Utrecht University

Master Thesis

Extensions of group varieties and l-adic cohomology

Victor de Vries

supervised by

Prof. Gunther Cornelissen

second reader:

Dr. Valentijn Karemaker

Acknowledgements

I would like to thank Prof. Cornelissen for supervising me on this project. He responded to my emails quickly with useful advice and discussing with him kept me enthusiastic, inspired and sometimes reassured that I was on the right track throughout the entire process. I also appreciate that I was given a lot of freedom to gravitate towards a subject that I would enjoy. I would also like to thank dr. Karemaker for being my second corrector and providing me with useful feedback. Last I would like to thank my family and Eeke for supporting me when things went well and when things were frustrating.

Introduction

The goal of this document is to study the l-adic cohomology of group varieties over an algebraically closed field k with $\mathrm{char}(k) \neq l$. A group variety is a connected variety whose functor of points is a group functor. Examples include $\mathbb{A}^n$, abelian varieties such as elliptic curves and linear algebraic groups (subvarieties of GL_n whose functors of points inherit the group law). In particular consider an exact sequence $e \to K \to G \to Q \to e$ of group varieties. We will consider the question how the cohomology rings $\mathrm{H}^*_{\text{ét}}(K, \mathbb{Q}_l)$ and $\mathrm{H}^*_{\text{ét}}(Q, \mathbb{Q}_l)$ are related to $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$.
Functoriality of l-adic cohomology gives that if $f : X \to Y$ is a morphism, there is a $\mathbb{Q}_l$-algebra homomorphism $f^* : \mathrm{H}^*_{\text{ét}}(Y, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_l)$. In the case that $f : X \to X$ is a self-morphism of a projective variety with finitely many fixed points one has the famous Groethendieck-Lefschetz trace formula which gives that the graded trace $\mathrm{tr}_X(f)$ of f^* equals the number of fixed points with multiplicity. This does not quite generalize to arbitrary varieties, however by the famous Weil-conjectures one can count the fixed points of the Frobenius morphism by using compactly supported cohomology. We will see in Chapter 3 that for an endomorphism $\sigma : G \to G$ of a group variety with finitely many fixed points the graded trace of σ^* relates directly to the fixed point count of σ. The cohomology ring of a general scheme is quite an abstract object and computing how the pullback morphism behaves may not be doable. We will see that for G a torus an abelian variety or a unipotent group variety, there is a functorial isomorphism $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l) \cong R$ where R is a graded ring on which we understand the pullback morphism 'much better' (more precisely R is functorially the exterior algebra on the vector space spanned by the characters, resp. the exterior algebra on the l-adic Tate module, resp. trivial). We will also consider the question if a similar result holds for a semisimple group variety.

In Chapter 1 an introduction to algebraic groups is given, which is largely based on Milne's book [32]. We give a few proofs and state several results that we will need in the later chapters. The theory of group varieties is in some sense quite close to the theory of groups as many concepts from the latter carry over to the former; actions, normal subgroups, quotients, isomorphism theorems... Some of the most important results in this chapter are certain structure theorems such as the Chevalley-Barsotti theorem, which states that any subgroup variety G over a perfect field has a largest normal linear group variety G_{lin} such that the quotient $G_{\text{ab}} := G/G_{\text{lin}}$ is an abelian variety. Similarly the linear group variety can then be decomposed further into a unipotent group variety, a torus and a semisimple group variety. We introduce the anti-affine group variety G_{ant} and give another important structure theorem due to Rosenlicht [37] stating that over a perfect field k the multiplication map $\mu : G_{\text{ant}} \times_k G_{\text{lin}} \to G$ is a quotient map.

In Chapter 2 we will give an introduction to étale cohomology. Most of the material in here is based on another book of Milne [29]. We begin by introducing the étale site of a scheme X and sheaves on it. After this we introduce the functors f_* and f^* associated to a morphism

$f : X \to Y$. We then introduce étale cohomology and compactly supported cohomology with their main properties before moving on to stating several classical results such as Poincaré duality, the Grothendieck-Lefschetz fixed point formula, the proper base change theorem, cohomological dimension and finiteness of étale cohomology. We then move on to proving the classical result that an isogeny $f : G \to H$ of group varieties induces an isomorphism $f^* : \mathrm{H}^*_{\text{ét}}(H, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ of which a proof is sketched in Srinivasan [42]. Finally we give a very brief introduction on the Chow ring $\mathrm{A}^*(X)$ of X based on [22] and we state the existence and properties of the l-adic cycle map $\mathrm{cl}^X : \mathrm{A}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to \mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_l)$.

Chapter 3 is devoted to describing the sequence $(\sigma_n)_n$, where σ_n is the cardinality of the set of fixed points of $\sigma^{\circ n}$ for $\sigma : G \to G$ is a surjective endomorphism such that all iterates have finitely many fixed points. It is largely based on work by Byszewski, Cornelissen and Houben [8]. We state a deep theorem by Steinberg [44] which says that the Lang-map $L_\sigma : g \mapsto g^{-1}\sigma(g)$ is surjective when σ is surjective and fixes finitely many points. Using the surjectivity of the Lang-map combined with a filtration of G by characteristic subgroups allows us to split up the fixed point count of σ into counting fixed points of the induced endomorphisms on the pieces in the decomposition of G. As a result, the authors of [8] were able to find the formula $\sigma_n = |d_n| c^n r_n |n|_p^{s_n} p^{-t_n |n|_p^{-1}}$ where r_n, s_n, t_n define gcd-sequences, i.e. they are sequences that have $a_n = a_{\gcd(n,\omega)}$ for some $\omega \in \mathbb{Z}_0$. The part d_n is linearly recurrent and comes from certain cohomological traces from the pieces in the decomposition of G. The question that remained is whether we actually have $d_n = \mathrm{tr}_G(\sigma)$. The authors of [8] solved this over $\overline{\mathbb{F}}_p$ by using a theorem of Arima [1], which implies that in this case $G_{\text{ab}} \times_{\overline{\mathbb{F}}_p} G_{\text{lin}}$ and G are isogenous and thus have isomorphic cohomology rings. However the general case remained open. This motivates studying how the l-adic cohomology of G_{lin} and G_{ab} relate to the l-adic cohomology of G.

In Chapter 4 we work towards proving two of Arima's theorems ([1], Theorem 1 and Theorem 2). We introduce the $\mathrm{Ext}(-,-)$ bifunctor, which classifies extensions of a commutative algebraic group B by a commutative algebraic group H up to a certain isomorphism. The methods used by Baer [4] give that $\mathrm{Ext}(B, H)$ is a commutative group. One of Arima's theorems states if G is an extension of a linear group variety L by an abelian variety A that $L \times_k A$ is isogenous to G if and only if $[G] \in \mathrm{Ext}(A, L)$ is of finite order. We show that this implies the second theorem of Arima, which states that in the case of $k = \overline{\mathbb{F}}_p$ that G is isogenous to $A \times_{\overline{\mathbb{F}}_p} L$. In the process we show that there is a natural map $\mathrm{Ext}(B, H) \to \mathrm{H}^1_{\text{ét}}(B, \underline{H})$ for $\underline{H}$ the étale sheaf $\mathrm{Mor}(-, H)$ on B in the case that H is smooth. We show that under suitable hypothesis on B, H the image of this homomorphism equals the primitive subspace of $\mathrm{H}^1_{\text{ét}}(B, \underline{H})$. Later in the chapter we exhibit an explicit example of a generalized Jacobian $J_{\mathfrak{m}}$, which in our case will be an extension of the multiplicative group $\mathbb{G}_m$ by an elliptic curve E such that $J_{\mathfrak{m}}$ and $E \times_k \mathbb{G}_m$ are not isogenous. This construction can be made whenever $E(k)$ has a point of infinite order.

In the first part of Chapter 5 we let k be an algebraically closed field and we consider the exact sequence $e \to G_{\text{lin}} \xrightarrow{\iota} G \xrightarrow{\pi} G_{\text{ab}} \to e$. The goal is showing that $\iota^* : \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}, \mathbb{Q}_l)$ admits a section s giving that $(s \otimes \pi^*) : \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ is an isomorphism of graded $\mathbb{Q}_l$-algebras. First we consider the case where G is commutative. By comparing the functors $\mathrm{H}^1_{\text{ét}}(-, \mathbb{Z}/l^n\mathbb{Z})$ and $\mathrm{Ext}(-, \mathbb{Z}/l^n\mathbb{Z})$ (done earlier by Miyanishi [33]) and using properties of Ext from Serre's book [39] we obtain $0 \to \mathrm{H}^1_{\text{ét}}(G_{\text{ab}}, \mathbb{Q}_l) \xrightarrow{\pi^*} \mathrm{H}^1_{\text{ét}}(G, \mathbb{Q}_l) \xrightarrow{\iota^*} \mathrm{H}^1_{\text{ét}}(G_{\text{lin}}, \mathbb{Q}_l) \to 0$ which is exact. Since $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ is a finite dimensional graded-commutative Hopf algebra we can use a cohomological dimension argument together with a structure theorem on such Hopf algebras by Hopf [5] to obtain an isomorphism $(s \otimes \pi^*) : \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ (also

see [7]).
In the general case we consider the quotient map $\mu : G_{\text{ant}} \times_k G_{\text{lin}} \to G$ whose kernel is up to finite index equal to $(G_{\text{ant}})_{\text{lin}}$. For the quotient Q, the fibration $(G_{\text{ant}})_{\text{lin}} \to G_{\text{ant}} \times_k G_{\text{lin}} \to Q$ satisfies the conditions needed to use the Leray-Hirsch principle. We note that Q and G are isogenous and thus that their cohomology is isomorphic. As G_{ant} is commutative we know that its cohomology has the above tensor product decomposition. Using these facts together with the fact that $(G_{\text{ant}})_{\text{ab}}$ and G_{ab} are isogenous we are able to obtain the desired isomorphism given by $(s \otimes \pi^*) : \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$. As far as we know this result is not stated anywhere in the literature.

In the second part of Chapter 5 we build upon the result of obtaining the previous isomorphism by showing that any exact sequence of group varieties $e \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to e$ over an algebraically closed field has that ι^* admits a section s such that $(s \otimes \pi^*) : \mathrm{H}^*_{\text{ét}}(K, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^*_{\text{ét}}(Q, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ is an isomorphism of graded $\mathbb{Q}_l$-algebras. For doing this the notion of an almost exact sequence of group varieties is introduced, which relaxes the conditions $\ker(\iota) = \ker(\pi)/\mathrm{Im}(\iota) = e$ to both being finite group schemes. This concept is introduced because for instance $e \to K_{\text{lin}} \to G_{\text{lin}} \to Q_{\text{lin}} \to e$ is almost exact when $e \to K \to G \to Q \to e$ is exact. We show that the cohomology of the almost exact sequences that we consider decomposes as above. We then conclude by the above result on $e \to G_{\text{lin}} \xrightarrow{\iota} G \xrightarrow{\pi} G_{\text{ab}} \to e$ and by certain other exact sequences splitting up to isogeny that we indeed have such an isomorphism $(s \otimes \pi^*) : \mathrm{H}^*_{\text{ét}}(K, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^*_{\text{ét}}(Q, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$.

Chapter 6 is devoted to proving the following result: Let G be a semisimple algebraic group and let $\sigma : G \to G$ be a surjective endomorphism with finitely many fixed points and let T be a σ-stable maximal torus of G. Let $S := \mathrm{Sym}(X(T) \otimes_{\mathbb{Z}} \mathbb{Q}_l)$ on which the Weyl group W acts linearly and denote by S^W_+ the invariants of positive degree and denote $\mathrm{J} := S^W_+/(S^W_+)^2$. The statement is that there is a functorial isomorphism $\bigwedge^* \mathrm{J}[\times 2 - 1] \cong \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$, where $[\times 2 - 1]$ means that the degrees are doubled and then lowered by 1 and where functorial means that the pullback morphisms σ^* on both sides are respected by the isomorphism. Note that the existence of an isomorphism follows from work done on Lie groups by Borel [5] together with G being defined over $\mathrm{Spec}(\mathbb{Z})$ with a comparison theorem on l-adic cohomology by Friedlander [21]. However this approach unfortunately does not give the desired functoriality. Altough the semisimple group variety G lifts to $\mathbb{C}$, the endomorphism $\sigma : G \to G$ need not lift.
We let (B, T) be a σ-stable Borel pair and begin by describing the cohomology of G/B. We use the cellular decomposition of G/B to obtain that $\mathrm{cl}^{G/B} : \mathrm{A}^*(G/B) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to \mathrm{H}^*_{\text{ét}}(G/B, \mathbb{Q}_l)$ is an isomorphism. By a result of Demazure [14] on $\mathrm{A}^*(G/B) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ we then obtain all the dimensions of $\mathrm{H}^r_{\text{ét}}(G/B, \mathbb{Q}_l)$. Using that $G/T \to G/B$ is a fibration over an affine space we obtain an isomorphism between $\mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ and $\mathrm{H}^*_{\text{ét}}(G/B, \mathbb{Q}_l)$ given by the pullback. By applying the Leray spectral sequence to the morphism $\pi : G \to G/T$ (also done for Lie groups by Leray [28]) we obtain, as the sheaves $R^q\pi_*(\mathbb{Z}/l^n\mathbb{Z})$ are equal to the constant sheaf $\mathrm{H}^q_{\text{ét}}(T, \mathbb{Z}/l^n\mathbb{Z})$, a homomorphism of rings $d_2^{0,1} : \mathrm{Sym}(\mathrm{H}^1_{\text{ét}}(T, \mathbb{Q}_l)) \to \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$. We show after the identification $S = \mathrm{Sym}(\mathrm{H}^1_{\text{ét}}(T, \mathbb{Q}_l))$ is made that $\ker(d_2^{0,1}) = S^W_+ \cdot S$ and that $d_2^{0,1}$ is surjective. We also consider the E_2-page as a complex with terms $E_2^{*,q} = \bigwedge^q (X(T) \otimes_{\mathbb{Z}} \mathbb{Q}_l) \bigotimes_{\mathbb{Q}_l} S/(S^W_+ \cdot S)$ and we show the existence of a homomorphism $\mathrm{J} \to h_1(E_2^{*,\bullet}) = E_3^{*,1}$ that extends to a graded $\mathbb{Q}_l$-algebra isomorphism $\bigwedge^* \mathrm{J} \to E_3^{*,*}$. After showing that this implies that the spectral sequence degenerates at the E_3-page, we obtain

a $\mathbb{Q}_l$-algebra isomorphism $\bigwedge^* \mathrm{J}[\times 2 - 1] \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ that has the crucial property for us, namely that it commutes with pulling back via σ^*. In particular we deduce from the results in Chapter 5 and Chapter 6 that the term d_n from Chapter 3 that was defined by the authors in [8] equals $\mathrm{tr}_G(\sigma^n)$.

Contents

Chapter 1

Algebraic groups

In this chapter we introduce algebraic groups over a general field k. We state a number of classical theorems such as the Chevalley-Barsotti theorem and the existence of largest normal subgroup varieties with certain properties. A good reference for the material covered in this chapter is [32]. Throughout this chapter k will be an arbitrary field.

Definition 1.1. An *algebraic group* is a scheme G of finite type over k with k-morphisms $\mu : G \times_k G \to G$, $\mathrm{inv} : G \to G$ and $e : \mathrm{Spec}(k) \to G$ such that the following diagrams commute:

$$\begin{array}{ccc} G \times_k G \times_k G & \xrightarrow{\mu \times \mathrm{Id}} & G \times_k G \\ \downarrow{\scriptstyle \mathrm{Id} \times \mu} & & \downarrow{\scriptstyle \mu} \\ G \times_k G & \xrightarrow{\mu} & G \end{array} \qquad \begin{array}{ccccc} \mathrm{Spec}(k) \times_k G & \xrightarrow{e \times \mathrm{Id}} & G \times_k G & \xleftarrow[\mathrm{Id} \times e]{} & G \times_k \mathrm{Spec}(k) \\ & \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\ & & G & & \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{inv},\mathrm{Id})} & G \times_k G & \xleftarrow{(\mathrm{Id},\mathrm{inv})} & G \\ \downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{e} & G & \xleftarrow{e} & \mathrm{Spec}(k) \end{array}$$

Remark 1.2. Let k be a field and let G be an algebraic scheme over k. Then G is an algebraic group if and only if $\mathrm{Hom}_k(\mathrm{Spec}(-), G) : k\text{-Alg} \to \mathrm{Set}$ factors through the category of groups via the forgetful functor $\mathrm{Grp} \to \mathrm{Set}$. This follows by applying the Yoneda Lemma.

Note that an algebraic group need not be connected or geometrically reduced and hence this is not assumed to be the case unless explicitly stated. Note that for any field k' containing k, we can base-change G to $G_{k'}$ to obtain an algebraic group over k'.

Definition 1.3. A *group variety* G over k is a connected geometrically reduced algebraic group over k.

That we only require connected and not irreducible in the definition has to do with the following.

Lemma 1.4. *([32], Summary 1.3.6) For G an algebraic group the following are equivalent:*

- *G is connected.*

- *G is geometrically irreducible.*

We also have that group varieties are smooth.

Proposition 1.5. *A geometrically reduced algebraic group G is smooth.*

Proof. We base-change the algebraic group G over k to one $G_{\bar{k}}$ over $\bar{k}$ whose underlying scheme is a variety by assumption. It is a classical fact ([23], Theorem I.5.3) that every variety over an algebraically closed field has a smooth point $x \in G_{\bar{k}}(\bar{k})$. Now for any $y \in G_{\bar{k}}(\bar{k})$, the left-translation map $\tau_{xy^{-1}} : G_{\bar{k}} \to G_{\bar{k}}$ is an isomorphism of schemes and hence an isomorphism $\mathcal{O}_{G_{\bar{k}},y} \to \mathcal{O}_{G_{\bar{k}},x}$ and hence we conclude that y is also a smooth point of $G_{\bar{k}}$. □

Now we define homomorphisms.

Definition 1.6. A *homomorphism* of algebraic groups over k is a k-morphism of schemes $\varphi : G \to H$ such that the maps $\varphi_R : G(R) \to H(R)$ are group homomorphisms for all R/k.

There is an equivalent definition involving diagrams as in the definition of an algebraic group.

Remark 1.7. By checking the axioms one verifies that there is a category of algebraic groups whose objects are algebraic groups and whose morphisms are homomorphisms.

Definition 1.8. Let G be an algebraic group over k. An *algebraic subgroup H* of G is a subscheme H of G such that μ, inv, e restrict to H, i.e. H inherits the structure of an algebraic group by G.

We introduce some examples of algebraic subgroups.

Definition 1.9. Denote the component of G that contains e by G°. It is called the *identity component.*

Lemma 1.10. *The connected component of G that contains e is an algebraic subgroup of G. When k is perfect the reduced subscheme G_{red} is an algebraic subgroup of G.*

We have the following lemma that says that algebraic subgroups are closed.

Lemma 1.11. *Let G be an algebraic group over k. An algebraic subgroup H of G is in particular a closed subscheme of G.*

The following special cases of homomorphisms are important to us.

Definition 1.12. A homomorphism $\varphi : G \to H$ is called an *embedding* if φ is a closed immmersion. It is called a *quotient map* if it is faithfully flat.

Just like in group theory there is the notion of normal subgroup.

Definition 1.13. Let H, N be algebraic subgroups of G. Say that H *normalizes* N if $H(R)$ normalizes $N(R)$ inside $G(R)$ for all R/k. In particular, say that N is a *normal algebraic subgroup* of G if G normalizes N.

Definition 1.14. A *flat sheaf* is a covariant functor $F : \{\text{finitely generated } k\text{-algebras}\} \to \mathrm{Grp}$ such that:

1. Whenever $\iota : R \hookrightarrow S$ is faithfully flat the sequence $F(R) \to F(S) \rightrightarrows F(S \otimes_R S)$ is exact, i.e. the elements in $F(S)$ that are mapped to the same element under $F(\iota \otimes 1)$ and $F(1 \otimes \iota)$ are precisely the elements that come from $F(R)$.

2. The projection maps induce an isomorphism $F(R_1 \times .... \times R_n) \cong F(R_1) \times ... \times F(R_n)$.

An algebraic group actually defines such a flat sheaf.

Lemma 1.15 ([32], Lemma 5.9)**.** *The functor* $\underline{G} := \mathrm{Hom}_k(\mathrm{Spec}(-), G)$ *is a flat sheaf.*

This is quite easy to show when G is affine and by covering G with affine opens subschemes the general case follows.

Remark 1.16. By the Yoneda-lemma we have that an algebraic group G is completely determined by its flat sheaf $\underline{G}$. Actually, giving a flat sheaf as we defined is the same as giving a sheaf for the 'flat topology' on $\mathrm{Spec}(k)$. For a covariant functor $F : \{\text{finitely generated } k\text{-algebras}\} \to \mathrm{Grp}$ there is a flat sheaf aF called the *sheafification* of F with a morphism $F \to aF$ that is universal among morphisms from F to a flat sheaf ([32], Prop 5.68). It is unique up to unique isomorphism.

We use the sheafification for the following.

Definition 1.17. Let $\varphi : G \to H$ be a homomorphism of algebraic groups. Define the *kernel sheaf* by $\ker : R \mapsto \ker(\varphi_R)$. Define the *image sheaf* Im by setting it to be the sheafification of the presheaf $R \mapsto \mathrm{Im}(\varphi_R)$.

Remark 1.18. Both of the above flat sheaves are representable by algebraic groups called the *kernel* of φ and the *image* of φ respectively. They are denoted by $\ker(\varphi)$ and $\varphi(G)$. It is easy to see that the kernel sheaf is represented by the following fibre product:

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & H \\ \uparrow & \urcorner & \uparrow e \\ \ker(\varphi) & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

(which also shows that it is in fact a flat sheaf). That the image sheaf is representable is much harder to show. It is done in Appendix B of [32].

Definition 1.19. A sequence $H \xrightarrow{f} G \xrightarrow{g} Q$ of algebraic groups is *exact* if $\mathrm{Im}(f) = \ker(g)$.

Remark 1.20. It follows from the definition of the kernel and image that a sequence $H \to G \to Q$ of algebraic groups is exact if and only if the sequence of flat sheaves on $\mathrm{Spec}(k)$, $\widetilde{H} \to \widetilde{G} \to \widetilde{Q}$, is exact. Note that this is in general **not** the same as $\underline{H}(R) \to \underline{G}(R) \to \underline{Q}(R)$ being exact for all R. For a counterexample, let $k = \mathbb{Q}$ and consider the n'th power map $\mathbb{G}_m \to \mathbb{G}_m$. Then $\mathbb{Q}^\times \to \mathbb{Q}^\times \quad x \mapsto x^n$ is not surjective, but for all $r \in R^\times$ one has the faithfully flat ring map $R \to R[X]/(X^n - r)$ and r is in the image of $(R[X]/(X^n - r))^\times \to (R[X]/(X^n - r))^\times$, hence $\widetilde{\mathbb{G}_m} \to \widetilde{\mathbb{G}_m}$ is a surjective morphism of sheaves.

Before introducing a fundamental exact sequence we make a definition.

Definition 1.21. A k-scheme X is *étale* if $X = \mathrm{Spec}(R)$ is finite over k and $R \otimes_k R \cong \prod_{i=1}^{n} k_i$ where k_i/k is a finite separable field extension.

The following is a basic example of an exact sequence called the 'connected-étale sequence'

Lemma 1.22 ([32], Prop 2.3.7)**.** *For G an algebraic group there is an algebraic group $\pi^0(G)$ that is étale such that $e \to G^\circ \to G \to \pi^0(G) \to e$ is exact.*

We also have the notion of an action of a group variety.

Definition 1.23. An *action* of an algebraic group G on a scheme X/k is a k-morphism $f : G \times_k X \to X$ such that $f_R : G(R) \times X(R) \to X(R)$ is an action for all R/k.

As earlier mentioned, showing that quotients exist is nontrivial.

Proposition 1.24 (Existence of quotients, [32] Appendix B)**.** *Let G be an algebraic group and let H be an algebraic subgroup of G.*

- *There exists a scheme X denoted G/H with an action $G \times_k X \to X$ and a point $o \in X(k)$ such that the orbit map $G \to X \quad g \mapsto go$ is faithfully flat and such that the flat sheaf* $\mathrm{Hom}(\mathrm{Spec}(-), X)$ *is the sheafification of the assignment $R \mapsto G(R)/H(R)$.*
- *If G acts on another scheme Y and $\varphi : G \to Y$ is G-equivariant such that φ is constant on H, then φ factors uniquely via G/H.*
- *In the case that $H = N$ is a normal subgroup of G we have that G/N inherits the structure of an algebraic group by G and that $\pi : G \to G/N$ has kernel N.*
- *For k' a separably closed field and G reduced with algebraic subgroup H we have an identification $(G/H)(k') = G(k')/H(k')$.*

We have the following variant of the first isomorphism theorem.

Proposition 1.25. *Let $\varphi : G \to H$ be a homomorhism of algebraic groups. Then φ factors as a quotient map followed by an embedding $\varphi : G \to G/\ker(\varphi) \xrightarrow{\sim} \varphi(G) \hookrightarrow H$.*

One can form the algebraic subgroup generated by two subgroups of an algebraic group.

Definition 1.26. Let G be an algebraic group and let H, N be algebraic subgroups of G. Then $H \cdot N$ is the algebraic subgroup of G whose flat sheaf is the sheafification of $R \mapsto H(R) \cdot N(R)$ (where by $N(R) \cdot H(R)$ we mean the smallest subgroup of $G(R)$ containing both $N(R)$ and $H(R)$).

The following is a variant of the second isomorphism theorem.

Proposition 1.27. *Let N, H be algebraic subgroups of G such that H normalizes N. There is a canonical isomorphism $\dfrac{H \cdot N}{N} \to \dfrac{H}{H \cap N}$.*

The above isomorphism can be checked on the level of R-points, as the corresponding sheaves are both sheafifications of the same functor. We now introduce some types of algebraic groups that are the main building blocks for algebraic groups.

Definition 1.28. Let $n > 0$. Define the *general linear group* GL_n to be the algebraic group over k that has functor of points $\mathrm{GL}_n(R) = \{M \in \mathrm{M}_{n\times n}(R) \mid \det(M) \in R^\times\}$.

It is easy to see that it is represented by $\mathrm{Spec}(k[\{X_{ij}\}_{1\leq i,j\leq n}]_{\det})$, where $\det = \det[(X_{ij})_{i,j}]$.

Definition 1.29. A homomorphism $\varphi : G \to \mathrm{GL}_n$ is called a *representation*. An algebraic group G is *linear* if it admits a representation that is an embedding.

Some examples of affine algebraic groups are the following.

Example 1.30. The scheme $\mathbb{A}^r$ is an algebraic group as its functor of points is $R \mapsto R^n$, which is a group functor. Denote the corresponding group scheme by $\mathbb{G}_a^r$. Such an algebraic group is called a *vector group*.

Example 1.31. For $\mathbb{G}_m := \mathrm{Spec}(k[X]_X)$ the scheme $\mathbb{G}_m^n$ is also an algebraic group as its functor of points is given by $R \mapsto (R^\times)^n$. Such an algebraic group is called a (split) *torus*.

Notice that a linear algebraic group is affine. It turns out that the converse is also true.

Theorem 1.32 ([32] Theorem 4.9)**.** *Let G be an affine algebraic group. There exists $n > 0$ and an embedding $\iota : G \hookrightarrow \mathrm{GL}_n$.*

When one has an affine scheme, the ring of global sections contains all the necessary information about it. The same is true for a linear algebraic group and in this case the global sections are a commutative Hopf algebra.

Definition 1.33. Let k be a field. A *k-Hopf algebra* is a k-algebra (not necessarily commutative) H equipped with k algebra morphisms *comultiplication* $\nabla : \mathrm{H} \to \mathrm{H} \otimes_k \mathrm{H}$, the *antipode* $\iota : \mathrm{H} \to \mathrm{H}$ and $\epsilon : \mathrm{H} \to k$ such that the following diagrams commute:

$$\begin{array}{ccccc} \mathrm{H} & \xleftarrow{(\iota,\mathrm{Id})} & \mathrm{H}\otimes_k\mathrm{H} & \xrightarrow{(\mathrm{Id},\iota)} & \mathrm{H} \\ \uparrow & & \nabla\uparrow & & \uparrow \\ k & \xleftarrow{\epsilon} & \mathrm{H} & \xrightarrow{\epsilon} & k \end{array} \qquad \begin{array}{ccc} \mathrm{H}\otimes_k\mathrm{H}\otimes_k\mathrm{H} & \xleftarrow[\nabla\otimes\mathrm{Id}]{} & \mathrm{H}\otimes_k\mathrm{H} \\ \mathrm{Id}\otimes\nabla\uparrow & & \nabla\uparrow \\ \mathrm{H}\otimes_k\mathrm{H} & \xleftarrow[\nabla]{} & \mathrm{H} \end{array} \qquad \begin{array}{ccccc} \mathrm{H}\otimes_k k & \xleftarrow{(\epsilon,\mathrm{Id})} & \mathrm{H}\otimes_k\mathrm{H} & \xrightarrow{(\mathrm{Id},\epsilon)} & k\otimes_k\mathrm{H} \\ & \nwarrow{\sim} & \nabla\uparrow & \nearrow{\sim} & \\ & & \mathrm{H} & & \end{array}$$

We have the following very natural example.

Example 1.34. For G an affine algebraic group over k we have a commutative Hopf-algebra $\mathcal{O}(G)$, where ∇ is induced by the multiplication of G, ι by the inversion map and ϵ by the neutral point $e : \mathrm{Spec}(k) \to G$. Specifically in the case $\mathrm{GL}_n = \mathrm{Spec}(k[\{X_{ij}\}_{1\leq i,j\leq n}, \frac{1}{\det}]$ the comultiplication is given by $X_{ij} \mapsto \sum_{l=1}^{n} X_{il} \otimes X_{lj}$ and the inversion map by $X_{ij} \mapsto \frac{1}{\det} \cdot (-1)^{i+j} \det(M_{ji})$, where $\det(M_{ji})$ is the (j,i) minor of the matrix $(X_{ml})_{m,l}$. The map ϵ is given by $X_{ij} \mapsto \delta_i^j \in k$.

Now we define a homomorphism of Hopf algebras.

Definition 1.35. A *homomorphism* of k-Hopf algebras $\varphi : \mathrm{H} \to \mathrm{H}'$ is a k-algebra homomorphism that satisfies $(\varphi \otimes \varphi) \circ \nabla = \nabla' \circ \varphi$ and $\epsilon = \epsilon' \circ \varphi$.

We have the following proposition on affine algebraic groups that can be checked by reversing all arrows and using the (contravariant) equivalence of categories between affine k-schemes and commutative k-algebras.

Proposition 1.36. *There is a contravariant equivalence of categories:*
$\{$*Finite type commutative k-Hopf algebras*$\} \to \{$*Affine algebraic groups over k*$\}$

We have the following classes of linear algebraic groups.

Definition 1.37. A linear algebraic group G is called *solvable* if it admits a filtration of normal algebraic subgroups $e = G_0 \trianglelefteq G_1 \trianglelefteq ... \trianglelefteq G_n = G$ such that all quotients G_i/G_{i+1} are commutative.

We give several examples below.

Example 1.38. Consider the algebraic group of *upper triangular matrices* $\mathbb{T}^n$ defined by:

$$\mathbb{T}_n(R) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{pmatrix} \right\} \leq \mathrm{GL}_n(R)$$

Example 1.39. Consider the algebraic group of *unipotent matrices* $\mathbb{U}_n$, which is defined by:

$$\mathbb{U}_n(R) = \left\{ \begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ 0 & 1 & \dots & a_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \right\} \leq \mathrm{GL}_n(R)$$

Example 1.40. Consider the algebraic group of *diagonal matrices* $\mathbb{D}_n$, which is defined by:

$$\mathbb{D}_n(R) = \left\{ \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{pmatrix} \right\} \leq \mathrm{GL}_n(R)$$

Remark 1.41. It follows that $\mathbb{U}^n$ is a solvable subgroup by considering its composition series with successive quotients isomorphic to $\mathbb{G}_a$ given in ([32], p.137). It also follows that $\mathbb{D}^n$ is solvable as it has a composition series with successive quotients isomorphic to $\mathbb{G}_m$.
It follows that $e \to \mathbb{U}_n \to \mathbb{T}_n \to \mathbb{D}_n \to 0$ is exact and that hence $\mathbb{T}_n$ is solvable.

The group variety $\mathbb{U}_n$ can be used to make the following definition.

Definition 1.42. A linear algebraic group G is *unipotent* if it admits an embedding $G \hookrightarrow \mathbb{U}_n$.

We also have the following class of algebraic groups, which are in some sense opposite to linear algebraic groups.

Definition 1.43. An algebraic group G over k is *anti-affine* if it has $\mathcal{O}_G(G) = k$.

A basic example of such a type of algebraic group is the following.

Definition 1.44. A group variety G over k is an *abelian variety* if it is complete.

We have the following basic examples.

Example 1.45. A smooth geometrically irreducible cubic curve E in $\mathbb{P}^2_k$ with a point $O \in E(k)$ gives an example of a one dimensional abelian variety (see [40]), called an elliptic curve.
Any abelian variety is an example of an anti-affine group variety. For an example of an anti-affine group variety that is not an abelian variety, consider an extension $e \to \mathbb{G}_m \to G \to E \to e$, where E is an elliptic curve. Clearly no such G is an abelian variety as it contains a closed affine subgroup of positive dimension. Over an infinite field k there exist such extensions such that G contains no abelian variety (we will see this in Chapter 4). This implies that by Theorem 1.60, we must necessarily have $G_{\text{ant}} = G$ as otherwise G_{ant} is isogenous (see Definition 1.57) to E and G_{ant} would be an abelian variety.

We now state a lemma about when an algebraic group has a largest smooth normal connected subgroup of a certain type.

Lemma 1.46 ([32], Section 6g)**.** *Let P be a property of algebraic groups such that: Any extension of an algebraic group with P by an algebraic groups with P has P, any quotient of an algebraic group with P has P and e has P. Then for any algebraic group G there exists a largest normal algebraic subgroup $N \subset G$ having P. The quotient G/N has no such normal subgroup. If G is smooth, and k is perfect then G has a largest normal subgroup variety with property P.*

The hypothesis that k is perfect above is such that one can find the largest normal connected N with P and then take the underlying reduced subscheme N_{red}.

Remark 1.47 ([32], p.61, p.135). Properties P satisfying the hypothesis of 1.46 include:

$$P \in \{\text{linear}, \text{complete}, \text{anti affine}, \text{solvable}, \text{unipotent}\}$$

The references given are for P being linear and unipotent respectively. For P being anti-affine and solvable it follows from the definitions. For P being complete we note that if G is an extension of N by Q, we may base-change to $\bar{k}$ and then apply ([32], Prop. 8.25) stating that $G_{\bar{k}}$ is proper over $\bar{k}$ variety if and only if it has no affine algebraic subgroup of positive dimension. This holds as $N_{\bar{k}}$ and $Q_{\bar{k}}$ are both complete. Now by faithfully flat descent and since $\text{Spec}(\bar{k}) \to \text{Spec}(k)$ is faithfully flat we obtain that G is proper over k.

Definition 1.48. For G an algebraic group we denote:

- The largest linear normal group variety of G by G_{lin}.
- The quotient G/G_{lin} by G_{ab}.

- The largest anti-affine normal group variety of G by G_{ant}.

In the case that G is linear, denote:

- The largest normal solvable group variety of G by $\mathrm{R}(G)$, called the *radical.*
- The largest normal unipotent group variety of G by $\mathrm{R_u}(G)$, called the *unipotent radical.*

Example 1.49. Let $G = \mathrm{GL}_n$ for $n \geq 2$ over an algebraically closed field k and consider the normal diagonal torus $\mathbb{G}_m \trianglelefteq \mathrm{GL}_n$. The quotient is PSL_n given by $\mathrm{PSL}_n(R) = \mathrm{GL}_n(R)/R^\times$ (as can be seen by using that the flat cohomology group $\mathrm{H}^1_{\text{flat}}(k, \mathbb{G}_m)$ vanishes, see [32] p.111). It is known that the group $\mathrm{PSL}_n(k)$ is simple for $n \geq 2$ and hence $\mathbb{G}_m$ is in fact the largest proper normal subgroup variety of GL_n, hence $\mathrm{R}(\mathrm{GL}_n) = \mathbb{G}_m$.

The following justifies the notation G_{ab}.

Theorem 1.50 (Chevalley-Barsotti, [32] Theorem 8.27)**.** *Let G be a group variety over a perfect field k. Then G has a largest linear normal subgroup variety G_{lin} such that G/G_{lin} is an abelian variety.*

In [32] (Theorem 8.27) it is shown that when k is perfect G containing no normal linear subgroup variety is equivalent to G being an abelian variety. Then the above theorem follows directly from Lemma 1.46. Combining the Chevalley-Barsotti theorem with the fact that an abelian variety is projective (see [30] Theorem 6.4) is used to show the following proposition.

Proposition 1.51 ([32] Theorem 8.45, Homogeneous spaces are quasi-projective)**.** *Let X be a separated scheme over k on which an algebraic group G acts such that $G \times_k X \to X \times_k X$ $(g, x) \mapsto (gx, x)$ is faithfully flat. Then X is quasi-projective.*

In particular any algebraic group is quasi-projective, which follows from the case $X = G$ above. The anti-affine algebraic group G_{ant} also has some nice properties.

Proposition 1.52 ([32], Cor 8.14 and Prop 8.37)**.** *The algebraic group G_{ant} is connected, smooth and contained in the centre of G.*

For checking that G_{ant} is smooth and connected k can be assumed to be algebraically closed and in this case the quotient $G_{\text{ant}}/(G_{\text{ant}})^\circ_{\text{red}}$ is finite, hence affine and hence trivial. For checking that it is central, note that $G/Z(G)$ is affine since $G/Z(G)$ acts faithfully on G by conjugation with fixed point e, (see Prop 8.9 in [32]).

Now we define other types of linear algebraic groups.

Definition 1.53. A linear algebraic group G is called *reductive* if $\mathrm{R_u}(G_{\bar{k}}) = e$ and *semisimple* if $\mathrm{R}(G_{\bar{k}}) = e$.

Notice that a semisimple algebraic group is reductive. The following gives an example of a semisimple algebraic group.

Definition 1.54. The *derived group* $\mathcal{D}(G)$ of G is the intersection of all normal $N \trianglelefteq G$ such that G/N is commutative.

It is a fully characteristic subgroup of G (See [32], p.129). In some cases we can say something about the normal subgroups of algebraic groups with a certain property.

Lemma 1.55. *Let G be a linear algebraic group. Then any algebraic subgroup of G is also linear. If G is solvable resp. unipotent then every algebraic subgroup of G is solvable resp. unipotent.*

The unipotent case follows by definition. The solvable case follows from the fact that if one has that $e = G_0 \trianglelefteq ... \trianglelefteq G$ is a subnormal series with commutative quotients and $H \leq G$ a normal algebraic subgroup that $e = G_0 \cap N \trianglelefteq ... \trianglelefteq N$ is a subnormal series for N with commutative quotients.

Definition 1.56. An algebraic group G is *almost simple* if it is non-commutative and it contains no normal algebraic subgroup of positive dimension.

We also make the following definition.

Definition 1.57. A homomorphism $\varphi : G \to H$ is called an *isogeny* if $\ker(\varphi)$ is finite and $H/\varphi(G)$ is finite. Say that G, H are *isogenous* if there exists $G = G_0 - ... - G_n = H$ where $-$ means an isogeny going in either direction.

Remark 1.58. If H is a group variety over a perfect field, then an isogeny is a quotient map with finite kernel. Such a morphism is a finite morphism and thus it has a degree.

We now have a lemma for quotients of algebraic groups with certain properties.

Lemma 1.59. *Let $G/\bar{k}$ be a group variety with property $P \in \{\text{complete}, \text{reductive}, \text{semisimple}, \text{solvable}\}$. Then any quotient of G has P.*

Proof. In the solvable case: Pick a filtration of G with commutative quotients, then the images give a filtration of H with commutative quotients.
For the complete case: The image of a connected complete variety is a connected complete variety.
An algebraic group G is semisimple if and only if for $G_1, ...G_r$ its minimal almost-simple normal subgroup varieties, the multiplication map $G_1 \times ... \times G_r \to G$ is an isogeny ([24], p.167). The quotient of an almost-simple algebraic group is almost-simple as it is either $\{e\}$ or a quotient by a finite group scheme. We get that multiplication $\pi(G_1) \times ... \times \pi(G_r) \to \pi(G)$ is an isogeny since the $\pi(G_i)$ are almost-simple, hence their intersection is finite, hence $\pi(G)$ is an almost direct product of almost-simple algebraic subgroups, thus $\pi(G)$ is semisimple.
A group variety is reductive if and only if (Proposition 21.60 [32]) it is an almost direct product of a torus with a semisimple algebraic group. In fact the multiplication map $\mathrm{R}(G) \times G_{\mathrm{der}} \to G$ is an isogeny. A quotient map $\pi : G \to H$ gives that the multiplication map $\pi(\mathrm{R}(G)) \times \pi(G_{\mathrm{der}}) \to H$ is surjective. A quotient of a solvable algebraic group is solvable and we have seen that the quotient of a semisimple group variety is semisimple. Hence $\pi(\mathrm{R}(G))$ is a torus and $\pi(G_{\mathrm{der}})$ is semisimple respectively. The kernel of the multiplication map $\pi(G_{\mathrm{der}}) \times \pi(R(G)) \to H$ is isomorphic to the intersection $\pi(G_{\mathrm{der}}) \cap \pi(R(G))$, which is necessarily finite since it is solvable and semisimple. Hence H is reductive as it is the almost direct product of a torus and a semisimple group variety. ☐

The following decomposition theorem is by Rosenlicht.

Theorem 1.60 (Corollary 5 [37]). *Let G be a group variety over a perfect field. The multiplication map $\mu : G_{\text{ant}} \times_k G_{\text{lin}} \to G$ is a surjective quotient map.*

Proof. That μ is a homomorphism follows from $G_{\text{ant}} \subset Z(G)$. Consider the image $G_{\text{ant}} \cdot G_{\text{lin}} \subset G$ and the quotient $Q := G/(G_{\text{ant}} \cdot G_{\text{lin}})$. There are quotient maps $G_{\text{ab}} \to Q$ and $G_{\text{aff}} \to Q$. Hence Q is both affine and complete, hence finite. Since G is a group variety it has no nontrivial algebraic subgroup of finite index, hence $G_{\text{ant}} \cdot G_{\text{lin}} = G$ and hence μ is surjective. □

The following lemma gives us that in some cases there are induced maps between the largest subgroup varieties of the above type.

Lemma 1.61. *Let G be an algebraic group and denote by G_* the largest normal subgroup variety with some property $P \in \{\text{linear}, \text{unipotent}, \text{solvable}\}$ (the properties unipotent and solvable are only considered when G is linear). Let $\varphi : G \to H$ be a homomorphism that is either a normal embdedding or a quotient map. Then $\varphi(G_*)$ is normal in H and φ induces a homomorphism $G_* \to H_*$.*

Proof. It suffices to show that $\varphi(G_*)$ is normal in H since it is a group variety (as G_* is a group variety) with property P, thus contained in H_* if it is normal. If φ is a quotient map, then because G_* is normal in G, $\varphi(G_*)$ is normal in $\varphi(G) = H$. If φ is a normal embedding, G is a normal subgroup of H. Let γ be any automorphism of H restricting to an automorphism on G (such as conjugation by an element of H). Then $\gamma(G_*) \subset \gamma(G) = G$ is a normal group variety with property P and it follows that it equals G_*. □

In the following proposition we keep the notation of Lemma 1.61. It is almost identical to the proof of (Lemma 4.7 [6]). Again the subscript $*$ will be put under G to denote the largest normal subgroup variety of G with property $P \in \{\text{linear}, \text{unipotent}, \text{solvable}\}$.

Proposition 1.62. *Let $e \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to e$ be an exact sequence of algebraic groups over an algebraically closed field k. Then there is a commuting diagram with exact rows and with q an isogeny:*

$$
\begin{array}{ccccccccc}
e & \longrightarrow & G_* \cap K & \longrightarrow & G_* & \longrightarrow & Q_* & \longrightarrow & e \\
 & & \big\uparrow & & \big\uparrow{\scriptstyle \mathrm{Id}} & & \big\uparrow{\scriptstyle q} & & \\
e & \longrightarrow & K_* & \longrightarrow & G_* & \longrightarrow & H & \longrightarrow & e
\end{array}
$$

Proof. By Lemma 1.61 we get that there is an induced sequence $e \to K_* \to G_* \to Q_* \to e$ which need not be exact. The hardest part is showing that the homomorphism $G_* \to Q_*$ is surjective. We start with showing this. The same lemma gives us a sequence $e \to K/K_* \to G/G_* \to Q/Q_* \to e$, which is also not necessarily exact. Note that there is a commuting square:

$$
\begin{array}{ccc}
G & \twoheadrightarrow & Q \\
\big\downarrow & & \big\downarrow \\
G/G_* & \longrightarrow & Q/Q_*
\end{array}
$$

As $G \to Q$ and $Q \to Q/Q_*$ are both quotient maps, so is $G/G_* \to Q/Q_*$.
We now compute $\ker(K/K_* \to G/G_*)$. This is equal to $\dfrac{G_* \cap K}{K_*}$ of which we claim that it is finite. Since the field k is perfect we have the algebraic subgroup $(G_* \cap K)^\circ_{\text{red}}$, which is normal in K as it is characteristic in $G_* \cap K$ and $G_* \cap K$ is normal in K. Since $(G_* \cap K)$ is also normal in G_*, so is $(G_* \cap K)^\circ_{\text{red}}$, hence by Lemma 1.55 we obtain that $(G_* \cap K)^\circ_{\text{red}}$ has property P. Thus we get $(G_* \cap K)^\circ_{\text{red}} \subset K_*$. The other inclusion also holds as K_* is characteristic in K and K is normal in G, thus K_* is normal in G and has property P, thus $K_* \subset G_*$. So we get $K_* = (G_* \cap K)^\circ_{\text{red}}$, which is of finite index in $G_* \cap K$ since the dimensions are the same.

We now show that $\dfrac{\ker(G/G_* \to Q/Q_*)}{\mathrm{Im}(K/K_* \to G/G_*)}$ is finite. The top algebraic group equals $\dfrac{\pi^{-1}(Q_*)}{G_*}$. The one on the bottom equals $\dfrac{K \cdot G_*}{G_*}$, hence $\dfrac{\ker(G/G_* \to Q/Q_*)}{\mathrm{Im}(K/K_* \to G/G_*)} \cong \dfrac{\pi^{-1}(Q_*)}{K \cdot G_*}$. We have an isomorphism $\pi^{-1}(Q_*)/K \cong Q_*$, which has property P. Hence as $\dfrac{\pi^{-1}(Q_*)}{K \cdot G_*}$ is a quotient of Q_* it has property P. It is also a quotient of $\dfrac{\pi^{-1}(Q*)}{G_*}$. This is a normal algebraic subgroup of G/G_*, which has the 'opposite property' of P, i.e. it is either complete, reductive or semisimple. A normal algebraic subgroup also has this 'opposite property'. Quotients also have this 'opposite property' by Lemma 1.59. So we see that as $\dfrac{\pi^{-1}(Q_*)}{K \cdot G_*}$ both has the property P as well as the opposite property, that it is finite.
From these computations we obtain that $\dim(G/G_*) = \dim(Q/Q_*) + \dim(K/K_*)$. We also have that $\dim(\pi(G_*)) = \dim(G_*) - \dim(\ker(\pi|_{G_*})) = \dim(G_*) - \dim(G_* \cap K) = \dim(G_*) - \dim(K_*)$ since we have already seen that K_* is a subgroup of finite index in $G_* \cap K$. The right hand side can be rewritten as $\dim(G) - \dim(G/G_*) - \dim(K) + \dim(K/K_*)$, which equals $\dim(Q) - \dim(Q/Q_*) = \dim(Q_*)$. Hence since $\pi(G_*) \subset Q_*$ is a closed subvariety of the same dimension it is the whole of Q_*. This shows the exactness of the top sequence in the proposition. That the bottom sequence is exact follows from the fact that K_* has finite index in $G_* \cap K$. □

Remark 1.63. It follows from the above proof that the algebraic group $\pi^{-1}(Q_*)$ actually equals $G_* \cdot K$. Since $G_* \cdot K$ is of finite index in $\pi^{-1}(Q_*)$ it suffices to show that $\pi^{-1}(Q_*)$ is a group variety. Connectedness follows from the exact sequence $\pi_0(K) \to \pi_0(\pi^{-1}(Q_*)) \to \pi_0(Q_*) \to e$ (exercise 5-9 [32]) and smoothness follows from the fact that $\ker(\pi)$ is smooth if and only if π is smooth (III.10.5 [23]) and that the base-change of π to $\pi|_{\pi^{-1}(Q_*)} : \pi^{-1}(Q_*) \to Q_*$ is smooth as π is smooth, thus $\pi^{-1}(Q_*) \to Q_* \to \mathrm{Spec}(k)$ is smooth since Q_* is smooth and smoothness is stable under composition.
However it does not always hold that $K \cap G_* = K_*$ although K_* is an algebraic subgroup of finite index. For a counterexample, take $G = \mathrm{GL}_n$ for $n \geq 2$, $K = \mathrm{SL}_n$ and $G_* = \mathbb{G}_m = \mathrm{R}(\mathrm{GL}_n)$. Then $G_* \cap K = \mu_n$, which is not a group variety, however $K_* = \{e\} \subset \mathrm{SL}_n$. So $G_* \cap K$ need not be reduced nor connected.

We now begin with introducing Borel subgroups and maximal tori, which are especially useful to study reductive group varieties.

Definition 1.64. Let G be a linear algebraic group over a field k. A *Borel subgroup* of G is a maximal solvable subgroup of G.

We have the following definition that we will use frequently.

Definition 1.65. Let G be a linear algebraic group over k. A *maximal torus* is a subtorus T of G that is not properly contained in any other subtorus of G.

Example 1.66. For $G = \mathrm{GL}_n$ over $k = \overline{k}$ we have that $\mathbb{T}_n$ is a Borel subgroup (Lie-Kolchin theorem 16.30 [32] and Proposition 16.2 [32]) and that $\mathbb{D}_n$ is a maximal torus.

Note that a torus is solvable, hence every maximal torus T has $T \subset B$ for B some Borel subgroup of G. We have the following structure theorem on Borel subgroups and maximal tori.

Proposition 1.67 (Theorem 17.9.(b) [32] and Theorem 17.10 [32])**.** *Let G be a linear group variety over an algebraically closed field k. Let B be a Borel subgroup of G and let T be a maximal torus of G. The Borel subgroups of G are precisely $\{gBg^{-1} \mid g \in G(k)\}$ and the maximal tori of G are precisely $\{gTg^{-1} \mid g \in G(k)\}$.*

We now introduce the Weyl group of a reductive group variety.

Definition 1.68. Let G be a reductive group variety over a perfect field k and let T be a maximal torus. Define the *Weyl group* with respect to T by $W(G,T) := N_G(T)/T$.

The Weyl group acts on the maximal torus T by conjugation.

Lemma 1.69 (Prop 21.1 [32])**.** *The Weyl group $W(G,T)$ is a finite étale group scheme.*

This implies that we can think of W as just being a finite group. By Proposition 1.67 it follows that over an algebraically closed field $W(G,T)$ does not depend on T as an abstract group, so just write $W = W(G,T)$ in this case

Example 1.70. Consider $G = \mathrm{GL}_n$ and the maximal torus $T = \mathbb{D}_n$ over an algebraically closed field. Consider the equation $(a_{ij}) \cdot (d_{ij}) = (e_{ij}) \cdot (a_{ij})$ for $(d_{ij}), (e_{ij}) \in \mathbb{D}_n(k)$. Then notice that comparing the relations from the first column of the resulting matrix gives that $d_{11} = e_{i_1 i_1}$ for some i_1. Continuing, it follows that there is $\sigma \in S_n$ such that $b_{ii} = a_{\sigma(i)\sigma(i)}$ for all $1 \leq i \leq n$. So this gives $W \leq S_n$. Any transposition between entries $(i, i+1)$ can be realized by conjugating with a matrix of the form:

$$\begin{pmatrix} 1 & \dots & & \dots & 0 \\ \vdots & \ddots & & \dots & \vdots \\ \vdots & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \dots & \vdots \\ \vdots & \dots & & \ddots & \vdots \\ 0 & \dots & & \dots & 1 \end{pmatrix}$$

Hence we conclude that the Weyl group of GL_n is precisely S_n.

We make the following definition.

Definition 1.71. Let G be an algebraic group. A *character* of G is a homomorphism $G \to \mathbb{G}_m$. Denote the character group of G by $X(G)$.

This is especially interesting for a split torus.

Lemma 1.72. *Let $T \cong \mathbb{G}_m^n$ be a split torus, then $X(T) \cong \mathbb{Z}^n$.*

The proof is done by observing that $X(T)$ has a basis consisting of the projections to $\mathbb{G}_m$. Note that as the Weyl group acts on T, it also acts on $X(T)$.

Definition 1.73. Let G be a reductive algebraic group over $k = \bar{k}$ and let T be a maximal torus. Define $S := \mathrm{Sym}(X(T) \otimes_{\mathbb{Z}} \mathbb{Q}_l)$ for $l \neq \mathrm{char}(k)$ prime. Then the *Weyl group invariants* of G are S^W.

We can easily compute this in the previously mentioned case.

Example 1.74. Let $G = \mathrm{GL}_n$ and $T = \mathbb{D}_n$, $W = S_n$. It follows that $S \cong \mathbb{Q}_l[X_1, ..., X_n]$ and that W acts by permuting the variables. Hence we obtain $S^W = \mathbb{Q}_l[e_1, ..., e_n]$, where the e_i are the elementary symmetric polynomials $e_i = \sum_{i_1+...+i_n=i} X_1^{i_1} \cdot ... \cdot X_n^{i_n}$. That S^W is a polynomial ring follows from the Chevalley-Shephard-Todd theorem [10].

To finish the chapter we state some general results about isogenies that we will use later.

Definition 1.75. We say that an isogeny is *separable* if $\ker(\varphi)$ is an étale group scheme. Say that φ is inseparable if this is not the case and that it is *purely inseparable* if $\ker(\varphi)$ is connected.

