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Master's thesis

Electromagnetic interactions near the black hole horizon

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"How wonderful that we have met this paradox. Now we have some hope of making progress."

Niels Bohr

Abstract

The thermal nature of black-hole radiation led Hawking to argue that black holes destroy quantum information, leading to the formulation of the so-called 'black-hole information paradox'. Dissatisfied with such conclusion, 't Hooft emphasized the importance of gravitational interactions, ignored in the derivation of the Hawking spectrum. In this thesis, while shortly reviewing and partly revisiting 't Hooft's gravitational S-matrix, we generalize his approach to the case of degrees of freedom carrying electric charge. In particular, we investigate how to expand the effect of the change of gauge of the electromagnetic field in partial waves. We proceed with computing the same effect via elastic $2 \to 2$ scattering diagrams by constructing the photon propagator near the event horizon in an angular momentum basis. More precisely, we calculate the S-matrix by summing over an infinite number of photon exchange diagrams in the high-energy limit. The S-matrix so-obtained agrees with the one derived within 't Hooft's approach, thus providing a generalized result. The techniques used in this work allow for straightforward generalizations to capture particle production, study higher-derivative corrections and include non-Abelian gauge fields, among many others. They open up possibilities for new research on black-hole scattering.

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Contents

Introduction

In 1965, it was proved that black holes results from the death of supermassive stars which, after exhausting all their fuel, collapse under the enormity of their own gravity, leaving a black hole as their final state [1]. At the center of a black hole, the curvature of space and time is so strong that even general relativity itself breaks down. From the outside, the only properties of a black hole that we can discern are mass, charge and state of rotation. Two static chargless black holes with the same mass are indistinguishable to us. This is the statement that black holes have no hair, or perhaps we should say black holes have only three hairs, one for each of the three properties. Therefore, a natural question arises: where does the information carried by objects thrown into a black hole go? From a general relativistic point of view, such information is not lost, but simply hidden behind the event horizon of the black hole. However, the situation completely changed after Hawking discovered, by combining general relativity with quantum field theory, that black holes emit radiation at a temperature[1] [2]

$$T_H = \frac{1}{8\pi M},$$

where M is the mass of the black hole. This discovery led to more puzzles and questions. In particular, it seems to have a catastrophic consequence. Indeed, the radiation emitted carries away mass from the black hole. Now, the radius of the black hole is proportional to its mass, so if the black hole radiates, it shrinks. Moreover, the temperature is inversely proportional to the black-hole mass. Thus, as the black hole shrinks, it gets hotter, and it shrinks even faster, eventually entirely evaporating. When this happens, all that is left is the thermal radiation it emitted, which in the case we are considering only depends on the initial mass of the black hole. This indicates that, besides this quantity, it does not matter what formed the black hole originally or what fell in later, the result is the same thermal radiation. Black hole evaporation is therefore irreversible: we cannot tell from the final state, namely the outcome of the evaporation, what the initial state was that formed the black hole. This contradicts the fundamentals of quantum mechanics, where processes in quantum theory are indeed always time-reversible. Thus, we set out to combine quantum mechanics with gravity, but we produce an outcome that contradicts what we started with. This contradiction, which was first pointed out by Hawking himself [3] and is now known as the "black hole information paradox", had troubled scientists for the last forty years and remains one of the biggest problems in theoretical physics [4–9]. For a review about the current status, the reader may refer to Ref. [10].

In this thesis we focus on 't Hooft's proposed resolution to the paradox. Such proposal have been presented long ago, in the nineties [11–13], but in the last few years significant progress has been made [14–18]. The idea is to consider gravitational interactions, ignored in the derivation of the Hawking spectrum, between the particles that fall into an already formed black hole and the ones that come out due to Hawking radiation. More precisely, 't Hooft studied how ingoing particles affect the outgoing ones, leading to a construction of a scattering matrix that maps in- to out-states. It is easy to check that this S-matrix is unitary, which therefore implies

[1]This result is valid for a Schwarzschild black hole. However, it can be extended to black holes carrying charge and angular momentum.

that all information falling into the black hole is entirely transferred to the outgoing Hawking particles. This remarkable result has been obtained in the context of quantum mechanics. However, in the last few years, a second-quantized approach has also been proposed, showing that the same S-matrix can be derived by taking into account gravitational interactions mediated by graviton exchange in $2 \to 2$ scattering processes near the horizon [19, 20]. Very recently, further progress has been made by considering the possibility for particle production [21] and by rewriting the theory as a scalar theory with a specific four-vertex to compute Feynman diagrams more efficiently [22].

In this work we generalize 't Hooft's approach, as well as its second-quantized extension, to the case of electromagnetic interactions. The thesis is organized as follows. In section 1 we will discuss the framework in which Hawking radiation has been derived, namely quantum field theory in curved spacetime. In section 2 we will carefully reproduce the original calculation performed by Hawking in the seventies, deriving the formula for the temperature introduced above. In section 3, while shortly reviewing and partly revisiting 't Hooft's gravitational S-matrix, we generalize his approach to the case of degrees of freedom carrying electric charge. In particular, we obtain an expression for the electromagnetic S-matrix in a partial-wave basis[2]. We proceed, in section 4, by constructing a scalar quantum electrodynamics near the black-hole horizon, thus defining the Feynman rules of the theory. Section 5 is devoted to the computation of all the scattering amplitudes of interest, leading to an expression for the scattering matrix which is in agreement with the one found in section 3. We end this section by considering a one-loop diagram with four vertices; the entire one-loop calculation is presented, showing that such diagram is sub-leading with respect to the corresponding one containing three-vertices only. Finally, we conclude by briefly summarizing all the results obtained in this work, as well as discussing some possible future directions.

[2]This section has been written in collaboration with N. Gaddam and N. Groenenboom. The author of this thesis primarily carried out all the field-theory calculations.

1 Preliminaries

Our discussion starts by introducing the concept of vacuum states in quantum field theory. We proceed by presenting the basic idea on how to quantize scalar fields on curved backgrounds, a necessary ingredient for the derivation of the Hawking spectrum. This first section primarily relies on the book written by Birrell and Davies [23], the book written by Wald [24], as well as the one by Mukhanov and Winitski [25]. In addition to that, the lecture notes on black-hole physics by Dowker [26] and Lambert [27] have been consulted. The reader is assumed to be familiar with the basics of quantum field theory and general relativity.

1.1 A subtle concept: the vacuum state

At the basis of the original formulation of general relativity there is the identification, from a physical point of view, between uniformly accelerated systems and uniform gravitational fields. That is, an observer in a free-falling reference system is in no way able to locally distinguish the effects of gravity from those produced by the acceleration of the system itself; in other words, a non-inertial reference system is locally equivalent to a uniform gravitational field. In Newtonian mechanics, in order to study accelerated (non-inertial) reference systems, it is necessary to add apparent forces to the treatment. This is no longer true in general relativity; in fact, here the principle of general covariance holds, according to which there is no distinction that favours inertial systems over non-inertial ones. These premises are indispensable to understand how the framework in which a theory is developed may influence its treatment. This point will be clear when we will discuss the Unruh effect, an analog of Hawking effect in special relativity.

Let us now suppose that we have a certain coordinate system and then we perform a transformation and change coordinates. With such an operation, the way of describing the physical system has been changed, but its “nature” has not. The purpose, therefore, is to be able to transform all the quantities that characterized the system in the initial coordinates into new quantities obtained using the new coordinates. Of course there are many types of transformation, depending on the mathematical objects involved. For example, we can consider a scalar quantity; its value at one point, say p, does not change if we evaluate it at the same point expressed in different coordinates. Differently, a vector or a tensor change under coordinate transformation since all its components may be altered. In summary, when a transformation is performed, the functions that describe the physical properties of a system change, and change differently depending on whether they are scalars, vectors or tensors.

Let us now consider the quantization of a scalar field in Minkowski spacetime. Moreover, let us suppose we apply a Lorentz boost and quantize again once the transformation is done. We certainly do not expect the operators and the other quantities thus obtained to be the same as the initial ones. For example, we may consider a plane-wave with wave vector $\mathbf{k}$, four-momentum p and a certain energy E. Since after the transformation the components of the four-momentum change, then both $\mathbf{k}$ and the energy will change too; we would eventually get some momentum p' and some energy E'. Therefore, in correspondence with the starting mode characterized by $\mathbf{k}$, p and E, we would obtain, according to the Lorentz transformation, a mode characterized by p', $\mathbf{k}'$ and E'. Now we would have particles with a different momentum, so it is not unreasonable to suspect that the annihilation and creation operators, usually denoted by a and $a^\dagger$, respectively, may have been changed too. In fact, it is really the case and there are transformations that allow us to describe how such operators change; these are called Bogoliubov transformations and they tell us how the creation and annihilation operators change between reference systems (not necessarily inertial ones). After these transformations have been performed, we are able to write a and $a^\dagger$ in the final reference system as a linear combination of the creation and annihilation operators in the initial system with appropriate coefficients (and vice versa). Moreover, we expect that the number operator $N := a^\dagger a$, which counts particles with momentum $\vec{k}$, might change too. But if the number operator changes, its eigenvalues will change, and that means that we might count particles differently depending on the reference system we are working in.

At this point, a natural question arises: how can quantum field theory in Minkowski spacetime be consistent? Indeed, based on our previous discussion, it seems that the definition of vacuum state depends on a particular choice of frame. Fortunately, it turns out that in this case some of the Bogoliubov coefficients are vanishing, and this happens in such a way that the concept of particle and, consequently, the one of vacuum state are invariant under Lorentz transformations; for this reason quantum field theory in flat spacetime is consistent in all inertial systems, and so the concept of particle.

Let us now come back to the Unruh effect, already mentioned before. The basic idea is fairly simple: this effect is a manifestation of the fact that observers with different notions of positive- and negative-frequency modes will disagree on the particle content of a given state. In other words, we start by considering an inertial reference frame in Minkowski spacetime. Then, by applying a certain transformation, we introduce a second reference frame which is uniformly accelerated; the initial vacuum state is classified by the observer in the new reference system as a state containing particles. Moreover, if we studied the distribution of these particles and plotted, e.g., the spectrum, we would obtain a black-body spectrum at a certain temperature which depends on the acceleration of the system. Therefore, in the accelerated system, the creation operators must have acted on the accelerated vacuum state, which is no longer empty (unlike what happened in the case of Lorentz transformations, in which the vacuum state was Lorentz invariant). To sum up, if we consider a massless scalar field, where an inertial observer sees a vacuum state, a non-inertial one will detect a thermal spectrum of field excitations. This paradox can be solved in a simple way, namely by considering that, according to special relativity, inertial systems are privileged; and it is precisely this adjective, “privileged”, that eliminates the contradiction. Since quantum field theory has been formulated in order to work in inertial systems, if one gets into a situation where there is a discrepancy between two predictions obtained in two different reference systems, one of which is non-inertial, then the prediction made in the inertial one can be certainly favoured. This is analogous to how we describe accelerated systems in Newtonian mechanics. In that case, we are forced to introduce the so-called “fictitious” or “apparent” forces. The adjectives we choose clearly denote that “real” forces are the ones that we perceive as real in an inertial reference system (which is privileged according to Newtonian theory). Even though it occurs in flat spacetime, the Unruh effect teaches us a very important lesson, namely the idea that “vacuum" and “particle" are observer-dependent notions rather than fundamental concepts. However, when we deal with black holes and curved spacetimes in general, the physics is much more subtle.

Hawking radiation is treated within the framework of general relativity (a field theory on curved spacetime). As we know, in this framework no system is privileged; therefore, if two observers draw different conclusions on the number of particles contained in a certain state, there is no way to establish which one of them is right; we will see that the root of this problem lies in the choice of the concept of particle we use to describe the physical system. In fact, choosing the concept of particle and, consequently, the number operator with the aim of characterizing the system of interest is in general inconsistent in quantum field theory on curved backgrounds. Then, how does Hawking compare an initial vacuum state with a final state in which the number of particles is different from zero? In order to answer this question, we need to define what a particle is. We can basically give two definitions. The first one is more operative: a particle is something we measure by means of a detector. The second one is more mathematical: a particle is something we obtain if we apply a creation operator. An identification between these two definitions should hold, or quantum field theory would not work as we know.

According to the equivalence principle, as already anticipated before, an accelerated system can be related to a uniform gravitational field. Although in reality gravitational fields are not globally uniform, we can consider a very small portion of spacetime where they may be considered as such. For example, the gravitational field produced by a black hole is globally non-uniform. Nevertheless, if we consider a very small time interval (say of the order of 10^{-12} s) and a very small spatial region (with linear dimension, say of the order of 10^{-15} m), in this portion of spacetime the gradient of the field could be small and so it will not be unreasonable to consider the field uniform there. Of course the time interval and the spatial dimensions should be chosen depending on the curvature and magnitude of the gravitational field. For instance, the larger is the curvature the smaller should be the region that we consider. Starting from these considerations, one expects to reduce Hawking radiation to Unruh radiation, at least from a theoretical point of view. At this point, however, an objection could be made. Precisely because of the "operative" definition of particle that we have given above, one could make a further step, which consists in placing a detector near the event horizon. Naively, one would detect both a number of particle and an energy which diverge. Thus, how is it possible to reduce conceptually this situation to the Unruh effect? The answer lies in the fact that in reality it is not correct to speak of a particle near the event horizon since this concept, as seen before, is consistent only in Minkowski space. Thus, what would be the meaning of placing a detector in a curved region if we are unable to count particles in such a region? Consider now the Schwarzschild solution in its maximal extension. As it is well-known, we can identify two asymptotically flat regions where we will be able to count particles, while we cannot do it near the event horizon; usual definition of a detector as we know it does not apply close to the event horizon, nor where the gravitational field is not negligible. In order to get a better understanding, let us just think about the way we study scattering processes: we prepare the incoming particle at infinite distance from the target before the collision and then we measure what happens at an infinite distance after the collision, but we do not specifically consider the region of interaction between the two particles. Similarly, Hawking counts particles in the two flat regions, away from the black hole, in the past and in the future. This allows him to start with a vacuum state and to obtain a final state in which particles are distributed according to a thermal spectrum, at a certain temperature T_H. To sum up, it is not possible to count particles near the horizon because the concept of particle does not make sense there. In fact, a particle is what makes the detector emit a signal, but we do not know what is really going on for our standard definitions near the horizon. Therefore, it is fundamental to clarify the origin of the apparent paradox. As often happens, it is nothing but an imprecise use of a language or a concept that in the considered context is somehow meaningless. In this case, the concept of particle and so that of particle number are not adequate near the black-hole event horizon, nor where there is a non-negligible gravitational field. For this reason, Hawking's argument is built as much as possible on the knowledge of quantum field theory and its application in Minkowski spacetime, so his purpose was to use as much of this theory as possible in an asymptotically flat region of the curved spacetime considered.

1.2 How to quantize in curved spacetime

General relativity treats the spacetime metric as a smooth changing field, which is therefore a dynamical quantity. Conversely, in quantum field theory the metric is a background. Therefore, in any physical situation in which gravity and another interaction appear, the role of the metric must be certainly clarified. The first approach one can think of is to quantize spacetime and all the other fields independent of the metric; unfortunately, no theory has yet proven successful in doing this using standard quantization methods. Technically speaking, we say that quantum gravity is a non-renormalizable theory (the reader may refer to Ref. [28] for further details).

Therefore, as a first step, it seems reasonable to consider the so-called semiclassical approximation, which is a simplified scheme in which we treat fields quantum-mechanically on a classical curved spacetime background. This is exactly the framework in which Hawking analyzed the problem of particle creation caused by the gravitational collapse of a body to form a black hole; a natural question arises: how accurate is this approximation? Of course, this question cannot be answered with certainty until we have a satisfactory quantum theory of the gravitational field (it is generally believed that an approximation of this kind is valid when the spacetime curvature is much less than Planckian). However, it is not unreasonable to suppose such an analysis to give a good indication of the kind of phenomena which will occur in an exact quantum treatment.

Let us start by introducing a fixed background, say $(\mathcal{M}, g_{\mu\nu})$. Furthemore, we assume this spacetime to have a Cauchy surface (definition of globally-hyperbolicity): once a set of initial data is defined, the solution to the field equations results entirely determined everywhere. So, in a global hyperbolic spacetime, it is possible to determine the entire past and future history of the universe from the conditions imposed at the instant of time defined implicitly by the Cauchy surface. In the present context, our background is represented by the Schwarzschild spacetime. To be more precise, two important results have to be mentioned; the first is a theorem due to Dieckmann and Geroch (see Ref. [29] for further details), while the second one is due to Hawking and Ellis (one may consult the book written by the same authors, Ref. [30]):

- **Theorem 1:** If $(\mathcal{M}, g_{\mu\nu})$ is globally hyperbolic with Cauchy surface Σ, then $\mathcal{M}$ has topology $\mathbb{R} \times \Sigma$. Furthermore, $\mathcal{M}$ can be foliated by a one-parameter family of smooth Cauchy surfaces Σ_t, where t acts as a type of time coordinate. All hypersurfaces of coordinate $t =$ const. form Cauchy surfaces.

- **Theorem 2:** For any globally hyperbolic spacetime $(\mathcal{M}, g_{\mu\nu})$ with Cauchy surface Σ, there exists a global solution φ to the Klein-Gordon equation valid on all of $\mathcal{M}$ for which $\varphi = \varphi_0$ and $n^\mu \nabla_\mu \varphi = \tilde{\varphi}_0$ on Σ, where n^μ is the future unit normal to Σ. In this sense the function φ_0 is the classical evolution of the pair of functions $(\varphi_0, \tilde{\varphi}_0)$ on Σ.

The above two theorems basically say that the presence of a Cauchy surface provides the framework to built a quantum field theory. By singling out a single timelike coordinate t, one can consistently define the momentum density conjugate to the field and so proceed with a phase-space formulation of the evolution of the scalar field φ. This is how we usually proceed in quantum field theory in flat spacetime, where we write the solution to the Klein-Gordon equation in terms of a complete, orthonormal set of modes. In order to make sense of "orthonormal", a precise binary operation (i.e., an inner product) is defined, expressed as an integral over a constant-time hypersurface Σ_t[3]. As it is well-known, if we restrict to the subspace of positive-frequency solutions, then the Klein-Gordon inner product is positive-definite, and so we can safely define an Hilbert space $\mathcal{H}$. Instead, the negative-frequency solutions can be put into linear correspondence with vectors of the complex conjugate Hilbert space $\mathcal{H}^*$ that is dual to $\mathcal{H}$. Thus, as elements of $\mathcal{H}^*$, the negative-frequency solutions adhere to the positive-definiteness condition necessary to build up a Hilbert space. Now, in order to define a vacuum state, we introduce a physically meaningful Hamiltonian by recalling Noether's theorem. Indeed, the Hamiltonian is nothing more than the conserved charge associated with time translations between Cauchy surfaces. Having said that, we have all the necessary tools to quantize our theory; applying the standard procedure, the field φ, which is now treated as an operator-valued distribution, can be written in terms of creation and annihilation operators. These operators define a bosonic Fock space of particles as excitations of a unique vacuum state $|0\rangle$, the ground state of the Hamiltonian, and satisfy the usual commutation relations. The main point of this discussion is that, in order for a globally valid quantum field theory (with a unique vacuum) to be defined, the spacetime in which we are working in must be "uniform".

Now, the question is: can we more or less easily generalize this scheme when the spacetime is curved? Or, equivalently, can we build a quantum field theory in globally hyperbolic spacetimes following the same procedure? Unfortunately, for general globally hyperbolic spacetimes, this is not as straightforward. The reason is that, even if Cauchy surfaces are present, it is impossible to define a universal concept of time on them. As a consequence, no conserved charge associated with time translation between Cauchy surfaces can be defined and, in turn, no single quantum field theory that is valid everywhere. Luckily, there is a way out if one more assumption is made. In fact, if we restrict to stationary spacetimes, then the situation completely changes. Let us now try to be more concrete. The metric signature we use is $(-,+,+,+)$.

Consider a free scalar field φ propagating on a given globally hyperbolic spacetime. The action is given by the following expression:

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi - \frac{1}{2} m^2 \varphi^2 \right) , \tag{1}$$

where ∇_μ denotes the covariant derivative. The equation of motion of the scalar field can be

[3]It is possible to prove that such an inner product is independent of the hypersurface Σ_t.

easily obtained in the usual way, namely by varying the action. We have:

$$\delta S = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} g^{\mu\nu}\partial_\nu\right)\varphi - m^2\varphi := \Box\varphi - m^2\varphi = 0, \tag{2}$$

where we defined the d'Alembertian operator. The above equation is the Klein-Gordon equation in curved spacetime, with no coupling to the curvature scalar R (however, the results we will obtain would be the same since in the case of Schwarzschild we have that $R = 0$). Let us now try to quantize the theory following the procedure outlined before. For a spacelike hypersurface Σ with induced metric γ_{ij} and unit normal vector n^μ, we define the scalar product as

$$(\varphi_1, \varphi_2) = -i\int_\Sigma \left(\varphi_1\nabla_\mu\varphi_2^* - \varphi_2^*\nabla_\mu\varphi_1\right) n^\mu\sqrt{\gamma} d^{n-1}x. \tag{3}$$

Working in four dimensions, namely setting $n = 4$ in the above expression, we can easily show that the above definition of the inner product is independent of the choice of Σ:

$$\begin{aligned}
(\varphi_1, \varphi_2)|_{\Sigma_1} - (\varphi_1, \varphi_2)|_{\Sigma_2} &= -i\int_{\Omega=\Sigma_1-\Sigma_2} \left(\varphi_1\nabla_\mu\varphi_2^* - \varphi_2^*\nabla_\mu\varphi_1\right)\sqrt{\gamma} n^\mu d^3x, \\
&= -i\int_{\partial\Omega} \nabla^\mu\left(\varphi_1\nabla_\mu\varphi_2^* - \varphi_2^*\nabla_\mu\varphi_1\right)\sqrt{-g} d^4x, \\
&= -i\int_{\partial\Omega} \left(\varphi_1 m^2\varphi_2^* - \varphi_2^* m^2\varphi_1\right)\sqrt{-g} d^4x \\
&= 0,
\end{aligned} \tag{4}$$

where Gauss' theorem has been used to get to the second equality and the equation of motion to get to the last one. So far, so good. Now, since the spacetime in question admits a Cauchy surface, it is well-known that a non-unique orthonormal basis of solutions can be found:

$$(u_i, u_j) = \delta_{ij}, \quad (u_i^*, u_j^*) = -\delta_{ij}, \quad (u_i, u_j^*) = 0, \tag{5}$$

where, for simplicity, the indices i, j have been chosen to be discrete. So, the good news is that we were able to define a consistent binary operation that in turn allowed us to find an orthonormal basis. The "bad news" is that, since such a basis is non-unique, if we did standard field quantization we would obtain a vacuum state without a basis-independent physical meaning. However, what bothers us most is not the fact that we have more than one basis (we already expected that after the discussion about the Unruh effect!), but that we cannot introduce a physically meaningful Hamiltonian due to the absence of a preferred time coordinate. Or, putting it differently, in curved spacetime we are not in general be able to find solutions to the Klein-Gordon equation that separate into time-dependent and space-dependent factors, and so we cannot classify modes as positive- or negative-frequency as in flat spacetime. To sum up, having at hand many sets of solutions, on what grounds we choose one of them over the others? It seems that the standard way of proceeding breaks down at this point.

Now, the assumption we were talking about before, namely the stationarity of the metric, comes into play. By definition, a spacetime $(\mathcal{M}, g_{\mu\nu})$ is stationary if there exists a timelike Killing vector field $K = \partial_t$ for the metric; why does this assumption is helpful? This can be understood by considering a couple of properties of the Killing vector field. The first one is that K commutes with the Klein-Gordon operator. Indeed, on one hand we have that

$$K\Box\varphi = \partial_t\Box\varphi = \partial_t\left(\nabla_\mu g^{\mu\nu}\partial_\nu\varphi\right) = g^{\mu\nu}\left(\partial_t\partial_\mu\partial_\nu\varphi - \Gamma^\sigma_{\mu\nu}\partial_t\partial_\sigma\varphi\right). \tag{6}$$

On the other hand, we can also write

$$\Box K\varphi = \Box\partial_t\varphi = \nabla_\mu g^{\mu\nu}\partial_\nu\partial_t\varphi = g^{\mu\nu}\left(\partial_\mu\partial_\nu\partial_t\varphi - \Gamma^\sigma_{\mu\nu}\partial_\sigma\partial_t\varphi\right), \tag{7}$$

from which the result immediately follows. The second property of K is antihermiticity. The proof is immediate; given two complex-valued functions f, g, we have:

$$\begin{aligned}(f, Kg) &= -i\int_\Sigma \left[f\nabla_\mu\left(\partial_t g^*\right) - \left(\partial_t g^*\right)\nabla_\mu f\right]\sqrt{\gamma}n^\mu d^3x \\ &= -i\int_\Sigma \left[\left(-\partial_t f\right)\nabla_\mu g^* - g^*\nabla_\mu\left(-\partial_t f\right)\right]\sqrt{\gamma}n^\mu d^3x = (-Kf, g),\end{aligned} \tag{8}$$

where we integrated by parts to get to the second equality. Now, the first property allows us to find simultaneous eigenmodes of these two operators, while the second property we have introduced, antihermiticity, tells us that the eigenvalues of K are purely imaginary, namely we can write $Kf_i = -i\omega f_i$, with $\omega \in \mathbb{R}_{\neq 0}$; we also recall that eigenfunctions corresponding to distinct eigenvalues must be orthogonal. We now basically have all the necessary tools to second-quantize our theory. The set of modes f_j are defined to be positive-frequency if

$$\partial_t f_j = -i\omega f_j, \quad \omega > 0. \tag{9}$$

If, instead, we have a set of modes f_{j^*} satisfying

$$\partial_t f_j^* = i\omega f_j^*, \quad \omega > 0, \tag{10}$$

then these are defined to be negative-frequency modes[4]. Having said that, we can now easily find a (unique) basis $\{u_i\}$ of positive-frequency eigenmodes (see the first of the two equations above) that are solutions of the wave equation with a purely positive-definite scalar product, $(u_i, u_j) = \delta_{ij}$. Therefore, in exactly the same way as in Minkowski flat spacetime, a legitimate Hilbert space can be defined; concerning the negative-frequency solutions, they can be put into linear correspondence with vectors of the complex conjugate Hilbert space. By applying the now familiar quantization procedure, we write the field operator φ in terms a and $a^\dagger$:

$$\varphi = \sum\nolimits_i \left(a_i f_i + a_i^\dagger f_i^*\right). \tag{11}$$

These operators, a and $a^\dagger$, define a bosonic Fock space of particles as excitations of a unique vacuum state which is the ground state of an Hamiltonian that can now be consistently introduced thanks to the presence of the time Killing vector field K.

What we want to do now is to consider a particular spacetime $(\mathcal{M}, g_{\mu\nu})$ divided into three regions, denoted as B, C and T, respectively, such that $\mathcal{M} = B \cup C \cup T$ (this is usually called

[4]Note that these modes are called negative-frequency modes even if $\omega > 0$.

sandwich spacetime). If we consider a scalar field φ propagating on $\mathcal{M}$, then we know that the wave equation holds in the entire spacetime (Theorem 2). Unlike regions B and T, where the metric is stationary (but timelike Killing vectors are different), region C is not stationary provided that the manifold remains globally hyperbolic. In regions B and T, the scalar field is quantized by choosing two sets of positive-frequency modes, say $\{f_i\}$ and $\{g_i\}$, respectively. Therefore, it can be expressed in terms of these modes as

$$\varphi = \sum\nolimits_i \left(a_i f_i + a_i^\dagger f_i^*\right) = \sum\nolimits_i \left(b_i g_i + b_i^\dagger g_i^*\right). \tag{12}$$

Moreover, thanks to the completeness property, we can express one set in terms of the other:

$$g_i = \sum\nolimits_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^*\right), \tag{13}$$

where α_{ij} and β_{ij} are called Bogoliubov coefficients. First of all, we need to find a way to compute these coefficients; concerning α_{ij}, this can be easily done by calculating the scalar product between g_i and f_j. Indeed, we have:

$$(g_i, f_j) = (\alpha_{ik} f_k + \beta_{ik} f_k^*, f_j) = \alpha_{ik}(f_k, f_j^*) + \beta_{ik}(f_k^*, f_j) = \alpha_{ij}. \tag{14}$$

Similarly we find $\beta_{ij} = -(g_i, f_j^*)$. Moreover, the coefficients α_{ij} and β_{ij} must satisfy their own normalization conditions. This can be see in the following way. We first compute the inner product between g_i and g_j, obtaining

$$\begin{aligned}(g_i, g_j) &= (\alpha_{ik} f_k + \beta_{ik} f_k^*, \alpha_{jl} f_l + \beta_{jl} f_l^*) \\ &= \alpha_{ik}\alpha_{jk}^* + \beta_{il}\beta_{jl}^*(-1) \\ &= \alpha_{ik}\alpha_{kj}^\dagger - \beta_{ik}\beta_{kj}^\dagger = \delta_{ij}.\end{aligned} \tag{15}$$

In matrix notation we can write this last expression as

$$\alpha\alpha^\dagger - \beta\beta^\dagger = \mathbb{1}, \tag{16}$$

where $\mathbb{1}$ is the identity matrix. We now compute the inner product between g_i and g_j^*:

$$\begin{aligned}\left(g_i, g_j^*\right) &= \left(\alpha_{ik} f_k + \beta_{ik} f_k^*, \beta_{jl}^* f_l + \alpha_{jl}^* f_l^*\right) \\ &= \alpha_{ik}\beta_{jk} + \beta_{il}\alpha_{jl}(-1) \\ &= \alpha_{ik}\alpha_{kj}^T - \beta_{ik}\alpha_{kj}^T = 0,\end{aligned} \tag{17}$$

where we used Eq. (16) to get to the last equality. In matrix notation we have

$$\alpha\beta^T - \beta\alpha^T = 0. \tag{18}$$

Now, we can also express f_i in terms of g_j and g_j^* as follows:

$$f_i = \sum\nolimits_j \left(\alpha'_{ij} g_j + \beta'_{ij} g_j^*\right), \tag{19}$$

where we denoted the new coefficients with α'_{ij} and β'_{ij}; by making use of the above expression, we can write (again using matrix notation)

$$g = \alpha \left(\alpha' g + \beta' g^*\right) + \beta \left(\alpha'^* g^* + \beta'^* g\right) = \left(\alpha\alpha' + \beta\beta'^*\right) g + \left(\alpha\beta' + \beta\alpha'^*\right) g^*. \tag{20}$$

This last expression is equal to g if and only if $\alpha' = \alpha^\dagger$ and $\beta' = -\beta^T$. Therefore,

$$f_i = \sum\nolimits_j \left(\alpha^*_{ji} g_j - \beta_{ji} g^*_j\right). \tag{21}$$

Also, α_{ij} and β_{ij} must satisfy the following conditions:

$$\alpha^\dagger \alpha - \beta^T \beta^* = \mathbb{1}, \quad \alpha^\dagger \beta - \beta^T \alpha^* = 0. \tag{22}$$

From these results we can easily find

$$a_i = (\varphi, f_i) = \sum\nolimits_j \left(\alpha_{ji} b_j + \beta^*_{ji} b^\dagger_j\right), \quad b_i = (\varphi, g_i) = \sum\nolimits_j \left(\alpha^*_{ij} a_j - \beta^*_{ij} a^\dagger_j\right). \tag{23}$$

We have thus expressed a_i in terms of b_j and $b^\dagger_j$, and vice versa.

The procedure outlined above allows us to define the in-vacuum as $a_i \left|0_{in}\right\rangle = 0 \ \forall i$; we will call this state in-vacuum. If we consider an observer together with a stationary reference frame in B, the vacuum will appear empty to such an observer. Now, a very natural questions arises: what happens in T? In order to answer this question, we should be able to compute the number of particles in the initial vacuum state as seen from T. If this number is different from zero, then some particles must have been created; consequently, the vacuum state, which we should call the out-vacuum, defined as $b_i \left|0_{out}\right\rangle = 0 \ \forall i$, would be different from the in-vacuum. In conclusion, there must have been a particle production due to a change of the spacetime geometry, i.e., due to the presence of region C. We therefore now compute N^{out}_i in the in-vacuum as follows:

$$\begin{aligned}
\left\langle 0_{in} \left| N^{out}_i \right| 0_{in} \right\rangle &= \left\langle 0_{in} \left| b^\dagger_i b_i \right| 0_{in} \right\rangle \\
&= \left\langle 0_{in} \left| \sum\nolimits_j \left(\alpha_{ji} a^\dagger_j - \beta_{ji} a_j\right) \sum\nolimits_k \left(\alpha^*_{ik} a_k - \beta^*_{ik} a^\dagger_k\right) \right| 0_{in} \right\rangle \\
&= \sum\nolimits_{jk} \left(-\beta_{ji}\right)\left(-\beta^*_{ik}\right) \left\langle 0_{in} \left| \hat{a}_j \hat{a}^\dagger_k \right| 0_{in} \right\rangle \\
&= \sum\nolimits_{jk} \beta_{ij}\beta^*_{ik} \left\langle 0_{in} \left| \left(\hat{a}^\dagger_k \hat{a}_j + \delta_{jk}\right) \right| 0_{in} \right\rangle \\
&= \sum\nolimits_{jk} \beta_{ij}\beta^*_{ik} \delta_{jk} \left\langle 0_{in} \mid 0_{in} \right\rangle \\
&= \sum\nolimits_j \left|\beta_{ij}\right|^2,
\end{aligned} \tag{24}$$

where we made use of the fact that $\beta_{ij} = \beta_{ji}$ and also used the commutation relation between a and $a^\dagger$. What we learn from the above relation is that the number of particles as seen from T is in general different from the number of particles measured from B.

2 Derivation of the Hawking spectrum

In the present section we will reproduce Hawking's calculation on black-hole radiance. Then, the same phenomenon will be derived by considering the so-called eternal black hole. This section is developed on the original paper written by Hawking in the seventies [2], as well as on the subsequent work written by Wald [31]. We will closely follow the lecture notes on black holes by Dowker [26] and Traschen [32], and also Refs. [33–35]. The reader is assumed to be familiar with the construction of Penrose diagrams.

2.1 Field quantization

Let us consider, in Fig. 1, the Penrose diagram corresponding to the spacetime of a spherically-symmetric collapsing star that is about to form a black hole. As is well-known, Schwarzschild spacetime is curved and globally hyperbolic; it is not dissimilar to the sandwich spacetime we have talked in the previous section. What we called region C corresponds to the star that is about the form a black hole, while regions B and T corresponds to the far asymptotic past (i.e., near I^-) and the far asymptotic future (i.e., near I^+), respectively.

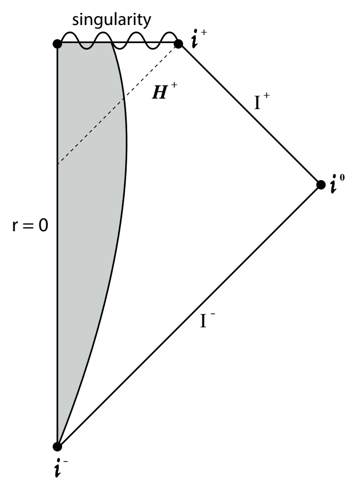


Figure 1: Penrose diagram of a collapsing star. Figure adapted from [26].

We can therefore proceed in the same way as before. A positive-frequency set of modes can be certainly defined on I^-. However, the same cannot be done for I^+, which is not a Cauchy surface. Indeed, at late times, some of the in-falling particles may cross the horizon H^+ and so never reach the far asymptotic future. Thus, the Cauchy surface needed for quantization is $\mathrm{I}^+ \cup H^+$. To sum up, we can write the set of modes as follows:

- $\{f_i\}$: positive-frequency modes on I^-;
- $\{p_i\}$: positive-frequency modes on I^+ and zero on H^+;
- $\{q_i\}$: "positive-frequency modes" on H^+ and zero on I^+.

As we see, quotation marks have been used when introducing positive-frequency modes on the horizon. The reason is simple: no timelike Killing vector exists on H^+. Putting it differently, we can also say that in absence of a timelike Killing vector, a physically meaningful Hamiltonian cannot be defined, as well as a consistent quantum field theory; Wald discussed this issue in his 1975 paper, proving that all predictions of the theory with regard to measurements at infinity are actually independent of the definition of positive-frequency modes on the horizon. Therefore, in order to perform the calculation, we can safely assume that the modes on the horizon are positive-frequency modes. We are now ready to expand our field φ in terms of the two different sets of modes $\{f_i, f_i^*\}$ and $\{p_i, p_i^*\} \cup \{q_i, q_i^*\}$:

$$\begin{aligned}\varphi &= \sum\nolimits_i \left(a_i f_i + a_i^\dagger f_i^*\right) \\ &= \sum\nolimits_i \left[\left(b_i p_i + b_i^\dagger p_i^*\right) + \left(c_i q_i + c_i^\dagger q_i^*\right)\right].\end{aligned} \tag{25}$$

The number of particles at I^+ will be given by

$$\langle 0_{in} \left| N_i^{out} \right| 0_{in} \rangle = (\beta\beta^\dagger)_{ii}, \tag{26}$$

where the in-vacuum is defined as $a_i \left|0_{in}\right\rangle = 0 \; \forall i$. Now, on the basis of our previous discussion, the modes $\{f_i, f_i^*\}$ are related to the modes $\{p_i, p_i^*\} \cup \{q_i, q_i^*\}$ as

$$p_i = \sum\nolimits_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^*\right), \tag{27}$$

$$q_i = \sum\nolimits_j \left(\gamma_{ij} f_j + \delta_{ij} f_j^*\right). \tag{28}$$

As we have seen before, the Bogoliubov coefficient β_{ij} in the p_i expression is needed if we want to obtain an expression for the particle number. To this end, as we will see, Hawking "traced the solution back in time", from the far future to the far past. We will soon understand what does it mean exactly. In the next paragraph we will analyze the form of the Klein-Gordon equation in Schwarzchild spacetime. In the following we will use geometric units ($c = G = 1$).

2.2 Equation of motion in a Schwarzchild background

In Schwarzschild coordinates, (t, r, θ, ϕ), the Schwarzschild metric reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{29}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The massless Klein-Gordon equation,

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu\right) \varphi = 0, \tag{30}$$

can be specialized to the spacetime in question, obtaining

$$\partial_t \left[-\left(1 - \frac{2M}{r}\right)^{-1} \partial_t \varphi\right] + \frac{1}{r^2} \partial_r \left[\left(1 - \frac{2M}{r}\right) r^2 \partial_r \varphi\right] + \frac{1}{r^2} \Delta_\Omega \varphi = 0, \tag{31}$$

where Δ_Ω is the Laplacian operator on the unit two-sphere:

$$\Delta_\Omega := \frac{1}{\sin\theta} \partial_\theta \left(\sin\theta \partial_\theta\right) + \frac{1}{\sin^2\theta} \partial_\phi^2. \tag{32}$$

In order to solve the above equation we can use the method of separation of variables. Also, the spherical-symmetry of the background allows us to make the following ansatz[5]:

$$\varphi(t, r, \theta, \phi) = \frac{\phi(r,t)}{r} Y_{lm}(\theta, \phi), \tag{33}$$

where an expansion in spherical harmonics has been considered[6]. Thus, Eq. (31) reduces to

$$-\left(1 - \frac{2M}{r}\right)^{-1} \partial_t^2 \phi + \frac{2M}{r^2}\left(\partial_r \phi - \frac{1}{r}\phi\right) + \left(1 - \frac{2M}{r}\right)\partial_r^2 \phi - \frac{l(l+1)}{r^2}\phi = 0. \tag{34}$$

Above, we used one of the main properties of the spherical harmonics, namely that

$$\Delta_\Omega Y_{lm}(\theta, \phi) = -l(l+1) Y_{lm}(\theta, \phi). \tag{35}$$

Now, by introducing the so-called tortoise coordinate,

$$r_* = r + 2M \ln\left|\frac{r}{2M} - 1\right|, \tag{36}$$

from which the following expressions can be easily derived:

$$\partial_r = \frac{\partial_{r*}}{1 - \frac{2M}{r}}, \tag{37}$$

$$\partial_r^2 = -\frac{2M}{r^2}\frac{\partial_{r_*}}{\left(1 - \frac{2M}{r}\right)^2} + \frac{\partial_{r_*}^2}{\left(1 - \frac{2M}{r}\right)^2}. \tag{38}$$

Eq. (34) can then be written as (note that now ϕ is a function of r_* and t)

$$\left(\partial_t^2 - \partial_{r_*}^2 + V_l\right)\phi = 0, \tag{39}$$

where the potential $V_l(r_*)$ in the above equation is given by[7]

$$V_l(r_*) := \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right]. \tag{40}$$

Now, the method of separation of variables can be used once again:

$$\phi(r_*, t) = e^{-i\omega t}\psi(r_*), \tag{41}$$

where $R_{l\omega}$ is the solution of the radial equation

$$\left(\partial_{r_*}^2 + \omega^2\right)\psi = V_l \psi. \tag{42}$$

All that remains is solving Eq. (42), which is an ordinary differential equation of second order. One can cast the above equation into the so-called confluent Heun equation; however, the solutions of such equation are rather complicated. Luckily, we can avoid considering the complete

[5]For simplicity, we will omit sums over the indices l and m.

[6]We will dedicate an entire paragraph to the spherical harmonics in the next section.

