



Universiteit Utrecht

Institute for Theoretical Physics

# ENTROPY AND REPLICA GEOMETRY IN GENERIC TWO-DIMENSIONAL DILATON GRAVITY THEORIES

MASTER THESIS

*Yueren Xing*

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*Supervisors:*

Dr. UMUT GÜRSOY  
Institute for Theoretical Physics

Prof. Dr. STEFAN VANDOREN  
Institute for Theoretical Physics

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## Abstract

We set up a new version of black hole information paradox in an eternal Narayan black hole, a generic two-dimensional dilaton gravity theory with non-trivial on-shell bulk action and a product of dimensional reduction from higher-dimensional AdS black brane, joined to Minkowski bath on both sides. We also report both similarities as well as important differences between our model and the famous model of JT gravity coupled with baths. The contradiction of Hawking's result of entanglement entropy with unitarity is resolved by including a new saddle in the Euclidean gravitational path integral. As part of ongoing and developing research, we attempt, and have had partial success, to explicitly construct the replica wormhole geometry for our model to fully justify the quantum extremal surface calculations with Euclidean gravitational path integral, without using holography.

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## 0 Introduction

In recent years, there have been many papers dedicated to the development of self-consistent solutions, built from first principles of theoretical physics, to certain versions of the black hole information paradox(puzzle) in various gravity theories, see [1] for a general and heuristic review on recent progress. Among them, a simple model of black holes in the two-dimensional Jackiw-Teitelbolm (JT) gravity [2][3][4] stands out as the playground of one version of the information paradox first formulated in [5] and [6] where a black hole in anti-de Sitter spacetime radiates into an attached non-gravitational flat region.

An answer to this version of the paradox was proposed using different theoretical tools in a series of paper [7] [8] [9][10][11][12][13]. Their results converge on the point that a quantum extremal island appears in the interior of the black hole after Page time through an extension of the generalized gravitational entropy formula [5] [6] [14] [15] [16] [17] [18] [19], namely the following 'island formula'[10],

$$S(R) = \min \text{ext}_I \left[ \frac{\text{Area}(\partial I)}{4G_N} + S_{\text{QFT}(I \cup R)} \right] \quad (0.1)$$

where  $R$  is the radiation,  $I$  is the island, and  $S_{\text{QFT}}$  is the traditional semi-classical entropy of quantum fields computed by quantum field theory in curved spacetime. In the context of two-dimensional dilaton gravity, 'Area' is represented by the value of the dilaton.

While the island formula (0.1) was first put forward based on - and a unitary Page curve [20] [21] is consistently produced derived through- holography, a calculation, without holography while recovering the same result, was done using the gravitational replica method (developed in [16] and [22]) originally by Almheiri et.al. [11] for eternal JT black holes joined by Minkowski space and later generalized into evaporating black hole scenario in [13]. As shown in [11], new saddles - replica wormholes - in the gravitational path integral dominate, at late time, over the Hawking saddle which gives the ever-increasing entanglement entropy of the radiation initially derived by Hawking [23] and thus inconsistent with unitarity.

Although in [11] the results were explicitly obtained only in some simple examples with regards to the information paradox for eternal black hole in JT gravity, the same method based on Euclidean path integral description nonetheless can, at least in principle, be applied to eternal black holes in other more general two-dimensional dilaton gravity theories, again producing large corrections to the fine-grained entropy from the effects brought by replica wormholes.

One interesting type of the more general two-dimensional gravity theories arises from the two-dimensional subsector of higher-dimensional gravity on spaces of the form  $\mathcal{M}_2 \times X^d$  on Kaluza-Klein compactification over the compact space  $X^d$  [24], while  $AdS_2$  arising in the near horizon geometry of extremal black holes and branes is one well-known example of it. [25] [26] [27]. Here we study one specific example of this type, two dimensional dilaton gravity theory as the product of dimensional reduction of higher dimensional gravity with a negative cosmological constant, or Narayan theories as we name them. The on-shell metric of Narayan theories is always conformally  $AdS_2$ , whatever higher dimension the gravity theory lives in before dimensional reduction. But unlike the constant curvature  $AdS_2$  arising from  $AdS_3$  by dimensional reduction, which is the simplest example of these conformally  $AdS_2$  metrics, it generally has an IR singularity - a curvature singularity where the dilaton is also vanishingly small, and a UV divergence in the **bulk** on-shell action where the dilaton grows large [24]. Both of these are, however, operationally curable if we introduce a black hole horizon to regulate the curvature singularity and add appropriate counterterms to regulate the on-shell action. The divergences the two-dimensional theories exhibit suggest they are best regarded as UV incomplete thermodynamic low energy effective theories universal to all UV completions  $AdS_D \times X$  and encode higher dimensional gravity intrinsically, quite different from the near extremal near horizon  $AdS_2 \times X$  throats in extremal objects, where the  $X$ -compactification leads to an intrinsically two-dimensional theory. What we are concerned with the most in this article, is to formulate a version of black hole information problem for eternal black holes in a model of generic two-dimensional dilaton gravity theory featuring both on-shell curvature singularity and a divergent bulk action as opposed to the constant non-singular curvature and a trivial on-shell bulk action in JT gravity, compute the fine-grained entropy of the black hole and Hawking radiation and then set up replica wormhole geometry to verify the calculations by recovering the same quantum extremal surface (QES) result. We also very much intend, in addition, to compare the results obtained in our version of black hole information problem and the underlying reasoning process with that in the context of JT gravity [11].

This article is organized as follows.

In section 1 we review the results presented in several seminal papers, especially [11], of solving the black hole information problem for eternal black holes in the simplest but physically rich 2D dilaton gravity theory — Jackiw-Teitelbolm gravity, or JT gravity in short, coupled to CFT matter. The way the authors in [11] investigated the role of replica wormhole played in leading to the island formula for the computation of the fine-grained gravitational entropy greatly inspired us in our own investigation into the case of generic two-dimensional dilaton gravity theory.

Section 2 is dedicated to setting the stage for our new version of black hole information paradox. In section 2.1 we review generic two-dimensional dilaton gravity theories arising from dimensional reduction of higher-dimensional gravity theories by writing down the bare action with an on-shell UV divergence and the metric and dilaton solutions bearing IR singularity. We then show the theory can be regulated via a black hole horizon leading to a black hole solution of the same theory and of non-zero Hawking temperature. We also show there is a renormalized action made up of the bulk and boundary terms where all the UV divergence of the action cancel out. We then, in section 2.2, proceed to maximally extend it to a black hole with two asymptotic regions connected by a wormhole and possessing spacelike singularities in the black hole and white hole regions cloaked by the event horizon. In order to let the black hole radiate away Hawking quanta, in section 2.3, we join the maximal extension of the black hole by two copies of Minkowski half-space, one on each side, and then couple it to a CFT with  $c \gg 1$  to build an eternal black hole assuming a global vacuum state, mimicking the setup in [10] and [11]. Of course, the boundary conditions at the physical boundary gluing the black hole spacetime and Minkowski space change accordingly, differing from those in JT gravity (see section 2 of [5] or [13]), and impose new kinematic constraint on the dynamics of 'boundary particle' [28] [29] [30] by identifying the Poincaré time with the boundary proper time to the leading order. This new kinematic constraint proves to be consistent with an eternal black hole solution in Narayan theory.

In section 3 we formulate our version of black hole information paradox and show it can be resolved after a calculation similar to that in [10] and [11]. In section 3.1, we begin with QES calculation in the simple case of a single interval tethered to the boundary of eternal black hole on one end, which constitutes a warm-up for the more physically relevant case of two-intervals in the eternal black hole in section 3.2 where our version of black hole information paradox is formally presented. We show, at late times, the island always exists and dominates the entropy in two-interval scenario, just like in JT gravity [10] [11], producing large non-perturbative corrections to the true entropy of the Hawking radiation and the two-side eternal black hole according to the QES prescription.

In section 4, we attempt to do the replica calculations in Euclidean signature in an effort to reproduce the results we obtained in our version of black hole information paradox. For the one-interval scenario built in section 3.1, we propose a central hypothesis about the on-shell replica geometry on the covering manifold, inspired by its parallel in JT gravity. Based on this hypothesis, we construct the on-shell action on the orbifold where conical singularities and twist fields are present. We then try to recover the extremity condition for the one-interval scenario by deriving the equation of motion for the boundary mode. Although this hypothesis proves not to account for the whole story and is thus incomplete, there is still much to discuss and the method employed in this subsection to sidestep conical singularities in the calculation of the bulk action can be easily generalized/modified to be used in other theories.

In section 5, we conclude the results we obtained and then share some thoughts about how we can fully recover the results we obtained in section 3 by accounting for the complete on-shell replica geometry. We also compare our version of the black hole information paradox and the JT version and discuss the underlying reason and list some possibilities of further research.

# 1 The black hole information problem in JT gravity

In this section we first review the formulation of the black hole information paradox in JT gravity in the specific setup of eternal black holes, in Lorentzian signature. After that, we proceed to present calculations reproducing the unitary answer to the information paradox following a direct, bulk derivation in Euclidean signature, rather than resorting to holography. We start with a review of an extension of the general replica trick for computing the von Neumann entropy of a system coupled with gravity and its application to JT gravity. We then continue to define the "conformal welding problem" in the most general form, which will characterize joining the replica geometry to the flat space in Euclidean signature. Having set up the basic formalism, we discuss the computation of the entropy for a single interval that contains degrees of freedom living at  $AdS$  boundary, standing for the entropy of a one-side eternal black hole in JT gravity, following [11]. The quantum extremal surface lies slightly outside the event horizon in this case. We will show, in the intermediate computation of the Renyi entropy of a single interval on the way to compute its true quantum gravity von Neumann entropy, the problem can be reduced to an integro-differential equation for a single function  $\theta(\tau)$ , namely the boundary mode, that relates the physical time  $\tau$  to the  $AdS$  time  $\theta$ . This equation is then solved for  $n \rightarrow 1$ , recovering the quantum extremal surface result. As for the two-interval case most relevant to the information problem for the eternal black hole in JT gravity, we invite the reader to refer to the introduction and section 5 of the paper [11] for succinct but clear discussion. This review section is mainly based on the paper [11].

## 1.1 Jackiw-Teitelboim gravity theory plus a CFT

In this subsection we review the two-dimensional Jackiw-Teitelboim or JT gravity, coupled to conformal matter and one specific setup of black hole information problem in JT gravity — eternal black hole coupled to an external bath system which was formulated in [5].

We begin with the Lorentzian action of JT gravity coupled with matter CFT,

$$I_{\text{JT+CFT}}[g_{\mu\nu}, \phi, \chi] = I_{\text{JT}}[g_{\mu\nu}, \phi] + I_{\text{CFT}}[g_{\mu\nu}, \chi] \quad (1.1)$$

where (in Lorentzian signature)

$$I_{\text{JT}}[g_{\mu\nu}, \phi] = \frac{\phi_0}{16\pi G_2} \left( \int_{M_2} R + 2 \int_{\partial M_2} K \right) + \frac{1}{16\pi G_2} \left[ \int_{M_2} \phi(R+2) + 2\phi_b \int_{\partial M_2} (K-1) \right]. \quad (1.2)$$

where we set  $l_{\text{AdS}} = 1$ .

It is notable that the dynamics of JT gravity are especially simple. The gravitational action can be thought of as the dimensional reduction of a higher dimensional theory describing the s-wave sector of the near horizon limit of near extremal black holes.[28] [29] [31] [32] The area of the transverse space in the higher dimensional theory becomes the dilaton  $\phi_0 + \phi$  in this picture, which implies the constant  $\phi_0 \gg \phi$ . The first term involving  $\phi_0$  in the JT action is a purely topological term and provides the extremal entropy of the black hole  $\phi_0/16\pi G_2$ . Apart from this pure topological term, the dynamics are governed by the second term in the JT action, which is not topological since  $\phi$  is dynamical. This dynamical term is easily solved by integrating out the dilaton along an imaginary contour which imposes the constraint on the spacetime to have a constant negative curvature via the delta function

$$\delta(R+2) \quad (1.3)$$

which requires that the two-dimensional metric is locally  $AdS_2$ . Alternatively, we can argue that, by varying the action with respect to  $\phi$ , we get the equation  $R+2=0$  which dictates the metric to be locally  $AdS_2$ . In the setting of eternal black hole that we are interested in, this locally  $AdS_2$  spacetime in the bulk is then joined with two non-gravitating Minkowski half spaces, one on each side as drawn in Figure 1. The matter CFT living on the two non-gravitating flat spaces is the same as that on the locally  $AdS$  spacetime in the bulk, and since the CFT living on both sides of the  $AdS_2$  boundary is the same, we impose transparent boundary conditions for the CFT at the boundary [5] [11], see Figure 2. We also require the matter CFT action to be independent of the dilaton  $\phi$  (namely, a minimally coupled dilaton  $\phi_0$ ). In Poincaré coordinates, the locally  $AdS_2$  metric in the interior region is

$$ds_{\text{in}}^2 = \frac{-dt^2 + dz^2}{z^2} = -\frac{4dx^+ dx^-}{(x^+ - x^-)^2}, \quad x^\pm = t \pm z, \quad (1.4)$$

and the flat metric in the two flat half spaces in the natural Minkowskian coordinates is

$$ds_{\text{out}}^2 = \frac{1}{\epsilon^2} (-du^2 + d\sigma^2) = -\frac{1}{\epsilon^2} dy^+ dy^-, \quad y^\pm = u \pm \sigma. \quad (1.5)$$

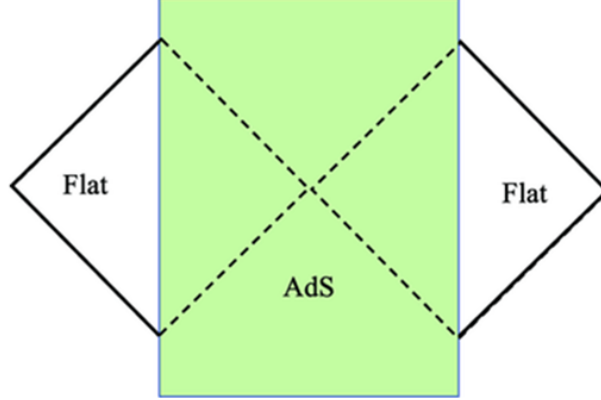


Figure 1: Eternal black hole glued to flat half-spaces, one on each side, in Lorentzian signature. This figure is adapted from [33].

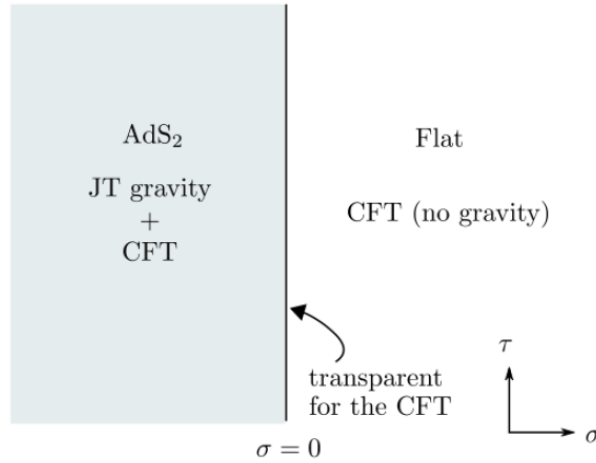


Figure 2: We consider nearly- $AdS_2$  gravity with a matter CFT. The same CFT lives in an exterior flat space with no gravity. We have transparent boundary conditions for the CFT. This figure is adapted from [11].

The two spacetimes are glued together near the  $AdS_2$  boundary according to a map  $t(u)$  that will be dynamically determined. Given  $t(u)$ , the gluing identifies the curve  $y^+ = y^- = u$  in the center of Minkowski spacetimes to the curve

$$x^+ = t(u - \epsilon) \approx t(u) - \epsilon t'(u), \quad x^- = t(u + \epsilon) \approx t(u) + \epsilon t'(u) \quad (1.6)$$

in  $AdS_2$  which we call the "physical boundary". In the limit  $\epsilon \rightarrow 0$  the gluing is simply along  $x^\pm = t(u)$ . At the interface, the boundary conditions for the metric and the dilaton are

$$g_{uu}|_{\text{bd}} = -\frac{1}{\epsilon^2}, \quad \phi|_{\text{bd}} = \phi_b = \frac{\phi_r}{\epsilon}, \quad (1.7)$$

where  $g_{uu}$  is the time-time component of the metric near the boundary along the physical boundary time  $u$ . The second equation in (1.7) indicates that we are interested in large  $\phi_b$  ( $\phi_b \sim 1/\epsilon$ ) with fixed constant coefficient  $\phi_r$  at the boundary, so it is natural to require that the dynamics of the dilaton  $\phi$  to freeze outside the physical boundary, so we set  $\phi = 0$  in the flat baths. A priori, the gluing map  $t(u)$  is defined only at the interface, so it is a real function of a real variable, and the  $y^\pm$  coordinates are only defined in two flat half spaces. But we can extend the  $y^\pm$  coordinate system into the  $AdS_2$  bulk by the coordinate change

$$x^+ = t(y^+), \quad x^- = t(y^-). \quad (1.8)$$

We can immediately tell from the original definition of the physical boundary (1.6), the coordinate transformation (1.8) and  $x^\pm = t \pm z$ ,  $y^\pm = u \pm \sigma$  that the physical boundary can also be identified with the curves parametrized by

$$\sigma = -\epsilon \quad (1.9)$$

and that

$$z = -\epsilon t'(u) + o(\epsilon^2) \quad (1.10)$$

at the physical boundary.

Varying the action with respect to the metric  $g$  yields the constraints and equations of motion that couple the bulk CFT to the dilaton on  $AdS_2$  background:

$$2\partial_{x^+}\partial_{x^-}\phi + \frac{4}{(x^+ - x^-)^2}\phi = 16\pi G_2 T_{x^+x^-}, \quad (1.11a)$$

$$-\frac{1}{(x^+ - x^-)^2}\partial_{x^\pm} \left( (x^+ - x^-)^2 \partial_{x^\pm} \phi \right) = 8\pi G_2 T_{x^\pm x^\pm}, \quad (1.11b)$$

or in covariant form

$$\nabla_a \nabla_b \phi - g_{ab} \nabla^c \nabla_c \phi + \phi g_{ab} = -8\pi G_2 T_{ab}. \quad (1.12)$$

where  $T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} I_{\text{CFT}}$ . We assume the central charge of the matter CFT satisfy  $c \gg 1$  so that we are working in the limit where the gravitational sector can be treated semi-classically, so the stress tensors  $T_{ab}$  should be identified with their expectation values  $\langle T_{ab} \rangle$ . Also, the trace  $T_{x^+x^-}$  is set by the Weyl anomaly and can be absorbed into the definition of  $\phi_0$ , so we will adopt this convention and drop it from the first equation in (1.11).