Lemma 1.76. *An isogeny $\varphi : G \to H$ factors as $\varphi = \phi \circ \psi$, where $\phi : Q \to H$ is a separable isogeny and $\psi : G \to H$ is a purely inseparable isogeny.*

Proof. Let $K = \ker(\varphi)^\circ$. Then φ factors via the quotient map $G \to G/K$. Since $\ker(G/K \to H)$ is isomorphic to $\ker(\varphi)/K$, which is an étale group scheme and since K is connected, this gives the factorization. □

For G, H commutative group varieties we can in some cases reverse the isogeny.

Lemma 1.77. *Let $\varphi : G \to H$ be an isogeny between commutative group varieties such that $[\deg(\varphi)] : H \to H$ is an isogeny. An isogeny $\psi : H \to G$ exists such that $\varphi \circ \psi = \psi \circ \varphi = [\deg(\varphi)]$.*

The proof of the lemma follows from the fact that a commutative finite group scheme G is killed by multiplication by its order $\dim_k(\mathcal{O}(G))$ (Corollary 2.2 [38]).

Chapter 2

Étale cohomology

In this chapter we introduce étale cohomology, which is related to ordinary sheaf cohomology, though with 'opens' in the étale topology. We will state some results without a proof in this chapter to not get too far astray from the purpose of this document. For a more thorough introduction to this topic we refer to [31] and the references in these notes.

First we recall the definition of an étale morphism.

Definition 2.1. A morphism of schemes $f : X \to Y$ is *étale* if it is locally of finite type, flat and unramified; unramified meaning that for all $x \in X$ with $f(x) = y$, the map $f^{\#} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induces a finite separable field extension $\kappa(x)/\kappa(y)$ and $\mathfrak{m}_X = f^{\#}(\mathfrak{m}_Y) \cdot \mathcal{O}_{X,x}$.

Remark 2.2. Étale morphisms have the following basic properties:

- **Equidimensionality:** A particular property that unramified morphisms have is that they have relative dimension 0, i.e. for $f : X \to Y$ unramified and $y \in Y$, $\dim(X_y) = 0$. To see this, reduce to the affine case $X = \mathrm{Spec}(S) \to \mathrm{Spec}(R) = Y$ with $x = [\mathfrak{q}]$, $y = [\mathfrak{p}]$ and note that $X_y = \mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p})_{\bar{\mathfrak{q}}} = S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) = \kappa(\mathfrak{q})$, which is finite over $\kappa(\mathfrak{p})$. Then we apply [43, Tag 00PK]. As an étale morphism $f : X \to Y$ is moreover flat, we have that $\dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y}) = \dim(X_y)$ by Corollary (21.10) in [26]. Thus if X and Y are integral schemes and $f : X \to Y$ is étale we have $\dim(X) = \dim(Y)$.

- **Openness:** An étale morphism is in particular flat and locally of finite presentation, which implies that it is open by Lemma 29.25.10 [43, Tag 01U2].

- **'Locally' standard étaleness:** For $f : X \to Y$ étale, $x \in X$ there exists $\mathrm{Spec}(R)$, an affine open neighbourhood of $f(x)$, and an affine open neighbourhood U of x together with an open immersion $j : U \to \mathrm{Spec}(R[T]_h/(f))$ for a monic polynomial $f \in R[T]$ such that f' is invertible in $R[T]_h$ such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\quad j \quad} & \mathrm{Spec}(R[T]_{f'}/(f)) \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & \mathrm{Spec}(R) & \end{array}$$

  Here the right arrow is the canonical one. For more background see [43, Tag 025A].

- **Stability under base-change:** For $f : X \to Y$ étale and $Z \to Y$ any morphism, the induced morphism $X \times_Y Z \to Z$ is étale. The flatness and locally of finite presentation properties are basic and follow from the fact that flat ring maps and ring maps of finite presentation are stable under taking the tensor product. That unramifiedness is stable under base-change is a bit harder to see and follow by using the characterisation; $f : X \to Y$ is unramified if and only if $\Omega^1_{X/Y} = 0$ [43, Tag 02G3].

- **Stability under composition:** For $f : X \to Y$ étale and $g : Y \to Z$ étale we have that $g \circ f : X \to Z$ is étale.

Example 2.3. We give some easy (non)examples of étale morphisms: Consider for k an algebraically closed field the ring map $\varphi : k[T] \to k[T] \quad T \mapsto T^n$, which gives $f : \mathbb{A}^1 \to \mathbb{A}^1$ and a restriction $f : \mathbb{G}_m \to \mathbb{G}_m$. It is clear that f is flat as φ is a flat ring map as $k[T]$ is a free $k[T^n]$-module and it is also clearly of finite presentation. However it is not unramified for $n \geq 2$ as the ring map $f_{(T)} : k[T]_{(T)} \to k[T]_{(T)}$ has $T \notin f_{(T)}((T)) \cdot k[T]_{(T)}$.
Whether the restriction to $\mathbb{G}_m$ is étale depends on $\mathrm{char}(k)$ and n. Clearly it is still flat and of finite presentation. For $(T-a)$ a prime ideal with pre-image $(T-a^n)$ we consider the induced map on the local rings $k[T]_{(T-a^n)} \to k[T]_{(T-a)}$. Under this map $(T-a^n)$ goes to $(T^n - a^n)$, which is equal to:

$$(T^n - a^n) = \begin{cases} (T-a)^n \text{ if } \mathrm{char}(k) | n \\ (T-a) \cdot Q \text{ with } (T-a) \nmid Q \text{ if } \mathrm{char}(k) \nmid n \end{cases}$$

Hence we see that $\varphi(T-a^n) \cdot k[T]_{(T-a)} = (T-a)_{(T-a)}$ is satisfied if and only if $\mathrm{char}(k) \nmid n$. The map on the residue fields is after the identification $k \cong k[T]/(T-b)$ by $T \mapsto b$ given by the identity map on k, thus indeed this is a finite separable extension.

We now have the following abstract definition of a site. It is a generalization of the notion of a topological space. The definition includes the analogies between a topological space and a site.

Definition 2.4. A *site* is given by the following data:

- A category $\mathcal{C}$
- For any object U of $\mathcal{C}$ a set of *coverings* $\{U_i \to U\}_i$ such that:
  1. (homeomorphism) Any isomorphism is a covering.
  2. (arbitrary union of opens is open) If $\{U_i \to U\}$ is a cover and $\{V_{ij} \to U_i\}_j$ are covers then $\{V_{ij} \to U\}_{i,j}$ is a cover.
  3. (finite intersection of opens is open) If $\{U_i \to U\}_i$ is a cover and $V \to U$ a morphism, then $U_i \times_U V$ exist for all i and $\{U_i \times_U V \to V\}_i$ is a cover.

The following are examples. The first one should be familiar while the second one will be of interest for us.

Example 2.5. Let X be a scheme and let $\mathcal{C}$ be the category whose objects are open sets of X and whose morphisms are the inclusion maps between these open sets. Define a covering to be a family $\{U_i \to U\}_i$ such that $U = \bigcup_i U_i$. This is the *Zariski site* of X and it satisfies the properties above.
We can also let $\mathcal{C}$ be the category whose objects are étale morphisms $\varphi_U : U \to X$ and whose

morphisms $\mathrm{Hom}(\varphi_U, \varphi_V)$ are the commuting triangles, where $U \to V$ is a morphism of schemes:

$$\begin{array}{ccc} U & \longrightarrow & V \\ & {\scriptstyle \varphi_U}\searrow \quad \swarrow{\scriptstyle \varphi_V} & \\ & X & \end{array}$$

Then we define the coverings $\{U_i \to X\}_i$ to be the families of étale morphisms such that $\bigsqcup_i U_i \to X$ is surjective. By stability of étale morphisms under composition and base-change it follows that what is written above defines a site, which we will call the *étale site* of X, denoted $X_{\text{ét}}$.
Another site to consider is the *flat site* of X, whose objects are flat morphisms of finite type $Y \to X$ and whose coverings are $\{Y_i \to X\}_i$ such that the images cover X.

It follows that one can define presheaves and sheaves for sites just as one can for a topological spaces. We let S be any site, but the reader is encouraged to only think about the three sites above.

Definition 2.6. Let S be a site with underlying category $\mathcal{C}$. A *presheaf* of abelian groups on S is a contravariant functor $\mathcal{F} : \mathcal{C} \to \mathrm{Ab}$. A *sheaf* on S is a presheaf $\mathcal{F}$ on S such that for any object U in $\mathcal{C}$ and any covering $(U_i \to U)_i$ there is an exact sequence:

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

Here the last map is induced by the natural restriction maps $\rho_i : \mathcal{F}(U_i) \to \mathcal{F}(U_i \times_U U_j)$ and $-\rho_j : \mathcal{F}(U_j) \to \mathcal{F}(U_i \times_U U_j)$.

Now we define certain objects attached to sheaves analogously to how they are defined for topological spaces.

Definition 2.7. A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ is a natural transformation. A *morphism of sheaves* is a morphism of the underlying presheaves. Denote by $\mathrm{Sh}(X_{\text{ét}})$ the category of étale sheaves on X.

Example 2.8. The following are commonly used sheaves on the étale site of a scheme X:

- **Hom-sheaves**: For G a commutative group scheme, one defines the sheaf $\underline{G}$ on $X_{\text{ét}}$, which assigns to an étale morphism $U \to X$ the abelian group $G(U) := \mathrm{Hom}(U, G)$ and assigns to a morphism $U \to V$ over X the natural map $G(V) \to G(U)$.
  To check that this is in fact a sheaf we have the following criterion for a presheaf to be a sheaf for the étale topology, which is ([31], p.44): A presheaf $\mathcal{F}$ on $X_{\text{ét}}$ is a sheaf if and only if the sheaf condition holds for arbitrary Zariski covers and for étale covers consisting of a single morphism $U \to V$ between affine schemes.
  The fact that it satisfies the sheaf condition for Zariski covers follows directly from the fact that one can glue morphisms from open sets to any scheme that agree on intersections. The case of an étale cover $V \to U$ is done by first reducing to the case in which all the schemes are affine. Then we note that if R is the ring corresponding to G, A to U and B to V we are requiring that the sequence $\mathrm{Hom}(R, A) \to \mathrm{Hom}(R, B) \rightrightarrows \mathrm{Hom}(R, B \otimes_A B)$ is exact. This follows from exactness of $0 \to A \to B \rightrightarrows B \otimes_A B$ for any B/A faithfully flat.

- **Constant sheaves:** We let Λ be a finite abelian group and then define $\Lambda(U) = \Lambda^{\pi^0(U)}$. For $\varphi : V \to U$ an X-morphism the induced map $\Lambda(U) \to \Lambda(V)$ has yet to be defined. For U_i a path-component of U and V_j one of V we set $\Lambda_{U_i} \to \Lambda_{V_j}$ to be the zero-map if $\varphi(V_j) \cap U_i = \emptyset$ and we set it to be the identity if $\varphi(V_j) \cap U_i \neq \emptyset$. This yields a map $\Lambda^{\pi^0(U)} \to \Lambda^{\pi^0(V)}$. For proving that this is a sheaf, notice that this assignment coincides with the assignment $\underline{G}$ defined by the group scheme $G := \bigsqcup_{\lambda \in \Lambda} \mathrm{Spec}(\mathbb{Z})$, which is a sheaf by what was written above.

- **Common examples of the above:**
  - $\underline{\mathbb{G}_a} := \mathrm{Hom}(-, \mathbb{A}^1_{\mathbb{Z}})$ given by the additive group $U \mapsto \mathcal{O}_U(U)$.
  - $\underline{\mathbb{G}_m} : \mathrm{Hom}(-, \mathbb{G}_{m,\mathbb{Z}})$ given by the multiplicative group $U \mapsto \mathcal{O}_U(U)^*$.
  - $\underline{\mathbb{Z}/l^n\mathbb{Z}}$, a constant sheaf.
  - $\underline{\mu_{l^n}}$, a sheaf defined by the group scheme $\mu_{l^n} = \mathrm{Spec}(\mathbb{Z}[X]/(X^{l^n} - 1)$. In the case that X/k is a variety and $\mathrm{char}(k) \nmid l$ this sheaf is isomorphic to $\underline{\mathbb{Z}/l^n\mathbb{Z}}$, but in general the sheaves are not isomorphic.

We underline the above sheafs in order to be able to distinguish them from the corresponding algebraic groups. In later chapters we may omit the underlining of $\mathbb{Z}/l^n\mathbb{Z}$. One can also define the stalk of an étale sheaf.

Definition 2.9. Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. The *stalk at a geometric point* $\bar{x} \in X$ is given by $\varinjlim_{(U,\bar{u})} \mathcal{F}(U)$, the colimit taken over all étale neighbourhoods $U \to X$ such that $\bar{u}$ maps to $\bar{x}$.

As in the Zariski case one can sheafify.

Proposition 2.10 (Sheafification exists, [31] Theorem 7.15)**.** *Let $\mathcal{F}$ be a presheaf on $X_{\acute{e}t}$, then there exsits a sheaf $a\mathcal{F}$ with a morphism of presheaves $\mathcal{F} \to a\mathcal{F}$ that is universal among all morphisms from $\mathcal{F}$ to a sheaf. The stalks of $\mathcal{F}$ and $a\mathcal{F}$ are isomorphic and $a\mathcal{F}$ is called the sheafification of $\mathcal{F}$.*

We have the following notion of an exact sequence of sheaves. There are other equivalent definitions, see [31].

Definition 2.11. A sequence of sheaves on $X_{\text{ét}}$, $\mathcal{K} \to \mathcal{G} \to \mathcal{Q}$, is *exact* if for any geometric point $\bar{x} \in X$ the sequence $\mathcal{K}_{\bar{x}} \to \mathcal{G}_{\bar{x}} \to \mathcal{Q}_{\bar{x}}$ is exact in the category of abelian groups.

Thus we can now define the following operation. This in particular implies that the inverse image functor is exact.

Definition 2.12. For $g : X \to Y$ a morphism and $\mathcal{F} \in \mathrm{Sh}(Y_{\text{ét}})$ let $g^*\mathcal{F}$ be the sheaf on $X_{\text{ét}}$ that is the sheafification of the assignment $(g^*\mathcal{F})^P(U \to X) \mapsto \varinjlim_{V \to Y} \mathcal{F}(V)$, the colimit taken over all $V \to Y$ étale that give a commuting square:

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow g \\ V & \longrightarrow & Y \end{array}$$

It follows that g^* defines a functor $\mathrm{Sh}(Y_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$, called the *inverse image* functor.

We have the following trivial example.

Example 2.13. Let $g : X \to Y$ be a morphism and let Λ_Y be a constant sheaf on Y. Then note that we have $(g^*\Lambda_Y)(U) = \varinjlim_{V \to Y} \Lambda(V)$. Notice that there is an identification $\Lambda(V) = \mathrm{Mor}_Y(V, \Lambda_{\mathbb{Z}})$, where $\Lambda_{\mathbb{Z}} = \bigsqcup_{\lambda \in \Lambda} \mathrm{Spec}(\mathbb{Z})_\lambda$ is a constant group scheme. For any such V appearing in the colimit there is a morphism $\mathrm{Mor}_Y(V, \Lambda_{\mathbb{Z}}) \to \mathrm{Mor}_X(U, \Lambda_{\mathbb{Z}})$ by postcomposition. This gives a morphism of presheaves $(g^*\Lambda_Y)^P \to \Lambda_X$. Note that it is injectice as $(U \to V \to \Lambda_{\mathbb{Z}})$ being the 0-element is if and only if the image lands in $\mathrm{Spec}(\mathbb{Z})_0 \subset \Lambda_{\mathbb{Z}}$, which means that we can take $V' \subset V$ such that V' equals the union of connected components of V to which U maps. Then note that $(V' \to \Lambda_{\mathbb{Z}}) = 0$ and hence $(V \to \Lambda_{\mathbb{Z}})$ is zero in the colimit. Note that it is also surjective at every U as we can take $V = \bigsqcup_{\pi^0(U)} Y$ such that $(U \to V \to \Lambda_{\mathbb{Z}})$ can be any element of $\Lambda^{\pi_0(U)}$. By the universal mapping property of sheafification we have hence constructed an isomorphism $g^*\Lambda_Y \to \Lambda_X$.

The above isomorphism of sheaves can be realized in a bit of an easier way by just thinking of Λ_Y, Λ_X as constant group schemes rather than Hom-sets, which will give the same result. The reason why we make the above strange computation is that we have the Kummer-sequence (see Proposition 2.21) in which the Hom-sheaves $\underline{\mathbb{G}_m}$ and $\underline{\mu_n}$ appear. We want to introduce sheaf cohomology for étale sheaves. Before we do this, we state the following important theorem.

Theorem 2.14 (existence of 'enough injectives', [31] p.61)**.** *For a sheaf $\mathcal{F}$ on $X_{ét}$, there exists a complex of injective sheaves $\mathcal{I}^\bullet = \mathcal{I}^0 \to \mathcal{I}^1 \to ...$ such that $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ is exact.*

The theorem above implies together and the fact that category of sheaves on $X_{\text{ét}}$ is an abelian category ([31], p.53) together with the theory of derived functors imply that any left-exact functor $F : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Ab}$ admits its right derived functors $\{R^tF : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Ab}\}_{t \in \mathbb{Z}_{\geq 0}}$. They have the following properties:

Proposition 2.15. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$, such that $\mathcal{A}$ has 'enough injectives', i.e. any object has an injective resolution (as in Theorem 2.14). Then there exist functors $\{R^tF : \mathcal{A} \to \mathcal{B}\}_{t \in \mathbb{Z}_{\geq 0}}$ such that:*

- $R^0F = F$
- *For a short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathcal{A}$ there exist 'boundary maps' $\{\delta_t\}_{t \in \mathbb{Z}_{\geq 0}}$ such that $... \to R^tF(A_1) \to R^tF(A_2) \to R^tF(A_3) \xrightarrow{\delta_t} R^{t+1}F(A_1) \to R^{t+1}F(A_2) \to ...$ is a long exact sequence in $\mathcal{B}$.*
- *For I an injective object of $\mathcal{A}$ we have $R^tF(I) = 0$ for $t > 0$.*
- *For A an object of $\mathcal{A}$ and $A \to I^\bullet$ an injective resolution, the objects $R^tF(A)$ can be calculated by $R^tFA \cong h^t(F(I^\bullet)) := \dfrac{\ker(F(I^t) \to F(I^{t+1}))}{Im(F(I^t) \to F(I^{t+1}))}$.*

The first example will be the crucial one for us, while the second one also occurs at times.

Example 2.16. For X a scheme the *global sections functor* $\Gamma(X,-) : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Ab}$, which sends a sheaf of abelian groups to its sections on X is left-exact. We can hence form its right-derived functors. For $\pi : X \to Y$ a morphism of schemes the *direct image functor* $\pi_* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$, which sends a sheaf $\mathcal{F}$ to the sheaf $\pi_*\mathcal{F}$, which assigns $(U \to Y) \mapsto \mathcal{F}(U \times_Y X \to X)$. This functor is also left-exact, hence we can form its right-derived functors.

Using the above we define étale cohomology.

Definition 2.17. For X a scheme and $n \geq 0$ we denote $R^n\Gamma(X,-) =: \mathrm{H}^n_{\text{ét}}(X,-)$.
The n'th *étale cohomology group* of X with values in $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ is $\mathrm{H}^n_{\text{ét}}(X,\mathcal{F})$.

There are many properties of étale cohomology that we will use. They are listed below.

Proposition 2.18. *Étale cohomology has the following properties:*

- *(δ-**functor**) Whenever $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{Q} \to 0$ is a short exact sequence in $\mathrm{Sh}(X_{\acute{e}t})$, there exist connecting homomorphism $\delta_n : \mathrm{H}^n_{\text{ét}}(X,\mathcal{Q}) \to \mathrm{H}^{n+1}_{\text{ét}}(X,\mathcal{K})$ such that there is a long exact sequence:*
$$... \to \mathrm{H}^n_{\text{ét}}(X,\mathcal{K}) \to \mathrm{H}^n_{\text{ét}}(X,\mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(X,\mathcal{Q}) \xrightarrow{\delta_n} \mathrm{H}^{n+1}_{\text{ét}}(X,\mathcal{K}) \to ...$$
This is functorial with respect to homomorphism of short exact sequences.
- ***(pullback homomorphism)** For $g : X \to Y$ a morphism we have, since $\Gamma(X,-) = \Gamma(Y,-) \circ g_*$, natural transformations $\mathrm{H}^r_{\text{ét}}(Y, g^*-) \to \mathrm{H}^r_{\text{ét}}(X,-)$. There is also a natural transformation $\mathrm{Id}(-) \to g_*g^*(-)$ of functors $\mathrm{Sh}(Y_{\acute{e}t}) \to \mathrm{Sh}(Y_{\acute{e}t})$. Thus we have $g^* : \mathrm{H}^r_{\text{ét}}(Y,\mathcal{F}) \to \mathrm{H}^r_{\text{ét}}(X, g^*\mathcal{F})$ by composing them, which is called the pullback by g. The pullback homorphisms satisfy $(g \circ f)^* = f^* \circ g^*$ and $\mathrm{Id}^* = \mathrm{Id}$.*

The last claim follows from the fact that given $X \xrightarrow{f} Y \xrightarrow{g} Z$ the two homomorphisms $(g \circ f)^*$ and $f^* \circ g^*$ $\mathrm{H}^r_{\text{ét}}(Z,\mathcal{F}) \rightrightarrows \mathrm{H}^r_{\text{ét}}(X, f^*g^*\mathcal{F})$ agree for $r = 0$ and $\mathrm{H}^r_{\text{ét}}(Z,-) \to \mathrm{H}^r_{\text{ét}}(X, g^*f^*-)$ is a morphism of δ-functors. Since $\mathrm{H}^r_{\text{ét}}(Z,-)$ is a universal δ-functor, the maps $\mathrm{H}^r_{\text{ét}}(Z,-) \to \mathrm{H}^r_{\text{ét}}(X, g^*f^*-)$ are determined by the ones with $r = 0$ and hence they are equal (see [43, Tag 010P]). Similarly we define the higher direct images.

Definition 2.19. For $\pi : X \to Y$ a morphism of schemes and $n \geq 0$, define the n'th *higher direct image* functor by $R^n\pi_*(-) : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$.

It turns out that there is an alternative way to describe the n'th higher direct images.

Proposition 2.20. *Let $\pi : X \to Y$ be a morphism and let $\mathcal{H}^n(-) : \mathrm{Sh}(X_{\acute{e}t}) \to \mathrm{Sh}(Y_{\acute{e}t})$ be the functor that takes $\mathcal{F}$ to $\mathcal{H}^n(\mathcal{F}) \in \mathrm{Sh}(Y_{\acute{e}t})$, which is defined to be the sheafification of the assignment $U \mapsto \mathrm{H}^n_{\text{ét}}(U \times_Y X, \mathcal{F})$. The functors $\mathcal{H}^n(-)$ and $R^n\pi_*(-)$ are canonically isomorphic.*

We are often interested in the cohomology of a variety with coefficients in the constant sheaf $\underline{\mathbb{Z}/n\mathbb{Z}}$. As mentioned earlier, the sheaf $\underline{\mu_n}$ agrees with $\underline{\mathbb{Z}/n\mathbb{Z}}$ in some cases and we have the following exact sequence of sheaves involving $\underline{\mu_n}$.

Proposition 2.21 (Kummer sequence)**.** *For X a scheme over a field k and n an integer not divisible by* char(k) *the following sequence of sheaves on $X_{ét}$ is exact:*

$$0 \to \underline{\mu_n} \xrightarrow{\iota} \underline{\mathbb{G}_m} \xrightarrow{[n]} \underline{\mathbb{G}_m} \to 0$$

Here $[n]$ denotes the morphism of sheaves induced by $\mathcal{O}(U)^* \to \mathcal{O}(U)^* \quad x \mapsto x^n$.

Proof. That $\ker([n]) = \iota(\underline{\mu_n})$ follows easily. Now we need to check that $[n] : \underline{\mathbb{G}_m} \to \underline{\mathbb{G}_m}$ is surjective, i.e. that for any section $s \in \underline{\mathbb{G}_m}(U)$ there is $V \to U$ étale such that $s|_V \in \underline{\mathbb{G}_m}(V)$ is in the image of $[n](V)$. First of all, as surjectivity can be checked locally, we may shrink U to make it affine, $U = \mathrm{Spec}(R)$. Take $V := \mathrm{Spec}(R[X]/(X^n - s)) \to \mathrm{Spec}(R) = U$ to be induced by the inclusion of R into $R[X]/(X^n - s)$. Under $[n](V)$ the element $X \in (R[X]/(X^n - s))^*$ is mapped to s. As the derivative of $X^n - s$ is $n \cdot X^{n-1}$, which is invertible in $R[X]/(X^n - s)$ we see that the morphism $V \to U$ is standard-étale, hence étale, so indeed $[n]$ is surjective. □

This gives us the following result.

Corollary 2.22. *For X a scheme over a field whose characteristic does not divide n there is a long exact sequence:* $... \to \mathrm{H}^r_{ét}(X, \underline{\mu_n}) \to \mathrm{H}^r_{ét}(X, \underline{\mathbb{G}_m}) \xrightarrow{[n]} \mathrm{H}^r_{ét}(X, \underline{\mathbb{G}_m}) \to \mathrm{H}^{r+1}_{ét}(X, \underline{\mu_n}) \to ...$

As in the case of the Zariski topology one has the Cech cohomology groups. First we define the Cech-complex.

Definition 2.23. Let $\mathcal{F}$ be a sheaf on $X_{ét}$ and let $\mathcal{U} = \{U_i \to X\}_i$ be an étale covering. We define the *Cech-complex* of $\mathcal{F}$ relative to $\mathcal{U}$ to be:

$$C(\mathcal{U}, \mathcal{F})^\bullet = 0 \to \mathcal{F}(X) \to \prod_i \mathcal{F}(U_i) \xrightarrow{d^0} \prod_{i,j} \mathcal{F}(U_i \times_X U_j) \xrightarrow{d^1} \prod_{i,j,k} \mathcal{F}(U_i \times_X \times U_j \times_X U_k) \to ...$$

Using the abbreviation $U_{i_1} \times_X ... \times_X U_{i_n} = U_{i_1 ..., i_n}$, the maps d^n are defined on $\prod\limits_{i_0,...,i_n} \mathcal{F}(U_{i_1 ... i_n})$ by $(d^n(s))_{i_0 ... i_{n+1}} = \sum\limits_{k=0}^{n+1} (-1)^k \mathrm{res}_k(s_{i_1 ... \hat{i_k} ... i_{n+1}})$, where res_k denotes the restriction map obtained from the projection $U_{i_0 ... i_{n+1}} \to U_{i_0 ... \hat{i_k} ... i_{n+1}}$ (the hat means 'omit').

One checks that this indeed defines a complex. Using this we can define the étale Cech cohomology groups.

Definition 2.24. Let $\mathcal{F}$ be a sheaf on $X_{ét}$ and $\mathcal{U} = \{U_i\}$ an étale cover of X. The r'th Cech cohomology group of $\mathcal{F}$ relative to $\mathcal{U}$ is defined to be $\check{\mathrm{H}}^r_{ét}(\mathcal{U}, \mathcal{F}) := h^r(C(\mathcal{U}, \mathcal{F})^\bullet)$.
Define the r'th *Cech cohomology group* of X with coefficients in $\mathcal{F}$ to be $\check{\mathrm{H}}^r_{ét}(X, \mathcal{F}) := \varinjlim\limits_{\mathcal{U}} \check{\mathrm{H}}^r_{ét}(\mathcal{U}, \mathcal{F})$, where the colimit is taken over the refinements of étale covers.

One easily that $\check{\mathrm{H}}^r_{ét}(X, -)$ is in fact a functor $\mathrm{Sh}(X_{ét}) \to \mathrm{Ab}$, it can even be defined as a functor from $\mathrm{PSh}(X_{ét})$ (the category of étale presheafs). In particular the first Cech group is interesting to us.

Remark 2.25. We consider $\check{\mathrm{H}}^1_{\text{ét}}(X, \mathcal{F})$. Take a cover $\mathcal{U} = (U_i)_i$ of X, then giving an element of $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{F})$ is the same as giving elements $g_{ij} \in \mathcal{F}(U_{ij})$ such that $g_{ij} + g_{jk} = g_{ik}$ up to changing g_{ij} to $g_{ij} + h|_{U_{ij}}$ for $h \in \mathcal{F}(X)$. Now consider the case $\mathcal{F} = \underline{\mathbb{G}_m}$, then the g_{ij} lie in $\mathcal{O}(U_{ij})^\times$. It follows from ([31] p.78) that $\check{\mathrm{H}}^1_{\text{ét}}(X, \underline{\mathbb{G}_m}) \cong \mathrm{L}_1(X)$, the isomorphism classes of étale locally trivial line bundles. By Proposition 4.32 any such line bundle is locally trivial for the Zariski topology and hence we obtain $\mathrm{Pic}(X) \cong \check{\mathrm{H}}^1_{\text{ét}}(X, \underline{\mathbb{G}_m})$.

The 0'th and first Cech groups always agree with the derived functor cohomology groups.

Proposition 2.26 ([29], III Corollary 2.10)**.** *Let $\mathcal{C}/X$ be a site and let $r \in \{0, 1\}$, then there are canonical iomorphisms* $\check{\mathrm{H}}^r_{\mathcal{C}}(X, \mathcal{F}) \cong \mathrm{H}^r_{\mathcal{C}}(X, \mathcal{F})$

This follows from the Cech to derived spectral sequence (see [29] III Prop. 2.7). Another consequence of this is the following.

Proposition 2.27 (Mayer Vietoris)**.** *Let X be a scheme with cover U_0, U_1 by open immersions and let $\mathcal{F} \in \mathrm{Sh}(X_{\acute{e}t})$. There is a long exact sequence where $\phi(s_0, s_1) = s_0 - s_1$:*

$$\dots \to \mathrm{H}^n_{\text{ét}}(U_0, \mathcal{F}) \bigoplus \mathrm{H}^n_{\text{ét}}(U_1, \mathcal{F}) \xrightarrow{\phi} \mathrm{H}^n_{\text{ét}}(U_0 \cap U_1, \mathcal{F}) \to \mathrm{H}^{n+1}_{\text{ét}}(X, \mathcal{F}) \to \mathrm{H}^{n+1}_{\text{ét}}(U_0, \mathcal{F}) \bigoplus \mathrm{H}^{n+1}_{\text{ét}}(U_1, \mathcal{F}) \to \dots$$

Our goal now is to construct the cup product in étale cohomology on a quasi-projective variety X. One can do this for any scheme X via the language of derived categories [43, Tag 0FKU] but derived categories lie beyond the scope of this thesis (see [36] for an intuitive introduction and [43, Tag 05QI] for a rigorous treatment).

Theorem 2.28 ([2] Theorem 4.2)**.** *Let X be a variety such that any finite set of geometric points of X is contained in an affine open subscheme of X. Then $\check{\mathrm{H}}^*_{\acute{e}t}(X, -)$ is a δ-functor $\mathrm{Sh}(X_{\acute{e}t}) \to \mathrm{Ab}$ and the isomorphism $\mathrm{H}^0_{\text{ét}}(X, -) \to \check{\mathrm{H}}^0_{\acute{e}t}(X, -)$ in degree 0 extends to an isomorphism of δ-functors $\mathrm{H}^*_{\text{ét}}(X, -) \to \check{\mathrm{H}}^*_{\acute{e}t}(X, -)$.*

Any variety X that is quasi-projective over a Noetherian ring meets the condition in the above theorem [2]. Thus by Proposition 1.51 we have that in particular for an algebraic group that étale Cech cohomology and derived functor cohomology agree.

Before introducing cup products we introduce the following notion.

Definition 2.29. Let Λ be a commutative ring and also denote by Λ the constant sheaf on $X_{\text{ét}}$. Then $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ is *sheaf of* Λ*-modules* if for each U, $\mathcal{F}(U)$ is a $\Lambda(U)$-module and the obvious diagramms commute.
For $\mathcal{F}, \mathcal{G}$ sheaves of Λ-modules, we define their *tensor product* $\mathcal{F} \otimes_\Lambda \mathcal{G}$ over Λ to be the sheafification of the assignment $U \mapsto \mathcal{F}(U) \otimes_{\Lambda(U)} \mathcal{G}(U)$.

Remark 2.30. The tensor product of two Λ-modules satisfies the usual universal mapping property in the category of Λ-modules. If $g : X \to Y$ is a morphism, then $g^*(\mathcal{F} \otimes_\Lambda \mathcal{G}) \cong g^*\mathcal{F} \otimes_\Lambda g^*\mathcal{G}$ canonically (by using that tensor products and filtered colimits commute).

We now give the properties of cup-products.

Proposition 2.31 (Cup products)**.** *Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X_{\acute{e}t})$ be sheaves of Λ-modules. For all $p, q \geq 0$ there exist Λ-linear maps called cup products $\cup : \mathrm{H}^p_{\text{ét}}(X, \mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(X, \mathcal{G}) \to \mathrm{H}^{p+q}_{\text{ét}}(X, \mathcal{F} \otimes_\Lambda \mathcal{G})$. They satisfy the following properties:*

- *The cup product is associative, i.e. for $x \in \mathrm{H}^p_{\text{ét}}(X, \mathcal{F})$, $y \in \mathrm{H}^q_{\text{ét}}(X, \mathcal{G})$ and $z \in \mathrm{H}^r_{\text{ét}}(X, \mathcal{H})$ we have $(x \cup y) \cup z = x \cup (y \cup z)$.*
- *If $\mathcal{F} \to \mathcal{F}'$ and $\mathcal{G} \to \mathcal{G}'$ are morphisms of Λ-modules then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^p_{\text{ét}}(X, \mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(X, \mathcal{G}) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\text{ét}}(X, \mathcal{F} \otimes_\Lambda \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{H}^p_{\text{ét}}(X, \mathcal{F}') \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(X, \mathcal{G}') & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\text{ét}}(X, \mathcal{F}' \otimes_\Lambda \mathcal{G}') \end{array}$$

- *If $g : X \to Y$ is a morphism, then the cup products of X and Y are related by the following commuting diagram:*

$$\begin{array}{ccc} \mathrm{H}^p_{\text{ét}}(X, g^*\mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(X, g^*\mathcal{G}) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\text{ét}}(X, g^*\mathcal{F} \otimes_\Lambda g^*\mathcal{G}) \\ {\scriptstyle g^* \otimes g^*}\uparrow & & {\scriptstyle g^*}\uparrow \\ \mathrm{H}^p_{\text{ét}}(Y, \mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(Y, \mathcal{G}) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\text{ét}}(X, \mathcal{F} \otimes_\Lambda \mathcal{G}) \end{array}$$

- *If $\mathcal{A} = \mathcal{F} = \mathcal{G}$ is a sheaf of Λ-algebras then $\mathrm{H}^*_{\text{ét}}(X, \mathcal{A}) := \bigoplus_n \mathrm{H}^n_{\text{ét}}(X, \mathcal{A})$ becomes a graded Λ-algebra, so for $x \in \mathrm{H}^p_{\text{ét}}(X, \mathcal{A})$ and $y \in \mathrm{H}^q_{\text{ét}}(X, \mathcal{A})$ we have $x \cup y = (-1)^{pq} y \cup x$.*

A proof of the existence of these producs is given in the Appendix. The case when $\mathcal{F} = \Lambda$ is the most interesting as in this case we in fact can construct a ring.

Definition 2.32. Let Λ be a constant sheaf of commutative rings on $X_{\text{ét}}$. The *cohomology ring* of X with coefficients in Λ is the set $\mathrm{H}^*_{\text{ét}}(X, \Lambda) = \bigoplus_{n \geq 0} \mathrm{H}^n_{\text{ét}}(X, \Lambda)$ with multiplication $\cup$.

Note that naturality of cup products in the second argument implies that if $f : X \to Y$ is a morphism, then $f^* : \mathrm{H}^*_{\text{ét}}(X, \Lambda) \to \mathrm{H}^*_{\text{ét}}(Y, \Lambda)$ is a Λ-algebra homomorphism.

Remark 2.33. By the first property of the cup products applied to the sheaves $\underline{\mathbb{Z}/l^n\mathbb{Z}}$, one can pass to the limit and apply $- \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ to obtain $\mathbb{Q}_l$-linear maps:

$$\cup : \mathrm{H}^p_{\text{ét}}(X, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^q_{\text{ét}}(X, \mathbb{Q}_l) \to \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{Q}_l)$$

This puts a $\mathbb{Q}_l$-algebra structure on $\mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_l) := \bigoplus_{n \geq 0} \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_l)$.

Note that for X, Y schemes and $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$, $\mathcal{G} \in \mathrm{Sh}(Y_{\text{ét}})$ sheaves of Λ-modules there are maps $\mathrm{H}^p_{\text{ét}}(X, \mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(Y, \mathcal{G}) \to \mathrm{H}^{p+q}_{\text{ét}}(X \times_k Y, \mathcal{F} \otimes_\Lambda \mathcal{G}) \quad x \otimes y \mapsto p_1^*(x) \cup p_2^*(y)$. These induce a map $\bigoplus_{p+q=n} \mathrm{H}^p_{\text{ét}}(X, \mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(Y, \mathcal{G}) \to \mathrm{H}^n_{\text{ét}}(X \times_k Y, p_1^*\mathcal{F} \otimes_\Lambda p_2^*\mathcal{G})$ is an isomorphism. Below we mean by a flat sheaf $\mathcal{F}$ that $\mathcal{F}(U)$ is flat over $\Lambda(U)$ for all U.

Theorem 2.34 (Künneth, [29] Cor. VI.8.13)**.** *Let $X, Y, \mathcal{F}, \mathcal{G}, \Lambda$ as above and assume that $\mathcal{F}$ is flat over Λ and for all $m \geq 0$ that $\mathrm{H}^m_{\text{ét}}(X, \mathcal{F})$ is flat over Λ. Then for all p, q the above maps $\bigoplus_{p+q=n} \mathrm{H}^p_{\text{ét}}(X, \mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(Y, \mathcal{G}) \to \mathrm{H}^n_{\text{ét}}(X \times_k Y, p_1^*\mathcal{F} \otimes_\Lambda p_2^*\mathcal{G})$ are isomorphisms of Λ-modules.*

A very useful tool in étale cohomology is the Leray spectral sequence. For stating the existence of this spectral sequence we give the definition of a spectral sequence. For background knowledge on spectral sequences, see [46].

Definition 2.35. A *spectral sequence* in an abelian category $\mathcal{A}$ consists of the following data:

1. A 'starting page' for $r_0 \in \mathbb{Z}_{>0}$; with objects $\{E_{r_0}^{p,q}\}_{p,q\in\mathbb{Z}}$ such that $E_{r_0}^{p,q} \neq 0$ implies $p, q \geq 0$.
2. Differentials on the starting page $d_r^{p,q} : E_{r_0}^{p,q} \to E^{p+r_0,q-r_0+1}$ such that $d_r^{p+r_0,q-r_0+1} \circ d_r^{p,q} = 0$.
3. The r'th pages for $r > r_0$, where $E_{r+1}^{p,q} = \dfrac{\ker(d_r^{p,q})}{\mathrm{Im}(d_r^{p-r,q+r-1})}$, i.e. the object in position (p, q) on the $r+1$'th page equal the homology in position (p, q) on the $r'th$ page.
4. Differentials on the r'th page $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ that satisfy the same condition as in 2.

We note that for the $(p, q) \in \mathbb{Z}^2_{\geq 0}$ and $r \geq q + 2$ that the differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ has to have $E_r^{p,q}$ as kernel as $q - r + 1 < 0$, hence $E_r^{p+r,q-r+1} = 0$. Moreover we note that for $r \geq p + 1$ the differential $E_r^{p-r,q+r-1} \to E_r^{p,q}$ is the zero-map as $E_r^{p-r,q+r-1} = 0$. So we see that for some $r_{p,q}$ large enough, the homology in position (p, q) on page r will be $E_{r_{p,q}}^{p,q}$ for all $r \geq r_{p,q}$. We will call this *convergence* of the spectral sequence.

Definition 2.36. We denote the entry $E_{r_{p,q}}^{p,q}$ above by $E_\infty^{p,q}$. We say that a spectral sequence *converges to* $(M_n)_{n\geq 0}$ if there is an isomorphism of graded objects $\bigoplus_{n\geq 0}(\bigoplus_{p+q=n} E_\infty^{p,q}) \cong \bigoplus_{n\geq 0} M_n$. We denote the convergence by $E_{r_0}^{p,q} \implies M_n$.

We now introduce the most fundamental spectral sequence of the ones that we will be using. In fact, the existence of the others are derived from this one.

Proposition 2.37 ([46], p.150 Grothendieck spectral sequence)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two additive left-exact functors between abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and assume that F takes injective objects to G-acyclic objects. Then there is a spectral sequence: $E_2^{p,q} = (R^pG \circ R^qF) \implies R^{p+q}(G \circ F)$*

One application of this is the existence of the Leray spectral sequence.

Proposition 2.38 (Leray spectral sequence)**.** *Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. There is a spectral sequence $E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(Y, R^q f_*\mathcal{F}) \implies H^{p+q}(X, \mathcal{F})$.*

Proof. We have that the composition of the functors $\Gamma(Y, -) \circ f_* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}}) \to \mathrm{Ab}$ equals $\Gamma(X, -)$. We note that the spectral sequence above is then just the Grothendieck spectral sequence for these specific functors. The only thing left to show is that f_* sends injective sheaves to $\Gamma(Y, -)$-acyclic sheaves. We do this by showing that $f_*\mathcal{I}$ is injective for $\mathcal{I}$ injective. This is equivalent to the functor $\mathrm{Hom}(-, f_*\mathcal{I})$ being exact, which by the adjunction $\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$ is equivalent to $\mathrm{Hom}(f^*(-), \mathcal{I})$ being exact. This follows since f^* is an exact functor and $\mathrm{Hom}(-, \mathcal{I})$ is an exact functor. So we conclude that the spectral sequence exists. □

Sometimes it is easier to work over a field of characteristic 0 than with an abelian group, so we make the following definition.

Definition 2.39. Define $\mathrm{H}^*_{\text{ét}}(X, \mathbb{Z}_l) := \varprojlim_n \mathrm{H}^*_{\text{ét}}(X, \underline{\mathbb{Z}/l^n\mathbb{Z}})$. It has the structure of a $\mathbb{Z}_l$-module. We define $\mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_l) := \mathrm{H}^*_{\text{ét}}(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, which we call the *$l$-adic cohomology of X*.

We can 'upgrade' the spectral sequences above to their l-adic versions under a suitable hypothesis.

Remark 2.40 (Upgrading spectral sequence to l-adic variant under conditions). Let $\{E(n)^{p,q}_r\}_{n\in\mathbb{Z}_{>0}}$ be spectral seqeuences of abelian groups starting at the same page $r = r_0$ and assume that $E(n)^{p,q}_{r_0}$ is a $\mathbb{Z}/l^n\mathbb{Z}$-module. Then notice that all the terms $E(n)^{p,q}_r$ have the structure of a $\mathbb{Z}/l^n\mathbb{Z}$-module and that the differentials $d(n)^{p,q}_r$ on each page are $\mathbb{Z}/l^n\mathbb{Z}$ linear (as they are $\mathbb{Z}$-linear). Suppose that we are given maps $... \leftarrow E(n)^{p,q}_{r_0} \leftarrow E(n+1)^{p,q}_{r_0} \leftarrow ...$ for all p, q that commute with the differentials and such that for all other pages $r \geq r_0$ the induce maps $E(n+1)^{p,q}_r \to E(n)^{p,q}_r$ also commute with the differentials. This implies that one can define an r_0'th page with objects $\varprojlim_n E(n)^{p,q}_{r_0}$ and differentials $\varprojlim_n d(n)^{p,q}_{r_0}$, which are now $\mathbb{Z}_l$-linear. If we assume that taking the inverse limit is exact on all the pages then we get $\dfrac{\ker(\varprojlim_n d(n)^{p,q}_{r_0})}{\mathrm{Im}(\varprojlim_n d(n)^{p,q}_{r_0})} = \varprojlim_n E(n)^{p,q}_{r_0+1}$. Similarly on all the higher pages, one gets the result above with r_0 replaced by r. Hence there is a spectral sequence $\{E^{p,q}_r\} := \{\varprojlim_n E_r(n)^{p,q}\}$ and as the convergence of $E(n)^{p,q}_r$ does not depend on n it converges to $E^{p,q}_\infty = \varprojlim_n E(n)^{p,q}_\infty$.
Once one has the spectral sequence $\{E^{p,q}_r\} := \{\varprojlim_n E_r(n)^{p,q}\}$, one can consider the terms $E^{p,q}_r \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. The differentials then extend to $\mathbb{Q}_l$-linear differentials. Notice that as $\mathbb{Q}_l$ is flat over $\mathbb{Z}_l$ that $E^{p,q}_{r+1} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ equals $\ker(d^{p,q}_r \otimes 1)/\mathrm{Im}(d^{p-r,q+r-1}_r \otimes 1)$, hence we get a spectral sequence $\{E^{p,q}_r \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\}$ with infinity page $E^{p,q}_\infty \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$.

We now define another operation that one can do on sheafs.

Definition 2.41. Let $U \subset X$ be an open subset of a scheme X and denote the inclusion map by j. Then for any $\mathcal{F} \in \mathrm{Sh}(U_{\text{ét}})$ and any étale $\varphi : V \to X$ we consider the assignment:

$$V \mapsto \begin{cases} \mathcal{F}(V) \text{ if } \varphi(V) \subset U \\ 0 \text{ else} \end{cases}$$

The corresponding sheaf $j!\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ is called the *extension by* 0 of $\mathcal{F}$.

For defining compactly supported cohomology we are interested in the case of embedding a variety X into a complete one. The fundamental theorem concering this is the following.

Theorem 2.42 (Nagata, [35]). *Let $f : X \to S$ be a finite type morphism between Noetherian schemes. Then there exists an open immersion $j : X \to \overline{X}$ and a proper morphism $\overline{f} : \overline{X} \to S$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & S \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle \overline{f}} & \\ & \overline{X} & \end{array}$$

Such $\overline{X}$ is called a *compactification* of X. In particular a compactification of a scheme of finite type over a field exists.

Definition 2.43. (Cohomology with compact support) Let X be a scheme that admits a compactification $j : X \hookrightarrow \overline{X}$ and let $\mathcal{F}$ be a torsion sheaf on $X_{\text{ét}}$. Define the n'th *cohomology group with compact support* to be $\mathrm{H}^n_{\mathrm{c}}(X, \mathcal{F}) := \mathrm{H}^n_{\text{ét}}(\overline{X}, j!\mathcal{F})$.

An important thing to mention is that the above definition does not depend on the compacification $j : X \to \overline{X}$.

Proposition 2.44 ([29], Prop. VI.3.1 and Prop. III.1.29)**.** *The following holds:*

- *The definition of cohomology with compact support does not depend on the choice of compactification $j : X \hookrightarrow \overline{X}$.*
- *The functor $\mathrm{H}^n_{\mathrm{c}}(X, -)$ defines a δ-functor, i.e. a short exact sequence of sheaves on $X_{\text{ét}}$ is sent to a long exact sequence functorially.*
- *For $f : X \to Y$ a morphism of finite type k-schemes and $\mathcal{F}$ a torsion sheaf on $Y_{\text{ét}}$ there exists a pullback homomorphism $\mathrm{H}^r_{\mathrm{c}}(Y, \mathcal{F}) \to \mathrm{H}^r_{\mathrm{c}}(X, f^*\mathcal{F})$.*

We use the Nagata compactification for the following definition.

Definition 2.45. For $\pi : X \to S$ a finite type morphism of Noetherian schemes, define $R^n_c\pi_*\pi(\mathcal{F})$ for $\mathcal{F}$ a torsion sheaf as follows: Choose a compactification $j : X \to \overline{X}$ such that π factors via a proper morphism $\overline{\pi} : \overline{X} \to S$. Then define $R^n_c\pi_*(\mathcal{F}) = R^n\overline{\pi}_*(j!\mathcal{F}) \in \mathrm{Sh}(S_{\text{ét}})$.

For a proof that the previous is well defined, see ([29], Prop VI.3.1). Just as in the case of étale cohomology, a commuting square as below and a torsion sheaf $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ gives a canonical base-change morphism $g^*R^n_c\pi_*\mathcal{F} \to R^n_c\pi_*f^*\mathcal{F}$.

$$\begin{array}{ccc} X' & \xrightarrow{\pi'} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\pi} & Y \end{array}$$

Compactly supported cohomology has the following nice feature.

Proposition 2.46 ([29], Cor VI.2.3 and Prop VI.3.2)**.** *If we have a commuting square as above that is cartesian (i.e. $X' = Y' \times_Y X$) and $\mathcal{F}$ is a torsion sheaef on $X_{\text{ét}}$ then:*

- *For étale cohomology: The base-change morphism $g^*R^n\pi_*\mathcal{F} \to R^n\pi_*f^*\mathcal{F}$ is an isomorphism when π is proper. This is called proper base change.*
- *For compactly supported cohomology: The base-change morphism $g^*R^n_c\pi_*\mathcal{F} \to R^n_c\pi_*f^*\mathcal{F}$ is an isomorphism.*

The proof of the second statement makes use of the first statement. The other important feature that compactly supported cohomology has is Poincaré duality. We present it here for constant torsion sheaves, but it also holds for 'constructible' sheaves, which we did not define.

Theorem 2.47 (Poincare duality, [29], Theorem VI.11.1). *Let X be a smooth variety of dimension d over an algebraically closed field and let $\Lambda \in \mathrm{Sh}(X_{ét})$ be a constant torsion sheaf. Then for all $n \leq 2d$ there are perfect pairings $\langle -,- \rangle : \mathrm{H}^n_{\text{ét}}(X,\Lambda) \times \mathrm{H}^{2d-n}_{\text{c}}(X,\Lambda) \to \mathrm{H}^{2d}_{\text{c}}(X,\Lambda) \cong \Lambda$ such that:*

- *For $f : Y \to X$ a morphism of smooth varieties of dimension d one has $\langle f^*x, f^*y\rangle = f^*\langle x,y\rangle$. The former equals $\deg(f) \cdot \langle x,y\rangle$ when f is a finite flat morphism.*
- *The pairings are functorial in the sheaf Λ and in particular, they induce a perfect pairing of $\mathbb{Q}_l$-vector spaces $\mathrm{H}^n_{\text{ét}}(X,\mathbb{Q}_l) \to \mathrm{H}^{2d-n}_{\text{c}}(X,\mathbb{Q}_l) \to \mathrm{H}^{2d}_{\text{c}}(X,\mathbb{Q}_l) \cong \mathbb{Q}_l$.*

Two other important finiteness theorems are the following:

Theorem 2.48 (Cohomological dimension, [29] Theorem VI.1.1 and SGA 4, §2,3). *Let X be a finite type scheme of dimension d over a separably closed field k and let $l \neq \mathrm{char}(k)$ be a prime such that l^n kills $\mathcal{F} \in \mathrm{Sh}(X_{ét})$. Then:*

- *In general $\mathrm{H}^r_{\text{ét}}(X,\mathcal{F}) = 0$ for $r > 2d$.*
- *If X is affine, then $\mathrm{H}^r_{\text{ét}}(X,\mathcal{F}) = 0$ for $r > d$.*

Theorem 2.49 (Finiteness of étale cohomology, SGA4½ Corollaire 1.10 p.236). *Let X be a finite type scheme over a separably closed field k and let $\mathcal{F}$ be a locally constant sheaf on X killed by l^n where $\mathrm{char}(k) \neq l$ is prime. Then $\mathrm{H}^r_{\text{ét}}(X,\mathcal{F})$ are finite groups for all r.*

It is time to use all of the above machinery above to actually calculate some étale cohomolgy groups. We begin with perhaps the most basic schemes and sheaves that one can consider.

Example 2.50. Let k be a separably closed field over which we consider the following varieties and let $l \neq \mathrm{char}(k)$ be a prime. Then we have:

- $\mathrm{H}^0_{\text{ét}}(\mathbb{A}^1,\underline{\mathbb{Z}/l^n\mathbb{Z}}) = \mathbb{Z}/l^n\mathbb{Z}$ and $\mathrm{H}^r_{\text{ét}}(\mathbb{A}^1,\underline{\mathbb{Z}/l^n\mathbb{Z}}) = 0$ for $r > 0$.
- $\mathrm{H}^1_{\text{ét}}(\mathbb{G}_m,\underline{\mathbb{Z}/l^n\mathbb{Z}}) = \mathrm{H}^0_{\text{ét}}(\mathbb{G}_m,\underline{\mathbb{Z}/l^n\mathbb{Z}}) = \mathbb{Z}/l^n\mathbb{Z}$ and $\mathrm{H}^r_{\text{ét}}(\mathbb{G}_m,\underline{\mathbb{Z}/l^n\mathbb{Z}}) = 0$ for $r > 1$.
- $\mathrm{H}^2_{\text{ét}}(\mathbb{P}^1,\underline{\mathbb{Z}/l^n\mathbb{Z}}) = \mathrm{H}^0_{\text{ét}}(\mathbb{P}^1,\underline{\mathbb{Z}/l^n\mathbb{Z}}) = \mathbb{Z}/l^n\mathbb{Z}$ and $\mathrm{H}^r_{\text{ét}}(\mathbb{P}^1,\underline{\mathbb{Z}/l^n\mathbb{Z}}) = 0$ for $r \neq 0,2$.

Proof. As $\mathbb{A}^1$ and $\mathbb{G}_m$ are affine of dimension 1 and connected it suffices to show by Theorem 2.48 that $\mathrm{H}^1_{\text{ét}}(\mathbb{A}^1,\underline{\mathbb{Z}/l^n\mathbb{Z}}) = 0$ and $\mathrm{H}^1_{\text{ét}}(\mathbb{G}_m,\underline{\mathbb{Z}/l^n\mathbb{Z}}) = \mathbb{Z}/l^n\mathbb{Z}$ for the first two points. Note that over a separably closed field k with $\mathrm{char}(k) \neq l$ the sheaves $\underline{\mu_{l^n}}$ and $\underline{\mathbb{Z}/l^n\mathbb{Z}}$ are isomorphic.

