl, "e\circ t"] \\
& \Gamma_1 &
\end{tikzcd} \tag{2.3.14}$$

▲

The discusion of super Lie groups is included in this definition by taking the base space Γ_0 to be the complete even point $\mathbb{R}^{0|0}$. Lie goupoids are obtained when taking all supermanifolds to be ordinary manifolds.

Similar to how (super) Lie groups can act on (super)manifolds, we can act with super Lie groupoids on supermanifolds. However, since the multiplication in the groupoid is defined on the fiber $\Gamma_{1\,t}\!\times_s \Gamma_1$, the action is defined on a similar fiber space.

Definition 2.51. Let $\Gamma = (\Gamma_1, \Gamma_0)$ a super Lie groupoid and M a supermanifold. A **right Γ-action on** M consists of an **anchor map** $\text{⚓}\colon M \to \Gamma_0$ and a multiplication map $\mu\colon M_{\text{⚓}}\!\times_s \Gamma_1$ on the fiber space in the diagram

$$\begin{tikzcd}
M_{\text{⚓}}\!\times_s \Gamma_1 \arrow[r] \arrow[d] & \Gamma_1 \arrow[d, "s"] \\
M \arrow[r, "\text{⚓}"] & \Gamma_0
\end{tikzcd} \tag{2.3.15}$$

such that the following diagram commute

$$\begin{tikzcd}
M_{\text{⚓}}\!\times_s \Gamma_{1\,t}\!\times_s \Gamma_1 \arrow[r, "\mathrm{Id}_M \times m"] \arrow[d, "\mu \times \mathrm{Id}_{\Gamma_1}"] & M_{\text{⚓}}\!\times_s \Gamma_1 \arrow[d, "\mu"] \\
M_{\text{⚓}}\!\times_s \Gamma_1 \arrow[r, "\mu"] & M
\end{tikzcd} \tag{2.3.16}$$

$$
\begin{tikzcd}
M {}_{\text{⚓}}\times_s \Gamma_1 \arrow[r, "\mu"] \arrow[d, "\mathrm{Pr}_{\Gamma_1}"] & M \arrow[d, "\text{⚓}"] \\
\Gamma_1 \arrow[r, "t"] & \Gamma_0
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\langle \mathrm{Id}_M, e\circ\text{⚓}\rangle"] \arrow[dr, "\mathrm{Id}_M"'] & M {}_{\text{⚓}}\times_s \Gamma_1 \arrow[d, "\mu"] \\
 & M
\end{tikzcd}
\tag{2.3.17}
$$

▲

2.4 Super Vector Bundles

In this section, we will consider super bundles over supermanifolds. These will be supermanifolds fibered over some supermanifold in some supermanifold. I.e., we simply superficate all spaces in consideration. Using this notion, we can consider super vector bundles. In particular, the *super tangent bundle* of a supermanifold. In turn, this gives rise to the *odd tangent bundle* which will be one of the main objects of study when considering field theories in later chapters.

We start by defining super fiber bundles.

Definition 2.52. A **super fiber bundle** $E \to M$ with fibers F for supermanifolds E, M and F is a smooth super map that locally looks like the projection $M \times F \to M$. ▲

The geometric notion of super vector bundles follows naturally:

Definition 2.53. A **super vector bundle** is a super fiber bundle with fibers $\mathbb{R}^{p|q}$ such that the transition functions lie in the supergroup $\mathrm{GL}(\mathbb{R}^{p|q})$. Here, we use the identification from Lemma 2.41 to view $\mathbb{R}^{p|q}$ as a super vector space. ▲

While the geometric description is very intuitive, it is cumbersome to work with. Instead, we will use an equivalent, more algebraic notion. For ordinary manifolds, this is closely related to the Swan's Theorem.

Theorem 2.54 ([DM99, Sections 3.1.2-4])**.** *Let M be a supermanifold. Taking the sheaf of sections $U \mapsto \mathit{SMfld}_M(U \times \mathbb{R}^{0|1}, E|_U)$ of a super vector bundle $E \to M$ defines a bijection between super vector bundles over M and locally free sheaves over $\mathcal{O}_M$.*

Proof. For a super vector bundle, the sheaf $U \mapsto \mathrm{SMfld}_M(U \times \mathbb{R}^{0|1}, E|_U)$ is a locally free since there are local trivializations $E|_U \cong U \times \mathbb{R}^{p|q}$. The space of sections $\mathrm{SMfld}_M(U \times \mathbb{R}^{0|1}, U \times \mathbb{R}^{p|q})$ can be identified with the super vector space $\mathbb{R}^{p|q}$, see Lemma 2.41.

A locally free sheaf $\mathcal{E}$ over $\mathcal{O}_M$ has by definition a cover $(U_i)_{i\in I}$ with trivializations $\phi_i\colon \mathcal{E}(U_i) \cong \mathcal{O}_{U_i} \times \mathbb{R}^{p|q}$. Here, we see $\mathbb{R}^{p|q}$ as a vector space. Using the identification from Lemma 2.41, we can glue together copies of the supermanifold $U_i \times \mathbb{R}^{p|q}$ along the transition functions induced by $\phi_i \circ \phi_j^{-1}$ on $U_i \cap U_j$ for $i, j \in I$. This gives us a suitable super vector bundle, whose sheaf of sections is isomorphic to $\mathcal{E}$. □

2.4.1 Super Tangent Bundle

A leading example when considering ordinary vector bundles is the tangent bundle. We will construct a super analogue of the tangent bundle. We follow [CF07, Section 3.2]. A tangent vector in the ordinary sense is a derivation on the function algebra. Therefore, we will consider derivations on the structure sheaves of supermanifolds. This will give us a locally free sheaf on the base manifold. Hence, by Theorem 2.54, it gives a vector bundle.

Definition 2.55. Let $(M, \mathcal{O}_M)$ be a supermanifold. Define for all opens $U \subseteq M$, **the super ring** $\mathrm{Der}_{\mathcal{O}_M}(U)$ **of derivations on** $\mathcal{O}_M(U)$. The even component of $\mathrm{Der}_{\mathcal{O}_M}(U)$ are linear maps $\partial \colon \mathcal{O}_M(U) \to \mathcal{O}_M(U)$ such that for all $f, g \in \mathcal{O}_M(U)$ there holds that

$$\partial(fg) = (\partial f)g + f(\partial g). \tag{2.4.1}$$

The odd component of $\mathrm{Der}_{\mathcal{O}_M}(U)$ consists of the linear maps $\partial \colon \mathcal{O}_M(U) \to \mathcal{O}_M(U)$ with the property that

$$\partial(fg) = (\partial f)g + (-1)^{p(f)} f(\partial g) \tag{2.4.2}$$

for all $f, g \in \mathcal{O}_M(U)$. Notice the use of the rule of signs here. We write $\mathrm{Der}_{\mathcal{O}_M}$ for the **sheaf of derivations** consisting of $\mathrm{Der}_{\mathcal{O}_M}(U)$ for all opens $U \subseteq M$. ▲

If we work on a superdomain $\mathbb{R}^{p|q}$, then we have a canonical basis for the super ring of derivations. We have the usual even derivatives $\frac{\partial}{\partial x_i}$ and the odd derivatives $\frac{\partial}{\partial \theta_j}$. These derivatives are explicitly given by

$$\frac{\partial}{\partial x_i}\left(f_I(x)\theta_I\right) = \frac{\partial f_I(x)}{\partial x_i}\theta_I, \qquad \frac{\partial}{\partial \theta_j}\left(f_I(x)\theta_j\theta_I\right) = f_I(x)\theta_I. \tag{2.4.3}$$

In the second identity, we assume that $j \notin I$.

Lemma 2.56. *For a supermanifold M, the sheaf of derivations $Der_{\mathcal{O}_M}$ is locally free.*

Proof. Locally, the supermanifold M looks like a super domain $\mathbb{R}^{p|q}$. As we have seen in the text above, the sheaf of derivations on $\mathbb{R}^{p|q}$ is free. □

With this lemma, we can rigorously define the super tangent bundle of a supermanifold:

Definition 2.57. Let M be a supermanifold. The **Super Tangent Bundle** TM of M is defined by the locally free sheaf $\mathrm{Der}_{\mathcal{O}_M}$.

Dually, the **Super Cotangent Bundle** T^*M of a supermanifold M is the dual bundle to the tangent bundle TM. Its sections are the **Super Differential 1-Forms** $\Omega^1(M)$. Taking exterior powers $\Lambda^\bullet T^*M$ and their sections, we have the **Super Differential Forms** $\Omega^\bullet(M)$. ▲

Notice that if we started with an ordinary manifold M (which we can see as a supermanifold with superdimension zero), then the definitions reduces to the usual definition of (co)tangent bundles and differential forms on ordinary manifolds. A De Rham operator can be constructed for supermanifolds too, by extending the usual formulae to odd coordinates.

The algebra $\Omega^\bullet(M)$ of (super) differential forms can be seen completely algebraically as the *universal differential envelope* of $\mathcal{O}(M)$, [BDM95, Section 2.2]. It is universal in the sense that there exists an algebra homomorphism $\mathcal{O}(M) \to \Omega^\bullet(M)$ and for any other graded commutative differential algebra $\mathcal{A}$ with a map $\mathcal{O}(M)\mathcal{A}^0$, there exists a unique homomorphism $\Omega^\bullet(M) \to \mathcal{A}$ of differential algebras making the triangle

$$\begin{tikzcd} & \mathcal{O}(M) \arrow[dl] \arrow[dr] & \\ \Omega^\bullet(M) \arrow[rr] & & \mathcal{A} \end{tikzcd} \tag{2.4.4}$$

commute. This point of allows us to generalize to n-differential algebras too. So instead of a single De Rham differential, we require n of them.

2.4.2 Odd Tangent bundle

On super vector bundles, we can perform all the usual constructions like direct sums, tensor products, duals etc. in a straightforward fashion. However, in the super setting, we can do more: we can perform a *change of parity* transformation. On the level of vector space and sheaves, the change of parity transformation does not seem too drastic. We simply switch the even and odd parts. However, on the level of spaces, the geometric change is immense.

Definition 2.58. Given a super vector bundle $E \to M$ over a supermanifold M given with sheaf of sections $\mathcal{E}$, the **change of parity vector bundle** ΠE is defined by the locally free sheaf constructed by changing the parity $U \times \mathbb{R}^{p|q} \to U \times \mathbb{R}^{q|p}$ in the local trivializations of $\mathcal{E}$. ▲

Notice that under this construction, the total space of the super vector bundle changes, even as a topological space. While the reduced manifold $|E|$ is a vector bundle over $|M|$ of rank p. After the change of parity, the vector bundle $|\Pi E|$ over $|M|$ has rank q.

Definition 2.59 (Odd tangent bundle)**.** For any supermanifold M, we define the **odd tangent bundle** ΠTM by the parity changed bundle of the tangent TM. By iterating the procedure of taking the tangent bundle and switching the parity δ times, we obtain the space $(\Pi T)^\delta M$. The elements of function sheaf $\mathcal{O}\left((\Pi T)^\delta M\right)$ are called **pseudo-differential forms**. ▲

The odd tangent bundle ΠTM of a (super)manifold has some intriguing properties. We will present some of them below. Inspiration is taken from [Hoh+11, Section 3] and [Koc04]. While studying the properties of ΠTM, we upgrade them to $(\Pi T)^\delta M$ where possible. Ideas, especially for the case $\delta = 2$, are taken from [KS03].

Lemma 2.60. *Let M be some supermanifold. The generalized supermanifold $S \mapsto$ SMfld$(S \times \mathbb{R}^{0|1}, M)$ is represented by the supermanifold ΠTM from Definition 2.59.*

Proof. We follow the proof of [DM99, Lemma 3.3.1b]. We need to show that the assignments $S \mapsto \mathrm{SMfld}(S \times \mathbb{R}^{0|1}, M)$ and $S \mapsto \mathrm{SMfld}(S, \Pi TM)$ are naturally isomorphic.

The induced map $\phi\colon \mathcal{O}(M) \to \mathcal{O}(S) \times \mathcal{O}(\mathbb{R}^{0|1}) \cong \mathcal{O}(S)[\theta]/(\theta^2 = 0)$ from an element in $\mathrm{SMfld}(S \times \mathbb{R}^{0|1}, M)$ can be written as

$$\phi = \alpha + \theta\beta\colon \mathcal{O}(M) \to \mathcal{O}(S) \oplus \theta\mathcal{O}(S). \tag{2.4.5}$$

For $f, g \in \mathcal{O}(M)$, there holds

$$\begin{aligned}
\alpha(fg) + \theta\beta(fg) &= \phi(fg) \\
&= \phi(f) \cdot \phi(g) \\
&= (\alpha(f) + \theta\beta(f)) \cdot (\alpha(g) + \theta\beta(g)) \\
&= (\alpha(f)\alpha(g) + \theta\left[\beta(f)\alpha(g) + (-1)^{p(\alpha(f))}\alpha(f)\beta(g)\right].
\end{aligned}$$

From this expansion, we see that $\alpha\colon \mathcal{O}(M) \to \mathcal{O}(S)$ is a map of algebras, while β is an odd S valued α-derivation. I.e., we have that $\beta \in \Gamma(S, \alpha^*TM)$.

On the other hand, notice that we can decompose $\mathcal{O}(\Pi TM)$ is locally generated respectively by the even and odd coordinates

$$x_1, \ldots, x_p, d\theta_1, \ldots, d\theta_q \text{ and } dx_1, \ldots, dx_p, \theta_1, \ldots, \theta_q. \tag{2.4.6}$$

The coordinates $x_1, \dots, x_p, \theta_1, \dots, \theta_q$ together generate a copy of $\mathcal{O}(M)$. Restricting an element of $\mathrm{SMfld}(S, \Pi TM)$ to $\mathcal{O}(M)$ leave us with an algebra map $\alpha\colon \mathcal{O}(M) \to \mathcal{O}(S)$. Restricting an element of $\mathrm{SMfld}(S, \Pi TM)$ to the piece of $\mathcal{O}(\Pi TM)$ locally generated by $d\theta_1, \dots, d\theta_q, dx_1, \dots, dx_p$ gives us precisely a section of the pullback bundle $\alpha^* TM$. This shows the claim. □

Proposition 2.61. *Let M be some supermanifold. The generalized supermanifold $S \mapsto SMfld(S \times \mathbb{R}^{0|\delta}, M)$ is represented by the supermanifold $(\Pi T)^\delta M$ for all non-negative integers δ.*

Proof. We proceed by induction on δ. The case for $\delta = 0$ is trivial. The case $\delta = 1$ is the claim of Lemma 2.60. Suppose that $S \mapsto \mathrm{SMfld}(S \times \mathbb{R}^{0|\delta}, M)$ is represented by the supermanifold $(\Pi T)^\delta M$ for some δ. By Lemma 2.60, we know that $S \mapsto \mathrm{SMfld}(S \times \mathbb{R}^{0|\delta} \times \mathbb{R}^{0|1}, M)$ is represented by the supermanifold $\Pi T (\Pi T)^\delta M = (\Pi T)^{\delta+1} M$. Since canonically, we have that $\mathbb{R}^{0|\delta} \times \mathbb{R}^{0|1} \cong \mathbb{R}^{0|\delta+1}$, the induction step is shown. □

The mapping space $\mathrm{SMfld}(S \times \mathbb{R}^{0|\delta}, M)$ admits an action of $\mathbb{R}^{0|\delta}$ via precomposition of the translation action of $\mathbb{R}^{0|\delta}$ on itself. In fact, we can act by any diffeomorphism $S \times \mathbb{R}^{0|\delta} \to S \times \mathbb{R}^{0|\delta}$. For such an action to define an action on the representables of the considered presheaves, we need to require that the diffeomorphisms $S \times \mathbb{R}^{0|\delta} \to S \times \mathbb{R}^{0|\delta}$ to respect the projections to S.

Definition 2.62. The **Generalized Supergroup of Diffeomorphisms** $\underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta})$ of a supermanifold X is the group valued functor $S \mapsto \mathrm{SMfld}_S(S \times X \to S \times X)$, which sends a supermanifold S to the set of diffeomorphisms $S \times X \to S \times X$ respecting the projections to S. I.e., the following diagram commutes.

$$\begin{array}{ccc} S \times X & \longrightarrow & S \times X \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mathrm{Id}_S} & S \end{array} \tag{2.4.7}$$

▲

We will be interested in the case of the diffeomorphism supergroup of the superpoints $\mathbb{R}^{0|\delta}$. This case is particularly nice, since the generalized supergroup has a chance to be representable by a Lie supergroup. In case, there was a nonzero even dimension, the representable would need to be infinite dimensional. It remains to show that $\underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta})$ can be represented by some supergroup $\mathrm{Diff}(\mathbb{R}^{0|\delta})$.

Generators of the generalized supergroup $\underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta})$ can be found by explicitly writing out the transformations in coordinates using Theorem 2.36. To aid us in notation, we will write I for an **ordered sequence** $(I_1, \dots I_k) \subseteq \{1, \dots, \delta\}$. For odd coordinates $\theta_1, \dots \theta_\delta$ of $\mathbb{R}^{0|\delta}$, we write $\theta^I = \theta_{I_1} \cdots \theta_{I_k}$. We write $|I|$ for the length of the sequence. Sequences of length 0 are allowed. We continue using the notation for the rest of this section.

Lemma 2.63. *Denote $\theta_1, \dots, \theta_\delta$ for the odd coordinates of $\mathbb{R}^{0|\delta}$. The generalized supergroup $\underline{\mathit{Diff}}(\mathbb{R}^{0|\delta})$ is for any index space S generated by the assignments*

$$\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_\delta \end{pmatrix} \mapsto A \cdot \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_\delta \end{pmatrix}, \quad A \in GL_\delta(\mathcal{O}(S)), \tag{2.4.8}$$

$$\theta_i \mapsto \theta_i + \eta_i^I \theta^I, \quad \eta_i^I \in \mathcal{O}(S),\ 0 < i \leq \delta \text{ and } |I| \neq 1. \tag{2.4.9}$$

Proof. By the Chart Theorem, Theorem 2.36, we can write a general map $S \times \mathbb{R}^{0|\delta} \to S \times \mathbb{R}^{0|\delta}$ respecting the projections to S as a pullback assignment of the form

$$\theta_i \mapsto \sum_I c_i^I \theta^I \tag{2.4.10}$$

for some elements $c_i^I \in \mathcal{O}(S)$. The condition of respecting the space S boils via the Chart Theorem down to asserting that the coordiantes of S are send to itself. Such an assignment represents an element of $\underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta})$ if and only if it is invertible. In turn, this is equivalent to requesting invertibility when shrinking to just the linear terms in $\theta_1, \ldots \theta_\delta$. I.e., the assignment from Eq. (2.4.10) is invertible if and only if the following assignment is invertible.

$$\theta_i \mapsto \sum_{j=1}^{\delta} c_i^j \theta_j \tag{2.4.11}$$

This is simply saying that the matrix $(c_i^j)_{0<i,j\leq\delta} \in \mathrm{GL}_\delta(\mathcal{O}(S))$ as in Eq. (2.4.8). Notice that the nonlinear terms of Eq. (2.4.10) can be recovered using transformations of the form in Eq. (2.4.9). □

Corollary 2.64. *The generalized supergroup $\underline{Diff}(\mathbb{R}^{0|\delta})$ represented by a finite dimensional super Lie group $Diff(\mathbb{R}^{0|\delta})$ with underlying supermanifold $GL_\delta \times \mathbb{R}^{\delta(2^{\delta-1}-\delta)|\delta 2^{\delta-1}}$. The group action of $Diff(\mathbb{R}^{0|\delta})$ is induced by Eqs.* (2.4.8) *and* (2.4.9)*.*

Proof. Notice that the GL_δ part of $\underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta})$ generated by Eq. (2.4.8) gives even transformations. The other generators each give a copy $\mathbb{R}^{1|0}$ or $\mathbb{R}^{0|1}$ depending on whether they give rise to an even or odd transformation. Counting the generators gives the precise dimension. □

Remark 2.65. The dimension of the Super Lie group $\mathrm{Diff}(\mathbb{R}^{0|\delta})$ is $\delta 2^{\delta-1}|\delta 2^{\delta-1}$. This should be of no surprise since space of maps $\mathbb{R}^{0|\delta} \to \mathbb{R}^{0|\delta}$ can be identified with $(\Pi T)^\delta \mathbb{R}^{0|\delta}$ which has dimension $\delta 2^{\delta-1}|\delta 2^{\delta-1}$. Indeed, if $\delta = 0$ there is nothing to show. Otherwise, we have that $\dim\left(\Pi T \mathbb{R}^{0|\delta}\right) = \delta|\delta$ and further acting with ΠT doubles the dimension every time.[1] ▽

Albeit, the supermanifolds they are defined over are the same, the Super Lie groups $\mathrm{Diff}(\mathbb{R}^{0|\delta})$ and $\mathrm{GL}_\delta \times \mathbb{R}^{\delta(2^{\delta-1}-\delta)|\delta 2^{\delta-1}}$ with its canonical Super Lie group structure are vastly different. Even if $\delta = 1$, we obtain a (slightly) more complicated group operation. Increasing the dimension further will worsen the situation further. For $\delta > 2$, Noam Shomron [Sho02] showed, in somewhat different notation, that the representation theory of $\mathrm{Diff}(\mathbb{R}^{0|\delta})$ is wild. For $\delta = 1$, we can write the diffeomorphism supergroup $\mathrm{Diff}(\mathbb{R}^{0|1})$ completely as a semi-direct product of scalar multiplications and translations.

Lemma 2.66 ([Hoh+11, Lemma 9])**.** *The diffeomorphism supergroup $Diff(\mathbb{R}^{0|1})$ is isomorphic to $\mathbb{R}^\times \ltimes \mathbb{R}^{0|1}$. The semi-direct product is defined by the scalar multiplication of $\mathbb{R}^\times$ on $\mathbb{R}^{0|1}$.*

Proof. We work in the generalized setting over an index space S and conclude the statement using the Yoneda Lemma. Lemma 2.63 tells us that $\underline{\mathrm{Diff}}(\mathbb{R}^{0|1})$ is generated by scalar multiplications $\mathrm{GL}_1(\mathcal{O}(S)) \cong \mathrm{SMfld}(S, \mathbb{R}^\times)$ and translations $\mathrm{SMfld}(S, \mathbb{R}^{0|1})$. The semi-direct product structure follows from composing two diffeomorphisms:

$$\theta \overset{(\alpha,\beta)}{\mapsto} \alpha\theta + \beta \overset{(\alpha',\beta')}{\mapsto} \alpha\alpha'\theta + \alpha\beta' + \beta. \tag{2.4.12}$$

[1] More precisely, if $\dim(M) = p|q$, then $\dim(\Pi TM) = p + q|p + q$. This reduces to doubling the dimension if $p = q$.

where $\alpha, \alpha' \in \mathrm{GL}_1(\mathcal{O}(S))$ are scalar multiplications and $\beta, \beta' \in \mathrm{SMfld}(S, \mathbb{R}^{0|1})$ are translations. $\square$

Increasing δ will give much richer diffeomorphism supergroups. However, the action of the general linear group GL_δ and the action of translations by $\mathbb{R}^{0|\delta}$ are still present. The same argument as in the lemma above, shows that the semi-direct product $\mathrm{GL}_\delta \ltimes \mathbb{R}^{0|\delta}$ is a subgroup of $\mathrm{Diff}(\mathbb{R}^{0|\delta})$ in a canonical way. We have shown the following.

Lemma 2.67. *The semi-direct product $GL_\delta \ltimes \mathbb{R}^{0|\delta}$ can be canonically embedded as a subgroup in $Diff(\mathbb{R}^{0|\delta})$.*

By construction, we have a precomposition action of the generalized supergroup $\underline{\mathrm{Diff}}(\mathbb{R}^{0|\delta})$ on the generalized supermanifold $S \mapsto \mathrm{SMfld}(S \times \mathbb{R}^{0|\delta})$. By Proposition 2.61, we obtain an action of $\mathrm{Diff}(\mathbb{R}^{0|\delta})$ on $(\Pi T)^\delta M$. This action induces an action on the structure sheaf $\mathcal{O}\left((\Pi T)^\delta M\right)$. For $\delta = 1$, the supermanifold ΠTM has locally even coordinates $x_1, \ldots, x_p, d\theta_1, \ldots, d\theta_q$ and odd coordinates $dx_1, \ldots, dx_p, \theta_1, \ldots, \theta_q$. Here, the coordinates $x_1, \ldots x_p, \theta_1, \ldots \theta_q$ are just the coordinates of M and $dx_1, \ldots dx_p, d\theta_1, \ldots d\theta_q$ are the coordinates in the fibers of the shifted tangent bundle. Inductively, we obtain local coordinates $d_I x_1, \ldots d_I x_p$ and $d_I\theta_1, \ldots d_I\theta_q$ of $(\Pi T)^\delta M$. Here, the x_i resp. θ_j are the even resp. odd coordinates of M and $d_I = d_{I_1} \ldots d_{I_k}$ for some ordered sequence $I \subseteq \{1, \ldots, \delta\}$.

The notation using the symbol d is suggestive in the sense that we obtain some kind of differential forms. In fact, the *differential* forms of M are a subset of the pseudo-differential forms. Namely, those forms that are polynomial in $d\theta_1, \ldots, d\theta_q$. Pseudo-differential forms could look like $e^{d\theta_j}$, while this clearly cannot be a differential form.

Definition 2.68 ([Ber13b, Definition 1.19])**.** Define for a supermanifold M the sheaf of **(homogeneous) polynomial functions on** $(\Pi T)^\delta M$ as

$$\mathcal{O}_{\mathrm{pol}}\left((\Pi T)^\delta M\right) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \{f \in \mathcal{O}\left((\Pi T)^\delta M\right) | r \cdot f = r^k f, r \in \mathbb{R}_{>0}\} \tag{2.4.13}$$

Here, we write $r \cdot f$ with $r \in \mathbb{R}_{>0}$ and $f \in \mathcal{O}\left((\Pi T^\delta M\right)$ for the dilatation action of $\mathbb{R}_{>0}$ on $(\Pi T)^\delta M$. This is the action by the elements $r \cdot \mathbb{1} \in \mathrm{GL}_\delta \subseteq \mathrm{Diff}(\mathbb{R}^{0|\delta})$.

A function $f \in \mathcal{O}\left((\Pi T^\delta M\right)$ satisfying $r \cdot f = r^k f$ is said to have **polynomial degree** k. We will denote them by $\mathcal{O}^k((\Pi T)^\delta M)$ ▲

Proposition 2.69 ([Del+99, Remark page 74], [Ber13b, Proposition 2.4])**.** *For any supermanifold M, there is an isomorphism of sheaves*

$$\Omega^\bullet(M) \cong \mathcal{O}_{pol}(\Pi TM) \tag{2.4.14}$$

between (super) differential forms and the polynomial functions on ΠTM.

Proof. Locally identify the coordinates $x_1, \ldots, x_p, d\theta_1, \ldots, d\theta_q, dx_1, \ldots, dx_p, \theta_1, \ldots, \theta_q$ of ΠTM with the (super) differential forms with the same notation. This extends to an isomorphism of sheaves as requested. $\square$

Corollary 2.70. *For an ordinary manifold M, the structure sheaf of ΠTM can be identified with $\Omega^\bullet M$, the sheaf of differential forms. I.e., we obtain the space constructed in Example 2.34.*

Proof. For an ordinary manifold M the dilatation action on ΠTM is trivial in the even coordinates and linear in the odd coordinates. This makes the function sheaf $\mathcal{O}(\Pi TM)$ equal to $\mathcal{O}_{\mathrm{pol}}(\Pi TM)$. $\square$

The De Rham operator d on differential forms can be recovered as an odd vector field from the $\mathrm{Diff}(\mathbb{R}^{0|1})$ action on ΠTM. It is simply the derivative at zero of the translation action of $\mathbb{R}^{0|1} \subseteq \mathrm{Diff}(\mathbb{R}^{0|1})$ on ΠTM. Following the computation in [Hoh+11, Section 3.1], we see that acting by an element $\eta \in \mathbb{R}^{0|1}$ on a map $S \times \mathbb{R}^{0|1}$ has the effect of *replacing* the odd coordinate ψ of $\mathbb{R}^{0|1}$ by $\psi + \eta$. A map $\phi\colon U \times \mathbb{R}^{0|1} \to M$ induces a map on structure sheaves locally given by

$$\phi^*(x_i) = y_i + \psi \hat{y}_i, \quad \phi^*(\theta_j) = \zeta_j + \psi \hat{\zeta}_j. \tag{2.4.15}$$

Hence, in coordinates of ΠTM, we obtain the action given by

$$\left(\eta, (x_i, \theta_j, dx_i, d\theta_j)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}\right) \mapsto (x_i + \eta dx_i, \theta_j + \eta d\theta_j, dx_i, d\theta_j)_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}. \tag{2.4.16}$$

The derivative to η evaluated at 0 becomes the odd vector field D which satisfies

$$Dx_i = dx_i, \quad D\theta_i = d\theta_i, \quad Ddx_i = 0 \text{ and } Dd\theta_i = 0. \tag{2.4.17}$$

I.e., it is the operator given by

$$D = \sum_{i=1}^{p} dx_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{q} d\theta_j \frac{\partial}{\partial \theta_j}. \tag{2.4.18}$$

Lemma 2.71 ([Hoh+11, Lemma 8])**.** *The restriction of the operator D to $\mathcal{O}(M) \subseteq \mathcal{O}(\Pi TM)$ yields the De Rham operator d on supermanifolds.*

Proof. From the formulae above, we easily deduce that D is an odd vector field on ΠTM. I.e., it satisfies

$$D(fg) = (Df)g + (-1)^{|f|} f Dg, \tag{2.4.19}$$

for all $f, g \in \mathcal{O}(\Pi TM)$. Moreover, it satisfies $D^2 = 0$. Using the identification of Proposition 2.69, we complete the proof by computing Df for $f \in \mathcal{O}(M)$ in coordinates:

$$Df = \sum_{i=1}^{p} dx_i \frac{\partial}{\partial x_i} f + \sum_{j=1}^{q} d\theta_j \frac{\partial}{\partial \theta_j} f = \sum_{i=1}^{p} (-1)^{|f|} \left(\frac{\partial}{\partial x_i} f\right) dx_i + \sum_{j=1}^{q} \left(\frac{\partial}{\partial \theta_j} f\right) d\theta_j. \tag{2.4.20}$$

$\square$

Each component of the translation action of $\mathbb{R}^{0|\delta} \subseteq \mathrm{Diff}(\mathbb{R}^{0|\delta})$ give just a copy of the case where $\delta = 1$. Hence, applying the above analysis to $(\Pi T)^\delta M$, we obtain operators $D_1, \dots, D_\delta$ on $\mathcal{O}\left((\Pi T)^\delta M\right)$. Inductively declaring local coordinates in the same way as we did for ΠTM, we obtain that local coordinates of $(\Pi T)^\delta M$ can be denoted by $d_I x_i$ and $d_I \theta_j$. Here, the x_i resp. θ_j are the even resp. odd coordinates of M and $d_I = d_{I_1} \cdots d_{I_k}$ for any ordered sequence $I \subseteq \{1, \dots, \delta\}$. Applying Lemma 2.71 to an operator D_i, we realize that on $\mathcal{O}(M)$, there holds $D_i = d_i$. Straightforward induction yields the following lemma.

Lemma 2.72 ([Ber13b, Lemma 2.6])**.** *An ordered sequence of operators D_I on $\mathcal{O}\left((\Pi T)^\delta M\right)$ reduces to d_I on $\mathcal{O}(M) \subseteq \mathcal{O}\left((\Pi T)^\delta M\right)$.*

Up to now, we kept the operators D_I ordered. We wish to understand their commutation relations.

Lemma 2.73. *The operators D_i are anti-commutative. I.e., they satisfy $D_k D_l = -D_l D_k$ for all $1 \leq l, k \leq \delta$.*

Proof. We consider $(\Pi T)^\delta M$ as a mapping space following Proposition 2.61. The induced map on structure sheaves of a morphism $\phi\colon S \times \mathbb{R}^{0|\delta} \to M$ takes the form

$$\phi^*(x_i) = \sum_{I \subseteq \{1,\ldots,\delta\}} y_I^i \psi_I, \quad \phi^*(\theta_j) = \sum_{I \subseteq \{1,\ldots,\delta\}} \zeta_I^j \psi_I. \tag{2.4.21}$$

For the commutation of D_k and D_l only the pieces proportional to $\psi_k \psi_l$ is relevant. Translating ψ_k by η_k and ψ_l by η_l yields

$$\psi_l \psi_k \mapsto (\psi_l + \eta_l)(\psi_k + \eta_k) = \psi_l \psi_k + \eta_l \psi_k + \psi_k \eta_l + \eta_l \eta_k. \tag{2.4.22}$$

Differentiating with $\frac{\partial}{\partial \eta_l} \frac{\partial}{\partial \eta_k}$ induces a sign change when commuting η_l with the derivative $\frac{\partial}{\partial \eta_k}$. Hence, the differentials $\frac{\partial}{\partial \eta_k} \frac{\partial}{\partial \eta_l}$ and $\frac{\partial}{\partial \eta_l} \frac{\partial}{\partial \eta_k}$ differ exactly by a sign. This implies that $D_k D_l = -D_l D_k$ as requested. □

The differential operators D_i map the space $\mathcal{O}_{\mathrm{pol}}\left((\Pi T)^\delta M\right)$ of polynomial functions, see Definition 2.68 on itself. Moreover, the D_i give $\mathcal{O}_{\mathrm{pol}}\left((\Pi T)^\delta M\right)$ the structure of an *n-differential graded commutative algebra*. In fact, making the same identifications as in Proposition 2.69, we can see that $\mathcal{O}_{\mathrm{pol}}\left((\Pi T)^\delta M\right)$ can be identified with the *universal* n-differential graded commutative algebra containing $\mathcal{O}(M)$, [KS03, Section 2].

The anti-commutativity of D_i in particular implies that $D_i^2 = 0$. Hence, we can consider the cohomology.

Proposition 2.74 ([KS03, Section 4.3])**.** *The cohomology of $\mathcal{O}_{pol}\left((\Pi T)^\delta M\right)$ with respect to any of the D_i is naturally isomorphic to the ordinary De Rham cohomology of M.*

Proof. The case for $\delta = 1$ is directly implied by Proposition 2.69 and Lemma 2.71. For $\delta > 1$ notice that $(\Pi T)^\delta M = (\Pi T)^{\delta-1} \Pi T M$ is a vector bundle over $\Pi T M$. Hence, it contracts to $\Pi T M$. Therefore, the cohomology of $(\Pi T)^\delta M$ is the cohomology of $\Pi T M$. This shows the claim. □

Instead of considering the operators D_i individually, we can also consider the action of all of them together. We come to the following definition.

Definition 2.75. A pseudo differential form $\omega \in \mathcal{O}\left((\Pi T)^\delta M\right)$ is **closed** if $D_i \omega = 0$ for all i. It is **exact** if there exists an $f \in \mathcal{O}\left((\Pi T)^\delta M\right)$ such that $D_1 \cdots D_\delta \omega$. ▲

From the commutation relations of the D_i it is immediately clear that exactness implies closedness. Moreover, when M is a superdomain iterative standard integration yields that all closed polynomial functions of nonzero degree are exact.

Thus far, we considered the translation action of $\mathbb{R}^{0|\delta}$ on $(\Pi T)^\delta M$. Recall that the whole supergroup $\mathrm{Diff}(\mathbb{R}^{0|\delta})$ acts on $(\Pi T)^\delta M$. In particular, we have an action of GL_δ on $(\Pi T)^\delta M$.

Computing similar to the case of odd translations, Eqs. (2.4.15) and (2.4.16), a map $\phi\colon U \times \mathbb{R}^{0|\delta} \to M$ is locally in coordinates given by

$$\phi^*(x_i) = \sum_I y_i^I \theta^I. \tag{2.4.23}$$

Here, we sum over all ordered sequences, the θ_k are the odd coordinates and $\theta^I = \theta_{I_1} \cdots \theta_{I_{|I|}}$. The I superscript on the y_i is only a label. Acting by an $(a_k^l)_{0\leq k,l\leq\delta} = A \in \mathrm{GL}_\delta$ has the effect of replacing θ_k with $\sum_l a_k^l \theta_l$. Passing to the action on $(\Pi T)^\delta M$ in coordinates, we see that A acts by replacing d_k by $\sum_l a_k^l d_l$.

In particular, a coordinate of the form $d_1 \dots d_\delta x_i$ will transform like

$$d_1 \dots d_\delta x_i \mapsto \sum_l a_1^l d_l \dots \sum_l a_\delta^l d_l x_i = \det(A) d_1 \dots d_\delta x_i. \tag{2.4.24}$$

Here, we used the anti-commutation relations of the D_k, Lemma 2.73, and Lemma 2.72 to obtain the equality with the determinant.

Definition 2.76. A pseudo differential form $\omega \in C^\infty((\Pi T)^\delta M)$ form has **determinant degree** k if for any $A \in \mathrm{GL}_\delta$, the precomposition action yields

$$A \cdot \omega = \det(A)^k \omega. \tag{2.4.25}$$

▲

Example 2.77. If $\delta = 1$, then the determinant degree agrees with the usual degree of differential forms. We know that locally a degree k form can be written as $f dx_1 \dots dx_k$. Acting with a linear scalar $A \in \mathrm{GL}_1$ gives

$$(A, f dx_1 \dots dx_k) \mapsto A^k f dx_1 \dots dx_k. \tag{2.4.26}$$

So indeed, we have the pseudo differential form has determinant degree k. Moreover, any element of $C^\infty(\Pi TM)$ is of determinant degree k for some k. △

2.4.3 Equivariant Super Vector Bundles

The equivariant case for super vector bundles is completely analogously to the ordinary case.

Definition 2.78. Let G be a super Lie group. A **G-equivariant super vector bundle** is a super vector bundle $\pi\colon E \to M$ with G actions on E and M such that $\pi(ge) = g\pi(e)$ for all $e \in E$ and $g \in G$ and G acts on E by linear transformation between the fibers. In other words, the following diagram commutes

$$\begin{array}{ccc} G \times E & \longrightarrow & M \\ {\scriptstyle \mathrm{Id}_G \times \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ G \times M & \longrightarrow & M \end{array} . \tag{2.4.27}$$

and the action in the fibers of E is linear. Here, the horizontal arrows are the relevant action maps. ▲

Example 2.79. A G-equivariant super vector bundle over the point $\mathbb{R}^{0|0}$ is nothing else than a super vector space V with a G action acting by linear transformations. E.g., it is the same data as a G-representation $G \to \mathrm{GL}(V)$. △

Example 2.80. More generally, a G-representation $\rho\colon G \to \mathrm{GL}(V)$ defines a vector bundle V_ρ on every G-manifold M by the action given by

$$G \times (M \times V) \ni (g, (m, v)) \overset{\mu}{\mapsto} (gm, \rho(g)(v)) \in M \times V. \tag{2.4.28}$$

The (equivariant) sections of this bundle can be identified with the equivariant maps $M \to V$:

$$\Gamma_G(M, V_\rho) \cong \{f \in \mathrm{SMfld}(M, V) | f \circ \mu = \rho \circ p_G \cdot f \circ p_M \in \mathrm{SMfld}(G \times M, V)\}. \tag{2.4.29}$$

Here, the maps $p_G\colon G \times M \to G$ and $p_M\colon G \times M \to M$ are the projections. The $\cdot$ on the right-hand side, denotes the action of the representation on V. △

From an equivariant vector bundle $E \to M$, we can consider the induced vector bundle $E//G \to M//G$ with the action groupoids $E//G$ and $M//G$. The action groupoid of a manifold M with a group action of G is defined to be the groupoid with objects the points of M and morphisms triples $(m, g, m') \in M \times G \times M$ such that $gm = m'$. The source and target maps are simply the projections to M. The fact that $E//G \to M//G$ is a vector bundle can be easily seen from trivializing charts of the equivariant vector bundle $E \to M$.

2.5 Super Model Geometries

In Chapter 2.2, we have established that superdomains $\mathbb{R}^{p|q}$ give us a local model for supermanifold. This was done in close analogy how ordinary manifold have $\mathbb{R}^n$ as local model and how $\mathbb{C}^n$ is for complex manifolds. However, all these local model exhibit much more structure than used for defining manifolds. We can remember much more of the geometry via the charts. We consider, following [HST10, Section 6.3], the case of supermanifolds, but the same analysis can be performed on ordinary manifolds, complex manifolds and other similar structures.

Definition 2.81 ([HST10, Definition 6.13])**.** A supermanifold $\mathbb{M}$ with a (left) action of a super Lie group G is called a **super model geometry**. An $(\mathbb{M}, G)$**-supermanifold** is a supermanifold Y together with a maximal atlas of **smooth equivariant charts**

$$Y \subseteq U_i \xrightarrow{\phi_i} V_i \subseteq \mathbb{M} \tag{2.5.1}$$

from an open $U_i \subseteq Y$ to an open $V_i \subseteq \mathbb{M}$ such that the following properties hold:

- The U_i's cover Y.
- The transition function

  $$\mathbb{M} \supseteq \phi_i(U_i \cap U_j) \xrightarrow{\phi_j \circ \phi_i^{-1}} \phi_j(U_i \cap U_j) \subseteq \mathbb{M} \tag{2.5.2}$$

  is given by the action of an element $g \in G$. I.e., it is (a restriction of) the map

  $$\mu(g, -)\colon \mathbb{M} \to \mathbb{M} \tag{2.5.3}$$

  for some $g \in G$ and $\mu\colon G \times \mathbb{M} \to \mathbb{M}$ the action map.

▲

Notice that in this definition, we can retrieve the supermanifold Y as a gluing of open subsets of $\mathbb{M}$. Therefore, we could instead work with just the transition function on which we impose a suitable cocycle condition.

Example 2.82. For any super Lie group G with subgroup H, we have a model geometry (G, H). A basic (G, H)-manifold is the group G itself. △

Example 2.83. In Example 2.47 above, we have constructed the supergroup of super Euclidean translation $\mathbb{E}_R^{p|q}$ for a paring $R\colon \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^p$. Suppose that we further have Lie group G acting on $\mathbb{R}^p$ and a G-representation of $\mathbb{R}^q$ such that the paring R is G-equivariant. With this data, we can form the semi-direct product $\mathbb{E}_R^{p|q} \rtimes G$. Here, the action of G on $\mathbb{E}_R^{p|q} = \mathbb{R}^p \times \Pi\mathbb{R}^q$ is given by the action of G on $\mathbb{R}^p$ and through the representation on $\Pi\mathbb{R}^q$. We call $\mathbb{E}_R^{p|q} \rtimes G$ the **super Euclidean isometry group for** G and the pair $(\mathbb{R}^{p|q}, \mathbb{E}_R^{p|q} \rtimes G)$ the **super Euclidean model geometry for** G.

Standard examples of groups G are any kind of matrix group acting naturally on $\mathbb{R}^p$, like $\mathrm{GL}(p), \mathrm{SO}(p)$ and $\mathrm{Spin}(p)$ etc. △

Example 2.84. The Möbius strip $M = ([0,1] \times \mathbb{R}) \,/\, ((0,t) \sim (1,-t))$ can be seen as an $(S^1 \times \mathbb{R}, \mathbb{Z}/2\mathbb{Z})$-manifold in the following way. Denote $\pi\colon M \to S^1 = [0,1]/(0 \sim 1)$ for the projection on the first coordinate. As charts take

$$\phi_1\colon (0, \frac{2}{3}) \times \mathbb{R} \to S^1 \times \mathbb{R}, \quad \phi_1(s,t) = (\pi(s), t), \tag{2.5.4}$$

$$\phi_2\colon (\frac{1}{3}, 1) \times \mathbb{R} \to S^1 \times \mathbb{R}, \quad \phi_2(s,t) = (\pi(s), t), \tag{2.5.5}$$

$$\phi_3\colon \left(\left([0, \frac{1}{3}) \cup (\frac{2}{3}, 1]\right) \times \mathbb{R}\right) /(0,t) \sim (1,-t) \to S^1 \times \mathbb{R}, \quad \phi_3(s,t) = \begin{cases} (\pi(s), t) & \text{if } s < \frac{1}{3} \\ (\pi(s), -t) & \text{if } s > \frac{2}{3} \end{cases}. \tag{2.5.6}$$

These charts are visualized in Fig. 1. The action of $\mathbb{Z}/2\mathbb{Z}$ on the cylinder $S^1 \times \mathbb{R}$ is given by flipping the sign in the second coordinate. Notice that the transition functions $\phi_1 \circ \phi_2^{-1}$ and $\phi_1 \circ \phi_3^{-1}$ are given by the action of the trivial element of $\mathbb{Z}/2\mathbb{Z}$, while $\phi_2 \circ \phi_3^{-1}$ is exactly given by the action of the nontrivial element. △

Remark 2.85. Notice that in the previous example, we cannot merge the charts ϕ_1 and ϕ_2 into some ϕ. If we would, the transition function does not act like an element of $\mathbb{Z}/2\mathbb{Z}$ anymore. The common domain of the charts ϕ and ϕ_3 now consists of two disjoint opens. On one component the action is trivial while, it is non-trivial on the other component. ▽

As a single object, an $(\mathbb{M}, G)$-supermanifold is simply a supermanifold, with a carefully chosen atlas such that locally it resembles $\mathbb{M}$ with its group action. The relevant notion of *(iso)morphims* in this situation must locally boil down to suitable (iso)morphisms on $\mathbb{M}$. The latter being the actions by elements of the group G.

Definition 2.86. The **Isometries** $f\colon X \to Y$ of $(\mathbb{M}, G)$-supermanifolds X and Y are diffeomorphisms which represented in charts are given by the action of an element $g \in G$ on $\mathbb{M}$. More explicitly, for any chart ϕ around $x \in X$ and any chart ψ around $f(x) \in Y$, the composition $\psi \circ f \circ \phi^{-1}$ equals $\mu(g, -)$ for some $g \in G$. Here, μ is the action map on $\mathbb{M}$.

The isometries of a single $(\mathbb{M}, G)$-supermanifold Y form canonically a group under composition. We will call this group $\mathrm{Iso}(Y)$. ▲

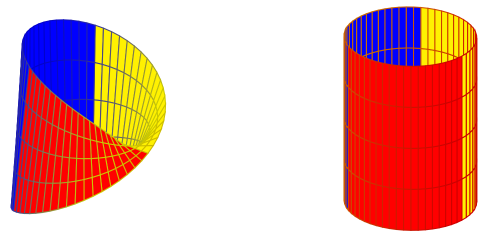

Figure 1: *The Möbius strip on the left and cylinder on the right. In different colors, we show the overlapping domains and codomains of the charts ϕ_1, ϕ_2 and ϕ_3 as in Example 2.84. Any single chart has as domain and codomain exactly two union of the regions indicated by two colors. The action by $\mathbb{Z}/2\mathbb{Z}$ is given by turning the cylinder upside down.*

2.5.1 Families of Model geometries

Instead of standalone $(\mathbb{M}, G)$-supermanifolds and their isometries, we can consider *families* of them. In fact, in the case of supermanifolds some sort of generalization must be made, since the standalone objects do not give us much of an advantage over the ordinary (non-super) case. This is due to the fact that any map $\mathbb{R}^{0|0} \to G$ for a supergroup G factors uniquely through the underlying manifold $|G|$. Therefore, a $(\mathbb{M}, G)$-supermanifold is nothing more than a $(\mathbb{M}, |G|)$-manifold.

