Symmetric effective and degenerate-preferring Kan complexes

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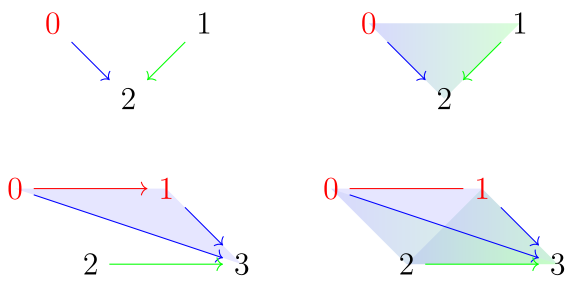

Abstract

Kan complexes and fibrations play a fundamental role in simplicial homotopy theory. Recently, in [BF22], the effective Kan fibrations were defined. These are part of a program to reformulate the foundations of simplicial homotopy theory in a constructive setting. For this thesis, we have compared results on Kan fibrations and complexes to the effective Kan fibrations and complexes. We introduce and study two subclasses of the effective Kan fibrations and complexes, namely the symmetric effective and degenerate-preferring Kan fibrations and complexes. We show that simplicial Malcev algebras have the structures of degenerate-preferring Kan complexes. We also show that the symmetric effective Kan fibrations fit into a lifting algebraic weak factorisation system as in [Bou23].

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I am incredibly thankful for the daily supervision of Benno van den Berg. I enjoyed his insights, feedback, guidance, good company and awesome research subject. He did it all while I was not even formally one of his students. I want to thank Jaap van Oosten for being my first supervisor, introducing me to logic and toposes, and for being a mentor throughout my studies. I want to thank Gijs Heuts for being my second supervisor and introducing me to higher category theory. I also want to thank Storm Diephuis and John Bourke for some particularly insightful discussions for this thesis.

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Contents

Chapter 1

Introduction

Kan fibrations and complexes play a fundamental role in simplicial homotopy theory, higher category theory and the Kan-Quillen model structure. Kan complexes, also known as ∞-groupoids, are simplicial sets which model topological spaces, and are used in [KL21] to model homotopy type theory.

In [BF22], a definition of effective Kan fibrations is given. This definition is part of a program which provides constructive formulations for the foundations of simplicial homotopy theory. By constructive, we mean that our proofs give us methods for explicit computations. If one does not care about constructivism, or explicit calculations in simplicial homotopy theory, the resulting structures are still interesting of themselves.

Where being a Kan fibration is a property, being an effective Kan fibration is a structure, allowing for these constructive proofs. For example, it is shown in [BCP15] that there is no constructive proof that Kan fibrations are closed under pushforwards, while in [BF22] it is shown constructively that the effective Kan fibrations are.

Effective Kan fibrations are thus a new subject, as are the effective Kan complexes. And there are some natural open questions. We know a lot about Kan complexes and fibrations and we might see if these facts also hold for effective Kan complexes and fibrations.

One of those facts is that all simplicial groups have the property of being a Kan complex. The question whether they also have the structure of an effective Kan complex was the starting point of this thesis. We shall answer a generalized version of this question in Chapter 6. In fact, simplicial groups fit into a specific subclass of the effective Kan complexes: namely the degenerate-preferring Kan complexes, which are newly introduced in this thesis in Definition 5.5.

Another fact concerns the role of Kan fibrations in Kan-Quillen model theory. There, Kan fibrations form the right class in the weak factorisation system. In this thesis, we will use a more explicit version of such a system, called a lifting algebraic weak factorisation system, from [Bou23]. Between the effective Kan complexes and the degenerate-preferring Kan complexes lie the symmetric effective Kan complexes, which are newly introduced in Definition 3.1. In Chapter 7, we will show that the symmetric effective Kan fibrations fit into a right category of a lifting algebraic weak factorisation system.

To summarize, the main contributions of this thesis are:

- The definitions of symmetric effective and degenerate-preferring Kan complexes and fibrations in Definitions 3.1 and 5.5.
- A generalization of the fact that simplicial groups are Kan complexes to degenerate-preferring Kan complexes in Chapter 6.
- A proof that the symmetric effective Kan fibrations fit into a lifting algebraic weak factorisation system in Chapter 7.

A shortcoming of this thesis is that for these newly introduced symmetric effective and degenerate-preferring Kan fibrations, it becomes an open question whether they are closed under pushforwards, which we do not address.

With the exceptions of Chapters 5 and 7, we shall restrict ourselves to constructive arguments. Chapter 7 can be read directly after Chapter 3, the other chapters are meant to be read in the given order. The chapters of this thesis are as follows:

- We recall the necessary preliminaries about Kan complexes in Chapter 2.
- We motivate the definition of symmetric effective Kan complexes in Chapter 3, using pictures.
- We discuss two basic examples of Kan complexes in Chapter 4.
- In Chapter 5, we will introduce some definitions we will need in Chapter 6. These definitions will be the constructivist version of some classical definitions, and we will discuss the constructivist nuance between these new definitions and the classical definitions.
- In Chapter 6, we will explain and prove the generalization of the statement that all simplicial groups are Kan complexes.
- In Chapter 7, we recall the definition of a lifting algebraic weak factorisation system and show how the symmetric effective Kan fibrations fit into a right class of such a system.
- One of the main motivations to study effective Kan complexes lies in homotopy type theory. In Appendix A, we shall define the effective Kan complexes and give some mathematical and historical context to the definition.
- The original question of this thesis was to show that simplicial groups carry the structure of an effective Kan complex. In studying this question, some Haskell code was developed, which is included in Appendix B.

Chapter 2

Preliminaries

In this chapter, we shall introduce some concepts from higher category theory relevant to this thesis. In particular, we want to introduce the notion of Kan complexes. There are many sources available on the subject. Standard references include [Lur22] and [May67]. All of the work presented here is folklore and can be found in most references. The exception is Proposition 2.1, which can be considered folklore, but is not usually mentioned.

We assume the reader has some knowledge on category theory. In particular, we assume familiarity with limits, colimits, functors, adjunctions, presheaf categories and the Yoneda lemma. Other concepts that are of interest will be explained in this chapter. Many text books and course notes on category theory are available, we mention [Rie16] as an example.

2.1 The simplex category Δ

For each natural number $n \in \mathbb{N}$, we define the category $[n]$ corresponding to the poset $\mathbb{N}_{\leq n}$, ordered under $\leq$. Functors between the category $[n]$ and $[m]$ are then order-preserving functions between $\mathbb{N}_{\leq n}$ and $\mathbb{N}_{\leq m}$.

Let Δ be the category with the objects $\{[n] | n \in \mathbb{N}\}$ and morphisms the order-preserving functions. Δ is called the **simplex category**.

The simplex category has two special classes of maps, called the degeneracy maps and the face maps.

- For any $n \in \mathbb{N}$, $0 \leq i \leq n$, we have a **degeneracy map** s_i hitting i twice.

$$s_i : [n+1] \to [n], \qquad s_i(k) = \begin{cases} k \text{ if } k \leq i \\ k-1 \text{ if } k > i \end{cases} \tag{2.1}$$

- For $0 \leq i \leq n$, we have a **face map** d_i skipping over i.

$$d_i : [n] \to [n+1], \qquad d_i(k) = \begin{cases} k \text{ if } k < i \\ k+1 \text{ if } k \geq i \end{cases} \tag{2.2}$$

Between the face and degeneracy maps, we have the following composition laws, called **simplicial identities**:

$$s_j \circ d_k = \begin{cases} d_{k-1} \circ s_j & \text{if } k > j+1 \\ 1 & \text{if } k \in \{j, j+1\} \\ d_k \circ s_{j-1} & \text{if } k < j \end{cases} \tag{2.3}$$

$$d_j \circ d_k = d_{k+1} \circ d_j \text{ if } k \geq j \tag{2.4}$$

$$s_j \circ s_k = s_k \circ s_{j+1} \text{ if } j \geq k \tag{2.5}$$

The special thing about these maps is that every map in Δ can be written as composition of face and degeneracy maps. If we require a strict order in this composition, it is unique.

Proposition 2.1. *Any morphism $f : [m] \to [n]$ of Δ, can be written as a composition*

$$f = d_{k_1} \circ \cdots \circ d_{k_a} \circ s_{j_1} \circ \cdots \circ s_{j_b} \tag{2.6}$$

such that $n \geq k_1 > \cdots > k_a \geq 0$ and $0 \leq j_1 < \cdots < j_b \leq m$. This normal form is unique.

Proof. Let $f : [m] \to [n]$ be a morphism in Δ. Let K be the set of elements in $[n]$ that are not in the image of f. Let J be the set of elements i in $[m]$ such that $f(i) = f(i+1)$.

We claim that K, J together uniquely determine f. First note that K uniquely determines the image of f.

Suppose $f' : [m] \to [n]$ shares the same image and J as above. I claim that $f(i) = f'(i)$ for all $0 \leq i \leq m$.

- If $i = 0$, then $f(0), f'(0)$ must both equal the least element of their image.
- Suppose that $f(i) = f'(i)$ for some $i < n$. As f is order-preserving, we must have $f(i+1) \geq f(i)$. There are two options:
  - If $f(i+1) = f(i)$, then $i \in J$. Hence $f'(i+1) = f'(i)$, which by induction hypothesis equals $f(i)$.
  - If $f(i+1) > f(i)$, then $i \notin J$, hence $f'(i+1) > f(i)$ as well. Then both $f'(i+1)$ and $f(i+1)$ must be equal to the least element above $f(i) = f'(i)$ in their image, as both f, f' are order-preserving.

  In both these cases, we have $f(i+1) = f'(i+1)$.

Hence K, J uniquely determine f.

Now take $k_1 > k_2 > \cdots > k_a$ by sorting K and $j_1 < j_2 < \cdots < j_b$ by sorting J. We claim that

$$d_{k_1} \circ \cdots \circ d_{k_a} \circ s_{j_1} \circ \cdots \circ s_{j_b} \tag{2.7}$$

equals f. First note that as $J \subseteq [m]$, $K \subseteq [n]$, the above is indeed a function $[m] \to [n]$.

Similar to before, define the sets K', J' for function 2.7. We shall show that $K' = K, J' = J$.

Remark that $d = d_{k_1} \circ \cdots \circ d_{k_a}$ is injective and that $s = s_{j_1} \circ \cdots \circ s_{j_b}$ is surjective. Hence to study K', we only care about the image of d. And to study J', we only care about the elements i such that $s(i) = s(i+1)$.

- $K = K'$. In general for any morphism f, we have that $d_k \circ f$ does not have k in its image.

  Now suppose that g does not have k in its image, and $i > k$. Then $d_i \circ g$ cannot have k in its image, as d_i is the identity for numbers $\leq k$ and order-preserving above k. So nothing can get sent to k.

  By repeating this argument, $d_{k_1} \circ \cdots \circ d_{k_a} \circ s$ cannot have any of the k_i in its image. Hence $K \subseteq K'$.

  Note that d is injective and must be a function $[n-a] \to [n]$. As such, K' must have exactly a elements. As K also has a elements, we conclude that $K = K'$.

- $J = J'$ Let $i \in J$. Note that for $j > i$, we have that $s_j(i+1) = i+1, s_j(i) = i$, so we can ignore the degeneracy maps until we encounter s_i. Also $s_i(i) = i = s_i(i+1)$, so

$$s(i+1) = s_{j_1} \circ \cdots \circ s_{j_b}(i+1) = \tag{2.8}$$
$$s_{j_1} \circ \cdots \circ s_i(i+1) = \tag{2.9}$$
$$s_{j_1} \circ \cdots \circ s_i(i) = \tag{2.10}$$
$$s_{j_1} \circ \cdots \circ s_{j_b}(i) = s(i) \tag{2.11}$$

  Note that s is surjective and must be a function $[n] \to [n-b]$. As such, J' must have exactly b elements, just like J. We conclude that $J = J'$.

So f can be written as composition 2.7.

Remark that if there are two such compositions both describing f, so

$$f = d_{k_1} \circ \cdots \circ d_{k_a} \circ s_{j_1} \circ \cdots \circ s_{j_b} = d_{k'_1} \circ \cdots \circ d_{k'_{a'}} \circ s_{j'_1} \circ \cdots \circ s_{j'_{b'}} \tag{2.12}$$

these compositions must then share the sets K and J as above. But the sets K, J uniquely determine a, b and the k_i, j_i sequences. So these compositions must be equal.

We conclude that the normal form is unique.

□

Remark 2.2. The above theorem is usually inferred from standard epi-mono factorisation and the remark that epimorphisms correspond to compositions of degeneracy maps and monomorphisms correspond to composition of face maps. We chose to present it explicitly as this normal form allows us to computationally check for equality between simplicial morphisms.

In the Haskell file `SimpId.hs` (see Appendix B.1) we have included some code to calculate this normal form. ◊

2.2 Simplicial sets

For $\mathcal{C}$ any category, a simplicial object of $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$.

A **simplicial set** is a simplicial object of the category of sets, which is the same as a presheaf on Δ. The category of simplicial sets is denoted $sSet$

Because morphisms in Δ are generated by face and degeneracy maps, to give a simplicial set X, we need to give

- for each $n \in \mathbb{N}$ a set X_n.
- An action of degeneracy maps $X(s_i) : X_n \to X_{n+1}$.
- An action of face maps $X(d_i) : X_{n+1} \to X_n$.

such that the actions satisfy the composition laws given by the simplicial identities.

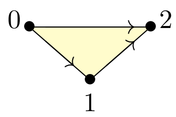
2.3 Simplices

In a presheaf category, the representable presheaves are of special interest. For the category of simplicial sets, a representable preseheaf is called a **simplex** (plural: simplices). The simplex corresponding to $[n]$ is denoted Δ^n and called the n-simplex.

For a simplicial set G, the Yoneda lemma allows us to identify maps $\Delta^n \to G$ as elements of G_n, which are also called n-simplices or n-cells of G.

We sometimes draw the simplex Δ^n in a figure, by explicitly drawing all non-identity arrows of the poset $[n]$.

For example, Δ^2 can be drawn as follows:



(2.13)

For any face map $d_i : [n-1] \hookrightarrow [n]$ in Δ, we have a corresponding face subobject $d_i : \Delta^{n-1} \hookrightarrow \Delta^n$ in sSet.

The faces of Δ^2 in (2.13) are 1-cells, which we draw as arrows, because they have an orientation due to the action on face maps. For example, the face d_1 corresponds with the arrow $0 \to 2$.

2.4 Lattice structure on subobjects

sSet is a presheaf category, and all presheaf categories are toposes, about which we will now say something more general. For any object C of a topos, there is a lattice structure on the subobjects of C. This lattice structure is given as follows: let $a : A \hookrightarrow C, b : B \hookrightarrow C$ represent two subobjects of C.

- We write $A \subseteq B$ if there is a map $c : A \to B$ such that $a = b \circ c$. We also say that A **factors through** B, or that a factors through b.
- The intersection $A \cap B$ is given by the pullback of a and b.
- The union $A \cup B$ can be given by the pushout along the arrows $A \cap B \to A, A \cap B \to B$.

$$\begin{tikzcd}
A \cap B \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, hook, "a"] & \\
B \arrow[r, hook] \arrow[drr, hook, "b"'] & A \cup B \arrow[dr, hook, dashed] & \\
& & C
\end{tikzcd} \tag{2.14}$$

Subobjects of representable presheaves are called **sieves**. The sieves of particular interest in defining Kan complexes are horn inclusions.

2.5 Horn inclusions

For all $n > 0$ and each $0 \leq m \leq n$, we can define the **horn** Λ^n_m as the union of all faces of Δ^n except the m'th face:

$$\Lambda^n_m = \bigcup_{\substack{0 \leq k \leq n \\ k \neq m}} d_k \tag{2.15}$$

and a horn inclusion is a representing mono $\Lambda^n_m \hookrightarrow \Delta^n$.

An **inner horn** is a horn Λ^n_m with $0 < m < n$.

For any simplicial set G, a **horn map** is a map $x : \Lambda^n_m \to G$. To define such a horn map, we can define it on all faces of the horns, but we need to make sure that the map is properly defined on the intersections $d_k \cap d_j$. So in order to give such a horn map, it is necessary and sufficient to give:

- For each $0 \leq k \leq n$ with $k \neq m$, an element $x^k \in G_{n-1}$,
- such that $G(d_j)(x^k) = G(d_{j'})(x^{k'})$ whenever $d_k \circ d_j = d_{j'} \circ d_{k'}$.

Because d_k factors through Λ^n_m for all $0 \leq k \leq n$ with $k \neq m$, we will abuse the notation and write $x \circ d_k$ for x^k.

2.6 Lifting property

Let $l : A \to B$ be a map. An object X is said to have the right lifting property (**RLP**) against l iff for any map $x : A \to X$, there exists a map $B \to X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "l"'] & X \\ B \arrow[ur, dashed] & \end{tikzcd} \tag{2.16}$$

The same definition can be made for maps instead of objects. We say that $r : X \to Y$ has the RLP against l if for any maps $x : A \to X, y : B \to Y$, there exists a map $B \to X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "l"'] & X \arrow[d, "r"] \\ B \arrow[ur, dashed] \arrow[r, "y"'] & Y \end{tikzcd} \tag{2.17}$$

If this is the case, we also say that l has the left lifting property (**LLP**) against r.

In both cases, the dashed arrow can be called an extension, lift or filler.

This allows us to make the following definitions:

- If a simplicial set G has the RLP against all horn inclusions, we call G a **Kan complex**.
- If a simplicial set G has the RLP against all inner horn inclusions, we call G an **∞-category**.

- If a morphism of simplicial sets r has the RLP against all horn inclusions, we call r a **Kan fibration**.

Remark that Δ^0 is a terminal object in the category of simplicial sets. Hence a simplicial set G is a Kan complex iff the unique morphism $G \to \Delta^0$ is a Kan fibration. Hence we generally study Kan fibrations, as we can translate the results to Kan complexes. However, as the definitions are similar, it is sometimes easier to explain something about Kan complexes. Therefore, we will sometimes use the terms interchangeably when outside of a formal mathematical context.

2.7 ∞-categories and Kan complexes

Suppose that $\mathcal{C}$ is an ∞-category, and let C, D, E be 0-cells of $\mathcal{C}$. Let f, g be 1-cells of G with $d_0(f) = d_1(g) = D$ and $d_1(f) = C, d_0(g) = E$. This means there is an inner horn map $y : \Lambda_1^2 \to G$ with $y \circ d_0 = g, y \circ d_2 = f$. Because G has the RLP against such maps, there is a simplex $z \in G_2$ with $z \circ d_0 = g, z \circ d_2 = f$. We define $h := z \circ d_1$ and because of simplicial identities, we have that $h \circ d_0 = E, h \circ d_1 = C$.

$$\begin{tikzcd} C \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & E \\ & D \arrow[ur, "g"'] & \end{tikzcd} \tag{2.18}$$

In this sense, an ∞-category indeed has a categorical structure.

- 0-cells can be interpreted as objects.
- 1-cells f with $d_0(f) = D, d_1(f) = C$ are interpreted as morphisms $f : C \to D$.
- A 2-cell z with $d_0(z) = g, d_1(z) = h$ and $d_2(z) = f$ is interpreted as a commuting triangle stating that $h = g \circ f$.
- For each 0-cell C, we have that $d_0(s_0(C)) = d_1(s_0(C))$, and we can interpret $s_0(C)$ as identity morphism $C \to C$.

Note however that the "composition" as defined above is not necessarily unique. In an ∞-category we require existence of a filler, and there might be more possible fillers. We say that "z witnesses that $g \circ f = h$".

If we require that G is a Kan complex, we also have fillers for the horns

$$\begin{tikzcd} C \arrow[rr, "s_0(C)"] \arrow[dr, "f"'] & & C \\ & D \arrow[ur, dashed] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D \arrow[rr, "s_0(D)"] \arrow[dr, dashed] & & D \\ & C \arrow[ur, "f"'] & \end{tikzcd} \tag{2.19}$$

giving that f has a right and left inverse, hence an inverse. For this reason, Kan complexes are also called ∞-groupoids.

So in a Kan complex, 1-cells can be composed and inverted, and for every 0-cell x there is a reflexive 1-cell with x at both of its endpoints. For this reason, 1-cells can be interpreted as paths. 2-cells can then be seen as paths between paths, also called homotopies. This point of view comes from homotopy type theory (see appendix A).