Then, the general solution to the equations (1.11) is [4]

$$\begin{aligned} \phi(x^+, x^-) = & \\ & -\frac{2\pi\phi_r}{\beta} \frac{x^+ + x^-}{x^+ - x^-} - \frac{8\pi G_2}{x^+ - x^-} \int_0^{x^-} dx (x^+ - x)(x^- - x) T_{x^-x^-}(x) + \frac{8\pi G_2}{x^+ - x^-} \int_0^{x^+} dx (x^+ - x)(x^- - x) T_{x^+x^+}(x) \end{aligned} \quad (1.13)$$

up to  $SL(2)$  transformations.

As argued in [28], it is convenient to express the dynamic of JT gravity in terms of the shape of the boundary curve, or gluing map,  $t(u)$ . This is simply a reparametrization between the bulk Poincaré time  $t$  and the physical boundary time  $u$ . The dynamics of  $t(u)$  are governed by the Schwarzian action. After setting  $R = -2$ , the JT action is a pure boundary term that evaluates to

$$I_{\text{JT}} = \frac{\phi_b}{8\pi G_2} \int_{\partial M_2} (K - 1) + \text{topological} = -\frac{\phi_r}{8\pi G_2} \int_{\partial M_2} du \{t(u), u\} + \text{topological}. \quad (1.14)$$

where the extrinsic curvature  $K$  at the physical boundary  $\sigma = -\epsilon$  is

$$K = \frac{t'(t'^2 - z'^2 - zz'') + zz't''}{(t'^2 - z'^2)^{\frac{3}{2}}} \stackrel{z \approx -\epsilon t' \text{ in (1.10)}}{\approx} 1 - \epsilon^2 \left[ \frac{t'''}{t'} - \frac{3}{2} \left( \frac{t''}{t'} \right)^2 \right] = 1 - \epsilon^2 \{t(u), u\}. \quad (1.15)$$

This is the Schwarzian action, which is invariant under  $SL_2(\mathbb{R})$  transformations of the gluing map — the trajectory of the 'boundary particle' —  $t(u)$ , as required by the isometries of the rigid  $AdS_2$  spacetime.

The ADM energy of the gravity region, defined as the Noether charge under physical time translations  $u \rightarrow u + \delta u$ , is given by

$$M(u) = -\frac{\phi_r}{8\pi G_2} \{t(u), u\}. \quad (1.16)$$

Conservation of energy relates  $dM/dt$  to the net flux of energy across the interface,

$$\frac{dM}{dt} = -\frac{d}{dt} \left( \frac{\phi_r}{8\pi G_2} \{t(u), u\} \right) = T_{y^+y^+} - T_{y^-y^-}. \quad (1.17)$$



This is the equation of motion for the gluing map  $t(u)$ .

In the eternal black hole case in question here, we demand the black hole have inverse temperature  $\beta$  and be in equilibrium with the baths at the same temperature on both sides, so the net flux across the  $AdS_2$  boundary is zero, and the Schwarzian equation of motion is

$$\partial_u \{t(u), u\} = 0. \quad (1.18)$$

The solution corresponding to inverse temperature  $\beta$  is

$$t = \frac{\beta}{\pi} \tanh \frac{\pi}{\beta} u. \quad (1.19)$$

We can also act on this with an  $SL(2)$  transformation, which corresponds to a different coordinate system in  $AdS_2$ .<sup>1</sup>

In the case of eternal black hole, we have in the bulk  $AdS_2$  spacetime, according to (1.8),

$$x^\pm = \frac{\beta}{\pi} \tanh \frac{\pi}{\beta} y^\pm. \quad (1.20)$$

Plugging the above equations into (1.4), we get the following physical metric inside the physical boundary

$$ds_{\text{in}}^2 = -\frac{4\pi^2}{\beta^2} \frac{dy^+ dy^-}{\sinh \frac{\pi}{\beta} (y^+ - y^-)^2}. \quad (1.21)$$

We further choose the state of the matter sector to be a global vacuum state — the Hartle-Hawking state where the stress-energy tensor of the matter CFT vanishes throughout the bulk  $AdS_2$  spacetime.<sup>2</sup> Therefore, according to (1.13), we have the following dilaton profile inside the physical boundary

$$\phi = \phi_r \frac{2\pi}{\beta} \coth \frac{\pi}{\beta} (y^+ - y^-). \quad (1.22)$$

Since we imposed the transparent condition for the matter CFT at the boundary, the stress tensor in the flat baths in the Hartle-Hawking state is given entirely by the Weyl anomaly assuming the bulk spacetime were "flat"

$$T_{y^\pm y^\pm, \text{bath}} = -\frac{c}{24\pi} \{x^\pm(y^\pm), y^\pm\} = \frac{\pi c}{12\beta^2}. \quad (1.23)$$

This means zero flux at the physical boundary, consistent with Schwarzian equation of motion (1.18) for eternal black holes.<sup>3</sup> Given the gluing map (1.19) for an eternal black hole of inverse temperature  $\beta$ , the constant ADM energy is given by

$$M = -\frac{\phi_r}{8\pi G_2} \{t(u), u\} = \frac{\phi_r}{4\pi G_2} \frac{\pi}{\beta^2}. \quad (1.24)$$

In Euclidean signature, the JT action becomes

$$I_{\text{JT}}[g_{\mu\nu}, \phi] = -\frac{\phi_0}{16\pi G_2} \left( \int_{M_2} R + 2 \int_{\partial M_2} K \right) - \frac{1}{16\pi G_2} \left[ \int_{M_2} \phi(R+2) + 2\phi_b \int_{\partial M_2} (K-1) \right]. \quad (1.25)$$

With regards to the Euclidean eternal black hole, see Figure 3, one can do a Wick rotation  $u \rightarrow i\tau$  and an analytical continuation by defining the Euclidean counterpart of coordinates  $\{y^+, y^-\} \rightarrow \{y, \bar{y}\}$ :

$$y = \sigma + i\tau, \quad \bar{y} = \sigma - i\tau, \quad (1.26)$$

thus the metric of an Euclidean eternal black hole of inverse temperature  $\beta$ , glued to flat space on both sides of the physical boundary  $\sigma = -\epsilon$ , is

<sup>1</sup>For example, another solution to (1.18) is  $t(u) = e^{2\pi u/\beta}$ .

<sup>2</sup>For the case conformal matter we are considering, the trace of the stress tensor is a constant determined by the Weyl anomaly  $T_\mu^\mu = \frac{c}{24\pi} R$ , which can be absorbed into the extremal value of the dilaton:  $\phi_0 \rightarrow \phi_0^{\text{Ren}} = \phi_0 + \frac{cG_2}{3}$ .

<sup>3</sup>Indeed, one can define global coordinates  $w^\pm$  covering both the left and the right baths, the entire portion of the bulk outside the event horizons and some part in the bulk inside the event horizon such that the vanishing physical stress tensor in the bulk  $AdS_2$  and the physical stress tensor in the baths (1.23) lead to vanishing unphysical stress tensor  $T_{w^\pm w^\pm}$  throughout the coordinate chart of  $w^\pm$  where the metric is  $ds^2 = dw^+ dw^-$  by Weyl anomaly, giving a global Hartle-Hawking state for the CFT in the spacetime which is flat in  $w^\pm$  coordinates. See Lecture 1 of the TASI 2021 course *Black Hole Information Paradox* given by Ahmed Almheiri: <https://sites.google.com/colorado.edu/tasi-2021-wiki/lecture-topics/black-hole-information-paradox> and [https://youtu.be/EfEqd1GG\\_5Q](https://youtu.be/EfEqd1GG_5Q).

$$ds_{\text{in}}^2 = \frac{4\pi^2}{\beta^2} \frac{dyd\bar{y}}{\sinh^2 \frac{\pi}{\beta}(y + \bar{y})}, \quad ds_{\text{out}}^2 = \frac{1}{\epsilon^2} dyd\bar{y}, \quad \tau \sim \tau + \beta.^4 \quad (1.27)$$

and the dilaton profile inside the boundary in Euclidean signature is

$$\phi = \frac{2\pi\phi_r}{\beta} \coth \frac{\pi}{\beta}(y + \bar{y}). \quad (1.28)$$

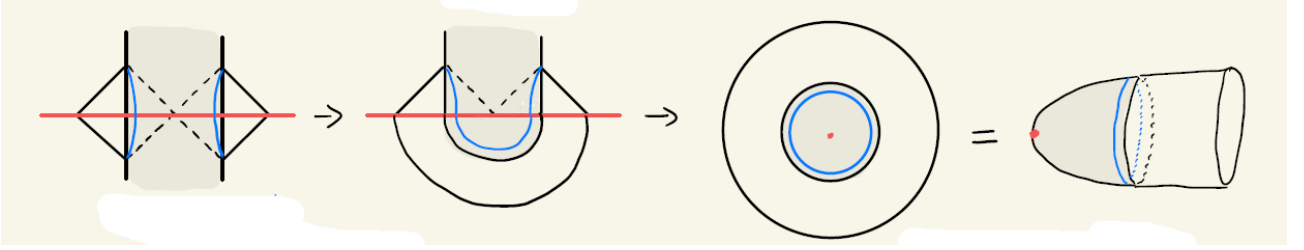


Figure 3: By doing the Wick rotation and analytically continuing the eternal black hole metric on both sides of the physical boundary (denoted as blue curves), (1.21) and (1.5), one gets an Euclidean eternal black hole joined to baths on the right of this figure. On the farthest right, the Euclidean eternal black hole, just like the Euclidean Schwarzschild black hole, can also be depicted as the famous "Euclidean cigar" but with a physical boundary (blue circle) interpolating the Euclidean  $AdS_2$  space outside the event horizon (grey area) in the interior and the Euclidean flat space (white area) in the exterior. Note that the event horizon in the Euclidean black hole is denoted by a single red dot at the tip of the "cigar". This figure is adapted from the lecture notes of the TASI 2021 course *Black Hole Information Paradox* given by Ahmed Almheiri.

After a similar derivation leading to (1.15) and (1.14) but now in Euclidean signature<sup>5</sup>, one can easily get the on-shell Euclidean JT action

$$I_{\text{JT, eternal}} = -\frac{1}{4G_2} \left( \phi_0 + \frac{\pi}{\beta} \phi_r \right), \quad (1.29)$$

which in turn allows us to calculate the Bekenstein-Hawking entropy of an eternal black hole

$$\begin{aligned} S_{\text{BH}} &= (1 - \beta\partial_\beta) \log Z \\ &\approx -(1 - \beta\partial_\beta) I_{\text{JT, eternal}} = \frac{1}{4G_2} \left( \phi_0 + \frac{2\pi}{\beta} \phi_r \right) = \frac{1}{4G_2} (\phi_0 + \phi|_{\text{horizon}}). \end{aligned} \quad (1.30)$$

We can see from (1.30) that the area of the event horizon in two-dimensional JT gravity is represented by the value of the dilaton  $\phi_0 + \phi$  at the event horizon.

## 1.2 The replica trick produces the generalized entropy and the "conformal welding problem" for replica geometry on the orbifold

In this section we review the ideas in [16] [18] [22] [34] for proving the holographic formula for the von Neumann entropy and generalize them to the case the system in question is coupled with gravity. We also show that the generalized entropy appears when we evaluate the  $n \sim 1$  total action based on simple assumptions and how it gives the von Neumann entropy of a subregion within a gravitationally non-dynamic region.

Suppose there is a subregion  $R$  inside a region where gravity is non-dynamic and one can associate a density matrix  $\rho_R$ , which we assume can be expressed as an Euclidean path integral, with the quantum state of  $R$  in full quantum gravity. Then, the von Neumann or fine-grained entropy of the subregion  $R$  can be calculated by evaluating and analytically continuing its  $n$ th-Renyi entropy to  $n \rightarrow 1$

$$S[\rho_R] = \lim_{n \rightarrow 1} \frac{1}{1-n} \text{Tr}(\hat{\rho}_R^n) = \lim_{n \rightarrow 1} \frac{1}{1-n} \frac{Z_n}{Z_1^n} = -\frac{\partial}{\partial n} \left( \frac{\log Z_n}{n} \right) \Bigg|_{n=1} = \lim_{n \rightarrow 1} \frac{1}{1-n} \delta \left( \frac{\log Z_n}{n} \right) \quad (1.31)$$

<sup>4</sup>This periodicity relation of the Euclidean time  $\tau$  can be verified by expanding the Euclidean eternal black hole metric  $ds_{\text{in}}^2$  near the event horizon in near-horizon coordinates  $\{\tau, r\}$ , where  $r = \log \tanh \frac{\pi\sigma}{\beta}$ , and demanding there be no conical singularity at the event horizon.

<sup>5</sup>One can also define Euclidean Poincaré coordinates with  $t \rightarrow i\theta$ :  $x = z + i\theta$ ,  $\bar{x} = z - i\theta$ . The Euclidean  $AdS_2$  metric is  $ds_{\text{in}}^2 = 4dx d\bar{x} / (x + \bar{x})^2$ . The relation between  $x$  and  $y$  takes the same form as in Lorentzian signature:  $x = \frac{\beta}{\pi} \tanh \frac{\pi y}{\beta}$ , which means  $\theta = -i \frac{\beta}{\pi} \tanh \frac{i\pi\tau}{\beta}$  at the physical boundary. The extrinsic curvature  $K$  at the physical boundary is  $K = [\theta'(\theta'^2 + z'^2 + zz'') - zz'\theta''] / (\theta'^2 + z'^2)^{\frac{3}{2}} \frac{z = -\epsilon\theta'}{z = -\epsilon\theta'} 1 + \epsilon^2 \{\theta(\tau), \tau\} = 1 + \frac{2\pi^2 \epsilon^2}{\beta^2}$ . Plugging it into (1.25), one gets (1.29).

where  $\hat{\rho}_R = \frac{\rho_R}{\text{Tr}[\rho_R]}$  is the normalized state of  $R$  and  $Z_n = Z[\tilde{\mathcal{M}}_n] = \text{Tr}(\hat{\rho}_R^n)$  can be viewed as the partition function of the theory on a topologically non-trivial manifold  $\tilde{\mathcal{M}}_n$  which is the product of cyclically identifying  $n$  copies of the original theory along the subregion  $R$ , see Figure 4 for a typical example.

The covering manifold (this name will be justified shortly)  $\tilde{\mathcal{M}}_n$  involved in the calculation of the  $n$ th-Renyi entropy has its geometry totally fixed in the non-gravitational region that the subregion in question -  $R$  is a part of. Meanwhile, since it is gravitational path integral involved in the evaluation of  $Z_n$ , we have to consider any geometry with any topology that obeys the appropriate boundary conditions characterizing the fundamental constructs of our model. Therefore, by assuming the gravitational sector of the theory being characterized the metric  $g$  and the dilaton  $\phi$  and adding minimally coupled matter now collectively denoted by  $\chi$ ,  $Z_n$  can be expressed as

$$\begin{aligned} Z_n &= \int \mathcal{D}g \mathcal{D}\phi \mathcal{D}\chi e^{-I_{\text{total}}[\tilde{\mathcal{M}}_n; g, \phi, \chi]} = \int \mathcal{D}g \mathcal{D}\phi \mathcal{D}\chi e^{-(I_{\text{grav}}[\tilde{\mathcal{M}}_n; g, \phi] + I_{\text{mat}}[\tilde{\mathcal{M}}_n; g, \chi])} \\ &= \int \mathcal{D}g \mathcal{D}\phi e^{-I_{\text{grav}}[\tilde{\mathcal{M}}_n; g, \phi]} \int \mathcal{D}\chi e^{-I_{\text{mat}}[\tilde{\mathcal{M}}_n; g, \chi]} = \int \mathcal{D}g \mathcal{D}\phi e^{-I_{\text{grav}}[\tilde{\mathcal{M}}_n; g, \phi]} Z_{\text{mat}}[\tilde{\mathcal{M}}_n; g] \quad (1.32) \\ &= \sum_{i, \text{saddles}} e^{-(I_{\text{grav}}[\tilde{\mathcal{M}}_n; g_i, \phi_i] - \log Z_{\text{mat}}[\tilde{\mathcal{M}}_n; g_i])} \end{aligned}$$

where it is presupposed that the entanglement entropy of matter is large for kinematic reasons such as a CFT with a large central charge  $c \gg 1$ , and thus the contribution to the total stress-energy tensor from the gravitons can be ignored. Subsequently, we evaluate the integral over geometries on semi-classical saddle points, namely classical solutions of metric and dilaton plus quantum fluctuations of matter QFT on the background of those classical geometries excluding the quantum aspects of gravity brought by graviton fluctuations.

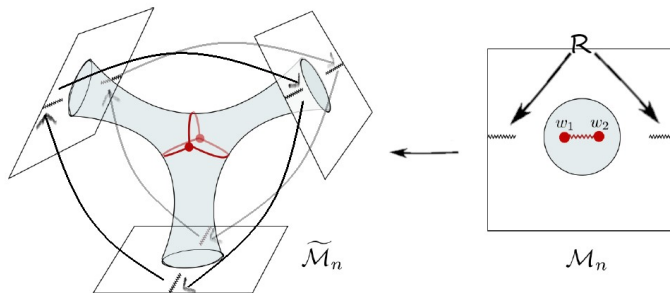


Figure 4: Here are the two representations of the same Euclidean replica geometry.  $\mathcal{M}_n$  on the right is the orbifold of the two-interval scenario in the eternal black hole. The two intervals  $R$  denote the radiation region in the two flat baths. We are interested to calculate  $R$ 's entanglement entropy using replica tricks. The replica/covering manifold  $\tilde{\mathcal{M}}_n$  on the left, here displayed as  $\tilde{\mathcal{M}}_3$ , is the product of cyclically gluing  $n$  copies of  $\mathcal{M}_n$  along  $R$ 's. In the center of  $\tilde{\mathcal{M}}_3$ , there is a replica wormhole geometry straddling glued  $\mathcal{M}_3$ 's, which can also be represented as two fixed points  $w_1$  and  $w_2$  on the ends of a branch cut in a disk at the center of  $\mathcal{M}_3$ . This figure is adapted from Figure 9 of [11].