For $\mathbb{A}^1$, consider the Kummer sequence: $0 \to \mu_{l^n}(k) \to k^\times \overset{x\mapsto x^{l^n}}{\to} k^\times \to \mathrm{H}^1_{\text{ét}}(\mathbb{A}^1,\underline{\mu_{l^n}}) \to \mathrm{H}^1_{\text{ét}}(\mathbb{A}^1,\underline{\mathbb{G}_m})$. We have $\mathrm{H}^1_{\text{ét}}(\mathbb{A}^1,\underline{\mathbb{G}_m}) = \check{\mathrm{H}}^1_{\text{ét}}(\mathbb{A}^1,\underline{\mathbb{G}_m}) = \mathrm{Pic}(\mathbb{A}^1) = \mathrm{Pic}(k[X]) = 0$ since $k[X]$ is a UFD and hence $\mathrm{H}^1_{\text{ét}}(\mathbb{A}^1,\underline{\mu_{l^n}}) = 0$ since $k^\times \to k^\times$ is onto.
For $\mathbb{G}_m$, note that $\mathrm{H}^1_{\text{ét}}(\mathbb{G}_m,\underline{\mathbb{G}_m}) = 0$ and hence $\mathrm{H}^1_{\text{ét}}(\mathbb{G}_m,\underline{\mu_{l^n}}) = \mathrm{coker}(\mathcal{O}(\mathbb{G}_m)^\times \to \mathcal{O}(\mathbb{G}_m)^\times)$, the map being the l^n'th power map. As we have $\mathcal{O}(\mathbb{G}_m)^\times = k^\times \cdot \{X^n\}_{n\in\mathbb{Z}}$ we see that this cokernel is $\mathbb{Z}/l^n\mathbb{Z}$.
For the third point, apply the Mayer-Vietoris sequence 2.27 to the standard open covering U_0, U_1 of $\mathbb{P}^1$ with intersection $\mathbb{G}_m$. □

Note that by the Künneth formula we can now calculate the l-adic cohomology of $\mathbb{A}^r$ and $\mathbb{G}_m^r$ for all $r \geq 1$. We also have the following useful example.

Example 2.51. Let A be an abelian variety of dimension g over a separably closed field k and let $l \neq \mathrm{char}(k)$ be prime. Then $\mathrm{H}^1_{\text{ét}}(A, \underline{\mathbb{Z}/l^n\mathbb{Z}}) = (\mathbb{Z}/l^n\mathbb{Z})^{2g}$.

Proof. We again apply the Kummer sequence. Since A is complete and connected we have $\mathcal{O}(A)^\times = k^\times$ and hence it follows that $\mathrm{H}^1_{\text{ét}}(A, \underline{\mathbb{Z}/l^n\mathbb{Z}}) = \ker(\mathrm{Pic}(A) \xrightarrow{[l^n]} \mathrm{Pic}(A))$. For $\tau_a : A \to A$ translation by $a \in A(k)$ we define $\mathrm{Pic}^0(A)(k) = \{\mathcal{L} \in \mathrm{Pic}(A) \,|\, \mathcal{L} \cong \tau_a^*\mathcal{L} \text{ for all } a \in k\}$. Then any torsion point of $\mathrm{Pic}(A)$ lies in $\mathrm{Pic}^0(A)(k)$ ([34] (v) p.75). It is a theorem ([34], Section 13) that $\mathrm{Pic}^0(A)(k)$ is isomorphic to $A^\vee(k)$, where $A^\vee$ is the dual abelian variety of A, which is an abelian variety of the same dimension g (see [30] or [34]). When $\mathrm{char}(k) \nmid n$ the n-torsion points of an abelian variety are isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$ ([34] p.64). So this gives the result. □

To understand the étale cohomology of group varieties, the following classical result is one of the most important ones.

Theorem 2.52. *Let G, H be connected group varieties over an algebraically closed field k and let $\varphi : G \to H$ an isogeny. Then $\varphi^* : \mathrm{H}^*_{\text{ét}}(H, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ is an isomorphism.*

To prove this statement, we begin with the following special case.

Lemma 2.53. *Let $\varphi : G \to H$ be a purely inseparable isogeny of group varieties over $k = \bar{k}$. Then $\varphi^* : \mathrm{H}^n_{\text{ét}}(H, \Lambda) \to \mathrm{H}^n_{\text{ét}}(G, \mathcal{F})$ is an isomorphism for any constant sheaf $\mathcal{F} \in \mathrm{Sh}(H_{\acute{e}t})$.*

Proof. As φ is purely inseparable, we have that $K := \ker(\varphi)$ is connected. An isogeny is in particular a finite morphism, which is proper. So the proper base change theorem 2.46 gives that the stalk of $R^q\varphi_*\mathcal{F} = 0$ at $\bar{y}$ is the cohomology of the fibre over $\bar{y}$ and the fibre has dimension 0. So for $q \geq 1$ the sheaves $R^q\varphi_*\mathcal{F}$ are 0. This causes the Leray Spectral sequence associated to $\mathcal{F}, \varphi$ to degenerate at the E_2-page. So this causes the edge morphism $\mathrm{H}^n_{\text{ét}}(H, \varphi_*\varphi^*\mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(G, \varphi^*\mathcal{F})$ to be an isomorphism. The pullback $\mathrm{H}^n_{\text{ét}}(H, \mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(G, \varphi^*\mathcal{F})$ is equal to the composition of homomorphisms $\mathrm{H}^n_{\text{ét}}(G, \mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(H, \varphi_*\varphi^*\mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(G, \varphi^*\mathcal{F})$, where the first map is induced by the morphism of sheaves $\mathcal{F} \to \varphi_*\varphi^*\mathcal{F}$ (indeed the edge map in the Leray spectral sequence equals the second map, see ([46], p.150)).
The second map in the composition is an isomorphism by what is written above. Now for $\mathcal{F} = \Lambda$ a constant sheaf we claim that $\Lambda \to \varphi_*\varphi^*\Lambda$ is an isomorphism. Indeed, note that $\varphi_*\varphi^*\Lambda$ is the sheaf associated to the presheaf $U \mapsto \varinjlim_{V \to H} \Lambda(V)$. Since φ is a purely inseparable isogeny, it is a homeomorphism, hence $\pi_0(G \times_H U) = \pi_0(U)$ for any $U \to H$ étale. For $U \to H$ étale, the inclusion $\Lambda(U) \to \varinjlim_{V \to H} \Lambda(V)$ defines the morphism of sheaves $\Lambda \to \varphi_*\varphi^*\Lambda$. Now note that we may take the limit over over those V for which $\pi_0(V) = \pi_0(U \times_G H) = \pi_0(U)$ as these V form a cofinal system. It hence follows that $\Lambda(U) \to \varinjlim_{V \to H} \Lambda(V)$ is an isomorphism, hence the morphism $\Lambda \to \varphi_*\varphi^*\Lambda$ is an isomorphism. We conclude that $\mathrm{H}^n_{\text{ét}}(H, \Lambda) \to \mathrm{H}^n_{\text{ét}}(H, \varphi_*\varphi^*\Lambda)$ is an isomorphism and hence that $\varphi^* : \mathrm{H}^n_{\text{ét}}(H, \Lambda) \to \mathrm{H}^n_{\text{ét}}(G, \varphi^*\Lambda)$ is an isomorphism. □

Now we want to show that the result also holds for a separable isogeny to conclude that the general case holds. This is quite involved and is written out below.

Proof. (of Theorem 2.52, starting with the case of a separable isogeny)

The transfer map $\mathrm{H}^n_{\text{ét}}(Y, f_*f^*\mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(Y, \mathcal{F})$

Let $Y = X/G$ for G a finite group and let $f : X \to Y$ be the natural quotient map. Then notice that f is a Galois cover, i.e. the natural map $G \times_k X \to X \times_Y X \quad (g, x) \mapsto (x, xg)$ is an isomorphism. Let $\mathcal{F} \in \mathrm{Sh}(Y_{\text{ét}})$. We consider $f_*f^*\mathcal{F} \in \mathrm{Sh}(Y_{\text{ét}})$. Notice that this assigns to $U \to Y$ an étale morphism $\varinjlim_{U \times_X Y \to V \to Y} \mathcal{F}(V)$. This actually equals $\mathcal{F}(X \times_Y U)$ as $X \times_Y U \to Y$ is étale, since being étale is stable under base-change and composition, hence $X \times_Y U$ is the cofinal object in the diagram corresponding to the colimit.
We will now define a left action of G on $f_*f^*\mathcal{F}$, i.e. we will define for all $g \in G$ an isomorphism $\gamma_g : f_*f^*\mathcal{F} \to f_*f^*\mathcal{F}$ such that $\gamma_e = \mathrm{Id}$ and such that $\gamma_g \circ \gamma_h = \gamma_{gh}$. To do this, notice that for any étale $U \to Y$ if we denote by $\tau_g : X \to X$ the map $x \mapsto xg$ the following diagram commutes:

$$\begin{array}{ccc} U \times_Y X & & \\ \big\downarrow{\scriptstyle (\mathrm{Id},\tau_g)} & \searrow & \\ & & Y \\ & \nearrow & \\ U \times_Y X & & \end{array}$$

Moreover, the map (Id, τ_g) is an isomorphism, hence étale, so by $\mathcal{F}$ being an étale sheaf, we get an induced map $\gamma_g := \mathcal{F}(\mathrm{Id} \times \tau_g) : f_*f^*\mathcal{F}(U) \to f_*f^*\mathcal{F}(U)$. It is clear that if $g = e$, then $\gamma_e(U) = \mathrm{Id}(U)$. To show that this is a morphism of sheaves, use that for $V \to U$ étale over Y that the following diagram commutes together with the functoriality of $\mathcal{F}$:

$$\begin{array}{ccc} V \times_Y X & \longrightarrow & U \times_Y X \\ \big\downarrow{\scriptstyle (\mathrm{Id},\tau_g)} & & \big\downarrow{\scriptstyle (\mathrm{Id},\tau_g)} \\ V \times_Y X & \longrightarrow & U \times_Y X \end{array}$$

Note that we have a natural morphism of sheaves $\iota : \mathcal{F} \to f_*f^*\mathcal{F}$, defined on U by applying functoriality of $\mathcal{F}$ giving $\mathcal{F}(U \times_Y X \to U) : \mathcal{F}(U) \to \mathcal{F}(U \times_Y X)$. In the case that $\mathcal{F}$ is a separated sheaf, this map is injective as $U \times_Y X \to U$ is an étale cover. We will now assume that $\mathcal{F}$ is **separated**.

Notice that the following triangle commutes:

$$\begin{array}{ccc} U \times_Y X & \xrightarrow{(\mathrm{Id},\tau_g)} & U \times_Y X \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

This implies that the image of $\mathcal{F}(U)$ inside $f_*f^*\mathcal{F}(U)$ is fixed by the action of G. Actually, what we will be showing now is that $\mathcal{F} \subset f_*f^*\mathcal{F}$ equals $(f_*f^*\mathcal{F})^G$. The isomorphism $G \times_k X \to X \times_Y X$ induces an isomorphism $G \times_k U \to U \times_Y X \quad (g, u) \mapsto (u, ug)$ for all étale $U \to X \to Y$. This isomorphism fits into the following commuting square:

$$\begin{array}{ccc} G \times_k U & \longrightarrow & U \times_Y X \\ \downarrow{\scriptstyle (\cdot g, \mathrm{Id})} & & \downarrow{\scriptstyle (\mathrm{Id}, \tau_g)} \\ G \times_k U & \longrightarrow & U \times_Y X \end{array}$$

As for $\bar{x} \to Y$ a geometric point, the étale neighbourhoods of $\bar{x}$ that factor via $f : X \to Y$ are cofinal in all of the étale neighbourhoods of $\bar{x}$ we obtain:

$$(f_* f^* \mathcal{F})_{\bar{x}} = \varinjlim_{\bar{x} \to U} f_* f^* \mathcal{F}(U) = \varinjlim_{\bar{x} \to U} \mathcal{F}(U \times_Y X) = \varinjlim_{\bar{x} \to U} \mathcal{F}(G \times_k U)$$

By the commuting square above and since G acts trivially on $\mathcal{F}(U)$ it follows that there is an isomorphism of G-modules $(f_* f^* \mathcal{F})_{\bar{x}} \cong \varinjlim_{\bar{x} \to U} \mathcal{F}(G \times_k U) = \bigoplus_{g \in G} \mathcal{F}_{\bar{x}}$, where G acts on $\bigoplus_{g \in G} \mathcal{F}_{\bar{x}}$ by $h \cdot (s_g)_g = (s_{hg})_g$, i.e. by permuting the entries. It follows from the fact that the relevant map $\mathcal{F}(U) \to \mathcal{F}(U \times_k G)$ is induced by the projection $U \times_k G \to U$ that $\mathcal{F} \to f_* f^* \mathcal{F}$ embeds $\mathcal{F}_{\bar{x}}$ into $(f_* f^* \mathcal{F})_{\bar{x}} = \bigoplus_{g \in G} \mathcal{F}_{\bar{x}}$ diagonally and hence we see that $\mathcal{F}_{\bar{x}} = (f_* f^* \mathcal{F}_{\bar{x}})^G$. Thus the morphism $\mathcal{F} \to (f_* f^* \mathcal{F})^G$ is an isomorphism on stalks, hence an isomorphism.
Now that we have established that $\mathcal{F} = (f_* f^* \mathcal{F})^G$, we will define a map $\mu : f_* f^* \mathcal{F} \to \mathcal{F}$ that is called the 'trace map' in the literature for more general f. It is a variant of the Reynolds operator. Namely, one defines $\mu(U) : f_* f^* \mathcal{F}(U) \to (f_* f^* \mathcal{F}(U))^G$ by $s \mapsto \sum_{g \in G} g \cdot s$. This gives a morphism of sheaves $\mu : f_* f^* \mathcal{F} \to \mathcal{F}$ after identifying $\mathcal{F} = (f_* f^* \mathcal{F})^G$. We note that $\mu \circ \iota : \mathcal{F} \to \mathcal{F}$ is just multiplication by $|G|$ and that $\iota \circ \mu$ is given by $\sum_{g \in G} \gamma_g$ (the sum taken in the abelian group $\mathrm{Hom}_{\mathrm{Sh}(Y_{\text{ét}})}(f_* f^* \mathcal{F}, f_* f^* \mathcal{F})$).
By the functoriality of étale cohomology we have a left-action of G on $\mathrm{H}^n_{\text{ét}}(Y, f_* f^* \mathcal{F})$ and we have that the natural map $\iota : \mathrm{H}^n_{\text{ét}}(Y, \mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(Y, f_* f^* \mathcal{F})$ lands inside $\mathrm{H}^n_{\text{ét}}(Y, f_* f^* \mathcal{F})^G$. By the functoriality in the second argument of $\mathrm{H}^n_{\text{ét}}(Y, -)$ together with the fact that this functor is additive we obtain that there is a morphism $\mu : \mathrm{H}^n_{\text{ét}}(Y, f_* f^* \mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(Y, \mathcal{F})$ such that $\mu \circ \iota = |G|$ and such that $\iota \circ \mu(x) = \sum_{g \in G} g \cdot x$.

The G actions on $\mathrm{H}^n_{\text{ét}}(Y, f_* f^* \mathcal{F})$ and $\mathrm{H}^n_{\text{ét}}(X, f^* \mathcal{F})$

Now we want to relate the G-actions on $\mathrm{H}^n_{\text{ét}}(Y, f_* f^* \mathcal{F})$ and $\mathrm{H}^n_{\text{ét}}(X, f^* \mathcal{F})$ with each other. We have that the G-action on $\mathrm{H}^n_{\text{ét}}(X, f^* \mathcal{F})$ is given by $\tau_g^* : \mathrm{H}^n_{\text{ét}}(X, f^* \mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(X, \tau_g^* f^* \mathcal{F}) = \mathrm{H}^n_{\text{ét}}(X, * f^* \mathcal{F})$ as $f \circ \tau_g = f$. There is an edge morphism in the Leray spectral sequence $\mathrm{H}^n_{\text{ét}}(Y, f_* f^* \mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(X, f^* \mathcal{F})$. Since f is a finite morphism, the functor f_* is exact, which implies that this homomorphism is in fact an isomorphism. By the pullback property of the Leray spectral sequence we have that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n_{\text{ét}}(Y, f_* f^* \mathcal{F}) & \xrightarrow{\text{Edge}} & \mathrm{H}^n_{\text{ét}}(X, f^* \mathcal{F}) \\ \downarrow{\scriptstyle \mathrm{BC}} & & \downarrow{\scriptstyle \tau_g^*} \\ \mathrm{H}^n_{\text{ét}}(Y, f_* f^* \mathcal{F}) & \xrightarrow{\text{Edge}} & \mathrm{H}^n_{\text{ét}}(X, f^* \mathcal{F}) \end{array}$$

Here BC denotes the homomorphism induced by the base-change morphism:
$f_*f^*\mathcal{F} = \mathrm{Id}^* f_*(f^*\mathcal{F}) \to f_*\tau_g^*(f^*\mathcal{F}) = f_*f^*\mathcal{F}$.
We claim that this base-change morphism is the isomorphism of sheaves $\gamma_g : f_*f^*\mathcal{F} \to f_*f^*\mathcal{F}$ making the Edge homomorphism a G-equivariant isomorphism. That this is true is quite easy to see, namely take any sheaf $\mathcal{G} \in \mathrm{Sh}(X_{\text{ét}})$, then the base change morphism is the natural map $f_*\mathcal{G} \to f_*\tau_g^*\mathcal{G}$ that on an étale open $U \to X$ is given by the map $\mathcal{G}(U \times_Y X \to X) \to \mathcal{G}(U \times_Y X \to X \overset{\tau_g}{\to} X)$ that one gets from applying the functoriality of $\mathcal{G}$ to $(\mathrm{Id} \times \tau_g) : U \times_Y X \to U \times_Y X$.
So the edge homomorphisms are G-equivariant isomorphisms. The pullback homomorphism is given by $f^* : \mathrm{H}^n_{\text{ét}}(Y, \mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(Y, f_*f^*\mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(X, f^*\mathcal{F})$. So we can apply the result from the previous subsection, which gives that there exists a homomorphism $\mu : \mathrm{H}^n_{\text{ét}}(X, f^*\mathcal{F}) \to \mathrm{H}^n_{\text{ét}}(Y, \mathcal{F})$ such that $\mu \circ f^*$ is multiplication by $|G|$ and such that $(f^* \circ \mu)(x) = \sum_{g \in G} g \cdot x$.
Now we are ready to show that $\mathrm{H}^n_{\text{ét}}(Y, \mathbb{Q}_l) = \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_l)^G$. Notice that for all $\Lambda_n := \underline{\mathbb{Z}/l^n\mathbb{Z}} \in \mathrm{Sh}(Y_{\text{ét}})$ we have such maps μ as above. This gives that after passing to the limit and applying $- \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ we have a map $\mu : \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_l) \to \mathrm{H}^n_{\text{ét}}(Y, \mathbb{Q}_l)$ such that $\mu \circ f^* = |G|$ and such that $(f^* \circ \mu)(x) = \sum_{g \in G} g \cdot x$.
It is clear that $f^* : \mathrm{H}^n_{\text{ét}}(Y, \mathbb{Q}_l) \to \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_l)^G$ is injective (as $\mathcal{F} \to f_*f^*\mathcal{F}$ is injective) and as $(f^* \circ \mu)$ restricted to $\mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_l)^G$ equals $|G|$ we have that μ is also injective, giving that the dimensions of $\mathrm{H}^n_{\text{ét}}(Y, \mathbb{Q}_l)$ and $\mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_l)^G$ are the same and hence $f^* : \mathrm{H}^n_{\text{ét}}(Y, \mathbb{Q}_l) \to \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_l)^G$ is an isomorphism.

Connected algebraic groups act trivially on cohomology

Now we want to show that if X is a variety over an algebraically closed field k and G is a connected algebraic group that acts on X, then $G(k)$ acts trivially on $\mathrm{H}^n_{\text{ét}}(X, \Lambda)$ for Λ some constant sheaf on X. As we will consider X to be **smooth** it will suffice to show it for the compactly supported cohomology $\mathrm{H}^n_{\mathrm{c}}(X, \Lambda)$ by the Poincare duality theorem.
The general strategy will be as follows: To show that $\tau_g^* : \mathrm{H}^n_{\mathrm{c}}(X, \Lambda) \to \mathrm{H}^n_{\mathrm{c}}(X, \Lambda)$ is the identity map we will consider $\mathrm{H}^n_{\mathrm{c}}(X, \Lambda)$ as a constant sheaf on $G_{\text{ét}}$. We will then construct a morphism of sheaves $\tilde{f} : \mathrm{H}^n_{\mathrm{c}}(X, \Lambda) \to \mathrm{H}^n_{\mathrm{c}}(X, \Lambda)$ such that on all stalks we have that $\tilde{f}_g : \mathrm{H}^n_{\mathrm{c}}(X, \Lambda)_g \to \mathrm{H}^n_{\mathrm{c}}(X, \Lambda)_g$ equals τ_g^* for all $g \in G(k)$. If we can do this, then as G is connected, we that $\tilde{f}$ is determined by $\tilde{f}(G)$ as $\mathrm{H}^n_{\mathrm{c}}(X, \Lambda)$ is a constant sheaf and as G is connected, $\tilde{f}(G)$ then equals τ_g^* for all $g \in G(k)$. So all τ_g^* are equal to $\tau_e^* = \mathrm{Id}$ and hence $G(k)$ acts trivially on $\mathrm{H}^n_{\mathrm{c}}(X, \Lambda)$.
We proceed with constructing $\tilde{f}$. Consider the morphism $f : G \times_k X \to G \times_k X \quad (g, x) \mapsto (g, gx)$. For $\pi : G \times_k X \to G$ the projection we have $\pi \circ f = \pi$. So we have a base-change homomorphism of sheaves $\tilde{h} : R^n_c\pi_*\Lambda \to R^n_c\pi_*f^*\Lambda = R^n_c\pi_*\Lambda$ (the last equality holding as Λ is constant). Now we consider the following the following square where the top arrow is the projection:

$$\begin{array}{ccc} G \times_k X & \longrightarrow & X \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ G & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

By applying the base-change theorem for compactly supported cohomology for the sheaf Λ on X we obtain that $R^n_c\pi_*\Lambda = \mathrm{H}^n_{\mathrm{c}}(X, \Lambda)$ as sheaves on G. For $g \in G(k)$ we note that by the base-change

theorem we have $\mathrm{H}^n_c(X \times \{g\}, \Lambda) \xrightarrow{\sim} R^n_c\pi_*\Lambda$. Now by definition of $\tilde{h}$, we note that it is induced by taking for all $U \to G$ étale, the induced map $h_U^* : \mathrm{H}^n_{\text{ét}}(U \times_k X) \to \mathrm{H}^n_{\text{ét}}(U \times_k X)$, where h_U is the map $(u, x) \mapsto (u, ux)$ (u identified with its image in G in the second argument). In particular we note that on the fibre over g, the map $\tilde{h}_g$ on the stalk $\mathrm{H}^n_c(X, \Lambda)_g$ is precisely τ_g^*. Now we have constructed the desired morphism of sheaves $\tilde{f} : \mathrm{H}^n_{\text{ét}}(X, \Lambda) \to \mathrm{H}^n_{\text{ét}}(X, \Lambda)$ described above. From this we conclude that $G(k)$ acts trivially on $\mathrm{H}^n_c(X, \Lambda)$ for Λ any constant sheaf.
The poincaré duality pairing satisfies $\langle f^*x, f^*y\rangle = f^*\langle x, y\rangle = \deg(f)\langle x, y\rangle$ for $f : X \to X$ a finite morphism. Now τ_g is finite of degree 1, hence we have for any $x \in \mathrm{H}^n_{\text{ét}}(X, \Lambda)$ and $y \in \mathrm{H}^{2d-n}_c(X, \Lambda)$ that $\langle \tau_g^* x, y\rangle = \langle x, \tau_{g^{-1}}^* y\rangle = \langle x, y\rangle$ as $G(k)$ acts trivially on $\mathrm{H}^{2d-n}_c(X, \Lambda)$. Hence as the pairing is perfect and y is arbitrary we obtain that $\tau_g^* x = x$ for all $x \in \mathrm{H}^n_{\text{ét}}(X, \Lambda)$. So we conclude that for X a smooth variety, Λ a constant sheaf on X and G a connected algebraic group acting on X that $G(k)$ acts trivially on $\mathrm{H}^n_{\text{ét}}(X, \Lambda)$. In particular $G(k)$ acts trivially on $\mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_l)$.

Isogenies induce isomorphisms on l-adic cohomology

Let $\varphi : G \to H$ be an isogeny of group varieties. Then φ factors as $G \to G/\ker(\varphi)$ composed with an isomorphism. Let $\ker(\phi)^0$ be the connected component of $\ker(\varphi)$. Then as $G \to G/\ker(\phi)^0$ is purely inseparable, it gives an isomorphism on the l-adic cohomology as seen in Lemma 2.53. Now let $\overline{G} = G/\ker(\phi)^0$ and $K = \ker(\varphi)/\ker(\varphi)^0$ and consider the natural map $\pi : \overline{G} \to \overline{G}/K$. As K is étale and we are working over an algebraically closed field we have that $K = \bigsqcup_{m \in M} \mathrm{Spec}(k)$ for M some finite group. Hence $\overline{G}/K$ is the quotient by a finite group M, hence $\pi^* : \mathrm{H}^n_{\text{ét}}(\overline{G}/K, \mathbb{Q}_l) \to \mathrm{H}^n_{\text{ét}}(\overline{G}, \mathbb{Q}_l)$ is injective and its image equals $\mathrm{H}^n_{\text{ét}}(\overline{G}, \mathbb{Q}_l)^M$. Now notice that $M \subset \overline{G}(k)$ and that the action of M is induced by the action of $\overline{G}(k)$ on $\mathrm{H}^n_{\text{ét}}(\overline{G}, \mathbb{Q}_l)$, which is induced by the action of $\overline{G}$ on itself. This implies by the previous part that M acts trivially as $\overline{G}$ is a (connected) group variety. So $\pi^* : \mathrm{H}^n_{\text{ét}}(\overline{G}/K, \mathbb{Q}_l) \to \mathrm{H}^n_{\text{ét}}(\overline{G}, \mathbb{Q}_l)$ is in fact an isomorphism. Hence $\varphi^* : \mathrm{H}^n_{\text{ét}}(H, \mathbb{Q}_l) \to \mathrm{H}^n_{\text{ét}}(G, \mathbb{Q}_l)$ is the composition of three isomorphisms, hence an isomorphism. This concludes the proof of Theorem 2.52. □

We now state the existence of two fundamental exact sequences. The first one is for compactly supported cohomology.

Proposition 2.54 ([29] Remark III.1.30)**.** *Let Z be a closed subscheme of X and let $\mathcal{F} \in \mathrm{Sh}(X_{\acute{e}t})$. Denote $U = X \setminus Z$, $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$. The sequence $0 \to j!\mathcal{F}|_U \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$ in $\mathrm{Sh}(X_{\acute{e}t})$ is exact and gives rise to a long exact sequence:*

$$... \to \mathrm{H}^r_c(U, j^*\mathcal{F}) \to \mathrm{H}^r_c(X, \mathcal{F}) \to \mathrm{H}^r_c(Z, i^*\mathcal{F}) \to \mathrm{H}^{r+1}_c(U, |^*F) \to ...$$

The second one is called the Gysin sequence.

Proposition 2.55 (Gysin sequence, [31] Corollary 16.2)**.** *Let (Z, X) be a smooth pair of $\overline{k}$-varieties of codimension c and let $U = X \setminus Z$ with inclusions $\iota : Z \to X$ and $j : U \to X$. Let $\mathcal{F}$ be a locally constant sheaf of Λ-modules on X. There is a long exact sequence:*

$$... \to \mathrm{H}^r_{\text{ét}}(X, \mathcal{F}) \to \mathrm{H}^r_{\text{ét}}(U, j^*\mathcal{F}) \to \mathrm{H}^{r-2c}_{\text{ét}}(Z, i^*\mathcal{F}) \to \mathrm{H}^{r+1}_{\text{ét}}(X, \mathcal{F}) \to ...$$

Note that in the sequence we put $\mathrm{H}^r_{\text{ét}}(Z, \mathcal{F}) = 0$ when $r < 0$. This sequence is called the Gysin sequence and $\mathrm{H}^r_{\text{ét}}(Z, \mathcal{F}) \to \mathrm{H}^{r+2c}_{\text{ét}}(X, \mathcal{F})$ is called the Gysin map.

Intersection theory

We give a brief introduction to intersection theory as we will use some results from this area later. For a more thorough introduction see [22]. Throughout this part X will be a scheme of finite type over a field k.

Definition 2.56. An integral subvariety Z of X is called a *prime cycle*.

Definition 2.57. The group of codimension *r-cycles* is $Z^r(X) := \bigoplus_{Z \subset X} \mathbb{Z} \cdot Z$, the sum being taken over all prime cycles of codimension r. We denote by $Z_r(XL) = Z^{\dim(X)-r}(X)$ the group of dimension r cycles. A codimension 1 cycle is called a *divisor* and the group of them is denoted $\mathrm{Div}(X)$.

When X is irreducible we can associate a divisor to a rational function.

Definition 2.58. Let X be irreducible and $V \subset X$ be irreducible of codimension 1 with generic point v. For $0 \neq c \in \mathcal{O}_{X,v}$ define $\mathrm{ord}_V(c) := \mathrm{length}_{\mathcal{O}_{X,v}}(\mathcal{O}_{X,v}/(c))$. Then for $f = \dfrac{a}{b}$ with $a, b \in \mathcal{O}_{X,v}$ define $\mathrm{ord}_V(f) = \mathrm{ord}_V(a) - \mathrm{ord}_V(b)$.

Notice that if X is smooth, then $\mathcal{O}_{X,v}$ is a DVR and hence $\mathrm{length}_{\mathcal{O}_{X,v}}(\mathcal{O}_{X,v}/(a)) = n$ for the unique $n \in \mathbb{Z}_{\geq 0}$ such that $t^n \cdot u = a$ for $u \in \mathcal{O}_{X,v}^{\times}$ and $(t) = \mathfrak{m}_v$.

Remark 2.59. From the relation $\mathrm{length}_A(A/(a)) + \mathrm{length}_A(A/(d)) = \mathrm{length}_A(A/(ad))$ whenever these are finite ([22], Lemma A.2.5) together with $\dim(\mathcal{O}_{X,v}) = 1$ and that $\mathcal{O}_{X,v}$ has no nonzero zero divisors, it follows that the definition of $\mathrm{ord}_V(f)$ makes sense. In fact from the above additivity, it follows that $\mathrm{ord}_V : K(X)^{\times} \to \mathbb{Z}$ is a homomorphism.

Definition 2.60. Let X be an irreducible variety and $f \in K(X)^{\times}$. The *principal divisor* correspdoning to f is $\mathrm{div}(f) := \sum_Z \mathrm{ord}_Z(f) \cdot Z$, the sum taken over all prime divisors.

Since $\mathrm{div} : K(X)^{\times} \to \mathrm{Div}(X)$ is a homomorphism, $\mathrm{div}(K(X)^{\times}) \subset \mathrm{Div}(X)$ is a subgroup.

Now we make a generalization of the previous definition to arbitrary codimension.

Definition 2.61. A cycle $Z \in Z^r(X)$ is *rationally equivalent* to 0 if there exist a finite number of prime cycles of codimension $r-1$, $\{W_i\}$, and $f_i \in K(W_i)^{\times}$ such that $Z = \sum_i \mathrm{div}(f_i)$.

Remark 2.62. Notice that by div being a homomorphism, it follows that the codimension r-cycles that are rationally equivalent to 0 form a subgroup of $Z^r(X)$.

This leads to the following definition.

Definition 2.63. The r'th *Chow group* of X is the quotient of $Z^r(X)$ by the cycles that are rationally equivalent to 0. Denote it by $\mathrm{A}^r(X)$. Denote $\mathrm{A}_r(X) = \mathrm{A}^{d-r}(X)$ for $d = \dim(X)$.

We have the following trivial example.

Example 2.64. The Chow group of $\mathbb{A}^1$ has $\mathrm{A}^0(\mathbb{A}^1) = \mathbb{Z}$, which is generated by the class $[\mathbb{A}^1]$ and $\mathrm{A}^1(\mathbb{A}^1) = 0$ as any closed point is the vanishing locus of some polynomial. By applying ([22] Proposition 1.9) it follows that $\mathrm{A}^0(\mathbb{A}^r) = \mathbb{Z}$ and $\mathrm{A}^i(\mathbb{A}^r) = 0$ for all $i > 0$. We will calculate the Chow groups of a complete variety X that admits a filtration $X = Y_n \supset .... \supset Y_0 = \emptyset$ of closed subschemes such that $Y_i \backslash Y_{i-1}$ is a disjoint union of affine spaces in Section 6. Note that an example of such a space X is $\mathbb{P}^n$.

It is also possible to attach a cycle to certain closed subschemes.

Definition 2.65. Let Z be a closed subscheme of X such that all the irreducible components Z_i with generic points η_i of Z have the same dimension. Let $m_i := \mathrm{length}_{\mathcal{O}_{X,\eta_i}}(\mathcal{O}_{Z,\eta_i})$. Then the *cycle corresponding to* Z is $[Z] := \sum_i m_i \cdot Z_i$.

Now we define the flat pullback of a cycle.

Definition 2.66. Let $f : X \to Y$ be a flat morphism of relative dimension d. Define the *flat pullback* $f^* : Z^r(Y) \to Z^{r-d}(X)$ by $f^*([Z]) = [f^{-1}(Z)]$.

We have the following basic functoriality of the Chow groups.

Proposition 2.67 ([22] p.19 Theorem 1.7)**.** *Let $f : X \to Y$ be a flat morphism of relative dimension d. The map f^* defined above is a homomorphism of abelian groups and descends to a homomorphism $f^* : \mathrm{A}^r(Y) \to A^{r-d}(X)$.*

Given a proper morphism such as a closed immersion, one gets covariant functoriality. In particular one has the functoriality for closed immersions.

Proposition 2.68. *Let $\pi : X \to Y$ be a proper morphism and let $\dim(Y) = d$. Define a map by $\pi_* : Z^r(X) \to Z^r(Y)$ $\quad \pi_*[Z] = \begin{cases} 0 \text{ if } \dim(f(Z)) < d - r \\ m \cdot [f(Z)] \text{ else} \end{cases}$ for $m = \deg(\pi : Z \to f(Z))$. The function π_* descends to a homomorphism of abelian groups $\pi_* : \mathrm{A}^r(X) \to \mathrm{A}^r(Y)$.*

It is possible to put all the Chow groups together and make a commutative ring.

Theorem 2.69 ([22] Chapter 6)**.** *Let X be a variety of dimension d. Let $0 \leq m, n \leq d$. There exists a $\mathbb{Z}$-bilinear map $\mathrm{A}^m(X) \times A^n(X) \to A^{m+n}(X)$ called the 'intersection product' such that:*

- *The abelian group $\mathrm{A}^*(X) := \bigoplus_{i=0}^{d} A^i$ becomes a graded ring that is commutative when put together with the intersection product.*
- *For $f : X \to Y$ a flat morphism, $f^* : \mathrm{A}^*(Y) \to \mathrm{A}^*(X)$ is a ring homomorphism.*

Definition 2.70. Call the ring $\mathrm{A}^*(X) := \bigoplus_i \mathrm{A}^i(X)$ the *Chow ring* of X.

There turns out to be a way to compare the l-adic cohomology of a smooth variety with its Chow ring.

Theorem 2.71 (Cycle class map, [29] Paragraph VI. 9)**.** *Let X be a smooth quasiprojective variety over an algebraically closed field k and let $\mathrm{char}(k) \neq l$ be prime. There exists a cycle map $\mathrm{cl}^X : \mathrm{A}^*(X) \to \mathrm{H}^*_{\text{ét}}(X, \underline{\mathbb{Z}/l^n\mathbb{Z}})$ which is a ring homomorphism and maps $\mathrm{A}^r(X)$ into $\mathrm{H}^{2r}_{\text{ét}}(X, \underline{\mathbb{Z}/l^n\mathbb{Z}})$ and has the following properties:*

- *For $[Z] \in \mathrm{A}^r(X)$ the class of a smooth prime cycle Z we have that $\mathrm{cl}_X([Z])$ is equal to the image of $1 \in \mathrm{H}^0_{\text{ét}}(Z, \Lambda)$ under the Gysin map $\mathrm{H}^0_{\text{ét}}(Z, \Lambda) \xrightarrow{\sim} \mathrm{H}^{2r}_Z(X, \Lambda) \to \mathrm{H}^{2r}_{\text{ét}}(X, \Lambda)$.*
- *If $f : X \to Y$ is flat then the pullbacks commute, i.e. $\mathrm{cl}^X \circ f^* = f^* \circ \mathrm{cl}^Y$.*
- *One may pass to the limit and apply $- \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ to obtain the l-adic cycle class map, which is a homomorphism of $\mathbb{Q}_l$-algebras $\mathrm{cl}^X : \mathrm{A}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to \mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_l)$.*

We use the following notation for the graded trace on l-adic cohomology.

Definition 2.72. For $f : X \to X$ a morphism over an algebraically closed field denote the *graded trace* of f by $\mathrm{tr}_X(f) := \sum_i (-1)^i \mathrm{tr}(f^* \,|\, \mathrm{H}^i_{\text{ét}}(X, \mathbb{Q}_l))$

The following theorem can is proved by applying properties of the cycle map.

Theorem 2.73 (Grothendieck-Lefschetz fixed point formula, [29] Theorem VI.12.3 and [31] Lemma 25.6)**.** *Let X be a smooth projective variety over $k = \bar{k}$ of dimension d and let $f : X \to X$ such that f fixes finitely many points. Then:*

$$\Gamma_f \cdot \Delta = \mathrm{tr}_X(f)$$

If for all $P \in X(k)$ that are fixed by f we have that $df_P : T_P(X) \to T_P(X)$ does not have 1 as an eigenvalue, then the above graded trace is the actual fixed point count of f.

To conclude the chapter we have the following calculation. We use Example 2.13 which gives the canonical isomorphism of the pullback of a constant sheaf.

Proposition 2.74. *Let T be a split torus and let A be an abelian variety both over a field k that contains the l^n'th roots unity ($l \neq \mathrm{char}(k)$). There are isomorphisms of abelian groups $\mathrm{H}^1_{\text{ét}}(T, \underline{\mathbb{Z}/l^n\mathbb{Z}}) \cong X(T)/(l^n)$ and $\mathrm{H}^1_{\text{ét}}(A, \underline{\mathbb{Z}/l^n\mathbb{Z}}) \cong \mathrm{Pic}^0(A)[l^n]$ that are functorial in the following sense: For $f : B \to A$ a homomorphism of abelian varieties and $g : S \to T$ of algebraic tori, the pullback maps on cohomology are the natural pullbacks $f^* : \mathrm{Pic}^0(A)[n] \to \mathrm{Pic}^0(B)[n]$ and $g^* : X(T)/(l^n) \to X(S)/(l^n)$.*

Proof. We begin with the case of the split tori. Since k contains the l'nth roots of unity, the sheaf $\underline{\mu_{l^n}}$ is constant and hence $g^* : \mathrm{H}^1_{\text{ét}}(T, \underline{\mu_{l^n,T}}) \to \mathrm{H}^1_{\text{ét}}(S, g^*\underline{\mu_{l^n,T}}) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(S, \mu_{n,S})$ is the pullback as in Example 2.13 Note that the last map is compatible with the Kummer sequence in the sense that there is a morphism of short exact sequences, with map below induced by $(g^*\underline{\mathbb{G}_{m,T}})^P \to \underline{\mathbb{G}_{m,S}}$, which at the level of $U \to S$ étale maps $V \to \mathbb{G}_m$ in the colimit to $U \to V \to \mathbb{G}_m$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & g^*\underline{\mu_{l^n,T}} & \longrightarrow & g^*\underline{\mathbb{G}_{m,T}} & \longrightarrow & g^*\underline{\mathbb{G}_{m,T}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\mu_{l^n,S}} & \longrightarrow & \underline{\mathbb{G}_{m,S}} & \longrightarrow & \underline{\mathbb{G}_{m,S}} & \longrightarrow & 0
\end{array}$$

Thus in particular the following square commutes:

$$\begin{array}{ccccccc}
\mathrm{Mor}(S,\mathbb{G}_m) & \xrightarrow{[l^n]} & \mathrm{Mor}(S,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(S,\underline{\mu_{l^n,S}}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} & & \\
\mathrm{Mor}(T,\mathbb{G}_m) & \xrightarrow{[l^n]} & \mathrm{Mor}(T,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(T,\underline{\mu_{l^n,T}}) & \longrightarrow & 0
\end{array}$$

The pullback maps $g^* : \mathrm{Mor}(S,\mathbb{G}_m) \to \mathrm{Mor}(T,\mathbb{G}_m)$ are postcomposition by g, which follows directly from considering the composition $\underline{\mathbb{G}_{m,S}}(S) \to g_*g^*\underline{\mathbb{G}_{m,S}}(S) = g^*\underline{\mathbb{G}_{m,S}}(T) \to \underline{\mathbb{G}_{m,T}}(T)$. So the isomorphism $\mathrm{H}^1_{\text{ét}}(S,\underline{\mu_{l^n,S}}) \cong \mathrm{Mor}(S,\mathbb{G}_m)/(l^n)$ is functorial in the sense that the pullback $\mathrm{H}^1_{\text{ét}}(S,\underline{\mu_{l^n,S}}) \to \mathrm{H}^1_{\text{ét}}(T,\underline{\mu_{l^n,T}})$ is postcomposition by g. Now the above works perfectly fine for any morphism g and any schemes S,T such that $\mathrm{Pic}(S) = \mathrm{Pic}(T) = 0$. For tori S,T and a homomorphism $g : S \to T$ there is a pullback map $X(T) \to X(S)$ and $X(T) \subset \mathrm{Mor}(T,\mathbb{G}_m)$ has $X(T)/(l^n) = \mathrm{Mor}(T,\mathbb{G}_m)/(l^n)$. Hence in this case we may take $\mathrm{H}^1_{\text{ét}}(S,\underline{\mu_{l^n,S}}) \cong \mathrm{Mor}(S,\mathbb{G}_m)/(l^n) = X(S)/(l^n)$ such that the following diagram commutes:

$$\begin{array}{ccc}
X(T)/(l^n) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(T,\underline{\mu_{l^n,T}}) \\
\downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} \\
X(S)/(l^n) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(S,\underline{\mu_{l^n,S}})
\end{array}$$

Now we look at the case of a homorphism $f : B \to A$ between abelian varieties. As above the Kummer sequence provides us with the following commuting diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^1_{\text{ét}}(A,\underline{\mu_{l^n,A}}) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(A,\underline{\mathbb{G}_{m,A}}) & \xrightarrow{[l^n]} & \mathrm{H}^1_{\text{ét}}(A,\underline{\mathbb{G}_{m,A}}) \\
 & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
0 & \longrightarrow & \mathrm{H}^1_{\text{ét}}(B,\underline{\mu_{l^n,B}}) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(B,\underline{\mathbb{G}_{m,B}}) & \xrightarrow{[l^n]} & \mathrm{H}^1_{\text{ét}}(B,\underline{\mathbb{G}_{m,B}})
\end{array} \tag{2.1}$$

The isomorphism $\check{\mathrm{H}}^1(A,\mathbb{G}_m) \to \mathrm{H}^1_{\text{ét}}(A,\underline{\mathbb{G}_{m,A}})$ (actually for an arbitrary scheme A and sheaf $\underline{\mathbb{G}_{m,A}}$) comes from the Cech to derived spectral sequence ([29] Prop III.2.7), which is compatible with the pullback homomorphism in the sense that, for $\mathcal{U} = (U_i \to A)_i$ a cover and $\mathcal{U}' = (U_i \times_A B \to B)_i$ the pulled back cover, the following diagram commutes (compare with [16] diagram (12.1.4.2):

$$\begin{array}{ccc}
\check{\mathrm{H}}^1(\mathcal{U},\underline{\mathbb{G}_{m,A}}) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(A,\underline{\mathbb{G}_{m,A}}) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
\check{\mathrm{H}}^1(\mathcal{U}',f^*\underline{\mathbb{G}_{m,A}}) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(B,f^*\underline{\mathbb{G}_{m,A}})
\end{array}$$

The Cech to Derived spectral sequence is a particular case of the Groehtendieck spectral sequence, which is functorial in the sheaf argument, hence it is compatible with the morphism $f^*\underline{\mathbb{G}_{m,A}} \to \underline{\mathbb{G}_{m,B}}$. This implies that there is a commuting diagram:

$$\begin{array}{ccc} \check{\mathrm{H}}^1(\mathcal{U}, \underline{\mathbb{G}_{m,A}}) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(A, \underline{\mathbb{G}_{m,A}}) \\ \downarrow f^* & & \downarrow f^* \\ \check{\mathrm{H}}^1(\mathcal{U}', \underline{\mathbb{G}_{m,B}}) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(B, \underline{\mathbb{G}_{m,B}}) \end{array} \tag{2.2}$$

The map f^* on Cech cohomology is induced by mapping $(g_{ij})_{ij} \in \prod_{i,j} \mathrm{Mor}(U_{ij}, \mathbb{G}_m)$ to its class in $([g_{ij}])_{i,j} \in \prod_{i,j} f^*\underline{\mathbb{G}_{m,A}}(U_{ij} \times_B A)$ and then to $(g_{ij} \circ f)_{ij} \in \prod_{i,j} \mathrm{Mor}(U_{ij} \times_B A, \mathbb{G}_m)$ (this follows since the pullback homomorphism is defined on the level of sections of the sheaves and as above we know that they are hence defined like this). In particular it is thus the usual pullback homomorphism $f^* : \mathrm{Pic}(A) \to \mathrm{Pic}(B)$. We see by Diagram 2.1 that $f^* : \mathrm{H}^1_{\text{ét}}(A, \underline{\mu_{l^n,A}}) \to \mathrm{H}^1_{\text{ét}}(B, \underline{\mu_{l^n,B}})$ corresponds is given by the map $f^* : \mathrm{H}^1_{\text{ét}}(A, \underline{\mathbb{G}_{m,A}}) \to \mathrm{H}^1_{\text{ét}}(B, \underline{\mathbb{G}_{m,B}})$ restricted to the l^n-torsion, which by Diagram 2.2 and the above corresponds to the map $f^* : \mathrm{Pic}(A) \to \mathrm{Pic}(B)$ restricted to the l^n-torsion. We know that for any abelian variety A and $n > 0$ we have $\mathrm{Pic}^0(A)[n] = \mathrm{Pic}(A)[n]$ ([34], p.75 (v)), which finishes of the proof. □

Chapter 3

Counting fixed points

Throughout this chapter let G be an algebraic group over an arbitrary algebraically closed field k and let $\sigma : G \to G$ be an endomorphism. Throughout this chapter we will be interested in counting fixed points of all the iterates of σ.

Definition 3.1. For $\sigma : G \to G$ an endomorphism, define the **fixed subscheme** of σ, $\mathrm{Fix}(\sigma)$, to be the subgroup scheme of G defined by $\mathrm{Fix}(\sigma)(R) = \{g \in G(R) \,|\, \sigma_R(g) = g\}$.

This is represented by the scheme $\mathrm{Eq}(\sigma, \mathrm{Id})$.

Definition 3.2. An endomorphism $\sigma : G \to G$ is **confined** if $\mathrm{Fix}(\sigma^n)$ is a finite scheme over k for all n or equivalently if σ^n fixes finitely many points in $G(k)$ for all n.

We are interested in the following sequence.

Definition 3.3. Let σ be a confined endomorphism. Define the **fixed point sequence** $(\sigma_n)_n$ by setting $\sigma_n = \#\mathrm{Fix}(\sigma^n)(k)$.

We have the following example.

Example 3.4. Let $G/\overline{\mathbb{F}}_p$ be an algebraic group (or an algebraic scheme, this also works fine here) defined over a finite field $\mathbb{F}_q$. Denote the algebraic group over $\mathbb{F}_q$ by $G_{\mathbb{F}_q}$. Then one has the q-Frobenius endomorphism $\mathrm{Frob}_q : G \to G$, which is defined on any affine open $U = \mathrm{Spec}(R) \subset G$ where we have $R = \overline{\mathbb{F}}_p[X_1, ..., X_n]/(f_1, ..., f_m)$ with $f_i \in \mathbb{F}_q[X_1, ..., X_n]$ by $\mathrm{Frob}_q(X_i) = X_i^q$. Note that $(a_1, ..., a_n) \in G(\overline{\mathbb{F}}_p)$ is fixed if and only $a_i \in \mathbb{F}_q$ for all i, hence if and only if this points lies in $G_{\mathbb{F}_q}(\mathbb{F}_q)$. So we see that understanding the sequence $(\mathrm{Frob}_{q^n})_n$ is equivalent to understanding the sequence $(G_{\mathbb{F}_q}(\mathbb{F}_{q^n}))_n$.

Remark 3.5. If one is interested in understanding the sequence $(\sigma_n)_n$, then one may assume that σ is surjective. Indeed, the images $\sigma(G) \supset \sigma^2(G) \supset \sigma^3(G) \supset ...$ form a descending chain of closed subschemes of G, which terminates at some j and one can replace G by $\sigma^j(G)$.

We have the following theorem by Steinberg.

Theorem 3.6 (Lang-Steinberg, Theorem 10.1 [44])**.** *Let $\sigma : G \to G$ be a surjective endomorphism of a group variety such that $\mathrm{Fix}(\sigma)$ is finite, then the Lang map $L_\sigma : G \to G \quad g \mapsto g^{-1}\sigma(g)$ is surjective.*

By applying this theorem we are able to split up the fixed point count.

Lemma 3.7. *Let H be a subgroup of an algebraic group G and let σ be a confined endomorphism of G that restricts to an endomorphism τ of H and hence induces an endomorphism φ of the quotient $Q := G/H$. Then $\sigma_n = \tau_n \cdot \varphi_n$.*

Proof. Note that it suffices to prove $n = 1$. As k is algebraically closed we have $Q(k) = G(k)/H(k)$. If $x \in G(k)$ is fixed, then the reduction $(\text{mod } H(k))$ is fixed. The fibre over $x \,(\text{mod } K(k))$ is $\{x \cdot y \,|\, y \in H(k)\}$ and a point $x \cdot y$ in this fibre is fixed if and only if y is fixed. So the number of fixed points in this fibre equals τ_1. Hence we see that it suffices to show that $\text{Fix}(\sigma_G)(k) \to \text{Fix}(\sigma_Q)(k)$ is surjective. Consider $x \in G(k)$ such that $x \,(\text{mod } H(k))$ is fixed by σ. Then $x^{-1}\sigma(x) \in H(k)$ and hence by surjectivity of the Lang map 3.6 we obtain $x^{-1}\sigma(x) = y^{-1}\sigma(y)$ for $y \in H(k)$. So xy^{-1} is a point fixed by σ that lies over $x \,(\text{mod } H(k))$. □

We see that if we can get a filtration of our algebraic group by characteristic subgroups that understanding the sequence $(\sigma_n)_n$ may be easier. To obtain such a filtration we assume that σ is surjective (note that after some $n > 0$ the map $\sigma^{\circ n}(G) \to \sigma^{\circ n+1}(G)$ is surjective as the algebraic subgroups of G satisfy the d.c.c. condition. Now note that by surjectivity of σ, $\sigma(G_{\text{lin}})$ is normal in G and hence contained in G_{lin}. For linear G, note that $\sigma \text{R}_\text{u}(G)$ is contained in $\text{R}_\text{u}(G)$ and note that for reductive G, $\sigma(\text{R}(G))$, which is a torus, is contained in $\text{R}(G)$. This gives the desired filtration. Note that instead of 'contained in' above we may actually write 'equals' because of dimension reasons.

Corollary 3.8. *Let G be a group variety and σ a surjective confined endomorphism. Then $\sigma_n = \sigma_n^{\text{A}} \cdot \sigma_n^{\text{U}} \cdot \sigma_n^{\text{T}} \cdot \sigma_n^{\text{ss}}$ for σ^{A} a confined endomorphism on an abelian variety, σ^{U} an endomorphism on a unipotent group variety, σ^{T} an endomorphism on a torus and σ^{ss} an endomorphism on a semisimple group variety.*

Proof. Apply the above lemma to G/G_{lin} being an abelian variety, $G_{\text{lin}}/\text{R}_\text{u}(G)$ being reductive and $G_{\text{red}}/\text{R}(G)$ being semisimple. □

It turns out that we can do slightly better than just any unipotent algebraic group.

Lemma 3.9. *([8], Prop 7.1.2) For σ a confined endomorphism on a unipotent group variety U we have $\sigma_n = \sigma_n^{V_1} \cdot ... \cdot \sigma_n^{V_r}$ for $V_1, ..., V_r$ vector groups.*

Proof. Consider the derived series of $U \supset \mathcal{D}(U) \supset \mathcal{D}(\mathcal{D}(U)) \supset ...$, which is a filtration of U by characteristic subgroups. As U is solvable the derived series terminates. The quotients Q_i in this series are commutative and unipotent, hence the Q_i admit a subnormal series with quotients $\mathbb{G}_a$ ([32], Corollary 16.6 together with the subnormal series of $\mathbb{T}^n$ that has quotients $\mathbb{G}_a$ or $\mathbb{G}_m$). So Q_i admits a finite filtration by the characteristic subgroups $p^m Q_i$, whose quotients are commutative unipotent algebraic groups in which every element has order p, which are vector groups ([39] p.177). □

We have the following additional proposition, which will also aid us later in Chapter 6. It is used in Chapter 5 of [8] to understand the fixed point count in the case of a semisimple group variety.

Proposition 3.10 (Steinberg, [44] Corollary 10.10). *Let $\sigma : G \to G$ be a surjective confined endomorphism of a linear algebraic group. There exists a σ-stable Borel pair (B, T) of G.*

Remark 3.11. It is easily checked that the above proposition is a direct consequence of the Lang Steinberg Theorem 3.6 (though one should note that Steinberg first shows the existence of a σ-stable Borel group for surjective $\sigma : G \to G$ [44] Theorem 7.2 before proving Theorem 3.6). The above proposition is used in [8] Chapter 5 to give a fixed point formula for a semisimple group variety G, which then reduces to giving one for B and one for G/B for B a σ-stable Borel subgroup. We will also use this Proposition heavily in chapter 6.