Chapter 3

Symmetric effective Kan complexes

In this chapter, we will introduce the symmetric effective Kan complexes. The definition of symmetric effective Kan complexes, while newly introduced in this thesis, is based on the definition of effective Kan complexes from [BF22]. We will not go into the effective Kan complexes in this chapter, but we will come back to them in Appendix A. Instead, we will introduce the symmetric effective Kan complexes as a standalone definition.

- We will start of with some intuition by sketching out the idea and working out an example with pictures.
- We will then work out what symmetric effective Kan complexes look like in general, using diagrams instead of pictures.
- We shall use these to give the formal definition.

3.1 The idea

Last chapter, we have seen that a Kan complex is a simplicial set G which has for any map from a horn into G a lift extending this map to the entire simplex.

$$\begin{tikzcd} \Lambda^n_m \arrow[r, "y"] \arrow[d, hook] & G \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{3.1}$$

Being a Kan complex is thus a property of a simplicial set. Being a symmetric effective Kan complex entails not only having lifts against horn inclusions, but having a specific choice of lifts. This choice of lifts is a structure. So being a symmetric effective Kan complex is a structure on a simplicial set. These lifts must then be compatible in some sense.

So instead of mere existence of extensions, we require that for each map y from a horn into G, we have a specific choice of extension $\mathrm{fil}(y)$.

$$\begin{tikzcd} \Lambda^n_m \arrow[r, "y"] \arrow[d, hook] & G \\ \Delta^n \arrow[ur, dashed, "\mathrm{fil}(y)"'] & \end{tikzcd} \tag{3.2}$$

This "choice function" fil should satisfy certain conditions. Inspired by Theorem 12.1 from [BF22], these conditions amount to stability along pullback along degeneracy maps. To understand what this means, consider any horn map

$$\begin{tikzcd} \Lambda^n_m \arrow[r, "y"] \arrow[d, hook] & G \\ \Delta^n & \end{tikzcd} \tag{3.3}$$

We can pull back this map along a degeneracy map s_j:

$$\begin{tikzcd} s_j^*(\Lambda^n_m) \arrow[r, "\sigma_j"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda^n_m \arrow[r, "y"] \arrow[d, hook] & G \\ \Delta^{n+1} \arrow[r, "s_j"] & \Delta^n & \end{tikzcd} \tag{3.4}$$

Note that $s_j^*(\Lambda^n_m)$ is not a horn. So we have not required that fil has lifts against $y \circ \sigma_j$. However, in some sense it does. If we have fil, there are multiple ways to extend the map $y \circ \sigma_j$ to all of Δ^{n+1}.

- We can take our extension of y and compose this with s_j:

$$\begin{tikzcd} \Lambda^n_m \arrow[r, "y"] \arrow[d, hook] & G \\ \Delta^n \arrow[ur, dashed, "\mathrm{fil}(y)"'] & \end{tikzcd} \tag{3.5}$$

  using this lift, we have that

$$\begin{tikzcd} s_j^*(\Lambda^n_m) \arrow[r, "\sigma_j"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda^n_m \arrow[r, "y"] \arrow[d, hook] & G \\ \Delta^{n+1} \arrow[r, "s_j"] & \Delta^n \arrow[ur, dashed, "\mathrm{fil}(y)"'] & \end{tikzcd} \tag{3.6}$$

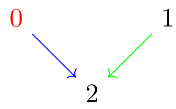
  commutes, giving that $\mathrm{fil}(y) \circ s_j$ extends $y \circ \sigma_j$.

- Using fil, we can extend $y \circ \sigma_j$ to a horn map $s_j^*(y) : \Lambda^{n+1}_{m^*} \to G$, and extend this map using fil again. This is a bit more complicated, and will be further explained in section 3.3, after we will have given an example in section 3.2.

If those two ways give the same result for any horn map $y : \Lambda^n_m \to G$, we shall call G a symmetric effective Kan complex with structure fil.

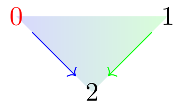
3.2 An example

Consider the horn Λ^2_1, of which we shall draw a picture with colors. These colors can be interpreted as values in a simplicial set G. We shall draw d_0 in green, d_1 in blue and $d_1 \circ d_1$ in red.



(3.7)

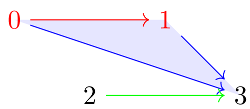
Let's call this horn map y. Consider now the situation where y has a filler, as drawn in the following figure:



(3.8)

We will pull back y along s_0. Now $s_0 : \Delta^3 \to \Delta^2$ sends the points $0, 1$ both to 0, sends 2 to 1 and 3 to 2. We get that the pullback of the point 0 is the diagram $0 \to 1$, the pullback of 1 is 2 and the pullback of 2 is 3.

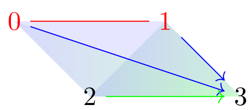
Geometrically, we interpret pulling back along s_0 as taking the red point, and stretching it out to a red line. We get a subset of a tetrahedron (which is a sieve, as the tetrahedon is the representable 3-simplex) as follows:



(3.9)

Note that when we stretch out the red point, we also stretch out a face of the blue line. As a consequence, the entire blue line should be stretched out along that face as well. This corresponds to the face d_2 of this tetrahedron being equal to $y \circ d_1 \circ s_0$.

Now suppose we want to fill this sieve to the entire tetrahedon. If we have a specific choice of fillers, it makes sense to pick the filler for y on the faces which look like y, which are d_0 and d_1, as their boundary contains the original horn (the blue and green arrows). We get a new horn:



(3.10)

Now so far, we have pulled back our original horn map y along s_0, we got a map with as domain a sieve. We then extended this map to two more faces, giving a higher-dimensional horn map, let's call it $s_0^*(y)$. If we are in a Kan complex, this new horn map also has a filler.

We also could have taken $\mathrm{fil}(y)$, which is defined on Δ^2 we could then precompose this map with s_0, and get something defined on the entire tetrahedron Δ^3.

Note that $s_0^*(y)$ equals $\mathrm{fil}(y) \circ s_0$ on its domain. Therefore $\mathrm{fil}(y) \circ s_0$ is a possible filler for $s_0^*(y)$. The condition on G being a symmetric effective Kan complex will be that it chooses exactly this filler.

3.3 The general case

Consider the situation of pulling back a horn map as in section 3.1

$$\begin{tikzcd}
s_j^*(\Lambda^n_m) \arrow[r, "\sigma_j"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda^n_m \arrow[r, "y"] \arrow[d, hook] & G \\
\Delta^{n+1} \arrow[r, "s_j"] & \Delta^n &
\end{tikzcd} \tag{3.11}$$

This section, we shall argue that the sieve inclusion $s_j^*(\Lambda^n_m) \hookrightarrow \Delta^{n+1}$ factors through a horn inclusion $\Lambda^{n+1}_{m^*} \hookrightarrow \Delta^{n+1}$. We shall show how to extend $y \circ \sigma_j$ to a map $s_j^*(y) : \Lambda^{n+1}_{m^*} \to G$, which we shall use to formalize our definition.

As we have seen in section 2.5, we can define a horn map facewise. We will therefore study $s_j^*(\Lambda^n_m)$ facewise. By the pullback property, we have that d_k factors through $s_j^*(\Lambda^n_m)$ iff $s_j \circ d_k$ factors through Λ^n_m. We will now use some simplicial identities.

- As $s_j \circ d_j = s_j \circ d_{j+1} = id$, we can see that the faces d_j and d_{j+1} do not factor through $s_j^*(\Lambda^n_m)$.
- If $j \neq m$, we have for
$$(j^*, m^*) = \begin{cases} (j-1, m) \text{ if } m < j \\ (j, m+1) \text{ if } m > j \end{cases} \tag{3.12}$$
that $s_j \circ d_{m^*} = d_m \circ s_{j^*}$. Now $d_m \circ s_{j^*}$ does not factor through Λ^n_m. So if $j \neq m$, we have that d_{m^*} does not factor through $s_j^*(\Lambda^n_m)$.
- For all other faces d_k, we have that $s_j \circ d_k = d_{k'} \circ s_{j'}$ for some $k' \neq m$ and some j', hence those d_k do factor through $s_j(\Lambda^n_m)$.

While d_j, d_{j+1} do not factor through $s_j^*(\Lambda^n_m)$, the faces do intersect $s_j^*(\Lambda^n_m)$. And we do know what this intersection looks like. If we use $d_{j\pm}$ as shorthand for both d_j and d_{j+1}, we can see that $s_j \circ d_{j\pm} = id$. Since taking intersection with $d_{j\pm}$ corresponds to taking a pullback along $d_{j\pm}$, and the pullback of the identity is always the identity, we get the following diagram:

$$\begin{tikzcd}
\Lambda^n_m \arrow[r, dotted] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "id"] & s_j^*(\Delta^n_m) \arrow[r, "\sigma_j"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda^n_m \arrow[r, "y"] \arrow[d, hook] & G \\
\Delta^n \arrow[r, "d_{j\pm}"] \arrow[rr, bend right, "id"'] & \Delta^{n+1} \arrow[r, "s_j"] & \Delta^n &
\end{tikzcd} \tag{3.13}$$

As said, an effective Kan complex should have a specific choice of filler for each horn map. So a natural choice as filler for the face $d_{j\pm}$ is $\mathrm{fil}(y)$. Therefore we can extend $y \circ \sigma_j$ to the domain $s_j^*(\Lambda^n_m) \cup d_{j\pm}$, with the value on face $d_{j\pm}$ given by $\mathrm{fil}(y)$. Note that this is properly defined at the intersection of $d_{j\pm}$ and $s_j^*(\Lambda^n_m)$, as that intersection is the horn Λ^n_m.

Now depending on whether m and j are equal, we can extend with either one or two faces and get a horn map.

- If $m \neq j$, then we missed three faces in $s_j^*(\Lambda_m^n)$, hence if we add the faces d_j, d_{j+1} as above, we get a new horn map $\Lambda_{m^*}^n \to G$.
- If $m = j$, then we only miss two faces, namely $m, m+1$, and we need only add one face to get a new horn map. So we get two possible horn maps: one where we add d_m, and one where we add d_{m+1}.

So to summarize, for all m^* with

$$m^* \in \begin{cases} \{m\} \text{ if } m < j \\ \{m, m+1\} \text{ if } m = j \\ \{m+1\} \text{ if } m > j \end{cases} \tag{3.14}$$

we have a map $s_j^*(y) : \Lambda_{m^*}^{n+1} \to G$ with the following values on its faces:

$$s_j^*(y) \circ d_k = \begin{cases} y \circ s_j \circ d_k \text{ if } k \neq j, j+1, m^* \\ \text{fil}(y) \text{ if } k \in \{j, j+1\} - \{m^*\} \end{cases} \tag{3.15}$$

And we can take $\text{fil}(s_j^*(y))$, which extends $y \circ \sigma_j$.

Definition 3.1. A simplicial set G is a **symmetric effective Kan complex** if it comes equiped with an operation fil which takes as input any horn map $y : \Lambda_m^n \to G$ and gives at output and extension $\text{fil}(y) : \Delta^n \to G$ in such a way that for any $0 \leq j \leq n$ and any m^* and $s_j^*(y)$ as described above, we have $\text{fil}(s_j^*(y)) = \text{fil}(y) \circ s_j$. $\Diamond$

Now just as in section 2.6, we can define the notions of a symmetric effective Kan fibration and a symmetric effective ∞-category.

Definition 3.2. A morphism of simplicial sets $\alpha : X \to Y$ is a **symmetric effective Kan fibration** if it comes equiped with an operation lift which takes as input any pair of maps $x : \Lambda_m^n \to X, y : \Delta^n \to Y$ as in the following diagram

$$\begin{tikzcd} \Lambda_m^n \arrow[r, "x"] \arrow[d, hook] & X \arrow[d, "\alpha"] \\ \Delta^n \arrow[r, "y"] & Y \end{tikzcd} \tag{3.16}$$

and gives as output a lifting map $\text{lift}(x, y) : \Delta^n \to X$ in such a way that for any $0 \leq j \leq n$ and any m^* and $s_j^*(x)$ as described above, we have $\text{lift}(s_j^*(x), y \circ s_j) = \text{lift}(x, y) \circ s_j$. $\Diamond$

Remark 3.3. As in section 2.6, note that the point Δ^0 is the terminal simplicial set. A simplicial set G is a symmetric effective Kan complex iff the unique arrow $G \to \Delta^0$ is a symmetric effective Kan fibration. $\Diamond$

Remark 3.4. If Λ_m^n is an inner horn, we have that $0 < m < n$. As $m^* \in \{m, m+1\}$, we also have that $0 < m^* < n+1$. Hence definition 3.1 makes sense if we restrict it to inner horns. This defines a **symmetric effective ∞-category**. $\Diamond$

Chapter 4

Basic examples

The following three classes of examplex of Kan complexes will be important for this thesis:

- Singular simplicial sets.
- Nerves of groupoids.
- Simplicial groups.

For these standard examples, the standard proofs give algorithms for finding a filler for horn inclusions. These proofs thus not only give existence of a filler, they actually show us how to compute fillers: they are constructive. So these proofs give us a functional choice of fillers: they actually give us a structure. A natural question is whethere this structure satisfies the conditions from the previous chapter.

We can use the proof that nerves of groupoids and simplicial groups are Kan complexes to actually give the structure of a symmetric effective Kan complex. For singular simplicial sets, this is not the case. This does not mean that there is no structure for a symmetric effective Kan complex on singular simplicial sets, but the standard method does not give us such a structure.

In this chapter, we shall discuss the proofs that singular simplicial sets and nerves of groupoids are Kan complexes. The proof for simplicial groups has been generalized to simplicial Malcev algebras, and we will discuss this generalization in Chapter 6. The standard proofs are of course not original work, but the question and answer of whether they also give the structure of a symmetric effective Kan complex is new.

4.1 Nerves of a groupoid

A classical example of Kan complexes are nerves of groupoids. For any category, the nerve of that category consists of sequences of composable morphisms.

Definition 4.1. Let $\mathcal{C}$ be a small category. The **nerve** of $\mathcal{C}$ is the simplicial set $\mathcal{N}_\bullet(\mathcal{C})$ which is defined as follows:

- For each object $[n]$ of Δ, $\mathcal{N}_n(\mathcal{C})$ is the set of diagrams $[n] \to \mathcal{C}$.

- Let $n \in \mathbb{N}$, $0 \leq k \leq n$ and suppose $f \in \mathcal{N}_n(\mathcal{C})$ is given by

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_k} C_k \xrightarrow{f_{k+1}} \cdots \xrightarrow{f_n} C_n \tag{4.1}$$

then $s_k(f)$ is given by

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_k} C_k \xrightarrow{id} C_k \xrightarrow{f_{k+1}} \cdots \xrightarrow{f_n} C_n \tag{4.2}$$

- Let $n \geq 1$ and $0 < k < n$ and let $f \in \mathcal{N}_n(\mathcal{C})$ be as follows:

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{k-1}} C_{k-1} \xrightarrow{f_k} C_k \xrightarrow{f_{k+1}} C_{k+1} \xrightarrow{f_{k+2}} \quad \xrightarrow{f_n} C_n \tag{4.3}$$

Then $d_k(f)$ is the element of $\mathcal{N}_{n-1}(\mathcal{C})$ given by

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{k-1}} C_{k-1} \xrightarrow{f_{k+1} \circ f_k} C_{k+1} \xrightarrow{f_{k+2}} \cdots \xrightarrow{f_n} C_n \tag{4.4}$$

- Let $n \geq 1$ and $f \in \mathcal{N}_n(\mathcal{C})$ be as follows:

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n \tag{4.5}$$

then $d_0(f)$ is given by

$$C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n \tag{4.6}$$

and $d_n(f)$ is given by

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1} \tag{4.7}$$

$\diamondsuit$

A classical result is that a nerve of a category is a Kan complex iff that category is a groupoid, see for example [Lur22, Proposition 0037]. This theorem generalizes to symmetric effective Kan complexes.

Theorem 4.2. *For any category $\mathcal{C}$, the nerve $\mathcal{N}_\bullet(\mathcal{C})$ is a symmetric effective Kan complex iff $\mathcal{C}$ is a groupoid.*

Remark 4.3. Note that any diagram $[n] \to \mathcal{C}$ is uniquely determined by its values on the objects in $[n]$ and the morphisms of the form $k \leq k+1$ in $[n]$. These correspond to the objects C_i and morphisms f_i in the definition above.

In particular, for any sieve S of Δ^n which contains all of the 1-cells corresponding to $k \leq k+1$, we have that for any map $S \to \mathcal{N}_\bullet(\mathcal{C})$ there is a unique extension $\Delta^n \to \mathcal{N}_\bullet(\mathcal{C})$. $\diamondsuit$

Example 4.4. If $f \in \mathcal{N}_3(\mathcal{C})$ given by

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} C_3, \tag{4.8}$$

when we consider f as a map $\Delta^3 \to \mathcal{N}_\bullet(\mathcal{C})$, the morphisms f_i correspond to the value of f on the colored 1-cells in the following picture of Δ^3:

$$\begin{tikzcd} 0 \arrow[rr, red] \arrow[dr] \arrow[drrr] & & 1 \arrow[dl, blue] \arrow[dr] & \\ & 2 \arrow[rr, green] & & 3. \end{tikzcd} \tag{4.9}$$

For example, the value of f on the 2-cell d_3, so the face (012), is already determined by vertices (01) and (12), namely

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \tag{4.10}$$

In particular, we can see that if we know the values of f on any horn $\Lambda^3_m \hookrightarrow \Delta^3$, there is exactly one extension to the entire simplex Δ^3. $\diamondsuit$

Proof. We start by showing that the nerve of any groupoid is a symmetric effective Kan complex.

First, by the remark above, for $n \geq 3$, and any $0 \leq m \leq n$, any map $y : \Lambda^n_m \to \mathcal{N}_\bullet(\mathcal{C})$ has a unique filler as all 1-cells of Δ^n are already contained in Λ^n_m. As a consequence, for $n \geq 2$, and $y, s_j^*(y)$ as in definition 3.1, we have that $\text{fil}(s_j^*(y)) = \text{fil}(y) \circ s_j$ for any $y : \Lambda^n_m \to \mathcal{N}_\bullet(\mathcal{C})$, as $\text{fil}(y) \circ s_j$ is always a filler for $s_{j^*}(y)$

So we only have to make choices in defining the fillers for horns Λ^n_m with $n \in \{1, 2\}$:

(n=1) Any horn of the form Λ^1_m equals Δ^0. Hence a map $\Lambda^1_m \to \mathcal{N}_\bullet(\mathcal{C})$ corresponds with an object C of $\mathcal{C}$. We define the filler by $C \xrightarrow{id_C} C$ regardless of m.

(n=2) Now we do need to make case distinctions on the missing face m:

(m=0) A map $f : \Lambda^2_0 \to \mathcal{N}_\bullet(\mathcal{C})$ can be extended as follows:

$$\begin{tikzcd} C_0 \arrow[r, "f.d_2"] \arrow[d, "f.d_1"'] & C_1 \arrow[dl, dashed, "(f.d_1)\circ(f.d_2)^{-1}"] \\ C_2 & \end{tikzcd} \tag{4.11}$$

(m=1) A map $f : \Lambda^2_1 \to \mathcal{N}_\bullet(\mathcal{C})$ can be extended as follows:

$$\begin{tikzcd} C_0 \arrow[r, "f.d_2"] \arrow[d, dashed, "f.d_0 \circ f.d_2"'] & C_1 \arrow[dl, "f.d_0"] \\ C_2 & \end{tikzcd} \tag{4.12}$$

(m=2) A map $f : \Lambda^2_0 \to \mathcal{N}_\bullet(\mathcal{C})$ can be extended as follows:

$$\begin{tikzcd} C_0 \arrow[r, dashed, "(f.d_0)^{-1}\circ f.d_1"] \arrow[d, "f.d_1"'] & C_1 \arrow[dl, "f.d_0"] \\ C_2 & \end{tikzcd} \tag{4.13}$$

Now suppose we pull back a map $y : \Lambda^1_m \to \mathcal{N}_\bullet(\mathcal{C})$ along s_j, where y corresponds to some object C of $\mathcal{C}$. From definition 3.1, we know that $s_j^*(y) \circ d_k$ is given by $y \circ s_j \circ d_k$ or by fil(y). But both $y \circ s_j \circ d_k$ and fil(y) must be the identity morphism $C \xrightarrow{id} C$.