Roughly, a *family of* $(\mathbb{M}, G)$*-supermanifolds* is a fiber bundle where the fibers are $(\mathbb{M}, G)$-supermanifolds.

Definition 2.87 ([HST10, Definition 6.14])**.** Let S be a supermanifold. An S**-family of** $(\mathbb{M}, G)$**-supermanifolds** is a morphism $p\colon Y \to S$ together with a maximal atlas of charts for Y which are diffeomorphism $\phi_i\colon U_i \to V_i$ between the opens $U_i \subseteq Y$ and $V_i \subseteq S \times \mathbb{M}$ such that the following properties hold:

- The U_i cover Y.
- For all i, the following diagram commutes

$$\begin{tikzcd} Y \supseteq U_i \arrow[rr, "\phi_i", "\cong"'] \arrow[dr, "p"'] & & V_i \subseteq S \times \mathbb{M} \arrow[dl, "p_1"] \\ & S & \end{tikzcd} . \tag{2.5.7}$$

- For all i, j, the transition function

$$S \times \mathbb{M} \supseteq \phi_i(U_i \cap U_j) \xrightarrow{\phi_j \circ \phi_i^{-1}} \phi_j(U_i \cap U_j) \subseteq S \times \mathbb{M} \tag{2.5.8}$$

takes the form

$$S \times \mathbb{M} \ni (s, m) \mapsto (s, g_{ij}(s)m) \in S \times \mathbb{M} \tag{2.5.9}$$

for some function $g_{ij} \colon p(U_i \cap U_j) \to G$.

▲

Notice that by the property listed second, we obtain local sections of p. We deduce that p is a submersion. As in the case of standalone $(\mathbb{M}, G)$-supermanifolds, we can retrieve the supermanifold Y from gluing together opens of $S \times \mathbb{M}$ along some suitable cocycles.

Also, the notion of isometries can be lifted to families. We simply require an isometry of a family to be a map factoring over the index space S, which is an isometry of the standalone $(\mathbb{M}, G)$-supermanifold in the fiber.

Definition 2.88. An **isometry** between two families $Y \to S$ and $Y' \to S'$ of $(\mathbb{M}, G)$-supermanifolds is a pair of maps $(\overline{f}, f) \colon (Y, S) \to (Y', S')$ such that the following holds:

- The diagram
$$\begin{array}{ccc} Y & \xrightarrow{\overline{f}} & Y' \\ \big\downarrow & & \big\downarrow \\ S & \xrightarrow{f} & S' \end{array} \tag{2.5.10}$$
commutes.
- For every chart ϕ of Y and ϕ' of Y' the chart representation $\phi' \circ \overline{f} \circ \phi^{-1}$ of $\overline{f}$, where defined, is of the form
$$S \times \mathbb{M} \supseteq V \ni (s, m) \mapsto (f(s), g(s)m) \in U \subseteq S' \times \mathbb{M} \tag{2.5.11}$$
for some smooth function $g \colon S \to G$.

We write $\mathrm{Iso}_S(Y)$ for the isometries on an S-family of $(\mathbb{M}, G)$-supermanifolds Y. ▲

These families of spaces with their isometries form a category in a canonical way. The projection to the index space S yields a functor to the category of smooth manifolds. This functor turns out to have very desirable properties. It will be a *stack*. In the next chapter, we will define stacks. In Proposition 3.21, we show that families of $(\mathbb{M}, G)$-supermanifolds give stacks.

3 Stacks

Stacks will form the language in which we can formulate field theories rigorously. They are certain categories with desirable properties. Especially, they assume certain descent data, as studied by Grothendieck [Gro60]. With this, Grothendieck observes that a stack is a solution to a moduli problem in case a moduli space does not exist due to the existence of automorphism. In this case, one can have a *moduli stack*.

The definition of a stack is originally due to Jean Giraud, [Gir71]. We will build up the definition following [BX06] starting with the notion of *Groupoid Fibrations*. Stacks then arise by imposing additional sheaf-like properties. The main goal in this chapter is to relate stacks, more precisely *differentiable* stacks, to Lie groupoids and vice versa. The transition from differentiable stacks to groupoid and reverse are the result of Proposition 3.25 and Proposition 3.38 respectively.

The categorical structures defined in this chapter can be used to formalize objects of study in physics. We will treat one such example in Chapter 4, where we define Functorial Field Theories in the language of stacks. More direct application exist. A number of them are summarized in [BS17]. In particular, one can obtain a version of Kaluza-Klein theory in this way.

In this chapter, I assume the reader is familiar with basic category theory. In particular, the Yoneda Lemma and slice categories are important. A good introduction to category theory can be found in [Rie14].

3.1 Groupoid Fibrations

To build up the notion of a stack, we will first consider a weaker notion. Namely, that of a category fibered in groupoids.

Definition 3.1. A **category fibered in groupoids** or simply a **groupoid fibration** is a functor $\pi\colon \mathfrak{X} \to \mathfrak{T}$ which has the following two properties:

- (**Pullback**) For every arrow $V \to U$ in $\mathfrak{T}$ and object $x \in \mathfrak{X}$ lying over U under π, there exists an object $y \in \mathfrak{X}$ and an arrow $y \to x$ in $\mathfrak{X}$ lying over the arrow $V \to U$.
- (**Cartesian arrows**) For every commutative triangle triple of arrows a, b, c in $\mathfrak{T}$ such that $b \circ a = c$ and arrows $\gamma\colon z \to x$ and $\beta\colon y \to x$ in $\mathfrak{X}$ lying over c and b respectively, there exists a unique arrow $\alpha\colon z \to y$ lying over a making the diagram such that $\beta \circ \alpha = \gamma$. Diagrammatically, we have the following commutative prism

$$\begin{tikzcd} & z \arrow[dl, dashed, "\alpha"'] \arrow[dr, "\gamma"] \arrow[dd, "\pi"] & \\ y \arrow[rr, "\beta"] \arrow[dd, "\pi"'] & & x \arrow[dd, "\pi"] \\ & W \arrow[dl, "a"'] \arrow[dr, "c"] & \\ V \arrow[rr, "b"] & & U \end{tikzcd} \quad . \tag{3.1.1}$$

  Here, the upper triangle lies in $\mathfrak{X}$, while the lower one lies in $\mathfrak{T}$. The structure morphism π of the groupoid fibration links the two diagrams.

▲

Remark 3.2. The condition $\pi' \circ F = \pi$ for morphisms is meant a strict equality of functors. It is too weak to require that $\pi' \circ F$ and π are isomorphic as functors. ▽

Example 3.3 ([BX06, Example 2.2])**.** For every object X of a category $\mathfrak{T}$, the slice category $\mathfrak{T}/X$ gives a category fibered in groupoids $\mathfrak{T}/X \to \mathfrak{T}$. The objects of $\mathfrak{T}/X$ are pairs (U, f) where U is an object in $\mathfrak{T}$ and $f\colon U \to X$ a morphism. The morphisms $(U, f) \to (V, g)$ are commuting triangles

$$\begin{tikzcd} U \arrow[rr] \arrow[dr, "f"'] & & V \arrow[dl, "g"] \\ & X & \end{tikzcd} \quad . \tag{3.1.2}$$

The functor $\mathfrak{T}/X \to \mathfrak{T}$ simply projects on the first component. The pullback and Cartesian arrows property hold canonically. △

Notice that in the example above, we have that the fibers $(\mathfrak{T}/X)_U$, on objects, can be identified with the set of morphisms $\mathrm{Hom}_{\mathfrak{T}}(U, X)$. This identification highlights that the category fibered in groupoids constructed just uses the presheaf represented by X. In fact, any presheaf gives rise to a category fibered in groupoids.

Example 3.4 ([BX06, Example 2.4])**.** Any presheaf, i.e., contravariant functor, $F\colon \mathfrak{T} \to \mathrm{Set}$ gives rise to a category fibered in groupoids $\mathfrak{X} \to \mathfrak{T}$ in the following way: The objects of $\mathfrak{X}$ are pairs (U, x) where U is an object in $\mathfrak{T}$ and $x \in FU$. The morphisms $(U, x) \to (V, y)$ in $\mathfrak{X}$ are arrows $a\colon U \to V$ such that $FU(y) = x$. The functor $\mathfrak{X} \to \mathfrak{T}$ is again just the projection on the first component.

To see this indeed defines a category fibered by groupoids notice that for any arrow $a\colon V \to U$ and $x \in FU$, we have the object $(V, Fa(x)) \in \mathfrak{X}$ and thus the arrow a in $\mathfrak{X}$ lying over a in $\mathfrak{T}$. Moreover, suppose we have a commuting triangle $\begin{tikzcd} & W \arrow[dl, "a"'] \arrow[dr, "c"] & \\ V \arrow[rr, "b"] & & U \end{tikzcd}$ in $\mathfrak{T}$ and arrows $\widetilde{b}\colon (V, y) \to (U, x)$ and $\widetilde{c}\colon (V, z) \to (U, x)$ lying over b and c respectively. Since the functor $\mathfrak{X} \to \mathfrak{T}$ is faithful by construction, we have that $Fb(x) = y$ and $Fc(x) = z$. Therefore, there holds $Fa(y) = Fa(Fb(x)) = F(b \circ a)(x) = Fc(x) = z$. Hence, we have found a suitable arrow in $\mathfrak{X}$ over a. It is unique by faithfulness of $\mathfrak{X} \to \mathfrak{T}$. △

The nomenclature category fibered in groupoids suggests there is a natural fibering of the category in groupoid. Indeed, there is: fibers along identity morphism form groupoids. These groupoids contain, as we shall see later on, much of the information of the entire category fibered in groupoids.

Proposition 3.5. *Let $\pi\colon \mathfrak{X} \to \mathfrak{T}$ be a category fibered in groupoids and $U \in \mathfrak{T}$ an object. The category $\mathfrak{X}_U$ consisting of all objects lying over U with morphisms, the morphisms lying over Id_U is a groupoid.*

Proof. We have to show that all morphisms of Γ admit an inverse. Suppose we have a morphism $\beta\colon x \to x'$ in Γ. By the Cartesian arrow property, applied to the triangle of identities on U, we obtain a morphism $\beta^{-1}\colon x' \to x$ lying over Id_U such the diagram

$$\begin{tikzcd}
 & x' \arrow[dl, dashed, "\beta^{-1}"'] \arrow[dr, "\mathrm{Id}_{x'}"] \arrow[dd, "\pi"] & \\
x \arrow[rr, "\beta"] \arrow[dd, "\pi"'] & & x' \arrow[dd, "\pi"] \\
 & U \arrow[dl, "\mathrm{Id}_U"'] \arrow[dr, "\mathrm{Id}_U"] & \\
U \arrow[rr, "\mathrm{Id}_U"] & & U
\end{tikzcd} . \tag{3.1.3}$$

commutes. This shows that every element has a right inverse, hence an inverse. □

Proposition 3.6. *Let $\pi\colon \mathfrak{X} \to \mathfrak{T}$ be a category fibered in groupoids, $b\colon V \to U$ an arrow in $\mathfrak{T}$ and x on object in $\mathfrak{X}$ lying over U. The object y whose existence is implied by the pullback property is unique up to unique isomorphism.*

Proof. Suppose objects $y, y' \in \mathfrak{X}$ lay over V with arrows $\beta\colon y \to x$ and $\beta'\colon y' \to x$ both lying over the arrow $b\colon V \to U$. By the Cartesian arrows property of a category fibered in groupoids, we obtain a unique arrow $\phi\colon y \to y'$ lying over id_V in the following diagram.

$$\begin{tikzcd}
 & y' \arrow[dl, dashed, "\phi"'] \arrow[dr, "\beta'"] \arrow[dd, "\pi"] & \\
y \arrow[rr, "\beta"] \arrow[dd, "\pi"'] & & x \arrow[dd, "\pi"] \\
 & V \arrow[dl, "\mathrm{Id}_V"'] \arrow[dr, "b"] & \\
V \arrow[rr, "b"] & & U
\end{tikzcd} \tag{3.1.4}$$

Since ϕ lies over the identity on V, by Proposition 3.5 it is in an isomorphism. The uniqueness of the isomorphism is implied by the uniqueness of the arrow ϕ. □

The (up to unique isomorphism) unique object y is called the **pullback** of x along the arrow $a\colon V \to U$. We denote it by $x|V$ or a^*x when including the explicit arrow. Similarly, a morphism $\phi\colon x \to y$ over the identity on U, by the Cartesian arrow property, there is a unique arrow $a^*\phi\colon a^*x \to a^*y$ over the identity on V making the following diagram commute.

$$\begin{tikzcd}
a^*x \arrow[r, "a^*\phi"] \arrow[d] & a^*y \arrow[d] \\
x \arrow[r, "\phi"] & y
\end{tikzcd} \tag{3.1.5}$$

In case the arrow a is understood, we denote the pullback morphism $a^*\phi$ by $\phi|_V$.

3.1.1 Morphisms of Groupoid Fibrations

Having considered groupoid fibrations and their internal structure, we turn our head to morphisms between them.

Definition 3.7. A **morphism** groupoid fibrations over $\mathfrak{T}$ from $\pi\colon \mathfrak{X} \to \mathfrak{T}$ to $\pi'\colon \mathfrak{Y} \to \mathfrak{T}$ is a functor $F\colon \mathfrak{X} \to \mathfrak{Y}$ such that $\pi' \circ F = \pi$. The morphism is an **isomorphism** if it is an equivalence of categories. ▲

Example 3.8. Given a groupoid fibration $\pi\colon \mathfrak{X} \to \mathfrak{T}$. For every object $x \in \mathfrak{X}$, we can define a morphism of groupoid fibrations $\mathfrak{T}/\pi x \to \mathfrak{X}$ as follows: On objects send an object $f\colon U \to \pi x$ of $\mathfrak{T}/\pi x$ to the pullback f^*x. The choice of pullback does not matter, just take any. This construction extends to morphisms of $\mathfrak{T}/\pi x$ by the Cartesian arrow property of groupoid fibrations. Indeed for a morphism $\phi\colon (U,f) \to (V,g)$ of $\mathfrak{T}/\pi x$, we can form the diagram

$$\begin{tikzcd}
& f^*x \arrow[dl, dashed] \arrow[dd, "\pi"] \arrow[dr] & \\
g^*x \arrow[rr] \arrow[dd, "\pi"] & & x \arrow[dd, "\pi"] \\
& U \arrow[dl, "\phi"'] \arrow[dr, "f"] & \\
V \arrow[rr, "g"] & & \pi x
\end{tikzcd} . \tag{3.1.6}$$

We obtain a suitable (and unique) arrow $f^*x \to g^*x$ as an image of ϕ. It is straightforward to check that this is indeed a morphism of groupoid fibrations. We will denote the obtained functor by $\underline{x}\colon \mathfrak{T}/\pi x \to \mathfrak{X}$.

Notice that from the functor $\underline{x}$ we can recover the object x as the image of $(\mathrm{Id}_{\pi x}\colon \pi x \to \pi x) \in \mathfrak{T}/\pi x$. △

As we have alluded to before, the groupoids as fibers of a groupoid fibration constructed in Proposition 3.5 contain much of the information of the entire groupoid fibration. To illustrate this fact, we show in the next proposition that fullness and faithfulness of a morphism is completely determined by the action on the fiber groupoids.

Proposition 3.9. *Let $F\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of groupoid fibrations over $\mathfrak{T}$. Denote $\mathfrak{X}_U(x,x')$ for the set of arrows $x \to x'$ in $\mathfrak{X}$ over the identity on U. The following assertions hold.*

(i) The morphism F is full if and only if for any two objects x, x' in $\mathfrak{X}$ over the same object $U \in \mathfrak{T}$ the map $\mathfrak{X}_U(x,x') \to \mathfrak{Y}_U(Fx,Fx')$ is surjective.

(ii) The morphism F is faithful if and only if for any two objects x, x' in $\mathfrak{X}$ over the same object $U \in \mathfrak{T}$ the map $\mathfrak{X}_U(x,x') \to \mathfrak{Y}_U(Fx,Fx')$ is injective.

Proof. In both cases, the condition is obviously necessary. We focus on the converses. Denote $\pi_{\mathfrak{X}}\colon \mathfrak{X} \to \mathfrak{T}$ and $\pi_{\mathfrak{Y}}\colon \mathfrak{Y} \to \mathfrak{T}$ for the groupoid fibrations. Let $x, \widetilde{x}$ be any objects in $\mathfrak{X}$. Denote $\widetilde{U}$ for $\pi_{\mathfrak{X}}\widetilde{x}$.

Assume the condition of (i). Let $\gamma\colon Fx \to F\widetilde{x}$ be an arrow in $\mathfrak{Y}$. By the pullback condition of groupoid fibrations, we obtain an arrow $\beta\colon x' \to \widetilde{x}$ in $\mathfrak{X}$ over $\pi_{\mathfrak{Y}}\gamma$. Here, the object x' must lie over U. Applying the Cartesian arrow axiom, we obtain an arrow $\alpha\colon Fx \to Fx'$ in $\mathfrak{Y}$ making the following prism commutes.

$$
\begin{tikzcd}
 & Fx \arrow[dl, dashed, "\alpha"'] \arrow[dr, "\gamma"] \arrow[dd, "\pi_{\mathfrak{Y}}"] & \\
Fx' \arrow[rr, "F\beta"] \arrow[dd, "\pi_{\mathfrak{Y}}"] & & F\widetilde{x} \arrow[dd, "\pi_{\mathfrak{Y}}"] \\
 & U \arrow[dl, "\mathrm{Id}_U"'] \arrow[dr, "\pi_{\mathfrak{Y}}\gamma"] & \\
U \arrow[rr, "\pi_{\mathfrak{Y}}\gamma"] & & \widetilde{U}
\end{tikzcd}
\tag{3.1.7}
$$

Using the assumed condition, we find an arrow $\widetilde{\alpha}\colon x \to x'$ such that $F\widetilde{\alpha} = \alpha$. By functoriality, we conclude that

$$F(\beta \circ \widetilde{\alpha}) = F\beta \circ F\widetilde{\alpha} = F\beta \circ \alpha = \gamma. \tag{3.1.8}$$

This shows the first claim.

Now suppose the condition of (ii). Let $\beta, \gamma\colon x \to \widetilde{x}$ be arrows such that $F\beta = F\gamma$. Since there holds

$$\pi_{\mathfrak{X}}\beta = \pi_{\mathfrak{Y}}F\beta = \pi_{\mathfrak{Y}}F\gamma = \pi_{\mathfrak{X}}\gamma, \tag{3.1.9}$$

we can apply the by the Cartesian arrow axiom to see there is just one arrow $Fx \to Fxs$ over Id_U in $\mathfrak{Y}$ making the following diagram commute.

$$
\begin{tikzcd}
 & Fx \arrow[dl, dashed, "\alpha"'] \arrow[dr, "F\gamma"] \arrow[dd, "\pi_{\mathfrak{Y}}"] & \\
Fx \arrow[rr, "F\beta"] \arrow[dd, "\pi_{\mathfrak{Y}}"] & & F\widetilde{x} \arrow[dd, "\pi_{\mathfrak{Y}}"] \\
 & U \arrow[dl, "\mathrm{Id}_U"'] \arrow[dr, "\pi_{\mathfrak{X}}\gamma"] & \\
U \arrow[rr, "\pi_{\mathfrak{X}}\beta"] & & \widetilde{U}
\end{tikzcd}
\tag{3.1.10}
$$

However, we know such arrow $Fx \to Fx$. Namely, the identity on Fx. Therefore, any arrow $\phi\colon x \to x$ over Id_U in $\mathfrak{X}$ making, the diagram

$$
\begin{tikzcd}
 & x \arrow[dl, "\phi"'] \arrow[dr, "\beta"] & \\
x \arrow[rr, "\alpha"] & & \widetilde{x}
\end{tikzcd}
\tag{3.1.11}
$$

commute, must have $F\phi = \mathrm{id}_{Fx}$. Hence, by the assumed condition (ii), we conclude that $\phi = \mathrm{id}_x$ and thus $\alpha = \beta$. This shows the claim. $\square$

In the same spirit, we can detect essentially surjectiveness of morphisms on the fiber groupoids.

Proposition 3.10. *A morphism $F\colon \mathfrak{X} \to \mathfrak{Y}$ of groupoid fibrations is essentially surjective if and only if for every $y \in \mathfrak{Y}$ lying over some object U, there exists an object $x \in \mathfrak{X}$ such that Fx is isomorphic to y with an isomorphism over the identity on U.*

Proof. Suppose that F is essentially surjective. Take an object $y \in \mathfrak{Y}$. Now there exists an x' such that $y \cong Fx'$ however this isomorphism need to lie over the identity on U. Nevertheless, denote $\phi \colon y \to Fx'$ for the isomorphism and suppose it lies over some isomorphism $\psi \colon U \to U$. Then by the pullback property of groupoid fibrations, we have a map $\psi^* x' \to x'$ and thus, by the Cartesian arrow property, an arrow $F\psi^* x' \to y$ over the identity on U making the diagram

$$\begin{tikzcd}
& F\psi^* x' \arrow[dl, dashed] \arrow[dr] \arrow[dd, "\pi_{\mathfrak{Y}}"] & \\
y \arrow[rr, "\phi"] \arrow[dd, "\pi_{\mathfrak{Y}}"'] & & Fx' \arrow[dd, "\pi_{\mathfrak{Y}}"] \\
& U \arrow[dl, "\mathrm{Id}_U"'] \arrow[dr, "\psi"] & \\
U \arrow[rr, "\psi"] & & U
\end{tikzcd} \tag{3.1.12}$$

commute. This provides us with the necessary isomorphism over Id_U.

The converse is obvious. $\square$

Hereby, we have seen that equivalence of groupoid fibrations is detected in the fiber groupoids. Therefore, the notion of isomorphism, which just assumes the functor is an equivalence of categories, can be detected simply in the fibers.

3.1.2 Representables

The groupoid fibrations arising as slices, like in Example 3.3, take a special place in the category of groupoid fibrations. We will call them *representable*:

Definition 3.11. A category fibered in groupoids $\mathfrak{X} \to \mathfrak{T}$ is **representable** if there exists an object $X \in \mathfrak{T}$ such that the slice $\mathfrak{T}/X \cong \mathfrak{X}$ as groupoid fibrations. We will use the notation $\underline{X}$ for the category fibered in groupoids represented by X. ▲

Definition 3.12. Suppose we have two categories fibered in groupoids $\mathfrak{X} \to \mathrm{Mfld}$ and $: \mathfrak{Y} \to \mathrm{Mfld}$ over the category of manifolds Mfld. We say that a morphism of categories fibered in groupoids $f \colon \mathfrak{X} \to \mathfrak{Y}$ is a **representable** if for every manifold U and morphism $\underline{U} \to \mathfrak{Y}$, the fibered product $\mathfrak{X} \times_{\mathfrak{Y}} \underline{U}$ is represented by some V. If additionally the induced map on manifolds $V \to U$ is a submersion, then the map f is a **representable submersion**. Moreover, we say that a morphism of categories fibered in groupoids $\mathfrak{X} \to \mathfrak{Y}$ is an **epimorphism** if for any manifold U and morphism $\underline{U} \to \mathfrak{X}$ there is a covering $(U_i \to U)_{i \in I}$ and there exist morphisms $\underline{U_i} \to \mathfrak{X}$ such that the following diagram 2-commutes for all $i \in I$.

$$\begin{tikzcd}
\underline{U_i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\!\!\swarrow"] & \underline{U} \arrow[d] \\
\mathfrak{X} \arrow[r] & \mathfrak{Y}
\end{tikzcd} \tag{3.1.13}$$

Putting everything together, we say that a morphism of categories fibered in groupoids $f \colon \mathfrak{X} \to \mathfrak{Y}$ is a **surjective representable submersion** if it is a representable submersion and an epimorphism. ▲

Lemma 3.13 ([BX06, Lemma 2.8]). *Let F be a sheaf over SMfld. Let X be a (super)manifold and $F \to \underline{X}$ a morphism and let $(U_i)_{i\in I}$ be a cover of $|X|$. If for every $i \in I$ the sheaf $U_i \times_X F$ is representable, then F is representable.*

Proof. Without loss of generality, we can assume that the cover $(U_i)_{i\in I}$ consists of opens of $|X|$. Otherwise, we can choose a refinement of opens. Suppose that $U_i \times_X F$ is represented by some F_i. Notice that for all $i, j \in I$, the fibered product $(U_i \cap U_j) \times_X F \subseteq U_i \times_X F$ is representable, say by some F_{ij}. Moreover, there are inclusions of open subsets $F_{ij} \to F_i$ and $F_{ij} \to F_j$. Therefore, we can glue the F_{ij} together. We obtain a (super)manifold representing F. □

3.2 Stacks

With some additional axiom imposed on a category fibered in groupoids, we obtain the notion of a stack.

Definition 3.14. Let $\mathfrak{X} \to \mathrm{Set}$ be a category fibered in groupoids. We say that $\mathfrak{X}$ is a **stack** over Set if the following axioms are satisfied for every object $S \in \mathrm{Set}$:

- (**Locality**) Let $x, y \in \mathfrak{X}$ be objects over S and $\phi, \psi \colon x \to y$ isomorphisms in $\mathfrak{X}$ over the identity. If for some cover $(U_i)_{i\in I}$ of X there holds $\phi|_{U_i} = \psi_{U_i}$ for all $i \in I$, then there holds $\phi = \psi$.
- (**Gluing**) Let $x, y \in \mathfrak{X}$ be objects over S and $(U_i)_{i\in I}$ a cover of S with isomorphisms $\phi_i \colon x|U_i \to y|U_i$ over the identity on U_i for all $i \in I$. If there holds $\phi_i|(U_i \cap U_j) = \phi_j|(U_i \cap U_j)$, then there is a morphism $\phi \colon x \to y$ such that $\phi|U_i = \phi_i$.
- (**Descent**) Let $(U_i)_{i\in I}$ be a cover of S with objects x_i over U_i and isomorphisms $(\phi_{ij} \colon x_i|(U_i \cap U_j) \to x_j|(U_i \cap U_j))_{i,j\in I}$ over the identity on $U_i \cap U_j$ satisfying the cocycles condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ taken in the fiber $\mathfrak{X}_{U_i \cap U_j \cap U_k}$. Then there exists an object x over S and isomorphisms $\phi_i \colon x|U_i \to x_i$ such that $\phi_{ij} \circ \phi_i = \phi_j$ in the fiber $\mathfrak{X}_{U_i \cap U_j}$.

A functor $\mathfrak{X} \to \mathfrak{Y}$ is a **morphism of stacks** if it is a morphism of groupoid fibrations. ▲

Remark 3.15. The pullbacks used in the axioms involve some choices. They are only unique up to unique isomorphism, see Proposition 3.6. However, none of the properties depend on the exact choice. ▽

Remark 3.16. Notice that the morphism ϕ obtained from gluing is unique by the locality axiom. Moreover, the object x obtained by the descent axiom is unique up to unique isomorphism by the other two axioms. ▽

Remark 3.17. Instead of landing in the category of sets, we can, similar to Remark 2.19, land in any category like rings, algebras or vector spaces. ▽

Remark 3.18. For generalizing the source of the stack, we can only use categories with a *Grothendieck Topology.* This gives us the suitable notion of covers. For more details on Grothendieck topologies, I refer to [Vis04]. When working on some catgeory with a Grothendieck topology, we should to interpret the intersection $U_i \cap U_j$ as the fibered product $U_i \times_S U_j$ along the inclusion maps. For us, the most important case is when we take the category of smooth (super) manifolds with smooth maps, obtaining **stacks of (super) manifolds**. ▽

Example 3.19. For a (super)manifold X, the slice category SMfld/X is a stack. In Example 3.3, we have seen it is a groupoid fibration. Notice that any morphism over the identity must be the identity itself. Therefore, the locality and gluing property hold trivially. The descent axiom holds, since we can glue opens together to a supermanifold when the given cocycle condition holds. We will use the same terminology and notation as for groupoid fibrations that the supermanifold X represents the stack $\underline{X}$ constructed by the slice category. △

Example 3.20 ([Hei05, Example 1.5])**.** Given a supermanifold X with an action of a super Lie group G. Denote $a\colon G \times X \to X$ for the action map. We define the **quotient stack** $[X/G]$ to have objects the pairs $(P \xrightarrow{p} S, P \xrightarrow{f} X)$. Here, p is a principal G-bundle and f an equivariant map for the relevant G action. The morphisms of $[X/G]$ are G-equivariant maps $P \to P'$ between principal bundles $P \to S$ and $P' \to S'$. This category is a stack over SMfld under the projection to S. △

More examples of stacks arise as categories of families of $(\mathbb{M}, G)$-(super)manifolds with their isometries as defined in Chapter 2.5.

Proposition 3.21. *Given a (super) model geometry $(\mathbb{M}, G)$ and S a (super)manifold. The category $\mathcal{M}(\mathbb{M}, G)$ with as objects the S-families of $(\mathbb{M}, G)$-(super)manifolds for all (super) manifolds S and as arrows the isometries between them is a stack under the canonical projection to the index space S.*

Proof. We first check that the functor $\pi\colon \mathcal{M}_S(\mathbb{M}, G) \to \mathrm{SMfld}$ is a groupoid fibration. Take an S-family of $(\mathbb{M}, G)$-manifolds $Y \to S$, a manifold S' and a map $f\colon S' \to S$. We can now consider the pullback manifold $f^*Y = Y \times_S S'$ in the diagram

$$\begin{tikzcd}
Y \times_S S' \arrow[r] \arrow[d] & Y \arrow[d] \\
S' \arrow[r, "f"] & S
\end{tikzcd} . \tag{3.2.1}$$

By pulling back the charts ϕ_i of Y over f, we obtain charts ψ_i of $Y \times_S S'$. If a transition function $\phi_j \circ \phi_i^{-1}$ takes in coordinates the form $S \times \mathbb{M} \ni (s, m) \mapsto (s, g_{ij}(s)m) \in S \times \mathbb{M}$, then the transition function $\psi_j \circ \psi_i^{-1}$ takes the form $S' \times \mathbb{M} \ni (s', m) \mapsto (s', g_{ij}(f(s'))m) \in S' \times \mathbb{M}$ for the same function $g_{ij}\colon S \to G$. The upper horizontal map in the diagram is an isometry. This shows the pullback property of groupoid fibrations.

To show the Cartesian arrow property take manifolds S, S' and S'' with $(\mathbb{M}, G)$ families Y, Y' and Y'' over them respectively and morphisms forming the diagram

$$\begin{tikzcd}
& Y'' \arrow[dd] \arrow[dr, "\phi"] & \\
Y' \arrow[rr, "\psi"] \arrow[ddd] & & Y \arrow[ddd] \\
& S'' \arrow[ddl, "f"'] \arrow[ddr] & \\
& & \\
S' \arrow[rr] & & S
\end{tikzcd} . \tag{3.2.2}$$

Since, the isometries ϕ and ψ are diffeomorphisms in the fibers over S, S' and S'', there exists a unique map $Y'' \to Y'$ over f which is an isometry in every fiber. Considering this map in local charts, we deduce smoothness of this map. This shows the required property. We have shown that $\mathcal{M}(\mathbb{M}, G)$ is a groupoid fibration.

Since isometries of families are completely governed by their local properties on the index manifold, the locality and gluing axioms on the morphism hold trivially. Since the families can be retrieved from gluings of opens in together opens of $S\times\mathbb{M}$, provided a suitable cocycle condition, restricting sufficiently, we deduce that the descent axiom holds. This completes the proof. □

3.2.1 Morphisms of Stacks

Morphisms of stacks are defined simply as morphisms of the underlying groupoid fibrations. However, the additional axioms imposed on the groupoid fibrations do enforce properties of the morphisms. The next proposition is particularly useful.

Proposition 3.22. *Let $F\colon \mathfrak{X}\to\mathfrak{Y}$ be a fully faithful morphism of stacks. The morphism F is essentially surjective if and only if for every object $y\in\mathfrak{Y}$ lying over some object U, there exists a cover $(U_i)_{i\in I}$ of U with for every $i\in I$ an object $x_i\in\mathfrak{X}$ lying over U_i such that $Fx_i\cong y|U_i$ over the identity on U_i.*

Proof. The implication left to right is directly implied by the similar result on groupoid fibrations, see Proposition 3.10. We show the converse statement.

Denote $U_{ij}=U_i\cap U_j$ for all $i,j\in I$. Fix an isomorphism $\phi_i\colon Fx_i\to y|U_i$ over the identity on U_i for every $i\in I$. Consider the objects $x_i|U_{ij}$. We have by construction a morphism $\iota_i\colon x_i|U_{ij}\to x_i$ over the inclusion $U_{ij}\to U_i$. Hence, the morphism $F\iota_i$ in $\mathfrak{Y}$ lies over the identity. By uniqueness of pullbacks, Proposition 3.6, there is a unique isomorphism $F(x_i|U_{ij})\cong F(x_i)|U_{ij}$ over the identity on U_{ij}. Using the isomorphism ϕ_i we obtain an isomorphism $F(x_i|U_{ij})\cong(y|U_i)|U_{ij}\cong y|U_{ij}$. Applying the same argument with the indices i and j switches, we obtain an isomorphism $F(x_j|U_{ij})\cong y|U_{ij}$. Composing the isomorphisms, we find an isomorphism $\psi_{ij}\colon F(x_i|U_{ij})\to F(x_j|U_{ij})$. Since the functor F is fully faithful, we can reflect this isomorphism to an isomorphism $\widetilde{\psi}_{ij}\colon x_i|U_{ij}\to x_j|U_{ij}$.

We check the cocycle condition on the isomorphisms ψ_{ij}. Notice that it suffices to check it on the $\widetilde{\psi}_{ij}$'s instead, because F is fully faithful. The composition $\widetilde{\psi}_{jk}\circ\widetilde{\psi}_{ij}$ is induced by the following chain of isomorphisms

$$\begin{aligned}F(x_i|U_{ijk})&\cong F(x_i)|U_{ijk}\cong y|U_i|U_{ijk}\cong y|U_{ijk}\cong y|U_j|U_{ijk}\cong F(x_j)|U_{ijk}\cong F(x_j|U_{ijk})\\&\cong F(x_j)|U_{ijk}\cong y|U_j|U_{ijk}\cong y|U_{ijk}\cong y|U_k|U_{ijk}\cong F(x_k)|U_{ijk}\cong F(x_k|U_{ijk})\end{aligned}. \tag{3.2.3}$$

Notice that the fourth term and the third to last term are equal. Therefore, the isomorphism over the identity on U_{ijk} in between is the identity on $y|U_{ijk}$. Hence, we can shrink the chain of isomorphisms to

$$F(x_i|U_{ijk})\cong F(x_i)|U_{ijk}\cong y|U_i|U_{ijk}\cong y|U_{ijk}\cong y|U_k|U_{ijk}\cong F(x_k)|U_{ijk}\cong F(x_k|U_{ijk}) \quad . \tag{3.2.4}$$

This corresponds precisely to the chain of isomorphisms giving the isomorphism $\widetilde{\psi}_{ik}$ restricted to U_{ijk}. This shows that the cocycle condition holds for the $\widetilde{\psi}_{ij}$'s. Hence, it holds for the ψ_{ij}'s.

The gluing property of stacks now gives us an object $x\in\mathfrak{X}$ over U and isomorphisms $\psi_i\colon x|U_i\to x_i$ such that $\psi_{ij}\circ\psi_i=\psi_j$. We show that Fx is isomorphic to y. Notice that $\phi_j|U_{ij}\circ\widetilde{\psi}_{ij}$ corresponds to the chain of isomorphisms

$$F(x_i|U_{ij})\cong F(x_i)|U_{ij}\cong y|U_i|U_{ij}\cong y|U_{ij}\cong y|U_j|U_{ij}\cong F(x_j)|U_{ij}\cong y|U_j|U_{ij}\cong y|U_{ij}. \tag{3.2.5}$$

Similar to the argument above, we can shrink this chain to

$$F(x_i|U_{ij}) \cong F(x_i)|U_{ij} \cong y|U_i|U_{ij} \cong y|U_{ij}. \tag{3.2.6}$$

We conclude that $\phi_j|U_{ij} \circ \widetilde{\psi}_{ij} = \phi_i|U_{ij}$. Hereby, we obtain the identity

$$\phi_i|U_{ij} \circ (F\psi_j)|U_{ij} = \phi_j|U_{ij} \circ \widetilde{\psi}_{ij} \circ (F\psi_i)|U_{ij} = \phi_j|U_{ij} \circ (F\psi_j)|U_{ij}. \tag{3.2.7}$$

By the gluing action of stacks, we obtain an isomorphism $Fx \to y$. As requested. □

Corollary 3.23. *A morphism of stacks $F\mathfrak{X} \to \mathfrak{Y}$ is an equivalence if and only if the following two conditions hold:*

- *For any two objects $x, x' \in \mathfrak{X}$ over the same object U the map $\mathfrak{X}_U(x, x') \to \mathfrak{Y}_U(Fx, Fx')$ is bijective.*
- *For every object $y \in \mathfrak{Y}$ lying over some object U, there exists a cover $(U_i)_{i\in I}$ of U with for every $i \in I$ an object $x_i \in \mathfrak{X}$ lying over U_i such that $Fx_i \cong y|U_i$ over the identity on U_i*

Proof. Apply Proposition 3.9, Proposition 3.22 and the usual construction of a pseudo-inverse for a fully faithful and essentially surjective functor, e.g., see [Rie14, Theorem 1.5.9]. □

3.2.2 Differentiable Stacks

In the case of stacks on (super) manifolds, as in Remark 3.18, we can make a further definition. We will say when stacks themselves are *differentiable*. Naturally, the stacks represented by a smooth manifold should be differentiable. Taking this as a guidance, we define the following.

Definition 3.24. A stack $\mathfrak{X} \to \mathrm{Mfld}$ is called **(super) differentiable** if there exists a (super) manifold X and a surjective representable submersion $\underline{X} \to \mathfrak{X}$. The (super) manifold X and the surjective representable submersion $\underline{X} \to \mathfrak{X}$ together are called a **(super) presentation** or **(super) atlas** of $\mathfrak{X}$. ▲

Given an atlas $\underline{X} \to \mathfrak{X}$ on a stack, we can form the 2-fiber product

$$\begin{tikzcd} \underline{X} \times_{\mathfrak{X}} \underline{X} \arrow[r] \arrow[d] & \underline{X} \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ \underline{X} \arrow[r] & \mathfrak{X} \end{tikzcd} \tag{3.2.8}$$

Since an atlas $s\colon \underline{X} \to \mathfrak{X}$ is a representable submersion by definition, we know that the fibered product $\underline{X} \times_{\mathfrak{X}} \underline{X}$ is representable. Hence, we can see it as a manifold. Inspecting the points of $\underline{X} \times_{\mathfrak{X}} \underline{X}$ more closely, we see that it is given by triples (y, ϕ, y'), where $y\colon U \to X$ and $y'\colon U \to X$ are objects in $\underline{X}$ and $\phi\colon s(y) \to s(y')$ an isomorphism in $\mathfrak{X}$ over the identity on X. From this description, we can distill a composition rule on the fibered product

$$(y, \phi, y') \circ (y', \psi, y'') = (y, \psi \circ \phi, y''). \tag{3.2.9}$$

This composition rule makes the $\underline{X} \times_{\mathfrak{X}} \underline{X}$ into a groupoid.

Proposition 3.25. *Let $\underline{X} \to \mathfrak{X}$ be a differentiable stack over some (super)manifold X. Then the fibered product $\underline{X} \times_{\mathfrak{X}} \underline{X}$ is represented by a (super) Lie groupoid.*

Proof. The fibered product is representable, since the atlas is a representable submersion. The composition rule Eq. (3.2.9) provides us with an associative composition. The projections $\underline{X} \times_{\mathfrak{X}} \underline{X} \to \underline{X}$ to the first and third component provide us with source and target maps respectively. All we have to show is that these maps are surjective submersions. The fact that the source and target maps are surjective submersions is implied by the fact that the atlas is an epimorphism. □

Example 3.26. A stack $\underline{X}$ represented by a manifold X defines an atlas on itself. The Lie groupoid associated is simply the groupoid with only identity morphism on the points of X. △

Example 3.27. For any Lie group G, the category BG consisting of all principal G bundle with G-equivariant maps is a stack over Mfld. The map $\underline{*} \to BG$ which sends

$$\underline{*} \ni (U \to *) \mapsto G \times U \in BG$$

to the trivial G bundle over U defines an atlas for BG. The Lie groupoid can be identified with the group G, seen as a Lie groupoid. △

Example 3.28. Similarly, the stack $\underline{\mathrm{VecBun}_n}^{\mathrm{Iso}}$ of rank n vector bundles with morphisms the morphisms of vector spaces which are fiberwise isomorphisms forms a stack. It has a presentation given by the group GL_n. △

Example 3.29. For a (super)manifold X with a (super) Lie group action by G, the quotient stack from Example 3.20 admits an atlas given by sending an object $f\colon S \to X$ to the trivial principal bundle $S \times G \to S$ and the G-equivariance $S \times G \to X$ given in local coordinates by

$$S \times G \ni s, g \mapsto g \cdot f(s) \in X.$$

The map on arrows is canonical. The (super) Lie groupoid is the action groupoid of G acting on M. △

We can reformulate the conditions imposed on a presentation into weaker conditions. The proof goes through the following lemma.

Lemma 3.30 ([BX06, Lemma 2.11])**.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of stacks over SMfld and let U be a (super)manifold with an epimorphism $\underline{U} \to \mathfrak{Y}$. If the fibered product $\mathfrak{X} \times_{\mathfrak{Y}} \underline{U}$ is represented by some (super)manifold V and the map $V \to U$ is a submersion, then f is a representable submersion.*

Proof. Let W be a (super)manifold and $\underline{W} \to \mathfrak{Y}$ a morphism. Firstly, we show that the fibered product $\mathfrak{F} = \mathfrak{X} \times_{\mathfrak{Y}} \underline{W}$ is representable. Since $U \to \mathfrak{Y}$ is an epimorphism, we can find a cover $(W_i \to W)_{i \in I}$ of W and morphisms $\phi_i\colon W_i \to U$ such that the following diagram 2-commutes for every $i \in I$.

$$\begin{tikzcd} \underline{W_i} \arrow[r, hook] \arrow[d, "\phi_i"'] \arrow[dr, phantom, "\swarrow"] & \underline{W} \arrow[d] \\ \underline{U} \arrow[r] & \mathfrak{Y} \end{tikzcd} \tag{3.2.10}$$

With this 2-commutative square, we can form a 2 commutative Cartesian cube.

$$\begin{tikzcd}[row sep=small, column sep=small] \underline{W_i} \times_{\underline{W}} \mathfrak{F} \arrow[rr] \arrow[dr] \arrow[dd] & & \underline{V} \arrow[dr] \arrow[dd] & \\ & \underline{W_i} \arrow[rr] \arrow[dd] & & \underline{U} \arrow[dd] \\ \mathfrak{F} \arrow[rr] \arrow[dr] & & \mathfrak{X} \arrow[dr] & \\ & \underline{W} \arrow[rr] & & \mathfrak{Y} \end{tikzcd} \tag{3.2.11}$$

Since, the left, right and bottom side of this cube are pullback squares, so is the top square. Hence, we have an isomorphism $\underline{W_i} \times_{\underline{W}} \mathfrak{F} \cong \underline{W_i} \times_{\underline{U}} \underline{V}$. Therefore, the fibered product $\underline{W_i} \times_{\underline{W}} \mathfrak{F}$ is representable and thus by Lemma 3.13 the stack $\mathfrak{F}$ is representable, say by some (super)manifold F.

Also since the top square of the cube is a pullback and the fact that $V \to U$ is a submersion, so is the map $\underline{W_i} \times_{\underline{W}} \mathfrak{F} \to W_i$ a submersion. Now $F \to W$ is a submersion, since being submersive is a local property. $\square$

Corollary 3.31. *A stack $\mathfrak{X}$ over SMfld is differentiable if and only if there exists a morphism $\underline{X} \to \mathfrak{X}$ with the following properties:*

- *The groupoid $\underline{X} \times_{\mathfrak{X}} \underline{X}$ is represented by some super Lie groupoid Γ and the projections $\Gamma \to X$ are submersions.*
- *The morphisms $\underline{X} \to \mathfrak{X}$ is an epimorphism.*

Proof. The conditions are obviously necessary. Lemma 3.30 immediately shows that they are sufficient. $\square$

3.3 From Lie Groupoids to Stacks

In the previous section, we have seen that we can associate a (super) Lie groupoid to any differentiable stack. In this section, we will go the reverse route and construct a suitable differentiable stack from a given (super) Lie groupoid. This construction goes by *torsors* over a groupoid.

3.3.1 Torsors

Torsors are for a Lie groupoid what principal bundles are for groups. We give a definition.

Definition 3.32 ([BX06, Definition 2.8])**.** Let Γ be a (super) Lie groupoid, see Definition 2.50, and B a (super) manifold. A **right Γ-torsor** over B is a manifold E with a right Γ action, see Definition 2.51, together with a surjective submersion $\pi\colon E \to B$ such that if $\pi(p) = \pi(p')$, then there exists a unique $\gamma \in \Gamma$ such that $p \cdot \gamma$ is defined and $p \cdot \gamma = p'$. Notice that the right action of the groupoid includes a anchor map ⚓: $E \to \Gamma_0$, where Γ_0 is the base of the Lie groupoid Γ. ▲

Example 3.33. A super Lie groupoid Γ with its target map $\Gamma \to \Gamma_0$ to the base Γ_0 is a Γ-torsor. △

Example 3.34. Any principal bundle with any structure group G is a G-torsor. Here, we see G as a groupoid. △

Example 3.35 (Trivial torsors, [BX06, Example 2.20])**.** Denote Γ_0 for the base manifold of a (super) Lie groupoid Γ. Let $f\colon B \to \Gamma_0$ be a smooth map. The fiber product $B \times_{\Gamma_0} \Gamma$ in the diagram

$$\begin{tikzcd} B \times_{\Gamma_0} \Gamma \arrow[r] \arrow[d, "\pi"] & \Gamma \arrow[d, "s"] \\ B \arrow[r, "f"] & \Gamma_0 \end{tikzcd} \tag{3.3.1}$$

is a Γ-torsor over B with projection π. Here, the map $s\colon \Gamma \to \Gamma_0$ is the source map. The Γ action on $B \times_{\Gamma_0} \Gamma$ is given in local coordinates by acting on the right in the second coordinate

$$((s, \gamma), \delta) \mapsto (s, \gamma \cdot \delta) \in B \times_{\Gamma_0} \Gamma. \tag{3.3.2}$$

The anchor map is the projection to the second coordinate composed with the target map of the (super) Lie groupoid. We call the torsors constructed in this fashion **trivial torsors**. △

Many more examples of torsors arise from differentiable stacks. The following lemma outlines the construction.