In the following theorem by a gcd-sequence we mean a sequence $(a_n)_n$ such that $a_n = a_{\gcd(n,\omega)}$ for some $\omega \in \mathbb{Z}$. Any gcd-sequence is periodic. Recall that S^W denote the Weyl group invariants. Let $S^W_+ \subset S^W$ be the ideal of those of positive degree.

Theorem 3.12 (Byszewski, Cornelissen, Houben [8]). *Let σ^A be an endomorphism on an abelian variety A, σ^V on a vector group V, σ^T on a torus T and σ^G on a semisimple group variety G, all surjective and confined. Assume that the field that we are working over has characteristic $p > 0$.*

- $\sigma^A_n = |d_n| \cdot r_n \cdot |n|_p^{s_n}$ *where $r_n \in \mathbb{Q}_{>0}$ and $s_n \in \mathbb{Z}_{\geq 0}$ define gcd-sequences and $d_n = \mathrm{tr}_A((\sigma^A)^n)$.*
- $\sigma^T_n = |d_n| \cdot r_n \cdot |n|_p^{s_n}$ *where $r_n \in \mathbb{Q}_{>0}$ and $s_n \in \mathbb{Z}_{\geq 0}$ define gcd-sequences and $d_n = \mathrm{tr}_T((\sigma^T)^n)$.*
- $\sigma^G_n = |d_n| \cdot c^n$ *where $d_n = \det(1 - \sigma^* \mid J)$, where $J = S^W_+/(S^W_+)^2$.*
- *When $k = \overline{\mathbb{F}}_p$, then $\sigma^V_n = c^n p^{-t_n |n|_p^{-1}}$ where $t_n \in \mathbb{Z}_{\geq 0}$ defines a gcd-sequence.*

As a consequence, a confined surjective endomorphism $\sigma : G \to G$ over $\overline{\mathbb{F}}_p$ has fixed point count given by:

$$\sigma_n = |d_n| c^n r_n |n|_p^{s_n} p^{-t_n |n|_p^{-1}}$$

Here d_n is linearly recurrent, $c = p^r$ and $r_n \in \mathbb{Q}_{>0}$, $s_n, t_n \in \mathbb{Z}_{\geq 0}$ all three define gcd*-sequences.*

Remark 3.13. It should be noted that in the first three cases the terms $|d_n|$ equal $\deg(\sigma^n - 1)$ and that other terms make up $\deg_i(\sigma^n - 1)^{-1}$. For the first three cases the methods used in [8] also apply to $\mathrm{char}(k) = 0$ and hence the fixed point count reduces to $|d_n|$ in all the three above cases when $\mathrm{char}(k) = 0$. For the fourth cases, note that any endomorphism of a vector group over k with $\mathrm{char}(k) = 0$ is linear, hence any confined endomorphism fixes only one point in this case.

Note that the last case of Theorem 3.12 is only over $\overline{\mathbb{F}}_p$.

Question 1. Let V be a vector group over an arbitrary algebraically closed field k with $\mathrm{char}(k) = p$. Does there exist a gcd-sequence $(t_n)_n \in \mathbb{Z}^{\mathbb{N}}$ such that $\sigma_n = c^n p^{-t_n |n|_p^{-1}}$?

Solving this problem would be very desirable. Unfortunately the efforts of the author came up short. Another interesting problem is understanding the terms $|d_n|$. They seem to have a cohomological interpretation.

Question 2. Let G be a group variety and let $\sigma : G \to G$ be a surjective confined endomorphism. Set $|d_n|_G$ to be the product of the three $|d_n|$ in Theorem 3.12 that one obtains by applying the filtration of G by fully characteristic subgroups. Do we have $|d_n|_G = \mathrm{tr}_G(\sigma^n)$?

In this thesis we show among other things that the above question has a positive answer. The main focus of the authors in [8] was the case $k = \overline{\mathbb{F}}_p$ and they found the following solution in this case after finding that $|d_n|_G$ equals $\mathrm{tr}_{G_{\mathrm{lin}}}(\sigma) \cdot \mathrm{tr}_{G_{\mathrm{ab}}}(\sigma)$.

Partial answer ([8] Lemma 12.3.6)**.** Let G be a group variety over $\overline{\mathbb{F}}_p$. There is an isogeny $G_{\mathrm{lin}} \times_{\overline{\mathbb{F}}_p} G_{\mathrm{ab}} \to G$ that commutes with the relevant morphisms induced by σ.

The above is a direct consequence of Arima's theorem [1], which we shall see in the next chapter. Theorem 2.52 then implies that $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G_{\mathrm{lin}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^*_{\text{ét}}(G_{\mathrm{ab}}, \mathbb{Q}_l)$ is an isomorphism that commutes with the action of σ^*. Thus it follows that $|d_n|_G = \mathrm{tr}_G(\sigma)$ in the case of $k = \overline{\mathbb{F}}_p$. For general k however there does **not** exist an isogeny $G_{\mathrm{lin}} \times_k G_{\mathrm{ab}} \to G$. We will see an explicit counterexample in the next chapter. In Chapter 5 and 6 we show by using different methods that Question 2 has a positive answer in general by studying the l-adic cohomology of group varieties.

Chapter 4

Arima's Theorem

In this chapter we let k be an algebraically closed field. We are interested in whether $G_{\text{lin}} \times_k G_{\text{ab}}$ and G are isogenous. At the beginning of this chapter we will see that it suffices to study this question when G is commutative and divisible. We then introduce the $\text{Ext}(-,-)$ bifunctor and we give a proof of Arima's theorem, which states that for commutative G we have that $[G] \in \text{Ext}(G_{\text{ab}}, G_{\text{lin}})$ is of finite order if and only if an isogeny $G_{\text{lin}} \times_k G_{\text{ab}} \to G$ exists. Using this we state and proof another one of Arima's theorems, which states that if we work over $\overline{\mathbb{F}}_p$, then $G_{\text{lin}} \times_{\overline{\mathbb{F}}_p} G_{\text{ab}}$ and G are isogenous. We conclude the section by give an example of an exact sequence $e \to G_{\text{lin}} \to G \to G_{\text{ab}} \to e$ where such that G and $G_{\text{lin}} \times_k G_{\text{ab}}$ are not isogenous.

First we state the two theorems of Arima.

Theorem 4.1 (Arima's Theorem 1, ([1] p.235))**.** *Let G be a commutative group variety. Then G is isogenous to $G_{\text{lin}} \times_k G_{\text{ab}}$ if and only if the class of G in $\text{Ext}(G_{\text{ab}}, G_{\text{lin}})$ is of finite order.*

We introduce the notion of $\text{Ext}(-,-)$ in Subsection 4.2.

Theorem 4.2 (Arima's Theorem 2, ([1] p.235))**.** *Let G be a group variety over $\overline{\mathbb{F}}_p$. Then there exists an abelian subvariety $A \subset G$ such that $A \cdot G_{\text{lin}} = G$.*

Although this is stated a bit differently from how it is written above, it will turn out to be equivalent to $G_{\text{lin}} \times_k G_{\text{ab}}$ and G being isogenous by Lemma 4.3 below.

4.1 Reducing to the case that G is commutative and divisible

In this subsection we show that G is isogenous to $G_{\text{ant}} \times_k G_{\text{lin}}$ if and only if G_{ant} is an abelian variety (and thus equal to $(G_{\text{ant}})_{\text{ab}}$). As G_{ant} commutative and divisible, this allows us to reduce to the case that G is commutative and divisible. We have the following equivalent characterization of splitting up to isogeny.

Lemma 4.3. *Let G be a connected group variety, then $G \simeq G_{\text{lin}} \times G_{\text{ab}}$ is equivalent to there existing an abelian subvariety $A \subset G$ such that $A \cdot G_{\text{lin}} = G$.*

Proof. Suppose that $A \cdot G_{\text{lin}} = G$. Any abelian subvariety A is contained in G_{ant}, which is contained in $Z(G)$ by Lemma 1.52. Hence the natural map $A \times G_{\text{lin}} \to G$ is a homomorphism of algebraic groups. We notice that the kernel is isomorphic as a scheme to $A \cap G_{\text{lin}}$, so it is complete and affine, hence finite, i.e. the map is an isogeny. On the other hand, the Chevally-Barsotti theorem gives us that $e \to G_{\text{lin}} \to G_{\text{lin}} \cdot A \to G_{\text{ab}} \to e$ is exact. Hence the restriction of $G \to G_{\text{ab}}$ to A is a quotient map with kernel $A \cap G_{\text{lin}}$, which is finite, hence this is an isogeny. This gives an isogeny $G_{\text{lin}} \times A \to G_{\text{lin}} \times G_{\text{ab}}$, which means that $G \simeq G_{\text{lin}} \times G_{\text{ab}}$.
Now suppose that $G \simeq G_{\text{lin}} \times G_{\text{ab}}$, i.e. we have a chain $G = G_0 - ... - G_n = G_{\text{lin}} \times G_{\text{ab}}$. We claim that it suffices to prove the following: There exists $B \trianglelefteq G$ such that $B/F = A$ is an abelian variety for F finite and $G/(B \cdot G_{\text{lin}})$ is finite. Indeed, the second condition implies $G = B \cdot G_{\text{lin}}$ as G is a connected group variety. As B and G_{lin} generate G we have $G_{\text{ant}} \subset B$, so $G_{\text{ant}} \to A'$ is an isogeny for A' an abelian subvariety of A. As G_{ant} is commutative and divisible, this isogeny reverses, hence G_{ant} is complete as it is also a connected group variety. So as $G = G_{\text{ant}} \cdot G_{\text{lin}}$ and G_{ant} is complete, we see that the statements are equivalent.
We will now show that G satisfies this equivalent condition by decreasing induction: Clearly $G_n = G_{\text{lin}} \times G_{\text{ab}}$ satisfies it. Suppose G_k satisfies it, then we need to show that G_{k-1} satisfies it. If $\varphi : G_k \to G_{k-1}$ is an isogeny, then up to finite index G_{k-1} equals $\varphi((G_k)_{\text{lin}}) \cdot \varphi(B)$ for $B \subset G_k$ such that $B/F = A$ an abelian variety. Now notice that $\varphi(B)/\varphi(F)$ is complete, and that we have an inclusion $\varphi((G_k)_{\text{lin}}) \subset (G_{k-1})_{\text{lin}}$, which shows the induction step in this case.
If $\varphi : G_{k-1} \to G_k$ is an isogeny then up to finite index G_{k-1} equals $\varphi^{-1}((G_k)_{\text{lin}}) \cdot \varphi^{-1}(B)$. Notice that $(G_{k-1})_{\text{lin}}$ is contained in $\varphi^{-1}((G_k)_{\text{lin}})$. It follows from $G_{k-1} = (G_{k-1})_{\text{lin}} \cdot (G_{k-1})_{\text{ant}}$ and that φ is an isogeny that it equals $\varphi^{-1}((G_k)_{\text{lin}})$ up to finite index. As $\varphi^{-1}(B) \to B$ has finite kernel, there exists finite F' such that $\varphi^{-1}(B)/F' \to A$ is a closed immersion, hence $\varphi^{-1}(B)$ is an abelian variety up to finite index. This proves the induction step, hence the lemma. □

Next we have the following lemma.

Lemma 4.4. *Let G be a group variety. Then G_{ant} is divisible. Moreover G_{ant} is an abelian variety if and only if G is isogenous to $G_{\text{lin}} \times_k G_{\text{ab}}$.*

Proof. For showing that G_{ant} is divisible we consider $[n] : G_{\text{ant}} \to G_{\text{ant}}$ and let H denote the cokernel. Then H_{ab} is an abelian variety such that $[n]$ is trivial on this abelian variety. This implies that the abelian variety is trivial as abelian varieties are divisible. So $H = H_{\text{lin}}$. Thus the image of the map $G_{\text{ant}} \to H$ must be trivial, so H is trivial. So $[n]$ is surjective and hence G_{ant} is divisible.

Let $A \subset G$ be an abelian subvariety variety, then since $G_{\text{ant}} = \ker(G \to \text{Spec}(\mathcal{O}(G)))$ we have $A \subset G_{\text{ant}}$. Notice that $G_{\text{ant}} \cdot G_{\text{lin}} = G$ as $G/G_{\text{ant}} \cdot G_{\text{lin}}$ admits quotient maps coming from G/G_{ant} and G/G_{lin} showing that it is a complete, affine, smooth and connected variety, which is trivial. Let D be another subgroup variety of G such that $D \cdot G_{\text{lin}} = G$. Then $\dfrac{D}{D \cap G_{\text{ant}}} \cong \dfrac{G_{\text{ant}} \cdot D}{G_{\text{ant}}}$ is linear as it is a subgroup of $\text{Spec}(\mathcal{O}(G))$. Notice that $D \cap G_{\text{ant}}$ is a normal subgroup of G as it is contained in $Z(G)$. Consider the quotient map $G \to \dfrac{G}{D \cap G_{\text{ant}}}$. Then the images of D and G_{lin} generate the quotient. As both the images are linear, the quotient is linear. So the image of G_{ant} is trivial in the quotient, hence $G_{\text{ant}} = G_{\text{ant}} \cap D$, so $G_{\text{ant}} \subset D$. In particular if $G_{\text{lin}} \times_k G_{\text{ab}}$ and G are isogenous we have by Lemma 4.3 that $A \cdot G_{\text{lin}} = G$ for A an abelian variety, giving $A \subset G_{\text{ant}} \subset A$ and hence $G_{\text{ant}} = A$.
Conversely if G_{ant} is an abelian variety, note that $\mu : G_{\text{ant}} \times_k G_{\text{lin}} \to G$ is an isogeny as it is a quotient map by Theorem 1.60 and its kernel is complete and affine. □

4.2 The Ext functor

In this section we introduce the Ext functor. Until the end of this chapter all algebraic groups are assumed to be commutative unless stated otherwise.

Proposition 4.5 ([32], p.115)**.** *The category of commutative algebraic groups is an abelian subcategory of the category of algebraic groups. Denote this subcategory by $\mathcal{C}$.*

This ensures that we can use some basic tools that apply in abelian categories such as pushouts, the Snake Lemma, etc.

Definition 4.6. Let B and L be algebraic groups. We define $\mathrm{Ext}(B, L)$ to be the set of exact sequences $[G] = 0 \to L \to G \to B \to 0$ modulo the equivalence relation $[G] \sim [G']$ whenever a commuting diagram as follows exists:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

Note that this indeed defines an equivalence relation. If such an f exists, it is necessarily an isomorphism by the Snake Lemma.

Remark 4.7. Note that if $[G] := 0 \to L \to G \to A \to 0$ is an exact sequence and $\varphi : L \to L'$ is a homomorphism, we can form another exact sequence as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\iota} & G & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & L' & \longrightarrow & G \times^L L' & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

Here $G \times^L L = \dfrac{G \times L}{N}$ and N is the algebraic group defined by $N(R) = \{(\varphi(l), -\iota(l)) \mid l \in L(R)\}$. We denote the bottom sequence that is obtain by this by $\varphi_*([G])$. We call φ_* the **pushforward**.

Now suppose that we have a homomorphism $\varphi : B' \to B$. Then we get from $[G]$ the following exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow & & \uparrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & L & \longrightarrow & G \times_B B' & \longrightarrow & B' & \longrightarrow & 0
\end{array}$$

We denote the bottom sequence by $\varphi^*([G])$. We call φ^* the **pullback**.

We note that both the pullback and the pushforward respect the relation defined in Definition 4.6.

Remark 4.8. Very useful characterizations of the pullback and the pushforward are the following (p. 162 [39]): For $[G] \in \mathrm{Ext}(B, L)$ and $\varphi : B' \to B$ and $\psi : L \to L'$ we have that $\varphi^*[G] \in \mathrm{Ext}(B', L)$

is the unique element represented by a sequence $e \to L \to C \to B' \to e$ such that there exists a commuting diagram:

$$\begin{array}{ccccccccc} e & \longrightarrow & L & \longrightarrow & G & \longrightarrow & B & \longrightarrow & e \\ & & {\scriptstyle \mathrm{Id}}\uparrow & & \uparrow & & {\scriptstyle \varphi}\uparrow & & \\ e & \longrightarrow & L & \longrightarrow & C & \longrightarrow & B' & \longrightarrow & e \end{array}$$

Indeed note that there is a natural map $C \to G \times_B B'$ giving that the relevant sequences are equivalent. Similarly we have that $\psi_*[G] \in \mathrm{Ext}(B, L')$ is the unique element represented by a sequence $e \to L' \to C \to B \to e$ such that there exists a commuting diagram:

$$\begin{array}{ccccccccc} e & \longrightarrow & L & \longrightarrow & G & \longrightarrow & B & \longrightarrow & e \\ & & {\scriptstyle \psi}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ e & \longrightarrow & L' & \longrightarrow & C & \longrightarrow & B & \longrightarrow & e \end{array}$$

Again a natural map $G \times^L L' \to C$ exists giving that the sequences $[C]$ and $[G \times^L L']$ are equivalent.

As a consequence we obtain that Ext is functorial in both arguments.

Lemma 4.9. *Let B be an algebraic group, then $\mathrm{Ext}(B, -)$, which sends an object L to the set $\mathrm{Ext}(B, L)$ and a homomorphism $\varphi : L \to L'$ to the pushforward $\varphi_* : \mathrm{Ext}(B, L) \to \mathrm{Ext}(B, L')$ defines a covariant functor $\mathcal{C} \to \mathrm{Set}$.*
For L an algebraic group, $\mathrm{Ext}(-, L)$, which sends an object A to the set $\mathrm{Ext}(B, L)$ and a morphism $\varphi : B' \to B$ to the pullback $\varphi^* : \mathrm{Ext}(B', L) \to \mathrm{Ext}(B, L)$ defines a contravariant functor $\mathcal{C} \to \mathrm{Set}$.

Proof. This follows directly from the uniqueness of the relevant extension classes in Remark 4.8. □

We give the construction of the Baer sum below.

Definition 4.10. Let $[G_1], [G_2] \in \mathrm{Ext}(B, L)$, then $[0 \to L \oplus L \to G_1 \oplus G_2 \to B \oplus B \to 0]$ is an element of $\mathrm{Ext}(B \oplus B, L \oplus L)$, which we call $\oplus([G_1], [G_2])$. As the diagonal map $\Delta : B \to B \oplus B$ is a homomorphism we get $\Delta^* \circ \oplus([G_1], [G_2]) \in \mathrm{Ext}(B, L \oplus L)$. As L is commutative, the multiplication map $\mu_L : L \oplus L \to L$ is a homomorphism, so $(\mu_L)_* \circ \Delta^* \circ \oplus([G_1], [G_2]) \in \mathrm{Ext}(L, B)$. This element is called the **Baer sum** of $[G_1]$ and $[G_2]$.

We now have the following lemma that we will not prove. Baer (Satz 1 p.395 [4]) has shown that the lemma below is true when B, L are abelian groups. His methods generalize to the category of commutative algebraic groups as claimed by Serre ([39] p.163).

Lemma 4.11. *The operation '+' defined above defines a group structure on $\mathrm{Ext}(B, L)$, where the inverse of a sequence $[G]$ is given by the same sequence with $\iota : L \to G$ replaced by $-\iota$ and with neutral element being the split sequence $[B \times_k L]$.*

Remark 4.12. Note that Baer works in slightly in more generality then we do as he considers group extensions $e \to N \to G \to Q \to e$ where Q need not be abelian and Q has some prediscribed action on N (called χ_g in [4]) which corresponds to the action induced by conjugation when one has an exact sequence $e \to N \to G \to Q \to e$. The case that Q is abelian and the action is trivial corresponds to the central extensions of Q by N. It follows that if $[G_1], [G_2]$ are given by commutative algebraic groups, then so is their Baer sum $[G_1] + [G_2]$, so $\mathrm{Ext}(Q, N)$ (as we defined it, i.e. the extension being commutative) inherits the group structure from the central extensions of Q by N.

To deduce other properties of Ext we prove a lemma.

Lemma 4.13. *Let* $[G] \in \operatorname{Ext}(B, L)$ *and* $[G'] \in \operatorname{Ext}(B', L')$ *and let* $\psi : L \to L'$ *and* $\varphi : B \to B'$ *be homomorphisms. Then* $\varphi^*[G'] = \psi_*[G]$ *in* $\operatorname{Ext}(B, L')$ *is equivalent to there existing a commuting diagram:*

$$\begin{array}{ccccccccc}
e & \longrightarrow & L & \longrightarrow & G & \xrightarrow{\pi} & B & \longrightarrow & e \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \varphi} & & \\
e & \longrightarrow & L' & \longrightarrow & G' & \longrightarrow & B' & \longrightarrow & e
\end{array}$$

Proof. If $\psi_*[G] = \varphi^*[G']$ then the existence of such a diagram follows by Remark 4.8. If such a diagram exists, then one can construct a homomorphism $\Psi : G \times^L L' \to G' \times_{B'} B$ which is on points given by $(g, l) \mapsto (F(g) \cdot l, \pi(g))$. We obtain the following commuting diagram, which implies that $\psi_*[G] = \varphi^*[G']$:

$$\begin{array}{ccccccccc}
e & \longrightarrow & L' & \longrightarrow & G \times^L L' & \longrightarrow & B & \longrightarrow & e \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
e & \longrightarrow & L' & \longrightarrow & G \times_{B'} B & \longrightarrow & B & \longrightarrow & e
\end{array}$$

□

We obtain the following corollary.

Corollary 4.14. *For* $\varphi : B' \to B$ *and* $\psi : L \to L'$ *homomorphisms we have* $\varphi^*\psi_* = \psi_*\varphi^*$.

Proof. Let $G \in \operatorname{Ext}(B, L)$ and write $[C] = \psi_*[G]$ and $D = \varphi^*[G]$. By the previous lemma we have that $\varphi^*\psi_*[G] = \psi_*\varphi^*[G]$, is equivalent to the existence of a commuting diagram:

$$\begin{array}{ccccccccc}
e & \longrightarrow & L & \longrightarrow & D & \longrightarrow & B' & \longrightarrow & e \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \\
e & \longrightarrow & L' & \longrightarrow & C & \longrightarrow & B & \longrightarrow & e
\end{array}$$

It is immediate that this diagram exists by putting the commuting diagram that features C, G and the one that features D, G together. □

We can upgrade the previous lemma to that Ext is a functor in both arguments to the category of groups.

Lemma 4.15. *The set-valued functors* $\operatorname{Ext}(B, -)$ *and* $\operatorname{Ext}(-, L)$ *factor through the category of abelian groups via the forgetful functor* $\mathrm{Grp} \to \mathrm{Set}$.

Proof. By the previous lemma we have that $\operatorname{Ext}(B, L)$ are abelian groups, hence it suffices to show that the pullback and pushforward are group homomorphisms. For showing this for the pullback we have to show that if $\varphi : B' \to B$ is a homomorphism then $\varphi^*[G_1 + G_2] = \varphi^*[G_1] + \varphi^*[G_2]$ for any $[G_1], [G_2] \in \operatorname{Ext}(B, L)$. So we want $\varphi^*(\mu_L)_*\Delta_B^*([G_1] \oplus [G_2]) = (\mu_L)_*\Delta_{B'}^*(\varphi^*[G_1] \oplus \varphi^*[G_2])$. As pullbacks and pushforwards commute by Corollary 4.14 we get that the left-hand side is equal to

$(\mu_L)_*(\Delta_B\circ\varphi)^*([G_1]\oplus[G_2])$, so it suffices to show that $(\Delta_B\circ\varphi)^*([G_1]\oplus[G_2]) = \Delta_{B'}^*(\varphi^*[G_1]\oplus\varphi^*[G_2])$. The left-hand side (denoted $[C]$) is the unique class in $\mathrm{Ext}(L\times_k L, B')$ such that a diagram as follows exists:

$$\begin{array}{ccccccccc} e & \longrightarrow & L\times_k L & \longrightarrow & C & \longrightarrow & B' & \longrightarrow & e \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow{\scriptstyle \Delta_B\circ\varphi} & & \\ e & \longrightarrow & L\times_k L & \longrightarrow & G\times_k G & \longrightarrow & B\times_k B & \longrightarrow & e \end{array}$$

Note that such a diagram also exists for $\Delta_{B'}^*(\varphi^*[G_1]\oplus\varphi^*[G_2])$ as $B' \stackrel{\Delta_{B'}}{\to} B'\times_k B' \stackrel{\varphi\times\varphi}{\to} B\times_k B$ equals $\Delta_B\circ\varphi$ and makes the relevant diagram commute by Remark 4.8.
Checking the functoriality in the first argument is done similarly. □

Another important property of the Ext functor is that it is additive in both arguments. We will only need the additivity in the second argument so for conciseness we restrict ourself to proving this case.

Lemma 4.16. *The* Ext *functor is additive in both arguments, i.e. for homomorphisms of algebraic groups $\varphi_1,\varphi_2 : L\to L'$ and $\pi_1,\pi_2 : B\to B'$ we have $(\varphi_1+\varphi_2)_* = (\varphi_1)_* + (\varphi_2)_*$ and we have $(\pi_1)^*+(\pi_2)^* = (\pi_1+\pi_2)^*$.*

Proof. Let $[G]\in\mathrm{Ext}(B,L)$ and let $f,g : B'\to B$ be homomorphisms. We need to show that $(f+g)^*[G] = f^*[G]+g^*[G] := (\mu_L)_*\Delta^*(f^*[G]\oplus g^*[G])$. Let $C := \Delta^*(f^*[G]\oplus g^*[G])$. By Lemma 4.13 it suffices to show that there exists a commuting diagram:

$$\begin{array}{ccccccccc} e & \longrightarrow & L\times_k L & \longrightarrow & C & \longrightarrow & B' & \longrightarrow & e \\ & & \downarrow{\scriptstyle \mu_L} & & \downarrow & & \downarrow{\scriptstyle f+g} & & \\ e & \longrightarrow & L & \longrightarrow & G & \longrightarrow & B & \longrightarrow & e \end{array}$$

Note that by Remark 4.8 there is a commuting diagram:

$$\begin{array}{ccccccccc} e & \longrightarrow & L\times_k L & \longrightarrow & G\times_k G & \longrightarrow & B\times_k B & \longrightarrow & e \\ & & {\scriptstyle \mathrm{Id}}\uparrow & & \uparrow & & {\scriptstyle (f\times g)\circ\Delta}\uparrow & & \\ e & \longrightarrow & L\times_k L & \longrightarrow & C & \longrightarrow & B' & \longrightarrow & e \end{array}$$

As we have $\mu_B\circ(f\times g)\circ\Delta = f+g$ we can consider compose the previous commuting diagram with the following commuting diagram to get that $(f+g)^*[G] = f^*[G]+g^*[G]$:

$$\begin{array}{ccccccccc} e & \longrightarrow & L\times_k L & \longrightarrow & G\times_k G & \longrightarrow & B\times_k B & \longrightarrow & e \\ & & \downarrow{\scriptstyle \mu_L} & & \downarrow{\scriptstyle \mu_G} & & \downarrow{\scriptstyle \mu_B} & & \\ e & \longrightarrow & L & \longrightarrow & G & \longrightarrow & B & \longrightarrow & e \end{array}$$

For showing additivity in the first argument, one has to show that $(f+g)_*[G] = (\mu_L)_*\Delta^*(f_*[G]\oplus g_*[G]) = \Delta^*(\mu_L)_*(f_*[G]\oplus g_*[G])$. Then one uses a similar argument, which is again based on Lemma 4.13. □

Remark 4.17. A slightly easier approach such as one takes in the category of R-modules would be to show that $\mathrm{Ext}(B,-)$ and $\mathrm{Ext}(-,L)$ equal $R^1\mathrm{Hom}(B,-)$ and $L^1\mathrm{Hom}(-,L)$ where these Hom functors are from the category of commutative algebraic groups to the category of abelian groups. The problem with this in our setting is shown by Brion ([6], p.40). He shows that there do not exist any nonzero injective objects in the category of commutative algebraic groups. So one can not define these derived functors.

4.3 Principal G-bundles

In this section we introduce principal G-bundles over some variety Y. For the general definition we make the exception that G may be a non-commutative algebraic group. We use this to study extensions of commutative algebraic groups B by L by thinking of them as principal L-bundles over B.

Definition 4.18. Let G be an algebraic group. A (left) **principal G-bundle** is a morphism $\pi : X \to Y$ that is faithfully flat and of finite type together with a (left) action $\rho : G \times_k X \to X$ such that $\pi \circ \rho = \pi \circ \pi_X$ for $\pi_X : X \times_k G \to X$ the projection. We also require that the morphism $(\rho, \mathrm{Id}_X) : L \times_k X \to X \times_Y X$ is an isomorphism.

Note that the last condition implies that the action is free, i.e. all stabilizers are trivial. A basic example that we will use a lot is the following:

Example 4.19. Let $e \to L \to G \to B \to 0$ be an exact sequence of algebraic groups. Then L acts on G with trivial stabilizers and the image of L in B is trivial. It follows by looking at the functor of points that $L \times_k X \to X \times_B X$ is an isomorphism.

We wish to consider principal bundles up to isomorphism.

Definition 4.20. Two G-bundles X, X' over Y are **isomorphic** if there is an isomorphism $X \to Y$ that respects the action of G and commutes with the maps to Y. We denote the set of principal G-bundles over Y up to isomorphism by $\mathrm{PB}_{Y,G}$.

Now we define sections.

Definition 4.21. Let $\pi : X \to Y$ be a principal G-bundle. A **section** to this bundle is a morphism $s : Y \to X$ such that $\pi \circ s = \mathrm{Id}_Y$.

It follows that a principal bundle $X \to Y$ is isomorphic to the trivial bundle $Y \times_k G \to Y$ with G acting on itself if and only if $X \to Y$ admits a section.

Definition 4.22. In the previous definition, we say that X is **locally trivial** for the Zariski (resp. étale) topology if there exists a covering $(U_i \to Y)_i$ such that $X_{U_i} \to U_i$ has a section for all i. For $\mathcal{T}$ either of these topologies, denote by $\mathrm{PB}^{\mathcal{T}}_{X,G}$ the classes of G-bundles over A that are locally trivial for $\mathcal{T}$.

It follows by the definition that any principal G-bundle is locally trivial in the flat topology.

Remark 4.23. We have the following operations that we can do on principal G-bundles $[X]$,$[X']$ over Y:

- We can take the product bundle $[X \times_k X']$ over $B \times B$, which is a principal $G \times G$-bundle, each copy of G acting componentwise.
- For a morphism $\varphi : B' \to B$ we can form the pullback $\varphi^*[X] = [X \times_B B']$, which is a principal G-bundle over B'.
- For a homomorphism $G \to G'$ we can form the pushforward $\varphi_*[X] = [X \times^G G']$, which is a principal G'-bundle over B. Here we considering $X \times G'$ and quotienting out by the anti-diagonal action given by $(g,(x,g')) \mapsto (g \cdot x, \varphi(g)^{-1} \cdot g')$ of G on $X \times G'$.

In particular we can imitate the Baer-sum, which was defined on $\mathrm{Ext}(B, L)$.

Lemma 4.24. *For G commutative, $\mathrm{PB}_{Y,G}$ is a commutative group with $(\mu_G)_* \circ \Delta^* \circ \times$ as its operation. The neutral element is given by $[Y \times_k G]$ with G acting on itself and the projection to Y. The inverse of an element $[X]$ with G-action ρ is given by the same element only with a new action ρ' being $\rho'(g,x) = \rho(-g,x)$.*
Moreover $\mathrm{PB}_{-,-}$ is functorial in both arguments by sending a morphism to its pullback, resp. its pushforward. The pullback and pushforward maps are group homomorphisms.

The proving that the group structure exists is the same as proving that the Baer-sum makes $\mathrm{Ext}(B, L)$ into an abelian group see [4] (Satz 1 p.395).

Remark 4.25. Showing that the pullback and pushforward are group homomorphisms can be done in a similar way as in Lemma 4.15, namely the pullback $\varphi^*[X]$ and pushforward $\psi_*[X]$ are characterized by similar diagrams as in Remark 4.8. This follows from the fact that a morphism between principle bundles over a base is always an isomorphism (this is true flat-locally and one 'descends down' the isomorphism).

Remark 4.26. Denote by $\mathrm{PB}_{-,G}|$ the restriction of $\mathrm{PB}_{-,G}$ to the category of commutative algebraic groups $\mathcal{C}$. The lemma implies that for B, L algebraic groups there are natural transformations $\mathrm{Ext}(-, L) \to \mathrm{PB}_{-,L}|$ and $\mathrm{Ext}(B, -) \to \mathrm{PB}_{B,-}$.

Definition 4.27. Given a cover $\mathcal{U} = (U_i \to Y)_i$ for the $\mathcal{T}$-topology, denote the set of principal G-bundles that trivialize over $\mathcal{U}$ by $\mathrm{PB}^{\mathcal{U}}_{Y,G}$.

Let G_Y denote the $\mathcal{T}$-sheaf $\mathrm{Hom}_k(-, G)$ on Y. For such an element $[X] \in \mathrm{PB}^{\mathcal{U}}_{Y,G}$ we can assign an element of $\check{H}^1(\mathcal{U}, G_Y)$ is follows: A trivializing cover $(U_i \to Y)_i$ can be taken to consist of a single element by taking the disjoint union (formally the fibre product over Y), so the cover is $U \to Y$ with a section $s : U \to U \times_Y X \to X$. For $k = 1, 2$ these can be pulled back to $p_k^* s : U \times_Y U \to U \to X$, thus defining a morphism $(p_1^*(s), p_2^*(s)) : U \times_Y U \to X \times_Y X$. Then using that as X is a principal G-bundle over Y there is the isomorphism $(\rho, p_2) : G \times_k X \to X \times_Y X$ we get that there is a unique morphism $g : U \times_Y U \to G$ such that $\rho(g, p_2^*(s)) = p_1^*(s)$. We can check that the element $s \in G_Y(U \times_Y U)$ gives an element that maps to 0 in $G_Y(U \times_A U \times_Y U)$, hence it defines an element of $\check{H}^1_{\mathcal{T}}(\mathcal{U}, G_Y)$.

Example 4.28. We go through this process in the example that $X \to Y$ is given by the n'th power map $\pi = [n] : \mathbb{G}_m \to \mathbb{G}_m$ and $G = \mu_n$ acting on $\mathbb{G}_m$ in the obvious way. We take $\mathcal{T}$ to be the étale topology and n not divisible by the characteristic of the field k. Then we have an étale cover $[n] : U = \mathbb{G}_m \to \mathbb{G}_m$ such that there is a section $s : U \to \mathbb{G}_m$ to $\pi : \mathbb{G}_m \to \mathbb{G}_m$, namely take s to be the identity map. Hence we get an induced map $(p_1, p_2) : \mathbb{G}_m \times_{[n]} \mathbb{G}_m \to \mathbb{G}_m \times_{[n]} \mathbb{G}_m$. Hence the element g of $\mu_n(\mathbb{G}_m \times_{[n]} \mathbb{G}_m)$ that we get is the unique element such that $g(x, y) \cdot x = y$ for all $x, y \in \mathbb{G}_m \times_{[n]} \mathbb{G}_m$, thus it equals $g(x, y) = \frac{y}{x}$. Note that $p_{12}^* g \cdot p_{23}^* g \cdot (p_{13}^* g)^{-1} \in \mu_n(\mathbb{G}_m^3)$ (omitting the fibre product subscript) equals the morphism $(x, y, z) \mapsto \frac{x}{y} \cdot \frac{y}{z} \cdot (\frac{x}{z})^{-1} = 1$ as needed.

By our above discussion, any isomorphism class of a principal bundle gives a 1-cocycle. It turns out that the converse is also true. By the discussion on Cech cohomology of Section 2 we obtain the following.

Proposition 4.29 ([31], p.77). *The assignment above* $\mathrm{PB}^{\mathcal{U}}_{Y,G} \to \check{\mathrm{H}}^1_{\mathcal{T}}(\mathcal{U}, \underline{G})$ *gives a bijection. It gives an isomorphism* $\mathrm{PB}^{\mathcal{T}}_{Y,G} \to \check{\mathrm{H}}^1_{\mathcal{T}}(Y, \underline{G})$.

In the future we write G instead of G_Y for the sheaf on Y. The following proposition is claimed in [11]. What we mean below by functorial in Y is that for $f : X \to Y$ a morphism, the below isomorphisms for X and Y commute with $\check{\mathrm{H}}^1_{\mathcal{T}}(Y, G_Y) \to \check{\mathrm{H}}^1_{\mathcal{T}}(X, f^*G_Y) \to \check{\mathrm{H}}^1_{\mathcal{T}}(X, G_X)$, which on a cocycle is given by componentwise precomposotion with f (see Proposition 2.74).

Proposition 4.30. *Let G be a commutative algebraic group. There is an isomorphism of abelian groups* $\mathrm{PB}^{\mathcal{T}}_{Y,G} \cong \check{\mathrm{H}}^1_{\mathcal{T}}(Y, \underline{G})$. *It is functorial in both G and in Y.*

Proof. We need to check that the the bijection above respects the group law induced by the Baer-sum and that it is functorial in both Y and G. First we check the addition law. Given principal G-bundles $[X]$ and $[X']$ over Y. We may pick a trivializing open cover $\mathcal{U} = (\varphi : U \to Y)$ that trivializes both $[X]$ and $[X']$. Let $s : U \to X$ and $\sigma : U \to X'$ denote the sections of $[X]$ and $[X']$ respectively and denote by (g_X) and $(g_{X'})$ the corresponding cocycles in $\check{H}^1_{\mathcal{T}}(\mathcal{U}, G_A)$. Taking the product $G \times_k G$-bundle $[X \times_k X']$ over $Y \times_k Y$ gives that $\varphi \times_k \varphi : U \times_k U \to Y \times_k Y$ is a trivializing open cover such that $(s \times \sigma) : U \times_k U \to X \times_k X'$ is a section. Then taking the pullback of $[X \times_k X']$ with respect to $\Delta : Y \to Y \times_k Y$ gives that $\varphi : U \to Y$ is a trivializing cover with section $(s, \sigma, \varphi) : U \to X \times_k X' \times_{Y \times_k Y} \times Y$. Then taking the pushforward $(\mu_G)_* \Delta^*[X \times_k X'] = X + X'$ gives that $(s, \sigma, \varphi, 0) : U \to X \times_k X' \times_{Y \times_k Y} \times Y \times^{G \times G} G$ is a section. We call this section Φ for simplicity. Taking the pullbacks $p_i^* \Phi$, we need to show that $p_2^* \Phi(x, y) + g_X(x, y) + g_{X'}(x, y) = p_1^* \Phi$. The left side reads $(s(y), \sigma(y), \varphi(y), g_X(x, y) + g_Y(x, y)) = (s(y) + g_X(x, y), \sigma(y) + g_Y(x, y), \varphi(y), 0) = (s(x), \sigma(x), \varphi(y), 0)$, which equals $p_1^* \Phi(x)$ as $\varphi(x) = \varphi(y)$ as the elements (x, y) are in $U \times_Y U$, where the fibre product taken over $\varphi : U \to Y$.

We now check that this identification is functorial in Y. Let $\psi : Y' \to Y$ be a morphism and $\varphi : U \to A$ be a cover. The pullback morpism then maps a cocycle $g : U \times_Y U \to G$ to $\psi^* g : U \times_Y U \times_Y Y' \to U \times_Y U \to G$. Let $[X]$ be a principal G-bundle over Y such that $s : U \to X$ is a section. Then $U \times_Y Y' \to Y'$ is a cover and $(s \times \mathrm{Id}) : U \times_Y Y' \to X \times_Y Y'$ is a section. Then $p_i^*(s \times \mathrm{Id}) : (U \times_Y Y') \times_{Y'} (U \times_Y Y') \to X \times_Y Y'$ are sections. Through the isomorphism $(U \times_Y Y') \times_{Y'} (U \times_Y Y') \to U \times_Y U \times_Y Y'$ these sections correspond to the sections given by $(p_i^* s, \mathrm{Id}) : U \times_Y U \times_Y Y' \to X \times_Y Y'$. Let g be the cocycle obtained by s and let $\psi^*(g)$ denote its pullback $U \times_Y U \times_Y Y' \to G$. We have $(p_2^* s, \mathrm{Id})(x, y, z) + g(x, y, z) = (s(y) + g(x, y), z) = (s(x), z)$, hence indeed $\psi^*(g)$ is the cocycle that corresponds to $\psi^*[X]$. Checking the functoriality in G is routine as well. □

Putting this together with Proposition 2.26 gives the following.

Corollary 4.31. *For B and G commutative algebraic groups such that any extension of B by G is locally trivial in the $\mathcal{T}$-topology there are natural homomorphisms:*

$$\mathrm{Ext}(B,G) \to \mathrm{PB}^{\mathcal{T}}_{B,G} \to \mathrm{H}^1_{\mathcal{T}}(B,\underline{G})$$

All of them are functorial in both B and G and the last map is an isomorphism.

If we assume that G is a linear group variety then we get more information. We omit underlining the flat sheaves to improve notation.

Proposition 4.32 ([39], p.169))**.** *Let X be a principal G-bundle over Y, where G is a commutative linear group variety. Then $X \to Y$ is locally trivial for the Zariski topology.*

Proof. Let $\eta \in B$ be the generic point and let $X_\eta \subset X$ denote the generic fibre. Let $\lambda \in X$ be some point sent to η, then there is a morphism of local rings $K(Y) := \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\lambda}$. Giving a rational section $Y \dashrightarrow X$ that maps η to λ is equivalent to giving a section to $K(Y) \to \mathcal{O}_{X,\lambda}$, which is a $K(Y)$-homomorphism. As $K(Y)$ is a field, the former is equivalent to $K(Y) \cong \mathcal{O}_{X,\lambda}/\mathfrak{m}_\lambda = \kappa(\lambda)$. The scheme X_η is by definition the pullback of X by the morpihism $\{\eta\} \to B$, thus it is a principal G-bundle over η. A section $\{\eta\} \to X_\eta$ is thus equivalent to X_η being the trivial G-bundle. Hence it suffices to prove the following statement: For G a commutative group variety any principal G-bundle over $\mathrm{Spec}(K)$ for K a field is trivial. The group $\mathrm{PB}_{\mathrm{Spec}(K),G}$ is isomorphic to $\mathrm{H}^1_{\mathrm{fl}}(\mathrm{Spec}(K),G)$ as any G-bundle trivializes in the flat topology.

As G is a linear commutative group variety it admits a composition series $e = G_0 \leq ... \leq G_m = G$ such that G_{i+1}/G_i is isomorphic to $H \in \{\mathbb{G}_a, \mathbb{G}_m\}$ ([6], prop 2.8). As exactness of a sequence of algebraic groups is preserved under an extension of the basefield ([32], p.24) we have the exact sequence of algebraic groups over $\{\eta\} = \mathrm{Spec}(K)$, which reads $0 \to (G_i)_K \to (G_{i+1})_K \to H_K \to 0$. We get an exact sequence of the corresponding flat sheaves on $\{\eta\}$, which gives the following long exact sequence in cohomology:

$$0 \to (G_i)_K(K) \to (G_{i+1})_K(K) \to H_K(K) \to \mathrm{H}^1_{\mathrm{fl}}(\{\eta\},(G_i)_K) \to \mathrm{H}^1_{\mathrm{fl}}(\{\eta\},(G_{i+1})_K) \to \mathrm{H}^1_{\mathrm{fl}}(\{\eta\},H_K) \to ...$$

The flat sheafs $\mathrm{Hom}_K(-,G_K)$ and $\mathrm{Hom}_k(-,G)$ on $\{\eta\}$ are equal, so their derived functors are also equal, hence it suffices to show that $\mathrm{H}^1_{\mathrm{fl}}(\{\eta\},G_K) = 0$. We see that by induction on the dimension i and by noting that $\dim((G_1)_K) = 1$, hence $G_1 \in \{\mathbb{G}_a,\mathbb{G}_m\}$, that it suffices to show that $\mathrm{H}^1_{\mathrm{fl}}(\mathrm{Spec}(K),H_K) = 0$ for $H_K \in \{(\mathbb{G}_m)_K,(\mathbb{G}_a)_K\}$. In the next part the K-subscripts are omitted.

We can show that this vanishes by using Cech-chomology. Let R be a K-algebra, then in the case $H = \mathbb{G}_a$ the Cech-complex is $R \to R \otimes_K R \to R \otimes_K R \otimes_K R \to ...$ where the maps are given by $r \mapsto r \otimes 1 - 1 \otimes r$ and $r \otimes r' \mapsto 1 \otimes r \otimes r' - r \otimes 1 \otimes r' + r \otimes r' \otimes 1$. This complex is always exact for R/K flat ([43, Tag 023M]), hence the cohomology groups vanish. In the case of $H = \mathbb{G}_m$ we have $\mathrm{H}^1_{\mathrm{fl}}(\mathrm{Spec}(K),\mathbb{G}_m) \cong \mathrm{H}^1_{\mathrm{zar}}(\mathrm{Spec}(K),\mathbb{G}_m)$ by (Theorem 11.4, [31]), thus it is isomorphic to $\mathrm{Pic}(\mathrm{Spec}(K))$), hence we get that this vanishes as the only invertible sheaf on $\mathrm{Spec}(K)$ is the structure sheaf. This concludes the proof of the proposition. □

The assumption that G is connected and smooth is crucial for the above to work as we have the following counterexample for disconnected or singlar G.

Example 4.33. Consider the quotient map $\mathbb{G}_m \to \mathbb{G}_m \quad x \mapsto x^n$. The kernel of this map equals μ_n, which is either disconnected or singular for $n > 1$. Suppose for a contradiction that this map has a rational section $\mathbb{G}_m \dashrightarrow \mathbb{G}_m$. This corresponds to a section to the map between the function fields $k(T) \to k(T) \quad T \mapsto T^n$. Note that this map having a section implies that there is an element $x \in k(T)$ such that $x^n = T$, which one can check by hand does not happen unless $n = 1$. So this principal μ_n-bundle is not Zariski locally trivial.

There is a variant of the above for local triviality for the étale topology.

Lemma 4.34. *Let $\pi : X \to Y$ be a principal G-bundle with G smooth (not necessarily commutative). Then $\pi : X \to Y$ is locally trivial for the étale topology.*

Proof. As $X \to Y$ is a principal G-bundle the following is a pullback square:

$$\begin{array}{ccc} G \times_k X & \xrightarrow{(g,x)\mapsto gx} & X \\ \big\downarrow{\scriptstyle p_2} & & \big\downarrow \\ X & \longrightarrow & Y \end{array}$$

As G is smooth over Spec(k) we have that p_2 is a smooth morphism. For any closed point $y \in Y$ we have that X_y is smooth as G is smooth. Hence by ([23] III.10.5) we obtain that π is smooth. This implies by ([43, Tag 02GH]) that for any point $y \in Y$ there is an open neigbbourhood V of y, an open neighbourhood U of X such that $\pi(U) \subset V$ and an étale map $\alpha : U \to \mathbb{A}^d_V$ such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow[\pi]{} & V \\ & {\scriptstyle \alpha}\searrow & \big\uparrow \\ & & \mathbb{A}^d_V \end{array}$$

Note that $\mathbb{A}^d_V \to V$ has a section s. Then note that $V \times_{\mathbb{A}^d_V} U \to V$ is an étale map. We have that $X \times_Y (U \times_{\mathbb{A}^d_V} V) \to U \times_{\mathbb{A}^d_V} V$ admits a section by $(u, v) \mapsto (u, u, v)$ and hence $\pi : X \to Y$ is étale locally trivial. □

4.4 The image of Ext(B, L) inside $\mathrm{H}^1_{\mathcal{T}}(B, \underline{L})$

We wish to determine the image of Ext(B, L) inside $\mathrm{H}^1_{\mathcal{T}}(B, L)$. Under certain assumptions on B and L we can find a description of this.

Lemma 4.35. *Let L, B be commutative algebraic groups such that any morphism $B \times B \to L$ is constant. Then* Ext$(B, L) \to \mathrm{PB}^{\mathcal{T}}_{B,L}$ *is injective.*

Proof. Let $[G]$ be in the kernel, so $G \to B$ has a section s. We may assume that $s(0) = 0$ by composing with a translation. Consider the morphism $B \times B \to L, \quad (x, y) \mapsto s(x+y) - s(x) - s(y)$. This morphism has 0 in the image and it is constant, hence s is a homomorphism. Thus $[G]$ is trivial. □

Remark 4.36. Let T be any of $\mathrm{H}^r_{\mathcal{T}}(G,\underline{L})$ for $r \geq 1$, L a commutative algebraic group and G an algebraic group. There are projections morphisms $p_i : G \times G \to G$ and inclusion morphisms $\iota_i : G \to G \times G$. They have that $p_j \circ \iota_i$ is constant if $i \neq j$ and it is the identity if $i = j$. This implies that on T we have that $\iota_i^* p_j^*$ is the zero map if $i \neq j$ and the identity if $i = j$. In particular both p_i^* are injective and the intersection of $p_i^*(T)$ and $p_j^*(T)$ in $\mathrm{H}^r_{\mathcal{T}}(G \times G,\underline{L})$ is $\{0\}$, thus $p_1^*(T) \bigoplus p_2^*(T) \subset \mathrm{H}^r_{\mathcal{T}}(G \times G,\underline{L})$ and (ι_1^*,ι_2^*) is a retraction to $(p_1^* + p_2^*)$, hence $p_1^*(T) \bigoplus p_2^*(T)$ is a direct factor of $\mathrm{H}^r_{\mathcal{T}}(G \times G,\underline{L})$. The multiplication map $\mu : G \times G \to G$ induces the map $\mu^* : T \to \mathrm{H}^r_{\mathcal{T}}(G \times G,\underline{L})$. As $\iota_i^* \mu^* x = \mathrm{Id}^* x = x$ for all $x \in T$ we see that projecting $\mu^*(x)$ onto the $p_1^*(T) \bigoplus p_2^*(T)$-component gives exactly $p_1^* x + p_2^* x$.

We now define the elements of T for which $\mu^* x$ and this component are equal.

Definition 4.37. Let G be an algebraic group. Let T be some $\mathrm{H}^r_{\mathcal{T}}(G,\underline{L})$ for some $r \geq 1$. An element $x \in T$ is **primitive** if $p_1^* x + p_2^* x = \mu^* x$. Denote the set of primitive elements by $P(T)$.

Remark 4.38. With T again as above, $P(T)$ forms a subgroup of T. Moreover notice that if $\varphi : G \to G'$ is a homomorphism of algebraic groups, then the pullback morphism $\varphi^* : T_G \to T_{G'}$ maps primitive elements to primitive elements. This makes $P(\mathrm{H}^r_{\mathcal{T}}(-,\underline{L}))$ into a functor from the category of commutative algebraic groups to the category of abelian groups.

Lemma 4.39 ([39], p.181)**.** *The functor* $\mathrm{P}(\mathrm{H}^r_{\mathcal{T}}(-,\underline{L}))$ *is additive.*

Proof. Let $\varphi + \psi = \theta$ as homomorphisms $G \to G'$. We must show that $\theta^* = \varphi^* + \psi^*$ to conclude the proof. Note that $\theta = \mu_{G'} \circ (\psi \times \varphi) \circ \Delta$, hence we get $\theta^*(x) = \Delta^*(\psi \times \varphi)^* \mu^*(x) = \Delta^* \circ (\psi \times \varphi)^*(p_1^*(x) + p_2^*(x))$. As we have $p_1 \circ (\psi \times \varphi) \circ \Delta = \psi$ and similarly $p_2 \circ (\psi \times \varphi) \circ \Delta = \varphi$ we get that $\theta^*(x) = \varphi^*(x) + \psi^*(x)$. □

We have the following proposition, which is based on ([39], p.181 Theorem 5). Actually the proof is identical, however the assumptions that are made here are slighly weaker than in the original proof, where B was required to be an abelian variety. These assumptions suffice though as the proof demonstrates.

Proposition 4.40. *([39], p.181) Let L, B be a commutative algebraic groups such that any morphism $B \to L$, $B \times B \to L$ and $B^3 \to L$ is constant. Then the image of* $\mathrm{Ext}(B,L)$ *in* $\mathrm{H}^1_{\mathcal{T}}(B,\underline{L})$ *are the primitive elements of* $\mathrm{H}^1_{\mathcal{T}}(B,\underline{L})$.

Proof. Let $[G] \in \mathrm{Ext}(B,L)$. Then we have by additivity of Ext and the fact that the natural map $\Phi : \mathrm{Ext}(B,L) \to \mathrm{H}^1_{\mathcal{T}}(B,\underline{L})$ is functorial in B that:

$$\mu^*\Phi([G]) - p_1^*(\Phi([G])) - p_2^*(\Phi([G])) = \Phi(\mu^*[G]) - \Phi(p_1^*[G]) - \Phi(p_2^*[G]) = \Phi(\mu^*[G] - (p_1 + p_2)^*[G]) = 0$$

So the image of any element of $\mathrm{Ext}(B,L)$ inside $\mathrm{H}^1_{\mathcal{T}}(B,\underline{L})$ is primitive.

Given a primitive element $x \in \mathrm{H}^1_{\mathcal{T}}(B,\underline{L})$, which we represent by some principal L-bundle $[\pi : X \to B]$, we need to show that there exists $[G] \in \mathrm{Ext}(B,L)$ such that $[G] = [X]$ as principal L-bundles. We have that $p_1^*[X] + p_2^*[X] = \mu^*[X]$. The left side equals $\nabla_*(\Delta^*(p_1^*[X] \times p_2^*[X]))$.

We claim that $\Delta^*(p_1^*[X] \times p_2^*[X])$ is isomorphic to $L \times L$-bundle $[\pi \times \pi : X \times X \to B \times B]$ with componentwise $L \times L$-action. We have that $p_1^*[X]$ is isomorphic to $[\pi \times \mathrm{Id}_B : X \times B \to B \times B]$

with L acting on X, which follows from its description as a fibre product. Similarly we see that $p_2^*[X] = [\mathrm{Id}_B \times \pi : B \times X \to B \times B]$ with L acting on X. This means that we get $p_1^*[X] \times p_2^*[X] = [\mathrm{Id}_B \times \pi \times \pi \times \mathrm{Id}_B : X \times B \times B \times X \to B^4]$ with $L \times L$ acting componentwise on $X \times X$. This implies that $\Delta^*(p_1^*[X] \times q_2^*[X]) = [q_2 : (X \times B \times B \times X) \times_{B^4} B^2 \to B^2]$ for q_2 the projection on the second factor with $L \times L$ acting componentwise on $X \times X$. This principal bundle is a closed subscheme of $X \times B \times B \times X$, which by looking at the points equals $X \times X$. Through this isomorphism the projection map becomes $\pi \times \pi : X \times X \to B \times B$, which shows that the claim holds.