For all Λ^2_m-horns we described above, when both faces of the Λ^2_m-horn are identity morphisms, the extension will be of the form $C \xrightarrow{id} C \xrightarrow{id} C$, which is also equal to $\text{fil}(y) \circ s_j$.

We conclude that with this filling method, the nerve of a groupoid is a symmetric effective Kan complex.

Conversely, for any morphism in $f : A \to B$ in $\mathcal{C}$, by considering the 2-horns

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "id"'] & B \\ A & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[d, "id"'] & A \arrow[dl, "f"] \\ B & \end{tikzcd} \tag{4.14}$$

we can see that if $\mathcal{N}_\bullet(\mathcal{C})$ has fillers for Λ^2_0 and Λ^2_2 horns, f has a left and a right inverse, which must then both be equal to the inverse of f.

So whenever $\mathcal{N}_\bullet(\mathcal{C})$ is a Kan complex, $\mathcal{C}$ must be a groupoid. $\square$

Remark 4.5. This proof also gives that nerves of categories are symmetric effective ∞-categories in the sense of Remark 3.4. $\Diamond$

Remark 4.6. Note that when a Λ^2_m-horn in the nerve of a groupoid has a possible degenerate solution, it must be unique. The reason is that when one of f, g is the identity in $fg = h$, we have that the other must equal h, and if h is the identity, f, g are each others inverses. This will actually mean that the nerve of a groupoid is a degenerate-preferring Kan complex. We will study these in Chapter 5. $\Diamond$

4.2 Singular simplicial sets

Another standard example of Kan complexes are singular simplicial sets, see for example [Lur22, Proposition 002K]. Singular simplicial sets are simplicial sets corresponding to a topological space, via the embeddings of topological simplices. This section, we shall recall what those singular simplicial sets are and why they are Kan complexes. We shall also show that this proof does not translate to symmetric effective Kan complexes.

We shall introduce the singular simplicial sets via an adjunction with the geometric realization. First we shall define the geometric realization of simplices as follows:

$$|\Delta^n| = \big\{(x_i)_{i\leq n} \in [0,1]^{n+1} \big| \sum_{i=0}^{n} x_i = 1\big\}. \tag{4.15}$$

Then for any morphism $f : [n] \to [m]$ in Δ, we can define a morphism $|f| : |\Delta^n| \to |\Delta^m|$, given by

$$\big(|f|(x_0, \cdots, x_n)\big)_i = \sum_{f(j)=i} (x_j) \tag{4.16}$$

In particular, we have that

$$|s_j|(x_0, \cdots, x_n) = (x_0, \cdots, x_{j-1}, x_j + x_{j+1}, x_{j+2}, \cdots, x_n) \tag{4.17}$$
$$|d_j|(x_0, \cdots, x_n) = (x_0, \cdots, x_j, 0, x_{j+1}, \cdots, x_n) \tag{4.18}$$

Definition 4.7. Let $\mathcal{T}$ be a topological space. We define the **singular simplicial set $\mathcal{S}_\bullet(\mathcal{T})$** as follows:

- For any object $[n]$ of Δ, we define $\mathcal{S}_n(\mathcal{T})$ to be the space of morphisms $|\Delta^n| \to \mathcal{T}$ in the category of topological spaces.

$$\mathcal{S}_n(\mathcal{T}) = Top\big[|\Delta^n|, \mathcal{T}\big] \tag{4.19}$$

- On morphisms $f : [n] \to [m]$ in Δ, we define $\mathcal{S}(f) : \mathcal{S}_m(\mathcal{T}) \to \mathcal{S}_n(\mathcal{T})$ by precomposition with $|f|$.

$\Diamond$

Now there is the classical result that singular simplicial sets are Kan complexes. The standard proof does however not translate to symmetric effective Kan complexes.

Theorem 4.8. *Every singular simplicial set is a Kan complex.*

Proof. We start of with a horn filling problem in the category of simplicial sets.

$$\begin{tikzcd} \Lambda^n_m \arrow[r] \arrow[d, hook] & \mathcal{S}_\bullet(\mathcal{T}) \\ \Delta^n & \end{tikzcd} \tag{4.20}$$

This corresponds to the following diagram in the category of topological spaces

$$\begin{tikzcd} |\Lambda^n_m| \arrow[r] \arrow[d, hook] & \mathcal{T} \\ |\Delta^n| & \end{tikzcd} \tag{4.21}$$

where we have implicitly defined

$$|\Lambda^n_m| = \{(x_i)_{i \leq n} \in [0,1]^{n+1} | \sum_{i=0}^{n} x_i = 1 \text{ and } x_i = 0 \text{ for some } i \neq m\}. \tag{4.22}$$

The proof works via the morphism $r^n_m : |\Delta^n| \to |\Lambda^n_m|$ given by

$$(r^n_m(x))_i = \begin{cases} x_i - \min\limits_{m \neq k}(x_k) \text{ if } i \neq m \\ x_i + n \cdot \min\limits_{m \neq k}(x_k) \text{ if } i = m \end{cases} \tag{4.23}$$

Note that if x is an element of $|\Lambda^n_m|$, then $x_i = 0$ for some $i \neq m$, thus $\min\limits_{m \neq k}(x_k) = 0$. so in that case $r^n_m(x) = x$. We conclude that r is a retraction.

Thus we can define our filler for diagram 4.21 precomposing the morphism $|\Lambda^n_m| \to \mathcal{T}$ with r^n_m.

To be precise, for a topological horn map $y : |\Lambda^n_m| \to \mathcal{T}$, we have a topological filler $\mathrm{fil}_{Top}(y) : |\Delta^n| \to \mathcal{T}$ given by

$$\mathrm{fil}_{Top}(y)(x) = y(r^n_m(x)) \tag{4.24}$$

$\square$

Proposition 4.9. *The lifting structure in the above proof does not give the structure of a symmetric effective Kan complex.*

Proof. Let $n \geq 1$, and let $0 \leq m \neq j \leq n$. Consider the space $\mathcal{T} = |\Delta^n|$ and the inclusion map $\iota : |\Lambda^n_m| \to |\Delta^n|$.

Then consider the point $x \in \Delta^{n+1}$ given by

$$x_i = \frac{1}{n+2} \text{ for all } 0 \leq i \leq n+2 \tag{4.25}$$

We shall show that

$$\mathrm{fil}_{Top}(s_j^*(\iota))(x) \neq (\mathrm{fil}_{Top}(\iota) \circ s_j)(x) \tag{4.26}$$

We shall prove this by showing that the square in following diagram does not commute

$$\begin{tikzcd}
{|\Lambda^{n+1}_{m^*}|} \arrow[rr, bend left, "s_j^*(\iota)"] & {|\Lambda^n_m|} \arrow[r, "\iota"] & {|\Delta^n|} \\
{|\Delta^{n+1}|} \arrow[u, "r^{n+1}_{m^*}"] \arrow[r, "|s_j|"] & {|\Delta^n|} \arrow[u, "r^n_m"'] &
\end{tikzcd} \tag{4.27}$$

First note that according to the proof above, we have

$$(\mathrm{fil}_{Top}(\iota) \circ s_j)(x) = (\iota \circ r^n_m)(s_j(x)) \tag{4.28}$$

And $s_j(x)$ is given as follows

$$(s_j(x))_i = \begin{cases} \frac{1}{n+2} \text{ if } i \neq j \\ \frac{2}{n+2} \text{ if } i = j \end{cases} \tag{4.29}$$

and we thus have that $\min\limits_{m \neq k} (s_j(x))_k = \frac{1}{n+2}$, so

$$(\iota \circ r^n_m(s_j(x)))_i = \begin{cases} 0 \text{ if } i \neq j, i \neq m \\ \frac{n+1}{n+2} \text{ if } i = m \\ \frac{1}{n+2} \text{ if } i = j \end{cases} \tag{4.30}$$

Now we shall calculate $\mathrm{fil}_{Top}(s_j^*(\iota))(x)$. By the proof above, this is given by $s_j^*(\iota) \circ r^{n+1}_{m^*}(x))$, with s as in 4.27. Note that

$$(r^{n+1}_{m^*}(x))_i = \begin{cases} 1 \text{ if } i = m^* \\ 0 \text{ if } i \neq m^* \end{cases} \tag{4.31}$$

And thus

$$(s_j(r^{n+1}_{m^*}(x)))_i = \begin{cases} 1 \text{ if } i = m \\ 0 \text{ if } i \neq m \end{cases} \tag{4.32}$$

which lies on Λ^n_m, so $r^{n+1}_{m^*}(x) \in s^*_j(\Lambda^n_m)$, and we have $s^*_j(\iota)(r^{n+1}_{m^*}(x)) = s_j(r^{n+1}_{m^*}(x))$ We conclude that

$$s_j(r^{n+1}_{m^*}(x)) = (s^*_j(\iota))(r^{n+1}_{m^*}x) = \mathrm{fil}_{Top}(s^*_j(\iota))(x) \tag{4.33}$$

However as 4.30 and 4.32 are not equal, we have that $\mathrm{fil}_{Top}(s^*_j(\iota))(x) \neq (\mathrm{fil}_{Top}(\iota) \circ s_j)(x)$. Thus the method presented in the above proof does not give us the structure of a symmetric effective Kan complex. □

This does not mean that there is no symmetric effective Kan complex structure on $\mathcal{S}(\mathcal{T})$. We shall see in the next chapter that we can actually assign the structure of a symmetric effective Kan complex to all Kan complexes using classical axioms.

We would like to explore in some more detail what the problem is for filling Λ^2_1-horns. We will also mention a different notion of topological 1-cells where that problem does not occur.

Remark 4.10. The 1-cells of $\mathcal{S}_\bullet(\mathcal{T})$ correspond with paths in $\mathcal{T}$, which are continuous functions $\gamma : [0,1] \to \mathcal{T}$. A path $\gamma : [0,1] \to \mathcal{T}$ corresponding to a degenerate 1-cell has the form $\gamma(t) = c$ for some point $c \in \mathcal{T}$.

Recall that a map $y : \Lambda^2_1 \to \mathcal{S}_\bullet(\mathcal{T})$ corresponds with two composable one-cells of $\mathcal{S}_\bullet(\mathcal{T})$. These composable one-cells can be seen as two composable paths $y_0, y_2 : [0,1] \to \mathcal{T}$ such that $y_2(1) = y_0(0)$. The face d_1 of filler for y then corresponds to the concatenation of these two paths, which is given by

$$y_1 : [0,1] \to \mathcal{T} \qquad y_1(t) = \begin{cases} y_2(2t) \text{ if } t \leq \frac{1}{2} \\ y_0(2t-1) \text{ if } t > \frac{1}{2} \end{cases} \tag{4.34}$$

So concatenation of two paths gives both paths equal weight.

We will see in the next chapter that if we were to assign no weight to degenerate paths, we would have the structure of a symmetric effective Kan complex. We would require that if y_0 was a degenerate path, then y_1 would be equal to y_2. However in general this is not the case for the path concatenation as above. ◊

There is a notion of path composition where we can weigh paths differently. Namely the path composition of Moore paths.

Remark 4.11. A **Moore path** of $\mathcal{T}$ consists of a length $r \in \mathbb{R}_{\geq 0}$ and a continuous function $\gamma : [0,r] \to \mathcal{T}$. Concatenation of Moore paths $y_0 : [0,r_0] \to \mathcal{T}$ and $y_2 : [0,r_2] \to \mathcal{T}$ with $y_2(r_2) = y_0(0)$ is then given by

$$y_1 : [0,r_1] \to \mathcal{T} \qquad y_1(t) = \begin{cases} y_2(t) \text{ if } t \leq r_2 \\ y_0(t-r_2) \text{ if } t > r_2 \end{cases} \tag{4.35}$$

There is no formal notion of a general Moore simplex, but suppose that for such a notion 0-cells would be points and 1-cells would be Moore paths and higher cells would be compositions of Moore paths. In such a setting, it would be intuitive to assign a length 0 to a degenerate Moore path. And in this case, the concatenation structure would assign no weight to degenerate paths.

The higher Moore simplices we have mentioned might lie too close to nerves to be interesting of itself in the context of symmetric effective Kan complexes. There might be different notions of higher Moore cells closer to singular simplicial sets.

Moore paths have been used in [vdBG10] and [OP19] to give models for homotopy type theory, (see Appendix A). It could be interesting to compare these models to potential models using symmetric effective Kan structures on Moore cells. $\lozenge$

Chapter 5

Degenerate simplices

In constructive mathematics, we often have the following “constructivist game”: We are interested in a statement s which has proof p. However, the proof p might not be constructive, or the statement s might not even be constructively valid. The constructive mathematician then attempts to find a similar statement s' and a proof p' of s' so that p' constructively proves s'. Playing this game and applying some constructive nuance between s and s' helps us to better understand the mathematical theory under consideration. In the case that s is about the existence of a certain mathematical object, s' allows us to actually compute it.

An example of such a statement is that for simplicial sets G, H, if G, H are both Kan complexes, so is G^H. This was already shown in [May67], Theorem 6.9. However, the proof relied on the a case distinction on whether or not certain simplices were degenerate. It is not constructively possible to always make this case distinction. Not only does that proof not work constructively, it was shown in [BCP15] that the statement is not constructively valid.

In this chapter, we will focus on these “degenerate” simplices. We will first give the definition of a degenerate simplex, and then explore how decidability of whether or not being a degenerate simplex helps us to give constructive nuance to the notion of a Kan complex.

Definition 5.1. Let G be a simplicial set, $n \in \mathbb{N}$ and $x \in G_n$. Then x is called **degenerate** if there is a natural number $m < n$, an m-simplex $y \in G_m$ and a surjection $s : [n] \to [m]$ in Δ such that $y \circ s = x$. $\Diamond$

With this definition, we shall:

- give a classical proof that all classical Kan complexes are symmetric effective Kan complexes.
- define a subclass of the symmetric effective Kan complexes which choose degenerate simplices whenever possible.
- give a direct proof that the case distinction described above cannot be made with constructive mathematics.
- introduce the notion of having a degeneracy-section, which will be a constructive replacement to the notion of being a surjection of simplicial sets. We will use this notion in Chapter 6.

For the proof that classical Kan complexes are symmetric effective Kan complexes, we use a proof from [BF22]. The rest is original work.

5.1 From Kan to symmetric effective Kan

It is straightforward to see that simplicial sets with the structure of a symmetric effective Kan complex have the property of being a Kan complex. A natural question is about the converse: Can we turn Kan complexes into symmetric effective Kan complexes?

For effective Kan complexes, the answer, given in Appendix C from [BF22], is yes. In this section, we will use that proof to show that all Kan complexes can (non-constructively) be assigned the structure of an effective Kan complexes. Let us state without proof the main result of this appendix, which the authors of [BF22] developed together with Christian Sattler.

Theorem 5.2. *Suppose that $x \circ s$ and $x' \circ s'$ are both fillers for the lifting problem*

$$\begin{tikzcd} \Lambda^n_m \arrow[r] \arrow[d, hook] & G \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{5.1}$$

where $s : \Delta^n \to \Delta^k, s' : \Delta^n \to \Delta^{k'}$ are surjections from Δ for some $k, k' < n$. Then $x \circ s = x' \circ s'$.

The theorem above can be summarised by saying that there is at most one degenerate solution to a horn lifting problem.

Corollary 5.3. *Using classical arguments, any Kan complex can be assigned the structure of a symmetric effective Kan complex.*

Proof. Let G be a Kan complex. We will define fil on each horn map y.

- Using the law of the excluded middle, there either is a degenerate filler for y or not.
- If there exists one, it is unique and we pick that for $\mathrm{fil}(y)$.
- For those y we have not yet defined fil on, use the axiom of choice and the property of G being a Kan complex to pick any solution.

Now for any $s_j^*(y)$ as in definition 3.1, $\mathrm{fil}(y) \circ s_j$ is a degenerate solution for the lifting problem. By the theorem, it must equal the chosen filler and $\mathrm{fil}(s_j^*(y)) = \mathrm{fil}(y) \circ s_j$.

We conclude that fil gives G the structure of a symmetric effective Kan complex. □

Remark 5.4. Note also that each solution to a lifting problem in the definition of a Kan fibration is of the form

$$\begin{tikzcd} \Lambda^n_m \arrow[r, "x"] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[ur, dashed] \arrow[r, "y"] & Y \end{tikzcd} \tag{5.2}$$

and such a lift is also a solution to the lifting problem

$$\begin{tikzcd} \Lambda^n_m \arrow[r, "x"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{5.3}$$

So the above proof can be used to show that all Kan fibrations can be assigned the structure of a symmetric effective Kan fibration. $\Diamond$

5.2 Degenerate-preferring Kan complexes

In the last section, we have shown that if fil always picks a degenerate solution if possible, it gives the structure of a symmetric effective Kan complex. Beware that the converse is not true. Not every symmetric effective Kan complex assigns degenerate solutions whenever possible. This is because not all horn lifting problems with a degenerate solution need to be of the form $s_j^*(y)$ as in definition 3.1. If non-degenerate fillers exist for such horns we are free to choose them.

For example, there are no Λ_0^1-horns of the form $s_j^*(y)$, as there are no 0-horns. As $\Lambda_0^1 \simeq \Delta^0$, a Λ_0^1 horn is just a 0-cell. Any Kan complex with a non-degenerate 1-cell $A \to B$ has $A \to B$ as a filler for the Λ_0^1-horn corresponding to B. We are free to pick this filler for that horn, and choose degenerate fillers whenever possible everywhere else, still giving us a symmetric effective Kan complex. Such an example works for any horn not of the form $s_j^*(y)$ with a non-degenerate filler, and we could imagine artificially adding non-degenerate cells to a Kan complex to create higher-dimensional examples.

However, this discussion does suggest another definition. Namely that of a Kan complex with a filler structure which picks degenerate horn fillers whenever possible. The proof given above actually shows that every Kan complex can be assigned the structure of such a "degenerate-preferring" Kan complex.

While "whenever possible" is not very constructive, we could describe such a structure in a slightly more constructive manner, inspired by a discussion with Storm Diephuis:

Definition 5.5. Let G be a simplicial set. Let fil give G the structure of a symmetric effective Kan complex. We say that fil gives the structure of a **degenerate-preferring Kan complex**, if for all $n \in \mathbb{N}$, all $g \in G_n$, all $0 \leq j \leq n$, all $0 \leq m \leq n+1$ and all horn inclusions $\iota : \Lambda_m^{n+1} \to \Delta^{n+1}$, we have

$$\mathrm{fil}(g \circ s_j \circ \iota) = g \circ s_j \tag{5.4}$$

$\Diamond$

Note that if we only make the above requirement for simplices of the form $g = \mathrm{fil}(y)$ for some horn map y, we recover the definition of a symmetric effective Kan complex. Conversely, if we replace $\mathrm{fil}(y)$ by any filler for the horn map y in the definition of $s_j^*(y)$ in definition 3.1, we arrive at the definition of a degenerate-preferring Kan complex.

So every degenerate-preferring Kan complex is a symmetric effective Kan complex. And as discussed above, the converse is not true.

Also by a similar definitions, we could define a degenerate-preferring Kan fibration, and a degenerate-preferring ∞-category.

5.3 Non-existence of a constructive proof for decidable degeneracy

We have now seen that if we use the axiom of choice and can make case distinction on whether a simplex is degenerate, we can assign every Kan complex the structure of a degenerate-preferring

Kan complex. We have based this assumption on the classical law of excluded middle. However, it might be possible that we didn't need that law, and that this case distinction can be made constructively. This is not the case, and in this section we will explain why.

It is known that we cannot prove the law of excluded middle with constructive mathematics. So to show that we cannot decide whether a simplex is degenerate with constructive mathematics, we can show that it implies the law of excluded middle.

This proof works in a meta-theory of constructive set theory, where we can take quotients of sets and use (bounded) separation.

Theorem 5.6. *If we can decide whether an arbitrary simplex is degenerate, we can show the law of the excluded middle.*

Proof. Let p be a proposition.

Consider the simplicial circle S, which has one non-degenerate 0-simplex v and one non-degenerate 1-simplex e.