Lemma 3.36. *Given a differentiable stack $\mathfrak{X}$ with atlas $s\colon \underline{X} \to \mathfrak{X}$, a manifold U and a morphism of stacks $p\colon \underline{U} \to \mathfrak{X}$. Then the 2-fibered product $\underline{X} \times_{\mathfrak{X}} \underline{U}$ in the diagram*

$$\begin{tikzcd} \underline{X} \times_{\mathfrak{X}} \underline{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \underline{U} \arrow[d, "p"] \\ \underline{X} \arrow[r, "s"] & \mathfrak{X} \end{tikzcd} \tag{3.3.3}$$

is represented by a torsor for the groupoid $\underline{X} \times_{\mathfrak{X}} \underline{X}$ as constructed in Proposition 3.25. The anchor maps is simply the projection onto $\underline{X}$

Proof. The fact that $\underline{X} \times_{\mathfrak{X}} \underline{U}$ is representable follows immediately from the fact that the atlas s is a representable submersion. Moreover, the induced map on manifolds by the morphism $\underline{X} \times_{\mathfrak{X}} \underline{U} \to \underline{U}$ is a surjective submersion (surjectivity follows from surjectivity of s).

Inspecting the objects of $\underline{X} \times_{\mathfrak{X}} \underline{U}$, we see that they are exactly triples (g, ϕ, f) where $f\colon V \to X$ and $g\colon V \to U$ are maps and an isomorphism $\phi\colon p(g) \to s(f)$ in $\mathfrak{X}$. Therefore, the groupoid $\underline{X} \times_{\mathfrak{X}} \underline{X}$, which has as objects the triples (f, ψ, f') with $f\colon V \to X$, $f'\colon V \to X$ maps and isomorphism $\psi\colon s(f) \to s(f')$ in $\mathfrak{X}$ over Id_U, has an obvious action on $\underline{X} \times_{\mathfrak{X}} \underline{U}$ from the right. Namely, the action given by

$$(g, \phi, f) \cdot (f, \psi, f') = (g, \psi \circ \phi, f'). \tag{3.3.4}$$

A straightforward check shows this is indeed a groupoid action. The anchor map is clearly as claimed. Moreover, if (g, ϕ, f) and (g, ϕ', f') are objects of $\underline{X} \times_{\mathfrak{X}} \underline{U}$, then the object $\gamma = (f, \phi' \circ \phi^{-1}, f')$ is the unique object $\gamma \in \underline{X} \times_{\mathfrak{X}} \underline{X}$ such that $(g, \phi, f) \cdot \gamma = (g, \phi', f')$. This shows that $\underline{X} \times_{\mathfrak{X}} \underline{U}$ is indeed represented by a torsor for the relevant groupoid. □

3.3.2 A Stack of Torsors

In Example 3.27, we have seen that principal bundles give a differentiable stack in a natural way. Since, torsors are the natural generalization of principal bundles from groups to groupoids, we will now generalize this example. Firstly, we need a suitable notion of morphism between stacks.

Definition 3.37. A **morphism of Γ-torsors** $\pi\colon E \to B$ and $\pi'\colon E' \to B'$ is a commutative diagram of the following form

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ B \arrow[r] & B' \end{tikzcd} \tag{3.3.5}$$

such that the top map is Γ-equivariant. ▲

With these morphisms, we can form the category $B\Gamma$ of Γ-torsors and their morphisms. Notice that the trivial torsors play a special role, since any torsor is locally isomorphic to a trivial torsor. Indeed, for a torsor $\pi\colon E \to B$, the map π is a surjective submersion. Hence, it admits local sections. A section $\sigma\colon B \to E$ constructs an isomorphism between E and the trivial torsor constructed using the map $⚓ \circ \sigma$ where $⚓\colon E \to \Gamma_0$ is the anchor map. The isomorphism is given by the assignment in local coordinates

$$B \times_{\Gamma_0} \Gamma \ni (b, \gamma) \mapsto \sigma(b) \cdot \gamma \in E. \tag{3.3.6}$$

Using this local triviality property, one can easily deduce that for any morphism of torsors the diagram in Eq. (3.3.5) is a pullback diagram.

The category $B\Gamma$ admits a canonical forgetful functor $B\Gamma \to$ SMfld mapping a torsor $E \to B$ to the base manifold B. This functor turns out to be a differentiable stack. We show this fact in several steps. We start by showing that $B\Gamma$ is a stack over SMfld, Proposition 3.38. Secondly, we show there is a suitable functor $\underline{\Gamma_0} \to B\Gamma$ from which we can recover the (super) Lie groupoid Γ, Lemma 3.39. Lastly, we deduce that the constructed functor in Lemma 3.39 is actually a presentation, Proposition 3.40.

Proposition 3.38 ([Blo07, Proposition 2.16])**.** *For any (super) Lie groupoid, the category $B\Gamma$ is a stack over SMfld.*

Proof. We first show that the functor $B\Gamma \to$ SMfld is a groupoid fibration. Notice that for any Γ-torsor $E' \to B'$ and a map $B \to B'$, the pullback $E' \times_{B'} B$ is naturally a Γ-torsor over B. Moreover, any morphism of Γ-torsors gives rise to a pullback diagram, as in Eq. (3.3.5). Hence, any arrow in $B\Gamma$ is Cartesian. This shows that $B\Gamma \to$ SMfld is a groupoid fibration.

To see that $B\Gamma \to$ SMfld is a stack, notice that the locality and gluing axiom are fulfilled since the morphisms of $B\Gamma$ are given by maps of supermanifold and Γ-equivariance is a local property on the base space of the torsor. For the descent axiom, notice that we can glue torsors from local pieces if the cocycle condition is satisfied. This is similar to the way one can glue vector bundles or principal bundles together from cocycles. Suppose that $\mathcal{U} = (U_i)_{i\in I}$ is an open cover of some space B with cocycles $\phi_{ij}\colon E_{\to} E_j$ as in the descent axiom. In particular, we have torsors $E_i \to U_i$ for all $i \in I$. Consider the pushout space

$$E := \sqcup_{i\in I} E_i / \sim\ . \tag{3.3.7}$$

Here, the pushout is taken over the cocycle maps ϕ_{ij}. Straightforward checks show that this indeed defines a torsor satisfying the requirements of the descent axiom. This shows that $B\Gamma \to$ SMfld is a stack. □

3.3.3 Presentation for the Torsor Stack

The stack of principal bundles, Example 3.27, was one of our prime examples of stacks admiting a presentation. Here, the presentation was given by embedding trivial principal bundles into BG. The stack of trivial bundles was then identified with the stack $\underline{*}$ (the only datum of a trivial principal bundle is the base space). The trivial torsors defined in Example 3.35, give us the natural generalization. Now, the stack $\underline{*}$ should be generalized to the base of the super Lie groupoid $\underline{\Gamma_0}$.

Lemma 3.39. *Let Γ be a (super) Lie groupoid. The assignment $s\colon \underline{\Gamma_0} \to B\Gamma$ by sending an object $(f\colon U \to \Gamma_0)$ to the trivial torsor $U \times_{\Gamma_0} \Gamma$ extends to morphisms and the 2-fiber product $\underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0}$ equivalent to $\underline{\Gamma}$ with an equivalence identifying the two projections $\underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0} \to \underline{\Gamma_0}$ with the source and target map of the (super) Lie groupoid.*

Proof. The extension of s to morphisms is straightforward. We focus on the other claims. We follow the proof of [Blo07, Proposition 2.17].

Objects of $\underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0}$ are given by triples (a, ϕ, b) where $a, b\colon U \to \Gamma_0$ are smooth maps of supermanifolds and an isomorphism of Γ-torsors $\phi\colon \hat{b} \to \hat{a}$. Here, $\hat{a}$ and $\hat{b}$ denote the trivial torsors $U \times_{\Gamma_0} \Gamma$ defined by the maps a and b respectively. Denote the isomorphism ϕ explicitly by $\phi(u, \gamma) = (\phi_U(u, \gamma), \phi_\Gamma(u, \gamma))$. For any object (a, ϕ, b) of $\underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0}$, we can define a map $\eta_{(a,\phi,b)}\colon U \to \Gamma$ in local coordinates by

$$U \ni u \mapsto \phi_\Gamma(u, \mathrm{id}_{b(u)}) \in \Gamma. \tag{3.3.8}$$

Conversely, for any map $\eta\colon U \to \Gamma$, we can define in local coordinates

$$\begin{aligned} &a_\eta\colon U \to \Gamma_0 && \phi_\eta\colon \hat{b} \to \hat{a} && b_\eta\colon U \to \Gamma_0 \\ &a_\eta(u) = s_\Gamma(\eta(u)) && \phi_\eta(u, \gamma) = (u, \eta(u)\gamma) && b_\eta(u) = t_\Gamma(\eta(u)) \end{aligned} \tag{3.3.9}$$

Here, s_Γ and t_Γ denote the source and target map $\Gamma \to \Gamma_0$ of the (super) Lie groupoid respectively. Notice that

$$\phi(u, \gamma\gamma') = (u, \eta(u)\gamma\gamma') = (u, \eta(u)\gamma)\gamma' = \phi(u, \gamma)\gamma'. \tag{3.3.10}$$

Moreover, notice that $\hat{a} \ni (u, \gamma) \mapsto (u, \eta(u)^{-1}\gamma) \in \hat{b}$ is an inverse of ϕ. Therefore, the map ϕ is an isomorphism of Γ-torsors. Hence, we have constructed maps on objects

$$\begin{aligned} &\underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0} \to \underline{\Gamma} && \underline{\Gamma} \to \underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0} \\ &(a, \phi, b) \mapsto \eta_{(a,\phi,b)} && \eta \mapsto (a_\eta, \phi_\eta, b_\eta) \end{aligned}. \tag{3.3.11}$$

These assignments extend straightforwardly to morphisms and become morphisms of stacks. We show that they are each other's pseudo-inverses (hence they form the required equivalence of stacks). Composing the two morphisms, we see that

$$\eta_{(a_\eta,\phi_\eta,b_\eta)}(u) = (\phi_\eta)_\Gamma\left(u, \mathrm{id}_{b_\eta(u)}\right) = \eta(u)\mathrm{id}_{b_\eta(u)} = \eta(u). \tag{3.3.12}$$

So on $\underline{\Gamma}$, there is nothing to show. Composing the other way around, we find

$$\begin{aligned} a_{\eta_{a,\phi,b}}(u) &= s_\Gamma(\eta_{a,\phi,b}(u)) = s_\Gamma(\phi_\Gamma(u, \mathrm{id}_{b(u)})) = a(\phi_U(u)) \\ \phi_{\eta_{a,\phi,b}}(u, \gamma) &= (u, \eta_{a,\phi,b}(u)\gamma) = (u, \phi_\Gamma(u, \mathrm{id}_{b(u)})\gamma) = (u, \phi_\Gamma(u, \gamma)). \\ b_{\eta_{a,\phi,b}}(u) &= t_\Gamma(\eta_{a,\phi,b}(u)) = t_\Gamma(\phi_\Gamma(u, \mathrm{id}_{b(u)})) = b(u) \end{aligned} \tag{3.3.13}$$

We observe that ϕ_U gives us an isomorphism between a and $a_{\eta_{a,\phi,b}}$. Hence, it induces an isomorphism between (a, ϕ, b) and $(a_{\eta_{a,\phi,b}}, \phi_{\eta_{a,\phi,b}}, b_{\eta_{a,\phi,b}})$. This gives us the required natural isomorphism on $\underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0}$. This shows that $\underline{\Gamma}$ and $\underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0}$ are equivalent as stacks. Therefore, the stack $\underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0}$ is represented by Γ.

Tracing through the definitions, under this equivalence, the two maps $\Gamma \to \Gamma_0$ induced by the projections $\underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0} \to \underline{\Gamma_0}$ correspond precisely to the source and target maps of the (super) Lie groupoid. $\square$

Proposition 3.40 ([BX06, Proposition 2.21])**.** *For any (super) Lie groupoid Γ, the category $B\Gamma$ is a differentiable stack with a presentation $\underline{\Gamma_0} \to B\Gamma$.*

Proof. Take the functor $s\colon \underline{\Gamma_0} \to B\Gamma$ from the previous lemma. We show that it is a presentation, i.e., it is a surjective representable submersion.

We show that s is an epimorphism. Let U be a (super) manifold and a morphism of stacks $\beta\colon \underline{U} \to B\Gamma$. Denote $E \to B$ for the Γ-torsor $\beta(\mathrm{Id}_U\colon U \to U)$. Since any Γ-torsor is locally trivial, we can find an open cover B_i of B such that the pullbacks $E_i \to B_i$ along the inclusion $B_i \to B$ are trivial. By the pullback property of groupoid fibrations, we can find inclusions $U_i \to U$ such that $\beta(U_i \to U) = (E_i \to B_i)$. For any object $\mu\colon T \to U_i$ in $\underline{U_i}$, we can form the following diagram

$$\begin{tikzcd} T \arrow[r, "\mu"] \arrow[dr, "\mu"'] & U_i \arrow[r, hook] \arrow[d, hook] & U \arrow[dl, "\mathrm{Id}_U"] \\ & U & \end{tikzcd} . \tag{3.3.14}$$

Applying β yields the commuting prism

$$\begin{tikzcd} & E_\mu \arrow[dl, "\beta(\mu)"'] \arrow[dr] \arrow[dd] & \\ E_i \arrow[rr, hook] \arrow[dd] & & E \arrow[dd] \\ & B_\mu \arrow[dl] \arrow[dr] & \\ U_i \arrow[rr, hook] & & B \end{tikzcd} . \tag{3.3.15}$$

Here, $E_\mu \to B_\mu$ denotes the torsor given by $\beta(T \xrightarrow{\mu} U_i \hookrightarrow U)$. Since, the torsor $E_\mu \to B_\mu$ factors through a trivial torsor, it is trivial itself. Hence, we can find a map $\sigma_\mu\colon B_\mu \to \Gamma_0$ giving rise to this trivial torsor. The assignment $\mu \to \sigma_\mu$ extends easily to a functor $\underline{U_i} \to \underline{\Gamma_0}$ making the following diagram 2-commutes.

$$\begin{tikzcd} \underline{U_i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \underline{U} \arrow[d, "\beta"] \\ \underline{\Gamma_0} \arrow[r] & B\Gamma \end{tikzcd} \tag{3.3.16}$$

This shows that the functor $\underline{\Gamma_0} \to B\Gamma$ is an epimorphism.

Lemma 3.39 shows that the 2-fibered product $\underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0}$ is represented by Γ and both projections to Γ_0 correspond to the source and target map of the (super) Lie groupoid. Since the latter two are submersions, so are the projections $\underline{\Gamma_0} \times_{B\Gamma} \underline{\Gamma_0} \to \underline{\Gamma_0}$. An application of Corollary 3.31 completes the proof. ☐

3.3.4 Morita Equivalence

We have now established ways to go back and forth between (super) Lie groupoids and differentiable stacks. To recall: For a (super) Lie groupoid Γ the category $B\Gamma$ gives rise to a differentiable stack, Proposition 3.40. Conversely, for a differentiable stack $\mathfrak{X}$ with atlas $s\colon \underline{X} \to \mathfrak{X}$, obtain a (super) Lie groupoid representing $\underline{X} \times_{\mathfrak{X}} \underline{X}$, Proposition 3.25.

Theorem 3.41 ([BX06, Theorem 2.22])**.** *Let $\pi\colon \mathfrak{X} \to$ Mfld be a differentiable stack with atlas $s\colon \underline{X} \to \mathfrak{X}$. Let Γ represent $\underline{X} \times_{\mathfrak{X}} \underline{X}$. Then there is an equivalence of stacks $F\colon \mathfrak{X} \to B\Gamma$.*

Proof. Recall from Example 3.8 that we can view an object $x \in \mathfrak{X}$ as a morphism of groupoid fibration (or stacks) $\underline{x}\colon \underline{\pi x} \to \mathfrak{X}$. Using this morphism, we obtain a Γ-torsor $\underline{X} \times_{\mathfrak{X}} \underline{\pi x}$ by Lemma 3.36. We assign on objects $Fx = \underline{X} \times_{\mathfrak{X}} \underline{\pi x}$.

The assignment F extends on morphisms. Indeed, given an arrow $a\colon x \to x'$ in $\mathfrak{X}$, then for any object $(g, \phi, f) \in Fx$ the inner two squares of the following diagram are pullbacks

$$\begin{tikzcd} g^*x \arrow[r] \arrow[d, "\pi"] & x \arrow[r, "a"] \arrow[d, "\pi"] & x' \arrow[d, "\pi"] \\ \pi(g^*x) \arrow[r] & \pi x \arrow[r, "\pi a"] & \pi x' \end{tikzcd} \tag{3.3.17}$$

By pullback pasting, the outer square is also a pullback square. We conclude that there exists a unique isomorphism $\chi_{a,g}\colon \underline{x}(g) = g^*x \to ((\pi a) \circ g)^*x' = \underline{x'}((\pi a) \circ g)$. Now we can define $F(a)(g, \phi, f) = ((\pi a) \circ g, \chi_{a,g} \circ \phi, f)$. This extends F to arrows.

Notice from the description on arrows that F is faithful. Indeed, if $a, a'\colon x \to x'$ are arrows in $\mathfrak{X}$ such that $Fa = Fa'$, then $\chi_{a,g} = \chi_{a',g}$ for all g. In particular, this holds if $g = \mathrm{Id}_U$. This immediately implies that $a^*x' = (a')^*x'$ and since pullbacks are unique up to unique isomorphism, this in turn shows that $a = a'$.

We show that F is an equivalence of stacks. We will use the simpler criterion, Corollary 3.23. Let $x, x' \in \mathfrak{X}$ be objects over some V and $\phi\colon Fx \to Fx'$ an isomorphism over Id_V. Since Fx and Fx' are torsors, we can choose a cover $(V_i)_{i\in I}$ of V trivializing both torsors. In the trivializations, the isomorphism ϕ becomes an isomorphism $\Gamma \times_{\Gamma_0} U_i \to \Gamma \times_{\Gamma_0} U_i$. Notice that we can identify $F(x|U_i)$ and $F(x'|U_i)$ with these trivializations. By equivariance, the isomorphism ϕ looks in local coordinates like

$$(\gamma, u) \mapsto (\phi_\Gamma(u)\gamma, u). \tag{3.3.18}$$

Here, the maps $\phi_\Gamma\colon U_i \to \Gamma$ is given by the Γ coordinate of $\phi(\mathrm{Id}_u, u)$, where Id_u is the identity map on the relevant base point. Applying the Yoneda embedding to the map ϕ_Γ, we obtain a map $G\colon \underline{U_i} \to \underline{X} \times_{\mathfrak{X}} \underline{X}$. Notice that the map G sends an object $g \in \underline{U_i}$ to an isomorphism $\underline{x|U_i(g)} \cong \underline{x'|U_i(g)}$. Evaluating G on the object $\mathrm{Id}_{U_i} \in \underline{U_i}$ yields a map $\theta_i\colon x|U_i \to x'|U_i$. Carefully tracing through the definitions, shows that $F\theta_i = \phi|U_i$. Since F is faithful, the maps θ_i are unique with this property. Therefore, on the overlaps $U_i \cap U_j$ in the covers, the θ_i and θ_j must agree. Hence, by the gluing property of stacks, we obtain a morphism $\theta\colon x \to x'$ such that $\theta|U_i = \theta_i$ for all $i \in I$. Clearly, we have that $F\theta = \phi$. This shows the first requirement of Corollary 3.23.

For the second requirement of Corollary 3.23, notice that for any Γ-torsor $T \to U$ there is a trivializing cover $(U_i)_{i\in I}$. Hence, there exists sections $s_i\colon U_i \to T$ of the torsor. Applying the Yoneda embedding, we obtain morphisms $\underline{s_i}\colon \underline{U_i} \to \underline{X} \times_{\mathfrak{X}} \underline{X}$. Hence, by projecting on the target of the groupoid $\underline{X} \times_{\mathfrak{X}} \underline{X}$ and composing with the atlas, we obtain morphisms $\underline{x_i}\colon \underline{U_i} \to \mathfrak{X}$. Let $x_i \in \mathfrak{X}$ be the object related to $\underline{x_i}$ as in Example 3.8. Now $Fx_i = \underline{X} \times_{\mathfrak{X}} \underline{\pi x_i}$. Notice that the section $\underline{s_i}$ gives a trivialization of this torsor and thus a canonical identification of Fx_i and $T|U_i$. This shows the second requirement of Corollary 3.23 and thus completes the proof. $\square$

As the previous theorem shows, one composition of the functors between (super) Lie groupoids and differentiable stacks leaves the stack untouched, up to equivalence of stacks. The other composition we can make, will also leave the Lie groupoids invariant, up to so called *Morita equivalence*. Morita equivalence is named after the Japanese mathematician Kiiti Morita, who introduced the concept in abstract algebra. In this setting, two rings are called *Morita equivalent* if their categories of modules are equivalent. [AH97]

Definition 3.42. Two (super) Lie groupoid Γ and Γ' are called **Morita Equivalent** if their associated stacks of torsors $B\Gamma$ and $B\Gamma'$ are equivalent. ▲

For simplicitly, we show the following theorem just in the case of ordinary (non-super) geometry. The proofs of $(i) \implies (ii)$ and $(ii) \implies (iii)$ can be lifted to the super case straightforwardly. The proof form $(iii) \implies (i)$ is more involved to lift, since it uses quotients of manifolds, which are non-trivial in the super setting, see Chapter 3.3.6.

Theorem 3.43 ([BX06, Theorem 2.26])**.** *Let Γ and Γ' be Lie groupoids. Then the following are equivalent:*

(i) The Lie groupoids Γ and Γ' are Morita equivalent.

(ii) There exists a manifold Q with commuting groupoid actions of both Γ and Γ' making Q a Γ-torsor over Γ'_0 and a Γ'-torsor over Γ_0 such that the anchor maps $Q \to \Gamma_0$ and $Q \to \Gamma_0$ coincide with the projections of the Γ' and Γ-torsor respectively.

(iii) There exists a third Lie groupoid Ξ and groupoid homomorphisms $\phi\colon \Xi \to \Gamma$ and $\phi'\colon \Xi \to \Gamma'$ such that the maps $\phi_0\colon \Xi_0 \to \Gamma_0$ and $\phi'_0\colon \Xi_0 \to \Gamma'_0$ are surjective submersions and the diagrams

$$\begin{tikzcd} \Xi \arrow[r] \arrow[d, "\phi"] & \Xi_0 \times \Xi_0 \arrow[d, "\phi_0\times\phi_0"] \\ \Gamma \arrow[r] & \Gamma_0 \times \Gamma_0 \end{tikzcd} \qquad \begin{tikzcd} \Xi \arrow[r] \arrow[d, "\phi'"] & \Xi_0 \times \Xi_0 \arrow[d, "\phi'_0\times\phi'_0"] \\ \Gamma' \arrow[r] & \Gamma'_0 \times \Gamma'_0 \end{tikzcd} \tag{3.3.19}$$

are pullbacks. Here, the horizontal maps are given by the product of the source and target map of the relevant Lie groupoid.

Proof. We first show $(i) \implies (ii)$. Take an equivalence of stacks $\phi\colon B\Gamma' \to B\Gamma$. From Proposition 3.40, we know that there are presentations $s\colon \underline{\Gamma_0} \to B\Gamma$ and $s'\colon \underline{\Gamma'_0} \to B\Gamma'$. We can now form the 2-fibered product

$$\begin{tikzcd} \underline{\Gamma'_0} \times_{B\Gamma} \underline{\Gamma_0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow"] & \underline{\Gamma_0} \arrow[d, "s"] \\ \underline{\Gamma'_0} \arrow[r, "\phi\circ s'"'] & B\Gamma \end{tikzcd} . \tag{3.3.20}$$

Lemma 3.39 shows that the fibered product $\underline{\Gamma'_0} \times_{B\Gamma} \underline{\Gamma_0}$ is represented by Γ. From Lemma 3.36, we immediately deduce that $\underline{\Gamma'_0} \times_{B\Gamma} \underline{\Gamma_0}$ is a $\underline{\Gamma}$-torsor over $\underline{\Gamma'_0}$ with anchor the projection to $\underline{\Gamma_0}$. To show that it also is a Γ'-torsor over Γ_0 by the same lemma, we need to show that $\phi \circ s'$ is an atlas on $B\Gamma$. Since s is an atlas, we can apply Lemma 3.30 to see that $\phi \circ s'$ is a representable submersion. We are left to show that $\phi \circ s'$ is an epimorphism. Denote $\psi\colon B\Gamma \to B\Gamma'$ for a pseudo-inverse of ϕ. Let U be some manifold with a morphism $f\colon \underline{U} \to B\Gamma$. Now, by the fact that s' is an epimorphism, we can find an open cover $(U_i)_{i\in I}$ such that for every $i \in I$ there is a morphism $\underline{U_i} \to \underline{\Gamma'_0}$ making the following diagram 2-commute.

$$\begin{tikzcd} \underline{U_i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow"] & \underline{U} \arrow[d, "\psi\circ f"] \\ \underline{\Gamma'_0} \arrow[r, "s'"] & B\Gamma' \end{tikzcd} \tag{3.3.21}$$

Since $\phi \circ \psi$ is naturally isomorphic to the identity on $B\Gamma$, the following diagram is also 2-commutative.

$$\begin{tikzcd} \underline{U_i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \underline{U} \arrow[d, "f"] \\ \underline{\Gamma_0'} \arrow[r, "\phi\circ s'"] & B\Gamma \end{tikzcd} \tag{3.3.22}$$

This shows that $\phi \circ s'$ is an epimorphism. Inspecting the resulting actions immediately shows that they commute. This completes the proof of $(i) \implies (ii)$.

Secondly, we show $(ii) \implies (iii)$. Take a manifold Q as in the assumption. Using the maps $p\colon Q \to \Gamma_0$ and $p'\colon Q \to \Gamma_0'$, we can form the fibered product $\Xi = \Gamma' \times_{\Gamma_0'} Q \times_{\Gamma_0} \Gamma$. Here, the maps $\Gamma' \to \Gamma_0'$ and $\Gamma_0 \to \Gamma$ are the source maps of the relevant Lie groupoid. Let the projection to Q be the source map s_Ξ. Since the projections are also the anchor maps, we can also define the target map $t_\Xi\colon \Gamma' \times_{\Gamma_0'} Q \times_{\Gamma_0} \Gamma \to Q$ by assigning

$$\Xi \ni (\gamma', q, \gamma) \mapsto (q \cdot_\Gamma \gamma) \cdot_{\Gamma'} \gamma' = (q \cdot_{\Gamma'} \gamma') \cdot_\Gamma \gamma \in Q. \tag{3.3.23}$$

Here, we used the commutativity of the actions. For any two points $(\gamma', q, \gamma), (\widetilde{\gamma}', \widetilde{q}, \widetilde{\gamma}) \in \Xi$ such that

$$t_\Xi(\gamma', q, \gamma) = (q \cdot_\Gamma \gamma) \cdot_{\Gamma'} \gamma' = (q \cdot_{\Gamma'} \gamma') \cdot_\Gamma \gamma = \widetilde{q} = s_\Xi(\widetilde{\gamma}', \widetilde{q}, \widetilde{\gamma}), \tag{3.3.24}$$

we can define the composition rule

$$(\gamma', q, \gamma) \cdot (\widetilde{\gamma}', \widetilde{q}, \widetilde{\gamma}) = \left(\gamma'\widetilde{\gamma}', q, \gamma\widetilde{\gamma}\right). \tag{3.3.25}$$

It is clear that this makes Ξ a Lie groupoid with base Q.

There are obvious groupoid homomorphisms $\phi\colon \Xi \to \Gamma$ and $\phi'\colon \Xi \to \Gamma$ such that the diagrams

$$\begin{tikzcd} \Xi \arrow[r] \arrow[d, "\phi"] & Q \times Q \arrow[d, "\phi_0\times\phi_0"] \\ \Gamma \arrow[r] & \Gamma_0 \times \Gamma_0 \end{tikzcd} \qquad \begin{tikzcd} \Xi \arrow[r] \arrow[d, "\phi'"] & Q \times Q \arrow[d, "\phi_0'\times\phi_0'"] \\ \Gamma' \arrow[r] & \Gamma_0' \times \Gamma_0' \end{tikzcd} \tag{3.3.26}$$

commute. We show that these diagrams are pullbacks. By symmetry, it suffices to show that just one of the diagrams is a pullback. Suppose we have a space E with maps $f\colon E \to Q \times Q$ and $g\colon E \to \Gamma$ making the diagram

$$\begin{tikzcd} E \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\ & \Xi \arrow[r] \arrow[d, "\phi"] & Q \times Q \arrow[d, "\phi_0\times\phi_0"] \\ & \Gamma \arrow[r] & \Gamma_0 \times \Gamma_0 \end{tikzcd} \tag{3.3.27}$$

commute. Denote f_1 and f_2 for the components of f. For any point $e \in E$, we have that $p(f_2(e)) = t_\Gamma(g(e)) = p(f_2(e) \cdot_\Gamma g(e))$. Therefore, there is a unique $\gamma_e' \in \Gamma'$ such that $f_2(e) = (f_2(e) \cdot_\Gamma g(e)) \cdot_{\Gamma'} \gamma_e'$. Now we can define the map $E \to \Xi$ by

$$E \ni e \mapsto (\gamma_e', f_1(e), g(e)) \in \Xi. \tag{3.3.28}$$

A straightforward verification shows that this map is well-defined and makes the diagram

$$\begin{tikzcd}
E \arrow[dr] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& \Xi \arrow[r] \arrow[d, "\phi"] & Q \times Q \arrow[d, "\phi_0 \times \phi_0"] \\
& \Gamma \arrow[r] & \Gamma_0 \times \Gamma_0
\end{tikzcd} \tag{3.3.29}$$

commute. The uniqueness of γ'_e implies that the map $E \to \Xi$ is unique with this property. This completes the proof of $(ii) \implies (iii)$.

Lastly, we show $(iii) \implies (i)$. Suppose we have a Lie groupoid Ξ with groupoid homomorphism $\phi \colon \Xi \to \Gamma$ and $\phi \colon \Xi \to \Gamma'$ as in (iii). Using the maps $\phi_0 \colon \Xi_0 \to \Gamma_0$, $\phi'_0 \colon \Xi_0 \to \Gamma'_0$, the source maps $s_\Gamma \colon \Gamma \to \Gamma_0$ and $s_{\Gamma'} \colon \Gamma' \to \Gamma'_0$ we can form the fibered product

$$\Gamma \times_{\Gamma_0} \Xi_0 \times_{\Gamma'_0} \Gamma'. \tag{3.3.30}$$

This fibered product has a canonical Ξ action by acting with a $\xi \in X$ with source $q \in \Xi_0$ according to

$$(\gamma, q, \gamma') \cdot \xi = \left(\phi(\xi)^{-1}\gamma, t_\Xi(\xi), \phi'(\xi)^{-1}\gamma'\right). \tag{3.3.31}$$

Hence, we can take the quotient

$$Q = \left(\Gamma \times_{\Gamma_0} \Xi_0 \times_{\Gamma'_0} \Gamma'\right)/\Xi. \tag{3.3.32}$$

We now construct a functor $F \colon B\Gamma \to B\Gamma'$. Define for a Γ-torsor $\pi \colon E \to U$ that

$$FE = \left(E \times_{\Gamma_0} Q\right)/\Gamma. \tag{3.3.33}$$

Here, the map $⚓_E \colon E \to \Gamma_0$ is the anchor map and $Q \to \Gamma_0$ is the composition of the projection to the first coordinate and the target map t_Γ. To see that this is well-defined, we should show two things: Firstly, we should show that there is a canonical Γ action on $E \times_{\Gamma_0} Q$. Secondly, the space FE should be a Γ'-torsor.

The Γ action on $E \times_{\Gamma_0} Q$ for $(e, [\gamma, q, \gamma']) \in E \times_{\Gamma_0} Q$ and $\alpha \in \Gamma$ such that $s_\Gamma(\alpha) = ⚓(e)$ is given by

$$(e, [\gamma, q, \gamma']) \cdot \alpha = (e \cdot \gamma, [\gamma\alpha, q, \gamma']). \tag{3.3.34}$$

Notice that this is a well-defined action, since it commutes with the Ξ action on $\left(\Gamma \times_{\Gamma_0} \Xi_0 \times_{\Gamma'_0} \Gamma'\right)$.

We now show that FE is a Γ'-torsor. For any points $[e, [\gamma, q, \gamma']] \in FE$ and $\alpha' \in \Gamma'$ with $t_{\Gamma'}(\gamma') = s_{\Gamma'}(\alpha')$ we have the action given by

$$[e, [\gamma, q, \gamma']] \cdot \alpha' = [e, [\gamma, q, \gamma'\alpha']]. \tag{3.3.35}$$

Again this action commutes with the action of Ξ on $\left(\Gamma \times_{\Gamma_0} \Xi_0 \times_{\Gamma'_0} \Gamma'\right)$ and it also commutes with the Γ action on $E \times_{\Gamma_0} Q$. Therefore, this Γ' action is well-defined. There is a clearly well-defined map $\pi' \colon FE \to U$ taking the torsor map $\pi \colon E \to U$ of the first coordinate of a representative. Suppose there are points $[e, [\gamma, q, \gamma']], [\widetilde{e}, [\widetilde{\gamma}, \widetilde{q}, \widetilde{\gamma'}]] \in FE$ such that

$\pi'([e,[\gamma,q,\gamma']]) = \pi'([\widetilde{e},[\widetilde{\gamma},\widetilde{q},\widetilde{\gamma'}]])$. Since there holds $\pi(e) = \pi(\widetilde{e})$, there is a unique $\alpha \in \Gamma$ such that $e \cdot \alpha = \widetilde{e}$. We obtain the equation

$$[\widetilde{e},[\widetilde{\gamma},\widetilde{q},\widetilde{\gamma'}]] = [e\cdot\alpha,[\widetilde{\gamma},\widetilde{q},\widetilde{\gamma'}]] = [e,[\widetilde{\gamma}\alpha^{-1},\widetilde{q},\widetilde{\gamma'}]]. \tag{3.3.36}$$

Now we can take the composition $\widetilde{\gamma}\alpha^{-1}\gamma^{-1}$ which has source $\widetilde{q}$ and target q. Since the diagram

$$\begin{tikzcd} \Xi \arrow[r] \arrow[d, "\phi"] & \Xi_0 \times \Xi_0 \arrow[d, "\phi_0\times\phi_0"] \\ \Gamma \arrow[r] & \Gamma_0 \times \Gamma_0 \end{tikzcd} \tag{3.3.37}$$

is a pullback, we find a unique $\xi \in \Xi$ such that $\phi(\xi) = \widetilde{\gamma}\alpha^{-1}\gamma^{-1}$ and $s_\Xi(\xi) = \widetilde{q}$ and $t_\Xi(\xi) = q$. We deduce the formula

$$\begin{aligned}[e,[\widetilde{\gamma}\alpha^{-1},\widetilde{q},\widetilde{\gamma'}]] &= [e,[\phi(\xi)^{-1}\widetilde{\gamma}\alpha^{-1},q,\phi'(\xi)^{-1}\widetilde{\gamma'}]] \\ &= [e,[\gamma\alpha\widetilde{\gamma}^{-1}\widetilde{\gamma}\alpha^{-1},q,\phi'(\xi)^{-1}\widetilde{\gamma'}]] = [e,[\gamma,q,\phi'(\xi)^{-1}\widetilde{\gamma'}]] \end{aligned} \tag{3.3.38}$$

We now find the element $\alpha' = \gamma'^{-1}\phi'(\xi)^{-1}\widetilde{\gamma'}$. Using Eqs. (3.3.36) and (3.3.38), we see that

$$[e,[\gamma,q,\gamma']]\cdot\alpha' = [\widetilde{e},[\widetilde{\gamma},\widetilde{q},\widetilde{\gamma'}]]. \tag{3.3.39}$$

Uniqueness of α' with this property is implied by the uniqueness of α and ξ and the fact that the diagram

$$\begin{tikzcd} \Xi \arrow[r] \arrow[d, "\phi'"] & \Xi_0 \times \Xi_0 \arrow[d, "\phi'_0\times\phi'_0"] \\ \Gamma' \arrow[r] & \Gamma'_0 \times \Gamma'_0 \end{tikzcd} \tag{3.3.40}$$

is a pullback. This shows that FE is a Γ'-torsor.

The assignment Eq. (3.3.33) straightforwardly extends to morphisms and it is clearly a morphism of stacks. We are left to show that F is an equivalence. We will apply Corollary 3.23. We check the conditions:

Choose a section $\sigma\colon \Gamma'_0 \to \Xi_0$ of ϕ'_0. Notice that for any representative $[e,[\gamma,q,\gamma']] \in FE$, we can find by the pullback diagram (3.3.40) a unique $\xi \in \Xi$ with source q and target $\sigma(t_{\Gamma'}(\gamma'))$ such that $\phi'(\xi) = \gamma'$. Therefore, we have that

$$[e,[\gamma,q,\gamma']] = [e,[\phi(\xi)^{-1}\gamma,t_\Xi(\xi),\phi'(\xi)^{-1}\gamma']] = [e\gamma^{-1}\phi(\xi),[\mathrm{Id}_{\phi_0(t_\Xi(\xi))},t_\Xi(\xi),\mathrm{Id}_{\phi'_0(t_\Xi(\xi))}]]. \tag{3.3.41}$$

Notice that the representative on the right hand side does not depend on the representative we started with (only on the class it represents). Hence, by putting elements of FE and similar for FE' in this standard form the maps $B\Gamma_U(E,E')$ and $B\Gamma'_U(FE,FE')$ are in bijective correspondence by F.

For any Γ-torsor E, we can take a cover $(U_i)_{i\in I}$ which trivializes both E and FE. There holds $FE_i = (FE)_i$ and therefore the second condition of is satisfied. Applying Corollary 3.23 completes the proof of $(iii) \implies (i)$. $\square$

Remark 3.44. The space Q as in (ii) in the theorem above is called a **Γ-Γ'-bitorsor**. Diagrammatically, we can summarize the data as follows:

$$\begin{tikzcd} \Gamma' \arrow[d, shift right, "t_{\Gamma'}"'] \arrow[d, shift left, "s_{\Gamma'}"] \arrow[r, bend left] & Q \arrow[dl] \arrow[dr] & \Gamma \arrow[d, shift right, "s_{\Gamma}"'] \arrow[d, shift left, "t_{\Gamma}"] \arrow[l, bend right] \\ \Gamma_0' & & \Gamma_0 \end{tikzcd} . \tag{3.3.42}$$

With the equivalence of (i) and (ii), we can see stacks as objects in the bicategory of Lie groupoids, bibundles and bibundle maps. ▽

3.3.5 The Dictionary Lemmas

The theorems in the previous section are merely the starting point of a dictionary between (super) Lie groupoids (up to Morita equivalence) and differentiable stacks. We can continue to consider the morphisms and 2-morphisms. The result is that morphisms of stacks correspond to Lie groupoid homomorphisms (up to the right notion of equivalence) and natural transformation between morphisms of stacks correspond to the following *natural equivalences* of groupoid homomorphisms.

Definition 3.45 ([BX06, Definition 2.28])**.** Let $\phi, \psi\colon \Gamma \to \Gamma'$ be two Lie groupoid homomorphisms. A **natural equivalence** $\theta\colon \phi \Rightarrow \psi$ is a smooth map $\theta\colon \Gamma_0 \to \Gamma_1'$ such that for all local coordinates $x \in \Gamma_1$ there holds that

$$\theta(s(x))\psi(x) = \phi(x)\theta(t(x)). \tag{3.3.43}$$

This notion of equivalence give the category of (super) Lie groupoids the structure of a 2-category. ▲

To build the dictionary, we first associate a morphism of stacks to a Lie groupoid homomorphism. We will omit proofs of these *Dictionary Lemmas* as they are only careful, tedious checks of all properties. More details of the proofs are given in [Car11, Chapter I.2.9].

Lemma 3.46 ([BX06, Lemma 2.29])**.** *Let $\phi\colon \Gamma \to \Gamma'$ be a Lie groupoid homomorphism and let $\mathfrak{X}$ and $\mathfrak{Y}$ be the differentiable stacks associated to Γ and Γ' via Proposition 3.40 respectively. Then there exists a morphism of stacks $f\colon \mathfrak{X} \to \mathfrak{Y}$ and natural isomorphism η making the diagram*

$$\begin{tikzcd} \underline{\Gamma_0} \arrow[r, "\phi_0"] \arrow[d] & \underline{\Gamma_0'} \arrow[d] \\ \mathfrak{X} \arrow[r, "f"'] \arrow[ur, phantom, "\Nearrow \eta"] & \mathfrak{Y} \end{tikzcd} \tag{3.3.44}$$

commutes and the cube

$$\begin{tikzcd} \underline{\Gamma_1} \arrow[rr, "\phi_1"] \arrow[dr, "t"] \arrow[dd, "s"] & & \underline{\Gamma_1'} \arrow[dr, "t"] \arrow[dd, "s"] & \\ & \underline{\Gamma_0} \arrow[rr, "\phi_0"] \arrow[dd] & & \underline{\Gamma_0'} \arrow[dd] \\ \underline{\Gamma_0} \arrow[rr] \arrow[dr] & & \underline{\Gamma_0'} \arrow[dr] & \\ & \mathfrak{X} \arrow[rr, "f"] & & \mathfrak{Y} \end{tikzcd} \tag{3.3.45}$$

2*-commutes. If there is another pair (f', η') satisfying these properties, then there is a unique natural isomorphism of stacks $\theta\colon f \Rightarrow f'$ such that $\theta \circ \eta' = \eta$.*

The other two Dictionary Lemmas presented here treat the converse. They associate to morphisms and natural isomorphisms of stacks homomorphisms and natural equivalences of Lie groupoids.

Lemma 3.47 ([BX06, Lemma 2.30])**.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of differentiable stacks and $\phi_0\colon \Gamma_0 \to \Gamma'_0$ a map between the bases of the (super) Lie groupoids associated to $\mathfrak{X}$ and $\mathfrak{Y}$ respectively and let η be a natural isomorphism making the square* (3.3.44) *commute. Then, there exists a unique Lie groupoid homomorphism $\phi_1\colon \Gamma_1 \to \Gamma'_1$ covering ϕ_0 and making the cube* (3.3.45) *commute.*

Lemma 3.48 ([BX06, Lemma 2.31])**.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of differentiable stacks. Let $\phi, \psi\colon \Gamma \to \Gamma'$ be Lie groupoid homomorphisms between the associated (super) Lie groupoids and let η, η' be natural isomorphisms of stacks such that the squares of the form* (3.3.44) *commute for the pairs (ϕ, η) and (ψ, η'). Then there exists a unique natural equivalence $\theta\colon \phi \Rightarrow \psi$ such that the following diagram 2-commutes.*

$$\begin{tikzcd}
 & \underline{\Gamma'_1} \arrow[ddl, bend right, "s"'] \arrow[ddr, bend left, "t"] & \\
 & \underline{\Gamma_0} \arrow[u, "\theta"] \arrow[dl, "\psi"] \arrow[d] \arrow[dr, "\phi"'] & \\
\underline{\Gamma'_0} \arrow[dr] & \mathfrak{X} \arrow[d, "f"] & \underline{\Gamma'_0} \arrow[dl] \\
 & \mathfrak{Y} &
\end{tikzcd} \tag{3.3.46}$$

3.3.6 Quotient Supermanifolds as Stacks

I will conclude the discussion of stacks by highlighting an application of them in the realm of supermanifolds, namely that of *quotient supermanifolds*. Consider some supermanifold M with an action of a Lie supergroup G. We did like to make precise the quotient M/G. Already for ordinary manifolds, it is well-known that the space M/G with the quotient topology need not be a manifold. For example, take the action of $\mathbb{Z}$ on S^1 by an irrational rotation. Every orbit will be dense in S^1. Hence, the quotient space is not even Hausdorff. Obviously, there is the Quotient Manifold Theorem asserting that the quotient of a manifold under a Lie group acting freely and properly can be given a unique smooth structure making the quotient map a submersion.[Lee12, Theorem 21.10]

For super Lie groups acting on supermanifolds the situation worsens in the odd dimensions. E.g., consider the superpoint $\mathbb{R}^{0|1}$ with the $\mathbb{Z}/2$ action given by $\theta \mapsto -\theta$. In the quotient, we did like to think that θ and $-\theta$ are equal but is should not remove the odd part completely. However, identifying the coordinate function θ with $-\theta$, would yield the completely even space $\mathbb{R}^{0|0}$. To accommodate these issues, we will work with stacks over supermanifolds instead. In particular, the quotient stacks considered in Examples 3.20 and 3.29.

We consider a supermanifold M with an action of a super Lie group G via an action $\mu\colon G \times M \to M$. We will denote $p\colon G \times M \to M$ for the projection. We will view the quotient stack $[M/G]$ as the incarnation of the "quotient supermanifold" M/G. We investigate its functions: From Example 3.29, we know that the quotient stack $[M/G]$ can be represented by the action groupoid of G acting on M. Recall from Corollary 2.39 that $C^\infty(M)^{\mathrm{ev}} = \mathrm{SMfld}(M, \mathbb{R}^1)$ and $C^\infty(M)^{\mathrm{odd}} = \mathrm{SMfld}(M, \mathbb{R}^{0|1})$.

Proposition 3.49 ([Hoh+11, Corollary 57])**.** *The morphisms of stacks $[M/G] \to \underline{\mathbb{R}}$ and $[M/G] \to \underline{\mathbb{R}}^{0|1}$ can be identified with the even and odd G-invariant functions on M respectively. I.e.,*

$$Fun_{SMfld}([M/G], \underline{\mathbb{R}}) \cong \{f \in C^\infty(M)^{ev} | \mu^*(f) = p^*(f) \in C^\infty(G \times M)^{ev}\}, \tag{3.3.47}$$

$$Fun_{SMfld}([M/G], \underline{\mathbb{R}}^{0|1}) \cong \{f \in C^\infty(M)^{odd} | \mu^*(f) = p^*(f) \in C^\infty(G \times M)^{odd}\}. \tag{3.3.48}$$

Proof. By the Yoneda Lemma, a functor between representables can be identified with a function between the representables. Recall that $\underline{\mathbb{R}}$ and $\underline{\mathbb{R}}^{0|1}$ are represented by themselves. I.e., they are represented by the Lie (super)groupoid with only identity morphisms. Considering the Dictionary Lemmas 3.46 to 3.48, we see indeed that the morphisms of stacks $[M/G] \to \underline{\mathbb{R}}$ and $[M/G] \to \underline{\mathbb{R}}^{0|1}$ can be identified with natural equivalences of super Lie groupoids. In turn, these are by definition the even and odd G-invariant functions on M respectively. ☐

On top of these quotient like objects of supermanifolds, we can consider vector bundles and their sections. Recall from Example 3.28 the differentiable stack $\underline{\mathrm{VecBun}_n}^{\mathrm{Iso}}$ of rank n vector bundles with morphisms the morphisms of vector bundles which are fiberwise isomorphisms. Its presentation is given by GL_n. Following the case of line bundle laid out in [Hoh+11, Section 7.4], we make the following definition.

