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Master Thesis: Probability and Statistics

Solving Stochastic Optimal Control Problems Using Fully Coupled FBSDEs

and its applications to Pension Funds

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Abstract

This thesis explores the application of stochastic control techniques in a parameter study to examine the implications of climate taxes on a pension fund. The problem at hand involves modeling a pension fund comprising various assets, with a specific focus on the portfolio's emissions and the associated tax implications. This research aims to find the optimal allocation strategies for managing the pension fund, considering both the portfolio's performance and the impact of taxes. By utilizing stochastic control, we explore how various factors, such as risk preferences, tax regulations, and emission considerations, influence the optimal investment and consumption policies. To solve this complex problem, the BCOS method is employed to numerically tackle the fully coupled Forward-Backward Stochastic Differential Equations (FBSDEs) arising from the control problem. The BCOS method, known for its effectiveness in solving coupled FBSDEs, is applied to tackle the complexity of the problem. This numerical approach enables the exploration of different control parameters, allowing for a comprehensive parameter study. Numerical experiments are presented to illustrate the different outcomes by varying the parameters, thus revealing the dynamic nature of the solutions.

Keywords: Stochastic control, forward-backward stochastic differential equations, extended Merton portfolio problem, pension fund, emissions, taxes, BCOS method.

Contents

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Chapter 1

Introduction

Climate risks in finance

The global transition towards sustainability is gaining momentum as the urgency to address climate change becomes increasingly evident. The Paris Agreement [16], a landmark international accord, plays a pivotal role in catalyzing this transformative process. It sets forth a framework to combat climate change by limiting global warming to well below 2 degrees Celsius above pre-industrial levels and pursuing efforts to limit the temperature increase to 1.5 degrees Celsius. However, translating the Paris Agreement into tangible actions requires the establishment of benchmarks that can guide and influence various sectors, including the financial industry. In particular, pension funds, entrusted with managing long-term investments, face the challenge of aligning their portfolios with sustainability goals.

The consequences of CO2 emissions extend beyond the environmental realm and significantly impact the financial market in both the short and long term. In the short term, the financial markets experience increased volatility and drift as climate-related events, such as extreme weather conditions and policy changes, affect investor sentiment and market dynamics. These fluctuations introduce uncertainties and risks that need to be managed effectively. Looking towards the long term, it becomes evident that maintaining constant emission levels will lead to a growing divergence from sustainability benchmarks. This divergence can result in escalating penalties for companies falling short of emission reduction targets and the introduction of carbon taxes.

Pension funds, responsible for securing retirees' financial well-being, currently face a significant knowledge gap regarding the impact of climate change on their investment portfolios. Identifying "brown" and "green" companies and assessing their vulnerability becomes crucial in this context [38]. The question arises: Will green businesses ultimately prevail, and what are the implications for pension funds? To address this knowledge gap, this thesis aims to develop a dynamic model and objective function that capture the evolving dynamics of stocks, accounting for their "greenness" and its influence on their future trajectory. The goal is to optimize the portfolio's wealth over time, considering diversification and transition risks.

Supervisory institutions, such as the Dutch Central Bank (DNB), play a crucial role in overseeing the operations of pension funds. Understanding the implications of climate change on investment portfolios is of great interest to these institutions as they seek to ensure the long-term financial stability of pension funds. This research provides valuable insights into the dynamics of sustainable investing, the potential influence of the Paris Benchmark on portfolios, and the potential future implementation of CO2 taxes. By exploring these aspects, the DNB can refine its regulatory framework and guide pension funds towards sustainable investment strategies.

In conclusion, this thesis aims to bridge the knowledge gap regarding the effects of climate change on pension fund portfolios. By developing a dynamic model and objective function that account for the "greenness" of companies, optimal investment strategies can be determined. The research also explores the role of sustainability benchmarks, potential CO2 taxes, and the utility shift towards sustainable and ESG-rated investments. The findings have implications for supervisory institutions like the DNB, assisting them in guiding pension funds towards sustainable and resilient financial practices.

Mathematical description

We aim to develop a simplified model that represents a pension fund and examine the potential impact of climate risks and taxes. However, addressing this issue is not straightforward, as financial assets themselves do not emit emissions. Nevertheless, the companies associated with these assets do contribute to CO_2 emissions, which we will consider. To simplify the problem, we generalize the portfolio to include only a green stock and a brown stock, reflecting their sustainability levels based on the Paris Alignment Benchmark. The key question becomes: how can we determine an optimal investment strategy? To address such inquiries, we turn to the field of stochastic control, utilizing its techniques in conjunction with other research to solve this allocation problem.

Through the application of Feynman-Kac relations and Itô's lemma, a link exists between stochastic control problems and Forward-Backward Stochastic Differential Equations (FBSDEs). Solving one problem yields an immediate solution for the other. Leveraging this relationship and our understanding that complex FBSDEs can be solved using numerical methods, we can determine the optimal control process over time.

While we can solve simple stochastic control problems using BSDEs and the BCOS method, real-world problems often exhibit a degree of coupling that remains challenging to solve. Therefore, it becomes essential to extend these numerical methods to a general framework capable of solving control problems for given deterministic drift and volatility functions.

Our contribution to the existing literature lies in combining the techniques for solving BSDEs using the BCOS method with stochastic control problems, thereby facilitating the computation of numerical solutions for these control problems. Additionally, we propose a novel and straightforward portfolio problem that captures the "greenness" of the underlying fund. Our objective is to determine the optimal ratio between green and brown stocks while simultaneously optimizing the return rate, accounting for the investor's evolving preference for green considerations over time. This extends Merton's portfolio problem and incorporates the dynamic nature of ESG preferences.

Organisation of this thesis

The structure of this thesis is as follows: Following this introduction, Chapter 2 serves as an introduction to the essential theories necessary for building the foundation of this thesis. In Section 2.2, BSDEs and establish the existence and uniqueness of solutions under general assumptions regarding the underlying randomness. Additionally, we briefly discuss the extension to FBSDE and present two assumptions guaranteeing the existence and uniqueness of solutions. This chapter concludes with an exploration of stochastic control problems and their resolution methods, specifically the Hamilton-Jacobi-Bellman (HJB) equation and Pontryagin's maximum principle. Furthermore, we examine the connection between solving stochastic control problems and BSDEs, highlighting the feedback map for the control variable, which facilitates numerical solutions for these control problems.

The subsequent chapter, Chapter 3, delves into the discussion of various mathematical control models, [27], [14] and [10], within the context of stochastic control and BSDE theories. We explore how these models can be applied to pension funds and analyze their effectiveness. The chapter ends in the development of a novel control problem centered around the "greenness" of a pension fund portfolio and its corresponding return on investment. We introduce a constant contribution rate and subsequently incorporate a tax based on the Paris Alignment Benchmark to incentivize emission mitigation.

Chapter 4 introduces the BCOS method from [18], which serves as the numerical solver for the fully coupled FBSDEs. Additionally, a short statement about the error analysis is provided to evaluate the accuracy of the method.

Moving on, Chapter 5 presents the numerical results obtained from our study. We showcase the impact of different contribution values on investment strategies and explore the diverse effects of increasing the penalty.

Finally, in Chapter 6, we summarize our findings and draw conclusions from our work. We also suggest potential avenues for future research and investigation.

Chapter 2

Preliminary Theory

The field of stochastic analysis has emerged as an important and strong tool in the study of complex financial systems, with applications ranging from portfolio management to option pricing. Among the numerous stochastic models that have been proposed in the literature, the Backward Stochastic Differential Equation (BSDE), Forward-Backward Stochastic Differential Equation (FBSDE), and stochastic control theories have proven to be particularly useful.

The theory of BSDEs provides a powerful framework for modeling and analyzing a wide range of stochastic phenomena, including option pricing, risk management, and stochastic control. BSDEs can be viewed as a natural extension of the classical theory of stochastic differential equations, where the solution process evolves in a reverse time direction. This reverse-time evolution provides a natural interpretation of the problem, allowing for a deeper understanding of the underlying dynamics.

In recent years, the FBSDE theory has attracted significant attention in the financial mathematics community due to its ability to model complex systems that involve multiple agents, feedback loops, and interactions between different time scales. FBSDEs can be viewed as a natural generalization of BSDEs, where the forward dynamics of the system are also taken into account. This allows for a more complete and accurate modeling of the underlying dynamics, providing new insights into the behavior of complex systems.

Finally, the stochastic control theory provides a general framework for optimizing the behavior of a stochastic system subject to various constraints. This theory has found numerous applications in the financial industry, where it is used to model and optimize complex financial systems, such as portfolio management and option pricing.

In this chapter, we will first give some definitions and notations used in the construction of the theories and lemmas we need to use. Then we start with the formulation of the theories of BSDEs and FBSDEs. Next, we provide a comprehensive overview of stochastic control theories, including methods to solve these problems and establishing their connection with the aforementioned BSDE/FBSDE systems. In this regard, we draw heavily on the work of Pham [32] as a primary reference for this chapter. Key results will be used in later chapters to understand and decompose known control papers, construct our own control problem and eventually use numerical methods to solve this.

2.1 Preliminaries

2.1.1 Notations

In order to start this chapter, we first need to define certain notations and sets that will be used to build the different definitions and theorems necessary. Throughout this thesis, we are concerned with a filtered probability space $W = (W_t)_{0\leq t\leq T}$ be a standard d-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ is the natural filtration of W, and T is a fixed horizon. For $x \in \mathbb{R}^d$, the Euclidean norm is denoted as $|x|$ and $\langle x_1, x_2 \rangle$ as the inner product. Furthermore,

Definition 1 (Spaces)
We introduce the following notations,

- $\mathbb{S}^2(0,T)$*: set of real-valued progressively measurable processes* Y *such that,*

$$\mathbb{E}[\sup_{0\le t\le T} |Y_t|^2] < \infty$$

- $\mathbb{H}^2(0,T)^d$*: set of* $\mathbb{R}^d$*-valued progressively measurable processes* Z *such that,*

$$\mathbb{E}[\int_0^T |Z_t|^2] < \infty$$

- $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$*: the space of all* $\mathcal{F}$*-measurable random variables* $X : [0,T] \times \Omega \to \mathbb{R}$ *such that for all* $t \in [0,T]$*:* $\mathbb{E}[|X_t|^p < \infty]$ *for* $p \in [1, \infty)$*.*
- $C^0(\mathbb{T} \times \mathcal{O})$*: the space of all real-valued continuous functions* f *on* $\mathbb{T} \times \mathcal{O}$*. We mostly use* $\mathbb{T} = [0,T]$ *and* $\mathcal{O}$ *is an open set of* $\mathbb{R}$*.*
- $C^k(\mathcal{O})$*: the space of all real-valued continuous functions* f *on* $\mathcal{O}$ *with continuous derivatives up to order* k*.*
- $C^{1,2}([0,T) \times \mathcal{O})$*: the space of real-valued functions* f *on* $[0,T) \times \mathcal{O}$ *whose partial derivatives:* $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j}$*,* $1 \le i,j \le n$*, exist and are continuous on* $[0,T)$*. If these partial derivatives can be extended by continuity on* $[0,T] \times \mathcal{O}$*, we write* $f \in C^{1,2}([0,T] \times \mathcal{O})$*. We define similarly for* $k \ge 3$ *the space* $C^{1,k}([0,T) \times \mathcal{O})$*.*

We will provide a short introduction to strong solutions for stochastic differential equations. The assumptions required for establishing uniqueness and existence of the strong solutions will also play a part in later definitions and theorems. By studying the concept of strong solutions, we aim to lay the groundwork for a rigorous mathematical framework and explore its implications for further analysis within this thesis.

Strong solutions of SDE

We consider the following simple forward SDE in $\mathbb{R}$,

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.1.1}$$

where X is a 1-dimensional random process, the drift function $\mu(t,x,\omega)$ and the diffusion function $\sigma(t,x,\omega)$ are defined on $\mathbb{T} \times \mathbb{R} \times \Omega$ and take values in $\mathbb{R}$ and $\mathbb{R}^d$ respectively. We assume that for all ω, the functions $\mu(\cdot,\cdot,\omega)$ and $\sigma(\cdot,\cdot,\omega)$ are Borelian on $\mathbb{T} \times \mathbb{R}$ and for all $x \in \mathbb{R}$, the processes $\mu(\cdot,x,\cdot) =$ and $\sigma(\cdot,x,\cdot)$ are progressively measurable. We omit writing the ω variable in the future, and write $\mu(t,x)$ and $\sigma(t,x)$.

Definition 2
(Strong solution of SDE) A strong solution of the SDE starting at time t is a progressively measurable process X *s.t,*

$$\int_t^s |\mu(u, X_u)|du + \int_t^s |\sigma(u, X_u)|^2 du < \infty, \forall t \le s a.s.,$$

and the following relation holds true a.s.,

$$X_s = X_t + \int_t^s \mu(u, X_u)du + \int_t^s \sigma(u, X_u)dW_u, t \le s.$$

Note, that a strong solution of a SDE is a continuous process. Existence and uniqueness of a strong solution to the SDE is ensured by Lipschitz and linear growth conditions. There exist a constant K and a real-valued process κ such that $\forall t \in \mathbb{T}, \omega \in \Omega, x, y \in \mathbb{R}$ we have,

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|,$$
$$|\mu(t,x) + \sigma(t,x)| \leq \kappa_t + K|x|,$$

with

$$\mathbb{E}\left[\int_0^t |\kappa_u|^2 du\right] < \infty \forall \in \mathbb{T}.$$

A natural choice for κ is $\kappa_t = |\mu(t,0)| + |\sigma(t,0)|$.

Additional theorems We assume familiarity with basic stochastic calculus and results from Itô calculus, which will be utilized without further introduction. Furthermore, throughout the remainder of this chapter, we will rely on several widely recognized theorems from functional analysis and stochastic calculus, which will be stated without proofs. These theorems and a definition are indispensable for constructing and proving subsequent definitions and theorems, and their significance may not be immediately evident to the reader.

Definition 3 (Local martingale)
Let X be a càdlàg adapted process. We say that X is a local martingale if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\lim_{n\to\infty} \tau_n = \infty$ a.s. and the stopped process X^{τ_n} is a martingale for all n.

Theorem 1 (Banach's fixed point method)
Let (S,d) be a non-empty complete metric space. A mapping $F : S \to S$ is a constraction mapping if,

$$d(F(x), F(y)) \leq K d(x,f), \quad x, y \in S,$$

for some $0 \leq K < 1$. Then there exists a unique fixed point of F, $F(x^*) = x^*$.

Proof. See, [20]. □

Theorem 2 (Martingale Representation theorem)
Let $\xi \in L^2(\mathbb{R})$. Then there exists a unique $Z \in \mathbb{H}^2(0,T)^d$ such that,

$$\xi = \mathbb{E}(\xi) + \int_0^T Z_s dW s.$$

As a consequence for M_t, a square integrable martingale with respect to $\mathcal{F}_t$, there exists $Z \in \mathbb{H}^2(0,T)^d$ such that.

$$M_t = M_0 + \int_0^t Z_s dW_s.$$

Proof. See, [40]. □

Theorem 3 (Doob's inequality)
Let $X = (X_t)t \in [0,T]$ be a nonnegative submartingale or a martingale, càdlàg. Then, for any stopping time τ valued in $\mathbb{T}$, we have:

$$\mathbb{P}\left[\sup_{0\leq t\leq\tau} |X_t| \geq \lambda\right] \leq \frac{\mathbb{E}|X_\tau|}{\lambda}, \quad \forall \lambda > 0,$$
$$\mathbb{E}\left[\sup_{0\leq t\leq\tau} |X_t|\right]^p \leq \left(\frac{p}{1-p}\right)^p \mathbb{E}[|X_\tau|^p], \quad \forall p > 1.$$

Proof. See, [32] □

Theorem 4 (Burkholder-Davis-Gundy inequality)
For any $p > 0$, there exist universal constants $0 < c_p < C_p$ such that for all continuous martingales, $M = (M_t)_{t\in\mathbb{T}}$, and all stopping times τ valued in $\mathbb{T}$,

$$c_p\mathbb{E}\left[(\int_0^T |\sigma_t|^2 dt)^{\frac{p}{2}}\right] \leq \mathbb{E}\left[|\sup_{0\leq t\leq T}\int_0^t \sigma_s dW_s|^p\right] \leq C_p\mathbb{E}\left[(\int_0^T |\sigma_t|^2 dt)^{\frac{p}{2}}\right]$$

Proof. See, [40]. □

2.2 Backward stochastic differential equations

The theory of BSDEs was pioneered by Pardoux and Peng [28], but was first introduced by Bismut in 1973 for linear backward stochastic differential equations [7]. Since then, BSDEs have become an important area of research in stochastic analysis, and have been used to study a variety of problems, such as stochastic optimal control problems, option pricing, and risk management. In this section we provide a concise introduction to BSDEs, including a proof for their well-posedness.

2.2.1 General formulation

While a SDE is a nonlinear extension of the stochastic integration, a Backward SDE can be regarded as a nonlinear counterpart of the martingale representation theorem. Given $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, it induces a martingale $Y_t := \mathbb{E}[\xi|\mathcal{F}_t]$. By the martingale representation theorem, there exists a unique $Z \in \mathbb{H}^2(0,T)^d$ such that,

$$dY_t = Z_t dW_t, \quad \text{or equivalent,} \quad Y_t = \xi - \int_t^T Z_s dW_s.$$

This is a linear SDE where we have a fixed terminal condition $Y_T = \xi$, instead of a fixed initial condition, hence the name backward stochastic differential equation. It is important to emphasize that the solution to a BSDE consists of a pair of adapted random processes, namely (Y, Z). The presence of Z ensure the $\mathcal{F}$-measurability of Y. In the following chapter we consider the following definition of a nonlinear BSDE,

Definition 4 (BSDE)
A BSDE is written in differential form as,

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T = \xi, \tag{2.2.1}$$

or equivalently,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{2.2.2}$$

where we assume,

- *$\xi : \Omega \to \mathbb{R}^d$ is called the terminal condition and is a $\mathcal{F}_T$-measurable random variable.*
- *$f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is called the generator and is $\mathbb{F}$-measurable.*

Definition 5 (Solution to the BSDE)
A solution to the BSDE Eq. (2.2.1) is a pair $(Y, Z) \in \mathbb{S}^2(0,T) \times \mathbb{H}^2(0,T)^d$ satisfying Eq. (2.2.2).

Note that a BSDE cannot be considered as a time-reversed FSDE, because at time t the pair (Y_t, Z_t) is $\mathcal{F}_t$ measurable, indicating that the process does not possess knowledge of the terminal condition at that particular time. We will now present a classical proof for the existence and uniqueness of a BSDE. There exists multiple proofs in literature, we will present the proof of Pham given in [32].

Theorem 5 (Existence and Uniqueness of solution)
Given a pair (ξ, f) *satisfying the following assumptions, there exist a unique solution* (Y, Z) *which solves the BSDE of Eq.* (2.2.1)*,*

- $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$
- *f is uniformly Lipschitz in (y,z), i.e. there exists a positive constant C_f such that for all* $y, z, y', z' \in \mathbb{R}$*:*

$$|f(t, y, z) - f(t, y', z') \leq C_f(|y - y'| + |z - z'|), \quad dt \otimes d\mathbb{P} a.e.$$

- $f(\cdot, 0, 0) \in \mathbb{H}^2(0, T)$.

Proof. This proof is from [32] and is based on the Banach's Fixed Point as given in Theorem 1, for which we can argue that a defined contraction mapping has a unique fixed point. Let us consider the function Φ on $\mathbb{S}^2(0, T) \times \mathbb{H}^2(0, T)^d$, which is a Banach space, mapping $(U, V) \in \mathbb{S}^2(0, T) \times \mathbb{H}^2(0, T)^d$ to $(Y, Z) = \Phi(U, V)$ defined by,

$$Y_t = \xi + \int_t^T f(s, U_s, V_s)ds - \int_t^T Z_s dW_s. \tag{2.2.3}$$

This pair (Y, Z) is constructed as such: we first consider the martingale $M_t = \mathbb{E}[\xi + \int_0^T f(s, U_s, V_s)ds|\mathcal{F}_t]$, which is square integrable under the assumptions of Theorem 5. It is possible to apply the Martingale Representation Theorem as given in Theorem 2, which gives the existence and uniqueness of $Z \in \mathbb{H}^2(0, T)^d$ such that,

$$M_t = M_0 + \int_0^t Z_s dW_s.$$

We then define the process Y as,

$$Y_t := \mathbb{E}\left[\xi + \int_t^T f(s, U_s, V_s)ds|\mathcal{F}_t\right] = M_t - \int_0^t f(s, U_s, V_s)ds, \ 0 \leq t \leq T,$$

which satisfies Eq. (2.2.3). This can be found by using the definition of M and noting that $Y_T = \xi$,

$$\begin{aligned} Y_t &= M_t - \int_0^t f(s, U_s, V_s)ds \\ &= M_T - \int_t^T Z_s dW_s - \int_0^t f(s, U_s, V_s)ds \\ &= \xi + \int_0^T f(s, U_s, V_s)ds - \int_t^T Z_s dW_s - \int_0^t f(s, U_s, V_s)ds \\ &= \xi + \int_t^T f(s, U_s, V_s)ds - \int_t^T Z_s dW_s. \end{aligned}$$

Observe that by Doob's inequality Theorem 3 that,

$$\mathbb{E}[\sup_{0 \leq t \leq T} |\int_t^T Z_s dW_s|^2] \leq 4\mathbb{E}[\int_0^T |Z_s|^2 ds] < \infty.$$

Under the assumptions of Theorem 5, we can deduce that Y lies in $\mathbb{S}^2(0, T)$ and therefore Φ is a well-defined function from $\mathbb{S}^2(0, T) \times \mathbb{H}^2(0, T)^d$ into itself. With this information we see that (Y, Z) is a solution to Eq. (2.2.1) if and only if it is a fixed point to mapping Φ, which is the next step.