Remark that there are now exactly two 1-simplices in S, namely e and $s_0(v)$. We define a simplicial subset $T \subset S$ with one (non-degenerate) 0-simplex $T_0^{nd} = T_0 = S_0 = \{v\}$ and non-degenerate 1-simplices $T_1^{nd} = \{x \in \{e\} | p\}$.

Now consider the quotient projection map $\pi : S \to S/T$.

Remark that $\pi(e)$ must be a 1-simplex. Suppose we could determine whether or not it is degenerate.

- If $\pi(e)$ is degenerate, it must be of the form $\pi(v) \circ s_0$ as $\pi(v)$ is the only 0-simplex in S/T. Hence $\pi(e) = \pi(v \circ s_0)$ and hence $e \sim (v \circ s_0)$. Hence $e \in T_1$ and p must hold.
- If $\pi(e)$ is non-degenerate, we cannot have that $\pi(e) = \pi(v) \circ s_0$, and hence we cannot have that $e \in T_1$. Thus p cannot hold and $\neg p$ holds.

Hence $p \vee \neg p$ holds for all propositions p and we have shown the law of excluded middle. □

Sometimes something slightly stronger than this decidability condition is used, namely the Eilenberg-Zilber lemma, which we shall discuss next section. It follows also that the Eilenberg-Zilber lemma cannot be proven in the meta-theory described above.

5.4 Degeneracy-sections

In [JP02], it is shown that any surjection $\alpha : X \to Y$ between simplicial objects of a Malcev algebra is a Kan fibration. We shall generalize this statement to degeneracy-picking Kan complexes in Chapter 6.

In this section, we will introduce the notion of a degeneracy-section, and show that classically, having a degeneracy-section is equivalent to being surjective.

Definition 5.7. Let $\alpha : X \to Y$ be a map of simplicial sets. Let $\beta = (\beta_n : Y_n \to X_n)_{n \in \mathbb{N}}$ be a collection of functions. We call β a **degeneracy-section** of α iff

- for all $n \in \mathbb{N}$ we have $\alpha_n \circ \beta_n = 1_{Y_n}$.
- for all $n \in \mathbb{N}$ and all $0 \leq k \leq n$, we have $\beta_{n+1}(y \circ s_k) = \beta_n(y) \circ s_k$.

$\Diamond$

Remark 5.8. If we would add the condition that $\beta_n(y \circ d_k) = \beta_{n+1}(y) \circ d_k$, β becomes a section of α. $\Diamond$

Remark 5.9. Consider Δ_{epi} to be the subcategory of Δ with all objects from Δ, but only the surjective morphisms. A morphism in $Sets^{\Delta^{op}}$ has a degeneracy-section iff it has a section and can be seen as a morphism in $Sets^{\Delta^{op}_{\mathrm{epi}}}$. $\Diamond$

We will give a classical proof showing that every surjection can be assigned a degeneracy section. This proof will make use of the **Eilenberg-Zilber lemma**. The Eilenberg-Zilber lemma is the following lemma, first introduced in [EZ50] in Theorem (8.3), which we shall state without proof.

Lemma 5.10 (Eilenberg-Zilber)**.** *For any simplicial set X, natural number n and $x \in X_n$, there is a unique natural number m, surjection $s : [n] \to [m]$ and non-degenerate $y \in X_m$ such that $y \circ s = x$.*

Note that if $m = n$, then x is non-degenerate, otherwise it is degenerate. So Eilenberg-Zilber implies decidability of being degenerate. With the previous section, we can conclude that the lemma cannot be shown constructively.

Proposition 5.11. *In a classical framework, all surjections of simplicial sets have degeneracy-sections.*

Proof. Let $\alpha : X \to Y$ be a surjection of simplicial sets. By the axiom of choice, there is a family of sections $\gamma = (\gamma_n : Y_n \to X_n)_{n \in \mathbb{N}}$ such that $\alpha_n \circ \gamma_n = 1_{Y_n}$ for all $n \in \mathbb{N}$.

For any $n \in \mathbb{N}, y \in Y_n$, let m, y', s be as in the Eilenberg-Zilber lemma and define

$$\beta_n(y) = \gamma_m(y') \circ s. \tag{5.5}$$

We shall show that β is a degeneracy-section of α.

- We claim that $\alpha_n \circ \beta_n = 1_{Y_n}$ for all $n \in \mathbb{N}$.

  Let $y \in Y_n$ be arbitrary. Let m, y', s be as in the Eilenberg-Zilber lemma. Because α is a morphism of simplical sets and s is a morphism in Δ, α respects s and we have that

  $$\alpha_n(\beta_n(y)) = \alpha_n(\gamma_m(y') \circ s) = \alpha_m(\gamma_m(y')) \circ s = y' \circ s = y \tag{5.6}$$

  as required.

- Let $n \in \mathbb{N}, y \in Y_n$ and $k \leq n$. We claim that $\beta_{n+1}(y \circ s_k) = \beta_n(y) \circ s_k$. Let m, y', s be as in the Eilenberg-Zilber lemma applied to $y \circ s_k$. Also let m', y'', s' be as in the Eilenberg-Zilber lemma applied to y. Note that $y'' \circ s' \circ s_k = y \circ s_k$. By uniqueness, we must have that $y'' = y', m' = m, s = s' \circ s_k$. Hence

  $$\beta_{n+1}(y \circ s_k) = \gamma_m(y') \circ s = \gamma_{m'}(y'') \circ s' \circ s_k = \gamma_n(y) \circ s_k \tag{5.7}$$

  as required.

We conclude that β is a degeneracy-section of α. Hence we have a classical proof that all surjective morphims of simplicial sets have a degeneracy-section. $\square$

Chapter 6

Simplicial Malcev Algebras

In the introduction of Chapter 4, we mentioned three standard examples of Kan complexes. In the rest of said chapter, we only considered two of them. We have not yet studied the third standard example, namely that of simplicial groups. The original proof that simplicial groups are Kan complexes, was given as Theorem 3.4 of [Moo58]. Since then, the proof has been generalized. We can actually classify all algebras for which the simplicial objects are Kan complexes, namely the Malcev algebras.

In this chapter, we will study a generalization of a proof that all simplicial Malcev algebras are Kan complexes in Section 6.2. In Section 6.3, we shall use this proof to show that all morphisms of simplicial Malcev algebras with a degeneracy-section are degeneracy-preferring Kan fibrations. Before we do so, we shall discuss the mathematical context on Malcev algebras and Kan fibrations in Section 6.1.

The work in Sections 6.1 and 6.2 comes mainly from [CKP93] and [JP02]. The work in Section 6.3 is original.

6.1 Mathematical context

Definition 6.1. A **Malcev algebra** is an algebra A with a ternary operation μ satisfying

$$\mu(x, x, y) = x \text{ and } \mu(x, y, y) = y \text{ for all } x, y \in A \tag{6.1}$$

μ is called a **Malcev operation**.

We call an algebraic theory $\mathbb{T}$ a **Malcev theory** if we can assign such an operation for each algebra of $\mathbb{T}$. ♢

Example 6.2. The theory of groups is a Malcev theory, because to each group, we can assign the Malcev operation

$$\mu(x, y, z) = x \cdot y^{-1} \cdot z \tag{6.2}$$

♢

Example 6.3. The theory of Heyting algebras is also a Malcev theory via the Malcev operation

$$((z \to y) \to x) \wedge ((x \to y) \to z). \tag{6.3}$$

◊

To explain the connection between Kan complexes and Malcev theories, let us state without proof a result following from Proposition 3 of [JP02], which was shown with a constructive proof.

Theorem 6.4. *For an algebraic theory $\mathbb{T}$, if any simplicial $\mathbb{T}$-model is a Kan complex, then $\mathbb{T}$ is a Malcev theory.*

The converse of this theorem is also known to be true, and was even generalized further in [JP02], Theorem 4 to the following:

Theorem 6.5. *Any surjective morphism $f : X \to Y$ of simplicial Malcev algebras is a Kan fibration.*

In section 6.2, we will recall the proof of this theorem from [JP02], only now applied to morphisms which have a degeneracy-section instead of morphisms which are surjections, and formulated constructively. In section 6.3, we will show that the proof actually gives us the structure of a degenerate-preferring Kan complex.

Remark 6.6. Note that any inhabited simplicial set G must have an element in G_0, which corresponds to a section of the morphism $G \to \Delta^0$. So it follows from the above Theorem that any inhabited simplicial Malcev algebra is a degenerate-preferring Kan complex. Also, we should mention that by vacuous proof any empty simplicial set is a degenerate-preferring Kan complex. So it follows that all inhabited and non-inhabited simplicial Malcev algebras are degenerate-preferring Kan complexes.

By Theorem 6.4, it follows that if all simplicial objects of an algebraic theory $\mathbb{T}$ are Kan complexes, the inhabited and non-inhabited simplicial objects of $\mathbb{T}$ are actually all degenerate-preferring Kan complexes.

In particular, as groups are inhabited by their unit, simplicial groups are degenerate-preferring Kan complexes. ◊

Remark 6.7. Where the classical mathematician might say that all simplicial sets are either inhabited or non-inhabited, there is some constructive nuance. For G a simplicial Malcev algebra, if we have a proof that G is empty or a proof that G is inhabited, we can transform such a proof into a proof that G is a degenerate-preferring complex. If we don't have proofs that G is empty or inhabited, this method does not give us a proof that G is a degenerate-preferring Kan complex. ◊

6.2 The original proof

Let $\alpha : X \to Y$ be a morphism of simplicial Malcev algebras, and let $(\beta_n : Y_n \to X_n)_{n \in \mathbb{N}}$ be a degeneracy-section of α.

Now consider maps x, y as in the following diagram:

$$\begin{tikzcd} \Lambda^n_m \arrow[r, "x"] \arrow[d, hook] & X \arrow[d, "\alpha"] \\ \Delta^n \arrow[r, "y"] & Y \end{tikzcd} \tag{6.4}$$

We shall recursively define an extension $\mathrm{lift}(x, y) : \Delta^n \to X$.

We shall first define for all $k \in \mathbb{Z}$ with $-1 \leq k < m$ or $m < k \leq n+1$ a helper function $w_k : \Delta^n \to X$.

Construction 6.8. To define w_k, we use that $x \circ d_k$ is well-defined for all $k \neq m$.

- We start with
$$w_{n+1} := \beta_n(y). \tag{6.5}$$
- And for $m < k \leq n$, given that w_{k+1} has been defined, define
$$w_k := \mu\big(w_{k+1}, w_{k+1} \circ d_k \circ s_{k-1}, x \circ d_k \circ s_{k-1}\big), \tag{6.6}$$
- When w_{m+1} has been defined, we can define
$$w_{-1} := w_{m+1} \tag{6.7}$$
- For $0 \leq k < m$, given that w_{k-1} has been defined, define
$$w_k := \mu\big(w_{k-1}, w_{k-1} \circ d_k \circ s_k, x \circ d_k \circ s_k\big). \tag{6.8}$$

And finally, we define
$$\text{lift}(x, y) := w_{m-1} \tag{6.9}$$
$\diamondsuit$

Remark 6.9. We want to remark that the construction above introduces an order on the numbers $0 \leq k \leq n$ with $k \neq m$. We also want to note that the construction has a more general form, which will help streamline induction arguments.

- We say that w_l is **encountered after** w_k or that l is **encountered after** k if in the above construction, w_l is defined after w_k. We also say that k or w_k is **encountered before** l or w_l respectively.
- For $0 \leq k \leq n$ and $k \neq m$, we have defined
$$w_k := \mu\big(w, w \circ d_k \circ s_{k'}, x \circ d_k \circ s_{k'}\big). \tag{6.10}$$
for some earlier defined $w \in X_n$ and $k' = \begin{cases} k \text{ if } k < m \\ k-1 \text{ if } k > m \end{cases}$.

$\diamondsuit$

Now we claim the following:

Theorem 6.10. *lift(x, y) as define above is a lift for diagram 6.4*

To show this, we need to show two things:

- $\alpha \circ \text{lift}(x, y) = y$
- $\text{lift}(x, y) \circ d_k = x \circ d_k$ for all $0 \leq k \leq n$ with $k \neq m$.

To showt that $\alpha \circ \text{lift}(x, y) = y$, we use the following lemma:

Lemma 6.11. *Let α, x, y be as above. Let $w \in X_n$ be such that $\alpha_n(w) = y$. Then*

$$\alpha_n(\mu(w, w \circ d_j \circ s_{j'}, x \circ d_j \circ s_{j'})) = y \tag{6.11}$$

as well for any j, j' with $0 \leq j, j' \leq n$ and $j \neq m$.

Proof. As α is a morphism of simplicial Malcev algebras, α_n must respect μ. Hence

$$\alpha_n(\mu(w, w \circ d_j \circ s_{j'}, x \circ d_j \circ s_{j'})) = \mu(\alpha_n(w), \alpha_n(w \circ d_j \circ s_{j'}), \alpha_n(x \circ d_j \circ s_{j'})) \tag{6.12}$$

We shall show that the last two entries in μ above are equal. Because μ is a Malcev operation, the above equation is then equal to $\alpha_n(w) = y$.

As α is a morphism of simplicial sets, we have that

$$\alpha_n(w \circ d_j \circ s_{j'}) = \alpha_n(w) \circ d_j \circ s_{j'} = y \circ d_j \circ s_{j'} \tag{6.13}$$

Note that as 6.4 commutes, we have for all $j \neq m$ that $\alpha \circ x \circ d_j = y \circ d_j$. So we see that

$$\alpha_n(x \circ d_j \circ s_{j'}) = (\alpha_{n-1}(x \circ d_j)) \circ s_{j'} = y \circ d_j \circ s_{j'} \tag{6.14}$$

And indeed the last two entries of μ in equation 6.12 are equal as they both equal $y \circ d_j \circ s_{j'}$. □

Because w_{n+1} was chosen such that $\alpha(w_{n+1}) = y$, by induction with the above lemma, we have that $\alpha(w_k) = y$ for all k. Hence also $\alpha \circ \mathrm{lift}(x, y) = y$.

We shall now prove the following lemma:

Lemma 6.12. *Let $k, l \neq m$ be as in construction 6.8 with $0 \leq l \leq n$. Whenever $l = k$ or l is encountered after k, we will have $w_k \circ d_l = x \circ d_l$.*

It will then follow that as $\mathrm{fil}(x, y)$ corresponds to w_{m-1}, we have $\mathrm{fil}(x, y) \circ d_k = x \circ d_k$ for all $0 \leq k \leq n$ with $k \neq m$ as required.

Proof. The proof goes by induction.

- For the base case, we need to show that $w_k \circ d_k = x \circ d_k$ for all $0 \leq k \leq n$ with $k \neq m$.
- For the induction case, we need that whenever $w_k \circ d_l = x \circ d_l$, we also have $w \circ d_l = x \circ d_l$ for w encountered after w_k.

For the base case, consider that for $0 \leq k \leq n$ and $k \neq m$, we have that

$$w_k = \mu(w, w \circ d_k \circ s_{k'}, x \circ d_k \circ s_{k'}) \text{ for some } w \in X_n, k' \in \{k, k-1\}. \tag{6.15}$$

And as we are in a simplicial Malcev algebra, d_k must respect μ, so

$$w_k \circ d_k = \mu(w \circ d_k, w \circ d_k \circ s_{k'} \circ d_k, x \circ d_k \circ s_{k'} \circ d_k) \tag{6.16}$$

But now for $k' \in \{k, k-1\}$, we have that $s_{k'} \circ d_k = id$. From this follows that the first two entries in μ above must be equal, and the last entry must equal $x \circ d_k$. Hence $w_k \circ d_k = x \circ d_k$.

Now for the induction case, we shall use the simplicial identities from section 2.1. Remark that if l occurs after k and $k' \in \{k, k-1\}$, we have that $l \neq k, k', k'+1$, so either $l < k'$ or $l > k$.

- If $l < k'$, we have that $s_{k'} \circ d_l = d_l \circ s_{k'-1}$ and that $d_k \circ d_l = d_l \circ d_{k-1}$ as $l < k' \leq k$. It follows that
$$d_k \circ s_{k'} \circ d_l = d_l \circ d_{k-1} \circ s_{k'-1} \tag{6.17}$$
- If $l > k$, we have that $l > k'$ and hence $s_{k'} \circ d_l = d_{l-1} \circ s_{k'}$. Also $d_l \circ d_k = d_k \circ d_{l-1}$, hence
$$d_k \circ s_{k'} \circ d_l = d_l \circ d_k \circ s_{k'} \tag{6.18}$$

In both cases, we have an identity of the following form:

$$d_k \circ s_{k'} \circ d_l = d_l \circ f \text{ for some morphism } f \text{ of } \Delta \tag{6.19}$$

If $w \circ d_l = x \circ d_l$, we can use this identity to show that

$$w \circ d_k \circ s_k \circ d_l = w \circ d_l \circ f = x \circ d_l \circ f = x \circ d_k \circ s_k \circ d_l \tag{6.20}$$

And it follows that if w_k is defined as in equation 6.10 and $w \circ d_l = x \circ d_l$ then

$$w_k \circ d_l = \mu\big(w \circ d_l, w \circ d_k \circ s_{k'} \circ d_l, x \circ d_k \circ s_{k'} \circ s_l\big) = w \circ d_l = x \circ d_l \tag{6.21}$$

as the latter two entries of μ are equal. The induction step follows. □

6.3 A degenerate-preferring structure

Theorem 6.13. *The lifting structure defined in the previous section gives the structure of a degnerate-preferring Kan fibration.*

Suppose the following diagram commutes:

$$\begin{tikzcd} \Lambda^{n+1}_m \arrow[rr, "x"] \arrow[d, hook] & & X \arrow[d, "\alpha"] \\ \Delta^{n+1} \arrow[r, "s_j"] & \Delta^n \arrow[ur, "g"] \arrow[r, "y"] & Y \end{tikzcd} \tag{6.22}$$

We claim that in this case $\mathrm{lift}(x, y \circ s_j) = g \circ s_j$.

We use two lemma's on what construction 6.8 looks like in this case:

- For k encountered before j and $j+1$, we will show that $w_k = z \circ s_j$ for some $z \in X_n$.
- For all k encountered in or after $\{j, j+1\}$ we will have that $w_k = g \circ s_j$.

We will also use that because the above diagram commutes, $x \circ d_k = g \circ s_j \circ d_k$ for all k with $0 \leq k \leq n+1$, $k \neq m$.

Lemma 6.14. *For k encountered before j and $j+1$, we have that $w_k = z \circ s_j$ for some $z \in X_n$.*

Proof. For the base case, consider that $w_{n+2} = \beta(y \circ s_j) = \beta(y) \circ s_j$.

Now suppose that $w = z \circ s_j$ in equation 6.10. We can thus write

$$\mu\big(w, w \circ d_k \circ s_{k'}, x \circ d_k \circ s_{k'}\big) = \mu\big(z \circ s_j, z \circ s_j \circ d_k \circ s_{k'}, g \circ s_j \circ d_k \circ s_{k'}\big) \tag{6.23}$$

For $k \neq j, j+1$, we can observe the following:

- If $k > j+1$, we have that $s_j \circ d_k = d_{k-1} \circ s_j$. Also it follows that $j < k'$, so $s_j \circ s_{k'} = s_{k'-1} \circ s_j$. In this case, we may conclude that

$$s_j \circ d_k \circ s_{k'} = d_{k-1} \circ s_{k'-1} \circ s_j \tag{6.24}$$

- If $k < j$, we have that $s_j \circ d_k = d_k \circ s_{j-1}$ and as $k' < j$, we have that $s_{j-1} \circ s_{k'} = s_{k'} \circ s_j$. Hence

$$s_j \circ d_k \circ s_{k'} = d_k \circ s_{k'} \circ s_j \tag{6.25}$$

Hence for $k \neq j, j+1$, there is some f in Δ such that

$$s_j \circ d_k \circ s_{k'} = f \circ s_j \tag{6.26}$$

From which it follows that

$$\mu\big(z \circ s_j, z \circ s_j \circ d_k \circ s_{k'}, g \circ s_j \circ d_k \circ s_{k'}\big) = \mu\big(z \circ s_j, z \circ f \circ s_j, g \circ f \circ s_j\big) = \mu\big(z, z \circ f, g \circ f\big) \circ s_j \tag{6.27}$$

By induction, it follows that for all k encountered before $j, j+1$, there is some $z \in X^n$ such that $w_k = z \circ s_j$. □

Now we will consider what happens when we first encounter $k \in \{j, j+1\}$.