Definition 3.50. We define the **strong vector bundles of rank n over a quotient stack** $[M/G]$ as the morphisms of stacks $[M/G] \to \underline{\mathrm{VecBun}_n}^{Iso}$. Given such a vector bundle $F\colon [M/G] \to \underline{\mathrm{VecBun}_n}^{Iso}$, we define its **sections** $\Gamma([M/G], V_\rho)$ to be the natural transformations

$$\begin{tikzcd} {[M/G]} \arrow[rr, bend left, "\mathbb{1}"] \arrow[rr, bend right, "F"'] \arrow[rr, phantom, "\Downarrow E"] & & \underline{\mathrm{VecBun}_n}^{\mathrm{Iso}} \end{tikzcd} . \tag{3.3.49}$$

Here, the functor $\mathbb{1}$ is given by sending all objects to the trivial vector bundle and all morphisms to the identity. ▲

Notice that a homomorphism $\rho\colon G \to \mathrm{GL}_n$, by Lemma 3.46 induces a morphism of stacks $V_\rho\colon [M/G] \to \underline{\mathrm{Vec}_n}^{\mathrm{Iso}}$. I.e., homomorphisms $\rho\colon G \to \mathrm{GL}_n$ give rise to vector bundles over $[M/G]$. In a particular, the vector bundle $\mathbb{1}$ is given by the trivial homomorphism. With a similar application of the Dictionary Lemmas as in Proposition 3.49, we can identify the sections of a vector bundle V_ρ with certain functions on M.

Proposition 3.51. *Let M be a (super)manifold with a G action μ and a homomorphism $\rho\colon G \to GL_n$. The sections of the strong vector bundle V_ρ over $[M/G]$ are given by*

$$\Gamma([M/G], V_\rho) = \{f \in C^\infty(M, GL_n) | \mu^*(f) = p_1^*(f) p_2^*(\rho) \in C^\infty(M \times G, GL_n)\} \tag{3.3.50}$$

When changing V_ρ to its odd partner ΠV_ρ the section will instead take values in the odd partner of GL_n, i.e., the invertible linear transformations of the purely odd n-dimensional vector space.

With this proposition, we can interpret vector bundles on the quotient stack as equivariant super bundles on M, see also Example 2.79.

Instead of working with the stack $\underline{\mathrm{Vec}_n}^{\mathrm{Iso}}$ we can also work with the category $\underline{\mathrm{Vec}_n}$ of rank n vector bundles with vector bundle morphisms. The difference between $\underline{\mathrm{Vec}_n}^{\mathrm{Iso}}$ and $\underline{\mathrm{Vec}_n}$ lies in the fact that in the former we assume that all morphisms are isomorphisms in the fibers. In the latter, all linear maps in the fibers are permitted. The category $\underline{\mathrm{Vec}_n}$ is not a stack. In fact, it is not a groupoid fibration as can easily been seen by considering the fibers over the identity morphisms (these do not form a groupoid, since there always is the zero map, cf. Proposition 3.5). However, we can still consider the functors $F\colon [M/G] \to \underline{\mathrm{VecBun}_n}$ as being vector bundles.

Definition 3.52. We define the **vector bundles of rank** n **over a quotient stack** $[M/G]$ the morphisms $[M/G] \to \underline{\mathrm{VecBun}_n}$ compatible with the projection to the base (super)manifolds. Similar to Definition 3.50, given such a vector bundle $F\colon [M/G] \to \underline{\mathrm{VecBun}_n}^{Iso}$, its **sections** $\Gamma([M/G], V_\rho)$ are the natural transformations

$$\begin{tikzcd} {[M/G]} \arrow[r, bend left, "\mathbb{1}"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow E"] & \underline{\mathrm{VecBun}_n}^{\mathrm{Iso}} \end{tikzcd} . \tag{3.3.51}$$

Here, the functor $\mathbb{1}$ is again given by sending all objects to the trivial vector bundle and all morphisms to the identity. ▲

Since, the category $\underline{\mathrm{VecBun}_n}$ is a subcategory of $\underline{\mathrm{Vec}_n}^{\mathrm{Iso}}$, a homomorphism $\rho\colon G \to \mathrm{GL}_n$ still induces a vector bundle $V_\rho\colon [M/G] \to \underline{\mathrm{Vec}_n}$. The additional morphism in $\underline{\mathrm{VecBun}_n}$ amount to the non-invertible linear transformations in the fibers. Therefore, the interplay between $\underline{\mathrm{Vec}_n}^{\mathrm{Iso}}$ and $\underline{\mathrm{Vec}_n}$ resembles the interplay between the group GL_n and the monoid of $n \times n$-matrices $\mathrm{Mat}(n \times n)$. A careful analysis, as done in [Hoh+11, Proposition 53] of this transition, yields the following result on sections of vector bundles.

Proposition 3.53. *Let M be a (super)manifold with a G action μ and a homomorphism $\rho\colon G \to GL_n$. The sections of the vector bundle, in the sense of Definition 3.52, V_ρ over $[M/G]$ are given by*

$$\Gamma([M/G], V_\rho) = \{f \in C^\infty(M, Mat(n \times n)) | \mu^*(f) = p_1^*(f)p_2^*(\rho) \in C^\infty(M \times G, Mat(n \times n))\} \tag{3.3.52}$$

When changing V_ρ to its odd partner ΠV_ρ the section will instead take values in the odd partner of $Mat(n \times n)$, i.e., the linear transformations of the purely odd n-dimensional vector space.

The characterization of the sections of Eqs. (3.3.50) and (3.3.52) are similar to the sections of equivariant vector bundles over supermanifolds, as in Example 2.80. This should also justify the terminology of vector bundles adopted above.

4 Field Theories

Most of Theoretical Physics is written in terms of *Field Theories*. *Quantum Field Theories*, *Statistical Field Theories* and *Conformal Field Theories* are obvious examples. In particular, the Standard Model in Particle Physics can be formulated as a field theory. In a very general sense, field theories in physics consists of a bunch of maps on which we define a functional, called the *Lagrangian* $\mathcal{L}$ (or *Hamiltonian* $\mathcal{H}$ if one uses the equivalent Hamiltonian description). From this Lagrangian, we then try to compute things like equations of motions and conserved currents.

A main object in a field theory computed from the Lagrangian is the *partition function*. This function collects virtually all physically relevant data of the theory. The main tool to compute partition functions are *path integrals*. Here, we "integrate" over all the fields. Since, the fields are maps, they generally form an infinite dimensional manifold. Hence, constructing a well-defined integration measure might be troublesome. Therefore, mathematicians certainly raise eyebrows when coming across path integrals, due to ill-definedness problems. This is mostly caused by the way physicist manipulate them. Physicist generally turn a blind eye and simply compute with them as ordinary integrals. At the end of the day, when comparing the results with experiments, nature will tell whether we messed up somewhere along the way. Nature doesn't make mistakes, does it? However, in many cases, the ill-definedness problems can be dodged to yield a mathematically rigorous theory.

In mathematics, there also is a notion of *(Topological Quantum) Field Theory, TQFT,* first considered by Edward Witten [Wit88] and axiomatized by Michael Atiyah [Ati88]. Over the years, Atiyah's axioms were reformulated in terms of functors out of bordism categories. This formulation will also be our mathematical viewpoint of field theories. An introduction to TQFT's is presented in [CR18]. Since, we will work in a smooth setting, in fact with Model geometries, we will use the more general term *Functorial Field Theories* instead of more common *Topological Field Theories* which only refers to the topological nature.

The main goal of this chapter is linking the physical and mathematical notions of field theories. We will start with the mathematical view in Chapters 4.1 and 4.2 and continue with the physical view Chapter 4.3. Throughout, this chapter, we will use the term *Functorial Field Theories* to indicate we are working in a mathematical framework. While we simply say *field theory* for the physical notion.

4.1 Bordism Categories

The building blocks of Functorial Field Theories are bordism categories. Roughly speaking, these are categories with objects the $d-1$-dimensional compact manifolds and morphisms the d-dimensional compact manifolds with boundary. A precise definition is given below. The manifolds can be assumed to have extra structure, such as being orientable. In particular, we can consider bordism categories consisting of $(\mathbb{M}, G)$-manifolds for some model geometry $(\mathbb{M}, G)$ as defined in Chapter 2.5. Following [Hoh+11, Section 4+5], we can lift the notion of bordism to families of supermanifolds.

We start by the most basic definition of a bordism category. Following this, we upgrade the definition to allow for easier addition of additional structure. Bordisms with extra structures are then described in the following sections.

Definition 4.1. For $(p-1)$-dimensional closed manifolds E and F consider the triples $(\iota_{\text{in}}, M, \iota_{\text{out}})$ with M a p-dimensional compact manifold with boundary and $\iota_{\text{in}}\colon E \to \partial M$ and $\iota_{\text{out}}\colon F \to \partial M$ embeddings with disjoint image together forming the whole boundary of M. I.e., there holds

$$\iota_{\text{in}}(E) \bigcap \iota_{\text{out}}(F) = \emptyset \quad \text{and} \quad \iota_{\text{in}}(E) \bigcup \iota_{\text{out}}(F) = \partial M. \tag{4.1.1}$$

Two triples $(\iota_{\text{in}}, M, \iota_{\text{out}})$ and $(\iota'_{\text{in}}, M', \iota'_{\text{out}})$ are considered equivalent if there exists a diffeomorphism $\psi\colon M \to M'$ such that $\psi \circ \iota_{\text{in}} = \iota'_{\text{in}}$ and $\psi \circ \iota_{\text{out}} = \iota'_{\text{out}}$. An equivalence class under this equivalence relation, we will call a **bordism**. We will denote a bordism $(\iota_{\text{in}}, M, \iota_{\text{out}})$ between closed manifolds E and F as $M\colon E \to F$.

We define the category of n-dimensional bordism as the category with objects the $(n-1)$-dimensional (real) closed manifolds and arrows the bordisms between the manifolds. The composition of two bordisms $M\colon E \to F$ and $M'\colon F \to G$ is the bordism obtained by gluing M and M' along their common boundary F to form a new bordism $M \sqcup_F M'\colon E \to G$. The identity bordisms are the cylinders $E \times [0,1]$. ▲

Remark 4.2. The topology on a composite bordism $M \sqcup_F M'$ is fixed, since it is a pushout in the category of topological spaces. However, we also demanded a smooth structure. This needs a little more care. We can construct a smooth structure on $M \sqcup_F M'$ by using *collar neighborhoods* around F. These are neighborhoods of F diffeomorphic to $F \times [0,1)$ in M and M'. It is clear that such neighborhoods can be glued together along $F \times \{0\}$ to obtain a smooth structure.

The collar neighborhoods of F in M (and similarly M') can be constructed from charts of M around the points of F.[Bro62] Compactness of F implies finitely many charts suffice. Hence, shrinking the charts sufficiently and putting them together will yield a collar neighborhood. Different collar neighborhood give diffeomorphic smooth structures. Notice that bordisms are manifolds consider up to diffeomorphisms, so this shows that the composition of bordisms is well-defined. ▽

This issue around the smooth structures when gluing bordisms together can be solved differently too: We can remember the collar neighborhoods in the objects by "fattening" the object manifolds and boundaries of the bordisms. In the simple case expositioned above, this is unnecessary extra data. However, when we want to add supersymmetry or additional structure to the manifolds, like some $(\mathbb{M}, G)$ structure of a model geometry, we will need this data. We obtain the following:

Definition 4.3 ([ST11, Definition 2.21])**.** We define the category n-Bord to be the category with objects the pairs $(E \times (-1,1), E \times \{0\})$ where E is a closed manifold of dimension $n-1$. Usually, we will refer to the object $(E \times (-1,1), E \times \{0\})$ with just E. Figure 2 illustrates these objects.

The morphism from E to F are equivalence classes of triples $(\iota_{\text{in}}, M, \iota_{\text{out}})$, called the **bordisms**. Here, we have that M is an n-dimensional manifold and $\iota_{\text{in}}\colon E \times (-1,1) \to M$ and $\iota_{\text{out}}\colon F \times (-1,1) \to M$ embeddings with the following properties.

(c) The **core** $M \setminus (E \times (-1,0) \cup F \times (0,1))$ of the bordism is compact.

(+) The images of $\iota_{\text{in}}(E \times (-1,0))$ and $\iota_{\text{out}}(F \times [0,1))$ are disjoint.

(b) The closures of the opens $\iota_{\text{in}}(E \times (-1,0))$ and $\iota_{\text{out}}(F \times (0,1))$ in M are $\iota_{\text{in}}(E \times (-1,0])$ and $\iota_{\text{out}}(F \times [0,1))$ respectively.

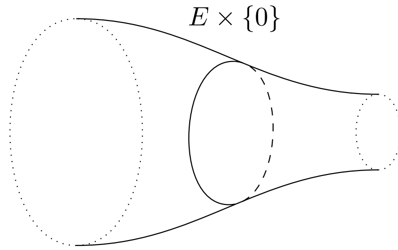

Figure 2: An illustration of an object E in the category $n-\underline{\text{Bord}}$. As usual in bordism theories, we will limit the pictures to the case for $n = 2$.

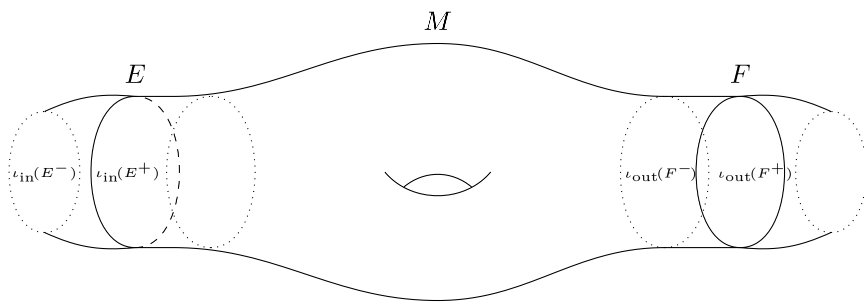

Figure 3: A generic example of a morphism $M\colon E \to F$ in the category n-$\underline{\text{Bord}}$. In the figure, we write E^- and E^+ for $E \times (-1, 0)$ and $E \times (0, 1)$ respectively and similar for F. The gluing action on such bordisms goes by cutting out the piece $\iota_{\text{out}}(F \times [0, 1))$ and the along gluing $\iota_{\text{in}}(F^-)$.

Two triples $(\iota_{\text{in}}, M, \iota_{\text{out}})$ and $(\iota'_{\text{in}}, M', \iota'_{\text{out}})$ are considered equivalent if there exists a diffeomorphism $\psi\colon M \to M'$ such that $\psi \circ \iota_{\text{in}} = \iota'_{\text{in}}$ and $\psi \circ \iota_{\text{out}} = \iota'_{\text{out}}$. We will usually write just M for the triple $(\iota_{\text{in}}, M, \iota_{\text{out}})$. A generic example of these bordisms is visualized in Fig. 3.

The composition of $M\colon E \to F$ and $M'\colon F \to G$ in this category is given by the gluing

$$(M \backslash \iota_{\text{out}}(F \times [0, 1))) \sqcup_{F \times (-1,0)} F \times (-1, 1) \sqcup_{F \times (0,1)} (M' \backslash \iota_{\text{in}'}(F \times (-1, 0])) . \tag{4.1.2}$$

▲

Remark 4.4. Compared to [ST11, Definition 2.21], there are two main differences. Firstly, their definition is in terms of *internal categories* where we remember all morphisms by unraveling the equivalence taken in the definition of bordisms. We will present this viewpoint shortly. Secondly, an omission has been fixed by assuming that the boundaries of $\iota_{\text{in}}(E \times (-1, 0))$ and $\iota_{\text{out}}(F \times (0, 1))$ in M are $\iota_{\text{in}}(E \times \{0\})$ and $\iota_{\text{out}}(F \times \{0\})$ respectively. This ensures that we can only glue on the edges of the manifold. In other words, the manifold cannot continue on both sides of the embedding. This is necessary to have identity bordisms (cylinders of the form $E \times (-1, 1)$). If we wouldn't assume this condition, then the gluing of a cylinder can leave some disconnected piece.

▽

Remark 4.5. At first, when trying to enforce the gluing to work on the edges, I was working with manifolds with boundary instead. The objects of the bordism category would stay the same, but

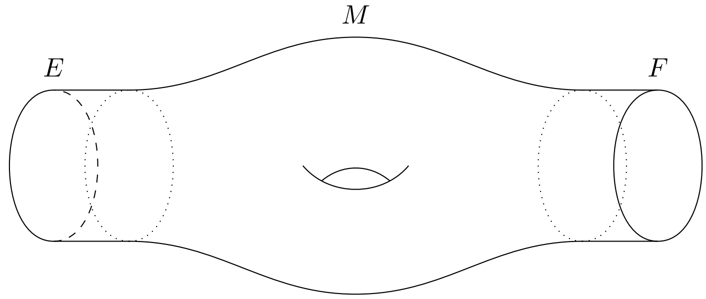

Figure 4: A version of a definition of bordisms with thickened boundaries to glue on, however with boundaries to ensure gluing work near the edges of the manifold.

the morphisms would turn into manifolds with boundary with embeddings $\iota_{\text{in}} \colon E \times [0,1) \to M$ and $\iota_{\text{out}} \colon F \times (-1,0] \to M$. Pictorially, these bordisms would look like Fig. 4. While such a definition is perfectly fine in this setting, it does add the fuss of shrinking and stretching the manifold when showing that the cylinders act like identity morphisms. This can turn out problematic if one adds further structure, such that the shrinking and stretching is not possible anymore, like with Model Geometries. $\triangledown$

The above definition makes manifestly sure that we obtain smooth structures on composite bordisms. Indeed, we glue along open subsets. Hence, all local properties are preserved. However, this comes at the price that Hausdorffness is not immediately clear anymore. E.g., gluing two copies of $\mathbb{R}$ along $(0,1)$ yields a non-Hausdorff space. Hausdorffness of composite bordisms follows from the following lemma in point-set topology.

Lemma 4.6 ([ST11, Lemma 2.23])**.** *Let X, Y be Hausdorff spaces and U an open subset of both X and Y. If the image of U under the natural map $U \to X \times Y$ is closed, then the space $X \sqcup_U Y$ is Hausdorff.*

Proof. Let $x, y \in X \sqcup_U Y$. We construct disjoint open neighborhoods of x and y respectively. If $x, y \in X$ or $x, y \in Y$, then there is nothing to show since both X and Y are Hausdorff itself. Hence, we assume that $x \in X$ and $y \in Y$. Notice that $U = X$ implies that $x \in Y$. Hence, we assume that $U \neq X$ and similarly $U \neq Y$. Moreover, if $x, y \in U$, then $x, y \in X$. Therefore, without loss of generality $x \in X \backslash U$. Now we know that the point (x, y) lies in the open $(X \times Y) \backslash U$. Hence, there exists opens $x \in I \subseteq X$ and $y \in I' \subseteq Y$ such that

$$(I \times I') \subseteq (X \times Y) \backslash U. \tag{4.1.3}$$

This implies that $I \cap I' \subseteq X \sqcup_U Y$ is empty and shows the claim. $\square$

In the definition of bordisms above, we carefully take equivalence classes of diffeomorphic manifolds. Taking the equivalence classes is required to have identity morphisms in the bordism category. However, we can unravel the equivalence and instead work with two groupoids: one for the object manifolds and one for the bordisms.

Definition 4.7. The **groupoid n-Bord$_0$ of bordism objects**, has objects the n-dimensional manifolds Y with a decomposition in disjoint submanifolds Y^c and $Y^{\pm}$. Here, the manifold Y^c is compact of codimension 1 called the **core** and $Y^{\pm}$ are open submanifolds of Y such that their closure in Y equals Y^c. An (iso)morphism $Y \to Y'$ in this groupoid is a germ of a diffeomorphism on open neighborhoods of Y^c respecting the decomposition (i.e., it sends Y^c

to $(Y')^c$ and $Y^\pm$ to $(Y')^\pm$). Two diffeomorphisms represent the same germ if they agree on a smaller open neighborhood of Y_c. Composition is given by the composition of the germs. ▲

Remark 4.8. Notice that every object of n-Bord$_0$ is isomorphic to an object of the form $Y = E \times (-1,1)$ with the decomposition $Y^c = E \times \{0\}$, $Y^+ = E \times (0,1)$ and $Y^- = E \times (-1,0)$. Hereby, we obtain the picture as in Fig. 2. We will only care about manifolds of this form. ▽

Definition 4.9. The **groupoid n-Bord$_1$ of bordism morphisms** has objects precisely the triples $(\iota_{\text{in}}, M, \iota_{\text{out}})$ of manifolds with certain embeddings $E \to M$ and $F \to M$ considered as bordisms in Definition 4.3. The morphism $M \to M'$ are the triples $(f_{\text{in}}, F, f_{\text{out}})$ of diffeomorphisms $f_{\text{in}} \colon E \times (-1,1) \to E' \times (-1,1)$, $F \colon M \to M'$ and $f_{\text{in}} \colon F \times (-1,1) \to F' \times (-1,1)$ such that f_{in} and f_{out} respect the decomposition as in the above remark and the following diagram commutes

$$\begin{tikzcd} E \times (-1,1) \arrow[r, "\iota_{\text{in}}"] \arrow[d, "f_{\text{in}}"] & M \arrow[d, "F"] & F \times (-1,1) \arrow[l, "\iota_{\text{out}}"'] \arrow[d, "f_{\text{out}}"] \\ E' \times (-1,1) \arrow[r, "\iota'_{\text{in}}"'] & M' & F' \times (-1,1) \arrow[l, "\iota'_{\text{out}}"] \end{tikzcd} . \tag{4.1.4}$$

▲

It should be clear that isomorphism classes of objects in the category n-Bord$_1$ can be identified with the morphisms of n-Bord.

This unraveling of the category shows that we can see the category n-Bord as a so called *internal category*. A full hands-on definition can be found in [ST11, Definition 2.4]. For more background on internal categories, I refer to [Mac71, Section XII.1] and [Fer06]. Notions of functors and natural transformations straightforwardly extend to internal categories.

4.1.1 Bordisms for Model geometries

As mentioned before, the thickening of the boundary achieved in Definition 4.3 not only solves the issue of smoothness around the gluing. It also allows for additional structure on the manifolds in question. Our prime example will be that of $(\mathbb{M}, G)$ model geometries as defined in Definition 2.81 for the supersymmetric case. For now, we consider just the model geometries for ordinary (not super) manifolds, in other words the odd dimension is taken to be zero. The supersymmetric case will follow in Chapter 4.1.3.

Definition 4.10. We define the category $(\mathbb{M}, G)$-$\underline{\text{Bord}}$ of bordisms for some model geometry $(\mathbb{M}, G)$ to have objects, the $(\mathbb{M}, G)$-manifolds of the form $E \times (-1,1)$ for some $n-1$-manifold Y.

The morphisms are equivalence classes $(\iota_{\text{in}}, M, \iota_{\text{out}})$, called the $(\mathbb{M}, G)$**-bordisms**. Here, we have that M is a $(\mathcal{M}, G)$-manifold and $\iota_{\text{in}} \colon E \times (-1,1) \to M$ and $\iota_{\text{out}} \colon F \times (-1,1) \to M$ isometric embeddings such that the conditions (c), (+) and (b) from Definition 4.3 hold. With an isometric embedding condition, we mean that the maps ι_{in} and ι_{out} are embeddings which are isometries on the image of $E \times (0,1)$ and $E \times (-1,0)$ respectively. Two triples are equivalent if there exists an isometry between them identifying the images of the relevant embeddings ι_{in}, ι_{out}.

The composition law is the same as in Definition 4.3, by gluing the bordism along their common (thickened) boundary. ▲

Remark 4.11. I wish to stress here that the product $Y \times (-1,1)$ needs to be an $(\mathbb{M}, G)$ manifold, not the manifold Y. ▽

Remark 4.12. The identity morphisms of this bordism category are the cylinders $E \times (-1, 1)$. Notice that gluing such a cylinder on a bordism, does not change the manifold at all. Indeed, we cut out a piece of the bordism isometric to $E \times [0, 1)$ and glue along $E \times (-1, 0)$ (for the left composition of the identity). So the obtained composition is indeed isometric to the original bordism. This illustrates the issue raised in Remark 4.5, since general model geometry does not have "stretching" as an isometry. Moreover, all gluing happens on opens of the relevant manifolds along isometric embeddings this makes sure that the composition of two $(\mathbb{M}, G)$-bordisms indeed again yields a $(\mathbb{M}, G)$-bordisms. $\triangledown$

In the same spirit as Definitions 4.7 and 4.9, we can unravel the equivalence taken in the definition of bordisms for model geometries. Obviously, the diffeomorphisms should be replaced by isometries of the relevant model geometry in this case. One obtains again two groupoids $(\mathbb{M}, G)$-$\underline{\text{Bord}}_0$ and $(\mathbb{M}, G)$-$\underline{\text{Bord}}_1$. As before, the morphisms of $(\mathbb{M}, G)$-$\underline{\text{Bord}}$ can be viewed as isomorphism classes of objects in $(\mathbb{M}, G)$-$\underline{\text{Bord}}_1$.

4.1.2 Families of Bordisms

In Chapter 2.5, we have generalized away from standalone (super) manifolds for a certain model geometry to families of them. For bordisms, we can do the same. This generalization is required when we want to consider *smoothness* of functors on this category. Since, Functorial Field Theories will be functors out of bordisms categories, we need families of bordisms for a notion of smoothness on Functorial Field Theories. [Hoh+11, Page 14]

The fact that families of manifolds can be used to describe smoothness of functors on the relevant categories follows from the fact that families of manifolds form (differentiable) stacks over the category of manifolds. We have shown this for model geometries in Proposition 3.21. Functors between differentiable stacks can, via de Yoneda Lemma, be seen as functions on the presenting Lie groupoids. These Lie groupoids give us an obvious notion of smoothness.

Since, we already established the notion of bordisms for $(\mathbb{M}, G)$-model geometries, we will now define bordisms of *families* of $(\mathbb{M}, G)$-manifolds. This definition is motivated by [ST11, Definition 4.46]. Remark 4.4 applies to this definition in a similar fashion.

Definition 4.13. An object in the category $(\mathbb{M}, G)$-$\underline{\text{Bord}}^{\text{fam}}$ of **families of $(\mathbb{M}, G)$-bordisms** has objects the families of $(\mathbb{M}, G)$-manifolds, see Definition 2.87, of the form $E \times (-1, 1) \to S$ such that the induced map $E \times \{0\} \to S$ is proper (i.e., the preimage of a compact set is compact).

A morphism from $E \times (-1, 1) \to S$ to $F \times (-1, 1) \to S$ in $(\mathbb{M}, G)$-$\underline{\text{Bord}}^{\text{fam}}$ is an equivalence class of triples $(\iota_{\text{in}}, M, \iota_{\text{out}})$. Here, we write M for a family of $(\mathbb{M}, G)$-manifolds over S such that the map $M \to S$ is proper and ι_{in} and ι_{out} are isometric embeddings of families $E \times (-1, 1) \to M$ and $F \times (-1, 1) \to M$ respectively such that the condition (+) of Definition 4.3 holds. Moreover, we assume that condition (b) of Definition 4.3 holds fiberwise over S. Two triples $(\iota_{\text{in}}, M, \iota_{\text{out}})$ and $(\iota'_{\text{in}}, M', \iota'_{\text{out}})$ are equivalent if there exists an isometry of families $M \to M'$ respecting the embeddings. An equivalence class of a triple $(\iota_{\text{in}}, M, \iota_{\text{out}})$ is called **a family of bordisms** and is usually denoted by just M

The composition in this category is given by gluing of families of $(\mathbb{M}, G)$-manifolds along the isometric embeddings. ▲

Remark 4.14. The properness assumption in this definition is a replacement of the compactness assumption in the none family case. For non-compact spaces S, it wouldn't make sense to request compactness on the domain. In case $S = \mathrm{Pt}$, we see by definition that properness is equivalent to compactness of the domain. $\triangledown$

Stacks come into this definition when we unravel the equivalence of bordisms under isometries of families. We can consider pairs of families of $(\mathbb{M}, G)$-manifolds and thus also spaces Y with the decomposition in a core Y^c and $Y^\pm$ as in Definition 4.7. Using the same proof as in Proposition 3.21, one can show that these pairs of families of $(\mathbb{M}, G)$-manifolds form a stack. With this one obtains the stack $(\mathbb{M}, G)$-$\underline{\mathrm{Bord}}_0^{\mathrm{fam}}$ consisting of families $(\mathbb{M}, G)$-manifolds Y decomposable in Y^c and $Y^\pm$ with their families of isometries. Furthermore, one obtains the stack $(\mathbb{M}, G)$-$\underline{\mathrm{Bord}}_1^{\mathrm{fam}}$ consisting of the triples $(\iota_{\mathrm{in}}, M, \iota_{\mathrm{out}})$ of families of bordisms with the relevant triples of isometries as in Definition 4.13.

Remark 4.15. The notion of *pairs of families of* $(\mathbb{M}, G)$*-manifolds* adopted here is relatively weak in the sense that we have not assumed any relation between the submanifold and the $(\mathbb{M}, G)$ structure. In the work of Stolz and Teichner [ST11, Definition 2.33], there are some compatibility condition. Roughly speaking, this boils down to assuming that for a pair (Y, Y') the submanifold $Y' \subseteq Y$ has some $(\mathbb{M}', G)$ structure for a submanifold $\mathbb{M}' \subseteq \mathbb{M}$ with a compatible G action. These extra restrictions are not required for our purposes. $\triangledown$

4.1.3 Supersymmetric Bordism Categories

Having done the details of ordinary bordism categories in the previous section, supersymmetry can be added straightforwardly. We only have to add a supersymmetric dimension to all the manifolds, both in the objects and in the arrows of the bordism category. The structure of model geometries and the notion of families of bordisms can be transferred to the super case immediately.

In Definition 2.42, we have defined supermanifolds with boundary, which are implicitly required in our notion of bordism. However, as we have seen in Example 2.43, boundaries of supermanifolds are not unique. Though, they are isomorphic by Proposition 2.45. In our definition of bordism, there will not be any ambiguity what the boundary of the supermanifold must be. Our bordisms come with embeddings around the topological boundaries, whose domains have a natural choice of boundary.

The definition below is the straightforward generalization of Definition 4.13 to the super world.

Definition 4.16. Let $(\mathbb{M}, G)$ be a super model geometry. An object in the category $(\mathbb{M}, G)$-$\underline{\mathrm{Bord}}^{\mathrm{fam}}$ of **families of** $(\mathbb{M}, G)$**-bordisms** has objects the families of $(\mathbb{M}, G)$-supermanifolds of the form $E \times (-1, 1) \to S$ such that the induced map on reduced manifolds $|E| \times \{0\} \to |S|$ is proper (i.e., the preimage of a compact set is compact).

A morphism from $E \times (-1, 1) \to S$ to $F \times (-1, 1) \to S$ in $(\mathbb{M}, G)$-$\underline{\mathrm{Bord}}^{\mathrm{fam}}$ is an equivalence class of triples $(\iota_{\mathrm{in}}, M, \iota_{\mathrm{out}})$. Here, we write M for a family of $(\mathbb{M}, G)$-supermanifolds over S and isometric embeddings $\iota_{\mathrm{in}} \colon E \times (-1, 1) \to M$ and $\iota_{\mathrm{out}} \colon F \times (-1, 1) \to M$ such that the following properties hold:

(c) The map on reduced manifolds $|M| \to |S|$ is proper.

(+) The reduced images of $|\iota_{\mathrm{in}}(E \times (-1, 0))|$ and $|\iota_{\mathrm{out}}(F \times [0, 1))|$ are disjoint.

(b) Boundaries of the closures of the opens $\iota_{\mathrm{in}}(E \times (-1, 0))$ and $\iota_{\mathrm{out}}(F \times (0, 1))$ in M fiberwise over S are $\iota_{\mathrm{in}}(E \times \{0\})$ and $\iota_{\mathrm{out}}(F \times \{0\})$ respectively.

Two triples $(\iota_{\text{in}}, M, \iota_{\text{out}})$ and $(\iota'_{\text{in}}, M', \iota'_{\text{out}})$ are equivalent if there exists an isometry of families $M \to M'$ respecting the embeddings. An equivalence class of a triple $(\iota_{\text{in}}, M, \iota_{\text{out}})$ is called **a family of bordisms** and is usually denoted by just M

The composition in this category is given by gluing of families of $(\mathbb{M}, G)$-supermanifolds along the isometric embeddings. ▲

Remark 4.17. This definition reduces to Definition 4.13 when $\mathbb{M}$ is an ordinary manifold. Therefore, we use the same notation and terminology for this category. ▽

As in the case for non supersymmetric manifolds unraveling the equivalence classes of bordisms, yields stacks $(\mathbb{M}, G)$-$\underline{\text{Bord}}_0^{\text{fam}}$ and $(\mathbb{M}, G)$-$\underline{\text{Bord}}_1^{\text{fam}}$. These stacks obviously lie over the category of supermanifolds instead of ordinary manifolds. The details are the same as the ordinary case.

4.1.4 Bordisms over a Space

The considered bordism theories can be enriched with the extra datum of a map into some fixed space X. This space X should, physically, be thought of as the background space-time manifold of the theory. Mathematically, it is the space we are probing using the bordisms. I.e., the (co)homological properties we will discover, are those of the space X.

The below definition applies to all bordism categories, supersymmetric, model geometry and family versions, constructed in the preceding sections.

Definition 4.18. Given any of the bordism categories $\underline{\text{Bord}}$ constructed before, we define the category of **bordisms over** X denoted by $\underline{\text{Bord}}(X)$ as the category with objects, the objects of $\underline{\text{Bord}}$ (which are certain manifolds $E \times (-1, 1)$) furnaced with a (smooth) map to $E \times (-1, 1) \to X$. The morphisms of $\underline{\text{Bord}}(X)$ are the relevant triples $(\iota_{\text{in}}, M, \iota_{\text{out}})$ defining a bordism in $\underline{\text{Bord}}$ together with a map $f \colon M \to X$. Two triples $(\iota_{\text{in}}, M, \iota_{\text{out}})$ and $(\iota'_{\text{in}}, M', \iota'_{\text{out}})$ with maps $f \colon M \to X$ and $f' \colon M' \to X$ define the same **bordism over** X if the triples define the same bordism in $\underline{\text{Bord}}$ and the isomorphism $\Phi \colon M \to M'$ inducing the equivalence makes the following diagram commute

$$\begin{tikzcd} M \arrow[rr, "\Phi"] \arrow[dr, "f"'] & & M' \arrow[dl, "f'"] \\ & X & \end{tikzcd} . \tag{4.1.5}$$

The source (and similarly the target) of a bordism $(\iota_{\text{in}}, M, \iota_{\text{out}})$ with map $f \colon M \to X$ is an object $E \times (-1, 1)$ with a map $\widetilde{f} \colon E \times (-1, 1) \to X$ such that $\iota_{\text{in}} \colon E \times (-1, 1) \to M$ is an embedding and $\widetilde{f} = f \circ \iota_{\text{in}}$. ▲

Remark 4.19. Notice that all the previous discussion of bordism categories is included in this definition when taking $X = Pt$ the one point space. A map to the point does not add any extra data, since the point is the terminal object in the category of (super)manifolds. ▽

Remark 4.20. The space X here can be taken a supermanifold if one wishes so. Also, it can be endowed with a (Riemannian) metric, which a common thing to do in physics. ▽

4.1.5 Some Properties of Bordism Categories

In this section, we investigate some basic properties which bordism categories posses. These results apply to all categories previously constructed. We will use the notation $\underline{\text{Bord}}$ for any such category and $\underline{\text{Bord}}(X)$ to explicitly refer to bordisms over a space X.

In all the bordism categories considered, there is a seeming discrepancy between the object and morphism. In each case, the objects are manifolds itself, while the morphisms are certain diffeomorphism classes of manifolds. Notice that we have remembered the source and target objects in the definition of bordisms. However, it doesn't make sense to expect that diffeomorphic objects behave any different in a bordism category and indeed they don't as a result of the following lemma.

Lemma 4.21. *Suppose that $E \times (-1,1)$ and $F \times (-1,1)$ are objects in a bordism category $\underline{\text{Bord}}$ and a diffeomorphism $\phi\colon E \to F$ (in case we have a model geometry in play, further assume that the map $\Phi \times Id_{(-1,1)}\colon E \times (-1,1) \to F \times (-1,1)$ is an isometry). Then, the objects $E \times (-1,1)$ and $F \times (-1,1)$ are isomorphic in the category $\underline{\text{Bord}}$.*

Proof. Consider the following bordism

$$M = (\mathrm{Id}_{E\times(-1,1)}, E \times (-1,1), \phi \times \mathrm{Id}_{(-1,1)})\colon E \times (-1,1) \to F \times (-1,1). \tag{4.1.6}$$

Similarly, we have the bordism given by

$$M' = (\phi \times \mathrm{Id}_{(-1,1)}), E \times (-1,1), \mathrm{Id}_{E\times(-1,1)})\colon F \times (-1,1) \to E \times (-1,1). \tag{4.1.7}$$

Notice that these indeed define bordisms. In particular, the (+) condition in the definition holds. It is clear that the composition $M' \circ M$ yields the identity morphism on $E \times (-1,1)$. The diffeomorphism $\phi^{-1} \times \mathrm{Id}_{(-1,1)}$ applied to the composition $M \circ N$ shows that $M \circ N$ defines the same bordism as the identity on $F \times (-1,1)$.

In case we work with bordisms over a space X, the same construction applies. We only need to process the map to X in every step. This shows the claim. □

A structure, which all the bordism categories considered posses, is a symmetric monoidal structure from the disjoint union. For any of the non-family, bordism categories $\underline{\text{Bord}}$, the disjoint union functor $\sqcup\colon \underline{\text{Bord}} \times \underline{\text{Bord}} \to \underline{\text{Bord}}$ acting by the disjoint union of manifolds and bordisms is canonically associative. The empty manifold is a left and right unit. The symmetry comes from the natural isomorphism $E \sqcup F \to F \sqcup E$. The symmetry bordism can be visualized as in Fig. 5. It is a straightforward, yet little tedious, check to show this satisfies all properties, [Eti+15, Definitions 2.1.1 and 8.1.1], of a symmetric monoidal category. I will leave these checks to the reader.

In the case of a category of families of bordisms, the disjoint union functor should be restricted to the fibers of the groupoid fibration. The disjoint union of two families can only be taken if the index space of the families coincides. This way, the disjoint union operator can be seen as a morphism of stacks.

4.2 Functorial Field Theories

The bordism theories constructed above are the main building block of *Functorial Field Theories (FFT's)*. Functorial Field Theories will be functors out of the bordism category to some category of vector spaces. The investigation has been initiated by Witten, [Wit88], attempting to construct topological invariants from field theories in physics. Shortly after, Michael Atiyah [Ati88], axiomatized the notion, which over the years, in more modern language, turned into the definition we present here.

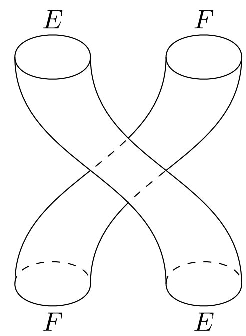

Figure 5: The bordism giving the symmetry isomorphism of the disjoint union operator in Bord.

4.2.1 Topological Vector Spaces and Algebras

Before we come to the definition of functorial field theories, we still need one ingredient: The target category of vector spaces. For ordinary bordism, we can simply use the category of vector spaces over some ground ring $\mathbb{K}$ with the linear maps between them. If required, these vector spaces can be topologized, yielding a category TV of *Topological Vectors Spaces*.

Definition 4.22 ([Sem03, Chapter 2.1])**.** A **Topological Field** is a field $\mathbb{K}$ endowed with a topology such that the addition, multiplication and inversion (both additive and multiplicative) maps are continuous. A **Topological Vector Space** is a vector space over a topological field $\mathbb{K}$ endowed with a topology making addition and scalar multiplication by $\mathbb{K}$ continuous. We will denote the **category of Topological Vector Spaces over** $\mathbb{K}$ with continuous linear maps by $\mathrm{TV}_{\mathbb{K}}$. ▲

Remark 4.23. If we wish to work in the smooth category, we replace the continuity assumption by smoothness. Moreover, this definition also lifts to the super world. Lifting the fields and vector spaces to superrings and super vector spaces. This way, we have **Topological Super Vector Spaces**. ▽

Example 4.24. Taking $\mathbb{K} = \mathbb{R}, \mathbb{C}$ with the Euclidean topology, the vector spaces $\mathbb{K}^n$ endowed with the Euclidean topology are topological vector spaces. In this fashion, the space $\mathbb{R}^{p|q}$ is a topological super vector space using the identification of Lemma 2.41. △

For ordinary bordism categories, like the ones defined in Definitions 4.3 and 4.10, it suffices to use just TV to define Functorial Field Theories. However, we need to extend the notion for the category of families of bordisms, as introduced in Definitions 4.13 and 4.16. For the families versions, we also need some kind of family version of TV. At first, one might expect to use vector bundles over the index space. However, as explained by [ST11, Remark 3.16] this turns out to be a too strict notion. Instead, we define the following.

Definition 4.25 ([ST11, Definition 2.47])**.** The category $\mathrm{TV}^{\mathrm{fam}}$ of **families of Topological Vector Spaces** has objects over a (super)manifold S, the sheaves V over S of topological monoids over the structure sheaf $\mathcal{O}_S$. I.e., the monoidal objects in the category of topological vector spaces. A **morphism of families of Topological Vector Spaces** is an $\mathcal{O}_S$ linear map of sheaves $V \to V'$ between objects V and V'. ▲

In the spirit of Definitions 4.7 and 4.9, we can restructure the above definition in terms of two stacks $\mathrm{TV}_0^{\mathrm{fam}}$ and $\mathrm{TV}_1^{\mathrm{fam}}$ of the objects and morphisms of TV respectively.

Definition 4.26. The objects of the category $\mathrm{TV}_0^{\mathrm{fam}}$ are the objects of $\mathrm{TV}^{\mathrm{fam}}$. A morphism of $\mathrm{TV}_0^{\mathrm{fam}}$ from a sheaf V over S to a sheaf V' over S' is a smooth map $f\colon S \to S'$ together with a $\mathcal{O}_{S'}(U)$ linear map $V(f^{-1}(U)) \to W(U)$ for every open subset $U \subseteq S'$. Here, the action of $\mathcal{O}_{S'}(U)$ on $V(f^{-1}(U))$ is given through the algebra homomorphism $f^*\colon \mathcal{O}_{S'}(U) \to \mathcal{O}_S(f^{-1}(U))$. ▲

Definition 4.27. The objects of the category $\mathrm{TV}_1^{\mathrm{fam}}$ are the morphisms of $\mathrm{TV}^{\mathrm{fam}}$. The morphisms of $\mathrm{TV}_1^{\mathrm{fam}}$ between objects $\phi\colon V_0 \to V_1$ and $\phi'\colon V_0' \to V_1'$ over S and S' respectively consists of a map $f\colon S \to S'$ together with $\mathcal{O}_{S'}(U)$ linear maps $V_0(f^{-1}(U)) \to V_0'(U)$ and $V_1(f^{-1}(U)) \to V_1'(U)$ for all opens $U \subseteq S'$ such that the diagram

$$\begin{tikzcd} V_0(f^{-1}(U)) \arrow[r, "\phi|_U"] \arrow[d] & V_1(f^{-1}(U)) \arrow[d] \\ V_0'(U) \arrow[r, "\phi'|_U"] & V_1'(U) \end{tikzcd} \tag{4.2.1}$$

commutes. As before, the action of $\mathcal{O}_{S'}(U)$ on $V_{0,1}(f^{-1}(U))$ is given through the algebra homomorphism $f^*\colon \mathcal{O}_{S'}(U) \to \mathcal{O}_S(f^{-1}(U))$. ▲

The fact that $\mathrm{TV}_0^{\mathrm{fam}}$ and $\mathrm{TV}_1^{\mathrm{fam}}$ are stacks follows easily from the properties of the sheaves.

We can enrich the notion of a Topological Vector Spaces to *Topological Algebras*. Similar to $\underline{\mathrm{TV}}$ the category $\underline{\mathrm{TA}}$ can be used as a target category for Functorial Field Theories. This will especially be important when considering twists in Chapter 4.2.5.

Definition 4.28 ([ST11, Definition 5.1])**.** The category of **Topological Algebras** $\underline{\mathrm{TA}}$ has objects the topological vector spaces A with a continuous associative multiplication $A \otimes A \to A$. The morphisms $A_0 \to A_1$ are the A_0, A_1-bimodules. I.e., they are topological vector spaces B with a multiplication $A_0 \otimes B \otimes A_1 \to B$ making B both a left A_0 and a right A_1 module.

The composition of two bimodules $B\colon A_0 \to A_1$ and $B'\colon A_1 \to A_2$ is the bimodule $B \otimes_{A_1} B'\colon A_0 \to A_2$.

Unraveling this category in the familiar style yields the groupoid of objects $\underline{\mathrm{TA}}_0$, whose morphisms are the continuous algebra isomorphisms and the groupoid of morphisms $\underline{\mathrm{TA}}_1$. The morphisms of the latter are triples (f_0, g, f_1) where $f_0\colon A_0 \to A_0'$ and $f_1\colon A_1 \to A_1'$ are algebra isomorphisms and $g\colon B \to B'$ a morphism of bimodules with B an A_0, A_1-bimodule and B' an A_0', A_1'-bimodule. ▲

The notion of topological algebras can be upgraded to families. Recall that we have defined families of topological vector spaces to be sheaves over the index space. We know that algebras are the monoidal objects in the category of vector spaces. Therefore, an **S-family of topological algebras** is a monoidal object in the category of S-families of topological vector spaces. Similarly, an **S-family of bimodules** is an object in the category of S-families of topological vector spaces with commuting left and right action by the modules. This way we obtain family versions of the categories $\underline{\mathrm{TA}}$, $\underline{\mathrm{TA}}_0$ and $\underline{\mathrm{TA}}_1$. The latter two are again turned into stacks with the projection to the index space.

Remark 4.29. These notions of families of topological vectors spaces and algebras introduced in this section apply mutatis mutantis for super vector spaces and algebras too. We simply assume a $\mathbb{Z}/2$ grading on all the vectors spaces involved. Families can also be taken over supermanifolds. ▽

4.2.2 Definition of Functorial Field Theories

We have now collected all the required ingredients to define Functorial Field Theories. We think of them as symmetric monoidal functors

$$Z\colon \underline{\mathrm{Bord}} \to \underline{\mathrm{TV}}. \tag{4.2.2}$$

from some bordims category with symmetric monoidal structure given by the disjoint union to the category of topological vectors spaces with symmetric monoidal structure given by the tensor product. More precisely we should work with the relevant unraveled internal categories, which are stacks, given that we work with family versions of bordisms and topological vector spaces.

