UTRECHT UNIVERSITY

MASTER THESIS

FACULTY OF SCIENCE

Graphical realizations of degree sequences, packing multiple colors and random graphs

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26th January, 2023

Abstract

Given a graph, one obtains its degree sequence by placing the degrees of the vertices in an ordered list. We are interested in the reverse problem: How do you go from a list of arbitrary natural numbers to a graphical realization with that list as its degree sequence? In 1960, Paul Erdős and Tibor Gallai solved the existence problem. By verifying a list of inequalities, one can determine whether a graphical realization exists. We continue the research on this question by first determining which lists have a unique graphical realization. We then investigate the problem of 'packing' multiple lists, that is verifying if there are edge-disjoint graphical realizations for each list. Although this problem is NP hard in general, we give some sufficient conditions. We also show that this problem can be solved in polynomial time for a specific case, when one packs a list of degrees with a given fixed graph. Finally, we investigate the problem of randomly generating graphical realizations. In 2010, a paper published by Mohsen Bayati, Jeong Han Kim and Amin Saberi gave a sequential algorithm. If the degrees are asymptotically not too large, then choosing the right bias on the edges results in an asymptotically uniform distribution of graphical realizations. This same approach has been applied to directed graphs and coloured graphs. We generalise these related results by bringing them under one framework for generating random maximum independent sets in a certain meta-graph.

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Introduction

Putting the degrees of the vertices of a graph $G = (V, E)$ with $V = [n] = \{1, \ldots, n\}$ in a list $\pi = (d_1, \ldots, d_n)$, we get the degree sequence of the graph. We will focus on the reverse, looking for graphical realizations of given degree sequences. We assume graphs to be simple, so we do not allow self-loops or double edges, making the existence problem non-trivial. In 1960, Paul Erdős and Tibor Gallai [29] stated precisely which n-tuples are graphical, that is, have a graphical realization. An n-tuple $\pi = (d_1, \ldots, d_n)$ of integers $d_1 \geq \ldots \geq d_n \geq 0$ is graphical if, and only if, $d_1 + \ldots + d_n$ is even, and for all $1 \leq k \leq n$ we have

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}.$$

This solves the existence problem with a simple parity check and a list of n inequalities.

Graphical degree sequences typically have many different graphical realizations. This can make it difficult to analyze the set of all graphical realizations. On the other end of the spectrum, there are some degree sequences with only one graphical realization. In [18] it is proven that these sequences are exactly the vertices of the convex polytope formed by the inequalities of the Erdős-Gallai Theorem, up to permutations, and it is concluded that there are 2^{n-1} such sequences of length n. In [22] more classifications are given for a graphical realization to be unique. An unlabeled variant is also considered, counting isomorphic graphs as the same. Many other publications such as [12, 17, 20, 16, 15, 14] study related questions. For example, in [19] bipartite graphs are considered. In Section 1 we give a recursive constructive characterization of degree sequences with a unique labeled graphical realization. This characterization gives an easy way to list all 2^{n-1} such sequences of length n, up to permutations. We also use this characterization to count all ordered such sequences, where permutations do matter. We also give a linear time verification algorithm for this characterization.

The problem of finding graphical realizations can be extended to graphs with colored edges. We say that multiple n-tuples $\pi_1, \ldots, \pi_k$ pack if they admit edge-disjoint graphical realizations. The different n-tuples can be thought to represent the different colors of the edges. Note in particular that the sum of any subset of the n-tuples needs to be graphical. However, this condition is not sufficient. Unlike with single colored graphs, there is no easy way to check whether n-tuples pack. In fact, in [6] it is proven that the problem is NP-hard even with just two colors. However, one can still look for sufficient conditions under which packing is guaranteed. For example, in [21] and [24] it is proven that a graphical n-tuple π_1 packs with $\pi_2 = (k, \ldots, k)$ exactly if $\pi_1 + \pi_2$ is graphical. In [4] a more general sufficient condition is given. If π_1, π_2 are graphical and Δ, δ are the maximum and minimum of $\pi_1 + \pi_2$ respectively, then π_1 and π_2 pack if $\Delta \leq \sqrt{2\delta n} - (\delta - 1)$, except that strict inequality is required when $\delta = 1$. In Section 2 we generalize this result to more than two colors. We also consider the scenario when certain specified edges are not allowed to be used.

Instead of specifying edges that are not allowed to be used, we can equivalently specify the edges that are allowed to be used. These edges form a graph, and any subgraph that is a graphical realization of a degree sequence f is called an f-factor. For instance, 1-factors correspond to perfect matchings. Tutte's factor theorems from [34, 35, 33] give characterizations for graphs with perfect matchings or f-factors. Other publications such as [8, 30, 27, 28] have also studied a directed graph analogue of this problem. In [11] a relaxation of f-factors is considered, which they call a partial f-factor. Just like how a general matching matches some but not necessarily all vertices, with a partial f-factor the degrees are only bounded by f and not necessarily equal. In Section 3 we show that a maximum partial f-factor can be constructed with a polynomial time algorithm. More specifically, we construct a related graph of which any maximum matching corresponds to a maximum f-factor of the original graph.

When a sequence has many graphical realizations, one might be interested in randomly sampling among those realizations. A straight forward approach would be to randomly match all half-edges and to apply rejection sampling if any self-loops or double edges are formed. However, in [3] this was shown to take exponential time with respect to the square of the average degree. However, in [25, 10, 9] a polynomial time exact uniform sampling algorithm is given. Instead of rejecting graphs with self-loops or double edges, they are corrected by swapping some of the edges. In [32, 2, 1] a sequential approach is used instead, matching half-edges one by one, making sure not to create any self-loops or double edges. With proper biases on the possible matches, this runs in practically linear time, however the sampling is only asymptotically exactly uniform, so this works best on very large graphs. In [13, 23] the sequential approach is applied to directed graphs and two-colored graphs instead. However, the proof does not generalize very easily, so in both cases, all calculations in the proof have to be done from scratch.

In Sections 4, 5, 6 and 7 we describe a way to generalize the concept of matching half-edges in such a way that it encapsulates all known cases. The main idea is to consider the line graph of the graph of all possible matches of half-edges. The line graph of a graph turns the edges into vertices and connects them if they share an endpoint. This concept turns matchings into independent sets, such that we need to sequentially sample a maximum independent set instead. In Section 4 we discuss the necessary regularity properties of a graph for this sequential approach to give a uniformly random maximum independent set. In Section 5 we discuss the performance of the sequential algorithm on graphs with the necessary regularity properties. In Section 6 we show how the algorithm can be applied to generate random graphs of different types. Finally, in Section 7 we give heuristics and we describe a framework for the proof of correctness.

1 Characterization of all uniquely graphical sequences

We determine which n-tuples are uniquely graphical, that is, admit exactly one graphical realization. Trivial examples are $(0,\dots,0)$ and $(n-1,\dots,n-1)$, only attaining the edgeless and complete graph respectively. Note that there are $2^{n(n-1)/2}$ labeled graphs on n vertices. Furthermore, since vertices have at most $n-1$ neighbors, there are at most n^n graphical n-tuples. It follows that a uniformly random graphical n-tuple has an average of at least $n^{-n}2^{n(n-1)/2} = 2^{n^2-O(n\log n)}$ graphical realizations. So uniquely graphical n-tuples are very unusual on this aspect.

Another example of a uniquely graphical n-tuple is the 4-tuple $(3,2,2,1)$. The following is the only graphical realization.

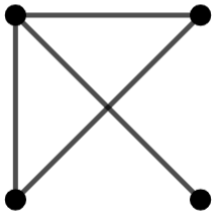

Figure 1.1: The only graph with degree sequence $(3,2,2,1)$.

All examples thus far have something in common. One of the entries is either 0 or $n-1$, forcing it to be connected to either nothing or everything respectively. For all other vertices it is thus already determined whether it is connected to the particular vertex. In order to find a graphical realization, we therefore might as well remove the vertex from the n-tuple and edit the remaining degrees accordingly. Inspired by this idea, we study the following definition.

Definition 1.1 (Dull n-tuples)**.** For an n-tuple $\pi = (d_1,...,d_n)$ with $n \geq 1$, we define the *null-addition* $(n+1)$-tuple $\pi^- = (d_1,...,d_n,0)$ and the *full-addition* $(n+1)$-tuple $\pi^+ = (d_1+1,...,d_n+1,n)$. We call π *dull* if it is a permutation of the null-addition or full-addition of some $(n-1)$-tuple.

Note that there is a symmetry between null-additions and full-additions. For any n-tuple $\pi = (d_1,\dots,d_n)$, there is a conjugate n-tuple $\pi^C = (n-1-d_1,\dots,n-1-d_n)$. Taking the complement of edges gives a bijection from graphical realizations of π to graphical realizations of π^C. So, π and π^C have the same number of graphical realizations. We find the symmetry $(\pi^-)^C = (\pi^C)^+$. So an n-tuple π is dull if, and only if, either π or π^C has a 0-entry, or equivalently, if π has either a 0-entry or an $(n-1)$-entry.

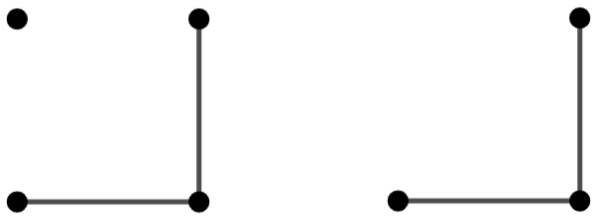

Figure 1.2: The complement of $(3,2,2,1)$ gives $(0,1,1,2)$, which is still uniquely graphical, as is $(1,1,2)$.

Since a null-addition only adds an isolated vertex to graphical realizations, we find that π, π^- and π^+ all have the same number of graphical realizations. It follows that we can turn any uniquely graphical n-tuple into a uniquely graphical $(n+1)$-tuple by taking either a null-addition or a full-addition. A natural question to ask is whether all uniquely graphical n-tuples can be constructed this way. This brings us to the following proposition.

Proposition 1.2 (Uniquely graphical n-tuples)**.** *Every uniquely graphical n-tuple is dull.*

This proposition lets us classify exactly which n-tuples are uniquely graphical, which is the main theorem of this section.

Theorem 1.3 (Classification of uniquely graphical n-tuples)**.** *Starting with the only uniquely graphical* 1*-tuple* $\pi = (0)$*, and iteratively taking null-additions and full-additions, we find every uniquely graphical* n*-tuple exactly once, up to permutations. For all* $n \geq 1$*, there are thus exactly* 2^{n-1} *uniquely graphical* n*-tuples, up to permutations.*

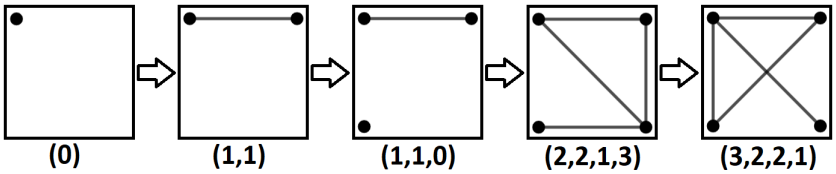

Figure 1.3: Constructing $(3, 2, 2, 1)$ with the procedure from Theorem 1.3.

Proof of Theorem 1.3. Since null-additions and full-additions do not change the number of graphical realizations, we indeed only find uniquely graphical n-tuples. So, what is left to show is that every uniquely graphical n-tuple is found exactly once. By induction, it suffices to show every uniquely graphical n-tuple with $n \geq 2$ is a permutation of the null-addition or full-addition of a uniquely graphical $(n-1)$-tuple in exactly one way, up to permutations.

Note that Proposition 1.2 provides the existence, so we only need to show uniqueness. For this, first note that null-additions, and thus also full-additions, are injective. Furthermore, null-additions always include a 0-entry, while full-additions never do. It follows that, even up to permutations, null-additions and full-additions are disjoint. Uniqueness follows. □

To prove Proposition 1.2, we take an arbitrary graphical realization of an n-tuple which is not dull, and we find a cyclic list of vertices where consecutive vertices alternate between having an edge between them or not. By shrinking this list if necessary, we find that we can switch which consecutive vertices have edges between them and which do not. Since this does not affect the overall degrees of the vertices, this will result in a different graphical realization of the same n-tuple.