Let $(U,V),(U',V') \in \mathbb{S}^2(0,T)\times\mathbb{H}^2(0,T)^d$ and $(Y,Z)=\Phi(U,V),(Y',Z')=\Phi(U',V')$. Furthermore set $(\Delta U,\Delta V=(U-U',V-V'),(\Delta Y,\Delta Z=(Y-Y',Z-Z')$ and $\Delta f_t = f(t,U_t,V_t)-f(t,U'_t,V'_t)$. Applying Itô's formula to $g(s,\Delta Y_s)=e^{\beta s}|\Delta Y_s|^2$, for some $\beta>0$, results in,

$$\begin{aligned} d(e^{\beta s}|\Delta Y_s|^2) &= \left(\frac{\partial g}{\partial s} - \Delta f\frac{\partial g}{\partial \Delta Y}_s + \frac{|\Delta Z_s|^2}{2}\frac{\partial^2 g}{\partial \Delta Y_s^2}\right)ds + \Delta Z_s\frac{\partial g}{\partial \Delta Y_s}dW_s \\ &= e^{\beta s}(\beta|\Delta Y_s|^2 - 2\Delta Y_s\Delta f_s)ds + e^{\beta s}|\Delta Z_s|^2 ds - 2e^{\beta s}\Delta Y_s\Delta Z_s dW_s. \end{aligned}$$

Integrating both sides from 0 to T results in,

$$\begin{aligned} |\Delta Y_0|^2 = &-\int_0^T e^{\beta s}(\beta|\Delta Y_s|^2 - 2\Delta Y_s\Delta f_s)ds \\ &-\int_0^T e^{\beta s}|\Delta Z_s|^2 ds - \int_0^T 2e^{\beta s}\Delta Y_s\Delta Z_s dW_S, \end{aligned} \tag{2.2.4}$$

where we used $|\Delta Y_T|^2 = |Y_T - Y'_T|^2 = |\xi-\xi|^2 = 0$. Observe that,

$$\mathbb{E}\left[(\int_0^T e^{2\beta t}|Y_t|^2|Z_t|^2 dt)^{1/2}\right] \le \frac{e^{\beta t}}{2}\mathbb{E}\left[\sup_{0\le t\le T}|Y_t|^2 + \int_0^T |Z_t|^2 dt\right] < \infty,$$

which shows that the local martingale $\int_0^t e^{\beta t}Y_t Z_t dt$ is a uniformly integrable martingale with the use of the Burkholder-Davis-Gundy inequality as given in Theorem 4. Taking the expectation of both sides of Eq. (2.2.4) yields,

$$\begin{aligned} \mathbb{E}|\Delta Y_0|^2 + \mathbb{E}[\int_0^T e^{\beta s}(\beta|\Delta Y_s|^2 + |\Delta Z_s|^2)ds] &= 2\mathbb{E}[\int_0^T e^{\beta s}\Delta Y_s\Delta f_s] \\ \le 2C_f\mathbb{E}[\int_0^T e^{\beta s}|\Delta Y_s|(|\Delta U_s| + |\Delta V_s|)ds] \\ \le 4C_f^2\mathbb{E}[\int_0^T e^{\beta s}|\Delta Y_s|^2 ds] + \frac{1}{2}\mathbb{E}[\int_0^T e^{\beta s}(|\Delta U_s|^2 + |\Delta V_s|^2)ds]. \end{aligned}$$

If we choose $\beta = 1 + 4C_f^2$, we obtain,

$$\mathbb{E}[\int_0^T e^{\beta s}(|\Delta Y_s|^2 + |\Delta Z_s|^2)] \le \frac{1}{2}\mathbb{E}[\int_0^T e^{\beta s}(|\Delta Y_s|^2 + |\Delta Z_s|^2)ds].$$

This shows that Φ is a contraction on the Banach space $\mathbb{S}^2(0,T)\times\mathbb{H}^2(0,T)^d$ endowed with the norm,

$$||(Y,Z)||_\beta = \left(\mathbb{E}[\int_0^T e^{\beta s}(|\bar{Y}_s|^2 + |\bar{Z}_s|^2)ds]\right)^{\frac{1}{2}}.$$

This concludes the proof and Φ admits a unique fixed point, which in our case is the solution to the BSDE. □

It is well known that BSDEs are closely connected to partial differential equations (PDEs). We state the linear Feynman-Kac theorem from [32],

Theorem 6 (Linear Feynman-Kac theorem)
On a finite horizon interval we consider the Cauchy linear PDE problem,

$$\begin{aligned} \rho u(t,x) - \frac{\partial u}{\partial t}(t,x) - (\mathcal{L}u)(t,x) - f(t,x) &= 0 \\ u(T,x) &= g(x) \end{aligned}$$

with differential operator

$$\mathcal{L}u = \mu(t,x)D_x u + \frac{1}{2}Tr(\sigma(t,x)\sigma'(t,x)D_x^2 u).$$

Recall the X denotes the solution to the diffusion Eq. (2.1.1). *Assume that the solution of the PDE satisfies $u \in C^{1,2}$ and Eq.* (2.1.1) *admits a unique strong solution. Then u admist the representation,*

$$u(t,x) = \mathbb{E}[\int_t^T e^{-\int_t^s \rho(r,X_r)dr} f(s,X_s)ds + e^{-\int_t^T \rho(r,X_r)dr} g(X_T)|X_t = x]$$

2.3 Forward-Backward Stochastic Differential Equations

In the remaining thesis we will consider a special class of BSDEs, namely the Forward-Backwards Stochastic Differential Equations. As the name suggests, this system comprises both a forward and a backward Stochastic Differential Equation. In further sections of this thesis we are going to see that FBSDEs naturally arise when solving stochastic control problems. The randomness driving this system comes from the Brownian Motion of the forward SDE. Another way to view this, is to consider a FBSDE as a two-point boundary value problem for stochastic differential equations, with extra requirement that its solution is adapted solely to the forward filtration [26]. The system describing the forward and backward equation can be stated by the following equations,

$$X_t = X_0 + \int_0^t \mu(s,X_s,Y_s,Z_s)ds + \int_0^t \sigma(s,X_s,Y_s,Z_s)dW_s, \quad 0 \le t \le T, \tag{2.3.1}$$

$$Y_t = g(X_T) + \int_t^T f(s,X_s,Y_s,Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T. \tag{2.3.2}$$

One thing that stands out is that the solutions of both equations depends on each other. In the case of Eq. (2.3.1) and Eq. (2.3.2), we say that they are *fully coupled*, which means that X_t is allowed to couple into the driver f and Y_t and Z_t are allowed to couple into the drift μ and the diffusion/volatility σ. If σ does not depend on Z we shall say the system is coupled, and if μ and σ do not depend on (Y,Z) the system is said to be decoupled. This degree of coupling is directly linked to the complexity of solutions to the corresponding FBSDE. For decoupled FBSDE there exist already efficient numerical algorithms but in upcoming chapter we will see that it is possible to solve fully coupled FBSDE, mainly based on the work of [18].

The well-posedness of fully coupled FBSDEs is a complex work, for which their have been a lot of studies around solving these systems, [29] [25] [1] [31]. The following definition comes from Hu and Peng [17] in which they study the existence and uniqueness of the solution to forward-backward stochastic differential equations where the σ does depend on Z.

Definition 6
A triple of processes $(X,Y,Z) : [0,T] \times \Omega \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$ is called an adapted solution to Eq. (2.3.1) *and Eq.* (2.3.2), *if $(X,Y,Z) \in \mathbb{H}^2(0,T) \times \mathbb{H}^2(0,T) \times \mathbb{H}^2(0,T)^d$*

The adaptedness of the solution will enable us to rewrite our definition of the FBSDE system into a differential form,

$$\begin{aligned} dX_s &= \mu(s,X_s,Y_s,Z_s)ds + \sigma(s,X_s,Y_s,Z_s)dW_s, \\ dY_s &= -f(s,X_s,Y_s,Z_s)ds + Z_s dW_s, \\ X_0 &= x. \quad Y_T = g(X_T). \end{aligned}$$

We will now state the two main standing assumptions of [17]. These assumptions will ensure existence and uniqueness of a FBSDE over an arbitrarily prescribed time duration. The conditions for solving a stochastic control problem will be sufficient for us to find a solution to their corresponding FBSDE, such as quadratic growth and concavity. We will use the following notations, for $u = (x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, we write,

$$k(t, u) = (-f(t, u), \mu(t, u), \sigma(t, u)).$$

Assumption 1
For each $u \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, $k(\cdot, u) \in \mathbb{H}^2(0, T)^d$, $k(\cdot, u) \in \mathbb{H}^2(0, T) \times \mathbb{H}^2(0, T) \times \mathbb{H}^2(0, T)^d$ *and for each* $x \in \mathbb{R}, g(x) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$*; there exists a constant* $c_1 > 0$*, such that*

$$|k(t, u^1) - k(t, u^2)| \leq c_1|u^1 - u^2|, \quad a.s., t \in [0, T]$$
$$\forall u^1 \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d, u^2 \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$$

and

$$|g(x_1) - g(x_2)| \leq c_1|x_1 - x_2|, a.s.,$$
$$\forall x^1 \in \mathbb{R},\ x^2 \in \mathbb{R}.$$

Assumption 2
There exists a constant $c_2 > 0$*, such that,*

$$\langle k(t, u^1) - k(t, u^2), u^1 - u^2\rangle \leq -c_2|u^1 - u^2|^2, a.s., t \in [0, T],$$
$$\forall u^1 \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d, u^2 \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$$

and

$$\langle g(x^1) - g(x^2), x^1 - x^2\rangle \geq c_2|x^1 - x^2|^2, a.s.,$$
$$\forall x^1 \in \mathbb{R},\ x^2 \in \mathbb{R}.$$

Theorem 7
Let Assumption 1 and Assumption 2 hold, then there exists a unique adapted solution (X, Y, Z) *for Eq.* (2.3.1) *and Eq.* (2.3.2)*.*

Proof. See, [17] □

In the remaining part of the master's thesis, we will not go into detail about the proof of Theorem 7. However, we will make use of these results on the well-posedness of FBSDEs to advance our study.

2.4 Stochastic Control

In the following section we will treat the general theory of stochastic control and optimization. This is a sub-field of classical control theory that deals with subjection to random perturbations that drive a dynamical system. The system can be controlled in order to find a optimum for some performance criterion. The aim is then to design a path through time for the controlled variables that performs the desired control task which is found in a performance criterion. This field has become more prominent in areas where the decision-making problems have an underlying uncertainty. Among various fields we mention economics, biology and finance, with applications such as finding necessary and sufficient conditions for optimality in the mathematical modelling of the spreading of infectious diseases [39], and for the design of a optimal timing of greenhouse gas emission abatement[3].
In this thesis, we will primarily focus on finite horizon problems due to the specific nature of climate control problems. Finite horizon problems involve optimizing a system's behavior over a fixed time interval. This choice is motivated by the inherent characteristics of climate control, and the looming the terminal time of climate mitigation. By considering finite horizon problems, we can effectively

address the time-sensitive aspects of climate control, such as ensuring emission reduction within a given timeframe. Therefore, we will exclude discussions related to infinite horizon problems, which involve optimizing over an unbounded time horizon, as they are less applicable to the specific context of climate control.

2.4.1 Problem Description for a Finite Horizon

Let us define a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Consider a stochastic control model where the state of the system is governed by the following controlled SDE valued in $\mathbb{R}$,

$$dX_s = \mu(s, X_s, \alpha_s)ds + \sigma(s, X_s, \alpha_s)dW_s, \tag{2.4.1}$$

here the coefficient functions $\mu(s, x, a)$ and $\sigma(s, x, a)$ depend also on time, because of the finite horizon nature of the control problem. The control process $\alpha = (\alpha_s)_{s\leq 0}$ is a progressively measurable process, taking values in A, subset of $\mathbb{R}^m$. The set of all admissible control processes α is denoted by $\mathcal{A}$.

The measurable function $\mu : [0, T] \times \mathbb{R} \times A \to \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R} \times A \to \mathbb{R}^d$ satisfy uniform Lipschitz conditions in $A : \exists K \geq 0, \forall x, y \in \mathbb{R}, \forall a \in A, \forall t \in [0, T]$,

$$|\mu(t, x, a) - \mu(t, y, a)| + |\sigma(t, x, a) - \sigma(t, y, a)| \leq K|x - y|. \tag{2.4.2}$$

Let's fix a finite horizon $0 < T < \infty$, and assume,

$$\mathbb{E}\left[\int_0^T |\mu(t, 0, \alpha_s)|^2 + |\sigma(t, 0, \alpha_s)|^2 dt\right] < \infty. \tag{2.4.3}$$

In the above condition Eq. (2.4.3), the element $x = 0$ is an arbitrary value of the diffusion, but any arbitrary value, as long as it lies in the support of the diffusion, will do. As indicated in Section 2.1.1, the conditions Eq. (2.4.2) and Eq. (2.4.3) guarantee the existence and uniqueness of a strong solution to the Stochastic Differential Equation (SDE) denoted by ${X_s^{t,x}}_{t\leq s\leq T}$ with a.s. continuous paths. This strong solution holds true for all control variables $\alpha \in \mathcal{A}$ and for any given initial condition $(t, x) \in [0, T] \in \mathbb{R}^n$. Under these conditions on μ, σ and α, it implies that,

$$\mathbb{E}[\sup_{t\leq s\leq T} |X_s^{t,x}|^2] < \infty. \tag{2.4.4}$$

Functional objective: Let $f : [0, T] \times \mathbb{R} \times A \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two measurable functions. We assume the following,

1. g is lower-bounded
2. g satisfies a quadratic growth condition: $|g(x)| \leq C(1 + |x|^2), \forall x \in \mathbb{R}$, for some constant C independent of x.

For $(t, x) \in [0, T] \times \mathbb{R}$, we denote by $\mathcal{A}(t, x)$ the subset of controls α in $\mathcal{A}$ such that,

$$\mathbb{E}[\int_t^T |f(s, X_s^{t,x}, \alpha_s)|ds] < \infty,$$

and assume that $\mathcal{A}(t, x)$ is not empty $\forall(t, x) \in [0, T] \times \mathbb{R}$. We can then define the gain functional under the assumption of Section 2.4.1,

$$J(t, x, \alpha) = \mathbb{E}[\int_t^T f(s, X_s^{t,x}, \alpha_s)ds + g(X_T^{t,x})], \tag{2.4.5}$$

$\forall(t,x) \in [0,T] \times \mathbb{R}$ and α in $\mathcal{A}(t,x)$. The objective is to maximize this gain functional over all admissible control processes. We introduce the associated value function,

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} J(t,x,\alpha). \tag{2.4.6}$$

If $v(t,x) = J(t,x,\hat{\alpha}_t)$ for a given initial condition $(t,x) \in [0,T] \times \mathbb{R}$, we say that $\hat{\alpha} \in \mathcal{A}(t,x)$ is an optimal control. A special case of control process is if α is of the form $\alpha_s = a(s, X_s^{t,x})$ for some measurable function $a : [0,T] \times \mathbb{R} \to A$, then it is called a Markovian control.

Remark 1 (Quadratic growth condition)
When f satisfies a quadratic growth condition in x, i.e. $\exists C \in \mathbb{R}_{\geq 0}$ and a positive function $\kappa : A \to \mathbb{R}_{\geq 0}$ such that,

$$|f(t,x,a)| \leq C(1 + |x|^2) + \kappa(a), \quad \forall(t,x,a) \in [0,T] \times \mathbb{R} \times A,$$

then the estimate of Eq. (2.4.4) *shows that $\forall(t,x) \in [0,T] \times \mathbb{R}$, for any constant control $\alpha = a \in A$,*

$$\mathbb{E}\left[\int_t^T |f(s, X_s^{t,x}, a)| ds\right] < \infty.$$

Hence, set of constant controls in A lies also in $\mathcal{A}(t,x)$. Moreover, if there exists a positive constant C such that: $\kappa(a) \leq C(1 + |\mu(t,0,a)|^2 + |\sigma(t,0,a)|^2)$, then $\forall a \in A$, the conditions Eq. (2.4.3) *and Eq.* (2.4.4) *show that $\forall(t,x) \in [0,T] \times \mathbb{R}$, for any control $\alpha \in \mathcal{A}$,*

$$\mathbb{E}\left[\int_t^T |f(s, X_s^{t,x}, \alpha_s)| ds\right] < \infty.$$

Concluding, in this case we have that $\mathcal{A}(t,x) = \mathcal{A}$.

Remark 2
We focus only on finite horizon control problems in this paper. The infinite horizon version of a stochastic control problem is formulated as,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho s} f(X_s, \alpha_s) ds\right],$$

where $\rho > 0$ is a large enough positive discount factor to ensure finiteness of the associated value function.

Having stated the basis of the control problem, we are going to look at two different but intertwined approaches to find the optimal control and corresponding value function.

2.4.2 The Hamilton-Jacobi-Bellman approach

In this section we look into the different steps we make to obtain a way to solve a stochastic control problem. We begin by stating the dynamic programming principle (DPP), a fundamental building block in the control theory. Then we look into a specific version of this principle which results in the Hamilton-Jacobi-Bellman equation.

The dynamic programming principle was first developed by Richard Bellman's in the 1950s in the field of mathematical optimization problems [4]. The first applications were for deterministic dynamic systems like graphs. Eventually, Bellman extended this technique with systems guided by an underlying uncertainty [5]. The DPP for controlled diffusion processes is formulated as follows,

Theorem 8 (Dynamic programming principle)
Finite horizon: let $(t, x) \in [0, T] \times \mathbb{R}$. The value function Eq. (2.4.6) is given by,
(i) $\forall \alpha \in \mathcal{A}(t, x)$ and $\theta \in \mathcal{T}_{t,T}$,

$$v(t, x) \geq \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)ds + v(\theta, X_\theta^{t,x})\right].$$

(ii) $\forall \epsilon < 0$, there exists $\alpha \in \mathcal{A}(t, x)$ such that $\forall \theta \in \mathcal{T}_{t,T}$,

$$v(t, x) - \epsilon \leq \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)ds + v(\theta, X_\theta^{t,x})\right].$$

$\mathcal{T}_{t,T}$ denotes, for $0 \leq t \leq T \leq \infty$, the set of stopping times valued in $[t, T]$.

A result of the dynamic programming principle is the Hamilton-Jacobi-Bellman equation (HJB). The HJB describes the local behavior of the value function when the stopping time θ goes to t. In optimal control theory, it will give us necessary conditions for optimality of a control with respect to the driver of the functional.

We first derive the formulation of the HJB. Then we prove that, given a smooth solution to the HJB equation, the candidate coincides with the value function v.

Deriving the HJB equation

Following the steps of Pham [32], we consider an infinitesimal time frame, with stopping time $\theta = t + h$, and a constant control $\alpha_s = a$, for some arbitrary a in A. The first relation of the dynamic programming principle becomes,

$$v(t, x) \geq \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, a)ds + v(t + h, X_{t+h}^{t,x})\right]. \tag{2.4.7}$$

Assuming that v is smooth enough, we can apply Itô's formula on $v(t, X_t^{t,x})$ between t and $t + h$,

$$v(t + h, X_{t+h}^{t,x}) = v(t, x) + \int_t^{t+h} \frac{\partial v}{\partial t}(s, X_s^{t,x}) + \mathcal{L}^a v(s, X_s^{t,x})ds + \int_t^{t+h} \sigma(s, X_s^{t,x}, a)D_x v(s, X_s^{t,x})dW_s,$$

note that the latter component is a (local) martingale, and vanishes when taking the expectation over the expressions. The differential operator, $\mathcal{L}^a$, is associated to the diffusion process for the constant control a,

$$\mathcal{L}^a v = \mu(t, x, a)D_x v + \frac{1}{2}Tr(\sigma(t, x, a)\sigma'(t, x, a)D_x^2 v).$$

Substituting this into Eq. (2.4.7), the local martingale will vanish, and we are left with,

$$0 \geq \mathbb{E}\left[\int_t^{t+h} \frac{\partial v}{\partial t}(s, X_s^{t,x}) + \mathcal{L}^a v(s, X_s^{t,x}) + f(s, X_s^{t,x}, a)ds\right].$$

Applying the mean-value theorem, first dividing by h and sending $h \to 0$, yields,

$$0 \geq \frac{\partial v}{\partial t}(t, x) + \mathcal{L}^a v(t, x) + f(t, x, a).$$

Since this must hold true for any $a \in A$, we obtain the inequality,

$$-\frac{\partial v}{\partial t}(t, x) - \sup_{a \in A}[\mathcal{L}^a v(t, x) + f(t, x, a)] \leq 0. \tag{2.4.8}$$

Now suppose that $\hat{\alpha} \in \mathcal{A}(t,x)$ is an optimal control. Then,

$$v(t,x) = \mathbb{E}\left[\int_t^{t+h} f(s,\hat{X}_s,\hat{\alpha}_s)ds + v(t+h, X^*_{t+h})\right],$$

where we denote with $\hat{X}$ the state system solution to Eq. (2.4.1) starting from x at t, with the control $\hat{\alpha}$. Then by similar steps as above, we arrive at,

$$-\frac{\partial v}{\partial t}(t,x) - \mathcal{L}^{\hat{\alpha}_t} v(t,x) - f(t,x,\hat{\alpha}_t) = 0. \tag{2.4.9}$$

Combining Eq. (2.4.8) and Eq. (2.4.9) suggests that the value function v should satisfy,

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a\in A}[\mathcal{L}^a v(t,x) + f(t,x,a)] = 0, \;\; \forall (t,x) \in [0,T) \times \mathbb{R},$$

if the supremum in a is finite. Rewriting the PDE, gives us the HJB equation of the form,

$$-\frac{\partial v}{\partial t}(t,x) - H(t,x,D_x v(t,x), D_x^2 v(t,x)) = 0, \;\; \forall (t,x) \in [0,T) \times \mathbb{R}, \tag{2.4.10}$$

where the function H is called the Hamiltonian. The terminal condition associated to the PDE of Eq. (2.4.10) reads,

$$v(T,x) = g(x), \;\; \forall x \in \mathbb{R},$$

which follows immediately from the definition of the value function v.