Lemma 6.15. *When and after we first encounter $k \in \{j, j+1\}$, we have that $w_k = g \circ s_j$.*

Proof. Whether we first encounter j or $j+1$ depends on whether $j < m$ or $j > m$:

- Note that if $j > m$, the first of $j, j+1$ we encounter must be $j+1$, and if $k > m$, we have $k' = k-1$, in particular $j' = j$.
- If $j < m$, the first of $j, j+1$ we encounter must be j, and if $k < m$, we have that $k' = k$, in particualr $j' = j$.

So in both cases, we have $j' = j$. Also remark that $s_j \circ d_j = s_j \circ d_{j+1} = id$.

Let $k \in \{j, j+1\}$ be the first we encounter. By the above discussion, $s_j \circ d_k = id$ and that $k' = j$. Hence

$$w_k = \mu(z \circ s_j, z \circ s_j \circ d_k \circ s_{k'}, g \circ s_j \circ d_k \circ s_{k'}) = \mu(z \circ s_j, z \circ \circ s_j, g \circ s_j) \tag{6.28}$$

And as the first two entries are equal, we get $w_k = g \circ s_j$.

Now if $w = g \circ s_j$, we have that for w_k as in equation 6.10 that

$$w_k = \mu(g \circ s_j, g \circ s_j \circ d_k \circ s_{k'}, g \circ s_j \circ d_k \circ s_{k'}) = g \circ s_j, \tag{6.29}$$

It follows that $w_k = g \circ s_j$ for all k encountered after we encounter $\{j, j+1\}$. Hence $w_{m-1} = g \circ s_j$ and we are done. □

Remark 6.16. We have now shown that simplicial models of an algebraic Malcev theory are symmetric effective Kan complexes. Because of Theorem 6.4, we have actually gone very general as far as algebras go: all simplicial models of algebraic theories are symmetric effective Kan iff the theory is a Malcev theory.

However, there are more general statements possible outside of algebras, in so-called Malcev categories. Let us mention Theorem 4.6, of [CKP93] which states that for any regular category $\mathcal{A}$, $\mathcal{A}$ is a Malcev category iff every simplicial object of $\mathcal{A}$ is a Kan complex.

A Malcev category is a category with a certain condition on the lattice of relations. Such categories are not studied in this work, but they might be of interest for future research. The reason such categories are called Malcev categories, is because Malcev has shown in [Mal54] [1] that for an algebra, satisfying the lattice condition is equivalent to having an operation μ as in Definition 6.1. $\Diamond$

[1] This reference is in Russian, and the author was unable to find a translation. The name Malcev is a translation, and has also been translated as Mal'cev, Maltsev and Mal'tsev.

Chapter 7

Lifting algebraic weak factorisation system

In Section 2.6, we defined Kan fibrations as maps of simplicial sets which have the right lifting property against horn inclusions. The notion of right lifting property comes from the theory of weak factorisation systems. These systems play a key role in Quillen model structures [Qui67]. In particular, Kan fibrations form the "right class" in the weak factorisation system of the Kan-Quillen model structure.

There is a more structured and constructive version of weak factorisation systems, namely the algebraic weak factorisation systems. Recently, in [Bou23], these algebraic weak factorisation systems were shown to be equivalent to lifting algebraic weak factorisation systems. The "right class" is now replaced by a "right double category". In this chapter we will show that the symmetric effective Kan complexes will fit into such a right double category.

In the first section of this chapter, we will recall what a double category is, which we will use in the second section to introduce the lifting algebraic weak factorisation systems. These first two sections are based on [BG16] and [Bou23]. In the later sections, which are original work, we will show how the symmetric effective Kan complexes fit into such a system.

7.1 Double categories

In this section, we will introduce the notion of a double category. Specifically, we will introduce double categories over a category $\mathcal{C}$. Double categories over $\mathcal{C}$ give more structure to classes of maps in $\mathcal{C}$.

Definition 7.1. A **double category** $\mathbb{D}$ is an internal category in the category of small categories. $\Diamond$

This means we have a small category of objects $\mathbb{D}_0$ and a small category of morphisms $\mathbb{D}_1$, with functors between them as in

$$\begin{tikzcd} \mathbb{D}_1 \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & \mathbb{D}_0 \arrow[l, "i" description] \end{tikzcd} \tag{7.1}$$

such that i is a section of both d, c. We call i, d and c the **identity**, **domain** and **codomain** functor respectively. If $f \in \mathbb{D}_1$, we write $f : A \Rightarrow B$ for $d(f) = A, c(f) = B$.

Furthermore, for $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ the category of composable morphisms as in

$$\begin{tikzcd} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_1 \arrow[d, "c"] \\ \mathbb{D}_1 \arrow[r, "d"] & \mathbb{D}_0, \end{tikzcd} \tag{7.2}$$

we have a **composition** functor

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{m} \mathbb{D}_1 \tag{7.3}$$

satisfying the following conditions

- m should respect domain and codomain, meaning that $c \circ m = c \circ \pi_1$ and $d \circ m = d \circ \pi_0$. So if $f : A \to B, g : B \to C$, we have $m(g, f) : A \to C$.
- m sees i as unit, meaning that $m \circ (1 \times i) : \mathbb{D}_1 \times \mathbb{D}_0$ and $m \circ (i \times 1) : \mathbb{D}_0 \times \mathbb{D}_1$ equal $m \circ \pi_0$ and $m \circ \pi_1$ respectively. So whenever $f : A \to B$, we have $m(i(B), f) = m(f, i(A)) = f$.
- m is associative, meaning that the following diagram commutes:

$$\begin{tikzcd} (\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1) \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[rr, "\simeq"] \arrow[dr, "m\circ(m\times 1)"'] & & \mathbb{D}_1 \times_{\mathbb{D}_0} (\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1) \arrow[dl, "m\circ(1\times m)"] \\ & \mathbb{D}_1 & \end{tikzcd} \tag{7.4}$$

  So for three composable morphisms $f : A \to B, g : B \to C, h : C \to D$, we have $m(f, (m(g, h))) = m(m(f, g), h)$.

An important example we will use is given by the category of squares corresponding to a small category:

Example 7.2. Given any small category $\mathcal{C}$, there is the double category of squares in $\mathcal{C}$ called $\mathbb{S}\mathbf{q}(\mathcal{C})$.

- The category of objects is given by $\mathbb{S}\mathbf{q}(\mathcal{C})_0 = \mathcal{C}$.
- The category of morphisms is given by $\mathbb{S}\mathbf{q}(\mathcal{C})_1 = \mathcal{C}^{[1]}$. So:
  - Objects correspond to morphisms in $\mathcal{C}$
  - Morphisms μ from $f : A \to B$ to $g : C \to D$ correspond with natural transformations:

$$\begin{tikzcd} A \arrow[r, "\mu_0"] \arrow[d, "f"] & C \arrow[d, "g"] \\ B \arrow[r, "\mu_1"] & D \end{tikzcd} \tag{7.5}$$

  - Composition of such morphisms corresponds to horizontal composition of the corresponding squares.
  - The identities correspond to squares as above where μ_0, μ_1 are identity morphisms and $f = g$.

- i is the functor sending objects to their corresponding identity morphisms and morphisms to their corresponding identity natural transformations.
- As $\mathcal{C}^{[0]} \simeq \mathcal{C}$, we can define $c, d : \mathcal{C}^{[1]} \to \mathcal{C}^{[0]}$ by via the exponential transpose of precomposition with d_1, d_0 respectively.
- Composition m is defined as normal composition of arrows, which are the objects of $\mathbb{S}\mathbf{q}(\mathcal{C})_1$ and as vertical composition of squares as above.

$\lozenge$

The above example also helps us to explain an alternative way to talk about double categories:

Remark 7.3. Often double categories are explained as having objects, horizontal arrows, vertical arrows and squares.

- The objects are the objects from $\mathbb{D}_0$.
- The horizontal arrows are the morphisms from $\mathbb{D}_0$.
- The vertical arrows are the objects from $\mathbb{D}_1$. The domain and codomain of the vertical arrows are objects given by d and c respectively.
- The squares are the morphisms from $\mathbb{D}_1$. These have a domain and codomain from d and c, which are horizontal arrows. And they have a domain and codomain because $\mathbb{D}_1$ is a category, which are the vertical arrows.

The compositions within $\mathbb{D}_0$ give us compositions of horizontal arrows, and the compositions within $\mathbb{D}_1$ give us horizontal compositions of squares. The functor m gives us compositions of vertical arrows via m_0 and vertical compositions of squares via m_1. $\lozenge$

There is a natural notion of a morphism between double categories, namely that of a double functor.

Definition 7.4. Given $\mathbb{A}, \mathbb{B}$ double categories, a **double functor** $\mathbb{F} : \mathbb{A} \Rightarrow \mathbb{B}$ consists of two functors $\mathbb{F}_0 : \mathbb{A}_0 \to \mathbb{B}_0$ and $\mathbb{F}_1 : \mathbb{A}_1 \to \mathbb{B}_1$ such that $\mathbb{F}$ respects c, d, i and m.

If we have an (implicit) double functor $\mathbb{A} \Rightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$, we will call $\mathbb{A}$ a **double category over** $\mathcal{C}$. $\lozenge$

In the light of Remark 7.3, we can formulate these double functors as follows:

Remark 7.5. Given $\mathbb{A}, \mathbb{B}$ double categories, a double functor from $\mathbb{A}$ to $\mathbb{B}$ is an operation sending objects, horizontal morphisms, vertical morphisms and squares of $\mathbb{A}$ to objects, horizontal morphisms, vertical morphisms and squares of $\mathbb{B}$ respectively, such that all domains, codomains and compositions are respected. $\lozenge$

7.2 Lifting structures

In this section, we will talk about a lifting algebraic weak factorisation system, or lifting awfs for short. These systems were introduced in [Bou23], and the notion was shown to be equivalent to the notion of an awfs. A lifting awfs consists of two double categories and a "lifting structure" between them, which satisfies certain axioms. We will introduce these axioms, but for the purposes of this thesis, our main interest is about so-called "cofibrantly generated" lifting awfs's. And [Bou23] provides us with a theorem on such lifting awfs's which we will use in the rest of this chapter to show that the symmetric effective Kan complexes fit into such a lifting awfs.

Definition 7.6. Let $\mathcal{C}$ be a category. Consider two double categories $\mathbb{L}$ and $\mathbb{R}$ with double functors $L : \mathbb{L} \Rightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$ and $R : \mathbb{R} \Rightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$.

A ($\mathbb{L}$,$\mathbb{R}$)**-lifting operation** is a family of operations ϕ which assigns for all vertical morphisms l of $\mathbb{L}$ and r of $\mathbb{R}$, and each commutative square in $\mathcal{C}$ of the form:

$$\begin{tikzcd} LA \arrow[r, "u"] \arrow[d, "Ll"'] & RX \arrow[d, "Rr"] \\ LB \arrow[r, "v"] & RY \end{tikzcd} \tag{7.6}$$

a choice of diagonal filler $\phi_{l,r}(u, v) : LB \to RX$. This lifting operation ϕ should satisfy some compatibility conditions:

- We have horizontal compatibilities, which is about horizontal composition of squares. For any

$$\begin{tikzcd} A' \arrow[r, "a"] \arrow[d, "l'"] & A \arrow[d, "l"] \\ B' \arrow[r, "b"] & B \end{tikzcd} \text{ square in } \mathbb{L} \text{ and } \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "r"] & X' \arrow[d, "r'"] \\ Y \arrow[r, "y"] & Y' \end{tikzcd} \text{ square in } \mathbb{R}, \tag{7.7}$$

  and any diagram in $\mathcal{C}$ of the form:

$$\begin{tikzcd} LA' \arrow[r, "La"] \arrow[d, "Ll'"] & LA \arrow[r, "u"] \arrow[d, "Ll"] & RX \arrow[r, "Rx"] \arrow[d, "Rr"] & RX' \arrow[d, "Rr'"] \\ LB' \arrow[r, "Lb"] & LB \arrow[r, "v"] & RY \arrow[r, "Ry"] & RY' \end{tikzcd} \tag{7.8}$$

  where the left square is the image of L of the square in $\mathbb{L}$ and the right square is the image of R of the square in $\mathbb{R}$, there are multiple ways of taking lifts $LB' \to RX'$. And we require that all these lifts will be equal. In particular:

  - If the right square is the identity morphism $r = r'$ in $\mathbb{R}_1$, we get the **left horizontal compatibility condition** stating that:

$$\phi_{l',r}(u \circ La, v \circ Lb) = \phi_{l,r}(u, v) \circ Lb. \tag{7.9}$$

  - If the left square is the identity morphism $l' = l$ in $\mathbb{L}_1$, we get the **right horizontal compatibility condition** stating that:

$$\phi_{l,r'}(Rx \circ u, Ry \circ v) = Ry \circ \phi_{l,r}(u, v) \tag{7.10}$$

- We have vertical compatibilities, which is about composition of vertical morphisms. For any composable vertical morphisms

$$\begin{tikzcd} A \arrow[d, "l"] \\ B \arrow[d, "l'"] \\ C \end{tikzcd} \text{ in } \mathbb{L} \text{ and } \begin{tikzcd} X \arrow[d, "r"] \\ Y \arrow[d, "r'"] \\ Z \end{tikzcd} \text{ in } \mathbb{R}, \tag{7.11}$$

and any diagram in $\mathcal{C}$ of the following form:

$$\begin{array}{ccc} LA & \xrightarrow{u} & RX \\ \Big\downarrow{\scriptstyle Ll} & & \Big\downarrow{\scriptstyle Rr} \\ LB & & RY \\ \Big\downarrow{\scriptstyle Ll'} & & \Big\downarrow{\scriptstyle Rr'} \\ LC & \xrightarrow{v} & RZ \end{array} \tag{7.12}$$

there are multiple ways of taking lifts $LC \to RX$. And we require that all these lifts will be equal. In particular:

- If l' comes from i, we have we have that Ll' is an identity morphism and we get the **right vertical compatibility condition** stating that

$$\phi_{l,r'\circ r}(u,v) = \phi_{l,r}\big(u, \phi_{l,r'}(Rr \circ u)\big) \tag{7.13}$$

- If r' comes from i, we have we have that Rr' is an identity morphism and we get the **left vertical compatibility condition** stating that

$$\phi_{l'\circ l,r}(u,v) = \phi_{l',r}\big(\phi_{l,r}(u, v \circ l'), v\big) \tag{7.14}$$

If ϕ is an $(\mathbb{L}, \mathbb{R})$-lifting operation, we call $(\mathbb{L}, \phi, \mathbb{R})$ a **lifting structure** on $\mathcal{C}$. ◊

For any class of maps $\mathcal{A}$ of $\mathcal{C}$, we can consider a class of maps haveing a right or left lifting property against $\mathcal{A}$. We can do something similar for double categories.

Definition 7.7. Given a double category $\mathbb{L}$ with double functor $L : \mathbb{L} \Rightarrow \mathbb{S}\mathbf{q}(\mathcal{C})$, there is a double category $\mathbf{RLP}(L)$.

- $(\mathbf{RLP}(\mathbb{L}))_0 = \mathcal{C}$, so the horizontal maps and objects are just those from $\mathcal{C}$.
- The vertical maps, thus objects of $(\mathbf{RLP}(\mathbb{L}))_1$ are given by tuples (f, ψ) with f a morphism of $\mathcal{C}$, and ψ is a family of operations assigning to each vertical map l of $\mathbb{L}$ an operation ψ_l, which assigns to diagrams of the form

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ \Big\downarrow{\scriptstyle l} & & \Big\downarrow{\scriptstyle f} \\ B & \xrightarrow{v} & Y \end{array} \tag{7.15}$$

in $\mathcal{C}$ a lift $\psi_l(u,v) : B \to X$. Furthermore, we require that ψ_l satisfies both the left horizontal and left vertical compatibility conditions from Definition 7.6.

- A morphism $(f, \psi) \to (f', \psi')$ in $(\mathbf{RLP}(\mathbb{L}))_1$ is a commuting square in $\mathcal{C}$ of the form

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f'} \\ Y & \xrightarrow{y} & Y' \end{array} \tag{7.16}$$

such that for all vertical maps l of $\mathbb{L}$ and all u, v as in Equation 7.15, we have that $\psi'_l(x \circ u, v \circ y) = x \circ \psi_l(u, v)$. compositions in $(\mathbf{RLP}\mathbb{L})_1$ correspond to horizontal compositions of such squares.

- The domain and codomain functors d and c send (f, ψ) to the domain and codomain of f respectively.
- The identity functor i assigns to each object C of $\mathcal{C}$ the tuple $(1_C, t)$, where $t_l(u, v) = v$.
- For the composition functor m, we use the right vertical compatibility condition from Definition 7.6. Thus m sends composable vertical morphisms (f, ψ) and (g, χ) to $m(g, f) = (g \circ f, \phi)$ where
$$\phi_l(u, v) = \psi_l\big(u, \chi_l(f \circ u)\big) \tag{7.17}$$
and squares as in Equation 7.16 are composed vertically.

There is a forgetful double functor $\mathbf{RLP}(\mathbb{L}) \Rightarrow \mathcal{C}$. So $\mathbf{RLP}(\mathbb{L})$ is a double category over $\mathcal{C}$.

Dually, given a double category $\mathbb{R}$, we have a double category $\mathbf{LLP}(\mathbb{R})$ over $\mathcal{C}$. $\Diamond$

Remark 7.8. Given a double category $\mathbb{A}$ over some category $\mathcal{C}$, there are canonical lifting operations ψ, χ such that $(\mathbb{A}, \psi, \mathbf{RLP}(\mathbb{A}))$ and $(\mathbf{LLP}(\mathbb{A}), \chi, \mathbb{A})$ are lifting structures.

Also whenever $(\mathbb{L}, \phi, \mathbb{R})$ is a lifting structure, there are canonical inclusion double functors $\mathbb{R} \hookrightarrow \mathbf{RLP}(\mathbb{L})$ and $\mathbb{L} \hookrightarrow \mathbf{LLP}(\mathbb{R})$. $\Diamond$

Definition 7.9. A lifting structure $(\mathbb{L}, \phi, \mathbb{R})$ on $\mathcal{C}$ is called a **lifting awfs** if it satisfies the following two axioms:

1. **Axiom of lifting**: The induced maps $\mathbb{R} \hookrightarrow \mathbf{RLP}(\mathbb{L})$ and $\mathbb{L} \hookrightarrow \mathbf{LLP}(\mathbb{R})$ are invertible.
2. **Axiom of factorisation**: each morphism $f : A \to B$ of $\mathcal{C}$ admits a factorisation
$$A \xrightarrow{Ll} C \xrightarrow{Rr} B \tag{7.18}$$
where l, r are vertical morphisms from $\mathbb{L}, \mathbb{R}$. This factorisation is bi-universal, meaning that
   - The square in $\mathbb{S}\mathbf{q}(C)$ of the form
$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "Ll"] & A \arrow[d, "f"] \\ C \arrow[r, "Rr"] & B \end{tikzcd} \tag{7.19}$$
corresponds to an arrow in $(\mathbb{S}\mathbf{q}(\mathcal{C}))_1$ from Ll to f. This arrow is co-universal for the functor $L_1 : \mathbb{L}_1 \to (\mathbb{S}\mathbf{q}(\mathcal{C}))_1$. So whenever we have a morphism in $\mathcal{C}^{[1]}$ of the form $Lx \to f$, there is a unique square in $\mathbb{L}$ with domain x and codomain l such that the morphism $Lx \to Ll$ makes everything commute.
   - Dually to the above situation, the square in $\mathbb{S}\mathbf{q}(\mathcal{C})$ of the form
$$\begin{tikzcd} A \arrow[r, "Ll"] \arrow[d, "f"] & C \arrow[d, "R_f"] \\ B \arrow[r, "1_B"] & B \end{tikzcd} \tag{7.20}$$
corresponds to a $\mathcal{C}^{[1]}$-morphism $f \to Rr$ which is universal for R_1.