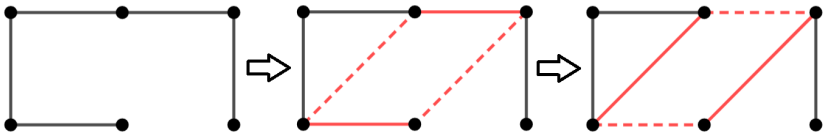

Figure 1.4: This graphical realization of $(2, 2, 2, 2, 1, 1)$ has an alternating cyclic walk. Swapping the edges gives a different graphical realization.

In order to make this argument precise, we use the following definitions.

Definition 1.4 (Walks)**.** Let G be a graph. A *walk* on the vertices of G is a list of vertices $(v_1, ..., v_k)$, where no two consecutive vertices are equal. In a *cyclic walk*, the last and first vertices are also considered consecutive in that order, and thus not equal. We say that a walk is *alternating* if consecutive pairs of vertices alternate between having an edge in G and having an edge in the complement G^C. We say that a walk is *simple* if all unordered pairs of consecutive vertices are distinct.

Proof of Proposition 1.2. We prove the contrapositive. Let G be a graphical realization of an n-tuple π which is not dull. The goal is to show that there is another graphical realization of π.

We start by showing that the graph G has an alternating cyclic walk. Since π is not dull, we find that any vertex has neighbours in both G and in the complement G^C. Therefore, we can start at any vertex, and we can indefinitely alternate walking over edges in G and over edges in G^C. Eventually there will be an edge over which we have walked twice in the same direction. At that point, we find an alternating cyclic walk.

Let $Z = (v_1, ..., v_k)$ be an alternating cyclic walk of minimal length. The goal will be to show that it follows that Z is simple. If we know Z is simple, then we can switch which consecutive vertices have an edge in G and which have an edge in G^C. Because Z is simple, we know that this modification does not change the degree of any vertex in G. The resulting graph is thus a different graphical realization of π.

Assume for the sake of contradiction that one edge appears twice in Z. Assume without loss of generality that the edge $\{v_1, v_2\}$ appears again as $\{v_i, v_{i+1}\}$, and that this edge is in G. By the alternating nature, we find that there must be an even and non-zero number of vertices, so at least two, on the walk from v_2 to v_i, and the same must hold on the walk from v_{i+1} back to v_1.

If $v_1 = v_i$ and $v_2 = v_{i+1}$, then $(v_1, ..., v_{i-1})$ is a shorter alternating cyclic walk, which is a contradiction. We conclude that $v_1 = v_{i+1}$ and $v_2 = v_i$, so the second time the edge $\{v_1, v_2\}$ is traversed in the opposite direction. Since there are at least two vertices on the walk from v_2 to v_i and on the walk from v_{i+1} back to v_1, we find that $2 < 3 < i$, and we find some vertex $v_j \neq v_3$ with $i + 1 < j \leq k$.

We split Z up in four major parts. Namely, walk W_1 from v_2 to v_3, walk W_2 from v_3 to v_i, walk W_3 from v_{i+1} to v_j, and walk W_4 from v_j back to v_1. See Figure 1.5 for an example of such a splitting. Let W^r denote the reverse of any walk W. Note that W_1, W_2^r, W_3 and W_4^r all start with an edge in G^C. Furthermore, exactly one of W_1 and W_2^r ends with an edge in G, while the other ends with an edge in G^C, and the same holds for W_3 and W_4^r.

Let W_A denote whichever walk of W_1 and W_2^r ends with an edge that would alternate with $\{v_3, v_j\}$, and let W_B denote whichever walk of W_3 and W_4^r would do the same. Consider the cyclic walk starting with $\{v_1, v_2\}$, then going over W_A, then going over $\{v_3, v_j\}$, and finally going over W_B^r. This walk alternates between edges in G and edges in G^C. It is shorter than Z, because we removed two non-empty sub-walks while we only added one edge. This is, again, a contradiction, so we conclude Z is simple.

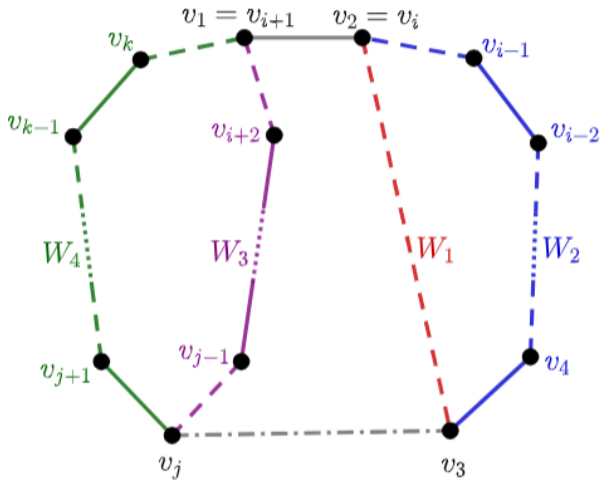

Figure 1.5: Example walk. If $\{v_3, v_j\} \in E(G)$, then $W_A = W_1$ and $W_B = W_3$. Otherwise, $W_A = W_2^r$ and $W_B = W_4^r$.

□

While the number of uniquely graphical n-tuples up to permutations has a very simple closed formula, the same is not true if we do count permutations as different n-tuples. However, it turns out that this number sequence can still be expressed with an exponential generating function.

Proposition 1.5 (Exponential generating function)**.** *Let a_n denote the number of ordered n-tuples which are uniquely graphical. Consider the convention $a_0 = a_1 = 1$ on the initial values. Then we have the following recurrence for $n \geq 2$:*

$$\begin{aligned} a_n &= 2\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}a_{n-k} \\ &= 2\left[na_{n-1} - \binom{n}{2}a_{n-2} + \binom{n}{3}a_{n-3} - \dots\right] \end{aligned}$$

It follows that the sequence (a_n) is A005840 on the OEIS, which is given by the following exponential generating function:

$$f(x) := \sum_{n=0}^{\infty}\frac{a_n}{n!}x^n = \frac{(1-x)e^x}{2-e^x}$$

Proof of Proposition 1.5. The recurrence formula is derived from Theorem 1.3. The factor 2 in the recurrence comes from the symmetry between null-additions and full-additions. So, without loss of generality, we only need to count the number of uniquely graphical ordered n-tuples that are a permutation of the null-addition of some uniquely graphical $(n-1)$-tuple. The formula follows from the principle of inclusion-exclusion on the set of 0-entries. For any set of $k \geq 1$ indices, exactly a_{n-k} sufficient n-tuples have zeros on all these indices, and there are exactly $\binom{n}{k}$ such sets.

With the recurrence relation established, we can determine the exponential generating function. We have the following exponential series:

$$\begin{aligned} 2e^{-x} - 1 &= 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n \\ &= 1 - 2x + \frac{2}{2!}x^2 - \frac{2}{3!}x^3 + \frac{2}{4!}x^4 - \dots \end{aligned}$$

By the initial values and the recurrence relation, we get the following Cauchy product:

$$\begin{aligned} f(x) \cdot \left(2e^{-x} - 1\right) &= \sum_{n=0}^{\infty} \left[a_n + 2\sum_{k=1}^{n} (-1)^k \binom{n}{k} a_{n-k} \right] \frac{x^n}{n!} \\ &= 1 - x \end{aligned}$$

Solving for the exponential generating function $f(x)$ gives

$$\begin{aligned} f(x) &= \frac{1-x}{2e^{-x} - 1} \\ &= \frac{(1-x)e^x}{2 - e^x}. \end{aligned}$$

□

While we have a classification of uniquely graphical n-tuples, it is not directly obvious how this translates into a decision problem algorithm. Theorem 1.3 seems to propose recursively trying to write the given n-tuple as a permutation of the null-addition or full-addition of an $(n-1)$-tuple. However, a naive implementation of this requires quadratic time, while linear time is possible.

Proposition 1.6 (Uniquely graphical algorithm)**.** *There is an algorithm that determines whether a given n-tuple is uniquely graphical in time linear with respect to n. In case the given n-tuple is uniquely graphical, the algorithm additionally constructs a sequence of null-additions and full-additions that, when applied to (0), result in a permutation of the given n-tuple.*

Proof of Proposition 1.6. First of all, for the given n-tuple, π to be graphical, all entries need to be in the range from 0 up and till $n-1$. This can be checked in linear time, after which we can use counting sort to sort π from large to small in linear time. With π sorted, write $\pi = (d_0, \dots, d_{n-1})$.

The algorithm will keep track of a contiguous subtuple of π, and a stack of null-additions and full-additions. We initialize the boundaries $i = 0$ and $j = n-1$ of the subtuple, and an empty stack S. We define $\pi(i,j) := (d_i - i, \dots, d_j - i)$. The algorithm will attempt to keep shrinking the subtuple while preserving the following invariant. From top to bottom, applying the stack S of null-additions and full-additions to $\pi(i,j)$ gives a permutation of π. Once we have $\pi(i,j) = (0)$, we know π is uniquely graphical, and we can terminate.

In order to shrink the subtuple, we use the fact that π, and thus also $\pi(i,j)$, is sorted. If $\pi(i,j)$ ends with a 0, we can simply decrease j by one and add a null-addition to stack S. Otherwise, if $\pi(i,j)$ starts with a $j-i$, we can instead increase i by one and add a full-addition to stack S. Using $|\pi(i,j)| - 1 = j - i$, it is easily verified that this procedure preserves the invariant. If neither scenario occurs, then by Proposition 1.2 we know $\pi(i,j)$ is not uniquely graphical, so by the invariant, neither is π. □

With uniquely graphical n-tuples fully classified, we finish this section with a slightly more general statement about restricting the number of graphical realizations. Namely, the following shows that any number of graphical realizations is possible.

Proposition 1.7 (Exact number of graphical realizations)**.** *For $n \geq k \geq 1$, the $(n+2)$-tuple $\pi = (\{n\}^k, \{n-1\}^{n-k}, k-1, 1)$ has exactly k graphical realizations.*

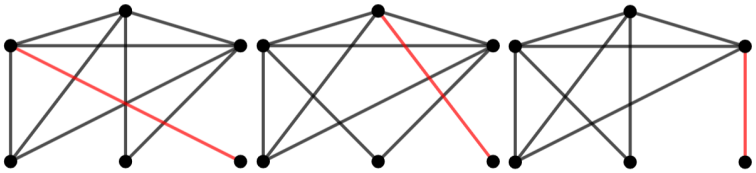

Figure 1.6: The parameters $n = 4, k = 3$ give the 6-tuple $(4, 4, 4, 3, 2, 1)$ which has exactly 3 graphical realizations.

Proof of Proposition 1.7. The first n vertices of any graphical realization G of π have $n(n-1)+k$ connections combined. Since at most k connections can be made with the remaining two vertices, at least $n(n-1)$ connections need to be between the first n vertices. This is the maximum possible number only attained if all of the first n vertices are connected with each other.

This only leaves the first k vertices with one more connection each with the last two vertices. Only one of these k vertices can get connected with the last vertex, while the remaining $k-1$ vertices will get connected with the second to last vertex. This leaves us with exactly k possibilities for G. □

2 Packing graphical sequences with small degrees

We give sufficient conditions for a number of graphical n-tuples to pack, so for there to be edge disjoint graphical realizations, based on the size of the degrees with respect to the number of vertices. Recall that the different n-tuples can be interpreted as the degree sequences of different colored edges. We also consider the possibility that there are some forbidden edges.

2.1 Packing more than two colors

We formalize the problem with the following definition.

Definition 2.1 (Packing of n-tuples)**.** We say that multiple graphical n-tuples $\pi_1, ..., \pi_k$ *pack* if there are graphical realizations $G_i = ([n], E_i)$ of each π_i respectively, such that the edge sets E_i are all disjoint. We call such graphical realizations a *packing* of $\pi_1, \dots, \pi_k$.

When the given degrees are small, one would expect there to be enough room for edges, such that you can avoid overlap between graphical realizations. The following is Theorem 2.2 from [4] which formalizes this intuition.

Theorem 2.2 (Packing two n-tuples with small degrees)**.** *Let π_1 and π_2 be graphical n-tuples. Let Δ and δ denote the maximum and minimum values in $\pi_1 + \pi_2$ respectively. If*

$$\Delta \leq \sqrt{2\delta n} - (\delta - 1),$$

then π_1 and π_2 pack, except that strict inequality is required when $\delta = 1$.

This theorem only applies to the case with two colors. We generalize the theorem to arbitrarily many colors. The proof is inspired by that from [4].