Remark 3 (Optimality)
If there is a measurable function $\hat{\alpha}(t,x)$ that satisfies,

$$\sup_{a\in A}[\mathcal{L}^a v(t,x) + f(t,x,a)] = \mathcal{L}^{\hat{\alpha}(t,x)} v(t,x) + f(t,x,\hat{\alpha}(t,x)),$$

i.e.,

$$\hat{\alpha}(t,x) \in \arg\max_{a\in A}[\mathcal{L}^a v(t,x) + f(t,x,a)],$$

then using the optimality argument of the DPP, we would get,

$$-\frac{\partial v}{\partial t} - \mathcal{L}^{\hat{\alpha}(t,x)} v(t,x) - f(t,x,\hat{\alpha}(t,x)) = 0, \quad v(T,\cdot) = g(\cdot).$$

By the Feynman-Kac formula Theorem 6 this gives,

$$v(t,x) = \mathbb{E}\left[\int_t^T f(\hat{X}_s, \hat{\alpha}(s,\hat{X}_s))ds + g(X^*_T)\right],$$

where $\hat{X}$ is the solution of,

$$\begin{aligned} d\hat{X}_s &= \mu(s,\hat{X}_s,\hat{\alpha}(s,\hat{X}_s))ds + \sigma(s,\hat{X}_s,\hat{\alpha}(s,\hat{X}_s))dW_s, \quad t \le s \le T, \\ X^*_t &= x. \end{aligned}$$

This shows that the optimal control $\hat{\alpha}(s,\hat{X}_s)$ is an optimal Markovian control.

The results found for the finite horizon problem given a gain functional 2.4.5 can be extended to a more general form for J. Here we assume that the discount factor β is a measurable function on $[0,T] \times \mathbb{R} \times A$. The general form of the functional is,

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T B(t,s) f(s, X_s^{t,x}, \alpha_s)ds + B(t,T)\right]$$

with,

$$B(t,s) = \exp(-\int_t^s \beta(u, X_u^{t,x}, \alpha_u)du), \quad t \le s \le T.$$

The Hamiltonian associated with this general control problem is given by,

$$H(t,x,v,p,M) = \sup_{a\in A}[-\beta(t,x,a)v + \mu(t,x,a)p + \frac{1}{2}Tr(\sigma(t,x,a)\sigma'(t,x,a)M) + f(t,x,a)].$$

This result can be used to formulate control problems with non-constant discount rates, e.g. an interest rate changing with time, which will be necessary in the world of mathematical finance.

The next step is to confirm that a smooth solution to the HJB equation coincides with the value function, as a candidate solution for the HJB equation does not necessarily solve the primal stochastic control problem. The general verification theorem for finite time horizon from Pham [32] is stated as follows,

Theorem 9 (Verification theorem for Finite Horizon)
Let w be a function in $C^{1,2}([0,T) \times \mathbb{R}]) \cap C^0([0,T] \times \mathbb{R}])$, satisfying a quadratic growth condition, i.e. $\exists C \in \mathbb{R}_{\le 0}$ such that,

$$|w(t,x)| \le C(1+|x|^2), \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

(i) Suppose that,

$$-\frac{\partial w}{\partial t}(t,x) - \sup_{a\in A}[\mathcal{L}^a w(t,x) + f(t,x,a)] \ge 0, \quad (t,x) \in [0,T) \times \mathbb{R},$$
$$w(T,x) \ge g(x), \quad x \in \mathbb{R},$$

then $w \ge v$ on $[0,T] \times \mathbb{R}$.
(ii) Suppose that $w(T,\cdot) = g(\cdot)$, and there exists a measurable function $\hat{\alpha}(t,x)$ valued in A such that,

$$-\frac{\partial w}{\partial t}(t,x) - \sup_{a\in A}[\mathcal{L}^a w(t,x) + f(t,x,a)] = -\frac{\partial w}{\partial t}(t,x) - \mathcal{L}^{\hat{\alpha}(t,x)} w(t,x) - f(t,x,\hat{\alpha}(t,x))$$
$$= 0,$$

then the SDE,

$$dX_s = \mu(s, X_s, \hat{\alpha}(s,X_s))ds + \sigma(s, X_s, \hat{\alpha}(s,X_s))dW_s,$$

admits a unique solution, $X_s^{\hat{t,x}}$, given initial condition $X_t = x$, and the process $\{\hat{\alpha}(s, X_s^{\hat{t,x}})\}$ lies in $\mathcal{A}(t,x)$. Then,

$$w = v \ \ on \ \ [0,T] \times \mathbb{R},$$

and $\hat{\alpha}$ is an optimal (Markovian) control.

Proof. See [32]. □

Furthermore, we are interested in the link between the HJB solution and BSDEs. As we have noticed, the HJB equation Eq. (2.4.10) is a form of non-linear PDE. We look at the extension of the Feynman-Kac theorem for semilinear PDE in the form,

$$-\frac{\partial u}{\partial t}(t,x) - (\mathcal{L}u)(t,x) - f(t,x,u,\sigma D_x u) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \tag{2.4.11}$$

$$u(T,x) = g(x), \quad x \in \mathbb{R}^n. \tag{2.4.12}$$

We shall represent the solution to this PDE by means of the BSDE,

$$-dY_s = f(s, X_s, Y_s, Z_s)ds - Z_s dW_s, \quad t \leq s \leq T, \quad Y_T = g(X_T). \tag{2.4.13}$$

The following theorem will show that a classical solution to the semilinear PDE provides a solution to the BSDE.

Theorem 10 (Solution to the BSDE)
Suppose that $v \in C^{1,2}([0,T) \times \mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ *is a classical solution to Eq.* (2.4.11) *and Eq.* (2.4.12)*, satisfying a linear growth condition and for some constants* $C, q \in \mathbb{R}_{\geq 0}$ *that* $|D_x v(t,x)| \leq C(1 + |x|^q), \forall x \in \mathbb{R}$*. Then, there exists a pair* (Y, Z) *defined by,*

$$(Y_t, Z_t) = (v(t, X_t), D_x v(t, X_t)\sigma(t, X_t, \alpha_t)),$$

which is the solution to the BSDE Eq. (2.4.13)*.*

Proof. This follows from applying Itô's lemma to the value function $v(t, X_t)$,

$$\begin{aligned} dv(t, X_t) &= (v_t(t, X_t) + \mathcal{L}v(t, X_t))dt + \sigma(t, X_t)D_x v(t, X_t)dW_t \\ &= -f(t, X_t, v(t, X_t), \sigma(t, X_t)D_x v(t, X_t))dt + \sigma(t, X_t)D_x v(t, X_t)dW_t. \end{aligned}$$

□

A similar relation, as the one mentioned above, will also play a crucial role in the method we are about to discuss. We will exploit this relationship when utilizing a numerical approach to approximate the solution of BSDEs and determine the optimal control for the corresponding stochastic control problem.

2.4.3 The Maximum Principle Approach

Another approach to solving stochastic control problems is via the Pontryagin's maximum principle, also known as the Stochastic Maximum Principle (SMP), which is based on optimal conditions for the controls. The maximum principle was first formulated in 1956 by Pontryagin and his students [13] and can be considered a specialization of the Hamilton-Jacobi-Bellman equation. It states that an optimal state trajectory must solve a Hamilton system together with a maximum condition of the generalized Hamiltonian. The original version studied by Pontryagin was deterministic of nature. The stochastic control case was extensively studied in in the 1970s by Bismut [8], Kushner [21], Bensoussan [6] and Haussmann [15]. However their results did not incorporate a control in the diffusion coefficient. This hurdle was eventually taken by Peng [30]. He obtained a maximum principle for degenerate and control-dependent diffusion.

The proof of the maximum principle, as presented in [33], is notably more intricate compared to the derivation of necessary conditions for the Hamilton-Jacobi-Bellman (HJB) equation. The maximum principle establishes a crucial requirement that must be satisfied along an optimal trajectory. In contrast, the HJB approach computes a general value function that corresponds to the application of the optimal action, denoted as $\hat{\alpha}$. This computation eliminates the optimization in the Hamiltonian, but it also forfeits the global properties associated with the HJB equation. SMP optimizes along a trajectory, as opposed to the value function $v(t, x)$ over the whole state space. Therefore, it can only be at most local optimal in the space of possible trajectories. Given our specific interest in determining the conditions for optimal control, our preference lies with the SMP over the HJB equation.

Let us consider the same framework of a stochastic control problem with an finite horizon as defined in Section 2.4.1, in which we have a forward SDE Eq. (2.4.1) and gain functional Eq. (2.4.5). Furthermore we assume that the driver $f : [0, T] \times \mathbb{R} \times A \to \mathbb{R}$ is continuous in (t, x) for all a in A, $g : \mathbb{R} \to \mathbb{R}$ is a concave C^1 function and f, g satisfy both the quadratic growth condition in x,

$$\left.\begin{array}{ll} |f(t,x,a)| & \leq C_f(1 + |x|^2) + \kappa_f(a) \\ |g(x)| & \leq C_g(1 + |x|^2) \end{array}\right\} \forall (t,x,a) \in [0,T] \times \mathbb{R} \times A,$$

for some C_f and C_g independent of x. Then we define the generalized Hamiltonian $H : [0,T] \times \mathbb{R} \times A \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ by,

$$H(t,x,a,y,z) = \mu(t,x,a)y + Tr(\sigma'(x,a)z) + f(t,x,a), \tag{2.4.14}$$

and we assume that H is differentiable in x with derivative denoted by $D_x H$. We consider for each $\alpha \in \mathcal{A}$ the following BSDE, called the adjoint equation,

$$-dY_t = D_x H(t, X_t, \alpha_t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T = D_x g(X_T). \tag{2.4.15}$$

The following theorem and proof for optimality comes from [32],

Theorem 11 (Optimal Control)
Let $\hat{\alpha} \in \mathcal{A}$ and $\hat{X}$ the associated controlled diffusion. Suppose that there exists a solution $(\hat{Y}, \hat{Z})$ to the associated BSDE such that,

$$H(t, \hat{X}_t, \hat{\alpha}_t, \hat{Y}_t, \hat{Z}_t) = \max_{a\in A} H(t, \hat{X}_t, a, \hat{Y}_t, \hat{Z}_t), \quad 0 \le t \le T\, a.s., \tag{2.4.16}$$

and

$$(x,a) \to H(t,x,a,\hat{Y}_t,\hat{Z}_t) \quad \textit{is a concave function } \forall t \in [0,T]. \tag{2.4.17}$$

Then, $\hat{\alpha}$ is an optimal control, i.e.,

$$J(\hat{\alpha}) = \sup_{\alpha\in\mathcal{A}} J(\alpha).$$

Proof. For any $\alpha \in \mathcal{A}$, we can write,

$$J(\hat{\alpha}) - J(\alpha) = \mathbb{E}\left[\int_0^T f(t,\hat{X}_t,\hat{\alpha}_t) - f(t,X_t,\alpha_t)dt + g(\hat{X}_T) - g(X_T)\right]. \tag{2.4.18}$$

$$\mathbb{E}\left[g(\hat{X}_T) - g(X_T)\right] \ge \mathbb{E}\left[(\hat{X}_T - g(X_t))D_x g(\hat{X}_T)\right] = \mathbb{E}\left[(\hat{X}_T - X_T)\hat{Y}_T\right] \tag{2.4.19}$$

$$= \mathbb{E}\left[\int_0^T (\hat{X}_t - X_t)d\hat{Y}_t + \int_0^T \hat{Y}_t(d\hat{X}_t - dX_t) + \int_0^T Tr[\sigma(t,\hat{X}_t,\hat{\alpha}_t) - \sigma(t,X_t,\alpha_t)'\hat{Z}_t]dt\right]$$

$$= \mathbb{E}\Bigg[\int_0^T (\hat{X}_t - X_t)(-D_x H(t,\hat{X}_t,\hat{\alpha}_t,\hat{Y}_t,\hat{Z}_t))dt + \int_0^T \hat{Y}_t(\mu(t,\hat{X}_t,\hat{\alpha}_t) - \mu(t,X_t,\alpha_t))dt$$

$$+ \int_0^T Tr[(\sigma(t,\hat{X}_t,\hat{\alpha}_t) - \sigma(t,X_t,\alpha_t))'\hat{Z}_t]dt\Bigg].$$

Furthermore, by definition of the Hamiltonian Eq. (2.4.14), we have,

$$\mathbb{E}\left[\int_0^T f(t,\hat{X}_t,\hat{\alpha}_t) - f(t,X_t,\alpha_t)dt\right] = \mathbb{E}\Bigg[\int_0^T H(t,\hat{X}_t,\hat{\alpha}_t,\hat{Y}_t,\hat{Z}_t) - H(t,X_t,\alpha_t,\hat{Y}_t,\hat{Z}_t)dt \tag{2.4.20}$$

$$- \int_0^T (\mu(t,\hat{X}_t,\hat{\alpha}_t) - \mu(t,X_t,\alpha_t))\hat{Y}_t dt$$

$$- \int_0^T (Tr[(\sigma(t,\hat{X}_t,\hat{\alpha}_t) - \sigma(t,X_t,\alpha_t))'\hat{Z}_t])dt\Bigg].$$

By adding Eq. (2.4.19) and Eq. (2.4.20) into Eq. (2.4.18), yields,

$$J(\hat{\alpha}) - J(\alpha) \ge \mathbb{E}\Bigg[\int_0^T H(t,\hat{X}_t,\hat{\alpha}_t,\hat{Y}_t,\hat{Z}_t) - H(t,X_t,\alpha_t,\hat{Y}_t,\hat{Z}_t)dt$$

$$- \int_0^T (\hat{X}_t - X_t)H(t,\hat{X}_t,\hat{\alpha}_t,\hat{Y}_t,\hat{Z}_t)D_x\Bigg].$$

Because of the concavity condition of Eq. (2.4.17) we know that the above relation is always nonnegative, which concludes the proof. ☐

We conclude this section by providing the connection between maximum principle and dynamic programming. The value function of the stochastic control problem considered above is defined by

$$v(t,x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[\int_t^T f(s, X_s^{t,x}, \alpha_s)ds + g(X_T^{t,x})], \tag{2.4.21}$$

where $\{X_s^{t,x}, t \leq s \leq T\}$ is the solution to Eq. (2.4.1) starting from x at t. Recall that Hamilton-Jacobi-Bellman equation but write now instead

$$-\frac{\partial v}{\partial t} - \sup_{a \in A}[\mathcal{G}(t, x, a, D_x v, D_x^2 v] = 0.$$

Then the following theorem states the relationship betweeen the optimal control and the solution to the adjoint BSDE.

Theorem 12 (Solution to the adjoint BSDE)
Suppose that $v \in C^{1,3}([0,T) \times \mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$, *and there exists an optimal control* $\hat{\alpha} \in \mathcal{A}$ *to Eq.* (2.4.21) *with associated controlled process* $\hat{X}$. *Then,*

$$\mathcal{G}(t, \hat{X}_t, \hat{\alpha}_t, D_x v(t, \hat{X}_t), D_x^2 v(t, \hat{X}_t)) = \max_{a \in A} \mathcal{G}(t, \hat{X}_t, a, D_x v(t, \hat{X}_t), D_x^2 v(t, \hat{X}_t)),$$

and the pair,

$$(\hat{Y}_t, \hat{Z}_t) = (D_x v(t, \hat{X}_t), D_x^2 v(t, \hat{X}_t)\sigma(t, \hat{X}_t, \alpha_t)),$$

is the solution to the adjoint BSDE.

Proof. See, [32]. □

Definition 7 (Feedback Map)
If $\hat{\alpha}$ *is an optimal control satisfying Theorem 11 and there exists a measurable function* $\delta : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to A$, *then*

$$\hat{\alpha} := \delta(t, x, y, z). \tag{2.4.22}$$

With this definition we can transform the adjoint equation of Eq. (2.4.15) to a fully coupled FBSDE. We use the concavity of the defined generalized Hamiltonian to ensure uniqueness of this mapping. Using Eq. (2.4.22) we transform the forward SDE into,

$$\begin{aligned} dX_s &= \mu(X_s, \delta(s, X_s, Y_s, Z_s))ds + \sigma(X_s, \delta(s, X_s, Y_s, Z_s))dW_s \\ &:= \bar{\mu}(s, X_s, Y_s, Z_s))ds + \bar{\sigma}(s, X_s, Y_s, Z_s)dW_s, \end{aligned}$$

and the adjoint BSDE becomes,

$$\begin{aligned} -dY_s &= D_x H(s, X_s, \delta(s, X_s, Y_s, Z_s))ds - Z_s dW_s, \quad Y_T = D_x g(X_T) \\ &:= D_x \bar{H}(s, X_s, Y_s, Z_s)ds - Z_s dW_s, \quad Y_T = D_x g(X_T) \end{aligned}$$

The end result is a FBSDE system with a coupling determined by the feedback mapping. In most cases, especially in situations which are not designed as toy models, we get a fully coupled FBSDE. In general there is no available explicit solution to this system. However in [18] they propose an extension of the BCOS method for coupled forward-backwards stochastic differential equations. This we can use for our gain to solve complex stochastic control problems and will be discussed in Chapter 4.

Chapter 3

Finding a fitting model

In order to construct our own model that focuses on emission rates of the underlying companies of a stock while still maximizing the overall wealth of the portfolio, we will need to address three key points in the mechanics of stochastic systems. Firstly, we will need to understand the dynamics of portfolios consisting of stocks, bonds, and riskless assets, such as cash. We will need to consider how these different asset classes behave under various market conditions and how their interaction affects portfolio returns. Secondly, we will need to investigate how DC pension funds behave in the market and how to eventually incorporate some degree of dependency on climate risks. Lastly, we will need to examine the relationship between emission levels and the total wealth of a firm or portfolio, which can be reflected in the relationship between emission levels and production rates, as well as in the discrepancy of a firm's mitigation rate with respect to the Paris Alignment Benchmark. By answering these questions, we will be able to build our own model that focuses on the emission rates of underlying stocks and determines the best allocation between two stocks with different emission rates, while still maximizing the overall wealth of the portfolio. For this, we have selected three insightful papers, aiming to thoroughly examine and comprehend their contents to extract the necessary insights and knowledge applicable to our research objective.

We begin this chapter by devoting to the notion of utility functions. That is a concept which appears often in literature when looking at control problem in finance. Then we look into the classical famous Merton's portfolio problem [27]. Thereafter, we examine a paper of Gao [14], in which he studies the portfolio problem of a pension fund manager with stochastic interest rates. Next, we investigate a paper from Gobet et al. [10] where they investigate the impact of transition risk on a firm's low-carbon production. Lastly, we are going to design a new control problem where the aim is to find, with a given utility, a optimal strategy regarding climate risks.

3.0.1 Utility functions

Utility functions play a crucial role in finance and economics, as they provide a way to quantify how much an individual values different outcomes or states of the world. The choice of a specific utility function is crucial, as it determines the optimal decision-making process for an individual, given their preferences and the constraints they face.

One popular class of utility functions is found in the constant relative risk (CRR) function. The CRR function assumes that individuals exhibit constant relative risk aversion (CRRA), which means that they are more averse to losses than they prefer gains, and the degree of aversion decreases as wealth increases. The class of CRRA utility functions can be expressed as,

$$\mathcal{U}(c) = \begin{cases} \frac{c^{1-\eta}-1}{1-\eta} & \text{if } \eta \geq 0, \eta \neq 1, \\ \log(c) & \text{if } \eta = 1, \end{cases}$$

where $\mathcal{U}(c)$ is the utility of consuming c units of a good or service, and η is the coefficient of relative risk aversion. The CRR function has been widely used in finance and economics, due to its tractability and intuitive interpretation. It is well known that pension funds are in general large companies who define their strategies with respect to the amount of money they are managing, approximately in a scaling way [14]. Moreover, the investment period is very long, generally from 20 to 40 years. Therefore, we consider the case where the pension fund manager will focus on the optimal growth portfolio. According to [22] and [9], there exists one logarithmic utility function which outperforms any other utility function in the long run in terms of growth of wealth. Hence, in this paper, we describe the pension fund manager's objective with a logarithmic utility function, However, the use of utility functions in economic models requires certain assumptions to hold. One such assumption is found in the Inada conditions, which ensure that the marginal utility of consumption is positive and approaches infinity as the consumption goes to zero, and vanishes as the consumption goes to infinity. Specifically, the Inada conditions require that,

$$\lim_{c\to 0}\mathcal{U}'(c) = \infty \quad \text{and} \quad \lim_{c\to\infty}\mathcal{U}'(c) = 0.$$

These conditions are important, as they ensure that individuals exhibit diminishing marginal utility of consumption, which is a reasonable assumption given the law of diminishing returns.

In conclusion, utility functions are a critical component of economic models, as they provide a way to capture individual preferences and decision-making processes. The CRR function, in particular, has been widely used in finance and economics, due to its simplicity and flexibility.