$\Diamond$

Now we will state without proof Propostion 18 from [Bou23]:

Proposition 7.10. *Let $\mathcal{C}$ be a locally presentable category and let $\mathbb{L}$ be a small double category over $\mathbb{S}\boldsymbol{q}(\mathcal{C})$. Then the canonical lifting structure $(\boldsymbol{LLP}(\boldsymbol{RLP}(\mathbb{L})), \phi, \boldsymbol{RLP}(\mathbb{L}))$ is a lifting awfs.*

Such lifting structures are called **cofibrantly generated**. It should be noted the proof for the above proposition uses a version of Quillen's small objects argument from [Gar08], where the axiom of choice was used. At the moment, it is not known whether a constructive proof for this argument exists.

In the next sections, we shall reintroduce the symmetric effective Kan fibrations as being of the form $\mathbf{RLP}(\mathbb{L})$ for some small category $\mathbb{L}$. Thus they will fit into a lifting awfs. We shall take for $\mathcal{C}$ the category *sSet*, which is locally presentable as it is the presheaf category of the small category Δ. $\mathbb{L}$ shall be a double category containing the horn inclusions in its vertical maps.

7.3 The double category of horn pushout sequences

In this section, we will introduce the double category $\mathbb{L}$ of horn pushout sequences. Before we do so, we will motivate our choice for this category by providing a wishlisht of properties that $\mathbb{L}$ should have:

- In order to apply Proposition 7.10, we need to have that $\mathbb{L}$ is small.
- In order for $\mathbf{RLP}(\mathbb{L})$ to contain the symmetric effective Kan fibrations, we need to require that $\mathbb{L}$ contains the horn inclusions.
- Horn inclusions cannot be composed, so we need to define some sensible composition structure on $\mathbb{L}$.
- In order for the definition of symmetric effective Kan fibrations to hold in $\mathbf{RLP}(\mathbb{L})$ we need the squares in $\mathbb{L}$ to include the pullback squares of horns along degeneracy maps from Section 3.3.

To make sure that $\mathbb{L}$ is small, we will let the objects of $\mathbb{L}$ be sieves in *sSet*, and will make sure that all morphisms correspond to diagrams of sieves. For technical reasons, we will actually only deal with decidable sieves:

Definition 7.11. A **decidable sieve** of Δ^n is a sieve $S \subseteq \Delta^n$ such that for any $0 \leq m \leq n$ and any $p : \Delta^m \to \Delta^n$, it is decidable whether or not p factors through S. $\diamondsuit$

To make sure we include pullback squares, the horizontal morphisms between decidable sieves $S \subseteq \Delta^a, T \subseteq \Delta^b$ will correspond to morphisms $f : [a] \to [b]$ in Δ such that $f^*(T) = S$.

Now to include the horn inclusions and still allow us to have composable morphisms, we will define the vertical arrows of $\mathbb{L}$ as inclusion sequences of sieves, where each sieve can be obtained from the previous one by filling in a horn.

Definition 7.12. A **horn pushout sequence** is a finite sequence of composable monomorphisms of the form

$$S_0 \xhookrightarrow{\sigma_1} S_1 \xhookrightarrow{\sigma_2} \cdots \xhookrightarrow{\sigma_k} S_k \hookrightarrow \Delta^a. \tag{7.21}$$

with length $k \geq 0$, where every σ_i is part of a chosen square of the form

$$\begin{tikzcd} \Lambda^n_m \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner" very near start] & S_{i-1} \arrow[d, hook, "\sigma_i"] \\ \Delta^n \arrow[r, hook] & S_i \end{tikzcd} \tag{7.22}$$

which is both a pushout and a pullback square with every object a decidable sieve of Δ^a. ♢

Remark 7.13. The squares above correspond with taking the union of subobjects in a topos, as all morphisms are monic. In this case, being a pushout implies being a pullback. ♢

Remark 7.14. A benefit of using decidable sieves inside a representable simplex and horn pushout sequences is that we can make a counting argument. Remark that for each $a \in \mathbb{N}$, and each $m \in \mathbb{N}$, there are only a finite amount of monomorphisms $[m] \to [a]$. Each of these correspond to a non-degenerate m-simplex of Δ^a. For a fixed $a \in \mathbb{N}$, we call the the non-degenerate m-simplices of Δ^a **small simplices**. For each $a \in \mathbb{N}$, there are only finitely many small simplices. The small simplices that factor through any decidable sieve $S \subseteq \Delta^a$ can be counted.

Furthermore, because all morphisms in Equation 7.22 are mono, we always have that S_i contains exactly two more small simplices than S_{i-1}, namely one nondegenerate n-simplex and one nondegenerate $n-1$-simplex, which was the missing face of the added n-simplex. Thus in a horn pushout sequence as in Equation 7.21, the amount of small simplices factoring through S_j is linearly increasing in the index j with steps of size 2. In particular in a horn pushout sequence we always add an even number of small simplices. ♢

Now for the squares, we want to include pullbacks of horn pushout sequences along degeneracy maps. In order to show the main theorem of this chapter, Theorem 7.19, it turns out we also need to add pullback along face maps. However, in Example 7.18 we shall see that the pullback of a horn pushout sequence along a face map is not necessarily a horn pushout sequence. Therefore we need to be a bit nuanced in what our squares will be exactly:

Definition 7.15. A **pullback square of horn pushout sequences** from a horn pushout sequence as in Definition 7.12 into a horn pushout sequence of the form

$$T_0 \xhookrightarrow{\tau_1} T_1 \xhookrightarrow{\tau_2} \cdots \xhookrightarrow{\tau_k} T_l \hookrightarrow \Delta^b. \tag{7.23}$$

consists of a morphism $f : [a] \to [b]$ in Δ and a nondecreasing function $\mu : \mathbb{N}_{\leq l} \to \mathbb{N}_{\leq k}$ such that $\mu(0) = 0$ and $\mu(l) = k$, and for each $0 \leq i \leq l$, we have that $f^*(T_i) = S_{\mu(i)}$.

We call the sequence in Equation 7.21 the **source sequence** and the sequence in Equation 7.23 the **target sequence**. ♢

Now by composing the underlying morphisms and nondecreasing functions, there is a natural horizontal composition structure of such pullback squares. So these pullback squares of horn pushout sequences are a good candidate for the squares in $\mathbb{L}$. However, it turns out that in order to show Theorem 7.19, we want our squares to be a bit more explicit.

Definition 7.16. A **1-step morphism of horn pushout sequences** is a pullback square of horn pushout sequences where the underlying morphism f in Δ is a face or degeneracy map.

A **morphism of horn pushout sequences** is a composition of 1-step morphisms of horn pushout sequences. ♢

By definition, all morphisms of horn pushout sequences are also pullback squares of horn pushout sequences. A natural question is about the converse. We shall see in Remark 7.17 that for any simplicial morphism f, any factorisation of f which corresponds to a morphism of horn pushout sequences gives the same morphism. However, in Example 7.18 we shall see that not all factorisations of f correspond to a morphism of horn pushout sequences. While the question whether a factorisation always exists is interesting, for the purpose of showing Theorem 7.19, it does not matter.

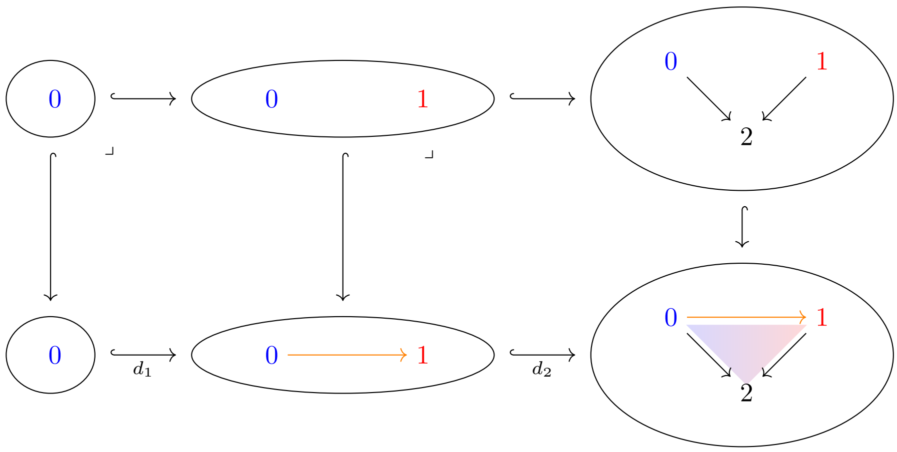
Remark 7.17. In Definition 7.15 the nondecreasing map μ is determined completely by the horn pushout sequences as in Equations 7.21 and 7.23 and the simplicial morphism f:

For each T_i, we have that $S_{\mu(i)}$ should equal $f^*(T_i)$ for all $i \in \mathbb{N}_{\leq l}$. By a counting argument, there is at most one index $j \in \mathbb{N}_{\leq k}$ such that S_j has the same amount of small simplices as $f^*(T_i)$. So there is at most one index j with $S_j = f^*(T_i)$. So if μ exists, then we must have that $\mu(i) = j$.

This does not mean that we can leave out μ in Definition 7.16. It is required that such an μ exists to have a morphism of horn pushout sequences. We have only shown that if it exists, it is unique. $\Diamond$

So if we can write f in two factorisations which give morphisms of horn pushout sequences, both compositions share their composite nondecreasing function μ. However, the existence of such factorisations is non-trivial, as the next example shows:

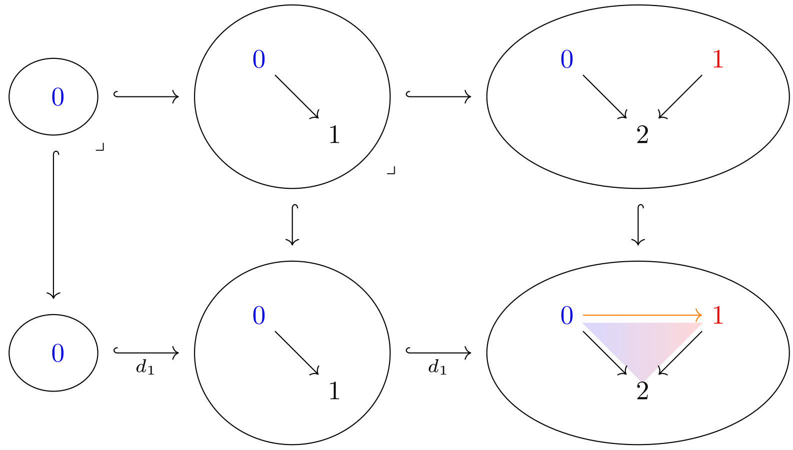
Example 7.18. Consider the horn inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$. We pull back this horn inclusion along the composition $d_2 \circ d_1 : \Delta^0 \hookrightarrow \Delta^2$, as in the following picture:



(7.24)

The leftmost vertical inclusion is the identity and thus can be written as a horn pushout sequence. The rightmost vertical inclusion is a horn inclusion and thus can also be written as a horn pushout sequence. However, the middle vertical inclusion cannot be written as a horn pushout sequence as we add 1 small simplex, which is an odd number. The composite morphism is a pullback square of horn pushout sequences, but the subsquares are not.

Note however that the simplicial morphism $d_2 \circ d_1$ is equal to $d_1 \circ d_1$. If we were to take this composition instead, we would get the following picture:



(7.25)

where we do have that all subsquares are pullback squares of horn pushout sequences.

Thus there exists a pullback square of horn pushout sequences with underlying simplicial morphism f, such that not all factorisations of f give a morphism of horn pushout sequences. But we have not shown that there is no factorisation of f which gives a morphism of horn pushout sequences. $\Diamond$

Now that we have motivated our definition for morphisms of horn pushout sequences, we can summarize how we define the double category $\mathbb{L}$ of horn pushout sequences as follows:

- $\mathbb{L}_0$ has:
  - Objects decidable sieves $S \subseteq \Delta^a$.
  - Morphisms from $S \subseteq \Delta^a$ to $T \subseteq \Delta^b$ are morphisms $f : [a] \to [b]$ from Δ such that $f^*(T) = S$.
  - Compositions correspond to ordinary compositions of such morphisms.
  - Identities correspond to the morphism $1_{[a]} : [a] \to [a]$.
- $\mathbb{L}_1$ has:
  - Objects are horn pushout sequences as in Definition 7.12.
  - Morphisms are as in Definition 7.16.
  - The compositions of two such morphisms corresponds to the composition of the underlying morphisms in Δ and the composition of the underlying nondecreasing functions.
  - Identities correspond to identity morphisms in Δ and the identity as nondecreasing function.
- The domain and codomain functors send a horn pushout sequence as in Definition 7.12 to $S_0 \subseteq \Delta^n$ and $S_k \subseteq \Delta^n$ respectively. For f as in Definition 7.15, f satisfies that $f^*(T_0) = S_0$ and $f^*(T_l) = S_k$. The domain and codomain functors send such a morphism to the corresponding canonical morphisms $S_0 \to T_0$ and $S_k \to T_l$.
- m works by concatenating sieve inclusions within a fixed Δ^a.

- The identity sends a sieve $S \subseteq \Delta^a$ to the unique horn pushout sequence of length $k = 0$.

And the double functor $L : \mathbb{L} \to sSet$ is the forgetful double functor. This means that

- The sieve $S \subseteq \Delta^a$ is sent to the simplicial set S.
- A morphism corresponding to the pullback $f^*(T) = S$ is sent to the morphism $S \to T$.
- A horn pushout sequence as in Definition 7.12 is sent to the composition $S_0 \hookrightarrow S_k$.
- A morphism of horn pushout sequences, seen as in Definition 7.16 is sent to the square

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & T_0 \arrow[d, hook] \\ S_k \arrow[r] & T_l \end{tikzcd} \tag{7.26}$$

7.4 Symmetric effective Kan fibrations

In this section, we shall prove the following theorem:

Theorem 7.19. *For $\mathbb{L}$ as in the previous section, and $\mathbb{R} = \boldsymbol{RLP}(\mathbb{L})$ as in Definition 7.7, there is a bijective correspondence between vertical morphisms of $\mathbb{R}$ and symmetric effective Kan fibrations.*

For the purposes of this theorem, an effective symmetric Kan fibration consists of both a map f and a filling structure fil. It is possible that the same map has multiple filling structures, and we consider these distinct symmetric effective Kan fibrations.

We shall do this in three parts:

- We shall first show that every symmetric effective Kan fibration gives a vertical morphism of $\mathbb{R}$ in Lemma 7.20.
- We shall then show that every vertical morphism of $\mathbb{R}$ gives a symmetric effective Kan fibration in Lemma 7.21.
- We shall then show that these two methods are each other's inverses in Lemma 7.22.

Lemma 7.20. *All symmetric effective Kan fibrations give vertical morphisms of $\mathbb{R}$.*

Proof. Let $\alpha : X \to Y$ be a symmetric effective Kan fibration with filling method lift. We shall define ϕ_l for each horn pushout sequence l. We shall use recursion on the length k of l:

- If $k = 0$, then Ll is an identity morphism, against which there is a unique lift.
- Suppose we have determined ϕ_l for l of length k. Consider the horn pushout sequence $m(l', l)$ for some a horn pushout sequence l' of length 1 with compatible domain. To define $\phi_{m(l',l)}$ consider the following diagram in $\mathcal{C}$ for any valid x, y:

$$\begin{tikzcd} & S_0 \arrow[r, "x"] \arrow[d, hook, "Ll"] & X \arrow[dd, "\alpha"] \\ \Lambda^n_m \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near end] & S_1 \arrow[d, hook, "Ll'"] & \\ \Delta^n \arrow[r, hook] & S_2 \arrow[r, "y"] & Y \end{tikzcd} \tag{7.27}$$

where the left square is the pushout corresponding to l'. Consider the map $\phi_l(x,y \circ Ll') : S_1 \to X$. This must fit into the above diagram in such a way that everyting commutes. Also it induces a map $\Lambda^n_m \to X$ giving a horn lifting problem we can solve with lift, giving a map $\Delta^n \to X$. This map $\Delta^n \to X$ can then be pushed out to a map $S_2 \to X$ which is a lifting solution for the right square. We define $\phi_{m(l',l)}$ as this map $S_2 \to X$.

By this inductive definition, ϕ satisfies the left vertical compatibility conditions. We now only need to verify that ϕ respects the left horizontal compatibility conditions for any morphism of pushout sequences γ .

We use double induction on the length of the target sequence of γ and the length of the factorisation of the underlying simplicial morphism of γ. The base cases are as follows:

- If the target sequence has length 0 and the source sequence length k, then the underlying nondecreasing map μ of the pullback square must satisfy that $\mu(0) = 0$ and $\mu(0) = k$, hence $k = 0$. Thus both source and target are identity vertical morphisms, and have a unique lift. This makes the compatibility condition trivial.
- If the underlying simplicial morphism has a factorisation of length 0, then γ is an identity morphism, and the compatibility condition is trivially satisfied.

Now for the induction step, we only need to consider γ a 1-step morphism of horn pushout sequences where the target sequence has length 1. There underlying simplicial morphism of γ is now either a face or a degeneracy map.

- Suppose the underlying simplicial morphism of γ is a face map d_j. Let $T_0 \hookrightarrow T_1$ be the target sequence of γ. As this inclusion is the pushout of a horn inclusion, T_1 contains exactly two non-degenerate simplices that T_0 does not. Remark that pulling back along a mono corresponds to taking an intersection in Δ^b. The inclusion $(T_0 \cap d_j) \hookrightarrow (T_1 \cap d_j)$ is also a horn pushout sequence in Δ^b. Note that it can not add more than two small simplices. As this inclusion is a horn pushout sequence, we must add an even number of small simplices, so either 0 or 2. The length of the source sequence of γ is thus either 0 or 1.
  - If the length of the source sequence is 0, we can use the same uniqueness argument as above.
  - If the length is 1, Δ^n factors through $T_1 \cap d_j =: S_1$ and Λ^n_m must factor through $T_0 \cap d_j =: S_0$. We thus have the following diagram in simplicial sets:

$$\begin{tikzcd}
& \Lambda^n_m \arrow[dl, hook'] \arrow[dr, hook] \arrow[dd, hook] & & \\
S_0 \arrow[rr, hook] \arrow[dd, hook] & & T_0 \arrow[r, hook] \arrow[dd, hook] & X \arrow[dd, "\alpha"] \\
& \Delta^n \arrow[dl, hook'] \arrow[dr, hook] & & \\
S_1 \arrow[rr, hook] \arrow[d, hook] & & T_1 \arrow[r, hook] \arrow[d, hook] & Y \\
\Delta^{b-1} \arrow[rr, hook, "d_j"] & & \Delta^b &
\end{tikzcd} \tag{7.28}$$

and both the fillers $S_1 \to X$ and $T_1 \to X$ are defined by pushing out the chosen filler of $\Lambda^n_m \hookrightarrow \Delta^n$. By the universal property of the pushout, the two maps $S_1 \to X$ we get must be equal.

So if γ has as underlying simplicial morphism a face map, the left horizontal compatibility condition for γ is satisified.

- Now suppose the underlying simplicial morphism of γ is a degeneracy map s_j. Let the target morphism of γ correspond to the pushout square

$$\begin{tikzcd}
\Lambda^n_m \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near end] & T_0 \arrow[d, hook] \\
\Delta^n \arrow[r, hook] & T_1 \arrow[d, hook] \\
 & \Delta^b
\end{tikzcd} \tag{7.29}$$

Consider the mono $\Delta^n \hookrightarrow \Delta^b$ in the above diagram. In our topological intuition, pulling back along s_j corresponds to taking the point j, seen as morphism $\Delta^0 \hookrightarrow \Delta^b$, and stretching it out. If the point j factors through Δ^n, we also stretch out our horn inclusion, as we did in Chapter 3. If j does not factor through Δ^n, the horn inclusion is not stretched out and we are in a setting as in Equation 7.28.

More formally, let N be the subset of $[b]$ of size n corresponding to the mono $\Delta^n \to \Delta^b$. Note that $s_j^*(\Delta^n) \hookrightarrow \Delta^{b+1}$ corresponds to the subset $s_j^{-1}(N) \subseteq [b+1]$, We can make a case distinction on whether the inclusion of the point $j : \Delta^0 \hookrightarrow \Delta^b$ factors through the decidable sieve $\Delta^n \hookrightarrow \Delta^b$, which it does iff $j \in N$.