Theorem 2.3 (Packing multiple n-tuples with small degrees)**.** *Let $\pi_1, \dots, \pi_k$ be graphical n-tuples. For $i < j$, consider the indices that are non-zero in $\pi_i + \pi_j$. Let $n_{i,j}$ be the number of these indices, let $\Delta_{i,j}$ be the maximum value of $\pi_1 + \ldots + \pi_k$ at these indices, and let $\delta_{i,j}$ be the minimum value of $\pi_i + \pi_j$ at these indices. If*

$$\Delta_{i,j} < \sqrt{2\delta_{i,j} n_{i,j}} - (\delta_{i,j} - 1)$$

for all $i < j$, then $\pi_1, \dots, \pi_k$ pack.

Proof of Theorem 2.3. We prove the contrapositive. So assume that $\pi_1, \dots, \pi_k$ do not pack. Let $G_1, ..., G_k$ be graphical realizations of $\pi_1, \dots, \pi_k$ respectively, such that the number of pairs of edges from different graphs that overlap is minimized.

By the assumption, there is at least one overlapping pair of edges, say $xy \in E(G_i) \cap E(G_j)$. For convenience, we write $n' = n_{i,j}$, $\Delta = \Delta_{i,j}$ and $\delta = \delta_{i,j}$. The idea of proof is to find conditions under which we can fix the overlap without creating any new overlap. The goal will be to show that, for all such conditions to fail, we need to have $\Delta \geq \sqrt{2\delta n'} - (\delta - 1)$, thus proving the theorem.

Let G be the unified graph of $G_1, ..., G_k$, restricted to the indices that are non-zero in $\pi_i + \pi_j$, such that $|V(G)| = n'$. Let $I = V(G) - (N_G(x) \cup N_G(y))$. Here, we use the notation $N_G(x) = \{x' \in V(G) : xx' \in E(G)\}$.

Let $\ell \in \{i, j\}$, and assume for the sake of contradiction that there exists an edge $uv \in E(G_\ell)$ with $u \in I$ and $v \notin N_G(x) \cap N_G(y)$. Assume without loss of generality that $v \notin N_G(y)$. Then we can fix the overlap by replacing $\{xy, uv\}$ with $\{xu, yv\}$ in G_ℓ, which does not affect the degrees. This contradicts minimality, so we conclude that $Q_\ell := N_{G_\ell}(I) \subseteq N_G(x) \cap N_G(y)$.

Note that it follows that Q_ℓ is disjoint from I, and we know that Q_ℓ does not contain x or y. Assume for the sake of contradiction that there exist two vertices $w, w' \in Q_\ell$ with $ww' \notin E(G)$. We find $z, z' \in I$ with $wz, w'z' \in E(G_\ell)$. Then we can fix the overlap by replacing $\{xy, wz, w'z'\}$ with $\{xz, yz', ww'\}$ in G_ℓ, which does not affect the degrees. This contradicts minimality again, so we conclude that Q_ℓ is a clique.

We find that the edge complement of $Q = Q_i \cup Q_j$ is bipartite in G. For $r = |E_G(Q)|$, writing $q = |Q|$, it follows that

$$r \geq \binom{q}{2} - \left(\frac{q}{2}\right)^2 = \frac{q^2}{4} - \frac{q}{2}. \tag{1}$$

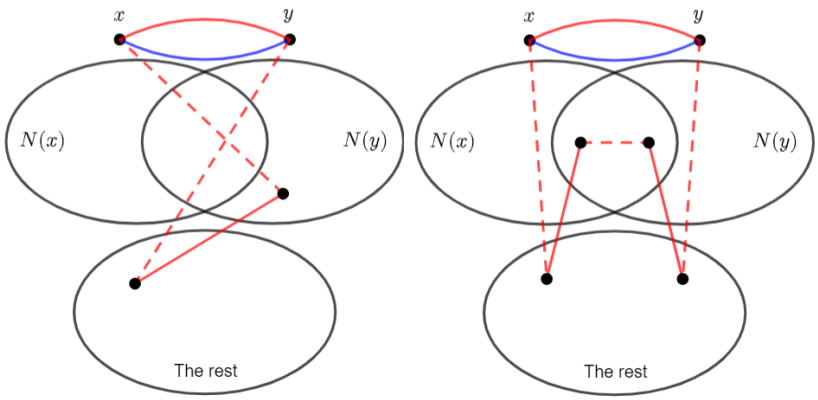

Figure 2.1: Left visualizes the argument that $Q_\ell \subseteq N_G(x) \cap N_G(y)$. Right visualizes the argument that Q_ℓ is a clique.

Note that $|I| = n' - |N_G(x) \cup N_G(y)| = n' - |N_G(x)| - |N_G(y)| + |N_G(x) \cap N_G(y)|$. Since xy is an overlapping edge, we have $|N_G(x)|, |N_G(y)| \leq \Delta - 1$. Since $Q \subseteq N_G(x) \cap N_G(y)$, we find

$$|I| \geq n' - 2(\Delta - 1) + q. \tag{2}$$

Each vertex $v \in I$ has at least δ incident edges in G_i and G_j combined, all connecting to a vertex in Q. Since every vertex in Q is connected to x and y, that leaves at most $(\Delta - 2)q - 2r$ edges with endpoints in Q and I. We find

$$\delta|I| \leq (\Delta - 2)q - 2r. \tag{3}$$

Combining (2) and (3) gives

$$\delta(n' - 2(\Delta - 1) + q) \leq (\Delta - 2)q - 2r.$$

Using (1) and simplifying gives

$$q(\Delta - \delta - 1 - q/2) \geq \delta(n' - 2\Delta + 2).$$

The left hand side is maximized when $q = \Delta - \delta - 1$, giving $(\Delta - \delta - 1)^2 \geq 2\delta(n' - 2\Delta + 2)$. Solving the quadratic inequality in Δ gives

$$\Delta \geq \sqrt{2\delta n'} - (\delta - 1).$$

□

2.2 Forbidden edges

We consider an additional restriction on packing n-tuples where we are not allowed to use certain forbidden edges. This can be interpreted as if one of the graphical n-tuples already has a given associated graph. If we can not modify this graph, then the edges in this graph can not be used for the graphical realizations of the remaining n-tuples. Again, the question is whether we can find graphical realizations of the remaining n-tuples, such that no edge appears in multiple graphs.

Definition 2.4 (Packing of a graph and n-tuples)**.** Let $G_0 = ([n], E_0)$ be a graph and let $\pi_1, \ldots, \pi_k$ be graphical n-tuples. We say that G_0 and $\pi_1, \ldots, \pi_k$ *pack* if there are graphical realizations $G_i = ([n], E_i)$ of each π_i for $i = 1, \ldots, k$ respectively, such that the edge sets E_i for $i = 0, \ldots, k$ are all disjoint. We call such graphical realizations a *packing* of G_0 and $\pi_1, \ldots, \pi_k$.

We want to know if a result similar to Theorem 2.3 also holds with this additional restriction. In the proof, we will need to take into account the additional possibility that an edge overlaps with the graph of forbidden edges. Although the forbidden edges can not be modified, we can still attempt the graph modifications in the proof of Theorem 2.3 to the other graph. We reach a similar conclusion with slightly modified inequalities.

Theorem 2.5 (Packing of a graph and n-tuples with small degrees)**.** *Let $G_0 = ([n], E_0)$ be a graph with degree sequence π_0 and let $\pi_1, \ldots, \pi_k$ be $k \geq 1$ graphical n-tuples. For integers $0 \leq i < j \leq k$, consider the indices that are non-zero in $\pi_i + \pi_j$. Let $n_{i,j}$ be the number of these indices, and let $\Delta_{i,j}$ be the maximum value of $\pi_0 + \ldots + \pi_k$ at these indices. Let $\delta_{i,j}$ be the minimum value at these indices of $\pi_i + \pi_j$ if $i > 0$, and of π_j if $i = 0$. If*

$$\Delta_{i,j} < \sqrt{2\delta_{i,j} n_{i,j}} - (\delta_{i,j} - 1) \qquad \forall 1 \leq i < j \leq k,$$
$$\Delta_{0,j} < 2\sqrt{\delta_{0,j}(n_{0,j} + 2\delta_{0,j})} - (3\delta_{0,j} - 1) \qquad \forall 1 \leq j \leq k,$$

then G_0 and $\pi_1, \ldots, \pi_k$ pack.

Proof of Theorem 2.5. Note that the proof of Theorem 2.3 still works, as long as we have $1 \leq i < j \leq k$ for the overlapping edge $xy \in E(G_i) \cap E(G_j)$. Therefore, we only need to consider what happens if $xy \in E(G_0) \cap E(G_j)$ for some $1 \leq j \leq k$. Again, we write $n' = n_{0,j}$, $\Delta = \Delta_{0,j}$ and $\delta = \delta_{0,j}$.

Choosing $\ell = j$ in the proof of Theorem 2.3, we ensure that we can still modify graph G_ℓ, such that we can reach the same conclusion that Q_j is a clique contained in $N_G(x) \cap N_G(y)$. Letting $Q = Q_j$, for $r = |E_G(Q)|$ and $q = |Q|$, we get $r = \binom{q}{2}$. Equations (2) and (3) still hold, giving

$$n\delta(n' - 2(\Delta - 1) + q) \leq (\Delta - 2)q - 2r.$$

Simplifying now gives

$$q(\Delta - \delta - 1 - q) \geq \delta(n' - 2\Delta + 2).$$

The left hand side is now maximized when $2q = \Delta - \delta - 1$, giving $(\Delta - \delta - 1)^2 \geq 4\delta(n' - 2\Delta + 2)$. Solving the quadratic inequality in Δ gives

$$\Delta \geq 2\sqrt{\delta(n' + 2\delta)} - (3\delta - 1).$$

□

3 Polynomial time algorithm for maximum partial f-factors

We give a polynomial time algorithm to find a maximum partial f-factor in a graph. The main idea will be to construct a related graph such that a maximum matching in the related graph can be turned into a maximum partial f-factor of the given graph. Finding a maximum matching in a graph can be done in polynomial time, for example with the blossom algorithm [7], so this allows us to find a maximum partial f-factor in polynomial time.

To start off, we formalize the problem with the following definition.

Definition 3.1 (Partial f-factor)**.** Let $G = (V, E)$ be a graph and let $f : V \to \mathbb{N}$ represent the desired non-negative degree of each vertex. A subgraph H of G is a *partial f-factor* if $d_H(x) \leq f(x)$ for all $x \in V$. If all inequalities are actually equalities, then H is an *f-factor*.

The following is the main theorem of this section.

Theorem 3.2 (Maximum partial f-factor algorithm)**.** *Given a graph $G = (V, E)$ and an function $f : V \to \mathbb{N}$. Let $f_{\max}$ be the maximum of $f(x)$ over $x \in V$. Then there is an algorithm that constructs a maximum partial f-factor of G in $O\left(f_{\max}|E|^{3/2}\right)$ time.*

In order to prove Theorem 3.2, we define a related graph K for which matchings correspond to partial f-factors of G. For all edges $xy \in E$, we add vertices $e_{x,y}$ and $e_{y,x}$ and connect them by an edge. Furthermore, for all vertices x of G and indices $1 \leq i \leq f(x)$, we add a vertex $v_{x,i}$ and connect it by an edge to $e_{x,y}$ for all neighbors $y \in N_G(x)$. We call the resulting graph K. We study matchings on K and their correspondence with subgraphs of G.

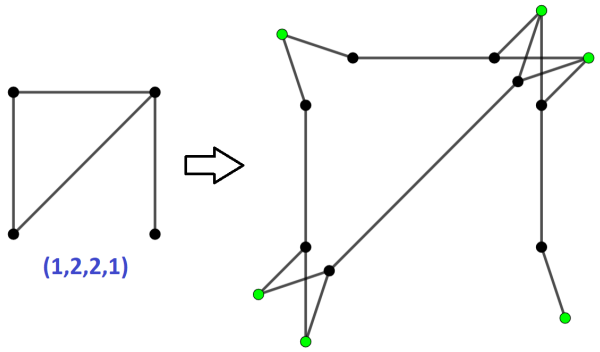

Figure 3.1: Example construction for graph K. The black vertices correspond to the edges in the original graph, while the green vertices correspond to the desired degrees of the vertices.

For any matching M on K, we define the corresponding partial f-factor H_M by taking those edges $xy \in E$ for which $e_{x,y}$ and $e_{y,x}$ are matched with vertices $v_{x,i}$ and $v_{y,j}$ respectively, for some i, j.