[7]The potential depends on r_* through r.

solutions if we look a bit more closely at $V_l(r_*)$; in fact, we can notice that the potential goes to zero both near the event horizon H^+, where $r \to 2M \iff r^* \to -\infty$, and near $\mathrm{I}^\pm$, where $r \to \infty \iff r^* \to \infty$. Starting from these considerations, Eq. (42) can be easily solved; the solution can then be plugged in Eq. (41) which, in turn, leads to the solutions of the Klein-Gordon equation in the limit $r \to \infty$. Since we are only interested in ingoing early modes and outgoing late modes, we can define

$$f_{\omega'} \sim \frac{1}{r\sqrt{2\pi\omega'}} e^{-i\omega' v}, \quad \text{(ingoing early modes)} \tag{43}$$

$$p_\omega \sim \frac{1}{r\sqrt{2\pi\omega}} e^{-i\omega u}, \quad \text{(outgoing late modes)} \tag{44}$$

where we also introduced the so-called light-cone coordinates u and v, defined as $u := t - r_*$ and $v := t + r_*$. Note that in the previous subsection we used the index i to denote the state, while here we are using ω (in principle we should have introduced also the "quantum numbers" l and m, but we are working in a spherically-symmetric setup, so we may safely drop them). Of course, the solutions should also contain the spherical harmonics, but since it is not relevant for the following discussion we decided not to write them. Concerning the frequencies ω' and ω, they are eigenvalues of the following eigenequations (see the discussion in subsection 1.2):

$$i\partial_t f_{\omega'} = \omega' f_{\omega'}, \tag{45}$$

$$i\partial_t p_\omega = \omega p_\omega. \tag{46}$$

Moreover, again referring to subsection 1.2, we know that each solution p_ω can be written as a linear combinations of the $f_{\omega'}$'s and their complex conjugates (continuous version):

$$p_\omega = \int_0^\infty d\omega' \left(\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^*_{\omega'}\right), \tag{47}$$

As already anticipated, in order to compute the coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$, we will trace the solution back in time; more specifically, we are going to consider a solution p_ω which propagates inwards from I^+ (towards decreasing values of r^*), with zero Cauchy data on the event horizon, until it reaches the potential barrier. Part of the wave, say $p_\omega^{(1)}$, will be reflected by the barrier and will end up on I^- preserving its frequency ω. This will give a coefficient $\alpha_{\omega\omega'}$ proportional to a delta function, namely $\delta(\omega - \omega')$. The remaining part of the wave, say $p_\omega^{(2)}$, will enter the collapsing matter, i.e., it will be transmitted through the potential barrier with the result of being distorted before eventually coming out towards past null infinity. Therefore, $p_\omega^{(2)}$ is the part of the wave we are interested in.

A couple of comments are in order here. We have said that we are interested in the part of the wave which enters the collapsing matter, but we did not specify any spacetime geometry there. Is this a serious problem? How can we trace back in time the solution if we have no idea about the internal geometry of the star? Fortunately, there is a way out; we will soon explain how we can perform the analysis without specifying the geometry inside the star. Before doing that, let us also notice that positive-frequency plane waves like p_ω are completely delocalized. This problem can be easily overcome if we consider a superposition of these waves, namely constructing a localized wave packet on I^+ (for example a Gaussian packet). In particular, we can construct it so that it is peaked in the neighborhood of some finite frequency ω_0 and some coordinate u_0; we will comment more on this later on. Now, concerning the problem of the internal geometry of the star, we know that u diverges at the horizon and so the effective frequency of the solution becomes arbitrarily large; the fact that near H^+ the frequency is so high will lead to a fundamental approximation, i.e., the geometrical optics approximation, which can be explained as follows. Consider a wave which propagates with a certain wavelength λ towards an obstacle. If the wavelength is of the order of the obstacle's characteristic dimension (or higher), we have to study interference and diffraction phenomena; on the contrary, if the wavelength is smaller with respect to the obstacle's dimension, these phenomena are negligible and we can safely consider the wave as if it is propagating along straight trajectories (rays). This logic applies to $p_\omega^{(2)}$ as well, which will be able to cross the body and escape towards I^+.

2.3 Tracing back in time

Let us now analyze the form of the solution at I^-. In order to do so, we refer to Fig. 2, where we indicated with y a point on the event horizon. Then, as we can notice, two null vectors have been introduced: l^μ, tangent to the horizon, and n^μ, a future-directed null vector pointing towards the singularity; we choose these two vectors to be normalized so that $l^\mu n_\mu = -1$. Looking at the figure, we notice that no path with an affine parameter v larger than v_0 would be able to arrive at I^+ because it would end into the newly-formed black hole. Furthermore, we introduce another vector, $-\epsilon n^\mu$, with ϵ small and positive.

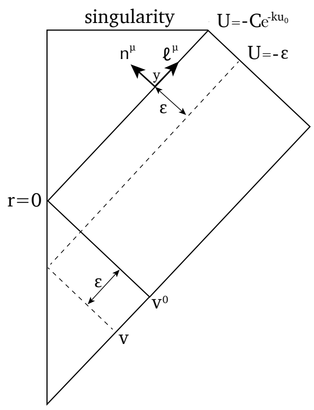


Figure 2: Penrose diagram useful to understand Hawking's analysis. Figure adapted from [33].

We can now parallel transport the two vectors n^μ and l^μ along γ_H, the null geodesic which travels backwards from I^+. Also, we notice that the vector $-\epsilon n^\mu$ generates another null geodesic, γ, which starts at some u at I^+ and ends on I^- at v. If we now transport the vectors l^μ and n^μ back to the point of intersection between the past and future horizons, then the vector $-\epsilon n^\mu$ introduced before will lie along the past event horizon. Let U be an affine parameter on the past event horizon. At the point where the two horizons intersect, we have that

$$U = 0, \quad n^\mu = \frac{dx^\mu}{dU}. \tag{48}$$

Moreover, U and the retarded time u are related by the following relation:

$$U = -Ce^{-\kappa u}, \tag{49}$$

where C is a constant and $\kappa = 1/4M$ is the surface gravity of the black hole. On the null geodesic γ, near the event horizon, the affine parameter is given by $U = -\epsilon$. Therefore, by inverting the above relation we get the expression for u on γ:

$$u = -\frac{1}{\kappa}\left(\ln \epsilon - \ln C\right). \tag{50}$$

So far, so good. Let us recall that our goal is to obtain the form of $p_\omega^{(2)}$ on I^-. We first notice that on I^- the vector n^μ is parallel to the Killing vector, say K^μ, which is tangent to the null geodesics generators of I^-. Therefore, we write

$$n^\mu = D\xi^\mu, \tag{51}$$

where D is some constant. Moreover, from the figure it is evident that $\epsilon = v_0 - v$ on I^-. Putting it all together, we finally obtain the phase of the solution:

$$\frac{\omega}{\kappa}\left(\ln\left(v_0 - v\right) - \ln D - \ln C\right). \tag{52}$$

At this point we need to distinguish two cases. For $v > v_0$, $p_\omega^{(2)}$ gives no contribution since particles would be trapped beyond the event horizon. However, if $v \leq v_0$, then we can write down the final expression for $p_\omega^{(2)}$ by using the result we just derived:

$$p_\omega^{(2)} \sim \exp\left[i\frac{\omega}{\kappa}\ln\left(\frac{v_0 - v}{CD}\right)\right], \quad v < v_0. \tag{53}$$

Let us now simplify a little bit our $p_\omega^{(2)}$. We may safely set $v_0 = 0$ since the spacetime in question is invariant under translations $v \to v + A$, where A is some constant. Moreover, we can define $\eta := (CD)^{-1}$. In this way, the solution can be written as

$$p_\omega^{(2)} \sim \exp\left[i\kappa^{-1}\omega\ln(-\eta v)\right], \quad v < 0,\ \|v\| \ll 1. \tag{54}$$

We now also recall that p_ω can be expressed as

$$p_\omega = \int_0^\infty d\omega'\left(\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^*_{\omega'}\right), \tag{55}$$

where we omitted the superscript (2) for simplicity. Our aim is find $\beta_{\omega\omega'}$ and, in turn, the particle number. In order to do so, we first substitute $f_{\omega'}$ in the above equation and then consider a Fourier transformation, resulting in

$$\begin{aligned}\int_{-\infty}^\infty dv e^{i\omega'' v} p_\omega(v) &\sim \int_{-\infty}^\infty dv e^{i\omega'' v}\int_0^\infty d\omega'\left(\alpha_{\omega\omega'} e^{-i\omega' v} + \beta_{\omega\omega'} e^{i\omega' v}\right)\\ &= 2\pi\int_0^\infty d\omega'\left[\alpha_{\omega\omega'}\delta\left(\omega' - \omega''\right) + \beta_{\omega\omega'}\delta\left(\omega' + \omega''\right)\right].\end{aligned} \tag{56}$$

Since $\omega' + \omega'' \neq 0$, from the above expression we get

$$\alpha_{\omega\omega'} \sim \int_{-\infty}^{\infty} dv e^{i\omega' v} p_\omega(v) := \tilde{p}_\omega(\omega'). \tag{57}$$

In a very similar way we can write

$$\begin{aligned}\int_{-\infty}^{\infty} dv e^{-i\omega'' v} p_\omega(v) &\sim \int_{-\infty}^{\infty} dv e^{-i\omega'' v} \int_0^{\infty} d\omega' \left(\alpha_{\omega\omega'} e^{-i\omega' v} + \beta_{\omega\omega'} e^{i\omega' v}\right) \\ &= 2\pi \int_0^{\infty} d\omega' \left[\alpha_{\omega\omega'} \delta\left(\omega' + \omega''\right) + \beta_{\omega\omega'} \delta\left(\omega' - \omega''\right)\right],\end{aligned} \tag{58}$$

from which we can deduce that

$$\beta_{\omega\omega'} \sim \int_{-\infty}^{\infty} dv e^{-i\omega' v} p_\omega(v) := \tilde{p}_\omega(-\omega'). \tag{59}$$

Now, if we are able to relate $\tilde{p}_\omega(\omega')$ to $\tilde{p}_\omega(-\omega')$, then we are done. Indeed, in this case the coefficient of interest can be easily found by using the orthonormality condition proved before (its continuous version to be precise), $\alpha\alpha^\dagger - \beta\beta^\dagger = I$. So, let try to find the relation between such quantities. First of all, we notice that

$$\tilde{p}_\omega(\omega') = \int_{-\infty}^{\infty} dv e^{i\omega' v} p_\omega(v) \approx \int_{-\infty}^{0} dv e^{i\omega' v} e^{i\kappa^{-1}\omega \ln(-\eta v)}. \tag{60}$$

The above approximation is valid since $|\omega'| \gg 1$; in fact, in this case $e^{i\omega' v}$ vary so rapidly that its oscillations cancel out any contribution to the integral when v is large. Concerning the integral over the positive range of v, it is zero since the solution p_ω vanishes in this case. Now, the reader may wonder why such approximation ($|\omega'| \gg 1$) is valid. Heuristically, only if the initial frequency is very high the photon can "come out from the horizon" and be seen by an observer at infinity as travelling with a "normal" frequency. The idea is that the photon loses a huge amount of energy while trying to escape the black hole and so, in order to succeed, it needs to have a lot of energy, i.e., a high frequency. Before proceeding, we first notice that the integral in Eq. (60) is not convergent; this is due to the fact we should have considered wave packets. Instead of doing so and making the procedure rigorous, we manipulate the solution as if it converged; the calculation will be much easier and the physics clear anyway. Above, we obtained an expression for $\tilde{p}_\omega(\omega')$. Similarly, find

$$\tilde{p}_\omega(-\omega') = \int_{\infty}^{\infty} dv e^{-i\omega' v} p_\omega(v) \approx \int_{-\infty}^{0} dv e^{-i\omega' v} e^{i\kappa^{-1}\omega \ln(-\eta v)}. \tag{61}$$

In order to get the relation between $\tilde{p}_\omega(\omega')$ and $\tilde{p}_\omega(-\omega')$, let us extend the above integrand into the complex v-plane; now, as we know, the complex logarithmic function that appears in the above integral is multiple-valued: if a complex number, say z, is written in polar form, namely as $z = re^{i\theta}$, then its logarithm $\ln z = \ln r + i(\theta + 2\pi n)$ has multiple possible values corresponding to different values of n. For this reason, let us take a brunch cut as in Fig. 3 (of course, this is not the only possible choice!). Consequently, the contour we choose to perform the integration must avoid the brunch cut; in order to do that, we simply deform the contour by considering a half circle of radius ϵ whose center is the origin of the complex plane, as shown below, and then send ϵ to zero. Moreover, we notice that the function we are integrating is holomorphic inside the closed contour since it contains no poles. Therefore, we can safely apply Cauchy's theorem:

$$\oint dv e^{-i\omega' v} e^{i\kappa^{-1}\omega \ln(-\eta v)} = \left\{ \int_{\gamma_1} + \int_{\gamma_2} + \int_{\Gamma} \right\} dv e^{-i\omega' v} e^{i\kappa^{-1}\omega \ln(-\alpha v)} = 0. \tag{62}$$

Notice that we did not write down the contribution coming from the semicircle of radius ϵ, which indeed vanishes when setting ϵ to zero. Now, by assuming $\omega' > 0$ (without losing generality), we see that the integrand in $\tilde{p}_\omega(-\omega')$ exponentially decays for $\mathrm{Im}(v) < 0$, implying that the integration over Γ gives no contribution as $R \to \infty$. Hence, we have:

$$\int_{\gamma_1} dv e^{-i\omega' v} e^{i\kappa^{-1}\omega \ln(-\eta v)} = -\int_{\gamma_2} dv e^{-i\omega' v} e^{i\kappa^{-1}\omega \ln(-\eta v)}, \tag{63}$$

where γ_1 and γ_2 extend to infinity now. So, in light of this, we can rewrite Eq. (61) as

$$\begin{aligned} \tilde{p}_\omega(-\omega') &\approx \int_{-\infty}^{0} dv e^{-i\omega' v} e^{i\kappa^{-1}\omega \ln(-\eta v)} \\ &= -\int_{0}^{\infty} dv e^{-i\omega' v} e^{i\kappa^{-1}\omega \ln(-\eta v)} \\ &= -\int_{-\infty}^{0} dv e^{i\omega' v} e^{i\kappa^{-1}\omega \ln(\eta v)} \\ &= -\int_{-\infty}^{0} dv e^{i\omega' v} e^{i\kappa^{-1}\omega [\ln(-\eta v) + i\pi]} \\ &= -e^{-\pi\omega/\kappa} \tilde{p}_\omega(\omega'). \end{aligned}$$

In the above expression, to get to the first equality, we used the result derived above, namely the fact that the integral along the curve γ_1 is equal to the one along γ_2 up to a minus sign. Moreover, in the next step we simply changed variable, $v \to -v$. Now, by recalling how the coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are related to $\tilde{p}_\omega(\omega')$ and $\tilde{p}_\omega(-\omega')$, respectively, we obtain $|\beta_{\omega\omega'}| = e^{-\omega\pi\kappa^{-1}} |\alpha_{\omega\omega'}|$.

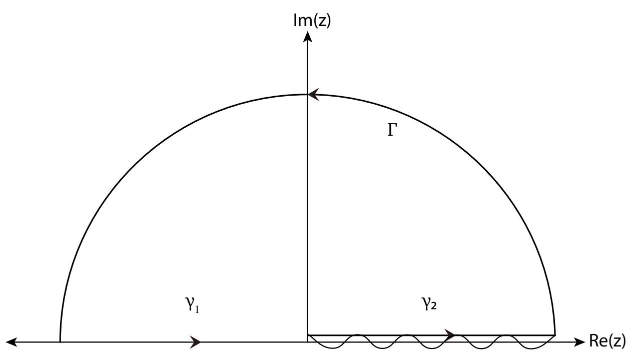


Figure 3: Contour chosen to perform the integration in Eq. (61). Figure adapted from [26].

By using the orthonormality condition (continuous version),

$$\int_0^\infty d\omega' \left(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2 \right) = 1, \tag{64}$$

we finally find the spectrum of particles:

$$N_\omega = \int_0^\infty d\omega' |\beta_{\omega\omega'}|^2 = \frac{1}{e^{2\pi\kappa^{-1}\omega} - 1}. \tag{65}$$

Eq. (65) shows that a black hole behaves like a black body, emitting radiation at the temperature

$$\boxed{T_H = \frac{\kappa}{2\pi}.} \tag{66}$$

Black holes are therefore not as black as they are painted, they are not the eternal prisons they were once thought. Below, we will derive Eq. (66) by considering a slightly different perspective.

2.4 The eternal black hole

From our previous discussion, we deduce that the features of the Hawking effect are actually independent of the nature of gravitational collapse, i.e., the effect is more a consequence of the causal and topological structure of spacetime rather than the specific geometry. Thus, one is tempted to ask: can we derive Hawking's result by considering an already formed black hole, a so-called eternal black hole? In other words, our goal is to examine quantum field theory on the maximally-extended manifold. The Penrose diagram of an eternal black hole is shown is Fig. 4.

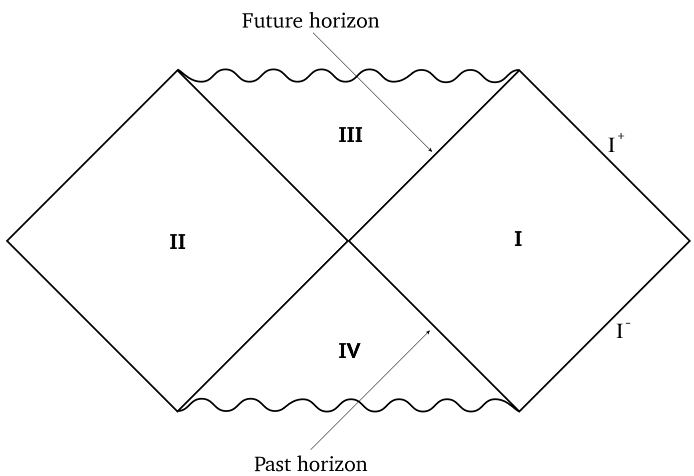


Figure 4: Penrose diagram of an eternal black hole. Figure adapted from [3].

How can we prove that eternal black holes emit radiation? The idea is the following. We consider two different observers: an inertial observer falling into a black hole in a finite proper time, say T, and an asymptotic observer. The relation between these two different points of view is exactly what is needed to derive Hawking's result. To be more precise, what we want to do it so quantize the massless scalar field ϕ in these two different coordinate frames and then compare the corresponding vacuum states. In the following, we will use Schwarzschild-tortoise coordinates for the asymptotic observer, while Kruskal coordinates for the freely-falling one.

Let us now write down the mode expansion in both these coordinates. For the asymptotic observer the procedure has already been outlined in subsection 2.2. Starting from the same ansatz, and introducing the tortoise coordinates r_*, we end up with the following ingoing and outgoing solutions to the Klein-Gordon equation[8]:

$$\begin{aligned}\varphi_{in} &\sim \mathcal{N}_{\omega lm} e^{-i\omega v} \frac{Y_{lm}}{r}, \\ \varphi_{out} &\sim \mathcal{N}_{\omega lm} e^{-i\omega u} \frac{Y_{lm}}{r},\end{aligned} \tag{67}$$

where with $\mathcal{N}_{\omega lm}$ we denoted the normalization. The outside observer can thus expand the fields in modes of given t-frequency ω as follows:

$$\varphi_R \propto \int_0^\infty d\omega \left(b_\omega e^{-i\omega u} + b_\omega^\dagger e^{i\omega u} \right), \tag{68}$$

where, for convenience, we omitted the angular part (spherical harmonics), the factor of $1/r$ and the normalization. Above, the subscript "R" stands for "right-moving", indicating that we are considering only outgoing solutions (this will turn out to be sufficient for our purposes). We also recall that the annihilation and creation operators, denoted as b_ω and $b_\omega^\dagger$, respectively, satisfy the following commutation relation :

$$[b_\omega, b_{\omega'}^\dagger] \propto \delta(\omega - \omega')\delta_{ll'}\delta_{mm'}. \tag{69}$$

The corresponding vacuum state, usually called Boulware vacuum, is defined as

$$b_\omega \left|0\right\rangle_B = 0. \tag{70}$$

Let us now consider the freely-falling observer in region I of the Kruskal diagram; we would like to proceed in exactly the same way as before. Our aim is to write down the massless Klein-Gordon equation in Kruskal coordinates, solve it and expand the field in modes of given T-frequency ν. Outside the event horizon, namely when $r > 2M$, Kruskal–Szekeres coordinates are defined in terms of Schwarzschild coordinates as follows:

$$T = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right), \tag{71}$$

$$R = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right). \tag{72}$$

In (T, R) coordinates, it is not hard to see that the Klein-Gordon equation reduces to

$$\left(\partial_T^2 - \partial_R^2 + \tilde{V}_l\right) \phi(T, R) = 0, \tag{73}$$

where, this time, the potential is given by

$$\tilde{V}_l(r) := \frac{4e^{-\frac{r}{2M}}}{r^3} \left[l(l+1) + \frac{2M}{r} \right]. \tag{74}$$

At this point, we immediately notice that the potential $\tilde{V}_l$ does not vanishes near the black-hole event horizon; however, it may be neglected anyway by assuming that we are dealing with a very

[8]For simplicity, we chose a slightly different notation with respect to the previous subsection.

large black hole (this assumption seems reasonable otherwise the semiclassical approximation would break down) and that the orbital angular momentum l of the Klein-Gordon particle is sufficiently small. Under these assumptions, in the proximity of the event horizon, the differential equation (73) takes the following form:

$$\left(\partial_T^2 - \partial_R^2\right)\phi(T,R) = 0. \tag{75}$$

The ingoing and outgoing positive-energy solutions can then be written as

$$\varphi_{in} \sim \mathcal{N}_{\nu lm} e^{-i\nu(T+R)}\frac{Y_{lm}}{r} = \mathcal{N}_{\nu lm} e^{-i\nu V}\frac{Y_{lm}}{r}, \tag{76}$$

$$\varphi_{out} \sim \mathcal{N}_{\nu lm} e^{-i\nu(T-R)}\frac{Y_{lm}}{r} = \mathcal{N}_{\nu lm} e^{-i\nu U}\frac{Y_{lm}}{r}, \tag{77}$$

respectively. Above, we introduced light-cone coordinates $U = T - R$ and $V = T + R$. Also, note that in this case we used ν to denote the frequency. Having found the ingoing and outgoing solutions, the freely-falling observer can now expand the fields in modes of given T-frequency ν in the usual way (again, we are considering only outgoing solutions):

$$\varphi_R \propto \int_0^\infty d\nu \left(a_\nu e^{-i\nu U} + a_\nu^\dagger e^{i\nu U}\right). \tag{78}$$

The creation operator $a_\nu^\dagger$ and the annihilation operator a_ν satisfy the commutation relation

$$[a_\nu, a_{\nu'}^\dagger] \propto \delta(\nu - \nu')\delta_{ll'}\delta_{mm'}. \tag{79}$$

The corresponding vacuum state, the Kruskal vacuum, is defined as

$$a_\nu \left|0\right\rangle_K = 0. \tag{80}$$

Now we can proceed as follows. To start, we can write down the relationship between the two outgoing solutions we have found, namely the Bogoliubov transformation. Then, we can express the Bogoliubov coefficients as Fourier integrals and find the relation between their absolute values, from which the result immediately follows. To be more explicit, we first recall the relation between the coordinates U and u:

$$\begin{aligned} U = -e^{-\kappa u} \quad &\Rightarrow \quad u = -\kappa^{-1}\ln(-U) \\ &\Rightarrow \quad e^{-i\omega u} = e^{i\omega\kappa^{-1}\ln(-U)}, \end{aligned} \tag{81}$$

where we recall that κ is the surface gravity. Now, the Bogoliubov transformation is given by

$$e^{i\omega\kappa^{-1}\ln(-U)} = \int_0^\infty d\nu \left(\alpha_{\omega\nu} e^{-i\nu U} + \beta_{\omega\nu} e^{i\nu U}\right), \tag{82}$$

where for simplicity we use the equality sign instead of the proportionality one; including all the proportionality factors would not alter our conclusions. The Bogoliubov coefficients $\alpha_{\omega\nu}$ and $\beta_{\omega\nu}$ can be written as Fourier integrals, in the same way as we did before. Starting from the above expression, we can write

$$\begin{aligned}\int_{-\infty}^{\infty} dU e^{i\nu' U} e^{i\kappa^{-1}\omega \ln(-U)} &\sim \int_{-\infty}^{\infty} dU e^{i\nu' U} \int_{0}^{\infty} d\nu \left(\alpha_{\omega\nu} e^{-i\nu U} + \beta_{\omega\nu} e^{i\nu U}\right) \\ &= 2\pi \int_{0}^{\infty} d\nu \left[\alpha_{\omega\nu}\delta\left(\nu - \nu'\right) + \beta_{\omega\nu}\delta\left(\nu + \nu'\right)\right].\end{aligned} \tag{83}$$

Since $\nu + \nu' \neq 0$, we immediately get

$$\alpha_{\omega\nu} \sim \int_{-\infty}^{0} dU e^{i\nu U} e^{i\omega\kappa^{-1}\ln(-U)}. \tag{84}$$

In a very similar way, we also obtain an expression for $\beta_{\omega\nu}$:

$$\beta_{\omega\nu} \sim \int_{-\infty}^{0} dU e^{-i\nu U} e^{i\omega\kappa^{-1}\ln(-U)}. \tag{85}$$

It is important to notice that the above integration is performed from $-\infty$ to 0 since the light-cone coordinate U is strictly negative in region I of the Kruskal diagram. Following a similar procedure as before, the above integrals can be manipulated by extending the integrand into the complex U-plane, finally obtaining the relation between the absolute values of $\beta_{\omega\nu}$ and $\alpha_{\omega\nu}$:

$$|\beta_{\omega\nu}| = e^{-\frac{\omega\pi}{\kappa}} |\alpha_{\omega\nu}|. \tag{86}$$

By proceeding as before, namely by using the orthonormality condition of the Bogoliubov coefficients, we obtain the spectrum of a blackbody at the temperature given by Eq. (66).

In these two last subsections we derived Hawking's famous result following two distinct procedures. As already explained in the introduction, such result poses a threat to the concept of unitarity in quantum mechanics. Dissatisfied with such conclusion, 't Hooft emphasized the importance of gravitational interactions which, as we have seen, have been ignored in the derivation of the Hawking spectrum. In the next section we will discuss how to treat such interactions (i.e., how information is transferred) in a curved background, constructing a scattering matrix which turns out to be unitary.

3 Shock wave of a charged particle in curved spacetime

In this section, we review 't Hooft's shock wave analysis in the case of a charged particle propagating in the background of a Schwarzschild black hole [13]. In four dimensions, the metric for the background is given by the following expression:

$$ds^2 = -2A\left(U,V\right)dUdV + r^2\left(U,V\right)d\Omega^2, \tag{87}$$

where U, V are light-cone coordinates. The functions $A\left(U,V\right)$ and $r\left(U,V\right)$ are defined as

$$A\left(U,V\right) = \frac{R}{r}\exp\left(1-\frac{r}{R}\right), \quad UV = 2R^2\left(1-\frac{r}{R}\right)\exp\left(\frac{r}{R}-1\right). \tag{88}$$

The line element $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, already introduced at the beginning of subsection 2.2, defines the round metric on the unit two-sphere and $R = 2GM$ is the Schwarzschild radius. In this section, as well as in all the remaining ones in this thesis, we will work in natural units (we will set $c = \hbar = 1$).

3.1 Gravitational backreaction and electromagnetic gauge rotation

Here we consider the backreaction of a highly boosted charged shock wave on a probe test particle [36]. The gravitational backreaction of the shock wave leaves an imprint on the gravitational field experienced by the probe. The probe then experiences geodesics that are shifted across the null surface traced out by the shock wave.

3.1.1 Backreaction on the gravitational field

The stress-energy tensor associated with a source carrying momentum p_{in} at a location $U = 0$ and a point on the sphere $\Omega_0 = (\theta_0, \phi_0)$ can be parametrised as

$$T^{\mu\nu} = 4p_{in}\delta\left(U\right)\delta\left(\Omega-\Omega_0\right)\delta^{\mu}_{V}\delta^{\nu}_{V}. \tag{89}$$

An ansatz for the backreacted geometry that solves the Einstein equations with the above source can be taken to be (see Appendix B in Ref. [36] for details)

$$ds^2 = -2A\left(U,V\right)dU\left(dV - \delta\left(U\right)\tilde{\lambda}_1\left(\Omega,\Omega_0\right)dU\right) + r^2\left(U,V\right)d\Omega^2, \tag{90}$$

where $\tilde{\lambda}_1(\Omega,\Omega_0)$ is a function of the angular separation between the points Ω and Ω_0: it essentially parametrizes the "kick". Outside of the location of the source shock, a probe particle experiences the background Schwarzschild solution with a vanishing $\tilde{\lambda}_1\left(\Omega,\Omega_0\right)$. At the location of the source, however, the Einstein equations reduce to[9] [15, 17]

$$\left(\Delta_\Omega - 1\right)\tilde{\lambda}_1\left(\Omega,\Omega_0\right) = -8\pi G p_{in}\delta\left(\Omega-\Omega_0\right), \tag{91}$$

[9]In this derivation, terms that are quadratic in $\delta\left(u\right)$ have been neglected. This implies that the calculation is only valid when the impact parameter between the probe and the shock, measured by the transverse distance on the sphere, is larger than Planck length. This is the regime of validity of this effective description. Beyond this regime, it is of course well-known that a point-particle description in gravity is problematic.

where Δ_Ω is the Laplacian on the unit two-sphere, namely Eq. (32). Now, by expanding (91) in spherical harmonics, we end up with the following expression for the coefficients:

$$\tilde{\lambda}_1^{lm} = \frac{8\pi G}{l^2+l+1} p_{in} := \lambda_1^{lm} p_{in}. \tag{92}$$

3.1.2 Backreaction on the electromagnetic field

In analogy to the gravitational backreaction discussed above, an electromagnetically charged shock wave leaves an imprint on the electromagnetic field of the probe. The probe then experiences a discontinuity in its electromagnetic field across the null surface traced out by the shock wave. By definition, since it is a probe, we will assume its electromagnetic field to be negligible in comparison to its response to the backreacting shock wave. Thus, before approaching a backreacting shockwave, the gauge field of a boosted probe particle near the horizon can be gauge fixed to zero. At the location of the shock wave, however, its field is affected by the source.

To see this, let us consider a localised source of charge q_{in} moving on the horizon, that is

$$J^\mu = q_{in} \frac{1}{\sqrt{-g}} \delta(U)\, \delta(\Omega - \Omega_0)\, \delta^\mu_V. \tag{93}$$

The ansatz for the electromagnetic field of the probe upon the introduction of the above source can be parametrised in the following way:

$$A^\mu = \delta(U)\, \tilde{\lambda}_2(\Omega, \Omega_0)\, \delta^\mu_V. \tag{94}$$

Therefore, in analogy with the gravitational case, we now must solve the Maxwell equations in curved spacetime at the location of the horizon $U = 0$:

$$\Box A^\mu - \nabla^\mu \nabla_\nu A^\nu = q_{in} \frac{1}{\sqrt{-g}} \delta(U)\, \delta(\Omega - \Omega_0)\, \delta^\mu_V, \tag{95}$$

The left-hand side can be simplified in the Schwarzschild background, obtaining

$$\begin{aligned} \Box A^V - \partial^V (\nabla \cdot A) = {} & g^{UV} (\partial_U \tilde{U}_V) A^V + 2\tilde{U}^U \partial_U A^V + 2\tilde{V}^U \partial_U A^V \\ & + 2\tilde{V}^V \tilde{U}_V A^V - 2\tilde{V}^V \tilde{V}_V A^V + \frac{1}{r^2} \Delta_\Omega A^V - \partial^V \left[\partial_V \log(A r^2) A^V \right], \end{aligned} \tag{96}$$

where we defined $\tilde{V}_a := \partial_a \log r$ and $\tilde{U}_a := \partial_a \log A(r)$. Notice that the Latin index a runs over the coordinates U and V. Now, upon integration over U, Eq. (95) reduces to an equation for the undetermined bilocal function $\tilde{\lambda}_2(\Omega, \Omega_0)$:

$$\Delta_\Omega \tilde{\lambda}_2(\Omega, \Omega_0) = q_{in} \delta(\Omega - \Omega_0), \tag{97}$$

which when expanded in partial waves results in the following solution:

$$\tilde{\lambda}_2^{lm} = -\frac{1}{l^2+l} q_{in} := \lambda_2^{lm} q_{in}. \tag{98}$$

In conclusion, while the electromagnetic field of the probe could be gauge-fixed to vanish in the absence of sources, the backreaction of a source shock results in a gauge rotation of the electromagnetic field of the probe.

3.2 An S-matrix for the wavefunction of a probe charged particle

The aim of this subsection is to calculate the S-matrix for the wavefunction of a charged particle in the presence of a gravitationally backreacting charged shock wave. To this end, let us first begin by writing the wavefunction of a charged particle as $\psi\left(p_{in}, q_{in}\right) = \langle\psi|p_{in}, q_{in}\rangle$. In order to label states as such, we may demand the existence of a charge operator which when acted on its eigenstate yields the charge of the state. Just as a superposition of momentum eigenstates yields a state of definite position, a superposition of charge eigenstates will yield a state with definite electric field. As we argued in the previous subsection, for boosted particles backreacting near the horizon of a black hole, this electric field approaches a pure gauge configuration and may be parameterised by a gauge parameter, say, Λ. Therefore, we may label states in the momentum-charge basis by $|p, q\rangle$ or by $|y, \Lambda\rangle$ in the position-gauge field basis. In terms of the momentum and charge eigenstates, the S-matrix can formally be written as

$$S\left(p_{in}, q_{in}; p_{out}, q_{out}\right) \coloneqq \langle p_{out}, q_{out}|p_{in}, q_{in}\rangle. \tag{99}$$

This allows us to write the wavefunction as follows:

$$\begin{aligned}
\psi\left(p_{in}, q_{in}\right) &= \langle\psi|p_{in}, q_{in}\rangle \\
&= \int dq_{out} \int \frac{dp_{out}}{2\pi} \langle\psi|p_{out}, q_{out}\rangle\langle p_{out}, q_{out}|p_{in}, q_{in}\rangle \\
&= \int dq_{out} \int \frac{dp_{out}}{2\pi} \langle\psi|p_{out}, q_{out}\rangle S\left(p_{in}, q_{in}; p_{out}, q_{out}\right) \\
&= \int d\Lambda_{out} \int dy \int dq_{out} \int \frac{dp_{out}}{2\pi} \psi\left(y, \Lambda_{out}\right) \langle y, \Lambda_{out}|p_{out}, q_{out}\rangle \\
&\qquad \times S\left(p_{in}, q_{in}; p_{out}, q_{out}\right),
\end{aligned} \tag{100}$$

where we used the completeness relations

$$\int dq_{out} \int \frac{dp_{out}}{2\pi} |p_{out}, q_{out}\rangle\langle p_{out}, q_{out}| = 1 \tag{101}$$

$$\int d\Lambda_{out} \int dy |y, \Lambda_{out}\rangle\langle y, \Lambda_{out}| = 1, \tag{102}$$

and the definition of the scattering matrix, Eq. (99). As we argued in the previous subsection, the gravitational backreaction implies that the position of the outgoing particle is determined by the momentum of the incoming particle. Similarly, the gauge parameter of the out-particle is determined by the charge of the in-particle. These relations (position $\leftrightarrow$ momentum and gauge parameter $\leftrightarrow$ charge) can be obtained from Eqs. (90) and (94), respectively[10]:

$$y = \lambda_1 p_{in}, \quad \Lambda_{out} = \lambda_2 q_{in}. \tag{103}$$

[10] Here, the "constants" λ_i are only constants along the longitudinal coordinates U, V, They indeed depend on the transverse distance on the horizon, between the backreacting shock and the probe outgoing particle.

We now insert in the previous expression for the wavefunction to find

$$\begin{aligned}\psi\left(p_{in},q_{in}\right) &= \int \lambda_2 dq'_{in} \int \lambda_1 dp'_{in} \int dq_{out} \int \frac{dp_{out}}{2\pi} \psi\left(\lambda_1 p'_{in}, \lambda_2 q'_{in}\right) \langle y, \Lambda_{out} | p_{out}, q_{out}\rangle \\ &\qquad \times S\left(p_{in}, q_{in}; p_{out}, q_{out}\right) \\ &= \int dq'_{in} \int dp'_{in} \int dq_{out} \int \frac{dp_{out}}{2\pi} \psi\left(p'_{in}, q'_{in}\right) \langle y, \Lambda_{out} | p_{out}, q_{out}\rangle \\ &\qquad \times S\left(p_{in}, q_{in}; p_{out}, q_{out}\right). \end{aligned} \tag{104}$$

The rescaling of integration variables to arrive at the second equality does not change the ranges of integration (which remain from $-\infty$ to ∞ for both the integrals.) This relation must hold for any wavefunction as (103) contains invertible basis transformations. Therefore, we find that

$$\int dq'_{in} \int dp'_{in} \langle y, \Lambda_{out} | p_{out}, q_{out}\rangle S^*\left(p_{in}, q_{in}; p_{out}, q_{out}\right) = \delta\left(p'_{in} - p_{in}\right) \delta\left(q'_{in} - q_{in}\right). \tag{105}$$

To invert this equation for the S-matrix, we now need an expression for $\langle y, \Lambda_{out} | p_{out}, q_{out}\rangle$. Writing the positions y in a momentum basis gives us a plane wave. Similarly, we know that the electric field and charge density are conjugate; therefore, we may write

$$\langle y, \Lambda_{out} | p_{out}, q_{out}\rangle = \exp\left(-iyp_{out} + i\Lambda_{out} q_{out}\right) = \exp\left(-i\lambda_1 p_{in} p_{out} + i\lambda_2 q_{in} q_{out}\right). \tag{106}$$

Plugging this into the previous expression, we see that it is a Fourier transform equation for the scattering matrix, which can easily be inverted to find

$$S\left(p_{in}, q_{in}; p_{out}, q_{out}\right) = \exp\left(i\lambda_1 p_{in} p_{out} - i\lambda_2 q_{in} q_{out}\right). \tag{107}$$

3.3 Generalization to many particles and the continuum

We would now like to generalise the previous results to the case of many particles in order to then take a continuum limit to describe a distribution of particles on the horizon. Since quantum mechanics does not allow for particle production, we may safely assume that the number of incoming and outgoing particles is equal; we call the number of incoming and outgoing particles as N_{in} and N_{out} respectively. We will label the i-th incoming particles by its longitudinal position x_i, angular position on the horizon Ω_i and momentum p^i_{in} such that $i \in N_{in}$. Similarly, outgoing particles are labelled by y_j, Ω_j, p^j_{out}, with $j \in N_{out}$. Assuming that there is no more than one particle at each angular position on the horizon, in the continuum limit $N_{in} = N_{out} \to \infty$, the positions of particles may be described by distributions $x\left(\Omega\right)$ and $y\left(\Omega\right)$. Let us start by writing the basis of states in the following way:

$$|p_{in,tot}, q_{in,tot}\rangle = \bigotimes_i |p^i_{in}, q^i_{in}\rangle, \tag{108}$$

$$|p_{out,tot}, q_{out,tot}\rangle = \bigotimes_j |p^j_{out}, q^j_{out,tot}\rangle, \tag{109}$$

where we assumed a factorised Hilbert space because all parallel moving particles are independent of each other. The completeness relations are now integrals defined with measures $dp_{out,tot} =$

$\prod_j dp_{out}^j$ and $dy_{tot} = \prod_j dy^j$. The S-matrix may formally be written as

$$S_{tot} \coloneqq S\left(p_{in,tot}, q_{in,tot}; p_{out,tot}, q_{out,tot}\right) \coloneqq \langle p_{out,tot}, q_{out,tot} | p_{in,tot}, q_{in,tot} \rangle. \tag{110}$$

This S-matrix is dictated by the backreaction relations which are now given in terms of invertible matrices that are in turn functions of the transverse distance between the in- and out-particles:

$$y^j = \lambda_1^{ij}\left(\Omega_i, \Omega_j\right) p_{in}^i, \quad \Lambda_{out}^j = \lambda_2^{ij}\left(\Omega_i, \Omega_j\right) q_{in}^i, \tag{111}$$

such that the out-state can be written as

$$|y_{tot}, \Lambda_{out,tot}\rangle = \bigotimes_j |\lambda_1^{ij}\left(\Omega_i, \Omega_j\right) p_{in}^i, \lambda_2^{ij}\left(\Omega_i, \Omega_j\right) q_{in}^i\rangle. \tag{112}$$

Since the scattering matrix is a basis transformation, it is necessarily bijective between the in- and out-Hilbert spaces. This implies that the matrices $\lambda_1\left(\Omega_i, \Omega_j\right)$ and $\lambda_2\left(\Omega_i, \Omega_j\right)$ are invertible, which in turn implies that there is no more than one particle entering (leaving) the horizon at any given angle. Moreover, we have the condition that $N_{in} = N_{out}$. Consequently, we may now adapt the procedure outlined above for the single particle case to the multiparticle case. We begin by writing the wavefunction as

$$\begin{aligned}
\psi\left(p_{in,tot}, q_{in,tot}\right) &= \langle \psi | p_{in,tot}, q_{in,tot} \rangle \\
&= \int dq_{out,tot} \int \frac{dp_{out,tot}}{2\pi} \langle \psi | p_{out,tot}, q_{out,tot} \rangle \langle p_{out,tot}, q_{out,tot} | p_{in,tot}, q_{in,tot} \rangle \\
&= \int dq_{out,tot} \int \frac{dp_{out,tot}}{2\pi} \langle \psi | p_{out,tot}, q_{out,tot} \rangle S_{tot} \\
&= \int d\Lambda_{out,tot} \int dy_{tot} \int dq_{out,tot} \int \frac{dp_{out,tot}}{2\pi} \psi\left(y_{tot}, \Lambda_{out,tot}\right) \\
&\qquad \times \langle y_{tot}, \Lambda_{out,tot} | p_{out,tot}, q_{out,tot} \rangle S_{tot}.
\end{aligned} \tag{113}$$

where we used the completeness relations

$$\int dq_{out,tot} \int \frac{dp_{out,tot}}{2\pi} |p_{out,tot}, q_{out,tot}\rangle \langle p_{out,tot}, q_{out,tot}| = 1 \tag{114}$$

$$\int d\Lambda_{out,tot} \int dy_{tot} |y_{tot}, \Lambda_{out,tot}\rangle \langle y_{tot}, \Lambda_{out,tot}| = 1. \tag{115}$$

Defining $\lambda_{1,2}^{ij} \coloneqq \lambda_{1,2}\left(\Omega_i, \Omega_j\right)$, we now insert the relations in (111), resulting in the measures

$$\prod_j dy^j = \det\left(\lambda_1^{ij}\right) \prod_i dp_{in}^i, \quad \prod_j d\Lambda_{out}^j = \det\left(\lambda_2^{ij}\right) \prod_i dq_{in}^i, \tag{116}$$

to write the wavefunction as

$$\begin{aligned}
\psi\left(p_{in,tot}, q_{in,tot}\right) = \det\left(\lambda_1^{ij}\right) \det\left(\lambda_2^{ij}\right) \int \prod_i dq'^i_{in} dp'^i_{in} \int dq_{out,tot} \\
\times \int \frac{dp_{out,tot}}{2\pi} \psi\left(\lambda_1^{ij} p_{in}^i, \lambda_2^{ij} q_{in}^i\right) \langle y_{tot}, \Lambda_{out,tot} | p_{out,tot}, q_{out,tot} \rangle S_{tot}.
\end{aligned} \tag{117}$$

For every j in the product, we have a sum over all incoming particles labelled by i. In each term of the sum, we rescale the integration variables p_{in} and q_{in} to neutralise the corresponding factors of λ_1 and λ_2, just as we did in the single particle case:

$$\psi\left(p_{in,tot}, q_{in,tot}\right) = \int dq'_{in,tot} \int dp'_{in,tot} \int dq_{out,tot} \int \frac{dp_{out,tot}}{2\pi} \psi\left(p'_{in,tot}, q'_{in,tot}\right) \\ \times \langle y_{tot}, \Lambda_{out,tot} | p_{out,tot}, q_{out,tot} \rangle S_{tot}. \quad (118)$$

In analogy to (106), we now write

$$\begin{aligned} \langle y_{tot}, \Lambda_{out,tot} | p_{out,tot}, q_{out,tot} \rangle &= \prod_j \langle y_j, \Lambda^j_{out} | p^j_{out}, q^j_{out} \rangle \\ &= \exp\left(-i \sum_j y_j p^j_{out} + i \sum_j \Lambda^j_{out} q^j_{out}\right). \end{aligned} \quad (119)$$

Therefore, we may invert the previous relation for the scattering matrix to find

$$S_{tot} = \exp\left(i\lambda_1^{ij} p^i_{in} p^j_{out} - i\lambda_2^{ij} q^i_{in} q^j_{out}\right). \quad (120)$$

In the above equation, a sum over all in- and out-particles is implicit. The continuum limit can be easily achieved. We first promote the momenta and charges to be distributions as smooth functions of the sphere coordinates and then replace the sum over in- and out-particles with integrals over the same coordinates, obtaining

$$S_{tot} = \exp\left[i \int d\Omega\, d\Omega' \left(\lambda_1\left(\Omega, \Omega'\right) p_{in}\left(\Omega\right) p_{out}\left(\Omega'\right) - \lambda_2\left(\Omega, \Omega'\right) q_{in}\left(\Omega\right) q_{out}\left(\Omega'\right)\right)\right]. \quad (121)$$

Now, by expanding the above expression in spherical harmonics, and substituting for λ_1 and λ_2 using Eqs. (92) and (98), we finally get

$$\boxed{S_{tot} = \exp\left[i\left(\frac{8\pi G p^{lm}_{in} p^{lm}_{out}}{l^2+l+1} - \frac{q^{lm}_{in} q^{lm}_{out}}{l^2+l}\right)\right].} \quad (122)$$

As one can easily check, the above S-matrix is manifestly unitary. We conclude this subsection by defining the electromagnetic part of the above scattering matrix as

$$S_{EM} \coloneqq \exp\left(-i\frac{q^{lm}_{in} q^{lm}_{out}}{l^2+l}\right). \quad (123)$$

This will turn out to be useful for the comparison with the scattering matrix we will obtain the context of quantum field theory in the next sections.