  - Suppose $j \notin N$, then $s_j^{-1}(N)$ has size n, and thus $s_j^*(\Delta^n) = \Delta^n$. As a consequence $s_j^*(\Lambda^n_m) = \Lambda^n_m$. So the pullback of the diagram in Equation 7.29 is as follows:

$$\begin{tikzcd}
\Lambda^n_m \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near end] & s_j^*(T_0) \arrow[d, hook] \\
\Delta^n \arrow[r, hook] & s_j^*(T_1) \arrow[d, hook] \\
 & \Delta^{b+1}
\end{tikzcd} \tag{7.30}$$

  So the source sequence of γ has length 1, and we have that the same horn inclusion $\Lambda^n_m \hookrightarrow \Delta^n$ is used to define the horn pushout sequence $T_0 \hookrightarrow T_1$ and $s_j^*(T_0) \hookrightarrow s_j^*(T_1)$. By a diagram similar as in Equation 7.28, (we only need to replace d_j by s_j) we can argue that we satisfy the left horizontal compatibility condition for γ.

  - Now suppose that $j \in N$, then $s_j^{-1}(N)$ has size $n+1$ and $s_j^*(\Delta^n) = \Delta^{n+1}$. Consider the pullback squares

$$\begin{tikzcd}
s_j^*(\Lambda^n_m) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda^n_m \arrow[d, hook] \\
\Delta^{n+1} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^n \arrow[d, hook] \\
\Delta^{a+1} \arrow[r, "s_j"] & \Delta^a
\end{tikzcd} \tag{7.31}$$

Now the bottom square in the above diagram corresponds to a square in the simplex category. If we write out this square with compositions of face and degeneracy maps using Proposition 2.1, we must have that the horizontal morphisms are compositions of degeneracy maps and the vertical morphisms of face maps. The bottom square in the above diagram must thus correspond to a square of the form

$$\begin{tikzcd}
{[n+1]} \arrow[r, "s_{j'}"] \arrow[d, hook] & {[n]} \arrow[d, hook] \\
{[a+1]} \arrow[r, "s_j"] & {[a]}
\end{tikzcd} \tag{7.32}$$

for some $0 \leq j' \leq n$, as the top morphism above is a composition of degeneracy maps $[n+1] \to [n]$. So $s_j|_{\Delta^{n+1}} = s_{j'}$. For ease of notation, we will assume that $j' = j$. Now we have studied the pullback square

$$\begin{tikzcd}
s_j^*(\Lambda^n_m) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda^n_m \arrow[d, hook] \\
\Delta^{n+1} \arrow[r, "s_j"] & \Delta^n
\end{tikzcd} \tag{7.33}$$

in Chapter 3, and we have seen that the pullback of a horn inclusion Λ^n_m along s_j contains all faces except d_j, d_{j+1} and d_{m^*}. Depending on whether $j = m$, we miss either 4 or 6 small simplices:

- ∗ If $j \neq m$, the small simplices $d_m \circ d_j$, $d_m \circ d_{j+1}$, d_j, d_{j+1}, d_{m^*} and Δ^{n+1} are missing. Every horn pushout sequence corresponding to the inclusion $s_j^*(\Lambda^n_m) \hookrightarrow \Delta^{n+1}$ must add all these small simplices. Because every horn pushout couples adds one small simplex and one of its faces, we must first add the faces d_j, d_{j+1} in any order, and then add all of Δ^{n+1}. We have seen that we can add the faces d_j, d_{j+1}, by taking the pushout of $\Lambda^n_m \hookrightarrow \Delta^n$.
- ∗ If $j = m$, the small simplices $d_m \circ d_j$, d_j, d_{j+1}, and Δ^{n+1}. There are two options to horn pushout sequences corresponding to the inclusion $s_j^*(\Lambda^n_m) \hookrightarrow \Delta^{n+1}$: either we first add d_j and then Δ^{n+1}, or we first add d_{j+1} and then Δ^{n+1}. As above, we add the face d_j or d_{j+1} by taking the pushout of $\Lambda^n_m \hookrightarrow \Delta^n$.

Our filling method assigns to any pushout of $\Lambda^n_m \hookrightarrow \Delta^n$ the pushout of $\text{lift}(x, y)$, hence we recover exactly all cases in Definition 3.1. In particular, we satisfy the left horizontal compatibility for γ.

So for all left squares γ with underlying simplicial morphism a degeneracy map, we satisfy the left horizontal compatibility condition for γ

We conclude that for all γ 1-step morphisms of horn pushout squares with target sequence of length 1, our filling method ϕ satisfies the left horizontal compatibility condition for γ. By induction, it follows that ϕ satisfy the left horizontal compatibility condition.

So ϕ satisfies all left compatibility conditions and (α, ϕ) is a vertical morphism of $\mathbb{R}$. $\square$

Lemma 7.21. *Every vertical morphism of $\mathbb{R}$ gives a symmetric effective Kan fibration.*

Proof. Let (α, ϕ) be a vertical morphism of $\mathbb{R}$. For any horn inclusion $\Lambda^n_m \hookrightarrow \Delta^n$, consider the horn pushout sequence l given by

$$\begin{tikzcd} \Lambda^n_m \arrow[r, "="] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \Lambda^n_m \arrow[d, hook] \\ \Delta^n \arrow[r, "="] & \Delta^n \arrow[d, "="] \\ & \Delta^n \end{tikzcd} \tag{7.34}$$

Now for any square in simplicial sets of the form

$$\begin{tikzcd} \Lambda^n_m \arrow[r, "x"] \arrow[d, hook] & X \arrow[d, "\alpha"] \\ \Delta^n \arrow[r, "y"] & Y \end{tikzcd} \tag{7.35}$$

Define $\mathrm{lift}(x, y) = \phi_l(x, y)$.

In our previous proof, we have studied what the pullback of a horn pushout sequence of length 1 along a degeneracy map looks like. In this case, we are always in the case where $j \in N$ and the left horizontal compatibility condition gives us that $\mathrm{lift}(s_j^*(x), y \circ s_j) = \mathrm{lift}(x, y) \circ s_j$ for any degeneracy map s_j. Hence (α, lift) is a symmetric effective Kan fibration. $\square$

Lemma 7.22. *The operations defined in Lemma's 7.21 and 7.20 are each other's inverses.*

Proof. We study the two ways of ordering the operations:

- Remark that the if we start out with a symmetric effective Kan fibration (α, lift) and apply the construction from Lemma 7.20 and then apply the construction from Lemma 7.21, we get the same morphism α. The new filling structure lift' is equal to the filling structure lift iff it assings the same lifts to all horn inclusions. If lift assigns g as filler for the horn inclusion $\Lambda^n_m \hookrightarrow \Delta^n$, then lift' assigns the filler given by the pushout of g along the identity, which is just g itself. Hence lift' = lift. So if we start out with a symmetric effective Kan fibration we recover the same symmetric effective Kan fibration.
- Conversely, suppose we start out with a vertical morphism (α, ϕ) of $\mathbb{R}$. If we first apply Lemma 7.21 and then Lemma 7.20, we get the same morphism α, with a lifting structure ϕ'. We shall show that $\phi' = \phi$ by showing that they are equal on all horn pushout sequences. It is sufficient to show that they are equal on horn pushout sequences of length 1: They are trivially equal on horn pushout sequences of length 0, and for the other sequences we can use induction and the left vertical compatibility condition.

Suppose we need to find a lift for α against pushout sequence of length 1 as in Equation 7.29 called l. Let d be the inclusion $\Delta^n \hookrightarrow \Delta^b$, and let ι be the horn pushout sequence as in Equation 7.34 (for the same n, m).

If $b > n$, then $d : \Delta^n \hookrightarrow \Delta^b$ must define an n-element strict subset of $[b]$. For any j outside of this subset we can pull back along d_j. And as Δ^n still lies in the intersection of $d_j \cap T$, we get a 1-step morphism of horn pushout sequences. If we repeat this argument, we can realize ι as a pullback of l along d. where the pullback square of horn pushout sequences is also a morphism of horn pushout sequences.

Hence we have the following diagram of simplicial sets:

$$
\begin{tikzcd}
\Lambda^n_m \arrow[r, hook, "t_0"] \arrow[d, hook, "L\iota"'] \arrow[dr, phantom, "\ulcorner", near end] & T_0 \arrow[r, "x"] \arrow[d, hook, "Ll"] & X \arrow[d, "\alpha"] \\
\Delta^n \arrow[r, hook, "t_1"] \arrow[d, "1_{\Delta^n}"'] \arrow[dr, phantom, "\lrcorner", very near start] & T_1 \arrow[r, "y"] \arrow[d, hook] & Y \\
\Delta^n \arrow[r, hook, "d"] & \Delta^b &
\end{tikzcd}
\tag{7.36}
$$

$\phi'_l(x, y)$ is defined as the pushout of $\phi_\iota(x \circ t_0, y \circ t_1)$. By the left horizontal compatibility condition, $\phi_l(x, y)$ satisfies the property by which this pushout is defined. Hence $\phi'_l(x, y) = \phi_l(x, y)$.

This holds for any horn pushout sequence l of length 1. By induction we conclude that $\phi' = \phi$ and hence if we start with a vertical morphism in $\mathbb{R}$ we recover the same vertical morphism from $\mathbb{R}$.

We conclude that the operations are each other's inverses. $\square$

These lemma's completes the proof for Theorem 7.19.

Appendix A

Mathematical history and context

In this thesis, we have studied alternative notions of Kan complexes. One of the motivations for Kan complexes is that they can be used in a model for homotopy type theory, as in [KL21]. In this appendix, we will discuss some of the history of homotopy type theory from this model to the introduction of effective Kan fibrations in [BF22].

A.1 Homotopy type theory

Homotopy type theory (or **HoTT**) is a foundation of constructive mathematics using principles from type theory and homotopy theory. Standard references for the subject are [Uni13] and [Rij22]. A full introduction to HoTT is beyond the scope of this thesis, but we would like to mention some aspects of homotopy type theory to motivate effective Kan complexes:

- As sets have elements, types have terms.
- Types can be interpreted as mathematical statements. Their terms are then interpreted as (constructive) proofs of those statements.
- Just as mathematical statements can be parameterized over some variable, types can be parametrized on terms of another type. These are called dependent types. A dependent type B over A gives for all terms a of type A a type $B(a)$.
- For any two terms a, b of the same type A, there is a type of called $a =_A b$. A term p of this type is called a path from a to b.
- For paths as above, the relation "there exists a path from a to b" is an equivalence relation.

HoTT allows us to formalize constructive proofs as mathematical objects. Homotopy type theory is itself a mathematical theory, and therefore we need to have models to make sure it is consistent. And indeed, there is a mathematical model for HoTT. In Section 2.7, we have discussed how Kan complexes have an interpretation for 1-cells as paths. This is one of the reasons Kan complexes are used to model types in [KL21]. In this model, the dependent types over A are modelled by Kan fibrations into A.

Now to make full use of the interpretation of types as mathematical statements, we would like our types to be closed under logical connectives and quantifiers. If we use Kan complexes and

Kan fibrations to be a model for HoTT, we need to have an interpretation for these connectives and quantifiers. In particular, we point out the following:

- Given A, B types, there is an exponential type $A \to B$.
- In the interpretation of types as mathematical statements, a term of $A \to B$ corresponds to a proof for the implication $A \implies B$. It is seen as a function taking proofs of the statement A to proofs of the statement B.
- To model the exponential of two types, we take the exponential of their corresponding Kan complexes.
- Let A be a type, and let B be a type over A. There is then a type $\prod_{a:A} B(a)$.
- A term of type $\prod_{a:A} B(a)$ corresponds has for each term a of A a term of $B(a)$. In the interpretation of types as statements, such a term corresponds to a proof that for all a of type A, the proposition $B(a)$ holds. This is universal quantification.
- Universal quantification over dependent types is modelled by pushforwards along their corresponding Kan fibrations.

For Kan complexes to work as a model for HoTT, it is thus important that Kan complexes are closed under exponentials. And indeed this is the case, see for example Theorem 6.9 of [May67]. However, as we have mentioned in Chapter 5, the proof uses case distinction on whether a simplex is degenerate, which cannot be shown constructively.

A.2 Constructive HoTT models

We are thus left with a non-constructive model for a theory used to formalize constructive mathematics. For constructivist mathematicians interested in using HoTT, this is a problem.

The first attempt to fix our problem would be to find a constructive proof that Kan complexes are closed under exponentials. However, in [BCP15], it was shown there is a Kripke model were Kan complexes are not closed under exponentials. Hence there is no constructive proof that Kan complexes are closed under exponentials.

A second attempt might be to make the definition of a Kan complex more constructive, and require not an extension property on horn inclusions, but a functional extension. These "algebraic Kan complexes" are however also not closed under exponentials, as was shown in [Par18].

There is a constructive model for HoTT which did not use simplicial sets, but so-called cubical sets. These were introduced in [BCH19] and [CCHM18], and are used in many other places. Types were modelled by "constructive Kan cubical sets", which are in some sense similar to Kan complexes. Constructive Kan cubical sets are defined as cubical sets having a lifting structure against certain inclusions. These inclusions correspond to a cube with its inner part and one face removed, which are similar to horn inclusions: those correspond to simplices with the inner parts and one face removed. In addition, this lifting structure needs to satisfy some uniformity conditions.

So we have a constructive HoTT model in cubical sets, and a non-constructive HoTT model in simplicial sets. The upshot of simplicial sets is that they are studied in mathematical fields outside of HoTT. One could ask whether the uniformity conditions from cubical sets could be transferred back to Kan complexes. Two affirmative answers were given, one in [GS17] in the form of uniform Kan complexes, the other in [BF22] in the form of effective Kan complexes. And indeed, both have constructively been shown to be closed under exponentials. The latter of these two, the effective Kan complexes, formed the starting point for the research thesis.

A.3 Effective Kan complexes

In this thesis, we have introduced symmetric effective Kan complexes. These were inspired by the definition of effective Kan complexes from [BF22]. In this section, we will recall what these effective Kan complexes are. We will only touch the surface of the subject, and refer the reader to [BF22] and [Fab19] for a proper treatment.

In order to introduce the effective Kan complexes, we will first introduce so-called "horn squares". The definition of horn squares uses sieves called boundary inclusions $\partial\Delta^n$. These are defined as the union of all face maps

$$\partial\Delta^n = \bigcup_{0\leq k\leq n} d_k. \tag{A.1}$$

As a sieve, $\partial\Delta^n$ corresponds to all non-surjective maps into Δ^n.

A horn Λ^n_m is parametrized by the dimension $n > 0$ and an index $0 \leq m \leq n$. A horn square is indexed by a natural number $n \geq 0$ and an index i, and also by a sign $\pm \in \{+, -\}$, called positive or negative. They are based on pulling back the boundary inclusion along the section-retraction pairs $s_i \circ d_{i_\pm}$, where $i_+ = i, i_- = i + 1$.

We have the following composition of pullback squares

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & s_i^*(\partial\Delta^n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \partial\Delta^n \arrow[d] \\
\Delta^n \arrow[r, "d_{i_\pm}"'] & \Delta^{n+1} \arrow[r, "s_i"'] & \Delta^n
\end{tikzcd} \tag{A.2}$$

Definition A.1. A **horn square** indexed by $(n, i, \pm)$ is the left hand square in the above diagram. ♢

Remark A.2. Note that $s_i^*(\partial\Delta^n)$ consists of those maps f into $[n]$ such that $s_i \circ f$ is not surjective, which is the case for all f which factor through d_k for some $k \neq i, i + 1$. Therefore, the inner pushout of a horn square corresponds to the sieve

$$d_{i\pm} \cup s_i^*(\partial\Delta^n) = d_{i\pm} \cup \Big(\bigcup_{\substack{0\leq k\leq n+1 \\ k\neq i,i+1}} d_k\Big) = \bigcup_{\substack{0\leq k\leq n+1 \\ k\neq d_{i_\mp}}} d_k = \Lambda^{n+1}_{i_\mp}, \tag{A.3}$$

where $i_\mp$ is the unique element of $\{i, i + 1\} - \{i_\pm\}$. So to summarize:

- The positive horn square with index i corresponds to the horn Λ^{n+1}_{i+1}.
- The negative horn square with index i corresponds to the with Λ^{n+1}_{i}.

This explains the name "horn" square. ♢

A map from a horn square into a simplicial set G consists of two maps (x, g) as in the following diagram:

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & s_i^*(\partial\Delta^n) \arrow[r, "x"] \arrow[d] & G \\
\Delta^n \arrow[r, "d_{i_\pm}"'] \arrow[urr, bend right, "g"'] & \Delta^{n+1} &
\end{tikzcd} \tag{A.4}$$

which via the inner pushout indeed corresponds with a map $\Lambda^{n+1}_{i_\mp} \to G$. And a lift against such a map from a horn square would be a map $\Delta^{n+1} \to G$ making the obvious diagrams commute.

Definition A.3. An **effective Kan complex** can be defined to be a simplicial set G together with a lifting structure against horn squares that is stable along pullback along degeneracy maps. $\Diamond$

As before, by replacing the simplicial set G with a map of simplicial sets, we can define an effective Kan fibration.

This stability under pullback along degeneracy maps means the same thing as it did in Section 3.3, only now the pullback construction preserves the sign of the horn squares. To be precise, if the map (x, g) from a horn square $(\Lambda^n_m, \pm)$ to G corresponds with the horn map $y : \Lambda^n_m \to G$, then $s_j^*(x, g)$ is a map from the horn square $(\Lambda^{n+1}_{m^*}, \pm)$ to G corresponding to $s_j^*(y) : \Lambda^{n+1}_{m^*} \to G$.

Remark A.4. Note that by Remark A.2, if Λ^n_m is an inner horn, so $0 < m < n$, there are two horn squares representing Λ^n_m: the horn squares indexed by $(n, m-1, +)$ and by $(n, m, -)$.

An effective Kan complex is allowed to have different lifts for these horn squares. If we require an effective Kan complex to assign the same lifts for two horn squares representing the same horn, it becomes a symmetric effective Kan complex. This is the sense in which they are “symmetric”. $\Diamond$

Appendix B

A model for simplicial groups in Haskell

This thesis started with the definition of effective Kan complexes in [BF22]. The original research question was about showing that simplicial groups are effective Kan complexes. Before we could give the proof in chapter 6, we needed to study what the pullback of a horn actually looks like and how the simplicial identities work out.

To do this study, we first wrote some Haskell code, which can be found on this github link. We shall also give the code in this appendix.