**Unlike in [11], we do NOT assume that all the off-shell geometries on  $\tilde{\mathcal{M}}_n$  in the above gravitational path integral necessarily have replica  $\mathbb{Z}_n$  symmetry.** In the spirit of [16], for on-shell geometries - the saddle points - we assume replica  $\mathbb{Z}_n$  symmetry extends into the bulk, arising from the Einstein equations and the  $\mathbb{Z}_n$  symmetry of the boundary conditions. Note that there are some off-shell geometries bearing  $\mathbb{Z}_n$  symmetry in the gravitational path integral since any geometry satisfying the boundary conditions is included in the gravitational path integral. The  $\mathbb{Z}_n$  symmetry allows us to consider everything on the orbifold  $\mathcal{M}_n = \tilde{\mathcal{M}}_n/\mathbb{Z}_n$  when it applies. When there is  $\mathbb{Z}_n$  symmetry on  $\tilde{\mathcal{M}}_n$ , there are conical singularities on  $\mathcal{M}_n$  with opening angle  $2\pi/n$  - the fixed points of the  $\mathbb{Z}_n$  action - because on the covering manifold  $\tilde{\mathcal{M}}_n$  the geometry is smooth at these points. We can add codimensional-two cosmic branes of tension

$$T_n = \frac{1}{4G_N} \left(1 - \frac{1}{n}\right) \quad (1.33)$$

to enforce the conical singularities. This permits us to express the gravitational part of the action by

$$\frac{1}{n} I_{\text{grav}}[\tilde{\mathcal{M}}_n] = I_{\text{grav}}[\mathcal{M}_n] + T_n \int_{\Sigma_{d-2}} \sqrt{\gamma} \quad (1.34)$$

where  $\gamma$  is the determinant of the induced transverse metric on the codimensional-2 cosmic brane. Note that in (1.34) we integrate the gravitational Lagrangian across the singularity where there is a  $\delta$ -function for the Ricci curvature scalar and the resulting contribution to the action from the neighborhood around this singularity can generally be evaluated by Gauss-Bonnet theorem. This contribution to the action coming from conical singularity is then cancelled by the cosmic brane term, consistent with the non-singular geometry in the covering manifold  $\tilde{\mathcal{M}}_n$ . Also, the position of cosmic branes in the gravitational region is determined by Einstein equations from the on-shell geometry on  $\tilde{\mathcal{M}}_n$ . They can not get into the non-gravitational region due to diffeomorphism symmetry. We then insert twist operators  $\mathcal{T}_n$  at all the conical singularities in the gravitational region and the end-points of the subregion  $R$  in the non-gravitational region to account for the partition function of the matter sector.

We then need to evaluate  $\delta(\log Z_n/n)$  in (1.31). When  $n = 1$ , we have the original solution to the problem on the manifold  $\mathcal{M}_1 = \tilde{\mathcal{M}}_1$  and thus

$$\log Z_1 = -I_{\text{grav}}[\tilde{\mathcal{M}}_1] + \log Z_{\text{mat}}[\tilde{\mathcal{M}}_1]. \quad (1.35)$$

Meanwhile, when  $n \sim 1$  we have generalized entropy coming out of the total action on any geometry on  $\tilde{\mathcal{M}}_n$  with replica  $\mathbb{Z}_n$  symmetry

$$\begin{aligned} I_{\text{total}}[\tilde{\mathcal{M}}_n] &= I_{\text{grav}}[\tilde{\mathcal{M}}_n] - \log Z_{\text{mat}}[\tilde{\mathcal{M}}_n] \\ &= n \left( I_{\text{grav}}[\mathcal{M}_n] + T_n \int_{\Sigma_{d-2}} \sqrt{\gamma} \right) - (1-n) \cdot \frac{1}{1-n} \log \frac{Z_{\text{mat}}[\tilde{\mathcal{M}}_n]}{Z_{\text{mat}}[\mathcal{M}_n]^n} - n \log Z_{\text{mat}}[\mathcal{M}_n] \\ &= n (I_{\text{grav}}[\mathcal{M}_n] - \log Z_{\text{mat}}[\mathcal{M}_n]) + n T_n \int_{\Sigma_{d-2}} \sqrt{\gamma} + (n-1) S_{\text{mat}}[R; \mathcal{M}_n] \\ &= n (I_{\text{grav}}[\mathcal{M}_1] - \log Z_{\text{mat}}[\mathcal{M}_1]) + n \underbrace{[(I_{\text{grav}}[\mathcal{M}_n] - \log Z_{\text{mat}}[\mathcal{M}_n]) - (I_{\text{grav}}[\mathcal{M}_1] - \log Z_{\text{mat}}[\mathcal{M}_1])]}_{=0 \text{ as } \delta I_{\text{total}} \text{ around the } n=1 \text{ original solution}} \\ &+ \frac{n-1}{4G_N} \sum_{\alpha, \text{Cosmic Branes}} \text{Area}(w_\alpha) + (n-1) S_{\text{mat}}[R; \mathcal{M}_1] + o(n-1)^2 \\ &= (n-1) \cdot \left( \sum_{\alpha, \text{Cosmic Branes}} \text{Area}(w_\alpha) + S_{\text{mat}}[R; \mathcal{M}_1] \right) - n \log Z_1 + o(n-1)^2 \\ &= (n-1) S_{\text{gen}}(w) - n \log Z_1 + o(n-1)^2 \end{aligned} \quad (1.36)$$

where we used (1.34) on the second line and the replica formula for entanglement entropy of matter in a QFT on a fixed geometry on the third line. Note also that on the fifth line of (1.36) we consider the cosmic branes and twist fields as living on the original  $n = 1$  solution on  $\mathcal{M}_1$  since these terms are already of order  $(n-1)$ . Next, we can express  $\log Z_n/n$ ,  $n \sim 1$  as

$$\begin{aligned} \frac{\log Z_n}{n} &= \frac{1}{n} \log \left[ \sum_{i, \text{saddles}} e^{-(I_{\text{grav}}[\tilde{\mathcal{M}}_n; g_i, \phi_i] - \log Z_{\text{mat}}[\tilde{\mathcal{M}}_n; g_i])} \right] \\ &= \frac{1}{n} \log \left[ \sum_{i, \text{saddles}} e^{-(I_{\text{grav}}[\tilde{\mathcal{M}}_n; g_i, \phi_i] - \log Z_{\text{mat}}[\tilde{\mathcal{M}}_n; g_i])} + \int_{\substack{\text{off-shell } g \text{ and } \phi \\ \text{also with replica} \\ \mathbb{Z}_n \text{ symmetry}}} \mathcal{D}g \mathcal{D}\phi e^{-(I_{\text{grav}}[\tilde{\mathcal{M}}_n] - \log Z_{\text{mat}}[\tilde{\mathcal{M}}_n])} \right] \\ &= \frac{1}{n} \log \left[ \sum_{i, \text{saddles}} e^{-[(n-1)S_{\text{gen}}(w_i) - n \log Z_1]} + \int_{\substack{\text{off-shell } g \text{ and } \phi \\ \text{also with replica} \\ \mathbb{Z}_n \text{ symmetry}}} \mathcal{D}g \mathcal{D}\phi e^{-[(n-1)S_{\text{gen}}(w) - n \log Z_1]} \right] \\ &= \frac{1}{n} \log \left( \sum_{i, \text{saddles}} e^{-[(n-1)S_{\text{gen}}(w_i) - n \log Z_1]} \right) \text{ where } \partial_{w_{i_\alpha}} S_{\text{gen}}(w_i) = 0, \forall w_{i_\alpha} \text{ for the } i\text{-th saddle,} \\ &= \frac{1}{n} \log \left( e^{-[(n-1)S_{\text{gen}}(w) - n \log Z_1]} \right) = (1-n)S_{\text{gen}}(w) + \log Z_1 + o(n-1)^2 \end{aligned} \quad (1.37)$$

where

$$S_{\text{gen}}(w) = \min_{i, \text{saddles}} S_{\text{gen}}(w_i). \quad (1.38)$$

We see, on the third line of (1.37), in the process of extremizing the total action on the covering manifold  $\tilde{\mathcal{M}}_n$ , we need to equivalently extremize the generalized entropy over the position of cosmic branes,  $w$  (we must include the off-shell geometries also with the replica symmetry to continuously extremize over  $w$ , but the contribution to the logarithm from these off-shell geometries is zero so they only play an auxiliary role here), and finally identify the solution where  $S_{\text{gen}}$  reaches its global minimum (when  $n \sim 1$ , it really means  $n > 1$ ) as the leading saddle point. Also, note that on the fourth line of (1.37), as already accounted for in (1.36), one saddle in the gravitational path integral can lead to multiple conical singularities on the orbifold. Finally, combining (1.31), (1.35) and (1.37), we have

$$S[\rho_R] = \lim_{n \rightarrow 1} \frac{1}{1-n} \delta \left( \frac{\log Z_n}{n} \right) = S_{\text{gen}}(w), \quad (1.39)$$

which manifestly gives the quantum extremal surface prescription of [14] [15] [19].

We proceed to specify the application of (1.39) to our theory, an eternal  $AdS_2$  black hole coupled to a CFT matter theory with  $c \gg 1$ .

As specified in subsection 1.1, the same CFT theory lives in the bulk  $AdS_2$  where the gravity is dynamic and flat baths where the gravity is non-dynamic, we impose transparent boundary conditions at the physical boundary. We thus have the following Euclidean full action on any smooth geometry

$$I_{\text{total}} = -\frac{\phi_0}{16\pi G_2} \left( \int_{M_2} R + 2 \int_{\partial M_2} K \right) - \frac{1}{16\pi G_2} \left[ \int_{M_2} \phi(R+2) + 2\phi_b \int_{\partial M_2} (K-1) \right] + \log Z_{\text{CFT}}[g]. \quad (1.40)$$

Instead of working on a smooth geometry on the covering manifold  $\tilde{\mathcal{M}}_n$ , as mentioned earlier, we choose to quotient  $\tilde{\mathcal{M}}_n$  by  $Z_n$  and work on the orbifold  $\mathcal{M}_n = \tilde{\mathcal{M}}_n/Z_n$  for our convenience. As a result, the total gravitational action on  $\mathcal{M}_n$  contains extra terms involving cosmic branes

$$\begin{aligned} I_{\text{JT}}[\mathcal{M}_n] &= \frac{1}{n} I_{\text{JT}}[\tilde{\mathcal{M}}_n] \\ &= -\frac{\phi_0}{16\pi G_2} \left( \int_{\mathcal{M}_n} R + 2 \int_{\partial \mathcal{M}_n} K \right) - \frac{1}{16\pi G_2} \left[ \int_{\mathcal{M}_n} \phi(R+2) + 2\phi_b \int_{\partial \mathcal{M}_n} (K-1) \right] \\ &\quad + \frac{1}{4G_2} \left( 1 - \frac{1}{n} \right) \sum_{\alpha, \text{Cosmic Branes}} [\phi_0 + \phi(w_\alpha)], \end{aligned} \quad (1.41)$$

In some senses, the above action can be regarded as a new gravity theory and we then add  $n$  copies of CFT on the orbifold  $\mathcal{M}_n$ . In addition, we place twist fields at the position of the cosmic branes,  $w_\alpha$ . Note that the diffeomorphism invariance of the action  $I_{\text{JT}}[\mathcal{M}_n]$ , (1.41), is not broken because all  $w_\alpha$ 's are dynamical variables which can be anywhere on the manifold and will be fixed by the equations of motion.

We define an interior complex coordinate  $w$  so that the metric on  $\mathcal{M}_n$  in the gravitational region inside the boundary can be written in conformal gauge

$$ds_{\text{in}}^2 = e^{2\rho} dw d\bar{w}, \quad |w| \leq 1 \quad (1.42)$$

where the boundary between the gravitational and non-gravitational region lies at  $|w| = 1$ ,  $w = e^{i\frac{2\pi}{\beta}\theta}$ . There are conical singularities at certain  $w_\alpha$  with an opening angle  $2\pi/n$ . By varying the action (1.41) with respect to the dilaton, the function  $\rho$  in the conformal factor in (1.42) is subject to the following dilaton equation of motion

$$-4\partial_w \partial_{\bar{w}} \rho + e^{2\rho} = 4\pi \left( 1 - \frac{1}{n} \right) \sum_{\alpha} \delta^{(2)}(w - w_\alpha). \quad (1.43)$$

The above equation is coupled with the Einstein equation derived by varying (1.41) with respect the metric

$$\partial_w^2 \phi - 2\partial_w \phi \partial_w \rho = -8\pi G_2 T_{ww}, \quad (1.44)$$

$$\partial_w \partial_{\bar{w}} \phi - \frac{1}{2} e^{2\rho} \phi = 8\pi G_2 T_{w\bar{w}}, \quad (1.45)$$

$$\partial_{\bar{w}}^2 \phi - 2\partial_{\bar{w}} \phi \partial_{\bar{w}} \rho = -8\pi G_2 T_{\bar{w}\bar{w}}. \quad (1.46)$$

Once we impose the above Einstein equations and dilaton EOM, the cosmic brane action terms in (1.41) are cancelled by the contribution from the singular Ricci curvature scalar which is, in turn, caused by the singular part in  $\rho$ .

Now that we have set up the coordinate system in the gravitational region, we need to do the same for the flat space outside the boundary. We consider a finite temperature configuration where  $\tau \sim \tau + \beta$ . We define the coordinate  $v = e^{\frac{2\pi}{\beta}y}$ ,  $y = \sigma + i\tau$  in the flat space such that

$$ds_{\text{out}}^2 = \frac{1}{\epsilon^2} dy d\bar{y}, \quad |v| \geq 1. \quad (1.47)$$

At the boundary  $|w| = |v| = 1$ , we have  $w = e^{i\frac{2\pi}{\beta}\theta(\tau)}$  and  $v = e^{i\frac{2\pi}{\beta}\tau}$ . Much noteworthy here is that, different from the  $n = 1$  original solution where  $\theta(\tau) \equiv \tau$  (as we will see at the end of this subsection), if  $n > 1$ , we generally can **NOT** directly extend the coordinate  $v$  on the outside to a holomorphic coordinate inside the boundary. Nevertheless, we can find another coordinate  $z$  so that there are holomorphic maps from both the interior  $|w| \leq 1$  and the exterior  $|v| \geq 1$  to the complex  $z$  plane, see Figure 5. The problem of finding this coordinate map  $z$  covering the whole orbifold  $\mathcal{M}_n$  can be framed as the ‘‘conformal welding problem’’, following [11]. For given data of  $\theta(\tau)$  at the boundary, we need to find two functions  $G$  and  $F$  such that

$$\begin{aligned} z &= G(w), \quad \text{for } |w| \leq 1 \\ z &= F(v), \quad \text{for } |v| \geq 1 \\ G(e^{i\frac{2\pi}{\beta}\theta(\tau)}) &= F(e^{i\frac{2\pi}{\beta}\tau}), \quad \text{for } |w| = |v| = 1. \end{aligned} \quad (1.48)$$

We require the functions  $G$  and  $F$  are holomorphic in their respective domains but they do not have to be holomorphic at the boundary.  $G$  and  $F$  depend non-locally on  $\theta(\tau)$  at the boundary and, respectively, they map the inside disk  $|w| \leq 1$  and the outside disk  $|v| \geq 1$  to the inside and outside of some irregular region on the complex  $z$  plane.

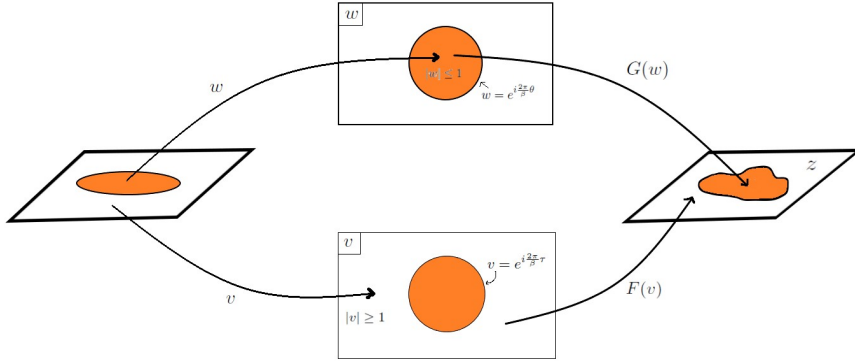


Figure 5: The conformal welding problem involves two disks, one parameterized by  $|w| \leq 1$  and another by  $|v| \geq 1$  with their boundaries glued according to a given function  $\theta(\tau)$  where  $w = e^{i\frac{2\pi}{\beta}\theta}$  and  $v = e^{i\frac{2\pi}{\beta}\tau}$ . What we need to find is holomorphic maps  $G$  from the  $|w| \leq 1$  disk to the inside of and  $F$  from the  $|v| \geq 1$  disk to the outside of some region on the complex  $z$  plane so that  $G$  and  $F$  are compatible at the boundary. It is not required that the  $G$  and  $F$  are holomorphic at the boundary, however.

Of course, when  $n = 1$ , the conformal welding problem of Figure 5 is trivial and we have

$$z = v = w = e^{2\pi y/\beta}, \quad y = \frac{\beta}{2\pi} \log w \quad (1.49)$$

The Euclidean solution is therefore the  $w$ -plane with gravity inside the unit disk,  $|w| < 1 - \frac{2\pi\epsilon}{\beta}$ . The metric is

$$ds_{\text{in}}^2 = \frac{4dw d\bar{w}}{(1 - |w|^2)^2}, \quad ds_{\text{out}}^2 = \frac{\beta^2}{4\pi^2\epsilon^2} \frac{dw d\bar{w}}{|w|^2}. \quad (1.50)$$

following (1.27).

The dilaton, which is defined only on the inside region, is rotationally invariant on the  $w$ -plane following (1.28),

$$\phi = \frac{2\pi\phi_r}{\beta} \frac{1 + |w|^2}{1 - |w|^2} = -\frac{2\pi\phi_r}{\beta} \coth \frac{2\pi\sigma}{\beta}, \quad (1.51)$$

with  $\phi = \phi_r/\epsilon$  at the physical boundary. In what follows, we set  $\epsilon = 0$  and rescale the exterior coordinate by  $\epsilon$  so that  $ds_{\text{out}}^2 = dy d\bar{y}$ .

### 1.3 One interval in eternal black hole in JT gravity coupled to CFT at finite temperature

We begin with the simple case of a single interval that contains one of the  $AdS_2$  boundaries, as shown in Figure 6(a). This is the interval  $B \equiv [0, b]$ .