The condition of $[X]$ being primitive implies that $(\mu_L)_*[X \times X] = (\mu_B)^*[X]$. This gives the following commuting diagram:

$$\begin{array}{ccc}
X \times X & \longrightarrow & B \times B \\
\big\downarrow & & \big\downarrow \\
\mu_B^*[X] & \longrightarrow & B \times B \\
\big\downarrow & & \big\downarrow{\scriptstyle \mu_B} \\
X & \xrightarrow{\ \ \pi\ \ } & B
\end{array}$$

Note that the map $f : X \times X \to \mu_B^*[X]$ is the induced map $X \times X \to (\mu_L)_*[X]$, which satisfies the equation $f(x + l, y + l') = f(x + y) + l + l'$ for any $l, l' \in L$, while $h : \mu_B^*[X] \to X$ satisfies $l + h(x) = h(x + l)$, hence the composition $g : X \times X \to X$ has that $g(x + l, x' + l) = g(x, x') + l + l'$. We pick a point $e \in X$ such that $\pi(e) = 0$. Then by the diagram above we see that $\pi(g(e, e)) = 0$, hence multiply g by an element of L to get $g(e, e) = e$. We now claim that (X, g, e) is an algebraic group, which is an extension of B by L.

From the commuting diagram, we see that g respects elements of L, i.e. $g(x + a, y + b) = g(x, y) + a + b$ for $a, b \in L$. This shows that if g defines a group law, then L is indeed an algebraic subgroup and will imply that π makes B into a quotient of G by L.

First we show that $g(x, e) = x$ for all $x \in G$. Note that $\pi(g(x, e)) = \pi(x)$, so the map $x \mapsto g(x, e)$ is given by $x \mapsto x + h(x)$ for $h : X \to L$ regular. Note that for $b \in L$ that $x + b + h(x + b) = g(x + b, e) = g(x, e) + b = b + x + h(x)$, so h is constant on L. Therefore h factors through $X/L = B$, which means that h is constant. As $g(e, e) = e$ we conclude that $h(x) = e$ for all x.

Now we show that $g(x, y) = g(y, x)$ for all $x, y \in X$. Note that $\pi(g(x, y)) = \pi(x) + \pi(y)$, hence the map $(x, y) \mapsto g(x, y)$ is given by $(x, y) \mapsto g(y, x) + \alpha(x, y)$ for $\alpha : X \times X \to L$ regular. For $(l, l') \in L \times L$ we have $g(x + l, y + l') = g(x, y) + l + l'$ and $g(y + l', x + l) = g(y, x) + l + l'$, hence we see that α is invariant under the action of $L \times L$. Thus α factors through $B \times B$ giving the result.

Now we show that $g(g((x, y), z) = g(x, g(y, z))$. By doing a similar computation we get that $g(g(x, y), z) = g(x, g(y, z)) + k(x, y, z)$ for $k(x, y, z) : X^3 \to L$ a regular map. By the same tricks as above this map is constant giving that it factors through B^3, hence it is constant, thus we obtain $g(g(x, y), z) = g(x, g(y, z))$.

Now we show that there is an inversion map. For ν_L the inversion map on L we have the principal L-bundle $(\nu_L)_*[X]$, which equals $-[X]$. On the other hand we have that as $[X]$ is primitive that for ν_B the inversion map on B we have $\nu_A^*[X] + [X] = (\mathrm{Id} + \nu_A)^*[X] = [0]^*[X]$, which is trivial, hence $\nu_A^*[X] = -[X] = (\nu_L)_*[X]$. This gives the following commuting diagram:

$$\begin{array}{ccc} X & \longrightarrow & B \\ \uparrow & & \nu_B \uparrow \\ \nu_B^*[X] & \longrightarrow & B \\ \uparrow & & \mathrm{Id} \uparrow \\ X & \longrightarrow & B \end{array}$$

Denote the vertical map $X \to X$ in the diagram by ν. As $X \to \nu_B^*[X]$ is the map $X \to (\nu_L)_*[X]$ we see that it maps $\nu(x + l) = \nu(x) - l$ for $l \in L$. We have $\pi(\nu(e)) = \nu_B(\pi(e)) = 0$, hence by multiplying ν by an element of L we may assume that $\nu(e) = e$. We need to check that $g(\nu(x), x)) = e$ for any $x \in X$. Note that $\pi(g(x, \nu(x))) = 0$, hence $g(x, \nu(x) = e + k(x)$ for $k : X \to L$ a regular map. Note that $g(x + l, \nu(x + l)) = g(x + l, \nu(x) - l) = g(x, \nu(x))$, hence k is invariant under L. Thus k factors via B, hence it is constant. □

There are two cases in which the proposition applies that we are specifically interested in, namely either L being a connected linear algebraic group and B an abelian variety or L a finite group scheme and B a connected group variety.

4.5 Extensions of abelian varieties by the multiplicative group

In this section we set $B = A$ to be an abelian variety and we set $L = \mathbb{G}_m$. As they satisfy the conditions of Proposition 4.40, there is a natural injection $\mathrm{Ext}(\mathbb{G}_m, A) \to \mathrm{H}^1_{\mathrm{zar}}(A, \mathbb{G}_m) \cong \mathrm{Pic}(A)$, the Picard group of A and the image equals the primitive elements. So the image consists of the classes of line bundles $[\mathcal{L}]$ satisfying $\mu_A^*(\mathcal{L}) \cong p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L})$.

Definition 4.41. For A an abelian variety and $l_a : A \to A$ the translation map by $a \in A(k)$ define $\mathrm{Pic}^0(A) := \{[\mathcal{L}] \in \mathrm{Pic}(A) \text{ s.t. for all } a \in A(k),\ l_a^*\mathcal{L} \cong \mathcal{L}\}$.

This is a subgroup of $(\mathrm{Pic}(X), \otimes)$ as pullbacks respect the tensor product. We state the following theorem, which we will assume. A proof can be found in ([34], p.123).

Theorem 4.42. *There exists an abelian variety $A^\vee$, called the* ***dual abelian variety*** *of A, such that for any field $k \subset k'$ we have $A^\vee(k') = \mathrm{Pic}^0(A_{k'})$ as abelian groups. Moreover, $A^\vee$ has the same dimension as A.*

The following proposition relates the primitive elements to this.

Proposition 4.43. *The primitive elements in* $\mathrm{Pic}(A)$ *are precisely* $\mathrm{Pic}^0(A)$.

Proof. Let $[\mathcal{L}]$ be a primitive element and consider $(\mu_A(\mathcal{L}) \otimes p_1^*(\mathcal{L}^\vee))|_{\{x\}\times A}$ for some $x \in A(\bar{k})$. After identifying $\{x\} \times A$ with A in the obvious way, we note that as an invertible sheaf on A this is given by $l_x^*(\mathcal{L}) \otimes \mathcal{O}_A$ as the map $(x, a) \mapsto x$ is constant, hence $p_1^*(\mathcal{L}^\vee)|_{\{x\}\times A}$ is trivial. On the other hand, as $[\mathcal{L}]$ is primitive, $(\mu_A(\mathcal{L}) \otimes p_1^*(\mathcal{L}^\vee))|_{\{x\}\times A}$ is isomorphic to $p_2^*(\mathcal{L})|_{\{x\}\times A}$, which is given

by $\mathcal{L}$ as $a \mapsto (x, a) \mapsto a$ is simply the identity map on A. Hence we see that the primitive elements are in $\mathrm{Pic}^0(A)$.
Now given an element $[\mathcal{L}] \in \mathrm{Pic}^0(A)$, then we want to show that $\mu_A^*(\mathcal{L}) \cong p_1^*(\mathcal{L}) \otimes p_2^*(\mathcal{L})$. By similar reasoning as before we have for all $a \in A(\bar{k})$ that $(\mu_A^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee)|_{A\times\{a\}} = l_a^*\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_A$ and we have that $(p_2^*\mathcal{L})|_{A\times\{a\}} = \mathcal{O}_A$. Moreover we have that $(\mu_A^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee)|_{\{0\}\times A} \cong \mathcal{L} \otimes \mathcal{O}_A$ and $(p_2^*\mathcal{L})|_{\{0\}\times A} = \mathcal{L}$, so $(p_2^*\mathcal{L})|_{\{0\}\times A}$ and $(\mu_A^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee)|_{\{0\}\times A}$ are isomorphic. This puts us into a position to use the Seesaw Lemma ([30], Corollary 5.18), which states that if $\mathcal{M}$ is a line bundle on $A \times A$ that is trivial when restricted to all $\{x\} \times A$ and trivial when restricted to $A \times \{y\}$ for one particular $y \in A(\bar{k})$, then $\mathcal{M}$ is the trivial bundle. This implies that $\mu_A^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee \cong p_2^*\mathcal{L}$, hence $[\mathcal{L}]$ is a primitive element. □

4.6 A proof of Arima's theorems

Using the machinery from the Ext functor as well as its comparison with principle bundles we prove the two theorems of Arima in this section. We begin with Theorem 4.1, which states that an isogeny $G = A \cdot G_{\mathrm{lin}}$ for A an abelian subvariety of G if and only if $[G] \in \mathrm{Ext}(G_{\mathrm{ab}}, G_{\mathrm{lin}})$ is of finite order.

Proof. (of Theorem 4.1)
Let $[G]$ be of finite order d. Then note that since $\mathrm{Ext}(-, G_{\mathrm{lin}})$ is an additive functor that we have $[d]^*([G]) = d \cdot [1]^*([G]) = d \cdot [G]$ is trivial. So we get a commuting pullback square:

$$\begin{array}{ccc} G & \longrightarrow & G_{\mathrm{ab}} \\ \uparrow & & \uparrow{\scriptstyle [d]} \\ G_{\mathrm{lin}} \times G_{\mathrm{ab}} & \longrightarrow & G_{\mathrm{ab}} \end{array}$$

So as $[d] : G_{\mathrm{ab}} \to G_{\mathrm{ab}}$ is an isogeny, it is surjective and since surjectivity is stable under base change we obtain that the homomorphism $G_{\mathrm{lin}} \times G_{\mathrm{ab}} \to G$ is surjective, therefore it is a quotient map. The dimension of the kernel is $\dim(G) - \dim(G_{\mathrm{lin}}) - \dim(G_{\mathrm{ab}}) = 0$, so the kernel is finite and an isogeny $G_{\mathrm{lin}} \times_k G_{\mathrm{ab}} \to G$ exists.
Conversely if the isogeny $G_{\mathrm{lin}} \times_k G_{\mathrm{ab}} \to G$ exists then $G = G_{\mathrm{lin}} \cdot A$ for A an abelian subvariety of G by Lemma 4.3. The map $\pi|_A : A \to G_{\mathrm{ab}}$ is surjective. We have $\ker(\pi|_A) = A \cap G_{\mathrm{lin}}$ which is affine and complete and thus finite, so $\pi|_A$ is an isogeny. Let the degree be d. Then there is an isogeny $\nu : G_{\mathrm{ab}} \to A$ such that the composition $G_{\mathrm{ab}} \to G_{\mathrm{ab}}$ is $[d]$. We consider $(\pi|_A)^*([G])$. This is trivial as $A \to G \times_{G_{\mathrm{ab}}} A, \quad a \mapsto (a, a)$ is a homomorphism section. By functoriality of Ext and the fact that the pullback maps are homomorphisms we have that $[d]^*([G]) = \nu^*(\pi|_A)^*([G])$ is trivial. Again using the additivity of Ext we get $[d]^*([G]) = d \cdot [G] = 0$, hence $[G]$ is of finite order. □

We can use the above theorem to prove the second theorem of Arima, which is Theorem 4.2. This theorem states that when $k = \overline{\mathbb{F}}_p$ there exists an isogeny $G_{\mathrm{lin}} \times_{\overline{\mathbb{F}}_p} G_{\mathrm{ab}} \to G$.

Proof. (of Theorem 4.2)
We have seen that for this it suffices to prove the theorem for commutative divisible group varieties over $\overline{\mathbb{F}}_p$ in Lemma 4.4. By the previous theorem of Arima we have that $G \simeq G_{\mathrm{lin}} \times G_{\mathrm{ab}}$ is equivalent

to $[G] \in \mathrm{Ext}(G_{\mathrm{ab}}, G_{\mathrm{lin}})$ being of finite order. Over an algebraically closed field, which is perfect we have that as G_{lin} is commutative that $G_{\mathrm{lin}} \cong T \times_k U$ (Theorem 16.13 (b) [32]) where U is unipotent and $T \cong (\mathbb{G}_m)^n$ as the field is algebraically closed. As the field has characteristic p and $\cdot p : G \to G$ is an isogeny as G is divisible, the unipotent group U is trivial, which follows by it admits a composition series with succesive quotients all being $\mathbb{G}_a$.

So the extension group in question is $\mathrm{Ext}(G_{\mathrm{ab}}, \mathbb{G}_m^n) \cong \bigoplus_{i-1}^{n} \mathrm{Ext}(G_{\mathrm{ab}}, \mathbb{G}_m)$ as $\mathrm{Ext}(G_{\mathrm{ab}}, -)$ is an additive functor. So it suffices to show that any element in $\mathrm{Ext}(G_{\mathrm{ab}}, \mathbb{G}_m)$ has finite order. By Lemma 4.43 this is isomorphic to $A(\overline{\mathbb{F}}_p)$, where we denote $A = G_{\mathrm{ab}}^{\vee}$. Now the theorem follows since any point $P \in A(\overline{\mathbb{F}}_p)$ is of finite order. This is true because we can find a finite field $\mathbb{F}_q$ such that A is defined over $\overline{\mathbb{F}}_p$. Then after possibly extending the base field by a finite extension we may assume that P is also defined over $\overline{\mathbb{F}}_p$. So P is an $\mathbb{F}_q$ point on an abelian variety $A_{\mathbb{F}_q}$ over $\mathbb{F}_q$ such that A is the base-change of $A_{\mathbb{F}_q}$ to $\overline{\mathbb{F}}_p$. As $A_{\mathbb{F}_q}$ is a projective variety there are only finitely points in $A_{\mathbb{F}_q}(\mathbb{F}_q)$ and hence P is of finite order in $A_{\mathbb{F}_q}$. The multiplication map on A is induced by the one on $A_{\mathbb{F}_q}$ and hence $P \in A$ is of finite order. □

4.7 Extensions of abelian varieties by unipotent groups

If we do not assume that the characteristic of our base field k is $p \neq 0$, then we can not conclude as in the proof above that as G is divisible and commutative that G_{lin} is a torus. As in the previous proof we have $G_{\mathrm{lin}} = (\mathbb{G}_m)^n \times U$, where U is a unipotent algebraic group. We want to describe $\mathrm{Ext}(A, U)$ from which we have an injection $\mathrm{Ext}(A, U) \to \mathrm{H}^1_{\mathrm{zar}}(A, \underline{U})$. We begin with the case $U = \mathbb{G}_a$.

Proposition 4.44. *([39], p. 185) The homomorphism* $\mathrm{Ext}(A, \mathbb{G}_a) \to \mathrm{H}^1_{\mathrm{zar}}(A, \underline{\mathbb{G}_a})$ *is an isomorphism.*

Proof. Recall that the image of $\mathrm{Ext}(A, \mathbb{G}_a)$ are the primitive elements of $\mathrm{H}^1_{\mathrm{zar}}(A, \underline{\mathbb{G}_a})$. Recall that there is an injection $\mathrm{H}^1_{\mathrm{zar}}(A, \underline{\mathbb{G}_a}) \times \mathrm{H}^1_{\mathrm{zar}}(A, \underline{\mathbb{G}_a}) \stackrel{(p_1^*+p_2^*)}{\to} \mathrm{H}^1_{\mathrm{zar}}(A \times A, \underline{\mathbb{G}_a})$ for which we have that an element of the form $\mu_A^*(x)$ has $\mathrm{H}^1_{\mathrm{zar}}(A, \underline{\mathbb{G}_a}) \times \mathrm{H}^1_{\mathrm{zar}}(A, \underline{\mathbb{G}_a})$-part equal to (x, x). So if $p_1^* + p_2^*$ is an isomorphism then we are done. As the sheaf $\underline{\mathbb{G}_a}$ is the structure sheaf, we have that the Künneth-formula for quasi-coherent sheafs applies [43, Tag 0BEC], hence this map is indeed an isomorphism. □

Now we want to treat the general unipotent case.

Proposition 4.45. *The injection* $\mathrm{Ext}(A, U) \to \mathrm{H}^1_{\mathrm{zar}}(A, \underline{U})$ *is an isomorphism.*

Proof. We will show this by induction on $\dim(U)$. The case $\dim(U) = 1$ has been treated above. For the general case: We have a composition series $0 = U_0 \leq ... \leq U_n = U$ where $U_i/U_{i-1} \cong \mathbb{G}_a$. The isomorphism $U/U_n \cong \mathbb{G}_a$ gives an exact sequence of flat sheaves on A: $0 \to \underline{U_n} \to \underline{U} \to \underline{\mathbb{G}_a} \to 0$. Hence we get an exact sequence in flat cohomology:

$$\mathrm{H}^0_{\mathrm{fl}}(A, \underline{U_n}) \to \mathrm{H}^0_{\mathrm{fl}}(A, \underline{U}) \to \mathrm{H}^0_{\mathrm{fl}}(A, \underline{\mathbb{G}_a}) \to \mathrm{H}^1_{\mathrm{fl}}(A, \underline{U_n}) \to \mathrm{H}^1_{\mathrm{fl}}(A, \underline{U}) \to \mathrm{H}^1_{\mathrm{fl}}(A, \underline{\mathbb{G}_a})$$

For any commutative group variety L there is an isomorphism $\mathrm{H}^1_{\text{fl}}(A, \underline{L}) \to \mathrm{PB}_{A,L}$. As L is linear in this case we have $\mathrm{PB}_{A,L} = \mathrm{PB}^{\text{zar}}_{A,L}$ by Proposition 4.32 and we have $\mathrm{PB}^{\text{zar}}_{A,L} = \mathrm{H}^1_{\text{zar}}(A, \underline{L})$. So the above exact sequence reads:

$$\mathrm{H}^0_{\text{zar}}(A, \underline{U_n}) \to \mathrm{H}^0_{\text{zar}}(A, \underline{U}) \to \mathrm{H}^0_{\text{zar}}(A, \underline{\mathbb{G}_a}) \to \mathrm{H}^1_{\text{zar}}(A, \underline{U_n}) \to \mathrm{H}^1_{\text{zar}}(A, \underline{U}) \to \mathrm{H}^1_{\text{zar}}(A, \underline{\mathbb{G}_a})$$

We have $U \cong \mathbb{A}^n$ as varieties because the field k is perfect, we note that there is no non-constant map $A \to U$ as A is complete and connected, so we have that the map $\mathrm{H}^0_{\text{zar}}(A, \underline{U}) \to \mathrm{H}^0_{\text{zar}}(A, \underline{\mathbb{G}_a})$ is given by $U(k) \to \mathbb{G}_a(k)$, which is onto. So the sequence $0 \to \mathrm{H}^1_{\text{zar}}(A, \underline{U_n}) \to \mathrm{H}^1_{\text{zar}}(A, \underline{U}) \to \mathrm{H}^1_{\text{zar}}(A, \underline{\mathbb{G}_a})$ is exact. As the maps on the cohomology defined by the morphisms of sheaves commute with pullbacks, we can do the following: Recall that by Remark 4.36 L a commutative algebraic group the map $(p_1^*, p_2^*) : \mathrm{H}^1_{\text{zar}}(A, \underline{L}) \times \mathrm{H}^1_{\text{zar}}(A, \underline{L})$ had a right inverse m^* defined by the injections $\iota_i : A \to A \times A$. Thus we get the following large commuting diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^1_{\text{zar}}(A \times A, \underline{U_n}) & \longrightarrow & \mathrm{H}^1_{\text{zar}}(A \times A, \underline{U}) & \longrightarrow & \mathrm{H}^1_{\text{zar}}(A \times A, \underline{\mathbb{G}_a}) \\
 & & \downarrow{\scriptstyle m^*} & & \downarrow{\scriptstyle m^*} & & \downarrow{\scriptstyle m^*} \\
0 & \longrightarrow & \mathrm{H}^1_{\text{zar}}(A, \underline{U_n}) \times \mathrm{H}^1_{\text{zar}}(A, \underline{U_n}) & \longrightarrow & \mathrm{H}^1_{\text{zar}}(A, \underline{U}) \times \mathrm{H}^1_{\text{zar}}(A, \underline{U}) & \longrightarrow & \mathrm{H}^1_{\text{zar}}(A \times A, \underline{\mathbb{G}_a})
\end{array}$$

We had assumed that $(p_1^* + p_2^*)$ was an isomorphism for U_n and $\mathbb{G}_a$, so its right inverse m^* is also an isomorphism for U_n and $\mathbb{G}_a$. By a variant of the four-lemma we conclude that the middle m^* is an injection, hence it is an isomorphism, the inverse of $(p_1^* + p_2^*)$. □

Remark 4.46. The above proposition implies that for L a linear group variety and A an abelian variety one has $\mathrm{Ext}(A, L) = \mathrm{Ext}(A, T \times_k U) = \mathrm{Ext}(A, T) \times \mathrm{Ext}(A, U) = \mathrm{Ext}(A, T) \times \mathrm{H}^1_{\text{zar}}(A, \underline{U})$. In particular we see that if the field has characteristic $p \neq 0$, then $\mathrm{H}^1_{\text{zar}}(A, \underline{U})$ is p-torsion as this is a k-vector space. Thus we see that in this case whether an element of $\mathrm{Ext}(A, L)$ has infinite order or not depends only on the $\mathrm{Ext}(A, T)$ part (making it logical that we could reduce to L being a torus in the proof of Theorem 4.2).

On the other hand if $\mathrm{char}(k) = 0$, then we have the following:

Corollary 4.47. *For* $\mathrm{char}(k) = 0$ *a commutative group variety G is isogenous to $G_{\text{lin}} \times G_{\text{ab}}$ if and only if $e \to U \to G/T \to G_{\text{ab}} \to e$ is split and $[G/U]$ has finite order in* $\mathrm{Ext}(G_{\text{ab}}, T)$.

Proof. We have $\mathrm{Ext}(G_{\text{ab}}, G_{\text{lin}}) = \mathrm{Ext}(G_{\text{ab}}, T) \times \mathrm{Ext}(G_{\text{ab}}, U)$. In order for G to be isogenous to $G_{\text{lin}} \times G_{\text{ab}}$ we need that $[G]$ is of finite order. By what is written above the corollary, $[G]$ is of finite order if and only if $[G/U]$ is of finite order and $[G/T]$ is of finite order and the second happens if and only if it is trivial as it is an element of a k-vector space, where $\mathrm{char}(k) = 0$. □

4.8 Examples of when G is not isogenous to $G_{\text{lin}} \times_k G_{\text{ab}}$

In this subsection we wish to give an explicit example of a case in which G and $G_{\text{lin}} \times_k G_{\text{ab}}$ are not isogenous. An excellent reference for this subsection is Chapter 5 from [39]. For simplicity we assume that we are working over some algebraically closed field k, so by points on a variety X we will mean elements of $X(k)$. For the example we will take $A = E$ an elliptic curve. It turns out that for an elliptic curve, $E^\vee$ is quite easy to describe.

Proposition 4.48. *An elliptic curve E is isomorphic to its own dual $E^\vee$. The isomorphism is given by $E \to E^\vee \quad P \mapsto \mathcal{O}([P]-[O])$.*

A proof can be found in ([40], p.61). Note that as E is smooth, we can also identify $E^\vee$ with the group of degree 0 divisors modulo linear equivalence. We denote the linear equivalence relation by $\sim$.

Definition 4.49. Let $\mathfrak{m}$ be an effective divisor on a smooth projective curve X. We then define the following equivalence relation on $\mathrm{Div}(X)$: We have $D \sim_{\mathfrak{m}} E$ if and only if $D - E = (g)$ where $g \in K(X)^*$ such that $\nu_P(g-1) \geq \mathfrak{m}(P)$ for all $P \in \mathrm{Supp}(\mathfrak{m})$.

So in particular we see that $\sim_{\mathfrak{m}}$ is a stronger equivalence relation than linear equivalence, $\sim$. In the case $\mathfrak{m} = 0$ we recover $\sim$.

Definition 4.50. For X a smooth projective curve, the **divisor class group** for $\mathfrak{m}$-equivalence is given by $\mathrm{Div}(X \setminus S)/\sim_{\mathfrak{m}}$. We denote it by $\mathrm{Cl}(X)_{\mathfrak{m}}$.

Note that in the case that $\mathfrak{m} = 0$ we recover the divisor class group of X.

Remark 4.51. Note that for any $[D] \in \mathrm{Cl}(X)$ and any effective divisor $\mathfrak{m}$ we may take a representative $D \in \mathrm{Div}(X)$ such that $\mathrm{Supp}(D) \cap \mathfrak{m} = \emptyset$. This relies on the fact that we may pick a uniformizer at P that is nonvanishing at given points $Q_1, .., Q_m$. This holds as we may pick an affine open $\mathrm{Spec}(R)$ containing all these points. Using the chinese remainder theorem on the map $R \to \bigoplus_{i=1}^{m} R/\mathfrak{m}_i \oplus R/\mathfrak{m}_P^2$ and quotienting in the second factor to $R_{\mathfrak{m}_P}/\mathfrak{m}_P^2$ gives that there exists an element t in R that has $\nu_P(t) = 1$ and $\nu_{Q_i}(t) = 0$ for all i. In partcicular this shows that we have a quotient map of groups $\mathrm{Cl}(X)_{\mathfrak{m}} \to \mathrm{Cl}(X)$. This map respects the degree of divisors, hence it factors through as a quotient map $\mathrm{Cl}^0(X)_{\mathfrak{m}} \to \mathrm{Cl}^0(X)$.

We now state the following theorem. It is proven in ([39], p.105).

Theorem 4.52. *Let X be a nonsingular curve. There exists a group variety $J_{\mathfrak{m}}$ such that as a group, $J_{\mathfrak{m}} = \mathrm{Cl}^0(X)_{\mathfrak{m}}$. Write J for J_0. After this identification, the map $J_{\mathfrak{m}} \to J$ induced by $\mathrm{Cl}^0(X)_{\mathfrak{m}} \to \mathrm{Cl}^0(X)$ is a homomorphism of algebraic groups.*

Then J is the *Jacobian* of the curve X and $J_{\mathfrak{m}}$ the *generalized Jacobian* of X.

Remark 4.53. As J is a quotient of X^g, which is complete, J is complete, hence an abelian variety. Serre ([39], p.96) shows that $L_{\mathfrak{m}} := \ker(J_{\mathfrak{m}} \to J)$ is a linear subgroup variety of $J_{\mathfrak{m}}$, hence we see that $0 \to L_{\mathfrak{m}} \to J_{\mathfrak{m}} \to J \to 0$ is the Chevalley sequence of $J_{\mathfrak{m}}$.

Now we set $X = E$ an elliptic curve and $\mathfrak{m} = P + Q$ for $P \neq Q$ and $P, Q \neq O$.

Proposition 4.54. *In the setting above we have* $\ker(J_{\mathfrak{m}} \to J) = \mathbb{G}_m$.

Proof. The kernel consists exactly of the divisor classes $(g)_{\mathfrak{m}}$ such that $P, Q \notin \mathrm{Supp}(g)$. This subgroup is isomorphic to $\dfrac{\{g \in K(X)^* \mid \nu_P(g) = \nu_P(g) = 0\}/k^*}{\{g \in K(X)^* \mid 0 \neq g(P) = g(Q)\}/k^*} \cong \dfrac{\mathcal{O}_{X,P}^* \cap \mathcal{O}_{X,Q}^*}{\{g \mid g(P) = g(Q)\}}$. Consider the homomorphism $\mathcal{O}^*_{X,P} \cap \mathcal{O}^*_{X,Q} \to k^*$ given by $g \mapsto \dfrac{g(P)}{g(Q)}$. It is surjective as we can find a uniformizer t_P at P such that $t_P(Q) = a - 1$ for any $a \neq 1$ giving that $g = t_p + a$ maps to a. Hence this homomorphism factors as an isomorphism $\dfrac{\mathcal{O}^*_{X,P} \cap \mathcal{O}^*_{X,Q}}{\{g \mid g(P) = g(Q)\}} \to k^*$. The inverse is given by $a \mapsto (g_a)_{\mathfrak{m}}$ where $g_a(P) = a$ and $g_a(Q) = 1$. It is shown in Theorem 3 of [39] that this map is a morphism, hence we conclude that $\ker(J_{\mathfrak{m}} \to J) = \mathbb{G}_m$. □

Now we have constructed an exact sequence of algebraic groups $0 \to \mathbb{G}_m \to J_{\mathfrak{m}} \to E \to 0$ (after identifying J with E). Note that as $\mathbb{G}_m$ is a linear group variety and J is an abelian variety (J is complete by Theorem 5.1 in III [30]) we have $(J_{\mathfrak{m}})_{\mathrm{lin}} = \mathbb{G}_m$ and $(J_{\mathfrak{m}})_{\mathrm{ab}} = J$. To determine whether $J_{\mathfrak{m}}$ is isogenous to $\mathbb{G}_m \times E$ we look at the class $[J_{\mathfrak{m}}] \in \mathrm{Ext}(E, \mathbb{G}_m)$.

Proposition 4.55 ([39], p.88)**.** *There is a rational section $s_{\mathfrak{m}} : E \dashrightarrow J_{\mathfrak{m}}$ given by $R \mapsto [R - O]_{\mathfrak{m}}$. It is regular on $E \setminus \{P, Q\}$.*

Note that it is indeed a section to $J_{\mathfrak{m}} \to E$. Rather than taking sections regular at P, Q, we first construct a map $\varphi_P : E \to \mathbb{G}_m$ and then show that the section $s_P := s_{\mathfrak{m}} - \varphi_P$ is regular at P (the same can be done for Q).

Definition 4.56. Define for $n \geq 1$ the variety $X^{(n)} := X^n / S_n$, where the symmetric group acts by permuting variables.

Notice that the k-points of $X^{(n)}$ may be identified with the effective degree n divisors of X.

Remark 4.57. The rational map $s_{\mathfrak{m}} : E \dashrightarrow J_{\mathfrak{m}}$ extends to one $s_{\mathfrak{m}} : E^{(n)} \dashrightarrow J_{\mathfrak{m}}$ by putting $s_{\mathfrak{m}}(P_1 + \dots + P_n) = s_{\mathfrak{m}}(P_1) + \dots + s_{\mathfrak{m}}(P_n)$. It is regular on the subset $\{D \in E^{(n)} \mid P, Q \notin \mathrm{Supp}(D)\}$.

We have the following lemma that will aid us in proving the next proposition.

Lemma 4.58. *(Lemma V.14, [39]) Let X be a smooth projective curve and let $f : X \to \mathbb{P}^1$ be a non-constant rational function of degree $d + 1$. For $\lambda \in \mathbb{P}^1$ define the divisor $H_\lambda := f^{-1}[\lambda]$. Then the map $X \to X^{(d)} \quad R \mapsto H_{f(R)} - R$ is regular.*

Recall that we have seen an isomorphism $\mathrm{Ext}(E, \mathbb{G}_m) \cong \mathrm{Pic}^0(E)$ by combining Propositions 4.43 and 4.40.

Proposition 4.59. *The element $[J_{\mathfrak{m}}] \in \mathrm{Ext}(E, \mathbb{G}_m)$ corressponds to $[P] - [Q] \in \mathrm{Pic}^0(E)$ under the isomorphism $\mathrm{Ext}(E, \mathbb{G}_m) \cong \mathrm{Pic}^0(E)$.*

Proof. The invertible sheaf that corresponds to the element $[J_{\mathfrak{m}}]$ is given by $\mathcal{O}(\mathrm{div}(s_{\mathfrak{m}}))$ as we have that $s_{\mathfrak{m}} : E \dashrightarrow J_{\mathfrak{m}}$ is a rational section (and hence the line bundle $J_{\mathfrak{m}} \times^{\mathbb{G}_m} \mathbb{A}^1 \to E$ also has $s_{\mathfrak{m}}$ as a rational section). Thus we need to compute the zeros and poles of $s_{\mathfrak{m}}$. Note that $s_{\mathfrak{m}}$ is nowhere vanishing on $E \setminus \{P, Q\}$ so it suffices to see what happens at P and Q (as being a regular section to the $\mathbb{G}_m$-bundle is equivalent to being a non-vanishing section on the induced line bundle). For this we follow p.190 of [39]. We let t_P be a uniformizer at P and t_Q a uniformizer at Q such that $t_Q(P) = t_P(Q) = 1$. We consider the partially defined function $\varphi_P : E \dashrightarrow J_{\mathfrak{m}} \quad R \mapsto (g_R)_{\mathfrak{m}}$ for $g_R := t_P - T_P(R)$. This is corresponds to a rational map into $\mathbb{G}_m \subset J_{\mathfrak{m}}$ given by $R \mapsto \dfrac{t_P(R)}{t_P(R) - 1}$

and hence $\nu_P(\varphi_P) = 1$.
We now consider the rational map $s_P := \varphi_P - s_{\mathfrak{m}} : E \dashrightarrow J_{\mathfrak{m}}$. If we can show that s_P is regular at P, then it will show that $s_{\mathfrak{m}} : E \dashrightarrow J_{\mathfrak{m}} \times^{\mathbb{G}_m} \mathbb{A}^1$ has $\nu_P(s_{\mathfrak{m}}) = \nu_P(\varphi_P) = 1$.
Write H_λ for the effective divisor $(t_P^{-1}(\lambda))$ for any $\lambda \in \mathbb{P}^1$. Notice that as a divisor we have $(g_R) = H_{t(R)} - H_\infty$ for all $R \in E$. We consider $H_{t(R)}$ as an element of $E^{(n+1)}$ for some $n \geq 0$ (the same n for all R). Then s_P is given by $R \mapsto s_{\mathfrak{m}}(R) - s_{\mathfrak{m}}(H_{t_P(R)}) + s_{\mathfrak{m}}(H_\infty)$ as we can extend $s_{\mathfrak{m}}$ to a rational map $E^{(n+1)} \dashrightarrow J_{\mathfrak{m}}$ as in Remark 4.57. Now notice that $H_{t_P(R)} = R + H'_R$ for $H'_R \in E^{(n)}$. By Lemma 4.58 the map $R \mapsto H'_R$ is a regular map $E \to E^{(n)}$. So our rational map s_P is given by $R \mapsto -s_{\mathfrak{m}}(H'_R) + s_{\mathfrak{m}}(H_\infty)$. We note that $s_{\mathfrak{m}}(H_\infty)$ is just a constant and does not influence whether s_P is regular at P. Thus it suffices to show that $R \mapsto s_{\mathfrak{m}}(H'_R)$ is regular at P. Note that $H'_P = H_{t(P)} - P$ and as P is a uniformizer at P, this divisor does not have P in the support. So $R \mapsto H'_R \mapsto s_{\mathfrak{m}}(H'_R)$ is regular at P by Remark 4.57.
We conclude that $s_P = s_{\mathfrak{m}} - \varphi_P$ is regular at P and hence $\nu_P(s_{\mathfrak{m}}) = \nu_P(\varphi_P) = 1$. Completely analogously it follows that $\nu_Q(s_{\mathfrak{m}}) = -1$ by doing the same steps but with $\varphi_Q : R \mapsto (t_Q - t_Q(R))_{\mathfrak{m}}$ and by showing that $s_Q := s_{\mathfrak{m}} - \varphi_Q$ is regular at Q. So we obtain that $\mathrm{div}(\varphi_{\mathfrak{m}}) = [P] - [Q]$ and hence the element corresponding to $[J_{\mathfrak{m}}]$ in $\mathrm{Pic}^0(E)$ is $[P] - [Q]$. □

Remark 4.60. This gives us many examples of extensions of E by $\mathbb{G}_m$ that do not split isogenously. A way to do this is to pick E an elliptic curve and $P \in E(k)$ of infinite order with $\mathfrak{m} = [P] + [-P]$. Then this leads to $\mathrm{Ext}(E, \mathbb{G}_m) \ni [J_{\mathfrak{m}}] \longleftrightarrow [2P - 2O] \in \mathrm{Pic}^0(E)$ implying that $J_{\mathfrak{m}}$ is not isogenous to $\mathbb{G}_m \times_k E$. One can do this construction over any algebraically closed field k that is not $\overline{\mathbb{F}}_p$ for any p since any abelian variety A over k of positive dimension has a point $x \in A(k)$ of infinite order (Theorem 10.1 [20]).

Chapter 5

The l-adic cohomology of group varieties

In this chapter we have $k = \bar{k}$ and we have $l \neq \mathrm{char}(k)$ prime. We will compute the l-adic cohomology of a group variety G in terms of the cohomology of G_{lin} and G_{ab}. The goal is showing that there is a natural isomorphism $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l) \cong \mathrm{H}^*_{\text{ét}}(G_{\mathrm{lin}}, \mathbb{Q}_l) \otimes \mathrm{H}^*_{\text{ét}}(G_{\mathrm{ab}}, \mathbb{Q}_l)$ of graded $\mathbb{Q}_l$-algebras. We begin with comparing Ext and cohomology after which we prove the statement in case of commutative G. After this we prove the general case. In the second section we show that there is a natural isomorphism $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l) \cong \mathrm{H}^*_{\text{ét}}(K, \mathbb{Q}_l) \otimes \mathrm{H}^*_{\text{ét}}(Q, \mathbb{Q}_l)$ whenever G is an extension of a group variety Q by a group variety K. We end the chapter with a section in which we calculate the l-adic cohomology of several semisimple group varieties.

First we consider the case of a principle G-bundle X over a scheme Y. It turns out that even in the following natural case we need not have $\mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_l) \cong \mathrm{H}^*_{\text{ét}}(Y, \mathbb{Q}_l) \otimes \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$.

Example 5.1. Consider $X = \mathbb{A}^2_k \backslash \{(0,0)\}$, which has $X(R)\{\varphi : k[x,y] \to R \mid \varphi^{-1}(\mathfrak{p}) \neq (x,y) \text{ for any } \mathfrak{p}\}$ as its R-poins. The condition $(x,y) \neq \varphi^{-1}(\mathfrak{p})$ for all $\mathfrak{p}$ is equivalent to $(\varphi(x)) + (\varphi(y)) = R$ and hence $X(R) = \{(r_1, r_2) \in R^2 \mid (r_1) + (r_2) = R\}$. Then $\mathbb{G}_m(R)$ acts freely on $X(R)$ by mapping $((x,y),\lambda) \mapsto (\lambda x, \lambda y)$. The flat presheaf defined by $R \mapsto \{(r_1, r_2) \in R^2 \mid (r_1) + (r_2) = R\}/R^\times$ is actually a flat sheaf as it is represented by $\mathbb{P}^1_k$. This implies that $\mathbb{P}^1$ is identified with the quotient $X/\mathbb{G}_m$ and that $X \to \mathbb{P}^1$ is a principle $\mathbb{G}_m$-bundle.
The codimension of $\{0\}$ in $\mathbb{A}^2$ is 2. By using the Gysin sequence (Proposition 2.55), we obtain:

$$\mathrm{H}^r_{\text{ét}}(X, \Lambda_n) = \begin{cases} \mathbb{Z}/l^n\mathbb{Z} \text{ if } r = 0, 3 \\ 0 \text{ else} \end{cases}$$

Thus $\mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_l) \neq \mathrm{H}^*_{\text{ét}}(\mathbb{P}^1, \mathbb{Q}_l) \otimes \mathrm{H}^*_{\text{ét}}(\mathbb{G}_m, \mathbb{Q}_l)$ as $\mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_l)$ has no degree 1 elements but by Example 2.50 $\mathrm{H}^*_{\text{ét}}(\mathbb{G}_m, \mathbb{Q}_l) \otimes \mathrm{H}^*_{\text{ét}}(\mathbb{P}^1, \mathbb{Q}_l)$ has elements in degree 1.

We now make an easy observation given the first two chapters.

Lemma 5.2. *Let G be a group variety over k. Then $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ is a finite dimensional graded-commutative $\mathbb{Q}_l$-Hopf algebra.*

We define $\mu^* : \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G \times G, \mathbb{Q}_l) \overset{\text{Künneth}}{\to} \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ to be the comultiplication. We set inv^* to be the antipode and we set ϵ to be $\mathbb{Q}_l \cong \mathrm{H}^0_{\text{ét}}(\mathrm{Spec}(k), \mathbb{Q}_l) \overset{e^*}{\to} \mathrm{H}^0_{\text{ét}}(G, \mathbb{Q}_l)$. One checks that by functoriality of étale cohomology all the diagrams in Definition 1.33 commute. The graded-commutativity comes from the cup-product and the finite dimensionality from Theorems 2.48 and 2.49.

5.1 Comparison of $\mathrm{H}^1_{\text{ét}}(G, \underline{\mathbb{Z}/l^n\mathbb{Z}})$ and $\mathrm{Ext}(G, \mathbb{Z}/l^n\mathbb{Z})$

In this section we let G be a commutative group variety. Consider $\mathbb{Z}/l^n\mathbb{Z} := \coprod_{a \in \mathbb{Z}/l^n\mathbb{Z}} \mathrm{Spec}(k)$, which is an étale algebraic group denoted the same as the abstract group $\mathbb{Z}/l^n\mathbb{Z}$. By Proposition 4.34 we get a homomorphism $\mathrm{Ext}(G, \mathbb{Z}/l^n\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(G, \underline{\mathbb{Z}/l^n\mathbb{Z}})$ that is functorial in G.

Lemma 5.3. *Let G be a commutative group variety. Then $\mathrm{Ext}(G, \mathbb{Z}/l^n\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(G, \underline{\mathbb{Z}/l^n\mathbb{Z}})$ is injective. The image of this homomorphism is exactly the primitive elements of $\mathrm{H}^1_{\text{ét}}(G, \underline{\mathbb{Z}/l^n\mathbb{Z}})$.*

Proof. The first claim is Lemma 4.35. The second claim follows from Proposition 4.40. □

Proposition 5.4. *([39], p.196) The sequence $0 \to \mathrm{Ext}(G_{\text{ab}}, \Lambda) \to \mathrm{Ext}(G, \Lambda) \to \mathrm{Ext}(G_{\text{lin}}, \Lambda) \to 0$ is exact for G a commutative group variety and Λ a finite algebraic group.*

Proof. There is an exact sequence for Λ any algebraic group stated in ([39], p.165), which comes from applying $\mathrm{Hom}(-, \Lambda)$ to the Chevalley sequence:

$$0 \to \mathrm{Hom}(G_{\text{ab}}, \Lambda) \to \mathrm{Hom}(G, \Lambda) \to \mathrm{Hom}(G_{\text{lin}}, \Lambda) \overset{d}{\to} \mathrm{Ext}(G_{\text{ab}}, \Lambda) \to \mathrm{Ext}(G, \Lambda) \to \mathrm{Ext}(G_{\text{lin}}, \Lambda)$$

As Λ is finite in our case we have $\mathrm{Hom}(G_{\text{lin}}, \Lambda) = 0$, so it suffices to show that the last arrow is surjective. We need to show that any isogeny $\varphi : H \to G_{\text{lin}}$ with kernel Λ extends to an isogeny $\varphi' : H' \to G$ with kernel Λ such that φ is the pullback morphism of φ'. Note that H is a linear algebraic group. By ([39] p.195) we have that for A an abelian variety that $\mathrm{Ext}(A, -)$ is an exact functor when restricted to the subcategory of commutative linear algebraic groups. So we can find some $[H'] \in \mathrm{Ext}(H, A)$ such that $\varphi_*([H']) = [G]$. This gives the following commuting diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & H' & \longrightarrow & G_{\text{ab}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} & & \downarrow & & \\
0 & \longrightarrow & G_{\text{lin}} & \overset{\iota}{\longrightarrow} & G & \longrightarrow & G_{\text{ab}} & \longrightarrow & 0
\end{array}$$

Upon applying the Snake Lemma we find that $H' \to G$ is an isogeny with kernel Λ. Any morphisms from a scheme to H' and G_{lin} that agree in G gives rise to a unique morphism through H by how G is constructed, so the left square is a pullback square. Hence we conclude that $\iota_*([H']) = [H]$ and so $\mathrm{Ext}(G, \Lambda) \to \mathrm{Ext}(G_{\text{lin}}, \Lambda)$ is surjective. □

The following lemma allows us to pass to take coefficients in $\mathbb{Q}_l$. Write $\underline{\mathbb{Z}/l^n\mathbb{Z}} = \Lambda_n$ for the constant sheaf. Recall the notation PH for the primitive elements in H.

Lemma 5.5. *For G an algebraic group we have $\mathrm{P}\varprojlim_k \mathrm{H}^1_{\text{ét}}(G,\Lambda_n) = \varprojlim_n \mathrm{PH}^1_{\text{ét}}(G,\Lambda_n)$.*
Moreover we have $(\mathrm{P}\varprojlim_n \mathrm{H}^1_{\text{ét}}(G,\Lambda_n)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l = \mathrm{P}(\varprojlim_n \mathrm{H}^1_{\text{ét}}(G,\Lambda_n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$.

Proof. By definition the pullback maps μ^*, p_1^*, p_2^* work on $\varprojlim_n \mathrm{H}^1_{\text{ét}}(G_{\text{ab}},\Lambda_n)$ componentwise. Hence we have $\mu^*((x_n)_n) = (\mu^* x_n)_n$ and $p_1^*(x_n)_k + p_2^*(x_n)_k = (p_1^* x_n + p_2^* x_n)_n$. Thus $(x_n)_n$ is primitive if and only if all components x_n are primitive, which takes care of the first claim.
We have that $\mathrm{H}^1_{\text{ét}}(G,\mathbb{Q}_l) = \mathrm{H}^1_{\text{ét}}(G,\mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is the localisation at the prime (l) of $\mathbb{Z}_l$. The pullbacks μ^*, p_1^*, p_2^* on $\mathrm{H}^1_{\text{ét}}(G,\mathbb{Q}_l)$ are those induced by localisation. We have that $\frac{a}{b}$ is primitive if and only if $\frac{\mu^*(a) - p_1^*(a) - p_2^*(a)}{b} = 0$. This vanishes if and only if $\mu^*(a) - p_1^*(a) - p_2^*(a)$ is l^n-torsion for some n. So then we have $\mu^*(a \cdot l^n) - p_1^*(a \cdot l^k) - p_2^*(a \cdot l^n) = 0$, hence $a \cdot l^n \in \mathrm{PH}^1_{\text{ét}}(G,\mathbb{Z}_l)$. Thus we obtain that $\frac{a}{b} = \frac{al^n}{bl^n}$ showing that $\mathrm{PH}^1_{\text{ét}}(G,\mathbb{Q}_l) \subset (\mathrm{PH}^1_{\text{ét}}(G,\mathbb{Z}_l)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. The other inclusion follows trivially. □

We want a certain condition for $\varprojlim_n(-)$ to preserve exactness.

Definition 5.6. An inverse system satisfies the **Mittag-Leffler condition** if for all i the descending chain $\mathrm{Im}(M_k \to M_i)_k$ stabilizes. We will refer to it as the **ML-condition**.

Remark 5.7. By [43, Tag 02MY] we have that for inverse systems $(A_i)_i$, $(B_i)_i$, $(C_i)_i$ and exact sequences $0 \to A_i \to B_i \to C_i \to 0$ that are compatible with the inverse systems that the sequence $0 \to \varprojlim_i A_i \to \varprojlim_i B_i \to \varprojlim_i C_i \to 0$ is exact if it satisfies the ML-condition. Note that if each A_i is a module of finite length, then the condition is automatically satsified. So by finiteness of étale cohomology (Theorem 2.49) we have that in all the relevant situations the ML-condition will be satisfied.

Lemma 5.8. *The sequence $0 \to \varprojlim_n \mathrm{PH}^1_{\text{ét}}(G_{\text{ab}},\Lambda_n) \to \varprojlim_n \mathrm{PH}^1_{\text{ét}}(G,\Lambda_n) \to \varprojlim_n \mathrm{PH}^1_{\text{ét}}(G_{\text{lin}},\Lambda_n) \to 0$ is exact for G a commutative group variety.*

Proof. It suffices to show that the inverse system $(\mathrm{PH}^1_{\text{ét}}(G_{\text{ab}},\Lambda_k))_k$ satisfies the ML condition. As each $M_k := \mathrm{PH}^1_{\text{ét}}(G_{\text{ab}},\Lambda_k)$ is a Λ_k-module of finite length and the sequence $\mathrm{Im}(M_i \to M_k)_i$ is a decscending chain of Λ_k-submodules this chain stabilizes, so combining this with the previous lemma gives the exact sequence. □

Proposition 5.9. *For G a commutative algebraic group the canonical sequence of $\mathbb{Q}_l$-vectorspaces $0 \to \mathrm{H}^1_{\text{ét}}(G_{\text{ab}},\mathbb{Q}_l) \to \mathrm{H}^1_{\text{ét}}(G,\mathbb{Q}_l) \to \mathrm{H}^1_{\text{ét}}(G_{\text{lin}},\mathbb{Q}_l) \to 0$ is exact.*

Proof. As $\mathbb{Q}_l$ is a flat $\mathbb{Z}_l$-module, we can take the exact sequence of the previous lemma to obtain an exact sequence:

$$0 \to \varprojlim_n \mathrm{PH}^1_{\text{ét}}(G_{\text{ab}},\Lambda_n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \to \varprojlim_n \mathrm{PH}^1_{\text{ét}}(G,\Lambda_n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \to \varprojlim_n \mathrm{PH}^1_{\text{ét}}(G_{\text{lin}},\Lambda_n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \to 0$$

As taking primitive elements commutes with inverse limits and $- \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ we get the exact sequence:

$$0 \to \mathrm{PH}^1_{\text{ét}}(G_{\text{ab}},\mathbb{Q}_l) \to \mathrm{PH}^1_{\text{ét}}(G,\mathbb{Q}_l) \to \mathrm{PH}^1_{\text{ét}}(G_{\text{lin}},\mathbb{Q}_l) \to 0$$

We have $\mathrm{H}^1_{\text{ét}}(B \times B, \mathbb{Q}_l) \cong \mathrm{H}^0_{\text{ét}}(B, \mathbb{Q}_l) \otimes \mathrm{H}^1_{\text{ét}}(B, \mathbb{Q}_l) \bigoplus \mathrm{H}^1_{\text{ét}}(B, \mathbb{Q}_l) \otimes \mathrm{H}^0_{\text{ét}}(B, \mathbb{Q}_l)$ for B any group variety by the Künneth formula induced from right to left by $(p_1^* \cup p_2^*)$. Since B is connected we have $\mathrm{H}^1_{\text{ét}}(B \times B, \mathbb{Q}_l) = p_1^* \mathrm{H}^1_{\text{ét}}(B, \mathbb{Q}_l) \bigoplus p_2^* \mathrm{H}^1_{\text{ét}}(B, \mathbb{Q}_l)$. Thus we see that any element of $\mathrm{H}^1_{\text{ét}}(B \times B, \mathbb{Q}_l)$ is primitive by Remark 4.36. The exact sequence of primitive elements then reads:

$$0 \to \mathrm{H}^1_{\text{ét}}(G_{\text{ab}}, \mathbb{Q}_l) \to \mathrm{H}^1_{\text{ét}}(G, \mathbb{Q}_l) \to \mathrm{H}^1_{\text{ét}}(G_{\text{lin}}, \mathbb{Q}_l) \to 0 \quad \square$$

We make the following definition.

Definition 5.10. A morphism $\pi : X \to Y$ of k-varieties is called a *fibration* if all the fibres over the points in $Y(k)$ are isomorphic to some k-scheme F. We say that the fibration *locally trivial* for $\mathcal{T}$ (the flat/étale/Zariski topology) if $\mathcal{T}$-locally there are sections to π.

In the case of certain fibrations we know something about the higher direct image sheaf which is useful for understanding the Leray spectral sequence associated the fibration.

Lemma 5.11. *Let $\pi : X \to Y$ be a fibration that is locally trivial for the étale topology. Assume that for all $q \geq 0$ the Λ-module $\mathrm{H}^q_{\text{ét}}(X_{\bar{y}}, \Lambda_n)$ is flat over $\mathbb{Z}/l^n\mathbb{Z}$. Then for all $q \geq 0$ the sheaves $R^q\pi_*\Lambda_n$ are finite locally constant sheaves on $Y_{\text{ét}}$ with stalks isomorphic to $(R^q\pi_*\Lambda_n)_{\bar{y}} \cong \mathrm{H}^q_{\text{ét}}(X_{\bar{y}}, \Lambda_n)$.*

Proof. The sheaf $R^q\pi_*\Lambda_n$ is the sheafifiction of the presheaf $U \mapsto \mathrm{H}^q_{\text{ét}}(U \times_Y X, \Lambda_n)$. Hence its stalk at $\bar{y}$ is given by $\varinjlim_U \mathrm{H}^q_{\text{ét}}(U \times_Y X, \Lambda_n)$, where the filtered direct limit is taken over all étale neighbourhoods $U \to Y$ that have $\bar{y}$ in the image. Note that the étale neighbourhoods V of $\bar{y}$ over which the fibration $V \times_Y X \to V$ is trivial form a cofinal system in the system of étale neighbourhoods of $\bar{y}$. For such étale neighbourhoods one has $V \times_Y X \cong X_{\bar{y}} \times V$. Hence the stalk is isomorphic to $\varinjlim_V \mathrm{H}^q_{\text{ét}}(V \times X_{\bar{y}}, \Lambda_n)$.

As we had assumed that $\mathrm{H}^q_{\text{ét}}(X_{\bar{y}}, \Lambda)$ is a flat Λ-module for all $q \geq 0$ the Künneth fornula applies, so $\bigoplus_{r+s=q} \mathrm{H}^r_{\text{ét}}(V, \Lambda) \otimes \mathrm{H}^s_{\text{ét}}(X_{\bar{y}}, \Lambda) \to \mathrm{H}^q_{\text{ét}}(X_{\bar{y}} \times V, \Lambda)$ is an isomorphism. We have the identification

$$\varinjlim_V \mathrm{H}^r_{\text{ét}}(V, \Lambda) = \begin{cases} \Lambda \text{ if } r = 0 \\ 0 \text{ else} \end{cases}$$

Combining this with the facts that filtered colimits are excat and commute with direct sums gives us that $(R^q\pi_*\Lambda_n)_{\bar{y}} = \mathrm{H}^q_{\text{ét}}(X_{\bar{y}}, \Lambda_{X_{\bar{y}}})$. Now we show that the sheaf is locally constant.