4 From quantum mechanics to quantum field theory

Our goal here is to show that a similar expression for the electromagnetic S-matrix computed by 't Hooft can be obtained in the context of quantum field theory. To this end, we build a scalar quantum electrodynamics near the horizon using the tools recently developed in Refs. [19, 20].

4.1 Scalar quantum electrodynamics in curved spacetime

Let us start by considering the following path integral:

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\phi e^{iS[\phi, A_\mu, g_{\mu\nu}]}, \tag{124}$$

where ϕ is a complex scalar field, A_μ is the electromagnetic vector potential and $g_{\mu\nu}$ is the spacetime metric; moreover, $S[\phi, A_\mu, g_{\mu\nu}]$ represents the action functional for a complex scalar field coupled to the electromagnetic field in an arbitrary background:

$$S[\phi, A_\mu, g_{\mu\nu}] = \int d^4x\sqrt{-g}\left[-\left(D_\mu\phi\right)^*\left(D^\mu\phi\right) + \left(\xi R - m^2\right)\phi\phi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]. \tag{125}$$

In the above expression, m is the mass of the complex scalar field, ξ is the scalar curvature coupling, R is the scalar curvature and D_μ is the gauge covariant derivative defined by

$$D_\mu = \nabla_\mu - iqA_\mu, \tag{126}$$

where q represents the coupling between the complex scalar field and the electromagnetic field. Moreover, we recall that the electromagnetic field tensor is given by

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu. \tag{127}$$

For simplicity, we consider a minimally coupled complex scalar field setting $\xi = 0$ in Eq. (125):

$$S[\phi, A_\mu, g_{\mu\nu}] \coloneqq S = \int d^4x\sqrt{-g}\left[-\left(D_\mu\phi\right)^*\left(D^\mu\phi\right) - m^2|\phi|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]. \tag{128}$$

The Lagrangian implicitly defined before can be easily manipulated by considering that

$$\begin{aligned}\left(D_\mu\phi\right)^*\left(D^\mu\phi\right) &= (\nabla_\mu\phi - iqA_\mu\phi)^*(\nabla^\mu\phi - iqA^\mu\phi) \\ &= \nabla_\mu\phi^*\nabla^\mu\phi - iqA^\mu\phi\nabla_\mu\phi^* + iqA_\mu\phi^*\nabla^\mu\phi + q^2A_\mu A^\mu\phi^*\phi \\ &= \nabla_\mu\phi^*\nabla^\mu\phi + iqA^\mu\left(\phi^*\nabla_\mu\phi - \phi\nabla_\mu\phi^*\right) + q^2A_\mu^2|\phi|^2.\end{aligned} \tag{129}$$

Therefore, Eq. (128) can now be written as

$$S = \int d^4x\sqrt{-g}\left[-|\nabla_\mu\phi|^2 - iqA^\mu\left(\phi^*\nabla_\mu\phi - \phi\nabla_\mu\phi^*\right) - \left(q^2A_\mu^2 + m^2\right)|\phi|^2 - \frac{1}{4}F_{\mu\nu}^2\right]. \tag{130}$$

We can split the action S into two terms:

$$S \coloneqq S_\gamma + S_M = S_\gamma[A_\mu] + S_M[A_\mu, \phi], \tag{131}$$

where S_γ is the action for the photon field, the last term in Eq. (128), while S_M is the matter action, given by the following expression:

$$\begin{aligned} S_M &:= \int d^4x\sqrt{-g}\left[-|\nabla_\mu\phi|^2 - iqA^\mu\left(\phi^*\nabla_\mu\phi - \phi\nabla_\mu\phi^*\right) - \left(q^2A_\mu^2 + m^2\right)|\phi|^2\right] \\ &= \int d^4x\sqrt{-g}\phi^*\left(\Box - m^2\right)\phi - q\int d^4x\sqrt{-g}A^\mu j_\mu - q^2\int d^4x\sqrt{-g}A_\mu^2|\phi|^2, \end{aligned} \tag{132}$$

where j_μ is the scalar field current, defined as

$$j_\mu := i\left(\phi^*\nabla_\mu\phi - \phi\nabla_\mu\phi^*\right). \tag{133}$$

Notice that in Eq. (132) an integration by parts has been performed to get to the last line. Let us now focus on the action for the photon field, S_γ. The first step is to rewrite it in terms of the vector potential; in particular, we want to write the Lagrangian in the form $A_\mu\mathcal{O}^{\mu\nu}A_\nu$ (to find the photon propagator in curved spacetime), where $\mathcal{O}^{\mu\nu}$ is a some operator. We have:

$$\begin{aligned} \mathcal{L}_\gamma := -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{4}\left[\left(\nabla_\mu A_\nu - \nabla_\nu A_\mu\right)\left(\nabla^\mu A^\nu - \nabla^\nu A^\mu\right)\right] \\ &= -\frac{1}{4}\left[\nabla_\mu A_\nu\nabla^\mu A^\nu - \nabla_\mu A_\nu\nabla^\nu A^\mu - \nabla_\nu A_\mu\nabla^\mu A^\nu + \nabla_\nu A_\mu\nabla^\nu A^\mu\right] \\ &= -\frac{1}{4}\left[\nabla_\nu A_\mu\nabla^\nu A^\mu - \nabla_\nu A_\mu\nabla^\mu A^\nu - \nabla_\nu A_\mu\nabla^\mu A^\nu + \nabla_\nu A_\mu\nabla^\nu A^\mu\right] \\ &= -\frac{1}{4}\left[2\nabla_\nu A_\mu\nabla^\nu A^\mu - 2\nabla_\nu A_\mu\nabla^\mu A^\nu\right] \\ &= -\frac{1}{2}\left[-A_\mu\nabla_\nu\nabla^\nu A^\mu + A_\mu\nabla_\nu\nabla^\mu A^\nu\right] \\ &= \frac{1}{2}A_\mu\left[g^{\mu\nu}\nabla_\sigma\nabla^\sigma - \nabla^\nu\nabla^\mu\right]A_\nu \\ &= \frac{1}{2}A_\mu\left[g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu - R_{\mu\nu}\right]A_\nu. \end{aligned} \tag{134}$$

Above, to get to the fifth line, we integrated by parts (omitting all the boundary terms). In analogy with the flat case, we managed to write down the Lagrangian in the form $A_\mu\mathcal{O}^{\mu\nu}A_\nu$ (we will not take into account $R_{\mu\nu}$ since the background we are considering is a vacuum solution):

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}A_\mu\left[g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu\right]A_\nu = \frac{1}{2}A_\mu\mathcal{O}^{\mu\nu}A_\nu, \tag{135}$$

where the operator in the middle has been defined as

$$\mathcal{O}^{\mu\nu} := g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu. \tag{136}$$

The action can then be easily written down as follows:

$$S_\gamma = \frac{1}{2}\int d^4x\sqrt{-g}A_\mu\left[g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu\right]A_\nu. \tag{137}$$

It is important to notice that we still need to fix a gauge. Indeed, when the path integral is written for the $U(1)$ invariant action S_γ, one can notice that the measure over the field is not well-defined (over-counting of gauge orbits). We will deal with this in the following section.

4.2 Intermezzo - analysis into spherical harmonics

The goal of this subsection is to introduce vector spherical harmonics, which have been widely used in different branches of physics; they have been defined in different ways, depending on the context. Here we will introduce a set of vector spherical harmonics which turned out to be extremely useful in classical electrodynamics. Here, we will closely follow Ref. [37]. In order to set notation, we will start by briefly reminding the reader how scalar spherical harmonics are defined. According to the expansion theorem, any scalar function $f(\theta,\phi)$ may be written as

$$f(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\alpha_{lm}Y_{lm}(\theta,\phi)\coloneqq\sum_{l,m}\alpha_{lm}Y_{lm}(\theta,\phi), \tag{138}$$

In the above expression, Y_{lm} represents the set of scalar spherical harmonics:

$$Y_{lm}(\theta,\phi)=\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}(-1)^m e^{im\phi}P_{lm}(\cos\theta)\coloneqq\mathcal{C}_{lm}e^{im\phi}P_{lm}(\cos\theta), \tag{139}$$

where $P_{lm}(x)$ are the associated Legendre polynomials of degree l and order m, defined as

$$P_{lm}(x)=\frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^{l+m}(x^2-1)^l. \tag{140}$$

One of the crucial properties of the scalar spherical harmonics is the orthonormality condition:

$$\int d\Omega Y^*_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi)=\delta_{ll'}\delta_{mm'}, \tag{141}$$

where $d\Omega\coloneqq\sin\theta d\theta d\phi$. The coefficients in Eq. (138) can be found by using Eq. (141):

$$\int d\Omega Y^*_{lm}(\theta,\phi)f(\theta,\phi)=\int d\Omega Y^*_{lm}(\theta,\phi)\sum_{l',m'}\alpha_{l'm'}Y_{l'm'}(\theta,\phi) \tag{142}$$

$$=\sum_{l',m'}\alpha_{l'm'}\delta_{ll'}\delta_{mm'}=\alpha_{lm}. \tag{143}$$

In general, the coefficients can be evaluated quite easily by exploiting the following symmetries:

$$\begin{aligned}Y_{lm}(\pi-\theta,\phi+\pi)&=(-1)^l Y_{lm}(\theta,\phi),\\ Y_{lm}(\theta,\phi+\pi)&=(-1)^m Y_{lm}(\theta,\phi),\\ Y_{lm}(\pi-\theta,\phi)&=(-1)^{l+m}Y_{lm}(\theta,\phi).\end{aligned} \tag{144}$$

As anticipated in the previous section, the analysis into scalar spherical harmonics can be extremely useful if one considers also another property, namely

$$\Delta_\Omega Y_{lm}(\theta,\phi)=-l(l+1)Y_{lm}(\theta,\phi), \tag{145}$$

where Δ_Ω is the Laplacian operator. There are other useful properties that should be mentioned, but we choose to write only the ones that will be of interest for us as we proceed. As already stated, the spherical harmonic expansion shown above greatly simplifies many problems. For example, let us consider the Poisson equation, $\Delta\Phi = -4\pi\rho$, where Φ is the electric potential and ρ the charge density. Now, by expanding Φ and ρ in spherical harmonics,

$$\Phi = \sum_{l,m} a_{lm}(r) Y_{lm}(\theta,\phi), \quad \rho = \sum_{l,m} b_{lm}(r) Y_{lm}(\theta,\phi), \tag{146}$$

the Poisson equation can then be written as

$$\begin{aligned}\Delta\Phi = \sum_{l,m} \Delta\left(a_{lm}Y_{lm}\right) &= \sum_{l,m} \left(\Delta a_{lm}\right) Y_{lm} + \sum_{l,m} a_{lm}\left(\Delta Y_{lm}\right) \\ &= \sum_{l,m}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}a_{lm}\right) - \frac{l(l+1)}{r^2}a_{lm}\right]Y_{lm} \\ &= -4\pi\rho = -4\pi\sum_{l,m} b_{lm}Y_{lm},\end{aligned} \tag{147}$$

finally resulting in the following simple equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}a_{lm}\right) - \frac{l(l+1)}{r^2}a_{lm} = -4\pi b_{lm}. \tag{148}$$

The effect of the spherical harmonic expansion is essentially to “cancel out" the angular dependence. Now, the question we would like to answer is: can we construct vector functions having the same useful properties as the scalar spherical harmonics? As explained in Ref. [37], it is tempting to try to consider the components of a vector (as an example we consider a vector in three dimensions) as a scalar field and then expand:

$$\begin{aligned}\mathbf{F}(r,\theta,\phi) &= \hat{\mathbf{e}}_r F^r + \hat{\mathbf{e}}_\theta F^\theta + \hat{\mathbf{e}}_\phi F^\phi \\ &= \hat{\mathbf{e}}_r \sum_{l,m} F^r_{lm}(r) Y_{lm} + \hat{\mathbf{e}}_\theta \sum_{l,m} F^r_{lm}(r) Y_{lm} + \hat{\mathbf{e}}_\phi \sum_{l,m} F^r_{lm}(r) Y_{lm}.\end{aligned} \tag{149}$$

Because of the completeness property of the scalar spherical harmonics, the above expansion is certainly allowed. Our main concern, however, is whether it is useful. Let us consider the partial differential equation $\nabla\cdot\mathbf{V} = g$, where g is some scalar function. Expanding the operator on the left-hand side of the equation, in spherical coordinates, we get

$$\nabla\cdot\mathbf{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F^r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial_\phi}F^\phi + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta F^\theta\right). \tag{150}$$

It is immediate to see that the second term in the above expression causes problems. Indeed, by expanding as in Eq. (149), we would get a term proportional to $Y_{lm}/\sin\theta$. Therefore, we would not be able to “cancel out" the angular dependence, and so we must conclude that Eq. (149) is not very useful for our purposes. Another strategy is therefore needed. A clever way to construct vector spherical harmonics, as already anticipated, has been presented in Ref. [37]. Here, the authors considered a scalar function, say $f = f(r,\theta,\phi)$, expanded it in spherical harmonics, and took its gradient, obtaining

$$\nabla f = \sum_{l,m}\left(Y_{lm}\nabla f_{lm} + f_{lm}\nabla Y_{lm}\right) = \sum_{l,m}\left(\frac{d}{dr}f_{lm}Y_{lm}\hat{\mathbf{e}}_{\mathbf{r}} + f_{lm}\nabla Y_{lm}\right), \tag{151}$$

where $f_{lm} = f_{lm}(r)$ are the expansion coefficients. Then, for the sake of simplicity, they assumed that (151) is itself a spherical harmonic expansion; this motivates the following definitions:

$$\mathbf{\Psi}_{lm} \coloneqq \nabla Y_{lm} = \frac{1}{r}\frac{\partial Y_{lm}}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial Y_{lm}}{\partial \phi}\hat{\mathbf{e}}_\phi, \tag{152}$$

$$\mathbf{Y}_{lm} \coloneqq Y_{lm}\hat{\mathbf{e}}_r. \tag{153}$$

Finally, they took into account another vector equation, namely[11]

$$\mathbf{E} = \mathbf{N} \times \hat{\mathbf{e}}_r. \tag{154}$$

If we now let $\mathbf{N} = \mathbf{\Psi}_{lm}$, then we can write

$$\mathbf{E} = \mathbf{\Psi}_{lm} \times \hat{\mathbf{e}}_r. \tag{155}$$

From the above equation one can easily notice that the vector $\mathbf{E}$ cannot be expanded neither in terms of $\mathbf{\Psi}_{lm}$ nor in terms of $\hat{\mathbf{e}}_r$ (it is orthogonal to both of these vectors). Therefore, the authors concluded that $\mathbf{E}$ is a new type of vector that has to be included in the expansion:

$$\mathbf{\Phi}_{lm} \coloneqq \mathbf{\Psi}_{lm} \times \hat{\mathbf{e}}_r = \nabla Y_{lm} \times \frac{\mathbf{r}}{r} = \frac{1}{r\sin\theta}\frac{\partial Y_{lm}}{\partial \phi}\hat{\mathbf{e}}_\theta - \frac{1}{r}\frac{\partial Y_{lm}}{\partial \theta}\hat{\mathbf{e}}_\phi. \tag{156}$$

We now have a set of three objects, namely $\{\mathbf{Y}_{lm}, \mathbf{\Psi}_{lm}, \mathbf{\Phi}_{lm}\}$, which can be shown to be orthogonal and complete, implying that any vector field can be expanded as follows:

$$\boxed{\mathbf{V}(r,\theta,\phi) = \sum\nolimits_{l,m}\left(V_{lm,r}\mathbf{Y}_{lm} + V_{lm,1}\mathbf{\Psi}_{lm} + V_{lm,2}\mathbf{\Phi}_{lm}\right).} \tag{157}$$

Above, V^r_{lm}, V^1_{lm} and V^2_{lm} are the expansion coefficients; as for the scalar case, the vector spherical harmonics have many interesting properties that should be definitely mentioned. However, for clarity reasons, in the following we shall write down only the ones we use. In order to check whether the above expansion is indeed useful, we want to consider again an equation like $\nabla \cdot V = g$. By using the above expansion, namely Eq. (157), we obtain

$$\nabla \cdot V = \nabla \cdot \sum_{l,m}\left[\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2 V_{lm,r}\right) - \frac{l(l+1)}{r}V_{lm,1}\right)\right]Y_{lm} = g = \sum_{l,m} g_{lm}(r)Y_{lm}, \tag{158}$$

where we also made use of the following properties:

$$\begin{aligned}\nabla \cdot (F(r)\mathbf{Y}_{lm}) &= \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 F(r)\right)\right]Y_{lm},\\ \nabla \cdot (F(r)\mathbf{\Psi}_{lm}) &= -\frac{l(l+1)}{r}F(r)Y_{lm}.\end{aligned} \tag{159}$$

Therefore, Eq. (158) reduces to the following ordinary differential equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 V_{lm,r}\right) - \frac{l(l+1)}{r}V_{lm,1} = g_{lm}(r). \tag{160}$$

[11] Actually, they considered $\mathbf{E} = \hat{\mathbf{e}}_r \times \mathbf{N}$. However, we defined it as in (154) for later convenience.

As hoped, the angular dependence has been "cancelled out". To perform computations, it is convenient to work with the covariant components of the vector harmonics (152) and (156). The component representation can be easily obtained by considering the scalar products $(\mathbf{\Psi}_{lm})_i \cdot \mathbf{e}_i$ and $(\mathbf{\Phi}_{lm})_i \cdot \mathbf{e}_i$, where $\{\mathbf{e}_i\}$ is the natural basis (to be distinguished from the physical basis $\hat{\mathbf{e}}_i$). From Eqs. (152) and (156) we then obtain the following components:

$$(\Psi_{lm})_i = \partial_i Y_{lm}, \tag{161}$$

$$(\Phi_{lm})_i = \epsilon_i^{\ j} \partial_j Y_{lm}, \tag{162}$$

where the labels i and j run over the coordinates θ and ϕ. In the above expressions, we defined the fully antisymmetric symbol on the sphere with non-vanishing components

$$\epsilon^{\theta\phi} = -\epsilon^{\phi\theta} = \frac{1}{r^2 \sin\theta}. \tag{163}$$

From Eqs. (161) and (162) we get the following four expressions:

$$(\Psi_{lm})_\theta = \partial_\theta Y_{lm}, \tag{164}$$

$$(\Psi_{lm})_\phi = \partial_\phi Y_{lm}, \tag{165}$$

$$(\Phi_{lm})_\theta = \frac{1}{\sin\theta} \partial_\phi Y_{lm}, \tag{166}$$

$$(\Phi_{lm})_\phi = -\sin\theta \partial_\theta Y_{lm}. \tag{167}$$

Another important remark we should make is the following. Under parity transformation,

$$\begin{aligned} r &\to r, \\ \theta &\to \pi - \theta, \\ \phi &\to \phi + \pi, \end{aligned} \tag{168}$$

the behaviour of the spherical harmonics comes into two different types: even-parity spherical harmonics, for which the transformation gives a factor $(-1)^l$, and odd-parity spherical harmonics, for which the transformation gives a factor $(-1)^{l+1}$. From the group of spherical harmonics introduced until now, Y_{lm} and $\partial_i Y_{lm}$ are multiplied by $(-1)^l$ under the above transformation, while $\epsilon_i^{\ j} \partial_j Y_{lm}$ is multiplied by $(-1)^{l+1}$. Our discussion so far can be quite easily extended to four-vectors. Indeed, our primary interest is to expand the electromagnetic vector potential $\mathbf{A}$ in Eq. (137) in spherical harmonics. In the previous section we did not specify the background geometry; however, here we are considering expansions in spherical harmonics, so we are implicitly assuming that the background is spherically symmetric. In particular, we want to consider a Schwarzschild black hole, described by Eq. (29); in these coordinates, the electromagnetic potential will be given by $\mathbf{A} = \mathbf{A}(t, r, \theta, \phi)$. Eq. (157) can be easily generalized as follows:

$$\boxed{\mathbf{A}(t, r, \theta, \phi) = \sum\nolimits_{l,m} \left(A_{lm,t} \mathbf{Y}^t_{lm} + A_{lm,r} \mathbf{Y}_{lm} + A_{lm,1} \mathbf{\Psi}_{lm} + A_{lm,2} \mathbf{\Phi}_{lm} \right).} \tag{169}$$

In the above expression, it should be noted that the expansions coefficients depend both on t and r. Moreover, we defined $\mathbf{Y}^t_{lm} \coloneqq Y_{lm} \hat{\mathbf{e}}_t = Y_{lm} \mathbf{e}_t$. Let us now work a bit on Eq. (169); based on our previous discussion, we have

$$\mathbf{A} = \sum_{l,m} [A_{lm,t}(t,r)Y_{lm}\mathbf{e}_t + A_{lm,r}(t,r)Y_{lm}\mathbf{e}_r + A_{lm,1}(t,r)\left(\partial_\theta Y_{lm}\mathbf{e}_\theta + \partial_\phi Y_{lm}\mathbf{e}_\phi\right) + A_{lm,2}(t,r)\left(\csc\theta\partial_\phi Y_{lm}\mathbf{e}_\theta - \sin\theta\partial_\theta Y_{lm}\mathbf{e}_\phi\right)], \quad (170)$$

which can also be rewritten in terms of its components in the following way:

$$A_\mu = \sum_{l,m} \begin{bmatrix} A_{lm,t}(t,r) \\ A_{lm,r}(t,r) \\ A_{lm,2}(t,r)\csc\theta\partial_\phi + A_{lm,1}(t,r)\partial_\theta \\ -A_{lm,2}(t,r)\sin\theta\partial_\theta + A_{lm,1}(t,r)\partial_\phi \end{bmatrix} Y_{lm}. \quad (171)$$

Now that we have an expression for the electromagnetic vector potential in terms of Y_{lm} and its derivatives, we can also split it into odd- and even-parity modes [38]:

$$A_\mu = A_\mu^- + A_\mu^+, \quad (172)$$

where the first term on the right-hand side (odd-parity mode) has been defined as

$$A_\mu^- = \sum_{l,m} \begin{bmatrix} 0 \\ 0 \\ A_{lm,2}(t,r)\csc\theta\partial_\phi \\ -A_{lm,2}(t,r)\sin\theta\partial_\theta \end{bmatrix} Y_{lm}, \quad (173)$$

while the second term (even-parity mode) as

$$A_\mu^+ = \sum_{l,m} \begin{bmatrix} A_{lm,t}(t,r) \\ A_{lm,r}(t,r) \\ A_{lm,1}(t,r)\partial_\theta \\ A_{lm,1}(t,r)\partial_\phi \end{bmatrix} Y_{lm}. \quad (174)$$

We can also work in Kruskal–Szekeres coordinates and so consider the line element [19]

$$ds^2 = -2A(U,V)dUdV + r^2(U,V)d\Omega^2, \quad (175)$$

where A and r are defined by the two equations

$$A = \frac{R}{r}\exp\left(\frac{r}{R} - 1\right), \quad UV = 2R^2\left(1 - \frac{r}{R}\right)\exp\left(\frac{r}{R} - 1\right), \quad (176)$$

respectively. Above, $R = 2GM$. In this case, Eqs. (173) and (174) become

$$A_\mu^- = \sum_{l,m} \begin{bmatrix} 0 \\ 0 \\ A_{lm,2}(U,V)\csc\theta\partial_\phi \\ -A_{lm,2}(U,V)\sin\theta\partial_\theta \end{bmatrix} Y_{lm}, \quad A_\mu^+ = \sum_{l,m} \begin{bmatrix} A_{lm,U}(U,V) \\ A_{lm,V}(U,V) \\ A_{lm,1}(U,V)\partial_\theta \\ A_{lm,1}(U,V)\partial_\phi \end{bmatrix} Y_{lm}. \quad (177)$$

We notice that the last two components in A^+_μ can be gauged away by the transformation[12]

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \tag{178}$$

where $\Lambda = \Lambda(U,V,\theta,\phi)$ is some function that has to be appropriately chosen; indeed, if we now choose $\Lambda(U,V,\theta,\phi) = -\sum_{l,m} A_{lm,1}(U,V)Y_{lm}(\theta,\phi)$, we have that[13]

$$\partial_\mu \Lambda(U,V,\theta,\phi) = \sum_{l,m} \begin{bmatrix} -\partial_U A_{lm,1}(U,V) \\ -\partial_V A_{lm,1}(U,V) \\ -A_{lm,1}(U,V)\partial_\theta \\ -A_{lm,1}(U,V)\partial_\phi \end{bmatrix} Y_{lm}. \tag{181}$$

Therefore, by redefining $A_{lm,U}(U,V)$ and $A_{lm,V}(U,V)$ as

$$A_{lm,U}(U,V) \to A_{lm,U}(U,V) + \partial_U A_{lm,1}(U,V), \tag{182}$$

$$A_{lm,V}(U,V) \to A_{lm,V}(U,V) + \partial_V A_{lm,1}(U,V), \tag{183}$$

we can write the even-parity mode in the following way:

$$A^+_\mu = \sum_{l,m} \begin{bmatrix} A_{lm,U}(U,V) \\ A_{lm,V}(U,V) \\ 0 \\ 0 \end{bmatrix} Y_{lm}. \tag{184}$$

Let us now set a new notation for later convenience; in particular, we introduce lower-case Latin indices a, b, c, etc., which run over U and V (light-cone coordinates), and upper-case Latin indices A, B, C, etc., which span the two-sphere. The odd- and even-parity modes can now be written in index notation. The odd mode is written as[14]

$$A^-_B = -\sum_{l,m} A_{lm,2}\epsilon_B{}^C \partial_C Y_{lm} \tag{185}$$

[12] The field strength is invariant under the gauge symmetry: $F_{\mu\nu} \to \partial_\mu(A_\nu + \partial_\nu\Lambda) - \partial_\nu(A_\mu + \partial_\mu\Lambda) = F_{\mu\nu}$.

[13] Here we are assuming that the gauge degree of freedom is in $A^+_{lm,\mu}$. This can be proved in the following way. We first treat $\partial_\mu\Lambda$ as a vector and expand it in spherical harmonics:

$$\partial_\mu \Lambda = \sum_{l,m} \left(A_{lm,U} Y_{lm} \mathbf{e}_U + A_{lm,V} Y_{lm} \mathbf{e}_V + A_{lm,1} \partial_i Y_{lm} \mathbf{e}_i + A_{lm,2} \epsilon_i{}^j \partial_j Y_{lm} \mathbf{e}_i \right) \tag{179}$$

Alternatively, we can first expand the function Λ in scalar spherical harmonics and then act with ∂_μ. We have:

$$\Lambda(U,V,\theta,\phi) = \sum_{l,m} \Lambda_{lm}(U,V)Y_{lm}(\theta,\phi) \;\Rightarrow\; \partial_\mu\Lambda(U,V,\theta,\phi) = \sum_{l,m} \begin{bmatrix} \partial_U \Lambda_{lm}(U,V) \\ \partial_V \Lambda_{lm}(U,V) \\ \Lambda_{lm}(U,V)\partial_\theta \\ \Lambda_{lm}(U,V)\partial_\phi \end{bmatrix} Y_{lm}. \tag{180}$$

By comparing the above two equations we understand that the most general gauge transformation is of the even-parity form, i.e., the gauge degree of freedom is indeed in $A^+_{lm,\mu}$; by choosing $\Lambda_{lm} = -A_{lm,1}$, we can indeed kill the desired degree of freedom.

[14] Note that for any diffeomorphism that acts on the light-cone and angular coordinates separately (this decomposition is always possible), we may transform $\epsilon_B{}^C \partial_C Y_{lm} \to \epsilon_B{}^{C'} \partial'_C Y_{lm}$ accordingly. Thus, in the new coordinates, A^-_B is still given by (185), meaning that the spherical harmonics decomposition is coordinate-independent. Of course, the same is also true for the even mode.

where we recall that the antisymmetric Levi-Civita tensor on the two-sphere is given by

$$\epsilon_{AB} = r^2 \sin\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{186}$$

Of course, we can raise and lower indices using the metric tensor. Therefore, for the more common form ${\epsilon_B}^C$ we obtain the following expression:

$${\epsilon_B}^C = \begin{pmatrix} 0 & \sin\theta \\ -\csc\theta & 0 \end{pmatrix}. \tag{187}$$

In a similar way, for the even mode we can write

$$A_a^+ = \sum_{l,m} A_{lm,a} Y_{lm}. \tag{188}$$

4.3 Odd- and even-parity decoupling

Here, following Ref. [19], we investigate the coupling between odd- and even-parity modes; in order to do that, let us recall the decomposition obtained in the previous subsection:

$$A_B^- = -\sum_{l,m} A_{lm,2} {\epsilon_B}^C \partial_C Y_{lm}, \quad A_a^+ = \sum_{l,m} A_{lm,a} Y_{lm}. \tag{189}$$

We also recall the Lagrangian for the photon extracted from the action (137):

$$\mathcal{L}_\gamma = \frac{1}{2} A_\mu \mathcal{O}^{\mu\nu} A_\nu, \tag{190}$$

where the operator in the middle has been defined as

$$\mathcal{O}^{\mu\nu} = g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu. \tag{191}$$

In the following we want to show the power of the covariant formalism introduced before; we will distinguish between operators on the light-cone and on the sphere. To see how it works, let us begin by explicitly writing down the operator (191):

$$\begin{aligned} \mathcal{O}^{\mu\nu} A_\nu &= \left[g^{\mu\nu}\nabla_\sigma\nabla^\sigma - \nabla^\mu\nabla^\nu\right] A_\nu \\ &= \left[g^{\mu\nu} g^{\sigma\rho}\nabla_\sigma\nabla_\rho - g^{\mu\sigma} g^{\rho\nu}\nabla_\sigma\nabla_\rho\right] A_\nu. \end{aligned} \tag{192}$$

In the above expression we have the action of a double covariant derivative on the vector potential A_ν. In terms of Christoffel symbols, we can write

$$\begin{aligned} \nabla_\sigma\nabla_\rho A_\nu &= \partial_\sigma(\nabla_\rho A_\nu) - \Gamma^\gamma_{\sigma\rho}\nabla_\gamma A_\nu - \Gamma^\delta_{\sigma\nu}\nabla_\rho A_\delta \\ &= \partial_\sigma(\partial_\rho A_\nu - \Gamma^\delta_{\rho\nu} A_\delta) - \Gamma^\gamma_{\sigma\rho}(\partial_\gamma A_\nu - \Gamma^\alpha_{\gamma\nu} A_\alpha) - \Gamma^\delta_{\sigma\nu}(\partial_\rho A_\delta - \Gamma^\alpha_{\rho\delta} A_\alpha) \\ &= \partial_\sigma\partial_\rho A_\nu - \left(\partial_\sigma \Gamma^\alpha_{\rho\nu}\right) A_\alpha - \Gamma^\delta_{\rho\nu}\partial_\sigma A_\delta - \Gamma^\gamma_{\sigma\rho}\partial_\gamma A_\nu + \Gamma^\gamma_{\sigma\rho}\Gamma^\alpha_{\gamma\nu} A_\alpha \\ &\quad - \Gamma^\delta_{\sigma\nu}\partial_\rho A_\delta + \Gamma^\delta_{\sigma\nu}\Gamma^\alpha_{\rho\delta} A_\alpha. \end{aligned} \tag{193}$$

By considering the fact that we can distinguish between upper- and lower-case Latin indices, and recalling the decomposition of the vector potential, we have

$$\begin{aligned}
\nabla_\sigma\nabla_\rho A_\nu &= \partial_\sigma\partial_\rho A_\nu - \left(\partial_\sigma\Gamma^g_{\rho\nu}\right)A_g - \left(\partial_\sigma\Gamma^B_{\rho\nu}\right)A_B - \Gamma^g_{\rho\nu}\partial_\sigma A_g - \Gamma^B_{\rho\nu}\partial_\sigma A_B - \Gamma^\gamma_{\sigma\rho}\partial_\gamma A_\nu \\
&\quad + \Gamma^\gamma_{\sigma\rho}\Gamma^g_{\gamma\nu}A_g + \Gamma^\gamma_{\sigma\rho}\Gamma^B_{\gamma\nu}A_B - \Gamma^g_{\sigma\nu}\partial_\rho A_g - \Gamma^B_{\sigma\nu}\partial_\rho A_B + \Gamma^\delta_{\sigma\nu}\Gamma^g_{\rho\delta}A_g + \Gamma^\delta_{\sigma\nu}\Gamma^B_{\rho\delta}A_B \\
&= \partial_\sigma\partial_\rho(A^+_\nu + A^-_\nu) - (\partial_\sigma\Gamma^g_{\rho\nu})(A^+_g + A^-_g) - \left(\partial_\sigma\Gamma^B_{\rho\nu}\right)(A^+_B + A^-_B) \\
&\quad - \Gamma^g_{\rho\nu}\partial_\sigma(A^+_g + A^-_g) - \Gamma^B_{\rho\nu}\partial_\sigma(A^+_B + A^-_B) - \Gamma^\gamma_{\sigma\rho}\partial_\gamma(A^+_\nu + A^-_\nu) \\
&\quad + \Gamma^\gamma_{\sigma\rho}\Gamma^g_{\gamma\nu}(A^+_g + A^-_g) + \Gamma^\gamma_{\sigma\rho}\Gamma^B_{\gamma\nu}(A^+_B + A^-_B) - \Gamma^g_{\sigma\nu}\partial_\rho(A^+_g + A^-_g) \\
&\quad - \Gamma^B_{\sigma\nu}\partial_\rho(A^+_B + A^-_B) + \Gamma^\delta_{\sigma\nu}\Gamma^g_{\rho\delta}(A^+_g + A^-_g) + \Gamma^\delta_{\sigma\nu}\Gamma^B_{\rho\delta}(A^+_B + A^-_B) \\
&= \partial_\sigma\partial_\rho(A^+_\nu + A^-_\nu) - \left(\partial_\sigma\Gamma^g_{\rho\nu}\right)A^+_g - \left(\partial_\sigma\Gamma^B_{\rho\nu}\right)A^-_B - \Gamma^g_{\rho\nu}\partial_\sigma A^+_g - \Gamma^B_{\rho\nu}\partial_\sigma A^-_B \\
&\quad - \Gamma^\gamma_{\sigma\rho}\partial_\gamma(A^+_\nu + A^-_\nu) + \Gamma^\gamma_{\sigma\rho}\Gamma^g_{\gamma\nu}A^+_g + \Gamma^\gamma_{\sigma\rho}\Gamma^B_{\gamma\nu}A^-_B - \Gamma^g_{\sigma\nu}\partial_\rho A^+_g - \Gamma^B_{\sigma\nu}\partial_\rho A^-_B \\
&\quad + \Gamma^\delta_{\sigma\nu}\Gamma^g_{\rho\delta}A^+_g + \Gamma^\delta_{\sigma\nu}\Gamma^B_{\rho\delta}A^-_B.
\end{aligned} \tag{194}$$

The above expression should be inserted in (192). However, to avoid confusion, let us consider the two pieces separately; the first one is given by $g^{\mu\nu}g^{\rho\sigma}\nabla_\sigma\nabla_\rho A_\nu$. Focusing on this term and considering first the case $\mu = a, \nu = b$, then Eq. (194) becomes

$$\begin{aligned}
g^{ab}g^{\rho\sigma}\nabla_\sigma\nabla_\rho A_b = g^{ab}g^{\rho\sigma}\Big[&\partial_\sigma\partial_\rho A^+_b - (\partial_\sigma\Gamma^g_{\rho b})A^+_g - \left(\partial_\sigma\Gamma^B_{\rho b}\right)A^-_B - \Gamma^g_{\rho b}\partial_\sigma A^+_g \\
&- \Gamma^B_{\rho b}\partial_\sigma A^-_B - \Gamma^\gamma_{\sigma\rho}\partial_\gamma A^+_b + \Gamma^\gamma_{\sigma\rho}\Gamma^g_{\gamma b}A^+_g + \Gamma^\gamma_{\sigma\rho}\Gamma^B_{\gamma b}A^-_B \\
&- \Gamma^g_{\sigma b}\partial_\rho A^+_g - \Gamma^B_{\sigma b}\partial_\rho A^-_B + \Gamma^\delta_{\sigma b}\Gamma^g_{\rho\delta}A^+_g + \Gamma^\delta_{\sigma b}\Gamma^B_{\rho\delta}A^-_B\Big].
\end{aligned} \tag{195}$$

Above, ρ and σ can be either upper- or lower-case Latin indices. For the sake of clarity, let us consider these two cases separately. We first set $\rho = c$ (and so $\sigma = d$ since $g^{aB} = 0$):

$$\begin{aligned}
g^{ab}g^{cd}\nabla_d\nabla_c A_b = g^{ab}g^{cd}\Big[&\partial_d\partial_c A^+_b - (\partial_d\Gamma^g_{cb})A^+_g - \left(\partial_d\Gamma^B_{cb}\right)A^-_B - \Gamma^g_{cb}\partial_d A^+_g - \Gamma^B_{cb}\partial_d A^-_B \\
&- \Gamma^e_{dc}\partial_e A^+_b - \Gamma^E_{dc}\partial_E A^+_b + \Gamma^e_{dc}\Gamma^g_{eb}A^+_g + \Gamma^E_{dc}\Gamma^g_{Eb}A^+_g \\
&+ \Gamma^e_{dc}\Gamma^B_{eb}A^-_B + \Gamma^E_{dc}\Gamma^B_{Eb}A^-_B - \Gamma^g_{db}\partial_c A^+_g - \Gamma^B_{db}\partial_c A^-_B \\
&+ \Gamma^g_{db}\Gamma^f_{cg}A^+_f + \Gamma^G_{db}\Gamma^g_{cG}A^+_g + \Gamma^e_{db}\Gamma^B_{ce}A^-_B + \Gamma^E_{db}\Gamma^B_{cE}A^-_B\Big].
\end{aligned} \tag{196}$$

For the sake of completeness, let us list the non-vanishing Christoffel symbols of the Schwarzschild metric in Kruskal-Szekeres coordinates [19]:

$$\begin{aligned}
&\Gamma^U_{UU} = \partial_U \log A, \\
&\Gamma^V_{VV} = \partial_V \log A, \\
&\Gamma^\theta_{\theta U} = \Gamma^\theta_{U\theta} = \Gamma^\phi_{\phi U} = \Gamma^\phi_{U\phi} = \partial_U \log r, \\
&\Gamma^\theta_{\theta V} = \Gamma^\theta_{V\theta} = \Gamma^\phi_{\phi V} = \Gamma^\phi_{V\phi} = \partial_V \log r, \\
&\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = -\sin^{-2}\theta\,\Gamma^\theta_{\phi\phi} = \cot\theta, \\
&\Gamma^U_{\theta\theta} = \sin^{-2}\theta\,\Gamma^U_{\phi\phi} = \frac{1}{2A}\partial_V r^2, \\
&\Gamma^V_{\theta\theta} = \sin^{-2}\theta\,\Gamma^V_{\phi\phi} = \frac{1}{2A}\partial_U r^2.
\end{aligned} \tag{197}$$