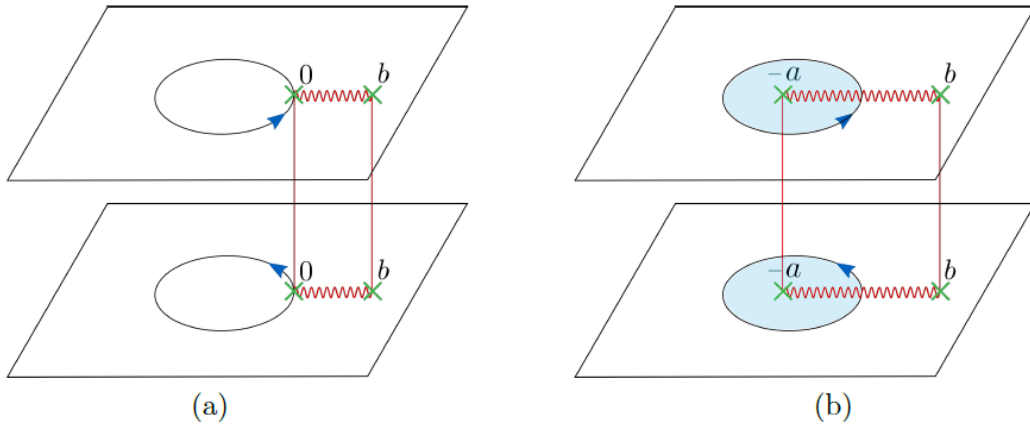


Figure 6: (a) We have a flat space field theory on the exterior of the disk. The disk is hollow in this picture, and will be filled in with gravitational configurations subject to the boundary conditions on the unit circle. This boundary is connected into a single long circle  $n$  times as long as the original one. This is indicated by the blue arrow which tells you how to go around the cut. (b) The disk is filled in with gravitational configurations with the topology of a disk which ends on the elongated unit circle. This configuration can be represented by adding a branch point inside. Note that the local geometry at the branch point "-a" is completely smooth.

To compute the entropy of this region we must consider the Euclidean path integral that evaluates the trace of powers of the density matrix  $\text{Tr}[\rho_B^n]$ . This is given by the path integral on  $n$  copies of the theory identified across the region  $B$ , as shown in Figure 6. The crucial point is that the presence of the branch point on the unit circle, which is where the asymptotic AdS boundary lives, elongates this circle by a factor of  $n$ . The Euclidean gravity configurations we must consider are all smooth manifolds with a single boundary that is identified with this elongated AdS boundary.

The simplest configuration to consider will be that with the topology of a disk. All other higher genus manifolds will be subleading since each extra handle will come with a cost of  $e^{-S_0}$ . Filling out the gravity region has the effect of extending the identification across the different sheets into the gravity region, which ends on some point "-a" in Figure 6. The location of the point "-a" will be dynamically determined by the saddle point of the path integral.

We will now construct replica wormholes explicitly for a single interval in the eternal black hole in  $AdS_2$ . The Lorentzian and Euclidean geometries are shown in Figure 7. To begin with, we will first review the result of the QES calculation [10].

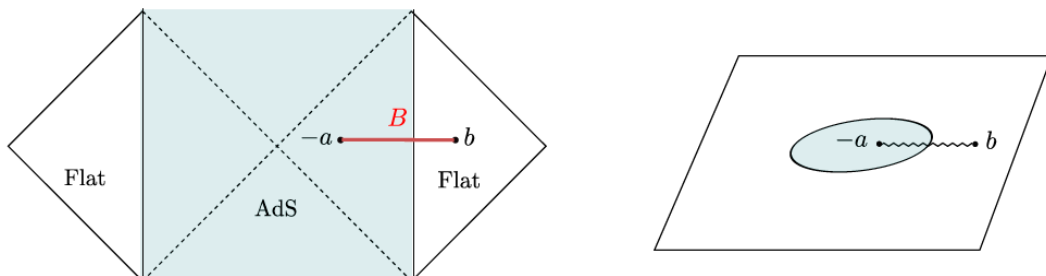


Figure 7: The single interval configuration in Lorentzian signature (left) and in Euclidean signature (right).

The computation of the entropy of the interval  $B \equiv [0, b]$  which includes the  $AdS_2$  boundary involves another interval  $[-a, b]$ , with  $a, b > 0$ , see Figure 7. The generalized entropy of the region  $[-a, b]$  is

$$S_{\text{gen}} = S_0 + \phi(-a) + S_{\text{CFT}}([-a, b]). \quad (1.52)$$

The entanglement entropy of a CFT on the interval  $[w_1, w_2]$  in the metric  $ds^2 = \Omega^{-2}dw d\bar{w}$  is

$$S_{\text{CFT}}([w_1, w_2]) = \frac{c}{6} \log \left( \frac{|w_1 - w_2|^2}{\epsilon_{1,\text{UV}} \epsilon_{2,\text{UV}} \Omega(w_1, \bar{w}_1) \Omega(w_2, \bar{w}_2)} \right). \quad (1.53)$$

Using the map  $w = e^{2\pi y/\beta}$  and the conformal factors in (1.50) this becomes

$$S_{\text{CFT}}([-a, b]) = \frac{c}{6} \log \left( \frac{2\beta \sinh^2 \left( \frac{\pi}{\beta} (a+b) \right)}{\epsilon_{-a,\text{UV}} \epsilon_{b,\text{UV}} \pi \sinh \left( \frac{2\pi a}{\beta} \right)} \right). \quad (1.54)$$

Then, using the dilaton in (1.51), (1.52) becomes

$$S_{\text{gen}}([-a, b]) = S_0 + \frac{2\pi\phi_r}{\beta} \frac{1}{\tanh \left( \frac{2\pi a}{\beta} \right)} + \frac{c}{6} \log \left( \frac{2\beta \sinh^2 \left( \frac{\pi}{\beta} (a+b) \right)}{\pi \sinh \left( \frac{2\pi a}{\beta} \right)} \right). \quad (1.55)$$

The UV divergence  $\epsilon_{-a,\text{UV}}$  was absorbed into  $S_0 = \phi_0/4G_2$  and we dropped the outside one at point  $b$ . The quantum extremal surface is defined by extremizing  $S_{\text{gen}}$  over  $a$

$$\partial_a S_{\text{gen}} = 0 \longrightarrow \sinh \left( \frac{2\pi a}{\beta} \right) = \frac{12\pi\phi_r}{\beta c} \frac{\sinh \left( \frac{\pi}{\beta} (a+b) \right)}{\sinh \left( \frac{\pi}{\beta} (a-b) \right)} \quad (1.56)$$

This is a cubic equation for  $e^{2\pi a/\beta}$  and there is always a single real solution to this cubic equation. For  $b \geq \beta/2\pi$  and  $\phi_r/(\beta c) \geq 1$ , the solution is

$$a \approx b + \frac{\beta}{2\pi} \log \left( \frac{24\pi\phi_r}{\beta c} \right). \quad (1.57)$$

Since we have restricted to one side of the black hole in this calculation, the configuration is invariant under translations in the Schwarzschild  $u$  direction. Therefore, the general extremal surface at  $u \neq 0$  is related by a time translation; for an interval that starts at  $u_b$  and  $\sigma_b = b$ , the other endpoint is at  $u_a = u_b$  and  $\sigma_a = -a$ , with  $a$  as in (1.57).

Next, we start to do the replica calculation in Euclidean signature, with  $a, b$  real. For convenience, we set  $\beta = 2\pi$  and reinforce it later by dimensional analysis. First, we will derive the on-shell JT action on the orbifold  $\mathcal{M}_n$  where there is a branch cut ending conical singularity/cosmic brane at  $-a$  inside the boundary and  $b$  outside the boundary. Then, we proceed to obtain the equation of motion of the boundary mode  $\theta(\tau)$  by doing local variation of the total action on the orbifold  $\mathcal{M}_n$  at the boundary. This equation of motion of boundary mode  $\theta(\tau)$  derived from replica wormhole will finally be reduced to an integro-differential equation for the boundary mode  $\theta(\tau)$ , from which we can extract the QES condition (1.56) for one-interval case in eternal black hole.

The replica manifold  $\tilde{\mathcal{M}}_n$  we seek is an  $n$ -fold cover of the Euclidean black hole shown as an orbifold  $\mathcal{M}_n$ , branched at the points  $a$  and  $b$ , see Figure 7. The orbifold  $\mathcal{M}_n$  will have a nontrivial gluing at the unit circle, unlike the  $n = 1$  eternal black hole itself. Therefore, it is more convenient to introduce different coordinates on the inside and outside on boundary mode  $\mathcal{M}_n$ . We use  $w$ , with  $|w| < 1$ , for the inside and  $v = e^y$ , with  $|v| > 1$  for the outside. The gluing function is  $\theta(\tau)$ , with  $w = e^{i\theta}$ ,  $v = e^{i\tau}$  on the unit circle, as in (1.48). We write the branch points as

$$w = A = e^{-a}, \quad v = B = e^b. \quad (1.58)$$

The Schwarzian equation is simplest in a different coordinate,

$$\tilde{w} = \left( \frac{w - A}{1 - Aw} \right)^{1/n}. \quad (1.59)$$

This coordinate transformation uniformizes  $n$  copies of the unit coordinate disk on  $\mathcal{M}_n$  to the unit coordinate disk inside the elongated boundary  $|\tilde{w}| = 1$  on the replica manifold  $\tilde{\mathcal{M}}_n$  where we need to fill in the leading saddle of  $I_{\text{JT}}[\tilde{\mathcal{M}}_n]$  as the replica geometry, see Figure 6. The leading saddle of the JT action on  $\tilde{\mathcal{M}}_n$  is a geometry with a disk topology and, of course, a constant curvature  $R = -2$ , which means the replica wormhole in question here is the Euclidean  $\text{AdS}_2$ /hyperbolic space with the standard hyperbolic metric

$$ds_{\text{in}}^2 = \frac{4|d\tilde{w}|^2}{(1 - |\tilde{w}|^2)^2}. \quad (1.60)$$



Plugging (1.59) into (1.60), back on the orbifold  $\mathcal{M}_n$  the above replica wormhole metric can be expand near the boundary as

$$ds_{\text{in}}^2 = \frac{4|dw|^2}{(1-|w|^2)^2} \left[ 1 - \frac{2}{3}(1-|w|)^2 U(\theta) + \dots \right], \quad \text{as } |w| \rightarrow 1 \quad (1.61)$$

with

$$U(\theta) = -\frac{1}{2} \left( 1 - \frac{1}{n^2} \right) \frac{(1-A^2)^2}{(e^{i\theta} - A)^2 (e^{-i\theta} - A)^2}, \quad (1.62)$$

which shows small deviation from the original  $n = 1$  standard hyperbolic metric on  $\mathcal{M}_1$  in (1.50).

We now define the spatial coordinate inside the unit disk,  $\gamma < 0$ , by  $w = e^{\gamma+i\theta}$  and expand the metric (1.61) in powers of  $\gamma$  as

$$ds_{\text{in}}^2 = \left[ \frac{1}{\gamma^2} - \frac{1}{3}(1+2U(\theta)) \right] (d\gamma^2 + d\theta^2), \quad \text{as } \gamma \rightarrow 0, \quad (1.63)$$

The boundary condition for the metric (1.7) becomes

$$g_{\tau\tau} = \frac{1}{\epsilon^2} \quad (1.64)$$

now in Euclidean signature, compatible with the flat metric (1.47) outside the boundary. At the physical boundary  $\sigma = -\epsilon$ , we set  $\theta = \theta(\tau)$  and then solve for  $\gamma$  in power series of  $\epsilon$  from (1.63) and (1.64)

$$\gamma = \epsilon\theta' \left\{ 1 + \epsilon^2 \left[ \frac{\theta''^2}{\theta'^2} - \frac{1}{6}(1+2U(\theta))\theta'^2 \right] + o(\epsilon^4) \right\} \quad (1.65)$$

where the prime means the derivative with respect to the physical boundary time  $\tau$ .

We can now compute the tangent vector to the curve  $t^\mu$  and the normal vector  $n^\mu$  and compute the extrinsic curvature at the physical boundary from

$$\begin{aligned} K &= t^\mu t^\nu \nabla_\mu n_\nu = -(t^\tau)^2 \Gamma_{\tau\tau}^\sigma n_\sigma = \frac{1}{2} (t^\tau)^2 n^\sigma \partial_\sigma g_{\tau\tau} \\ &= \frac{1}{2} \epsilon^3 \partial_\sigma \left\{ \left[ \frac{1}{\gamma^2} - \frac{1}{3}(1+2U(\theta)) \right] (\gamma'^2 + \theta'^2) \right\} \\ &= \frac{1}{2} \epsilon^3 \left\{ \left( -\frac{2\theta'}{\gamma^3} + \frac{2}{3} \gamma' \frac{dU(\theta)}{d\theta} \right) (\gamma'^2 + \theta'^2) + \left[ \frac{1}{\gamma^2} - \frac{1}{3}(1+2U(\theta)) \right] (2\gamma'\theta'' - 2\theta'\gamma'') \right\} \\ &= 1 + \epsilon^2 \left[ \{\theta, \tau\} + \left( \frac{1}{2} + U(\theta) \right) \theta'^2 \right] + o(\epsilon^4). \end{aligned} \quad (1.66)$$

Note that, at the physical boundary, the derivatives of  $\theta$  and  $\gamma$  with respect to  $\sigma$  on the second line of (1.66) can be converted to derivatives with respect to  $\tau$ :

$$\frac{\partial\theta}{\partial\sigma} = -\frac{\partial\gamma}{\partial\tau}, \quad \frac{\partial\gamma}{\partial\sigma} = \frac{\partial\theta}{\partial\tau} \quad (1.67)$$

by applying chain rule of derivatives to  $\theta(w(v(\tau, \sigma)), \bar{w}(\bar{v}(\tau, \sigma)))$  and  $\gamma(w(v(\tau, \sigma)), \bar{w}(\bar{v}(\tau, \sigma)))$ , we will meet with a similar situation in (A.2).

Up to the purely topological term, the on-shell JT action on the orbifold  $\mathcal{M}_n$  (1.41) reduces to the extrinsic curvature term

$$I_{\text{JT}}[\mathcal{M}_n] = -\frac{\phi_r}{8\pi G_2 \epsilon} \int_{\partial\mathcal{M}_n} \frac{d\tau}{\epsilon} (K - 1) = -\frac{\phi_r}{8\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \left[ \{\theta(\tau), \tau\} + \left( \frac{1}{2} + U(\theta(\tau)) \right) \theta'^2 \right] + o(\epsilon^2). \quad (1.68)$$

The conserved energy of the system is given by

$$E = \frac{\phi_r}{8\pi G_2} \left[ \{\theta(\tau), \tau\} + \left( \frac{1}{2} + U(\theta(\tau)) \right) \theta'^2 \right]. \quad (1.69)$$

Having derived the on-shell replica JT action, we need to do a local variation of the total action on  $\mathcal{M}_n$ ,  $I_{\text{JT+CFT}}[\mathcal{M}_n]$ , over  $\theta(\tau)$  at/around the physical boundary to obtain the equation of motion for the boundary

mode  $\theta(\tau)$ . Given that  $I_{\text{JT}}[\mathcal{M}_n]$  explicitly involves  $\theta(\tau)$ , we first do a variation  $\theta \rightarrow \hat{\theta} = \theta + \delta\theta$  and the variation in  $I_{\text{JT}}[\mathcal{M}_n]$  is simple

$$\delta I_{\text{JT}}[\mathcal{M}_n] = -\frac{\phi_r}{8\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \frac{[\{\theta(\tau), \tau\} + (\frac{1}{2} + U(\theta(\tau)))\theta'^2]'}{\theta'} \delta\theta. \quad (1.70)$$

Next, we still need to do the variation of the CFT action  $I_{\text{CFT}}[\mathcal{M}_n]$  to obtain the EOM for  $\theta(\tau)$ . Although  $I_{\text{CFT}}[\mathcal{M}_n]$  does not explicitly involve  $\theta(\tau)$ , we will show that a variation over  $\theta(\tau)$  at the physical boundary gives rise to local variation of the metric at the boundary if we demand the metric on both sides always match at the boundary. By the definition  $w = e^{\gamma+i\theta}$ , the same as above (1.63), we have

$$\log w = \gamma + i\theta \quad (1.71)$$

inside and on the boundary  $|w| = 1 \Leftrightarrow \gamma = 0$ . We also have the outside coordinates  $y = \sigma + i\tau$  and we can locally think of the relation between the two in terms of

$$\log w = i\theta(\tau) = i\theta(-iy) \quad (1.72)$$

at the boundary. Now imagine we do a small change  $\theta \rightarrow \hat{\theta} = \theta + \delta\theta$  with  $\delta\theta$  having compact support. This would change the relation between the two sides in (1.72), so we need to define a new coordinate  $\hat{w} = e^{\hat{\gamma}+i\hat{\theta}}$  inside the boundary such that

$$\log \hat{w} = i\hat{\theta}(\tau) = i\hat{\theta}(-iy) \quad (1.73)$$

at the boundary.<sup>6</sup> From the active point of view of infinitesimal transformation, this leads to a change in the metric near and at the boundary

$$\widehat{ds^2}_{\text{in}} = \frac{4|d\hat{w}|^2}{(1-|\hat{w}|^2)^2} \left[ 1 - \frac{2}{3}(1-|\hat{w}|^2)^2 U(\hat{\theta}) + \dots \right], \quad \text{as } |\hat{w}| \rightarrow 1, \quad (1.74)$$

which no longer matches  $\frac{1}{\epsilon^2} dy d\bar{y}$  at the boundary. However, if we instead redefine the outside coordinate by an infinitesimal reparametrization at the boundary,  $\hat{y} = y + \zeta^y$  in such a way that the relation set by  $\theta(\tau)$  is kept fixed, namely

$$\hat{\theta}(-iy) = \theta(-i\hat{y}) \Rightarrow \theta(-iy) + \delta\theta(-iy) = \theta(-i(y + \zeta^y)) = \theta(-iy) - i\zeta^y \theta'(-iy) \quad \rightarrow \quad \zeta^y = i \frac{\delta\theta(-iy)}{\theta'(-iy)} \quad (1.75)$$

and we have the complex conjugate expression for  $\zeta^{\bar{y}}$ . We can then extend this reparametrization in a non-holomorphic way in the region outside the boundary, defining

$$\tilde{\zeta}^y = i \frac{\delta\theta(-iy)}{\theta'(-iy)} h(\sigma), \quad \tilde{\zeta}^{\bar{y}} = -i \frac{\delta\theta(i\bar{y})}{\theta'(i\bar{y})} h(\sigma) \quad (1.76)$$

where  $h(\sigma)$  is one for  $\sigma = 0$  and quickly goes to zero when  $\sigma$  increases. An example is  $h(\sigma) = \theta(\sigma_0 - \sigma)$  for a small  $\sigma_0$ . Then, from the constraints (1.67) that  $\hat{\gamma}$  and  $\hat{\theta}$  are subject to at the boundary (with  $\sigma$  and  $\tau$  replaced by  $\hat{\sigma}$  and  $\hat{\tau}$  if we define  $\hat{y} = \hat{\sigma} + i\hat{\tau}$ ), we also have

$$\hat{\gamma}(-iy) = \gamma(-i\hat{y}), \quad (1.77)$$

the metric (1.74) should match  $\frac{1}{\epsilon^2} d\hat{y} d\hat{\bar{y}}$  at the boundary since the relations among the hatted coordinates,  $\hat{w}$  and  $\hat{y}$ , are unchanged from those among the original hatless coordinates,  $w$  and  $y$ . The new metric at the boundary differs from the original metric by some terms that are localized near the point where we are doing the variation

$$\widehat{ds^2} = \frac{1}{\epsilon^2} d\hat{y} d\hat{\bar{y}} = \frac{1}{\epsilon^2} \left( dy d\bar{y} + 2\partial_{(\alpha} \tilde{\zeta}_{\beta)} dy^\alpha d\bar{y}^\beta \right). \quad (1.78)$$