For $U \to Y$ an étale neighbourhood such that the fibration $U \times_Y X \to Y$ is trivial, one has for F the constant fibre and for $V \to U$ étale a morphism $\mathrm{H}^q_{\text{ét}}(F, \Lambda_n) \to \mathrm{H}^q_{\text{ét}}(F \times V, \Lambda_n) = \mathrm{H}^q_{\text{ét}}(V \times_X Y, \Lambda_n)$ defined by pulling back along the projection $F \times V \to V$. This homomorphism is natural in V, hence it defines a morphism of presheaves on $U_{\text{ét}}$, $\mathrm{H}^q_{\text{ét}}(F, \Lambda_n)^P \to (R^q\pi_*\Lambda_n|_U)^P$, where $(R^q\pi_*\Lambda_n|_U)^P$ is the presheaf defined by the assignment $V \mapsto \mathrm{H}^q_{\text{ét}}(V \times_X Y, \Lambda_n)$. By taking the sheafifications on both sides, we get a morphism of sheaves on $U_{\text{ét}}$, $\mathrm{H}^q_{\text{ét}}(F, \Lambda_n)_U \to R^q\pi_*\Lambda_n|_U$, which by what is written above is an isomorphism on the stalks, hence $\mathrm{H}^q_{\text{ét}}(F, \Lambda_n)_U \to R^q\pi_*\Lambda_n|_U$ is an isomorphism and thus $R^q\pi_*\Lambda_n$ is a locally constant sheaf. $\square$

We have the following basic result on the cohomological dimension of a semi-abelian variety.

Lemma 5.12. *Let G be a semi-abelian variety, i.e. $0 \to (\mathbb{G}_m)^n \to G \to A \to 0$ is exact for some n and A an abelian variety of dimension g. Then $\mathrm{H}^r_{\text{ét}}(G, \mathbb{Q}_l) = 0$ for all $r > 2g + n$.*

Proof. As G is a principle $\mathbb{G}_m^n$ bundle over A we have that $G \to A$ is locally trivial for the Zariski topology. Applying Lemma 5.11 gives that the sheaves $R^q\pi_*(\Lambda_n)$ are locally constant with stalks at $\bar{x}$ equal to $\mathrm{H}^q_{\text{ét}}(\mathbb{G}_m^n, \Lambda_n)$. As $\mathbb{G}_m^n$ is affine we have by Theorem 2.48 that $\mathrm{H}^q_{\text{ét}}(\mathbb{G}_m^n, \Lambda_n) = 0$ for $q > n$ and hence $R^q\pi_*(\Lambda_n) = 0$ for $q > n$. We obtain by Theorem 2.48 that $\mathrm{H}^p_{\text{ét}}(A, R^q\pi_*(\Lambda_n)) = 0$ for $p > 2g$ as $R^q\pi_*(\Lambda_n)$ is locally constant and killed by l^n. Consider the Leray spectral sequence $E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(A, R^q\pi_*(\Lambda_n)) \implies \mathrm{H}^{p+q}_{\text{ét}}(G, \Lambda)$. The only nonzero terms on the E_2-page have $p \leq 2g$ and $q \leq n$, hence $p + q \leq 2g + n$. This gives that $E_\infty^{p,q} = 0$ for all p, q with $p + q > 2g + n$ and hence $\mathrm{H}^r_{\text{ét}}(G, \mathbb{Q}_l) = 0$ for all $r > 2g + n$. □

We have the following structure theorem on graded commutative Hopf algebras.

Lemma 5.13. *(p.191, [39]) Let H be a graded-commutative finite dimensional Hopf-algebra over a field k with $\mathrm{H}^0 = k$. Let $r > 0$ such that for all $i > r$ we have $\mathrm{H}^i = 0$. If $\dim_k \mathrm{H}^1 = r$ then $\mathrm{H} = \bigwedge^* \mathrm{H}^1$.*

Proof. By the structure theorem of Hopf-algebras of Hopf-Borel ([5], p.141) we have an isomorphism of graded $\mathbb{Q}_l$-vectorspaces $\mathrm{H} \cong \bigotimes_i k[x_i]/(f_i)$, where the tensor product is taken over k and f_i is a polynomial of degree > 1 in x_i. Let $n_i = \deg(x_i)$, then $\prod_i x_i \in \mathrm{H}^{\sum_i n_i}$ is a nonzero element, hence we have $\sum_i n_i \leq r$, which implies $n_i = 1$ for all i. Also we have that $x_i^2 = 0$ for all i as otherwise $x_i^2 \cdot \prod_{i \neq j} x_j$ is a nonzero element of degree $> r$. Together with the graded commutativity this implies $\mathrm{H} = \bigwedge^* \mathrm{H}^1$. □

A lemma that will help us prove the concluding theorem is the following.

Lemma 5.14. *Let Λ be a finite group such that $\mathrm{char}(k) \nmid \#\Lambda$. Let $\pi : X \to Y$ be a fibration that is locally trivial for the étale topology and assume that the fibre F is connected and that $\mathrm{H}^r_{\text{ét}}(F, \Lambda) = 0$ for all $r > 0$. Then $\pi^* : \mathrm{H}^*_{\text{ét}}(Y, \Lambda_n) \to \mathrm{H}^*_{\text{ét}}(X, \Lambda_n)$ is an isomorphism.*

Proof. The sheaf $R^q\pi_*\Lambda$ is locally constant and hence 0 for $q > 0$ by Lemma 5.11. So the Leray spectral sequence degenerates associated to π at the E_2-page. The pullback morphism is given by $\mathrm{H}^r_{\text{ét}}(Y, \Lambda) \to \mathrm{H}^r_{\text{ét}}(X, \pi_*\pi^*\Lambda) \to \mathrm{H}^r_{\text{ét}}(Y, \pi^*\Lambda)$ where the second map is the edge map from the Leray spectral sequence. This second map is an isomorphism as $R^q\pi_*\pi^*\Lambda = 0$ for $q > 0$. The first map is induced by $\Lambda \hookrightarrow \pi_*\pi^*\Lambda$. We have $(\pi_*\pi^*\Lambda)_{\bar{x}} = \varinjlim_{(U,\bar{u})} \pi_*\pi^*\Lambda(U) = \varinjlim_{(U,\bar{u})} \pi^*\Lambda(U \times_Y X)$. By étale local triviality, the étale $(U, \bar{u})$ with U of the form $V \times_k F$ with $V \to Y$ étale form a cofinal system. Hence we obtain $\varinjlim_{(V,\bar{v})} \pi^*\Lambda(V \times_k F) = \varinjlim_{(V,\bar{v})} \pi^*\Lambda(V)$ as $\pi^*\Lambda$ is consant and F is connected. Hence we note that the obtained limit is exactly $\Lambda_{\bar{x}}$ and hence the inclusion $\Lambda \to \pi_*\pi^*\Lambda$ is an equality. Thus the map $\mathrm{H}^r_{\text{ét}}(Y,\Lambda) \to \mathrm{H}^r_{\text{ét}}(Y,\pi_*\pi^*\Lambda)$ is an isomorphism and hence the pullback π^* is an isomorphism. □

We can use this to conclude that the desired isomorphism holds in the case that G is commutative.

Theorem 5.15. *Let G be a commutative group variety. Then $\mathrm{H}^*_{\text{ét}}(G,\mathbb{Q}_l) \cong \bigwedge^* H^1(G,\mathbb{Q}_l)$ as graded $\mathbb{Q}_l$-algebras. Also there is an isomorphism of $\mathbb{Q}_l$-algebras $\mathrm{H}^*_{\text{ét}}(G,\mathbb{Q}_l) \cong \mathrm{H}^*_{\text{ét}}(G_{\text{lin}},\mathbb{Q}_l) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}},\mathbb{Q}_l)$.*

Proof. Let $\mathrm{R_u}(G_{\text{lin}})$ be the unipotent radical of G_{lin}. Then as a scheme, $\mathrm{R_u}(G_{\text{lin}}) \cong \mathbb{A}^s$ for some s ([32], Prop 14.32). Hence the fibration $G \to G/\mathrm{R_u}(G_{\text{lin}})$ is Zariski locally trivial by Proposition 4.32. So Lemma 5.14 applies which gives that $\mathrm{H}^*_{\text{ét}}(G/\mathrm{R_u}(G_{\text{lin}}),\mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G,\mathbb{Q}_l)$ is an isomorphism of $\mathbb{Q}_l$-algebras. Hence it suffices to understand $\mathrm{H}^*_{\text{ét}}(G/\mathrm{R_u}(G_{\text{lin}}),\mathbb{Q}_l)$.
We have that $G' := G/\mathrm{R_u}(G)$ is a semi-abelian variety. So by applying Proposition 5.9 we get $\dim \mathrm{H}^1_{\text{ét}}(G',\mathbb{Q}_l) = n + 2g$ for n the dimension of the maximal torus of G' and g the dimension of G'_{ab}. By Lemma 5.12, $\mathrm{H}^r_{\text{ét}}(G',\mathbb{Q}_l) = 0$ for $r > 2g+n$.Then by applying Lemma 5.13 we get $\mathrm{H}^*_{\text{ét}}(G',\mathbb{Q}_l) \cong \bigwedge^* \mathrm{H}^1_{\text{ét}}(G',\mathbb{Q}_l)$. As $\mathrm{H}^1_{\text{ét}}(G,\mathbb{Q}_l) \cong \mathrm{H}^1_{\text{ét}}(G',\mathbb{Q}_l)$ this implies $\mathrm{H}^*_{\text{ét}}(G,\mathbb{Q}_l) \cong \bigwedge^* \mathrm{H}^1_{\text{ét}}(G,\mathbb{Q}_l)$ as $\mathbb{Q}_l$-algebras.
The second statement follows directly from the split exact sequence of Proposition 5.9 combined with that we know that the ring structure on $\mathrm{H}^*_{\text{ét}}(G,\mathbb{Q}_l)$ is determined by the first degree. □

We have seen applied the identification $\mathrm{Ext}(G,\mathbb{Z}/l^n\mathbb{Z}) = \mathrm{PH}^1_{\text{ét}}(G,\Lambda_n)$ together with the Künneth formula to obtain that $\mathrm{H}^1_{\text{ét}}(G,\mathbb{Q}_l) = \varprojlim_n \mathrm{Ext}(G,\mathbb{Z}/l^n\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ (by Lemma 5.5) and which allowed us to use properties of Ext to obtain the results about $\mathrm{H}^1_{\text{ét}}(G,\mathbb{Q}_l)$. A natural question would be whether $\mathrm{Ext}(G,\mathbb{Z}/l^n\mathbb{Z}) = \mathrm{H}^1_{\text{ét}}(G,\Lambda_n)$ holds. This is a actually true by a result of Miyanishi (Theorem 2 [33]).

5.2 The Chevalley sequence for non-commutative G

We have the following intermediate lemma.

Lemma 5.16. *Let G be a group variety and let $D := G_{\text{ant}}$. Then $D \cap G_{\text{lin}}$ contains D_{lin} and they are equal up to finite index. More precisely we have $D_{\text{lin}} = (D \cap G_{\text{lin}})^\circ_{\text{red}}$.*

Proof. This follows directly from the proof of Remark 1.63 applied to the case $D = K$ and $*$ being the property 'linear'. □

We will now prove a lemma on fibre bundles originally from algebraic topology but it also applies to algebraic geometry. As we could not find a good source for the lemma we modify the proof in the context of algebraic topology given in ([25] p.245). We omit writing the $\mathbb{Q}_l$-coefficients in the proof below.

Lemma 5.17 (The Leray-Hirsch principle)**.** *Let $\pi : E \to B$ fibration that is locally trivial for the Zariski topology and let F be the fibre. Assume that the inclusion $\iota : F \to E$ induces a surjection $\iota^* : \mathrm{H}^*_{\text{ét}}(E, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(F, \mathbb{Q}_l)$. Then there is an isomorphism of $\mathrm{H}^*_{\text{ét}}(B, \mathbb{Q}_l)$-modules: $\mathrm{H}^*_{\text{ét}}(B, \mathbb{Q}_l) \otimes \mathrm{H}^*_{\text{ét}}(F, \mathbb{Q}_L) \cong \mathrm{H}_{\text{ét}}(E, \mathbb{Q}_l)$. It is given as follows: Pick classes $d^r_{r_j} \in \mathrm{H}^r_{\text{ét}}(E, \mathbb{Q}_l)$ such that the $\iota^*(d^r_{r_j})$ form a basis of $\mathrm{H}^r_{\text{ét}}(F, \mathbb{Q}_l)$. Then the isomorphism maps $a \otimes \iota^*(d^r_{r_j}) \mapsto \pi(a) \cup d^r_{r_j}$.*

Proof. We pick a finite trivializing cover $\{U_i\}_{1 \leq i \leq n}$ of B and we denote $\pi^{-1}(U) = E_U$ for any open U of B. We will prove the result by induction on the number of elements in the cover. Note that for any E_U we still have a surjection $\mathrm{H}^*_{\text{ét}}(E_U) \to \mathrm{H}^*_{\text{ét}}(F)$ as $F \to E$ factors via E_U, hence we may indeed proceed by induction. The case $n = 1$ is covered by the Künneth formula. So it suffices to show that if $\mathrm{H}^*_{\text{ét}}(U) \otimes \mathrm{H}^*_{\text{ét}}(F) \to \mathrm{H}^*_{\text{ét}}(E_U)$ and $\mathrm{H}^*_{\text{ét}}(V) \otimes \mathrm{H}^*_{\text{ét}}(F) \to \mathrm{H}^*_{\text{ét}}(E_V)$ are isomorphisms then $\mathrm{H}^*_{\text{ét}}(U \cup V) \otimes \mathrm{H}^*_{\text{ét}}(F) \to \mathrm{H}^*_{\text{ét}}(E_{U \cup V})$ is an isomorphism. Note that we can assume that $\mathrm{H}^*_{\text{ét}}(U \cap V) \otimes \mathrm{H}^*_{\text{ét}}(F) \to \mathrm{H}^*_{\text{ét}}(E_{U \cap V})$ is an isomorphism as $E_{U \cap V} \to U \cap V$ has a trivializing cover with no more elements than $E_U \to U$. We have the following exact sequence, which is induced by the Mayer-Vietoris sequence for $U \cup V$:

$$... \to \bigoplus_{n+m=r} (\mathrm{H}^n_{\text{ét}}(U) \oplus \mathrm{H}^n_{\text{ét}}(V)) \otimes \mathrm{H}^m_{\text{ét}}(F) \to \bigoplus_{n+m=r} \mathrm{H}^n_{\text{ét}}(U \cap V) \otimes \mathrm{H}^m_{\text{ét}}(F) \to \bigoplus_{n+m=r+1} \mathrm{H}^n_{\text{ét}}(U \cup V) \otimes \mathrm{H}^m_{\text{ét}}(F) \to ...$$

We also have the following exact sequence, which is the Mayer-Vietoris sequence for $E_{U \cup V}$:

$$... \to \mathrm{H}^r_{\text{ét}}(E_U) \oplus \mathrm{H}^r_{\text{ét}}(E_V) \to \mathrm{H}^r_{\text{ét}}(E_{U \cap V}) \to \mathrm{H}^{r+1}_{\text{ét}}(E_{U \cup V}) \to ...$$

There is a natural map from the first exact sequence to the second one. For W any of the opens $U, V, U \cap V$ or $U \cup V$ it is defined by $\mathrm{H}^n_{\text{ét}}(W) \otimes \mathrm{H}^m_{\text{ét}}(F) \to \mathrm{H}^{n+m}_{\text{ét}}(E_W)$ sending $\alpha \otimes \iota^* \beta^{\sigma_m}_m \mapsto \pi^*(\alpha) \otimes \beta^{\sigma_m}_m$ for homogeneous elements $\beta^{\sigma_m}_m \in \mathrm{H}^m_{\text{ét}}(E_W)$ such that $\iota^* \beta^{\sigma_m}_m$ generate $\mathrm{H}^m_{\text{ét}}(F)$ freely over $\mathbb{Q}_l$ and then it is extended to the whole of the elements in the above exact sequence. These maps commute with the horizontal maps in the exact sequences as the Mayer-Vietoris sequence is compatible with pullbacks. Now we use that we assumed that for U, V and $U \cap V$ this map is an isomorphism to conclude by the 'five lemma' that $\bigoplus_{n+m=r} \mathrm{H}^n_{\text{ét}}(U \cup V) \otimes \mathrm{H}^m_{\text{ét}}(F) \to \mathrm{H}^r_{\text{ét}}(E_{U \cup V})$ is an isomorphism. □

In the following theorem we exploit Lemma 1.60, which says that $G_{\text{lin}} \cdot G_{\text{ant}} = G$ for a group variety G together with the fact that G_{ant} is commutative. We may omit writing the $\mathbb{Q}_l$ coefficients for l-adic cohomology in the upcoming proofs.

Theorem 5.18. *Let G be a group variety. Then $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ and $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}}, \mathbb{Q}_l) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}, \mathbb{Q}_l)$ are isomorphic as graded $\mathbb{Q}_l$-vectorspaces.*

Proof. Let $D := G_{\text{ant}}$, which is a commutative group variety. We have an exact sequence of algebraic groups $0 \to \ker \xrightarrow{\iota} D \times G_{\text{lin}} \xrightarrow{m} G \to 0$, where $\ker \cong D \cap G_{\text{lin}}$ and the map $D \cap G_{\text{lin}} \to D \times G_{\text{lin}}$ is given by $a \mapsto (-a, a)$. Note that by Lemma 5.16 we have that D_{lin} is a normal subgroup of $D \times G_{\text{lin}}$, hence we can form the following commuting diagram with exact rows:

$$\begin{array}{ccccccccc} e & \longrightarrow & D_{\text{lin}} & \xrightarrow{j'} & D \times G_{\text{lin}} & \xrightarrow{m'} & Q & \longrightarrow & e \\ & & \Big\downarrow{\scriptstyle \iota_{\text{lin}}} & & \Big\downarrow{\scriptstyle \text{Id}} & & \Big\downarrow{\scriptstyle q} & & \\ e & \longrightarrow & D \cap G_{\text{lin}} & \xrightarrow{j} & D \times G_{\text{lin}} & \xrightarrow{m} & G & \longrightarrow & e \end{array}$$

Note that Q is a group variety as it is a quotient of $D \times G_{\text{lin}}$. Also note that q is an isogeny as D_{lin} and $D \cap G_{\text{lin}}$ are equal up to finite index by Lemma 5.16. Composing $D_{\text{lin}} \to D \times G_{\text{lin}}$ with the projection $p_1 : D \times G_{\text{lin}} \to D$ gives an exact sequence $0 \to D_{\text{lin}} \overset{-\iota_D}{\to} D \overset{\pi_D}{\to} D/D_{\text{lin}} \to 0$. By Corollary 5.15 we have the canonical isomorphism $\mathrm{H}^*_{\text{ét}}(D) \cong \mathrm{H}^*_{\text{ét}}(D_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(D/D_{\text{lin}})$, hence the induced map $\mathrm{H}^*_{\text{ét}}(D) \to \mathrm{H}^*_{\text{ét}}(D_{\text{lin}})$ is surjective. This gives that $(j')^* : \mathrm{H}^*_{\text{ét}}(D \times G_{\text{lin}}) \to \mathrm{H}^*_{\text{ét}}(D_{\text{lin}})$ is surjective.
Since D_{lin} is a commutative linear group variety we have by Proposition 4.32 that the $D \times G_{\text{lin}} \to Q$ is a Zariski locally trivial fibre bundle. Thus the Leray-Hirsch principle 5.17 applies, which gives an $\mathrm{H}^*_{\text{ét}}(Q)$-module isomorphism $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}} \times D) \cong \mathrm{H}^*_{\text{ét}}(Q) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{lin}})$. By the Künneth formula we have $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}} \times D) \cong \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(D)$ and by the previous Theorem 5.15 this is isomorphic to $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(D/D_{\text{lin}})$. Because we have $G = D \cdot G_{\text{lin}}$ we obtain $G_{\text{ab}} = G/G_{\text{lin}} = \dfrac{D \cdot G_{\text{lin}}}{G_{\text{lin}}} = \dfrac{D}{D \cap G_{\text{lin}}}$, which is admits an isogeny q_{ab} to $D_{\text{ab}} = \dfrac{D}{D_{\text{lin}}}$. As isogenies induce isomorphisms on the cohomology by Theorem 2.52, we can put this together to obtain an isomorphism of graded $\mathbb{Q}_l$-vectorspaces $\mathrm{H}^*_{\text{ét}}(G) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{lin}}) \cong \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{lin}})$. As $\mathrm{H}^*_{\text{ét}}(D_{\text{lin}})$ is a nonzero graded $\mathbb{Q}_l$-vectorspace we see by induction on the grading that there is an isomorphism of graded $\mathbb{Q}_l$-vectorspace isomorphism $\mathrm{H}^*_{\text{ét}}(G) \cong \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{lin}})$ by 'dimension-counting'. □

This is a first step but not quite yet the result that we want as we want an isomorphism of graded $\mathbb{Q}_l$-algebras.

Remark 5.19. It is useful to trace the isomorphisms used in the theorem above. We put them into the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^*_{\text{ét}}(G_{\text{lin}} \times D) & \xleftarrow{\quad\text{Künneth}\quad} & \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(D) \\
\uparrow{\scriptstyle \text{LH}} & & \uparrow{\scriptstyle 1\otimes\text{LH}} \\
\mathrm{H}^*_{\text{ét}}(Q) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{lin}}) & & \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{ab}}) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{lin}}) \\
\uparrow{\scriptstyle q^*\otimes 1} & & \uparrow{\scriptstyle 1\otimes q^*_{\text{ab}}\otimes 1} \\
\mathrm{H}^*_{\text{ét}}(G) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{lin}}) & & \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{lin}})
\end{array}$$

For the left Leray-Hirsch map we have that as $(j')^*$ is surjective and that $j' \circ p_1 = -\iota_D$ as homomorphisms $D_{\text{lin}} \to D$ that we can pick homogeneous classes of weight k, $p_1^* d_k^{\sigma_k} \in \mathrm{H}^*_{\text{ét}}(G_{\text{lin}} \times D)$, such that $\{(-\iota_D)^* d_k^{\sigma_k}\}$ generates $H^k(D_{\text{lin}})$ and the Leray-Hirsch map maps $(-\iota_D)^* d_k^{\sigma_k} \mapsto p_1^* d_k^{\sigma_k}$. By the commuting diagram of the previous theorem we have that $m = m' \circ q$, hence the map $\mathrm{H}^*_{\text{ét}}(G) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{lin}}) \to \mathrm{H}^*_{\text{ét}}(G_{\text{lin}} \times D)$ maps an element $x \in \mathrm{H}^*_{\text{ét}}(G)$ to $m^* x$. The Leray-Hirsch map on the right sends an element of $x \in \mathrm{H}^*_{\text{ét}}(D_{\text{ab}})$ to $\pi_D^* x$. As we have $q_{\text{ab}} \circ \pi_D = \pi$, the restriction of the map $G \to G_{\text{ab}}$, hence on the right side the map is π^* when restricted to $\mathrm{H}^*_{\text{ét}}(G_{\text{ab}})$.

Another useful observation to make is that the maps LH and $1 \otimes \text{LH}$ are injective graded ring homomorphisms when restricted to $\mathrm{H}^*_{\text{ét}}(Q)$ and $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{ab}})$. The isomorphisms induced by the isogenies are isomorphisms of graded rings. This implies that $\mathrm{H}^*_{\text{ét}}(G) \to \mathrm{H}^*_{\text{ét}}(G_{\text{lin}} \times D)$ and $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}) \to \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(D)$ are injective homomorphisms of graded rings.

Now we will use the above to show that the desired isomorphism indeed exists.

Theorem 5.20. *There is an isomorphism* $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}}, \mathbb{Q}_l) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ *of graded* $\mathbb{Q}_l$*-algebras. On* $\mathrm{H}^*_{\text{ét}}(G_{\text{ab}}, \mathbb{Q}_l)$ *it is given by* $a \mapsto \pi^*(a)$*. On* $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}}, \mathbb{Q}_l)$ *it is given by a section to* $\iota^* : \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}, \mathbb{Q}_l)$.

Proof. We consider the following composition of isomorphisms of graded $\mathbb{Q}_l$-vectors spaces:

$$\mathrm{H}^*_{\text{ét}}(G) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{lin}}) \to \mathrm{H}^*_{\text{ét}}(D \times G_{\text{lin}}) \to \mathrm{H}^*_{\text{ét}}(D) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \to \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}) \otimes \mathrm{H}^*_{\text{ét}}(D_{\text{lin}})$$

An element x of $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}})$ corresponds to $p_2^* x \in \mathrm{H}^*_{\text{ét}}(D \times G_{\text{lin}})$. This is then equal to $\sum m^*\beta_k \cup p_1^*\alpha_r^{n_r}$, where r and k denote the homogeneous degrees. Hence we obtain that $x = \sum \iota_2^* m^* \beta_k \cup \iota_2^* p_1^* \alpha_r^{n_r}$, which equals the same sum but with all the terms taken out that have $r \neq 0$. Hence $p_1^* x$ equals an element of the form $m^* y$ for $y \in \mathrm{H}^*_{\text{ét}}(G)$. For an element in $a \in \mathrm{H}^*_{\text{ét}}(G_{\text{ab}})$, we have that it corresponds to the element $p_2^* \pi_D^*(a) = m^* \pi^*(a)$ in $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}} \times D)$, so we see that $\mathrm{H}^*_{\text{ét}}(G_{\text{ab}})$ is also contained in the image of $\mathrm{H}^*_{\text{ét}}(G)$. Hence the image of $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}})$ in $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}} \times D)$ is a sub $\mathbb{Q}_l$-algebra of $m^*\mathrm{H}^*_{\text{ét}}(G)$. As their dimensions over $\mathbb{Q}_l$ are the same by Proposition 5.18, we get that these rings are equal. The middle arrow above is an isomorphism of rings. The left arrow restricted to $\mathrm{H}^*_{\text{ét}}(G)$ is an isomorphism of rings $\mathrm{H}^*_{\text{ét}}(G) \to m^*\mathrm{H}^*_{\text{ét}}(G)$. The inverse of the right arrow induces an isomorphism of rings $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}) \to \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \pi_D^* \mathrm{H}^*_{\text{ét}}(G_{\text{ab}})$. Putting this together we find that $\mathrm{H}^*_{\text{ét}}(G) \to \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}})$ is a graded ring isomorphism.
An element $a \in \mathrm{H}^*_{\text{ét}}(G_{\text{ab}})$ corresponds to $m^*\pi^*(a) \in m^*\mathrm{H}^*_{\text{ét}}(G)$ under the above isomorphisms and so it maps to $\pi^*(a)$. For an element b of $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}})$ we have that it maps to $p_1^*(b) \in m^*\mathrm{H}^*_{\text{ét}}(G)$, so the corresponding element is some $c \in \mathrm{H}^*_{\text{ét}}(G)$ such that $m^*(c) = p_1^*(b)$. This means that our element b is $\iota_1^* p_1^*(b) = \iota_1^* m^*(c) = \iota^*(c)$, so indeed the induced map $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \to \mathrm{H}^*_{\text{ét}}(G)$ is a section to ι^*. □

Notation 5.21. Let $\sigma : G \to G$ be an endomorphism. Then it restricts to the fully characteristic subgroup G_{lin}. We denote $\sigma_{\text{lin}} : G_{\text{lin}} \to G_{\text{lin}}$ and $\sigma_{\text{ab}} : G_{\text{ab}} \to G_{\text{ab}}$ for the endomorphisms induced by σ. For f an endomorphism of a finite dimensional graded vectorspace V^*, write $\mathrm{tr}(f|_{V^r}) =: \mathrm{tr}(f, r)$.

We apply the result of Theorem 5.20 to compare the graded traces (see Definition 2.72) of the above endomorphisms .

Proposition 5.22. *Let $\sigma : G \to G$ be an endomorphism. Then* $\mathrm{tr}(\sigma) = \mathrm{tr}(\sigma_{\text{lin}}) \cdot \mathrm{tr}(\sigma_{\text{ab}})$.

Proof. Let $\sigma^* : \mathrm{H}^r_{\text{ét}}(G) \to \mathrm{H}^r_{\text{ét}}(G)$. By the isomorphism $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}) \to \mathrm{H}^*_{\text{ét}}(G)$, there is a unique graded $\mathbb{Q}_l$-algebra endomorphism $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}) \to \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}})$ giving a commuting diagram:

$$\begin{array}{ccc} \mathrm{H}^*_{\text{ét}}(G) & \xrightarrow{\sigma^*} & \mathrm{H}^*_{\text{ét}}(G) \\ \uparrow & & \uparrow \\ \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}) & \longrightarrow & \mathrm{H}^*_{\text{ét}}(G_{\text{lin}}) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}) \end{array}$$

We claim that this endomorphism is $\sigma^*_{\text{lin}} \otimes \sigma^*_{\text{ab}}$. Indeed for $a \in \mathrm{H}^*_{\text{ét}}(G_{\text{ab}})$ we have $\sigma^*\pi^*(a) = \pi^*\sigma^*_{\text{ab}}(a)$ and we have for an element $\iota^*(b)$ of $\mathrm{H}^*_{\text{ét}}(G_{\text{lin}})$ that $\sigma^*_{\text{lin}}\iota^*(b) = \iota^*\sigma^*(b)$, which is mapped to $\sigma^*(b)$ by s, so the diagram commutes for $\sigma^*_{\text{lin}} \otimes \sigma^*_{\text{ab}}$. Thus $\mathrm{tr}(\sigma^*, r) = \mathrm{tr}(\sigma^*_{\text{lin}} \otimes \sigma^*_{\text{ab}}, r) = \sum_{i+j=r} \mathrm{tr}(\sigma^*_{\text{lin}}, i) \cdot \mathrm{tr}(\sigma^*_{\text{ab}}, j)$.

Hence we obtain:

$$\mathrm{tr}(\sigma) = \sum_r (-1)^r \sum_{i+j=r} \mathrm{tr}(\sigma^*_{\text{lin}}, i) \cdot \mathrm{tr}(\sigma^*_{\text{ab}}, j) = \sum_{i,j} (-1)^{i+j} \mathrm{tr}(\sigma^*_{\text{lin}}, i) \cdot \mathrm{tr}(\sigma^*_{\text{ab}}, j) = \mathrm{tr}(\sigma_{\text{lin}}) \cdot \mathrm{tr}(\sigma_{\text{ab}}) \quad \square$$

5.3 General extensions of group varieties and l-adic cohomology

We introduce the following notion for this subsection.

Definition 5.23. A sequence of group varieties $e \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to e$ is called **almost exact** if $\ker(\iota)$ is finite, π is a quotient map and $\ker(\pi)/\mathrm{Im}(\iota)$ is finite.

We have seen several examples of this.

Example 5.24.
- For any exact sequence of group varieties $e \to K \to G \to Q \to e$, there is an almost exact sequence $e \to K_{\mathrm{lin}} \xrightarrow{\iota} G_{\mathrm{lin}} \xrightarrow{\pi} Q_{\mathrm{lin}} \to e$ by Proposition 1.62. In this case we even have $\ker(\iota) = e$. By the proof of the same proposition we see that $e \to K_{\mathrm{ab}} \xrightarrow{\iota} G_{\mathrm{ab}} \to Q_{\mathrm{ab}} \to e$ is also almost exact but now ι need not be injective.
- Similarly if $e \to K \to G \to K \to e$ is an exact sequence of linear group varieties then all of the above holds with $-_{\mathrm{lin}}$ replaced by $R_u(-)$ and $-_{\mathrm{ab}}$ by $-_{\mathrm{red}}$. If $e \to K \to G \to Q \to e$ is a sequence of reductive group varieties, then all of the above holds with $-_{\mathrm{lin}}$ replaced by $R(-)$ and $-_{\mathrm{ab}}$ by $-_{\mathrm{ss}}$.

We now want to show that any exact sequence of connected group varieties over an algebraically closed field $e \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to e$, there exists a Künneth type isomorphism as in Proposition 5.20.

Definition 5.25. An almost exact sequence of group varieties $e \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to e$ has *the property* $(*)$ if there exists a section s to ι^* such that $(s \otimes \pi^*) : \mathrm{H}^*_{\text{ét}}(K) \otimes \mathrm{H}^*_{\text{ét}}(Q) \to \mathrm{H}^*_{\text{ét}}(G)$ is an isomorphism of graded $\mathbb{Q}_l$-algebras.

The strategy will be to decompose the exact sequence $e \to N \to G \to Q \to e$ into almost exact sequences where the group varieties are of a special type (all semisimple, all tori or all abelian varities). After proving that these sequences have $(*)$ we put the pieces back together to conclude that the original sequence has $(*)$.

Lemma 5.26. *Consider the commuting diagram of group varieties with exact rows and all columns isogenies:*

$$\begin{array}{ccccccccc} e & \longrightarrow & K_1 & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q_1 & \longrightarrow & e \\ & & \uparrow{\scriptstyle q_K} & & \uparrow{\scriptstyle q_G} & & \uparrow{\scriptstyle q_Q} & & \\ e & \longrightarrow & K_2 & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q_2 & \longrightarrow & e \end{array}$$

Then the top row has $(*)$ if and only if the bottom row has $(*)$.

Proof. Suppose that the bottom row has $(*)$. Denote by s_2 the homomorphism section to ι_2^*. As isogenies induce isomorphisms on the cohomology we can make an isomorphism as follows:

$$\mathrm{H}^*_{\text{ét}}(K_1) \otimes \mathrm{H}^*_{\text{ét}}(Q_1) \overset{q_K^* \otimes q_Q^*}{\to} \mathrm{H}^*_{\text{ét}}(K_2) \otimes \mathrm{H}^*_{\text{ét}}(Q_2) \overset{s_2 \otimes \pi_2^*}{\to} \mathrm{H}^*_{\text{ét}}(G_2) \overset{(q_G^*)^{-1}}{\to} \mathrm{H}^*_{\text{ét}}(G_1)$$

On $\mathrm{H}^*_{\text{ét}}(Q_1)$ this equals $(q_G^*)^{-1}\pi_2^* q_Q^*$ and as the diagram commutes this is $(q_G^*)^{-1} q_G^* \pi_1^* = \pi_1^*$ as desired. On $\mathrm{H}^*_{\text{ét}}(K_1)$ applying ι_1^* to the map gives $\iota_1^*(q_G^*)^{-1} s_2 \pi_K^*$. As the diagram commutes we have $\iota_2^* q_G^* = q_K^* \iota_1^*$, hence $\iota_1^* = (q_K^*)^{-1} \iota_2^* q_G^*$. This implies that the map $\mathrm{H}^*_{\text{ét}}(K_1) \to \mathrm{H}^*_{\text{ét}}(G_1)$ followed by ι_1^* is the identity as desired. So the top row has $(*)$. The proof of the other direction is the same but with some inverse signs changed. □

Following our strategy mentioned above we look at the case of semisimple group varieties.

Lemma 5.27. *Let $e \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to e$ be an exact sequence of semisimple group varieties. Then this sequence has $(*)$.*

Proof. Let $G_1, ..., G_{n+m}$ be the normal minimal almost simple subgroup varieties of G. As K is a normal subgroup variety of G we have by ([24], p.167) that $K = G_1 \cdot ... \cdot G_m$. Putting this together with the fact that G is the almost direct product of $G_1, ..., G_{n+m}$ under the multiplication map gives the following commuting diagram with exact rows, where ι_1^m is the injection on the first μ factors and p_{m+1}^{m+n} is the projection on the last n factors.

$$\begin{array}{ccccccccc} e & \longrightarrow & K & \xrightarrow{\iota} & G & \xrightarrow{\pi} & Q & \longrightarrow & e \\ & & \uparrow{\scriptstyle \mu|} & & \uparrow{\scriptstyle \mu} & & \uparrow{\scriptstyle \mu_Q} & & \\ e & \longrightarrow & \prod_{i=1}^{m} G_i & \xrightarrow{\iota_1^m} & \prod_{i=1}^{n+m} G_i & \xrightarrow{p_{m+1}^{m+n}} & \prod_{i=m+1}^{n} G_i & \longrightarrow & e \end{array}$$

By the snake lemma we have that m_Q is an isogeny as both μ and $\mu|$ are isogenies. The bottom sequence splits and has $(*)$ by the Künneth isomorphism. Hence the top sequence has $(*)$ by Lemma 5.26. □

Now treat the case of an exact sequence of tori.

Lemma 5.28. *Let $e \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to e$ be an exact sequence of tori (so all algebraic groups are $(\mathbb{G}_m)^n$ for some n). Then this sequence splits, which implies that it has $(*)$.*

Proof. Let $K = \mathbb{G}_m^{n_k}$ and $Q = \mathbb{G}_m^{n_q}$. We consider the class $[G] \in \mathrm{PB}_{K,Q}^{\mathrm{zar}}$ (see Chapter 4). Then there is a homomorphism $\mathrm{PB}_{K,Q}^{\mathrm{zar}} \to \mathrm{H}_{\mathrm{zar}}^1(Q, K) = \mathrm{H}_{\mathrm{zar}}^1(Q, \mathbb{G}_m^{n_k}) = \bigoplus_{n_k} \mathrm{H}_{\mathrm{zar}}^1(Q, \mathbb{G}_m) = \bigoplus_{n_k} \mathrm{Pic}(Q)$. As Q is an affine variety, $\mathrm{Pic}(Q)$ equals $\mathrm{Cl}(\mathcal{O}(Q))$ and as $\mathcal{O}(Q) = k[X_1, ..., X_{n_q}]_{X_1,...,X_{n_q}}$ is a localization of a UFD it is a UFD, hence its class group is trivial. So we see that the image of $[G]$ in $\mathrm{H}_{\mathrm{zar}}^1(Q, K)$ is 0, hence the fibration $\pi : G \to Q$ admits a section s. By applying translations we may assume that $s(e) = e$. By ([32], Prop. 12.49) we have that any regular map φ from a connected group variety to a group of multiplicative type satisfying $\varphi(e) = e$ is a homomorphism of algebraic groups. A torus is of multiplicative type, hence we conclude that $s : Q \to G$ is a homomorphism, hence the sequence splits. Thus we have a Künneth isomorphism on the l-adic cohomology induced by ι and π, so the sequence has $(*)$. □

And last we treat the case of an exact sequence of abelian varieties.

Lemma 5.29. *An exact sequence of abelian varieties $e \to K \to G \to Q \to e$ has $(*)$.*

Proof. For any abelian variety A, the simple abelian subvarieties $A_1, ..., A_n$ (finitely many) of A satisfy that the multiplication map $A_1 \times ... \times A_n \to A$ is an isogeny ([30], p.42). This implies that for any abelian subvariety $K \subset G$ there exists an abelian subvariety $H \subset G$ such that the multiplication map $K \times H \to G$ is an isogeny. The proof is now just the same proof as the one of Lemma 5.27. □

We now show that the following sequence also has the property.

Proposition 5.30. *For G reductive the exact sequence $e \to R(G) \to G \to G_{\mathrm{ss}} \to e$ has $(*)$.*

Proof. As G is reductive, the multiplication map $\mu : R(G) \times G_{\mathrm{der}} \to G$ is an isogeny. As the image of $R(G)$ inside G_{ss} is trivial, the homomorphism $R(G) \times G_{\mathrm{der}} \to G_{\mathrm{ss}}$ factors via G_{der}. This gives the following commuting diagram with exact rows:

$$\begin{array}{ccccccccc} e & \longrightarrow & R(G) & \longrightarrow & G & \longrightarrow & G_{\mathrm{ss}} & \longrightarrow & e \\ & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle \mu} & & \uparrow & & \\ e & \longrightarrow & R(G) & \xrightarrow{\iota_1} & R(G) \times G_{\mathrm{der}} & \xrightarrow{p_2} & G_{\mathrm{der}} & \longrightarrow & e \end{array}$$

The bottom row has $(*)$ by the Künneth isomorphism. The homomorphism $G_{\mathrm{der}} \to G_{\mathrm{ss}}$ is an isogeny by the Snake lemma. So the top row has $(*)$ by Lemma 5.26. $\square$

Using the results on the exact sequences of tori and semisimple group varieties we can prove the following.

Proposition 5.31. *Let $e \to K \stackrel{\iota}{\to} G \stackrel{\pi}{\to} Q \to e$ be an exact sequence of reductive group varieties. Then the almost exact sequences $e \to R(K) \stackrel{\iota_R}{\to} R(G) \stackrel{\pi_R}{\to} R(Q) \to e$ and $e \to K_{\mathrm{ss}} \stackrel{\iota_s}{\to} G_{\mathrm{ss}} \stackrel{\pi_s}{\to} Q_{\mathrm{ss}} \to e$ both have $(*)$.*

Proof. This follows almost immediately from Proposition 1.62. As K, G, Q are reductive there is an exact sequence of group varieties $e \to R(K) \stackrel{\iota_R}{\to} R(G) \stackrel{\pi'}{\to} R' \to e$ and an isogeny $q : R' \to R(Q)$ such that $q \circ \pi' = \pi_R$. As G, K are reductive we have that $R(G), R(K)$ are tori and as quotients of tori are tori, so is R'. Hence the sequence $e \to R(K) \stackrel{\iota_R}{\to} R(G) \stackrel{\pi'}{\to} R' \to e$ has $(*)$ by Lemma 5.28. So there is an isomorphism of $\mathbb{Q}_l$-algebras $\mathrm{H}^*_{\text{ét}}(R(Q)) \otimes \mathrm{H}^*_{\text{ét}}(R(K)) \stackrel{q^* \otimes 1}{\to} \mathrm{H}^*_{\text{ét}}(R') \otimes \mathrm{H}^*_{\text{ét}}(R(K)) \stackrel{\pi'^* \otimes s}{\to} \mathrm{H}^*_{\text{ét}}(G)$. As $q^* \pi'^* = \pi_R^*$ we conclude that the sequence of radicals has $(*)$.
The case of the sequence of semisimple algebraic groups is very much the same and follows from the exact sequence $e \to K_{\mathrm{ss}}/F \to G_{\mathrm{ss}} \to Q_{\mathrm{ss}} \to e$ for F a finite algebraic group combined with Lemma 5.27. $\square$

We now have a general statement on finitely generated graded-commutative Hopf-Algebras by Hopf [5] also shown in ([9], Sections 2.3, 2.4).

Proposition 5.32. *Let H be a graded-commutative Hopf-algebra that is finitely generated over a field k of characteristic 0 such that $\mathrm{H}^0 = k$. Then there is an isomorphsim of graded k-algebras $\mathrm{H} \cong \bigwedge^* \mathrm{PH}$, thus H is identified with the exterior algebra on the primitive elements.*

Note that for G a connected group variety the previous proposition applies to the Hopf-algebra $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$. Also note that for $f : G \to H$ a homomorphism of group varieties the pullback $f^* : \mathrm{H}^*_{\text{ét}}(H, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ preserves the subspace of primitive elements.

Lemma 5.33. *Let $e \to K \stackrel{\iota}{\to} G \stackrel{\pi}{\to} Q \to e$ be a sequence of group varieties (it need not be exact). Then the sequence satisfies $(*)$ if and only if $0 \to \mathrm{PH}^*_{\text{ét}}(Q) \stackrel{\pi^*}{\to} \mathrm{PH}^*_{\text{ét}}(G) \stackrel{\iota^*}{\to} \mathrm{PH}^*_{\text{ét}}(K) \to 0$ is an exact sequence of $\mathbb{Q}_l$-vectorspaces.*

Proof. If $0 \to \mathrm{PH}^*_{\text{ét}}(Q) \xrightarrow{\pi^*} \mathrm{PH}^*_{\text{ét}}(G) \xrightarrow{\iota^*} \mathrm{PH}^*_{\text{ét}}(K) \to 0$ is exact, then the sequence induced by the pullbacks $\mathrm{H}^*_{\text{ét}}(Q) \to \mathrm{H}^*_{\text{ét}}(G) \to \mathrm{H}^*_{\text{ét}}(K)$ is induced by this sequence, hence as $\mathrm{H}^*_{\text{ét}}(G) = \bigwedge^* \mathrm{PH}^*_{\text{ét}}(G)$, it is identified with $\bigwedge^*(\mathrm{PH}^*_{\text{ét}}(K) \oplus \mathrm{PH}^*_{\text{ét}}(Q)) = \mathrm{H}^*_{\text{ét}}(K) \otimes \mathrm{H}^*_{\text{ét}}(Q)$.

Now suppose that s is a section to ι^* such that $s \otimes \pi^* : \bigwedge^* \mathrm{PH}^*_{\text{ét}}(K) \otimes \bigwedge^* \mathrm{PH}^*_{\text{ét}}(Q) \to \bigwedge^* \mathrm{PH}^*_{\text{ét}}(G)$ is an isomorphism. After identifying the left side with $\bigwedge^*(\mathrm{PH}^*_{\text{ét}}(K) \oplus \mathrm{PH}^*_{\text{ét}}(Q))$ it follows that $s \otimes \pi^*$ is induced by $s \oplus \pi^* : \mathrm{PH}^*_{\text{ét}}(K) \oplus \mathrm{PH}^*_{\text{ét}}(Q) \to \mathrm{PH}^*_{\text{ét}}(G)$ extended to the exterior algebra. As the extension to the exterior algebra is an isomorphism of $\mathbb{Q}_l$-algebras the map $s \oplus \pi^*$ is an isomorphism of $\mathbb{Q}_l$-vectorspaces. Hence the sequence $0 \to \mathrm{PH}^*_{\text{ét}}(Q) \to \mathrm{PH}^*_{\text{ét}}(G) \to \mathrm{PH}^*_{\text{ét}}(K) \to 0$ is exact. □

Lemma 5.34. *Any exact sequence $e \to K \to G \to Q \to e$ of reductive group varieties has $(*)$. Moreover for $e \to K \to G \to Q \to e$ an exact sequence of linear group varieties, the induced almost exact sequence of reductive group varieties $e \to K_{\text{red}} \to G_{\text{red}} \to Q_{\text{red}} \to e$ has $(*)$.*

Proof. There is a large commuting diagram with exact columns:

$$\begin{array}{ccccccccc}
 & & e & & e & & e & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e & \longrightarrow & R(K) & \longrightarrow & R(G) & \longrightarrow & R(Q) & \longrightarrow & e \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & e \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
e & \longrightarrow & K_{\text{ss}} & \longrightarrow & G_{\text{ss}} & \longrightarrow & Q_{\text{ss}} & \longrightarrow & e \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & e & & e & & e & &
\end{array}$$

The diagram above gives another large commuting diagram of primitive elements, where the arrows are the pullbacks.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(R(K)) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(R(G)) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(R(Q)) & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(K) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(G) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(Q) & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(K_{\text{ss}}) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(G_{\text{ss}}) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(Q_{\text{ss}}) & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Using Lemma 5.33 we see that all of the vertical sequences are exact by Proposition 5.30 and the top horizontal row and bottom horizontal row are exact by Proposition 5.31. Applying the snake

Lemma twice gives that the maps $\mathrm{PH}^*_{\text{ét}}(Q) \to \mathrm{PH}^*_{\text{ét}}(G)$ and $\mathrm{PH}^*_{\text{ét}}(G) \to \mathrm{PH}^*_{\text{ét}}(K)$ are injective resp. surjective. As $K \to G \to Q$ is constant, we have that $\mathrm{PH}^*_{\text{ét}}(Q)$ is contained in $\ker(\mathrm{PH}^*_{\text{ét}}(G) \to \mathrm{PH}^*_{\text{ét}}(K))$. Using exactness of the columns and top and bottom row implies that the dimension of the kernel and $\mathrm{PH}^*_{\text{ét}}(Q)$ are the same, hence they coincide. Hence the exact sequence $e \to K \to G \to Q \to e$ has $(*)$.
For an exact sequence of linear group varieties $e \to K \to G \to Q \to e$ we have the almost exact sequence $e \to K_{\text{red}} \to G_{\text{red}} \to Q_{\text{red}} \to e$. Completely analogously to the proof of Proposition 5.31 it thus follows that $e \to K_{\text{red}} \to G_{\text{red}} \to Q_{\text{red}} \to e$ has $(*)$. □

Lemma 5.35. *An exact sequence $e \to K \to G \to Q \to e$ of linear group varieties has $(*)$.*

Proof. There is a commuting diagram where the top row is exact (here the subscript refers to reductive and not reduced):

$$\begin{array}{ccccccccc} e & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & e \\ & & \downarrow & & \downarrow & & \downarrow & & \\ e & \longrightarrow & K_{\text{red}} & \xrightarrow{\iota} & G_{\text{red}} & \xrightarrow{\pi} & Q_{\text{red}} & \longrightarrow & e \end{array}$$

For any linear group variety L the homomorphism $\pi : L \to L_{\text{red}}$ has constant fibre $R_u(B)$, which as a scheme is isomorphic to $\mathbb{A}^s$ for some s, hence the cohomology of the fibre is concentrated in degree 0. This implies that $\pi : B \to B_{\text{red}}$ induces an isomorphism on the cohomology for all group varieties B. Similarly to how Lemma 5.26 was proven we therefore see that it suffices to show that the sequence $e \to K_{\text{red}} \to G_{\text{red}} \to Q_{\text{red}} \to e$ has $(*)$. By Proposition 1.62 there is an exact sequence $e \to K_{\text{red}}/F \xrightarrow{\iota'} G_{\text{red}} \xrightarrow{\pi} Q_{\text{red}} \to e$ for F finite and an isogeny $q : K_{\text{red}} \to K_{\text{red}}/F$ such that $\iota = q \circ \iota'$. We have seen in Lemma 5.34 that the bottom sequence of reductive group varieties has $e \to K_{\text{red}} \to G_{\text{red}} \to Q_{\text{red}} \to e$ has $(*)$. This concludes the proof. □

Proposition 5.36. *Let $e \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to e$ be an exact sequence of group varieties. The almost exact sequences $e \to K_{\text{lin}} \xrightarrow{\iota_l} G_{\text{lin}} \xrightarrow{\pi_l} Q_{\text{lin}} \to e$ and $e \to K_{\text{ab}} \xrightarrow{\iota_a} G_{\text{ab}} \xrightarrow{\pi_a} Q_{\text{ab}} \to e$ both have $(*)$.*

Proof. This is the same proof as Proposition 5.31. First use Proposition 1.62 and then for the linear case use Lemma 5.35 and for the abelian variety case use Lemma 5.29. □

We conclude the chapter with the following theorem, putting everything together.

Theorem 5.37. *Let $e \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to e$ be an exact sequence of group varieties over an algebraically closed field. Then $\iota^* : \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(K, \mathbb{Q}_l)$ admits a section s that is a $\mathbb{Q}_l$-algebra homomorphism such that the induced map $s \otimes \pi^* : \mathrm{H}^*_{\text{ét}}(K, \mathbb{Q}_l) \otimes \mathrm{H}^*_{\text{ét}}(Q, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ is an isomorphism of graded $\mathbb{Q}_l$-algebras.*

Proof. This is mainly the proof of Lemma 5.34 repeated. We have a large commuting diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(K_{\text{lin}}) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(G_{\text{lin}}) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(Q_{\text{lin}}) & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(K) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(G) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(Q) & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(K_{\text{ab}}) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(G_{\text{ab}}) & \longleftarrow & \mathrm{PH}^*_{\text{ét}}(Q_{\text{ab}}) & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Proposition 5.36 the top and bottom horizontal sequences are exact. By Proposition 5.20 all of the vertical sequences are exact. Hence by doing the same as in the proof of Lemma 5.34 we conclude that the middle horizontal sequence is exact and that there thus exists a section s to ι^* such that the induced map $s \otimes \pi^* : \mathrm{H}^*_{\text{ét}}(K, \mathbb{Q}_l) \otimes \mathrm{H}^*_{\text{ét}}(Q, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ is an isomorphism of graded $\mathbb{Q}_l$-algebras. □

Remark 5.38. One might wonder if the condition that $K \subset G$ is normal can be relaxed to K being any subgroup variety of G. This fails already in some very simple cases. For instance, for $n \geq 2$ we can take $G = \mathrm{GL}_n$ and $K = \{\mathrm{diag}(t_1, ..., t_n) \,|\, t_1, ..., t_n \in k^*\} \cong \mathbb{G}_m^n$. We will see in the next section that $\mathrm{H}^*_{\text{ét}}(\mathrm{SL}_n, \mathbb{Q}_l)$ has no degree 1 elements, hence as GL_n is an extension of $\mathbb{G}_m$ by SL_n we see that $\mathrm{H}^1_{\text{ét}}(\mathrm{GL}_n, \mathbb{Q}_l) = \mathrm{H}^1_{\text{ét}}(\mathbb{G}_m, \mathbb{Q}_l)$, which is not isomorphic to $\mathrm{H}^1_{\text{ét}}(\mathbb{G}_m^n, \mathbb{Q}_l)$ when $n \geq 2$.

Question 3. The proof of Theorem 5.37 relies on various structure theorems for algebraic groups and being able to compare certain exact to a split exact sequences. One may ask if there is a more intrinsic proof than ours. For example, does the l-adic Leray spectral sequence of $G \to Q$ have $E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(Q, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^q_{\text{ét}}(K, \mathbb{Q}_l)$ and does it degenerate at the E_2-page?