3.1 Merton's Portfolio Problem

The allocation of assets in a portfolio is a well-known problem in stochastic control theory. One of the most widely studied portfolio allocation problems is the Merton Portfolio Problem, which was first introduced by Merton in 1969 [27]. The objective of the Merton Portfolio Problem is to determine the optimal investment strategy for an investor who seeks to maximize the expected lifetime utility of the portfolio wealth, subject to constraints on the wealth and risk tolerance. The model assumes that the investor has a CRRA utility function and faces a market consisting of one risk-free asset and a single risky asset that follows a general geometric Brownian motion process. The problem is of great practical significance as it provides a framework for investors to make informed decisions about asset allocation in their investment portfolios.

3.1.1 Problem formulation

Let us consider a stochastic model for a portfolio consisting of one risk-free asset $S^0 = (S^0_t)_{t\geq 0}$ with prices evolving as,

$$dS^0_t = S^0_t r_t dt,$$

and one risky asset $S = (S_t)_{t\geq 0}$, whose dynamics are given by,

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{3.1.1}$$

where W is an 1-dimensional Brownian motion, and the coefficients $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}$. Both the drift and diffusion of Eq. (3.1.1) are progressively measureable 1-dimensional processes. The control of the problem is $\alpha = (\alpha_t)_{t\in[0,T]}$, where α_t represents the fraction of wealth invested in the risky asset and $1 - \alpha_t$ in the riskless asset at time t. The wealth process of such portfolio at time t is denoted by,

$$\begin{aligned} dX_t &= X_t(1-\alpha_t)\frac{dS^0_t}{S^0_t} + X_t\alpha_t\frac{dS_t}{S_t}, \\ &= ((r + \alpha_t(\mu - r))X_t)dt + X_t\alpha_t\sigma dW_t. \end{aligned}$$

For the gain functional we will use the logarithmic utility function for the investor and apply a discounting factor $e^{-\rho t}, \rho \geq 0$. Thus, we want to maximize,

$$J(\alpha) = \mathbb{E}[\int_0^T e^{-\rho s} \ln(X_s) ds + e^{-\rho T} \ln(X_T)].$$

3.1.2 General case with Pontryagin

Instead of using the HJB equation which is traditionally used, we look at the SMP route because in our research we are mostly interested in the optimal control variable. Through the use of SMP, the general Hamiltonian of this problem is given by,

$$H(t, X_t, \alpha_t, c_t, Y_t, Z_t) = ((r + \alpha_t(\mu - r))X_t)Y_t + X_t \alpha_t \sigma Z_t + e^{-\rho t} \ln(X_t).$$

As we can see this expression is linear in the control variable. So conditions Eq. (2.4.16) and Eq. (2.4.17) will be satisfied if and only if,

$$(\mu - r)\hat{Y}_t + \sigma \hat{Z}_t = 0, \tag{3.1.2}$$

as we can see, maximizing the Hamiltonian does yield a feedback map Eq. (2.4.22) for the control variable, but this map is constant and the optimal control variable is not dependent on Y or Z. We know the terminal conditions for Y_t and Z_t for SMP. Therefore, we are looking for a candidate solution in the the form,

$$\hat{Y}_t = D_x\big(\phi(t) \ln(\hat{X}_t)\big) = \phi(t)\frac{1}{\hat{X}_t},$$

for some positive function ϕ. This gives us immediately a candidate solution for Z_t,

$$\begin{aligned} \hat{Z}_t &= \sigma(t, \hat{X}_t, \alpha_t) D_x^2\big(\phi(t) \ln(\hat{X}_t)\big) \\ &= -\hat{X}_t \alpha_t \sigma \phi(t) \frac{1}{\hat{X}_t^2} = -\frac{\alpha_t \sigma \phi(t)}{\hat{X}_t}. \end{aligned}$$

Together with Eq. (3.1.2) we get an explicit value for the control variable, i.e.,

$$\hat{\alpha_t} = \frac{\mu - r}{\sigma^2}.$$

3.1.3 Take away

Having laid the foundation with the Merton portfolio problem is beneficial for our objective of incorporating emission strategies into portfolio allocation problems. The Merton portfolio problem explores dynamic portfolio optimization in continuous time, aiming to maximize the expected utility of an investor's terminal wealth. It takes the form of a stochastic control problem, and the associated paper presents the necessary optimality conditions for this problem. One key takeaway is that the solution to the Merton portfolio problem provides a closed-form expression for the optimal investment policy. This policy is influenced by the investor's initial wealth and risk aversion.

Furthermore, the Merton portfolio problem highlights the crucial trade-off between risk and return in investment decision-making. It emphasizes the need for investors to carefully balance these factors in their investment strategies. This trade-off is of particular relevance to our study as we aim to evaluate the trade-off between a basic green and brown asset.

3.2 Optimizing a DC pension fund framework

The world of pension funds can be broadly classified into two main types: the defined benefit (DB) scheme and the defined contribution (DC) scheme. In a DB scheme, the benefits are fixed in advance, and the contributions are adjusted in the future to maintain the fund's balance. On the other hand, in a DC scheme, the contributions are fixed, and the benefits depend on the returns on the fund's portfolio. Currently, the Dutch government and especially the DNB are working on a transition from a DB to a DC framework. However, the financial risk in a DC plan lies with the contributors. Hence, for contributors to receive a good pension at retirement, a successful DC scheme with a sound investment strategy is essential. One way to find such optimal strategies is through stochastic control theory, as first applied by Merton in his portfolio problem.

In this section, we focus on the research of Gao [14], where he extended the portfolio problem by considering stochastic interest rates instead of constant rates. Since the contribution period in a pension fund is generally more than 20 years, assuming constant rates for DC plans is not realistic. We first introduce the model and its dynamics, followed by stating the optimal control problem. Finally, we attempt to find an analytical solution using Legendre transforms.

It will be interesting to explore a stochastic pension fund framework that can provide insight and enables us to extract elements to construct our own pension fund model. However, we are unable to utilize stochastic interest rates, as done in the work of Gao. Instead, we will use constant interest rates, which is worth noting, but will not pose any problems for our analysis.

3.2.1 Problem formulation

As in the cases treated above, many other studies assume that the short rate is a constant. In practice the contribution rate of pensions funds is generally long, for instance 20+ years. Therefore it is preferred to look at stochastic interest rates, for example the Cox-Ingersoll-Ross (CIR) model or Vašiček model [14]. We assume a framework for a simplified pension fund portfolio composed of three different kinds of assets and add stochasticity to the interest rate. First, the riskless asset, i.e. cash, is discussed. The price at time t evolves according to,

$$dS_t^0 = r_t S_t^0 dt, \quad S_0^0 = 1,$$

where the short rate process r_t in the general form is given by,

$$dr_t = (a - br_t)dt - \sqrt{k_1 r_t + k_2} dW_t^r, \quad t \geq 0,$$

with the coefficients $a, b, r_0, k_1, k_2 \in \mathbb{R}_{\geq 0}$. Note that setting $k_2 = 0$ gives back the Vašiček model and $k_1 = 0$ the CIR model.
The two risky assets in [14] are the stock, $\{S_t\}_{t\geq 0}$, and zero-coupon bond $\{B_t^T\}_{t\geq 0}$. Their dynamics are given by

$$\begin{cases} dS_t = S_t \left[r_t dt + \sigma_1 (dW_t^s + \lambda_1 dt) + \sigma\sqrt{k_1 r_t + k_2}(dW_t^r + \lambda_2 \sqrt{k_1 r_t + k_2} dt) \right], & S_0 = 1 \\ dB_t^T = B_t^T \left[r_t dt + \sigma_B(T-t, r_t)(dW_t^r + \lambda_2 \sqrt{k_1 r_t + k_2} dt) \right], & B_T^T = 1, \end{cases}$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \mathbb{R}_{\geq 0}$. The volatility of the bond is defined as,

$$\sigma_B(T-t, r_t) = h(T-t)\sqrt{k_1 r_t + k_2},$$

with

$$h(t) = \frac{2(e^{mt} - 1)}{m - (b - k_1\lambda_2) + e^{mt}(m + b - k_1\lambda_2)},$$
$$m = \sqrt{(b - k_1\lambda_2)^2 + 2k_1}.$$

We assume that the contribution rate, $c(t)$, is a constant. The composition of the pension fund portfolio comprises of these three key assets, namely cash, stocks, and bonds, alongside the contribution rate. Let $(X_t)_{t\in[0,T]}$ be the pension wealth, and its corresponding dynamics are given by,

$$dX_t = \alpha_t^r \frac{dS_t^0}{S_t^0} + \alpha_t^S \frac{dS_t}{S_t} + \alpha_t^B \frac{dB_t^T}{B_t^T} + c_t dt,$$

where α_t^r, α_t^S and α_t^B are, respectively, the amount of money invested in the riskless asset, stocks and bonds.

3.2.2 The optimal control problem

The value function for the stochastic optimal control problem is defined as,

$$v(t,r,x) = \sup_{\alpha_S,\alpha_B} \mathbb{E}(\mathcal{U}(X_T^{r,x})), \quad 0 < t < T.$$

Note that the value function now also depends on the interest rate due to its stochastic nature. The associated Hamilton-Jacobi-Bellman equation leads to,

$$\begin{aligned} & v_t + a(b-r)v_r + [xr + (\lambda_1\sigma_1 + \lambda_2\sigma_2\sigma_r^2)\alpha_s + \lambda_2\sigma_B\sigma_r\alpha_B + c_t]v_x \\ & + \frac{1}{2}[\sigma_1^2\alpha_s^2 + (\sigma_B\alpha_B + \sigma_2\sigma_r\alpha_S)^2]v_{xx} + \frac{\sigma_r^2}{2}v_{rr} - (\sigma_B\sigma_r\alpha_B + \sigma_2\sigma_r^2\alpha_S)v_{rx} = 0. \end{aligned} \tag{3.2.1}$$

We want to get the first-order maximization conditions for the optimal strategies α_S and α_B. Maximizing the Hamiltonian of Eq. (3.2.1) with respect to α_S and α_B gives,

$$[\lambda_1\sigma_1 + \lambda_2\sigma_2\sigma_r^2]v_x + [\sigma_1^2\alpha_S + \sigma_B\alpha_B\sigma_2\sigma_r + \sigma_2^2\sigma_r^2\alpha_S]v_{xx} - \sigma_2\sigma_r^2 v_{rx} = 0,$$

and for α_B,

$$[\lambda_2\sigma_B\sigma_r]v_x + [\sigma_B^2\alpha_B + \sigma_B\sigma_2\sigma_r\alpha_S]v_{xx} - \sigma_B\sigma_r v_{rx} = 0.$$

These expressions result in the following maximum values for the control variables,

$$\hat{\alpha}_S = \frac{\sigma_2\sigma_r^2 v_{rx} - [\lambda_1\sigma_1 + \lambda_2\sigma_2\sigma_r^2]v_x - \sigma_B\alpha_B\sigma_2\sigma_r v_{xx}}{(\sigma_1^2 + \sigma_2^2\sigma_r^2)v_{xx}}, \tag{3.2.2}$$

$$\hat{\alpha}_B = \frac{-\lambda_2\sigma_r v_x + \sigma_r v_{rx} - \sigma_2\sigma_r\alpha_S v_{xx}}{\sigma_B v_{xx}}. \tag{3.2.3}$$

Substituting Eq. (3.2.3) into Eq. (3.2.2) and subsequently reversing the substitution, gives us the optimal strategies $\hat{\alpha}_S$ and $\hat{\alpha}_B$,

$$\hat{\alpha}_S = -\frac{\lambda_1}{\sigma_1}\frac{v_x}{v_{xx}} \quad \text{and} \quad \hat{\alpha}_B = \frac{\sigma_r(\lambda_1\sigma_2 - \lambda_2\sigma_1)v_x + \sigma_1\sigma_r v_{rx}}{\sigma_1\sigma_B v_{xx}}.$$

Putting everything together, we obtain a partial differential equation for the value function v,

$$v_t + (a - br)v_r + \frac{\sigma_r^2}{2}v_{rr} + [xr + c_t]v_x - \frac{\lambda_1^2}{2}\frac{v_x^2}{v_{xx}} - \frac{(\lambda_2\sigma_r v_x - \sigma_r v_{rx})^2}{2v_{xx}} = 0. \tag{3.2.4}$$

We notice that this problem is difficult to solve because it is a semi-linear second order partial differential equation.

3.2.3 Analytical solution

In the paper, Gao uses Legendre transforms and dual theory to transform a non-linear second partial differential equation to a linear partial differential equation. We will not go into detail in the theories and notions used but rather comment on the end results. For the interested reader we refer to [14]. After applying the Legendre transform and turning the primary problem into a dual problem, we end up with the following partial differential equation,

$$
\begin{aligned}
&g_t + a(b-r)g_r - rg - c_t - \lambda_2^2 - \sigma_r^2 g_r - rzg_z + \frac{\sigma_r^2}{2} g_{rr} \qquad (3.2.5)\\
&- (\lambda_2\sigma_r^2 - \lambda_1^2)zg_z - \frac{1}{2}(\lambda_2^2\sigma_r^2 - \lambda_1^2)z^2 g_{zz} - \lambda_2^2\sigma_r^2 z g_{rz} = 0.
\end{aligned}
$$

Notice that the non-linear second-order partial differential equation of Eq. (3.2.4) has been transformed into a linear partial differential equation Eq. (3.2.5). Going back to our optimal strategies α_B and α_S, expressing these in terms of the dual function g, yields,

$$\alpha_B = \frac{\sigma_r[(\lambda_1\sigma_2 - \lambda_2\sigma_1)zg_z + \sigma_1 g_r]}{\sigma_1\sigma_B} \quad \text{and} \quad \alpha_S = -\frac{\lambda_1}{\sigma_1} z g_z. \qquad (3.2.6)$$

The remaining problem is now to solve the linear partial differential equation Eq. (3.2.5) for g and inserting it into Eq. (3.2.6),to end up with the optimal strategies. Eventually, the solutions for these optimal strategies are given by,

$$\hat{\alpha}_t^B = \frac{\sigma_r(\lambda_2\sigma_1 - \lambda_1\sigma_2)X_t}{\sigma_1\sigma_B} - \sigma_r c_t \left[\frac{(\lambda_1\sigma_2 - \lambda_2\sigma_1)\frac{1-e^{-r(T-t)}}{r}}{\sigma_1\sigma_B} + \frac{\frac{1-e^{-r(T-t)}}{r} - (T-t)e^{-r(T-t)}}{r\sigma_B} \right],$$

and

$$\hat{\alpha}_t^S = \frac{\lambda_1}{\sigma_1}\left[X_t + c_t \frac{1-e^{-r(T-t)}}{r} \right].$$

The optimal strategy invested in the riskless asset is then,

$$\hat{\alpha}_t^r = X_t - \hat{\alpha}_t^B - \hat{\alpha}_t^S.$$

3.2.4 Take away

We learned about a dynamic control problem for a DC pension fund in which the contribution rate and investment policy are chosen to maximize expected utility of the pension's holder. One key result is the explicit solution for a special case where the investment returns are i.i.d. and follow a normal distribution. The solution provides a closed-form expression for the optimal investment allocations, which depend on the pension fund's current wealth and the stochastic interest rate.

While the paper considers a more general case where the interest rate is also stochastic, we will not focus on it in this master thesis as it is nontrivial to solve two-dimensional control problems with numerical methods. Nonetheless, the paper highlights the importance of taking into account the randomness of investment returns and interest rates in the pension fund management, and provides a solid foundation for future research in the area.

3.3 The impact of emission levels on the production profit

In the search of a model that shows a correlation between CO2 emissions and return of assets, we studied a paper from Gobet et al. [10]. The authors were interested in the impact of transition risk on a firm's low-carbon production. Due to immanent climate change and its irreducible effects, the Intergovernmental Panel on Climate Change (IPCC) has set the idealized carbon-neutral scenario around 2050. Gobet et al. aimed to find the optimal emission path for a production company while maximizing its production profit and respecting emission mitigation scenarios. These scenarios are projections of socioeconomic global changes up to 2100, known as Share Socioeconomic Pathways (SSPs) [34]. Solving this optimization problem provides the optimal emission according to a given SSP benchmark.

To start off, we first give the mathematical problem formulation. To solve this stochastic control problem we will use Pontryagin's maximum principle Section 2.4.3, and extend this work to finite horizons which we are more interested in. Then we show how to solve this problem in the general case. Lastly we look at an explicit example and retrieve the optimal emission strategy.

3.3.1 Problem formulation

To begin, let us reiterate a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a natural filtration generated by 1-dimensional Brownian motion $W = (W_t)_{t\geq 0}$. We consider a firm whose production process, $P = (P_t)_{t\geq 0}$, depends on the energy consumption and in particular on its effective CO_2-emission rate $\gamma = (\gamma_t)_{t\geq 0}$, i.e. the control variable. This production process then solves the following SDE,

$$dP_t = P_t(\mu(t, P_t, \gamma_t)dt + \sigma dW_t), \tag{3.3.1}$$

where σ is a positive constant volatility parameter. The function $\mu : (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is the drift characterizing the production rate and satisfies the local Lipschitz condition on x, i.e., $\forall t \geq 0$,

$$\forall y \in \mathbb{R}, \forall x, x' \in \mathbb{R}_+, \quad |\mu(t, x, y) - \mu(t, x', y)| \leq K|x - x'|,$$

for a positive constant K.

The firm's goal is to maximize its production profit against the losses and costs of its excessive emission level with respect to the SSP benchmark. The stochastic control problem is then as follows: given a benchmark emission projection, a firm needs to choose its optimal emission pathway to maximize the expected discounted profit while controlling the related production π, costs $\mathcal{C}$ and losses ℓ. The objective gain functional is,

$$J(\gamma) := \mathbb{E}[\int_0^T e^{-rt}(\pi(P_t) - \mathcal{C}(\gamma_t) - \ell(\gamma - e_t))dt],$$

where $r > 0$ is the constant discount rate, and we aim to solve

$$\hat{J} = \sup_{\gamma \in \mathcal{A}} J(\gamma), \tag{3.3.2}$$

where $\mathcal{A}$ is the set of all admissible controls for the positive progressively measurable processes γ such that for some $\eta \in (0, r)$,

$$\mathbb{E}[\int_0^\infty e^{-\eta t}\gamma_t^2 dt] < \infty.$$

To guarantee existence and uniqueness and economic applicability we will need some assumptions and conditions on our profit, cost and loss functions.

- The instantaneous profit of the firm is described by $\pi : \mathbb{R}_+ \to \mathbb{R}$. We suppose that it is concave and increasing, and belongs to C^1. In addition, π satisfies the Inada conditions, i.e. $\lim_{x\to 0_+} \pi'(x) = \infty$ and $\lim_{x\to\infty} \pi'(x) = 0$. These are standard assumptions on production functions in economic growth theory.
- The firm's production cost function is described by $\mathcal{C} : \mathbb{R}_+ \to \mathbb{R}_+$, which introduces the firm's energy-related costs. We suppose that $\mathcal{C}$ is increasing and convex, meaning that higher emission rates will become increasingly more expensive for firms to maintain.
- The loss function of the firm is described by $\ell : \mathbb{R} \to \mathbb{R}$, which is the penalty related to regulation risk constraint. We suppose that ℓ is an increasing and convex function and assume that it is of quadratic growth, i.e. $\ell(x) = \mathcal{O}(|x|^2)$ as $|x| \to \infty$.

For simplicity, we exclude any other forms of costs for the firm. It is worth noting that all of these conditions align with the conditions outlined in Theorem 5.

3.3.2 General case with Pontryagin's maximum principle

We consider the optimal control problem Eq. (3.3.2) in its general form. From Eq. (3.3.1), the log-production $p := \log P$ solves the following SDE,

$$dp_t = \bar{\mu}(t, p_t, \gamma_t)dt + \sigma dW_t,$$

where we applied Itô's lemma. The general drift coefficient satisfies $\bar{\mu}(t, p_t, \gamma_t) = \mu(t, p_t, \gamma_t) - \frac{1}{2}\sigma^2$ $\forall t \in \mathbb{R}_+, x \in \mathbb{R}, y \in \mathbb{R}_+$. We define an auxiliary function, $\bar{\pi}(x) := \pi(e^x)$, that adapts to our log-transformation, i.e.

$$\pi(P_t) = \bar{\pi}(p_t).$$

The driver of the gain functional transforms,

$$f(t, p_t, \gamma_t) = e^{-rt}[\bar{\pi}(p_t) - \mathcal{C}(\gamma_t) - \ell(\gamma_t - e_t)].$$

With all the necessary components in hand for the general Hamiltonian, it takes the following form,

$$\begin{aligned} H(t, p, \gamma, y, z) &= \bar{\mu}(t, p, \gamma)y + \sigma z + f(t, p, \gamma) \\ &= \bar{\mu}(t, p, \gamma)y + \sigma z + e^{-rt}[\bar{\pi}(p_t) - \mathcal{C}(\gamma_t) - \ell(\gamma_t - e_t)]. \end{aligned} \tag{3.3.3}$$

The adjoint BSDE for any admissible strategy $\gamma \in \mathcal{A}$ is given by,

$$\begin{aligned} dY_t &= -(e^{-rt}\bar{\pi}'(p_t) + \partial_x\bar{\mu}(t, p_t, \gamma_t)\hat{Y}_t)dt + dM_t, \qquad (3.3.4) \\ Y_T &= D_x g(X_T) = 0, \end{aligned}$$

here we set $dM_t = Z_t dW_t$ and M is an $\mathbb{F}$-martingale. Next, we will show how to solve this BSDE using differential and Itô calculus. After we carry out an explicit model to show some results and what these say about the emission strategy of a company.