We see from the above equation that the variation of  $\theta \rightarrow \theta + \delta\theta$  leads to the variation of metric localized at the boundary

$$\delta g_{\alpha\beta} = \frac{1}{\epsilon^2} \cdot 2\partial_{(\alpha} \tilde{\zeta}_{\beta)} \Leftrightarrow \delta g^{\alpha\beta} = -\frac{1}{\epsilon^2} \cdot 2\partial^{(\alpha} \tilde{\zeta}^{\beta)}. \quad (1.79)$$

Using the definition of the stress tensor,  $T_{\alpha\beta} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\alpha\beta}} I_{\text{CFT}}[\mathcal{M}_n]$ , we get

<sup>6</sup> $\hat{\gamma}$  is not necessarily equal to  $\gamma$  since  $\hat{\gamma}$  and  $\hat{\theta}$  have to obey similar constraints (1.67) like  $\gamma$  and  $\theta$  do at the boundary, following the same reasoning.

$$\delta I_{\text{CFT}}[\mathcal{M}_n] = \frac{1}{2} \int d\tau d\sigma (T_{yy} \delta g^{yy} + T_{\bar{y}\bar{y}} \delta g^{\bar{y}\bar{y}}) = -2 \int d\tau d\sigma (T_{yy} \partial_{\bar{y}} \zeta^y + T_{\bar{y}\bar{y}} \partial_y \zeta^{\bar{y}}), \quad (1.80)$$

where we used that the background metric is flat and that the trace of the stress tensor is zero. We now evaluate the derivatives

$$\partial_{\bar{y}} \zeta^y = \frac{i}{2} \frac{\delta\theta(-iy)}{\theta'(-iy)} h'(\sigma), \quad \partial_y \zeta^{\bar{y}} = -\frac{i}{2} \frac{\delta\theta(i\bar{y})}{\theta'(i\bar{y})} h'(\sigma), \quad h'(\sigma) = -\delta(\sigma - \sigma_0). \quad (1.81)$$

Here we used that the arguments of  $\delta\theta$  and  $\theta'$  are holomorphic or anti-holomorphic, so the derivative receives only a contribution from  $h$ , which is just a delta function. Inserting this into (1.80), intergrating over  $\sigma$ , and taking  $\sigma_0 \rightarrow 0$ , we get

$$\delta I_{\text{CFT}}[\mathcal{M}_n] = i \int d\tau (T_{yy} - T_{\bar{y}\bar{y}}) \frac{\delta\theta}{\theta'}. \quad (1.82)$$

Finally, from  $\delta(I_{\text{JT}}[\mathcal{M}_n] + I_{\text{CFT}}[\mathcal{M}_n]) = 0$  and the expression of  $I_{\text{JT}}[\mathcal{M}_n]$  (1.70) and of  $I_{\text{CFT}}[\mathcal{M}_n]$  (1.82), we get the following equation of motion for the boundary mode  $\theta(\tau)$  on the orbifold  $\mathcal{M}_n$

$$\frac{\phi_r}{8\pi G_2} \frac{d}{d\tau} \left[ \{\theta(\tau), \tau\} + \left( \frac{1}{2} + U(\theta(\tau)) \right) \theta'^2 \right] = i(T_{yy} - T_{\bar{y}\bar{y}}) \quad (1.83)$$

where  $U(\theta(\tau))$  takes the form in (1.62).

The stress tensor appearing on the right-hand side of (1.83) is obtained through the conformal welding. Namely, we define the  $z$  coordinate by the map  $G$  on the inside and  $F$  on the outside as in (1.48). These maps each have an ambiguity under  $SL(2, C)$  transformations of  $z$ , which we may use to map the twist operator at  $w = A$  to  $z = 0$ , and the twist operator at  $v = B$  to  $z = \infty$ . The symmetries of the conformal welding problem is further discussed in Appendix B of [11].

The  $z$ -coordinate covers the full plane holomorphically. It has twist points at the origin and at infinity, which can be removed by the standard mapping,  $\tilde{z} = z^{1/n}$ . On the  $\tilde{z}$  plane, the stress tensor vanishes, so on the  $z$ -plane,

$$T_{zz}(z) = -\frac{c}{24\pi} \{z^{\frac{1}{n}}, z\} = -\frac{c}{48\pi} \left( 1 - \frac{1}{n^2} \right) \frac{1}{z^2}. \quad (1.84)$$

Finally, the stress tensor  $T_{yy}$  comes from inverting the conformal welding map to return to the  $v$ -plane, and using  $v = e^y$ :

$$T_{yy}(y) = e^y \left[ F'(v)^2 T_{zz}(F(v)) - \frac{c}{24\pi} \{F, v\} + \frac{c}{48\pi} \right]. \quad (1.85)$$

Putting it all together, the equation of motion (1.83) is

$$\frac{24\pi\phi_r}{c\beta} \partial_\tau \left[ \{e^{i\theta(\tau)}, \tau\} + \frac{1}{2} \left( 1 - \frac{1}{n^2} \right) R(\theta(\tau)) \right] = ie^{2i\tau} \left[ -\frac{1}{2} \left( 1 - \frac{1}{n^2} \right) \frac{F'(e^{i\tau})^2}{F(e^{i\tau})^2} - \{F, e^{i\frac{2\pi}{\beta}\tau}\} \right] + c.c. \quad (1.86)$$

where

$$R(\theta(\tau)) = -\frac{(1 - A^2)^2 \theta'(\tau)^2}{|1 - Ae^{i\theta(\tau)}|^4}. \quad (1.87)$$

This equation originated on the smooth replica manifold  $\tilde{\mathcal{M}}_n$ , but has now been written entirely on the quotient manifold  $\mathcal{M}_n = \tilde{\mathcal{M}}_n/\mathbb{Z}_n$ . We have restored the nontrivial temperature dependence.<sup>7</sup> In particular, note that  $\theta(\tau + 2\pi) = \theta + 2\pi$ . The  $\tau \rightarrow -\tau$  symmetry of the insertions allows us to choose a function  $\theta(-\tau) = -\theta(\tau)$  which will automatically obey  $\theta(0) = 0$ ,  $\theta''(0) = 0$ . In addition, we should then impose  $\theta(\pi) = \pi$  and  $\theta''(\pi) = 0$ . The problem now is such that  $n$  appears as a continuous parameter and there is no difficulty in analytically continuing in  $n$ .

This is our final answer for the equation of motion at finite  $n$ . It is rather complicated because the welding map  $F$  depends implicitly on the boundary mode or gluing function  $\theta(\tau)$ . However, as we will show below, the equation of motion (1.83) reproduces the extremity condition in one-interval case (1.56).

<sup>7</sup>The temperature dependence is restored by  $\tau \rightarrow \frac{2\pi}{\tau} \tau_{\text{phys}}$ , with  $\tau_{\text{phys}}$  the physical Euclidean time with period  $\beta$ .

We start with the solution for  $n = 1$ . In this case, the welding problem is trivial and we can set  $w = v$  everywhere. It is convenient to set

$$z = F(v) = \frac{v - A}{B - v} = G(w), \quad w = v. \quad (1.88)$$

At  $n = 1$  any choice of  $A$  can do. Different choices of  $A$  can be related by an  $SL(2, R)$  transformation acting on  $w$ . It will be convenient for us to choose  $A$  so that when we go to  $n \sim 1$ , it corresponds to the position of the conical singularity.

We now expand in the limit  $n \sim 1$

$$e^{i\theta} = e^{i\tau} + ie^{i\tau}\delta\theta(\tau), \quad (1.89)$$

where  $\delta\theta$  is of order  $n - 1$ . We then proceed to find the welding map perturbatively in  $(n - 1)$ . It is shown in the Appendix B of [11] using Fourier transformation that

$$e^{2i\tau}\{F, e^{i\tau}\} = -\delta\{e^{i\theta}, \tau\}_- = -(\delta\theta''' + \delta\theta')_- \quad (1.90)$$

where the following identity is used

$$\delta\{e^{i\theta}, \tau\} \equiv \{e^{i\tau+i\delta\theta}\} - \{e^{i\tau}, \tau\} = \delta\theta''' + \delta\theta'. \quad (1.91)$$

The minus subscript indicates that this is projected onto negative-frequency modes. This can be written neatly using the Hilbert transform,  $H$ , which is defined by the action  $H \cdot e^{im\tau} = -\text{sgn}(m)e^{im\tau}$  and  $H \cdot 1 = 0$ . Then

$$e^{2i\tau}\{F, e^{i\tau}\} = -\frac{1}{2}(1 + H)(\delta\theta''' + \delta\theta'). \quad (1.92)$$

Wherever else  $F$  appears in (1.83), it is multiplied by  $(n - 1)$ , so there we can set  $F = \frac{v-A}{B-v}$ , as in (1.88). Therefore, the equation of motion for the perturbation is

$$\partial_\tau(\delta\theta''' + \delta\theta') + \frac{ic}{12\phi_r} H \cdot (\delta\theta''' + \delta\theta') = (n - 1) \left[ \frac{c}{12\phi_r} \mathcal{F} - \partial_\tau R(\tau) \right] \quad (1.93)$$

where

$$\mathcal{F} = -i \frac{e^{2i\tau}(A - B)^2}{(e^{i\tau} - A)^2(e^{i\tau} - B)^2} + c.c. \quad (1.94)$$

Equation (1.93) is nonlocal due to the Hilbert transform. We can solve it by expanding both sides in a Fourier series. The important observation is that, due to the structure of derivatives in each term of the left hand side of (1.93), namely

$$\delta\theta''' + \delta\theta' = \sum_{l=-\infty}^{\infty} -il(l^2 - 1)c_l e^{il\tau} \quad (1.95)$$

if we expand  $\theta(\tau)$  in Fourier series

$$\theta(\tau) = \tau + \sum_{l=-\infty}^{\infty} c_l e^{il\tau}. \quad (1.96)$$

(For more details, see Appendix B in [11]). The Fourier modes of the form  $e^{il\tau}$  for  $l = 0, \pm 1$  are automatically zero in the left hand side. Therefore, in order to solve this equation, we must impose the same condition on the right-hand side. The  $k = 1$  mode requires

$$\int_0^{2\pi} d\tau e^{-i\tau} \left( \frac{c}{12\phi_r} \mathcal{F} - \partial_\tau R(\tau) \right) = 0. \quad (1.97)$$

After doing the integrals, we recover the extremity condition in one-interval case (1.56) that came from extremizing the generalized entropy,

$$\frac{c}{6\phi_r} \frac{\sinh \frac{a-b}{2}}{\sinh \frac{a+b}{2}} = \frac{1}{\sinh a} \quad (1.98)$$

up to restoration of the temperature dependence. The term with  $k = 0$  is automatically zero in the right hand side, as  $\partial_\tau \mathcal{R}$  is explicitly a total derivative and  $\int_0^{2\pi} d\tau \mathcal{F} = 0$ .

Thus the QES has been reproduced directly from the equation of motion. Once the QES condition is imposed, it is straightforward to solve for the rest of the Fourier modes of  $\delta\theta$  to confirm that there is indeed a solution.

## 2 Generic dilaton gravity theories in 2D: Narayan theories

The bulk action of the most general two-dimensional dilaton gravity theory (without kinetic terms of the dilaton) takes the form

$$I = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} (\phi \mathcal{R} - U(\phi)), \quad (2.1)$$

including the two-dimensional dilaton  $\phi$  coupled to gravity and a general dilaton potential  $U(\phi)$ . (For the definition of more general dilaton gravity theories in 2D, see [35].) The equations of motion of the above general 2D dilaton gravity theory are

$$g_{\mu\nu} \nabla^2 \phi - \nabla_\mu \nabla_\nu \phi + \frac{1}{2} g_{\mu\nu} U = 0, \quad \mathcal{R} - \frac{\partial U}{\partial \phi} = 0. \quad (2.2)$$

In JT gravity, the potential takes the simple form  $U(\phi) = -2\phi$  in (2.1). The EOMs (2.2) dictate the metric part of the classical solution is a constant negative curvature spacetime where  $\mathcal{R} = -2$ , identified as AdS<sub>2</sub> spacetime, and the on-shell bulk action vanishes. JT gravity, as the simplest 2D dilaton gravity theory of physical relevance, has been under active investigation following discussions of the nearly AdS<sub>2</sub> holography. [4] [28] [29] [30].

It is widely known that AdS<sub>2</sub> arises in the near horizon geometry of extremal black holes and branes [25] [26] [27]. Nonetheless in its own right, upon Kaluza-Klein compactification over the compact space  $X^d$ , 2D dilaton gravity theories arise generically from the 2D subsector of higher dimensional gravity on spaces of the form  $\mathcal{M}_2 \times X^d$ . We now review an interesting example of the dilaton gravity theories produced by dimensional reduction from well-behaved theories in higher dimensions.

### 2.1 Basics of Narayan theory and its regularization

The dimensional reduction of higher dimensional gravity theories with a negative cosmological constant is a prototypical example of arising generic dilaton gravity. The gravity sector of such higher dimensional gravity theories gives the action

$$I_D = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g^{(D)}} (\mathcal{R}^{(D)} - 2\Lambda), \quad (2.3)$$

$$\Lambda = -\frac{1}{2} d_i (d_i + 1), \quad D = d_i + 2.$$

Following [24], we adopt the reduction conventions of [36] [37] [38]. The higher dimensional space takes the form  $\mathcal{M}_2 \times X^{d_i}$  by assuming translational and rotational invariance in the  $d_i$  spatial dimensions on the boundary. If  $X^{d_i}$  is taken to be  $T^{d_i}$  - a torus and is compactified with the following reduction ansatz

$$ds_D^2 = g_{\mu\nu}^{(2)} dx^\mu dx^\nu + \phi^{\frac{2}{d_i}} d\sigma_{d_i}^2; \quad g_{\mu\nu} = \phi^{\frac{d_i-1}{d_i}} g_{\mu\nu}^{(2)}, \quad (2.4)$$

the following 2D dilaton gravity theory or *Narayan theory*, as we call it, with a monomial potential comes out of the reduction,

$$I_{\text{bulk}} = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} (\phi \mathcal{R} - U(\phi)), \quad U = 2l^{-2} \Lambda \phi^{\frac{1}{d_i}} = -l^{-2} d_i (d_i + 1) \phi^{\frac{1}{d_i}}, \quad (2.5)$$

where  $l$  is a constant length scale, similar to the AdS<sub>2</sub> radius in JT gravity. See the appendix of [24] for how  $T^{D-2}$  reduction of the  $D$ -dimensional space with  $D = d_i + 2$  gives (2.5). The  $d_i = D - 2$  spatial coordinates defines the transverse part of the bulk spacetime, we also identify the dilaton defined in (2.4) as the transverse area in  $D$ -dimensional space,  $g_{ii}^{(D-2)/2} = \phi$ . Note that if we set  $d_i = 1$ , AdS<sub>3</sub> leads to  $U = -2l^{-2}\phi$  in (2.5) marking the JT action.

We now present one simple solution to the theory (2.5). We assume the 2D space here is expressed in "Poincaré" coordinates - a time coordinate,  $t$  and a radial coordinate,  $r$  - and the vector field parameterized by  $t$  is a Killing symmetry, then in conformal gauge, the metric ansatz can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{h(r)} (-dt^2 + dr^2) \quad (2.6)$$

and the Einstein equations can be written as

$$-\partial_r^2 \phi + \partial_r h \partial_r \phi = 0, \quad \partial_r^2 \phi + e^h U = 0, \quad -\partial_r^2 h - e^h \partial_\phi U = 0. \quad (2.7)$$

It can be shown ([24]) that the following conformally AdS<sub>2</sub> solution is a time-independent solution to the above Einstein equations

$$ds^2 = \frac{l^{d_i+1}}{r^{d_i+1}}(-dt^2 + dr^2) = \frac{l^{d_i-1}}{r^{d_i-1}} ds_{\text{AdS}_2}^2, \quad \phi = \frac{l^{d_i}}{r^{d_i}}. \quad (2.8)$$

where  $l$  is still a constant length scale. According to (2.2), the curvature is

$$\mathcal{R} = -(d_i + 1) l^{-(d_i+1)} r^{d_i-1}, \quad (2.9)$$

displaying an curvature (IR) singularity in the deep interior as  $r \rightarrow \infty$  for  $d_i > 1$ . Note that if we set  $d_i = 1$ , the metric is exactly AdS<sub>2</sub> with a constant curvature  $\mathcal{R} = -2$ . However, if  $d_i > 1$ , the geometry is only conformally AdS<sub>2</sub>. As one moves towards the boundary  $r = 0$ , the conformal factor and the value of dilaton both blow up while the Ricci curvature scalar  $\mathcal{R} \rightarrow 0$ , showing the space "flattens" near the boundary. Another thing worth much attention is that the on-shell bulk action of the solution (2.8) has a UV-divergence if  $d_i \neq 1$ . The reasoning is follows. Starting from (2.5) and (2.2), we have

$$\begin{aligned} I_{\text{bulk, on-shell}} &= \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left( \phi \frac{\partial U}{\partial \phi} - U(\phi) \right) \\ &= \frac{(d_i^2 - 1)}{16\pi G_2} \int d^2x \sqrt{-g} \phi^{\frac{1}{d_i}} \\ &\propto (d_i^2 - 1) \int dt \int_0 dr \frac{1}{r^{d_i+1}} \cdot \frac{1}{r} = (d_i^2 - 1) \int dt \int_0 dr \frac{1}{r^{d_i+2}}. \end{aligned} \quad (2.10)$$

While in JT case,  $d_i = 1$ , so the on-shell action vanishes due to the same equation (2.10).