5.4 Calculating the l-adic cohomology in explicit examples

We have seen that for any group variety G, that $\mathrm{H}^*_{\text{ét}}(G) \cong \mathrm{H}^*_{\text{ét}}(G_{\text{ab}}) \otimes \mathrm{H}^*_{\text{ét}}(\mathrm{R}(G_{\text{red}})) \otimes \mathrm{H}^*_{\text{ét}}(G_{\text{ss}})$ as $\mathbb{Q}_l$-algebras in a natural way. Here $\mathrm{H}^*_{\text{ét}}(G_{\text{ab}})$ and $\mathrm{H}^*_{\text{ét}}(\mathrm{R}(G_{\text{red}}))$ are well understood as $\mathrm{R}(G_{\text{red}}) \cong \mathbb{G}_m^n$ for some n, so $\mathrm{H}^*_{\text{ét}}(G_{\text{ab}})$ and $\mathrm{H}^*_{\text{ét}}(\mathrm{R}(G_{\text{red}}))$ both only depend on the dimension of G_{ab} and G_{red} respectively (see Examples 2.50 and 2.51). The more complicated part is $\mathrm{H}^*_{\text{ét}}(G_{\text{ss}})$. Write $G_{\text{ss}} = Q$. We have by ([24], p.167) that for $Q_1, ..., Q_n$ the minimal normal subgroup varieties of Q that the multiplication map $\mu : Q_1 \times ... \times Q_n \to Q$ is an isogeny. The pullback map μ^* is then an isomorphism $\mathrm{H}^*_{\text{ét}}(Q) \cong \bigotimes_{1 \leq i \leq n} \mathrm{H}^*_{\text{ét}}(Q_i)$. The subgroups Q_i are almost-simple algebraic groups that are classified up to isogeny (which thus classifies their cohomology) by their root data (see [41] Chapter 17). By the former decomposition of the cohomology of Q we see that it suffices to understand the cohomology of almost-simple algebraic groups. Omitting the case that $\mathrm{char}(k) = 2$ we have that the isogeny classes of almost-simple group varieties are as follows:

- The groups of type A_n ($n \geq 2$). The isogeny class is represented by SL_n, which is the algebraic group given by $\mathrm{SL}_n(R) = \{M \in \mathrm{GL}_n(R) \mid \det(M) = 1\}$.

- The group varieties of type B_n ($n \geq 2$). The isogeny class is represented by SO_{2n+1}, defined as follows: Let $\mathcal{J}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$ and define $\mathrm{O}_{2n+1}(R) := \{M \in \mathrm{GL}_{2n+1}(R) \mid M^T \mathcal{J}_0 M = \mathcal{J}_0\}$. Then we set $\mathrm{SO}_{2n+1} := \ker(\det : \mathrm{O}_{2n+1} \to \mathbb{G}_m)$.

- The group varieties of type C_n ($n \geq 3$). The isogeny class is represented by Sp_{2n}, which is defined as follows: Define $\mathcal{J}_1 := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and then $\mathrm{Sp}_{2n}(R) := \{M \in \mathrm{GL}_{2n} \mid M^T \mathcal{J}_1 M = \mathcal{J}\}$.

- The group varieties of type D_n ($n \geq 4$). The isogeny class is given by SO_{2n}, defined as follows: First define $\mathcal{J}_2 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $\mathrm{O}_{2n}(R) := \{M \in \mathrm{GL}_{2n}(R) \mid M^T \mathcal{J}_2 M = \mathcal{J}_2\}$. Then we set $\mathrm{SO}_{2n} := \ker(\det : \mathrm{O}_{2n+1} \to \mathbb{G}_m)$.

- The exceptional group varieties of isogeny class G_2, F_4, E_6, E_7, E_8. These are harder to describe explicitly. Their root data is given in ([41] Chapter 17).

Remark 5.39. The same classification holds in the case that $\mathrm{char}(k) = 2$, the only thing being that one should define Sp_{2n} and SO_n a bit differently (see Section 24 of [32]).

We have the following proposition that is stated in (SGA 4½ p.230-231). Note that the degree below is well-defined by the Chevalley-Shephard-Todd theorem [10].

Proposition 5.40. *Let G be a reductive group variety with maximal torus T and Weyl group W. There is an isomorphism of graded $\mathbb{Q}_l$-algebras $\bigwedge^* J \cong \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$, where $J = S_+^W/(S_+^W)^2$ and the degree d of a polynomial $F \in J$ is set to be $2d-1$.*

Functoriality of the above isomorphism will be discussed in the next chapter. Note that it makes sense to write $\mathrm{Sym}(X(T) \otimes \mathbb{Q}_l)^W$ as W acts on $X(T)$, so W also acts on $\mathrm{Sym}(X(T) \otimes \mathbb{Q}_l)$. We will use this to calculate the cohomology rings of the almost simple groups of type A_n, B_n, C_n, D_n. For this we use the following results on the Weyl groups of the almost-simple algebraic groups, which is given in the Section 21.j of [32].

Proposition 5.41. *The Weyl groups and maximal tori of the groups listed above are given by:*

- SL_n*: A maximal torus is given by* $T = \{\mathrm{diag}(t_1, ..., t_n) \mid \prod_i t_i = 1\}$*. The Weyl group is isomorphic to S_n and acts on T by* $\sigma \cdot \mathrm{diag}(t_1, ..., t_n) = \mathrm{diag}(t_{\sigma(1)}, ... t_{\sigma(n)})$.

- Sp_{2n}*: A maximal torus is given by* $T = \{\mathrm{diag}(t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1})\}$*. The Weyl group is* $\langle -1 \rangle^n \rtimes S_n$ *with S_n acting by* $\sigma \cdot \mathrm{diag}(t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1}) = \mathrm{diag}(t_{\sigma(1)}, ..., t_{\sigma(n)}, t_{\sigma(1)}^{-1}, ..., t_{\sigma(n)}^{-1})$ *and* $\langle -1 \rangle^n$ *acting by* $(\epsilon_1, ..., \epsilon_n) \cdot \mathrm{diag}(t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1}) = \mathrm{diag}(t_1^{\epsilon_1}, ..., t_n^{\epsilon_n}, (t_1^{\epsilon_1})^{-1}, ..., (t_n^{\epsilon_n})^{-1})$.

- SO_{2n}*: A maximal torus is given by* $T = \{\mathrm{diag}(t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1})\}$*. The Weyl group is* $H_{n-1} \rtimes S_n$ *with* $H_{n-1} := \ker(\mu : \langle -1 \rangle^n \to \langle -1 \rangle)$ *acting in the same way as* $\langle -1 \rangle^n$ *above and* S_n *also acting as above.*

- SO_{2n+1}: *A maximal torus is given by* $T = \{\mathrm{diag}(1, t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1}\}$. *The Weyl group is isomorphic to* $\langle -1\rangle^n \rtimes S_n$, *which acts the same as in the case of* Sp_{2n}.

For proving that any of the above tori T is maximal, note that any of the above group varieties G is contained in GL_r for some r and the tori in $\mathbb{D}_r$ for r either n or $2n+1$. For T' a maximal torus of G that contains T we have that any element of T' commutes with all elements of T. It can be shown though that the elements of GL_r that commute with any element of T are exactly $\mathbb{D}_r$. So $T' \subset \mathbb{D}_r \cap G$, which can be checked is equal to T in all cases.
For showing that the Weyl groups are the correct ones, one can either calculate explicitly or note that all the above permutations can be realized by conjugation and then compare the cardinality with the Weyl groups of the corresponding Dynkin-diagrams ([41], Chapter 17).
Using this proposition and Proposition 5.40 we can calculate the cohomology rings of the groups above. First of all notice that cohomology rings of SO_{2n+1} and Sp_{2n} are isomorphic.

First we prove a short lemma on invariants.

Lemma 5.42. *Let G be a finite group acting on a finitely generated k-algebra A such that* $\mathrm{char}(k) = 0$. *Let $I \subset A$ be an ideal spanned by $f_1, ..., f_n \in A^G$. Then the natural map $A^G \to (A/I)^G$ is surjective.*

Proof. The assumption that I is spanned by invariant polynomials allows the map $A^G \to (A/I)^G$ to exist. Let $\bar{g} \in (A/I)^G$. Pick any $g \in A$ that reduces to $\bar{g}$ in A/I. As $\mathrm{char}(k) = 0$ we have the Reynolds-operator $\rho : A \to A^G$ that maps $f \mapsto \frac{1}{|G|}\sum_{\sigma\in G} \sigma \cdot f$. The assumption that $\bar{g}$ is invariant implies that $\sigma \cdot g = g + h_\sigma$, where $h_\sigma \in I$. Thus we see that $\rho(g) = \frac{1}{|G|}\sum_\sigma g + h_\sigma = g + \frac{1}{|G|}\sum_\sigma h_\sigma$. Hence the image of $\rho(g)$ inside $(A/I)^G$ equals $\bar{g}$, which as $\rho(g)$ is invariant under G implies that $A^G \to (A/I)^G$ is surjective. □

In the following proposition we write $\mathbb{Q}_l^{(k)} \subset H$ for H a graded algebra to indicate that the elements of $\mathbb{Q}_l^{(k)}$ are in degree k.

Proposition 5.43. *For $n \geq 2$ there is an isomorphism of graded $\mathbb{Q}_l$-algebras:*

$$\mathrm{H}^*_{\text{ét}}(\mathrm{SL}_n, \mathbb{Q}_l) \cong \bigwedge^*\Big(\bigoplus_{2\leq k\leq n} \mathbb{Q}_l^{(2k-1)}\Big)$$

Proof. We use the identification $\bigwedge^* J \cong \mathrm{H}^*_{\text{ét}}(\mathrm{SL}_n, \mathbb{Q}_l)$ from Proposition 5.40 with Weyl group W and maximal torus given in Proposition 5.41. As the maximal torus $T = \{(\mathrm{diag}(t_1, ..., t_n) \mid \prod_i t_i = 1\}$ is a maximal torus, the character group is given by $X(T) = \bigoplus_{1\leq i\leq n} \mathbb{Z}\cdot x_i/(\sum_i x_i)$ where $W = S_n$ acts by permuting the variables. Hence $\mathrm{Sym}(X(T)\otimes\mathbb{Q}_l) \cong \mathbb{Q}_l[X_1, ..., X_n]/(\sum_i X_i)$. This action is inherited by the action of S_n on $\mathbb{Q}_l[X_1, ..., X_n]$ by permuting variables. The ring of invariants under this action is $\mathbb{Q}_l[e_1, ..., e_n]$, where $\{e_j\}_{1\leq j\leq n}$ are the elementary symmetric polynomials. We need the ring of invariant of $\mathbb{Q}_l[X_1, ..., X_n]/(e_1)$. By Lemma 5.42 this algebra is generated by the images of $e_2, ..., e_n$, so for $R = \mathbb{Q}_l[e_1, ..., e_n]$ it is given by $R/(e_1 \cdot R)$. We claim that $e_2, ..., e_n$ form a set of indecomposable generators of $(\mathbb{Q}_l[X_1, ..., X_n]/(e_1))^G$, i.e. we need to show that no e_j is a polynomial in the other e_i. Since R is isomorphic as a ring to a polynomial ring in variables e_i we have that $(\mathbb{Q}_l[X_1, ..., X_n]/(e_1))^G = R/(e_1 \cdot R)$ is isomorphic to a polynomial ring in $e_2, .., e_n$, hence $e_2, ..., e_n$ are algebraically independent, which in particular implies that they form a set of indecomposable generators of $(\mathbb{Q}_l[X_1, ..., X_n]/(e_1))^G$. Therefore we conclude that $J \cong \langle e_2, ..., e_n\rangle_{\mathbb{Q}_l}$. As e_i is a polynomial of degree i we have by Proposition 5.40 that $\mathrm{H}^*_{\text{ét}}(\mathrm{SL}_n, \mathbb{Q}_l) \cong \bigwedge^*(\bigoplus_{2\leq k\leq n} \mathbb{Q}_l^{(2k-1)})$. □

Now we deal with the case Sp_{2n}.

Proposition 5.44. *For $n \geq 2$ there is an isomorphism of graded $\mathbb{Q}_l$-algebras:*

$$\mathrm{H}^*_{\text{ét}}(\mathrm{Sp}_{2n}, \mathbb{Q}_l) \cong \bigwedge^*(\bigoplus_{1\leq k\leq n} \mathbb{Q}_l^{(4k-1)})$$

Proof. By Proposition 5.41 a maximal torus of Sp_{2n} is $T = \{\mathrm{diag}(t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1})\}$ and the Weyl group W is $\langle -1\rangle^n \rtimes S_n$ with S_n acting by permuting $t_1, ..., t_n$ and the action of $\langle -1\rangle^n$ is $(\epsilon_1, ..., \epsilon_n) \cdot \mathrm{diag}(t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1}) = \mathrm{diag}(t_1^{\epsilon_1}, ..., t_n^{\epsilon_n}, (t_1^{\epsilon_1})^{-1}, ..., (t_n^{\epsilon_n})^{-1})$. This implies that $\mathrm{Sym}(X(T) \otimes \mathbb{Q}_l)$ is isomorphic to $\mathbb{Q}_l[X_1, ..., X_n]$ with S_n acting on it by permuting variables and $\langle -1\rangle^n$ acting on it by $(\epsilon_i)_i \cdot X_j = \epsilon_j \cdot X_j$. Clearly a polynomial f is invariant under W if and only if it is both invariant under S_n and $\langle -1\rangle^n$. Hence $\mathbb{Q}_l[X_1, ..., X_n]^W \subset \mathbb{Q}_l[X_1, ..., X_n]^{\langle -1\rangle^n}$. As the action of $\langle -1\rangle^n$ preserves monomials, f is invariant if and only if all monomials of f are invariant. So the ring of invariants for $\langle -1\rangle^n$ is exactly $\mathbb{Q}_l[X_1^2, ..., X_n^2]$. This subring inherits the action of S_n. As $\mathbb{Q}_l[X_1^2, ..., X_n^2]$ is a polynomial ring in variables $\{X_i^2\}$, the indecomposable invariants are the elementary symmetric functions $\{e_j\}$ in variables $\{X_i^2\}$. Thus we get $\mathbb{Q}_l[X_1, ..., X_n]^W = \mathbb{Q}_l[X_1^2, ..., X_n^2]^{S_n} = \mathbb{Q}_l[e_1(X_1^2, ..., X_n^2), ..., e_n(X_1^2, ..., X_n^2)]$. Using Proposition 5.40 this then gives $\mathrm{H}^*_{\text{ét}}(\mathrm{Sp}_{2n}, \mathbb{Q}_l) \cong \bigwedge^*(\bigoplus_{1\leq k\leq n} \mathbb{Q}_l^{(4k-1)})$. □

Remark 5.45. As we saw in Proposition 5.41 that Sp_{2n} and SO_{2n+1} have isomorphic maximal tori and isomorphic Weyl groups that act in the same way on the torus, we obtain:

$$\mathrm{H}^*_{\text{ét}}(\mathrm{SO}_{2n+1}, \mathbb{Q}_l) \cong \bigwedge^*(\bigoplus_{1\leq k\leq n} \mathbb{Q}_l^{(4k-1)})$$

Last we deal with the case SO_{2n}.

Proposition 5.46. *For $n \geq 4$ there is an isomorphism of graded $\mathbb{Q}_l$-algebras:*

$$\mathrm{H}^*_{\text{ét}}(\mathrm{SO}_{2n}, \mathbb{Q}_l) \cong \bigwedge^*((\bigoplus_{1\leq k\leq n-1} \mathbb{Q}_l^{4k-1}) \oplus \mathbb{Q}_l^{(2n-1)})$$

Proof. By Proposition 5.41 a maximal torus of SO_{2n} is given by $T = \{\mathrm{diag}(t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1})\}$ and the Weyl group W is isomorphic to $H_{n-1} \rtimes S_n$ with S_n acting in the same way as in the

case of Sp_{2n} and $H_{n-1} := \ker(\mu : \langle -1\rangle^n \to \langle -1\rangle)$ also acting in the same way as $\langle -1\rangle^n$ in the case of Sp_{2n}. This means that $\mathrm{Sym}(X(T)\otimes\mathbb{Q}_l) \cong \mathbb{Q}_l[X_1,...,X_n]$ with S_n acting by permuting the variables and H_{n-1} by $(\epsilon_i)_i \cdot X_j = \epsilon_j \cdot X_j$. Notice that $\mathbb{Q}_l[X_1,...,X_n]^{\langle -1\rangle^n \rtimes S_n} \subset \mathbb{Q}_l[X_1,...,X_n]^W$. Hence $\mathbb{Q}_l[e_1(X_1^2,...,X_n^2),...,e_n(X_1^2,...,X_n^2)]$ is contained in the ring of invariants for W. We also have an invariant polynomial e_n as for any $(\epsilon_i)_i \in H_{n-1}$ an even number of ϵ_i are -1. Hence $\mathbb{Q}_l[e_1(X_1^2,...,X_n^2),...,e_{n-1}(X_1^2,...,X_n^2),e_n] \subset \mathbb{Q}_l[X_1,...,X_n]^W$. We claim that this is an equality.
Let f be any invariant polynomial and let $X_1^{\lambda_1}\cdot ... \cdot X_n^{\lambda_n}$ be a monomial occuring in f. Then this monomial is invariant under H_{n-1}. And hence $(X_1^{\lambda_1}\cdot ... \cdot X_n^{\lambda_n})/e_n^{\min_i\{\lambda_i\}} =: X_1^{\mu_1}\cdot ... \cdot X_n^{\mu_n}$ is invariant under H_{n-1}. Since at least one μ_i is 0 this implies that all μ_j are even by invariance under H_{n-1}. Thus f is in the ring $\mathbb{Q}_l[X_1^2,...,X_n^2,e_n]$. This ring inherits the S_n action of $\mathbb{Q}_l[X_1,...,X_n]$ and $\mathbb{Q}_l[X_1^2,...,X_n^2,e_n] \cong \mathbb{Q}_l[Y_1,...,Y_n,Z]/(Z^2 - Y_1\cdot ... \cdot Y_n)$ is an isomorphism, which preserves the action of S_n by permuting the Y_i and acting trivially on Z. Since $Y^2 - Y_1 \cdot ... \cdot Y_n$ is invariant under S_n we can use Lemma 5.42 which provides us with a surjection $\mathbb{Q}_l[Y_1,...,Y_n,Z]^{S_n} \to \mathbb{Q}_l[X_1^2,...,X_n^2,e_n]^{S_n}$. The S_n invariants of $\mathbb{Q}_l[Y_1,...,Y_n,Z]$ are given by $\mathbb{Q}_l[e_1(Y),...,e_n(Y),Z]$ which gives us the equality:

$$\mathbb{Q}_l[X_1,...,X_n]^W = \mathbb{Q}_l[X_1^2,...,X_n^2,e_n]^{S_n} = \mathbb{Q}_l[e_1(X_1^2,...,X_n^2),...,e_{n-1}(X_1^2,...,X_n^2),e_n]$$

The last thing that we do is that we show that $e_1(X_1^2,...,X_n^2),...,e_n$ are algebraically independent. Notice that $\mathbb{Q}_l[e_1(X_1^2,...,X_n^2),...,e_n]$ is a finite module over $\mathbb{Q}_l[e_1(X_1^2,...,X_n^2),...,e_n(X_1^2,..,X_n^2)]$, hence their dimension is the same. Hence the dimension of $\mathbb{Q}_l[e_1(X_1^2,...,X_n^2),...,e_n]$ is n, thus as $e_1(X_1^2,...,X_n^2),...,e_n$ generate it, they are algebraically independent. By Proposition 5.40 we have an isomorphism of graded $\mathbb{Q}_l$-algebras:

$$\mathrm{H}^*_{\text{ét}}(\mathrm{SO}_{2n},\mathbb{Q}_l) \cong \bigwedge^*((\bigoplus_{1\le k\le n-1}\mathbb{Q}_l^{(4k-1)}) \oplus \mathbb{Q}_l^{2n-1})$$

□

Remark 5.47. As a sanity check we compute the highest j with $\mathrm{H}^j_{\text{ét}}(G,\mathbb{Q}_l) \neq 0$ for G any of the three groups in the preceeing calculations. We know that an affine variety X has no cohomology above degree $\dim(X)$.

- SL_n: The dimension of SL_n is n^2-1 as it is the kernel of the quotient map $\det : \mathrm{GL}_n \to \mathbb{G}_m$. The highest j such that $\mathrm{H}^j_{\text{ét}}(\mathrm{SL}_n,\mathbb{Q}_l) \neq 0$ is $j = \sum_{k=2}^n (2k-1) = \sum_{k=1}^n (2k-1) - 1 = 2\cdot\frac{n\cdot(n+1)}{2} - n - 1 = n^2 - 1$.
- Sp_{2n}: The dimension of Sp_{2n} is $2n^2+n$.
The highest j such that $\mathrm{H}^j_{\text{ét}}(\mathrm{Sp}_{2n},\mathbb{Q}_l) \neq 0$ is $j = \sum_{k=1}^n (4k-1) = 4\cdot\frac{n(n+1)}{2} - n = 2n^2+n$.
- SO_n: The dimension of SO_n is $n(n-1)/2$. The highest j such that $\mathrm{H}^j_{\text{ét}}(\mathrm{SO}_{2n},\mathbb{Q}_l) \neq 0$ is $j = \sum_{k=1}^{n-1}(4k-1) + 2n - 1 = 2\cdot n\cdot(n-1) - (n-1) + 2n - 1 = (2n^2-2n) + (1-n) + (2n-1) = 2n^2 - n$.
By the previous item we can also tick off SO_{2n+1}.

The sanity check indeed holds.

The claims about the dimensions can be checked by looking at the Lie algebras (tangent spaces at identity element). For $k[\epsilon] = k[X]/(X^2)$ we have $T_e(G) = \{g_\epsilon \in G(k[\epsilon]) \mid g_0 = e \in G(k)\}$ where g_0 is the image of g_ϵ in $G(k)$ under the induced map $G(k[\epsilon]) \to G(k)$ by $\epsilon \to 0$. Recall how $\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2$ were defined in the beginning of this section.
We note that $T_{I_n}(\mathrm{Sp}_{2n})$ consists $A = \epsilon M + I_{2n}$ with $M \in M_{2n}(k)$ such that $A^T \mathcal{J}_1 A = \mathcal{J}_1$, which as k-vectorspace is isomorphic to $\{M \in M_{2n}(k), \mid M^T \mathcal{J}_1 + \mathcal{J}_1 M = 0\}$ from which follows that the dimension of Sp_{2n} is $T_{I_n}(\mathrm{Sp}_{2n}) = 2n^2 + n$.
Two similar computations show that $T_{I_n}(\mathrm{SO}_{2n+1}) \cong \{M \in M_{2n+1}(k) \mid M^T \mathcal{J}_0 + \mathcal{J}_0 M = 0\}$ and that $T_{I_n}(\mathrm{SO}_{2n}) \cong \{M \in M_{2n}(k) \mid M^T \mathcal{J}_2 + \mathcal{J}_2 M = 0\}$ from which it follows that the dimensions are indeed what they are claimed to be in Remark 5.47.

Chapter 6

The l-adic cohomology of semisimple group varieties

We have by Theorem 5.37 that the cohomology of a group variety can be decomposed functorially into the cohomology of a semisimple group variety, a torus and an abelian variety. The l-adic cohomology of the abelian variety and the torus are relatively easy to understand (see Proposition 2.74). In this chapter we focus on the l-adic cohomology of a semisimple group variety and throughout this chapter G will be a semisimple group variety over an algebraically closed field k. For such G one has that there exists a group scheme $G_{\mathbb{Z}}$ over $\mathrm{Spec}(\mathbb{Z})$ such that $G = G_{\mathbb{Z}} \times_{\mathbb{Z}} \mathrm{Spec}(k)$, shown in SGA3 XXV, p.268 [3] and one can consider $G_{\mathbb{C}} := G_{\mathbb{Z}} \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{C})$. There is a comparison isomorphism between $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ and $\mathrm{H}^*_{\text{ét}}(G_{\mathbb{C}}, \mathbb{Q}_l)$ as shown in Friedlander [21]. As for any variety X over $\mathbb{C}$ there is a comparison isomorphism $\mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_l) \cong \mathrm{H}^*_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$, it suffices to calculate the singular cohomology of the complex lie group $G_{\mathbb{C}}(\mathbb{C})$. This was done by Borel [5], who gave the following description where $T \subset G_{\mathbb{C}}(\mathbb{C})$ is a maximal torus:

$$\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l) \cong \mathrm{H}^*_{\text{sing}}(G_{\mathbb{C}}(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \cong \bigwedge^{*}(S^W_+/(S^W_+)^2)[\times 2 - 1] \tag{6.1}$$

So we know all the Betti numbers and the algebra structure of $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$.
Suppose that $\sigma : G \to G$ is an endomorphism of G. Sometimes one wants to compute the pullback morphism σ^* on $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ (for instance we will see that the graded trace of σ^* relates to counting fixed points of σ) and although G lifts to a group variety $G_{\mathbb{C}}$ over $\mathbb{C}$, it need not happen that σ lifts to an endomorphism of $G_{\mathbb{C}}$. This section is devoted to reproving (6.1) but in a functorial manner, i.e. such that one keeps track of what endomorphisms do.

Notation 6.1. Throughout this section we denote:

- $\sigma : G \to G$ will be a fixed surjective endomorphism and (B, T) is a σ-stable Borel pair (which we assume exists). We denote $W = N_G(T)/T$, which may be regarded as a finite group and which acts by conjugation on T.
- We denote $S = \mathrm{Sym}(X(T) \otimes_{\mathbb{Z}} \mathbb{Q}_l)$ and S^W the W-invariants of S. Denote $\mathrm{J} := S^W_+/(S^W_+)^2$.

Note that this σ-stable Borel pair exists if $\mathrm{Fix}(\sigma)$ is finite and σ is surjective by Proposition 3.10.

We must define a pullback morphism σ^* on $\bigwedge^* \mathrm{J}$. This is possible when σ is surjective and T is σ-stable.

Proposition 6.2. *For $\sigma : G \to G$ surjective and T σ-stable there is a pullback homomorphism $\sigma^* : \bigwedge^* \mathrm{J} \to \bigwedge^* \mathrm{J}$.*

Proof. As T is σ-stable we have a pullback homomorphism $\sigma^* : X(T) \to X(T)$ which is defined by $\sigma^*(\chi) = \chi \circ \sigma$. So this defines a linear endomorphism of $\mathbb{Q}_l$-algebras $\sigma^* : S \to S$. Now it suffices to show that σ^* preserves W-invariants in order to show that J inherits the pullback morphism from S. Let $f = f(\chi_1, ..., \chi_n)$ be W-invariant and consider $\sigma^* f = f(\sigma^*\chi_1, ..., \sigma^*\chi_n)$. For $w \in W$ we have $w \cdot \sigma^* f = f((\sigma \circ \gamma_w)^*\chi_1, ..., (\sigma \circ \gamma_w)^*\chi_n)$, where γ_w denotes conjugation by w. Note that $(\sigma \circ \gamma_w) = \gamma_{\sigma(n)} \circ \sigma$ for $n \in N_G(T)$ representing w. As σ is surjective we have that $\sigma(n)T\sigma(n)^{-1} = \sigma(n)\sigma(T)\sigma(n)^{-1} = \sigma(T) = T$ as n normalizes T. So we have $\sigma(n) \in N_G(T)$, so $\gamma_{\sigma(n)}$ equals γ_v for some $v \in W$. Hence we obtain $w \cdot \sigma^* f = \sigma^*(v \cdot f) = \sigma^* f$, so σ^* preserves W-invariants. □

6.1 The cohomology of G/B

In this subsection we do not require that (B, T) is a σ-stable Borel pair. A first step in understanding $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ will be understanding $\mathrm{H}^*_{\text{ét}}(G/B, \mathbb{Q}_l)$. To understand this object, the main tool is the **cellular decomposition** of G/B. For more background on these tools we refer to ([41], Section 8).

Theorem 6.3. *For G a semisimple group variety we have the* ***Bruhat decomposition****:*

$$G = \bigsqcup_{w \in W} BwB$$

Moreover we have the ***cellular decomposition****:*

$$G/B = \bigsqcup_{w \in W} BwB/B$$

The double cosets BwB are open in their closures and for all $w \in W$ there are $v \in W$ such that $\overline{BwB} = \bigsqcup_{v} BvB$.

We now introduce some notation and state several results from ([41], Section 8).

Notation 6.4. We use the following notation in this subsection:

- Denote the set of roots of (G, T) by $\mathcal{R}$ and denote a set of positive roots by $\mathcal{R}^+$. For $w \in W$ we denote $\mathcal{R}(w) := \{\alpha \in \mathcal{R}^+ \,|\, w \cdot \alpha \in -\mathcal{R}^+\}$, which has cardinality $l(w)$.
- For $\alpha \in \mathcal{R}$ we denote by $U_\alpha \subset G$ the unique algebraic subgroup of G for which there is an isomorphism $u_\alpha : \mathbb{G}_a \to U_\alpha$ that satisfies $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for all $t \in T(k)$, $x \in \mathbb{G}_a(k)$.

- For $w \in W$ we denote $U_w = U_{\alpha_1} \cdot ... \cdot U_{\alpha_n}$ where the product is taken over all $\alpha_{i_j} \in \mathcal{R}(w)$. It is isomorphic as a variety to $\mathbb{A}^{l(w)}$.
- The double cosets BwB will be denoted by $C(w)$ and the cells BwB/B by $X(w)$.

We have the following proposition, which is Lemma 8.3.6.(ii) in [41].

Proposition 6.5. *Denote $\dot{w}$ for a lift of w to $N_G(T)$. The morphism*

$$U_w \times B \to C(w) \quad (u, b) \mapsto u\dot{w}b$$

is an isomorphism of varieties. In particular $X(w)$ is isomorphic to $\mathbb{A}^{l(w)}$.

We have the following lemma, which states how we can obtain a filtration of G/B by closed subvarieties.

Lemma 6.6. *The cellular decomposition induces a filtration of G/B by smooth closed subvarieties $Y_m \subset ... \subset Y_0 = G/B$ and such that Y_m and for all $n < m$, $Y_n \setminus Y_{n+1}$ are disjoint unions of affine spaces.*

Proof. For exactly one w_0 we must have $\dim(X(w_0)) = \dim(G/B)$ (this is called the 'big cell'). Hence $\overline{X(w_0)} = G/B$ and hence $X(W_0)$ is an open cell of G/B. Set $W_0 = \{w_0\}$, then the complement $Y_1 := \bigsqcup_{w \notin W_0} X(w)$ is closed. Let Z_1^i be the irreducible components of Y_1. As the $X(w)$ with $w \neq w_0$ are irreducible, so are their closures, hence each of them is contained in at least one Z_1^i. Now each Z_1^i contains some $X(w_1^i)$ of the same dimension and as the $X(w_1^i)$ are open in their closures, we have that $X(w_1^i)$ is open in Z_1^i. Set $W_1 = W_0 \cup \{w_1^i\}$, then we get that $Y_2 := \bigsqcup_{w \notin W_1} X(w)$ is closed inside Y_1 and $Y_2 \setminus Y_1 = \bigsqcup_i X(w_1^i)$. Repeating the process for Y_2 as we did for Y_1 and so on yields the filtration of G/B by closed subvarieties Y_n and the complements are given by $\bigsqcup_i X(w_n^i)$, which are disjoint unions of affine spaces by Proposition 6.5. □

If all the Y_j would be smooth then we would be in a position to use the Gysin sequence and obtain a complete description of the cohomology of G/B. Unfortunately matters are not that simple as the varieties Y_n can in general have singularities. We can use compactly supported cohomology to remedy the situation together with the following statement.

Proposition 6.7. *The variety G/B is projective.*

Proof. It is shown in ([32], p.368) that there is an isomorphism of varieties $G/B \to \mathcal{B}$, where $\mathcal{B}$ is the flag variety of G. In ([32], Proposition 7.30) it is shown that flag varieties are complete. A homogeneous space is quasi-projective and hence smooth by Theorem 1.51. □

Now we calculate the cohomology of G/B.

Proposition 6.8. *For Λ a finite group with $p \nmid \#\Lambda$ we have $\mathrm{H}^r_{\text{ét}}(G/B,\Lambda) = 0$ for every odd r. When $\Lambda \in \{\mathbb{Z}/l^n\mathbb{Z}\}$ we have $\mathrm{H}^r_{\text{ét}}(G/B,\Lambda) = \bigoplus_w \Lambda_{2d_w}$, where $d_w = \dim(X(w))$ and the Λ_{2d_w} are in degree $2d_w$.*

Proof. We consider for $n \geq 2$ the exact sequence of compactly supported cohomology of the pair (Y_n, U_n) ([29], Remark III.1.30) where $U_n = Y_n \setminus Y_{n-1}$, which is a disjoint union of r_n affine spaces:

$$... \to \mathrm{H}^r_{\mathrm{c}}(U_n,\Lambda) \to \mathrm{H}^r_{\mathrm{c}}(Y_n,\Lambda) \to \mathrm{H}^r_{\mathrm{c}}(Y_{n-1},\Lambda) \to \mathrm{H}^{r+1}_{\mathrm{c}}(U_n,\Lambda) \to ...$$

As U_n is a disjoint union of affine spaces $X(w^j_n)$ of dimensions d^j_i where $1 \leq j \leq r_n$ we have that $\mathrm{H}^*_{\mathrm{c}}(U_n,\Lambda) = \prod_j \mathrm{H}^*_{\mathrm{c}}(X(w^j_n),\Lambda) = \prod_j \Lambda_{2d^j_i}$ where the subscript $2d^j_i$ means that the cohomology is concentrated in degree $2d^j_i$. For proving the the vanishing when r is odd: Note that by Poincaré duality it suffices to show that $\mathrm{H}^r_{\mathrm{c}}(G/B,\Lambda) = 0$ for all odd r. Now we may prove the first statement by decreasing induction on the Y_n. Notice that Y_m, the smallest proper closed subset in the filtration, is a disjoint union of affine spaces, hence it indeed has $\mathrm{H}^r_{\mathrm{c}}(Y_m,\Lambda) = 0$ for all odd r. For the induction step, consider for odd r the part of the exact sequence that reads $\mathrm{H}^r_{\mathrm{c}}(U_n,\Lambda) \to \mathrm{H}^r_{\mathrm{c}}(Y_n,\Lambda) \to \mathrm{H}^r_{\mathrm{c}}(Y_{n-1},\Lambda)$, which by the induction hypothesis for Y_{n-1} directly implies that $\mathrm{H}^r_{\mathrm{c}}(Y_n,\Lambda) = 0$.
For the second statement we will show by decreasing induction on the Y_n that $\mathrm{H}^*_{\mathrm{c}}(Y_n) = \bigoplus_{X(w)\subset Y_n} \Lambda_{2d_w}$, where $d_w = \dim(X(w))$. This is clear for the case $n = m$. For the induction step, consider for even r the exact sequence of Λ-modules $0 \to \mathrm{H}^r_{\mathrm{c}}(U_n,\Lambda) \to \mathrm{H}^r_{\mathrm{c}}(Y_n,\Lambda) \to \mathrm{H}^r_{\mathrm{c}}(Y_{n-1},\Lambda) \to 0$. By the induction step this sequence reads $0 \to \bigoplus_{\substack{X(w)\subset U_n \\ 2d_w = r}} \Lambda \to \mathrm{H}^r_{\mathrm{c}}(Y_n,\Lambda) \to \bigoplus_{\substack{X(w)\subset Y_{n-1} \\ 2d_w = r}} \Lambda \to 0$. This exact sequence splits and note that the $X(w)$ inside U_n and those inside Y_{n-1} precisely make up those inside Y_n. Hence we obtain $\mathrm{H}^r_{\mathrm{c}}(Y_n,\Lambda) = \bigoplus_{\substack{2d_w = r \\ X(w)\subset Y_n}} \Lambda$, which proves the induction step. Now using that G/B is projective gives the expression $\mathrm{H}^r_{\text{ét}}(G/B,\Lambda) = \bigoplus_w \Lambda_{2d_w}$. □

We want to compare $\mathrm{H}^*_{\text{ét}}(G/B,\mathbb{Q}_l)$ with the cycle class group $A^*(G/B)$. Actually we will show that the cycle morphism $\mathrm{cl}^{G/B}$ is an isomorphism.

Notation 6.9.

- We denote $U_i := X \setminus Y_i$. This gives a filtration $X = U_0 \supset ... \supset U_n = \emptyset$ of X by open subschemes such that $U_i \setminus U_{i+1} = V_{i+1}$.
- To ease notation somewhat we will write $\mathrm{A}^r(S)$ (resp. $\mathrm{A}_r(S)$) for $\mathrm{A}^r(S)\otimes_{\mathbb{Z}}\mathbb{Q}_l$ (resp. $\mathrm{A}_r(S)\otimes_{\mathbb{Z}} \mathbb{Q}_l$). Moreover we write $\mathrm{H}^r_{\text{ét}}(S) = \mathrm{H}^r_{\text{ét}}(S,\mathbb{Q}_l)$.

Our main strategy for showing that the Chow ring and cohomology ring are isomorphic will be as follows: We will consider the sequence in étale cohomology that is associated to the pair (U_i, V_{i+1}) that is given by: $... \to \mathrm{H}^{2r-1}_{\text{ét}}(U_{i+1}) \to \mathrm{H}^{2r}_{V_{i+1}}(U_i) \to \mathrm{H}^{2r}_{\text{ét}}(U_i) \to \mathrm{H}^{2r}_{\text{ét}}(U_{i+1}) \to ...$
We will then construct a map $\mathrm{A}_{d-r}(V_{i+1}) \to \mathrm{H}^r_{V_{i+1}}(U_i)$ such that the following diagram commutes:

$$\begin{array}{ccccccc} \mathrm{H}^{2r-1}_{\text{ét}}(U_{i+1}) & \longrightarrow & \mathrm{H}^{2r}_{V_{i+1}}(U_i) & \longrightarrow & \mathrm{H}^{2r}_{\text{ét}}(U_i) & \longrightarrow & \mathrm{H}^{2r}_{\text{ét}}(U_{i+1}) & \\ & & \uparrow & & \uparrow {\scriptstyle \mathrm{cl}^{U_i}} & & \uparrow {\scriptstyle \mathrm{cl}^{U_{i+1}}} & \\ & & \mathrm{A}_{d-r}(V_{i+1}) & \longrightarrow & \mathrm{A}_{d-r}(U_i) & \longrightarrow & \mathrm{A}_{d-r}(U_{i+1}) & \longrightarrow 0 \end{array} \tag{6.2}$$

Then we show that $\mathrm{H}^j_{\text{ét}}(U_i) = 0$ for all odd j. We will apply decreasing induction on i together with the fact that our constructed map $\mathrm{A}_{d-r}(V_{i+1}) \to \mathrm{H}^r_{V_i}(U_i)$ will be an isomorphism to conclude by the Snake lemma that cl^{U_i} is an isomorphism. By induction this then implies that cl^X is an isomorphism. We begin with constructing the map $\mathrm{A}_{d-r}(V_{i+1}) \to \mathrm{H}^r_{V_i}(U_i)$, which is the hardest part.

The map $\mathrm{A}_{d-r}(V_{i+1}) \to \mathrm{H}^r_{V_{i+1}}(U_i)$

As we have $V_{i+1} = \bigsqcup_j X(w^j_{i+1})$, we get that the inclusion maps $X(w_{i+1}) \to V_{i+1}$ induce an isomorphism $\bigoplus_j \mathrm{A}_{d-r}(X(w^j_{i+1})) \to \mathrm{A}_{d-r}(V_{i+1})$. Let c_j be the codimension of $X(w^j_{i+1})$ inside V_{i+1}. Then note that for $V^r_{i+1} := \bigsqcup_{c_j = d-r} X(w^j_{i+1})$ we have that $\mathrm{A}_r(V^r_{i+1}) \to \mathrm{A}_r(V_{i+1})$ is an isomorphism.
On the other hand, the exact sequence associated to the triple of opens ([29], III remark 1.26) $(U_i, U_i \setminus V^r_{i+1}), U_i \setminus V_{i+1})$ gives a long exact sequence

$$... \to \mathrm{H}^{2r-1}_{V_{i+1} \setminus V^r_{i+1}}(U_i \setminus V^r_{i+1}) \to \mathrm{H}^{2r}_{V^r_{i+1}}(U_i) \to \mathrm{H}^{2r}_{V_{i+1}}(U_i) \to \mathrm{H}^{2r}_{V_{i+1} \setminus V^r_{i+1}}(U_i \setminus V^r_{i+1}) \to ...$$

We claim that $\mathrm{H}^{2r}_{V_{i+1} \setminus V^r_{i+1}}(U_i \setminus V^r_{i+1})$ and $\mathrm{H}^{2r-1}_{V_{i+1} \setminus V^r_{i+1}}(U_i \setminus V^r_{i+1})$ vanish for all r. Note that we have $V_{i+1} = \bigsqcup_s V^s_{i+1}$. For $s \neq r$ the triple $(U_i \setminus V^r_{i+1}, U_i \setminus (V^r_{i+1} \sqcup V^s_{i+1}), U_i \setminus V_{i+1})$ gives an exact sequence:

$$... \to \mathrm{H}^p_{V^s_{i+1}}(U_i \setminus V^r_{i+1}) \to \mathrm{H}^p_{\bigsqcup_{s \neq r} V^s_{i+1}}(U_i \setminus V^r_{i+1}) \to \mathrm{H}^p_{\bigsqcup_{j \neq s,r} V^j_{i+1}}(U_i \setminus (V^r_{i+1} \sqcup V^s_{i+1})) \to ...$$

As V^s_{i+1} is smooth of codimension $d - s \neq d - r$ we have $\mathrm{H}^p_{V^s_{i+1}}(U_i \setminus V^r_{i+1}) \cong \mathrm{H}^{p-2(d-s)}_{\text{ét}}(V^s_{i+1})$ by the Gysin isomorphism and as V^s_{i+1} is a disjoint union of affine spaces this vanishes for all $p \neq 2(d-s)$ and so in particular for all odd p and $p = 2r$. Now the statement that $\mathrm{H}^{2r-1}_{V_{i+1} \setminus V^r_{i+1}}(U_i \setminus V^r_{i+1})$ and $\mathrm{H}^{2r}_{V_{i+1} \setminus V^r_{i+1}}(U_i \setminus V^r_{i+1})$ vanish follows by induction on cutting out the V^j_{i+1} out of $\bigsqcup_{j \neq s,r} V^j_{i+1}$ and repeatedly applying the sequence of triples together with the above arugment. We conclude that $\mathrm{H}^{2r}_{V^r_{i+1}}(U_i) \to \mathrm{H}^{2r}_{V_{i+1}}(U_i)$ is an isomorphism.
As the sequence of triples is functorial in the triples (see Remark 6.11) and as the sequence of the pair (U_i, V_{i+1}) is a special case of a sequence of triples we have that the following diagram commutes:

$$\begin{array}{ccc} & \mathrm{H}^{2r}_{\text{ét}}(U_i) & \\ \nearrow & & \nwarrow \\ \mathrm{H}^{2r}_{V_{i+1}}(U_i) & \longleftarrow & \mathrm{H}^{2r}_{V^r_{i+1}}(U_{i+1}) \end{array}$$

As V_i^r is smooth of codimension r inside U_i, there is a Gysin isomorphism $\mathrm{H}^{2r}_{V^r_{i+1}}(U_i) \cong \mathrm{H}^0(V^r_{i+1})$ such that the following diagram commutes:

$$\begin{array}{ccc} & \mathrm{H}^{2r}_{\text{ét}}(U_i) & \\ \nearrow & & \nwarrow \\ \mathrm{H}^0_{\text{ét}}(V^r_{i+1}) & \longleftarrow & \mathrm{H}^{2r}_{V^r_{i+1}}(U_{i+1}) \end{array}$$

By ([29], VI Prop 9.3) the cycle maps of V^r_{i+1} and U_i are compatible, i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathrm{A}^0(V^r_{i+1}) & \longrightarrow & \mathrm{H}^0_{\text{ét}}(V^r_{i+1}) \\ \downarrow & & \downarrow \\ \mathrm{A}^r(U_i) & \longrightarrow & \mathrm{H}^r_{\text{ét}}(V^r_{i+1}) \end{array}$$

Now as $\mathrm{A}_{d-r}(V_{i+1}) = \mathrm{A}^0(V^r_{i+1})$ we simply define $\mathrm{A}_{d-r}(V_{i+1}) \to \mathrm{H}^{2r}_{V_{i+1}}(U_i)$ by taking the map induced from the cycle map $\mathrm{A}^0(V^r_{i+1}) \to \mathrm{H}^0_{\text{ét}}(V^r_{i+1})$. After showing that in (6.2) we have $\mathrm{H}^j_{\text{ét}}(U_i) = 0$ for all odd j this will imply that (6.2) commutes by all the above diagrams. It is also clear that as $\mathrm{A}^0(V^r_{i+1}) \to \mathrm{H}^0_{\text{ét}}(V^r_{i+1})$ that $\mathrm{A}_{d-r}(V_{i+1}) \to \mathrm{H}^{2r}_{V_{i+1}}(U_i)$ is an isomorphism.

The last missing piece is showing that the odd cohomology of U_i is trivial for all i. As the U_i are smooth this is equivalent to showing that the odd compactly supported is trivial. We will prove this by using descreasing induction. We have already done the case $i = n-1$ (the highest i) as $U_{n-1} = X \setminus Y_{n-1}$ is the big cell (an affine space). From $\mathrm{H}^r_{\mathrm{c}}(U_{i+1}) \to \mathrm{H}^r_{\mathrm{c}}(U_i) \to \mathrm{H}^r_{\mathrm{c}}(V_{i+1})$, which is exact, the statement now follows from induction.

Theorem 6.10. *The l-adic cycle map* $\mathrm{cl}^X : \mathrm{A}^*(X) \to \mathrm{H}^*_{\text{ét}}(X)$ *is an isomorphism of graded rings (where the grading on* $\mathrm{A}^*(X)$ *is doubled).*

Proof. The l-adic cycle map is a homomorphism of graded rings, so it suffices to show that it is bijective on all the graded pieces. Note that $\mathrm{cl}^{U_{n-1}} : \mathrm{A}^*(U_{n-1}) \to \mathrm{H}^*_{\text{ét}}(U_{n-1})$ is an isomorphism as U_{n-1} is an affine space. We now assume that the cycle map is an isomorphism for U_{i+1} for some i. Then consider the diagram (6.2) which exists as we have shown the existence of the isomorphism $\mathrm{A}_{d-r}(V_{i+1}) \to \mathrm{H}^{2r}_{V_{i+1}}(U_i)$ and that $\mathrm{H}^{2r-1}_{\text{ét}}(U_{i+1}) = 0$. By the snake lemma we conclude that cl^{U_i} is also an isomorphism. So by induction cl^X is an isomorphism. □

Remark 6.11. Associated to a triple of opens (X, U, V) with $V \subset U \subset X$ and $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ there is a long exact sequence in cohomology that is contravariantly functorial in triples $(X, U, V) \hookrightarrow (X, U', V')$ and covariantly in $\mathcal{F}$:

$$\ldots \to \mathrm{H}^r_{X\setminus U}(X, \mathcal{F}) \to \mathrm{H}^r_{X\setminus V}(X, \mathcal{F}) \to \mathrm{H}^r_{U\setminus V}(U, \mathcal{F}|_U) \to \mathrm{H}^{r+1}_{X\setminus V}(X, \mathcal{F}) \to \ldots$$

Its construction is based on ([29] III Prop 1.25, Remark 1.26). Use the notation $j_U : U \to V$, $i : U \setminus V \to U$, $i_V : X \setminus V \to X$ and $i_U : X \setminus U \to X$. Let $\mathbb{Z}$ be the constant sheaf on $X_{\text{ét}}$. Then there is a short exact sequence of sheafs on $X_{\text{ét}}$:

$$0 \to j!i_*i^*j^*\mathbb{Z} \to (i_V)_*i_V^*\mathbb{Z} \to (i_U)_*i_U^*\mathbb{Z} \to 0 \tag{6.3}$$

To check that it is exact, look at the stalks, where it shows that $j!i_* i^* j^* \mathbb{Z}$ has support on $U \setminus V$, $(i_V)_* i_V^* \mathbb{Z}$ on $X \setminus V$ and $(i_U)_* i_U^* \mathbb{Z}$ on $X \setminus U$. We consider the long exact sequence in cohomology that we get from applying $\mathrm{Hom}_{\mathrm{Sh}(X_{\acute{e}t})}(-, \mathcal{F})$:

$$... \to \mathrm{Ext}^r((i_U)_* i_U^* \mathbb{Z}, \mathcal{F}) \to \mathrm{Ext}^r((i_V)_* i_V^* \mathbb{Z}, \mathcal{F}) \to \mathrm{Ext}^r(j!i_* i^* j^* \mathbb{Z}, \mathcal{F}) \to \mathrm{Ext}^{r+1}((i_U)_* i_U^* \mathbb{Z}, \mathcal{F}) \to ... \tag{6.4}$$

Then by the proof of ([29] III Prop 1.25) this is after some identifications the sequence:

$$... \to \mathrm{H}^r_{X \setminus U}(X) \to \mathrm{H}^r_{X \setminus V}(X) \to \mathrm{H}^r_{U \setminus V}(U) \to \mathrm{H}^{r+1}_{X \setminus U}(X) \to ...$$

A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ on $X_{\acute{e}t}$ defines a morphism of δ-functors $\mathrm{Ext}^*(-, \mathcal{F}) \to \mathrm{Ext}^*(-, \mathcal{G})$ and hence one gets the functoriality in $\mathcal{F}$. An inclusion of triples $(\mathrm{Id}, \iota_1, \iota_2) : (X, U, V) \hookrightarrow (X, U_1, V_1)$ gives a morphism of short exact sequences in (6.3) (from the one from (X, U, V) to the one from (X, U_1, V_1)). Now applying that $\mathrm{Ext}^*(-, \mathcal{F})$ is a δ-functor gives that there is an induced morphism of long exact sequences in (6.4) (now from the one from (X, U_1, V_1) to the one from (X, U, V)) which gives the desired functoriality.

6.2 The Leray spectral sequence of $G \to G/T$

For this section we switch back to the notation introduced in Notation 6.1, so (B, T) is a σ-stable Borel pair. We will investigate the Leray spectral sequence associated to $\pi : G \to G/T$. This was also done by Leray [28] in the case where G is a compact Lie group. This is a Zariski locally trivial fibration as it is a principle T-bundle and T is linear, commutative.

We have the following general properties on the Leray spectral sequence. We will just state them without giving a proof. A proof of the statements concerning cup products can be found in ([45], p.202). The statements for the pullbacks are formulated in the language of derived categories. Refences on this topic are ([12], Section 1) and ([13], Section 7).

Proposition 6.12. *Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{F} \in \mathrm{Sh}(X)$ be a sheaf of Λ-modules. The Leray spectral sequence associated to f has the following properties:*

1. ***Cup products:*** *The E_2-page of the Leray spectral sequence has a product:*

$$\mathrm{H}^p_{\acute{e}t}(Y, R^q f_* \mathcal{F}) \otimes_\Lambda \mathrm{H}^{p'}_{\acute{e}t}(Y, R^{q'} f_* \mathcal{F}) \to \mathrm{H}^{p+p'}_{\acute{e}t}(Y, R^{q+q'} f_* \mathcal{F})$$

It induces a product on $\mathrm{Tot}(E_\infty^{*,*})$ which equals the product on $\mathrm{H}^*_{\acute{e}t}(X, \mathcal{F})$. The product on the E_2-page is given by first applying the cup product of Y:

$$\cup : \mathrm{H}^p_{\acute{e}t}(Y, R^q f_* \mathcal{F}) \otimes_\Lambda \mathrm{H}^{p'}_{\acute{e}t}(Y, R^{q'} f_* \mathcal{F}) \to \mathrm{H}^{p+p'}_{\acute{e}t}(Y, R^q f_* \mathcal{F} \otimes_\Lambda R^{q'} f_* \mathcal{F})$$

and then applying $\mathrm{H}^{p+p'}_{\acute{e}t}(Y, \phi)$, where $\phi : R^q f_* \mathcal{F} \otimes_\Lambda R^{q'} f_* \mathcal{F} \to R^{q+q'} f_* \mathcal{F}$ is the morphism of sheaves that is induced by the cup products on $U \times_Y X$ (note that indeed this induces cup products as $R^q f_* \mathcal{F}$ is the sheafification of $U \mapsto \mathrm{H}^q_{\acute{e}t}(U \times_Y X, \mathcal{F})$).
Then this composition is multiplied by $(-1)^{qp'}$ to give a product $E_2^{p,q} \otimes_\Lambda E_2^{p',q'} \to E_2^{p+p',q+q'}$.

2. ***Differentials and cup products:*** *The differentials and the products satisfy the following relation: Denote the differentials by $d_r^{s,t} : E_r^{s,t} \to E_r^{s+r,t-(r-1)}$. Then for $x \in E_r^{p+p'}$ and $y \in E_r^{q+q'}$ we have $d_r^{p+p',q+q'}(x \cdot y) = d_r^{p,q}(x) \cdot y + (-1)^{p+q} x \cdot d_r^{p',q'}(y)$.*

3. ***Pullback morphism:*** *Suppose that we have a commuting diagram of the form* $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow h \\ X' & \xrightarrow{f'} & Y' \end{array}$ *and that we have* $\mathcal{F} \in \mathrm{Sh}(X')$*. Denote by* E_r *the Leray spectral sequence associated to* f *and* $g^*\mathcal{F}$ *and* E'_r *the one associated to* f' *and* $\mathcal{F}$*. Then there is a morphism of Leray spectral sequences* $E'_r \to E_r$ *such that the induced morphism* $\mathrm{Tot}(E'_\infty) \to \mathrm{Tot}(E_\infty)$ *coincides with the normal pullback morphism* $g^* : \mathrm{H}^*_{\text{ét}}(X', \mathcal{F}) \to \mathrm{H}^*_{\text{ét}}(X, g^*\mathcal{F})$*.*
   The morphism of the spectral sequences is induced by how it is defined on the E_2*-page, where one defines* $E_2^{p,q} \to E_2'^{p,q}$ *as follows:*
   First one takes the pullback $h^* : \mathrm{H}^p_{\text{ét}}(Y', R^q f'_*\mathcal{F}) \to \mathrm{H}^p_{\text{ét}}(Y, h^* R^q f'_*\mathcal{F})$*. Then one applies* $\mathrm{H}^p_{\text{ét}}(Y, \phi)$*, where* ϕ *denotes the base-change morphism of sheafs:* $\phi : h^* R^q f'_*\mathcal{F} \to R^q f_* g^*\mathcal{F}$ *(this base-change morphism exists by the commuting diagram, as* g *maps* $U \times_Y X$ *into* $V \times_{Y'} X'$ *for any* $V \to Y'$ *étale such that there is a commuting square*

$$\begin{array}{ccc} U & \longrightarrow & Y \\ \uparrow & & h\uparrow \\ V & \longrightarrow & Y' \end{array} \;).$$

4. ***Fibrations with trivial local system:*** *Let* $\Lambda \in \{\mathbb{Z}/l^n\mathbb{Z}\}$ *and let* $\pi : X \to Y$ *be an étale locally trivial fibration with constant fibre* F *such that* $R^q\pi_*\Lambda = \mathrm{H}^q_{\text{ét}}(F, \Lambda)$ *is a free* Λ*-module. For* E_r *the Leray spectral sequence corresponding to* π *and* Λ *we then have an isomorphism* $E_2^{p,q} \cong \mathrm{H}^p_{\text{ét}}(Y, \Lambda) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(F, \Lambda)$ *such that:*

   - *The product on* $E_2^{p,q}$ *is equal to:*
     $(\mathrm{H}^p_{\text{ét}}(Y, \Lambda) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(F, \Lambda)) \otimes_\Lambda (\mathrm{H}^{p'}_{\text{ét}}(Y, \Lambda) \otimes_\Lambda \mathrm{H}^{q'}_{\text{ét}}(F, \Lambda)) \to \mathrm{H}^{p+p'}_{\text{ét}}(Y, \Lambda) \otimes_\Lambda \mathrm{H}^{q+q'}_{\text{ét}}(F, \Lambda)$
     $(a_p \otimes b_q) \otimes (c_{p'} \otimes d_{q'}) \mapsto (-1)^{p'q}(a_p \cup c_{p'}) \otimes (b_q \cup d_{q'})$
     In particular $E_2^{*,*}$ *is then a differential bigraded algebra.*
   - *If we have a morphism of fibrations such as in the second point*

     $$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & Y \\ \downarrow f & & \downarrow & & \downarrow g \\ F' & \longrightarrow & X' & \longrightarrow & Y' \end{array}$$

     then morphism of Leray spectral sequences induced by this is given on the E_2*-page by* $\mathrm{H}^p_{\text{ét}}(Y', \Lambda) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(F', \Lambda) \overset{g^* \otimes f^*}{\to} \mathrm{H}^p_{\text{ét}}(Y, \Lambda) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(F, \Lambda)$*.*

We have knowledge of the on the cohomology of the fibre T of the fibration. It turns out that we also have knowledge on the cohomology of G/T.