3.3.3 Analytical solution to the optimization problem

Let $\hat{\gamma}$ be a candidate for the optimal control and $(\hat{X}, \hat{Y}, \hat{Z})$ the corresponding processes, then through maximization of the Hamiltonian of Eq. (3.3.3) we get the following expression,

$$\partial_y\bar{\mu}(t, \hat{p}_t, \hat{\gamma}_t)\hat{Y}_t e^{rt} = \mathcal{C}'(\hat{\gamma}) + \ell'(\hat{\gamma} - e_t). \tag{3.3.5}$$

Thus, if $\hat{\gamma} \in \mathcal{A}$, then Eq. (3.3.5) must hold $dt \otimes d\mathbb{P}$ a.e. Observing Eq. (3.3.5), one notices the dependence of the $\hat{Y}$ process. Consequently, in order to obtain an expression for the optimal control, we must first derive a general expression for $\hat{Y}$. This can be accomplished through either a numerical method, which we will delve into in Chapter 5, or can be done analytical, which we will demonstrate right away.

By utilizing our understanding of differential and Itô calculus, we employ the subsequent substitution: $f(Y_u) = e^{\int_0^u \partial_x\bar{\mu}(s, p_s, \gamma_s)ds}Y_u$. Then Itô's lemma yields,

$$\begin{aligned} df(\hat{Y}_u) &= \partial_x\bar{\mu}(u, p_u, \gamma_u)e^{\int_0^u \partial_x\bar{\mu}(s, p_s, \gamma_s)ds}\hat{Y}_u du + e^{\int_0^u \partial_x\bar{\mu}(s, p_s, \gamma_s)ds}d\hat{Y}_u \\ &= \partial_x\bar{\mu}(u, p_u, \gamma_u)e^{\int_0^u \partial_x\bar{\mu}(s, p_s, \gamma_s)ds}\hat{Y}_u du + e^{\int_0^u \partial_x\bar{\mu}(s, p_s, \gamma_s)ds}(-(e^{-ru}\bar{\pi}'(p_u) \\ &+ \partial_x\bar{\mu}(u, p_u, \gamma_u)\hat{Y}_u)du + dM_u) \\ &= -e^{\int_0^u \partial_x\bar{\mu}(s, p_s, \gamma_s)ds}(e^{-ru}\bar{\pi}'(p_u)du - dM_u), \end{aligned} \tag{3.3.6}$$

where we used the definition of the adjoint equation Eq. (3.3.4). Integrating the left side from initial time, t, to terminal time, T, results in,

$$\int_t^T df(Y_u) = e^{\int_0^T \partial_x\bar{\mu}(s, p_s, \gamma_s)ds}\hat{Y}_T - e^{\int_0^t \partial_x\bar{\mu}(t, p_t, \gamma_t)ds}\hat{Y}_t = -e^{\int_0^t \partial_x\bar{\mu}(t, p_t, \gamma_t)}\hat{Y}_t, \tag{3.3.7}$$

where we used the fact that $Y_T = 0$. Substituting Eq. (3.3.7) into the left side of Eq. (3.3.6) we get,

$$-e^{\int_0^t \partial_x \bar{\mu}(t,p_t,\gamma_t)}\hat{Y}_t = -\int_t^T e^{-ru+\int_0^u \partial_x \bar{\mu}(s,p_s,\gamma_s)ds}\bar{\pi}'(p_u)du + \int_t^T e^{\int_0^u \partial_x \bar{\mu}(s,p_s,\gamma_s)ds}dM_u$$
$$\hat{Y}_t = \int_t^T e^{-ru+\int_t^u \partial_x \bar{\mu}(s,p_s,\gamma_s)ds}\bar{\pi}'(p_u)du - \int_t^T e^{\int_t^u \partial_x \bar{\mu}(s,p_s,\gamma_s)ds}dM_u.$$

As stochastic integrals have vanishing expectations and M_t is an $\mathbb{F}$-martingale by definition, we can reduce the expression to,

$$\hat{Y}_t(\gamma) = \mathbb{E}\left[\int_t^T e^{-ru+\int_t^u \partial_x \bar{\mu}(t,p_s,\gamma_s)ds}\bar{\pi}'(p_u)du|\mathcal{F}_t\right], \quad t \geq 0. \tag{3.3.8}$$

Once Eq. (3.3.5) and Eq. (3.3.8) are known, and provided that the appropriate conditions and assumptions are met for the drift $\bar{\mu}$, as well as cost and loss functions $\mathcal{C}$ and ℓ, we can determine the optimal control $\hat{\gamma}_t$.

3.3.4 Explicit model

We now apply the general solution to an explicit model using some basic functions. This case comes directly from the paper [10] and consider again the logarithmic production $\pi(x) = \log x$, which due to the substitution of $p_t = \log(P_t)$ becomes $\bar{\pi}(p_t) = p_t$. The resulting forward SDE is then,

$$dp_t = \bar{\mu}(t, p_t, \gamma_t)dt + \sigma dW_t,$$

where the drift coefficient, $\bar{\mu}$, satisfies: $\bar{\mu}(t, p_t, \gamma_t) = \mu(t, p_t, \gamma_t) - \frac{1}{2}\sigma^2$. We suppose that this drift function has an affine form of,

$$\bar{\mu}(t, x, y) = a + bx + cy, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, y \in \mathbb{R}_+,$$

where the coefficient $a \geq 0$ corresponds to an average production level, $b \leq 0$ is a mean-reverting parameter with the negative sign meaning that over-production may deteriorate the production ability and $c \geq 0$ represents the dependence of the firm's production upon CO_2 emission. Furthermore, let the cost and penalty functions be given respectively by,

$$\mathcal{C}(x) = \frac{x^2}{2} \quad \text{and} \quad \ell(x) = \omega\frac{(x_+)^2}{2},$$

where $\omega \geq 0$ is a constant coefficient characterizing the penalty force of the CO_2 emission constraint and the function x_+ denotes: $\max(x, 0)$. The reason for selecting a quadratic penalty function is to amplify the impact of excessive emissions beyond the benchmark level.
Inserting the explicit expression of the drift, cost and loss functions into Eq. (3.3.5) and Eq. (3.3.8), we obtain the following,

$$e^{-rt}(\hat{\gamma}_t + \omega(\hat{\gamma}_t - e_t)_+) = c\int_t^T e^{-ru+\int_t^u b\, ds}du$$
$$(\hat{\gamma}_t + \omega(\hat{\gamma}_t - e_t)_+) = ce^{-(b-r)t}\int_t^T e^{(b-r)u}du$$
$$(\hat{\gamma}_t + \omega(\hat{\gamma}_t - e_t)_+) = \frac{c}{b-r}e^{-(b-r)t}[e^{(b-r)T} - e^{(b-r)t}] = \frac{c}{r-b}[1 - e^{(b-r)(T-t)}]. \tag{3.3.9}$$

Rewriting 3.3.9 gives us an optimal emission strategy of

$$\hat{\gamma} = \min\left\{\frac{c}{r-b}[1 - e^{(b-r)(T-t)}], \frac{1}{1+\omega}\left(\omega e + \frac{c}{r-b}[1 - e^{(b-r)(T-t)}]\right)\right\}. \tag{3.3.10}$$

We will define the time-varying constant value as,

$$\bar{\gamma} := \frac{c}{r-b}[1 - e^{(b-r)(T-t)}],$$

which corresponds to the desired emission level without carbon penalty, i.e., when $\omega = 0$.
We can conclude two things from Eq. (3.3.10). If a company's optimal emission strategy, $\bar{\gamma}$, is better than the benchmark, i.e. a company produces less emission than the benchmark imposes, the optimal strategy is to retain the same constant level. That means that no effort needs to be provided. But if their emission strategy is worse than the benchmark, i.e. $\bar{\gamma}$ is higher than the benchmark e_t, regulation requires a stricter emission reduction plan. The optimal strategy is then to proceed as an affine function of the benchmark and the penalty weight.

3.3.5 Take away

The paper by Gobet provides valuable insights into incorporating a penalty or tax that impacts the overall wealth of a company. While the paper primarily focuses on the direct influence of this penalty/tax on a company's production, it is crucial to consider the potential long-term effects on pension fund portfolios. The emissions generated by the underlying companies within the portfolio stocks can eventually affect the portfolio's returns. Therefore, it becomes essential to incorporate a tax mechanism in a control problem that penalizes the total emissions of the portfolio, aligning with the reduction targets outlined in the Paris Agreement. This approach allows us to proactively address climate-related risks, even if high-emitting "brown" stocks outperform low-emitting "green" stocks. By implementing such a tax, we can better prepare for future climate shocks and imminent taxes related to emissions.

3.4 Isolating Green from Brown

The motivation for this thesis stems from the increasing pressure on pension funds to consider climate change risks and the need to meet environmental targets by 2050. With the expectation that green stocks will be less vulnerable to climate-related shocks than brown stocks, the composition of a portfolio will have a significant impact on its resilience to such risks. However, it is commonly believed that portfolios that favor green stocks will underperform in the short term but offer greater stability in the long run. Therefore, the challenge is to design a dynamic system that models a portfolio consisting of green and brown stocks over a period of time until 2050, while also optimizing the allocation of wealth between the two types of assets.

To tackle this challenge, we propose an extension of the Merton optimization problem that considers a portfolio consisting of green and brown stocks, where each stock is modeled by a geometric Brownian Motion Eq. (3.1.1). The SDE for green stocks represents every stock with a CO2 footprint below the Paris Alignment Benchmark, while the SDE for brown stocks represents those with a CO2 footprint above the benchmark. For simplicity, both SDEs are generated by the same Wiener process, but differ in their time dependence of drift and volatility.

Additionally, we introduce the possibility of a tax on brown stocks due to the potential introduction of a CO2 tax by the EU. This tax function, denoted as $\tau(t)$, is a deterministic function that depends solely on time and is yet to be determined. By incorporating this tax, we can incentivize a shift towards green stocks and discourage investments in brown stocks that have not taken adequate measures to reduce their greenhouse gas emissions. Overall, our aim is to develop a model that can assist pension funds in making optimal investment decisions while simultaneously considering climate change risks and meeting environmental targets.

We split up the problem formulation in three parts. First, we will extend Merton's portfolio problem by changing the two asset to two stock which we label "green" and "brown", characterizing two assets with, respectively, lower and higher emission than the Paris alignment benchmark prescribes. For this problem there still is an easy to find analytical solution. Second, we will add a contribution rate which characterizes the added wealth for the pension fund. This contribution will not be related to the total wealth of the portfolio and is constant in time. Lastly, we will look at different taxes we can implement to push investors in the more green future, when the EU passes legislature concerning climate change.

3.4.1 A portfolio of green and brown stocks

Let us begin by considering a portfolio consisting of one green stock, $S^G = (S_t^G)_{t\in[0,T]}$, and one brown stock, $S^B = (S_t^B)_{t\in[0,T]}$, whose prices evolve respectively as,

$$\frac{dS_t^G}{S_t^G} = \left[\mu_G(t)dt + \sigma_G(t)dW_t\right],$$
$$\frac{dS_t^B}{S_t^B} = \left[\mu_B(t)dt + \sigma_B(t)dW_t\right],$$

where W is an 1-dimensional Brownian motion and μ^G, μ^B, σ^G and σ^B are all real positive constants. Both the drifts and volatilities are progressively measureable 1-dimensional processes. The total wealth of our portfolio, X_t, at time t is modelled by,

$$\begin{aligned} dX_t &= \theta_t^G \frac{dS_t^G}{S_t^G} + \theta_t^B \frac{dS_t^B}{S_t^B} \\ &= \theta_t^G \left[\mu_G(t)dt + \sigma_G(t)dW_t\right] + \theta_t^B \left[\mu_B(t)dt + \sigma_B(t)dW_t\right] \end{aligned}$$

where θ^G and θ^B denote the amount of money invested in the green and brown stocks and c_t the contribution rate of the pension fund at each time $t \in [0,T]$. Furthermore, we assume $\theta_t^G + \theta_t^B = X_t$,

$$\begin{aligned} dX_t &= \left[\theta_t^G \mu_G(t) + (X_t - \theta_t^G)\mu_B(t)\right] dt + \left[(X_t - \theta_t^G)\sigma_B(t) + \theta_t^G \sigma_G(t)\right] dW_t \\ &= \left[\theta_t^G(\mu_G(t) - \mu_B(t)) + X_t\mu_B(t)\right] dt + \left[\theta_t^G(\sigma_G(t) - \sigma_B(t)) + X_t\sigma_B(t)\right] dW_t. \end{aligned}$$

To simplify the SDE of X_t we use a substitution to the control variable $\alpha_t^{(G,B)} = \theta_t^{(G,B)}/X_t$, where $\alpha_t^{(G,B)}$ denotes the fraction of the wealth put in green or brown stocks respectively. Using this substitution we obtain,

$$dX_t = X_t\left[\alpha_t^G(\mu_G(t) - \mu_B(t)) + \mu_B(t)\right] dt + X_t\left[\alpha_t^G(\sigma_G(t) - \sigma_B(t)) + \sigma_B(t)\right] dW_t. \qquad (3.4.1)$$

We have our forward process describing the dynamics of a two asset portfolio. Now we need an objective for our portfolio strategy. This objective is the same as for Merton's portfolio problem, namely that we want to determine the optimal investment strategy for an investor who seeks to maximize the expected lifetime utility of the portfolio wealth. We choose to use the logarithmic utility, $\mathcal{U}(x) = \ln(x)$, which results in the following gain functional,

$$J(x) = \sup_{\alpha^G} \mathbb{E}\left[\int_0^T \ln(X_s)ds + \ln(X_T)\right], \quad 0 < t < T. \qquad (3.4.2)$$

Comparing with Eq. (3.4.1), we have obtained a slightly more complex version of the wealth value from Merton's portfolio problem. This makes finding an analytical solution also harder. As we will see in the next chapter, where we will use a numerical method to solve BSDEs, we need to first find the adjoint equation and the feedback map of the formulated problem. Because of the linearity of α and the dependence of the diffusion on α, we will first apply Itô's lemma to aid us. This results in,

$$\begin{aligned} dx_t = &\left[\alpha_t^G(\mu_G(t) - \mu_B(t)) + \mu_B(t) - \frac{1}{2}(\alpha_t^G(\sigma_G(t) - \sigma_B(t)) + \sigma_B(t))^2\right] dt \\ &+ \left[\alpha_t^G(\sigma_G(t) - \sigma_B(t)) + \sigma_B(t)\right] dW_t. \end{aligned}$$

We can use SMP to find an analytical solution to this control problem. The general Hamiltonian of the process is,

$$H(t,x_t,\alpha_t^G,Y_t,Z_t) = \left[\alpha_t^G(\mu_G(t)-\mu_B(t))+\mu_B(t)-\frac{1}{2}(\alpha_t^G(\sigma_G(t)-\sigma_B(t))+\sigma_B(t))^2\right]Y_t \\ + \left[\alpha_t^G(\sigma_G(t)-\sigma_B(t))+\sigma_B(t)\right]Z_t + x_t.$$

We notice that this Hamiltonian is not linear anymore in the control variable. So conditions Eq. (2.4.16) and Eq. (2.4.17) will be satisfied if and only if,

$$(\mu_G(t)-\mu_B(t)-(\hat{\alpha}_t^G(\sigma_G(t)-\sigma_B(t))+\sigma_B(t))(\sigma_G(t)-\sigma_B(t)))\hat{Y}_t + (\sigma_G(t)-\sigma_B(t))\hat{Z}_t = 0.$$

The feedback map is then,

$$\hat{\alpha}_t^G = \frac{(\mu_G(t)-\mu_B(t))\hat{Y}_t + (\sigma_G(t)-\sigma_B(t))\hat{Z}_t - \sigma_B(t)(\sigma_G(t)-\sigma_B(t))\hat{Y}_t}{(\sigma_G(t)-\sigma_B(t))^2\hat{Y}_t}.$$

The adjoint BSDE is given as,

$$-d\hat{Y}_t = dt - \hat{Z}_t dW_t,$$

with terminal conditions,

$$\hat{Y}_T = D_x g(\hat{X}_T) = \frac{1}{\hat{X}_T}, \tag{3.4.3}$$
$$\hat{Z}_T = \sigma(t,X_T,\hat{\alpha}_T^G)D_x^2 g(\hat{X}_T) = -\frac{(\hat{\alpha}_t^G(\sigma_G(t)-\sigma_B(t))+\sigma_B(t))}{\hat{X}_T}.$$

3.4.2 Concerning the contribution rate

When considering a portfolio consisting of green and brown stocks, the SDE can be extended to incorporate a contribution rate term. The contribution rate represents the additional funds injected into the portfolio, associated with a defined contribution (DC) pension fund as mentioned in [14], typically corresponding to a fixed proportion of the individual's income. This rate can be assumed to follow a stochastic process, capturing the uncertainties and fluctuations associated with the individual's income and contributions over time, but in our case we will consider a constant contribution for the commencement of the pension until the terminal time at which the individual will be paid out.

By incorporating the contribution rate into the SDE, the dynamics of the portfolio evolve not only based on market forces but also as a result of the continuous inflow of new funds. This has implications for investment decisions, asset allocation strategies, and risk management in the context of sustainable investing. The presence of green stocks, which align with environmentally friendly and socially responsible criteria, further adds to the complexity of the model, as the dynamics of these stocks may exhibit distinct features compared to traditional brown stocks.

By considering the interplay between the contribution rate, the dynamics of green and brown stocks, and potentially other factors such as market trends and risk preferences, it becomes possible to analyze the long-term growth and performance of the portfolio. This incorporation of a contribution rate in the SDE framework provides a more realistic representation of the dynamics of a pension fund or similar investment vehicles, enabling more accurate assessments of risk exposure, sustainability considerations, and long-term financial planning. We extend the SDE given by Eq. (3.4.1) to simulate a DC pension fund and obtain.

$$dX_t = X_t\left[\alpha_t^G(\mu_G(t)-\mu_B(t))+\mu_B(t)\right]dt + X_t\left[\alpha_t^G(\sigma_G(t)-\sigma_B(t))+\sigma_B(t)\right]dW_t + c_t dt.$$

The objective stays the same which is to determine the optimal investment strategy for an investor who seeks to maximize the expected lifetime utility of the portfolio wealth and the gain functional is given by Eq. (3.4.2). Again we apply Itô's lemma to circumvent the linearity of α and the dependence of the diffusion on α, but contribution rate does not depend on the forward process X. This results in,

$$dx_t = \left[\alpha_t^G(\mu_G(t) - \mu_B(t)) + \mu_B(t) + \frac{c_1}{e^x} - \frac{1}{2}(\alpha_t^G(\sigma_G(t) - \sigma_B(t)) + \sigma_B(t))^2\right] dt \qquad (3.4.4)$$
$$+ \left[\alpha_t^G(\sigma_G(t) - \sigma_B(t)) + \sigma_B(t)\right] dW_t,$$

where $x_t = \log(X_t)$. Note that we still have a dependence on the log-transformed forward process. We can use SMP to find an analytical solution to this control problem. The general Hamiltonian of the process is,

$$H(t, x_t, \alpha_t^G, Y_t, Z_t) = \left[\alpha_t^G(\mu_G(t) - \mu_B(t)) + \mu_B(t) + \frac{c_t}{e^x} - \frac{1}{2}(\alpha_t^G(\sigma_G(t) - \sigma_B(t)) + \sigma_B(t))^2\right] Y_t$$
$$+ \left[\alpha_t^G(\sigma_G(t) - \sigma_B(t)) + \sigma_B(t)\right] Z_t + x_t.$$

The conditions Eq. (2.4.16) and Eq. (2.4.17) will be satisfied if and only if,

$$(\mu_G(t) - \mu_B(t) - (\hat{\alpha}_t^G(\sigma_G(t) - \sigma_B(t)) + \sigma_B(t))(\sigma_G(t) - \sigma_B(t)))\hat{Y}_t + (\sigma_G(t) - \sigma_B(t))\hat{Z}_t = 0,$$

which will give us the following feedback map,

$$\hat{\alpha}_t^G = \frac{(\mu_G(t) - \mu_B(t))\hat{Y}_t + (\sigma_G(t) - \sigma_B(t))\hat{Z}_t - \sigma_B(t)(\sigma_G(t) - \sigma_B(t))\hat{Y}_t}{(\sigma_G(t) - \sigma_B(t))^2\hat{Y}_t}.$$

Notice that the feedback map for the optimal control does not depend on X, but when we look at the adjoint BSDE,

$$-d\hat{Y}_t = \left(-\frac{c_t}{e^{\hat{X}_t}}\hat{Y}_t + 1\right)dt - \hat{Z}_t dW_t,$$

we can deduce that this wealth process of the portfolio will influence the allocation of the green and brown stock. We have now obtained a control problem that is not solvable with analytical methods, as a candidate solution for Y cannot be found. The terminal conditions of $\hat{Y}_T$ and $\hat{Y}_T$ stay the same and are given by Eq. (3.4.3).

3.4.3 Climate taxes

We propose the incorporation of climate taxes within the stochastic control problem. These climate taxes not only account for the environmental impact of the portfolio but also capture climate-related risks associated with transition and physical factors, which can be correlated to the relative amount of emissions the underlying company of the stock produces.