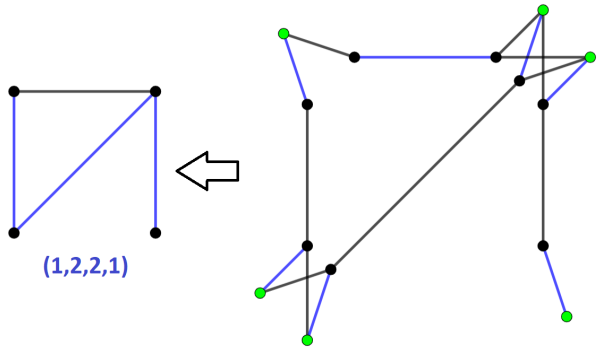

Figure 3.2: Example conversion from matching to partial f-factor.

The following lemma implies that a maximum matching M on K always gives a maximum partial f-factor H_M.

Lemma 3.3 (Maximum partial f-factor)**.** *We have the following correspondence between matchings and partial f-factors.*

(i) If M is a maximum matching on K, then H_M is a partial f-factor and $|M| = |E| + |H_M|$.

(ii) For any partial f-factor H there exists a matching M on K such that $H_M = H$ and $|M| = |E| + |H|$.

Proof of Lemma 3.3. (i) We have $d_{H_M}(x) \leq f(x)$ for all $x \in V$, because there are only $f(x)$ vertices of the form $v_{x,i}$ that vertices of the form $e_{x,y}$ can be connected with, so H_M is a partial f-factor. Since M is a maximum matching, for any edge $xy \in E$, there is at least one match in M connecting at least one of $e_{x,y}$ and $e_{y,x}$. There are two such matches exactly if $xy \in E(H_M)$, so $|M| = |E| + |H_M|$ follows.

(ii) We first match $e_{x,y}$ with $e_{y,x}$ for any $xy \in E \setminus E(H)$. Then any $x \in V$ has at most $f(x)$ neighbors in H, so for every such neighbor y, we can match $e_{x,y}$ with $v_{x,i}$ for some i. We find a matching M on K such that $H_M = H$. The equality $|M| = |E| + |H|$ follows a similar reasoning as in (i). □

To see why this lemma implies that a maximum matching M on K always gives a maximum partial f-factor H_M, let M be a maximum matching. By (i) H_M is a partial f-factor and $|M| = |E| + |H_M|$. Then for any partial f-factor H, by (ii) there exists a matching M' on K such that $|M'| = |E| + |H|$. Since M is a maximum matching, we have $|H| = |M'| - |E| \leq |M| - |E| = |H_M|$, so H_M is a maximum partial f-factor. The only thing left to prove for Theorem 3.2 is its running time.

Proof of Theorem 3.2. Note that we may assume without loss of generality that $f(x) \leq d_G(x)$ for all $x \in V$, since we are looking for a subgraph of G. It follows that $\sum_{x \in V} f(x) \leq 2|E|$ and $f_{\max} \leq n - 1$.

We first construct K. For all edges $xy \in E$ we introduce two vertices and one edge, resulting in $2|E|$ vertices of the form $e_{x,y}$ with $|E|$ edges. Then, for any $x \in V$, we introduce $f(x)$ more vertices and $f(x)d_G(x)$ more edges. In total, this adds at most $2|E|$ vertices of the form $v_{x,i}$ with at most $2f_{\max}|E|$ additional edges. We find that the constructed graph K has $O(|E|)$ vertices and $O(f_{\max}|E|)$ edges.

In [26] an algorithm is given that can construct a maximum matching on a graph $G' = (V', E')$ in $O\left(\sqrt{|V'|}|E'|\right)$ time. Applying this algorithm to the constructed graph K, we thus find a maximum matching M on K in asymptotic time $O\left(f_{\max}|E|^{3/2}\right)$. Then translating to H_M gives a maximum partial f-factor of G by Lemma 3.3. □

4 Independent set regularity

The sequential random graph generating algorithm from [2] attempts to find a uniformly random graphical realization of a given n-tuple $\pi = (d_1, \ldots, d_n)$. Their analysis relies on the following configuration model. Let $W = \cup_{i=1}^{n} W_i$ be a set of $2m = \sum_{i=1}^{n} d_i$ mini-vertices with $|W_i| = d_i$. The algorithm sequentially generates a random perfect matching on W. By combining the mini-vertices of each W_i into a single vertex i, the perfect matching is turned into a graphical realization of π.

In order to end up with a perfect matching, the generated matches need to be disjoint. Furthermore, consider sequentially adding a uniformly random match that is disjoint from all pairings added thus far. Since all mini-vertices are symmetrically equivalent in this setting, this procedure generates a uniformly random perfect matching on W. This symmetry argument appears to be the most fundamental argument behind the analysis.

In the case of generating a directed graph instead, the configuration model becomes bipartite. The mini-vertices are no longer symmetrically equivalent, because they are either an in-degree or an out-degree. However, the matches between in-degrees and out-degrees are still symmetrically equivalent. In order to generalize the symmetry argument, we thus need to focus on the matches and forget about the mini-vertices. The only thing we need to take into account with respect to the mini-vertices is which matches overlap.

We model a generalization of the configuration models with a meta-graph. The matches of mini-vertices become the vertices, and the edges indicate which pairings overlap. This turns the problem of generating a perfect matching of mini-vertices into the problem of finding a maximum independent set of vertices in the meta-graph. Since independent sets of the same size are symmetrically equivalent in the meta-graph, we can sequentially add uniformly random independent vertices to end up with a uniformly random maximum independent set.

This concept successfully generalizes the configuration model, however this does not yet take into account the problem of self-loops and double edges. It turns out that we can generalize these concepts in this setting as well. Self-loops are essentially a set of forbidden vertices, and double edges essentially form an equivalence relation on the vertices. It turns out that this is enough information to run the sequential algorithm. This will be further justified in Section 5.

4.1 Definition

Recall that an independent set of a graph is a set of vertices, no two of which are adjacent. The important symmetry induced by the pairings of mini-vertices is that independent sets of the same size are symmetrically equivalent. We formalize this idea by defining the following graph property.

Definition 4.1 (Independent set regularity)**.** We say that a graph $G = (V, E)$ has *independent set regularity* if any two independent sets S, T of G of the same size also have neighborhoods of the same size. Let $d_k(G)$ denote this size for independent sets of size k and $d(G) = d_1(G)$.

Remark 4.2. Note that a collection of disjoint cliques of the same size has independent set regularity. We refer to these examples as trivial, and are mostly interested in non-trivial graphs with independent set regularity. Note that this includes empty, edgeless and complete graphs.

This property allows us to make the fundamental symmetry argument for generating uniformly random maximum independent sets. The number of vertices that can be added to an independent set is only dependent on its size. It follows that all maximum independent sets are generated with equal probability, if you sequentially add uniformly random independent vertices.

We explore some basic properties of graphs with independent set regularity. First, note that G is $d(G)$-regular. If the graph is not connected, then two vertices can have disjoint neighborhoods, so $d_2(G) = 2d(G)$. Then no two non-adjacent vertices have a neighbor in common, so every connected component is a clique, so by regularity the graph is trivial. By contraposition, any non-trivial graph is connected, and any non-adjacent pair of vertices have overlapping neighborhoods, so the graph has a diameter of 2. In particular, it follows that $d(G)^2 + 1 \geq n(G) = |V|$, so the graph can not be sparse.

4.2 Examples

The simplest examples of non-trivial graphs with independent set regularity are regular complete multipartite graphs.

Example 4.3 (Regular complete multipartite graph)**.** Given positive integers r, k, we define the $r(k-1)$*-regular complete* k*-partite graph*, denoted $K_{\{r\}^k}$, as the edge complement of k disjoint cliques of size r. This graph has independent set regularity, because any independent set has a neighborhood of size $r(k-1)$. △

For our purposes of sequentially generating random graphs, the two most important examples are the meta-graphs of the configuration models. Recall that the meta-graphs turn edges into vertices, so they are instances of line graphs.

Definition 4.4 (Line graph)**.** The *line graph* of a (hyper)graph $G = (V, E)$, denoted $L(G)$, has vertex set E, and has two vertices adjacent if they intersect.

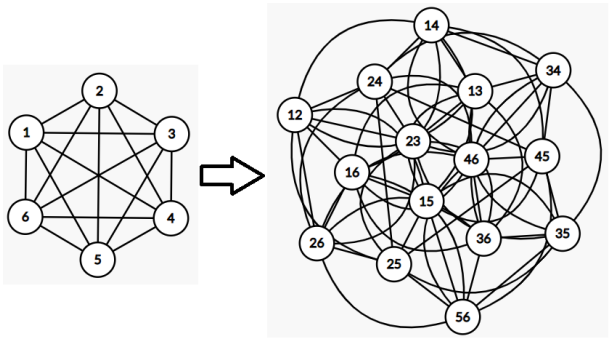

Figure 4.1: The line graph of a complete graph on 6 vertices.

The motivation behind line graphs is that they turn matchings into independent sets. Indeed, a set of edges of G is a matching if, and only if, it is an independent set of $L(G)$. Thus, the following two examples turn the problem of generating a perfect matching of mini-vertices into the problem of finding a maximum independent set of vertices in a meta-graph.

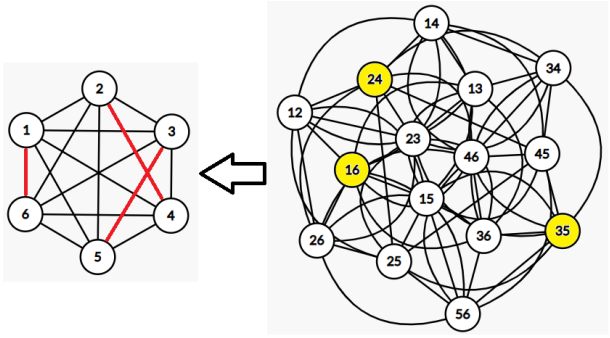

Figure 4.2: A maximum independent set in the line graph gives a maximum matching in the original graph.

Example 4.5 (Matching graph)**.** Given a set X, we define the *matching graph on* X, denoted $G_2(X)$, as the line graph of the complete graph on X. This graph is also known as the triangular graph, which is an instance of a Johnson graph [31]. This graph has independent set regularity, because all independent sets of the same size are equivalent by a bijection on X. △

The following definition is related to the bipartite case.

Definition 4.6 (Cartesian product graph)**.** For graphs G, H, the *Cartesian product graph of* G *and* H, denoted $G \square H$, is defined on the Cartesian product of the vertices, by connecting vertices if one entry has equality and the other has adjacency.

Example 4.7 (Bipartite matching graph)**.** Given sets X_1, X_2, we define the *bipartite matching graph on X_1 and X_2*, denoted $G_{1,1}(X_1, X_2)$, as the line graph of the complete bipartite graph on X_1 and X_2. This graph is also known as the rook's graph from chess, which is also the Cartesian product of two complete graphs. This graph has independent set regularity, because all independent sets of the same size are equivalent by bijections on X_1 and X_2. △

We can generalize the concept behind these two examples to hypergraphs as well. This gives quite a large family of graphs with independent set regularity. In the following definition, the notation $\binom{S}{k}$ is used to denote the set of all subsets of the set S of size k.

Example 4.8 (Generalized matching graph)**.** Given sets $X_1, \ldots, X_k$ and positive integers $r_1, \ldots, r_k$, we define the *generalized matching graph on $X_1, \ldots, X_k$ with parameters $r_1, \ldots, r_k$*, denoted $G_{r_1,\ldots,r_c}(X_1, \ldots, X_k)$, as the line graph of the hypergraph with vertices $X_1 \cup \ldots \cup X_k$ and edges $\binom{X_1}{r_1} \times \ldots \times \binom{X_k}{r_k}$. This graph has independent set regularity, because all independent sets of the same size are equivalent by bijections on $X_1, \ldots, X_k$. △

Finally, we have a construction to generate larger graphs with independent set regularity from smaller ones.

Example 4.9 (Complement Cartesian product)**.** If the edge complement G^C of a graph G has independent set regularity, then $(G \square G)^C$ also has independent set regularity, because independent sets of multiple vertices are constant in one entry. In particular, since $G_{1,1}(X, X)$ is the Cartesian product of two complete graphs on X, the edge complement $G_{1,1}(X, X)^C$ has independent set regularity. △

4.3 Uniform independent set regularity

The generalization to generalized matching graphs may seem promising, allowing us to sequentially generate random hypergraphs. However, when we carefully study the analysis of sequentially generating random graphs, we run into an issue. In order to deal with self-loops, we will have to forbid certain vertices from the generated maximum independent set. The problem we run into is that not all vertices are equivalent with respect to a maximum independent set, which makes the effect of forbidding certain vertices unpredictable.