The IR curvature singularity deep inside the solution (2.8) can nonetheless be regulated by introducing a black hole horizon as follows,

$$ds^2 = e^{h(r)}(-dt^2 + dr^2) \longrightarrow e^{h(r)} \left( -H(r)dt^2 + \frac{1}{H(r)}dr^2 \right); \quad (2.11)$$

with  $H \rightarrow 1$  as  $r \rightarrow 0$  and  $H \rightarrow 0$  as  $r \rightarrow r_0$ , where  $r = r_0$  is the radius of the event horizon. The Einstein equations now become

$$-\partial_r^2 \phi + \partial_r h \partial_r \phi = 0, \quad \partial_r(H \partial_r \phi) + e^h U = 0, \quad H \partial_r^2 h + \partial_r H \partial_r h + \partial_r^2 H + e^h \frac{\partial U}{\partial \phi} = 0. \quad (2.12)$$

Combining (2.12) and the boundary condition for  $H(r)$  below (2.11), we get the regulated black hole solution to Narayan theory (2.5)

$$ds^2 = \frac{l^{d_i+1}}{r^{d_i+1}} \left( -H(r)dt^2 + \frac{1}{H(r)}dr^2 \right), \quad H(r) = 1 - \frac{r^{d_i+1}}{r_0^{d_i+1}}; \quad \phi = \frac{l^{d_i}}{r^{d_i}}, \quad (2.13)$$

and  $\mathcal{R} = -(d_i + 1) l^{-(d_i+1)} r^{d_i-1}$  still holds. Essentially, this is not the dimensional reduction of global AdS<sub>D</sub> spacetime but the AdS<sub>D</sub> black brane [24].

To obtain the temperature of the black hole solution (2.13), we do a Wick rotation  $t \rightarrow i\theta$ ,

$$ds_E^2 = \frac{l^{d_i+1}}{r^{d_i+1}} \left( H(r)d\theta^2 + \frac{1}{H(r)}dr^2 \right), \quad H(r) = 1 - \frac{r^{d_i+1}}{r_0^{d_i+1}}. \quad (2.14)$$

We then expand the Euclidean black hole metric (2.14) at the horizon  $r = r_0$

$$ds_{\text{near horizon}}^2 = -l^{d_i+1} \left[ \frac{(d_i + 1)(r - r_0)}{r_0^{d_i+2}} d\theta^2 + \frac{1}{(d_i + 1)r_0^{d_i}(r - r_0)} dr^2 \right], \quad (2.15)$$

and define the near-horizon radial coordinate

$$\rho = \frac{2}{r_0^{\frac{d_i}{2}} \sqrt{d_i + 1}} \sqrt{r - r_0}. \quad (2.16)$$

In  $\rho$  and Euclidean time  $\theta$ , the near-horizon geometry (2.15) takes the form

$$ds_{\text{near horizon}}^2 = -l^{d_i+1} \left[ d\rho^2 + \frac{(d_i + 1)^2}{4r_0^2} \rho^2 d\theta^2 \right]. \quad (2.17)$$

That there is no conical singularity at the horizon  $r = r_0$  in (2.17) gives the Hawking temperature of the black hole (2.13) in Narayan theory

$$T = \frac{1}{\beta} = \frac{d_i + 1}{4\pi r_0}. \quad (2.18)$$

As shown in (2.10), the on-shell bulk action for the original conformally AdS<sub>2</sub> solution (2.8) and the black hole solution (2.13) in Narayan theory has a UV divergence for  $d_i \neq 1$ . This can be fixed by adding the Gibbons-Hawking-York boundary term  $I_{\text{GHY}}$  and a holographic counterterm  $I_{\text{ct}}$  [24]

$$\begin{aligned} I_{\text{ren}} &= I_{\text{bulk}} + I_{\text{GHY}} + I_{\text{ct}} \\ &= \frac{1}{16\pi G_2} \left[ \int d^2x \sqrt{-g} (\phi \mathcal{R} - U(\phi)) - 2 \int dt \sqrt{-\gamma} \phi K - 2d_i \int dt \sqrt{-\gamma} \phi^{\frac{d_i+1}{2d_i}} \right], \quad U = -l^{-2} d_i (d_i + 1) \phi^{\frac{1}{d_i}}, \end{aligned} \quad (2.19)$$

where  $\gamma_{\mu\nu}$  is the induced metric on the timelike boundary  $\partial\mathcal{M}$  at  $r = 0$  and  $K$  is the extrinsic curvature at  $\partial\mathcal{M}$  for the outward pointing normal with  $n^r > 0$ . It can be easily proved that the UV divergence in the action disappears as

$$I_{\text{ren}}^{\text{div}} \propto \frac{1}{16\pi G_2} \left( \frac{d_i - 1}{\epsilon^{d_i+1}} + \frac{d_i + 1}{\epsilon^{d_i+1}} - \frac{2d_i}{\epsilon^{d_i+1}} \right) \rightarrow 0 \quad (\epsilon \rightarrow 0), \quad (2.20)$$

therefore the action is renormalized. The variation of the renormalized action (2.20) with the metric can be written as [39]

$$\begin{aligned} \delta I_{\text{ren}} &= \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left( g_{\mu\nu} \nabla^2 \phi - \nabla_\mu \nabla_\nu \phi + \frac{1}{2} g_{\mu\nu} U \right) \delta g^{\mu\nu} \\ &\quad + \left( \frac{\delta I_{\text{ct}}}{\delta \gamma^{\mu\nu}} - \frac{1}{16\pi G_2} \int dt \sqrt{-\gamma} \gamma_{\mu\nu} n^\rho \nabla_\rho \phi \right) \delta \gamma^{\mu\nu}, \end{aligned} \quad (2.21)$$

which gives the correct Einstein equation (2.2) in the bulk.

## 2.2 The maximally extended black holes in Narayan theories when $d_i = 3$

Now that we have the black hole solution (2.13) with a regulated metric and the renormalized action, (2.19) of the Narayan theory, we are ready to construct the playground of our version of black hole information paradox. **For algebraic simplicity and without loss of generality, we set  $d_i = 3$  throughout the rest of this article.**

To maximally extend the black hole solution in Narayan theory, we start from the metric in the black hole solution of inverse temperature  $\beta = \pi r_0$ , expressed in "Poincaré" coordinates  $\{t, r\}$  with  $d_i = 3$ ,

$$ds^2 = \frac{l^4}{r^4} \left[ - \left( 1 - \frac{r^4}{r_0^4} \right) dt^2 + \left( 1 - \frac{r^4}{r_0^4} \right)^{-1} dr^2 \right] = \frac{l^4}{r^4} \left( 1 - \frac{\pi^4 r^4}{\beta^4} \right) \left[ - dt^2 + \left( 1 - \frac{\pi^4 r^4}{\beta^4} \right)^{-2} dr^2 \right]. \quad (2.22)$$

The above metric has a coordinate singularity at the horizon  $r = r_0 = \frac{\beta}{\pi}$  and therefore  $r \in (0, r_0)$  in the Poincaré patch.

We first define the tortoise coordinate in the Poincaré patch

$$\begin{aligned} r_* &= r_0 \left( \frac{1}{2} \arctan \frac{r}{r_0} + \frac{1}{4} \log \left| \frac{r + r_0}{r - r_0} \right| \right) = \frac{1}{2} r_0 \left( \arctan \frac{r}{r_0} + \text{arctanh} \frac{r}{r_0} \right) \\ &= \frac{\beta}{2\pi} \left( \arctan \frac{\pi r}{\beta} + \text{arctanh} \frac{\pi r}{\beta} \right), \end{aligned} \quad (2.23)$$

where  $r_*$  is a monotonically non-decreasing function of  $r$  and  $r_* = 0$  when  $r = 0$ ,  $r_* \rightarrow +\infty$  when  $r \rightarrow r_0$ . It is not feasible to analytically invert (2.23). The black hole metric in (2.22) can thus be written in tortoise coordinates as

$$ds^2 = l^4 \left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right) (-dt^2 + dr_*^2). \quad (2.24)$$

Then, based on (2.23), we define the Kruskal(-like) coordinates

$$V = -e^{-\frac{2\pi}{\beta}(t+r_*)}, \quad U = e^{\frac{2\pi}{\beta}(t-r_*)}, \quad (2.25)$$

where  $V \in (-\infty, 0)$ ,  $U \in (0, \infty)$ . The (implicit) inverse coordinate transformations in "Poincaré" patch are

$$VU = -e^{-\frac{4\pi}{\beta}r_*} = e^{-2 \arctan \frac{\pi r}{\beta}} \frac{r - \frac{\beta}{\pi}}{r + \frac{\beta}{\pi}} \quad (2.26)$$

and

$$t = -\frac{\beta}{4\pi} \log \left( -\frac{V}{U} \right). \quad (2.27)$$

The black hole metric can now be cast into the following form in Kruskal coordinates

$$ds^2 = -\frac{l^4(r^2 + r_0^2)(r + r_0)^2}{4r_0^2r^4} e^{2 \arctan \frac{r}{r_0}} dV dU = -\frac{\pi^2 l^4 \left( r^2 + \frac{\pi^2}{\beta^2} \right) \left( r + \frac{\pi}{\beta} \right)^2}{4\beta^2 r^4} e^{2 \arctan \frac{\pi r}{\beta}} dV dU, \quad (2.28)$$

which is regular at the event horizon  $r = r_0 = \beta/\pi$  and can thus be extended beyond the event horizon to a maximally extended black hole spacetime with two asymptotic regions,  $A_R$  and  $A_L$ , connected by a wormhole and two spacelike singularities, one in the black hole region  $BH$  and the other in the white hole region  $WH$ , cloaked by the event horizons  $N_1$  and  $N_2$ . see Figure 8.

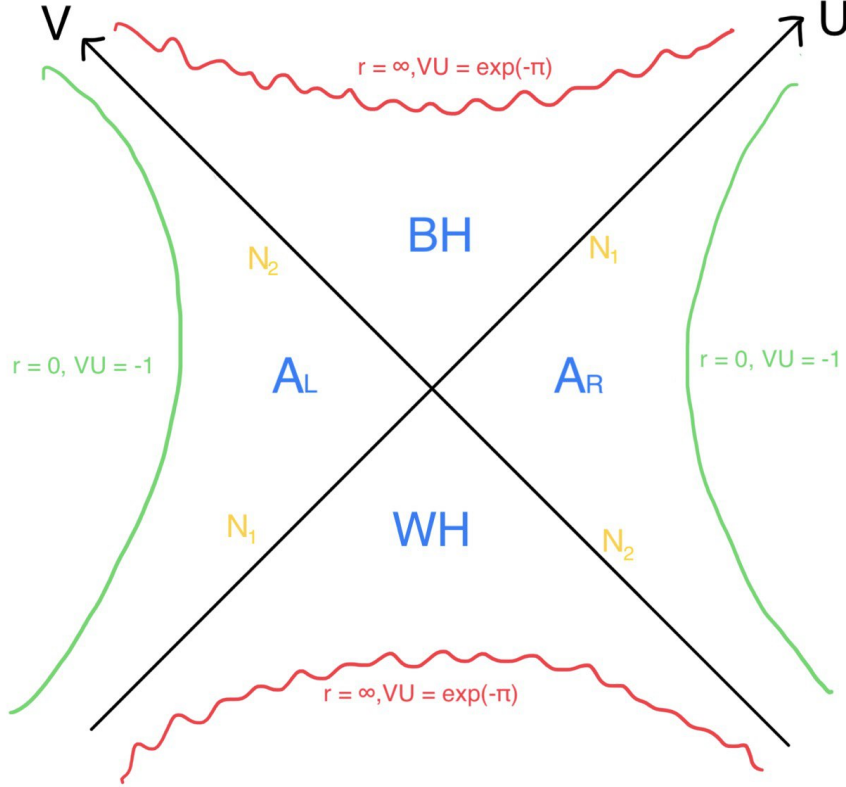


Figure 8: Maximally extended Narayan black hole in Kruskal coordinates. The timelike boundaries are denoted as two green curves and the spacelike singularities are shown as red wavy lines. Meanwhile, the event horizons comprise two diagonal black lines where  $r = r_0 = \frac{\beta}{\pi}$ ,  $VU = 0$ .

Next, we list the relation among Poincaré coordinates, tortoise coordinates and Kruskal coordinates in all 4 regions

$$A_R : \begin{cases} V = -e^{-\frac{2\pi}{\beta}(t+r_*)}, \\ U = e^{\frac{2\pi}{\beta}(t-r_*)}, \end{cases} \quad r_* = \frac{\beta}{2\pi} \left( \arctan \frac{\pi r}{\beta} + \operatorname{arctanh} \frac{\pi r}{\beta} \right); \quad (2.29)$$

$$A_L : \begin{cases} V = e^{-\frac{2\pi}{\beta}(t+r_*)}, \\ U = -e^{\frac{2\pi}{\beta}(t-r_*)}, \end{cases} \quad r_* = \frac{\beta}{2\pi} \left( \arctan \frac{\pi r}{\beta} + \operatorname{arctanh} \frac{\pi r}{\beta} \right); \quad (2.30)$$

$$BH : \begin{cases} V = e^{-\frac{2\pi}{\beta}(t+r_*)}, \\ U = e^{\frac{2\pi}{\beta}(t-r_*)}, \end{cases} \quad r_* = \frac{\beta}{2\pi} \left( \arctan \frac{\pi r}{\beta} + \operatorname{arctanh} \frac{\beta}{\pi r} \right); \quad (2.31)$$



$$WH : \begin{cases} V = -e^{-\frac{2\pi}{\beta}(t+r_*)}, \\ U = -e^{\frac{2\pi}{\beta}(t-r_*)}, \end{cases} \quad r_* = \frac{\beta}{2\pi} \left( \arctan \frac{\pi r}{\beta} + \operatorname{arctanh} \frac{\beta}{\pi r} \right); \quad (2.32)$$

while we stress that following two identities regarding the relations between the Kruskal and the Poincaré coordinates hold throughout the maximally extended spacetime:

$$VU = e^{-2 \arctan \frac{\pi r}{\beta}} \cdot \frac{r - \frac{\beta}{\pi}}{r + \frac{\beta}{\pi}}, \quad (2.33)$$

and

$$t = -\frac{\beta}{4\pi} \log \left| \frac{V}{U} \right|. \quad (2.34)$$

Also, from (2.33), we see the boundary is located at  $r = 0, r_* = 0, VU = -1$ . Note that we cannot extend the metric beyond the boundary even though, in principle, the metric could still be extended if we disregarded the dilaton. However, the dilaton  $\phi = l^3/r^3$  diverges when approaching the boundary, and if the metric is extended beyond the boundary where  $r < 0$ , the value of the dilaton will suddenly change from plus infinity to minus infinity once we cross the boundary, leading to an incurable singularity. Since the dilaton is an integral part of the Narayan black hole solutions, we cannot just consider the extension of the metric so extension beyond the boundary  $r = 0, r_* = 0, VU = -1$  is a no-go. Meanwhile, the event horizons  $N_1$  and  $N_2$  lie at  $r = r_0 = \beta/\pi, r_* = \infty, VU = 0$ ; the spacelike singularities where  $R = -4l^{-4}r^2$  blows up in  $BH$  and  $WH$  are found to be at  $r = \infty, VU = e^{-\pi}$ .

Based on the Kruskal coordinates  $V$  and  $U$ , we can further define the conformal coordinates

$$\xi = \arctan U + \arctan V, \quad \chi = \arctan U - \arctan V, \quad (2.35)$$

such that the maximally extended black hole spacetime can be illustrated as the Penrose diagram Figure 9. Note that this maximally extended black hole spacetime bears some resemblance to the maximally extended Schwarzschild spacetime though we have two wormhole-connected asymptotically conformally  $\text{AdS}_2$  regions each with a timelike conformal boundary rather than two asymptotically flat regions.

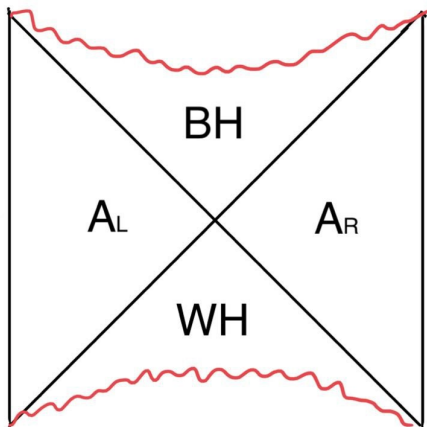


Figure 9: The Penrose diagram of the maximally extended Narayan black hole in conformal coordinates  $\xi$  and  $\chi$ . Again, the spacelike singularities are denoted by red wavy lines.

### 2.3 Narayan theory plus a CFT: eternal black hole coupled to baths

To set up an example of the black hole information problem, we intend to allow some portion of the Hawking radiation to escape away from but remains entangled with the black hole. We thus glue two copies of Minkowski half space, in which the gravity is totally irrelevant, to the maximally extended black hole in Narayan theory, one copy on each side in the spirit of [11], see Figure 10.

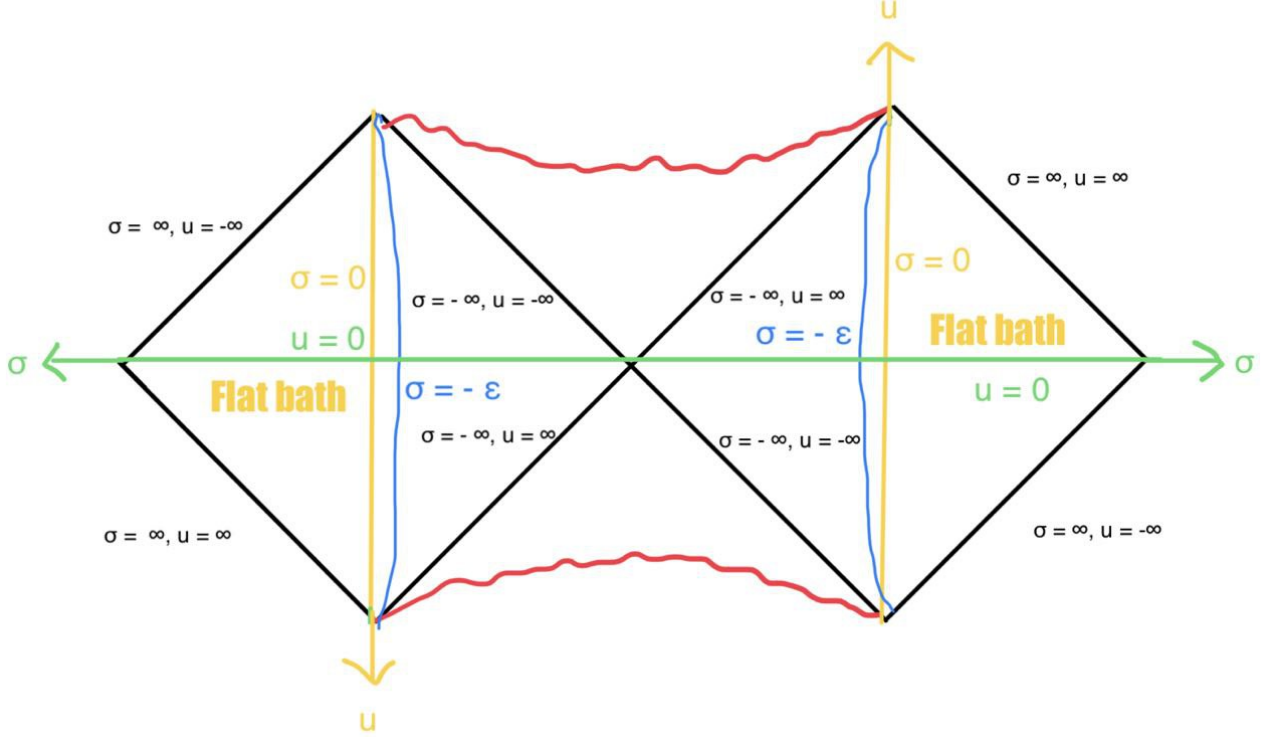


Figure 10: The maximally extended Narayan black hole is joined by a flat baths on both sides. Our choice of Rindler coordinates are also shown in this figure.