B.1 Simplicial identities

```
module SimpId where

{- This code is supposed to help calculate simplicial identities
 - In order to do so, we introduce the face and degeneracy maps
 - And we write some code to calculate the normal form of such maps
 - and equality of compositions of such maps
 -
 - the data type of Generators model the face and degeneracy maps
 - the function normalformgenerators calculates the normal form of a list of such maps,
 - read as composition
-}


type Nat = Int

-- We define the maps as functions Nat -> Nat.
-- It should however be noted that these maps come with a domain and codomain
-- (d i : [n] -> [n+1], s i : [n+1]\to [n])
-- We do not include these as it makes the code easier to write.
--
```

```haskell
-- It might help to do sanity-checks in which we do keep track of the domain and codomain.
-- But for the purposes of the thesis I only want to do quick calculations.

-- definition of face maps
d :: Nat -> Nat -> Nat
d  i  x | x < i     = x
        | otherwise = x + 1

-- definition of degeneracy maps
s :: Nat -> Nat -> Nat
s  i  x | x <= i     = x
        | otherwise = x - 1

--how to show maps [n] -> [m], and check for equality between those maps, given a domain [n].
shown :: Nat -> (Nat -> Nat) -> String
shown n f = show( map (\x ->(x, f x)) [0..n])

check :: Nat -> (Nat -> Nat) -> (Nat -> Nat) -> Bool
check n f g = (shown n f) ==  (shown n g)

-- We introduce abstract notions of degeneracy and face maps, which are not functions.
-- We call these Generators as they generate all maps in the simplex category.
data Generators = D Nat | S Nat deriving (Eq, Show)

-- We then introduce a way to go from a list of generators to their function composition.
tomap :: Generators -> (Nat -> Nat)
tomap (D n) = d n
tomap (S n) = s n

concatter :: [a -> a] -> a -> a
concatter []     = id
concatter (x:xs) = x . (concatter xs)

lstomap :: [Generators] -> (Nat -> Nat)
lstomap gs = concatter(map tomap gs)

-- Now it can be shown that every map in \Delta (every increasing map [n] \to [m])
-- can be written as, d_i \circ \cdots \circ d_j \circ s_k \circ \cdots \circ s_l
-- If we furthermore require that the s_i are taken in decreasing order,
-- and the d_i in increasing order,
-- this form is unique. We use this normal form to help us study simplicial identities.

helpnormalformer :: Nat -> Nat ->  [Generators] ->  Nat -> (Nat -> Nat) -> [Generators]
helpnormalformer input expectedoutput gs n f
-- We assume that f is an increasing function and that
-- on [0.. input -1], the function f equals the composition of the generators gs
-- this function takes that assumption and add generators untill this also holds on [0..input]
-- untill we reach input > n.
--
```

```
-- First we check if we have reached n already
  | input > n = gs
-- Now we keep track of an expectedoutput, which will be the output the generators will give.
-- If this is indeed what f also gives, we continue,
  | f input == expectedoutput = helpnormalformer (input + 1 ) (expectedoutput + 1) gs n f
-- If f gives a bigger output, it skips over the epected output.
-- skipping over a number corresponds to D
  | f input >  expectedoutput = helpnormalformer input (expectedoutput + 1)
                                                ((D expectedoutput):gs) n f
-- If f gives a lower output, it can only be equal to expectedoutput -1
-- Because by assumption f (input - 1) = expectedoutput -1 and f is increasing
-- If this is the case, we double on expectedoutput-1,
-- doubling over a number corresponds to S
  | f input == expectedoutput - 1 = helpnormalformer (input+1) (expectedoutput)
                                        (gs ++ [S (input - 1) ]) n f
-- If this does not happen, we do not have an increasing function.
  | otherwise = error "I don't think this function is increasing."

normalformer :: Nat -> (Nat -> Nat) -> [Generators]
normalformer = helpnormalformer 0 0 []

checknf n f = check n f ( lstomap (normalformer n f))

helpnormalformgenerators :: Nat -> [Generators] -> [Generators]
helpnormalformgenerators n gs = normalformer n (lstomap gs)

-- After a while, any finite composition f of generators
-- should satisfy (f (k+1) == f k + 1)  for all k\geq N
-- Where is this N? Well it should be at least the maximum of the indices we encounter,
-- and all other maps could make it one bigger, so maximum + length is a good estimate,
-- but we add 2 to be sure.

normalformgenerators :: [Generators] -> [Generators]
normalformgenerators gs = helpnormalformgenerators maxn gs where
                            dismantle :: Generators -> Nat
                            dismantle (D n) = n
                            dismantle (S n) = n
                            ns = map dismantle gs
                            maxn | ns /= [] = (length gs) + (maximum ns) +  2
                                 | ns == [] = 2
```

B.2 Simplicial free groups

```
module SimpFreeGroup where

{- This code is supposed to model the behaviour of a simplicial group
 - We model the simplicial group as follows:
    - We start of with some basic elements at each level. (In this case strings)
    - We can apply group morphisms to these basic elements,
      these are the degeneracy and face maps (called generators)
    - We then allow for multiplication and taking inverse
-}

import SimpId

data Element a = N a | I a deriving (Eq, Show)
-- An atomof a free group is a generator in either its Inverted form or Non-inverted form
-- (N used to stand for normal, but that is also overloaded in these pieces of code)

instance Functor Element where
  fmap f ( N x ) = N (f x)
  fmap f ( I x ) = I (f x)


-- Now a free group on such atoms has as elements list of Inverted or Non-inverted atoms
-- where we can normalize such lists by
-- letting subsequent Inverted and Non-inverted elements cancel each other out.
normalizestep :: (Eq a) => [Element a] -> [Element a]
normalizestep ( ( N x ) : (I y) : xs ) | x == y = normalizestep xs
                                   | otherwise = ( N x ) : (normalizestep ( (I y) : xs))
normalizestep ( ( I x ) : (N y) : xs ) | x == y = normalizestep xs
                                   | otherwise = ( I x ) : (normalizestep ( (N y) : xs))

normalizestep [x] = [x]
normalizestep [] = []


-- We should iterate such normilzation steps a couple of times
--length xs /2 + 1 should be enough steps because when we cancel, we remove 2 elements
normalize :: (Eq a ) => [Element a] -> [Element a]
normalize xs = iterate normalizestep xs !! ((div (length xs) 2) + 1)


-- now we add the generators to make it a simplicial group element.
-- (Note there is no level specificiation, we do not specify that x\in X_n,
-- we can get away with this for the limited application we have)
type SimpGroupElt = [Element (String , [Generators])]

-- and we give the generators normal form.
normalgeneratorsinGroupElt :: SimpGroupElt -> SimpGroupElt
```

```
normalgeneratorsinGroupElt =  map (fmap (\(x,gs) -> (x,normalformgenerators gs)))

-- and we combine this with group normalization as seen.
normalformGroupElt :: SimpGroupElt -> SimpGroupElt
normalformGroupElt = normalize . normalgeneratorsinGroupElt

-- This is how we compose with more generators.
compose :: SimpGroupElt -> [Generators] -> SimpGroupElt
compose els gs = map (fmap (\(x,hs) -> (x,hs ++ gs ))) els

multiply :: SimpGroupElt -> SimpGroupElt -> SimpGroupElt
multiply = (++)
-- At this moment, this allows us to multiply two elements from different groups
-- because we have no level specified.
-- For example f and f\circ d1 can be multiplied in the code,
-- but these are not elements of the same group.
-- Again we can get away with it for the limited application.

-- also we want to have inverses
inverseElement :: Element a -> Element a
inverseElement (N a) = I a
inverseElement (I a) = N a

inverse :: SimpGroupElt -> SimpGroupElt
inverse = reverse . (map inverseElement)

-- and an unit
unit :: SimpGroupElt
unit = []

-- We now have composition with generators (so application of G on any function in \Delta)
-- multiplication, inverses and unit
-- Hence we have all we want out of a simplicial group
```

B.3 Filler algorithm

```
module FillerAlgo where

import SimpId
import SimpFreeGroup

{-
 - This code is supposed to model the filler algorithm for simplicial groups
 - We assume we are given a map y : \Lambda^n_i \to G where
 -   G is a simplicial group
 -   \Lambda^n_i is a horn, which is an n-simplex with one face and the inner part removed
 -}
```

```
--   We represent such horns y by a Nat i to indicate which face is missing and then a
--   list of n-1 group elements called
--   $ y \circ d_0 , \cdots, \hat{ y \circ d_i} \cdots y\circ d_n$.
--   (the hat means this index is omitted)

type Horn = (Nat, [SimpGroupElt])

toHorn :: String -> Int -> Int -> Horn
-- toHorn y n i is how we represent the sieve on an n-simplex
-- with the i'th face missing, and name y.
toHorn y n i = (i , ys) where
                 ys = [ [N (y, [D k])] |k <- ([0..i-1] ++ [i+1..n])]

normalHorn :: Horn -> Horn
normalHorn (i,ys) = (i, map normalformGroupElt ys)

-- We can multiply such a horn with a group element. This multiplication goes facewise.
multiplyHorn :: Horn -> SimpGroupElt -> Horn
multiplyHorn (i,ys) x = (i, zs) where
                          xs = [compose x [D k] | k <-([0..i-1] ++ [i+1..length(ys)+1]) ]
                          zs= [multiply y x | (y,x)<- zip ys xs]

-- This algorithm works by induction in an unusual order
-- first we start at the top and go down until we reach the missing face (called the i'th face)
-- then we continue at 0 and go up

suc :: Nat -> Nat -> Nat
-- S :([n] -\{i\}) -> [n]
suc i k | k >  i + 1 = k-1
      | k == i + 1 = 0
      | k <i = k+1

-- Furthermore because we make a distinction to what happens before i and after i
-- we want to use the ' notation which in Haskell we call prime

prime :: Nat -> Nat -> Nat
prime i k | k < i = k
          | k > i = k - 1
          | otherwise = error "i' should not be called anywhere, but it is called here"

-- The k'th face of a horn can now be called as follows:
takeFace :: Horn -> Nat -> SimpGroupElt
takeFace (i, ys) k = ys !! prime i k
-- Note that in our abuse of notation for a horn we write
-- y \circ d_k \equiv ("y", [D k]) == takeFace y k here.

takeGap :: Horn -> Nat
takeGap (i,_) = i
```

```
takelength:: Horn -> Nat
takelength (_,ys) = length ys

-- The algorithm consists of repeating one step along the successor function defined above.
-- It uses a helper map tau, which gives a simplex of level n (which is a group element).

tau :: Horn -> Nat -> SimpGroupElt
tau (i, ys) k = compose ( takeFace (i,ys) k ) ([ S ( prime i k)])

algoStep :: (Horn, Nat) -> (Horn, Nat)
-- This is the formula for y^{S(k}}.
algoStep (y, k) = ( multiplyHorn y (inverse $ tau y k) , suc (takeGap y) k )


sol :: (Horn, Nat) -> SimpGroupElt
sol (y, k) | k == takeGap y = unit
           | otherwise = multiply ( sol(algoStep (y,k)) ) (tau y k )

fil :: Horn -> SimpGroupElt
fil y = sol (y, suc (takeGap y) ((takelength y) + 1) )
-- We cannot just takelenght y for our starting point if our missing face is at the end.

-- below are some methods used to study the solutions

studyElt :: SimpGroupElt -> Nat -> [SimpGroupElt]
--This code prints all faces of an n-simplex
-- The nat is needed because in our method, a group element does not carry
--     in which layer of the simplicial group it is an element
--     or, equivalently, it's domain.  Maybe doing the theory from domains is also very useful.
studyElt x n = [normalformGroupElt $ compose x [ D k ] | k <- [0..n]]

studyFaces :: Horn -> [SimpGroupElt]
studyFaces (_,ys) = ys

toElement :: String -> SimpGroupElt
toElement f =  [ N (f, [])]
```

B.4 Horn squares

```
module HornSquares where
{- This module models horn squares in the sense of Berg and Faber
 -}

import Data.List

import SimpId
import SimpFreeGroup
```

```
import FillerAlgo


data PM = Plus | Min deriving (Eq, Show)

type HornSquare = (Nat, PM, SimpGroupElt , [SimpGroupElt])

-- the Hornsquare (i, Min, f , gs) is the square
--
-- let n = length gs + 1
--
-- \partial \Delta^n  ----------> s_i^*(\partial \Delta^{n+1})
--     |                                |
--     |                                |   all faces except d_i, d_i+1
--     |                                |
--    \ /                              \ /
--     .                                .
-- \Delta^n    -------------------> \Delta^{n+1}
--                        d_i
--  and a map f from the bottom left corner to G
--  and a map g from the top right corner to G
--  where g\circ d_k | k < i = gs !! k
--                   | k > i+2 = gs !! k-2
--
-- and the Hornsquare (i,Plus, f, gs) changes the bottom arrow from d_i to d_{i+1}.
--
--
--We want a method to take a face of such a horn square
takesqFace :: HornSquare -> Nat -> SimpGroupElt
takesqFace (i, Min, f, gs) k | k < i = gs !! k
                             | k == i = f
                             | k == i+ 1 = error "You are trying to take a misssing face"
                             | k > i+1 = gs !! (k-2)

takesqFace (i, Plus, f, gs) k | k < i = gs !! k
                              | k == i+1 = f
                              | k == i = error "You are trying to take a misssing face"
                              | k > i+1 = gs !! (k-2)

--Some terminology on which face is the added face and which is the missing face
missingFace :: Nat -> PM -> Nat
missingFace i Min  = i+1
missingFace i Plus = i

addedFace :: Nat -> PM -> Nat
addedFace i Min  = i
addedFace i Plus = i + 1

--How to deal with these horn squares in relation to horns.
```

```
squareToHorn :: HornSquare -> Horn
squareToHorn (i,pm,f,gs) =  (hole,simpElements) where
                  hole = missingFace i pm
                  newface = addedFace i pm
                  (gs1, gs2) = splitAt i gs
                  simpElements = gs1 ++ [f] ++ gs2

hornToSquare :: Horn -> PM -> HornSquare
hornToSquare (hole, hs) pm = (i,pm,f,gs) where
    i | pm == Min = hole - 1
      | pm == Plus = hole
    (gs1,f:gs2) = splitAt i hs
    gs = gs1 ++ gs2

toHornSquare :: String -> PM -> Int -> Int -> HornSquare
toHornSquare y pm n i = (i, pm, f, gs) where
  f = [ N (y, [D (addedFace i pm) ]) ]
  gs = [ [N (y, [D k])] |k <- ([0..i-1] ++ [i+2..n+1])]
  -- the horn square should end at n+1

--What does the pullback of these horn squares along s_j look like:
pullBackHorn :: String -> PM -> Nat ->  Nat -> Nat -> Nat -> Nat -> HornSquare
pullBackHorn ys pm n i j i' j' = (i', pm, f, zs) where

-- this gives the correct pulled back horns,
  y = squareToHorn $ toHornSquare ys pm n i
  fily = fil y
  z :: Nat -> SimpGroupElt
  z k | (k == i' || k == i'+1 ) = error "z should not be called on i', i'+1"
      | (k == j' || k == j'+1  ) = fily
      | otherwise =  [ N (ys, [S j' , D k])]
  zs = [ z k | k <- [0.. i'-1] ++ [i'+2 .. n+2]]
  -- we use n+2 as the literature has n+1 horns,
  -- so the pullbacks are n+2 horns
  f | (pm == Plus && j == i && i' == i && j'==i+1) = fily
    | (pm == Min  && j == i && i' == i+1 && j'==i) = fily
    | otherwise = [N(ys, [ D (addedFace i pm), S j])]
  --the case distinction says that the added face might have fil y as value as well.
```

B.5 Generating examples

```
module GenerateExamples where

{- This module was used to check whether with our definitions
 - for effective Kan complexes and simplicial groups work out such that
 - simplicial groups are effective Kan complexes.
 - If it does, wrongcases should always return an empty list.
 -
```

```
 - This code was used to play around with examples of freely generated groups.
 - It was used to find the correct formulation of the pulled back horn in Hornsquares.hs
 - That formulation is now correct, and hence this code does not give new information.
-}

import SimpId
import SimpFreeGroup
import FillerAlgo
import HornSquares

-- the valid combinations for i*,j* given i,j
validcombination :: Nat ->  Nat  -> [(Nat, Nat)]
validcombination i j | j < i = [(i+1,j)]
                     | j > i = [(i,j+1)]
                     | j == i = [(i+1,i),(i,i+1)]
                     | otherwise = error "no ijcomparison"

-- the set of i,j,i*,j* combinations given some n
mathcalA :: Nat -> [(Nat,Nat,Nat,Nat)]
mathcalA n = [ (i,j,i+1,j)| i<-[0..n],j<-[0..i] ] ++
             [ (i,j,i,j+1)| i<-[0..n],j<-[i..n] ] ++
             [ (i,i,i,i+1)| i<-[0..n]] ++
             [ (i,i,i+1,i)| i<-[0..n]]

-- we can check that for a certain horn square, we have that the
-- effective condition holds:
effectiveCheck :: PM -> Nat -> Nat -> Nat -> Nat -> Nat -> Bool
effectiveCheck pm n i j i' j'=
    normalformGroupElt solz == normalformGroupElt expectedsolz where
          y = toHornSquare "y" pm n i
          z = pullBackHorn "y" pm n i j i' j'
          solz = fil (squareToHorn z )
          expectedsolz = compose (fil (squareToHorn (toHornSquare "y" pm n i))) [S j']

-- now we generate given n all possible horn squares and apply the check above
lotsofcases :: Nat -> [(Bool, PM, Nat, Nat, Nat , Nat )]
lotsofcases n = [(effectiveCheck pm n i j i' j',pm,i,j, i', j') |
                    pm <- [Min,Plus], (i,j,i',j') <- mathcalA n]

-- now we filter when this goes wrong
wrongcases :: Nat -> [(Bool, PM, Nat, Nat, Nat ,Nat)]
wrongcases n = filter (not.first) (lotsofcases n) where first (b, _,_,_, _ , _) = b
```

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Glossary

($\mathbb{L}$,$\mathbb{R}$)-lifting operation An operation ϕ assigning lifts between maps in $\mathbb{L}$ and $\mathbb{R}$ satisfying some compatibility conditions. 40

Δ See simplex category. 6

Δ^n See simplex. 9

∞-category Simplicial set with the RLP against inner horn inclusions. 10

$\mathbb{S}\mathbf{q}(\mathcal{C})$ The double category of squares in some small category $\mathcal{C}$. 38

$\mathcal{S}_\bullet$ See singular simplicial set. 21

$\partial\Delta^n$ The boundary of a simplex. 55

d_i See face map. 6

$sSet$ The category of simplicial sets. 8

s_i See degeneracy map. 6

$|\Delta^n|$ The convex hull of the unit vectors of $\mathbb{R}^{n+1}$. 20

decidable sieve A sieve with decidable inclusions of simplices. 43

degeneracy map Maps in Δ hitting one element twice, denoted s_i. 6

degeneracy-section A level-wise inverse of a simplicial morphism that respects degeneracy maps. 28

degenerate A simplex that factors through a degeneracy map is called degenerate. 25

degenerate-preferring Kan complex A structure of lifts against horn inclusions on a simplicial set, which always picks a degenerate solution if it exists. 27

double category An internal category in the category of categories. 37

double functor A morphism of double categories. 39

effective Kan complex A simplicial set with a lifting structure against horn squares that is stable along pullback along degeneracy maps. . 56

Eilenberg-Zilber lemma The notion that every simplex can be written uniquely as a degeneracy of a non-degenerate simplex. 29

encountered We say that k is encountered after l if we define w_k after w_l in construction 6.8 . 32

face map Maps in Δ skipping over one element, denoted d_i. 6

factors through For f, g maps, we say that f factors through g if there is a map h such that $f = g \circ h$. 9

horn Union of all faces of a simplex except for one face, denoted Λ^n_m. 10

horn map A map from a horn to a simplicial set. 10

horn pushout sequence A finite sequence of sieve inclusions which correspond to pushouts of horn inclusions. . 43

horn square A commutative square for which the inner pushout corresponds to a horn. 55

HoTT Homotopy Type Theory. 53

inner horn A horn Λ^n_m with $0 < m < n$. 10

Kan complex Simplicial set with the RLP against horn inclusions. 10

Kan fibration Map of simplicial sets with the RLP against horn inclusions. 11

lifting awfs A lifting algebraic weak factorisation system is a lifting structure satisfying the axiom of lifting and the axiom of factorisation. 42

lifting structure A tuple $(\mathbb{L}, \phi, \mathbb{R})$ with ϕ a $(\mathbb{L}, \mathbb{R})$-lifting operation.. 41

LLP Left Lifting Property. 10, 42

Malcev algebra An algebra with a Malcev operation. 30

Malcev operation An operation μ satisfying $\mu(x, x, y) = \mu(y, x, x) = x$. 30

Malcev theory An algebraic theory with a Malcev operation. 30

Moore path A path with a specified path length $r \in \mathbb{R}_{\geq 0}$. 23

nerve The nerve of a category consists of sequences of composable morphisms. 17

RLP Right Lifting Property. 10, 41

sieve Subobject of a representable presheaf. 10

simplex Representable simplicial set, denoted Δ^n. 9

simplex category The category of finite non-empty linear orders, denoted Δ. 6

simplicial identities The way in which face and degeneracy maps interact. 7

simplicial set Presheaf on Δ. 8

singular simplicial set A simplicial set corresponding to a topological space, denoted $\mathcal{S}_\bullet$. . 21

small simplices Those simplices of Δ^a corresponding to a mono $[n] \hookrightarrow [a]$. 44

symmetric effective ∞-category A simplicial sets having chosen lifts against inner horn inclusions satisfying the same conditions as a symmetric effective Kan complex . 16

symmetric effective Kan complex A simplicial set with a structure of lifts against horn inclusions which is stable under pullback along degeneracy maps. 16

symmetric effective Kan fibration A map of simplicial sets with a structure of lifts against horn inclusions which is stable under pullback along degeneracy maps. 16