Consider for example the generalized matching graph $G_3([9])$ and consider the maximum independent set $S = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}\}$. Then vertex $v = \{1,2,4\}$ is adjacent to two elements from S, while $w = \{1,4,7\}$ is adjacent to three elements from S. We find that forbidding v has a larger effect on the probability of generating S compared to forbidding w. Whichever of the vertices of S is added first, vertex w is adjacent and thus no longer worth considering, while the same is not true for vertex v.

We are going to prevent this issue by conditioning on the elements outside a maximum independent set being equivalent with respect to the maximum independent set.

Definition 4.10 (Uniform independent set regularity)**.** Let G be a graph with independent set regularity. If, for any maximum independent set S, all vertices outside S are adjacent to the same number of elements of S, then we denote this number by $\varsigma(G)$, and we say that G has $\varsigma(G)$*-uniform independent set regularity*.

Remark 4.11. Note that $\varsigma(G)$ is unique, and thus well-defined, unless G is edgeless, in which case any value of $\varsigma(G)$ is valid.

We find that the matching graph $G_2(X)$ has 2-uniform independent set regularity if $|X|$ is even, because then any maximum independent set is a perfect matching on X, and any other vertex connects two elements of X from different matches. For the same reason, the bipartite matching graph $G_{1,1}(X_1, X_2)$ has 2-uniform independent set regularity if $|X_1| = |X_2|$. However, generalized matching graphs usually do not have uniform independent set regularity, as we have seen with $G_3([9])$ for example.

Before we explore some basic properties of graphs with uniform independent set regularity, we introduce some notation. Recall that independent set regularity means that independent sets of the same size have the same size neighborhoods. By induction, using the inclusion-exclusion principle, it follows that the size of the intersection of the neighborhoods, defined by $\cap_{s \in S} N(s)$ for the independent set S, also only depends on the size of S. Indeed, if $\ell_k(G)$ denotes this size for independent sets of size k, then $d_k(G) = \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \ell_i(G)$.

If a graph G has uniform independent set regularity, we write $\ell(G) = \ell_{\varsigma(G)}(G)$ and note that $\ell_{\varsigma(G)+1}(G) = 0$. For example, the matching graph has $\ell(G_2(X)) = 4$, because common neighbors of non-adjacent vertices simply connect one endpoint of one vertex to one endpoint of the other vertex, giving $2 \times 2 = 4$ options. Similarly, the

bipartite matching graph has $\ell(G_{1,1}(X_1, X_2)) = 2$. Furthermore, let $\alpha(G)$ denote the independence number of G, that is the size of a maximum independent set of G.

Remark 4.12. If G is edgeless, then $\ell(G)$ depends on the choice of $\varsigma(G)$. If $\varsigma(G) \leq \alpha(G)$, then $\ell(G) = 0$, otherwise any value of $\ell(G)$ is valid.

Consider a graph G with uniform independent set regularity. For a maximum independent set S, consider the mapping from $V \setminus S$ to $\binom{S}{\varsigma(G)}$, mapping v to $N(v) \cap S$. By the definitions of $\varsigma(G)$ and $\ell(G)$, this is an $\ell(G)$ to one relation, which means that $n(G) = \ell(G)\binom{\alpha(G)}{\varsigma(G)} + \alpha(G)$.

For any subset $T \subset S$ of size k, we can also consider the mapping from $\cap_{t \in T} N(t)$ to $\binom{S \setminus T}{\varsigma(G)-k}$ mapping x to $N(x) \cap S \setminus T$. We find another $\ell(G)$ to one relation, this time giving $\ell_k(G) = \ell(G)\binom{\alpha(G)-k}{\varsigma(G)-k}$. In particular, this gives $d(G) = \ell(G)\binom{\alpha(G)-1}{\varsigma(G)-1}$.

Conversely, consider a graph G with independent set regularity with $n(G) = \ell_k(G)\binom{\alpha(G)}{k} + \alpha(G)$ and $\ell_{k+1}(G) = 0$. Consider a maximum independent set S. For any subset $T \subset S$ of size k, the intersection of neighborhoods of T has size $\ell_k(G)$, and all such sets are disjoint. We find that all $\ell_k(G)\binom{\alpha(G)}{k}$ vertices outside S are adjacent to exactly k elements of S, so G has k-uniform independent set regularity.

4.4 Classification

We briefly investigate the problem of classifying graphs with independent set regularity. This is mostly unrelated to the goal of generalizing the random graph generating algorithm, but it does give some insight as to how widely the generalization may apply.

Recall that a graph is called vertex transitive if any vertex can be mapped to any other vertex by a graph automorphism. An edge transitive graph is defined similarly. A natural alternative to independent set regularity would be independent set transitivity, when any two independent sets of the same size can be mapped to each other by a graph automorphism. An even stronger alternative would be bijective independent set transitivity, when graph automorphisms can induce any bijection between two indendent sets of the same size. It is worth looking into how these definitions compare. It is clear that bijective independent set transitivity implies independent set transitivity, which implies independent set regularity. However, none of the reverse implications hold.

A graph with independent set regularity, but no vertex transitivity, is $(C_4 + C_5)^C$, the edge complement of a disjoint union of a four-cycle and a five-cycle. Indeed, maximum independent sets correspond to edges on one of the two cycles, but vertices between the cycles are not transitive.

A graph with independent set regularity and vertex transitivity, but no independent set transitivity, is $(C_4 \square C_5)^C$, the edge complement of the Cartesian product of a four-cycle and a five-cycle. Indeed, maximum independent sets again correspond to edges on one of the cycles, but maximum independent sets between the cycles are not transitive.

A graph with independent set transitivity, but no bijective independent set transitivity, is the edge complement of the Holt graph [5]. This is because the Holt graph is half-transitive. Since the Holt graph is triangle-free, independent sets in its edge complement are either singletons or edges in the Holt graph.

A graph with bijective independent set transitivity, but no edge transitivity, is C_6^C, the edge complement of a six-cycle. Indeed, maximum independent sets correspond to the edges on the six-cycle, but vertex pairs that are two apart form edges that are not transitive with vertex pairs that are three apart.

We finally investigate uniform independent set regularity. We already know that the values $\varsigma(G)$, $\ell(G)$ and $\alpha(G)$ determine the number of vertices and edges, using $2|E(G)| = d(G)n(G)$, so maybe the entire graph is determined by just these three values.

Conjecture 4.13 (Uniform independent set regularity characterization)**.** *If a non-trivial graph G has uniform independent set regularity, then $\varsigma(G)|\ell(G)$ and $\alpha(G) \geq \varsigma(G) \geq 2$. Furthermore, for any integers α, ς, ℓ with $\varsigma|\ell$ and $\alpha \geq \varsigma \geq 2$, there is a unique non-trivial graph G with uniform independent set regularity and with $(\alpha(G), \varsigma(G), \ell(G)) = (\alpha, \varsigma, \ell)$, up to graph isomorphisms. This graph also has bijective independent set transitivity and edge transitivity.*

This would lead to a complete characterization of graphs with uniform independent set regularity. We find that there would be infinitely many more families of graphs on which our algorithm can be applied. However, the uniqueness with respect to only three values would also emphasize how restrictive uniform independent set regularity is. Moreover, uniform independent set regularity would imply both bijective independent set transitivity and edge transitivity. The rest of this section discusses the evidence for this conjecture.

4.4.1 Deletion of neighborhoods

We investigate the hereditary nature of uniform independent set regularity.

Definition 4.14 (Deletion of a neighborhood)**.** For a graph $G = (V, E)$ with an independent set S, the *deletion of the neighborhood of* S is the induced subgraph H on $V \setminus S \setminus N(S)$.

First, note that independent set regularity is hereditary on the deletion of neighborhoods. Consider an independent set T of H. Note that $T \cup S$ is an independent set of G. We find that $d_{|T|}(H) = |N_H(T)| = |N_G(T \cup S) \setminus N_G(S)| = d_{|T|+|S|}(G) - d_{|S|}(G)$ only depends on the size of T. Also note that $\alpha(H) = \alpha(G) - |S|$.

Furthermore, if G has uniform independent set regularity, then $\varsigma(H) = \varsigma(G)$ and $\ell(H) = \ell(G)$. Consider a maximum independent set T of H. Indeed, vertices outside T in H are adjacent to $\varsigma(G)$ elements of $S \cup T$, none of which can be in S. We thus also find that the intersection of neighborhoods of $\varsigma(G)$ elements of T is not adjacent to S and thus contained in H.

Second, note that bijective independent set transitivity is also hereditary on the deletion of neighborhoods. Any bijection $f : T \to R$ between two independent sets of H can be extended to a bijection $f^+ : T \cup S \to R \cup S$ by simply sending S to itself. Then the corresponding automorphism of G is closed on $N(S)$, so it can be restricted to an automorphism of H.

If we consider going in reverse, we run into a problem. Consider a path of length three. The deletion of any neighborhood gives a graph with bijective independent set transitivity, yet the graph is not even regular. However, if we assume the deletion of the neighborhoods of different individual vertices to be similar, then we can make useful deductions.

Proposition 4.15. *Let G be a d-regular graph.*

(i) Assume the deletion of the neighborhood of any vertex gives a subgraph H with independent set regularity, and with the same values of $\alpha(H) = \alpha$ and $d_k(H) = d_k$ for all k. Then G has independent set regularity with $d_{k+1}(G) = d + d_k$, and $\alpha(G) = \alpha + 1$.

(ii) Assume furthermore that any such subgraph H has uniform independent set regularity, and with the same values of $\varsigma(H) = \varsigma \leq \alpha$ and $\ell(H) = \ell$. If furthermore $n(G) = \ell\binom{\alpha(G)}{\varsigma} + \alpha(G)$, then G also has ς-uniform independent set regularity and $\ell(G) = \ell$.

Remark 4.16. Recall that the values $\alpha(H)$ and $d_k(H)$ determine whether H has uniform independent set regularity, so either all deletions of neighborhoods H are non-trivial with uniform independent set regularity, and with the same values of $\varsigma(H) = \varsigma$ and $\ell(H) = \ell$, or none of them are. Furthermore, the condition $n(G) = \ell\binom{\alpha(G)}{\varsigma} + \alpha(G)$ is necessary to avoid counter examples such as the disjoint union of two isomorphic graphs with uniform independent set regularity, with additional edges connecting all vertices from one graph with the other.

Proof of Proposition 4.15. (i) For any vertex v, the maximum independent set not adjacent to v has size α, so the maximum independent set including v has size $\alpha + 1$, so $\alpha(G) = \alpha + 1$. Consider an independent set S of size $k + 1$ with $v \in S$. Then $S \setminus \{v\}$ is an independent set of size k in the deletion of the neighborhood of v, so $N(S \setminus \{v\}) \setminus N(v)$ has size d_k, so $|N(S)| = |N(v)| + |N(S) \setminus N(v)| = d + d_k$.

(ii) Consider a maximum independent set S of G. Consider a vertex v outside S and not adjacent to some $s \in S$. Then v is in the deletion of the neighborhood of s, so it is adjacent to exactly ς elements of $S \setminus \{s\}$, and thus of S. So all vertices outside S are adjacent to either all of S, or exactly ς elements of S.

Consider a subset $T \subset S$ of size ς. Consider the intersection of the neighborhoods of T in the deletion of the neighborhood of some $s \in S \setminus T$. Any element v can not be adjacent to all elements of S, so we find $N(v) \cap S = T$. It follows that all such sets are disjoint and of size ℓ. We thus find $\ell\binom{\alpha(G)}{\varsigma}$ vertices outside S that are adjacent to

exactly ς elements of S. Since $n(G) = \ell\binom{\alpha(G)}{\varsigma} + \alpha$, it follows that G has ς-uniform independent set regularity and $\ell(G) = \ell$. □

Note that this does not work for bijective independent set transitivity. Consider again the graphs $(C_4 + C_5)^C$, $(C_4 \square C_5)^C$, and the edge complement of the Holt graph. None of these graphs have bijective independent set transitivity, but the deletion of the neighborhood of any vertex always gives a complete graph on the same number of vertices, which has bijective independent set transitivity.