The first tax mechanism that we will include focuses on the reduction process of companies represented by their stock. We look at the excess emissions of the portfolio using the Paris alignment benchmark. This tax term acts as a penalty for the utility of the total wealth of the portfolio and can be found in the driver part of the gain functional. We introduce,

$$\Gamma_t := \lambda\frac{(\alpha_t^G\gamma_t^G + \alpha_t^B\gamma_t^B - e_t^{PAB})_+^2}{2},$$

where γ^G and γ^B are respectively the emission reduction pathways of the green and the brown stock, e^{PAB} is the proposed reduction scenario by the PAB, λ is a constant coefficient characterizing the penalty force of the CO_2 emission constraint and the function $(x)_+$ denotes $\max(x, 0)$. We set $e_0^{PAB} = 1$. The transformed gain functional, after the application of Itô's lemma, is given by,

$$J(x) = \sup_{\alpha^G} \mathbb{E}\left[\int_0^T x_s - \lambda\frac{(\alpha_s^G\gamma_s^G + \alpha_s^B\gamma_s^B - e_s^{PAB})_+^2}{2} ds + x_T\right], \quad 0 < t < T.$$

The general Hamiltonian becomes,

$$H(t,x_t,\alpha_t^G,Y_t,Z_t) = \left[\alpha_t^G(\mu_G(t)-\mu_B(t))+\mu_B(t)+\frac{c_t}{e^x}-\frac{1}{2}(\alpha_t^G(\sigma_G(t)-\sigma_B(t))+\sigma_B(t))^2\right]Y_t$$
$$+\left[\alpha_t^G(\sigma_G(t)-\sigma_B(t))+\sigma_B(t)\right]Z_t + x_t - \lambda\frac{(\alpha_t^G\gamma_t^G+(1-\alpha_t^G))\gamma_t^B-e_t^{PAB})_+^2}{2}.$$

Solving for α, the conditions Eq. (2.4.16) and Eq. (2.4.17) will be satisfied if and only if,

$$\hat{\alpha}_t = \max\Bigg(\frac{(\mu_G(t)-\mu_B(t))\hat{Y}_t+(\sigma_G(t)-\sigma_B(t))\hat{Z}_t-\sigma_B(t)(\sigma_G(t)-\sigma_B(t))\hat{Y}_t}{(\sigma_G(t)-\sigma_B(t))^2\hat{Y}_t}, \quad (3.4.5)$$
$$\frac{(\mu_G(t)-\mu_B(t))\hat{Y}_t+(\sigma_G(t)-\sigma_B(t))\hat{Z}_t-\sigma_B(t)(\sigma_G(t)-\sigma_B(t))\hat{Y}_t}{(\sigma_G(t)-\sigma_B(t))^2\hat{Y}_t+\lambda(\gamma_t^G-\gamma_t^B)^2}$$
$$-\frac{\lambda(\gamma_t^B-e_t^{PAB})(\gamma_t^G-\gamma_t^B)}{(\sigma_G(t)-\sigma_B(t))^2\hat{Y}_t+\lambda(\gamma_t^G-\gamma_t^B)^2}\Bigg).$$

Chapter 4

Numerical methods

In many cases, complex FBSDE systems lack well-defined analytical solutions, necessitating the utilization of numerical methods. In this chapter we will explore particular method, which is based on the use of Fourier cosine series expansion, called the COS method [12]. This method serves as a rapid and accurate alternative to other approaches such as the Longstaff-Schwartz method [23] or the use of binomial trees [24]. Initially developed as an option pricing technique for European options [12], it was later extended to solve BSDEs [18], giving rise to the BCOS method. Distinct from classical forward SDEs, discretizing a BSDE results in conditional expectations, which stems from the problem of adaptedness we run into if we don't apply these conditional expectation. When implementing the BCOS method, we efficiently approximate these conditional expectations by employing the characteristic function of the transitional density.

The COS method has already been applied to solving stochastic control problems [37]. However, our approach differs from theirs in that we aim to find the solution through the adjoint BSDE and Pontryagin's maximum principle, while they utilize the dynamic programming principle to determine the value function. Our research extends the application of the BCOS method to fully coupled FBSDEs and using this result to recover the optimal control variable at the end. In the problems we will examine, we restrict our focus only to one-dimensional Lévy processes, thereby excluding any analysis of multi-dimensional control problems in this chapter.

This chapter is structured as follows. Firstly, we outline the framework of the dynamic stochastic system that we aim to solve, along with the resulting adjoint FBSDE. Solving this system ensures that the associated control is optimal. To adapt this framework for the BCOS method, we discretize the adjoint coupled FBSDE. Subsequently, we state the BCOS method and illustrate how it approximates the conditional expectations arising from the discretization of a BSDE. Finally, we discuss the errors encountered throughout the process and provide an analysis of the computational complexity of the algorithm.

The setting For convenience of the reader, let us reiterate the compact framework presented in Chapter 2. Our interest lies only in in the optimal control and thus we will use Pontryagin's maximum principle is necessary to use. The feedback map that this approach induces coincides better with the solution pair of (Y, Z) which we will be helpful in the numerical scheme. As we have seen in Chapter 2 the stochastic control problem is a forward SDE, (X_s), revolving around a control variable, (α_s). The dynamics are defined as,

$$dX_s = \bar{\mu}(s, X_s, \alpha_s)ds + \bar{\sigma}(s, X_s, \alpha_s)dW_s. \tag{4.0.1}$$

For each control problem we need to define a functional to optimize,

$$J(\alpha) = \mathbb{E}[\int_0^T h(s, X_s, \alpha_s, Y_s, Z_s)ds + g(X_T)],$$

and the value function is,

$$v(t,x) = \sup_{\alpha \in \mathcal{A}} J(\alpha),$$

The general Hamiltonian of the SMP is given by,

$$H(s, X_s, \alpha_s, Y_s, Z_s) = \bar{\mu}(s, X_s, \alpha_s)Y_s + \bar{\sigma}(s, X_s, \alpha_s)Z_s + h(s, X_s, \alpha_s, Y_s, Z_s),$$

and the adjoint BSDE,

$$-dY_s = D_x H(s, X_s, \alpha_s, Y_s, Z_s)dt - Z_s dW_s, \quad Y_T = D_x g(X_T) \tag{4.0.2}$$

Maximizing the Hamiltonian with respect to α gives us the feedback map of the optimal control variable, $\hat{\alpha}$,

$$\begin{aligned}\hat{\alpha}_s &:= \arg\max H(t, \hat{X}_s, a, \hat{Y}_s, \hat{Z}_s) \\ &= \delta(t, \hat{X}_s, \hat{Y}_s, \hat{Z}_s).\end{aligned}$$

For convenience of the writer we will use the notation $(X, Y, Z) = (\hat{X}, \hat{Y}, \hat{Z})$. Inserting the feedback map in Eq. (4.0.1) and Eq. (4.0.2) gives,

$$\begin{aligned}X_t &= X_0 + \int_0^t \bar{\mu}(s, X_s, \delta(t, X_s, Y_s, Z_s))ds + \int_0^t \bar{\sigma}(s, X_s, \delta(t, X_s, Y_s, Z_s))dW_s, \quad 0 \le t \le T, \\ &= X_0 + \int_0^t \mu(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dW_s, \quad 0 \le t \le T, \\ Y_t &= D_x g(X_T) + \int_t^T h(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T,\end{aligned} \tag{4.0.3}$$

where we write $h(s, X_s, Y_s, Z_s) = D_x H(s, X_s, Y_s, Z_s)$. The end result is a FBSDE with the coupling determined by the optimal control mapping. In most cases, if the given system is not a predetermined toy model or the diffusion has control dependency, we get a fully coupled FBSDE. In general there is no available explicit solution to this system. However in [18] they propose an extension of the BCOS method for coupled forward-backwards stochastic differential equations. We will use this to our gain to solve complex stochastic control problems.

4.1 Discretization of a coupled FBSDE

This section is a repetition of the work of Huijskens, Ruijter and Oosterlee [18]. In their paper they propose a discretization of a coupled FBSDE. We will extend this to a fully coupled FBSDE where the backward component is also coupled in the diffusion σ.

For the discretization we will use the well-known Euler-Maruyama approximation. We define a partition Π for the time steps: $0 = t_0 < t_1 < t_2 < \cdots < t_m < \cdots < t_M = T$, with fixed time step $\Delta t = t_{m+1} - t_m$ for m $= M-1, \ldots, 0$. For notational convenience we write X_m to denote the value of any arbitrary stochastic process X_{t_m} at time t_m and define $\Delta W_m = W_{t_{m+1}} - W_{t_m}$, where the Brownian Motion W_m has norminally distributed increments, $\Delta W_m \sim \mathcal{N}(0, \Delta t)$. The evolution of the forward SDE on a small time interval $[t_m, t_{m+1})$ is given by,

$$X_{m+1} = X_m + \int_{t_m}^{t_{m+1}} \mu(s, X_s, Y_s, Z_s)ds + \int_{t_m}^{t_{m+1}} \sigma(s, X_s, Y_s, Z_s)dW_s.$$

To circumvent the problem of coupling we will use [11] to obtain an explicit scheme for coupled FBSDEs. We calculate backwards the BSDE numerically while assuming that the forward SDE is a grid of points. We know the characteristic function of this grid, so we know the probability of going from one point on the grid to another for each time-step. The problem with this is that due to the coupling, X also depends on (Y, Z). That is why instead of the standard Euler-Maruyama approximation of a forward SDE, we assume that the time step $\Delta = t_{m+1} - t_m$ is small, so the approximations $y(t_{m+1}, \cdot)$ and $z(t_{m+1}, \cdot)$ on interval $[t_m, t_{m+1})$ can be interpreted as predictors of the solution at time t_m. This will give us the following approximation of the forward process X,

$$\begin{aligned} X^{\Delta}_{m+1} &= x + \mu(t_{m+1}, x, y(t_{m+1}, x), z(t_{m+1}, x))\Delta t + \sigma(t_{m+1}, x, y(t_{m+1}, x), z(t_{m+1}, x))\Delta W_{m+1}, \\ X^{\Delta}_m &= x, \end{aligned}$$

this is a right-point approximation for the y, z and t components in the drift and volatility as opposed to the conventional left-point approximation where take m instead of $m + 1$. It can be interpreted as a backward Euler approximation of the forward SDE.

The local evolution of the backward SDE Eq. (4.0.3) of the coupled FBSDE is as follows,

$$Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} h(s, X_s, Y_s, Z_s)ds - \int_{t_m}^{t_{m+1}} Z_s dW_s. \tag{4.1.1}$$

To obtain an approximation of the Y process, conditional expectation are taken on both sides with respect to the underlying filtration $\mathcal{F}$. Using the theta method to approximate the integrals, the fact that Brownian motion has independent increments and that an Itô integral has zero expectation, we end up with the following approximation,

$$\begin{aligned} Y_m &= \mathbb{E}_m[Y_{m+1}] + \int_{t_m}^{t_{m+1}} \mathbb{E}_m[h(s, X_s, Y_s, Z_s)]ds \\ &\approx \mathbb{E}_m[Y_{m+1}] + \Delta t\theta_1 h(t_m, X_m, Y_m, Z_m) + \Delta t(1 - \theta_1)\mathbb{E}_m[h(t_{m+1}, X_{m+1}, Y_{m+1}, Z_{m+1})], \end{aligned}$$

where $\theta_1 \in [0, 1]$ and $\mathbb{E}_m[\cdot]$ is denoted for the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_m]$. Likewise we can obtain the approximation of the Z_t process, by first multiplying Eq. (4.1.1) with ΔW_{m+1} and then take the conditional expectation. This results in,

$$\begin{aligned} Z_m = &-\frac{1}{\theta_2}(1 - \theta_2)\mathbb{E}_m[Z_{m+1}] + \frac{1}{\Delta t\theta_2}\mathbb{E}_m[Y_{m+1}\Delta W_{m+1}] \\ &+ \frac{1 - \theta_2}{\theta_2}\mathbb{E}_m[h(t_{m+1}, X_{m+1}, Y_{m+1}, Z_{m+1})\Delta W_{m+1}], \end{aligned}$$

where $\theta_2 \in [0, 1]$. As a consequence of the Pontryagin's maximum principle, the terminal values Y_M and Z_M are given as,

$$\begin{aligned} Y_M &= D_x g(X_M) \\ Z_M &= \sigma(t_M, X_M, Y_M, Z_M)D_x^2 g(X_M), \end{aligned}$$

note that the terminal condition for Z_M presents a self-referencing issue, which poses certain difficulties in our analysis. However, we will address this matter later in the chapter, providing some comments on how to overcome this problem.

Since the terminal conditions are deterministic function of time t and the Markov process X^{Δ}, by the induction argument, we also get that there are deterministic functions $y(t_m, x)$ and $z(t_m, x)$ such that,

$$Y^{\Delta}_m = y(t_m, x), \quad Z^{\Delta}_m = z(t_m, x).$$

The random variables Y^{Δ}_m and Z^{Δ}_m are functions of X^{Δ}_m and therefore the conditional expectation can be replaced by,

$$\mathbb{E}_m[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_m] = \mathbb{E}[\cdot|X_m = x] \approx \mathbb{E}[\cdot|X^{\Delta}_m = x],$$

which we define as $E_m^x[\cdot] := \mathbb{E}[\cdot|X_m^\Delta = x]$. Summarizing the scheme gives us a discrete-time approximation (Y^Δ, Z^Δ) for (Y, Z):

$$\begin{aligned} Y_M^\Delta &= D_x g(X_M^\Delta), \quad Z_M^\Delta = \sigma(t_M, X_M^\Delta, D_x g(X_M^\Delta)) D_x^2 g(X_M^\Delta) \\ Z_m^\Delta &= -\theta_2^{-1}(1-\theta_2)\mathbb{E}_m[Z_{m+1}^\Delta] + \frac{1}{\Delta t}\theta_2^{-1}\mathbb{E}_m[Y_{m+1}^\Delta \Delta W_{m+1}] \\ &\quad + \theta_2^{-1}(1-\theta_2)\mathbb{E}_m[h(t_{m+1}, X_{m+1}^\Delta)\Delta W_{m+1}] \\ Y_m^\Delta &= \mathbb{E}_m[Y_{m+1}^\Delta] + \Delta t \theta_1 b(t_m, X_{m+1}^\Delta) + \Delta t(1-\theta_1)\mathbb{E}_m[h(t_{m+1}, X_{m+1}^\Delta)], \end{aligned} \tag{4.1.2}$$

for $m = M-1, \ldots, 0$. There are now several choices of the parameters θ_1 and θ_2, for which the behaviour and computational effort of our numerical methods will vary. If $\theta_1 = 0$, the scheme becomes explicit for Y_m^Δ, whereas if $\theta_1 \in (0, 1]$ results in an implicit scheme. For this implicit scheme we end up with a fixed-point problem for Y_m^Δ, which we will approximate by using P Picard iterations.

4.2 BCOS Method

In this section we state the COS approximation for conditional expectations. We first derive the COS formulas and the Fourier cosine coefficients. Then we approximate the functions z and y and the recursive recovery of the Fourier coefficients. Lastly, we show how to implement the numerical method for stochastic control problems and elaborate on the errors and convergence rates.

Transitional density function

Since we use the right-point approximation of the forward process X_t, the conditional characteristic function of X_m^Δ will be as follows,

$$\phi(u|X_m^\Delta = x) = \exp\left(iux + iu\mu(t_{m+1}, x, y(t_{m+1}, x), z(t_{m+1}, x))\Delta t - \frac{1}{2}u^2\sigma^2(t_{m+1}, x, y(t_{m+1}, x), z(t_{m+1}, x))\Delta t\right).$$

Furthermore we define two transition matrices as,

$$\Phi_k(x) := \mathrm{R}\left(\phi(\frac{k\pi}{b-a}|x)\exp(ik\pi\frac{-a}{b-a})\right),$$

and,

$$\hat{\Phi}_k(x) := \mathrm{R}\left(\frac{ik\pi}{b-a}\phi(\frac{k\pi}{b-a}|x)\exp(ik\pi\frac{-a}{b-a})\right).$$

Fourier Cosine Coefficients

We define the Fourier cosine coefficients of the deterministic functions $y(t_m, X_m^\Delta) = Y_m^\Delta$ and $z(t_m, X_m^\Delta) = Z_m^\Delta$ from the FBSDE discretization in Eq. (4.1.2), as such,

$$\begin{aligned} \mathcal{Y}_k(t_{m+1}) &= \frac{2}{b-a}\int_a^b y(t_m, x)\cos\left(k\pi\frac{x-a}{b-a}dx\right), \\ \mathcal{Z}_k(t_{m+1}) &= \frac{2}{b-a}\int_a^b z(t_m, x)\cos\left(k\pi\frac{x-a}{b-a}dx\right), \end{aligned} \tag{4.2.1}$$

and $\mathcal{H}_k(t_{m+1})$, which is the Fourier cosine coefficient of $h(t_{m+1}, x, y(t_{m+1}, x), z(t_{m+1}, x))$ is,

$$\mathcal{H}_k(t_{m+1}) = \frac{2}{b-a}\int_a^b h(t_{m+1}, x, y(t_{m+1}, x), z(t_{m+1}, x))\cos\left(k\pi\frac{x-a}{b-a}\right)dx.$$

We will approximate these Fourier cosine coefficients using the discrete Fourier-cosine transforms (DCT). From [35], the idea is take a grid and create N grid-points, defined as,

$$x_n := a + (n + \frac{1}{2})\frac{b-a}{N} \quad \text{and} \quad \Delta x := \frac{b-a}{N}.$$

We can calculate then the value of the function $y(t_m, x)$, from Eq. (4.2.1), on these grid-points. The midpoint-rule integration gives us then our final result,

$$\mathcal{Y}_k(t_{m+1}) \approx \sum_{n=0}^{N-1} \frac{2}{b-a} y(t_m, x) \cos\left(k\pi \frac{x_n - a}{b-a}\right) \Delta x = \frac{2}{N} \sum_{n=0}^{N-1} y(t_m, x) \cos\left(k\pi \frac{2n+1}{2N}\right).$$

This DCT is of Type II and can be calculated efficiently with certain packages in Python.

COS approximations

Now we have all the pieces to construct the COS approximation of $z(t_m, x)$. We derive the following approximations of the three different expectations,

$$\begin{aligned}
\mathbb{E}_m^x[Z_{m+1}^\Delta] &\approx \sum_{k=0}^{N-1}{}' \mathcal{Z}_(t_{m+1}) \Phi_k(x), \\
\mathbb{E}_m^x[Y_{m+1}^\Delta \Delta\omega_{m+1}] &\approx \sum_{k=0}^{N-1}{}' \mathcal{Y}_(t_{m+1}) \sigma(t_{m+1}, x) \Delta t \hat{\Phi}_k(x), \\
\mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)\Delta\omega_{m+1}] &\approx \sum_{k=0}^{N-1}{}' \mathcal{H}_(t_{m+1}) \sigma(t_{m+1}, x) \Delta t \hat{\Phi}_k(x).
\end{aligned}$$

Together to give the subsequent COS approximation,

$$z(t_m, x) \approx -\frac{1-\theta_2}{\theta_2} \sum_{k=0}^{N-1}{}' \mathcal{Z}_k(t_{m+1}) \Phi_k(x) + \sum_{k=0}^{N-1}{}' \left(\frac{1}{\Delta t \theta_2} \mathcal{Y}_k(t_{m+1}) + \frac{1-\theta_2}{\theta_2} \mathcal{H}_k(t_{m+1}) \right) \sigma(t_m, x) \Delta t \hat{\Phi}_k(x).$$

For the computation of the function $y(t_m, x)$ there are two conditional expectation to be approximated, and are given by,

$$\begin{aligned}
\mathbb{E}_m^x[Y_{m+1}^\Delta] &\approx \sum_{k=0}^{N-1}{}' \mathcal{Y}_k(t_{m+1}) \Phi_k(x) \\
\mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)] &\approx \sum_{k=0}^{N-1}{}' \mathcal{H}_k(t_{m+1}) \Phi_k(x).
\end{aligned}$$

We then write the COS approximation of $y(t_m, x)$ as,

$$y(t_m, x) \approx \Delta t \theta_1 h(t_m, x, y(t_m, x), z(t_m, x)) + \sum_{k=0}^{N-1}{}' \big(\mathcal{Y}_k(t_{m+1}) + \Delta t(1-\theta_1) \frac{b-a}{2} \mathcal{H}_k(t_{m+1}) \big) \Phi_k(x).$$

Because of the self-referencing part of this expression we will perform P Picard iterations, starting with initial guess, $y^0(t_m, x) := \mathbb{E}_m^x[Y_{m+1}^\Delta]$.