Therefore, from the above expressions we deduce that

$$\Gamma^{A}_{ab} = \Gamma^{a}_{bA} = 0, \tag{198}$$

which implies that Eq. (196) reduces to

$$\begin{aligned} g^{ab}g^{cd}\nabla_d\nabla_c A_b = g^{ab}g^{cd}\big[\partial_d\left(\partial_c A^+_b - \Gamma^g_{cb}A^+_g\right) - \Gamma^e_{dc}\left(\partial_e A^+_b - \Gamma^g_{eb}A^+_g\right) \\ -\Gamma^g_{db}\left(\partial_c A^+_g - \Gamma^f_{cg}A^+_f\right)\Big]\,. \end{aligned} \tag{199}$$

By defining differential operators with a tilde to represent those on the light-cone, the above expression can be compactly written as

$$\begin{aligned} g^{ab}g^{cd}\nabla_d\nabla_c A_b &= g^{ab}g^{cd}\left[\partial_d\tilde{\nabla}_c A^+_b - \Gamma^e_{dc}\tilde{\nabla}_e A^+_b - \Gamma^g_{db}\tilde{\nabla}_c A^+_g\right] \\ &= g^{ab}g^{cd}\tilde{\nabla}_d\tilde{\nabla}_c A^+_b\,. \end{aligned} \tag{200}$$

Let us come back to Eq. (195). We now set ρ and σ to be upper-case Latin indices, say $\rho = B$ and $\sigma = C$. Proceeding in a similar way as before, we have:

$$\begin{aligned} g^{ab}g^{BC}\nabla_B\nabla_C A_b = g^{ab}g^{BC}\big[&\partial_C\partial_B A^+_b - (\partial_C\Gamma^g_{Bb})A^+_g - \left(\partial_C\Gamma^L_{Bb}\right)A^-_L - \Gamma^g_{Bb}\partial_C A^+_g - \Gamma^E_{Bb}\partial_C A^-_E \\ &- \Gamma^E_{BC}\partial_E A^+_b - \Gamma^d_{BC}\partial_d A^+_b + \Gamma^E_{BC}\Gamma^g_{Eb}A^+_g + \Gamma^d_{BC}\Gamma^g_{db}A^+_g \\ &+ \Gamma^E_{BC}\Gamma^L_{Eb}A^-_L + \Gamma^d_{BC}\Gamma^L_{db}A^-_L - \Gamma^g_{Cb}\partial_B A^+_g - \Gamma^L_{Cb}\partial_B A^-_L \\ &+ \Gamma^E_{Cb}\Gamma^g_{BE}A^+_g + \Gamma^e_{Cb}\Gamma^g_{Be}A^+_g + \Gamma^D_{Cb}\Gamma^E_{BD}A^-_E + \Gamma^d_{Cb}\Gamma^E_{Bd}A^-_E\Big]\,. \end{aligned} \tag{201}$$

By using again (198) and the fact that Γ^L_{Bb}, when evaluated, depends only on the light-cone coordinates, the above equation reduces to

$$\begin{aligned} g^{ab}g^{BC}\nabla_B\nabla_C A_b = g^{ab}g^{BC}\Big[&\partial_C\partial_B A^+_b - \Gamma^E_{Bb}\partial_C A^-_E - \Gamma^E_{BC}\partial_E A^+_b - \Gamma^d_{BC}\partial_d A^+_b + \Gamma^d_{BC}\Gamma^g_{db}A^+_g \\ &+\Gamma^E_{BC}\Gamma^L_{Eb}A^-_L - \Gamma^L_{Cb}\partial_B A^-_L + \Gamma^E_{Cb}\Gamma^g_{BE}A^+_g + \Gamma^D_{Cb}\Gamma^E_{BD}A^-_E\big]\,. \end{aligned} \tag{202}$$

As we can notice, we have four coupling terms. Let us show that they are zero; the first two coupling terms can be paired. Explicitely, the first term is given by[15]

$$\begin{aligned} g^{BC}\Gamma^E_{Bb}\partial_C A^-_E &= g^{\theta\theta}\Gamma^E_{\theta b}\partial_\theta A^-_E + g^{\phi\phi}\Gamma^E_{\phi b}\partial_\phi A^-_E \\ &= g^{\theta\theta}\Gamma^\theta_{\theta b}\partial_\theta A^-_\theta + g^{\phi\phi}\Gamma^\phi_{\phi b}\partial_\phi A^-_\phi\,, \end{aligned} \tag{203}$$

while the second coupling term is

$$-g^{BC}\Gamma^E_{BC}\Gamma^L_{Eb}A^-_L = -g^{\theta\theta}\Gamma^E_{\theta\theta}\Gamma^L_{Eb}A^-_L - g^{\phi\phi}\Gamma^E_{\phi\phi}\Gamma^L_{Eb}A^-_L = -g^{\phi\phi}\Gamma^\theta_{\phi\phi}\Gamma^\theta_{\theta b}A^-_\theta\,. \tag{204}$$

We now realize that the sum of the above two terms gives the following quantity:

$$g^{BC}\Gamma^E_{Bb}\hat{\nabla}_C A^-_E, \tag{205}$$

[15]We are ignoring the metric g^{ab} for a moment.

where the hat denotes operators on the sphere. Indeed, by explicitly evaluating it, we have

$$
\begin{aligned}
g^{BC}\Gamma^{E}_{Bb}\hat{\nabla}_C A^-_E &= g^{BC}\Gamma^{E}_{Bb}\left(\partial_C A^-_E - \Gamma^{L}_{CE}A^-_L\right) \\
&= g^{BC}\Gamma^{E}_{Bb}\partial_C A^-_E - g^{BC}\Gamma^{E}_{Bb}\Gamma^{L}_{CE}A^-_L \\
&= g^{\theta\theta}\Gamma^{E}_{\theta b}\partial_\theta A^-_E + g^{\phi\phi}\Gamma^{E}_{\phi b}\partial_\phi A^-_E - g^{\theta\theta}\Gamma^{E}_{\theta b}\Gamma^{L}_{\theta E}A^-_L - g^{\phi\phi}\Gamma^{E}_{\phi b}\Gamma^{L}_{\phi E}A^-_L \\
&= g^{\theta\theta}\Gamma^{\theta}_{\theta b}\partial_\theta A^-_\theta + g^{\phi\phi}\Gamma^{\phi}_{\phi b}\partial_\phi A^-_\phi - g^{\phi\phi}\Gamma^{\phi}_{\phi b}\Gamma^{L}_{\phi\phi}A^-_L \\
&= g^{\theta\theta}\Gamma^{\theta}_{\theta b}\partial_\theta A^-_\theta + g^{\phi\phi}\Gamma^{\phi}_{\phi b}\partial_\phi A^-_\phi - g^{\phi\phi}\Gamma^{\phi}_{\phi b}\Gamma^{\theta}_{\phi\phi}A^-_\theta \\
&= g^{\theta\theta}\Gamma^{\theta}_{\theta b}\partial_\theta A^-_\theta + g^{\phi\phi}\Gamma^{\phi}_{\phi b}\partial_\phi A^-_\phi - g^{\phi\phi}\Gamma^{\theta}_{\theta b}\Gamma^{\theta}_{\phi\phi}A^-_\theta .
\end{aligned}
\tag{206}
$$

Now, from Eqs. (166) and (167) it can be explicitly checked that $\nabla^i\left(\Phi_{lm}\right)_i = 0$, where we recall that i runs over θ and ϕ, from which we deduce that the above term is zero:

$$
g^{BC}\Gamma^{E}_{Bb}\hat{\nabla}_C A^-_E = \Gamma^{E}_{Bb}\hat{\nabla}^B A^-_E \propto \hat{\nabla}^B\left(\epsilon_B{}^C\partial_C Y_{lm}\right) = 0, \tag{207}
$$

The last two coupling terms in Eq. (202) can also be paired. By renaming indices, we can write

$$
\begin{aligned}
-g^{BC}\Gamma^{L}_{Cb}\partial_B A^-_L + g^{BC}\Gamma^{D}_{Cb}\Gamma^{E}_{BD}A^-_E &= g^{BC}\Gamma^{D}_{Cb}\hat{\nabla}_B A^-_D \\
&= \Gamma^{D}_{Cb}\hat{\nabla}^C A^-_D = 0,
\end{aligned}
\tag{208}
$$

for the same reason as before. Therefore, we are left with the following expression:

$$
g^{ab}g^{BC}\nabla_B\nabla_C A_b = g^{ab}g^{BC}\left[\partial_C\partial_B A^+_b - \Gamma^{E}_{BC}\partial_E A^+_b - \Gamma^{d}_{BC}\tilde{\nabla}_d A^+_b + \Gamma^{E}_{Cb}\Gamma^{g}_{BE}A^+_g\right]. \tag{209}
$$

Let us focus on the first two terms for a moment. It is easy to see that they give the two-sphere Laplacian. Indeed, we have

$$
\begin{aligned}
g^{BC}\left(\partial_C\partial_B - \Gamma^{E}_{BC}\partial_E\right) &= g^{\theta\theta}\partial^2_\theta + g^{\phi\phi}\partial^2_\phi - g^{\theta\theta}\Gamma^{E}_{\theta\theta}\partial_E - g^{\phi\phi}\Gamma^{E}_{\phi\phi}\partial_E \\
&= \frac{1}{r^2}\partial^2_\theta + \frac{1}{r^2\sin^2\theta}\partial^2_\phi - \frac{1}{r^2\sin^2\theta}\Gamma^{\theta}_{\phi\phi}\partial_\theta \\
&= \frac{1}{r^2}\partial^2_\theta + \frac{1}{r^2\sin^2\theta}\partial^2_\phi - \frac{1}{r^2\sin^2\theta}\left(-\sin^2\theta\right)\cot\theta\partial_\theta \\
&= \frac{1}{r^2}\partial^2_\theta + \frac{1}{r^2\sin^2\theta}\partial^2_\phi - \frac{1}{r^2}\cot\theta\partial_\theta = \frac{1}{r^2}\Delta_\Omega,
\end{aligned}
\tag{210}
$$

Now, we want to express the Christoffel symbols of the form Γ^a_{AB} and Γ^A_{aC} in terms of the vector potential $V_a \coloneqq 2\partial_a \log r$. By noting that[16]

$$
\partial^a g_{BD} = \partial^a\left(r^2\Omega_{BD}\right) = g^{ab}\partial_b\left(r^2\Omega_{BD}\right) = 2rg^{ab}\Omega_{BD}\partial_a r = r^2 g^{ab}\Omega_{BD}V_b = g_{BD}V^a, \tag{211}
$$

we can easily deduce the following two relations:

$$
\Gamma^a_{AB} = \frac{1}{2}g^{ab}\left(\partial_A g_{Bb} + \partial_B g_{bA} - \partial_b g_{AB}\right) = -\frac{1}{2}g^{ab}\partial_b g_{AB} = -\frac{1}{2}\partial^a g_{AB} = -\frac{1}{2}g_{AB}V^a, \tag{212}
$$

$$
\Gamma^A_{aC} = \frac{1}{2}g^{AB}\left(\partial_a g_{CB} + \partial_C g_{Ba} - \partial_B g_{aC}\right) = \frac{1}{2}g^{AB}\partial_a g_{CB} = \frac{1}{2}g^{AB}g_{CB}V^a = \frac{1}{2}\delta^A_C V^a. \tag{213}
$$

[16]In Eq. (211), Ω^{BC} is the metric of the unit two-sphere.

Therefore, the last two terms appearing in (209) can be written in terms of the vector potential defined above in the following way:

$$-g^{BC}\Gamma^{d}_{BC} = -g^{BC}\left(-\frac{1}{2}g_{BC}V^{d}\right) = V^{d}, \tag{214}$$

$$g^{BC}\Gamma^{E}_{Cb}\Gamma^{g}_{BE} = g^{BC}\left(\frac{1}{2}\delta^{E}_{C}V_{b}\right)\left(-\frac{1}{2}g_{BE}V^{g}\right) = -\frac{1}{2}V_{b}V^{g}, \tag{215}$$

respectively. Eq. (209) finally becomes

$$g^{ab}g^{BC}\nabla_{B}\nabla_{C}A_{b} = \left[\frac{1}{r^{2}}g^{ab}\Delta_{\Omega} + g^{ab}V^{d}\tilde{\nabla}_{d} - \frac{1}{2}g^{ac}V_{c}V^{b}\right]A^{+}_{b}. \tag{216}$$

We now focus on the second piece in Eq. (192), namely $-g^{\mu\sigma}g^{\rho\nu}\nabla_{\sigma}\nabla_{\rho}A_{\nu}$. Since we are considering the case $\mu = a, \nu = b$, then σ and ρ must be lower-case Latin indices, say $\sigma = c$ and $\rho = b$. We can then write

$$\begin{aligned} g^{ac}g^{bd}\nabla_{c}\nabla_{d}A_{b} = g^{ac}g^{bd}\big[&\partial_{c}\partial_{d}A^{+}_{b} - (\partial_{c}\Gamma^{g}_{db})A^{+}_{g} - \left(\partial_{c}\Gamma^{B}_{db}\right)A^{-}_{B} - \Gamma^{g}_{db}\partial_{c}A^{+}_{g} - \Gamma^{B}_{db}\partial_{c}A^{-}_{B} \\ &- \Gamma^{e}_{cd}\partial_{e}A^{+}_{b} + \Gamma^{e}_{cd}\Gamma^{g}_{eb}A^{+}_{g} + \Gamma^{e}_{cd}\Gamma^{B}_{eb}A^{-}_{B} - \Gamma^{g}_{cb}\partial_{d}A^{+}_{g} - \Gamma^{B}_{db}\partial_{d}A^{-}_{B} \\ &+ \Gamma^{e}_{cb}\Gamma^{g}_{de}A^{+}_{g} + \Gamma^{e}_{db}\Gamma^{B}_{de}A^{-}_{B}\big]. \end{aligned} \tag{217}$$

Above, when summing over γ, we already took into account of the identities (198). Furthermore, for exactly the same reason, we can immediately notice that all the coupling terms vanish. What is left is the following expression:

$$\begin{aligned} g^{ac}g^{bd}\nabla_{c}\nabla_{d}A_{b} = g^{ac}g^{bd}\big[&\partial_{c}\partial_{d}A^{+}_{b} - (\partial_{c}\Gamma^{g}_{db})A^{+}_{g} - \Gamma^{g}_{db}\partial_{c}A^{+}_{g} - \Gamma^{e}_{cd}\partial_{e}A^{+}_{b} \\ &+ \Gamma^{e}_{cd}\Gamma^{g}_{eb}A^{+}_{g} - \Gamma^{g}_{cb}\partial_{d}A^{+}_{g} + \Gamma^{e}_{cb}\Gamma^{g}_{de}A^{+}_{g}\big], \end{aligned} \tag{218}$$

which can be written more compactly as

$$\begin{aligned} g^{ac}g^{bd}\nabla_{c}\nabla_{d}A_{b} &= g^{ac}g^{bd}\left[\partial_{c}\tilde{\nabla}_{d}A^{+}_{b} - \Gamma^{e}_{cd}\tilde{\nabla}_{e}A^{+}_{b} - \Gamma^{g}_{cb}\tilde{\nabla}_{d}A^{+}_{g}\right] \\ &= g^{ac}g^{bd}\tilde{\nabla}_{c}\tilde{\nabla}_{d}A^{+}_{b}. \end{aligned} \tag{219}$$

Putting it all together, namely by considering Eqs. (200), (216) and (219), we obtain a compact expression for the case $\mu = a, \nu = b$:

$$A_{a}\mathcal{O}^{ab}A_{b} = A^{+}_{a}\left[g^{ab}g^{cd}\tilde{\nabla}_{c}\tilde{\nabla}_{d} - g^{ac}g^{bd}\tilde{\nabla}_{c}\tilde{\nabla}_{d} + \frac{1}{r^{2}}g^{ab}\Delta_{\Omega} + g^{ab}V^{d}\tilde{\nabla}_{d} - \frac{1}{2}g^{ac}V_{c}V^{b}\right]A^{+}_{b}. \tag{220}$$

Let us now consider the case $\mu = a, \nu = B$; the first term in (192) vanishes since $g^{\mu\nu} = g^{aB} = 0$. Therefore, we only need to consider the second term:

$$\begin{aligned} g^{a\sigma}g^{B\rho}\nabla_{\sigma}\nabla_{\rho}A_{B} = g^{a\sigma}g^{B\rho}\Big[&\partial_{\sigma}\partial_{\rho}A^{-}_{B} - (\partial_{\sigma}\Gamma^{g}_{\rho B})A^{+}_{g} - \left(\partial_{\sigma}\Gamma^{E}_{\rho B}\right)A^{-}_{E} - \Gamma^{g}_{\rho B}\partial_{\sigma}A^{+}_{g} \\ &- \Gamma^{E}_{\rho B}\partial_{\sigma}A^{-}_{E} - \Gamma^{\gamma}_{\sigma\rho}\partial_{\gamma}A^{-}_{B} + \Gamma^{\gamma}_{\sigma\rho}\Gamma^{g}_{\gamma b}A^{+}_{g} + \Gamma^{\gamma}_{\sigma\rho}\Gamma^{E}_{\gamma B}A^{-}_{E} \\ &- \Gamma^{g}_{\sigma B}\partial_{\rho}A^{+}_{g} - \Gamma^{E}_{\sigma B}\partial_{\rho}A^{-}_{E} + \Gamma^{\delta}_{\sigma B}\Gamma^{g}_{\rho\delta}A^{+}_{g} + \Gamma^{\delta}_{\sigma B}\Gamma^{E}_{\rho\delta}A^{-}_{E}\Big]. \end{aligned} \tag{221}$$

From the above expression we immediately see that σ must be a lower-case Latin index, while ρ an upper-case Latin index. Let us set $\sigma = b, \rho = C$:

$$\begin{aligned}
g^{ab}g^{BC}\nabla_b\nabla_C A_B = g^{ab}g^{BC} &\left[\partial_b\partial_C A_B^- - (\partial_b\Gamma^g_{CB})A_g^+ - (\partial_b\Gamma^E_{CB})A_E^- - \Gamma^g_{CB}\partial_b A_g^+ - \Gamma^E_{CB}\partial_b A_E^- \right.\\
&- \Gamma^\gamma_{bC}\partial_\gamma A_B^- + \Gamma^\gamma_{bC}\Gamma^g_{\gamma B}A_g^+ + \Gamma^\gamma_{bC}\Gamma^E_{\gamma B}A_E^- - \Gamma^g_{bB}\partial_C A_g^+ - \Gamma^E_{bB}\partial_C A_E^-\\
&\left.+\Gamma^\delta_{bB}\Gamma^g_{C\delta}A_g^+ + \Gamma^\delta_{bB}\Gamma^E_{C\delta}A_E^-\right]\\
= g^{ab}g^{BC} &\left[\partial_b\partial_C A_B^- - (\partial_b\Gamma^g_{CB})A_g^+ - \Gamma^g_{CB}\partial_b A_g^+ - \Gamma^E_{CB}\partial_b A_E^- \right.\\
&- \Gamma^D_{bC}\partial_D A_B^- + \Gamma^D_{bC}\Gamma^g_{DB}A_g^+ + \Gamma^D_{bC}\Gamma^E_{DB}A_E^- - \Gamma^E_{bB}\partial_C A_E^-\\
&\left.+\Gamma^D_{bB}\Gamma^g_{CD}A_g^+ + \Gamma^D_{bB}\Gamma^E_{CD}A_E^-\right]. \qquad (222)
\end{aligned}$$

As we can notice, in the above expression we have six coupling terms. Let us show that they are zero. Considering the first two, we have:

$$\begin{aligned}
g^{ab}g^{BC}\left[\partial_b\partial_C A_B^- - \Gamma^E_{CB}\partial_b A_E^-\right] &= g^{ab}g^{BC}\partial_b\left(\partial_C A_B^- - \Gamma^E_{CB}A_E^-\right)\\
&= g^{ab}g^{BC}\tilde{\nabla}_b\hat{\nabla}_C A_B^-\\
&= r^2 g^{ab}\Omega^{BC}\tilde{\nabla}_b\hat{\nabla}_C A_B^- = 0. \qquad (223)
\end{aligned}$$

The same holds for the other coupling terms. Therefore, we are left with the following expression:

$$g^{ab}g^{BC}\nabla_b\nabla_C A_B = g^{ab}g^{BC}\left[-\partial_b\left(\Gamma^g_{CB}A_g^+\right) + \Gamma^D_{bC}\Gamma^g_{DB}A_g^+ + \Gamma^D_{bB}\Gamma^g_{CD}A_g^+\right]. \qquad (224)$$

As before, we want to express our results in terms of V^a. The first term is given by

$$\begin{aligned}
\partial_b\left(\Gamma^g_{CB}A_g^+\right) &= -\frac{1}{2}\partial_b\left(g_{CB}V^g A_g^+\right)\\
&= -\frac{1}{2}\Omega_{CB}\partial_b\left(r^2 V^g A_g^+\right)\\
&= -\frac{1}{2}\Omega_{CB}\left[2r(\partial_b r)V^g A_g^+ + r^2\partial_b(V^g A_g^+)\right]\\
&= -\frac{1}{2}r^2\Omega_{CB}V_b V^g A_g^+ - \frac{1}{2}r^2\Omega_{CB}\partial_b\left(V^g A_g^+\right)\\
&= -\frac{1}{2}g_{CB}\left[V_b V^g A_g^+ + \tilde{\nabla}_b\left(V^g A_g^+\right)\right], \qquad (225)
\end{aligned}$$

where we made use of Eq. (212) and the fact that $\partial_b r = (r/2)V_b$. Therefore, by also considering the minus sign and the metric g^{BC} in front of the first term, we get

$$-g^{BC}\partial_b\left(\Gamma^g_{CB}A_g^+\right) = \tilde{\nabla}_b\left(V^g A_g^+\right) + V_b V^g A_g^+. \qquad (226)$$

Regarding the last two terms, we have

$$g^{BC}\Gamma^D_{bC}\Gamma^g_{DB}A_g^+ = g^{BC}\Gamma^D_{bB}\Gamma^g_{CD}A_g^+ = -\frac{1}{2}V_b V^g A_g^+. \qquad (227)$$

We can now write down the final expression:

$$A_a\mathcal{O}^{aB}A_B = -g^{ab}A_a^+\tilde{\nabla}_b\left(V^g A_g^+\right) = -A_a^+\tilde{\nabla}^a\left(V^g A_g^+\right). \qquad (228)$$

We still need to consider two cases: $\mu = B, \nu = a$ and $\mu = A, \nu = B$. If we consider the first case, we just need to deal with the second piece in (192) since $g^{Ba} = 0$. We have:

$$\begin{aligned}
g^{B\sigma} g^{a\rho} \nabla_\sigma \nabla_\rho A_a = g^{B\sigma} g^{a\rho} & \left[\partial_\sigma \partial_\rho A_a^+ - (\partial_\sigma \Gamma^g_{\rho a}) A_g^+ - (\partial_\sigma \Gamma^E_{\rho a}) A_E^- - \Gamma^g_{\rho a} \partial_\sigma A_g^+ - \Gamma^E_{\rho a} \partial_\sigma A_E^- \right. \\
& - \Gamma^\gamma_{\sigma\rho} \partial_\gamma A_a^+ + \Gamma^\gamma_{\sigma\rho} \Gamma^g_{\gamma a} A_g^+ + \Gamma^\gamma_{\sigma\rho} \Gamma^E_{\gamma a} A_E^- - \Gamma^g_{\sigma a} \partial_\rho A_g^+ - \Gamma^E_{\sigma a} \partial_\rho A_E^- \\
& \left. + \Gamma^\delta_{\sigma a} \Gamma^g_{\rho\delta} A_g^+ + \Gamma^\delta_{\sigma a} \Gamma^E_{\rho\delta} A_E^- \right] \\
= g^{ab} g^{BC} & \left[\partial_C \partial_b A_a^+ - (\partial_C \Gamma^g_{ba}) A_g^+ - (\partial_C \Gamma^E_{ba}) A_E^- - \Gamma^g_{ba} \partial_C A_g^+ - \Gamma^E_{ba} \partial_C A_E^- \right. \\
& - \Gamma^L_{Cb} \partial_L A_a^+ + \Gamma^L_{Cb} \Gamma^g_{La} A_g^+ + \Gamma^L_{Cb} \Gamma^E_{La} A_E^- - \Gamma^g_{Ca} \partial_b A_g^+ - \Gamma^E_{Ca} \partial_b A_E^- \\
& \left. + \Gamma^L_{Ca} \Gamma^g_{bL} A_g^+ + \Gamma^L_{Ca} \Gamma^E_{bL} A_E^- \right] \\
= g^{ab} g^{BC} & \left[\partial_C \partial_b A_a^+ - \Gamma^g_{ba} \partial_C A_g^+ - \Gamma^L_{Cb} \partial_L A_a^+ + \Gamma^L_{Cb} \Gamma^E_{La} A_E^- \right. \\
& \left. - \Gamma^E_{Ca} \partial_b A_E^- + \Gamma^L_{Ca} \Gamma^E_{bL} A_E^- \right] . \qquad (229)
\end{aligned}$$

To get to the last line, we made use of (198) and the fact that Γ^g_{ba} only depends on the light-cone coordinates. Therefore, the above expression gives rise to the following three coupling terms:

$$A_B^- g^{ab} g^{BC} \partial_C \partial_b A_a^+, \quad A_B^- g^{ab} g^{BC} \Gamma^g_{ba} \partial_C A_g^+, \quad A_B^- g^{BC} \Gamma^L_{Cb} \partial_L A_a^+. \qquad (230)$$

In the following we will show that they vanish by explicitly plugging in both equations in (189). Concerning the first of these terms, we can write

$$\begin{aligned}
A_B^- g^{ab} g^{BC} \partial_C \partial_b A_a^+ &= \left(-\sum\nolimits_{l,m} A_{lm,2} \epsilon_B{}^D \partial_D Y_{lm}\right) g^{ab} g^{BC} \partial_C \partial_b \left(\sum\nolimits_{l'',m''} A_{l''m'',a} Y_{l''m''}\right) \\
&= -\sum\nolimits_{l,m} \sum\nolimits_{l'',m''} g^{ab} \left(\partial_b A_{l''m'',a}\right) A_{lm,2} g^{BC} \epsilon_B{}^D \partial_D Y_{lm} \partial_C Y_{l''m''}. \qquad (231)
\end{aligned}$$

As we can notice, there is a clear splitting between the light-cone part (depending on U and V only), and the spherical part, depending on ϕ and θ. Focusing on the angular part, the above term, when inserted in the action, gives the following integral:

$$\mathcal{S}_{lm;l'm'} := \int d\phi d\theta \sqrt{g_{S_2}} g^{BC} \epsilon_B{}^D \partial_D Y_{lm} \partial_C Y_{l''m''} \propto \int d\phi d\theta \sqrt{g_{S_2}} \epsilon^{CD} \partial_D Y_{lm} \partial_C Y^*_{l'm'}, \qquad (232)$$

where we recall that $\sqrt{g_{S_2}} = r^2 \sin^2\theta$, i.e., the volume element on the 2-sphere. Moreover, we expressed the integral in terms of Y^* and defined $m' := -m''$ for the sake of simplicity. The last remark we want to make is that the quantity $\mathcal{S}_{lm;l'm'}$ is coordinate-independent. This can be understood from the fact that the integrand above contains contractions over C, D and is integrated over the sphere. Let us explicitly write down (232):

$$\mathcal{S}_{lm;l'm'} = \int d\phi d\theta \left(\partial_\phi Y_{lm} \partial_\theta Y^*_{l'm'} - \partial_\theta Y_{lm} \partial_\phi Y^*_{l'm'}\right), \qquad (233)$$

where we recalled the definition of the Levi-Civita antisymmetric tensor, i.e., (186). The above integral can be written in terms of the associated Legendre polynomials by making use of (139), i.e., the definition of Y_{lm}. Furthermore, from the same relation, we can easily deduce that

$$\partial_\phi Y_{lm} = -im Y_{lm}, \qquad (234)$$

since the only dependence on ϕ comes from the exponential. Therefore, we can write

$$\begin{aligned}\mathcal{S}_{lm;l'm'} &= \int d\phi d\theta \left[(-im)Y_{lm}\partial_\theta Y^*_{l'm'} - \partial_\theta Y_{lm}(im')Y^*_{l'm'}\right] \\ &\propto \int_0^\pi d\theta \int_0^{2\phi} d\phi e^{i(m-m')\phi}\left[mP_{lm}(\cos\theta)\partial_\theta P_{l'm'}(\cos\theta) + m'P_{l'm'}\partial_\theta P_{lm}(\cos\theta)\right]. \end{aligned} \tag{235}$$

Now, by solving the integral in ϕ, namely

$$\int_0^{2\pi} d\phi e^{i(m-m')\phi} = 2\pi\delta_{mm'}, \tag{236}$$

we arrive at the following expression:

$$\begin{aligned}\mathcal{S}_{lm;l'm'} &\propto m\delta_{mm'}\int_0^\pi \left[P_{lm}(\cos\theta)\partial_\theta P_{l'm}(\cos\theta) + P_{l'm}\partial_\theta P_{lm}(\cos\theta)\right] \\ &= m\delta_{mm'}\int_0^\pi d\theta\partial_\theta\left[P_{l'm}(\cos\theta)P_{lm}(\cos\theta)\right] \\ &= m\delta_{mm'}\left[P_{l'm}(1)P_{lm}(1) - P_{l'm}(-1)P_{lm}(-1)\right]. \end{aligned} \tag{237}$$

If $m=0$, then $\mathcal{S}_{lm;l'm'}=0$. However, it is zero also for $m\neq 0$, as one can understand from the definition of the associated Legendre polynomials, (140). Thus, we have shown that $\mathcal{S}_{lm;l'm'}=0$. The same reasoning applies for the other two terms in (230) since Γ^a_{gb} and Γ^L_{Cb} depend only on the light-cone coordinates. Moreover, it is important to notice that Γ^L_{Cb} is non-vanishing if and only if $L=C$. Now, starting from these considerations, we can write

$$A_B\mathcal{O}^{Ba}A_a = -A^-_B g^{ab}g^{BC}\left[\Gamma^L_{Cb}\Gamma^E_{La}A^-_E - \Gamma^E_{Ca}\partial_b A^-_E + \Gamma^L_{Ca}\Gamma^E_{bL}A^-_E\right]. \tag{238}$$

We can now express the various Christoffel symbols in terms of the vector potential V^a:

$$g^{ab}g^{BC}\Gamma^L_{Cb}\Gamma^E_{La} = g^{ab}g^{BC}\Gamma^L_{Ca}\Gamma^E_{bL} = \frac{1}{4}g^{BE}V_aV^a, \tag{239}$$

$$g^{BC}\Gamma^E_{Ca} = \frac{1}{2}g^{BE}V_a. \tag{240}$$

Therefore, we can finally write down our result:

$$A_B\mathcal{O}^{Ba}A_a = -\frac{1}{2}g^{BE}A^-_B\left[-V_a\tilde{\nabla}^a A^-_E + V_aV^aA^-_E\right] = \frac{1}{2}g^{BE}A^-_B\left[V_a\tilde{\nabla}^a - V_aV^a\right]A^-_E. \tag{241}$$

The case that still needs to be considered is $\mu=A, \nu=B$. Considering the first piece in (192), and setting $\rho=E, \sigma=F$, we can write

$$\begin{aligned}g^{AB}g^{EF}\nabla_F\nabla_E A_B = g^{AB}g^{EF}\Big[&\partial_F\partial_E A^-_B - (\partial_F\Gamma^a_{EB})A^+_a - \left(\partial_F\Gamma^L_{EB}\right)A^-_L - \Gamma^a_{EB}\partial_F A^+_a \\ &- \Gamma^L_{EB}\partial_F A^-_L - \Gamma^L_{EF}\partial_L A^-_B - \Gamma^a_{EF}\partial_a A^-_B + \Gamma^L_{EF}\Gamma^a_{LB}A^+_a \\ &+ \Gamma^b_{EF}\Gamma^a_{bB}A^+_a + \Gamma^P_{EF}\Gamma^L_{PB}A^-_L + \Gamma^b_{EF}\Gamma^L_{bB}A^-_L - \Gamma^a_{FB}\partial_E A^+_a \\ &- \Gamma^L_{FB}\partial_E A^-_L + \Gamma^P_{FB}\Gamma^a_{EP}A^+_a + \Gamma^b_{FB}\Gamma^a_{EB}A^+_a \\ &+ \Gamma^P_{FB}\Gamma^L_{EP}A^-_L + \Gamma^b_{FB}\Gamma^L_{Eb}A^-_L\Big]. \end{aligned} \tag{242}$$

Instead, by setting $\rho = e, \sigma = f$, we get

$$
\begin{aligned}
g^{AB} g^{ef} \nabla_f \nabla_e A_B = g^{AB} g^{ef} \big[& \partial_f \partial_e A_B^- - (\partial_f \Gamma^a_{eB}) A_a^+ - \left(\partial_f \Gamma^L_{eB}\right) A_L^- - \Gamma^a_{eB} \partial_f A_a^+ \\
& - \Gamma^L_{eB} \partial_f A_L^- - \Gamma^L_{ef} \partial_L A_B^- - \Gamma^a_{ef} \partial_a A_B^- + \Gamma^L_{ef} \Gamma^a_{LB} A_a^+ \\
& + \Gamma^b_{ef} \Gamma^a_{bB} A_a^+ + \Gamma^P_{ef} \Gamma^L_{PB} A_L^- + \Gamma^b_{ef} \Gamma^L_{bB} A_L^- - \Gamma^a_{fB} \partial_e A_a^+ \\
& - \Gamma^L_{fB} \partial_e A_L^- + \Gamma^P_{fB} \Gamma^a_{eP} A_a^+ + \Gamma^b_{fB} \Gamma^a_{eB} A_a^+ \\
& + \Gamma^P_{fB} \Gamma^L_{eP} A_L^- + \Gamma^b_{fB} \Gamma^L_{eb} A_L^- \big] .
\end{aligned} \tag{243}
$$

If we now consider the second piece in (192), then σ and ρ can only be upper-case Latin indices. We obtain the same expression as in (242) but with indices contracted in a different way:

$$
\begin{aligned}
g^{AF} g^{BE} \nabla_F \nabla_E A_B = g^{AF} g^{BE} \big[& \partial_F \partial_E A_B^- - (\partial_F \Gamma^a_{EB}) A_a^+ - \left(\partial_F \Gamma^L_{EB}\right) A_L^- - \Gamma^a_{EB} \partial_F A_a^+ \\
& - \Gamma^L_{EB} \partial_F A_L^- - \Gamma^L_{EF} \partial_L A_B^- - \Gamma^a_{EF} \partial_a A_B^- + \Gamma^L_{EF} \Gamma^a_{LB} A_a^+ \\
& + \Gamma^b_{EF} \Gamma^a_{bB} A_a^+ + \Gamma^P_{EF} \Gamma^L_{PB} A_L^- + \Gamma^b_{EF} \Gamma^L_{bB} A_L^- - \Gamma^a_{FB} \partial_E A_a^+ \\
& - \Gamma^L_{FB} \partial_E A_L^- + \Gamma^P_{FB} \Gamma^a_{EP} A_a^+ + \Gamma^b_{FB} \Gamma^a_{EB} A_a^+ \\
& + \Gamma^P_{FB} \Gamma^L_{EP} A_L^- + \Gamma^b_{FB} \Gamma^L_{Eb} A_L^- \big] .
\end{aligned} \tag{244}
$$

We explicitly wrote down all these expressions in order to show that all the coupling terms cancel. Indeed, this case is a bit different with respect to the other ones in the sense that some of the coupling terms do not vanish separately. In other words, we need to consider the sum of Eqs. (242) and (244) to show complete decoupling, namely

$$
A_A \left(g^{AB} g^{EF} - g^{AF} g^{BE}\right) \nabla_F \nabla_E A_B . \tag{245}
$$

Most of the terms vanish because of (198) or because of the fact that the Christoffel symbol under consideration does not depend on the coordinate with respect we are differentiating; thus, let us focus only on those terms that deserve greater attention. In (242), one of them is[17]

$$
1_{221} := -A_A^- g^{AB} g^{EF} \Gamma^a_{EB} \partial_F A_a^+ = -A_\theta^- g^{\theta\theta} g^{\theta\theta} \Gamma^a_{\theta\theta} \partial_\theta A_a^+ - A_\phi^+ g^\phi g^\phi \Gamma^a_{\phi\phi} \partial_\phi A_a^+ . \tag{246}
$$

The corresponding term in (244) gives

$$
\begin{aligned}
1_{223} := A_A^- g^{AF} g^{BE} \Gamma^a_{EB} \partial_F A_a^+ = & A_\theta^- g^{\theta\theta} g^{\theta\theta} \Gamma^a_{\theta\theta} \partial_\theta A_a^+ + A_\theta^- g^{\theta\theta} g^{\phi\phi} \Gamma^a_{\phi\phi} \partial_\theta A_a^+ \\
& + A_\phi^- g^{\phi\phi} g^{\theta\theta} \Gamma^a_{\theta\theta} \partial_\phi A_a^+ + A_\phi^- g^{\phi\phi} g^{\phi\phi} \Gamma^a_{\phi\phi} \partial_\phi A_a^+ .
\end{aligned} \tag{247}
$$

By summing over these last two expressions, we obtain

$$
1_{221} + 1_{223} = A_\theta^- g^{\theta\theta} g^{\phi\phi} \Gamma^a_{\phi\phi} \partial_\theta A_a^+ + A_\phi^- g^{\phi\phi} g^{\theta\theta} \Gamma^a_{\theta\theta} \partial_\phi A_a^+ . \tag{248}
$$

It is easy to see that the angular part of above sum, when inserted in the action, exactly gives the integral (233), which we proved to be zero. Always referring to (242), we can now consider another term, namely $A_A^- g^{AB} g^{EF} \Gamma^L_{EF} \Gamma^a_{LB} A_a^+$. By writing it explicitly, and by recalling (198)

[17] We also consider the A_μ in front of the expression, which is of course necessary to show decoupling.

and the expressions for the non-vanishing Christoffel symbols, we can write

$$\begin{aligned}A^-_A g^{AB} g^{EF} \Gamma^L_{EF} \Gamma^a_{LB} A^+_a &= A^-_\theta g^{\theta\theta} g^{\phi\phi} \Gamma^\theta_{\phi\phi} \Gamma^a_{\theta\theta} A^+_a \\ &= A^-_\theta g^{\theta\theta} g^{\phi\phi} \Gamma^\theta_{\phi\phi} \left(\Gamma^U_{\theta\theta} A^+_U + \Gamma^V_{\theta\theta} A^+_V \right) \\ &= -A_{lm,2} \epsilon_\theta{}^\phi \left(\partial_\phi Y_{lm} \right) g^{\theta\theta} g^{\phi\phi} \Gamma^\theta_{\phi\phi} \left(\Gamma^U_{\theta\theta} A_{l'm',U} + \Gamma^V_{\theta\theta} A_{l'm',V} \right) Y_{l'm'}, \quad (249)\end{aligned}$$

where we ignored for a moment the sum over l, l', m and m'. By inserting the above expression in the action, and by considering the spherical part only, we obtain the following integral:

$$\int d\phi d\theta \cot\theta \left(\partial_\phi Y_{lm} \right) Y^*_{l'm'}. \quad (250)$$

A couple of comments are in order here. First of all, the minus sign in Eq. (249) is canceled by the minus sign coming from the Christoffel symbol $\Gamma^\theta_{\phi\phi}$. Moreover, we also notice that $\Gamma^U_{\theta\theta}$ and $\Gamma^V_{\theta\theta}$ depend on the light-cone coordinates only and so they do not contribute to the angular part. If we tried to explicitly compute this term, we would obtain an expression which is in general non-vanishing; we did not explicitly write down the light-cone piece, but it can be easily verified that it is in general non-vanishing too. Therefore, the first thing one can think of is to look at the corresponding term in Eq. (244) and check if they cancel each other. The term we want to look at can be massaged to give

$$\begin{aligned}-A_A g^{AF} g^{BE} \Gamma^L_{EF} \Gamma^a_{LB} A^+_a &= -A^-_\theta g^{\theta\theta} g^{\phi\phi} \Gamma^\phi_{\phi\theta} \Gamma^a_{\phi\phi} A^+_a \\ &= -A^-_\theta g^{\theta\theta} g^{\phi\phi} \left(-\sin^2\theta \right) \Gamma^\theta_{\phi\phi} \left(\sin^2\theta \right) \Gamma^a_{\theta\theta} A^+_a \\ &= A^-_\theta g^{\theta\theta} g^{\phi\phi} \Gamma^\theta_{\phi\phi} \Gamma^a_{\theta\theta} A^+_a, \quad (251)\end{aligned}$$

which is equal to (249). Thus, these two terms add up instead of canceling each other. However, if we look at (244) more carefully, we will notice that there is a term of the form

$$A_A g^{AF} g^{BE} \left(\partial_F \Gamma^a_{EB} \right) A^+_a = A^-_\theta g^{\theta\theta} g^{\phi\phi} \left(\partial_\theta \Gamma^a_{\phi\phi} \right) A^+_a. \quad (252)$$

Now, by recalling that $\Gamma^a_{\phi\phi} = \sin^2\theta \Gamma^a_{\theta\theta}$, we have $\partial_\theta \Gamma^a_{\phi\phi} = \partial_\theta \left(\sin^2\theta \Gamma^a_{\theta\theta} \right) = 2\sin\theta\cos\theta \Gamma^a_{\theta\theta}$ since $\Gamma^a_{\theta\theta} = \Gamma^a_{\theta\theta}(U,V)$. Therefore, we obtain

$$A^-_\theta g^{\theta\theta} g^{\phi\phi} \left(\partial_\theta \Gamma^a_{\phi\phi} \right) A^+_a = -2 A^-_\theta g^{\theta\theta} g^{\phi\phi} \Gamma^\theta_{\phi\phi} \Gamma^a_{\theta\theta} A^+_a. \quad (253)$$

Therefore, (249), (251) and (252), when added up, cancel each other. Now, one can wonder what happens at the term corresponding to (252) in Eq. (242). We can easily show that this term is zero by simply writing down the full expression:

$$\begin{aligned}-A^-_A g^{AB} g^{EF} \left(\partial_F \Gamma^a_{EB} \right) A^+_a &= -A^-_\theta g^{\theta\theta} g^{EF} \left(\partial_F \Gamma^a_{E\theta} \right) A^+_a - A^-_\phi g^{\phi\phi} g^{EF} \left(\partial_F \Gamma^a_{E\phi} \right) A^+_a \\ &= -A^-_\theta g^{\theta\theta} g^{\theta\theta} \left(\partial_\theta \Gamma^a_{\theta\theta} \right) A^+_a - A^-_\phi g^{\phi\phi} g^{\phi\phi} \left(\partial_\phi \Gamma^a_{\phi\phi} \right) A^+_a = 0, \quad (254)\end{aligned}$$

where we made use of the fact that $\Gamma^a_{E\theta}$ is non-vanishing if and only if $E = \theta$. Lastly, we can immediately check that the term $-A_A g^{AB} g^{EF} \Gamma^a_{FB} \partial_E A^+_a$ appearing in (242) is canceled by the corresponding term in (244). All the other coupling terms vanish because of the identities (198), as already mentioned. Now that we have proved decoupling, we can write down the expression for $A_A\mathcal{O}^{AB}A_B$ in terms of covariant derivatives and the vector potential defined before; in particular, by summing all of the terms, making use of the fact that the Christoffel symbols can be written in terms of V^a, and lastly considering that Γ^a_{ef}, when contracted with g^{ef}, vanishes, we end up with the following expression:

$$A_A\mathcal{O}^{AB}A_B = A^-_A\left[g^{AB}g^{EF}\hat{\nabla}_E\hat{\nabla}_F - \frac{1}{4}g^{AB}V^aV_a + g^{AB}\tilde{\Box}\right]A^-_B - \frac{1}{2}A^-_A g^{AB}\tilde{\nabla}^a\left(V_aA^-_B\right). \quad (255)$$

Above, based on (192), we would have expected to obtain a term of the form $g^{AF}g^{BE}\hat{\nabla}_F\hat{\nabla}_E A^-_B$. However, it can be easily shown that this term vanishes by again making use of one of the properties of the vector spherical harmonics, namely $\nabla^i\left(\Phi_{lm}\right)_i = 0$:

$$g^{AF}g^{BE}\hat{\nabla}_F\hat{\nabla}_E A^-_B = g^{AF}g^{BE}\hat{\nabla}_F\hat{\nabla}^B A^-_B \propto \hat{\nabla}^B\left(\epsilon_B{}^C\partial_C Y_{lm}\right) = 0. \quad (256)$$

We are now ready to write down the full photon Lagrangian:

$$\mathcal{L}_\gamma = \frac{1}{2}A_\mu\mathcal{O}^{\mu\nu}A_\nu = \frac{1}{2}\left(A_a\mathcal{O}^{ab}A_b + A_a\mathcal{O}^{aB}A_B + A_B\mathcal{O}^{Ba}A_a + A_A\mathcal{O}^{AB}A_B\right), \quad (257)$$

where the four terms above are given by Eqs. (220, (228), (241) and (255), respectively.