Definition 4.30. A **Functorial Field Theory** is a pair of symmetric monoidal functor of stacks

$$Z_0\colon \underline{\mathrm{Bord}}_0 \to \underline{\mathrm{TV}}_0, \tag{4.2.3}$$
$$Z_1\colon \underline{\mathrm{Bord}}_1 \to \underline{\mathrm{TV}}_1. \tag{4.2.4}$$

Satisfying suitable compatibility properties. Here, the category $\underline{\mathrm{Bord}}_0$ and $\underline{\mathrm{Bord}}_1$ are the stacks which together form an internal category of bordisms $\underline{\mathrm{Bord}}$. This is just any of the constructed bordism categories. This can in particular includes supersymmetry, maps to a background space X and model geometry structures. The categories $\underline{\mathrm{TV}}_0$ and $\underline{\mathrm{TV}}_1$ are the unraveled versions of the category of topological vector spaces as in Definitions 4.26 and 4.27.

The compatibility property of the functors asserts that the functors cannonically induce a functor $Z\colon \underline{\mathrm{Bord}} \to \underline{\mathrm{TV}}$ in the ununraveled case. For a precise definition of internal categories and their morphisms see [ST11, Definition 2.4]. ▲

This definition in terms of the internal categories is the precise object we are looking at. However, for many purposes it suffices to look at the induced functor $Z\colon \underline{\mathrm{Bord}} \to \underline{\mathrm{TV}}$ instead.

Example 4.31. The *trivial* functorial field theory is given by sending every object of a bordism category $\underline{\mathrm{Bord}}$ to the ground field $\mathbb{K}$ of the vector spaces and all morphisms of $\underline{\mathrm{Bord}}$ to the identity of $\mathbb{K}$. For a category of families of bordisms $\underline{\mathrm{Bord}}^{\mathrm{fam}}$, we can send every object to (the section sheaf of) the trivial line bundle $S \times \mathbb{K}$, where S is the relevant index space. Again, all morphisms are sent to the identity on $S \times \mathbb{K}$. △

The funcotrial field theories, being functors $\underline{\mathrm{Bord}} \to \mathrm{TV}$ for some bordism category $\underline{\mathrm{Bord}}$, form the category of symmetric monoidal functors. The morphisms in this category are the natural transformations between the symmetric monoidal functors. We will denote the category of functorial field theories by $\underline{\mathrm{FFT}}$. When working over a space X, we can make this explicit with the notation $\underline{\mathrm{FFT}}(X)$.

The adopted nomenclature of *field theories*, which is a widely used term in physics, for this mathematical object hints at a direct link between mathematics and physics. The physical interpretation of functorial field theories is as follows. The objects of the bordism categories should be thought of as “particles” of the theory. In case, we have a 1-dimensional bordism category (so the bordism manifolds have dimension 1), the objects are indeed disjoint unions of points, the 0-dimensional manifolds. This reflects point like particles. For 2-dimensional bordism categories, the objects become circles, resembling strings. When working over a manifold X the “particles” are placed into some background manifold X, which can be interpreted as the relevant space-time. A field theory is now a functor which assigns to every such "particle" a vector space, which could be refined to resemble the Hilbert space of the particle.

The morphisms (bordisms) can be interpreted as a geometric description of a sort of time evolution of the system. The bordism tells us how the particles transform and interact. When working over a background manifold X, the map to X also tells us how the particles "move" in the background X. The field theory functor assigns a linear map between the relevant Hilbert spaces corresponding to the time evolution of the states under the given transformation of the particles.

The monoidal structures on the bordism category given by disjoint unions allows us to consider several particles simultaneously. The monoidal structure of the tensor product on the category of vector spaces gives us the quantum notion of "placing states together". Namely, the states get entangled using the tensor product structure.

4.2.3 Functorial Field Theories in Purely Odd Dimensions

In this section, we will consider functorial field theories on bordism categories of supermanifold in purely odd dimensions. I.e., the dimension of the bordism will be $0|q$ for any $q \geq 0$. The objects of any bordism category consists only of the empty set, as this is the only -1-dimensional manifold. The morphisms are disjoint unions of (super)point $\mathbb{R}^{0|q}$, potentially furnaced with extra datum like a map into a background manifold X. Notice that for the morphisms, we do need to take equivalence classes over the relevant notion of diffeomorphism. The diffeomorphism group of a one point topological space is obviously trivial. However, the superpoint $\mathbb{R}^{0|q}$ has a nontrivial diffeomorphism group, as explored in Lemma 2.63.

The above observations lead to the following theorem. We write $[(\Pi T)^q X/G]$ for the quotient stack of $(\Pi T)^q X$ by the action of $G \subseteq \mathrm{Diff}(\mathbb{R}^{0|q})$ as explained in Chapter 2.4.2, where $(\Pi T)^q X$ can be seen as the mapping space $\mathrm{Map}(\mathbb{R}^{0|q}, X)$, Proposition 2.61.

Theorem 4.32. *Let X be a supermanifold, $G \subseteq \mathit{Diff}(\mathbb{R}^{0|q})$ a subgroup and $\mathbb{K}$ a topological field. Consider the bordism category $(\mathbb{R}^{0|q}, G)$-$\underline{\mathit{Bord}}^{fam}(X)$ of families of bordisms over X for the model geometry $(\mathbb{R}^{0|q}, G)$. Then, the functorial field theories $(\mathbb{R}^{0|q}, G)$-$\underline{\mathit{Bord}}^{fam}(X) \to TV_{\mathbb{K}}^{fam}$ are in natural bijection with the $\mathbb{K}$ valued functions on $(\Pi T)^q X$ invariant under G.*

Proof. As observed, the category $(\mathbb{R}^{0|q}, G)$-$\underline{\mathrm{Bord}}^{\mathrm{fam}}(X)$ has a single object, namely the empty set, which is the unit for the monoidal structure given by the disjoint union. Therefore, every FFT on $0|q$-$\underline{\mathrm{Bord}}^{\mathrm{fam}}(X)$ sends all objects to the trivial family of topological vector spaces given by the trivial line $S \times \mathbb{K}$ for the relevant index space S. In particular, this means that the image of a morphism under a FFT is a vector bundle map between trivial line bundles. I.e., it can be identified with an map $S \to \mathbb{K}$.

Since a FFT is a symmetric monoidal functor, it is completely determined by its values on connected families of bordisms. Therefore, it suffices to remember just the values on the bordisms represented by a map $S \times \mathbb{R}^{0|q} \to X$. Two such maps $f, f' \colon S \times \mathbb{R}^{0|q} \to X$ define the same bordism precisely if there exist a $g \in G \subseteq \mathrm{Diff}(\mathbb{R}^{0|q})$ such that $f \circ (\mathrm{Id}_S \times g) = f'$.

We have obtained the following bijection

$$\mathrm{Fun}^{\otimes,\mathrm{fam}}((\mathbb{R}^{0|q}, G)\text{-}\underline{\mathrm{Bord}}^{\mathrm{fam}}(X), \mathrm{TV}^{fam}) \cong \mathrm{Map}^{\mathrm{fam}}(\mathrm{Map}(- \times \mathbb{R}^{0|q}, X)/G, \mathbb{K}). \tag{4.2.5}$$

Here, the left-hand side is precisely the set of FFT's on $(\mathbb{R}^{0|q}, G)$-$\underline{\text{Bord}}^{\text{fam}}(X)$. While, the right hand side are for every index space S the $\mathbb{K}$ valued S-families of functions on $\text{Map}(S\times\mathbb{R}^{0|q}, X)/G$. By Proposition 2.61, we know that $\text{Map}(S \times \mathbb{R}^{0|q}, X) = \text{Map}(S, (\Pi T)^q X)$. Hence, by Proposition 3.49 we see that the right hand side of Eq. (4.2.5) are the $\mathbb{K}$ valued functions on $(\Pi T)^q X$ invariant under G. We obtain the result. □

Corollary 4.33. *Taking $G = GL_\delta \ltimes \mathbb{R}^{0|\delta}$, we obtain a canonical bijection between the $(\mathbb{R}^{0|q}, G)$-FFT's with ground field $\mathbb{R}$ over X and the closed pseudo differential forms of determinant degree* 0.

Proof. By the theorem, we need to consider the functions on the quotient stack $[(\Pi T)^q X/G]$. Invariance under $\mathbb{R}^{0|\delta}$ is by definition equivalent with closedness. Invariance under GL_δ is by definition equivalent with having determinant degree 0. This shows the claim. □

Corollary 4.34 ([Hoh+11, Lemma 20])**.** *For a supermanifold X and topological field $\mathbb{K}$, the* $0|0$*-dimensional functorial field theories are in natural bijection with the $\mathbb{K}$ valued functions on X.*

Corollary 4.35 ([Hoh+11, Proposition 30])**.** *Let X be an ordinary manifold and $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then, the FFT's $0|1$-$\underline{Bord}^{fam}(X) \to TV_{\mathbb{K}}^{fam}$ are in bijection with the $\mathbb{K}$ valued closed differential forms on X of degree zero $\Omega^0_{cl}(X, \mathbb{K})$. I.e., the locally constant function on X.*

Proof. By the theorem above, we know that the FFT's are in bijection with the $\mathbb{K}$ valued functions on ΠTX invariant under $\text{Diff}(\mathbb{R}^{0|1})$. These are precisely the elements of the function sheaf $C^\infty(\Pi TX)$ invariant under $\text{Diff}(\mathbb{R}^{0|1})$ with values in $\mathbb{K}$, Proposition 3.49. The function sheaf of $C^\infty(\Pi TX)$ are the differential forms on X, Corollary 2.70. We know by Lemma 2.66 that $\text{Diff}(\mathbb{R}^{0|1}) = \mathbb{R}^\times \ltimes \mathbb{R}^{0|1}$, the translations and scalar multiplications on $\mathbb{R}^{0|1}$. The derivative of the action by translations is the De Rham operator, Lemma 2.71. A scalar multiplication by some $\alpha \in \mathbb{R}^\times$ acts on the function sheaf by multiplying a form ω with α^{-n} with n the degree of ω. Therefore, the invariant elements of the function sheaf $C^\infty(\Pi TX)$ are the degree zero closed differential forms on X. As requested. □

It seems a little sad we only see closed differential forms of degree zero. We know much more about differential forms and like to use their computative power to learn more about functorial field theories and bordism categories. The reason why we only obtain closed differential forms can be seen from the proof. The closedness and degree come respectively from the invariance under translation and scalar multiplication in $\text{Diff}(\mathbb{R}^{0|1}) = \mathbb{R}^\times \ltimes \mathbb{R}^{0|1}$. The invariance in turn followed from the fact that bordisms in $0|1$-$\underline{\text{Bord}}^{\text{fam}}(X)$ are equivalent if they are tied together by an element in $\text{Diff}(\mathbb{R}^{0|1})$. When we restrict this equivalence, we have less invariance assumptions on the forms. Hence, more forms will survive.

Allowing fewer bordisms to be equivalent can be done by enforcing the diffeomorphisms to be isometries in a model geometry. The interesting cases are the model geometries $(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ and $(\mathbb{R}^{0|1}, \{\pm 1\} \ltimes \mathbb{R}^{0|1})$.

Corollary 4.36 ([Hoh+11, Section 5.2])**.** *Let X be an ordinary manifold and $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then, the FFT's $(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$-$\underline{Bord}^{fam}(X) \to TV_{\mathbb{K}}^{fam}$ and $(\mathbb{R}^{0|1}, \{\pm 1\} \ltimes \mathbb{R}^{0|1})$-$\underline{Bord}^{fam}(X) \to TV_{\mathbb{K}}^{fam}$ are respectively in bijection $\Omega^\bullet_{cl}(X, \mathbb{K})$ and $\Omega^{ev}_{cl}(X, \mathbb{K})$.*

Proof. The proof is the same as in Corollary 4.35, only the invariance under scalar multiplication is limited to multiplying by 1^n and $(-1)^n$ in the respective cases with n the degree of the form. □

4.2.4 Concordance

As seen in the previous section, functorial field theories over certain bordism categories make up closed differential forms. From closed differential forms, it is a small step to De Rham cohomology. In this section, we aim to pass directly from functorial field theories to cohomology. I.e., we need to find a notion of equivalence on FFT's which sets closed forms differing by an exact form equal. The relevant notion will be *concordance* of functorial field theories. A concordance generally forgets geometrical data, while it retains topological data.

A map $f\colon X \to Y$ on the (super)manifolds induce by postcomposition a pushforward functor $f_*\colon \underline{\mathrm{Bord}}(X) \to \underline{\mathrm{Bord}}(Y)$ between the bordism categories over X and Y. Hence, by precomposition, we obtain a pullback functor $f^*\colon \underline{\mathrm{FFT}}(Y) \to \underline{\mathrm{FFT}}(X)$.

Definition 4.37 ([ST11, Definition 1.12])**.** Two functorial field theories $Z_\pm\colon \mathrm{Bord}(X) \to \mathrm{TV}$ for any of the bordism categories over a manifold X are **concordant** if there exists a functorial field theory $Z\colon \mathrm{Bord}(X \times \mathbb{R}) \to \mathrm{TV}$ such that $\pi_\pm^* Z_\pm$ and $i_\pm^* Z$ are naturally isomorphic as symmetric monoidal functors. Here, the functions $\pi_\pm\colon X \times (\pm 1, \pm\infty) \to X$ are the projection and the functions $i_\pm\colon X \times (\pm 1, \pm\infty) \to X \times \mathbb{R}$ are the inclusions. ▲

Remark 4.38. The term *concordance* is used in various mathematical fields, roughly meaning the same. All what really goes into the definition is a notion of pullback and isomorphism for some kind of structure defined over geometric objects. For example, we can say that two vector bundles $E_\pm$ over a base B are concordant if there exists a vector bundle E over $B \times \mathbb{R}$ such that $\pi_\pm^* E_\pm \cong i_\pm^* E$. By standard vector bundle theory, this notion of concordance is equivalent with isomorphisms of vector bundles.

Another example of concordance of (closed) differential forms. By applying Stokes' Theorem and the homotopy invariance of De Rham cohomology, we see that closed differential forms are concordant if and only if they represent the same cohomology class. ▽

We will now process the notion of concordance of functorial field theories through the proof of Theorem 4.32. Let X be a (super)manifold, $G \subseteq \mathrm{Diff}(\mathbb{R}^{0|q})$ a subgroup and $\mathbb{K}$ a topological field. By the theorem a concordance $Z \in \underline{\mathrm{FFT}}(X \times \mathbb{R})$ between two $(\mathbb{R}^{0|q}, G)$-FFT's $Z_\pm$ corresponds to a $\mathbb{K}$ valued function f on $(\Pi T)^q(X \times \mathbb{R})$ invariant under G. Since, the bijection of Theorem 4.32 is natural with respect to changing the background space, we deduce that there must hold

$$\pi_\pm^* f_\pm = i_\pm^* f. \tag{4.2.6}$$

Here, we write

$$\pi_\pm^*\colon C^\infty((\Pi T)^q(X)/G, \mathbb{K}) \to C^\infty((\Pi T)^q(X \times (\pm 1, \pm\infty))/G, \mathbb{K}) \quad \text{and} \tag{4.2.7}$$
$$i_\pm^*\colon C^\infty((\Pi T)^q(X \times \mathbb{R})/G, \mathbb{K}) \to C^\infty((\Pi T)^q(X \times (\pm 1, \pm\infty))/G, \mathbb{K}) \tag{4.2.8}$$

for the pullbacks given by precomposition of the relevant projections and inclusions respectively. The functions $f_\pm$ are the elements of $C^\infty((\Pi T)^q(X)/G, \mathbb{K})$ classifying $Z_\pm$.

We now consider concordance classes of functorial field theories in the low dimensions.

Proposition 4.39 ([Hoh+11, Corollary 21])**.** *There is a single concordance class for* $0|0$*-FFT's over a supermanifold* X*.*

Proof. Any smooth function on $X \times ((-\infty, -1) \cup (1, \infty))$ extends to a smooth function on $X \times \mathbb{R}$. Using Corollary 4.34, the result follows. □

Proposition 4.40. *The concordance classes of* $0|1$*-FFT's,* $(\mathbb{R}^{0|1}, \{\pm 1\} \ltimes \mathbb{R}^{0|1})$*-FFT's and* $(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$*-FFT's over an ordinary manifold* X *can be identified naturally with De Rham cohomology classes in* $H^0_{dR}(X)$*,* $H^{ev}_{dR}(X)$ *and* $H^{\bullet}_{dR}(X)$ *respectively.*

Proof. Observe that the equivalence induced, given by Corollaries 4.35 and 4.36 on the differential forms from concordance on functorial field theories, yields precisely concordance of closed differential forms. As by Remark 4.38, closed differential forms are concordant if and only if they represent the same cohomology class. This shows the claim in each of the cases. □

The way we have arrived here at De Rham cohomology in all degrees is different from [Hoh+11]. We have modified the model geometry to obtain the result, while [Hoh+11] introduce a notion of degree on the functorial field theories, using *twists*. We will present this approach in the next section. Our approach considered above is simpler in terms of getting just the identification with the total cohomology $\Omega^{\bullet}(X)$. However, using twists allows us to find the FFT's concentrated in a single degree.

4.2.5 Twisted Functorial Field Theories

In this section, we consider *twisted* Functorial Field Theories. These will allow us to define a suitable notion of degree, materializing the degree in cohomology. Twists will be a general framework in which we can define the degree. The name *twist* is used since it is believed that twisted field theories correspond to twisted cohomology, [ST11, Section 5]. From a physics perspective, the twisted theories are the conformal field theories with nonzero central charge. This correspondence to physics is due to Greame Segal, [Seg88].

Definition 4.41 ([ST11, Definition 5.2])**.** Let X be a smooth manifold and $(\mathbb{M}, G)$ a model geometry. A **twist** is a symmetric monoidal functor

$$T\colon (\mathbb{M}, G)\text{-}\underline{\mathrm{Bord}}(X) \to \underline{\mathrm{TA}}. \tag{4.2.9}$$

▲

A simple, yet fundamental, example is the *trivial* or *constant* twist

Example 4.42. The **trivial** or **constant twist**

$$T^0\colon (\mathbb{M}, G)\text{-}\underline{\mathrm{Bord}}(X) \to \underline{\mathrm{TA}} \tag{4.2.10}$$

sends all objects and morphism of $(\mathbb{M}, G)$-$\underline{\mathrm{Bord}}(X)$ to the ground field $\mathbb{K}$ seen as a topological algebra and $\mathbb{K}, \mathbb{K}$ bimodule respectively. This defines indeed a twist, since $\mathbb{K}$ is the monoidal unit in both $\underline{\mathrm{TA}}_0$ and $\underline{\mathrm{TA}}_1$. △

Definition 4.43 ([ST11, Definition 5.2])**.** A T**-twisted** Functorial Field Theory for a $(\mathbb{M}, G)$ is a natural transformation

$$\begin{tikzcd} (\mathbb{M}, G)\text{-}\underline{\mathrm{Bord}}(X) \arrow[r, bend left=40, "T^0"] \arrow[r, bend right=40, "T"'] \arrow[r, phantom, "\Downarrow Z"] & \underline{\mathrm{TA}} \end{tikzcd} \tag{4.2.11}$$

from the trivial twist to T. ▲

For this definition to make sense compared to the previous definition of functorial field theories, Definition 4.30, we need to recover the previous notion from the new one. This is indeed possible, as the following lemma shows. However, for this, it is important we work with internal categories explicitly.

Lemma 4.44 ([ST11, Lemma 5.7])**.** *The functorial field theories twisted by the trivial twist T^0 are in natural bijective correspondence with functorial field theories in the sense of Definition 4.30.*

Proof. A natural transformation Z between T^0 and itself amounts to an assignment $Z_0\colon (\mathbb{M}, G)\text{-}\underline{\mathrm{Bord}}(X)_0 \to \underline{\mathrm{TA}}_1$ on objects and an assignment Z_1 from the objects of $(\mathbb{M}, G)\text{-}\underline{\mathrm{Bord}}(X)_1$ to the morphisms of $\underline{\mathrm{TA}}_1$. Since, the trivial twist T^0 sends all objects of $(\mathbb{M}, G)\text{-}\underline{\mathrm{Bord}}(X)_{0,1}$ to the ground field, the assignment Z_0 sends every object to a topological vector space and Z_1 sends a bordism to a linear map between the topological vector spaces. The compatibility conditions of natural transformations enforce functoriality on the described assignment $Z\colon (\mathbb{M}, G)\text{-}\underline{\mathrm{Bord}}(X) \to \underline{\mathrm{TV}}$. This establishes the requested identification. □

A *degree* on a functorial field theory is given by a twist, which does not depend on the background manifold X. Defining these twists is non-trivial. A general construction of a degree in all dimensions has not been established. For purely bosonic dimensions $p|0$, a construction of twists T^n is given in [ST11, Section 5.5]. Below, we will consider the other extreme of purely fermionic dimensions $0|q$, following [Hoh+11, Section 6].

The diagram in (4.2.11) should remind us the sections of vector bundles over quotient stacks defined in Definition 3.52. The quotient stack here is that of the manifold $(\Pi T)^\delta X$ by the group G. Here, the group G needs to be some subgroup of $\mathrm{Diff}(\mathbb{R}^{0|\delta})$. This quotient stack detects precisely the connected bordisms in $(\mathbb{R}^{0|\delta}, G)\text{-}\underline{\mathrm{Bord}}(X)_1$, which is all relevant information for FFT's. In Corollary 2.64, we have identified the underlying supermanifold of $\mathrm{Diff}(\mathbb{R}^{0|\delta})$to be $\mathrm{GL}_\delta \times \mathbb{R}^{\delta(2^{\delta-1}-\delta)|\delta 2^{\delta-1}}$. Hence, at least on the level of supermanifolds, there is a projection map

$$\mathrm{Diff}(\mathbb{R}^{0|\delta}) \to \mathrm{GL}_\delta. \tag{4.2.12}$$

This projection removes all odd directions. In particular, all the odd translations are projected out. Recall from Lemma 2.67 that the translations and rotations in $\mathrm{Diff}(\mathbb{R}^{0|\delta})$ form a copy of $\mathrm{GL}_\delta \ltimes \mathbb{R}^{0|\delta}$. All other generators of $\mathrm{Diff}(\mathbb{R}^{0|\delta})$, see Lemma 2.63, do not affect the rotations. In conclusion, the projection map (4.2.12) is a group homomorphism. Postcomposing with the determinant map, we obtain a homomorphism

$$\rho\colon \mathrm{Diff}(\mathbb{R}^{0|\delta}) \to \mathrm{GL}_\delta \overset{\det}{\to} \mathbb{R}^\times. \tag{4.2.13}$$

Recall from Chapter 3.3.6 that a homomorphism $\rho_G\colon G \hookrightarrow \mathrm{Diff}(\mathbb{R}^{0|\delta}) \to \mathbb{R}^\times$ gives rise to a line bundle V_ρ over the quotient stack $[(\Pi T)^\delta X/G]$. From the proof of Theorem 4.32, we deduce that any symmetric monoidal functor $(\mathbb{R}^{0|\delta}, G)\text{-}\underline{\mathrm{Bord}}(X) \to \underline{TA}$ must take values in the line bundles. The stack of (real) line bundles can be identified with the quotient stack $[\mathrm{Pt}/\mathbb{R}]$ (this is just saying that line bundles are locally trivial). The bordism category can be identified with $(\Pi T)^\delta X/G$. Therefore, a line bundle bundle over $[(\Pi T)^\delta X/G]$ defines an $(\mathbb{R}^{0|\delta}, G)$-FFT twist. In particular, we have the line bundle V_{ρ_G} and its odd partner ΠV_{ρ_G}.

Definition 4.45. The **degree** 1 **twist** of $(\mathbb{R}^{0|\delta}, G)$-FFT's is the odd partner ΠV_{ρ_G} of the line bundle V_{ρ_G} over $(\Pi T)^\delta X/G$. Here, the homomorphism ρ_G is the restriction of the homomorphism of Eq. (4.2.13) to $G \subseteq \mathrm{Diff}(\mathbb{R}^{0|\delta})$. The **degree** n**-twist** is the n-fold tensor product $(\Pi V_{\rho_G})^{\otimes n}$ where we used the tensor product of $\underline{\mathrm{TA}}$.

The $(\mathbb{R}^{0|\delta}, G)$**-FFT's over** X **of degree** n are the natural transformations

$$\begin{tikzcd} (\mathbb{R}^{0|\delta}, G)\text{-}\underline{\mathrm{Bord}}(X) \arrow[r, bend left, "T^0"] \arrow[r, bend right, "(\Pi V_{\rho_G})^{\otimes n}"'] \arrow[r, phantom, "\Downarrow z"] & \underline{\mathrm{TA}} \end{tikzcd} . \tag{4.2.14}$$

▲

Proposition 4.46. *The* $(\mathbb{R}^{0|\delta}, G)$*-FFT's over* X *of degree* n *are in canonical bijection with*

$$\left\{ f \in C^\infty((\Pi T)^\delta X) | \mu^*(f) = p_1^*(f) p_2^*(\rho_{V_G}) \in C^\infty((\Pi T)^\delta X) \times G \right\}. \tag{4.2.15}$$

Proof. Using the same analysis of bordism categories and symmetric monoidal functors on them as in Theorem 4.32, we obtain that the $(\mathbb{R}^{0|\delta}, G)$-FFT's over X of degree n are in canonical bijection with the section of the vector bundle $(\Pi V_{\rho_G})^{\otimes n}$ over the quotient stack $[(\Pi T)^\delta X/G]$. Proposition 3.53 now implies the claim. □

Corollary 4.47. *Let* $G = GL_\delta \ltimes \mathbb{R}^{0|\delta}$*. Then, the* $(\mathbb{R}^{0|\delta}, G)$*-FFT's over* X *of degree* n *are in canonical bijection with the closed pseudo-differential forms of determinant degree* n*.*

Proof. By the proposition, we have to analyze the set

$$\left\{ f \in C^\infty((\Pi T)^\delta X) | \mu^*(f) = p_1^*(f) p_2^*(\rho_{V_G}) \in C^\infty((\Pi T)^\delta X) \times G \right\} \tag{4.2.16}$$

with $G = \mathrm{GL}_\delta \ltimes \mathbb{R}^{0|\delta}$. The required invariance under the $\mathbb{R}^{0|\delta}$ translation action is by definition equivalent with closedness of the pseudo-differential forms. The condition required by the GL_δ action boils down to the equality

$$A \cdot f = \det(A)^n f \tag{4.2.17}$$

for all $A \in \mathrm{GL}_\delta$. This is by definition equivalent with asking the form to be of determinant degree n. □

Restricting to the case of $\delta = 1$, we obtain, similar to Corollary 4.35, differential forms.

Corollary 4.48 ([Hoh+11, Proposition 35])**.** *The* $(\mathbb{R}^{0|1}, \mathit{Diff}(\mathbb{R}^{0|1}))$*-FFT's over* X *of degree* n *are in canonical bijection with the closed degree* n *differential forms* $\Omega^n_{cl}(X)$ *on* X*.*

Proof. Apply the previous corollary, using Lemma 2.66. The result now follows from Lemma 2.71 and Example 2.77. □

4.3 Physical Field Theories

Having consider the mathematical notion of functorial field theories, we will now turn to the physics side of things. In this section, we will relate the mathematical structures considered to certain field theories, in fact Sigma Models, common in physics. The models used will be taken from Edward Witten's chapter titled "*Index of Dirac Operators*" from [Del+99]. This section has a more "physicist" tone than before. We adopt notations and conventions common in the theoretical physics community.

4.3.1 Sigma Models

Field theories come in many variations in physics. The simplest are probably the scalar field theories. Here, we define a Lagrangian on top of maps from some manifold to $\mathbb{R}$ (or $\mathbb{C}$ when dealing with complex scalars). Naturally, we can extend this to vector fields, which are a bunch of scalars combined. I.e., we now have a Lagrangian from some manifold to $\mathbb{R}^n$ (or $\mathbb{C}^n$). However, we can go much more general by allowing our fields to land in any space X.

Definition 4.49 ([Lin18, Section 2]). A **Sigma Model** is a field theory whose fields take value in some manifold X. ▲

The name *Sigma* Model originates from the fact that initially the target manifold X was called σ, [GL60, Section 5]. In our application, the fields will take value in the space-time X.

Obviously, any of the (complex) scalar or vector field theories are included in the discussion of Sigma Models, by simply taking $X = \mathbb{R}^n$ (or $\mathbb{C}^n$). More interestingly, we can consider the Sigma Model of a (relativistic) point particle in a space-time X. This is described by maps $\Psi\colon \mathbb{R} \to X$, where the domain represents the time evolution. We write g_{ij} for a metric on X, a free Lagrangian for this model is given by

$$S(\Psi) = \int dt \langle \dot{\Psi}, \dot{\Psi} \rangle = \int dt g_{ij}(\Psi) \dot{\Psi}^i \dot{\Psi}^j. \tag{4.3.1}$$

Taking the dimension up one further, we obtain (relativistic) strings $\Psi\colon \Sigma \to X$ in a space-time X. In general, a Sigma Model with 2-dimensional domain can be given the Lagrangian

$$S(\Psi) = \int d^2x g_{ij}(\Phi) \eta^{ab} \partial_a \Psi^i \partial_b \Phi^j + B_{ij}(\Psi) \epsilon^{ab} \partial_a \Phi^i \partial_b \Psi^j. \tag{4.3.2}$$

Here, both the metric and the B field should be thought of as generalized coupling constants. After quantization, they will generically scale with energy.

The connection of these Sigma Models to the functorial field theories is as follows. We have defined functorial field theories over a space X on top of bordism categories over X. The data of both the object and the morphism in these categories contain a map to X, cf. Definition 4.18. Precisely this map resembles the fields in the Sigma Models. Obviously, in the topological case, we consider several manifolds as domain (depending on the structures we assume to be present). This can be incorporated in the physics story too. When taking path integrals, we need to integrate over all fields anyway. Hence, we can also integrate over a suitable moduli space to account for the various possible domains. Fixing ingoing and outgoing boundaries X_{in} and X_{out} and a FFT Z, the amplitude is computed by, [BM22, Equation 1.2],

$$\mathcal{A}(X_{\text{in}}, X_{\text{out}}) = \sum_{Y\colon X_{\text{in}} \to X_{\text{out}}} \frac{1}{|\text{Aut}(Y)|} Z(Y). \tag{4.3.3}$$

The automorphism group $\text{Aut}(Y)$ is the group of automorphisms of Y relative to the boundary. The fraction taking over the automorphism group is necessary to mitigate over counting. The result of this sum if obviously an element of $\text{Hom}(Z(X_{\text{in}}), Z(X_{\text{out}}))$.

4.3.2 Supersymmetric Sigma Models

Supersymmetry can be added to sigma models in two places. Both the domain and the target manifold of the model can be taken supermanifolds. We will respectively refer to the two cases as **world sheet supersymmetry** and **space-time supersymmetry**. We will mostly be concerned with world sheet supersymmetry.

Consider a world sheet[2] Σ whose dimension is $p|q$ and an ordinary manifold X. General fields $\Psi\colon \Sigma \to X$ in this Sigma Model take the form

$$\Psi_i(\vec{x}, \vec{\theta}) = \sum_{I \subseteq \{1,\ldots,q\}} a_i^I(\vec{x})\theta^I. \tag{4.3.4}$$

Here, the i index on the coefficients is there to account for the coordinates in X. The coefficient functions $a_i^I(\vec{x})$ are necessary even or odd functions, depending on the parity of the cardinality of I. This fact should raise some eyebrows, since what does it mean to have an *odd* resp. *even* function. The solution lies in the fact that we should work with families of Sigma Models instead. This is the analogue of the families of bordisms and their functorial field theories considered from a mathematical perspective. Explicitly, we will be working with world sheets which are fiber bundles over some index space S, whose fiber is Σ. Since, the index space S can be taken a supermanifold, function on this fiber bundle come with a notion of parity. We will tacitly assume the index space to be present in all our computation, but make no explicit reference to it.

We will consider the supersymmetric Sigma Model with an action of the form

$$S(\Psi) = -\frac{1}{2}\int_\Sigma d^p x d^q \theta \langle \partial_\mu \Psi, D^\mu \Psi\rangle. \tag{4.3.5}$$

Here, the brackets denote the pullback of the metric g_{ij} on X along the field Ψ and D^μ is a some vector field on Σ. The integral over the world sheet includes integration in both the even and odd coordinates. The integration of odd variables are Berezinian integrations. Berezinian integrals are the standard way in physics to integrate odd variables. For more details see [Wip19]. The μ index runs over the even directions on the worldsheet.

The equations of motion for such an action are most easily computed by varying the action once. We follow the computation in [Ber13a, Sections 4.5 and 5.5].

$$\begin{aligned}
\delta_v S(\Psi) &= -\frac{1}{2}\int_\Sigma d^p x d^q\theta \Big[\langle \nabla_v \partial_\mu \Psi, D^\mu \Psi\rangle + \langle \partial_\mu\Psi, \nabla_v D^\mu \Psi\rangle\Big] && (4.3.6)\\
&= -\frac{1}{2}\int_\Sigma d^p x d^q\theta \Big[\langle \nabla_\mu \delta_v \Psi, D^\mu \Psi\rangle + \langle \partial_\mu\Psi, \nabla_{D^\mu} \delta_v \Psi\rangle\Big] && (4.3.7)\\
&= -\frac{1}{2}\int_\Sigma d^p x d^q\theta \Big[- \langle \delta_v\Psi, \nabla_\mu D^\mu \Psi\rangle + \partial^\mu \langle \delta_v \Psi, D^\mu\Psi\rangle \\
&\qquad - \langle \nabla_{D^\mu}\partial_\mu \Psi, \delta_v\Psi\rangle + D^\mu\langle \partial_\mu \Psi, \delta_v \Psi\rangle\Big] && (4.3.8)\\
&= \int_\Sigma d^p x d^q\theta \langle \delta_v \Psi, \nabla_\mu D^\mu\Psi\rangle && (4.3.9)
\end{aligned}$$

Here, in the second step, we used that the connection is torsion free. In the third step, we have partially integrated. In the last step, we have discarded the total derivatives, which integrate to zero. From this equation, we see that the equation of motion is

$$\nabla_\mu D^\mu \Psi = 0. \tag{4.3.10}$$

[2]I will use the term world sheet for the domain of the Sigma Model, irrespective of its dimension. So it can be a world line, volume or any higher dimensional analogue

Continuing the computation in the same spirit, we obtain the second order variation on the classical solution.

$$\delta_w\delta_v S(\Psi) = \int_\Sigma d^p x d^q\theta \Big[\langle \nabla_w \delta_v \Psi, \nabla_\mu D^\mu \Psi\rangle + \langle \delta_v\Psi, \nabla_w \nabla_\mu D^\mu \Psi\rangle\Big] \tag{4.3.11}$$

$$= \int_\Sigma d^p x d^q\theta \langle \delta_v \Psi, \nabla_\mu \nabla_{D^\mu} \delta_w \Psi\rangle \tag{4.3.12}$$

We will now construct an explicit operator D for which the action (4.3.5) applies. For now, we will assume all function have complex values. I.e., the function sheaves of our supermanifolds will be tensored with $\mathbb{C}$. This has the effect that the odd generators θ_i obtain an additional spinor index α. We will use the shorthand $\theta_i^2 = (\theta_i)_\alpha(\theta_i)^\alpha$. We now can define an operator D^μ as

$$D^\mu = \theta_1^2 \cdots \theta_k^2 \partial^\mu - \sum_{i=1}^{k} \theta_1^2 \cdots \theta_{i-1}^2 \theta_{i+1}^2 \cdots \theta_k^2 (\theta_i)^\alpha \epsilon_{\alpha'}^{\alpha''} \sigma^\mu_{\alpha\alpha''} \frac{\partial}{\partial \theta_{\alpha'}}. \tag{4.3.13}$$

Here, the $\sigma^\mu_{\alpha\alpha'}$ are the sigma matrices. The $\epsilon_{\alpha'}^{\alpha''}$ is there to correct the sign in case the θ_i^α are ordered in reverse. We write the fields Ψ from Eq. (4.3.4) as

$$\Psi_i(\vec{x}, \vec{\theta}) = x_i(\vec{x}) + (\psi_i)^\mu_\alpha\, \theta^\alpha_\mu + \text{higher order}. \tag{4.3.14}$$

Substituting the operator D^μ and these fields in the action (4.3.5) and performing the Berezinian integrals similar to [Del+99, pages 651-652], we obtain the following:

$$S(\Psi) = -\frac{1}{2}\int_\Sigma d^p x d^q \theta \langle \partial_\mu \Psi, D^\mu \Psi\rangle \tag{4.3.15}$$

$$= -\frac{1}{2}\int_\Sigma d^p x d^q \theta g_{ij}(\Psi) \partial_\mu \Psi^i D^\mu \Psi^j \tag{4.3.16}$$

Since all terms coming from $D^\mu\Psi^j$ contain at least $2p-1$ factors of θ_i^α, the only terms surviving must have at most 1 factor of θ_i^α from $g_{ij}(\Psi)$ and $\partial_\mu \Psi^i$. Taylor expanding g_{ij} in the θ_i yields

$$= -\frac{1}{2}\int_\Sigma d^p x d^q\theta \left(g_{ij}(x(\vec{x})) + \theta^\nu_\alpha (\psi_k)^\alpha_\nu \partial^k g_{ij}(\vec{x})\right) \partial_\mu \Psi^i D^\mu \Psi^j \tag{4.3.17}$$

$$= \frac{1}{2}\int_{|\Sigma|} d^p x \bigg[g_{ij}(x(\vec{x})) \left(\partial_\mu x^i \partial^\mu x^j - \partial_\mu \left(\psi^i\right)^\alpha_\nu \sigma^\mu_{\alpha\alpha'} \left(\psi^j\right)^{\nu\alpha'}\right)$$
$$- (\psi_k)^\alpha_\nu \partial^k g_{ij}(\vec{x}) \partial_\mu x^i \sigma^\mu_{\alpha,\alpha'} \left(\psi^j\right)^{\nu\alpha'}\bigg] \tag{4.3.18}$$

$$= \frac{1}{2}\int_{|\Sigma|} d^p x \langle \partial_\mu x, \partial^\mu x\rangle - \langle \sigma^\mu_{\alpha\alpha'} \psi^\alpha_\nu, \nabla_\mu \psi^{\nu\alpha'}\rangle \tag{4.3.19}$$

In all, we see that we have obtained the free Lagrangian of a supersymmetric Sigma Model with q complex supersymmetries. Moreover, we can identify the Hamiltonian function with

$$\frac{1}{2}\langle \partial_\mu x, \partial^\mu x\rangle. \tag{4.3.20}$$

For the rest of this chapter we will specify the dimension to low dimension. We will just have one fermionic dimension and 1 resp. 2 bosonic dimensions. As we will see, in these dimensions the notation greatly simplifies. In particular, we do not have to work with complex odd generators anymore. Real ones will suffice.

4.3.3 1|1-dimensional Sigma Model and the $\hat{A}$-genus

The Sigma Model with one bosonic and one fermionic dimension gives rise to the $\hat{A}$-genus. From a mathematical perspective, the $\hat{A}$-genus is a homomorphism

$$\hat{A}\colon \Omega_{\bullet}^{\mathrm{SO}} \to \pi_{\bullet}(KO) \otimes \mathbb{Q} \tag{4.3.21}$$

from the bordism ring of orientable manifolds to the spectrum of K-theory.[Van13] This places the $\hat{A}$-genus in K-theory, which is a generalized cohomology theory. The characteristic series for the $\hat{A}$-genus is given by, [AHR10, Proposition 10.2],

$$\frac{x/2}{\sinh(x/2)} = \exp\left(\sum_{k=1}^{\infty} \frac{x^{2k}}{2k(2\pi i)^{2k}} 2\zeta(2k)\right). \tag{4.3.22}$$

As we will see below, we can construct the $\hat{A}$ genus from the 1|1-dimensional Sigma Model using Physical arguments. This argument is taken from Witten's chapter in [Del+99]. A mathematically rigorous construction following the same ideas will be given in Chapter 5.5. Moreover, the $\hat{A}$-genus itself has applications in physics itself. Firstly, the genus is integral part of the Atiyah-Singer index theorem, [AS71], which is widely used to compute indices of operators. Secondly, the genus is used for describing effective actions on Dp-branes in super string theory. For example, see [BLT13, Equation 16.171].

The Sigma Model we have constructed takes as central input the vector field D^{μ}. Since there is just one bosonic dimension, we can drop the index and use the following operator

$$D = \frac{\partial}{\partial\theta} - \theta\frac{d}{dt}. \tag{4.3.23}$$

As before, we take the action functional

$$\begin{aligned} S(\Psi) &= -\int dt d\theta \langle \partial_t \Psi, D\Psi\rangle && (4.3.24)\\ &= \int dt \Big[\langle \dot{x}, \dot{x}\rangle - \langle \nabla_t \psi, \psi\rangle\Big] && (4.3.25)\end{aligned}$$

where we have written the field $\Psi(t) = x(t) + \psi(t)\theta$ and we have used Eq. (4.3.19) to obtain the second line. The equations of motion remain $\nabla_t D\Psi = 0$ as by Eq. (4.3.10). We now investigate the second order variation of the action. From Eq. (4.3.12), we know that the second order variation evaluated at the classical solutions takes the form

$$\delta_w \delta_v S(\Psi) = \int dt d\theta \langle \delta_v \Psi, \nabla_t \nabla_D \delta_w \Psi\rangle. \tag{4.3.26}$$

Denote $i_0\colon \mathbb{R} \to \mathbb{R}^{1|1}$ for the inclusion in the even coordinate. We have the identities

$$i_0^*(\Psi) = x, \qquad i_0^*(D\Psi) = \psi, \tag{4.3.27}$$

$$i_0^* \nabla_D \nabla_t \nabla_D = i_0^* \nabla_t \nabla_D \nabla_D = \frac{1}{2} i_0^* \nabla_t (R(D,D) - \nabla_{[D,D]}) = \frac{1}{2}\mathcal{R}(D,D)\nabla_t i_0^* + \nabla_t^2 i_0^*. \tag{4.3.28}$$

Here, we used in the first equality the fact that the connection is torsion free, in the second equality that D is an odd vector field and thus

$$\nabla_D^2 = \frac{1}{2}\left(\nabla_D \nabla_D + \nabla_D \nabla_D\right) = \frac{1}{2}\left(\mathcal{R}(D,D) - \nabla_{[D,D]}\right) \tag{4.3.29}$$

and in the last equality that $[D, D] = 2\nabla_D D = -2\partial_t$. The $\mathcal{R}(D, D)$ denotes the curvature.

Decomposing the variation $\delta_v \Psi = a + \theta\eta$ and using these identities, we obtain that the Hessian takes the form

$$\begin{aligned}
\mathrm{Hess}(a, \eta) &= \delta_v \delta_v S(\Psi) && (4.3.30)\\
&= \int dt d\theta \langle a + \theta\eta, \nabla_t \eta + \theta \left(\nabla_t^2 + \frac{1}{2}\mathcal{R}(D, D)\nabla_t \right) a\rangle && (4.3.31)\\
&= \int dt \left[\langle \eta, \nabla_t \eta\rangle + \langle a, \left(\nabla_t^2 + \frac{1}{2}\mathcal{R}(D, D)\nabla_t \right) a\rangle \right] && (4.3.32)
\end{aligned}$$

where in the second equality we have taken the Berezinian integral. In particular, we have obtained the operators

$$\Delta_{\mathrm{ev}}^{1|1} = -\nabla_t^2 - \frac{1}{2}\mathcal{R}(D, D)\nabla_t \quad \text{and} \quad \Delta_{\mathrm{odd}}^{1|1} = -\nabla_t. \tag{4.3.33}$$

The $\hat{A}$-genus arises as a determinant of the operator $\Delta_{\mathrm{ev}}^{1|1}$. One can also add the contributions from the odd operator, however their contribution is only an irrelevant constant. We ignore the odd part and focus on the even compartment. Notice that the a used in the computation above is a map $a \colon \mathbb{R} \to TX$. Since, we work with a Riemannian metric on X, we can decompose TX, for any linear operator $R \in \mathrm{SO}(TX)$ (in particular the curvature $\mathcal{R}$), in two-dimensional subspaces such that R is block diagonal with blocks R_i of the form

$$\begin{pmatrix} 0 & x_i \\ -x_i & 0 \end{pmatrix}. \tag{4.3.34}$$

Therefore, a basis of eigenfunctions is given by the functions

$$t \mapsto e^{2\pi i k t}. \tag{4.3.35}$$

Hereby, we obtain that the eigenvalues of $\Delta_{\mathrm{ev}}^{1|1}$ are $(2\pi k)^2 \pm \frac{2\pi i k x_i}{2}$. Evaluating the determinant of $\Delta_{\mathrm{ev}}^{1|1}$ is some straightforward linear algebra:

$$\begin{aligned}
\sqrt{\det(\Delta_{\mathrm{ev}}^{1|1})} &= \prod_i \prod_{k \neq 0} \sqrt{(2\pi k)^2 \pm \frac{2\pi i k x_i}{2}} && (4.3.36)\\
&= \prod_i \prod_{k=1}^{\infty} (2\pi k)^4 \left(1 + \frac{x_i}{2\pi k}\right)^2 && (4.3.37)\\
&= C \prod_i \left[\frac{\frac{x_i}{2}}{\sinh\left(\frac{x_i}{2}\right)} \right]^{-1} && (4.3.38)
\end{aligned}$$

for some (infinite) constant C. We have obtained precisely the inverse of the characteristic series of the $\hat{A}$-genus, Eq. (4.3.22). The infinite constant is the obvious root cause of this computation not being mathematically rigorous. We will aid this problem in the next chapter by comparing to a flat background.

4.3.4 2|1-dimensional Sigma Model and the Witten Genus

The 2|1-dimensional Sigma Model gives rise to the Witten genus. This genus, first constructed by Edward Witten in his paper, [Wit87]. The Witten genus takes a similar place in *elliptic* *cohomology* as the $\hat{A}$-genus does in K-theory.

In the 2|1-dimensional Sigma Model, we consider has action functional

$$S(\Psi) = -\frac{1}{2}\int dz d\bar{z} d\theta \langle \partial_z \Psi, D\Psi\rangle \tag{4.3.39}$$

$$= \frac{1}{2}\int dz d\bar{z}\Big[\langle \partial_z x, \partial_{\bar{z}} x\rangle - \langle \partial_z \psi, \psi\rangle\Big] \tag{4.3.40}$$

where $D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\bar{z}}$. For the second line, we used Eq. (4.3.19).

The equations of motion are obtained immediately from Eq. (4.3.10),

$$\nabla_z D\Psi = 0. \tag{4.3.41}$$

Varying the action once more, we obtain the second order variation

$$\delta_w\delta_v S(\Psi) = \int_\Sigma dz d\bar{z} d\theta \langle \delta_v\Psi, \nabla_z \nabla_D \delta_w \Psi\rangle. \tag{4.3.42}$$

We analyze this equation with the same ideas as in the 1|1-dimensional case. We write $i_0 \colon \mathbb{R}^2 \to \mathbb{R}^{2|1}$ for the inclusion in the even coordinates. We have the identities

$$i_0^*(\Psi) = x, \qquad i_0^*(D\Psi) = \psi, \tag{4.3.43}$$

$$[D,D] = 2\nabla_D D = -2\partial_{\bar{z}}, \qquad \nabla_D^2 = \frac{1}{2}\left(\nabla_D\nabla_D + \nabla_D\nabla_D\right) = \frac{1}{2}\left(\mathcal{R}(D,D) - \nabla_{[D,D]}\right), \tag{4.3.44}$$

$$i_0^*\nabla_D\nabla_z\nabla_D = i_0^*\nabla_z\nabla_D\nabla_D = \frac{1}{2} i_0^*\nabla_z(R(D,D) - \nabla_{[D,D]}) = \frac{1}{2}\mathcal{R}(D,D)\nabla_z i_0^* + \nabla_z\nabla_{\bar{z}} i_0^*. \tag{4.3.45}$$

Here, we write $\mathcal{R}(D,D)$ for the curvature.