Finally, we have one more interesting counter example, showing that vertex transitivity is not hereditary on the deletion of neighborhoods, even for graphs with independent set regularity. Consider the edge complement of the Cartesian product of the octahedron graph with the icosahedron graph. This graph is vertex transitive and has independent set regularity, with maximum independent sets corresponding to faces of the platonic solids. However, the deletion of the neighborhood of any vertex gives a graph isomorphic to $(C_4 + C_5)^C$, which is not vertex transitive.

4.4.2 Partial solution for Conjecture 4.13

We first prove $\alpha(G) \geq \varsigma(G) \geq 2$. Consider a graph $G = (V, E)$ with uniform independent set regularity. Let S be a maximum independent set. If G is not edgeless, then there exists a vertex v outside S, giving $\alpha(G) = |S| \geq |N(v) \cap S| = \varsigma(G)$. If G is non-trivial, then we can pick two distinct elements $s, t \in S$, and they must have a common neighbor v. We find that $\varsigma(G) = |N(v) \cap S| \geq 2$.

We prove the conjecture for the case $\alpha(G) = \varsigma(G)$. First note that the regular complete multipartite graph $G = K_{\{r\}^k}$ has $\alpha(G) = \varsigma(G) = r$ and $\ell(G) = r(k-1)$, which proves the existence, since G is non-trivial for $r, k \geq 2$. Note that G indeed has bijective independent set transitivity and edge transitivity.

Uniqueness follows from the hereditary nature of uniform independent set regularity. The deletion of the neighborhood of any vertex gives a subgraph H with $\alpha(H) < \varsigma(H)$, which means that H is edgeless. Note that the hereditary nature of the values $\varsigma(G)$ and $\ell(G)$ also means that this suffices to prove that $\varsigma(G)|\ell(G)$ holds in general.

We prove the conjecture for the case $\alpha(G) - 1 = \varsigma(G) = \ell(G)$. First note that the complement of the bipartite matching graph $G = (G_{1,1}(X, X))^C$ has $\alpha(G) = |X|$ and $\varsigma(G) = \ell(G) = |X| - 1$, which proves the existence, since G is non-trivial for $|X| \geq 3$. Note that G indeed has bijective independent set transitivity and edge transitivity.

To prove uniqueness, note that the deletion of the neighborhood of any vertex v is isomorphic to the regular complete bipartite graph $K_{\{\alpha(G)-1\}^2}$. So v is part of exactly two maximum independent sets S, T, of which it is the intersection. We can map S and T to $[\alpha] \times \{1\}$ and $\{1\} \times [\alpha]$ respectively, with v mapped to $(1, 1)$. We can map the remaining vertices to $[2, \alpha] \times [2, \alpha]$, indicating which elements of S and T respectively they are not adjacent to. Since the deletion of the neighborhood of any vertex from S or T is again isomorphic to $K_{\{\alpha(G)-1\}^2}$, every row and column forms a maximum independent set. Since the deletion of the neighborhood of any other vertex is again isomorphic to $K_{\{\alpha(G)-1\}^2}$, no cell is occupied twice. We conclude that we have an isomorphism to $G = (G_{1,1}([\alpha], [\alpha]))^C$.

Finally, the cases $(\varsigma(G), \ell(G)) = (2, 2)$ and $(\varsigma(G), \ell(G)) = (2, 4)$ have existence given by the bipartite matching graph and the matching graph respectively, which indeed have bijective independent set transitivity and edge transitivity.

5 Sequential random maximum independent set algorithm

Consider again the configuration model used in [2] to generate a random graphical realization of a given n-tuple $\pi = (d_1, \dots, n)$. A perfect matching of mini-vertices is generated sequentially. Considering all pairs of unmatched mini-vertices i, j that do not create a self-loop or double edge, one of them is chosen at random with probability proportional to $1 - b_{i,j}$ for some small positive bias $b_{i,j}$. Self-loops essentially give us a set of mini-vertex pairings that are not allowed, and double edges are essentially an equivalence relation on mini-vertex pairings such that no two chosen pairings are allowed to be equivalent. We use this idea to generalize the sequential algorithm to generating maximum independent sets.

5.1 The algorithm

Suppose a non-trivial graph $G = (V, E)$ with uniform independent set regularity is given. We write $\alpha = \alpha(G)$, $\varsigma = \varsigma(G)$, $\ell = \ell(G)$, $n = n(G)$, and $d = d(G)$. Suppose furthermore that we are given a set $F \subset V$ of forbidden vertices and an equivalence relation $R \subset V \times V$.

Definition 5.1 (Feasible set)**.** We say that an independent set S of G is *feasible* if it contains no forbidden vertices and no two elements are R-equivalent.

Finally, suppose we are given a bias function $b : V \to [0, 1)$. The following is an algorithm for sampling a maximum independent set of G that is feasible.

Algorithm 1 Random feasible maximum independent set

1: Initialize the feasible set $S = \emptyset$
2: **while** There exist $v \in V \setminus S$ such that $S \cup \{v\}$ is feasible **do**
3: Add a random such v to S with probability proportional to $1 - b(v)$
4: **end while**
5: **return** S

Note that we do not specify how such v is randomly generated, so we can not make any conclusions about the running time of this algorithm. We are solely interested in the probability that this algorithm generates a maximum independent set, and the distribution of maximum independent sets generated by this algorithm. Efficient implementation of this algorithm can be discussed on a case by case basis, based on the specific graph G that is given.

5.2 Performance

In order for this algorithm to properly generate feasible maximum independent sets, we need to have some bounds on the number of forbidden vertices and the number of equivalences between vertices. We thus define the quantity $d^F_{\max}$ as the maximum of $d^F(v) = |F \cap N(v)|$ over all $v \in V$, and we define $b_{\max}$ as the maximum of $b(v)$ over all $v \in V$.

In order to quantify equivalences, let $R(v) = \{w \in V \setminus \{v\} \setminus N(v) \setminus F : (v, w) \in R\}$ be the set of vertices that any vertex $v \in V$ can not appear with due to R-equivalence. For a set of vertices $S \subset V$, let $R(S) = \cup_{s \in S} R(s)$. We define the quantity $d^R_{\max}$ as the maximum of $|R(s)| + |R(S) \cap N(s)|$ over all maximum independent sets S and elements $s \in S$. The bound on $|R(S) \cap N(s)|$ will play a similar role as the bound on $|F \cap N(v)|$, while the bound on $|R(s)|$ just limits the size of equivalence classes.

In order to precisely state the distribution of independent sets generated by the algorithm, we define some final quantities. For a set of vertices $S \subset V$, let $b(S) = \sum_{s \in S} b(s)$. Let $P_G = \alpha/n$ be the proportion of vertices of G present in any maximum independent set. Finally, let $\mathcal{I}$ denote the set of all maximum independent sets of G.

The following is our main theorem. It essentially states that, if $d^F_{\max}$, $d^R_{\max}$, and $b_{\max}$ are sufficiently bounded, then with probability close to 1, the algorithm will generate a random feasible maximum independent set, the distribution of which can be approximated by a formula based on the bias terms.

Theorem 5.2 (Random maximum independent set)**.** *Consider ς, ℓ to be constant and $\alpha \to \infty$. If $d^R_{\max}, d^F_{\max} = O(\alpha^{\varsigma - 3/2 - \tau})$ and $b_{\max} = O(\alpha^{-1/2-\tau})$ for some $\tau > 0$, then with probability $1 - o(1)$, the algorithm outputs a feasible*

maximum independent set. Furthermore, any feasible maximum independent set $S \subset V$ will be the output of the algorithm with the following probability:

$$\mathbb{P}(S) = \exp\left(P_G|F| + \frac{1}{2}P_G|R(S)| + P_G b(V) - b(S) + o(1)\right)/|\mathcal{I}| \tag{4}$$

The value $|\mathcal{I}|$ can be calculated with the following formula:

$$|\mathcal{I}| = \frac{1}{\alpha!}\prod_{r=0}^{\alpha-1}\left(\ell\binom{\alpha-r}{\varsigma} + \alpha - r\right)$$

Indeed, consider generating a sequence of α independent vertices, of which there are $\alpha!|\mathcal{I}|$. After generating r independent vertices, the next vertex has to be chosen from the deletion of the neighborhood of the first r vertices. This is a subgraph H with $\alpha(H) = \alpha - r$, $\varsigma(H) = \varsigma$, and $\ell(H) = \ell$, so there are $\ell\binom{\alpha-r}{\varsigma} + \alpha - r$ options.

Note that $n = \ell\binom{\alpha}{\varsigma} + \alpha = \frac{\ell}{\varsigma!}\alpha^{\varsigma} + O(\alpha^{\varsigma-1})$, since $\varsigma \geq 2$. It follows that $P_G = \frac{\varsigma!}{\ell}\alpha^{1-\varsigma} + O(\alpha^{-\varsigma})$ and $b(V) \leq nb_{\max} = O(\alpha^{\varsigma-1/2-\tau})$. Furthermore, for any feasible maximum independent set S, we have $b(S) \leq \alpha b_{\max} = O(\alpha^{1/2-\tau})$ and $|R(S)| \leq \alpha d^R_{\max} = O(\alpha^{\varsigma-1/2-\tau})$. Any element of F is adjacent to ς elements of S, which finally gives $|F| \leq \frac{1}{\varsigma}\alpha d^F_{\max} = O(\alpha^{\varsigma-1/2-\tau})$.

We find that the argument of the exponent is of order $O(\alpha^{1/2-\tau})$. Furthermore, replacing P_G with $\frac{\varsigma!}{\ell}\alpha^{1-\varsigma}$ changes the argument by at most $O(\alpha^{-1/2-\tau})$. This is negligible due to the $o(1)$ term, so we find that the theorem would still hold. Finally, note that $P_G b(F) \leq P_G|F|b_{\max} = O(\alpha^{-2\tau})$ is negligible, which indicates that the biases of forbidden vertices are negligible, which is to be expected.

6 Application to random graphs

We show how Theorem 5.2 can be used to generate asymptotically uniformly random graphs with a given degree sequence, as the algorithm from [2] does. We show that Theorem 5.2 can be used for undirected graphs, directed graphs and colored graphs. While generalizing these cases was the main goal of Theorem 5.2, as discussed in Section 4.4, there might very well be many more applications to still discover.

6.1 Undirected graphs

Recall the configuration model. Given the n-tuple $\pi = (d_1, \dots, d_n)$, let $W = \cup_{i=1}^n W_i$ be a set of $2m = \sum_{i=1}^n d_i$ mini-vertices with $|W_i| = d_i$. By combining the mini-vertices of each W_i into a single vertex i, a perfect matching on W can be turned into a graph on $[n]$. For this to be a graphical realization of π, there need to be no self-loops or double edges. A self-loop occurs when two mini-vertices vertices from one set W_i are matched, and a double edge occurs when two mini-vertices from one set W_i are matched with mini-vertices from the same other set W_j.

We apply Theorem 5.2 on the matching graph $G = G_2(W) = (V, E)$. We let F contain all pairs of mini-vertices from the same set W_i, and we say that two pairs of mini-vertices are R-equivalent if they connect mini-vertices from the same sets W_i and W_j. Then combining the mini-vertices of each W_i into a single vertex i turns a maximum independent set of G into a graphical realization of π if, and only if, the independent set is feasible.

For $v \in V$ connecting mini-vertices from W_i and W_j, let $b(v) = \frac{(d_i-1)(d_j-1)}{4m}$. Finally, let $d_{\max}$ be the maximum of d_i over all $i \in [n]$, and let $\lambda = \lambda(\pi) = \frac{1}{2m}\sum_{i=1}^n \binom{d_i}{2}$. We use the algorithm for generating a random feasible maximum independent set of G to generate an asymptotically uniformly random graphical realization of π.

Theorem 6.1 (Random undirected graph)**.** *Let $m \to \infty$. If $d_{\max} = O(m^{1/4-\tau})$ for some $\tau > 0$, then with probability $1 - o(1)$ a graphical realization is generated. Furthermore, any graphical realization H of π will be generated with the following probability:*

$$\mathbb{P}(H) = \exp\Big(\lambda + \lambda^2 + o(1)\Big)\, m! \prod_{i=1}^n d_i! \prod_{r=0}^{m-1} \frac{1}{\binom{2m-2r}{2}}$$

Proof of Theorem 6.1. For $v \in V$ connecting mini-vertices $w_i \in W_i$ and $w_j \in W_j$, we have $b(v) = \frac{(d_i-1)(d_j-1)}{4m}$, so $b_{\max} \le \frac{1}{4m} d_{\max}^2 = O(m^{-1/2-2\tau})$. The set $F \cap N(v)$ can be partitioned in those elements containing w_i, of which there are $d_i - 1$, and those elements containing w_j, of which there are $d_j - 1$. We find $d_{\max}^F \le 2d_{\max} = O(m^{1/4-\tau})$.