4.4 A two-dimensional field theory for the photon

In the previous section we split the spacetime into two components, the light-cone g_{ab} and the two-sphere g_{AB}, and showed that we have decoupling between even- and odd-parity modes. Finally, we ended up with the Lagrangian (257). Now, the question is: can we integrate the sphere out and obtain a two-dimensional field theory for the photon? This would clearly be an enormous simplification. We would obtain a two-dimensional description of the problem with the metric g_{ab} only. Last but not least, by doing so we will explicitly see that the different l, m modes decouple, i.e., we will obtain an infinite tower of decoupled actions, one for each partial wave; this is to be expected due to the spherical symmetry of the background. In order to see if what we have said so far can be achieved, let us analyze each term in (257); in particular, let us insert the decomposition (189) in the Lagrangian. For clarity, we will consider each term separately. The first one can be written as

$$A_a\mathcal{O}^{ab}A_b = \sum_{l,m;l',m'} A_{lm,a}Y_{lm}\left[g^{ab}\tilde{\Box} - \tilde{\nabla}^a\tilde{\nabla}^b - g^{ab}\frac{\lambda'}{r^2} + g^{ab}V^d\tilde{\nabla}_d - \frac{1}{2}V^aV^b\right]A_{l'm',b}Y^*_{l'm'}, \quad (258)$$

where we have defined $\lambda' := l(l+1)$. Moreover, we introduced the box operator on the light-cone, namely $\tilde{\Box} = \tilde{\nabla}^a\tilde{\nabla}_a$. At this point we can immediately see that the sphere can be integrated out. Indeed, the operators in the middle of the above expression are light-cone quantities and so the indices never sum over the coordinates θ and ϕ. We can therefore safely move around the spherical $Y^*_{l',m'}$ to the left, obtaining the following expression:

$$A_a\mathcal{O}^{ab}A_b = \sum_{l,m;l',m'} Y_{lm}Y^*_{l'm'}A_{lm,a}P_1^{ab}A_{l'm',b}, \quad (259)$$

where the operator in the middle has been defined as

$$P_1^{ab} := g^{ab}\tilde{\Box} - \tilde{\nabla}^a\tilde{\nabla}^b - g^{ab}\frac{\lambda'}{r^2} + g^{ab}V^d\nabla_d - \frac{1}{2}V^aV^b. \tag{260}$$

The same can be done for the second term in (257):

$$\begin{aligned} A_a\mathcal{O}^{aB}A_B &= -\sum_{l,m;l',m'} A_{lm,a}Y_{lm}\left[\tilde{\nabla}^aV^b + V^b\tilde{\nabla}^a\right]A_{l'm',b}Y^*_{l'm'} \\ &= -\sum_{l,m;l',m'} Y_{lm}Y^*_{l'm'}A_{lm,a}P_2^{ab}A_{l'm',b}, \end{aligned} \tag{261}$$

with $P_2^{ab} := \tilde{\nabla}^aV^b + V^b\tilde{\nabla}^a$. The above two terms can be combined, giving

$$A_a\mathcal{O}^{ab}A_b + A_a\mathcal{O}^{aB}A_B = \sum_{l,m;l',m'} Y_{lm}Y^*_{l'm'}A_{lm,a}\tilde{P}^{ab}A_{l'm',b}, \tag{262}$$

where the operator $\tilde{P}^{ab}$ is defined as

$$\boxed{\tilde{P}^{ab} := P_1^{ab} - P_2^{ab} = g^{ab}\tilde{\Box} - \tilde{\nabla}^a\tilde{\nabla}^b - g^{ab}\frac{\lambda'}{r^2} + g^{ab}V^d\tilde{\nabla}_d - \frac{1}{2}V^aV^b - \tilde{\nabla}^aV^b - V^b\tilde{\nabla}^a.} \tag{263}$$

Let us now focus on the last two terms in (257). The first one gives

$$\begin{aligned} A_B\mathcal{O}^{Ba}A_a &= g^{AB}A^-_A\left[\frac{1}{2}V_a\tilde{\nabla}^a - \frac{1}{2}V_aV^a\right]A^-_B \\ &= \sum_{l,m;l',m'} g^{AB}\left(\Phi_{lm}\right)_A\left(\Phi^*_{l'm'}\right)_B A_{lm,2}P_3A_{l'm',2}, \end{aligned} \tag{264}$$

where $P_3 := \frac{1}{2}V_a\tilde{\nabla}^a - \frac{1}{2}V_aV^a$. Note that above we used the same notation as in (162) for the vector spherical harmonics. Finally, for the last one we have

$$A_A\mathcal{O}^{AB}A_B = \sum_{l,m;l',m'} g^{AB}\left(\Phi_{lm}\right)_A\left(\Phi^*_{l'm'}\right)_B A_{lm,2}P_4A_{l'm',2}, \tag{265}$$

where the operator P_4 has been defined as

$$P_4 := \tilde{\Box} + \frac{1-\lambda'}{r^2} - \frac{1}{4}V^aV_a - \frac{1}{2}\tilde{\nabla}^aV_a - \frac{1}{2}V_a\tilde{\nabla}^a. \tag{266}$$

The $(1-\lambda')r^{-2}$ factor above comes from the action of the operator $\hat{\nabla}^F\hat{\nabla}_F$ on $(\Phi_{l'm'})_B$. As we did for the first two terms, let us combine these two last contributions:

$$A_B\mathcal{O}^{Ba}A_a + A_A\mathcal{O}^{AB}A_B = \sum_{l,m;l',m'} g^{AB}\left(\Phi_{lm}\right)_A\left(\Phi^*_{l'm'}\right)_B A_{lm,2}\tilde{P}A_{l'm',2}, \tag{267}$$

where the operator $\tilde{P}$ is defined as

$$\boxed{\tilde{P} := P_3 + P_4 = \tilde{\Box} + \frac{1-\lambda'}{r^2} - \frac{1}{2}\tilde{\nabla}^aV_a - \frac{3}{4}V_aV^a.} \tag{268}$$

The full Lagrangian can thus be written as

$$\mathcal{L}_\gamma = \frac{1}{2}\sum_{l,m;l',m'} Y_{lm}Y^*_{l'm'}A_{lm,a}\tilde{P}^{ab}A_{l'm',b} + \frac{1}{2}\sum_{l,m;l',m'} g^{AB}\left(\Phi_{lm}\right)_A\left(\Phi^*_{l'm'}\right)_B A_{lm,2}\tilde{P}A_{l'm',2}, \quad (269)$$

where the two terms represent the even- and odd-parity contributions, respectively. The corresponding action for the even part is given by

$$\begin{aligned} S_{\gamma,even} &= \frac{1}{2}\sum_{l,m;l',m'}\int d\Omega Y_{lm}Y^*_{l'm'}\int d^2x A(r)r^2 A_{lm,a}\tilde{P}^{ab}A_{l'm',b} \\ &= \frac{1}{2}\sum_{l,m}\int d^2x A(r)r^2 A_{lm,a}\tilde{P}^{ab}A_{l'm',b}, \end{aligned} \quad (270)$$

where we simply used the orthogonality relation for the scalar spherical harmonics:

$$\int d\Omega Y_{lm}Y^*_{l'm'} = \delta_{ll'}\delta_{mm'}. \quad (271)$$

The odd-parity contribution is instead given by

$$\begin{aligned} S_{\gamma,odd} &= \frac{1}{2}\sum_{l,m;l',m'}\int d\Omega g^{AB}\left(\Phi_{lm}\right)_A\left(\Phi^*_{l'm'}\right)_B\int d^2x A(r)r^2 A_{lm,2}\tilde{P}A_{l'm',2} \\ &= \frac{1}{2}\sum_{l,m}\int d^2x A(r)r^2\lambda' A_{lm,2}\tilde{P}A_{l'm',2} \\ &= \frac{1}{2}\sum_{l,m}\int d^2x A(r)r^2 A_{lm,2}\tilde{P}A_{l'm',2}, \end{aligned} \quad (272)$$

where this time we used the orthogonality relation for the vector spherical harmonics:

$$\int d\Omega g^{AB}\left(\Phi_{lm}\right)_A\left(\Phi^*_{l'm'}\right)_B = \lambda'\delta_{ll'}\delta_{mm'}. \quad (273)$$

Notice that in last step λ' has been absorbed in the operator $\tilde{P}$. To avoid clutter of notation, we did not give it a new name. Therefore, the sum of these two contributions, S_γ, is

$$S_\gamma = \frac{1}{2}\sum_{l,m}\int d^2x A(r)r^2 A_{lm,a}\tilde{P}^{ab}A_{l'm',b} + \frac{1}{2}\sum_{l,m}\int d^2x A(r)r^2 A_{lm,2}\tilde{P}A_{l'm',2}. \quad (274)$$

The above expression represents an infinite tower of decoupled actions, one for each l and m, as anticipated before. We have been able to integrate the sphere out and obtain a two-dimensional field theory for the photon. It is important to notice that the only residual curvature components arising from the two-sphere are embedded in the potential V_a present both in $\tilde{P}^{ab}$ and $\tilde{P}$. At this point we recall that our aim is to find the photon propagator, which can be obtained directly from the action by finding the inverse of the kinetic term operator. Starting from these considerations, we can immediately notice that there is a problem in the above action, namely the presence of the factor r^2, which comes from the two-sphere Jacobian; the best thing we can do in order to read off the propagator from the Lagrangian would be to absorb r^2 into the fields. This is not immediate since the operators $\tilde{P}^{ab}$ and $\tilde{P}$ do not commute with r^2, so a little more work is needed. Looking at (274), the idea is to remove one r to the left of $A_{lm,a}$ (or $A_{lm,2}$) and introduce an r on the right of $A_{l'm',b}$ (or $A_{l'm',2}$). If we are able to do this, then we can redefine the fields appropriately and obtain the form of the action that we want. In order to achieve this, we can first recall how the vector potential V_a has been defined: $V_a = 2\partial_a \log r$. This leads to the definition of a new operator:

$$\mathcal{D}_a(\cdot) \coloneqq \tilde{\nabla}_a(\cdot) + \frac{1}{2}V_a(\cdot) = \frac{1}{r}\tilde{\nabla}_a(r\,\cdot), \tag{275}$$

where with the symbol $\cdot$ we indicate the fact that we are not specifying the type of mathematical object we are acting on. The strategy is to now replace every covariant derivative in Eqs. (262) and (268) with this new operator that has been introduced, $\mathcal{D}_a$. Let us start from the operator $\tilde{P}^{ab}$, which can rewritten as follows:

$$\tilde{P}^{ab} = g^{ab}\left[\tilde{\Box} + V_d\tilde{\nabla}^d - \frac{\lambda'}{r^2}\right] - \tilde{\nabla}^a\tilde{\nabla}^b - \tilde{\nabla}^a V^b - V^b\tilde{\nabla}^a - \frac{1}{2}V^aV^b. \tag{276}$$

The box operator, in terms of $\mathcal{D}_a$, becomes[18]

$$\begin{aligned}\tilde{\Box}f = \tilde{\nabla}^a\tilde{\nabla}_a f &= \tilde{\nabla}^a\left(\mathcal{D}_a f - \frac{1}{2}V_a f\right)\\ &= \mathcal{D}^a\left(\mathcal{D}_a f - \frac{1}{2}V_a f\right) - \frac{1}{2}V^a\left(\mathcal{D}_a f - \frac{1}{2}V_a f\right)\\ &= \mathcal{D}^a\mathcal{D}_a f - \frac{1}{2}\mathcal{D}^a(V_a f) - \frac{1}{2}V^a\mathcal{D}_a f + \frac{1}{4}V^aV_a f\\ &= \mathcal{D}^2 f - \frac{1}{2}(\mathcal{D}^a V_a) f - V_a\mathcal{D}^a f + \frac{1}{2}V_aV^a f,\end{aligned} \tag{277}$$

where we defined $\mathcal{D}^2 \coloneqq \mathcal{D}^a\mathcal{D}_a$ and used the following identity[19]:

$$\mathcal{D}_a(V^a f) = (\mathcal{D}_a V^a) f + V^a\mathcal{D}_a f - \frac{1}{2}V_aV^a f. \tag{278}$$

The second term in (276), $V_d\tilde{\nabla}^d$, immediately gives

$$V_d\tilde{\nabla}^d f = V_a\tilde{\nabla}^a f = V_a\mathcal{D}^a f - \frac{1}{2}V_aV^a f. \tag{279}$$

Therefore, by summing these two contributions, we get

$$\tilde{\Box}f + V_d\tilde{\nabla}^d f = \mathcal{D}^2 f - \frac{1}{2}(\mathcal{D}^a V_a) f = \mathcal{D}^2 f - F^a_a f, \tag{280}$$

where, for later convenience, a new quantity has been introduced[20]:

$$F_{ab} \coloneqq \frac{1}{2}\mathcal{D}_{(a}V_{b)} = \frac{1}{r}\tilde{\nabla}_a\tilde{\nabla}_b r = \frac{1}{2}\tilde{\nabla}_{(a}V_{b)} + \frac{1}{4}V_aV_b. \tag{281}$$

[18] Note that, for the sake of simplicity, we are applying the operator to some generic function f. Indeed, it is not important to distinguish the object we are acting on here since we never work out the Christoffel symbols in this computation.

[19] The Leibniz rule does not hold for the operator $\mathcal{D}_a$.

[20] Here we are using the standard notation for symmetric tensors: $T_{(ab)} \coloneqq \frac{1}{2}(T_{ab} + T_{ba})$. The fact that F_{ab} is a symmetric tensor follows from the commutativity of covariant derivatives when acting on a scalar [19].

Let us now focus on the other terms in (276). The first term in parentheses is

$$
\begin{aligned}
\tilde{\nabla}^a\tilde{\nabla}^b f &= \tilde{\nabla}^a\left(\mathcal{D}^b - \frac{1}{2}V^b f\right) \\
&= \mathcal{D}^a\left(\mathcal{D}^b f - \frac{1}{2}V^b f\right) - \frac{1}{2}V^a\left(\mathcal{D}^b f - \frac{1}{2}V^b f\right) \\
&= \mathcal{D}^a\mathcal{D}^b f - \frac{1}{2}\left(\mathcal{D}^a V^b\right) f - \frac{1}{2}V^b\mathcal{D}^a f + \frac{1}{4}V^a V^b f - \frac{1}{2}V^a\mathcal{D}^b f + \frac{1}{4}V^a V^b f, \qquad (282)
\end{aligned}
$$

while the second and third terms are given by

$$\left(\tilde{\nabla}^a V^b\right) f = \left(\mathcal{D}^a V^b\right) f - \frac{1}{2}V^a V^b f, \qquad (283)$$

$$V^b\tilde{\nabla}^a f = V^b\mathcal{D}^a f - \frac{1}{2}V^b V^a f, \qquad (284)$$

respectively. Thus, by considering (282), (283), (284) and the last term in (276) we get

$$-\tilde{\nabla}^a\tilde{\nabla}^b - \tilde{\nabla}^a V^b - V^b\tilde{\nabla}^a - \frac{1}{2}V^a V^b = -\mathcal{D}^a\mathcal{D}^b f - F^{ab} f - V^{[b}\mathcal{D}^{a]} f. \qquad (285)$$

Finally, putting it all together, we can write down $\tilde{P}^{ab}$ in terms of the operator (275):

$$\tilde{P}^{ab} = g^{ab}\left[\mathcal{D}^2 - F_c^c - \frac{\lambda'}{r^2}\right] - \mathcal{D}^a\mathcal{D}^b - F^{ab} - V^{[b}\mathcal{D}^{a]}. \qquad (286)$$

We can do the same with $\tilde{P}$. Following the same steps, we can write

$$\tilde{\Box} f - \frac{1}{2}\left(\tilde{\nabla}^a V_a\right) f - \frac{3}{4}V_a V^a f = \mathcal{D}^2 f - 2F_a^a f - V_a\mathcal{D}^a f. \qquad (287)$$

The operator $\tilde{P}$ can then be written as

$$\tilde{P} = \lambda'\mathcal{D}^2 + \frac{\lambda'(1-\lambda')}{r^2} - 2\lambda' F_a^a - \lambda' V_a\mathcal{D}^a. \qquad (288)$$

Now we recall that $\mathcal{D}_a(\cdot) = \frac{1}{r}\tilde{\nabla}_a(r\,\cdot)$; starting from this, we want to write all the operators appearing in (286) and (288) in the same form. The following identities can be easily proven:

$$\mathcal{D}^a\mathcal{D}_a(\cdot) = \frac{1}{r}\tilde{\Box}(r\,\cdot), \qquad (289)$$

$$\mathcal{D}^a\mathcal{D}^b(\cdot) = \frac{1}{r}\tilde{\nabla}^a\tilde{\nabla}^b(r\,\cdot), \qquad (290)$$

$$V^{[b}\mathcal{D}^{a]}(\cdot) = \frac{1}{r}V^{[b}\tilde{\nabla}^{a]}(r\,\cdot). \qquad (291)$$

Therefore, by making use of the above relations, and ignoring the subscripts lm, $l'm'$ for a moment, we can rewrite the two integrands in (274) as follows:

$$r^2 A_a\tilde{P}^{ab}A_b \xrightarrow{(289)-(291)} rA_a\left[g^{ab}\left(\tilde{\Box} - F_c^c - \frac{\lambda'}{r^2}\right) - \tilde{\nabla}^a\tilde{\nabla}^b - F^{ab} - V^{[b}\tilde{\nabla}^{a]}\right] rA_b, \qquad (292)$$

$$r^2 A_2\tilde{P}A_2 \xrightarrow{(289)-(291)} rA_2\left[\lambda'\tilde{\Box} + \frac{\lambda'(1-\lambda')}{r^2} - 2\lambda' F_a^a - \lambda' V_a\tilde{\nabla}^a\right] rA_2. \qquad (293)$$

We can now safely make the following field redefinitions:

$$\tilde{A}_a \coloneqq rA_a, \quad \tilde{A} \coloneqq rA_2. \tag{294}$$

The photon action can then be written in the following form[21] [39]:

$$S_\gamma = S_{\gamma,even} + S_{\gamma,odd} = \frac{1}{2}\int d^2x\sqrt{-\tilde{g}}\tilde{A}^a\tilde{\Delta}^{-1}_{ab}\tilde{A}^b + \frac{1}{2}\int d^2x\sqrt{-\tilde{g}}\tilde{A}\tilde{\Delta}^{-1}\tilde{A}, \tag{295}$$

where $\sqrt{-\tilde{g}} = A(r)$ and the operators $\tilde{\Delta}^{-1}_{ab}$ and $\tilde{\Delta}^{-1}$ are defined as follows:

$$\tilde{\Delta}^{-1}_{ab} \coloneqq g_{ab}\left(\tilde{\Box} - F^c_c - \frac{\lambda'}{r^2}\right) - \tilde{\nabla}_a\tilde{\nabla}_b - F_{ab} - V_{[b}\tilde{\nabla}_{a]}, \tag{296}$$

$$\tilde{\Delta}^{-1} \coloneqq \lambda'\tilde{\Box} + \frac{\lambda'\left(1-\lambda'\right)}{r^2} - 2\lambda' F^a_a - \lambda' V_a\tilde{\nabla}^a. \tag{297}$$

In principle we can now insert the metric in our expressions and invert (296) and (297) in order to find the photon propagator. However, an important remark concerning the metric can be made at this point. Indeed, even if in section (4.2) we explicitly mentioned that we would have worked in Kruskal–Szekeres coordinates, we have not yet used a particular form of the function $A(r)$, which is in principle still arbitrary. In particular, the metric we are working with (i.e., the light-cone metric) is now given by

$$\tilde{g}_{ab} = A(r)\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = A(r)\eta_{ab}, \tag{298}$$

where η_{ab} is the two-dimensional Minkowski metric in light-cone coordinates. From the form of the metric we immediately deduce that our theory is conformally flat; at this point, one can ask: can we appropriately redefine the fields such that one obtains a flat theory? This would of course be extremely advantageous since, as we have seen in section (1.2), working with a quantum field theory in curved spacetime can be very challenging. The next section will be devoted to such redefinition.

4.5 From curved to flat spacetime

As already anticipated, the goal of this section is to exploit the fact that the metric is written as $\tilde{g}_{ab} = A(r)\eta_{ab}$, and appropriately redefine all the fields present in order to work with a flat theory. In particular, we will rescale the fields in such a way to make sure that the kinetic term in the action contains no more $r-$dependent terms. Let us first work on the odd part, which is much easier as we will see. Explicitly, we have:

$$\begin{aligned} S_{\gamma,odd} = \frac{1}{2}\int d^2x\sqrt{-\tilde{g}}\tilde{A}\tilde{\Delta}^{-1}\tilde{A} &= \frac{1}{2}\int d^2x A(r)\tilde{A}\tilde{\Delta}^{-1}\tilde{A} \\ &= \frac{\lambda'}{2}\int d^2x A(r)\tilde{A}\left[\tilde{\Box} + \frac{1-\lambda'}{r^2} - 2F^a_a - V_a\tilde{\nabla}^a\right]\tilde{A}. \end{aligned} \tag{299}$$

[21] We are again ignoring the subscripts lm, $l'm'$ as well as the sum. Moreover, for later convenience, we raised the indices of the vector potentials and lowered those of the operator in the middle in the even-parity action.

We now redefine the scalar as $\tilde{A} := \mathcal{A}$, i.e, we just give it a new name for later convenience. By also lowering all indices with the metric, we get

$$\begin{aligned} S_{\gamma,odd} &= \frac{\lambda'}{2}\int d^2x A(r)\mathcal{A}\left[\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b + \frac{1-\lambda'}{r^2} - 2\tilde{g}^{ab}F_{ab} - \tilde{g}^{ab}V_a\tilde{\nabla}_b\right]\mathcal{A} \\ &= \frac{\lambda'}{2}\int d^2x A(r)\mathcal{A}\left[\frac{1}{A(r)}\eta^{ab}\tilde{\nabla}_a\tilde{\nabla}_b + \frac{1-\lambda'}{r^2} - \frac{2}{A(r)}\eta^{ab}F_{ab} - \frac{1}{A(r)}\eta^{ab}V_a\tilde{\nabla}_b\right]\mathcal{A}. \end{aligned} \tag{300}$$

By simply considering the fact that

$$\eta^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\mathcal{A} = \eta^{ab}\partial_a\partial_b\mathcal{A} - \eta^{ab}\Gamma^c_{ab}\partial_c\mathcal{A} = \eta^{ab}\partial_a\partial_b\mathcal{A}, \tag{301}$$

we are left with the following expression:

$$S_{\gamma,odd} = \frac{\lambda'}{2}\int d^2x\mathcal{A}\left[\partial^2 + A(r)\frac{1-\lambda'}{r^2} - 2F^a_a - V^b\partial_b\right]\mathcal{A}, \tag{302}$$

where we defined $\partial^2 := \eta^{ab}\partial_a\partial_b$. It is important to notice that all index manipulations are performed with the flat metric; in particular, above we used that $F^a_a = \eta^{ab}F_{ab}$ and $V^b = \eta^{ab}V_a$. Actually, a further comment about the symmetric tensor F_{ab} must be made at this point. Indeed, we need to consistently redefine it in terms of partial derivatives and curvature potentials since the light-cone derivative in terms of which it has been originally defined does not hold anymore after the rescaling. By recalling how we defined F_{ab}, namely Eq. (281), we can write

$$F_{ab} = \frac{1}{4}\tilde{\nabla}_a V_b + \frac{1}{4}\tilde{\nabla}_b V_a + \frac{1}{4}V_aV_b = \frac{1}{4}\left(\partial_a V_b - \Gamma^e_{ab}V_e\right) + \frac{1}{4}\left(\partial_b V_a - \Gamma^e_{ba}V_e\right) + \frac{1}{4}V_aV_b. \tag{303}$$

Now, in order to write F_{ab} in terms of residual curvature components only, we first express the Christoffel symbols of the form Γ^e_{ab} in the following way[22]:

$$\Gamma^e_{ab} = 2\delta^e_{(a}U_{b)} - \tilde{g}_{ab}U^e = 2\delta^e_{(a}U_{b)} - \tilde{g}_{ab}\tilde{g}^{de}U_d = 2\delta^e_{(a}U_{b)} - \eta_{ab}\eta^{de}U_d = 2\delta^e_{(a}U_{b)} - \eta_{ab}U^e, \tag{304}$$

where the new potential U_a has been defined as

$$U_a := \frac{1}{2A(r)}\partial_a A(r). \tag{305}$$

Therefore, from Eq. (303) we obtain

$$\begin{aligned} F_{ab} &= \frac{1}{2}\partial_{(a}V_{b)} - \frac{1}{2}\left(\delta^e_a U_b + \delta^e_b U_a - \eta_{ab}U^e\right)V_e + \frac{1}{4}V_aV_b \\ &= \frac{1}{2}\partial_{(a}V_{b)} - \frac{1}{2}U_bV_a - \frac{1}{2}U_aV_b + \frac{1}{2}\eta_{ab}U^eV_e + \frac{1}{4}V_aV_b \\ &= \frac{1}{2}\partial_{(a}V_{b)} - U_{(a}V_{b)} + \frac{1}{2}\eta_{ab}U^eV_e + \frac{1}{4}V_aV_b. \end{aligned} \tag{306}$$

The last equality in the above expression is our new definition of F_{ab}, after the rescaling (we continue to call it F_{ab} to avoid clutter of notation). As we can notice from this discussion, the

[22]Here we use the same logic as before. We first find an expression for the Christoffel symbols in terms of the curvature potential; then we exploit the fact that $\tilde{g}_{ab} = A(r)\eta_{ab}$ and finally we define $U^e = \eta^{de}U_d$.

fact that our theory is conformally flat allowed us to reduce it to a flat theory by appropriate redefinitions. Of course, this does not affect the physics, i.e, the action we have obtained is equivalent to the one we started from. Let us now consider the even part of the action:

$$S_{\gamma,even} = \frac{1}{2}\int d^2x A(r)\tilde{A}^a \left[g_{ab}\left(\tilde{\Box} - F^c_c - \frac{\lambda'}{r^2}\right) - \tilde{\nabla}_a\tilde{\nabla}_b - F_{ab} - V_{[b}\tilde{\nabla}_{a]}\right]\tilde{A}^b. \tag{307}$$

By redefining $\tilde{A}_a \coloneqq \sqrt{A(r)}\mathcal{A}_a$ and by exploting the fact $\tilde{g}_{ab}$ is conformally invariant, we get

$$S_{\gamma,even} = \frac{1}{2}\int d^2x\sqrt{A(r)}\mathcal{A}^a \left[\eta_{ab}\left(\eta^{cd}\tilde{\nabla}_c\tilde{\nabla}_d - \eta^{cd}F_{cd} - A(r)\frac{\lambda'}{r^2}\right)\right.$$
$$\left. -\tilde{\nabla}_a\tilde{\nabla}_b - F_{ab} - \frac{1}{2}V_b\tilde{\nabla}_a + \frac{1}{2}V_a\tilde{\nabla}_b\right]\frac{1}{\sqrt{A(r)}}\mathcal{A}^b. \tag{308}$$

Therefore, we need to consider the action of the operator in the middle on $\mathcal{A}^b/\sqrt{A(r)}$; let us first work on the action of the double covariant derivative on such quantity[23]. We have:

$$\begin{aligned}\nabla_c\nabla_d\left(\frac{1}{\sqrt{A}}\mathcal{A}^b\right) &= \nabla_c\left[\left(\nabla_d\frac{1}{\sqrt{A}}\right)\mathcal{A}^b + \frac{1}{\sqrt{A}}\nabla_d\mathcal{A}^b\right] \\ &= \nabla_c\left[\left(\partial_d\frac{1}{\sqrt{A}}\right)\mathcal{A}^b\right] + \nabla_c\left[\frac{1}{\sqrt{A}}\nabla_d\mathcal{A}^b\right] \\ &= -\nabla_c\left[\frac{1}{\sqrt{A}}U_d\mathcal{A}^b\right] + \nabla_c\left[\frac{1}{\sqrt{A}}\nabla_d\mathcal{A}^b\right].\end{aligned} \tag{309}$$

By simply applying the product rule, we get

$$\nabla_c\nabla_d\left(\frac{1}{\sqrt{A}}\mathcal{A}^b\right) = -\left(\partial_c\frac{1}{\sqrt{A}}\right)U_d\mathcal{A}^b - \frac{1}{\sqrt{A}}\left(\nabla_cU_d\right)\mathcal{A}^b - \frac{1}{\sqrt{A}}U_d\nabla_c\mathcal{A}^b$$
$$+\left(\partial_c\frac{1}{\sqrt{A}}\right)\nabla_d\mathcal{A}^b + \frac{1}{\sqrt{A}}\nabla_c\nabla_d\mathcal{A}^b. \tag{310}$$

By explicitly writing down the action of the covariant derivative on $\mathcal{A}^b$, and recalling that

$$\nabla_c\nabla_d\mathcal{A}^b = \partial_c\left(\nabla_d\mathcal{A}^b\right) + \Gamma^b_{ce}\nabla_d\mathcal{A}^e - \Gamma^e_{cd}\nabla_e\mathcal{A}^b, \tag{311}$$

we obtain the following expression:

$$\nabla_c\nabla_d\left(\frac{1}{\sqrt{A}}\mathcal{A}^b\right) = \frac{1}{\sqrt{A}}\left[U_cU_d\mathcal{A}^b - \left(\partial_cU_d\right)\mathcal{A}^b + \Gamma^e_{cd}U_e\mathcal{A}^b - U_d\partial_c\mathcal{A}^b - U_d\Gamma^b_{ce}\mathcal{A}^e\right.$$
$$- U_c\partial_d\mathcal{A}^b - U_c\Gamma^b_{de}\mathcal{A}^e + \partial_c\partial_d\mathcal{A}^b + \left(\partial_c\Gamma^b_{de}\right)\mathcal{A}^e + \Gamma^b_{de}\partial_c\mathcal{A}^e$$
$$\left. + \Gamma^b_{ce}\partial_d\mathcal{A}^e + \Gamma^b_{ce}\Gamma^e_{df}\mathcal{A}^f - \Gamma^e_{cd}\partial_e\mathcal{A}^b - \Gamma^e_{cd}\Gamma^b_{ef}\mathcal{A}^f\right]. \tag{312}$$

Looking at the above equation, we can now express every term that contains a Christoffel symbol in terms of curvature potentials. This can be easily done by using the identity (304). By

[23]For the sake of simplicity, we will denote the covariant derivative on the light-cone without the tilde. There is no need to distinguish between light-cone coordinates and angular coordinates from the moment that these latter have been integrated out. Moreover, we will omit the $r-$dependence in A.

excluding those terms with Γ^e_{cd} (they do not contribute since the first piece in the even action (308) is contracted with η^{cd}), we find

$$\eta_{ab}\eta^{cd}U_d\Gamma^b_{ce}\mathcal{A}^e = \eta_{ab}\eta^{cd}U_c\Gamma^b_{de}\mathcal{A}^e = -\eta_{ab}\eta^{cd}U_cU_d\mathcal{A}^b, \tag{313}$$

$$\eta_{ab}\eta^{cd}\left(\partial_c\Gamma^b_{de}\right)\mathcal{A}^e = \left(\partial_a U_e\right)\mathcal{A}^e + \eta_{ab}\eta^{cd}\left(\partial_c U_d\right)\mathcal{A}^b - \left(\partial_c U_a\right)\mathcal{A}^c, \tag{314}$$

$$\eta_{ab}\eta^{cd}\Gamma^b_{de}\partial_c\mathcal{A}^e = \eta_{ab}\eta^{cd}\Gamma^b_{ce}\partial_c\mathcal{A}^e = U_e\partial_a\mathcal{A}^e + \eta_{ab}\eta^{cd}U_d\partial_c\mathcal{A}^b - U_a\partial_c\mathcal{A}^c. \tag{315}$$

Furthermore, we have that $\eta^{cd}\Gamma^b_{ce}\Gamma^e_{df} = 0$. Therefore, putting it all together, namely by inserting Eqs. (313), (314) and (315) in (312) and contracting with the two metrics, we end up with

$$\eta_{ab}\eta^{cd}\nabla_c\nabla_d\left(\frac{1}{\sqrt{A}}\mathcal{A}^b\right) = \frac{1}{\sqrt{A}}\left[\eta_{ab}\left(\partial^2 - U_cU^c\right) + 2U_b\partial_a - 2U_a\partial_b\right]\mathcal{A}^b. \tag{316}$$

Now, from Eq. (312) we can immediately deduce that

$$\begin{aligned}\nabla_a\nabla_b\left(\frac{1}{\sqrt{A}}\mathcal{A}^b\right) = \frac{1}{\sqrt{A}}\Big[&U_aU_b\mathcal{A}^b - \left(\partial_a U_b\right)\mathcal{A}^b + \Gamma^e_{ab}U_e\mathcal{A}^b - U_b\partial_a\mathcal{A}^b - U_b\Gamma^b_{ae}\mathcal{A}^e \\ &- U_a\partial_b\mathcal{A}^b - U_a\Gamma^b_{be}\mathcal{A}^e + \partial_a\partial_b\mathcal{A}^b + \left(\partial_a\Gamma^b_{be}\right)\mathcal{A}^e + \Gamma^b_{be}\partial_a\mathcal{A}^e \\ &+ \Gamma^b_{ae}\partial_b\mathcal{A}^e + \Gamma^b_{ae}\Gamma^e_{bf}\mathcal{A}^f - \Gamma^e_{ab}\partial_e\mathcal{A}^b - \Gamma^e_{ab}\Gamma^b_{ef}\mathcal{A}^f\Big].\end{aligned} \tag{317}$$

The third and fifth term, as well as the last four terms of the above expression cancel each other. The ones that contain Γ^b_{be} can be written as

$$U_a\Gamma^b_{be}\mathcal{A}^e = 2U_aU_b\mathcal{A}^b, \tag{318}$$

$$\left(\partial_a\Gamma^b_{be}\right)\mathcal{A}^e = 2\left(\partial_a U_b\right)\mathcal{A}^b, \tag{319}$$

$$\Gamma^b_{be}\partial_a\mathcal{A}^e = 2U_b\partial_a\mathcal{A}^b. \tag{320}$$

Therefore, Eq. (317) becomes

$$\nabla_a\nabla_b\left(\frac{1}{\sqrt{A}}\mathcal{A}^b\right) = \frac{1}{\sqrt{A}}\left[-U_aU_b + \partial_a U_b + U_b\partial_a - U_a\partial_b + \partial_a\partial_b\right]\mathcal{A}^b. \tag{321}$$

The last two terms in (308) can be easily written in terms of the two vector potentials we have defined. By following the same logic as before, for the first of these two we have

$$\begin{aligned}\frac{1}{2}V_b\nabla_a\left(\frac{1}{\sqrt{A}}\mathcal{A}^b\right) &= \frac{1}{2}V_b\left[\nabla_a\left(\frac{1}{\sqrt{A}}\right)\mathcal{A}^b + \frac{1}{\sqrt{A}}\nabla_a\mathcal{A}^b\right] \\ &= \frac{1}{2\sqrt{A}}V_b\left[-U_a\mathcal{A}^b + \partial_a\mathcal{A}^b + \left(\delta^b_aU_e + \delta^b_eU_a - \eta_{ae}U^b\right)\mathcal{A}^b\right] \\ &= \frac{1}{2\sqrt{A}}\left(-V_bU_a + V_b\partial_a + V_aU_b + V_bU_a - \eta_{ab}V_cU^c\right)\mathcal{A}^b.\end{aligned} \tag{322}$$

Similarly, we find that the second term is given by

$$\frac{1}{2}V_a\nabla_b\left(\frac{1}{\sqrt{A}}\mathcal{A}^b\right) = \frac{1}{2\sqrt{A}}\left(V_a\partial_b + V_aU_b\right). \tag{323}$$

Now, considering that the two terms above are opposite to each other in (308), we finally obtain

$$-\frac{1}{2}V_b\nabla_a\left(\frac{1}{\sqrt{A}}\mathcal{A}^b\right)+\frac{1}{2}V_a\nabla_b\left(\frac{1}{\sqrt{A}}\mathcal{A}^b\right)=\frac{1}{\sqrt{A}}\left(-V_{[b}\partial_{a]}+\frac{1}{2}\eta_{ab}V_cU^c\right)\mathcal{A}^b. \tag{324}$$

The last term we need to analyze before putting it all together is $\eta^{cd}F_{cd}$, where we recall that F_{cd} has been defined in Eq. (306). By simply contracting with the metric, we obtain

$$F^c_c=\frac{1}{2}\partial_cV^c+\frac{1}{4}V_cV^c. \tag{325}$$

The even part of the action can thus be written as

$$S_{\gamma,even}=\frac{1}{2}\int d^2x\mathcal{A}^a\left[\eta_{ab}\left(\partial^2-U_cU^c+\frac{1}{2}V_cU^c-\frac{1}{2}\partial_cV^c-\frac{1}{4}V_cV^c-A(r)\frac{\lambda'}{r^2}\right)\right.$$
$$\left.+2U_{[b}\partial_{a]}+U_aU_b-\partial_aU_b-\partial_a\partial_b-V_{[b}\partial_{a]}-F_{ab}\right]\mathcal{A}^b. \tag{326}$$

To sum up, the total photon action S_γ is now given by

$$S_\gamma=S_{\gamma,even}+S_{\gamma,odd}=\frac{1}{2}\int d^2x\mathcal{A}^a\Delta^{-1}_{ab}\mathcal{A}^b+\frac{1}{2}\int d^2x\mathcal{A}\Delta^{-1}\mathcal{A}, \tag{327}$$

where the operators after the rescaling have been defined as

$$\Delta^{-1}_{ab}:=\eta_{ab}\left(\partial^2-U_cU^c+\frac{1}{2}V_cU^c-\frac{1}{2}\partial_cV^c-\frac{1}{4}V_cV^c-A(r)\frac{\lambda'}{r^2}\right)$$
$$+2U_{[b}\partial_{a]}+U_aU_b-\partial_aU_b-\partial_a\partial_b-V_{[b}\partial_{a]}-F_{ab}, \tag{328}$$

$$\Delta^{-1}:=\lambda'\partial^2+A(r)\frac{\lambda'(1-\lambda')}{r^2}-2\lambda'F^a_a-\lambda'V^b\partial_b. \tag{329}$$

The attentive reader will not fail to notice that in the previous two sections all factors of $(-1)^m$ have been neglected; they arise when one uses the well-known relation $Y_{lm}=(-1)^mY^*_{l(-m)}$, which in turn is needed in order to make use of the orthogonality relation for complex spherical harmonics. These factors can easily be reintroduced once we will obtain the final expression for the propagators. However, keeping track of these m-dependent factors will become trickier when starting to compute scattering amplitudes; the way out is to consider a real basis of spherical harmonics $X_{lm}:S^2\to\mathbb{R}$, which can be defined in terms of the Y_{lm}'s in the following way (for further details the reader can refer to Ref. [40]):

$$X_{lm}=\begin{cases}\sqrt{2}(-1)^m\mathrm{Im}\left[Y_{l|m|}\right] & \text{if } m<0,\\ Y_{l0} & \text{if } m=0,\\ \sqrt{2}(-1)^m\mathrm{Re}\left[Y_{lm}\right] & \text{if } m>0,\end{cases} \tag{330}$$

where Im and Re denote the real and imaginary parts, respectively. Most of the properties of the real spherical harmonics can be quite easily deduced from the properties of the complex spherical harmonics. In particular, here the one of interest is the orthogonality relation

$$\int d\Omega X_{lm}(\theta,\phi)X_{l'm'}(\theta,\phi)=\delta_{ll'}\delta_{mm'}. \tag{331}$$

Moreover, from (330) it is not difficult to see that the real spherical harmonics also satisfy (145). Starting from the X_{lm}'s, it is possible to define a set of three vector spherical harmonics in a similar way as before. We denote this set as $\{\mathbf{X}_{lm}, \mathbf{\Psi}^r_{lm}, \mathbf{\Phi}^r_{lm}\}$, where the superscript "r" stands for "real". In this case, the orthogonality relation we want to use is

$$\int d\Omega g^{AB} \left(\Phi^r_{lm}\right)_A \left(\Phi^r_{l'm'}\right)_B = \lambda' \delta_{ll'} \delta_{mm'}. \tag{332}$$

The next step would be to expand the vector potential into real vector spherical harmonics, obtaining two expressions (odd- and parity-modes) analogous to (185) and (188):

$$A^-_B = -\sum\nolimits_{l,m} A_{lm,2} \epsilon_B{}^C \partial_C X_{lm}, \tag{333}$$

$$A^+_a = \sum\nolimits_{l,m} A_{lm,a} X_{lm}. \tag{334}$$

Now, by repeating the same computations of the previous sections, we would end up with the same results as above, but with no m-dependent factors. Therefore, from now on we can simply consider Eqs. (328) and (329) as the exact results. As we can see, we have successfully reduced the four-dimensional Schwarschild spacetime to a flat two-dimensional Minkowski spacetime; this has been possible thanks to the high degree of symmetry of the background and the spherical-harmonics expansion. Moreover, it is important to stress that, up to now, the procedure we have followed is exact: we have not lost any piece of information; we can in principle find the photon propagator on the Schwarzschild background starting from the action (327). However, even if we are working in flat spacetime now, it is still challenging to invert the operators given by (328) and (329). Luckily, in this work we are only interested in a particular region of the Schwarzschild spacetime, namely the event horizon. Indeed, this is exactly the region where high-energy scattering processes take place. We can thus build our quantum field theory by restricting our attention to this region; in particular, our aim is to approximate the above operators in order to be able to invert them. The next subsection will be devoted to what we call the "near-horizon approximation".