Decomposing the variation $\delta_v\Psi = a + \theta\eta$ and using these identities, we obtain that the Hessian takes the form

$$\mathrm{Hess}(a,\eta) = \delta_v\delta_v S(\Psi) \tag{4.3.46}$$

$$= \int dz d\bar{z} d\theta \langle a + \theta\eta, \nabla_z\eta + \theta\left(\nabla_z\nabla_{\bar{z}} + \frac{1}{2}\mathcal{R}(D,D)\nabla_z\right) a\rangle \tag{4.3.47}$$

$$= \int dz d\bar{z}\Big[\langle \eta, \nabla_t \eta\rangle + \langle a, \left(\nabla_z\nabla_{\bar{z}} + \frac{1}{2}\mathcal{R}(D,D)\nabla_z\right) a\rangle\Big] \tag{4.3.48}$$

where in the second equality we have taken the Berezinian integral. In particular, we have obtained the operators

$$\Delta_{\mathrm{ev}}^{2|1} = -\nabla_z\nabla_{\bar{z}} - \frac{1}{2}\mathcal{R}(D,D)\nabla_z \quad \text{and} \quad \Delta_{\mathrm{odd}}^{2|1} = -\nabla_z. \tag{4.3.49}$$

The Sigma Model in consideration is a version of Type II supersymmetric string theory. We will consider the sector with Ramond right movers and NS left movers. After quantization, we obtain the supercurrent

$$Q = \int d\sigma \langle \partial_+ x, \psi_+\rangle. \tag{4.3.50}$$

Its square is the Hamiltonian operator L_0. We will compute the index of the operator Q closely following [Wit87].

Naively, one would expect the index of Q to be the number of zero eigenvalues of Q with $(-1)^{F_R} = +1$ minus the number of eigenvalues with $(-1)^{F_R} = -1$. Here, the operator $(-1)^{F_R}$ counts the number of right movers modulo 2. However, we need a more refined character valued index.

We define the momentum operator

$$P = \widetilde{L_0} - L_0 \tag{4.3.51}$$

which commutes with Q. Hence, the eigenspaces of P and Q coincide. For every eigenvalue λ of P denote b_λ for the index of Q restricted to the eigenspace H_λ of P with eigenvalue λ. Define the character valued index of Q as

$$F(q) = \sum_\lambda b_\lambda q^\lambda. \tag{4.3.52}$$

Recall from standard supersymmetric string theory that the momentum operator P picks up a factor of $(-1)^{F_L}$ under a transformation $\tau \to \tau + 2\pi$ and that the global anomaly in the fermion determinant under the same transformation makes the states pick up a factor $e^{-i\pi\frac{d}{8}}$. Here, the dimension of the background is d. Hence, we have the operator statement

$$e^{2\pi i P} = (-1)^{F_L} e^{-i\pi\frac{d}{8}}. \tag{4.3.53}$$

We conclude that the possible eigenvalues for P are

$$\lambda = -\frac{d}{16} + \frac{l}{2} \tag{4.3.54}$$

where l is an integer.

We evaluate Eq. (4.3.52) with the found eigenvalues.

$$F(q) = q^{-\frac{d}{16}} \sum_{l\in\mathbb{Z}} b_l q^{\frac{l}{2}} \tag{4.3.55}$$

$$= q^{-\frac{d}{16}} \sum_{l=0}^{\infty} \mathrm{Index}(R_l) q^{\frac{l}{2}} \tag{4.3.56}$$

Here, the operators R_l are defined by the generating function

$$\sum_k q^{\frac{k}{2}} R_k = \otimes_{k\in\mathbb{N}+\frac{1}{2}} \Lambda_{q^k} T \otimes_{l\in\mathbb{N}} S_{q^l} T \tag{4.3.57}$$

where we have used the antisymmetric and symmetric expansions of T

$$\Lambda_t T = 1 + tT + +t^2 \Lambda^2 T + \dots \tag{4.3.58}$$

$$S_t T = 1 + tT + +t^2 S^2 T + \dots \tag{4.3.59}$$

The operator T denotes the fundamental representation of $SO(N)$ and the equality in Eq. (4.3.56) holds because the operator R_k gives us precisely the symmetry transformation of the k'th mass level of the theory. E.g., denote $|\Omega\rangle$ for the ground state, which is a spin(N) singlet. Now, the first excited states are created by applying the fermion creation operators

ψ^i_k for negative halfintegers k and boson creation operators a^i_l for negative integers l. At the first excited level, we only find the states $\psi^i_{-\frac{1}{2}}|\Omega\rangle$, which transforms like T. At the second level, we find the states $\psi^i_{-\frac{1}{2}}\psi^i_{-\frac{1}{2}}|\Omega\rangle$ and $a^i_{-1}|\Omega\rangle$, which transforms like $\Lambda^2 T \oplus T$. Continuing to the third level, we find the states $\psi^i_{-\frac{1}{2}}\psi^i_{-\frac{1}{2}}\psi^i_{-\frac{1}{2}}|\Omega\rangle$, $\psi^i_{-\frac{1}{2}}a^i_{-1}|\Omega\rangle$ and $\psi^i_{-\frac{3}{2}}|\Omega\rangle$. Together these transform like $\Lambda^3 T \oplus (T \oplus T) \oplus T$. Notice that these are precisely the first factors in the expansion of Eq. (4.3.57) and we can reasonably assume this continues to all orders.

By formally applying the Atiyah-Singer Index Theorem, [AS71], we can compute the index F following the computations in [SW86].

$$F(q) = q^{-\frac{d}{16}}\hat{A}(X)\mathrm{ch}\left(\otimes_{k\in\mathbb{N}+\frac{1}{2}}\Lambda_{q^k}T \otimes_{l\in\mathbb{N}} S_{q^l}T\right)[X] \tag{4.3.60}$$

$$= \left(\frac{\eta(-q^{\frac{1}{2}})}{\eta(q)\eta(-q)}\right)^d \Phi(q) \tag{4.3.61}$$

Here, the function Φ can be shown to be a modular form of weight $\frac{d}{2}$ and the η's are Dedekind η functions. The modular form Φ is referred to as the **Witten Genus**. In the next chapter, we will construct this genus in a more mathematically rigorous setting. The main issue in this computation is the fact we only have *formally* applied the Atiyah-Singer Index Theorem. This theorem has only been shown for finite dimensional manifolds, while here we apply it over a loop space, which generically is infinite dimensional.

5 One Loop Field Theories

In Chapter 4.2.3, we have analyzed field theories in purely odd dimensions. The goal now is to allow for non-trivial even dimensions. This makes the theories highly non-trivial in general. Therefore, we will perturbatively expand and just consider the theories up to having one loop. I.e., we restrict our manifolds to be (higher dimensional) tori. As we will see, we can construct some interesting objects already from these restricted theories.

We start by defining lattices and their tori and family versions of both. We continue in Chapter 5.2 by organizing the lattices in a stack of tori. By adding the datum of a map to a background, we lift to a stack of Fields. To make this stack manageable, we need to restrict the field maps to be constant in the even coordinates. We use the hereby obtained stacks of vacuum fields to construct complexified cohomology using similar ideas as in Chapter 4.2.3. We finish by constructing the $\hat{A}$ and Witten class in the present language.

5.1 Lattices and Tori

The tori will be defined in terms of lattices, so we define those first.

Definition 5.1. A p**-dimensional lattice** is a subgroup of $\mathbb{R}^p$ of the form $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_p$, where $v_1, \cdots, v_p$ are a linear basis of $\mathbb{R}^p$. We say that a lattice is **oriented** if

$$\det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{pmatrix} > 0. \tag{5.1.1}$$

We denote the set of all oriented p-lattices by $\mathcal{L}_p$. ▲

The space $\mathcal{L}_p$ can canonically be identified with the collection of oriented p-frames in $\mathbb{R}^p$, i.e., p-tuples of linearly independent vectors in $\mathbb{R}^p$ with orientation. In other words, $\mathcal{L}_p$ can be identified with the oriented **Stiefel-manifold** $V_p(\mathbb{R}^p)$ in canonical fashion. This oriented Stiefel manifold is an open submanifold of $(\mathbb{R}^p)^p$, the p-fold product of $\mathbb{R}^p$.[MS74, Paragraph 5, page 56]

Figure 6 visualizes some basic examples of lattices. With the notion of oriented lattices, we do not lose much compared to unoriented lattices. From an unoriented lattice, we can change the sign of one of its generators v_i and obtain an oriented one. Moreover, the lattice obtained by flipping the sign, should really be thought of the same as the original lattice, since the subgroup $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_p$ of $\mathbb{R}^p$ is invariant under this transformation. Obviously, there are many more transformations leaving the subgroup invariant. Indeed, the action of the general linear group with integer coefficients $GL_p(\mathbb{Z})$ does. However, the orbits of this action are somewhat harder to parameterize. Therefore, we stick with the oriented lattices. What we have achieved by passing to oriented lattices is that we reduced the symmetry group form $GL_p(\mathbb{Z})$ to $SL_p(\mathbb{Z})$, the special general linear group with integer coefficients.

In small dimensions, this notion of orientation takes the familiar form. If $p = 1$, then a lattice is given by a nonzero scalar $r \in \mathbb{R}^\times$. Orientability asserts that $r \in \mathbb{R}_{>0}$. If $p = 2$, then suppose that $v_1 = (a, b)$ and $v_2 = (c, d)$. Hence, there holds

$$\det \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \tag{5.1.2}$$

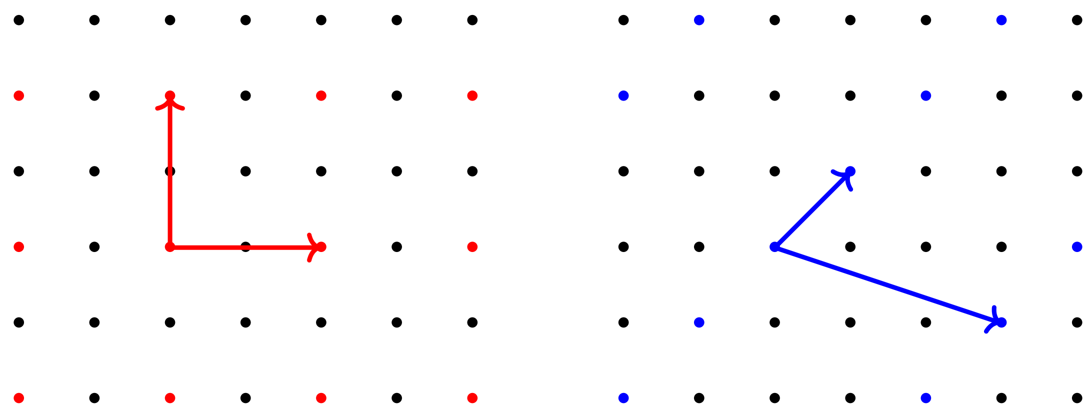

Figure 6: *Two lattices in* $\mathbb{R}$. *On the left is the lattice generated by the vectors* $(0,2)$ *and* $(2,0)$. *On the right are the lattice generated by* $(1,1)$ *and* $(4,-1)$. *In each plot, the image points of the related homomorphism* $\mathbb{Z}^2 \to \mathbb{R}^2$ *are indicated in red and blue respectively. The resulting torus in each case is obtained by identifying all points in* $\mathbb{R}^2$ *which only differ by a linear combination with integer coefficients of the generating vectors. In particular, the red and blue points are identified in each case.*

Identifying $\mathbb{R}^2 \cong \mathbb{C}$, we can consider

$$\frac{v_2}{v_1} = \frac{c+di}{a+bi} = \frac{(c+di)(a-bi)}{a^2+b^2} = \frac{ac+bd+(ad-bc)i}{a^2+b^2} \tag{5.1.3}$$

So the lattice $\mathbb{Z}v_1 + \mathbb{Z}v_2$ is oriented if the quotient $\frac{v_2}{v_1}$ lies in the upper half plane $\mathbb{H}$.

Notice that for any lattice we have a canonical homomorphism $\Lambda\colon \mathbb{Z}^p \to \mathbb{R}^p$ such that the images of the generators $(1,0,\cdots,0),\cdots,(0,0,\cdots,1)$ form a basis of $\mathbb{R}^p$. The condition for an oriented lattice now becomes explicitly

$$\det \begin{pmatrix} \Lambda(1,0,\cdots,0) \\ \Lambda(0,1,\cdots,0) \\ \vdots \\ \Lambda(0,0,\cdots,1) \end{pmatrix} > 0. \tag{5.1.4}$$

In this description of lattices, we can straightforwardly consider families of them. An illustration of a family of lattices is in Fig. 7.

Definition 5.2. For a manifold S, an **(oriented)** S**-family** of p-lattices is an equivariance $\Lambda_S\colon S \times \mathbb{Z}^p \to \mathbb{R}^p$ such that for any $s \in S$ the restriction to $\{s\} \times \mathbb{Z}$ is a(n oriented) lattice. Here, we take the $\mathbb{Z}^p$ action on $S \times \mathbb{Z}^p$ which just acts in the $\mathbb{Z}^p$ coordinate. ▲

Using the inclusion $\mathbb{R}^p \hookrightarrow \mathbb{R}^{p|q}$ as a completely even space, we can define a p-lattice on $\mathbb{R}^{p|q}$. Upon taking quotients by lattices, we obtain (super)tori in any dimension.

Definition 5.3. For a(n oriented) p-lattice $\Lambda\colon \mathbb{Z}^p \to \mathbb{R}^p$, we define the $p|q$**-dimensional super torus** $\mathbb{T}^{p|q}$ as the superspace $\mathbb{R}^{p|q}/\mathbb{Z}^p$, where the $\mathbb{Z}^p$ action if given by

$$(\vec{t},\vec{\theta}) \cdot (\vec{n}) = (\vec{t} + \Lambda(\vec{n}), \vec{\theta}) \tag{5.1.5}$$

▲

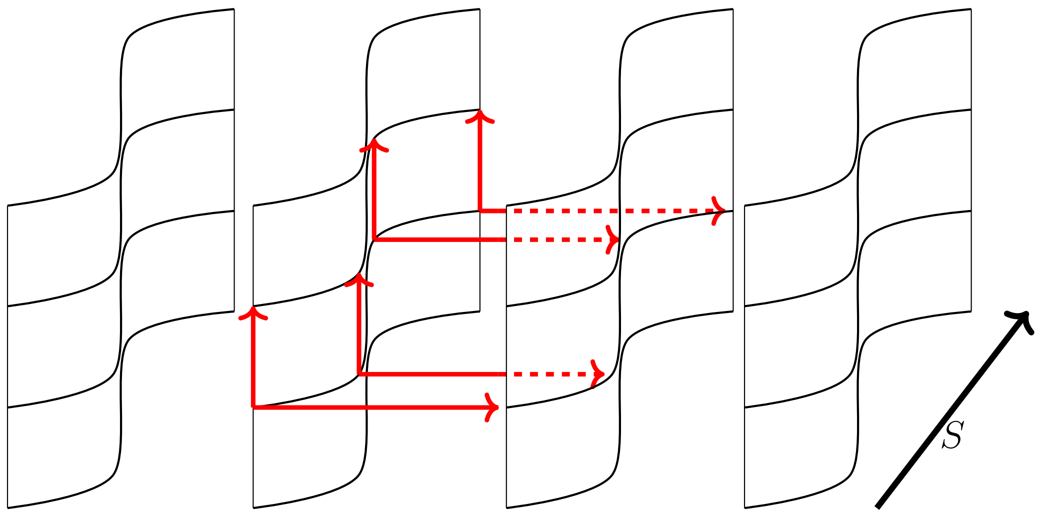

Figure 7: *An illustration of a family of 2-dimensional lattices for some space S. The space S runs in the as indicated with the arrow on the right. At every S-point one gets a lattice in the other two dimensions, as indicated by the red arrows.*

The underlying ordinary manifold of the torus $\mathbb{T}^{p|q}$ is simply the p-torus $\mathbb{T}^p$. In fact, from a (super)manifolds point of view, any choice of lattice gives up to diffeomorphism the same space. However, for now we will remember the lattice. This can be motivated by complex analysis, where in the same construction the Riemann surfaces can be non-holomorphic for different lattices.

As for lattices, we can consider families of (super)tori. Moreover, the manifold S parameterizing the family can be taken a supermanifold too.

Definition 5.4. For a (super)manifold S, the **S-family of $p|q$-supertori** $S \times_\Lambda \mathbb{R}^{p|q}$ associated to an S-family of (oriented) latices $\Lambda\colon S \times \mathbb{Z}^p \to \mathbb{R}^{p|q}$ is the space $\left(S \times \mathbb{R}^{p|q}\right)/\mathbb{Z}^p$ with the $\mathbb{Z}^p$-action

$$S \times \mathbb{R}^{p|q} \times \mathbb{Z}^p \ni \left(s, \vec{v}, \vec{\theta}, \vec{n}\right) \mapsto \left(s, \vec{v} + \Lambda(s, \vec{n}), \vec{\theta}\right) \in S \times \mathbb{R}^{p|q} \tag{5.1.6}$$

▲

These families of super tori are examples of $(\mathbb{R}^{p|q}, \mathbb{E}_R^{p|q} \rtimes G)$-supermanifolds for the model geometry $(\mathbb{R}^{p|q}, \mathbb{E}_R^{p|q} \rtimes G)$ we have seen before in Example 2.83. To recall, the group $\mathbb{E}_R^{p|q} \rtimes G$ acts on $\mathbb{R}^{p|q}$ by translations and the action of G on $\mathbb{R}^p$ on the even part directly and through the G-equivariant pairing R on the odd part. We can define an equivariant chart around every point $(s, x) \in |S \times \mathbb{R}^{p|q}|$ which lies inside the (closed) family of parallelepiped spanned by the generating vectors of the lattice Λ, i.e., a fundamental domain for the lattice action. This is done by taking a sufficiently small open neighborhood of (s, x) such that it contains no pairs of different points equivalent under the lattice action. These opens give equivariant charts of the torus $S \times_\Lambda \mathbb{R}^{p|q}$ by taking the inclusion into $S \times \mathbb{R}^{p|q}$. This construction of the charts is illustrated by Fig. 8. Notice that the hereby constructed charts also show that the projection $S \times \mathbb{R}^{p|q} \to S \times_\Lambda \mathbb{R}^{p|q}$ is a $\mathbb{Z}^p$-sheeted normal covering space or equivalently a $\mathbb{Z}^p$-principal bundle.

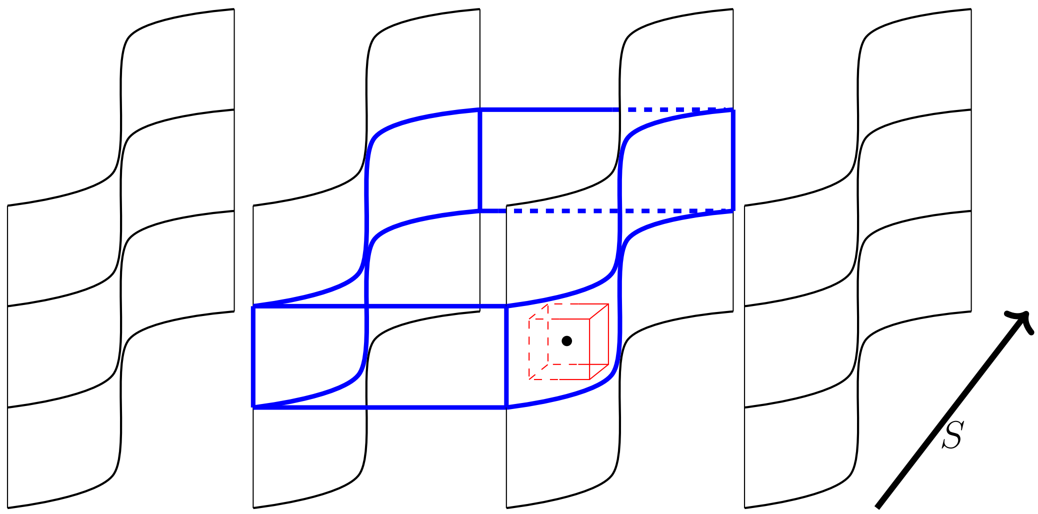

Figure 8: *Illustrated example of a fundamental domain of an S-family of lattices and the construction of charts for the associated S-family of tori. The fundamental domain is the area inside the blue family of parallelograms (including the boundary). A chart can be constructed by taking a point in the fundamental domain and taking a sufficiently small open (the red box) around it, such that there are no two different points inside the open equivalent under the lattice action. The torus can be seen as a gluing the top and bottom face and left and right face of the blue family of parallelograms.*

Remark 5.5. Up to this point, we have allowed all (orientable) lattices in the families. Obviously, we can be more strict in which lattices we allow. We can, e.g., assume that the individual lattices must be an element of some subset $L \subseteq \mathcal{L}_p$. The notion of tori for these restricted families of lattices follows immediately. One point to keep in mind when restricting the lattices is that the group $\mathrm{SL}_p(\mathbb{Z})$ might not act on the set L anymore. However, we can just restrict to any subgroup $\mathrm{SL}_p(\mathbb{Z})$. An example of these restriction would be the Siegel Lattices (Definition A.13) with the Siegel modular group (Definition A.9). $\triangledown$

Definition 5.6. We will say that a group K is a **full symmetry group** of a collection of lattices $L \subseteq \mathcal{L}_p$ if for $\Lambda, \Lambda' \in L$ and $A \in \mathrm{SL}_p(\mathbb{Z})$ with $\Lambda(A^{-1}\vec{n}) = \Lambda'(\vec{n})$ for all $\vec{n} \in \mathbb{Z}^p$, there holds $A \in K$. $\blacktriangle$

Example 5.7. The set of all lattice $\mathcal{L}_p$ has full symmetry group $\mathrm{SL}_p(\mathbb{Z})$. $\triangle$

5.2 A Stack of Tori

We have seen that the families of super tori from Definition 5.4 are families of $(\mathbb{R}^{p|q}, \mathbb{E}_R^{p|q} \rtimes G)$-supermanifolds. Therefore, we can consider the isometries between the families of tori for this model geometry. I.e., we look at the full subcategory tori in the stack of families of $(\mathbb{R}^{p|q}, \mathbb{E}_R^{p|q} \rtimes G)$-supermanifolds. The fact that the families of $(\mathbb{R}^{p|q}, \mathbb{E}_R^{p|q} \rtimes G)$-supermanifolds form a stack over SMfld is shown in Proposition 3.21. In fact, if we remember the family of lattices, the full subcategory of tori is a stack itself too. Considering Remark 5.5, we will take the lattices in some arbitrary but fixed subset of all (orientable) lattices.

Lemma 5.8. *Let $L \subseteq \mathcal{L}_p$ be a submanifold. The category $\mathcal{M}^{p|q}_{R,G,L}$ with objects, the (oriented) families of lattices $\Lambda\colon S\times\mathbb{Z}^p \to \mathbb{R}^p$ in L and morphisms, the $(\mathbb{R}^{p|q}, \mathbb{E}^{p|q}_R \rtimes G)$-isometries between the super tori $S\times_\Lambda \mathbb{R}^{p|q} \to S'\times_{\Lambda'} \mathbb{R}^{p|q}$ is a stack over SMfld under projection to the index space.*

Proof. The fact that all properties just related to the arrows of the categories hold is directly implied by the Proposition 3.21 and the fact that we are morally speaking about a full subcategory of the stack of families of supermanifolds for a certain model geometry. I.e., the Cartesian arrows property of groupoid fibrations and the locality and gluing properties of stacks hold. We are left to show the pullback axiom of groupoid fibrations and descent axiom of stacks.

For the pullback property of groupoid fibration notice that for an S-family of lattices $\Lambda\colon S\times\mathbb{Z}^p \to \mathbb{R}^p$ in L and a map $f\colon S' \to S$, we can canonically form an S'-family of lattices Λ' in L by precomposing the family Λ by f in the first coordinate. Moreover, the map $\overline{f}\colon S'\times_{\Lambda'}\mathbb{R}^{p|q} \to S\times_\Lambda \mathbb{R}^{p|q}$ by just acting with f in the first coordinate is a well-defined isometry over f. This shows the pullback property.

For the descent axiom of stacks, notice that the families of tori glue together, provided the given cocycle condition. This immediately gives us suitable isometries between the families. Using this data, in particular the fact that isometries of families are isometries in the fibers, we can retrieve the way we need to glue the underlying family of lattices to obtain the glued family of tori. This way, we construct a suitable family of lattices. This shows the descent axiom. ☐

5.2.1 Differentiability of the Tori Stack

Our goal is now to analyze the stack we just constructed. A natural question to ask is whether it is differentiable. We will answer this question in Corollary 5.27. Along the way, we will progressively put some constraints on the action of the group G on the Euclidean super space $\mathbb{R}^{p|q}$. We firstly will take a close inspection of the morphisms of $\mathcal{M}^{p|q}_{R,G,L}$ provided that the action of the group G is *free on opens*.

Definition 5.9. An action of a group G on a manifold M is **free on opens** if for any open $U\subseteq |M|$ the action map $G \mapsto \mathrm{Map}(M|_U \to M)$ is injective. ▲

Lemma 5.10. *Suppose that G acts on $\mathbb{R}^{p|q}$ freely on opens. Let Λ and Λ' be S and S' families of lattices respectively. A $\overline{\phi}\colon S\times_\Lambda\mathbb{R}^{p|q} \to S'\times_{\Lambda'}\mathbb{R}^{p|q}$ over a map $\psi\colon S\to S'$ lifts to a smooth map $\psi\colon S\times\mathbb{R}^{p|q} \to S'\times\mathbb{R}^{p|q}$ which is of the form*

$$S\times\mathbb{R}^{p|q} \ni \left(s, \left(\vec{v},\vec{\theta}\right)\right) \mapsto \left(\phi(s), g(s)\cdot\left(\vec{v},\vec{\theta}\right)\right) \in S'\times\mathbb{R}^{p|q} \tag{5.2.1}$$

for some smooth function $g\colon S\to \mathbb{E}^{p|q}_R \rtimes G$. Moreover, the map ψ is unique up to choosing a suitable family of base points $S \to \mathbb{R}^{p|q}$.

Proof. Locally, the isometry $\overline{\phi}$ is given by an action of $\mathbb{E}^{p|q}\rtimes G$ on $\mathbb{R}^{p|q}$. Moreover, the projection $S\times\mathbb{R}^{p|q} \to S\times_\Lambda\mathbb{R}^{p|q}$ is a covering space. By the standard lifting criterion, [Hat15, Proposition 1.33], we obtain a lift $\psi\colon S\times\mathbb{R}^{p|q}\to S'\times\mathbb{R}^{p|q}$ in the diagram

$$\begin{array}{ccc} S\times\mathbb{R}^{p|q} & \xrightarrow{\psi} & S'\times\mathbb{R}^{p|q} \\ \downarrow & & \downarrow \\ S\times_\Lambda\mathbb{R}^{p|q} & \xrightarrow{\overline{\phi}} & S'\times_{\Lambda'}\mathbb{R}^{p|q} \end{array} \tag{5.2.2}$$

Notice that the conditions of [Hat15, Proposition 1.33] are indeed met: Any loop in $S \times \mathbb{R}^{p|q}$ is homotopic to a loop in S (the space $\mathbb{R}^{p|q}$ is contractible[3]) and thus postcomposing with the map $\phi\colon S \to S'$ induced by $\overline{\phi}$ we obtain a loop in S'. Projecting down to $S' \times_{\Lambda'} \mathbb{R}^{p|q}$ with obtain the condition on fundamental groups. The conditions on path-connectedness and locally path-connectedness are trivial, since we can work in the connected components of the supermanifold S separately.

The fact that $\overline{\phi}$ is locally given by an action of $\mathbb{E}_R^{p|q} \rtimes G$ on $\mathbb{R}^{p|q}$ implies that ψ is too. By the assumption that $\mathbb{E}_R^{p|q} \rtimes G$ acts freely on opens, we deduce that ψ is globally of the form

$$S \times \mathbb{R}^{p|q} \ni \left(s, \left(\vec{v}, \vec{\theta}\right)\right) \mapsto \left(\phi(s), g(s) \cdot \left(\vec{v}, \vec{\theta}\right)\right) \in S' \times \mathbb{R}^{p|q} \tag{5.2.3}$$

for some smooth function $g\colon S \to \mathbb{E}_R^{p|q} \rtimes G$.

The uniqueness of ψ follows from the uniqueness property of lifts, [Hat15, Proposition 1.34]. □

The objects of $\mathcal{M}_{R,G,L}^{p|q}$ are certain families of lattice. Morphisms in $\mathcal{M}_{R,G,L}^{p|q}$ are given by group actions by $\mathbb{E}_R^{p|q} \rtimes G$. Therefore, we did like that this action also acts in some nice way on the lattices. The translations of $\mathbb{E}_R^{p|q}$ create no real issue. However, we will need G to act by linear transformations. I.e., we assume it acts via a representation. The condition of *freeness on opens* can be reformulated in terms of the representation.

Definition 5.11. A (super) representation $\rho\colon G \to \mathrm{GL}(V)$ is called **faithful** if ρ is injective (i.e., a monomorphism). ▲

Lemma 5.12. *A faithful representation $\rho\colon G \to GL(V)$ of a (super) Lie group G on a (super) vector space V gives rise to an action free on opens of $V \rtimes G$ on V. Here, both the semi-direct product and the action on V are taken via the representation ρ. The action of V on V is by translations.*

Proof. The action of doing first a translation $v \in V$ and followed by an invertible linear transformation $\rho(g)$ for $g \in G$ has precisely one neutral transformation. For all other transformations, the fixed points in V form a strict affine subspace. Since ρ is injective, the action of $V \rtimes G$ is free on opens. □

We come back to the stack of families of super tori. We wish to know whether it is differentiable. There is a natural candidate for a presentation, namely the submanifold L of the Stiefel-manifold of p-lattices $\mathcal{L}_p$. The data of a family of lattices $\Lambda\colon S \times \mathbb{Z}^p \to \mathbb{R}^{p|q}$ is the same as a map $S \to L$. Therefore, we obtain, on objects, an obvious functor $s_{\mathcal{M}_{R,G,L}^{p|q}}\colon \underline{L} \to \mathcal{M}_{R,G,L}^{p|q}$. On morphisms, a morphism $f\colon \Lambda \to \Lambda'$ in $\underline{L}$ with Λ and Λ' an S and S' family of lattices in L respectively (so under the hood f is a map $S \to S'$) gets mapped to the isometry $S \times_\Lambda \mathbb{R}^{p|q} \to S' \times_{\Lambda'} \mathbb{R}^{p|q}$ given by

$$S \times_\Lambda \mathbb{R}^{p|q} \ni [s, x] \mapsto [f(s), x] \in S' \times_{\Lambda'} \mathbb{R}^{p|q}. \tag{5.2.4}$$

A straightforward check shows that this is indeed a functor. Notice that this functor is bijective on objects by construction. Therefore, it gives a promising candidate to be an epimorphism of stacks.

[3]For this contractability, what we actually want is that $\mathbb{R}^{p|q}$ and $\mathbb{R}^{0|0}$ are homotopic as supermanifolds. This is easily seen to be true by using the convexity of $\mathbb{R}^{p|q}$.

Lemma 5.13. *Suppose that the group G acting via a faithful representation on $\mathbb{R}^{p|q}$. Then the functor $s_{\mathcal{M}^{p|q}_{R,G,L}} : \underline{L} \to \mathcal{M}^{p|q}_{R,G,L}$ defined in the text above is an epimorphism.*

Proof. For any supermanifold U with functor $F\colon \underline{U} \to \mathcal{M}^{p|q}_{R,G,L}$, we can find the relevant functor $H\colon \underline{U} \to \underline{L}$ on objects using the bijectivity of $s_{\mathcal{M}^{p|q}_{R,G,L}}$ on objects. On arrows, the functor is fixed by demanding it is a morphism of stacks. The assignment on arrows is well-defined, as can be shown by considering the inclusion of points: Let $f\colon S \to U$ and $f'\colon S' \to U$ be objects and $\phi\colon S \to S'$ an arrow in $\underline{U}$ and let $s \in S$. Now the lattice obtained by acting with F on $\{s\} \hookrightarrow S \xrightarrow{f} U$ is exactly the restriction of $s_{\mathcal{M}^{p|q}_{R,G,L}}(f)$ to the point s. Since, the map $\{s\} \hookrightarrow S \xrightarrow{f} U$ equals the map $\{s\} \xrightarrow{\psi} \{\psi(s)\} \hookrightarrow S' \xrightarrow{f'} U$, we deduce that the restriction of $s_{\mathcal{M}^{p|q}_{R,G,L}}(f)$ to the point s equals the restriction of $s_{\mathcal{M}^{p|q}_{R,G,L}}(f')$ to the point $\psi(s)$ for all $s \in S$. This shows that the functor H is well-defined.

We are left to check that the functors F and $s_{\mathcal{M}^{p|q}_{R,G,L}} \circ H$ are naturally isomorphic. Notice that on objects the two functors coincide.

Any object $f\colon S \to U$ in $\underline{U}$ can be seen as a morphism from f to Id_U:

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "f"'] & & U \arrow[dl, "\mathrm{Id}_U"] \\ & U & \end{tikzcd} . \tag{5.2.5}$$

Applying Lemma 5.10, we obtain that Ff can be lifted to a map $S \times \mathbb{R}^{p|q} \to U \times \mathbb{R}^{p|q}$ which is of the form

$$S \times \mathbb{R}^{p|q} \ni \left(s, \left(\vec{v}, \vec{\theta}\right)\right) \mapsto \left(f(s), g_f(s) \cdot \left(\vec{v}, \vec{\theta}\right)\right) \in U \times \mathbb{R}^{p|q} \tag{5.2.6}$$

for some smooth function $g_f\colon S \to \mathbb{E}^{p|q}_R \rtimes G$. As we have seen while showing the well-definedness of the functor H on arrows, the isometry Ff pointwise does not change the lattice. Therefore, the action of element $g_f(s) \in \mathbb{E}^{p|q}_R \rtimes G$ does not change the relevant lattice. Hence, using this action of $\mathbb{E}^{p|q}_R \rtimes G$ on $\mathbb{R}^{p|q}$, the map g_f defines a map $S \times \mathbb{R}^{p|q} \to S \times \mathbb{R}^{p|q}$ by sending

$$S \times \mathbb{R}^{p|q} \ni (s, x) \mapsto (s, g_f(s) \cdot x) \in S \times \mathbb{R}^{p|q}. \tag{5.2.7}$$

This map descents to a self-isometry η_f over the identity on the family of tori $S \times_{Ff} \mathbb{R}^{p|q}$. Here, we see f as an object of $\underline{U}$ so Ff is a family of lattices. We show that the assignment $f \mapsto \eta_f$ gives a natural isomorphism between F and $s_{\mathcal{M}^{p|q}_{R,G,L}} \circ H$.

Suppose that $f\colon S \to U$ and $f'\colon S' \to U$ are objects and $\phi\colon S \to S'$ a morphism from f to f' in $\underline{U}$. We can form the following diagram:

$$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "f"] \arrow[dd, "f"'] & & S' \arrow[dl, "f'"'] \arrow[dd, "f'"] \\ & U & \\ U \arrow[ur, "\mathrm{Id}_U"] \arrow[rr, "\mathrm{Id}_U"'] & & U \arrow[ul, "\mathrm{Id}_U"'] \end{tikzcd} . \tag{5.2.8}$$

Denote g_f and g_ϕ for the maps $S \to \mathbb{E}_R^{p|q} \rtimes G$ and $g_{f'}$ for the map $S' \to \mathbb{E}_R^{p|q} \rtimes G$ associated to the maps Ff, $F\phi$ and Ff' under Lemma 5.10. The map $S \to \mathbb{E}_R^{p|q} \rtimes G$ associated to the composite $\eta_{f'} \circ F\phi$ under Lemma 5.10 is the product $g_{f'}g_\phi$ in $\mathbb{E}_R^{p|q} \rtimes G$. Since $f' \circ \phi = f$, we deduce that we have obtained the map g_f.

On the other hand, the function $S \to \mathbb{E}_R^{p|q} \rtimes G$ associated to $(s_{\mathcal{M}^{p|q}_{R,G,L}} \circ H)(\phi)$ can be chosen to be constantly the identity in the $\mathbb{E}_R^{p|q} \rtimes G$. Therefore, the map g_f gives a lift of the composite $(s_{\mathcal{M}^{p|q}_{R,G,L}} \circ H)(\phi) \circ \eta_f$. This shows the claim. $\square$

The last lemma gives us an epimorphism. To see that it actually is an atlas, we need, by virtue of Corollary 3.31, to show that the groupoid $\underline{L} \times_{\mathcal{M}^{p|q}_{R,G,L}} \underline{L}$ is representable by some super Lie groupoid with base L. Moreover, we need that the source and target maps of the super Lie groupoid correspond to the projections to $\underline{L}$. For the dimensions $1|1$ and $2|1$ and a specific group G (a product of the Spin group and dilatations $\mathbb{R}_{>0}$) and pairing R and $L = \mathcal{L}_p$, this super Lie groupoid has been constructed and identified by Daniel Berwick Evans in [Ber13a, Lemma 2.5 and Lemma 3.3]. We follow the same intuition for the more general case.

Lemma 5.14. *Let $L \subseteq \mathcal{L}_p$ be a submanifold with a full symmetry group K. Assume that G acts on $\mathbb{R}^{p|q}$ via a faithful representation and L is closed under the G action. Then, the stack $\underline{L} \times_{\mathcal{M}^{p|q}_{R,G,L}} \underline{L}$ in the pullback diagram*

$$\begin{tikzcd} \underline{L} \times_{\mathcal{M}^{p|q}_{R,G,L}} \underline{L} \arrow[r] \arrow[d] & \underline{L} \arrow[d, "s_{\mathcal{M}^{p|q}_{R,G,L}}"] \\ \underline{L} \arrow[r, "s_{\mathcal{M}^{p|q}_{R,G,L}}"'] & \mathcal{M}^{p|q}_{R,G,L} \end{tikzcd} \tag{5.2.9}$$

is representable by the super Lie groupoid with base L and morphisms the quotient $\left(\mathbb{E}_R^{p|q} \rtimes G \times K \times L\right)/\mathbb{Z}^p$ by the $\mathbb{Z}$ action

$$\begin{aligned} \mathbb{E}_R^{p|q} \rtimes G \times K \times L \times \mathbb{Z}^p &\longrightarrow \mathbb{E}_R^{p|q} \rtimes G \times K \times L \\ (\vec{v}, \vec{\theta}, g, \mu, A, \Lambda, \vec{n}) &\longmapsto (\vec{v} + \Lambda(\vec{n}), \vec{\theta}, g, \mu, A, \Lambda). \end{aligned} \tag{5.2.10}$$

Here, the source map is the projection on L and the target map is induced by the action of $G \times K$) on L given by the left action of G on lattices via the representation on $\mathbb{R}^{p|q}$ and the right precomposition action of K on lattices. Explicitly the action is

$$(g, A) \cdot \Lambda(-) \mapsto g\Lambda(A^{-1}-). \tag{5.2.11}$$

The unit maps are given by the unit element of $\mathbb{E}_R^{p|q} \rtimes G \times K$.

Proof. The stack $\mathcal{M}^{p|q}_{R,G,L}$ has objects $\underline{L}$ by definition: The data of a family of lattices $\Lambda\colon S \times \mathbb{Z}^p \to \mathbb{R}^{p|q}$ in L is the same as a map $S \to L$. We are left to understand the morphisms.

A morphism over the identity in $\mathcal{M}^{p|q}_{R,G,L}$ is an isometry $\overline{\phi}\colon S \times_\Lambda \mathbb{R}^{p|q} \to S \times_{\Lambda'} \mathbb{R}^{p|q}$ over the identity on S. By Lemma 5.10, we can lift the map $\overline{\phi}$ to a map $\psi\colon S \times \mathbb{R}^{p|q} \to S \times \mathbb{R}^{p|q}$ of $\overline{\phi}$. Moreover, the map ψ is globally of the form

$$S \times \mathbb{R}^{p|q} \ni \left(s, \left(\vec{v}, \vec{\theta}\right)\right) \mapsto \left(s, g(s) \cdot \left(\vec{v}, \vec{\theta}\right)\right) \in S \times \mathbb{R}^{p|q} \tag{5.2.12}$$

for some smooth function $g\colon S \to \mathbb{E}^{p|q}_R \rtimes G$. Notice that this implies that ψ is a diffeomorphism. Notice that the isometry is completely determined by this function g. Since the group G acts by linear transformations, every such map determines an isometry to a relevant family of tori. Moreover, since $\mathbb{E}^{p|q} \rtimes G \times L \to \left(\mathbb{E}^{p|q} \rtimes G \times L\right)/\mathbb{Z}^p$ with the quotient as in the statement, is a covering space, there exist local sections. With these sections, we can lift a map $S \to \left(\mathbb{E}^{p|q} \rtimes G \times L\right)/\mathbb{Z}^p$ locally to $\mathbb{E}^{p|q} \rtimes G \times L$. From here, we can use the projection to obtain a map to $\mathbb{E}^{p|q} \rtimes G$. Since the resulting isometry on families of tori does not depend on the chosen section, we obtain an isometry from a map $S \to \left(\mathbb{E}^{p|q} \rtimes G\right)/\mathbb{Z}^p$.

A map $g\colon S \to \mathbb{E}^{p|q}_R \rtimes G$ gives rise to the identity isometry precisely when $g(s) \in \Lambda(s) \times \{e\}$ for all $s \in X$ with e the identity of G. Therefore, an isometry completely determines a map $S \to \left(\mathbb{E}^{p|q} \rtimes G \times L\right)/\mathbb{Z}^p$. The L component is determined by the lattice of the source object.

Notice that the existence of the map ψ in the diagram (5.2.2) implies that $\psi(\Lambda) \subseteq S \times \mathbb{R}^{p|q}$ must be contained in a potentially translated version of $\Lambda' \subseteq S' \times \mathbb{R}^{p|q}$. However, we exactly know the fiberwise translation. Namely, it must be the image of $\psi(s,0)$ or equivalently, the first, $\mathbb{E}^{p|q}$, component of $g(s)$ for all $s \in S$.

Since ϕ is a (fiberwise) diffeomorphism ψ maps the lattice Λ bijectively to the translated version of Λ'. Indeed, injectivity is implied by injectivity of ψ. For surjectivity, take $s \in S$ and $x \in \Lambda'(s) + \psi(s,0) \subseteq \mathbb{R}^{p|q}$. We have that $\psi(s, g(s)^{-1}x) = (s,x)$. Since $\overline{\phi}$ is an isometry, hence diffeomorphism, in the fibers, we see that $g(s)^{-1}x$ must be contained in $\Lambda(s)$.

We have obtained that $\psi(\Lambda)$ is a translated version of Λ' as sets. However, the diffeomorphism ψ may have permuted or changed the generators. I.e., there is for every $s \in S$ a unique element of $\mathrm{SL}_p(\mathbb{Z})$ restoring the generators. By the assumption made, we know that this element lies in K. Since K is a discrete space, we obtain a smooth map $A\colon S \to K$ assigning to every $s \in S$ the corresponding matrix restoring the generators.

Notice that the maps g and A together completely determine the lattice Λ' from the lattice Λ. Namely, we obtain Λ' from Λ by acting (fiberwise) with the second component of $g\colon S \to \mathbb{E}^{p|q} \rtimes G$ from the left and acting similarly by A from the right by precomposition.

In summary, we have obtained that the isometry $\overline{\phi}\colon S \times_\Lambda \mathbb{R}^{p|q} \to S \times_\Lambda \mathbb{R}^{p|q}$ is completely determined by a map $g\colon S \to \left(\mathbb{E}^{p|q}_R \rtimes G\right)/\mathbb{Z}^p$ and a map $A\colon S \to K$. Conversely, the maps g and A determine the isometry. Moreover, we can recover Λ' from g, A and Λ. Since the map A is unique and g is unique up to transformations in the lattice, we see that isometries of tori over the identity correspond exactly to the objects in the slice stack of the quotient $\left(\mathbb{E}^{p|q}_R \rtimes G \times K \times L\right)/\mathbb{Z}^p$. We have shown that $\underline{L} \times_{\mathcal{M}^{p|q}_{R,G,L}} \underline{L}$ is represented by a super Lie groupoid.

Notice that under the constructed equivalence, the source map of the groupoid $\underline{L} \times_{\mathcal{M}^{p|q}_{R,G,L}} \underline{L}$ corresponds to the projection $\left(\mathbb{E}^{p|q}_R \rtimes G \times K \times L\right)/\mathbb{Z}^p \to L$ and the target map corresponds to the map induced by the "action" of $G \times K$ on L. $\square$

Corollary 5.15. *The stack $\mathcal{M}^{p|q}_{R,G,L}$ is differentiable given that G acts on $\mathbb{R}^{p|q}$ via a faithful representation and L admits a full symmetry group.*

Proof. Apply Corollary 3.31 to the functor $s_{\mathcal{M}^{p|q}_{R,G,L}}$. The conditions of Corollary 3.31 are met due to Lemmas 5.13 and 5.14. □

Remark 5.16. The condition of the set of lattices needing to admit a full symmetry group arises from the fact we have allowed all isometries between all tori in the model geometry. Instead, one could work with any subgroup of $\mathrm{SL}_p(\mathbb{Z})$ acting on L. This will define a super Lie groupoid, and thus a stack via Proposition 3.40, in the same way as in the lemma above. However, one loses the canonical geometric interpretation in terms of the tori. ▽

5.3 A Stack of Fields

The Stack of Tori defined in the section above collects the data of a piece of the plain bordism category for the $(\mathbb{R}^{p|q}, \mathbb{E}^{p|q}_R \rtimes G)$ model geometry. Namely, we consider the piece of the category with just the tori as objects. What is not present in the discussion in the previous section is any kind of field maps to some background space X. In this section, we will add such field maps to obtain a *stack of fields*.

The following definition is a generalization of [Ber13a, Definitions 2.7 and 3.5] which contain the case of dimension $1|1$ and $2|1$ for a specific model geometry.