Lemma 6.13. *Let* Λ *be a finite group such that* $p \nmid \#\Lambda$*. The natural map* $\pi : G/T \to G/B$ *induces an isomorphism of graded rings* $\pi^* : \mathrm{H}^*_{\text{ét}}(G/T, \Lambda) \to \mathrm{H}^*_{\text{ét}}(G/B, \Lambda)$*.*

Proof. The fibre of $G/T \to G/B$ is $R_u(B)$, which is isomorphic to an affine space and has no higher cohomology. In particular this gives that $\pi : G/T \to G/B$ is étale locally trivial by Proposition 4.34. By applying Lemma 5.14 we get that $\pi^* : \mathrm{H}^*_{\text{ét}}(G/T, \Lambda) \to \mathrm{H}^*_{\text{ét}}(G/B, \Lambda)$ is an isomorphism. ☐

We have the following lemma on locally constant sheafs.

Lemma 6.14 (Stacks[43, Tag 0DV4]). *There is a bijection:*

$$\{\textit{finite locally constant sheafs on } X_{\acute{e}t}\} \longleftrightarrow \{\textit{finite } \pi^1_{\acute{e}t} \textit{ sets}\}$$

We now investigate the locally constant sheaf $R^q\pi_*\Lambda$ for $\pi : G \to G/T$.

Lemma 6.15. *Let G be a semisimple group variety and let $\pi : G \to G/T$ be the canonical map and $\Lambda_G \in \{(\mathbb{Z}/l^n\mathbb{Z})_G\}$. The sheafs $R^q\pi_*\Lambda_G$ are equal to the constant sheaf $\mathrm{H}^q_{\acute{e}t}(T, \Lambda)_{G/T}$ for all $q \geq 0$.*

Proof. We have $\mathrm{H}^1_{\text{ét}}(\mathbb{G}_m, \Lambda) = \Lambda$, which follows from the Kummer sequence. Hence flatness of Λ over itself implies by the Künneth-formula that $\mathrm{H}^q_{\text{ét}}(T, \Lambda) = \bigwedge^q(\oplus_n\Lambda)$ for $T = \mathbb{G}^n_m$. This is a flat Λ-module, hence $R^q\pi_*\Lambda$ is a finite locally constant sheaf for all $q \geq 0$ by Lemma 5.11. By Lemma 6.14 it suffices to show that $\pi^1_{\text{ét}}(G/T) = 0$. We have $\mathrm{H}^1_{\text{ét}}(G/T, \Lambda) = 0$ for all finite Λ with $\#\Lambda$ coprime to p by Lemma 6.13 and Proposition 6.8. As $\mathrm{H}^1_{\text{ét}}(G/T, \Lambda) = \mathrm{Hom}(\pi^1_{\text{ét}}(G/T), \Lambda) = 0$ for all finite Λ we obtain $\pi^1_{\text{ét}}(G/T) = 0$ and hence we obtain the result. □

Using this we obtain information about the differentials on the E_2-page. As all the terms inside the Leray spectral sequences associated to $\pi : G \to G/T$ and $\mathbb{Z}/l^n\mathbb{Z}$ are finite (as sets) we may pass to the limit in the following lemma.

Lemma 6.16. *On the E_2-page of the Leray spectral sequence, the differentials are completely determined by an isomorphism $d_2^{0,1} : \mathrm{H}^1_{\text{ét}}(T, \mathbb{Q}_l) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(G/T, \mathbb{Q}_l)$ and are given by:*

$$d_2^{p,q} : \mathrm{H}^p_{\text{ét}}(G/T, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \bigwedge^q \mathrm{H}^1_{\text{ét}}(T, \mathbb{Q}_l) \to \mathrm{H}^{p+2}_{\text{ét}}(G/T, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \bigwedge^{q-1} \mathrm{H}^1_{\text{ét}}(T, \mathbb{Q}_l)$$
$$b \otimes s_1 \wedge ... \wedge s_q \mapsto \sum_{i=1}^q (-1)^{i+1} b \cdot d_2^{0,1}(s_i) \otimes s_1 \wedge ... \wedge \hat{s_i} \wedge ... \wedge s_q$$

Proof. Notice that by Propositition 6.12 we have $E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(G/T, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^q_{\text{ét}}(T, \mathbb{Q}_l)$ and the product structure is induced by the ones on $\mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ and $\mathrm{H}^*_{\text{ét}}(T, \mathbb{Q}_l)$. We also know that $\bigwedge^* \mathrm{H}^1_{\text{ét}}(T, \mathbb{Q}_l) = \mathrm{H}^*_{\text{ét}}(T, \mathbb{Q}_l)$. Note that for $r \geq 3$ all the outgoing differentials from $E_3^{0,1}$ are zero. All the incoming differentials to $E_r^{0,1}$ are zero for $r \geq 2$. So $\ker(d_2^{0,1}) = E_\infty^{0,1}$. As G is semisimple we have $\mathrm{H}^1_{\text{ét}}(G, \mathbb{Q}_l) = 0$. Indeed we have $\mathrm{H}^1_{\text{ét}}(G, \mathbb{Z}/l^n\mathbb{Z}) = \mathrm{Ext}(G, \mathbb{Z}/l^n\mathbb{Z})$ by (Theorem 2 [33]) and $\mathrm{Ext}(G, \mathbb{Z}/l^n\mathbb{Z})$ is finite as the semisimple group variety G admits a universal covering map with finite kernel (Ch18 [32]). So this gives $\ker(d_2^{0,1}) = 0$. Notice that all outgoing differentials from $E_r^{2,0}$ are zero for $r \geq 2$ and all incoming differentials are zero for $r \geq 3$. So $E_\infty^{2,0} = \mathrm{coker}(d_2^{0,1})$. As $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ is a finite $\mathbb{Q}_l$-Hopf algebra we have that it is generated by elements of odd degree. Hence $\mathrm{H}^2_{\text{ét}}(G, \mathbb{Q}_l) = 0$. So $d_2^{0,1}$ is surjective, hence it is an isomorphism of $\mathbb{Q}_l$-algebras.
By Proposition 6.12, we know how the differentials and the product on the E_2-page interact and the $(-1)^{pq}$ term that appears when taking the product (see the same proposition) can be taken to be 1 as $\mathrm{H}^p_{\text{ét}}(G/T, \mathbb{Q}_l) = 0$ when p is odd. We have $d(b) = 0$ (omitting the super/sub-scripts here) for all $b \in \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$. So an element $b \otimes s_1 \wedge ... \wedge s_q \in \mathrm{H}^p_{\text{ét}}(G/T, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \bigwedge^* \mathrm{H}^1_{\text{ét}}(T, \mathbb{Q}_l)$ is sent to $0 + (-1)^{p+q} b \cdot d(s_1 \wedge ... \wedge s_q)$ by the differential. Repeatedly applying the second point in Proposition 6.12 (which lowers the degree of whatever is inside $d(-)$ by 1, which flips the sign) gives that $d(b \otimes s_1 \wedge ... \wedge s_q) = \sum_{i=1}^q (-1)^{i+1} b \cdot d_2^{0,1}(s_i) \otimes s_1 \wedge ... \wedge \hat{s_i} \wedge ... \wedge s_q$. □

To extract information out of this we first state a helpful theorem by Demazure.

Theorem 6.17 (Demazure [15], 4.6 p.79)**.** *Let G be a semisimple group variety. There exists a homomorphism $X(T) \to \mathrm{Pic}(G/B)$ that extends to a homomorphism $S \to \mathrm{A}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ of graded $\mathbb{Q}_l$-algebras with kernel $S_+^W \cdot S$.*

Ideally one would prove that $d_2^{0,1} : S \to \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ is induced by this map, but this unfortunately does not follow directly. Knowing that $\dim_{\mathbb{Q}_l}(S/S_+^W \cdot S) = \dim_{\mathbb{Q}_l} \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ is very helpful in the proof of the upcoming proposition.

Remark 6.18. There is an identification $\mathrm{H}^1_{\text{ét}}(T, \mathbb{Q}_l) \cong X(T) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ that is functorial in pullbacks $\sigma : G \to G$ by endomorphisms. As G/T has no odd cohomology we have that $\mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ is a commutative $\mathbb{Q}_l$-algebra. So $d_2^{0,1}$ can be extended to a homomorphism of graded $\mathbb{Q}_l$-algebras $d_2^{0,1} : S[\times 2] \to \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$.
We have that $N_G(T)$ acts on T by conjugation: $n \cdot t = \gamma_n(t) := ntn^{-1}$. This gives the following commuting diagram:

$$\begin{array}{ccccc} T & \longrightarrow & G & \longrightarrow & G/T \\ \downarrow{\scriptstyle \gamma_n} & & \downarrow{\scriptstyle \gamma_n} & & \downarrow{\scriptstyle \gamma_n} \\ T & \longrightarrow & G & \longrightarrow & G/T \end{array}$$

The last point of Proposition 6.12 this gives a morphism of Leray spectral sequences such that the following diagram commutes:

$$\begin{array}{ccc} X(T) \otimes_{\mathbb{Z}} \mathbb{Q}_l & \xrightarrow{d_2^{0,1}} & \mathrm{H}^2_{\text{ét}}(G/T, \mathbb{Q}_l) \\ {\scriptstyle \gamma_n^*}\uparrow & & {\scriptstyle \gamma_n^*}\uparrow \\ X(T) \otimes_{\mathbb{Z}} \mathbb{Q}_l & \xrightarrow{d_2^{0,1}} & \mathrm{H}^2_{\text{ét}}(G/T, \mathbb{Q}_l) \end{array}$$

Of course γ_n^* extends to automorphisms of graded $\mathbb{Q}_l$-algebras on $\mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ and S. As T is a connected algebraic group, $T(k) \subset N_G(T)(k)$ acts trivially on $\mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$, so the action factors as an action of $\dfrac{N_G(T)(k)}{T(k)} \cong W(k)$ on $\mathrm{H}^*_{\text{ét}}(G/T)$. Putting this together gives that the homomorphism $d_2^{0,1} : S \to \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ is W-equivariant.

To obtain more information on $d_2^{0,1} : S \to \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ we compute its kernel. We write N for $N_G(T)$ next.

Proposition 6.19. *The kernel of $d_2^{0,1} : S \to \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ is $S_+^W \cdot S$ and $d_2^{0,1}$ is surjective.*

Proof. We first show that any element in $\mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ that is invariant under W is of degree 0. We may as well show that the same holds for any invariant element of $\mathrm{H}^*_{\text{ét}}(G/T, \overline{\mathbb{Q}_l})$ For this we want to compute $\mathrm{tr}(\tau_w^*)$. Note that $N(k)$ acts on G/T by $gT \mapsto ngT$. This action is trivial on $\mathrm{H}^*_{\text{ét}}(G/T, \overline{\mathbb{Q}_l})$ as it is induced by the action of $G(k)$ on $\mathrm{H}^*_{\text{ét}}(G/T, \overline{\mathbb{Q}_l})$ and G is a connected group variety. So we may as well consider the action of N on G/T by $\tau_n(gT) = gn^{-1}T$ for $n \in N(k)$. Note that N acts analogously on G/B giving the following commuting diagram:

$$\begin{array}{ccc} \mathrm{H}^*_{\text{ét}}(G/T, \overline{\mathbb{Q}_l}) & \longleftarrow & \mathrm{H}^*_{\text{ét}}(G/B, \overline{\mathbb{Q}_l}) \\ {\scriptstyle \tau_n^*}\uparrow & & {\scriptstyle \tau_n^*}\uparrow \\ \mathrm{H}^*_{\text{ét}}(G/T, \overline{\mathbb{Q}_l}) & \longleftarrow & \mathrm{H}^*_{\text{ét}}(G/B, \overline{\mathbb{Q}_l}) \end{array}$$

As $\tau_n^* = \gamma_n^*$ as automorphisms of $\mathrm{H}^*_{\text{ét}}(G/T, \overline{\mathbb{Q}_l})$ and as $\mathrm{H}^*_{\text{ét}}(G/T, \overline{\mathbb{Q}_l}) \cong \mathrm{H}^*_{\text{ét}}(G/B, \overline{\mathbb{Q}_l})$ we may as well compute $\mathrm{tr}_{G/B}(\tau_n^*)$. Again, note that τ_n^* only depends on the class of n in $W(k)$. Of course we have for $n \in T$ that τ_n^* is the identity map. For $n \notin T$ we have that $\tau_n : G/B \to G/B$ has no fixed point as else we would have $Bn = B$, so $n \in B$ contradicting the fact that W acts simply transitively on $\mathcal{B}^T$, the set of Borel subgroups containing T ([32], Prop 17.53). Combining the above with the fact that G/B is projective so that the Grothendieck-Lefschetz trace formula applies gives that:

$$\mathrm{tr}_{G/B}(\tau_n^*) = \begin{cases} \chi(G/B) \text{ if } n \in T \\ 0 \text{ else} \end{cases}$$

We have seen in Proposition 6.8 that $\chi(G/B) = |W|$. Now we consider the representation $W \to \mathrm{GL}(\overline{\mathbb{Q}_l}[W])$. Notice that the elements of the $\overline{\mathbb{Q}_l}$-basis $\{w \in W\}$ are permuted by all $v \in W$. If $v \neq e$ then this permutation fixes no such base element. Hence for v a nontrivial element of W, the matrix representing v has only zeros on the diagonal giving that:

$$\mathrm{tr}(v \,|\, \overline{\mathbb{Q}_l}[W]) = \begin{cases} |W| \text{ if } v = e \\ 0 \text{ else} \end{cases}$$

As the the traces of all $v \in W$ on $\mathrm{H}^*_{\text{ét}}(G/T, \overline{\mathbb{Q}_l})$ and $\overline{\mathbb{Q}_l}[W]$ are the same and because the field $\overline{\mathbb{Q}_l}$ is algebraically closed we have that the representations $W \to \mathrm{GL}(\overline{\mathbb{Q}_l}[W])$ and $W \to \mathrm{GL}(\mathrm{H}^*_{\text{ét}}(G/T, \overline{\mathbb{Q}_l})$ are equivalent as they have the same character ([17], p.869). It is easily follows that the fixed subspace of equivalent representations is isomorphic. If $\sum_w a_w \cdot w \in \overline{\mathbb{Q}_l}[W]$ is fixed by W then as W acts transitively on the basis $\{w \in W\}$ this implies that $a_w = a_v$ for all $w, v \in W$. It follows that any element $\sum_w a \cdot w$ is invariant under W, hence $\dim_{\overline{\mathbb{Q}_l}}(\overline{\mathbb{Q}_l}[W]^W) = 1$. As we have for all $w \in W$ that $\tau_w^* : \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l) \to \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ is a $\overline{\mathbb{Q}_l}$-algebra isomorphism this gives that all τ_w^* fix $\mathrm{H}^0_{\text{ét}}(G/T, \overline{\mathbb{Q}_l})$. So by dimension counting we get $\mathrm{H}^*_{\text{ét}}(G/T, \overline{\mathbb{Q}_l})^W = \mathrm{H}^0_{\text{ét}}(G/T, \overline{\mathbb{Q}_l})$ and hence $\mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)^W = \mathrm{H}^0_{\text{ét}}(G/T, \mathbb{Q}_l)$.

Thus by W-equivariance of $d_2^{0,1} : S \to \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ that $S_+^W \cdot S \subset \ker(d_2^{0,1})$. By Proposition 6.17, which implies $\dim_{\mathbb{Q}_l}(S/S_+^W \cdot S) = \dim_{\mathbb{Q}_l} \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ we get that $\ker(d_2^{0,1}) = S_+^W \cdot S$ and that $d_2^{0,1}$ is surjective. □

The identification $\mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l) \cong S/(S_+^W \cdot S)$ is very useful as the ideal $S_+^W \cdot S$ is well understood.

Theorem 6.20 ([10] Chevalley-Shephard-Todd)**.** *The ideal $S_+^W \cdot S \subset S$ is generated by a regular sequence $f_1, ..., f_n$ of homogeneous polynomials, where $n = \dim(S)$.*

6.3 Degeneration at the E_3-page

The goal of this subsection is to show that the Leray spectral sequence of $G \to G/T$ degerates at E_3 and to conclude that the isomorphism $\bigwedge^* \mathrm{J}[\times 2 - 1] \cong \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ commutes with pulling back via σ on both sides.

Notation 6.21. Let $E_r^{*,*}$ be the r'th page of the Leray spectral sequence of $G \to G/T$.

- Write $E_r^{*,\bullet}$ for the differential graded algebra with terms $E_r^{*,n} := \bigoplus_p E_r^{p,n}$ and differential of degree $-r+1$; $d : E_r^{*,n} \to E_r^{*,n-r+1}$ induced by the differential on the E_r-page.
- Write $h_n(E_r^{*,\bullet})$ for the homology of the complex and write $H(E_r^{*,\bullet}) = \bigoplus_n h_n(E_r^{*,\bullet})$. Note that $H(E_r^{*,\bullet})$ is also a differential graded algebra, equal to $E_{r+1}^{*,\bullet}$.
- Write $V = X(T) \otimes_{\mathbb{Z}} \mathbb{Q}_l \cong \mathrm{H}^1_{\text{ét}}(T, \mathbb{Q}_l)$ and $I := S_+^W \cdot S$.

The first step will be using homological methods to calculate $\dim_{\mathbb{Q}_l} h_q(E_2^{*,\bullet})$. Then we will show that $H(E_r^{*,\bullet})$ has algebra generators in degree 1, which will imply the degeneration of the spectral sequence at E_3.

Lemma 6.22. *There is an isomorphism of $\mathbb{Q}_l$-vectorspaces $h_q(E_2^{*,\bullet}) \cong \mathrm{Tor}_q^S(S/I, \mathbb{Q}_l)$, where $\mathbb{Q}_l$ is an S-module by $\mathbb{Q}_l = S/(x_1, ..., x_n)$.*

Proof. We note that the complex $E_2^{*,\bullet}$ can be obtained by taking the complex $\bigwedge^{\bullet} V \bigotimes_{\mathbb{Q}_l} S$ with differential $d(e_{i_1} \wedge ... \wedge e_{i_q} \otimes s) = \sum_{j=1}^{q} (-1)^{j+1} e_{i_1} \wedge ... \wedge \hat{e_{i_j}} \wedge ... \wedge e_{i_q} \otimes s \cdot x_{i_j}$ and then applying $- \bigotimes_S S/I$. The S-module complex $\bigwedge^{\bullet} V \bigotimes_{\mathbb{Q}_l} S$ is exact ([27], p.856), hence it gives a free resulotion of $S/(x_1, ..., x_n)$ as an S-module. Thus applying $- \bigotimes_S S/I$ and taking $h_q(-)$ indeed gives $\mathrm{Tor}_q^S(S/I, \mathbb{Q}_l)$. □

Using a property of $\mathrm{Tor}_q^S(-,-)$ we can calculate the dimensions of $E_3^{*,q}$.

Lemma 6.23. *We have an isomorphism of $\mathbb{Q}_l$-vectorspaces $\bigwedge^q V \cong \mathrm{Tor}_q^S(S/I, \mathbb{Q}_l)$.*

Proof. By symmetry of the $\mathrm{Tor}_q^S(-,-)$ bifunctor we can also take a projective resolution $P^\bullet \to S/I$ and then apply $- \bigotimes_S \mathbb{Q}_l$ to calculate $\mathrm{Tor}_q^S(S/I, \mathbb{Q}_l)$. By Theorem 6.20 that $I = (f_1, ..., f_n)$ where the f_i are homogeneous polynomials that form a regular sequence for S. Hence by ([27], p.856) the complex $\bigwedge^{\bullet} V \bigotimes S \to S/I$ provides a free resolution, where the map $S \to S/I$ is the canonical map and the differential is given by:

$$\bigwedge^q V \bigotimes S \to \bigwedge^{q-1} V \bigotimes S \quad e_{i_1} \wedge ... \wedge e_{i_q} \otimes s \mapsto \sum_{j=1}^{q} (-1)^{j+1} e_{i_1} \wedge ... \wedge \hat{e_{i_k}} \wedge ... \wedge e_{i_q} \otimes f_{i_j} \cdot s$$

Tensoring this complex over S with $\mathbb{Q}_l = S/(x_1, ..., x_n)$ gives that all the differentials vanish as the f_k are homogeneous polynomials in $x_1, ..., x_n$. Thus we obtain the isomorphisms of $\mathbb{Q}_l$-vectorspaces:

$$\mathrm{Tor}_q^S(S/I, \mathbb{Q}_l) = h_q(\bigwedge^{\bullet} V \bigotimes_S \mathbb{Q}_l) = \bigwedge^q V \bigotimes_S \mathbb{Q}_l \cong \bigwedge^q V$$

□

For the following proposition denote $Df = \sum_{k=1}^{n} e_k \otimes \frac{\partial f}{\partial x_k}$.

Proposition 6.24. *The $\mathbb{Q}_l$-linear map $\varphi^1 : \mathrm{J} \to h_1(E_2^{*,\bullet}) \quad f \mapsto Df$ is well-defined and induces a homomorphism of $\mathbb{Q}_l$-algebras $\varphi^* : \bigwedge^* \mathrm{J} \to H(E_2^{*,\bullet})$.*

Proof. It is clear that the map is $\mathbb{Q}_l$-linear if it is well-defined. We need to show that $(S_+^W)^2$ is killed by this map. Take any $f \cdot g \in (S_+^W)^2$. It is sent to $\sum_{k=1}^{n} e_k \otimes \frac{\partial fg}{\partial x_k} = \sum_{k=1}^{n} e_k \otimes (g\frac{\partial f}{\partial x_k} + f\frac{\partial g}{\partial x_k)} = 0$. We also need to show that for $f \in S_+^W$, Df is sent to 0 by $d : V \otimes_{\mathbb{Q}_l} S/I \to S/I$. We have $d(Df) = \sum_{k=1}^{n} x_k \frac{\partial f}{\partial x_k} = n \cdot f = 0$ by the Euler formula. So indeed there is a well-defined map $\mathrm{J} \to h_1(E_2^{*,\bullet})$.

To get the homomorphism of $\mathbb{Q}_l$-algebras $\varphi^* : \bigwedge^* \mathrm{J} \to H(E_2^{*,\bullet})$ it suffices to show that we have $\varphi^1(f) \cdot \varphi^1(g) + \varphi^1(g) \cdot \varphi^1(f) = 0$ for all $f, g \in \mathrm{J}$ by the universal mapping property of exterior algebra. We compute:

$$\varphi^1(f) \cdot \varphi^1(g) + \varphi^1(g) \cdot \varphi^1(f) = \sum_{1 \le i,j \le n} e_i \wedge e_j \otimes \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} + \sum_{1 \le i,j \le n} e_j \wedge e_i \otimes \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} = 0$$

Hence φ^1 extends to a $\mathbb{Q}_l$-algebra homomorphism $\varphi^* : \bigwedge^* \mathrm{J} \to \bigwedge^* V \bigotimes S/I$. □

Now we show that φ^* is an isomorphism.

Proposition 6.25. *The homomorphism of $\mathbb{Q}_l$-algebras $\varphi^* : \bigwedge^* \mathrm{J} \to H(E_2^{*,\bullet})$ is an isomorphism.*

Proof. By Lemma 6.23 we see that it suffices to show that φ^* is injective. This implies that it suffices to show that $\varphi^n : \bigwedge^n \mathrm{J} \to h_n(E_2^{\bullet,*})$ is injective or equivalently nonzero. The basis element $f_1 \wedge ... \wedge f_n$ is sent to $\varphi_1(f_1) \cdot ... \cdot \varphi_1(f_n) = \sum_{j_1=1}^{n} e_{j_1} \otimes \frac{\partial f_1}{\partial x j_1} \cdot ... \cdot \sum_{j_n=1}^{n} e_{j_n} \frac{\partial f_n}{x_{j_n}} = \sum_{1 \le j_1,...,j_n \le n} e_{j_1} \wedge ... \wedge e_{j_n} \otimes \frac{\partial f_1}{\partial x_{j_1}} \cdot ... \cdot \frac{\partial f_n}{\partial x_{j_n}}$

The terms in the last sum are nonzero if and only if the j_i are pairwise distinct. If they are pairwise distinct, then $(j_1, ..., j_n) = \tau \cdot (1, ..., n)$ for some $\tau \in S_n$ and $e_{j_1} \wedge ... \wedge e_{j_n} = (-1)^{\mathrm{sign}(\tau)} \cdot e_1 \wedge ... \wedge e_n$. So the last sum becomes $e_1 \wedge ... \wedge e_n \otimes \sum_{\tau \in S_n} \mathrm{sign}(\tau) \cdot \frac{\partial f_1}{\partial x_{\tau(1)}} \cdot ... \cdot \frac{\partial f_n}{\partial x_{\tau(n)}} = e_1 \wedge ... \wedge e_n \otimes \det(\frac{\partial f_i}{\partial x_j})$ by the Leibniz determinant formula. So we see that the map φ^* is injective if and only if $\det(\frac{\partial f_i}{\partial x_j}) \notin I$.

Denote $\Delta := \frac{\partial f_i}{\partial x_j}$ and $M = (\frac{\partial f_i}{\partial x_j})$. As $(S/I)^n \overset{\cdot M}{\to} (S/I)^n \overset{e_i \mapsto dx_i}{\to} \Omega^1_{S/I} \to 0$ is an exact sequence of S/I-modules we have that $\det(\Delta)$ may be identified with the 0'th fitting ideal of $\Omega^1_{S/I}$ (the Kähler differentials of S/I over $\mathbb{Q}_l$). It is then shown in ([19], Theorem 8.3) that the 0'th fitting ideal of $\Omega^1_{S/I}$ (also denoted $\mathfrak{D}_K((S/I)/\mathbb{Q}_l)$ in this text) is equal to the Noether different $\mathfrak{D}_N((S/I)/\mathbb{Q}_l)$ which is defined to be $\mu(\mathrm{Ann}(\mathbb{D}))$ where we have that $\mu : S/I \otimes_k S/I \to S/I$ is the multiplication map and $\mathbb{D}$ its kernel. As I is generated by a regular sequence of length n we have $\dim(S/I) = 0$, thus $S/I = \prod_{I\subset\mathfrak{m}} S_\mathfrak{m}/I_\mathfrak{m}$, the product taken over maximal ideals in S. By standard properties of the Noether different [43, Tag 0BVK] we have $\mathfrak{D}_N(S/I) = \prod_{I\subset\mathfrak{m}} \mathfrak{D}_N((S_\mathfrak{m}/I_\mathfrak{m})/\mathbb{Q}_l)$. It is then shown in ([19], Theorem 8.5) that $\mathfrak{D}_N((S_\mathfrak{m}/I_\mathfrak{m})/\mathbb{Q}_l)$ is equal to $\mathrm{soc}(S_\mathfrak{m}/I_\mathfrak{m}) := \bigcap_{E\subset S_\mathfrak{m}/I_\mathfrak{m}} E$, the intersection taken over all essential submodules of $S_\mathfrak{m}/I_\mathfrak{m}$. As $S_\mathfrak{m}/I_\mathfrak{m}$ is a local ring that is a complete intersection, it is in particular Gorenstein ([43, Tag 0DW6], Lemma 47.21.6) and hence by ([18], Theorem 21.5) the socle of $S_\mathfrak{m}/I_\mathfrak{m}$ is a simple submodule of $S_\mathfrak{m}/I_\mathfrak{m}$, hence it is in particular nonzero. This shows that $(\Delta) \subset S/I$ is not the zero-ideal and hence we conclude that the map $\varphi^* : \bigwedge^* V \to H_*(\bigwedge^\bullet V \bigotimes S/I)$ is an isomorphism of $\mathbb{Q}_l$-algebras. □

Now we can show that the Leray spectral sequence of $G \to G/T$ degenerates at E_3.

Corollary 6.26. *The Leray spectral sequence associated to $\pi : G \to G/T$ degenerates at the E_3-page, where it gives an isomorphism of $\mathbb{Q}_l$-algebras $\mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l) \cong \bigwedge^* \mathrm{J}[\times 2 - 1]$.*

Proof. We have by the previous proposition an isomorphism between the bigraded algebras $E_3^{*,*}$ and $\bigwedge^* \langle Df_k\rangle^{1\leq k\leq n}_{\mathbb{Q}_l}$. Note that the second bigraded algebra is generated over $\mathbb{Q}_l$ by the Df_k that are in bidegrees $(2k-2, 1)$. These are all sent to 0 by the differential on the E_3-page as they land in $E_3^{2k+1,-1}$. Hence the differential on the E_3-page is 0 and so $E_3 = E_\infty$. □

This can be used to achieve the goal of this section.

Theorem 6.27. *Let G be a semisimple group variety and $\sigma : G \to G$ a surjective endomorphism and choose a σ-stable maximal torus T. There is an isomorphism $\bigwedge^* \mathrm{J}[\times 2 - 1] \cong \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ of graded $\mathbb{Q}_l$-algebras that commutes with the pullbacks of σ.*

Proof. As the Leray spectral sequence associated to $\pi : G \to G/T$ degenerates at the E_3-page, there is an isomorphism of graded $\mathbb{Q}_l$-algebras $\mathrm{Tot}(E_3^{*,*}) \cong \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ that commutes with the pullback of σ^* (see Proposition 6.12). There is an isomorphism of $\mathbb{Q}_l$-algebras $\bigwedge^* \mathrm{J} \to \mathrm{Tot}(E_3^{*,*})$ defined by $f \mapsto Df$ as described above. As Df has bidegree $(2\deg(f)-2, 1)$ we should change the grading on J to $\mathrm{J}[\times 2-1]$ to get an isomorphism of graded $\mathbb{Q}_l$-algebras $\bigwedge^* \mathrm{J}[\times 2-1] \to \mathrm{Tot}(E_3^{*,*})$. It remains to be checked that this isomorphism commutes with the pullbacks of σ. The pullback of σ on $\mathrm{Tot}(E_3^{*,*})$ is induced by the pullback on the E_2-page. This pullback homomorphism is given by $\sigma^* \otimes \sigma^* : \mathrm{H}^q_{\text{ét}}(T, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^p_{\text{ét}}(G/T, \mathbb{Q}_l) \to \mathrm{H}^q_{\text{ét}}(T, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathrm{H}^p_{\text{ét}}(G/T, \mathbb{Q}_l)$. The isomorphism $V \cong \mathrm{H}^*_{\text{ét}}(T, \mathbb{Q}_l)$ commutes with the pullbacks of σ^* by Proposition 2.74. The isomorphism $S/I \to \mathrm{H}^*_{\text{ét}}(G/T, \mathbb{Q}_l)$ was induced by a differential from the Leray spectral sequence, so this isomorphism also commutes with the pullback of σ. The map $\bigwedge^* \mathrm{J} \to \mathrm{Tot}(E_3^{*,*})$ was initially defined as a map $\bigwedge^* \mathrm{J} \to \bigwedge^* V \otimes S/I$, which was induced by $\mathrm{J} \to V \otimes S/I$, so it suffices to show that this last map commutes with the pullbacks of σ, i.e. we need to show that $D(\sigma^*(f)) = \sigma^*(Df)$ for all $f \in \mathrm{J}$. As σ^* acts linearly on S we can use the chain rule to get $D(\sigma^*(f)) = \sum_{i=1}^{n} e_i \otimes \frac{\partial(f(\sigma^* x_1, ..., \sigma^* x_n))}{\partial x_i} = \sum_{i=1}^{n} \sigma^* e_i \otimes \sigma^*(\frac{\partial f}{\partial x_i})$.

This thus concludes the proof that the isomorphism $\bigwedge^* \mathrm{J}[\times 2-1] \cong \mathrm{H}^*_{\text{ét}}(G, \mathbb{Q}_l)$ commutes with the pullbacks of σ. □

We obtain the following immediate corollary.

Corollary 6.28. *Let $\sigma : G \to G$ be an endomorphism and let (B, T) be a σ-stable Borel pair of G. Define $d_1 := \mathrm{tr}_T(\sigma) \cdot \mathrm{tr}_{G/B}(\sigma)$. We have $d_1 = \mathrm{tr}_G(\sigma)$.*

Proof. The proof is more or less reciting the functoriality above. We have that σ^* on $\mathrm{H}^*_{\text{ét}}(G/B, \mathbb{Q}_l)$ may be identified with σ^* on S/I. By ([8] Lemma 3.5.3) we have $P_{S/I,\sigma}(t) = \frac{\det(1 - t\sigma^* \mid \mathrm{J})}{\det(1 - t\sigma^* \mid V)}$, hence we obtain $d_1 = \det(1 - \sigma^* \mid \mathrm{J}) = \mathrm{tr}_G(\sigma)$ by the functoriality above and as the generators of $\mathrm{J}[\times 2 - 1]$ are in odd degree. □

We use this to calculate the graded trace in a simple example.

Example 6.29. Let $G = \mathrm{SL}_n$ over a field of characteristic p and let σ be the endomorphism induced by $\mathcal{O}(\mathrm{SL}_n) \to \mathcal{O}(\mathrm{SL}_n) \quad X_{ij} \mapsto X_{ij}^p$. Then note that on points $\mathrm{SL}_n(k) \to \mathrm{SL}_n(k)$ it maps $(m_{ij}) \mapsto (m_{ij}^p)$, so its fixed points are $\mathrm{SL}_n(\mathbb{F}_p)$ and it is surjective on the k-points as k is perfect. The maximal torus $\mathbb{D}_n$ is σ-stable and hence one gets the pullback $\sigma^* : \mathrm{J}[\times 2-1] \to \mathrm{J}[\times 2-1]$. A character χ_i is mapped to χ_i^p under σ, hence $\sigma^* : S \to S$ is given by $X_i \mapsto pX_i$. As $S^W = \mathbb{Q}_l[e_2, ..., e_n]$ and the e_i are homogeneous of degree i we see that σ^* acts via the diagonal matrix (d_{ij}) with entries $d_{ii} = p^i$ on J with respect to the basis $\{e_2, ..., e_n\}$ of J. This gives $\mathrm{tr}_G(\sigma) = \det(1 - \sigma^* \mid \mathrm{J}) = \prod_{i=2}^{n}(p^i - 1)$.

By [8] Theorem 5.4.2 and [44] Remark 11.19 we have $\#\mathrm{Fix}(\sigma)(k) = \#\mathrm{SL}_n(\mathbb{F}_q) = c \cdot \mathrm{tr}_G(\sigma)$ where we have $c = \sigma^*\mathrm{Jac}/\mathrm{Jac}$ with Jac the Jacobian determinant of the indecomposable invariants w.r.t $X_2, ..., X_n$. The term $|\det(1 - \sigma^* \mid V)|_p$ equals 1 as σ^* acts as $\cdot p$ and since $\frac{\partial e_j}{\partial x_i}$ is homogeneous of degree $j - 1$ we have that $\sigma^*\mathrm{Jac}$ equals $\prod_{j=1}^{n-1} p^j \mathrm{Jac}$. So we get $\#\mathrm{SL}_n(\mathbb{F}_p) = \prod_{i=1}^{n-1} p^i \cdot \prod_{i=2}^{n}(p^i - 1)$. This may very well be the hardest way to calculate $\#\mathrm{SL}_n(\mathbb{F}_p)$, but the use of the above method lies in the generality in which it can be applied.

Chapter 7

Appendix: Construction of cup products

We describe how to construct the cup products using Cech cohomology. This is mainly based on ([43, Tag 01FP]). Note that on the Stacks page the construction is done for a complex $\mathcal{F}^\bullet$. Our construction is the special case $(... \to 0 \to \mathcal{F} \to 0 \to ...)$. Note that by a result of Artin [2] we have that étale and Cech cohomology agree for schemes quasiprojective over a Noetherian ring, which is more than sufficient for our purposes. The cup products can be defined for even more general schemes though [45].

Proposition 7.1. *Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X_{\acute{e}t})$ be sheafs of Λ-modules. For all $p, q \geq 0$ there exist Λ-linear maps called* ***cup products*** $\cup : \mathrm{H}^p_{\text{ét}}(X, \mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(X, \mathcal{G}) \to \mathrm{H}^{p+q}_{\text{ét}}(X, \mathcal{F} \otimes_\Lambda \mathcal{G})$*. They satisfy the following properties:*

- *The cup product is associative, i.e. for x_p, y_q, z_n homogeneous we have $x_p \cup (y_q \cup z_n) = (x_p \cup y_q) \cup z_n$.*
- *If $\mathcal{F} \to \mathcal{F}'$ and $\mathcal{G} \to \mathcal{G}'$ are morphisms of Λ-modules then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^p_{\text{ét}}(X, \mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(X, \mathcal{G}) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\text{ét}}(X, \mathcal{F} \otimes_\Lambda \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{H}^p_{\text{ét}}(X, \mathcal{F}') \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(X, \mathcal{G}') & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\text{ét}}(X, \mathcal{F}' \otimes_\Lambda \mathcal{G}') \end{array}$$

- *If $g : X \to Y$ is a morphism, then the cup products of X and Y are related by the following commuting diagram:*

$$\begin{array}{ccc} \mathrm{H}^p_{\text{ét}}(X, g^*\mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(X, g^*\mathcal{G}) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\text{ét}}(X, g^*\mathcal{F} \otimes_\Lambda g^*\mathcal{G}) \\ {\scriptstyle g^*\otimes g^*}\uparrow & & {\scriptstyle g^*}\uparrow \\ \mathrm{H}^p_{\text{ét}}(Y, \mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(Y, \mathcal{G}) & \xrightarrow{\cup} & \mathrm{H}^{p+q}_{\text{ét}}(X, \mathcal{F} \otimes_\Lambda \mathcal{G}) \end{array}$$

- *When $\mathcal{A} = \mathcal{F} = \mathcal{G}$ is a sheaf of commutative Λ-algebras, so it is equipped with a multiplication map $\wedge : \mathcal{A} \otimes_\Lambda \mathcal{A} \to \mathcal{A}$, the cup product is a graded commutative ring, so for $x \in \mathrm{H}^p_{\text{ét}}(X, \mathcal{A})$ and $y \in \mathrm{H}^q_{\text{ét}}(X, \mathcal{A})$ we have $\wedge(x \cup y) = (-1)^{pq} \wedge (y \cup x)$.*

In the next part we give the construction for Cech cohomology.

Construction

Let $\mathcal{F}, \mathcal{G}, \Lambda$ be as above. Given a cover $\mathcal{U} = (U_i \to X)_i$ of X and $(h) := (h_{i_0 \ldots i_p})_{i_0 \ldots i_p} \in \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F})$ and $(h) := (g_{j_0 \ldots j_q})_{j_0 \ldots j_q} \in \check{\mathrm{H}}^q(\mathcal{U}, \mathcal{F})$. Then we define $(g) \cup (h) \in \check{\mathrm{H}}^{p+q}(\mathcal{U}, \mathcal{F} \otimes_\Lambda \mathcal{G})$ by setting:

$$((g) \cup (h))_{i_0 \ldots . i_{p+q}} = a(g_{i_0 \ldots i_p} \otimes h_{i_p \ldots i_{p+q}}) \tag{7.1}$$

where $a : (\mathcal{F} \otimes_\Lambda \mathcal{G})^P \to \mathcal{F} \otimes_\Lambda \mathcal{G}$ is the sheafification map. Before getting to the above properties we have to check that this assignment is well-defined, i.e. if the class representing (g) or (h) is in the image of d, then the class representing $(g) \cup (h)$ should also have this and we should have $d((g) \cup (h)) = 0$. We have the formula

$$d((g) \cup (h)) = d(g) \cup (h) + (-1)^{\deg(g)}(g) \cup d(h)$$

which can be checked componentwise. The first point follows from the fact that one can form the total complex $\mathrm{Tot}^n := \bigoplus_{p+q=n} \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \otimes_\Lambda \mathcal{C}^p(\mathcal{U}, \mathcal{G})$ with $d : \mathrm{Tot}^n \to \mathrm{Tot}^{n+1} : d(a \otimes b) = a \otimes d(b) + (-1)^{\deg(a)} \otimes b$ for homogeneous a, b. By the above formula we have that $\mathrm{Tot}^\bullet \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F} \otimes_\Lambda \mathcal{G})$ defined by $a \otimes b \mapsto a \cup b$ is a morphism of complexes. Hence by passing to homology we obtain a map $h_n(\mathrm{Tot}^\bullet) \to \check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F} \otimes_\Lambda \mathcal{G})$. For $p + q = n$ there is the obvious map ([43, Tag 068G]) $\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \otimes_\Lambda \check{\mathrm{H}}^q(\mathcal{U}, \mathcal{F}) \to \mathrm{Tot}^n$ and notice that the composition with $\mathrm{Tot}^n \to \check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F} \otimes_\Lambda \mathcal{G})$ is precisely equal to equation (7.1), which gives that the map in equation (7.1) is well defined.

The properties

The cup product defined as above has all the properties.

Proof. We first check the properties for Cech cohomology w.r.t. some cover $\mathcal{U}$. Associativity follows as whenever $\deg(x) = p$, $\deg(y) = q$, $\deg(z) = n$ we have that $((x \cup y) \cup z)_{i_0 \ldots i_{p+q+n}} = (x \cup (y \cup z))_{i_0 \ldots i_{p+q+n}}$ equals the image of $x_{i_0 \ldots i_p} \otimes y_{i_p \ldots i_{p+q}} \otimes z_{i_{p+q} \ldots i_{p+q+n}}$ inside $\mathcal{F} \otimes_\Lambda \mathcal{G} \otimes_\Lambda \mathcal{H}$.
Functoriality in the sheaf argument holds because $\mathcal{F} \otimes_\Lambda \mathcal{G} \to \mathcal{F}' \otimes_\Lambda \mathcal{G}'$ is induced by the morphism of the corresponding presheafs.
For naturality with respect to a morphism $f : X \to Y$, let $\mathcal{U} = (U_i \to Y)_i$ be a cover and form the pulled back cover $\mathcal{U}' = (U_i \times_Y X \to X)_i$. Given $(g) = (g_{i_0 \ldots i_p})_{i_0 \ldots i_p} \in \check{\mathrm{H}}^p(Y, \mathcal{F})$, then the pullback f^* is given by $f^*(g)_{i_0 \ldots i_p} = a([g_{i_0 \ldots i_p}])$ where $a : (f^*\mathcal{F})^P \to f^*\mathcal{F}$ is the sheafification and $[g_{i_0 \ldots i_p}]$ is the class inside $\varinjlim_{V \to Y} \mathcal{F}(V) = (f^*\mathcal{F})(U_{i_0 \ldots i_p} \times_X Y)$. Hence for $(g) \in \check{\mathrm{H}}^p(Y, \mathcal{F})$ and $(h) \in \check{\mathrm{H}}^q(Y, \mathcal{F})$ we have that

$$(f^*(g) \cup f^*(h))_{i_0 \ldots i_{p+q}} = [g_{i_0 \ldots i_p}] \otimes [h_{i_p \ldots i_{p+q}}] \in \check{\mathrm{H}}^{p+q}(\mathcal{U}', f^*\mathcal{F} \otimes f^*\mathcal{G})^P$$

and we have

$$f^*((g) \cup (h))_{i_0 \ldots i_{p+q}} = [g_{i_0 \ldots i_p} \otimes h_{i_p \ldots i_{p+q}}] \in \check{\mathrm{H}}^{p+q}(\mathcal{U}', f^*(\mathcal{F} \otimes \mathcal{G}))^P$$

-more precisely they are the images of these elements after sheafiffying-. As colimits and tensor products commute, the above presheafs are isomorphic by mapping $[a] \otimes [b] \mapsto [a \otimes b]$, so indeed through this canonical identification we have that $f^*((g) \cup (h)) = f^*(g) \cup f^*(h)$.
For the last statement of the proposition, define $\tau : \mathcal{F}(U_{i_0 \dots i_p}) \to \mathcal{F}(U_{i_0 \dots i_p}) \quad g_{i_0 \dots i_p} \mapsto (-1)^{p(p+1)/2} g_{i_p \dots i_0}$. It follows that τ commutes with d and hence defines an endomorphism of the complex $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$. A calculation ([43, Tag 01FP]) shows that $h : \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \quad h(\alpha)_{i_0 \dots i_p} = \sum_{j=0}^{p} \epsilon_p(\alpha) \alpha_{i_0 \dots i_j \dots i_p i_j}$

where we set $\epsilon_p(\alpha) = (-1)^{\frac{(p-a)(p-a-1)}{2} + p}$ is a homotopy from τ to the identity, hence τ induces the identity map on the homology of $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$. We now let $\mathcal{F} = \mathcal{A}$ be a sheaf of Λ-algebras with multiplication $\wedge$. Then for $(g) \in \check{\mathrm{H}}^p(Y, \mathcal{A})$ and $(h) \in \check{\mathrm{H}}^q(Y, \mathcal{A})$ we have

$$g_{i_0 \dots i_p} \otimes h_{i_p \dots i_{p+q}} = ((g) \cup (h))_{i_0 \dots i_{p+q}} = \tau(\wedge(\tau(g) \cup \tau(h)))_{i_0 \dots i_{p+q}} = (-1)^N h_{i_0 \dots i_q} \otimes g_{i_p \dots i_{p+q}}$$

where $N = \dfrac{p(p+1) + q(q+1) + (p+q)(p+q+1)}{2}$. It follows that $N \equiv pq \pmod 2$ and hence we obtain the graded commutativity.

By passing to the colimit over refinements of covers $\mathcal{U}$ as in ([43, Tag 01FP]) we obtain cup products on $\check{\mathrm{H}}^p(X, \mathcal{F}) \otimes_\Lambda \check{\mathrm{H}}^q(X, \mathcal{G}) \to \check{\mathrm{H}}^{p+q}(X, \mathcal{F} \otimes_\Lambda \mathcal{G})$ with all the properties of Proposition 7.1. If we assume Cech cohomology computes étale cohomology, then we obtain the cup products $\mathrm{H}^p_{\text{ét}}(X, \mathcal{F}) \otimes_\Lambda \mathrm{H}^q_{\text{ét}}(X, \mathcal{G}) \to \mathrm{H}^{p+q}_{\text{ét}}(X, \mathcal{F} \otimes_\Lambda \mathcal{G})$, which by the functoriality of the Cech to derived sequence in both arguments has all the properties of Proposition 7.1. $\square$

Bibliography

[1] Satoshi Arima. “Commutative group varieties.” *Journal of the Mathematical Society of Japan* 12.3 (1960), pp. 227–237.

[2] Michael Artin. “On the joins of hensel rings”. *Advances in Mathematics* 7.3 (1971), pp. 282–296. ISSN: 0001-8708.

[3] Michael Artin et al. *Schémas en groupes.* Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques. Paris: Institut des Hautes Études Scientifiques, 1963/1966.

[4] B. Baer. “Erweiterung von Gruppen und ihren Isomorphismen”. *Mathematische Zeitschrift* 38 (1934), pp. 375–416.

[5] Armand Borel. “Sur La Cohomologie des Espaces Fibres Principaux et des Espaces Homogenes de Groupes de Lie Compacts”. *Annals of Mathematics* 57 (1953), p. 115.

[6] Michel Brion. *Commutative algebraic groups up to isogeny.* 2016.

[7] Michel Brion and Tamás Szamuely. *Prime-to-p étale covers of algebraic groups.* 2012. arXiv: `1109.2802`.

[8] Jakub Byszewski, Gunther Cornelissen, and Marc Houben. *Dynamics of endomorphisms of algebraic groups and related systems.* 2022.

[9] Pierre Cartier. “A Primer of Hopf Algebras”. In: *Frontiers in Number Theory, Physics, and Geometry II: On Conformal Field Theories, Discrete Groups and Renormalization.* Ed. by Pierre Cartier et al. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007, pp. 537–615. ISBN: 978-3-540-30308-4.

[10] Claude C. Chevalley. “Invariants of Finite Groups Generated by Reflections”. *American Journal of Mathematics* 77 (1955), p. 778.

[11] Brian Conrad. *étale cohomology notes.* URL: `http://math.stanford.edu/~conrad/Weil2seminar/Notes/etnotes.pdf`.

[12] Pierre Deligne. “Théorie de Hodge : II”. fre. *Publications Mathématiques de l'IHÉS* 40 (1971), pp. 5–57.

[13] Pierre Deligne. “Théorie de Hodge : III”. fre. *Publications Mathématiques de l'IHÉS* 44 (1974), pp. 5–77.

[14] Michel Demazure. “Groupes Réductifs : Déploiements, Sous-Groupes, Groupes-Quotients”. In: *Structure des Schémas en Groupes Réductifs.* Berlin, Heidelberg: Springer Berlin Heidelberg, 1970, pp. 156–262.

[15] Michel Demazure. “Désingularisation des variétés de Schubert généralisées”. *Annales Scientifiques De L Ecole Normale Superieure* 7 (1974), pp. 53–88.

[16] Jean Dieudonné and Alexander Grothendieck. "Éléments de géométrie algébrique III". *Inst. Hautes Études Sci. Publ. Math.* 4, 8, 11, 17, 20, 24, 28, 32 (1961–1967).

[17] David S. Dummit and Richard M. Foote. *Abstract Algebra*. Wiley, 1999.

[18] David Eisenbud. *Commutative Algebra: With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1995.

[19] David Eisenbud and Bernd Ulrich. *Duality and Socle Generators for Residual Intersections*. 2013.

[20] Gerhard Frey and Moshe Jarden. "Approximation Theory and the Rank of Abelian Varieties Over Large Algebraic Fields". *Proceedings of the London Mathematical Society* s3-28.1 (1974), pp. 112–128.

[21] Eric M. Friedlander and Brian Parshall. "Etale cohomology of reductive groups". In: *Algebraic K-Theory Evanston 1980*. Ed. by Eric M. Friedlander and Michael R. Stein. Berlin, Heidelberg: Springer Berlin Heidelberg, 1981, pp. 127–140.

[22] William Fulton. *Intersection Theory:* Ergebnisse der Mathematik und ihrer Grenzgebiete : a series of modern surveys in mathematics. Folge 3. Springer-Verlag, 1984.

[23] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977.

[24] James E. Humphreys. *Linear Algebraic Groups*. Graduate Texts in Mathematics. Springer, 1975.

[25] Dale Husemoller. *Fibre Bundles*. Graduate Texts in Mathematics. Springer, 1994.

[26] Steven Kleiman and Allen Altman. *A Term of Commutative Algebra*. Worldwide Center of Mathematics, LLC, 2013.

[27] Serge Lang. *Algebra*. Graduate Texts in Mathematics. Springer New York, 2005.

[28] Jean Leray. "Sur l'homologie des groupes de Lie, des espaces homogenès et des espaces fibrés principaux" (1950).

[29] James S. Milne. *Etale Cohomology (PMS-33)*. Princeton Legacy Library. Princeton University Press, 1980.

[30] James S. Milne. *Abelian Varieties (v2.00)*. Available at www.jmilne.org/math/. 2008.

[31] James S. Milne. *Lectures on Etale Cohomology (v2.21)*. 2013.

[32] James S. Milne. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2017.

[33] Masayoshi Miyanishi. "On the algebraic fundamental group of an algebraic group". *Journal of Mathematics of Kyoto University* 12.2 (1972), pp. 351–367.

[34] David. Mumford and Chakravarti P. Ramanujam. *Abelian Varieties*. Studies in mathematics. Hindustan Book Agency, 2008.

[35] Masayoshi Nagata. "A generalization of the imbedding problem of an abstract variety in a complete variety". *Journal of Mathematics of Kyoto University* 3.1 (1963), pp. 89–102.

[36] *Poincaré duality for Étale cohomology*.

[37] Maxwell Rosenlicht. "Some Basic Theorems on Algebraic Groups". *American Journal of Mathematics* 78 (1956), p. 401.

[38] René Schoof. "Finite Flat Group Schemes over Local Artin Rings". *Compositio Mathematica* 128.1 (2001), pp. 1–15.

[39] Jean Pierre Serre. *Algebraic Groups and Class Fields*. Graduate Texts in Mathematics. Springer New York, 2012.

[40] Joseph H. Silverman. *The arithmetic of elliptic curves*. Vol. 106. Graduate Texts in Mathematics. New York: Springer-Verlag, 1986, pp. xii+400.

[41] Tonny A. Springer. *Linear Algebraic Groups*. Modern Birkhäuser Classics. Birkhäuser Boston, 2008.

[42] Bhama Srinivasan. *Representations of Finite Chevalley Groups: A Survey*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2006.

[43] The Stacks project authors. *The Stacks project*. `https://stacks.math.columbia.edu`. 2022.

[44] Robert Steinberg. *Endomorphisms of Linear Algebraic Groups*. Memoirs of the American Mathematical Society. American Mathematical Society, 1968.

[45] Richard Swan. "Cup products in sheaf cohomology, pure injectives, and a substitute for projective resolutions". *Journal of Pure and Applied Algebra - J PURE APPL ALG* 144 (Dec. 1999), pp. 169–211.

[46] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.