4.6 Physics near the horizon

As already anticipated in the previous subsection, it is difficult to find the photon propagator on the entire spacetime; thus, we will restrict to the near-horizon region. To do this, we first rewrite the quantities defining (328) and (329) in terms of the coordinates U and V [19]

$$V_a = \frac{A}{rR} x_a, \tag{335}$$

$$U_a = -\frac{A}{4rR}\left(1 + \frac{r}{R}\right) x_a, \tag{336}$$

$$\partial_a V_b = \frac{A}{rR}\eta_{ab} - \frac{A^2}{2R^2 r^2}\left(2 + \frac{r}{R}\right) x_a x_b, \tag{337}$$

$$\partial_a U_b = -\frac{A}{4rR}\left(1 + \frac{r}{R}\right)\eta_{ab} + \frac{A^2}{8R^2 r^2}\left(2 + 2\frac{r}{R} + \frac{r^2}{R^2}\right) x_a x_b, \tag{338}$$

$$F_{ab} = \frac{AR}{2r^3}\eta_{ab}, \tag{339}$$

where we recall that the Schwarzschild background is specified by

$$A(r) = \frac{R}{r} e^{1-\frac{r}{R}}, \quad UV = 2R^2 \left(1 - \frac{r}{R}\right) e^{\frac{r}{R}-1}. \tag{340}$$

The approximation of interest, keeping the spherical nature of the horizon intact, is such that the light-cone coordinates are much smaller than the Schwarzschild radius: $U, V \ll R$[24]. Indeed, we expect interactions to occur where the rays naturally accumulate[25] and not near an arbitrary point on the future horizon where the density of modes is sparse. From the second of (340), one can deduce that the radial coordinate can be written as [19]

$$r = R + R\mathcal{O}\left(\frac{UV}{R^2}\right). \tag{341}$$

Thus, to linear order, we can simply write $r = R$, $A(r) = 1$. Consequently, Eqs. (335)–(339) can be approximated in the following way:

$$V_a \approx \mu^2 x_a, \tag{342}$$

$$U_a \approx -\frac{\mu^2}{2} x_a, \tag{343}$$

$$\partial_a V_b \approx \mu^2 \eta_{ab}, \tag{344}$$

$$\partial_a U_b \approx -\frac{\mu^2}{2} \eta_{ab}, \tag{345}$$

$$F_{ab} \approx \frac{\mu^2}{2} \eta_{ab}. \tag{346}$$

We can now finally insert these approximated quantities into the quadratic operators defined by (328) and (329). The first one, (328), gives

$$\begin{aligned}
\Delta^{-1}_{ab,hor} = {} & \eta_{ab} \left(\partial^2 - \frac{1}{2}\eta^{cd}\mu^2\eta_{cd} - \mu^2\lambda'\right) + 2\left(-\frac{1}{2}\frac{\mu^2}{2}x_b\partial_a + \frac{1}{2}\frac{\mu^2}{2}x_a\partial_b\right) \\
& + \frac{\mu^2}{2}\eta_{ab} - \partial_a\partial_b - \left(\frac{1}{2}\mu^2 x_b\partial_a - \frac{1}{2}\mu^2 x_a\partial_b\right) - \frac{\mu^2}{2}\eta_{ab} \\
= {} & \eta_{ab}\left(\partial^2 - \mu^2 - \mu^2\lambda'\right) + \mu^2 x_{[a}\partial_{b]} + \frac{\mu^2}{2}\eta_{ab} - \partial_a\partial_b + \mu^2 x_{[a}\partial_{b]} - \frac{\mu^2}{2}\eta_{ab} \\
= {} & \eta_{ab}\left[\partial^2 - \mu^2\left(\lambda' + 1\right)\right] + 2\mu^2 x_{[a}\partial_{b]} - \partial_a\partial_b,
\end{aligned} \tag{347}$$

where we implicitly defined $\Delta^{-1}_{ab,hor} := \Delta^{-1}_{ab}\big|_{U,V \ll R}$. Concerning the second one, (329), we get

$$\Delta^{-1}_{hor} = \lambda'\partial^2 + \mu^2\lambda'(1-\lambda') - 2\mu^2\lambda' - \mu^2\lambda' x^b\partial_b = \lambda'\partial^2 - \mu^2\lambda'\left(\lambda'+1\right) - \mu^2\lambda' x^b\partial_b, \tag{348}$$

where $\Delta^{-1}_{hor} := \Delta^{-1}\big|_{U,V \ll R}$. Let us focus for a moment on the last term in the above expression. When inserted into the action (327), it can be rewritten in the following way:

$$\mu^2\lambda' \int d^2x \mathcal{A} x^b \partial_b \mathcal{A} = \frac{1}{2}\mu^2\lambda' \int d^2x x^b \partial_b \mathcal{A}^2. \tag{349}$$

[24]Recall that, in terms of the light-cone coordinates, the horizon is defined by the equation $UV = 0$.

[25]In Fig. 9 of Ref. [19], all wave fronts received on future null infinity appear to emanate from the central causal diamond in a collapsing scenario.

Now, integration by parts with vanishing boundary conditions yields

$$\frac{1}{2}\mu^2\lambda' \int d^2x x^b \partial_b \mathcal{A}^2 = -\frac{1}{2}\mu^2\lambda' \int d^2x \mathcal{A}^2 \partial_b x^b = -\mu^2\lambda' \int d^2x \mathcal{A}^2. \tag{350}$$

We have thus obtained another mass term. Putting it all together, Eq. (348) becomes

$$\Delta^{-1}_{hor} = \lambda' \left(\partial^2 - \mu^2\lambda'\right). \tag{351}$$

We finally write down the total photon action within the near-horizon approximation:

$$S_{\gamma,hor} = S^{even}_{\gamma,hor} + S^{odd}_{\gamma,hor} = \frac{1}{2}\int d^2x \mathcal{A}^a \Delta^{-1}_{ab,hor}\mathcal{A}^b + \frac{1}{2}\int d^2x \mathcal{A}\Delta^{-1}_{hor}\mathcal{A}, \tag{352}$$

where the quadratic operators are given by

$$\Delta^{-1}_{ab,hor} = \eta_{ab}\left[\partial^2 - \mu^2\left(\lambda'+1\right)\right] + 2\mu^2 x_{[a}\partial_{b]} - \partial_a\partial_b, \tag{353}$$

$$\Delta^{-1}_{hor} = \lambda'\left(\partial^2 - \mu^2\lambda'\right). \tag{354}$$

As we can easily notice, Eqs. (328) and (329) have been simplified significantly. In the next subsection we will try to invert (353) and (354) in order to find the photon propagator.

4.7 The photon propagator

This subsection is dedicated to the computation of the photon propagator. We will consider the even-harmonics and odd-harmonics cases separately.

4.7.1 Propagator for the even-harmonics

As a first step, we write the operator (353) in Fourier space:

$$\hat{\Delta}^{-1}_{ab,hor} = -\eta_{ab}\left[k^2 + \mu^2\left(\lambda'+1\right)\right] + k_a k_b + 2\mu^2 k_{[a}\partial^k_{b]}. \tag{355}$$

To obtain such expression, we used that

$$\begin{aligned}
\int d^2x \mathcal{A}^a \left(\mu^2 x_{[a}\partial_{b]}\right)\mathcal{A}^b &= \int d^2x \int \frac{d^2k}{(2\pi)^2}\int \frac{d^2k'}{(2\pi)^2}\hat{\mathcal{A}}^a(k)\hat{\mathcal{A}}^b\left(k'\right) e^{ik\cdot x}\left(2\mu^2 x_{[a}\partial_{b]}\right)e^{ik'\cdot x}\\
&= \int d^2x \int \frac{d^2k}{(2\pi)^2}\int \frac{d^2k'}{(2\pi)^2}\hat{\mathcal{A}}^a(k)\hat{\mathcal{A}}^b\left(k'\right) e^{ik\cdot x}\left(2i\mu^2 x_{[a}k'_{b]}\right)e^{ik'\cdot x}\\
&= \int d^2x \int \frac{d^2k}{(2\pi)^2}\int \frac{d^2k'}{(2\pi)^2}\hat{\mathcal{A}}^a(k)\hat{\mathcal{A}}^b\left(k'\right) e^{ik\cdot x}\left(2\mu^2 k'_{[b}\partial^{k'}_{a]}\right)e^{ik'\cdot x}\\
&= \int d^2x \int \frac{d^2k}{(2\pi)^2}\int \frac{d^2k'}{(2\pi)^2}e^{i(k+k')\cdot x}\hat{\mathcal{A}}^a(k)\left(-2\mu^2\partial^{k'}_{[a}k'_{b]}\right)\hat{\mathcal{A}}^b\left(k'\right)\\
&= \int \frac{d^2k'}{(2\pi)^2}\hat{\mathcal{A}}^a\left(-k'\right)\left(2\mu^2 k'_{[a}\partial^{k'}_{b]}\right)\hat{\mathcal{A}}^b\left(k'\right).
\end{aligned} \tag{356}$$

Above, in the second equality we simply acted with the partial derivative on the exponential. Moreover, to get to the third line, we simply rewrote x as a partial derivative with respect to k. Finally, integration by parts and antisymmetry have been used in the last two passages, respectively. In momentum space, the equation we want to solve is

$$\hat{\Delta}^{-1}_{ab,hor}\hat{\Delta}^{bc}_{hor} = \delta^c_a. \tag{357}$$

Now, Lorentz invariance suggests the following form for $\hat{\Delta}^{bc}_{hor}$:

$$\hat{\Delta}^{bc}_{hor} = A\left(k^2\right)\left(\eta^{bc} + B\left(k^2\right)k^b k^c\right), \tag{358}$$

where A and B have to be determined. Acting with $\hat{\Delta}^{-1}_{ab,hor}$ on $\hat{\Delta}^{bc}_{hor}$ gives

$$\begin{aligned}\hat{\Delta}^{-1}_{ab,hor}\hat{\Delta}^{bc}_{hor} &= -Ak^2\delta^c_a - ABk_a k^c k^2 - \mu^2 A\left(\lambda'+1\right)\delta^c_a - \mu^2 AB(\lambda'+1)k_a k^c \\ &\quad + Ak_a k^c + ABk_a k^c k^2 + 2\mu^2 k_{[a}\partial^{c]}_k A + 2\mu^2 k_{[a}\partial^k_{b]}\left(ABk^b k^c\right) \\ &= -A\left[k^2 + \mu^2\left(\lambda'+1\right)\right]\delta^c_a - A\left[B\mu^2\left(\lambda'+1\right) - 1\right]k_a k^c \\ &\quad + 2\mu^2\left[k_{[a}\partial^{c]}_k A + k_{[a}\partial^k_{b]}\left(ABk^b k^c\right)\right]. \end{aligned} \tag{359}$$

Let us now have a closer look at the two terms in the last square brackets. We can easily show that the first one vanishes. Indeed, we have

$$k_{[a}\partial^{c]}_k A = \frac{1}{2}k_a\partial^c_k A - \frac{1}{2}k_c\partial^a_k A = \frac{1}{2}k_a\left(2A'k^c\right) - \frac{1}{2}k_c\left(2A'k^a\right) = A'\left(k_a k^c - k_c k^a\right) = 0. \tag{360}$$

Concerning the second one, we get

$$\begin{aligned}k_{[a}\partial^k_{b]}\left(ABk^b k^c\right) &= ABk_{[a}\delta^b_{b]}k^c + ABk_{[a}\delta^c_{b]}k^b + k^b k^c k_{[a}\partial^k_{b]}(AB) \\ &= \frac{1}{2}FG\left(k_a k^c\delta^b_b - k_b k^c\delta^b_a + k_a k^b\delta^c_b - k^2\delta^c_a\right) \\ &= ABk_a k^c - \frac{1}{2}ABk^2\delta^c_a. \end{aligned} \tag{361}$$

Given the above, Eq. (357) turns out to be

$$\hat{\Delta}^{-1}_{ab,hor}\hat{\Delta}^{bc}_{hor} = -A\left[k^2 + \mu^2\left(\lambda'+1\right) + \mu^2 Bk^2\right]\delta^c_a - Ak_a k^c\left[B\mu^2\left(\lambda'-1\right) - 1\right] = \delta^c_a, \tag{362}$$

which can be immediately solved for A and B:

$$A = -\frac{1}{k^2\left(1+\frac{1}{\lambda'-1}\right) + \mu^2\left(\lambda'+1\right)}, \quad B = \frac{1}{\mu^2\left(\lambda'-1\right)}. \tag{363}$$

We finally obtained the photon propagator for the even-harmonics:

$$\hat{\Delta}^{ab}_{hor}(k) := \mathcal{P}^{ab}_{even} = -\frac{\lambda'-1}{\lambda'}\frac{1}{k^2 + \mu^2\left(\lambda'+1\right)\frac{\lambda'-1}{\lambda'} - i\epsilon}\left[\eta^{ab} + \frac{k^a k^b}{\mu^2\left(\lambda'-1\right)}\right]. \tag{364}$$

As we can notice, the shape of the above propagator is the one of a massive spin-1 particle. A final remark concerns the case $l = 0$. For this value of l, the odd-mode A^-_μ contains no degrees of freedom, signalling an additional gauge redundancy in the even sector. Thus, Eq. (364) is only valid for $l \geq 1$. In the next paragraph we will focus on the monopole-mode case.

4.7.2 A special case: the monopole mode

Before fixing any gauge, let us specialize the even-parity mode, namely the second expression in (177), to the case $l = 0$. We have[26]:

$$A_\mu^+ = \sum_{l,m} \begin{bmatrix} A_{lm,U}(U,V) \\ A_{lm,V}(U,V) \\ A_{lm,1}(U,V)\partial_\theta \\ A_{lm,1}(U,V)\partial_\phi \end{bmatrix} X_{lm} \xrightarrow{l=0} A_\mu^+|_{l=0} = \begin{bmatrix} A_{00,U}(U,V) \\ A_{00,V}(U,V) \\ 0 \\ 0 \end{bmatrix} X_{00} = A_{00,a}X_{00}. \tag{365}$$

Since we are dealing with a spin one field, a convenient choice of gauge to fix the redundant degree of freedom is a Lorenz gauge. In its covariant form, it reads

$$\nabla_\mu A^\mu = 0. \tag{366}$$

However, this does not fully specify the gauge condition because one may still transform the vector potential. We therefore impose the following condition:

$$\nabla_\mu A'^\mu = \nabla_\mu \left(A^\mu + \nabla^\mu \Lambda\right) = 0. \tag{367}$$

Above, Λ is an arbitrary function of the coordinates U and V[27]. In order to start, we compute the action of the covariant derivative on the vector potential A^μ, obtaining

$$\begin{aligned}
\nabla_\mu A^\mu &= \partial_\mu A^\mu + \Gamma^\mu_{\mu\rho}A^\rho \\
&= \partial_a A^a + \partial_A A^A + \Gamma^\mu_{\mu a}A^a + \Gamma^\mu_{\mu A}A^A \\
&= \partial_a A^a + \Gamma^b_{bU}A^U + \Gamma^b_{bV}A^V + \Gamma^A_{AU}A^U + \Gamma^A_{AV}A^V \\
&= \partial_a A^a + \Gamma^U_{UU}A^U + \Gamma^V_{VV}A^V + \Gamma^\theta_{\theta U}A^U + \Gamma^\phi_{\phi U}A^U + \Gamma^\theta_{\theta V}A^V + \Gamma^\phi_{\phi V}A^V \\
&= \partial_a A^a + \left(\partial_U \log A(r) + 2\partial_U \log r\right)A^U + \left(\partial_V \log A(r) + 2\partial_V \log r\right)A^V.
\end{aligned} \tag{368}$$

Moreover, from the expressions in (340), we deduce that

$$\partial_a \log r = \frac{1}{r}\partial_a r \propto \frac{x_a}{rR}, \tag{369}$$

$$\partial_a \log A(r) = \frac{1}{A(r)}\partial_a A(r) \propto \frac{x_a}{R}\left(\frac{1}{r} + \frac{1}{R}\right), \tag{370}$$

where we introduced the two-vector x^a, with the two components given by $x^U = U$ and $x^V = V$, respectively. Now, within the near-horizon approximation, i.e., $x_a \ll R$, we immediately notice that (369) and (370) give a negligible contribution. Thus, the near-horizon version of Eq. (366) reduces to the following simple expression:[28]:

$$\nabla_\mu A^\mu\big|_{x_a \ll R} = \partial_a \mathcal{A}^a = 0. \tag{371}$$

[26] We recall that, when writing down (177), we were still working with the complex spherical harmonics. Here we consider the real basis, as explained at the end of section 5.5.

[27] In the following we will determine which equation Λ needs to satisfy.

[28] Notice that we implicitly rescaled the field as we did in section 5.5.

Next, we explicitly compute the second term in Eq. (367), giving

$$\begin{aligned}\nabla_\mu \nabla^\mu \Lambda &= \nabla^\mu \partial_\mu \Lambda \\ &= g^{\mu\nu} \nabla_\nu \partial_\mu \Lambda \\ &= g^{\mu\nu} \left(\partial_\nu \partial_\mu \Lambda - \Gamma^\rho_{\mu\nu} \partial_\rho \Lambda\right) \\ &= \partial_a \partial^a \Lambda + \partial_A \partial^A \Lambda - g^{\mu\nu} \Gamma^a_{\mu\nu} \partial_a \Lambda - g^{\mu\nu} \Gamma^A_{\nu\mu} \partial_A \Lambda \\ &= \partial_a \partial^a \Lambda + \partial_A \partial^A \Lambda - g^{b\nu} \Gamma^a_{b\nu} \partial_a \Lambda - g^{A\nu} \Gamma^a_{A\nu} \partial_a \Lambda - g^{b\nu} \Gamma^A_{b\nu} \partial_A \Lambda - g^{B\nu} \Gamma^A_{B\nu} \partial_A \Lambda. \end{aligned} \tag{372}$$

Moreover, by recalling that $\Gamma^A_{ab} = \Gamma^a_{bA} = 0$, we end up with

$$\nabla_\mu \nabla^\mu \Lambda = \partial_a \partial^a \Lambda + \partial_A \partial^A \Lambda - g^{AB} \Gamma^a_{AB} \partial_a \Lambda - g^{BC} \Gamma^A_{BC} \partial_A \Lambda = \partial_a \partial^a - g^{AB} \Gamma^a_{AB} \partial_a \Lambda, \tag{373}$$

where the last equality follows from the fact that $\Lambda = \Lambda(U, V)$, as already specified before. Now, by making use of Eq. (212), we get

$$\nabla_\mu \nabla^\mu \Lambda = \partial_a \partial^a \Lambda - g^{AB} \Gamma^a_{AB} \partial_a \Lambda = \partial_a \partial^a \Lambda - g^{BC} \left(-\frac{1}{2} g_{AB} V^a\right) \partial_a \Lambda = \partial^2 \Lambda + V^a \partial_a \Lambda, \tag{374}$$

which, when inserted in (367), finally give

$$\partial^2 \Lambda + V^a \partial_a \Lambda = -\partial_a \mathcal{A}^a. \tag{375}$$

The function Λ can thus be found by solving the above equation. The action for the monopole mode can be quickly obtained from (352) by setting $l = 0$:

$$S^{even}_{\gamma,hor}\big|_{l=0} = \frac{1}{2} \int d^2x \mathcal{A}^a \left(\eta_{ab} \partial^2 - \partial_a \partial_b + \mu^2 x_a \partial_b - \mu^2 x_b \partial_a - \mu^2 \eta_{ab}\right) \mathcal{A}^b. \tag{376}$$

From (371) we can immediately conclude the second and third terms vanish identically. The fourth term can be integrated by parts, resulting in

$$\int d^2x \mathcal{A}^a x_b \partial_a \mathcal{A}^b = \int d^2x \mathcal{A}^a \partial_a \left(x_b \mathcal{A}^b\right) - \int d^2x \mathcal{A}^a \left(\partial_a x_b\right) \mathcal{A}^b. \tag{377}$$

Now, by noticing that $\partial_a x_b = \partial_a \left(\eta_{bc} x^c\right) = \eta_{bc} \partial_a x^c = \eta_{bc} \delta^c_a = \eta_{ab}$, and integrating by parts again, we finally obtain

$$\int d^2x \mathcal{A}^a x_b \partial_a \mathcal{A}^b = -\int d^2x \left(\partial_a \mathcal{A}^a\right) x_b \mathcal{A}^b - \int d^2x \mathcal{A}^a \eta_{ab} \mathcal{A}^b = -\int d^2x \mathcal{A}^a \eta_{ab} \mathcal{A}^b. \tag{378}$$

In the last step, we again made use of the Lorenz-gauge condition. Eq. (376) reduces to

$$S^{even}_{\gamma,hor}\big|_{l=0} = \frac{1}{2} \int d^2x \mathcal{A}^a \eta_{ab} \partial^2 \mathcal{A}^b. \tag{379}$$

In momentum space, the propagator is then found to be

$$\mathcal{P}^{ab}_{even\_0} = -\frac{\eta^{ab}}{k^2 - i\epsilon}. \tag{380}$$

4.7.3 Propagator for the odd-harmonics

In this case, the equation we want to solve is

$$\lambda' \left(\partial^2 - \mu^2\lambda'\right) \Delta_{hor}\left(x, x'\right) = \delta^{(2)}\left(x - x'\right), \tag{381}$$

where we directly inserted the operator (354). The strategy we adopt is exactly the same as before; we perform a Fourier transformation, namely we write

$$\Delta_{hor}\left(x, x'\right) = \int \frac{d^2k}{(2\pi)^2} e^{ik_a(x-x')^a} \hat{\Delta}_{hor}(k), \tag{382}$$

$$\delta^{(2)}\left(x - x'\right) = \int \frac{d^2k}{(2\pi)^2} e^{ik_a(x-x')^a}, \tag{383}$$

ending up with the following expression:

$$\hat{\Delta}_{hor}(k) := \mathcal{P}_{odd} = -\frac{1}{\lambda'}\frac{1}{k^2 + \mu^2\lambda' - i\epsilon}. \tag{384}$$

It is worth noting that the operator (354) vanishes identically for $l = 0$.

4.8 The scalar propagator

In this subsection our goal is to find the propagator for the scalar field by proceeding in exactly the same way as we did for the photon propagator. The procedure will be of course much simpler due to the scalar nature of the object in question. The kinetic term in (132) is given by

$$S_M^{kin} := \int d^4x\sqrt{-g}\phi^*\Box\phi, \tag{385}$$

where, for the sake of simplicity, we temporarily neglected the mass of the scalar field. The action of the box operator on ϕ is simply given by

$$\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = g^{\mu\nu}\nabla_\mu\partial_\nu\phi = g^{\mu\nu}\left(\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho\right)\phi. \tag{386}$$

Setting $\mu = a, \nu = b$, we get

$$g^{ab}\nabla_a\nabla_b\phi = g^{ab}\left(\partial_a\partial_b - \Gamma^c_{ab}\partial_c - \Gamma^C_{ab}\partial_C\right)\phi = g^{ab}\left(\partial_a\partial_b - \Gamma^c_{ab}\partial_c\right) = \tilde{\Box}\phi. \tag{387}$$

Then, setting $\mu = A, \nu = B$ gives

$$\begin{aligned} g^{AB}\nabla_A\nabla_B\phi &= g^{AB}\left(\partial_A\partial_B - \Gamma^C_{AB}\partial_C - \Gamma^a_{AB}\partial_a\right)\phi \\ &= \frac{1}{r^2}\Delta_\Omega\phi - g^{AB}\left(-\frac{1}{2}g_{AB}V^a\partial_a\right)\phi \\ &= \frac{1}{r^2}\Delta_\Omega\phi + V^a\tilde{\nabla}_a\phi. \end{aligned} \tag{388}$$

Therefore, the box operator can be compactly written as

$$\Box = \tilde{\Box} + V^a\tilde{\nabla}_a + \frac{1}{r^2}\Delta_\Omega. \tag{389}$$

Expanding ϕ in spherical harmonics, using that $\Delta_\Omega X_{lm}(\theta,\phi) = -l(l+1)X_{lm}(\theta,\phi)$ as well as the orthogonality relation, leads to the following action:

$$S_M^{kin} = \sum\nolimits_{l,m} \int d^2x A(r) r^2 \phi^*_{lm} \left(\tilde{\Box} + V^a \tilde{\nabla}_a - \frac{\lambda'}{r^2} \right) \phi_{lm}. \tag{390}$$

Now, it is possible to follow the same procedure explained in subsections 5.4 and 5.5. We have:

$$S_M^{kin} \xrightarrow{(280),(289),\ \varphi_{lm}:=r\phi_{lm}} S_M^{kin} = \sum\nolimits_{l,m} \int d^2x A(r) \varphi^*_{lm} \left(\tilde{\Box} - F^a_a - \frac{\lambda'}{r^2} \right) \varphi_{lm}. \tag{391}$$

The action of the box operator on φ_{lm} is given by

$$\begin{aligned} \tilde{\Box}\varphi_{lm} = \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\varphi_{lm} &= \frac{1}{A(r)}\eta^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\varphi_{lm} \\ &= \frac{1}{A(r)}\eta^{ab}\left(\partial_a\partial_b - \Gamma^c_{ab}\partial_c\right)\varphi_{lm} \\ &= \frac{1}{A(r)}\eta^{ab}\partial_a\partial_b\varphi_{lm} \\ &= \frac{1}{A(r)}\partial^2\varphi_{lm}, \end{aligned} \tag{392}$$

while from the definition of the tensor F_{ab} we get

$$F^a_a = \tilde{g}^{ab}F_{ab} = \tilde{g}^{ab}\frac{1}{r}\tilde{\nabla}_a\tilde{\nabla}_b r = \frac{1}{A(r)}\frac{1}{r}\eta^{ab}\left(\partial_a\partial_b r - \Gamma^c_{ab}\partial_c r\right) = \frac{1}{A(r)}\frac{1}{r}\partial^2 r. \tag{393}$$

Putting it all together, and taking into account the near-horizon approximation, results in

$$S_{M,hor}^{kin} = \sum\nolimits_{l,m} \int d^2x \varphi^*_{lm} \left[\partial^2 - \mu^2\left(\lambda' + 1\right)\right] \varphi_{lm}, \tag{394}$$

where the following relation has been used[29]:

$$\partial^2 r\big|_{hor} = \frac{1}{R} := \mu. \tag{395}$$

Performing a Fourier transformation finally leads to

$$\mathcal{P}_\varphi = -\frac{1}{p^2 + \mu^2\left(\lambda' + 1\right) + m^2 - i\epsilon}, \tag{396}$$

where we reinserted the mass of the scalar field.

4.9 Feynman vertices

In the previous two subsections we computed all the propagators we need, namely the photon propagator (both for even- and odd-harmonics) and the one for the complex scalar field. In this subsection we will deal with the interaction vertices. As we can notice from the action (132), the scattering process of interest are mediated by both three- and four-vertices. We will treat the two cases separately in the next two paragraphs. Once again, we follow the same procedure;

[29] Such relation can be easily deduced from the second expressions in (340)

we first expand the fields in spherical harmonics and integrate the sphere out, reducing our four-dimensional theory to a two-dimensional one. Then, we rescale the fields and take into account the horizon approximation introduced before. However, as we will see, integrating the sphere out is not trivial in this case. We will therefore rely on what we call the “spherical-symmetry approximation". Let us start with the three-vertex.

4.9.1 Three-point vertex

More explicitly, the three-vertex in (132) can be written as

$$\begin{aligned} S^{\{3\}} &:= -q \int d^4x \sqrt{-g} g^{\mu\nu} A_\mu j_\nu \\ &= -q \int d^4x \sqrt{-g} \left(g^{ab} A_a j_b + g^{AB} A_A j_B + g^{aB} A_a j_B + g^{Ba} A_B j_a \right). \end{aligned} \tag{397}$$

By recalling that $g^{aB} = g^{Ba} = 0$ and how the current has been previously defined, we get

$$\begin{aligned} S^{\{3\}} &= -q \int d^4x \sqrt{-g} \left(g^{ab} A_a j_b + g^{AB} A_A j_B \right) \\ &= -iq \int d^4x \sqrt{-g} \left[g^{ab} A_a \left(\phi^* \partial_b \phi - \phi \partial_b \phi^* \right) + g^{AB} A_A \left(\phi^* \partial_B \phi - \phi \partial_B \phi^* \right) \right]. \end{aligned} \tag{398}$$

Now, as already discussed before, in this work we are neglecting transverse-momentum effects[30], meaning that we can safely set $p_A = 0$ (in Fourier space). This, in turn, implies that the second term in (398) is negligible. Moreover, expanding in spherical harmonics gives

$$\begin{aligned} S^{\{3\}} &= -iq \int d^4x \sqrt{-g} g^{ab} A_a \left(\phi^* \partial_b \phi - \phi \partial_b \phi^* \right) \\ &= -iq \int d\Omega \int d^2x A(r) r^2 g^{ab} A_a \left(\phi^* \partial_b \phi - \phi \partial_b \phi^* \right) \\ &= -iq \sum_{l,m} \sum_{l_1,m_1} \sum_{l_2,m_2} I_{3X} \int d^2x A(r) r^2 g^{ab} A_{a,lm} \left(\phi^*_{l_1m_1} \partial_b \phi_{l_2m_2} - \phi_{l_2m_2} \partial_b \phi^*_{l_1m_1} \right), \end{aligned} \tag{399}$$

where the following integral has been defined:

$$I_{3X} := \int d\Omega X_{lm} X_{l_1m_1} X_{l_2m_2}. \tag{400}$$

It is certainly possible to compute the integral above and integrate the sphere out, resulting in a coupling between the various partial waves. However, it is reasonable to assume that this mixing of partial waves is a sub-leading effect when one considers a large spherically symmetric background, as in our case. Essentially, we are assuming that the effects of spherical-symmetry breaking are mild. The minimal assumption we need in order to make the action diagonal in the partial-waves indices is to consider one of the two scalar fields at $l = 0$ (we will refer to this as the “spherical-symmetry approximation"). One possible choice is to set $l_2 = m_2 = 0$, giving

$$I_{3X} = \int d\Omega X_{lm} X_{l_1m_1} X_{00} = \frac{1}{\sqrt{4\pi}} \int d\Omega X_{lm} X_{l_1m_1} = \frac{1}{\sqrt{4\pi}} \delta_{ll_1} \delta_{mm_1}, \tag{401}$$

[30]Transverse-momentum transfer is a Planckian effect.

which, in turn, leads to

$$S^{\{3\}} = -\frac{iq}{\sqrt{4\pi}} \sum\nolimits_{l,m} \int d^2x A(r) r^2 \tilde{g}^{ab} A_{a,lm} \left(\phi^*_{lm}\partial_b\phi_0 - \phi_0\partial_b\phi^*_{lm}\right) =: S^{\{3\}}_1. \tag{402}$$

Now, as already explained in the introduction to subsection 5.9, we rescale the fields and consider the near-horizon approximation. We have:

$$S^{\{3\}}_1 \xrightarrow{\text{rescaling, } x_a \ll R} -\frac{i\mu q}{\sqrt{4\pi}} \sum\nolimits_{l,m} \int d^2x \mathcal{A}^b_{lm} \left(\varphi^*_{lm}\partial_b\varphi_0 - \varphi_0\partial_b\varphi^*_{lm}\right) =: S^{\{3\}}_{1,hor}, \tag{403}$$

where $\mu = 1/R$. The second option is to set $l_1 = m_1 = 0$; in this case, we get

$$I_{3Y} = \int d\Omega X_{lm} X_{00} X_{l_2 m_2} = \frac{1}{\sqrt{4\pi}} \int d\Omega X_{lm} X_{l_2 m_2} = \frac{1}{\sqrt{4\pi}} \delta_{ll_2}\delta_{mm_2}. \tag{404}$$

By inserting the above expression into (399) finally gives

$$S^{\{3\}}_{2,hor} = -\frac{i\mu q}{\sqrt{4\pi}} \sum\nolimits_{l,m} \int d^2x \mathcal{A}^b_{lm} \left(\varphi^*_0\partial_b\varphi_{lm} - \varphi_{lm}\partial_b\varphi^*_0\right). \tag{405}$$

We have thus obtained two three-vertices for our two-dimensional effective field theory. However, one can wonder why we did not consider also the possibility of setting the partial-waves indices associated to $A_{a,lm}$ to zero. The reason is that we want to make contact with the result coming from the first-quantized picture (section 3), where all information about the ingoing particle is transferred to the outgoing particle. The only way this can happen is if the carrier (the photon in our case) carries angular momentum. In a future work it would be interesting to include this type of three-vertex in our theory.

4.9.2 Four-point vertex

We again start by looking at the action (132). We have:

$$\begin{aligned}
S^{\{4\}} &:= -q^2 \int d^4x \sqrt{-g} g^{\mu\nu} A_\mu A_\nu |\phi|^2 \\
&= -q^2 \int d^4x \left(g^{ab} A_a A_b |\phi|^2 + g^{AB} A_A A_B |\phi|^2\right) \\
&= -q^2 \int d\Omega \int d^2x A(r) r^2 g^{ab} A_a A_b |\phi|^2 - q^2 \int d\Omega \int d^2x A(r) r^2 g^{AB} A_A A_B |\phi|^2.
\end{aligned} \tag{406}$$

Expanding in spherical harmonics, the first term in the above expression gives

$$\begin{aligned}
S^{\{4\}}_{even} &:= -q^2 \int d\Omega \int d^2x A(r) r^2 g^{ab} A_a A_b |\phi|^2 \\
&= -q^2 \sum_{l,m} \sum_{l_1,m_1} \sum_{l_2,m_2} \sum_{l_3,m_3} I_{4X} \int d^2x A(r) r^2 g^{ab} A_{a,lm} A_{b,l_1 m_1} \phi_{l_2,m_2} \phi^*_{l_3,m_3},
\end{aligned} \tag{407}$$

where this time the integral over $d\Omega$ has been defined as

$$I_{4X} := \int d\Omega X_{lm} X_{l_1 m_1} X_{l_2 m_2} X_{l_3 m_3}. \tag{408}$$

We now want to consider again the spherical-symmetry approximation.The choice we make is to set the two complex scalar fields at $l = 0$[31], getting

$$I_{4X} = \int d\Omega X_{lm} X_{l_1 m_1} X_{00} X_{00} = \frac{1}{4\pi} \int d\Omega X_{lm} X_{l_1 m_1} = \frac{1}{4\pi} \delta_{ll_1} \delta_{mm_1}, \tag{409}$$

which, in turn, leads to the following expression:

$$S^{\{4\}}_{even} = -\frac{q^2}{4\pi} \sum\nolimits_{l,m} \int d^2x A(r) r^2 g^{ab} A_{a,lm} A_{b,lm} |\phi_0|^2. \tag{410}$$

As is now customary, we rescale the fields and consider the horizon approximation. We have:

$$S^{\{4\}}_{even} \xrightarrow{\text{rescaling, } x_a \ll R} -\frac{\mu^2 q^2}{4\pi} \sum\nolimits_{l,m} \int d^2x \mathcal{A}^2_{a,lm} |\phi_0|^2 =: S^{\{4\}}_{even,hor}. \tag{411}$$

The second term in (406) instead gives

$$\begin{aligned} S^{\{4\}}_{odd} &:= -q^2 \int d\Omega \int d^2x A(r) r^2 g^{AB} A_A A_B |\phi|^2 \\ &= -q^2 \sum_{l,m} \sum_{l_1,m_1} \sum_{l_2,m_2} \sum_{l_3,m_3} I_{4X\Phi} \int d^2x A(r) r^2 A_{lm,2} A_{l_1 m_1,2} \phi_{l_2 m_2} \phi^*_{l_3 m_3}, \end{aligned} \tag{412}$$

where the integral over angular coordinates has been defined as

$$I_{4X\Phi} := \int d\Omega g^{AB} \left(\Phi^r_{lm}\right)_A \left(\Phi^r_{l_1 m_1}\right)_B X_{l_2 m_2} X_{l_3 m_3}. \tag{413}$$

Now, setting $l_2 = m_2 = l_3 = m_3 = 0$ leads to[32]

$$I_{4X\Phi} = \int d\Omega g^{AB} \left(\Phi^r_{lm}\right)_A \left(\Phi^r_{l_1 m_1}\right)_B X_{00} X_{00} = \frac{\lambda'}{4\pi} \delta_{ll_1} \delta_{mm_1}. \tag{414}$$

Inserting the above expression in (412), we get

$$S^{\{4\}}_{odd} = -\lambda' \frac{q^2}{4\pi} \sum\nolimits_{l,m} \int d^2x A(r) r^2 A^2_{lm,2} |\phi_0|^2. \tag{415}$$

Finally, by rescaling $A_{lm,2}$ and considering the near-horizon limit, we end up with

$$S^{\{4\}}_{odd} \xrightarrow{\text{rescaling, } x_a \ll R} -\lambda' \frac{\mu^2 q^2}{4\pi} \sum\nolimits_{l,m} \int d^2x \mathcal{A}^2_{lm} |\varphi_0|^2 =: S^{\{4\}}_{odd,hor}. \tag{416}$$

Putting it all together, we write

$$S^{\{4\}} = -\frac{\mu^2 q^2}{4\pi} \sum\nolimits_{l,m} \int d^2x \left(\mathcal{A}^2_{a,lm} + \lambda' \mathcal{A}^2_{lm}\right) |\varphi_0|^2. \tag{417}$$

[31] In principle, other choices are possible. Differently from before, in this case there is no physical reason to not consider the possibility to set the partial-waves indices associated to $A_{a,lm}$ to zero. It would be certainly interesting to include the corresponding four-vertex in the theory. However, towards the end of the thesis, we will show that the one-loop diagram containing only four-vertices of the type obtained in this paragraph is sub-leading with respect to the one containing only three-vertices. Our expectation is that also the one-loop diagram in which the gauge field is at $l = 0$ is sub-leading. We hope to verify such expectation in a future work.

[32] In the odd-case there is no confusion about possible different choices. Indeed, we have that $\left(\Phi^r_{00}\right)_A = 0$.