Recovery of coefficients

At terminal time t_M we compute the following Fourier-cosine coefficients by using the terminal conditions from Pontryagin's maximum principle,

$$\begin{aligned}
\mathcal{Y}_k(t_M) &= \frac{2}{b-a}\int_a^b g(x)\cos\left(k\pi\frac{x-a}{b-a}\right)\\
&\approx \frac{2}{N}\sum_{n=0}^{N-1}{}' g(x_n)\cos\left(k\pi\frac{2n+1}{N}\right)\\
\mathcal{Z}_k(t_M) &= \frac{2}{b-a}\int_a^b \sigma(t_M,x,g(x))D_x g(x)\cos\left(k\pi\frac{x-a}{b-a}\right)\\
&\approx \frac{2}{N}\sum_{n=0}^{N-1}{}' \sigma(t_M,x_n,g(x_n))D_x g(x_n)\cos\left(k\pi\frac{2n+1}{N}\right)\\
\mathcal{H}_k(t_M) &= \frac{2}{b-a}\int_a^b h(t_M,x,g(x),\sigma(t_M,x,g(x))D_x g(x))\cos\left(k\pi\frac{x-a}{b-a}\right)\\
&\approx \frac{2}{N}\sum_{n=0}^{N-1}{}' h(t_M,x_n,g(x_n),\sigma(t_M,x_n,g(x_n))D_x g(x_n))\cos\left(k\pi\frac{2n+1}{N}\right).
\end{aligned}$$

From these equations we calculate backwards for $m \in [0,\ldots,M-1]$ the functions $z(t_m,x)$, $y(t_m,x)$ and $h(t_m,x,y(t_m,x),z(t_m,x))$ and the Fourier coefficients $\mathcal{Z}_k(t_m)$, $\mathcal{Y}_k(t_m)$ and $\mathcal{H}_k(t_m)$. We only need to store the Fourier coefficients for each time step because we need these coefficients to retrieve the pair (Y_t^Δ, Z_t^Δ). This is done by applying another Fourier cosine transformation and results in,

$$\begin{aligned}
Y_m \approx y(t_m,X_m) &\approx \sum_{k=0}^{N-1}{}' \mathcal{Y}_k(t_m)\cos\left(k\pi\frac{X_m-a}{b-a}\right),\\
Z_m \approx z(t_m,X_m) &\approx \sum_{k=0}^{N-1}{}' \mathcal{Z}_k(t_m)\cos\left(k\pi\frac{X_m-a}{b-a}\right).
\end{aligned}$$

In order to retrieve our approximations for (Y,Z) we need to simultaneously simulate the forward diffusion. Instead of our earlier Euler-Maruyama discretization where we applied a right-point approximation, we now apply a left-point approximation because we go forward in time rather then going backwards in time. So the new evolution of the forward SDE will be denoted by $\tilde{X}_m$ and is given by,

$$\tilde{X}_{m+1} = x + \mu(t_m,x,y(t_m,x),z(t_m,x))\Delta t + \sigma(t_m,x,y(t_m,x),z(t_m,x))\Delta W_m, \quad \tilde{X}_m = x.$$

In the end we find the triple processes $(\tilde{X},Y,Z)$, which we can use to find the optimal control,

$$\hat{\alpha}_m = \delta(m,\tilde{X}_m,Y_m,Z_m).$$

4.2.1 Error Analysis and Computational Complexity

Working with numerical methods entails considering various approximation errors. When utilizing the BCOS method to solve FBSDEs, these errors can be categorized into three parts. Firstly, we have discretization errors resulting from approximating the stochastic processes (X,Y,Z) in discrete time. Secondly, there are errors associated with the COS formulas and the Picard iterations. Lastly, when retrieving the control variable, we employ another discretization to simultaneously determine (X,Y,Z) to obtain the optimal control.

The discretization error in FBSDEs is nuanced, necessitating differentiation between the fully coupled and decoupled cases. The discretization of the coupled FBSDE has been addressed in

Algorithm 1 BCOS method

```
Compute the terminal functions y(t_M, x) and z(t_M, x) by using the terminal condition.
Using discrete Fourier-cosine transformation approximate the terminal coefficients Z_k(t_M), Y_k(t_M) and H_k(t_M).
for m = M, ..., 0 do do
    Compute the functions z(t_m, x), y(t_m, x) and h(t_m, x, y(t_m, x), z(t_m, x)).
    Using these functions approximate the corresponding coefficients Z_k(t_m), Y_k(t_m) and H_k(t_m) by using the discrete Fourier-cosine transformation.
end for
for m = 0, ..., M do do
    Compute the functions z(t_m, X̃_m) and y(t_m, X̃_m).
    Compute X̃_{m+1}.
end for
Retrieve α̂ = δ(·, X̃, Y, Z).
```

previous work [35] [36]. These papers demonstrate that in the decoupled case, Y exhibits second-order convergence, while Z converges at first-order. However, in the coupled framework, achieving a convergence rate better than that of the Euler-Maruyama scheme used to discretize the forward process is not possible due to coupling in the drift and volatility. Consequently, the error cannot be improved beyond $\mathcal{O}(\Delta t^{1/2})$.

The error in the COS formulas arises from three steps: truncation of the integration range, substitution of the density by its cosine series expansion within the truncated range, and substitution of the series coefficients by the characteristic function approximation. The use of discrete cosine transformation to approximate the Fourier cosine coefficients yields an error with second-order convergence in N.

Lastly, the convergence of the Picard iterations primarily depends on the Lipschitz constant L_f of the driver function. For sufficiently small Δt, specifically when $L_f \Delta t \theta_1 < 1$, a unique fixed point exists, and the Picard iterations converge towards that point regardless of the initial guess. The fixed-point technique converges to the true solution at the geometric rate $\Delta t \theta_1 L_2$.

In conclusion, the total error is the sum of these three components. By selecting a sufficiently high value for N, we can neglect the error from the COS method, leaving us with the error of the discretization scheme.

The computation time of the BCOS method is linear in the number of time steps M. For each time-step t_m the following computations are done to get the approximations,

- Computation of the conditional characteristic function on a x-grid in $\mathcal{O}(N)$ operations.
- Computation of $z(t_m, x)$ and $h(t_m, x)$ on a x-grid in $\mathcal{O}(N)$ operations.
- Initial guess for Picard iterations. Computation of $\mathbb{E}_m^x[y(t_{m+1}, X_{m+1}^{\Delta})]$ in $\mathcal{O}(N)$ operations.
- Computation of the P_{picard} Picard iterations for $y(t_m, x)$ in $\mathcal{O}(P_{picard}N)$ operations.
- Computation of the Fourier coefficients $\mathcal{Z}_k(t_m)$, $\mathcal{Y}_k(t_m)$ and $\mathcal{H}_k(t_m)$ with the DCT in $\mathcal{O}(N\log(N))$ operations.

Summarizing, the total complexity of the BCOS method is $\mathcal{O}(M(N^2 + P_{picard}N + N\log(N)))$.

Chapter 5

Results

This chapter presents the numerical results obtained using the BCOS method for various applications. The computations were performed using Python (Jupyter Notebook) on a system equipped with an AMD Ryzen 7 4800H CPU @2.90 GHz and 16.0 GB of RAM. In our numerical experiments, we make use of four different discretization schemes,

$$\text{Scheme A:}\quad \theta_1 = 1 \text{ and } \theta_2 = 1 \qquad \text{Scheme C:}\quad \theta_1 = \frac{1}{2} \text{ and } \theta_2 = \frac{1}{2}$$
$$\text{Scheme B:}\quad \theta_1 = 0 \text{ and } \theta_2 = 1 \qquad \text{Scheme D:}\quad \theta_1 = 0 \text{ and } \theta_2 = \frac{1}{2}.$$

5.1 Test the Method

To validate the numerical method used in this study, we perform two sets of analyses. Firstly, we demonstrate its capability to solve fully coupled FBSDEs with a diffusion coefficient dependent on Z_s, providing evidence of its effectiveness. This verification process involves solving a toy model derived from [41].

Secondly, we aim to establish the applicability of the BCOS method in solving stochastic control problems. For this purpose, we utilize the earlier studied paper by Gobet [10], which presents a well-defined stochastic control problem with an analytically solvable solution. By comparing our numerical results with the clean analytical solution provided by Gobet, we assess the accuracy and reliability of our method.

Through these validation steps, we ensure that our code performs well, delivering the expected results for both fully coupled FBSDEs and stochastic control problems.

5.1.1 Fully coupled FBSDE

A concrete example illustrating a type of fully coupled FBSDE can be found in [41] by Zhao et al. Specifically, Example 5 in the paper showcases such a problem, where the diffusion coefficient in the forward SDE depends on Z_t. This forms a class of problems where an efficient numerical method is not yet available. The fully coupled FBSDE is defined as,

$$\begin{aligned} X_t = x &- \int_0^t \frac{1}{2}\sin(s + X_s)\cos(s + X_s)(Y_s^2 + Z_s)ds \\ &+ \int_0^t \frac{1}{2}\cos(s + X_s)(Y_s \sin(s + X_s) + Z_s + 1)dW_s, \end{aligned} \tag{5.1.1}$$

$$Y_t = \sin(T + X_T) + \int_t^T Y_s Z_s - \cos(s + X_s)ds - \int_t^T Z_s dW_s, \tag{5.1.2}$$

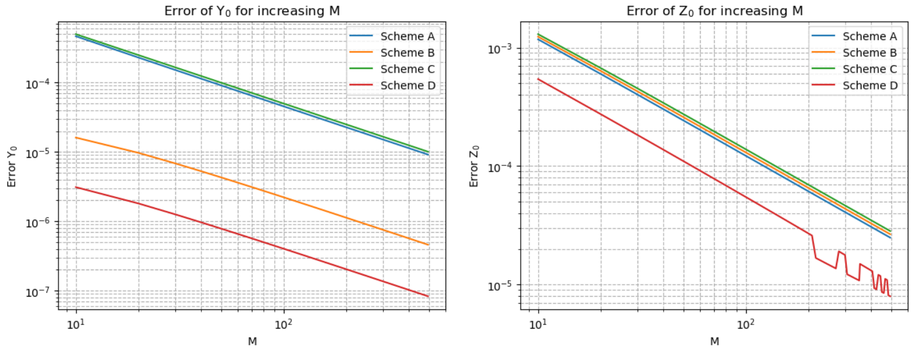

Figure 5.1: Plot of the error at t=0 for (Y,Z)

with terminal conditions,

$$Y_T = \sin(T + X_T)$$
$$Z_T = \frac{1}{2}\cos(T + X_T)(Y_T \sin(T + X_T) + Z_T + 1)\cos(T + X_T).$$

Notice that when the diffusion depends on z, we cannot know the terminal coefficients $\mathcal{Z}_k(t_\mathcal{M})$ and $\mathcal{F}_k(t_\mathcal{M})$. Therefore, we need to set $\theta_2 = 1$ in the first time step to circumvent this problem, after that we can use all other schemes. It is also possible that we set immediately $\mathcal{Z}_k(t_\mathcal{M}) = 0$ to circumvent this problem, but we have to remember this when recovering the control variable. The exact solutions to Eq. (5.1.1) and Eq. (5.1.2) are given by,

$$y(t, x) = \sin(t + x), \quad \text{and} \quad z(t, x) = \cos^2(t + x).$$

For our numerical experiments, we set the terminal time as $T = 0.1$, the initial value of the process as $X_0 = 1.5$, and choose the interval parameters as $a = -2\pi$ and $b = 2\pi$. In this case, the diffusion coefficient depends on Z_t, which introduces complexities in calculating the terminal coefficients due to self-referencing. To overcome this challenge, we either set $Z_T = 0$ or $\theta_2 = 1$ in the first time step for all schemes. By doing so, the numerical approximation becomes independent of $\mathcal{Z}_k(t_M)$ and $\mathcal{H}_k(t_M)$.

The results obtained from the numerical method are presented in Figures 5.1 and 5.2. We observe that the method converges for all schemes in $t = 0$. The error through time is sampled for a single Brownian motion, and one can approximate the expectation of this error using Monte Carlo simulations.

In conclusion, we have successfully solved fully coupled FBSDEs using the BCOS method, which will be instrumental in determining the control variables for the allocation of the green-brown portfolio problems.

5.1.2 Gobet

In order to evaluate the BCOS method for a stochastic control problem, our focus lies on the specific case discussed by Gobet et al. in their paper [10]. However, instead of considering an infinite time horizon, we investigate the finite horizon version as addressed in Section 3.3, primarily due to our interest in finite time intervals, which align with the terminal time for climate mitigation.

The analytical solution for the control variable, denoted as γ, is already known. Nevertheless, in order to obtain a numerical solution, it is essential for us to determine the optimal feedback map. By analyzing the optimization criterion presented in Gobet et al.'s work Eq. (3.3.5), along with the provided expressions for the drift, production, cost, and penalty functions, we are able to

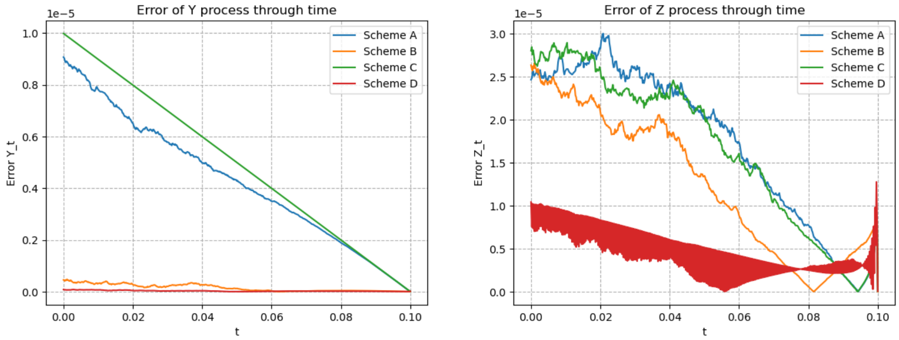

Figure 5.2: Plot of the error through time of the (Y,Z) for one simulation

deduce the following,

$$\hat{\gamma}_t = \min\left(c\hat{Y}e^{rt}, \frac{1}{1+\omega}(\omega e_t + c\hat{Y}e^{rt})\right).$$

SSP benchmarks In line with the research conducted in [10], we examine various CO_2 emission scenarios that correspond to different socioeconomic reference pathways provided by CMIP6 (Coupled Model Intercomparison Project Phase 6). Specifically, we consider the following scenarios: SSP1-2.6, SSP2-4.5, SSP3-LowNTCF, SSP4-6.0, and SSP5-3.4-OS. These scenarios, available in the SSP Public Database at https://tntcat.iiasa.ac.at/SspDb, are valuable resources that facilitate our understanding of and ability to anticipate the mid-term and long-term consequences of near-term decisions, particularly in light of the noticeable impacts of climate change.

For the purpose of comparison, we select the transportation sector and set the year 2015 as the starting point. The benchmark data required for our analysis can be obtained from the SSP Public Database. It is important to note that while the effects of climate change are already perceptible, these scenarios serve as indispensable tools for comprehending the potential ramifications of our present choices in the medium and long term.

Among the five SSPs, two baseline scenarios stand out. Firstly, SSP1-2.6 represents the most mitigated scenario, roughly corresponding to the previous scenario generation known as the Representative Concentration Pathway (RCP) 2.6. Secondly, SSP2-4.5 depicts a moderate scenario akin to RCP-4.5. Additionally, we consider three supplementary scenarios. These include SSP3-LowNTCF, which provides a comparison scenario with high Near-Term Climate Forcing (NTCF) emissions, notably sulfur oxides (SOx) and methane; SSP4-6.0, which focuses on a socioeconomic context characterized by inequality; and SSP5-3.4-OS (Overshoot), allowing for a significant overshoot in emissions until mid-century, followed by substantial policy interventions in the latter half of the century.

To facilitate the numerical computations, we plot these emission benchmarks and normalize them such that $e_0 = \hat{\gamma}_0 = \frac{c}{r-b}$. However, it is worth mentioning that due to limited data points, we employ an interpolation function to smoothen the graph. While this introduces some error, it is disregarded when assessing the convergence of the BCOS method error. Note that we receive different outcomes than in [10]. This is evidently because of the fact that we used a finite time horizon and used numerical solution to try to solve it. As one can observe, the scenario SSP1-2.6 is the hardest benchmark which imposes immediate reduction from the beginning 2015. For the transportation sector, the scenario SSP2-4.5, SSP3-LowNTCF and SSP4-6.0 impose no or very little emission constraints that the optimal emission remains at the level of $\bar{\gamma}$. In Fig. 5.4, it shows the impact of the parameter ω. More precisely, the optimal emission is decreasing with respect to the penalty force. In other words, a stronger penalty policy will induce larger emission reduction from the firm, which is as expected.

Based on the successful outcomes observed, we can confidently assert that the BCOS method is

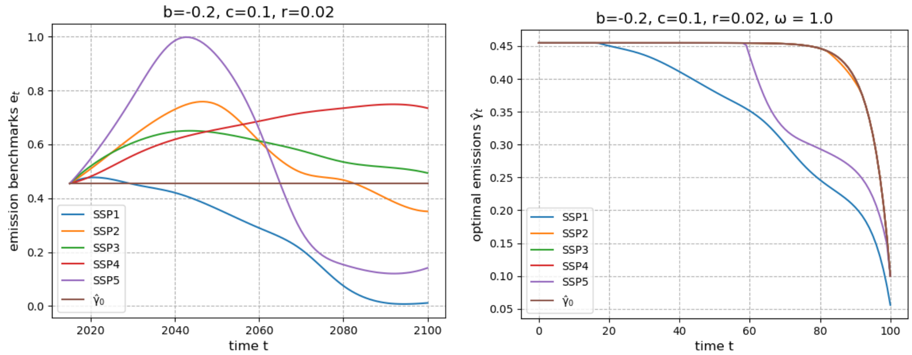

Figure 5.3: The SSPs emission scenarios v.s. associated optimal emission.

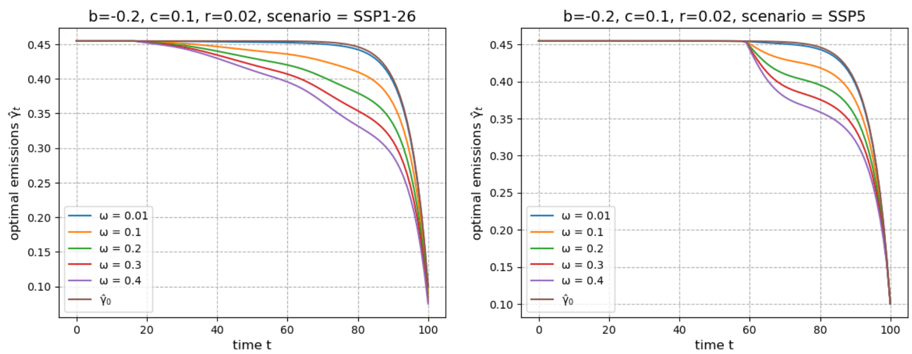

Figure 5.4: Different optimal emissions w.r.t. ω for scenarios SSP1 and SSP5.

effective in solving stochastic control problems, including the one investigated in this study. With this knowledge in hand, we are equipped to apply the BCOS method to determine solutions for the problems outlined in Section 3.4.

5.2 Portfolio based on Greenness

In the following analysis, we focus on our portfolio problem, specifically aiming to solve the allocation between the green and brown stocks. Through a parameter study, we explore different scenarios involving varying drifts and volatilities, providing a comprehensive analysis of the outcomes and discussing our findings.

To conduct the numerical study, we employ the BCOS method for the control problem outlined in Section 3.4 For the upcoming numerical results, we adopt Scheme D of the BCOS method, where $\theta_1 = 0.5$ and $\theta_2 = 0.5$. The initial wealth process is set at $X_0 = 1$, and for the Fourier cosine discretization, we choose $a = 0$, $b = 10$, $N = 2^8$, and $M = 100$ time steps. To achieve smoother graphs and reduce noise, we perform 500 Monte Carlo simulations to approximate the expectations.

In order to obtain a clearer understanding of the dynamics of the different processes, we set the terminal time $t_M = 31$, representing the time interval from 2020 to 2050. The drift of the assets is based on an annual increase of 6.8% for green stocks and 7% for brown stocks, while the corresponding volatilities are set as $\sigma^G = 0.19$ and $\sigma^B = 0.20$.

Through this parameter study and numerical simulations, we aim to gain insights into the behavior and dynamics of the portfolio allocation between Green and Brown stocks, considering the specified drifts and volatilities.

5.2.1 Contribution only

Let us consider the problem statement of Section 3.4.2. For analysis of the parameter study, we are interested in outcomes when using different constant contribution rates. For $c = 0, c = 0.01, c = 0.05$ and $c = 0.1$ we have plotted both the control variables α^G and α^B and the wealth process X in one graph to show that,

First of all, we notice that for an increasing contribution rate, an investor is more willing to invest in brown stocks than in green stocks, which is remarkable as opposed to the case when $c = 0$. The inclusion of a constant rate of contribution in a stochastic control problem has an impact on the outcome, even when the utility function is of a CRRA form. While this utility function considers the investor's risk aversion, the constant rate of contribution introduces a dynamic element, as seen in Eq. (3.4.4), that affects the overall wealth accumulation process, interacts with risk aversion and subsequently influences the optimal control strategy. There could be few reasons why the constant rate of contribution can influence the results of a stochastic control problem.

Time-varying wealth accumulation: The constant rate of contribution injects additional funds into the portfolio over time. This leads to an increasing wealth trajectory, which in turn affects the overall dynamics of the optimization problem. As the wealth grows, the relative importance of the control variables and the impact of investment decisions can change, potentially leading to different optimal strategies compared to a scenario without a contribution rate.

Impact on risk exposure: The constant rate of contribution alters the risk exposure of the portfolio. When new funds are injected into the portfolio, the allocation between risky and risk-free assets may change. This can affect the risk level and risk-adjusted performance of the portfolio, potentially leading to different control strategies to balance risk and return compared to a scenario without contributions.

Accumulation of compounding returns: The constant rate of contribution allows for compounding returns over time. As the portfolio grows with the accumulated contributions and investment returns, the compounding effect can magnify the impact of investment decisions and potentially result in different outcomes compared to a scenario without contributions.

The CRRA utility function captures risk preferences, but the contribution rate impacts wealth accumulation, introducing variables like time-varying wealth, evolving risk exposure, and compounded returns. These factors collectively contribute to discrepancies in numerical outcomes when comparing scenarios with and without the contribution rate.

It is worth noting that over time, for $c = 0.05$ and $c = 0.1$, the investor gradually shifts towards a higher allocation in green stocks until the terminal time, where the portfolio predominantly consists of green stocks. This behavior can be attributed to the nature of the problem's terminal condition. At the beginning of the investment strategy, it is advantageous to adopt a more aggressive approach, aiming for higher returns. This is because early on, there is room to absorb potential losses. However, as the terminal time approaches, the focus shifts towards preserving the accumulated savings. Consequently, investment strategies become increasingly conservative, aiming to mitigate any potential losses when cashing out at the end. Therefore, the observed trend of a transitioning portfolio allocation towards a larger proportion of green stocks can be understood as a strategic response to the evolving time horizon and the need to safeguard investments.