For a maximum independent set S of G and an element $s \in S$ connecting mini-vertices $w_i \in W_i$ and $w_j \in W_j$, we have $|R(s)| = (d_i - 1)(d_j - 1)$. The set $R(S) \cap N(s)$ can be partitioned in those elements containing w_i, and those elements containing w_j. Those containing w_i are double edges with some $t \in S \setminus \{s\}$ connecting another mini-vertex from W_i. There are $d_i - 1$ such t, all giving at most $d_{\max} - 1$ double edges containing w_i. We find $d_{\max}^R \le 3d_{\max}^2 = O(m^{1/2-2\tau})$.

Since $\alpha = m$, by Theorem 5.2 with probability $1 - o(1)$, a feasible maximum independent set of G is found. Any specific such set S is found with probability given by equation (4). For any graphical realization H of π, there are $\prod_{i=1}^n d_i!$ such sets S that generate H. Since $\varsigma = 2$ and $\ell = 4$, we have $\ell\binom{\alpha-r}{\varsigma} + \alpha - r = \binom{2m-2r}{2}$, so $|\mathcal{I}| = \frac{1}{m!}\prod_{r=0}^{m-1}\binom{2m-2r}{2}$.

We can replace P_G with $\frac{\varsigma!}{\ell}\alpha^{1-\varsigma} = \frac{1}{2m}$. We have $|F| = \sum_{i=1}^n \binom{d_i}{2}$, so $P_G|F| = \lambda$. Furthermore, for $v \in V$ connecting mini-vertices from W_i and W_j, we have $|R(v)| = (d_i - 1)(d_j - 1)$, so $\frac{1}{2}P_G|R(S)| = b(S)$. Since the biases of forbidden vertices are negligible, it suffices to show that $P_G b(V \setminus F) = \lambda^2 + o(1)$.

$$\begin{aligned} P_G b(V \setminus F) &= \frac{1}{2m} \sum_{1 \le i < j \le n} d_i d_j \frac{(d_i - 1)(d_j - 1)}{4m} \\ &= \frac{1}{16m^2}\left(\left(\sum_{i=1}^n d_i(d_i - 1)\right)^2 - \sum_{i=1}^n d_i^2(d_i - 1)^2\right) \end{aligned}$$

Indeed, we have $\sum_{i=1}^n d_i^2(d_i-1)^2 \le d_{\max}^3 \sum_{i=1}^n d_i = 2md_{\max}^3 = O(m^{7/4-3\tau})$. □

Remark 6.2. The bias $b(v) = \frac{(d_i-1)(d_j-1)}{4m}$ can be replaced by the bias $b(v) = \frac{d_i d_j}{4m}$. Indeed, summing the difference $\frac{d_i+d_j-1}{4m}$ over $V \setminus F$ gives

$$\sum_{1\le i<j\le n} d_i d_j \frac{d_i+d_j-1}{4m} = \frac{1}{4m}\left(\sum_{1\le i,j\le n} d_i^2 d_j - \sum_{i=1}^n d_i^3 - n + |F|\right),$$

which is $\frac{1}{2}\sum_{i=1}^n d_i^2 - \frac{n}{4m} + O(m^{1/2-\tau})$. Summed over a feasible maximum independent set, this gives $\frac{1}{4m}\sum_{i=1}^n d_i^2 - \frac{1}{4}$, so they cancel each other out in $P_G b(V) - b(S)$.

6.2 Directed graphs

Given n-tuples $\pi^+ = (d_1^+, \dots, d_n^+)$ and $\pi^- = (d_1^-, \dots, d_n^-)$ with $m = \sum_{i=1}^n d_i^+ = \sum_{i=1}^n d_i^-$. We want to generate a uniformly random directed graphical realization of (π^+, π^-), that is, a simple directed graph on $[n]$ with out-degrees given by π^+ and in-degrees given by π^-.

Let $W^+ = \cup_{i=1}^n W_i^+$ and $W^- = \cup_{i=1}^n W_i^-$ be sets of m mini-vertices with $|W_i^+| = d_i^+$ and $|W_i^-| = d_i^-$. A perfect bipartite matching between W_i^+ and W_i^- can be turned into a directed graph on $[n]$, by considering the matches to be directed from W_i^+ to W_i^-, and by combining the mini-vertices of each $W_i^+ \cup W_i^-$ into a single vertex i. A self-loop occurs when a mini-vertex from W_i^+ is matched with a mini-vertex from W_i^-, and a double edge occurs when two mini-vertices from one set W_i^+ are matched with mini-vertices from the same other set W_j^-.

We apply Theorem 5.2 on the bipartite matching graph $G = G_{1,1}(W^+, W^-) = (V, E)$. We let F contain all pairs of mini-vertices from W_i^+ and W_i^- for some i, and we say two pairs of mini-vertices are R-equivalent if they connect mini-vertices from the same sets W_i^+ and W_j^-. Combining the mini-vertices of each $W_i^+ \cup W_i^-$ into a single vertex i turns a maximum independent set of G into a directed graphical realization of (π^+, π^-) if, and only if, the independent set is feasible.

For $v \in V$ connecting mini-vertices from W_i^+ and W_j^-, let $b(v) = \frac{(d_i^+-1)(d_j^--1)}{2m}$. Let $d_{\max}$ be the maximum of d_i^+ and d_i^- over all $i \in [n]$, and let $\lambda^\pm = \lambda(\pi^\pm) = \frac{1}{m}\sum_{i=1}^n \binom{d_i^\pm}{2}$. We use the algorithm for generating a random feasible maximum independent set of G to generate an asymptotically uniformly random directed graphical realization of (π^+, π^-).

Theorem 6.3 (Random directed graph)**.** *Let $m \to \infty$. If $d_{\max} = O(m^{1/4-\tau})$ for some $\tau > 0$, then with probability $1 - o(1)$ a directed graphical realization is generated. Furthermore, any directed graphical realization H of (π^+, π^-) will be generated with the following probability:*

$$\mathbb{P}(H) = \exp\left(\frac{1}{m}\sum_{i=1}^n d_i^+ d_i^- + \lambda^+\lambda^- + o(1)\right)\frac{1}{m!}\prod_{i=1}^n d_i^+!\,d_i^-!$$

Proof of Theorem 6.3. The proof follows the same structure as for Theorem 6.1. Only some computations are different. We have $b_{\max} = O(m^{-1/2-2\tau})$, $d_{\max}^F = O(m^{1/4-\tau})$, and $d_{\max}^R \le 3d_{\max}^2 = O(m^{1/2-2\tau})$, using the same arguments.

Since $\alpha = m$, by Theorem 5.2 with probability $1 - o(1)$, a feasible maximum independent set of G is found. Any specific such set S is found with probability given by equation (4). For any graphical realization H of π, there are $\prod_{i=1}^n d_i^+!\,d_i^-!$ such sets S that generate H. Since $\varsigma = 2$ and $\ell = 2$, we have $\ell\binom{\alpha-r}{\varsigma} + \alpha - r = (m-r)^2$, so $|\mathcal{I}| = \frac{1}{m!}\prod_{r=0}^{m-1}(m-r)^2 = m!$.

We can replace P_G with $\frac{\varsigma!}{\ell}\alpha^{1-\varsigma} = \frac{1}{m}$. We have $P_G|F| = \frac{1}{m}\sum_{i=1}^n d_i^+ d_i^-$ and $P_G b(V) = \lambda^+\lambda^-$. Furthermore, for $v \in V$ connecting mini-vertices from W_i^+ and W_j^-, we have $|R(v)| = (d_i^+ - 1)(d_j^- - 1)$, so $\frac{1}{2}P_G|R(S)| = b(S)$. □

Remark 6.4. By a similar reasoning as with Remark 6.2, the bias $b(v) = \frac{(d_i^+-1)(d_j^--1)}{4m}$ can be replaced by the bias $b(v) = \frac{d_i^+ d_j^-}{4m}$.

6.3 Forbidden edges and colored graphs

In order to generate colored graphs, we will generate them one color at a time. We first randomly generate a graph of the first color with the desired degree sequence. We then randomly generate a graph of the second color with the desired degree sequence, but we have to forbid all the edges from the graph of the first color. We continue randomly generating graphs of all subsequent colors, each time forbidding all the edges from all previously generated graphs, until the graphs of all colors are generated.

Forbidding the edges from the graph of the first color in all subsequent graphs affects the distribution of all subsequently generated graphs. In order to cancel out this effect, we have to introduce a bias on the generated graphs of the first color. In order to determine this bias, we will first consider the problem of generating a random graph with forbidden edges, and we will keep the bias function unspecified.

Consider an n-tuple $\pi = (d_1, \dots, d_n)$ and a set of forbidden edges D. Let $W = \cup_{i=1}^n W_i$ be a set of $2m = \sum_{i=1}^n d_i$ mini-vertices with $|W_i| = d_i$, like in Section 6.1. We apply Theorem 5.2 on the matching graph $G = G_2(W) = (V, E)$ again. We define F, R, $d_{\max}$ and $\lambda = \lambda(\pi)$ in the same way, except that we add to F all pairs of mini-vertices from sets W_i and W_j with $ij \in D$.

We use the algorithm for generating a random feasible maximum independent set of G. For an unspecified bias function $\widetilde{b} : \binom{[n]}{2} \to [0,1)$, we use the bias function $b : V \to [0,1)$ on G, defined by $b(v) = \widetilde{b}(ij)$ if $v \in V \setminus F$ connects mini-vertices from W_i and W_j with $i \neq j$ and $b(v) = 0$ otherwise. Note that $b_{\max} = \widetilde{b}_{\max}$. Finally, let $d_{\max}^D$ be the maximum of $d_i^D = |\{j \in [n] : ij \in D\}|$ over all $i \in [n]$, and let $\delta_D(\pi) = \frac{1}{2m}\sum_{ij\in D} d_i d_j$ and $\beta(\widetilde{b}) = \frac{1}{2m}\sum_{1\leq i<j\leq n} d_i d_j \widetilde{b}(ij)$.

Theorem 6.5 (Random undirected graph with forbidden edges)**.** *Let $m \to \infty$. If $d_{\max}, d_{\max}^D = O(m^{1/4-\tau})$ and $b_{\max} = O(m^{-1/2-\tau})$ for some $\tau > 0$, then with probability $1 - o(1)$ a graphical realization with no edges from D is generated. Furthermore, any such graphical realization H of π will be generated with the following probability:*

$$\mathbb{P}(H) = \exp\Big(\lambda(\pi) + \delta_D(\pi) + \gamma_H + \beta(\widetilde{b}) - \widetilde{b}(E(H)) + o(1)\Big)\, C(\pi)$$

Here $\gamma_H = \frac{1}{4m}\sum_{ij\in E(H)}(d_i-1)(d_j-1)$ and $C(\pi) = m!\prod_{i=1}^n d_i! \prod_{r=0}^{m-1} \frac{1}{\binom{2m-2r}{2}}$.

Proof of Theorem 6.5. The proof follows the same structure as for Theorem 6.1 again. We have $d_{\max}^R \leq 3d_{\max}^2 = O(m^{1/2-2\tau})$ using the same argument. For $v \in V$, the set $F \cap N(v)$ contains at most $2d_{\max}$ self-loops and at most $2d_{\max}(d_{\max}-1)$ forbidden edges, so $d_{\max}^F \leq 2d_{\max}^2 = O(m^{1/2-2\tau})$.

Since $\alpha = m$, by Theorem 5.2 with probability $1 - o(1)$, a feasible maximum independent set of G is found. Any specific such set S is found with probability given by equation (4). For any graphical realization H of π with no edges from D, there are $\prod_{i=1}^n d_i!$ such sets S that generate H, and we have $|\mathcal{I}| = \frac{1}{m!}\prod_{r=0}^{m-1}\binom{2m-2r}{2}$. Replacing P_G with $\frac{1}{2m}$, we get $P_G|F| = \lambda(\pi) + \delta_D(\pi)$, $\frac{1}{2}P_G|R(S)| = \gamma_H$, $P_G b(V) = \beta(\widetilde{b})$ and $b(S) = \widetilde{b}(E(H))$. □

We want to apply Theorem 6.5 to generate colored graphs one color at a time. Let $\pi_1 = (d_1^1, \dots, d_n^1), \dots, \pi_k = (d_1^k, \dots, d_n^k)$ be n-tuples with $2m_c = \sum_{i=1}^n d_i^c$. We first generate a graph H_1 with degree sequence π_1 and no forbidden edges, using an unspecified bias functions $\widetilde{b}^1 : \binom{[n]}{2} \to [0,1)$. We then generate a graph H_2 with degree sequence π_2 and forbidden edges $E(H_1)$, using an unspecified bias functions $\widetilde{b}^2 : \binom{[n]}{2} \to [0,1)$. We continue this process, each time forbidding edges $E(H_1) \cup \ldots \cup E(H_{c-1})$ when generating graph H_c, until all c graphs are generated.