4.10 Near-horizon Feynman rules for scalar quantum electrodynamics

We are now ready to write down the full near-horizon action for our two-dimensional effective field theory. In momentum space, we have[33]

$$S_{hor} := S_{\gamma}^{even} + S_{\gamma}^{odd} + S_{M}^{kin} + S_{1}^{\{3\}} + S_{2}^{\{3\}} + S_{even}^{\{4\}} + S_{odd}^{\{4\}}, \tag{418}$$

where the single terms are given by

$$S_{\gamma}^{even} = \frac{1}{2}\int d\Gamma (2\pi)^2 \delta^{(2)}(k+k')\hat{\mathcal{A}}^a(k)\mathcal{P}^{-1}_{ab,even}\hat{\mathcal{A}}^b(k') = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\hat{\mathcal{A}}^a(k)\mathcal{P}^{-1}_{ab,even}\hat{\mathcal{A}}^b(-k), \tag{419}$$

$$S_{\gamma}^{odd} = \frac{1}{2}\int d\Gamma (2\pi)^2 \delta^{(2)}(k+k')\hat{\mathcal{A}}(k)\mathcal{P}^{-1}_{odd}\hat{\mathcal{A}}(k') = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\hat{\mathcal{A}}(k)\mathcal{P}^{-1}_{odd}\hat{\mathcal{A}}(-k), \tag{420}$$

$$S_{M}^{kin} = \int d\Phi (2\pi)^2 \delta^{(2)}(p-p')\hat{\varphi}^*(p)\mathcal{P}^{-1}_{\varphi}\hat{\varphi}(p') = \int \frac{d^2p}{(2\pi)^2}\hat{\varphi}^*(p)\mathcal{P}^{-1}_{\varphi}\hat{\varphi}(p), \tag{421}$$

$$S_{1}^{\{3\}} = \int dV_3 (2\pi)^2 \delta^{(2)}(k+p_1-p_2)\hat{\mathcal{A}}^b(k)\hat{\varphi}_0(p_1)\hat{\varphi}^*(p_2)\frac{\mu q}{\sqrt{4\pi}}(p_b^1+p_b^2), \tag{422}$$

$$S_{2}^{\{3\}} = \int dV_3 (2\pi)^2 \delta^{(2)}(k+p_2-p_1)\hat{\mathcal{A}}^b(k)\hat{\varphi}_0^*(p_1)\hat{\varphi}(p_2)\frac{\mu q}{\sqrt{4\pi}}\left(p_b^1+p_b^2\right), \tag{423}$$

$$S_{even}^{\{4\}} = -\int dV_4 (2\pi)^2 \delta^{(2)}(k+k'+p_2-p_1)\hat{\mathcal{A}}^a(k)\hat{\mathcal{A}}^b(k')\hat{\varphi}_0^*(p_1)\hat{\varphi}_0(p_2)\frac{\mu^2 q^2}{4\pi}\eta_{ab}, \tag{424}$$

$$S_{odd}^{\{4\}} = -\lambda'\int dV_4 (2\pi)^2 \delta^{(2)}(k+k'+p_2-p_1)\hat{\mathcal{A}}(k)\hat{\mathcal{A}}(k')\hat{\varphi}_0^*(p_1)\hat{\varphi}_0(p_2)\frac{\mu^2 q^2}{4\pi}. \tag{425}$$

Notice that we also defined the following quantities:

$$d\Gamma := \frac{d^2k}{(2\pi)^2}\frac{d^2k'}{(2\pi)^2}, \tag{426}$$

$$d\Phi := \frac{d^2p}{(2\pi)^2}\frac{d^2p'}{(2\pi)^2}, \tag{427}$$

$$dV_3 := \frac{d^2k}{(2\pi)^2}\frac{d^2p_1}{(2\pi)^2}\frac{d^2p_2}{(2\pi)^2}, \tag{428}$$

$$dV_4 := \frac{d^2k}{(2\pi)^2}\frac{d^2k'}{(2\pi)^2}\frac{d^2p_1}{(2\pi)^2}\frac{d^2p_2}{(2\pi)^2}. \tag{429}$$

Furthermore, in order to facilitate the reading, we also summarize below the results that have been obtained concerning the propagators:

$$\mathcal{P}^{ab}_{even} = -\frac{\lambda'-1}{\lambda'}\frac{1}{k^2+\mu^2(\lambda'+1)\frac{\lambda'-1}{\lambda'}-i\epsilon}\left[\eta^{ab}+\frac{k^a k^b}{\mu^2(\lambda'-1)}\right], \tag{430}$$

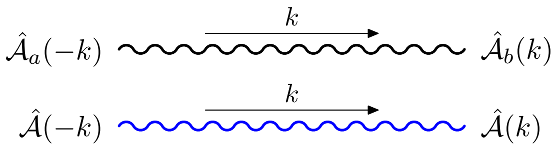
$$\mathcal{P}^{ab}_{even\_0} = -\frac{\eta^{ab}}{k^2-i\epsilon}, \tag{431}$$

$$\mathcal{P}_{odd} = -\frac{1}{\lambda'}\frac{1}{k^2+\mu^2\lambda'-i\epsilon}, \tag{432}$$

$$\mathcal{P}_{\varphi} = -\frac{1}{p^2+\mu^2(\lambda'+1)+m^2-i\epsilon}. \tag{433}$$

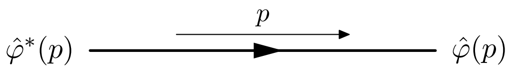
[33]Note that, for simplicity, we suppressed the partial-waves indices (and so all the sums). Moreover, we also suppressed the labels "*hor*" in the single terms since all the results we obtain are only valid in this limit.

Now, as is usually done in quantum field theory, we can read off the Feynman rules from the action written in momentum space. Concerning the photon propagators, we have



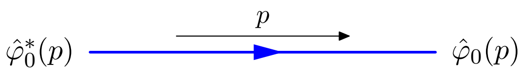
$$= i\mathcal{P}_{ab,even}(k),$$
$$= i\mathcal{P}_{odd}(k).$$

where we used different colors to distinguish between even- and odd-contributions, respectively. As for the complex scalar field, we may write



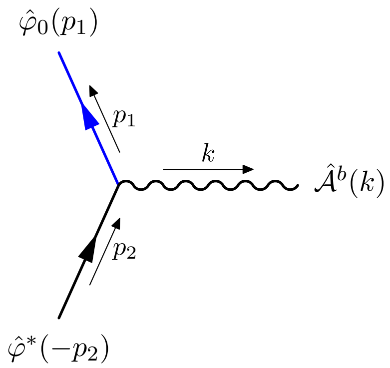
$$= i\mathcal{P}_\varphi(p).$$

Moreover, we recall that we distinguished between fields at $l = 0$ and at $l = 0$ (this is due to the spherical-symmetry approximation). Graphically, we again use different colors to make this distinction. The scalar propagator at $l = 0$ will be denoted with a solid blue line:

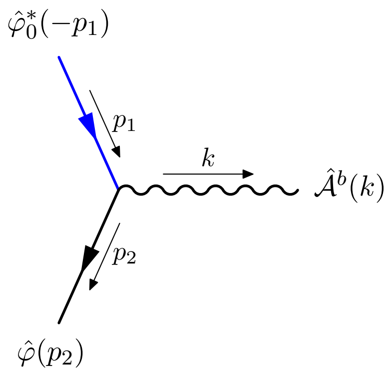


$$= i\mathcal{P}_\varphi(p)\big|_{l=0}.$$

Notice that the arrows that are outside the scalar lines indicate the direction that momentum is flowing, while the arrows superimposed on the lines corresponds to the flow of electric charge. Let us now consider the vertices. The Feynman rules for the three-point vertices are

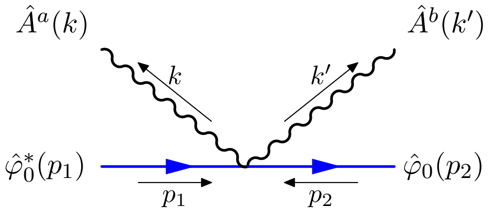


$$= i\frac{\mu q}{\sqrt{4\pi}}\left(p_b^1 + p_b^2\right),$$

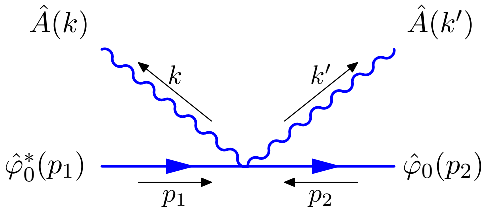


$$= i\frac{\mu q}{\sqrt{4\pi}}\left(p_b^1 + p_b^2\right).$$

Concerning the two four-vertices, we have[34]:



$$= -2i\frac{\mu^2 q^2}{4\pi}\eta_{ab},$$



$$= -2i\frac{\mu^2 q^2}{4\pi}.$$

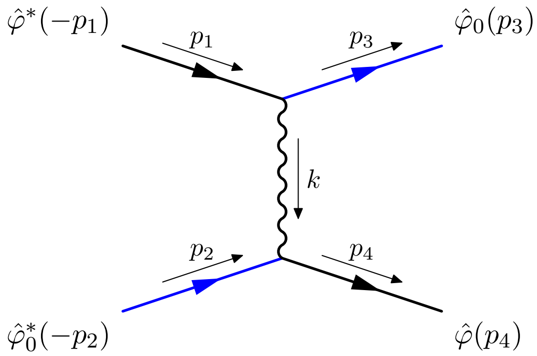
[34]Notice that, due to the fact that the two gauge fields can be interchanged, a factor of 2 is present in the two four-point vertices below. Actually, we already implicitly included all symmetry factors in the above rules. The photon propagators have been multiplied by a factor of 2 since we can always interchange the gauge fields; this cancels the factor of 1/2 coming from the photon action. As for the scalar propagator, we do not need to multiply by any factor in this case since it connects two different fields. The same is true for the three-point vertices (they consist of three different fields).

5 Scattering amplitudes on the horizon

In this section we will compute all the scattering amplitudes of interest.

5.1 Tree-level diagrams

At tree level, the diagrams we want consider are essentially three. The first one is the following:



$= i\mathcal{M}_{t,1}.$

By using the Feynman rules introduced in the previous section, we can write

$$i\mathcal{M}_{t,1} = i\frac{\mu q}{\sqrt{4\pi}}\left(p_1^a + p_3^a\right)\frac{\lambda' - 1}{\lambda'}\frac{-i}{k^2 + \mu^2\left(\lambda' + 1\right)\frac{\lambda'-1}{\lambda'}}\left[\eta_{ac} + \frac{k_a k_c}{\mu^2\left(\lambda' - 1\right)}\right] i\frac{\mu q}{\sqrt{4\pi}}\left(p_2^c + p_4^c\right), \quad (434)$$

where the following two-vector has been defined:

$$k_a \coloneqq p_a^1 - p_a^3. \quad (435)$$

We also notice that external particles are on-shell, which implies that

$$\begin{aligned} p_1^2 &= -m^2 - \mu^2\left(\lambda' + 1\right), & p_3^2 &= -m^2 - \mu^2, \\ p_2^2 &= -m^2 - \mu^2, & p_4^2 &= -m^2 - \mu^2\left(\lambda' + 1\right). \end{aligned} \quad (436)$$

Calculating the product between $\left(p_1^a + p_3^a\right)$ and k_a immediately gives

$$\left(p_1^a + p_3^a\right)k_a = \left(p_1^a + p_3^a\right)\left(p_a^1 - p_a^3\right) = m^2 - \mu^2\left(\lambda' + 1\right) + m^2 + \mu^2 = -\mu^2\lambda', \quad (437)$$

which, in turn, allows us to write (434) as

$$i\mathcal{M}_{t,1} = \frac{\lambda' - 1}{\lambda'}\frac{i\mu^2 q^2/4\pi}{k^2 + \mu^2\left(\lambda' + 1\right)\frac{\lambda'-1}{\lambda'}}\left[\left(p_1^a + p_3^a\right)\left(p_a^2 + p_a^4\right) + \frac{\lambda'}{\lambda' - 1}\left(p_a^3 - p_a^1\right)\left(p_2^a + p_4^a\right)\right]. \quad (438)$$

It is convenient to express amplitudes in terms of the so-called Mandelstam variables [41]:

$$s \coloneqq -\left(p_1 + p_2\right)^2 = -\left(p_3 + p_4\right)^2, \quad (439)$$

$$t \coloneqq -\left(p_1 - p_3\right)^2 = -\left(p_2 - p_4\right)^2, \quad (440)$$

$$u \coloneqq -\left(p_1 - p_4\right)^2 = -\left(p_2 - p_3\right)^2. \quad (441)$$

By definition, the Mandelstam variable s can be written as

$$\begin{aligned} s &= -p_1^2 - p_2^2 - 2p_1 \cdot p_2 \\ &= m^2 + \mu^2 \left(\lambda' + 1\right) + m^2 + \mu^2 - 2p_1 \cdot p_2 \\ &= 2m^2 + \mu^2 \left(\lambda' + 2\right) - 2p_1 \cdot p_2, \end{aligned} \tag{442}$$

and so for the scalar product we have

$$p_1 \cdot p_2 = p_1^a p_a^1 = -\frac{s}{2} + m^2 + \frac{1}{2}\mu^2 \left(\lambda' + 2\right) = p_3 \cdot p_4. \tag{443}$$

The same can be done for t and u, obtaining

$$t = 2m^2 + \mu^2 \left(\lambda' + 2\right) + 2p_1 \cdot p_3, \tag{444}$$

$$u = 2m^2 + 2\mu^2 \left(\lambda' + 1\right) + 2p_1 \cdot p_4. \tag{445}$$

From (445) we immediately get

$$p_1 \cdot p_4 = \frac{u}{2} - m^2 - \mu^2 \left(\lambda' + 1\right) = p_2 \cdot p_4. \tag{446}$$

We now want to express u in terms of s and t. First we notice that

$$s + t + u = 6m^2 + 6\mu^2 + 4\mu^2\lambda' - 2p_1 \cdot \left(p_2 - p_3 - p_4\right). \tag{447}$$

Furthermore, momentum conservation gives

$$p_1 + p_2 = p_3 + p_4 \;\Rightarrow\; -p_1 = p_2 - p_3 - p_4 \;\Rightarrow\; s + t + u = 4m^2 + 2\mu^2 \left(\lambda' + 2\right), \tag{448}$$

finally leading to the following expression for u:

$$u = 4m^2 + 2\mu^2 \left(\lambda' + 2\right) - s - t. \tag{449}$$

By inserting the above expression in (446) we obtain

$$p_1 \cdot p_4 = \frac{1}{2}\left[4m^2 + 2\mu^2 \left(\lambda' + 2\right) - s - t\right] - m^2 - \mu^2 \left(\lambda' + 1\right) = m^2 + \mu^2 - \frac{1}{2}\left(s + t\right). \tag{450}$$

From the definition of u we can also compute the scalar product between p_2 and p_3:

$$p_2 \cdot p_3 = m^2 + \mu^2 \left(\lambda' + 1\right) - \frac{1}{2}\left(s + t\right). \tag{451}$$

Therefore, the first term in square brackets in (438) gives

$$\left(p_1^a + p_3^a\right)\left(p_a^2 + p_a^4\right) = p_1 \cdot p_2 + p_1 \cdot p_4 + p_3 \cdot p_2 + p_3 \cdot p_4 = -2s - t + 4m^2 + 2\mu^2 \left(\lambda' + 2\right), \tag{452}$$

while the second term simply reduces to

$$\left(p_a^3 - p_a^1\right)\left(p_2^a + p_4^a\right) = p_3 \cdot p_2 + p_3 \cdot p_4 - p_1 \cdot p_2 - p_1 \cdot p_4 = \mu^2\lambda'. \tag{453}$$

Putting it all together, Eq. (438) can be written as follows:

$$i\mathcal{M}_{t,1} = \frac{\lambda'-1}{\lambda'}\frac{i\mu^2q^2/4\pi}{-t+\mu^2\left(\lambda'+1\right)\frac{\lambda'-1}{\lambda'}}\left[-2s-t+4m^2+2\mu^2\left(\lambda'+2\right)+\mu^2\frac{\lambda'^2}{\lambda'-1}\right]$$
$$= \frac{\lambda'-1}{\lambda'}\frac{-i\mu^2q^2s/2\pi}{-t+\mu^2\left(\lambda'+1\right)\frac{\lambda'-1}{\lambda'}}\left[1+\frac{t}{2s}-\frac{2m^2}{s}-\frac{\mu^2}{s}\left(\lambda'+2\right)-\frac{\mu^2}{s}\frac{\lambda'^2}{\lambda'-1}\right]. \tag{454}$$

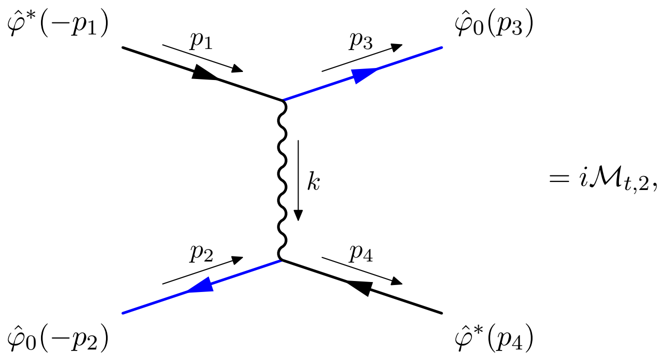
As already explained before, in our theory we are looking at the situation in which the center-of-mass-energy of collision is extremely high and where the momentum transfer is negligible. To be more precise, we are considering the following limits:

$$s \gg t, \quad s \gg \mu^2, m^2, \quad t \to 0. \tag{455}$$

In the above approximations, Eq. (454) becomes

$$i\mathcal{M}_{t,1} = -\frac{iq^2s}{2\pi\left(l^2+l+1\right)}. \tag{456}$$

The second diagram we want to consider is the following:



The computation is very similar to the previous one, so we will proceed more rapidly. We have:

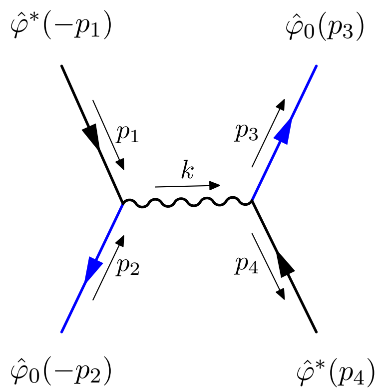
$$i\mathcal{M}_{t,2} = i\frac{\mu q}{\sqrt{4\pi}}\left(p_1^a+p_3^a\right)\frac{\lambda'-1}{\lambda'}\frac{-i}{k^2+\mu^2\left(\lambda'+1\right)\frac{\lambda'-1}{\lambda'}}\left[\eta_{ac}+\frac{k_ak_c}{\mu^2\left(\lambda'-1\right)}\right]i\frac{\mu q}{\sqrt{4\pi}}\left(-p_2^c-p_4^c\right)$$
$$= \frac{\lambda'-1}{\lambda'}\frac{-i\mu^2q^2/4\pi}{-t+\mu^2\left(\lambda'+1\right)\frac{\lambda'-1}{\lambda'}}\left[\left(p_1^a+p_3^a\right)\left(p_a^2+p_a^4\right)+\frac{\lambda'}{\lambda'-1}\left(p_a^3-p_a^1\right)\left(p_2^a+p_4^a\right)\right]$$
$$= \frac{\lambda'-1}{\lambda'}\frac{-i\mu^2q^2/4\pi}{-t+\mu^2\left(\lambda'+1\right)\frac{\lambda'-1}{\lambda'}}\left[-2s-t+4m^2+2\mu^2\left(\lambda'+2\right)+\mu^2\frac{\lambda'^2}{\lambda'-1}\right]$$
$$= \frac{\lambda'-1}{\lambda'}\frac{i\mu^2q^2s/2\pi}{-t+\mu^2\left(\lambda'+1\right)\frac{\lambda'-1}{\lambda'}}\left[1+\frac{t}{2s}-\frac{2m^2}{s}-\frac{\mu^2}{s}\left(\lambda'+2\right)-\frac{\mu^2}{s}\frac{\lambda'^2}{\lambda'-1}\right]. \tag{457}$$

By taking into account the same approximations as before, we finally get

$$i\mathcal{M}_{t,2} = \frac{iq^2s}{2\pi\left(l^2+l+1\right)}. \tag{458}$$

As we can notice, the only difference with respect to the previous result is the sign in front.

Finally, the last tree-level diagram of interest is



$= i\mathcal{M}_s.$

Proceeding in exactly the same way as before, we have

$$\begin{aligned} i\mathcal{M}_s &= i\frac{\mu q}{\sqrt{4\pi}}\left(p_1^a - p_2^a\right)\frac{\lambda' - 1}{\lambda'}\frac{-i}{k^2 + \mu^2\left(\lambda' + 1\right)\frac{\lambda'-1}{\lambda'}}\left[\eta_{ac} + \frac{k_a k_c}{\mu^2\left(\lambda' - 1\right)}\right] i\frac{\mu q}{\sqrt{4\pi}}\left(p_3^c - p_4^c\right) \\ &= \frac{\lambda' - 1}{\lambda'}\frac{i\mu^2 q^2/4\pi}{-s + \mu^2\left(\lambda' + 1\right)\frac{\lambda'-1}{\lambda'}}\left[\left(p_1^a - p_2^a\right)\left(p_a^3 - p_a^4\right) + \frac{\lambda'}{\lambda' - 1}\left(p_a^1 + p_a^2\right)\left(p_4^a - p_3^a\right)\right], \quad (459) \end{aligned}$$

where this time $k_a \coloneqq p_a^1 + p_a^2$. Proceeding as before, we easily find

$$i\mathcal{M}_s = -\frac{i\mu^2 q^2}{4\pi l\left(l + 1\right)}. \quad (460)$$

Let us now compare the results we have obtained so far. The amplitudes (456) and (458) are proportional to the center-of-mass energy of collision s, while the one above, (460), is proportional to the effective mass μ. Thus, within the regime we are working in, the contribution coming from the s-channel diagram is certainly negligible. This concludes the computation of all the scattering amplitudes of interest at tree-level. Once again, we remark that tree-level diagrams as the one shown in Fig. 5 below, although certainly allowed by the the theory, have not been considered for physical reasons (in the quantum-mechanics picture, all information about the in-particle is transferred to the out-particle). It would be interesting to understand what is the role they play in a future work. In what follows we will consider loop diagrams.

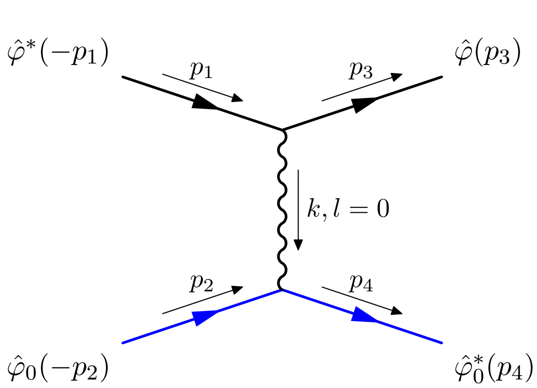


Figure 5: Example of a possible tree-level diagram obtained by setting the partial-waves indices associated to the gauge field to zero in the three-vertex calculation.

5.2 Loop diagrams with three-vertices

In 1969, Lévy and Sucher studied the Feynman amplitude describing the scattering of two particles with no spin, interacting by the exchange of mesons (with no spin) [42]. In particular, they showed that the contribution to the amplitude at arbitrary loop order can be evaluated in closed form within the eikonal approximation. Their computation can be readily adapted to our case, with only a few differences to be considered. The first one is that we are working in two dimensions, and not four. However, their analysis is dimension-independent, so all the results can be immediately adjusted. The second difference concerns the external legs; here we are dealing with a complex scalar field, allowing for more diagrams with respect to their case. In order to have an idea of the types of diagrams we consider at one loop, see Fig. 6.

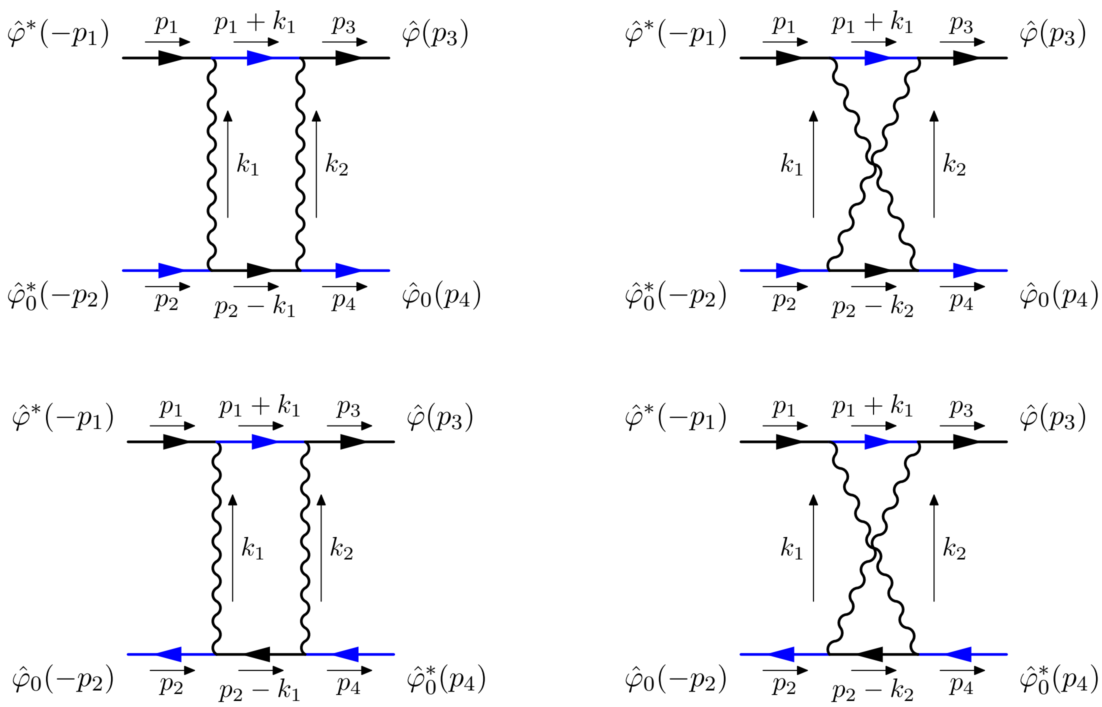


Figure 6: All leading one-loop diagrams in the $x_a \ll R$, $s \gg \mu^2$ limits.

5.2.1 Eikonal amplitude: particle-particle case

Let us consider the case where the charge and momentum arrows point in the same direction (see upper diagrams in Fig. 6). From now on, we will refer to this as "particle-particle case"[35]. As already anticipated, we can readily adapt the calculation in Ref. [42] to our scenario, obtaining the following expression for the n-order amplitude (n counts the number of photons exchanged):

$$\begin{aligned} i\mathcal{M}_{n,p-p} &= \left(i\frac{\mu q}{\sqrt{4\pi}}\right)^{2n} \int \prod_{j=1}^{n} \left[\frac{d^2k_j}{(2\pi)^2} 4ip_a^1 p_b^2 \mathcal{P}^{ab}_{even}(k_j)\right] \times I \times (2\pi)^2 \delta^{(2)}\left(\sum\nolimits_{j=1}^{n} k_j\right) \\ &= \left(i\frac{\mu q}{\sqrt{4\pi}}\right)^{2n} \left(\frac{s}{2}\right)^{n} \int \prod_{j=1}^{n} \left[\frac{d^2k_j}{(2\pi)^2} 4i\mathcal{P}^{UV}_{even}(k_j)\right] \times I \times (2\pi)^2 \delta^{(2)}\left(\sum\nolimits_{j=1}^{n} k_j\right), \end{aligned} \tag{461}$$

[35] It is important to notice that here we are talking about particles even if we did not specify a Fock-space basis yet. However, this will turn out to be a good choice, see subsection 5.3.

where "$p-p$" stands for "particle-particle". The above equation is the two-dimensional analog of Eq. (3.1) in Ref. [42], with different vertices and with $q = p_1 - p_3 = p_2 - p_4 = 0$. Above, the second equality follows from the fact that we are considering one particle going into the black hole and the other one going out, in orthogonal orbits. Due to the approximations made, the two momenta can be considered to be light-like: $p_1 = \left(p_U^1, 0\right)$, $p_2 = \left(0, p_V^2\right)$. Thus, s reduces to

$$s = -(p_1 + p_2)^2 = 2p_U^1 p_V^2. \tag{462}$$

Therefore, the quantity in square brackets in the first step of (461) can be written as

$$p_a^1 p_b^2 \mathcal{P}_{even}^{ab} = p_U^1 p_U^2 \mathcal{P}_{even}^{UU} + p_V^1 p_V^2 \mathcal{P}_{even}^{VV} + p_U^1 p_V^2 \mathcal{P}_{even}^{UV} + p_V^1 p_U^2 \mathcal{P}_{even}^{VU} = \frac{s}{2}\mathcal{P}_{even}^{UV}. \tag{463}$$

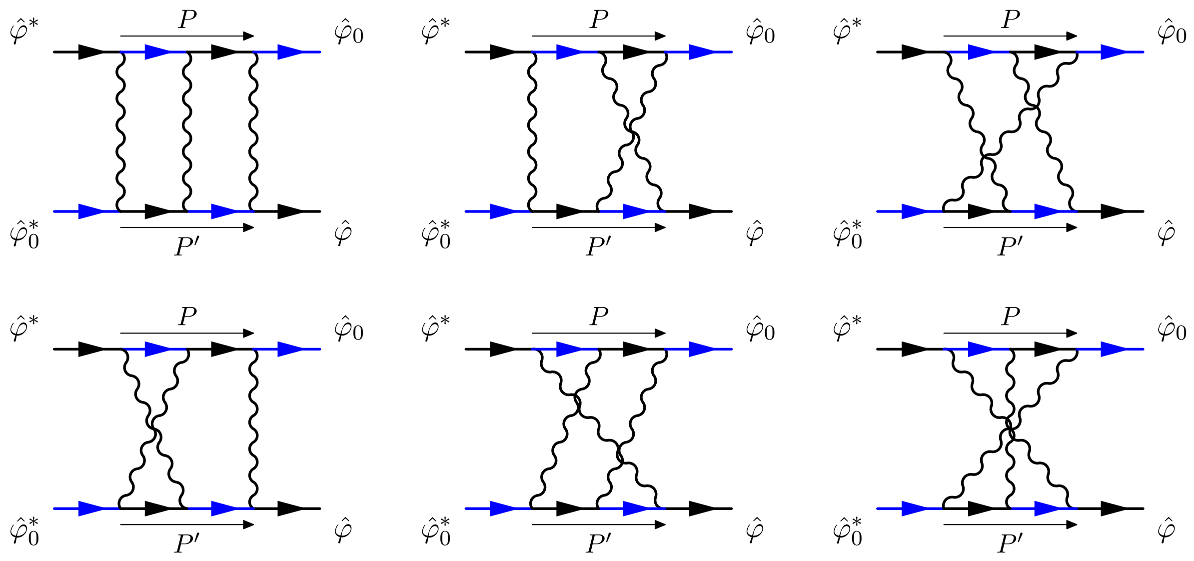


Figure 7: All leading two-loop diagrams ($p-p$ case) in the $x_a \ll R$, $s \gg \mu^2$ limits.

Let us now make a couple of comments more in regard to Eq. (461). The first one concerns an approximation that has been implicitly made. To understand this, let us set $n = 1$ in (461) and look again at the upper diagrams in Fig. 6; it is clear that, when writing down the amplitude, we should have included contributions coming from the additional k's (internal photon momenta). However, within the near-horizon approximation that we have already made, we are interested in effects from impact parameters of the order of the Schwarzschild radius or less, but much larger than the Planck scale. Thus, it is reasonable to neglect these k-terms in Eq. (461). The second comment concerns the quantity I that appears in the same equation[36], which contains the information regarding the matter propagators; for a fixed n, say $n = 3$, one needs to consider all possible permutations of k_j, with $j \in [1, 3]$ (see Fig. 7). Within the eikonal approximation, a simple expression for I has been derived, resulting in the general loop amplitude [42]

$$i\mathcal{M}_{n,p-p} = -\frac{\mu^2 q^2 s}{8\pi n!} \int \frac{d^2k}{(2\pi)^2} 4i\mathcal{P}_{even}^{UV}(k) \int d^2x e^{-ik\cdot x} (i\chi)^{n-1}, \tag{464}$$

[36]All other ingredients are the usual ones: vertex-factors, integration over all internal photon propagators and insertion of Dirac-delta functions to ensure conservation of internal momentum.

where the quantity χ has been defined as

$$\chi := -\frac{i\mu^2 q^2 s}{8\pi}\int\frac{d^2k}{(2\pi)^2}4i\mathcal{P}^{UV}_{even}(k)e^{-ik\cdot x}\times\Bigg[\frac{1}{-2p_1\cdot k-i\epsilon}\frac{1}{2p_2\cdot k-i\epsilon} + \frac{1}{-2p_1\cdot k-i\epsilon}\frac{1}{-2p_2\cdot k-i\epsilon}+\frac{1}{2p_1\cdot k-i\epsilon}\frac{1}{2p_2\cdot k-i\epsilon} + \frac{1}{2p_1\cdot k-i\epsilon}\frac{1}{-2p_2\cdot k-i\epsilon}\Bigg]. \quad (465)$$

The above expression in square brackets can be rewritten in a more convenient form, giving

$$\chi = -\frac{i\mu^2 q^2 s}{8\pi}\int\frac{d^2k}{(2\pi)^2}4i\mathcal{P}^{UV}_{even}(k)e^{-ik\cdot x}\left(\frac{1}{2p_1\cdot k+i\epsilon}-\frac{1}{2p_1\cdot k-i\epsilon}\right) \times\left(\frac{1}{2p_2\cdot k+i\epsilon}-\frac{1}{2p_2\cdot k-i\epsilon}\right). \quad (466)$$

Now, by using the identity (we recall that ϵ is an infinitesimal regulator)

$$\frac{1}{x+i\epsilon}-\frac{1}{x-i\epsilon} = -2\pi i\delta(x), \quad (467)$$

we arrive at a very simple expression for χ:

$$\begin{aligned}\chi &= -\frac{i\mu^2 q^2 s}{8\pi}\int\frac{d^2k}{(2\pi)^2}4i\mathcal{P}^{UV}_{even}(k)e^{-ik\cdot x}\left(-2\pi i\right)^2\delta(2p_1\cdot k)\delta(2p_2\cdot k)\\ &= -\frac{\mu^2 q^2}{4\pi}\mathcal{P}^{UV}_{even}(0)\\ &= -\frac{q^2}{4\pi\left(\lambda'+1\right)}.\end{aligned} \quad (468)$$

We can clearly see that χ is not dependent on spacetime coordinates. Thus, we can write

$$\begin{aligned}i\mathcal{M}_{n,p-p} &= -\frac{\mu^2 q^2 s}{8\pi n!}(i\chi)^{n-1}\int\frac{d^2k}{(2\pi)^2}4i\mathcal{P}^{UV}_{even}(k)\int d^2x e^{-ik\cdot x}\\ &= -\frac{\mu^2 q^2 s}{8\pi n!}(i\chi)^{n-1}\int\frac{d^2k}{(2\pi)^2}4i\mathcal{P}^{UV}_{even}(k)(2\pi)^2\delta^{(2)}(k)\\ &= -\frac{i\mu^2 q^2 s}{2\pi n!}\left(i\chi\right)^{n-1}\mathcal{P}^{UV}_{even}(0)\\ &= 2s\frac{(i\chi)^n}{n!},\end{aligned} \quad (469)$$

We can now find the total amplitude by summing over all odd and even n, resulting in

$$\begin{aligned}i\mathcal{M}_{p-p} &= i\mathcal{M}_{odd,pp}+i\mathcal{M}_{even,pp}\\ &= 2s\left[\sum\nolimits_{odd\ n}^{\infty}\frac{(i\chi)^n}{n!}+\sum\nolimits_{even\ n}^{\infty}\frac{(i\chi)^n}{n!}\right]\\ &= 2s\left[\sum\nolimits_{m=0}^{\infty}\frac{(i\chi)^{2m+1}}{(2m+1)!}+\sum\nolimits_{m=1}^{\infty}\frac{(-1)^m\chi^{2m}}{(2m)!}\right]\\ &= 2s\left[\exp(i\chi)-1\right].\end{aligned} \quad (470)$$

By inserting (468) in (470) and recalling that $\lambda' = l(l+1)$, we finally obtain

$$i\mathcal{M}_{p-p} = 4p_{in}p_{out}\left[\exp\left(-\frac{i}{4\pi}\frac{q^2}{l^2+l+1}\right)-1\right]. \tag{471}$$

For the sake of clarity, we also relabelled the external momenta as p_{in} and p_{out}.

5.2.2 Eikonal amplitude: particle-antiparticle case

Let us now consider slightly different types of loop diagrams with respect to the ones presented in the previous paragraph. In the present case[37], the charge and momentum arrows of the bottom vertices point in opposite directions, resulting in a factor of $(-1)^n$ in the amplitude. Again, as an example, we show the case $n = 3$ (Fig. 8). The general loop amplitude is given by

$$\begin{aligned} i\mathcal{M}_{n,p-a} &= \left(i\frac{\mu q}{\sqrt{4\pi}}\right)^{2n}(-1)^n\int\prod_{j=1}^{n}\left[\frac{d^2k_j}{(2\pi)^2}4ip_a^1p_b^2\mathcal{P}^{ab}_{even}(k_j)\right]\times I\times(2\pi)^2\delta^{(2)}\left(\sum_{j=1}^{n}k_j\right) \\ &= \left(i\frac{\mu q}{\sqrt{4\pi}}\right)^{2n}\left(-\frac{s}{2}\right)^n\int\prod_{j=1}^{n}\left[\frac{d^2k_j}{(2\pi)^2}4i\mathcal{P}^{UV}_{even}(k_j)\right]\times I\times(2\pi)^2\delta^{(2)}\left(\sum_{j=1}^{n}k_j\right). \end{aligned} \tag{472}$$

By repeating the same procedure as before, the quantity χ turns out to be

$$\chi = \frac{q^2}{4\pi(\lambda'+1)}, \tag{473}$$

which, in turn, leads to the following result for the amplitude:

$$i\mathcal{M}_{p-a} = 4p_{in}p_{out}\left[\exp\left(\frac{i}{4\pi}\frac{q^2}{l^2+l+1}\right)-1\right]. \tag{474}$$

As we can notice, the only difference with respect to Eq. (471) is a minus sign in the exponent.

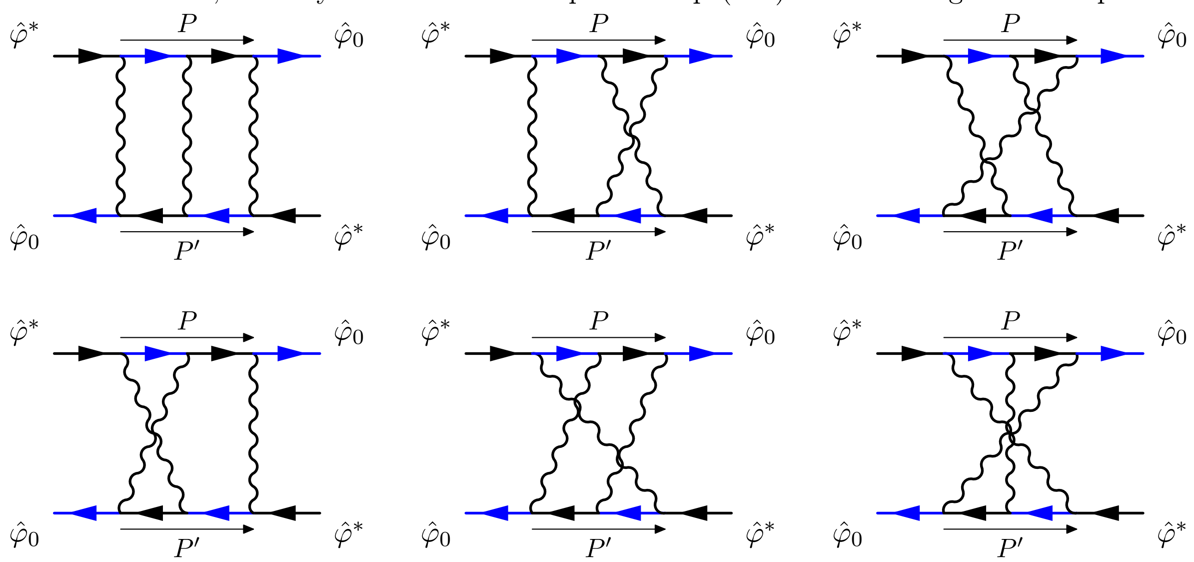


Figure 8: All leading two-loop diagrams ($p-a$ case) in the $x_a \ll R$, $s \gg \mu^2$ limits.

[37]From now on, even if we did not specify a Fock-space basis yet, we will refer to this case as "particle-antiparticle" case or, in abbreviated form, "p-a case".

5.3 From fields to physical particles

Let us now specify a Fock-space basis, i.e., let us define what a particle/antiparticle is. This can be done by expanding the field operator φ (and its conjugate of course) in terms of creation and annihilation operators. For a specific l, m, the on-shell expansions are given by[38] [41]

$$\varphi_{lm}(x) = \int \frac{dp}{2\pi\sqrt{2p}} \left(a_{lm}(p) e^{-ipx} + b^{\dagger}_{lm}(p) e^{ipx} \right), \tag{475}$$

$$\varphi^{*}_{lm}(x) = \int \frac{dp}{2\pi\sqrt{2p}} \left(a^{\dagger}_{lm}(p) e^{ipx} + b_{lm}(p) e^{-ipx} \right), \tag{476}$$

$$\varphi_{0}(x) = \int \frac{dp}{2\pi\sqrt{2p}} \left(a_{0}(p) e^{-ipx} + b^{\dagger}_{0}(p) e^{ipx} \right), \tag{477}$$

$$\varphi^{*}_{0}(x) = \int \frac{dp}{2\pi\sqrt{2p}} \left(a^{\dagger}_{0}(p) e^{ipx} + b_{0}(p) e^{-ipx} \right). \tag{478}$$

A φ_{lm} in the interaction implies the annihilation of an antiparticle or the creation of a particle at position x. A φ^{*}_{lm} implies the creation of an antiparticle or the annihilation of a particle. The same is true for φ_0 and φ^{*}_0, except for the fact that the particles and antiparticles that are created or annihilated are at $l = 0$. Therefore, we can now safely talk about "particle-particle" and "particle-antiparticle" cases referring to subsections 5.2.1 and 5.2.2, respectively. Let us now combine the two results we have obtained, Eqs. (471) and (474), into a single formula:

$$i\mathcal{M} = 4p_{in}p_{out} \left[\exp\left(-\frac{i}{4\pi} \frac{q_{in}q_{out}}{l^2 + l + 1} \right) - 1 \right]. \tag{479}$$

In the above expression, q_{in} and q_{out} are the asymptotic charges of the in-particle and out-particle, respectively, where $q_{in/out} = -q$ for particles and $q_{in/out} = q$ for antiparticles. The exponent in Eq. (479) naturally displays the emergence of repulsion/attraction for equal/opposite charged particles, respectively.

5.4 From the scattering amplitude to the S-matrix

In the previous section the final result for the amplitude has been found, namely Eq. (479). Compactly, the amplitude can be written as follows:

$$i\mathcal{M} = 2s \left(e^{i\chi} - 1 \right), \tag{480}$$

with χ given by (468). Here, to compare the results obtained in the two formalisms analyzed in this thesis, we aim at finding the scattering matrix that relates the in- and out-states. The relation between the S-matrix and the scattering amplitude is in our case given by [41]

$$\langle out | S - \mathbb{1} | in \rangle = (2\pi)^2 \delta^{(2)} \left(p_1 + p_2 - p_3 - p_4 \right) i \langle out | \mathcal{M} | in \rangle, \tag{481}$$

where the in- and out-states have been defined as (we consider only a_{lm} and $a^{\dagger}_{lm}$ for simplicity)

$$|in\rangle := a^{\dagger}(p_1) a^{\dagger}_0(p_2) |0\rangle, \quad |out\rangle := \langle 0 | a(p_3) a_0(p_4). \tag{482}$$

[38]Note that we are using the convention where we have a factor of $\sqrt{2p}$ in the field expansion but not in the commutation relation between a and $a^{\dagger}$: $[a_{lm}(p), a^{\dagger}_{l'm'}(p')] = 2\pi\delta(p - p')\delta_{ll'}\delta_{mm'}$.