We couple the Narayan theory to a CFT with a large central charge  $c \gg 1$  living in the maximally extended black hole. The action of the Narayan theory coupled to a CFT is

$$I_{\text{Narayan+CFT}}[g_{\mu\nu}, \phi, \chi] = I_{\text{Narayan}}[g_{\mu\nu}, \phi] + I_{\text{CFT}}[g_{\mu\nu}, \chi] \quad (2.36)$$

where (in Lorentzian signature)

$$I_{\text{Narayan}}[g_{\mu\nu}, \phi] = \frac{\phi_0}{16\pi G_2} \left( \int_{M_2} R + 2 \int_{\partial M_2} K \right) + \frac{1}{16\pi G_2} \left[ \int_{M_2} (\phi R + 12l^{-2} \phi^{-\frac{1}{3}}) + 2 \int_{\partial M_2} \phi K - \frac{6}{l} \int_{\partial M_2} \phi^{\frac{2}{3}} \right]. \quad (2.37)$$

where  $M_2$  denotes the maximally extended black hole in the Narayan theory and, in addition to the renormalized Narayan action (2.19), we add a purely topological term in the first line which provides the extremal entropy of the black hole  $S_0 = \frac{\phi_0}{16\pi G_2}$ . We assume the CFT matter does **NOT** couple to the dilaton  $\phi$  in the Narayan theory. We also set the Narayan black hole in our model to be in a global vacuum state, namely the expectation value of the CFT stress-energy tensor is zero throughout, because we can always absorb the trace of the CFT stress-energy tensor  $T_{\mu\nu}^{\text{CFT}} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} I_{\text{CFT}}[g_{\mu\nu}, \chi]$ ,  $T_{\mu, \text{CFT}}^{\mu} = \frac{c}{24\pi} R$ , into the extremal value of the dilaton:  $\phi_0 \rightarrow \phi_0^{\text{Ren}} = \phi_0 + \frac{1}{3} c G_2$  [5]. We further couple the above system (2.36) (2.37) to the exactly same CFT living in the baths, i.e. the two Minkowski half spaces, and impose transparent boundary conditions for the CFT at the interface [5] [11]. We use Poincaré coordinates  $\{t, r\}$  (or the null coordinates  $x^{\pm} = t \mp r_*$  whenever at our convenience) for the vacuum Narayan black hole and the natural Minkowskian or Rindler coordinates  $\{u, \sigma\}$  (or  $y^{\pm} = u \pm \sigma$ ) for the flat spaces/baths,

$$ds_{\text{in}}^2 = \frac{l^4}{r^4} \left( 1 - \frac{\pi^4 r^4}{\beta^4} \right) \left[ -dt^2 + \left( 1 - \frac{\pi^4 r^4}{\beta^4} \right)^{-2} dr^2 \right] = l^4 \left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right) (-dt^2 + dr_*^2), \quad (2.38)$$

$$ds_{\text{out}}^2 = \frac{l^4}{\epsilon^4} (-du^2 + d\sigma^2), \quad \epsilon \rightarrow 0. \quad (2.39)$$

Instead of directly gluing the  $\sigma = 0$  vertical boundary of half flat space and the  $r = 0$  boundary of the maximally extended Narayan black hole together, we choose to glue the two spaces together along the curve called the

"physical boundary" near the "absolute"  $\sigma = 0$  vertical boundary of half flat space, according to a map defined on the boundary  $\sigma = 0$  from  $u$  to  $f(u)$ : given  $f(u)$  on  $\sigma = 0$ , the physical boundary is identified to be the curve in the Narayan black hole near the  $\sigma = 0$  boundary, parameterized in  $\{x^+, x^-\}$  as

$$x^+ = f(u - \epsilon) \approx f(u) - \epsilon f'(u), \quad x^- = f(u + \epsilon) \approx f(u) + \epsilon f'(u), \quad (2.40)$$

where the exact form of  $f(u)$  needs to be determined by the proper boundary conditions for the metric and the dynamics at the boundary. Admittedly, the gluing map  $t = f(u)$  is defined only on the absolute boundary  $\sigma = 0$  and the Rindler coordinates  $y^\pm$  are defined only in the Minkowski region up until this point, however, we can extend the Rindler coordinates into the Narayan black hole by the following coordinate transformation

$$x^\pm = f(y^\pm) \quad (2.41)$$

which, in principle, can be done for any gluing maps. Combining the parametric equation of the physical boundary (2.40), the coordinate transformation (2.41) and the definition of the null Rindler coordinates  $y^\pm = u \pm \sigma$ , we see the physical boundary can be exactly identified as the curve  $\sigma = -\epsilon$ .

The natural boundary conditions for the asymptotically conformally AdS<sub>2</sub> black hole metric (2.24) are

$$ds_{\text{bd}}^2 \sim \frac{l^4}{\epsilon^4} (-du^2 + d\sigma^2) \quad (2.42)$$

which matches the flat metric outside the boundary. Therefore, from the line element on the physical boundary  $\sigma = -\epsilon$  and the inverse expansion of (2.23) near the physical boundary

$$r = r_* - \frac{\pi^4 r_*^5}{5\beta^4} + \frac{4\pi^8 r_*^9}{45\beta^8} + \dots, \quad (2.43)$$

we have

$$l^4 \left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right) \left[ - \left( \frac{\partial t}{\partial u} \right)^2 + \left( \frac{\partial r_*}{\partial u} \right)^2 \right] = - \frac{l^4}{\epsilon^4} \longrightarrow r_* \sim \epsilon \left( \frac{\partial t}{\partial u} \right)^{\frac{1}{2}} \quad (2.44)$$

to the leading order in  $\epsilon$ . Meanwhile, from (2.40) and the definition of null Poincare coordinates  $x^\pm = t \mp r_*$  we have at the physical boundary

$$t \approx f(u), \quad r_* \approx \epsilon f'(u) = \epsilon \left( \frac{\partial t}{\partial u} \right), \quad (2.45)$$

to the leading order in  $\epsilon$ . Just as in JT gravity,  $t = f(u)$  here also marks the trajectory of the 'boundary particle'. Note that in JT, in principle,  $t = f(u)$  can be any real function of  $u$  without contradicting the boundary condition  $r = \epsilon \partial t / \partial u$  there, as long as it is a solution to the equation of motion dictated by the conservation of ADM energy. However, here in the case of Narayan black hole, by comparing (2.44) and (2.45), we see there is a new kinematic (not dynamical) constraint we have to impose on the trajectory of the boundary particle to reconcile (2.44) and (2.45), which does not exist in the near AdS<sub>2</sub> spacetime, namely

$$\frac{\partial t}{\partial u} = f'(u) \sim 1, \quad (2.46)$$

to the leading order (zeroth order) in  $\epsilon$  at the physical boundary  $\sigma = -\epsilon$ . At first glance, this 'kinetic constraint' indicates bad choice of gluing the black hole and the baths and may lead to inconsistencies, but we will show it is good for eternal black holes. As a result, at the physical boundary  $\sigma = -\epsilon$ , the appropriate boundary condition for the dilaton should be

$$\phi_{\text{bd}} \sim \frac{l^3}{\epsilon^3}. \quad (2.47)$$

We see dilaton goes to the infinity at the boundary of Narayan black hole which is consistent with freezing gravity in the bath outside the boundary [5][29].

Now, we would like to derive the correct form of function  $f$  for eternal Narayan black holes, i.e. vacuum Narayan black hole solutions of constant ADM energy, and especially, we want to see if imposing constraint (2.46) will produce consistent results in our model, so a more rigorous analysis of the boundary dynamics is warranted and presented as follows.

For the sake of deriving the ADM energy of our model - a vacuum Narayan black hole joined by two baths, and the equation of motion of the boundary particle which determines the form of gluing function  $f$  for eternal

Narayan black holes, we need to calculate the on-shell gravitational action of a vacuum Narayan black hole (2.37) and turn it into a boundary integral over boundary time  $u$  so that we can extract the ADM energy. As a heads-up for the following calculation, we have to take a closer inspection of the boundary conditions for the metric to keep all the higher-order terms vital to the calculation of relevant boundary quantities, such as the extrinsic curvature and the value of dilaton on the physical boundary.

At the physical boundary  $\sigma = -\epsilon$ , we take the boundary condition for the metric to be exactly that in (2.42) but **to all orders of  $\epsilon$** . Therefore, we have according to (2.44)

$$\left(\frac{1}{r^4} - \frac{\pi^4}{\beta^4}\right) \left[-\left(\frac{\partial t}{\partial u}\right)^2 + \left(\frac{\partial r_*}{\partial u}\right)^2\right] \equiv -\frac{1}{\epsilon^4} \quad (2.48)$$

where  $r$  can be expanded in  $r_*$  as in (2.43) at the physical boundary. We take the following ansatz for the asymptotic expansion of  $r_*$  in  $\epsilon$  (we denote  $\frac{\partial t}{\partial u}$  as  $t'$  and so on)

$$r_*(\sigma = -\epsilon) = \epsilon t'^{\frac{1}{2}} [1 + \epsilon f_1(u) + \epsilon^2 f_2(u) + \epsilon^3 f_3(u) + \epsilon^4 f_4(u) + o(\epsilon^5)]. \quad (2.49)$$

It then follows that  $f_1(u) = 0$  as we plug the above ansatz into (2.48), and due to the kinetic constraint (2.46) and the relation (A.2), we have

$$t' = 1 + o(\epsilon^2), \quad (2.50)$$

which then gives  $f_2(u) = 0$ . Finally, we arrive at the following asymptotic expansion of  $r_*$

$$r_*(\sigma = -\epsilon) = \epsilon t'^{\frac{1}{2}} \left(1 - \frac{\pi^4 t'^2}{20\beta^4} \epsilon^4 + o(\epsilon^6)\right) \quad (2.51)$$

and of  $r$  in  $\epsilon$

$$r(\sigma = -\epsilon) = \epsilon t'^{\frac{1}{2}} \left(1 - \frac{\pi^4 t'^2}{4\beta^4} \epsilon^4 + o(\epsilon^6)\right) \quad (2.52)$$

After a lengthy calculation in Appendix A, we obtained the extrinsic curvature on any curve where  $\sigma$  is constant (A.8)

$$K = l^{-2} (t'^2 - r_*'^2)^{-\frac{3}{2}} r^{-1} \left(\frac{1}{r^4} - \frac{\pi^4}{\beta^4}\right)^{-\frac{1}{2}} (2t'^3 - 2t' r_*'^2 + t'' r r_*' - t' r r_*''). \quad (2.53)$$

Plugging in (2.51) and (2.52), we get the extrinsic curvature at the physical boundary

$$K = \epsilon t'^{\frac{1}{2}} l^{-2} \left[2 + \epsilon^2 \left(\frac{t''^2}{t'^3} - \frac{t'''}{2t'^2}\right) + \epsilon^4 \left(\frac{\pi^4 t'^2}{2\beta^4} + \frac{21t''^4}{64t'^6} - \frac{3t'' t''^2}{16t'^5}\right) + o(\epsilon^6)\right] \quad (2.54)$$

which agrees with the Lorentzian version of the result obtained in equation (38) of [24] up to an extra term arising from black hole regulation and several opposite signs.

We are now ready to compute the on-shell gravitational action of a vacuum Narayan black hole in Lorentzian signature,

$$\begin{aligned} I_{\text{Narayan, on-shell}} &= \frac{\phi_0}{16\pi G_2} \left( \int_{M_2} R + 2 \int_{M_2} K \right) + \frac{1}{16\pi G_2} \left[ \int_{M_2} (\phi R + 12l^{-2} \phi^{\frac{1}{3}}) + 2 \int_{\partial M_2} \phi K - \frac{6}{l} \int_{\partial M_2} \phi^{\frac{2}{3}} \right] \\ &= \frac{1}{16\pi G_2} \cdot 2 \left\{ \int dt \int_{r(\sigma=-\epsilon)}^{+\infty} dr \frac{l^4}{r^4} \left[ \frac{l^3}{r^3} * \frac{-4r^2}{l^4} + 12l^{-2} \left( \frac{l^3}{r^3} \right)^{\frac{1}{3}} \right] + 2 \int du \frac{l^2}{\epsilon^2} * \frac{l^3}{r(\sigma=-\epsilon)^3} * K \right. \\ &\quad \left. - \frac{6}{l} \int du \frac{l^2}{\epsilon^2} * \frac{l^2}{r(\sigma=-\epsilon)^2} \right\} + \text{topological} \\ &= \frac{l^3}{8\pi G_2} \left[ \int dt \left( -\frac{2}{r^4} \Big|_{r(\sigma=-\epsilon)}^{+\infty} \right) + \frac{2l^2}{\epsilon^2} \int du \frac{K}{r(\sigma=-\epsilon)^3} - \frac{6}{\epsilon^2} \int du \frac{1}{r(\sigma=-\epsilon)^2} \right] + \text{topological} \\ &= \frac{l^3}{8\pi G_2} \int du \left( \frac{2t'}{r(\sigma=-\epsilon)^4} + \frac{2l^2 K}{\epsilon^2 r(\sigma=-\epsilon)^3} - \frac{6}{\epsilon^2 r(\sigma=-\epsilon)^2} \right) + \text{topological} \\ &= \frac{l^3}{8\pi G_2} \int du \left[ \frac{1}{\epsilon^2} \left( \frac{2t''^2 - t' t'''}{t'^4} \right) + \frac{3\pi^4 t'}{\beta^4} + \frac{21t''^4}{32t'^7} - \frac{3t''^2 t'''}{8t'^6} \right] + \text{topological} \\ &= \frac{l^3}{16\pi G_2} \int_{\partial M_2} du \left( \frac{3\pi^4 t'}{\beta^4} + \frac{21t''^4}{32t'^7} - \frac{3t''^2 t'''}{8t'^6} \right) + \text{topological} \end{aligned} \quad (2.55)$$

where the overall factor of 2 in the second line accounts for the fact that our black hole has two asymptotic regions and two baths and by the same token, the factor of  $\frac{1}{2}$  is restored on the last line since  $\partial M_2$  contains both left and right boundaries, we also throw off the  $\frac{1}{\epsilon^2}$  divergence in the last line because it is supposed to be formally cancelled by another counterterm [24]

$$- \int_{\partial M_2} \left( (\nabla\phi)^2 + \phi\nabla^2\phi \right) \propto \int \frac{du}{\epsilon^2} \left( \frac{t't'' - 2t''^2}{t'^4} \right). \quad (2.56)$$

We see the on-shell gravitational action of a vacuum Narayan black hole is a pure boundary term and thus the ADM energy of a vacuum Narayan black hole of inverse temperature  $\beta$  is

$$M(u) = \frac{l^3}{16\pi G_2} \left( \frac{3\pi^4 t'}{\beta^4} + \frac{21t''^4}{32t'^7} - \frac{3t''^2 t'''}{8t'^6} \right). \quad (2.57)$$

Conservation of ADM energy relates  $\frac{dM(u)}{du}$  to the net flux of energy across the physical boundary,

$$\frac{dM(u)}{du} = \frac{l^3}{16\pi G_2} \left( \frac{3\pi^4}{\beta^4} - \frac{3t''^2}{4t'^6} - \frac{147t''^4}{32t'^8} - \frac{3t^{(4)}t''}{8t'^6} + \frac{39t''t'''}{8t'^7} \right) t'' = T_{y^+y^+} - T_{y^-y^-} \quad (2.58)$$

This is the equation of motion for the gluing map  $f(u) = t(u)$  which, at the same time, must satisfy the kinematic constraint (2.46) of Narayan black hole joined by flat baths. For eternal black holes, we demand the conservation of its ADM energy

$$\frac{dM(u)}{du} = T_{y^+y^+} - T_{y^-y^-} = 0, \quad (2.59)$$

which immediately gives

$$t'' = 0 \quad \text{or} \quad \frac{3\pi^4}{\beta^4} - \frac{3t''^2}{4t'^6} - \frac{147t''^4}{32t'^8} - \frac{3t^{(4)}t''}{8t'^6} + \frac{39t''t'''}{8t'^7} = 0. \quad (2.60)$$

Keeping (2.50) in mind, we can only have  $t'' = 0$  because there is no way that derivatives of  $t$  that are at least  $o(\epsilon^2)$  satisfy  $\frac{3\pi^4}{\beta^4} - \frac{3t''^2}{4t'^6} - \frac{147t''^4}{32t'^8} - \frac{3t^{(4)}t''}{8t'^6} + \frac{39t''t'''}{8t'^7} = 0$ . Therefore, at the physical boundary,

$$f(u) \equiv t(u) = g(\sigma = -\epsilon)u \quad (2.61)$$

where we set the initial time to zero and  $g$  is an undetermined function of  $\sigma$  which satisfies  $g(\sigma = -\epsilon) \sim 1 + o(\epsilon^2)$ . Now that we have obtained the gluing function, we can extend  $y^\pm$  from the two Minkowski baths, respectively, to the two asymptotic regions  $A_R$  and  $A_L$ , according to (2.61) at the boundary and (2.41). We get a simple coordinate transformation in the bulk

$$x^\pm = g(\sigma = -\epsilon)y^\pm, \quad (2.62)$$

which amounts to

$$t = g(\sigma = -\epsilon) * u, \quad r_* = -g(\sigma = -\epsilon) * \sigma. \quad (2.63)$$