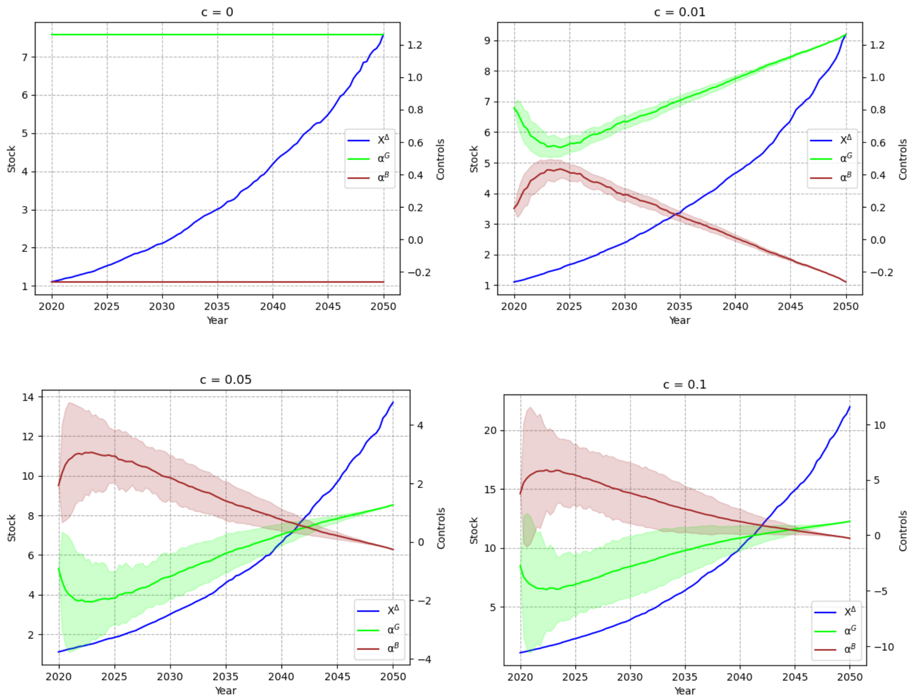

Figure 5.5: The average and standard deviation of the allocations of the stocks for different values of c, plotted against the discretization of wealth process X

5.2.2 PAB tax with contribution

Now, we will introduce the tax based on the Paris Alignment Benchmark, as discussed in Section 3.4. This benchmark will be a reduction of 7.6% each year. We will present plots that illustrate three distinct scenarios, each characterized by different reduction pathways.

Following the Paris reduction

Let's begin by examining a situation where both the green and brown stocks reduce their emissions by the same percentage as prescribed by the benchmark. However, they start at different values, specifically $\gamma^G = 0.9$ for the green stock and $\gamma^B = 1.1$ for the brown stock. In this case, the green

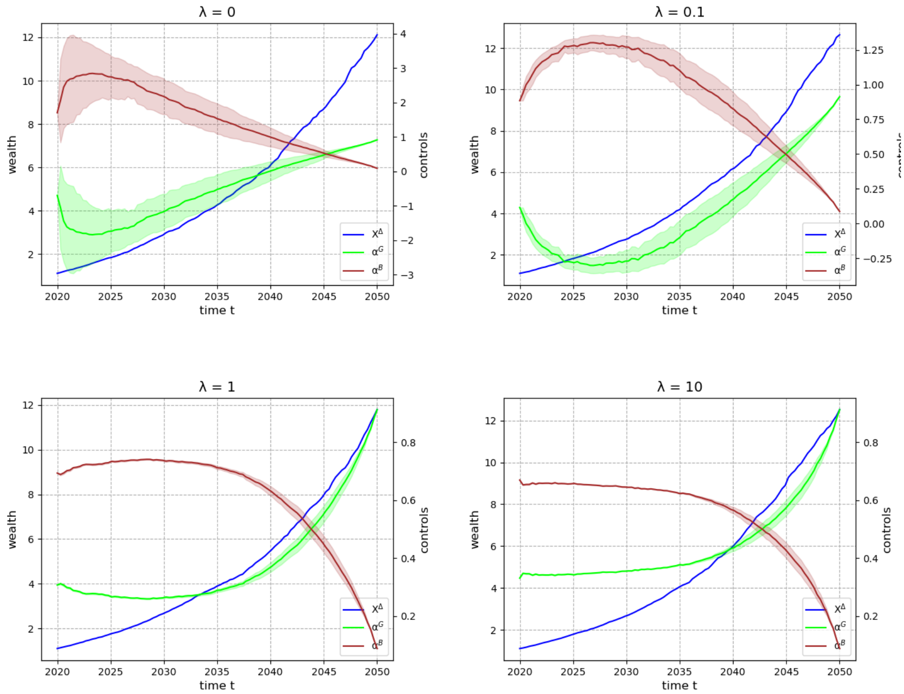

Figure 5.6: The average and standard deviation of the control variable for different values of λ, plotted with the discretization of wealth process X

company achieves a reduction of 7.8%, while the brown company manages to reduce by 7.4%. To explore the impact of the penalty parameter, denoted as λ, we have plotted the control variable of the allocation for various values of λ.

In Figure 5.6, observe that when $\lambda = 0$, the control variables for allocating green and brown stocks correspond to the scenario where only the contribution with $c = 0.05$ is considered, as expected. As λ increases, we observe an increase in the allocation of the green stock and a decrease in the allocation of the brown stock. When the investor is not constrained by penalties, they have more freedom in their investment strategies. However, as λ increases, we see a convergence towards the optimal control described in Equation 3.4.5, given by,

$$\hat{\alpha}_t^G \approx \frac{(e_t^{PAB} - \gamma_t^B)}{(\gamma_t^G - \gamma_t^B)}.$$

Consequently, we observe a convergence to $\alpha_0^G = 0.5$ as λ grows, which is due to their initial values at t_0.

Furthermore, we notice that as the terminal time approaches, there is a preference for the green stock in the allocation strategy. This preference can be attributed to the higher volatility associated with the brown stock. As investors near the end of the investment horizon, they tend to opt for safer, less volatile assets to protect their accumulated wealth, which is also seen in the scenario where we only added a contribution rate.

Shifting our focus to the emissions themselves, depicted in Figure 5.7, we can observe a convergence towards the PAB benchmark as the penalty parameter increases. This convergence indicates that the penalty mechanism effectively steers the total emissions of the portfolio towards the desired reduction targets set by the PAB. As the penalty becomes more severe, investors adjust their investment allocations to align with the emissions reduction goals, ultimately leading to emissions levels that fall below or closely match the PAB benchmark.

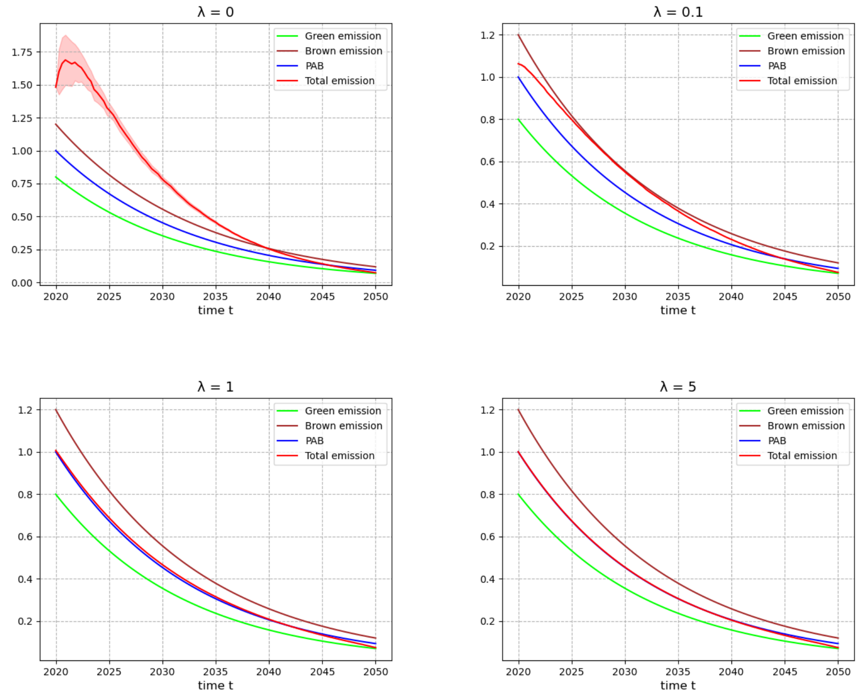

Figure 5.7: The average and standard deviation of the total emission of the portfolio for different values of λ, plotted with the PAB, Green and Brown emission paths.

Initial reduction of the Brown stock

Next, we explore a different scenario where the brown stock initially undergoes a reduction. We set $\gamma^G = 0.8$ and $\gamma^B = 0.9$, with annual reductions of 7.6% for the green stock and 5.6% for the brown stock. Consequently, the green stock aligns with the PAB benchmark, while the brown stock falls behind and incurs penalties. In Figure 5.8, we present the allocation controls for various values of λ.

Intuitively, it makes sense to invest primarily in the brown stock at the outset, as it still demonstrates favorable emission levels. In fact, the preference for the brown stock is strong enough to go short on green stocks, resulting in increased total emissions when the penalty has little to no influence. However, as the penalty parameter λ increases, it gradually aligns the total emissions with the PAB benchmark, as depicted in Figure 5.9. Notably, we observe some degree of asymptotic convergence when λ reaches a sufficiently high value, leading to a decrease in the standard deviation of the allocation controls.

Improving halfway through

Finally, let's consider a scenario where the brown stock demonstrates improvement and manages to reduce its emissions below the PAB benchmark. We begin with $\gamma^G = 0.9$ and $\gamma^B = 1.1$, and the annual reductions for green and brown stocks are 9.2% and 9%, respectively. The optimal allocation paths under these conditions are depicted in Figure 5.10.

When $\lambda = 0$, the investment strategy is not particularly wise, as it initially focuses heavily on the brown stock and then shifts towards a greener portfolio as the terminal time approaches. However, as we increase the penalty, we observe an increase in the allocation of the brown stock. This shift occurs because the investor recognizes the decreasing trend in brown stock emissions and takes advantage of the potential for higher returns. Ultimately, after falling below the PAB, the brown stock reaches its peak and achieves the highest return. Interestingly, in this scenario,

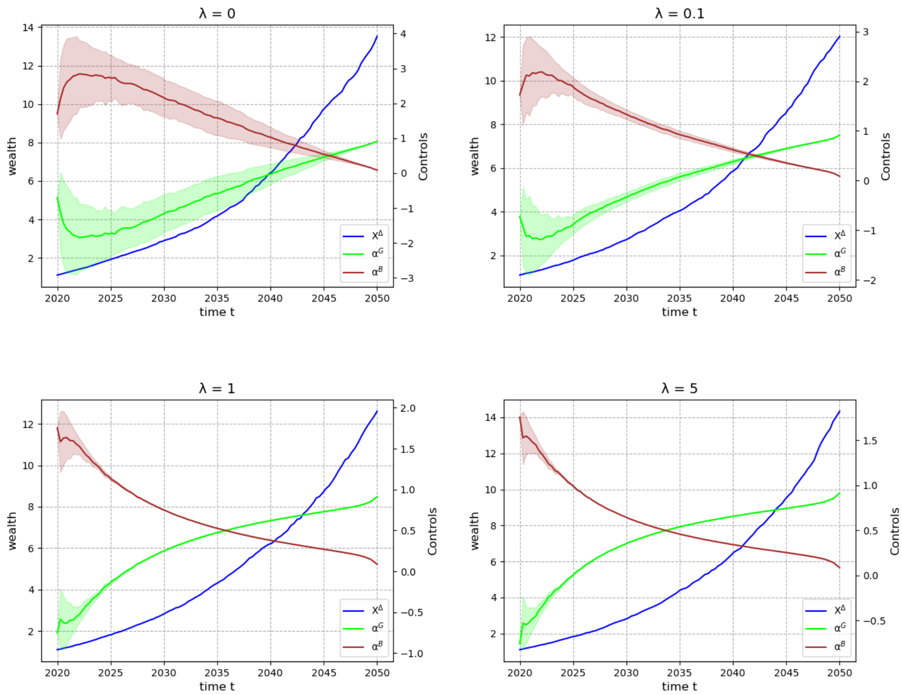

Figure 5.8: The average and standard deviation of the control variable for different values of λ, plotted with the discretization of wealth process X

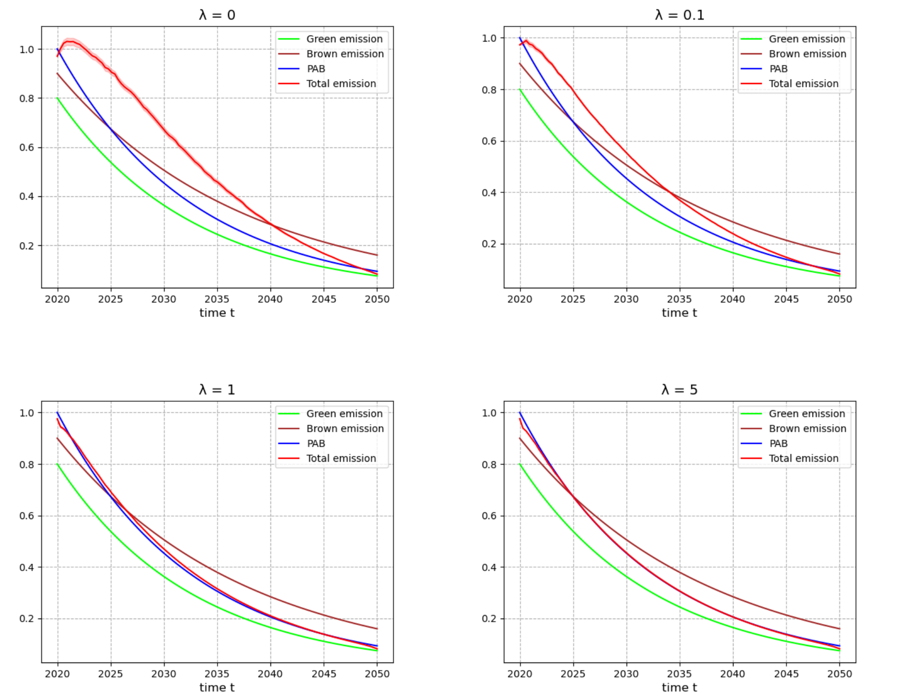

Figure 5.9: The average and standard deviation of the total emission of the portfolio for different values of λ, plotted with the PAB, Green and Brown emission paths.

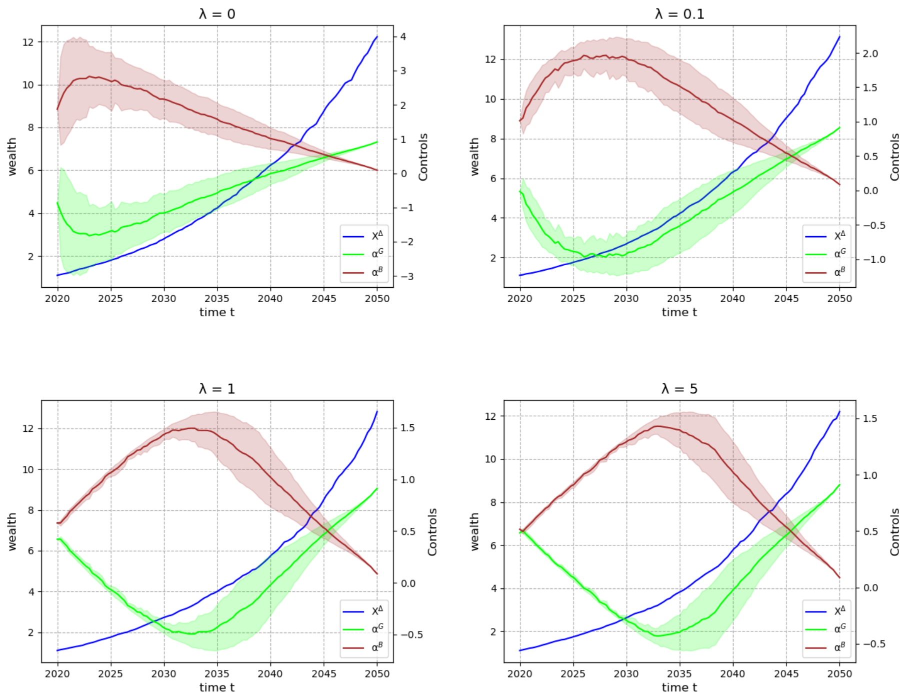

Figure 5.10: The average and standard deviation of the control variable for different values of λ, plotted with the discretization of wealth process X

the wealth process X does not decrease significantly with increasing penalty λ. This is due to the investor taking on additional risk by incurring penalties at the beginning to secure higher returns in the middle. However, similar to the previous situation, the higher return is eventually outweighed by higher volatility, prompting a shift towards the safer and less volatile green stock. This shift is also reflected in the emission paths, as shown in Figure 5.11, where the total emissions decrease as the penalty increases, eventually falling below the PAB for $\lambda > 5$.

It is worth noting that the standard deviation of the controls does not diminish as the penalty increases. This can be attributed to the absence of asymptotic convergence towards the terminal time, as observed in the previous setting. Due to the willingness to shift preferences towards the brown stock and subsequently transition to a greener portfolio, uncertainties are introduced, which may not be desirable for investors.

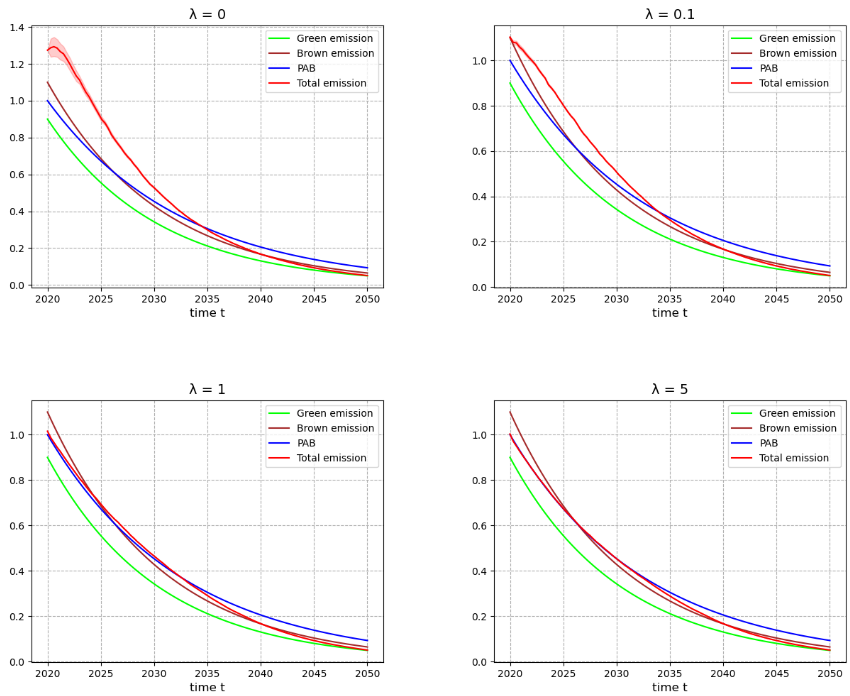

Figure 5.11: The average and standard deviation of the control variable for different values of λ, plotted with the discretization of wealth process X

Chapter 6

Conclusion

In the chapter we will conclude this thesis. We will first summarize our findings and the main results of this work. Finally, we will express some extension that can be made for further development, and list other problems which we are able to tackle now, for which it can be useful.

6.1 Concluding summary

Throughout this thesis, we have delved into the challenging task of solving fully coupled FBSDEs arising from stochastic control problems, specifically in the context of pension fund management. Our findings underscore the complexity of these problems and the ongoing research being conducted in this area.

In our parameter study in Chapter 5, we have explored the impact of a tax based on the total emissions of a portfolio. As anticipated, we observed that higher penalty values prompt investors to transition to a greener portfolio at an earlier stage. This aligns with the objective of incentivizing environmentally friendly investments and promoting sustainability. However, it is noteworthy that despite the appeal of greener assets from an environmental standpoint, the higher return rates and volatility associated with brown stocks still render them financially preferable. Consequently, when considering the utility of overall wealth, investors may still opt to allocate a significant portion of their portfolio to more polluting companies.

These results highlight the intricate trade-offs faced by investors, as they navigate the intersection of financial returns, environmental considerations, and regulatory constraints. It becomes evident that achieving an optimal balance between financial profitability and sustainability goals remains a complex challenge, requiring ongoing research and the development of sophisticated investment strategies.

6.2 Future work

There are different ways in which our approach and findings could be improved or extended. One potential direction is to incorporate stochastic interest rates into the analysis as in [14]. By introducing randomness to interest rate dynamics, we can capture the uncertainty and fluctuations in financial markets, which can significantly influence investment decisions and portfolio performance. It would be also interesting to look into the difference in brown and green bonds as the green bond market still represents less than 1 percent of the global bond market [19].

Additionally, incorporating stochastic emission paths of the corresponding stocks would enhance the realism of the model. By considering the variability and unpredictability of emission levels, we can better account for the inherent uncertainties associated with climate-related factors. This addition would enable a more comprehensive analysis of the effects of emissions on portfolio returns and the implications for optimal investment strategies.

The investigation conducted by [2] examined the implications of climate change concerns and observed that green firms tend to outperform brown firms when these concerns unexpectedly increase. Incorporating these findings into a parameter study would add an intriguing dimension to the optimization of portfolio allocations. Exploring how changes in climate change perceptions impact the performance of green and brown firms could provide valuable insights for optimizing investment strategies within a climate-conscious framework.

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