Definition 5.17. Let X be a standard manifold and $L \subseteq \mathcal{L}_p$ a submanifold. The **stack of fields** $\Phi^{p|q}_L(X)$ has objects over S the pairs (Λ, ψ) with Λ an S-family of lattices in L and $\psi\colon S \times_\Lambda \mathbb{R}^{p|q} \to X$ a map. We call the map ψ the **field map**. The morphisms of $\Phi^{p|q}_L(X)$ are commuting triangles

$$\begin{tikzcd} S \times_\Lambda \mathbb{R}^{p|q} \arrow[rr] \arrow[dr, "\psi"'] & & S' \times_{\Lambda'} \mathbb{R}^{p|q} \arrow[dl, "\psi'"] \\ & X & \end{tikzcd} . \tag{5.3.1}$$

Here, the horizontal arrow is a family of isometries in the $(\mathbb{R}^{p|q}, \mathbb{E}^{p|q}_R \rtimes G)$ model geometry, i.e., a morphism in the stack $\mathcal{M}^{p|q}_{R,G,L}$.

▲

Remark 5.18. In the notation of the stack of fields, we dropped the group G and pairing R. They obviously are still present in the background. From now onwards, we assume their existence implicitly. ▽

This definition needs a proof. We need to verify that the defined stack of fields is actually a stack.

Lemma 5.19. *The stack of fields $\Phi^{p|q}_L(X)$ from Definition 5.17 is a stack in the sense of Definition 3.14.*

Proof. We run the same proof as in Lemma 5.8, which inherits properties from Proposition 3.21. Only every time we construct an object, we need additionally to construct a map into X and when we construct an arrow, we need to verify this forms the triangle (5.3.1). We cycle through the properties:

For the pullback property of groupoid fibrations in Lemma 5.8, we construct the pullback lattice by precomposition and obtain a relevant family of isometries. The relevant map to X is given by precomposition by the family of isometries.

For the Cartesian arrow property, we only need to show that the constructed arrow makes the triangle (5.3.1) commute. Suppose that we have any two commuting triangle

$$\begin{tikzcd} S' \times_{\Lambda'} \mathbb{R}^{p|q} \arrow[r, "\beta"] \arrow[dr, "\psi'"'] & S \times_\Lambda \mathbb{R}^{p|q} \arrow[d, "\psi"] & S'' \times_{\Lambda''} \mathbb{R}^{p|q} \arrow[l, "\gamma"'] \arrow[dl, "\psi''"] \\ & X & \end{tikzcd} \tag{5.3.2}$$

and a map $\alpha\colon S'' \times_{\Lambda''} \mathbb{R}^{p|q} \to S' \times_{\Lambda'} \mathbb{R}^{p|q}$ such that $\beta \circ \alpha = \gamma$. Then, upon noticing that the outer boundary and two of the inner triangle of the following diagram commute, we can chase the diagram to see that the third inner triangle commutes too.

$$\begin{tikzcd} S'' \times_{\Lambda''} \mathbb{R}^{p|q} \arrow[rr, "\gamma"] \arrow[dr, "\psi''"'] \arrow[ddrr, bend right, "\alpha"'] & & S \times_\Lambda \mathbb{R}^{p|q} \arrow[dl, "\psi"] \\ & X & \\ & & S' \times_{\Lambda'} \mathbb{R}^{p|q} \arrow[ul, "\psi'"] \arrow[uu, "\beta"'] \end{tikzcd} \tag{5.3.3}$$

This shows the required property and hence that $\Phi_L^{p|q}(X)$ is a groupoid fibration.

We now focus on the properties of a stack, Definition 3.14. For locality, there is nothing to show. Maps over a space X glue to maps over a space X. This shows the gluing axiom. In Lemma 5.8, we have seen that families of lattices can be glued along cocycles. The same holds for maps to X, as long as the cocycles make the triangles like (5.3.1) commute. This completes the proof. □

For this stack to be differentiable, there is no hope, unless $p = 0$. For a single index space S and S-family of lattices Λ, we need to be able to retrieve (at least up to the notion of isomorphism in $\Phi_L^{p|q}(X)$) all field maps $S \times_\Lambda \mathbb{R}^{p|q} \to X$ from the atlas. If this is not the case, we cannot have an epimorphism. Generically, this becomes infinite dimensional. So since the presentation must be a functor from some slice category, the representing manifold must be infinite dimensional. Though, it might be worthwhile to generalize the notion of differentiability of stacks to the infinite dimensional case, we will stick to the finite dimensional case.

5.3.1 Vacuum Fields

We have seen that the full stack of fields is too big to be manageable for our tools. In order to have something manageable, we have to find a sensible way to restrict the field maps in consideration. The motivation stems from the relevant field theories in physics considered in Chapter 4.3. We there have found the equation of motion, (4.3.10), and the Hamiltonian density (4.3.20). From these together, we see that the classical (on shell) energy zero solutions are precisely the field maps $S \times_\Lambda \mathbb{R}^{p|q} \to X$ factoring through to $S \times \mathbb{R}^{0|q}$.

Another reason why this factorization is important, is that for a these field theories to admit Mayor-Vietoris sequences, we need some locality property. This means that we should be able to recover a field theory from the restrictions to an open cover $\{U, V\}$ of X. Assuming that the fields factor through $\mathbb{R}^{0|q}$ enforces them to be constant in the even coordinates. Therefore, every field must lie in at least one of the opens U and V. This certainly makes the theory sufficiently local. Notice that factoring $\mathbb{R}^{0|q}$ only makes the even coordinates constant. We can still have nontrivial behavior in the odd coordinates.

Hereby motivated, we generalize [Ber13a, Definitions 2.8 and 3.6] to higher dimensions and more general model geometries. In particular, it also includes [Ber15, Definition 3.2].

Definition 5.20. Let $L \subseteq \mathcal{L}_p$ be a submanifold. The **stack of classical π vacua** $\Phi^{p|q}_{L,0}(X)$ with values in a standard manifold X is the full substack of $\Phi^{p|q}_L(X)$ generated by the objects (Λ, ψ) over S, where $\psi\colon S \times_\Lambda \mathbb{R}^{p|q} \to X$ factors through the projection $\pi_{S,\Lambda}\colon S \times_\Lambda \mathbb{R}^{p|q} \to S \times \mathbb{R}^{0|\delta}$ over the identity on S induced by an epimorphism $\pi\colon \mathbb{R}^{p|q} \to \mathbb{R}^{0|\delta}$, the **vacuum projection**. We call the induced map $\psi_0\colon S \times_\Lambda \times \mathbb{R}^{0|\delta} \to X$ the **vacuum field map**. ▲

Again, this definition needs the verification that $\Phi^{p|q}_{L,0}(X)$ is indeed a stack.

Lemma 5.21. *The stack of classical vacua $\Phi^{p|q}_{L,0}(X)$ from Definition 5.20 is a stack in the sense of Definition 3.14.*

Proof. As $\Phi^{p|q}_{L,0}(X)$ is a full substack of the stack of fields $\Phi^{p|q}_L(X)$, all properties of groupoid fibrations and stacks just relating to arrows are automatically fulfilled. We only need to check that $\Phi^{p|q}_{L,0}(X)$ is closed under pullbacks and descents. The pullback on an object (Λ, ψ) over S of $\Phi^{p|q}_L(X)$ under some map $\phi\colon S' \to S$ is of the form $(\Lambda', \psi \circ \overline{\phi})$ with $\overline{\phi}$ acting trivially in the $\mathbb{R}^{p|q}$ component. Therefore, if ψ factors through $S \times_\Lambda \mathbb{R}^{p|q} \to S \times \mathbb{R}^{0|\delta}$, so does $\psi \circ \overline{\phi}$ (mutatis mutantis the index space). This shows that $\Phi^{p|q}_{L,0}(X)$ is a groupoid fibration.

The descent axiom holds, since if maps individually factor through the relevant projections, then their gluing factors through the gluing of the projections. Projections $U_i \times_{\Lambda_i} \mathbb{R}^{p|q} \to U_i \times \mathbb{R}^{0|\delta}$ obviously glue to the projection $S \times_\Lambda \mathbb{R}^{p|q} \to S \times \mathbb{R}^{0|\delta}$ if $\bigcup_i U_i = S$ and the families of lattices Λ_i glue to Λ. This completes the proof. □

Remark 5.22. If $q = 1$, then there is a canonical choice of vacuum projection $\pi\colon \mathbb{R}^{p|1} \to \mathbb{R}^{0|\delta}$. Namely, the canonical projection. For higher dimensions, we can obviously project to any of the factors or mix them. Also, we could factor through chosen maps $\pi_{S,\Lambda}\colon S \times_\Lambda \mathbb{R}^{p|q} \to S \times \mathbb{R}^{0|\delta}$ which potentially differ for different lattices. Do notice that in this case, the maps need to satisfy certain gluing relations in order for the classical vacua to form a stack. An example of such a field projection is worked out in [Ber15, Section 3.2]. ▽

The properties of the vacuum projection have significant influence on the stack of vacua. For us, it will be important when we can obtain the vacuum field map of the target from the vacuum field map of the source for some isometry. The condition on the vacuum projection required should morally say that for any coordinates $x, y \in \mathbb{R}^{p|q}$ and $g \in G$ there holds that $\pi(x) = \pi(y)$ implies $\pi(gx) = \pi(gy)$. However, we work in supermanifolds, so we don't have points explicitly. Hence, we need some more fancy condition on families making use of the functor of points viewpoint.

Definition 5.23. The vacuum projection is **G-covariant** if for any map $g\colon S \to \mathbb{E}^{p|q}_R \rtimes G$, there is a unique diffeomorphism $S \times \mathbb{R}^{0|\delta} \to S \times \mathbb{R}^{0|\delta}$ such that the following diagram commutes

$$\begin{tikzcd} S \times \mathbb{R}^{p|q} \arrow[r, "(s,x)\mapsto(s,g(s)x)"] \arrow[d, "\mathrm{Id}_S\times\pi"] & S \times \mathbb{R}^{p|q} \arrow[d, "\mathrm{Id}_S\times\pi"] \\ S \times \mathbb{R}^{0|\delta} \arrow[r] & S \times \mathbb{R}^{0|\delta} \end{tikzcd} . \tag{5.3.4}$$

▲

5.3.2 Differentiability of the Vacuum Stack

We can now consider the differentiability of the stack $\Phi^{p|q}_{L,0}(X)$. Since, we assume that the field map $S \times_\Lambda \mathbb{R}^{p|q} \to X$ must factor through $S \times \mathbb{R}^{0|\delta}$ by a chosen map, we are essentially just looking at maps $S \times \mathbb{R}^{0|\delta} \to X$. Recall from Proposition 2.61 that the generalized supermanifold $S \mapsto \mathrm{SMfld}(S \times \mathbb{R}^{0|\delta}, X)$ is represented by the odd tangent bundle $(\Pi T)^\delta X$. Therefore, compared to the case of just the stack of tori of the previous section, the stack of classical vacua is obtained by taking a product with the odd tangent bundle.

Lemma 5.24. *Let $L \subseteq \mathcal{L}_p$ be a submanifold with full symmetry group K. Consider the quotient $\left(\mathbb{E}^{p|q}_R \rtimes G \times L\right)/\mathbb{Z}^p$ with the $\mathbb{Z}^p$ action given by*

$$\begin{aligned} \mathbb{E}^{p|q}_R \rtimes G \times L \times \mathbb{Z}^p &\longrightarrow \mathbb{E}^{p|q}_R \rtimes G \times L \\ (\vec{v}, \vec{\theta}, g, \mu, \Lambda, \vec{n}) &\longmapsto (\vec{v} + \Lambda(\vec{n}), \vec{\theta}, g, \mu, \Lambda). \end{aligned} \tag{5.3.5}$$

If the vacuum projection $\pi\colon \mathbb{R}^{p|q} \to \mathbb{R}^{0|\delta}$ is G-covariant, then any pair of maps $g\colon S \to \left(\mathbb{E}^{p|q}_R \rtimes G \times L\right)/\mathbb{Z}^p$ and $\psi_0\colon S \times \mathbb{R}^{0|\delta} \to X$ induces the diagram

$$\begin{tikzcd} S \times \mathbb{R}^{p|q} \arrow[rr] \arrow[d] & & S \times \mathbb{R}^{p|q} \arrow[d] \\ S \times_\Lambda \mathbb{R}^{p|q} \arrow[rr, "\phi"] \arrow[d, "\pi_{S,\Lambda}"'] & & S \times_{\Lambda'} \mathbb{R}^{p|q} \arrow[d, "\pi_{S,\Lambda'}"] \\ S \times \mathbb{R}^{0|\delta} \arrow[dr, "\psi_0"'] & & S \times \mathbb{R}^{0|\delta} \arrow[dl, dashed, "\psi_0'"] \\ & X & \end{tikzcd} . \tag{5.3.6}$$

Here, the family of lattices Λ is obtained from g by projecting to the L component, i.e., the source map of Lemma 5.14. The family of lattices Λ' is obtained by acting with G on Λ, i.e., the target map of Lemma 5.14. The second horizontal arrow is the isometry of families of tori associated to g induced by the construction of Lemma 5.14. In particular, we obtain a vacuum field map ψ_0' from g and ψ_0. This induces a surjective submersion $\left(\mathbb{E}^{p|q}_R \rtimes G \times L\right)/\mathbb{Z}^p \times (\Pi T)^\delta X \to (\Pi T)^\delta X$.

Proof. The upper square of diagram (5.3.6) is induced by Lemma 5.14. The precise factor of K is unimportant for the constructed maps. Explicitly, we could simply take the identity element. Notice that the choice of upper horizontal map is not unique. However, since the composition of the vertical arrows of diagram (5.3.6) on either side gives $\mathrm{Id}_S \times \pi$, applying Definition 5.23 to any choice gives the same diffeomorphism $S \times \mathbb{R}^{0|\delta} \to S \times \mathbb{R}^{0|\delta}$. The map ψ_0' is now given by a composition of the inverse of this diffeomorphism and ψ_0.

From Lemma 2.60, we know that the generalized supermanifold $S \mapsto \mathrm{SMfld}(S \times \mathbb{R}^{0|\delta}, X)$ is represented by $(\Pi T)^\delta X$. Therefore, the assignment $(g, \psi_0) \mapsto \psi_0'$ given in the argument above induces a smooth map $\left(\mathbb{E}_R^{p|q} \rtimes G \times L\right)/\mathbb{Z}^p \times (\Pi T)^\delta X \to (\Pi T)^\delta X$. One can see it is a surjective submersion by considering the maps $g\colon S \to \left(\mathbb{E}_R^{p|q} \rtimes G \times L\right)/\mathbb{Z}^p$ which are constantly the identity element in the $\mathbb{E}_R^{p|q} \rtimes G$ coordinate. □

We construct an atlas $s_{\Phi_{L,0}^{p|q}(X)}\colon \underline{L \times (\Pi T)^\delta X} \to \Phi_{L,0}^{p|q}(X)$ for the stack $\Phi_{L,0}^{p|q}(X)$ of vacua. We perform essentially the same construction as spelled out directly above Lemma 5.13. The functor on objects is given by sending a map $S \times L \times (\Pi T)^\delta X$ to the pair (Λ, ψ), where the family of lattices Λ is defined by the projection to the L coordinate and the map ψ by the projection to the $(\Pi T)^\delta X$. The construction of the lattice Λ is exactly the same as for the stack of tori. The field map ψ is given by composing the map $S \times_\Lambda \mathbb{R}^{p|q} \to S \times \mathbb{R}^{0|\delta}$ associated to π intrinsic to the stack of vacua, with the map $S \times \mathbb{R}^{0|\delta} \to X$ associated with the map $S \to (\Pi T)^\delta X$ via Lemma 2.60. On morphisms nothing changes compared to the tori case, we send a map $f\colon S \to S'$ to the isometry

$$S \times_\Lambda \mathbb{R}^{p|q} \ni [s, x] \mapsto [f(s), x] \in S' \times_{\Lambda'} \mathbb{R}^{p|q}. \tag{5.3.7}$$

To show this is well-defined, we need to shows that the triangle

$$\begin{tikzcd} S \times_\Lambda \mathbb{R}^{p|q} \arrow[rr] \arrow[dr] & & S' \times_{\Lambda'} \mathbb{R}^{p|q} \arrow[dl] \\ & X & \end{tikzcd} \tag{5.3.8}$$

commute. Since the diagonal arrows factor through $S \times \mathbb{R}^{0|\delta}$ respectively $S' \times \mathbb{R}^{0|\delta}$, we can equivalently look at the diagram

$$\begin{tikzcd} S \times_\Lambda \mathbb{R}^{p|q} \arrow[rr] \arrow[d] & & S' \times_{\Lambda'} \mathbb{R}^{p|q} \arrow[d] \\ S \times \mathbb{R}^{0|\delta} \arrow[rr] \arrow[dr] & & S' \times \mathbb{R}^{0|\delta} \arrow[dl] \\ & X & \end{tikzcd} . \tag{5.3.9}$$

Here, the second horizontal arrow acts like f in the first coordinate and trivially in the $\mathbb{R}^{0|\delta}$ coordinate. Since, the triangle

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr] & & S' \arrow[dl] \\ & X & \end{tikzcd} \tag{5.3.10}$$

commute by definition, the diagram (5.3.9) commutes.

It is easy to see that $s_{\Phi_{L,0}^{p|q}(X)}$ is a functor.

Lemma 5.25. *Suppose that the group G acts via a faithful representation on $\mathbb{R}^{p|q}$. Then the functor $s_{\Phi^{p|q}_{L,0}(X)}\colon \underline{L\times(\Pi T)^\delta X}\to\Phi^{p|q}_{L,0}(X)$ defined in the text above is an epimorphism of stacks.*

Proof. The proof is the same as the proof of Lemma 5.13. We only need to carry the extra weight of the field maps to X around. □

With the epimorphism $s_{\Phi^{p|q}_{L,0}(X)}$, to show that $\Phi^{p|q}_{L,0}(X)$ is differentiable, it suffices to show that the groupoid $\underline{L\times(\Pi T)^\delta X}\times_{\Phi^{p|q}_{L,0}(X)}\underline{L\times(\Pi T)^\delta X}$ is represented by some super Lie groupoid with base $L\times(\Pi T)^\delta X$ with the source and target maps corresponding to the projections (Corollary 3.31). The cases for dimensions $1|1$ and $2|1$ for a specific group G, pairing R and $L=\mathcal{L}_p$ are shown in [Ber13a, Proposition 2.10 and Proposition 3.8].

Proposition 5.26. *Let $L\subseteq\mathcal{L}_p$ be a submanifold with full symmetry group K. Assume that G acts on $\mathbb{R}^{p|q}$ via a faithful representation such that L is closed under the G-action and that the vacuum projection is G-covariant. The stack $\underline{L\times(\Pi T)^\delta X}\times_{\Phi^{p|q}_{L,0}(X)}\underline{L\times(\Pi T)^\delta X}$ in the pullback diagram*

$$\begin{tikzcd}
\underline{L\times(\Pi T)^\delta X}\times_{\Phi^{p|q}_{L,0}(X)}\underline{L\times(\Pi T)^\delta X} \arrow[r] \arrow[d] & \underline{L\times(\Pi T)^\delta X} \arrow[d, "s_{\Phi^{p|q}_{L,0}(X)}"] \\
\underline{L\times(\Pi T)^\delta X} \arrow[r, "s_{\Phi^{p|q}_{L,0}(X)}"'] & \Phi^{p|q}_{L,0}(X)
\end{tikzcd} \tag{5.3.11}$$

is represented by the super Lie groupoid with as base the space $L\times(\Pi T)^\delta X$ and as morphisms the space $\left(\mathbb{E}^{p|q}\rtimes G\times K\times L\right)/\mathbb{Z}^p\times(\Pi T)^\delta X$. Here, the $\mathbb{Z}^p$ quotient is taken under the action

$$\begin{aligned}
\mathbb{E}^{p|q}_R\rtimes G\times K\times L\times\mathbb{Z}^p &\longrightarrow \mathbb{E}^{p|q}_R\rtimes G\times K\times L\\
(\vec v,\vec\theta,g,\mu,A,\Lambda,\vec n) &\longmapsto (\vec v+\Lambda(\vec n),\vec\theta,g,\mu,A,\Lambda).
\end{aligned} \tag{5.3.12}$$

The source map is the projection to $L\times(\Pi T)^\delta X$. The target map is obtained by looking at the components L and $(\Pi T)^\delta X$ separately. On the L, the target map is given by the "action" of $G\times K$ on L as in Lemma 5.14. The target map is the submersion obtained from Lemma 5.24. The unit maps are induced by the unit element of $\mathbb{E}^{p|q}\rtimes G\times K$.

Proof. The objects of $\Phi^{p|q}_{L,0}(X)$ are pairs (Λ,ψ). We know that $\Lambda\in\underline{L}$. Since the map π is assumed to be an epimorphism, we obtain a unique map $\widetilde\psi\colon S\times\mathbb{R}^{0|\delta}\to X$ such that $\psi=\pi_{S,\Lambda}\circ\widetilde\psi$. This verifies the claim on the base. The (iso)morphisms in $\Phi^{p|q}_{L,0}(X)$ over the identity are isometries of families of tori. These, we computed to be the quotient $\left(\mathbb{E}^{p|q}\rtimes G\times K\times L\right)/\mathbb{Z}^p$ under the mentioned action in Lemma 5.14. Notice that from the map of fields $\psi\colon S\times_\Lambda\mathbb{R}^{p|q}\to X$ of the source and the isometry, the map of fields in the target is fixed. This is due to the fact that the following triangle commutes for morphisms in $\Phi^{p|q}_{L,0}(X)$:

$$\begin{tikzcd}
S\times_\Lambda\mathbb{R}^{p|q} \arrow[rr] \arrow[dr] & & S\times_{\Lambda'}\mathbb{R}^{p|q} \arrow[dl] \\
& X &
\end{tikzcd}. \tag{5.3.13}$$

This verifies the claim on the morphism. The claim on unit elements is trivial. The source map can easily be identified with the projection $\underline{L\times(\Pi T)^\delta X}\times_{\Phi^{p|q}_{L,0}(X)}\underline{L\times(\Pi T)^\delta X}\to\underline{L\times(\Pi T)^\delta X}$ on the first coordinate. The target map is identified with the submersion obtained from Lemma 5.24 by construction. The extra factor of K in the domain can be projected out. $\square$

Corollary 5.27. *The stack $\Phi^{p|q}_{L,0}(X)$ is differentiable given that G acts on $\mathbb{R}^{p|q}$ via faithful representations for which L is closed and L admits a full symmetry group.*

Proof. Apply Corollary 3.31 to the functor $s_{\Phi^{p|q}_{L,0}(X)}$. The conditions of Corollary 3.31 are met due to Lemma 5.25 and Proposition 5.26. $\square$

5.4 Cohomologies from Tori Stacks

In Chapter 4.2.5, we have used the dictionary lemmas of stacks, Lemmas 3.46 to 3.48, to link (twisted) functorial field theories to sections of vector bundles over the relevant stack. In turn these section were shown to give differential forms. Recall that a G-representation $G \to \mathrm{GL}(V)$ gives rise to a vector bundle of the quotient stack $[\mathbb{R}^{0|0}/G]$. Recall that in the data of the Euclidean model geometry, Example 2.83, we have assumed a G-representation of $\mathbb{R}^q$. Alternatively, we assumed in the previous section that G acts by linear transformations on $\mathbb{R}^{p|q}$. Under this assumption, we obtain a G-representation on $\mathbb{R}^{p|q}$. This gives us two ways to construct (super) vector bundles on the quotient stack $[\mathbb{R}^{0|0}/G]$.

5.4.1 Vector Bundles over Tori Stacks

We define a suitable vector bundle of the stack of vacuum fields, extending the cases considered in [Ber13a, Definitions 2.6 and 3.4].

Definition 5.28. Consider the groupoid presentation $\left(\mathbb{E}^{p|q}_R \rtimes G \times K) \times L\right)/\mathbb{Z}^p$ of the stack of tori $\mathcal{M}^{p|q}_{R,G,L}$ from Lemma 5.14. We have a projection

$$\left(\mathbb{E}^{p|q}_R \rtimes G \times K) \times L\right)/\mathbb{Z}^p \to G. \tag{5.4.1}$$

This defines a map of (groupoid presentation of) stacks $\mathcal{M}^{p|q}_{R,G,L} \to [\mathbb{R}^{0|0}/G]$. Pulling the vector bundle given by the representation $\rho\colon G \to \mathrm{GL}(\mathbb{R}^q)$, intrinsic to the Euclidean model geometry, along this map of stacks gives us a vector bundle $\omega^{\frac{1}{2}}$ over $\mathcal{M}^{p|q}_{R,G,L}$.

There holds $\mathcal{M}^{p|q}_{R,G,L} \cong \Phi^{p|q}_L(\mathbb{R}^{0|0}) \cong \Phi^{p|q}_{L,0}(\mathbb{R}^{0|0})$. Hence, we also have vector bundles $\omega^{\frac{1}{2}}$ over these stacks. The unique map $X \to \mathbb{R}^{0|0}$ induces a functor $\Phi^{p|q}_{L,0}(X) \to \Phi^{p|q}_{L,0}(\mathbb{R}^{0|0})$. Pulling back further along this functor, we obtain a vector bundle over $\Phi^{p|q}_{L,0}(X)$.

We will denote $\omega^{\bullet/2}$ vector bundle obtained with using the representation $\rho^\bullet$ in the construction instead. ▲

We now compute the sections of this vector bundle. We adopt the method of [Ber13a, Proposition 3.9].

Theorem 5.29. *The sections $\Gamma(\Phi^{p|q}_{L,0}(X), \omega^{\bullet/2})$ are in canonical bijection with sums of functions $F \otimes \alpha \in C^\infty(L, Mat(q \times q)) \otimes C^\infty\left((\Pi T)^\delta X\right)$ such that F is invariant under the K action, α is closed and the sum Q transforms like $g \cdot Q = \rho(g)^k Q$ for all $g \in G$.*

Proof. From Propositions 5.26 and 3.53, we know that the sections $\Gamma(\Phi^{p|q}_{L,0}(X), \omega^{\bullet/2})$ can be identified with matrix valued functions on $L \times (\Pi T)^\delta X$ satisfying equivariant properties. I.e.,

$$\Gamma(\Phi^{p|q}_{L,0}(X), \omega^{\frac{k}{2}}) = \left\{ f \in C^\infty\left(L \times (\Pi T)^\delta X, \mathrm{Mat}(q \times q)\right) | \mu^*(f) = p_1^*(f) p_G^*(\rho^k) \right\}. \tag{5.4.2}$$

Here, we have written μ for the action of $\mathbb{E}^{p|q}_R \rtimes G \times K$ on $L \times (\Pi T)^\delta X$ and we used the projection

$$p_1 \colon \mathbb{E}^{p|q}_R \rtimes G \times K \times L \times (\Pi T)^\delta X \to L \times (\Pi T)^\delta X \qquad \text{and} \tag{5.4.3}$$

$$p_G \colon \mathbb{E}^{p|q}_R \rtimes G \times K \times L \times (\Pi T)^\delta X \to G. \tag{5.4.4}$$

Consider the homogeneous elements of $\Gamma(\Phi^{p|q}_{L,0}(X), \omega^{\frac{k}{2}})$. I.e., the elements that can be written as

$$F \otimes \alpha \in C^\infty(L, \mathrm{Mat}(q \times q)) \otimes C^\infty\left((\Pi T)^\delta X\right) \subseteq C^\infty\left(L \times (\Pi T)^\delta X, \mathrm{Mat}(q \times q)\right). \tag{5.4.5}$$

The equivariance condition in Eq. (5.4.2) in particular says that F is invariant under the action of K and α is invariant under the action of $\mathbb{E}^{p|q}_R$, i.e., it is closed due to Lemma 2.71. The leftover equivariance under the G action, precisely gives the requested transformation rule in the statement. This shows the claim. □

This theorem gives us a powerful tool to transition from field theories to closed pseudo-differential forms with values in the matrix valued function on lattices, satisfying a compatibility condition. In Appendix A, we have identified functions on lattices certain, at least in even dimensions, with functions of Siegel Modular weights.

Corollary 5.30. *Let $L = \mathcal{L}^S_{2g}$ be the set of Siegel Lattices. Let $K = Sp_{2g}(\mathbb{Z})$ be the Siegel modular group and $G = GL_g(\mathbb{C})$ acting naturally on $\mathbb{R}^{2g} \cong \mathbb{C}^g$ and through the representation $\rho \colon G \to GL_q$ on $\mathbb{R}^q$ given by multiplication with the real determinant $\det(g)$ for all $g \in G \subseteq GL_{2g}(\mathbb{R})$. Take as vacuum projection the canonical projection to $\mathbb{R}^{0|q}$. Then, taking Remark 5.16 into account, there is an isomorphism*

$$\bigoplus_{i+j=k} \mathcal{O}^j_{cl}((\Pi T)^q X) \otimes SMFu^i \to \Gamma(\Phi^{p|q}_{L,0}(X), \omega^{\frac{k}{2}}), \qquad \alpha \otimes vol^{2j} F \mapsto \alpha \otimes H. \tag{5.4.6}$$

Here, we write $vol \in C^\infty(\mathcal{L}_p)$ for the volume of the torus defined by the lattice and we have identified elements of $\Gamma(\Phi^{p|q}_{L,0}(X), \omega^{\frac{k}{2}})$ with functions on $L \times (\Pi T)^q X$. Recall that $\mathcal{O}^j_{cl}((\Pi T)^q X)$ are the closed pseudo differential forms on X of polynomial degree j.

Proof. Similar to the proof of the theorem, we consider the homogeneous elements $F \otimes \alpha \colon C^\infty(L) \otimes \mathcal{O}^j_{cl}((\Pi T)^q X)$ which define an element in $\Gamma(\Phi^{p|q}_{L,0}(X), \omega^{\frac{k}{2}})$. Using the theorem, we only need to check that the equivariance under the G action is equivalent with the modularity property of Eq. (A.1.34). To see this, notice that acting with an element $g \in G$ on α transforms as $g \cdot \alpha = \rho(g)^{-j}\alpha$. The equivariance requirement of the G-action asserts the transformation $g \cdot (\mathrm{vol}^{2j} F \otimes \alpha) = \rho(g)^k \mathrm{vol}^{2j} F \otimes \alpha$. We can assume this transformation on the homogeneous elements, since the action on the pseudo-differential form is by scalar multiplication. Since $g \cdot \mathrm{vol}^{2j} = \rho(g)^{2j}\mathrm{vol}^{2j}$, we must have that F transforms as $g \cdot F = \rho(g)^{k-j} F$. Using Proposition A.14, this precisely means that F is a function of Siegel modular weight ρ^{-i}, which implies the claim. □

It can be of interest when the functions of some Siegel modular weight become Siegel modular forms. I.e., the function on the lattices becomes holomorphic. The corresponding sections $\mathcal{H}(\Phi^{p|q}_{L,0}(X), \omega^{\frac{k}{2}}) \subseteq \Gamma(\Phi^{p|q}_{L,0}(X), \omega^{\frac{k}{2}})$ are called the **holomorphic sections**.

5.4.2 Cohomology Theories from Field Theories

Specifying Theorem 5.29 to some specific settings allows us to identify some complexified cohomology theories. The language might seem a bit overloaded since complexified cohomology theories are just ordinary cohomology theories with suitable coefficients, Proposition B.7. However, we have some benefits. E.g., it is possible to replace the space X with a quotient stack $[X/H]$ to obtain a version of equivariant cohomology, see [Ber20]. Also, as we will see in the next section, we can use some operators from the physics in Chapter 4.3 to construct the $\hat{A}$ and Witten genera analytically.

Throughout the rest of this chapter, we assume that the function sheaves of supermanifold take complex values. We consider the case of 1|1-dimensional field theories and construct complexified K-theory. We take the super Lie group $\mathbb{E}^{1|1}$ with group action

$$(t,\theta)\cdot(t',\theta') = (t+t'+i\theta\theta', \theta+\theta'). \tag{5.4.7}$$

The model geometry we will use, is the Euclidean structure $(\mathbb{R}^{1|1}, \mathbb{E}^{1|1} \rtimes \mathbb{R}^\times)$ with $\mathbb{R}^\times$ acting on $\mathbb{R}^{1|1}$ in coordinates by

$$\mu\cdot(t,\theta) = (\mu^2 t, \mu\theta). \tag{5.4.8}$$

We will denote $\omega^{\frac{k}{2}}$ for the k fold tensor product of the line bundle over $\Phi^{1|1}_{\mathcal{L}_1,0}$ induced by the map $\Phi^{1|1}_{\mathcal{L}_1,0}(X) \to \mathbb{R}^\times \to \{\pm 1\}$ using the construction of Definition 5.28.

Proposition 5.31 ([Ber13a, Proposition 2.1])**.** *We have a natural isomorphism of sheaves of graded algebras over $\mathbb{C}$,*

$$\begin{cases}\omega^{ev}_{cl}(-) & \text{if } \bullet = even\\ \omega^{odd}_{cl}(-) & \text{if } \bullet = odd\end{cases} \to \Gamma\left(\Phi^{1|1}_{\mathcal{L}_1,0}(-), \omega^{\frac{\bullet}{2}}\right), \quad \Omega^k_{cl}(X) \ni \alpha \mapsto (2\pi r)^{\frac{k}{2}} \otimes \alpha \in \Gamma\left(\Phi^{1|1}_{\mathcal{L}_1,0}(X), \omega^{\frac{\bullet}{2}}\right). \tag{5.4.9}$$

Proof. Theorem 5.29 tells us that elements of $\Gamma\left(\Phi^{1|1}_{\mathcal{L}_1,0}(-), \omega^{\frac{\bullet}{2}}\right)$ can be identified with sums of elements $F\otimes\alpha \in C^\infty(\mathcal{L}_p)\otimes\Omega_{\mathrm{cl}^k} = C^\infty(\mathbb{R}_{>0})\otimes\Omega_{\mathrm{cl}^k}$ which are equivariant under the $\mathbb{R}^\times$ action. The $\mathbb{R}^\times$ equivariance asserts that $F\otimes\alpha$ is transformed by flipping the sign according to the power of k. We can consider the $\mathbb{R}^\times$ action on the homogenous elements since we act by scalar multiplications on the differential forms. Since, we know the transformation of α to be $\mu\times\alpha = \mu^{-k}\alpha$ for $\mu\in\mathbb{R}^\times$, we must require that $F = (2\pi r)^{\frac{k}{2}}$ (up to a constant, which can be absorbed in α). Notice that we need the square root to counter the square in the action on the even part of $\mathbb{R}^{1|1}$.

Since, the stack $\Phi^{1|1}_{\mathcal{L}_p,0}(X)$ is natural in X, we know that $\Gamma\left(\Phi^{1|1}_{\mathcal{L}_1,0}(-), \omega^{\frac{\bullet}{2}}\right)$ is a presheaf. The argument above shows that this presheaf is isomorphic to a sheaf. Hence, the constructed isomorphism is one of sheaves. □

This realizes the sections $\Gamma\left(\Phi^{1|1}_{\mathcal{L}_1,0}(-), \omega^{\frac{\bullet}{2}}\right)$ as a cocyclic model for complexified K-theory. To see this, use Proposition B.7 and Example B.8.

We crank up the dimension to 2|1 to obtain tmf. We have the super Lie group $\mathbb{E}^{2|1}$ with the group action

$$(z,\overline{z},\theta)\cdot(z',\overline{z}',\theta') = (z+z', \overline{z}+\overline{z}'+\theta\theta', \theta+\theta'). \tag{5.4.10}$$

The model geometry, we consider will be the Euclidean $(\mathbb{R}^{2|1}, \mathbb{E}^{2|1} \rtimes \mathbb{C}^\times)$ model geometry where the action of $\mathbb{C}^\times$ on $\mathbb{R}^{2|1}$ is given in coordinates by

$$\mu \cdot (z, \overline{z}, \theta) = (\mu^2 z, \mu^2 \overline{z}, \overline{\mu}\theta). \tag{5.4.11}$$

Using Definition 5.28, we obtain complex line bundles $\omega^{\frac{k}{2}}$ over $\Phi^{2|1}_{\mathcal{L}_2,0}$ induced by the projection $\Phi^{2|1}_{\mathcal{L}_2,0} \to \mathbb{C}^\times$. For convenience, we will denote vol $\in C^\infty(\mathcal{L}_2)$ for the function assigning the volume of a torus defined by the lattice $(l,l') \in \mathcal{L}_2$. Explicitly, we have that

$$\mathrm{vol}(l,l') = \frac{1}{2i}(l\bar{l}' - \bar{l}l'). \tag{5.4.12}$$

The projection $\Phi^{2|1}_{\mathcal{L}_2,0}(X) \to \mathbb{C}^\times$ induces using the construction of Definition 5.28 a complex line bundle $\omega^{\frac{k}{2}}$ over the stack $\Phi^{2|1}_{\mathcal{L}_2,0}(X)$.

Proposition 5.32 ([Ber13a, Proposition 3.9])**.** *We have a natural isomorphism of sheaves of graded algebras over $\mathbb{C}$,*

$$\bigoplus_{i+j=k} \Omega^j_{cl}(-; MFu^i) \to \Gamma\left(\Phi^{2|1}_{\mathcal{L}_2,0}(-), \omega^{\frac{k}{2}}\right), \qquad F \otimes \alpha \mapsto (vol)^{\frac{j}{2}} F \otimes \alpha. \tag{5.4.13}$$

Here, the assignment goes from homogeneous elements $F \otimes \alpha \in MFu^i \otimes \Omega^j_{cl}(X)$ to functions on $\mathcal{L}_2 \times \Pi TX$ regarded as elements in $\Gamma\left(\Phi^{2|1}_{\mathcal{L}_2,0}(X), \omega^{\frac{k}{2}}\right)$.

Proof. Given a homogeneous element $f \in C^\infty(\mathcal{L}_2) \otimes \Omega^j(X)$. We need to consider the conditions implied by the equivariance under the $\mathbb{C}^\times$-action on the functions. For convenience, we write

$$f = \mathrm{vol}^{\frac{j}{2}} F \otimes \alpha, \qquad F \in C^\infty(\mathcal{L}_2) \text{ and } \alpha \in \Omega^j(X). \tag{5.4.14}$$

The covariance under $\mathbb{C}^\times$ now implies that

$$\mu^k \mathrm{vol}^{\frac{j}{2}} F \otimes \alpha = (\mu, \overline{\mu}) \cdot (\mathrm{vol}^{\frac{j}{2}} F \otimes \alpha) = (\mu^2 \overline{\mu}^2)^{\frac{j}{2}} \mathrm{vol}^{\frac{j}{2}} (\mu, \overline{\mu}) \cdot F \otimes \overline{\mu}^{-j} \alpha = \mu^j \mathrm{vol}^{\frac{j}{2}} (\mu, \overline{\mu}) \cdot F \otimes \alpha. \tag{5.4.15}$$

Therefore, Proposition A.7 implies that F must be of modular weight $j-k$. By Theorem 5.29, we obtain a natural isomorphism of graded algebras as in Eq. (5.4.17). Similar to the 1|1 case, we argue that both sides must be sheaves and thus we have an isomorphism of sheaves, as requested. □

Demanding that the functions $F \in C^\infty(\mathcal{L}_2)$ in the previous proof are holomorphic, requires precisely that $\partial_{\overline{l}} F = \partial_{\overline{l}'} F = 0$. Tracing this condition through the previous proof, these conditions are equivalent with demanding that the functions $f \in C^\infty(\mathcal{L}_2) \otimes C^\infty(\Pi TX)$ as in Eq. (5.4.14) satisfy

$$\left(2\bar{l}\partial_{\overline{l}} - \frac{\bar{l}l'}{\bar{l}l' - \bar{l}'l \deg}\right) f = \left(2\bar{l}'\partial_{\overline{l}'} - \frac{\bar{l}'l}{\bar{l}l' - \bar{l}'l \deg}\right) f = 0. \tag{5.4.16}$$

Here, we writedeg for the degree derivation on ΠTX coming from the dilatation action. This is the same derivation as used in Definition 2.68 to define polynomial degrees of pseudo differential forms.

Definition 5.33. We will call the elements of $\Gamma\left(\Phi^{2|1}_{\mathcal{L}_2,0}(X), \omega^{\frac{\bullet}{2}}\right)$ **holomorphic sections** if they satisfy Eq. (5.4.16). We denote the subspace of holomorphic sections by $\mathcal{H}\left(\Phi^{2|1}_{\mathcal{L}_2,0}(X), \omega^{\frac{\bullet}{2}}\right)$. ▲

Theorem 5.34 ([Ber13a, Theorem 1.1]). *We have a natural isomorphism of sheaves of graded algebras over $\mathbb{C}$,*

$$\bigoplus_{i+j=k} \Omega^j_{cl}(-; MF^i) \to \mathcal{H}\left(\Phi^{2|1}_{\mathcal{L}_2,0}(-), \omega^{\frac{k}{2}}\right). \tag{5.4.17}$$

Proof. We apply Proposition 5.32. The holomorphicity conditions on the functions F and f in the proof of are Proposition 5.32 equivalent. □

This theorem realized the space of holomorphic sections as a cocyclic model for complexified tmf. This fact follows from Proposition B.7 and Example B.9.

5.5 Genera from Field Theories

We have established constructions for both complexified K-theory and complexified tmf in the language of field theories on tori in the previous section. In these cohomology theories, there exists certain genera, respectively the $\hat{A}$-genus and Witten Genus for K-theory and tmf. We will construct these classes in the present language as ζ-determinants of operators found in the related physical sigma models. Integrations of the classes would return the respective genus. We will closely follow [Ber13a].

Definition 5.35 ([Ber13a, Section 1.7.3]). Let D be a Fredholm operator with discrete spectrum $\{\lambda_k\}_{k\in\mathbb{Z}}$. We define the **$\zeta$-function**

$$\zeta_D(s) = \sum \lambda_k^s. \tag{5.5.1}$$

Assume that this function defines a holomorphic function for all s with real part smaller than -1, which can be analytically extended to a meromorphic function on $\mathbb{C}$ which is regular at $s = 0$. We define the **ζ-determinant** as

$$\det_\zeta(D) = \exp(\zeta_D'(0)). \tag{5.5.2}$$

We define the **Pfaffian** as a square root of the ζ-determinant. I.e., we can take $\mathrm{pf}_\zeta(D) = \exp(\frac{1}{2}\zeta_D'(0))$.

In case, we have operators acting in the super world, i.e., on $\mathbb{Z}/2$ graded vector spaces, we define the **ζ-super determinant** as the quotient

$$\mathrm{sdet}_\zeta(D) = \frac{\mathrm{pf}_\zeta(D|_{\mathrm{odd}})}{\mathrm{pf}_\zeta(D|_{\mathrm{even}})}. \tag{5.5.3}$$

▲

The operators from which we can obtain the $\hat{A}$ and Witten class are motivated by the corresponding physical sigma models. They are the operators from Eq. (4.3.33) and Eq. (4.3.49) popping up in the second order variation of the Lagrangian. In the computation of the genera in Chapter 4.3, we had some infinite constants around, making the constructions there not entirely rigorous. We will resolve this problem by comparing to the trivial bundle over the space X. Compared to Chapter 4.3, we will work in a Wick rotated version of the theory. This will slightly change the operators.

Proposition 5.36 ([Ber13a, Proposition 4.1])**.** *Let X be a Riemannian manifold. Define operators*

$$\left(\Delta_X^{1|1}\right)^{ev} = -\nabla_t^2 + \frac{i}{2}\mathcal{R}\nabla_t \text{ and } \left(\Delta_X^{1|1}\right)^{odd} = i\nabla_t \text{ and } \Delta_X^{1|1} = \left(\Delta_X^{1|1}\right)^{ev} \oplus \left(\Delta_X^{1|1}\right)^{odd} \tag{5.5.4}$$

on the sections of the tangent bundle of $\Phi^{1|1}_{\mathcal{L}_1,0}(X)$, where $\mathcal{R}$ is the endomorphism valued curvature two form. Also define

$$\left(\Delta_n^{1|1}\right)^{ev} = -\partial_t^2 \text{ and } \left(\Delta_n^{1|1}\right)^{odd} = i\partial_t \tag{5.5.5}$$

on the trivial vector bundle. Then, the quotient of ζ-super determinants

$$\frac{sdet_\zeta(\Delta_X^{1|1})}{sdet_\zeta(\Delta_n^{1|1})} \tag{5.5.6}$$

gives a function in $C^\infty(\Phi^{1|1}_{\mathcal{L}_1,0}(X))$ whose cohomology class agrees with the $\hat{A}$-class under the isomorphism of Proposition 5.31

Proof. We will perform a similar computation as in Chapter 4.3.3. The $\hat{A}$-class has characteristic series, [AHR10, Proposition 10.2],

$$\frac{\frac{x}{2}}{\sinh\left(\frac{x}{2}\right)} = \exp\left(\sum_{k=1}^{\infty} \frac{x^{2k}}{2k(2\pi i)^{2k}} 2\zeta_R(2k)\right). \tag{5.5.7}$$

Here, we have written ζ_R for the Riemann ζ-function.

We pull the operators $\Delta_X^{1|1}$ and $\Delta_n^{1|1}$ back along the map $\pi\colon \mathbb{R}_{>0} \times \Pi TX \to \Phi^{1|1}_{\mathcal{L}_2,0}(X)$. This produces operators on the bundles whose fiber at $r \in \mathbb{R}_{>0}$ is $C^\infty(\mathbb{R}/r\mathbb{Z}) \otimes \Gamma(\mathrm{ev}^* TX)$ respectively $C^\infty(\mathbb{R}/r\mathbb{Z}) \otimes \Gamma(\mathrm{ev}^* X \times \mathbb{R}^n)$ with $\mathrm{ev}\colon \mathbb{R}^{0|1} \otimes \Pi TX \to X$ the evaluation map. In this framework, have that $(\Pi^*\phi^*\nabla)_t = \frac{d}{dt} \otimes \mathrm{Id}_{TX}$. Hence, the pulled back operators become

$$\pi^*\left(\Delta_X^{1|1}\right)^{\mathrm{ev}} = -\frac{d^2}{dt^2} \otimes \mathrm{Id}_{TX} + i\frac{d}{dt} \otimes \mathcal{R}, \qquad \pi^*\left(\Delta_X^{1|1}\right)^{\mathrm{odd}} = i\frac{d}{dt} \otimes \mathrm{Id}_{TX}. \tag{5.5.8}$$

To compute the ζ-function, we take the basis of functions on $\mathbb{R}/r\mathbb{Z}$ given by $F_n = e^{\frac{2\pi i n t}{r}}$. This gives the ζ-functions from the definition as

$$\zeta_X^{\mathrm{ev}}(s) = \sum_{n\neq 0} \mathrm{Tr}\left(\frac{4\pi^2 n^2}{r^2} \otimes \mathrm{Id}_{TX} + \frac{2\pi i n}{r} \otimes i\mathbb{R}\right)^s, \tag{5.5.9}$$

$$\zeta_X^{\mathrm{odd}}(s) = \sum_{n\neq 0} \mathrm{Tr}\left(-\frac{2\pi n}{r} \otimes \mathrm{Id}_{TX}\right)^s. \tag{5.5.10}$$

A similar computation for the operator $\Delta_n^{1|1}$ gives the ζ-functions

$$\zeta_n^{\mathrm{ev}}(s) = \sum_{n\neq 0} \mathrm{Tr}\left(\frac{4\pi^2 n^2}{r^2} \otimes \mathrm{Id}_{X\times\mathcal{R}^n}\right)^s, \tag{5.5.11}$$

$$\zeta_n^{\mathrm{odd}}(s) = \sum_{n\neq 0} \mathrm{Tr}\left(-\frac{2\pi n}{r} \otimes \mathrm{Id}_{X\times\mathcal{R}^n}\right)^s. \tag{5.5.12}$$

Notice that the ζ-functions of the odd parts does not depend on the Riemannian metric, but only depend on the dimensions of the vector bundles TX and $X \times \mathbb{R}^n$. Therefore, their contributions are equal and thus they cancel in taking the quotient (5.5.5). We continue with just the even operators.