Notice that in Eq. (480) we used a shorthand notation: with $i\mathcal{M}$ we usually mean $i\langle out|\mathcal{M}|in\rangle$. The operators $a^{\dagger}_{lm}(p)$ and $a_{lm}(p)$ create and annihilate on-shell quantum perturbations, respectively. Moreover, we recall that they obey the commutation relation [41]

$$[a_{lm}(p), a^{\dagger}_{l'm'}(p')] = 2p(2\pi)\delta(p-p')\delta_{ll'}\delta_{mm'}. \tag{483}$$

Now, in order to find the S-matrix we also need to look at the free theory, for which we have

$$\langle out|S|in\rangle = \langle out|in\rangle = \langle 0|a_{lm}(p_3)a_0(p_4)a^{\dagger}_{lm}(p_1)a^{\dagger}_0(p_2)|0\rangle. \tag{484}$$

By making use of the commutation relation (483), it is straightforward to get

$$\langle out|in\rangle = \langle 0|a_{lm}(p_3)a_0(p_4)a^{\dagger}_{lm}(p_1)a^{\dagger}_0(p_2)|0\rangle = 2s(2\pi)^2\delta(p_1-p_3)\delta(p_2-p_4), \tag{485}$$

which is indeed the expected result. The two delta functions tell us that the four-momenta of the two outgoing particles are equal to the four-momenta of the two incoming particles. On the other hand, by making use of Eq. (480), we can write the interacting piece as

$$(2\pi)^2\delta^{(2)}(p_1+p_2-p_3-p_4)i\langle out|\mathcal{M}|in\rangle = 2s(2\pi)^2\delta(p_1-p_3)\delta(p_2-p_4)(e^{i\chi}-1). \tag{486}$$

Putting it all together, Eq. (481) gives

$$\langle out|S-\mathbb{1}|in\rangle = \left\langle out|\mathbb{1}(e^{i\chi}-1)|in\right\rangle. \tag{487}$$

In the operator notation, we can then finally write the S-matrix as

$$S = \mathbb{1} + \mathbb{1}(e^{i\chi}-1) = \mathbb{1}e^{i\chi} = \mathbb{1}\exp\left(-\frac{i}{4\pi}\frac{q_{in}q_{out}}{l^2+l+1}\right). \tag{488}$$

5.5 A careful comparison

In the previous subsection we considered a $2\to 2$ scattering process, computed the corresponding amplitude, and finally obtained an expression for the S-matrix, Eq. (488). On the other hand, in subsection 3.3 we computed its quantum mechanics analog, Eq. (123). A couple of comments are in order here. The first one has to do with the presence of the identity operator in (488), which ensures that the S-matrix is indeed an operator and not a function. The second comment concerns the factor of $1/4\pi$ appearing in the same equation, showing a mismatch with respect to Eq. (123), excluding the slightly different l-dependence[39]. The purpose of this subsection is to shed light on this matter.

The idea is the following[40]. Two S-matrices that are very similar to each other have been obtained. It is thus reasonable to suspect that the only difference between the two resides in the sources. Therefore, we can quite safely conclude that the two scattering matrices are identical provided that the sources are the same. To be more precise, we will demand the charge current densities in the two frameworks to be the same, expecting to obtain the correct rescaling.

[39] A similar discrepancy was noticed in Ref. [19] and it deserves further attention.

[40] The author is greatly indebted to Nico Groenenboom for sharing his ideas about this topic.

In the quantum-mechanics case the V-component of the current, at a specific l, m, reads

$$J_V^{lm} = -\mu^2 q_{in}^{lm} \delta(U). \tag{489}$$

The above equation has been obtained from (93) by expanding in spherical harmonics; it contains information about the charges going into the black hole. On the other hand, in the field-theory side, the V-component of the current density is given by

$$j_V = -iq_{in}\left(\phi^* \partial_V \phi - \phi \partial_V \phi^*\right), \tag{490}$$

where, to make contact with the canonical current density that functions as source for the equations of motion, in the definition we also included the minus sign and the charge coming from the second term in (132). Moreover, the charge q has been renamed q_{in} for obvious reasons. Now, expanding in spherical harmonics, inserting the rescaling $\phi = \mu\varphi$ as we did in section 4.8, as well as taking into account the spherical-symmetry approximation, results in

$$j_V^{lm} = -iq_{in}\mu^2 Y_{00}\left(\varphi_{00}^* \partial_V \varphi_{lm} - \varphi_{lm}\partial_V \phi_{00}^* + \varphi_{lm}^* \partial_V \varphi_{00} - \varphi_{00}\partial_V \phi_{lm}^*\right), \tag{491}$$

where, again, we considered the current density at a specific l, m. The main difference between Eqs. (490) and (491) is that the current density in quantum field theory is an operator. Therefore, in order to properly compare these two quantities, we need to consider the expectation value of j_V^{lm} in an appropriately defined initial state. In order to start, we recall that the complex scalar field is expanded in terms of creation and annihilation operators as in (475). Moreover, we also recall that the only non-vanishing commutation relations are [41]

$$\left[a_{lm}(p), a_{l'm'}^\dagger(p')\right] = \left[b_{lm}(p), b_{l'm'}^\dagger(p')\right] = 2\pi\delta(p-p')\delta_{ll'}\delta_{mm'}. \tag{492}$$

As already anticipated, we now need to specify the initial state in order to compute the expectation value of the operator j_V^{lm}. Such state is written as follows:

$$|in\rangle = \int \frac{dp}{2\pi}\Phi(p) \times \frac{1}{\sqrt{2}}\left(a_{lm}^\dagger(p) + a_{00}^\dagger(p)\right)|0\rangle, \tag{493}$$

where $\Phi(p)$ is a normalized test function localized around a specific momentum, say, $p = p_1$. The expression above, Eq. (493), represents a one-particle state where we have a superposition of two spherical shells at equal momentum. Indeed, in computing the eikonal amplitude, we assumed one ingoing particle to be at a specific l, m first, and later at $l = m = 0$. From the moment that this choice is symmetric, this results in a superposition of both cases, i.e., a superposition of l, m and $l = m = 0$. We are now ready to compute the expectation value of the current density. For clarity, we will consider each term in Eq. (491) separately. Concerning the first term, already including all prefactors, we get

$$\begin{aligned} -iq_{in}\mu^2 Y_{00}\langle in|\varphi_{00}^*\partial_V\varphi_{lm}|in\rangle = -\frac{\mu^2}{2}Y_{00}\int \frac{dpdp'}{(2\pi)^2\sqrt{4pp'}}p' \int \frac{dkdk'}{(2\pi)^2}\Phi^*(k)\Phi(k') \\ \times \langle 0|a_{00}(k)a_{00}^\dagger(p)a_{lm}^\dagger(p')a_{lm}^\dagger(k')|0\rangle e^{i(p-p')x}, \end{aligned} \tag{494}$$

where we made use of the commutation relations, as well as of the definition of the vacuum state, i.e., $a_{lm}|0\rangle = b_{lm}|0\rangle = 0$. By using once again the commutation relation between a and $a^\dagger$, the above expression can be written as

$$\begin{aligned}-iq_{in}\mu^2 Y_{00}\langle in|\varphi^*_{00}\partial_V\varphi_{lm}|in\rangle = -\frac{\mu^2}{4}Y_{00}\int\frac{dpdp'}{(2\pi)^2}\frac{p'}{\sqrt{pp'}}\int\frac{dkdk'}{(2\pi)^2}\Phi^*(k)\Phi(k')\\ \times(2\pi)^2\delta(k-p)\delta(k'-p')e^{i(p-p')x}.\end{aligned} \tag{495}$$

Let us now consider the expectation value of the second term in (491). In this case, we have

$$\begin{aligned}iq_{in}\mu^2 Y_{00}\langle in|\varphi_{lm}\partial_V\varphi^*_{00}|in\rangle = -\frac{\mu^2}{2}Y_{00}\int\frac{dpdp'}{(2\pi)^2\sqrt{4pp'}}p'\int\frac{dkdk'}{(2\pi)^2}\Phi^*(k)\Phi(k')\\ \times\langle 0|a_{00}(k)a_{lm}(p)a^\dagger_{00}(p')a^\dagger_{lm}(k')|0\rangle e^{-i(p-p')x},\end{aligned} \tag{496}$$

finally resulting in

$$\begin{aligned}iq_{in}\mu^2 Y_{00}\langle in|\varphi_{lm}\partial_V\varphi^*_{00}|in\rangle = -\frac{\mu^2}{4}Y_{00}\int\frac{dpdp'}{(2\pi)^2}\frac{p'}{\sqrt{pp'}}\int\frac{dkdk'}{(2\pi)^2}\Phi^*(k)\Phi(k')\\ \times(2\pi)^2\delta(k-p')\delta(p-k')e^{-i(p-p')x}.\end{aligned} \tag{497}$$

By following the exact same logic, it is easy to realize that the expectation value of third term in Eq. (491) gives the same contribution as the expectation value of the first one, and the same is true concerning the second and fourth terms. Putting it all together, we can write

$$\begin{aligned}\langle in|j_V^{lm}|in\rangle = -\frac{\mu^2}{2}q_{in}Y_{00}\int\frac{dpdp'}{(2\pi)^2}\frac{p'}{\sqrt{pp'}}\int\frac{dkdk'}{(2\pi)^2}\Phi^*(k)\Phi(k')\\ \times(2\pi)^2\left[\delta(k-p)\delta(k'-p')e^{i(p-p')x}+\delta(k-p')\delta(p-k')e^{-i(p-p')x}\right].\end{aligned} \tag{498}$$

By taking into account that the particle falling into the black hole is localized around a specific momentum p_1, we obtain[41]

$$\begin{aligned}\langle j_V^{lm}\rangle &= -\frac{\mu^2}{2}q_{in}Y_{00}\int\frac{dpdp'}{(2\pi)^2}\left(\Phi^*(p)\Phi(p')e^{i(p-p')x}+\Phi^*(p')\Phi(p)e^{-i(p-p')x}\right)\\ &= -\mu^2 q_{in}Y_{00}\int\frac{dpdp'}{(2\pi)^2}\Phi^*(p)\Phi(p')e^{i(p-p')x}\\ &= -\mu^2 q_{in}Y_{00}|\Phi(x)|^2,\end{aligned} \tag{499}$$

where, to get to the last equality, the following Fourier transform has been defined:

$$\Phi(x) = \int\frac{dp}{2\pi}\Phi(p)e^{-ipx}. \tag{500}$$

Now, interpreting $|\Phi(x)|^2$ as a probability distribution, we assume the particle with momentum

[41] Here we are implicitly defining a region, a small interval around p_1, in which the test function is non-zero. Therefore, the entire integrand will be non-zero only in this interval, allowing us to approximate all functions obeying $|f(p_1)/f'(p_1)| \ll 1$ around p_1.

p_1 to be localized at $U = 0$, finally resulting in

$$\langle J_V^{lm} \rangle = -\mu^2 q_{in} Y_{00} \delta(U) = -\mu^2 \frac{q_{in}}{\sqrt{4\pi}} \delta(U). \tag{501}$$

By comparing Eqs. (490) and (501), we immediately get $q_{in}^{lm} = q_{in}/\sqrt{4\pi}$. Therefore, by inserting such rescaling in Eq. (123), we finally obtain

$$S_{EM} = \exp\left(-\frac{i}{4\pi} \frac{q_{in} q_{out}}{l^2 + l} \right), \tag{502}$$

in agreement with the result obtained in the field-theory calculation.

Up to now, we only considered loop diagrams with three-vertices. Indeed, the scattering matrix (488) is the result of a resummation of an infinite number of diagrams of the type shown in Fig. 6. However, we remind the reader that we have two types of four-vertex in our theory. In the next subsection we will consider two one-loop cases with four-vertices, showing that these are sub-leading with respect to the one-loop diagrams with three vertices only.

5.6 One-loop diagrams with “seagull" vertices

As anticipated, here we consider one-loop diagrams of the following type:

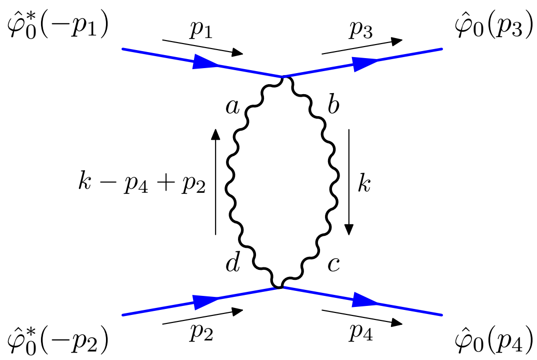


Figure 9: One-loop diagram containing the so-called “seagull" vertices.

We want to show that the above diagram is sub-leading with respect to the one-loop diagram containing three-vertices. Using the Feynman rules derived in subsection 4.10, we write the amplitude as follows[42]:

$$\begin{aligned} i\mathcal{M}_{seagull}^{even} = \int \frac{d^2k}{(2\pi)^2} \Bigg[&\frac{\lambda' - 1}{\lambda'} \frac{-i}{k^2 + \tilde{m}^2 - i\epsilon} \left(\eta^{bc} + \frac{1}{\mu^2} \frac{k^b k^c}{\lambda' - 1} \right) \left(-2i \frac{\mu^2 q^2}{4\pi} \eta_{ab} \right) \\ &\times \frac{\lambda' - 1}{\lambda'} \frac{-i}{k'^2 + \tilde{m}^2 - i\epsilon} \left(\eta^{da} + \frac{1}{\mu^2} \frac{k^d k^a}{\lambda' - 1} \right) \left(-2i \frac{\mu^2 q^2}{4\pi} \eta_{cd} \right) \Bigg], \end{aligned} \tag{503}$$

with $\tilde{m}^2$ and k' given by

$$\tilde{m}^2 := \mu^2 \frac{\lambda'^2 - 1}{\lambda'}, \quad k' := k - \tilde{p}, \quad \tilde{p} := p_4 - p_2. \tag{504}$$

[42]Note that momentum conservation implies $k - p_4 + p_2 - k + p_3 - p_1 = 0$.

It is easy to see that Eq. (503) can be split into four contributions, namely

$$i\mathcal{M}^{even}_{seagull} = I_1 + I_2 + I_3 + I_4, \tag{505}$$

where the following quantities have been defined:

$$I_1 \coloneqq \frac{\mu^4 q^4}{2\pi^2}\left(\frac{\lambda'-1}{\lambda'}\right)^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{\left(k^2+\tilde{m}^2-i\epsilon\right)\left(k'^2+\tilde{m}^2-i\epsilon\right)}, \tag{506}$$

$$I_2 \coloneqq \frac{\mu^2 q^4}{4\pi^2}\frac{\lambda'-1}{\lambda'^2} \int \frac{d^2k}{(2\pi)^2} \frac{k'^2}{\left(k^2+\tilde{m}^2-i\epsilon\right)\left(k'^2+\tilde{m}^2-i\epsilon\right)}, \tag{507}$$

$$I_3 \coloneqq \frac{\mu^2 q^4}{4\pi^2}\frac{\lambda'-1}{\lambda'^2} \int \frac{d^2k}{(2\pi)^2} \frac{k'^2}{\left(k^2+\tilde{m}^2-i\epsilon\right)\left(k'^2+\tilde{m}^2-i\epsilon\right)}, \tag{508}$$

$$I_4 \coloneqq \frac{q^4}{4\pi^2\lambda'^2} \int \frac{d^2k}{(2\pi)^2} \frac{\eta_{ab}\eta_{cd}k^b k^c k'^d k'^a}{\left(k^2+\tilde{m}^2-i\epsilon\right)\left(k'^2+\tilde{m}^2-i\epsilon\right)}. \tag{509}$$

Before computing such quantities, let us recall that we are working in light-cone coordinates:

$$ds^2 = -2dUdV. \tag{510}$$

In the following we will perform a Wick rotation; in order to do that we change coordinates from light-cone to Cartesian: $(U,V) \to \left(x^0, x^1\right)$. The two sets of coordinates are related by

$$U = \frac{1}{\sqrt{2}}\left(x^0+x^1\right), \;\; V = \frac{1}{\sqrt{2}}\left(x^0-x^1\right) \quad \Rightarrow \quad ds^2 = -\left(dx^0\right)^2 + \left(dx^1\right)^2. \tag{511}$$

A quick inspection of the expressions above shows that $i\mathcal{M}_{seagull}$ contains integrals that diverge. If there is some physics in the expressions we have written down, then we need a prescription to handle these infinities. As is well-known, such a procedure exists and it is called *renormalization*. In order to carry out this procedure in an explicit fashion, it is of course necessary to deal with the infinities in some well-defined mathematical way. To ensure that this is done without introducing spurious inconsistencies one usually employs a so-called *regularization* method, which renders the potentially divergent integrals finite. Such a regularization procedure is defined in terms of a parameter which, at the end of the computation, is taken to a certain limit in which the divergences will again become manifest. This limiting procedure enables us to unambiguously calculate the various Feynman diagrams. There exist many different regularization procedures; the one we will use here in this thesis is called *dimensional regularization* and it is due to 't Hooft and Veltman [43]. It is based on the observation that the degree of divergence of Feynman integrals depends on the number of spacetime dimensions; one analytically continues these integrals to arbitrary complex dimensions, where poles in the complex plane will emerge at certain integer values of the dimension, indicating that at those dimensions the integral will diverge. We assume the reader to be familiar with both the regularization and renormalization.

Let us start by considering (506), temporarily suppressing the $i\epsilon$'s for notational convenience. Even if it is convergent, in order to be consistent, we shift to n dimensions anyway:

$$\int \frac{d^2k}{(2\pi)^2}\frac{1}{\left(k^2+\tilde{m}^2\right)\left(k'^2+\tilde{m}^2\right)} \;\; \to \;\; \int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2+\tilde{m}^2\right)\left(k'^2+\tilde{m}^2\right)}, \;\; d = 2+\varepsilon. \tag{512}$$

In general, in order to bring the denominators into a form that allows us to perform the momentum integrals, we can make use of a class of identities introduced by Feynman. In the particular case we are considering what we need is the following expression [41]:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A + (B - A)x]^2}. \tag{513}$$

Looking at (512), we see that the role of A is played by $k^2 + \tilde{m}^2$ while the role of B by $k'^2 + \tilde{m}^2 = (k - \tilde{p})^2 + \tilde{m}^2$; therefore, we have

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \tilde{m}^2)\left(k'^2 + \tilde{m}^2\right)} = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(k - xq)^2 + \tilde{p}^2 x(1-x) + \tilde{m}^2]^2}. \tag{514}$$

Shifting $k \to k + x\tilde{p}$ and performing a Wick rotation (we substitute $k^0 = ik_E^0$) lead to [41]

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \tilde{m}^2)\left(k'^2 + \tilde{m}^2\right)} = i \int_0^1 dx \int \frac{d^n k_E}{(2\pi)^n} \frac{1}{\left(k_E^2 + \Delta\right)^2}, \tag{515}$$

where $\Delta \coloneqq \tilde{p}^2 x(1-x) + \tilde{m}^2$. As is well-known, the momentum integral above can be expressed in terms of gamma functions; the more general result is given by [41]

$$\int \frac{d^n k_E}{(2\pi)^n} \frac{1}{\left(k_E^2 + \Delta\right)^\alpha} = \frac{1}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} \Delta^{\frac{n}{2} - \alpha}. \tag{516}$$

In our case, $\alpha = 2$, we then obtain

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \tilde{m}^2)\left(k'^2 + \tilde{m}^2\right)} = \frac{i}{(4\pi)^{\frac{n}{2}}} \Gamma\left(2 - \frac{n}{2}\right) \int_0^1 dx \Delta^{\frac{n}{2} - \alpha}. \tag{517}$$

In terms of ε, the above expression becomes

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \tilde{m}^2)\left(k'^2 + \tilde{m}^2\right)} = \frac{i}{4\pi} (4\pi)^{-\frac{\epsilon}{2}} \Gamma\left(1 - \frac{\varepsilon}{2}\right) \int_0^1 dx \Delta^{\frac{\varepsilon}{2} - 1}. \tag{518}$$

Now we want to expand in powers of ε, considering terms up to zeroth-order. However, since we can only expand dimensionless quantities and Δ clearly has dimensions, we introduce an auxiliary mass parameter, say M, writing

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \tilde{m}^2)\left(k'^2 + \tilde{m}^2\right)} = \frac{i}{4\pi} (4\pi)^{-\frac{\epsilon}{2}} \Gamma\left(1 - \frac{\varepsilon}{2}\right) M^{\varepsilon - 2} \int_0^1 dx \left(\frac{\Delta}{M^2}\right)^{\frac{\varepsilon}{2} - 1}. \tag{519}$$

By considering the expansions

$$(4\pi)^{-\frac{\varepsilon}{2}} = 1 - \frac{\varepsilon}{2} \ln(4\pi) + \ldots, \tag{520}$$

$$\Gamma\left(1 - \frac{\varepsilon}{2}\right) = -\frac{\varepsilon}{2} \Gamma\left(-\frac{\varepsilon}{2}\right) = -\frac{\varepsilon}{2}\left(-\frac{2}{\varepsilon} - \gamma_E + \ldots\right), \tag{521}$$

$$\left(\frac{\Delta}{M^2}\right)^{\frac{\varepsilon}{2} - 1} = \frac{M^2}{\Delta} + \frac{\varepsilon}{2} \frac{M^2 \ln\left(\Delta / M^2\right)}{\Delta} + \ldots, \tag{522}$$

where $\gamma_E \approx 0.5772$ is the Euler-Mascheroni constant, we end up with the following expression[43]:

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+\tilde{m}^2)\left(k'^2+\tilde{m}^2\right)} = \frac{i}{4\pi}\int_0^1 dx \frac{1}{\Delta} = \frac{iM^\varepsilon}{4\pi}\int_0^1 dx \frac{1}{\tilde{p}^2 x(1-x)+\tilde{m}^2}. \tag{523}$$

In principle, we should consider various cases, depending on the values of $\tilde{p}^2$; however, we are interested in the limit $\tilde{p} \to 0$, so the most convenient approach is to directly expand the integrand and consider the first term of the expansion, which is $\tilde{p}$-independent[44]. We have:

$$\frac{1}{\tilde{p}^2 x(1-x)+\tilde{m}^2} = \frac{1}{\tilde{m}^2} + \frac{\tilde{p}^2(x-1)x}{\tilde{m}^4} + \frac{\tilde{p}^4(x-1)^2x^2}{\tilde{m}^6} + \mathcal{O}(\tilde{p}^6). \tag{524}$$

Therefore, in this specific limit the result of the above integral is

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+\tilde{m}^2)\left(k'^2+\tilde{m}^2\right)} = \frac{iM^\varepsilon}{4\pi\tilde{m}^2} + \mathcal{O}(\varepsilon). \tag{525}$$

Before proceeding, let us make another observation about dimensions. For this discussion, as well as for the subsequent one, we will closely follow Ref. [44]. When calculating Feynman diagrams in $2+\varepsilon$ spacetime dimensions, the coupling constants will carry the dimension that is appropriate for the theory in $2+\varepsilon$ dimensions. For the scalar quantum electrodynamics built here, the dimension of the effective coupling constant turns out to be equal to $1-\varepsilon/2$ in mass units. On the other hand, in the 2-dimensional case we have that $[\mu q] = 1$ (integrating the sphere out does not change the dimensions of the quantity q). Therefore, in order to ensure that dimensional counting remains consistent throughout the calculations, we make again use of the auxiliary parameter M and write the effective coupling constant as $M^{-\varepsilon/2}\mu q$. Putting it all together (taking into account the various prefactors), we now write down the final expression for I_1 in $2+\varepsilon$ spacetime dimensions, in the limit $\tilde{p} \to 0$:

$$I_1^d\big|_{\tilde{p}\to 0} = \frac{iM^{-\varepsilon}\mu^2 q^4}{8\pi^3} \frac{\lambda'-1}{\lambda'(\lambda'+1)} + \mathcal{O}(\varepsilon). \tag{526}$$

Let us now consider the second contribution, namely I_2. Ignoring the prefactors for a moment, shifting to n dimensions, and writing $k' = k' + \tilde{m}^2 - \tilde{m}^2$, leads to

$$\int \frac{d^dk}{(2\pi)^d} \frac{k'^2}{(k^2+\tilde{m}^2)\left(k'^2+\tilde{m}^2\right)} = \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+\tilde{m}^2} - \tilde{m}^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)}. \tag{527}$$

The first term can be quite easily computed by performing a Wick rotation to use (516) with $\alpha = 1$, where the role of Δ is now played by $\tilde{m}^2$. We have:

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+\tilde{m}^2} = \frac{i}{(4\pi)^{\frac{n}{2}}}\Gamma\left(1-\frac{n}{2}\right)\left(\tilde{m}^2\right)^{\frac{n}{2}-1}. \tag{528}$$

We now substitute $d = 2+\varepsilon$ and expand in powers of ε, keeping track of possible poles at $\varepsilon = 0$.

[43]Expanding before performing the integral is allowed in this case since each term in the expansion, when integrated, converges.

[44]Concerning the expansion, the same reasoning explained in the previous footnote is applied here: each term in the expansion, when integrated, converges.

In terms of ε, Eq. (528) becomes

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+\tilde{m}^2} = \frac{i}{4\pi}\left(4\pi\right)^{-\frac{\varepsilon}{2}}\Gamma\left(-\frac{\varepsilon}{2}\right)\left(\tilde{m}^2\right)^{\frac{\varepsilon}{2}}. \tag{529}$$

Introducing M as before and rearranging, we get

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+\tilde{m}^2} = \frac{iM^\varepsilon}{4\pi}\left(4\pi\right)^{-\frac{\varepsilon}{2}}\Gamma\left(-\frac{\varepsilon}{2}\right)\left(\frac{\tilde{m}^2}{M^2}\right)^{\frac{\varepsilon}{2}}. \tag{530}$$

We can now safely expand, obtaining

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+\tilde{m}^2} = -\frac{iM^\varepsilon}{2\pi}\left[\frac{1}{\varepsilon}+\frac{1}{2}\gamma_E+\frac{1}{2}\ln\left(\frac{1}{4\pi}\frac{\tilde{m}^2}{M^2}\right)+\mathcal{O}(\varepsilon)\right], \tag{531}$$

which is now dimensionally consistent. Concerning the second term in (527), it has already been computed. Putting it all together, we obtain the final result for I_2:

$$I_2^d\Big|_{\tilde{p}\to 0} = -\frac{iM^{-\varepsilon}\mu^2q^4}{8\pi^3}\frac{\lambda'-1}{\lambda'^2}\left[\frac{1}{\varepsilon}+\frac{1}{2}\left(\gamma_E+1\right)+\frac{1}{2}\ln\left(\frac{\lambda'^2-1}{4\pi\lambda'}\frac{\mu^2}{M^2}\right)+\mathcal{O}(\varepsilon)\right]. \tag{532}$$

Now, looking at the third contribution to the amplitude, (508), we immediately notice that it is equal to (507) upon shifting the momentum k, $k\to k+\tilde{p}$. Therefore, we can directly focus on the fourth contribution, I_4. The strategy is to reduce this apparently complicated expression to well-known integrals. Let us first consider the numerator of the integrand. By recalling how k' is defined, it can be split as

$$\begin{aligned}\eta_{ab}\eta_{cd}k^bk^ck'^dk'^a &= k_ak'^ak_ck'^c = (k\cdot k')^2\\ &= k^2k'^2 - k'^2(\tilde{p}\cdot k)+k^2(\tilde{p}\cdot k')-(\tilde{p}\cdot k)(\tilde{p}\cdot k'). \end{aligned}\tag{533}$$

Thus, shifting to $2+\varepsilon$ spacetime dimensions, we obtain

$$\int \frac{d^dk}{(2\pi)^d}\frac{\eta_{ab}\eta_{cd}k^bk^ck'^dk'^a}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)} = \int\frac{d^dk}{(2\pi)^d}\frac{k^2k'^2-k'^2(\tilde{p}\cdot k)+k^2(\tilde{p}\cdot k')-(\tilde{p}\cdot k)(\tilde{p}\cdot k')}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)}. \tag{534}$$

As we can see, I_4 has been split into four contributions. The first gives

$$\int \frac{d^dk}{(2\pi)^d}\frac{k^2k'^2}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)} = \int\frac{d^dk}{(2\pi)^d}\frac{k'^2}{k'^2+\tilde{m}^2} - \tilde{m}^2\int\frac{d^dk}{(2\pi)^d}\frac{k'^2}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)}. \tag{535}$$

The second piece has already been calculated before. The first one can be easily computed by Wick rotating and making use of the following result [41]:

$$\int\frac{d^nk_E}{(2\pi)^n}\frac{k_E^2}{\left(k_E^2+\Delta\right)^\alpha} = \frac{n}{2}\frac{1}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma\left(\alpha-\frac{n}{2}-1\right)}{\Gamma(\alpha)}\Delta^{\frac{n}{2}-\alpha+1}. \tag{536}$$

In terms of ε, by setting $\alpha=1$ and shifting $k\to k+\tilde{p}$, we can write

$$\int\frac{d^dk}{(2\pi)^d}\frac{k'^2}{k'^2+\tilde{m}^2} = \frac{i\tilde{m}^2}{4\pi}(1+\varepsilon)(4\pi)^{-\frac{\varepsilon}{2}}\Gamma\left(-1-\frac{\varepsilon}{2}\right)\left(\tilde{m}^2\right)^{\frac{\varepsilon}{2}}. \tag{537}$$

Moreover, inserting M and expanding in powers of ε, we end up with

$$\int \frac{d^dk}{(2\pi)^d} \frac{k'^2}{k'^2+\tilde{m}^2} = \frac{iM^\varepsilon \tilde{m}^2}{2\pi} \left[\frac{1}{\varepsilon} + \frac{1}{2}(\gamma_E+1) + \frac{1}{2}\ln\left(\frac{1}{4\pi}\frac{\tilde{m}^2}{M^2}\right) + \mathcal{O}(\varepsilon) \right]. \tag{538}$$

Therefore, Eq. (535) gives

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^2 k'^2}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)} = \frac{iM^\varepsilon \tilde{m}^2}{\pi} \left[\frac{1}{\varepsilon} + \frac{1}{2}(\gamma_E+1) + \frac{1}{2}\ln\left(\frac{1}{4\pi}\frac{\tilde{m}^2}{M^2}\right) + \mathcal{O}(\varepsilon) \right]. \tag{539}$$

Let us now consider the second piece coming from (534). By writing $k' = k' + \tilde{m}^2 - \tilde{m}^2$, we get

$$\int \frac{d^dk}{(2\pi)^d} \frac{k'^2(\tilde{p}\cdot k)}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)} = \int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}\cdot k}{k^2+\tilde{m}^2} - \tilde{m}^2 \int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}\cdot k}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)}. \tag{540}$$

The first integral vanishes since the integrand is antisymmetric under $k \to -k$. Concerning the second one, upon shifting $k \to k + x\tilde{p}$ and combining the denominator by using Feynman's trick once again, we have (we ignore for a moment the factor in front, $\tilde{m}^2$)

$$\int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}\cdot k}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)} = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}\cdot k}{[(k-xq)^2+\tilde{p}^2x(1-x)+\tilde{m}^2]^2} \tag{541}$$

$$= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}\cdot(k+x\tilde{p})}{[k^2+\tilde{p}^2x(1-x)+\tilde{m}^2]^2}. \tag{542}$$

The first term, the one proportional to $\tilde{p}\cdot k$, vanishes. The remaining one leads to

$$\int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}\cdot k}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)} = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{x\tilde{p}^2}{[k^2+\tilde{p}^2x(1-x)+\tilde{m}^2]^2}. \tag{543}$$

We can now proceed in the same way as before, see below Eq. (514). However, we immediately notice that the first term of the expansion (524) would be multiplied by $\tilde{p}^2$, and so we can safely conclude that, in this specific limit, the above integral vanishes. The next contribution in (534) can be also shown to be vanishing. We have:

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^2(\tilde{p}\cdot k')}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)} = \int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}\cdot k'}{k'^2+\tilde{m}^2} - \tilde{m}^2 \int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}\cdot k'}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)}$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}\cdot k}{k^2+\tilde{m}^2} - \tilde{m}^2 \int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}\cdot(k-\tilde{p})}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)}. \tag{544}$$

The first piece in the above equation vanishes, again because the integrand is antisymmetric under $k \to -k$. In the second term we can easily recognize two expressions we already proved to be zero in the limit we are considering. Indeed, from the previous discussion we have that

$$\int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}\cdot k}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)} = 0, \tag{545}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}^2}{(k^2+\tilde{m}^2)(k'^2+\tilde{m}^2)} = 0, \tag{546}$$

where we made use of Eq. (525). We can now finally consider the last contribution in Eq. (534), which can be written as follows:

$$\int \frac{d^dk}{(2\pi)^d} \frac{(\tilde{p}\cdot k)(\tilde{p}\cdot k')}{(k^2+\tilde{m}^2)\,(k'^2+\tilde{m}^2)} = \int \frac{d^dk}{(2\pi)^d} \frac{\tilde{p}_a k^a \tilde{p}_b k'^b}{(k^2+\tilde{m}^2)\,(k'^2+\tilde{m}^2)}$$
$$= \tilde{p}_a\tilde{p}_b \int \frac{d^dk}{(2\pi)^d} \frac{k^a (k-\tilde{p})^b}{(k^2+\tilde{m}^2)\,(k'^2+\tilde{m}^2)}. \tag{547}$$

The second term in the numerator of the above expression vanishes since, neglecting the factor of $\tilde{p}^2$, it is exactly the same integral as in Eq. (543). Concerning the first piece, we have

$$\tilde{p}_a\tilde{p}_b \int \frac{d^dk}{(2\pi)^d} \frac{k^a k^b}{(k^2+\tilde{m}^2)\,(k'^2+\tilde{m}^2)} = \tilde{p}_a\tilde{p}_b \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(k+x\tilde{p})^a (k+x\tilde{p})^b}{(k^2+\Delta)^2}, \tag{548}$$

where the definition of Δ is the same as the one below Eq. (515). Splitting the numerator we immediately notice that the above expression gives rise to integrals that vanish as long as $\tilde{p}\to 0$. Thus, the only non-vanishing contribution in Eq. (534) is the first one. Putting it all together[45], we now write down the final result for I_4:

$$I_4^d\big|_{\tilde{p}\to 0} = \frac{iM^{-\varepsilon}\mu^2 q^4}{4\pi^3}\frac{\lambda'-1}{\lambda'^3}\left[\frac{1}{\varepsilon}+\frac{1}{2}\left(\gamma_E+1\right)+\frac{1}{2}\ln\left(\frac{\lambda'^2-1}{4\pi\lambda'}\frac{\mu^2}{M^2}\right)+\mathcal{O}(\varepsilon)\right]. \tag{549}$$

We are now ready to write down the final result for $i\mathcal{M}^{even}_{seagull}$ by summing each of these terms. To be more precise, the expression below does not yet represent the final result since it is ε-dependent. However, as we will see, it is possible to conclude that it is indeed sub-leading even without the need of renormalizing. Up to zeroth-order in ε, we have:

$$i\mathcal{M}^{even,d}_{seagull} = \frac{iM^{-\varepsilon}\mu^2 q^4}{8\pi^3}\frac{\lambda'-1}{\lambda'\left(\lambda'+1\right)} + \frac{iM^{-\varepsilon}\mu^2 q^4}{4\pi^3}\frac{\lambda'^2-1}{\lambda'^3}\left[\frac{1}{\varepsilon}+\frac{1}{2}\left(\gamma_E+1\right)\right.$$
$$\left.+\frac{1}{2}\ln\left(\frac{\lambda'^2-1}{4\pi\lambda'}\frac{\mu^2}{M^2}\right)\right]. \tag{550}$$

We now want to argue that the above contribution is sub-leading with respect to the one-loop amplitudes arising from the type of diagrams shown in subsection 5.2, which are given by

$$i\mathcal{M}_{2,p-p} = i\mathcal{M}_{2,p-a} = -\frac{q^4 s}{16\pi^2\left(\lambda'+1\right)^2}. \tag{551}$$

This last expression has been obtained by setting $n=2$ in our previous eikonal calculation, where we recall that n counts the number of virtual photon exchanged. In principle we still cannot compare the above results since the amplitude (550) is ε-dependent while (551) is not (we cannot set $\varepsilon=0$ in (550) because of the presence of the term $1/\varepsilon$). However, we know that this term will cancel upon renormalizing[46], and so we only have to worry about the finite pieces. One last comment concerns the presence of the parameter M in Eq. (550), which appears in the form of a logarithm[47]; this parameter has been introduced for dimensional reasons. However,

[45] Essentially we are considering the prefactors in (509) as well as the fact that the effective coupling constant has to be written as $M^{-\varepsilon/2}\mu q$.

[46] One-loop renormalizability of quantum electrodynamics in a general curved spacetime has been extensively discussed in Ref. [45].

[47] The $M^{-\varepsilon}$ term in front of Eq. (550) will vanish in the limit $\varepsilon\to 0$.

physical quantities should not depend on an auxiliary parameter in the end. This proves to be the case indeed, noticing that the renormalized parameters will implicitly depend on M as well. Of course, it is impossible to know the functional form of the amplitude without following the entire procedure, but we can argue that renormalization will not introduce any power of s in the calculation[48]. Thus, we can safely conclude that the amplitude (550) is sub-leading with respect to (551). This is not the end of the story since, as we previously deduced by Fourier transforming Eq. (417), the odd-parity photon led to another type of four-vertex, the last one shown on page 71. There is, therefore, another one-loop diagram that we can consider[49]: Mathematically, the

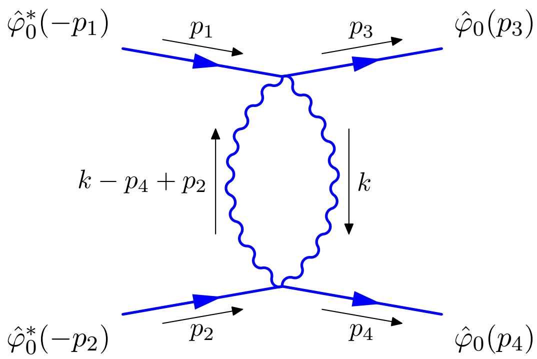


Figure 10: One-loop diagram containing the so-called “seagull" vertices (odd case).

above diagram corresponds to the following expression:

$$\begin{aligned} i\mathcal{M}^{odd}_{seagull} &= \int \frac{d^2k}{(2\pi)^2}\frac{1}{\lambda'}\frac{-i}{k^2+\tilde{m}^2}\left(-2i\frac{\mu^2 q^2}{4\pi}\right)\frac{1}{\lambda'}\frac{-i}{k'^2+\tilde{m}^2}\left(-2i\frac{\mu^2 q^2}{4\pi}\right) \\ &= \frac{\mu^4 q^4}{4\pi^2\lambda'^2}\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2+\tilde{m}^2}\frac{1}{k'^2+\tilde{m}^2} \\ &= \frac{i\mu^2 q^4}{16\pi^3}\frac{1}{\lambda'\left(\lambda'^2-1\right)}, \end{aligned} \tag{552}$$

where we made use of the result obtained in (525), in the limit $\varepsilon \to 0$. By applying the same argument as above, we can conclude that the amplitude (552) is sub-leading with respect to the one-loop amplitude with three-vertices only.

[48]This represents the key point since the sub-leading argument comes from the limit $s \gg \mu^2$.

[49]Note that, as before, momentum conservation at the top vertex implies $k - p_4 + p_2 - k + p_3 - p_1 = 0$.

Conclusions and future work

In Refs. [12–22], gravitational interactions on the black-hole horizon have been considered, both in the context of quantum mechanics and quantum field theory. One of the possible routes towards improvement is to include other fields in the picture, in principle everything known from the Standard Model of particle physics. In particular, we aim to understand how Standard-Model information falling into a black hole can be retrieved from scattering. The simplest field-theoretical interaction we can consider is the electromagnetic one, which is exactly what has been done in the present work.

Within 't Hooft's approach, in analogy to the gravitational backreaction discussed in the references above, we see that an electromagnetically charged shock wave leaves an imprint on the electromagnetic field of the probe, which indeed experiences a discontinuity in its electromagnetic field across the null surface traced out by the shock wave; we investigated how to expand the effect of the change of gauge of the electromagnetic field in partial waves and obtained an expression for the scattering matrix.

In the field-theory side, within a path-integral approach, the strategy that has been used is the following: starting from the action functional for a complex scalar field coupled to the electromagnetic field in a Schwarzschild background, we first considered the kinetic terms and inverted the quadratic operators in order to find the propagators; then, we looked at the interaction terms and finally defined the Feynman rules of the theory to compute all the scattering amplitudes of interest. However, being in curved space significantly complicates the procedure just outlined. Indeed, computing the propagator of even a scalar field is not analytically possible in the presence of a black hole. Luckily, by taking into account the high degree of symmetry of the Schwarzschild background, expanding in spherical harmonics, and restricting our attention to the near-horizon region, we have been able to reduce our four-dimensional theory to an infinite number of flat two-dimensional theories with potentials that capture the curvature effects, finally allowing us to invert the quadratic operators to find all the propagators of interest. Therefore, we proceeded with computing the effect of the change of gauge of the electromagnetic field via elastic $2 \to 2$ photon exchange diagrams. More precisely, we obtained an expression for the S-matrix by summing over an infinite number of such diagrams in the high-energy limit. The S-matrix so-obtained is in agreement with the one found within 't Hooft's approach. In this thesis, the theory presented in Refs. [19, 20] has been successfully extended to the electromagnetic case. Further, our model contains a four-vertex, leading to more types of loop-diagrams. In the last section we showed that the one-loop diagram with four-vertices is sub-leading with respect to the one with three-vertices only.

The black-hole scattering program developed in these last few years appears to be very rich. We conclude by listing some of the possible ideas that can be explored:

- Up to now, tree-level scattering amplitudes and one entire class of loop corrections have been computed. It would be certainly interesting to study higher-order derivative corrections, as well as include non-Abelian gauge fields in the picture;

- It would be of interest to understand and revisit the issue of antipodal identification [46,47] and quantum clones [18];

- So far, scattering matrix elements that arise from gravitational/electromagnetic interactions have been explicitly obtained, pretending the asymptotic states to be well-defined. However, it is not clear how to properly define the S-matrix in the presence of a black hole. Trying to understand this could allow us to combine Hawking's leading answer (free scalar fields in curved spacetime) and our corrections into one formula;

- Weinberg's soft graviton/photon theorem relates the matrix elements of a Feynman diagram with an external soft graviton/photon insertion to that of the same diagram without an external soft graviton/photon [48, 49]. An intriguing possibility is to try to understand if new soft theorems emerge near the black-hole horizon because of these gravitational/electromagnetic interactions;

- Analyzing the observational consequences of our research, especially in relation to gravitational waves, would be certainly of interest. A first step in this direction has been done in Ref. [50].

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