We consider k to be constant and we let $m_{\min} \to \infty$ with $m_{\min}$ the minimum of m_c over all c. By Theorem 6.5, if there is some $\tau > 0$ such that $d_{\max}^c = O(m_{\min}^{1/4-\tau})$ and $b_{\max}^c = O(m_{\min}^{-1/2-\tau})$ for all c, then with probability $1 - o(1)$ edge-disjoint graphical realization are generated. Furthermore, any such graphical realizations $H_1, \dots, H_k$ are generated with the following probability:

$$\begin{aligned}\mathbb{P}(H_1,\dots,H_k) =&(1+o(1))\prod_{c=1}^{k}\exp\left(\lambda(\pi_c)+\beta(\widetilde{b}^c)\right)C(\pi_c)\\ &\cdot\prod_{c=1}^{n}\exp\left(\gamma_{H_c}+\sum_{c'=c+1}^{k}\delta_{E(H_c)}(\pi_{c'})-\widetilde{b}^c(E(H_c))\right)\end{aligned}$$

In order for this probability to be asymptotically uniform, it follows that we need the term $\widetilde{b}^c(E(H_c))$ to cancel with the term $\gamma_{H_c}+\sum_{c'=c+1}^{k}\delta_{E(H_c)}(\pi_{c'})$ for all c. This is achieved by defining

$$\widetilde{b}^c(ij)=\frac{(d_i^c-1)(d_j^c-1)}{4m_c}+\sum_{c'=c+1}^{k}\frac{d_i^{c'}d_j^{c'}}{2m_{c'}}=O(m_{\min}^{-1/2-2\tau}).$$

The final probability can be calculated by evaluating $\beta(\widetilde{b}^c)$. This is similar to evaluating $P_Gb(V\setminus F)$ in the proof of Theorem 6.1.

$$\beta(\widetilde{b}^c)=\frac{1}{8m_c^2}\sum_{1\le i<j\le n}d_i^cd_j^c(d_i^c-1)(d_j^c-1)+\sum_{c'=c+1}^{k}\frac{1}{4m_cm_{c'}}\sum_{1\le i<j\le n}d_i^cd_j^cd_i^{c'}d_j^{c'}$$

Defining $\lambda(\pi_c,\pi_{c'})=\frac{1}{4m_cm_{c'}}\sum_{1\le i<j\le n}d_i^cd_j^cd_i^{c'}d_j^{c'}$, we get $\beta(\widetilde{b}^c)=\lambda(\pi_c)^2+o(1)+\sum_{c'=c+1}^{k}\lambda(\pi_c,\pi_{c'})$. We end up with the final probability

$$\begin{aligned}\mathbb{P}(H_1,\dots,H_k) =&(1+o(1))\prod_{c=1}^{k}\exp\left(\lambda(\pi_c)+\lambda(\pi_c)^2\right)C(\pi_c)\\ &\cdot\prod_{1\le c<c'\le k}\exp\left(\lambda(\pi_c,\pi_{c'})\right).\end{aligned}$$

Note that, by a similar reasoning as with Remark 6.2 again, the term $\frac{(d_i^c-1)(d_j^c-1)}{4m_c}$ in $\widetilde{b}^c(ij)$ can be replaced by the term $\frac{d_i^cd_j^c}{4m_c}$.

7 Idea of proof

We want to calculate for a given feasible maximum independent set $S \subset V$ the probability $\mathbb{P}(S)$ that it will be the output of the algorithm. We give a heuristic calculation and a framework for the proof of the equation. Both the heuristics and the proof framework follow a similar structure as [2], where the same statements are proven for the specific case of generating uniformly random undirected graphs with a given degree sequence, as described in Section 6.1. The framework for the proof will divide the work into a few lemmas that together prove Theorem 5.2. However, while [2] contains proofs for the equivalent versions of those lemmas in the specific case of generating uniformly random undirected graphs with a given degree sequence, we do not give proofs for these lemmas in this more general setting.

7.1 Heuristics

Consider step $r = 0, \ldots, \alpha - 1$ where we go from S_r to S_{r+1}. Let $\widetilde{V}_r \subset V$ contain all vertices $v \in V$ for which $S_r \cup \{v\}$ is still an independent set, and let $V_r \subset \widetilde{V}_r$ contain all vertices $v \in V$ for which $S_r \cup \{v\}$ is still a feasible set. Note that the maximum independent set symmetry on G implies that the restriction of G to $\widetilde{V}_r$ also has maximum independent set symmetry. It follows that $n_r = |\widetilde{V}_r| = \ell\binom{\alpha - r}{\varsigma} + \alpha - r$ and $|\mathcal{I}| = \frac{1}{\alpha!} n_0 \ldots n_{\alpha-1}$.

Let $p_r = r/\alpha$ denote the proportion of vertices already added to S_r, and let $q_r = 1 - p_r$ denote the proportion of vertices still to be added. By maximum independent set symmetry, any forbidden vertex is adjacent to exactly ς elements of S, so on average there are $q_r^\varsigma |F|$ forbidden vertices in $\widetilde{V}_r$. By a similar argument, on average there are $p_r q_r^\varsigma |R(S)|$ vertices in $\widetilde{V}_r$ that can not be added to S_r due to R-equivalence. Ignoring the bias function for now, we get the following probability:

$$\mathbb{P}(S) \approx \alpha! \prod_{r=0}^{\alpha-1} \frac{1}{|V_r|} \approx \alpha! \prod_{r=0}^{\alpha-1} \frac{1}{n_r - q_r^\varsigma |F| - p_r q_r^\varsigma |R(S)|}$$

Taking out a factor n_r and approximating $n_r/n \approx q_r^\varsigma$, so $q_r^\varsigma \approx \frac{1}{\alpha} P_G n_r$, gives

$$\mathbb{P}(S) \approx \alpha! \prod_{r=0}^{\alpha-1} \frac{1}{n_r} \frac{1}{1 - \frac{1}{\alpha} P_G |F| - p_r \frac{1}{\alpha} P_G |R(S)|}.$$

Approximating $\frac{1}{1-x} \approx e^x$ for small x and approximating $\sum_{r=0}^{\alpha-1} p_r \approx \alpha/2$ gives

$$\mathbb{P}(S) \approx \exp\left(P_G |F| + \frac{1}{2} P_G |R(S)| \right) / |\mathcal{I}|.$$

We can consider how the bias affects this product. The numerator will be multiplied by $1 - b(s) \approx e^{-b(s)}$ for all $s \in S$, which ends up giving an extra factor $e^{-b(S)}$. The denominator currently counts elements of V_r, all of which need to be multiplied by their bias. On average, the denominator is multiplied by $1 - \frac{1}{n} b(V) \approx e^{-\frac{1}{n} b(V)}$, which ends up giving an extra factor $e^{P_G b(V)}$. Equation (4) follows.

7.2 Proof framework

We give a framework for the proof of equation (4). We divide the work up into a few lemmas, and show that these lemmas together suffice to prove Theorem 5.2.

Let $\mathcal{S}$ be the set of all permutations of S, so $\mathbb{P}(S) = \sum_{\mathcal{N} \in \mathcal{S}} \mathbb{P}(\mathcal{N})$. Fix some $\mathcal{N} = (s_1, \ldots, s_\alpha) \in \mathcal{S}$. Let $S_r(\mathcal{N})$, $\widetilde{V}_r(\mathcal{N})$ and $V_r(\mathcal{N})$ be as in the heuristics. By the chain rule, we have

$$\begin{aligned}\mathbb{P}(\mathcal{N}) &= \prod_{r=0}^{\alpha-1} \mathbb{P}(S_{r+1} = \{s_1, \ldots, s_{r+1}\} | S_r = \{s_1, \ldots, s_r\}) \\ &= \prod_{r=0}^{\alpha-1} \frac{1 - b(s_{r+1})}{\sum_{v \in V_r(\mathcal{N})} 1 - b(v)}.\end{aligned}$$

Let $\Delta_r(\mathcal{N}) = |\widetilde{V}_r(\mathcal{N}) \setminus V_r(\mathcal{N})|$ and $\Psi_r(\mathcal{N}) = \Delta_r(\mathcal{N}) + b(V_r(\mathcal{N}))$ such that the denominator becomes $n_r - \Psi_r(\mathcal{N})$. For the numerator, we apply $1 - x = e^{-x+O(x^2)}$ as $x \to 0$ and $b_{\max} = O(\alpha^{-1/2-\tau})$, which gives

$$\mathbb{P}(\mathcal{N}) = \exp(-b(S) + o(1)) \prod_{r=0}^{\alpha-1} \frac{1}{n_r - \Psi_r(\mathcal{N})}. \tag{5}$$

For fixed r, we can model $\Psi_r(\mathcal{N})$ as a random variable Ψ_r depending on a uniform distribution of permutations $\mathcal{N} \in \mathcal{S}$. Equivalently, one might consider S_r to be a uniformly random subset of S of size r. The main goal is to show that Ψ_r is sharply concentrated around its expected value.

We can approximate the distribution on S_r by independently adding the elements of S with probability p_r. We use a p_r subscript to emphasise the use of this model, for example with S_{p_r} and Ψ_{p_r}. The first main step to proving equation (4) is to prove the following formula for the expected value of Ψ_{p_r}.

Lemma 7.1 (Expected value)**.** *The expected value $\psi_{p_r}(S) = \mathbb{E}(\Psi_{p_r})$ is given by the following formula:*

$$\psi_{p_r}(S) = q_r^{\zeta} n \left(\frac{1}{\alpha} P_G |F| + p_r \frac{1}{\alpha} P_G |R(S)| + \frac{1}{\alpha} P_G b(V) + \rho_{p_r}(S) \right)$$

Here $\rho_{p_r}(S) = O\left(\alpha^{-1-\tau} + \frac{r}{\alpha - r} \alpha^{-3/2-\tau}\right)$ is a negligible error term.

The second main step to proving equation (4) is to evaluate the product in equation (5) with $\Psi_r(\mathcal{N})$ replaced by $\psi_{p_r}(S)$.

Lemma 7.2 (Product)**.** *We have the following product equality:*

$$\prod_{r=0}^{\alpha-1} \frac{1}{n_r - \psi_{p_r}(S)} = \exp\left(P_G |F| + \frac{1}{2} P_G |R(S)| + P_G b(V) + o(1) \right) \prod_{r=0}^{\alpha-1} \frac{1}{n_r}$$

The third, final and most important step to proving equation (4) is to prove the following concentration result about Ψ_r.

Lemma 7.3 (Concentration)**.** *We have the following concentration equality:*

$$\sum_{\mathcal{N} \in \mathcal{S}} \prod_{r=0}^{\alpha-1} \frac{1}{n_r - \Psi_r(\mathcal{N})} = (1 + o(1)) \alpha! \prod_{r=0}^{\alpha-1} \frac{1}{n_r - \psi_r(S)}$$

In order to finally prove equation (4), we take $\mathbb{P}(S) = \sum_{\mathcal{N} \in \mathcal{S}} \mathbb{P}(\mathcal{N})$ and we apply equation (5), then Lemma 7.3 and finally Lemma 7.2. So, the only remaining ingredient to prove Theorem 5.2 is the probability of ending up with a feasible maximum independent set.

Lemma 7.4 (Termination)**.** *The algorithm has an $o(1)$ probability of terminating before S has α elements.*

In [2] proofs are given for statements similar to these four lemmas, but for the specific case of generating uniformly random undirected graphs with a given degree sequence, as described in Section 6.1. Specifically, in [2], Section 5 proves the equivalent of Lemma 7.4, Section 7.1 proves the equivalent of Lemma 7.1, Section 7.2 proves the equivalent of Lemma 7.2 and the rest of Section 7 proves the equivalent of Lemma 7.3.

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