A binomial expansion of Eq. (5.5.9) gives

$$\zeta_X^{\mathrm{ev}}(s) = \sum_{n\neq 0} \mathrm{Tr}\left(\mathrm{Id} - \frac{r}{2\pi n} \otimes \mathcal{R}\right)^s \left(\frac{4\pi^2 n^2}{r^2}\right)^s \tag{5.5.13}$$

$$= \sum_{n\neq 0}\sum_{k=0}^{\infty} \mathrm{Tr}\left(\mathcal{R}^k \frac{s(s-1)\cdots(s-k-+1)}{k!(2\pi n)^k} r^k\right)\left(\frac{4\pi^2 n^2}{r^2}\right)^s. \tag{5.5.14}$$

Here, the sum over k is finite since $\mathcal{R}$ is nilpotent as it is a differential form. The contribution of $k = 0$ is identical to the contribution of $\zeta_n^{\mathrm{ev}}(s)$ therefore it cancels in the quotient of super ζ-determinants. By differentiating under the sum and evaluating at $s = 0$, we see that the relevant contribution to $\zeta'(0)$ is

$$\sum_{k=0}^{\infty} \mathrm{Tr}\left(\mathcal{R}^k\right) \frac{(-1)^{k-1}}{k(2\pi)^k} r^k 2\zeta_R(k) = -\sum_{k=1}^{\infty} \frac{\mathrm{Tr}\left(\mathcal{R}^{2k}\right) r^{2k}}{k(2\pi^{2k})} \zeta_R(2k). \tag{5.5.15}$$

In the second step, we used that traces of odd powers of $\mathcal{R}$ vanish. We conclude that the quotient in Eq. (5.5.5) becomes

$$\frac{\mathrm{sdet}_\zeta(\Delta_X^{1|1})}{\mathrm{sdet}_\zeta(\Delta_n^{1|1})} = \exp\left(\sum_{k=1}^{\infty} \frac{\mathrm{Tr}\left(\mathcal{R}^{2k}\right) r^{2k}}{2k(2\pi i)^{2k}} \zeta_R(2k)\right). \tag{5.5.16}$$

From standard differential geometry, we know that we can write the Pontryagin character $\mathrm{ph}_k(TX)$ in terms of the curvature under the isomorphism of Proposition 5.31. In our case, the result is

$$r^{2k}\mathrm{Tr}(\mathcal{R})^{2k} = 2(2k)!\mathrm{ph}_k(TX). \tag{5.5.17}$$

Substituting this in Eq. (5.5.16) yields

$$\frac{\mathrm{sdet}_\zeta(\Delta_X^{1|1})}{\mathrm{sdet}_\zeta(\Delta_n^{1|1})} = \exp\left(\sum_{k=1}^{\infty} \frac{2(2k)!\mathrm{ph}_k(TX)}{2k(2\pi i)^{2k}} \zeta_R(2k)\right). \tag{5.5.18}$$

which we identify as the $\hat{A}$-class of X. □

Using similar methods, we can recover the Witten Genus from the case of 2|1-dimensional field theories. The relevant operators $\Delta_X^{2|1}$ are now the wick rotated versions of Eq. (4.3.49).

Theorem 5.37 ([Ber13a, Theorem 1.3]). *Let X be a Riemannian manifold. Define operators*

$$\left(\Delta_X^{2|1}\right)^{ev} = -\nabla_z\nabla_{\overline{z}} + \frac{1}{2}\mathcal{R}\nabla_z \;\; and \;\; \left(\Delta_X^{2|1}\right)^{odd} = \nabla_z \;\; and \;\; \Delta_X^{2|1} = \left(\Delta_X^{2|1}\right)^{ev} \oplus \left(\Delta_X^{2|1}\right)^{odd} \tag{5.5.19}$$

on the sections of the tangent bundle of $\Phi^{2|1}_{\mathcal{L}_2,0}(X)$, where $\mathcal{R}$ is the endomorphism valued curvature two form. Also define

$$\left(\Delta_n^{2|1}\right)^{ev} = -\partial_z\partial_{\overline{z}} \;\; and \;\; \left(\Delta_n^{2|1}\right)^{odd} = \partial_z \tag{5.5.20}$$

on the trivial vector bundle. Then, the quotient of ζ-super determinants

$$\frac{sdet_\zeta(\Delta_X^{2|1})}{sdet_\zeta(\Delta_n^{2|1})} \tag{5.5.21}$$

defines a function in $C^\infty(\Phi^{2|1}_{\mathcal{L}_2,0}(X))$ whose cohomology class which agrees with the non-holomorphic Witten class under the isomorphism of Proposition 5.32.

Proof. The non-holomorphic Witten genus has characteristic series

$$\exp\left(\frac{E_2^*}{2(2\pi i)^2}z^2 + \sum_{k=2}^{\infty}\frac{E_{2k}(q)}{2k(2\pi i)^{2k}z^2k}\right). \tag{5.5.22}$$

The E_{2k} for $k \geq 2$ are the holomorphic Eisenstein Series of weight $2k$ and E_2^* is the non-holomorphic, see also Example A.6.

We pull back the operator $\Delta_X^{2|1}$ and $\Delta_n^{2|1}$ along the map $\mathcal{L}_2 \times \Pi TX \to \Phi^{2|1}_{\mathcal{L}_2}(X)$ to obtain operators on bundles whose fibers are $C^\infty(\mathbb{R}/\Lambda) \otimes \Gamma(\mathrm{ev}^*TX)$ and $C^\infty(\mathbb{R}/\Lambda) \otimes \Gamma(\mathrm{ev}^*(X \times \mathbb{R}^n))$ respectively. In this framework, we have that

$$(\pi^*\phi^*\nabla)_z = \partial_z \otimes \mathrm{Id}_{TX}, \qquad (\pi^*\phi^*\nabla)_{\overline{z}} = \partial_{\overline{z}} \otimes \mathrm{Id}_{TX}. \tag{5.5.23}$$

Therefore, the components of $\Delta_X^{2|1}$ become

$$\Pi^*\left(\Delta_X^{2|1}\right)^{\mathrm{ev}} = -\partial_z\partial_{\overline{z}} \otimes \mathrm{Id}_{TX} + \partial_z \otimes \mathcal{R}, \qquad \Pi^*\left(\Delta_X^{2|1}\right)^{\mathrm{odd}} = \partial_z \otimes \mathrm{Id}_{TX}. \tag{5.5.24}$$

To compute the ζ-function, we take the basis of functions on $\mathbb{R}^2/(l\mathbb{Z} \oplus l'\mathbb{Z})$ for a lattice (l, l') given by the functions

$$F_{n,m}(z, \overline{z}) = \exp\left(\frac{\pi}{\mathrm{vol}}\left(-z(n\bar{l} + m\bar{l}') + \overline{z}(nl + ml')\right)\right) \tag{5.5.25}$$

for all $n, m \in \mathbb{Z}$ and $\mathrm{vol} = \frac{l\bar{l}' - \bar{l}l'}{2i}$ the volume of the induced torus. Evaluating the ζ-functions for the operators $\left(\Delta_X^{2|1}\right)^{\mathrm{ev}}$ and $\left(\Delta_X^{2|1}\right)^{\mathrm{odd}}$ yields

$$\zeta_X^{\mathrm{ev}}(s) = \sum_{(n,m)\in\mathbb{Z}^2_*} \mathrm{Tr}\left(\frac{\pi^2}{\mathrm{vol}^2}\left|nl + ml'\right|^2 \otimes \mathrm{Id}_{TX} + \frac{\pi}{\mathrm{vol}}(n\bar{l} + m\overline{l'}) \otimes \mathcal{R}\right)^s, \tag{5.5.26}$$

$$\zeta_X^{\mathrm{odd}}(s) = \sum_{(n,m)\in\mathbb{Z}^2_*} \mathrm{Tr}\left(\frac{\pi}{\mathrm{vol}}(n\bar{l} + m\overline{l'}) \otimes \mathrm{Id}_{TX}\right)^s. \tag{5.5.27}$$

Performing the same computation for the operators $\left(\Delta_n^{2|1}\right)^{\mathrm{ev}}$ and $\left(\Delta_n^{2|1}\right)^{\mathrm{odd}}$, we get the ζ-functions

$$\zeta_n^{\mathrm{ev}}(s) = \sum_{(n,m)\in\mathbb{Z}^2_*} \mathrm{Tr}\left(\frac{\pi^2}{\mathrm{vol}^2}\left|nl + ml'\right|^2 \otimes \mathrm{Id}_{X\times\mathbb{R}^n}\right)^s, \tag{5.5.28}$$

$$\zeta_n^{\mathrm{odd}}(s) = \sum_{(n,m)\in\mathbb{Z}^2_*} \mathrm{Tr}\left(\frac{\pi}{\mathrm{vol}}(n\bar{l} + m\overline{l'}) \otimes \mathrm{Id}_{X\times\mathbb{R}^n}\right)^s. \tag{5.5.29}$$

Again, the contributions of the odd operators only depend on the dimensions of the vector bundle. Hence, their contribution to the quotient (5.5.21) is trivial. We continue with just the even operators. Binomially expanding Eq. (5.5.26) yields

$$\zeta_X^{\mathrm{ev}}(s) = \sum_{(n,m)\in\mathbb{Z}_*^2} \mathrm{Tr}\left(\left(\mathrm{Id} + \frac{\mathrm{vol}}{\pi}(nl+ml')^{-1}\otimes\mathcal{R}\right)^s \left(\frac{\pi^2}{\mathrm{vol}^2}\left|nl+ml'\right|^2\right)^s\right) \tag{5.5.30}$$

$$= \sum_{(n,m)\in\mathbb{Z}_*^2}\sum_{k=0}^{\infty} \mathrm{Tr}\left(\left(\mathrm{vol}^k\frac{s(s-1)\cdots(s-k+1)}{k!(2\pi)^k(nl+ml')^k}\otimes\mathcal{R}^k\right)\left(\frac{\pi}{\mathrm{vol}}\left|nl+ml'\right|\right)^{2s}\right) \tag{5.5.31}$$

Where the sum over k is finite, since $\mathcal{R}$ is nilpotent. The $k=0$ term is cancelled by the contribution from ζ_n^{ev} in the quotient (5.5.21), so we will ignore it. For odd k the terms vanish since traces of odd powers of $\mathcal{R}$ vanish. For $k=2$ differentiating and taking the limit $s\to 0^-$ gives

$$\lim_{s\to 0^-} -\mathrm{Tr}\left(\sum_{(n,m)\in\mathbb{Z}_*^2}\frac{1}{2}\left(\frac{\mathrm{vol}^2|nl+ml'|^{2s}}{(2\pi)^2}\right)\right) = -\frac{\mathrm{vol}^2 E_2^*}{2(2\pi)^2}\mathrm{Tr}\left(\mathcal{R}\right) \tag{5.5.32}$$

where we introduced the non-holomorphic but modular Eisenstein series E_2^* of weight 2. Now focusing on the terms $k\geq 3$, the contribution to $\zeta'(0)$ is

$$\sum_{(n,m)\in\mathbb{Z}_*^2}\sum_{k=3}^{\infty}\mathrm{Tr}\left(\frac{(-1)^{k-1}\mathrm{vol}^k}{k(2\pi)^k}(nl+ml')^{-k}\otimes\mathcal{R}^k\right) = -\sum_{k=2}^{\infty}\frac{\mathrm{vol}^{2k}E_{2k}}{2k(2\pi)^{2k}}\mathrm{Tr}\left(\mathcal{R}^2\right), \tag{5.5.33}$$

where we used that the odd terms vanish. Putting all together, we can evaluate Proposition 5.31 as

$$\frac{\mathrm{sdet}_\zeta(\Delta_X^{2|1})}{\mathrm{sdet}_\zeta(\Delta_n^{2|1})} = \exp\left(\frac{\mathrm{vol}^2 E_2^*}{4(2\pi i)^2}\mathrm{Tr}\left(\mathcal{R}\right) + \sum_{k=2}^{\infty}\frac{\mathrm{vol}^{2k}E_{2k}}{4k(2\pi i)^{2k}}\mathrm{Tr}\left(\mathcal{R}^2\right)\right). \tag{5.5.34}$$

The isomorphism of Proposition 5.32 identifies the Pontryagin character $\mathrm{ph}_k(X)$ as

$$2(2k)!\mathrm{ph}_k(X) = \mathrm{vol}^{2k}\mathrm{Tr}(\mathcal{R}^{2k}). \tag{5.5.35}$$

Substituting this into Eq. (5.5.34), we obtain

$$\frac{\mathrm{sdet}_\zeta(\Delta_X^{2|1})}{\mathrm{sdet}_\zeta(\Delta_n^{2|1})} = \exp\left(\frac{\mathrm{ph}_1(X)}{(2\pi i)^2}E_2^* + \sum_{k=2}^{\infty}\frac{(2k)!\mathrm{ph}_k(X)}{2k(2\pi i)^{2k}}E_{2k}\right). \tag{5.5.36}$$

This function induces the non-holomorphic Witten class under the isomorphic of Proposition 5.31 as an element in the De Rham cohomology with values in the functions with some modular weight. □

The non-holomorphic Witten Genus is closely related to the **Holomorphic Witten Genus**, which has characteristic series, [AHR10, Section 15],

$$(e^{\frac{z}{2}} - e^{-\frac{z}{2}})\Pi\frac{\left(1-q^n e^{\frac{z}{2}}\right)\left(1-q^n e^{-\frac{z}{2}}\right)}{\left(1-q^n\right)^2} = \exp\left(\sum_{k=1}^{\infty}\frac{E_{2k}(q)}{2k(2\pi i)^{2k}z^2k}\right) \tag{5.5.37}$$

The equality of the two formulae was shown by Don Zagier in [Zag88]. If the manifold has a **rational string structure**, meaning that there exists a 3-form $H\in\Omega^3(X)$ such that

$dH = -\mathrm{p}_1(X)$ is the first Pontryagin class, then the non-holomorphic and holomorphic Witten classes are concordant. The concordance is given by

$$\frac{\mathrm{sdet}_\zeta(\Delta_X^{2|1})}{\mathrm{sdet}_\zeta(\Delta_n^{2|1})} \exp\left(\frac{d(tH)}{(2\pi i)^2} E_2^*\right) \tag{5.5.38}$$

At $t = 0$, by Theorem 5.37, we have the non-holomorphic Witten class, while for $t = 1$ we obtain the holomorphic Witten class.

6 Conclusion

In this thesis, we have built a bridge between Generalized Cohomologies and Supersymmetric Field Theories. We started by introducing supermanifolds, which are the fundamental objects in supersymmetry. In particular, we studied the *odd tangent bundle* ΠTX, Definition 2.59, and its symmetries. This is a very important object, since it can be interpreted as the mapping space $\mathbb{R}^{0|1} \to X$. We continued by introducing *stacks*. These give a categorical foundation for the afterward considered notions. We have seen that differentiable stacks can be interpreted as Lie groupoids up to Morita equivalence, Theorem 3.43.

With the notions of supermanifolds and stacks in place, we went on with defining Field Theories in both mathematical and physical sense. The mathematical notion is defined in terms of functors out of bordism categories. When working over a background space, the field theories can be interpreted as certain Sigma Models in physics. We were able to classify the field theories in dimension $0|1$ with differential forms, Corollary 4.35 and their concordance classes with De Rham cohomology, Proposition 4.40. On the physics side, we constructed the related Sigma Models. The equations of motion and certain operators motivate some definitions in the mathematical theory.

Lastly, we studied the mathematical notions of field theories, up to one loop. I.e., with just the (higher dimensional) tori. Using all the build up machinery of supermanifolds and stacks, we are able to relate the field theories here to closed (pseudo) differential forms with values in lattice functions satisfying an equivariance property, Theorem 5.29. In this setting, the $1|1$-dimensionsal field theories can be identified with complexified K-theory, Proposition 5.31 and the $2|1$-dimensional field theories with complexified tmf, Theorem 5.34. Moreover, we are able to obtain a connection to Siegel modular forms, Corollary 5.30. It is an interesting question whether Theorem 5.29 can be used in more settings to obtain other (complexified) cohomologies. Furthermore, one can try to find the integral versions of the cohomology theories from the full field theories. For K-theory this has been shown. For tmf, it is only conjectured.

Computing ζ-determinants of the operators identified in the physical theory give rise to the genera of the relevant cohomology theory. We have computed the $\hat{A}$-genus, Proposition 5.36, and Witten Genus, Theorem 5.37, in our setting of fields over tori. The question this poses is whether the same method can be applied in higher dimensions. For this, one would first need to identify suitable operators. Studies of the Sigma Models can prove useful.

Another question that remains is the following: One way to construct generalized cohomologies is via Formal Group Laws. In fact, singular cohomology and K-theory can be defined in this way. One could ask how these formal group laws show up in the setting of field theories. This might prove more insight in the relation between field theories and generalized cohomology.

A Appendix: Modular Forms

Modular Forms are holomorphic functions on the upper half plane $\mathbb{H}$ with certain symmetry properties. They arose from the study of elliptic functions and their moduli space, hence the name *modular* form. In this Appendix, we will define modular forms and derive some properties required in this thesis. Also, we extend the notion of modular form to *Siegel Modular Forms*, which are the analogue of modular forms on higher dimensional domains, Siegel Upper Half Planes.

Definition A.1 ([Bro17, Definition 2.1]). A smooth function $f\colon \mathbb{H} \to \mathbb{C}$ is **modular of weight** k if for every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, there holds

$$f\left(\frac{ac+b}{cz+d}\right) = (cz+d)^k f(z). \tag{A.0.1}$$

▲

Modular forms are obtained by imposing additional holomorphicity assumptions

Definition A.2. A **weak modular form of weight** k is a holomorphic function on the upper half plane $\mathbb{H}$ which is modular of weight k. It becomes a **modular form of weight** k if it is also holomorphic at ∞. ▲

Remark A.3. In the definition of functions modular some weight, we have not assumed any conditions involving a Laplacian or conditions of the asymptotic growth. Therefore, this definition is different from *Maass forms*. See e.g., [Liu07, Definition 2.1] for a definition of Maass Forms. ▽

Remark A.4. The modularity condition for the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ for a function f of modular weight k implies that $f(z) = (-1)^k f(z)$ for all $z \in \mathbb{Z}$. In particular, this tells us that the only functions of odd weight is the zero function. ▽

Remark A.5. The definition can be extended to cover half-integer weights too, [GMR20, Section 2]. We define a function to be modular of weight $k \in \mathbb{Z} + \frac{1}{2}$ to satisfy

$$f\left(\frac{ac+b}{cz+d}\right) = \frac{1}{\left(\frac{c}{d}\right)} \epsilon_d^{2k} \sqrt{cz+d}^{2k} f(z), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{with} c \equiv 0 \text{ Mod } 4. \tag{A.0.2}$$

Here, we have denoted $\left(\frac{c}{d}\right)$ for the extended Jacobi Symbol in the sense of [Shi73], the square root is taken in the branch with $-\frac{\pi}{2} < \arg(\sqrt{z}) \leq \frac{\pi}{2}$ and the ϵ_d are defined by

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \text{ Mod } 4 \\ 1 & \text{if } d \equiv 3 \text{ Mod } 4 \end{cases}. \tag{A.0.3}$$

▽

Example A.6 ([Sut17, Example 25.4]). The prime example of modular forms are the **Eisenstein Series**. These are the functions defined by the infinite sum

$$E_k(z) = \sum_{(m,n)\in\mathbb{Z}^2} \frac{1}{(mz+n)^k} \qquad z \in \mathbb{H}. \tag{A.0.4}$$

For odd k the terms (m,n) and $(-m,-n)$ cancel each other, making the Eisenstein series vanish. For even k the function becomes modular of weight k. It is a modular form over weight k if $k > 2$. For $k = 2$ the function fails to be holomorphic. △

The functions of some modular weight (or modular forms) can be put together in a graded ring with grading according to their weight. Multiplying two functions with some modular weight will return a function whose modular weight is the sum of the original two. We will denote MFu^{-k} and MF^{-k} for the functions and modular forms of weight k respectively. Hereby, we have the graded rings $\mathrm{MFu}^{\bullet}$ and $\mathrm{MF}^{\bullet}$.

The modularity condition in Definition A.1 can be formulated equivalently in terms of lattices. We take the construction from [Lac15, Page 4] to arrive at [Cal13, Definition 1.2.1]. Denote $\mathcal{L}_2$ for the set of orientable lattices in $\mathbb{C}$, i.e.

$$\mathcal{L}_2 = \left\{ (z_1, z_2) \in \mathbb{C}^\times \times \mathbb{C}^\times \middle| \mathrm{im}\left(\frac{z_1}{z_2}\right) > 0 \right\}. \tag{A.0.5}$$

We have the action of $\mathbb{C}^\times$ on $\mathcal{L}_2$ given by the assignment $(\lambda, (z1_1, z_2)) \mapsto (\lambda z_1, \lambda z_2)$ for $\lambda \in \mathbb{C}^\times$. Moreover, there is a natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{L}_2$ by assigning

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (z_1, z_2) \right) \mapsto (az_1 + bz_2, cz_2 + dz_2), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{A.0.6}$$

To see that this is indeed well-defined, notice that if $\mathrm{im}(\frac{z_1}{z_2}) > 0$, then $\mathrm{im}(z_1\overline{z_2}) > 0$. Since for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have that $ad - bc = 1$, we obtain that $\mathrm{im}(adz_1\overline{z_2} + bc\overline{z_1}z_2) > 0$. This in turn implies that the imaginary part of $(az_1 + bz_2)(c\overline{z_2} + d\overline{z_2})$ and thus also of $\frac{az_1+bz_2}{cz_1+dz_2}$ is positive.

Proposition A.7. *The smooth functions $f \colon \mathbb{H} \to \mathbb{C}$ of modular weight k are in canonical bijection with the functions $F \colon \mathcal{L}_2 \to \mathbb{C}$ invariant under the $SL_2(\mathbb{Z})$ action and satisfying*

$$F(\lambda \cdot \Gamma) = \lambda^{-k} F(\Gamma) \qquad \text{for all } \lambda \in \mathbb{C}^\times \text{ and } \Gamma \in \mathcal{L}_2. \tag{A.0.7}$$

Proof. Given a smooth function $f \colon \mathbb{H} \to \mathbb{C}$ of modular weight k define the map $F \colon \mathcal{L}_2 \to \mathbb{C}$ given by

$$F(z_1, z_2) = z_2^{-k} f\left(\frac{z_1}{z_2}\right), \qquad (z_1, z_2) \in \mathcal{L}_2. \tag{A.0.8}$$

Notice that $\frac{z_1}{z_2} \in \mathbb{H}$. We show that F is invariant under the $\mathrm{SL}_2(\mathbb{Z})$ action and satisfies Eq. (A.0.7). For any $\lambda \in \mathbb{C}^2$ and $(z_1, z_2) \in \mathcal{L}_2$, we have that

$$F(\lambda z_1, \lambda z_2) = \lambda^{-k} z_2^{-k} f\left(\frac{\lambda z_1}{\lambda z_2}\right) = \lambda^{-k} z_2^{-k} f\left(\frac{z_1}{z_2}\right) = \lambda^{-k} F(z_1, z_2). \tag{A.0.9}$$

Similarly, we have that for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ there holds by the modularity of f that

$$\begin{aligned}
F(az_1 + bz_2, cz_1 + dz_2) &= (cz_1 + dz_2)^{-k} f\left(\frac{az_1 + bz_2}{cz_1 + dz_2}\right) && \text{(A.0.10)} \\
&= (cz_1 + dz_2)^{-k} f\left(\frac{a\frac{z_1}{z_2} + b}{c\frac{z_1}{z_2} + d}\right) && \text{(A.0.11)} \\
&= (cz_1 + dz_2)^{-k} (c\frac{z_1}{z_2} + d)^k f\left(\frac{z_1}{z_2}\right) && \text{(A.0.12)} \\
&= z_2^{-k} f\left(\frac{z_1}{z_2}\right) && \text{(A.0.13)} \\
&= F(z_1, z_2). && \text{(A.0.14)}
\end{aligned}$$

Conversely, given a function $F\colon \mathcal{L}_2 \to \mathbb{C}$ invariant under the $\mathrm{SL}_2(\mathbb{Z})$ action and satisfying Eq. (A.0.7), we can recover the function of modular weight k by defining

$$f(z) = F(z, 1), \qquad z \in \mathbb{H}. \tag{A.0.15}$$

Since $z \in \mathbb{H}$, we know that $(z, 1)$ indeed defines a lattice. We show that f has modular weight k. Given a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have the following:

$$\begin{aligned} f\left(\frac{az+b}{cz+d}\right) &= F\left(\frac{az+b}{cz+d}, 1\right) && \text{(A.0.16)} \\ &= (cz+d)^k F\left(az+b, cz+d\right) && \text{(A.0.17)} \\ &= (cz+d)^k F\left(z, 1\right) && \text{(A.0.18)} \\ &= (cz+d)^k f(z) && \text{(A.0.19)} \end{aligned}$$

Here, we used the property of Eq. (A.0.7) for the second line and the $\mathrm{SL}_2(\mathbb{Z})$ invariance for the third.

Notice that the two constructions are each other's inverse. Hence, we have found the requested bijection. □

To pass to modular forms, we only need to request holomorphicity on all functions.

Corollary A.8. *The weak modular forms of weight k are in canonical bijection with the holomorphic functions $F\colon \mathcal{L}_2 \to \mathbb{C}$ invariant under the $SL_2(\mathbb{Z})$ action and satisfying Eq.* (A.0.7).

A.1 Siegel Modular Forms

The functions considered in the discussion above are all of the form $\mathbb{H} \to \mathbb{C}$. In a sense, this is the lowest possible dimension for the functions to live: both the source and target have complex dimension 1. We will now discuss extensions to higher dimensions. We will both extend the source, obtaining *Siegel Modular forms*, and the target, obtaining *vector valued modular forms*. In fact, we will do both simultaneously. Moreover, we need to up the dimension of the modular group $\mathrm{SL}_2(\mathbb{Z})$. One might expect to simply use $\mathrm{SL}_p(\mathbb{Z})$. However, this group turns out to be too big and we will need to take a suitable subgroup instead, the *Siegel Modular Group*.

For extending the domains, we need to find a suitable notion of *Upper Half Plane* in higher dimensions. These will go by the name of *Siegel Upper Half plane*. The extension of the target is relatively simple. We replace just $\mathbb{C}$ by a complex vector space V. The only thing to worry about is the multiplication of the factor $(cz+d)^k$ in the one dimensional case. This will be aided by choosing a representation

$$\rho\colon \mathrm{GL}_g(\mathbb{C}) \to \mathrm{GL}(V). \tag{A.1.1}$$

Following [Gee07, Section 1 and 2], we will construct the Siegel Modular Group, Upper Half Plane and Modular Forms.

Definition A.9. The **Siegel Modular Group** in dimension g is the **Symplectic group** $\mathrm{Sp}(2g, \mathbb{Z})$. This is the automorphism group of $\mathbb{Z}^{2g}$ respecting the standard symplectic form $\langle , \rangle$. More explicitly $\mathrm{Sp}(2g, \mathbb{Z})$ consists of the matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A.1.2}$$

where A, B, C and D are $g \times g$ matrices with integral coefficients which satisfy

$$AB^t = BA^t, \qquad CD^t = DC^t \quad \text{and} \quad AD^t - BC^t = \mathrm{Id}_g. \tag{A.1.3}$$

▲

From the conditions in Eq. (A.1.3) it is easy to see that in the case of $g = 1$, we obtain the group $\mathrm{SL}_2(\mathbb{Z})$. To see the fact that an element of the automorphism group of $\mathbb{Z}^{2g}$ respecting the standard symplectic form is equivalent to a matrix satisfying Eq. (A.1.3), we provide the next lemma.

Lemma A.10 ([Koh07, Lemma 3]). *For a matrix* $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2g, \mathbb{Z})$*, the following three assertions are equivalent.*

1. *The transformation γ respects the standard symplectic form.*
2. *The conditions of Eq.* (A.1.3) *hold.*
3. *The following conditions hold*

$$A^tC = C^tA, \qquad B^tD = D^tB \quad \textit{and} \quad A^tD - B^tC = Id_g. \tag{A.1.4}$$

Proof. Condition 1 says that

$$\gamma^t \begin{pmatrix} 0 & \mathrm{Id}_g \\ -\mathrm{Id}_g & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & \mathrm{Id}_g \\ -\mathrm{Id}_g & 0 \end{pmatrix}. \tag{A.1.5}$$

Working out the left-hand side, we obtain

$$\begin{aligned} \gamma^t \begin{pmatrix} 0 & \mathrm{Id}_g \\ -\mathrm{Id}_g & 0 \end{pmatrix} \gamma &= \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & \mathrm{Id}_g \\ -\mathrm{Id}_g & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} && \text{(A.1.6)} \\ &= \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} C & D \\ -A & -B \end{pmatrix} && \text{(A.1.7)} \\ &= \begin{pmatrix} A^tC - C^tA & A^tD - C^tB \\ B^tC - D^tA & B^tD - D^tB \end{pmatrix}. && \text{(A.1.8)} \end{aligned}$$

Imposing the condition of Eq. (A.1.5) is equivalent with imposing Eq. (A.1.4). This shows that 1 and 3 are equivalent.

To show that also 2 is equivalent, we take the inverse of Eq. (A.1.5) to obtain the equality

$$\gamma^{-1} \begin{pmatrix} 0 & -\mathrm{Id}_g \\ \mathrm{Id}_g & 0 \end{pmatrix} \left(\gamma^t\right)^{-1} = \begin{pmatrix} 0 & -\mathrm{Id}_g \\ \mathrm{Id}_g & 0 \end{pmatrix}. \tag{A.1.9}$$

Multiplying with γ and γ^t on either side, yields the equation

$$\begin{pmatrix} 0 & -\mathrm{Id}_g \\ \mathrm{Id}_g & 0 \end{pmatrix} = \gamma \begin{pmatrix} 0 & -\mathrm{Id}_g \\ \mathrm{Id}_g & 0 \end{pmatrix} \gamma^t. \tag{A.1.10}$$

By working out the right-hand side the same way as before, we see that 1 and 2 are equivalent. □

With the modular group in place, we turn to generalizing the upper half plane.

Definition A.11. The **Siegel Upper Half Plane** $\mathbb{H}_g$ is defined as

$$\mathbb{H}_g = \left\{\tau \in \mathrm{Mat}(g \times g, \mathbb{C}) | \tau^t = \tau, \mathrm{im}(\tau) > 0\right\}. \tag{A.1.11}$$

Here, the condition $\mathrm{im}(\tau) > 0$ amounts to requiring that the imaginary part of each entry is positive. ▲

Again, it should be clear that if $g = 1$, then this definition reduces to the usual upper half plane $\mathbb{H}$.

The Siegel Modular group admits an action on the Siegel Upper Half Plane generalizing the standard case in $g = 1$. We define

$$\mathrm{Sp}(2g, \mathbb{Z}) \times \mathbb{H}_g \ni \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tau\right) \mapsto (A\tau + B)(C\tau + D)^{-1} \in \mathbb{H}_g. \tag{A.1.12}$$

This definition requires a check of well-definedness. I.e., we need to check that $(C\tau + D)$ is invertible and that we land in $\mathbb{H}_g$. To check the former, write $\tau = x + iy$ with x and y symmetric real $g \times g$ matrices. The condition $\mathrm{im}(\tau) > 0$ implies that y is positive definite. Using the formulae from Lemma A.10, we compute:

$$\begin{aligned}
&(C\overline{\tau} + D)^t(A\tau + B) - (A\overline{\tau} + B)^t(C\tau + D) && \text{(A.1.13)}\\
&\quad = (\overline{\tau}C^t + D^t)(A\tau + B) - (\overline{\tau}A^t + B^t)(C\tau + D) && \text{(A.1.14)}\\
&\quad = \overline{\tau}C^tA\tau + D^tA\tau + \overline{\tau}C^tB + D^tB - (\overline{\tau}A^tC\tau + B^tC\tau + \overline{\tau}A^tD + B^tD) && \text{(A.1.15)}\\
&\quad = \tau - \overline{\tau} && \text{(A.1.16)}\\
&\quad = 2iy && \text{(A.1.17)}
\end{aligned}$$

From here we deduce that if $(C\tau + D)\xi = 0$ for any $\xi \in \mathbb{C}^g$, then $\overline{\xi}^t y\xi = 0$. This is a contradiction with the fact that y is positive definite. We conclude that $(C\tau + D)^{-1}$ is well-defined.

A similar computation yields

$$\begin{aligned}
&(C\overline{\tau} + D)^t\left(\left((A\tau + B)(C\tau + D)^{-1}\right)^t - (A\tau + B)(C\tau + D)^{-1}\right)(C\tau + D) && \text{(A.1.18)}\\
&\quad = (A\tau + B)^t(C\tau + D) - (C\tau + D)^t(A\tau + B) && \text{(A.1.19)}\\
&\quad = (\tau A^t + B^t)(C\tau + D) - (\tau C^t + D^t)(A\tau + B) && \text{(A.1.20)}\\
&\quad = \tau A^tC\tau + B^tC\tau + \tau A^tD + B^tD - (\tau C^tA\tau + D^tA\tau + \tau C^tB + D^tB) && \text{(A.1.21)}\\
&\quad = B^tC\tau + \tau A^tD - (D^tA\tau + \tau C^tB) && \text{(A.1.22)}\\
&\quad = \tau - \tau && \text{(A.1.23)}\\
&\quad = 0. && \text{(A.1.24)}
\end{aligned}$$

This implies that the matrix $(A\tau + B)(C\tau + D)^{-1}$ is symmetric. Lastly, writing $(A\tau + B)(C\tau + D)^{-1} = x' + iy'$, the same computation yields

$$\begin{aligned}
&(C\overline{\tau} + D)^t y'(C\tau + D) && \text{(A.1.25)}\\
&= (C\overline{\tau} + D)^t\left(\left((A\tau + B)(C\tau + D)^{-1}\right)^t - (A\overline{\tau} + B)(C\overline{\tau} + D)^{-1}\right)(C\tau + D) && \text{(A.1.26)}\\
&= y. && \text{(A.1.27)}
\end{aligned}$$

Using that y is positive definite, this implies that y' is positive definite. We now have shown that the assignment Eq. (A.1.12) is well-defined. The fact that it defines an action follows from the identity

$$\left(A'(A\tau+B)(C\tau+D)^{-1}+B'\right)\left(C'(A\tau+B)(C\tau+D)^{-1}+D'\right)^{-1}= \tag{A.1.28}$$

$$\left(A'(A\tau+B)+B'(C\tau+D)\right)\left(C'(A\tau+B)+D'(C\tau+D)\right)^{-1}= \tag{A.1.29}$$

$$\left((A'A+B'C)\tau+A'B+B'D\right)\left((C'A+D'C)\tau+C'B+D'D\right)^{-1}. \tag{A.1.30}$$

Having piled up all the necessary ingredients, we can now define Siegel Modular Forms.

Definition A.12 ([Gee07, Definition 3.1])**.** Let V be a complex vector space and $\rho\colon \mathrm{GL}_g(\mathbb{C}) \to \mathrm{GL}(V)$ a representation. A smooth function $f\colon \mathbb{H}_g \to V$ is of **Siegel Modular weight** ρ if for any $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z})$ and $\tau \in \mathbb{H}_g$ there holds

$$f\left((A\tau+B)(C\tau+D)^{-1}\right)=\rho(C\tau+D)f(\tau). \tag{A.1.31}$$

Additionally assuming that f is holomorphic (and if $g=1$ assume also that f is holomorphic at ∞), we obtain a **Siegel Modular Form**. We write $\mathrm{SMFu}^{\rho^{-1}}$ for the set of smooth functions of Siegel Modular weight ρ and $\mathrm{SMF}^{\rho^{-1}}$ for the Siegel Modular forms of weight ρ. ▲

The condition of being of weight ρ can be reformulated in the same fashion as in the $g=1$ case, Proposition A.7. In the $g=1$ case, we could work with all orientable lattices. We need to restrict the lattices a little more here.

Definition A.13. The set of **Siegel Lattices** $\mathcal{L}^S_{2g}$ is defined as the following set of lattices in $\mathbb{C}^{2g}$.

$$\mathcal{L}^S_{2g}=\left\{(v_1,\dots,v_{2g})\in(\mathbb{C}^g)^{2g}\,\middle|\,\begin{matrix}(v_1,\dots,v_{2g})\text{ real linearly independent and}\\ (v_1,\dots,v_g)(v_{g+1},\dots,v_{2g})^{-1}\text{ symmetric with positive imag. part}\end{matrix}\right\} \tag{A.1.32}$$

▲

The set of Siegel lattices $\mathcal{L}^S_{2g}$ admits an action by the Siegel Modular group $\mathrm{Sp}_{2g}(\mathbb{Z})$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\cdot(v_1,\dots,v_{2g})=(A(v_1,\dots,v_g)+B, C(v_{g+1},\dots,v_{2g})+D) \tag{A.1.33}$$

for $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\mathrm{Sp}_{2g}(\mathbb{Z})$ and $(v_1,\dots,v_{2g})\in\mathcal{L}^S_{2g}$. The discussion of the Siegel Upper Half Plane above shows that this is a well-defined action.

Proposition A.14. *Let $\rho\colon GL_g(\mathbb{C})\to GL(V)$ be a representation. The smooth functions $f\colon \mathbb{H}_g\to V$ of Siegel modular weight ρ are in canonical bijection with the smooth functions $F\colon \mathcal{L}^S_{2g}\to V$ invariant under the action of $SP_{2g}(\mathbb{Z})$ and satisfying*

$$F(\lambda(v_1,\dots,v_g),\lambda(v_{g+1},\dots,v_{2g}))=\rho(\lambda)^{-1}F(v_1,\dots,v_{2g}) \tag{A.1.34}$$

for all $\lambda\in GL_g(\mathbb{C})$.

Proof. The proof is analogous to the proof of Proposition A.7. Given a function $f\colon \mathbb{H}_g \to V$ of Siegel modular weight ρ, we define for all $(v_1, \dots, v_{2g}) \in \mathcal{L}_{2g}^S$

$$F(v_1, \dots, v_{2g}) = \rho(v_{g+1}, \dots, v_{2g})^{-1} f\left((v_1, \dots, v_g)(v_{g+1}, \dots, v_{2g})^{-1}\right). \tag{A.1.35}$$

This F is well-defined and f, invariant under $\mathrm{SP}_{2g}(\mathbb{Z})$ and satisfies (A.1.34). The proof is similar to the case for $g = 1$ and left as an exercise to the reader.

Conversely, let $F\colon \mathcal{L}_{2g}^S \to V$ be a function invariant under the action of $\mathrm{SP}_{2g}(\mathbb{Z})$ and satisfying (A.1.34). We define a smooth function $f\colon \mathbb{H}_g \to \mathbb{V}$ of Siegel modular weight ρ by

$$f(\tau) = F(\tau, \mathrm{Id}_g) \qquad \tau \in \mathbb{H}_g. \tag{A.1.36}$$

Here, we interpret the columns of τ together with the columns of the identity matrix as a Siegel lattice. Notice that this indeed defines a Siegel lattice. In particular, notice that since the entries of τ have positive imaginary part, we indeed have found $2g$ real linear independent vectors. The fact this is a Siegel modular form follows similar to the case $g = 1$. The details are left as an exercise. $\square$

With additionally imposing holomorphicity conditions, we obtain the following corollary.

Corollary A.15. *Let $\rho\colon GL_g(\mathbb{C}) \to GL(V)$ be a representation. The Siegel modular forms of weight ρ (for $g = 1$, weak modular forms) are in canonical bijection with the holomorphic function $F\colon \mathcal{L}_{2g}^S \to V$ invariant under the action of $SP_{2g}(\mathbb{Z})$ and satisfying Eq.* (A.1.34)*.*

B Appendix: Generalized Cohomology

One of the most studied invariant of topological spaces in algebraic topology is Singular (Co)homology. Singular cohomology consists of a contravariant functor

$$H^\bullet\colon \mathcal{T}op^{\mathrm{op}}_{\mathrm{pair}} \to \mathrm{Ab}^{\mathbb{Z}} \tag{B.0.1}$$

from pairs of topological spaces to the category of $\mathbb{Z}$ graded Abelian groups together with **connecting homomorphisms** $\delta\colon H^\bullet(A,\emptyset) \to H^{\bullet+1}(X,A)$. Singular cohomology satisfies the celebrated *Eilenberg-Steenrod* axioms [Eil+72]. These axioms assert homotopy invariance, excision, exactness and dimensionality. The last axiom of dimensionality assumes that the cohomology of the one point space is concentrated in degree zero. Eilenberg and Steenrod show in the mentioned paper that these axioms uniquely determine the cohomology theory (at least on CW complexes) up to isomorphism. This uniqueness property is one way to show that cellular cohomology and De Rham cohomology are isomorphic to singular cohomology. In fact, the reason Eilenberg and Steenrod came up with their axioms is to classify cohomology theories.

To generalize cohomology theories away from just singular cohomology, we need to drop one of the Eilenberg-Steenrod axioms. The most natural choice is to drop the dimensionality axiom. We obtain a notion of *generalized (Eilenberg-Steenrod) cohomology.*

Definition B.1. A **Generalized Cohomology** is a contravariant functor

$$E^\bullet\colon \mathcal{T}op^{\mathrm{op}}_{\mathrm{pair}} \to \mathrm{Ab}^{\mathbb{Z}} \tag{B.0.2}$$

from some category of pairs of topological spaces to the category of $\mathbb{Z}$ graded Abelian groups together with **connecting homomorphisms** $\delta\colon E^\bullet(A,\emptyset) \to E^{\bullet+1}(X,A)$ such that the following properties hold:

- (Homotopy invariance) For pairs (X,X') and (Y,Y') homotopy equivalent as pairs, the homotopy equivalence induces an isomorphism $E^\bullet(X,X') \cong E^\bullet(Y,Y')$.
- (Exactness) The following sequence is exact

$$\cdots \to E^\bullet(X,A) \to H^\bullet(X,\emptyset) \to E^\bullet(A,\emptyset) \xrightarrow{\delta} E^{\bullet+1}(X,A) \to \cdots \tag{B.0.3}$$

- (Excision) For a pair of spaces (X,X') and $Y \subseteq X'$ such that closure$(Y) \subseteq$ interior(X') then the inclusion $X\backslash Y \to X$ induces an isomorphism

$$E^\bullet(X,X') \cong E^\bullet(X\backslash Y, X'\backslash Y). \tag{B.0.4}$$

▲

Remark B.2. In this definition, we work with "some category of topological spaces". The precise category is not relevant for our purposes. Sensible choices include the category of all topological spaces, CW complexes and smooth manifolds. ▽

Remark B.3. Instead of the excision axiom, one can equivalently assume that the Mayor-Vietoris sequence is exact. The Mayor-Vietoris sequence is the sequence

$$\cdots \to E^\bullet(X,X') \xrightarrow{(j_U^*, j_V^*)} E^\bullet(U,U') \oplus E^\bullet(V,V') \xrightarrow{(i_U^* - i_V^*)} E^\bullet(U\cap V, U'\cap V') \xrightarrow{\partial} E^{\bullet+1}(X,X') \to \cdots \tag{B.0.5}$$

The equivalence of excision and the Mayor-Vietoris sequence follows from the exactness axiom, see e.g., [Swi02, Theorem 7.19]. ▽

We list some examples of generalized cohomologies.

Example B.4. Singular cohomology is a generalized cohomology where the cohomology of the one point space is concentrated in the zero degree. △

Example B.5. For a compact Hausdorff space X consider the set of isomorphism classes of real/complex vector bundles over X. We can give this a monoid structure with the direct product of vector bundles. Now define $K^0(X)$ to be the Grothendieck group completion of this monoid.

$$K^0(X) = (\{\text{isomorphism classes of vector bundles over } X\}, \oplus)^{\mathrm{Grp}} \tag{B.0.6}$$

Using suspensions and Bott periodicity, [Hat03, Theorem 2.11], this can be turned into a cohomology theory $K^\bullet(X)$ called **K-Theory of X**. For details, see [Hat03, Chapter 2]. △

Example B.6. A third example of a generalized cohomology theory is *elliptic cohomology*. Here, one constructs the cohomology using the Landweber Exact Functor Theorem, [Lan76], from a *formal group law* coming from an elliptic curve. The construction is laid out in detail in [Mei22, Section 4]. A formal group law is a power series $F \in R[\![x_1, x_2]\!]$ of the form $F = x_1 + x_2 +$ higher order which is symmetric and associative in the variables.

In fact, the previous singular cohomology and K-theory can be recovered using the same method too. One only start with a different formal group law. The formal group law for singular cohomology is the additional law $F(x_1, x_2) = x_1 + x_2$. For K-theory one needs the multiplicative law $F(x_1, x_2) = x_1 + x_2 + ux_1x_2$.

In elliptic cohomology, there is a universal cohomology theory called tmf for *topological modular forms*. It can be obtained as a homotopy limit from all elliptic cohomologies. This cohomology, or at least its spectrum, is discussed in [Mei22, Section 5]. △

Generalized cohomology theories reduce to ordinary cohomology upon killing all torsion.

Proposition B.7 ([DAS72, Corollary 4])**.** *For any generalized cohomology theory $E^\bullet$ defined on the category of pairs of CW-complexes, there is a natural isomorphism*

$$E^\bullet \otimes \mathbb{Q} \cong H^\bullet \otimes E^\bullet(Pt) \otimes \mathbb{Q}. \tag{B.0.7}$$

Here, we write $H^\bullet$ for the ordinary cohomology with $\mathbb{Z}$ coefficients.

Proof. The statement is usually shown for homology usual dualization arguments give the same statement in cohomology. See [DAS72] for the original argument on finite CW-complexes. More generally see [Rud98, Theorem II.7.13]. □

This proposition highlights once more that the cohomology groups of the one point space is fundamental to the generalized cohomology theory. We will give the cohomologies of the point for the examples above.

Example B.8. A vector bundle over a point is just a vector space. Therefore, we know that $K^0(\mathrm{Pt}) = \mathbb{Z}$. Using Bott periodicity, one can show that $K^n(\mathrm{Pt}) = \mathbb{Z}$ for n even and $K^n(\mathrm{Pt}) = 0$ for n odd. △

Example B.9. The topological modular forms over the point with complex coefficients $tmf^\bullet(\mathrm{Pt}) \otimes \mathbb{C}$ can be identified with modular forms, [Mei22, page 60]. We obtain a natural isomorphism

$$tmf^{2\cdot\bullet}(\mathrm{Pt}) \otimes \mathbb{C} \cong \mathrm{MF}^\bullet. \tag{B.0.8}$$

△

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