To pin down  $g(\sigma = -\epsilon)$ , we notice that at  $\sigma = -\epsilon$ , we have according to (2.63) and (2.51)

$$r_*(\sigma = -\epsilon) = \epsilon * g(\sigma = -\epsilon) = \epsilon g(\sigma = -\epsilon)^{\frac{1}{2}} \left[ 1 + \epsilon^4 \left( -\frac{\pi^4 g(\sigma = -\epsilon)^2}{20\beta^4} \right) + o(\epsilon^6) \right], \quad (2.64)$$

by self-consistency we have the following expansion of  $g(\sigma = -\epsilon)$  in  $\epsilon$

$$g(\sigma = -\epsilon) = 1 - \frac{\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8). \quad (2.65)$$

Hence, at the physical boundary  $\sigma = -\epsilon$ , the asymptotic expansions of  $r_*$  and  $r$  in the eternal Narayan black hole of inverse temperature  $\beta$  are, according to (2.51), (2.52), (2.61) and (2.65),

$$r_*(\sigma = -\epsilon) = \epsilon \left( 1 - \frac{\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8) \right) \quad (2.66)$$

and

$$r(\sigma = -\epsilon) = \epsilon \left( 1 - \frac{3\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8) \right). \quad (2.67)$$

Finally, the Rindler coordinates  $\{u, \sigma\}$  can be extended into the eternal Narayan black hole as

$$t = \left(1 - \frac{\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8)\right) u, \quad r_* = - \left(1 - \frac{\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8)\right) \sigma. \quad (2.68)$$

We can now express the metric and the dilaton in an eternal Narayan black hole of inverse temperature  $\beta$  in the extended Rindler coordinates,

$$ds_{\text{in}}^2 = l^4 \left(\frac{1}{r^4} - \frac{\pi^4}{\beta^4}\right) (-dt^2 + dr_*^2) = l^4 \left(\frac{1}{r(\sigma)^4} - \frac{\pi^4}{\beta^4}\right) \left(1 - \frac{\pi^4}{5\beta^4} \epsilon^4 + o(\epsilon^8)\right) (-du^2 + d\sigma^2), \quad (2.69)$$

$$\phi = \frac{l^3}{r(\sigma)^3}, \quad (2.70)$$

where  $r(\sigma)$  is the implicit function of  $\sigma$  determined by (2.23), (2.68) and thus

$$\left(1 - \frac{\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8)\right) \sigma = -\frac{\beta}{2\pi} \left[ \arctan\left(\frac{\pi}{\beta} r(\sigma)\right) + \operatorname{arctanh}\left(\frac{\pi}{\beta} r(\sigma)\right) \right]. \quad (2.71)$$

Our choice of coordinates is depicted in Figure 10.

Note that we assume the Narayan black hole coupled to a CFT is in a global vacuum state (Hartle-Hawking state) where  $T_{y^\pm y^\pm, ds_{\text{in}}^2} \equiv 0$  throughout inside the physical boundary, therefore the CFT stress-energy tensor in an eternal Narayan black hole where the metric were flat, i.e.  $ds^2 = \frac{l^4}{\epsilon^4} (-du^2 + d\sigma^2) = \Omega^2 ds_{\text{in}}^2$ , would be entirely given by the Weyl anomaly,

$$\begin{aligned} T_{y^\pm y^\pm, \text{flat}} &= \frac{c}{12\pi} \Omega^{-1} \partial_{y^\pm}^2 \Omega \\ &= \frac{c}{12\pi} \left[ \frac{1}{\epsilon^2} \left(1 + \frac{\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8)\right) \left(\frac{1}{r(\sigma)^4} - \frac{\pi^4}{\beta^4}\right)^{-\frac{1}{2}} \right]^{-1} \partial_{y^\pm}^2 \left[ \frac{1}{\epsilon^2} \left(1 + \frac{\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8)\right) \left(\frac{1}{r(\sigma)^4} - \frac{\pi^4}{\beta^4}\right)^{-\frac{1}{2}} \right] \\ &= \frac{c}{12\pi} \left( \frac{1}{r(\sigma = \frac{1}{2}(y^+ - y^-))^4} - \frac{\pi^4}{\beta^4} \right)^{\frac{1}{2}} \partial_{y^\pm}^2 \left( \frac{1}{r(\sigma = \frac{1}{2}(y^+ - y^-))^4} - \frac{\pi^4}{\beta^4} \right)^{-\frac{1}{2}} \\ &= \frac{c}{24\pi} \cdot \frac{\pi^4 r(\sigma)^4 + \beta^4}{\beta^4 r(\sigma)^2}. \end{aligned} \quad (2.72)$$

At the physical boundary  $\sigma = -\epsilon$ , according to (2.72) and (2.67) we have

$$T_{y^\pm y^\pm, \text{flat}} \Big|_{\sigma=-\epsilon} = \frac{c}{24\pi \epsilon^2} + \frac{c \pi^3 \epsilon^2}{15 \beta^4} + o(\epsilon^6), \quad (2.73)$$

which, under the transparent boundary conditions imposed on the CFT, means that we have a constant CFT stress-energy tensor equal to that in (2.73) across the entire baths,

$$T_{y^\pm y^\pm, ds_{\text{out}}^2} = \frac{c}{24\pi \epsilon^2} + \frac{c \pi^3 \epsilon^2}{15 \beta^4} + o(\epsilon^6). \quad (2.74)$$

We see from (2.73) and (2.74) that the flux across the physical boundary is zero, which is consistent with the conservation of ADM energy of an eternal black hole (2.59). Note that the jump of the induced metric  $[h_{uu}]$  is zero across the boundary and, since the extrinsic curvature on the outside of the boundary is zero, the jump of extrinsic curvature is  $[K_{uu}] = [K]h_{uu}$ . As a result, Israel's junction conditions give rise to vanishing total surface stress-energy tensor on the boundary, which adds another layer on the consistency of eternal black hole construction.

The ADM energy of an eternal Narayan black hole of inverse temperature  $\beta$ , according to (2.57) and (2.68), is

$$M_{\text{eternal}} = \frac{3\pi^3 l^3}{16G_2 \beta^4}. \quad (2.75)$$

To derive the gravitational entropy of eternal Narayan black holes, we first switch to Euclidean signature and calculate the on-shell action of an Euclidean eternal Narayan black hole of inverse temperature  $\beta$ . After a Wick rotation  $u \rightarrow i\tau$ , we have

$$y = \sigma + i\tau, \quad \bar{y} = \sigma - i\tau. \quad (2.76)$$

and the Euclidean black hole by doing analytical continuation, see Figure 11.

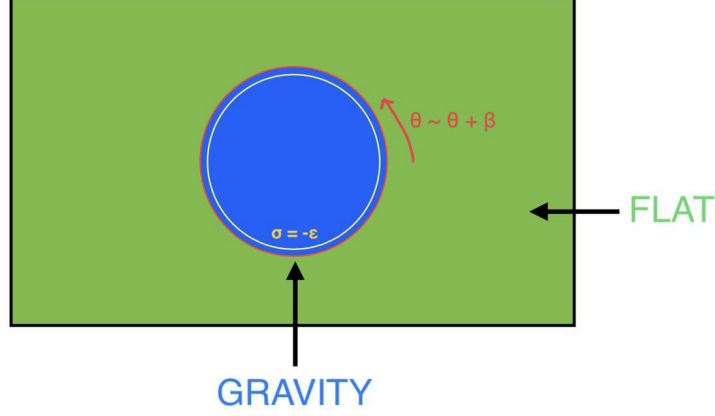


Figure 11: The Euclidean maximally extended Narayan black hole.

The eternal Narayan black hole metric in Euclidean signature is

$$ds_{\text{in}}^2 = \frac{l^4}{r(\sigma)^4} \left[ \left( 1 - \frac{\pi^4 r(\sigma)^4}{\beta^4} \right) d\theta^2 + \left( 1 - \frac{\pi^4 r(\sigma)^4}{\beta^4} \right)^{-1} dr^2 \right] = l^4 \left( \frac{1}{r(\sigma)^4} - \frac{\pi^4}{\beta^4} \right) \left( 1 - \frac{\pi^4}{5\beta^4} \epsilon^4 + o(\epsilon^8) \right) (d\tau^2 + d\sigma^2). \quad (2.77)$$

At the physical boundaries  $\sigma = -\epsilon$ , the tangent and outward-pointing normal vectors are

$$t^a = \frac{1}{l^2 \sqrt{\frac{1}{r(\sigma=-\epsilon)^4} - \frac{\pi^4}{\beta^4}}} \left( 1 + \frac{\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8) \right) \left( \frac{\partial}{\partial \tau} \right)^a, \quad n^a = \frac{1}{l^2 \sqrt{\frac{1}{r(\sigma=-\epsilon)^4} - \frac{\pi^4}{\beta^4}}} \left( 1 + \frac{\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8) \right) \left( \frac{\partial}{\partial \sigma} \right)^a. \quad (2.78)$$

The extrinsic curvature with respect to  $n^a$  at the physical boundary can now be calculated as follows

$$\begin{aligned} K &= g^{ab} \nabla_a n_b = t^a t^b \nabla_a n_b = \frac{2}{l^2 r(\sigma = -\epsilon) \sqrt{\frac{1}{r(\sigma=-\epsilon)^4} - \frac{\pi^4}{\beta^4}}} (1 + o(\epsilon^8)) \\ &= 2\epsilon l^{-2} \left( 1 + \frac{\pi^4}{5\beta^4} \epsilon^4 + o(\epsilon^8) \right). \end{aligned} \quad (2.79)$$

The gravitational part of the on-shell action of our Euclidean eternal black hole of inverse temperature  $\beta$  is

$$\begin{aligned} I_{\text{eternal, on-shell}} &= -\frac{\phi_0}{16\pi G_2} \left( \int_{M_2} R + 2 \int_{M_2} K \right) - \frac{1}{16\pi G_2} \left[ \int_{M_2} (\phi R + 12l^{-2} \phi^{\frac{1}{3}}) + 2 \int_{\partial M_2} \phi K - \frac{6}{l} \int_{\partial M_2} \phi^{\frac{2}{3}} \right] \\ &= -\frac{\phi_0}{4G_2} - \frac{1}{16\pi G_2} \left\{ \int_0^\beta d\theta \int_{\epsilon \left( 1 - \frac{3\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8) \right)}^{\frac{\beta}{\pi}} dr \left[ \frac{l^4}{r^4} \left( \frac{l^3}{r^3} * \frac{-4r^2}{l^4} + 12l^{-2} \left( \frac{l^3}{r^3} \right)^{\frac{1}{3}} \right) \right] \right. \\ &\quad \left. + 2 \int_0^{\left( 1 + \frac{\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8) \right)^\beta} d\tau \left[ \frac{l^2}{\epsilon^2} * \frac{l^3}{\left[ \epsilon \left( 1 - \frac{3\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8) \right) \right]^3} * 2l^{-2} \epsilon \left( 1 + \frac{\pi^4}{5\beta^4} \epsilon^4 + o(\epsilon^8) \right) \right] \right. \\ &\quad \left. - \frac{6}{l} \int_0^{\left( 1 + \frac{\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8) \right)^\beta} d\tau \left[ \frac{l^2}{\epsilon^2} * \frac{l^2}{\left[ \epsilon \left( 1 - \frac{3\pi^4}{10\beta^4} \epsilon^4 + o(\epsilon^8) \right) \right]^2} \right] \right\} \\ &= -\frac{\phi_0}{4G_2} - \frac{\beta l^3}{16\pi G_2} \left[ \left( -\frac{2}{r^4} \Big|_{\epsilon \left( 1 - \frac{3\pi^4}{10\beta^4} \epsilon^4 \right)}^{\frac{\beta}{\pi}} \right) + \left( \frac{4}{\epsilon^4} + \frac{24\pi^4}{5\beta^4} \right) - \left( \frac{6}{\epsilon^4} + \frac{21\pi^4}{5\beta^4} \right) + o(\epsilon^4) \right] \end{aligned} \quad (2.80)$$

$$\begin{aligned}
&= -\frac{\phi_0}{4G_2} - \frac{\beta l^3}{16\pi G_2} \left[ \left( \frac{2}{\epsilon^4} + \frac{2\pi^4}{5\beta^4} \right) + \left( \frac{4}{\epsilon^4} + \frac{24\pi^4}{5\beta^4} \right) - \left( \frac{6}{\epsilon^4} + \frac{21\pi^4}{5\beta^4} \right) + o(\epsilon^4) \right] \\
&= -\frac{1}{4G_2} \left( \phi_0 + \frac{\pi^3 l^3}{4\beta^3} \right),
\end{aligned}$$

where we used the expansion of  $r$  in  $\sigma = -r_* = -\epsilon$  (2.67) multiple times in our derivation. Using the definition of entropy in thermodynamics and the definition of partition function of the gravitational path integral in our case

$$Z = \int \mathcal{D}g \mathcal{D}\phi e^{-I_{\text{Narayan}}[g_{\mu\nu}, \phi]} Z_{\text{CFT}, g_{\mu\nu}} \quad (2.81)$$

we derive the gravitational entropy of an eternal Narayan black hole of inverse temperature  $\beta$

$$\begin{aligned}
S_{\text{grav}} &= (1 - \beta \partial_\beta) \log Z \\
&\approx -(1 - \beta \partial_\beta) I_{\text{Narayan}} = \frac{1}{4G_2} \left( \phi_0 + \frac{\pi^3 l^3}{\beta^3} \right) = \frac{1}{4G_2} \left( \phi_0 + \phi|_{\text{horizon}} \right),
\end{aligned} \quad (2.82)$$

which matches the normal expectation for the gravitational entropy in a two-dimensional dilaton gravity theory.

Meanwhile, we can calculate the gravitational free energy  $F_{\text{grav}}$  from (2.82) and the ADM energy of the eternal black hole (2.75)

$$F_{\text{grav}} = M - TS_{\text{grav}} = \frac{3\pi^3 l^3 T^4}{16G_2} - T \cdot \frac{1}{4G_2} \left( \phi_0 + \pi^3 l^3 T^3 \right) = -\frac{1}{4G_2} \left( \phi_0 T + \frac{1}{4} \pi^3 l^3 T^4 \right) \quad (2.83)$$

and then the gravitational entropy can be recovered

$$S_{\text{grav}} = -\frac{\partial F_{\text{grav}}}{\partial T} = \frac{1}{4G_2} \left( \phi_0 + \pi^3 l^3 T^3 \right) = \frac{1}{4G_2} \left( \phi_0 + \phi|_{\text{horizon}} \right) \quad (2.84)$$

which is exactly (2.82).

**We can now conclude that we have established an eternal black hole of inverse temperature  $\beta$ .**



### 3 Black hole information problem in Narayan theories

In this section, we set up a version of black hole information problem situated in the eternal Narayan black hole coupled with two baths that we constructed in section 2.3. In subsection 3.1, using (0.1), we do the QES calculation in the case of a single interval in eternal black hole in Narayan theory as a warm-up. Then in subsection 3.2, we turn to the case of two intervals in Narayan eternal black hole which exhibits Hawking's information paradox and do the QES calculation again using (0.1). Of course, in order to fully justify the results of entropy computations obtained in this section without resorting to holography, we must calculate the  $n$ th-Rényi entropy of the region in question and take the  $n \rightarrow 1$  limit, which involves the replica trick and the evaluation of the Euclidean gravitational path integral representing the trace of powers of the density matrix. In this section, however, we do not carry out any replica calculation and everything is calculated in Lorentzian signature, but we do plan to explicitly construct replica geometry in section 4 and legitimize the results derived here in subsection 3.1.

#### 3.1 Single interval in the Narayan eternal black hole

For future convenience, we first develop a coordinate system  $\{w^+, w^-\}$  covering the entirety of the eternal Narayan black hole and the baths, based on the Kruskal coordinates (2.25), (2.29) - (2.32) and the identifications across the physical boundary (2.68)

$$w^+ = U, \quad w^- = -V, \quad \text{taking } \epsilon \rightarrow 0 \quad (3.1)$$

For future reference, we also list the relation between  $\{w^+, w^-\}$  and Rindler coordinates  $\{u, \sigma\}$  in all parts of Narayan eternal black holes plus the right bath,  $B_R$  and the left bath,  $B_L$ .

$$A_R + B_R : \begin{cases} w^+ = e^{\frac{2\pi}{\beta} y^+} = e^{\frac{2\pi}{\beta}(u+\sigma)}, \\ w^- = e^{-\frac{2\pi}{\beta} y^-} = e^{-\frac{2\pi}{\beta}(u-\sigma)}. \end{cases} \quad (3.2)$$

$$A_L + B_L : \begin{cases} w^+ = -e^{\frac{2\pi}{\beta} y^+} = -e^{\frac{2\pi}{\beta}(u+\sigma)}, \\ w^- = -e^{-\frac{2\pi}{\beta} y^-} = -e^{-\frac{2\pi}{\beta}(u-\sigma)}. \end{cases} \quad (3.3)$$

$$BH : \begin{cases} w^+ = e^{\frac{2\pi}{\beta} y^+} = e^{\frac{2\pi}{\beta}(u+\sigma)}, \\ w^- = -e^{-\frac{2\pi}{\beta} y^-} = -e^{-\frac{2\pi}{\beta}(u-\sigma)}. \end{cases} \quad (3.4)$$

$$WH : \begin{cases} w^+ = -e^{\frac{2\pi}{\beta} y^+} = -e^{\frac{2\pi}{\beta}(u+\sigma)}, \\ w^- = e^{-\frac{2\pi}{\beta} y^-} = e^{-\frac{2\pi}{\beta}(u-\sigma)}. \end{cases} \quad (3.5)$$

In the new coordinates system  $\{w^+, w^-\}$ , the metric on both sides of the physical boundary can be written as

$$\begin{aligned} ds_{\text{in}}^2 &= -l^4 \left( \frac{1}{r(\sigma(w^+, w^-))^4} - \frac{\pi^4}{\beta^4} \right) dy^+ dy^- = \frac{\beta^2 l^4}{4\pi^2} \left( \frac{1}{r(\sigma(w^+, w^-))^4} - \frac{\pi^4}{\beta^4} \right) \frac{1}{w^+ w^-} dw^+ dw^- \\ &= \Omega_{\text{in}}^{-2} dw^+ dw^-, \quad \Omega_{\text{in}} = \frac{2\pi}{\beta l^2} \sqrt{w^+ w^-} \left( \frac{1}{r(\sigma(w^+, w^-))^4} - \frac{\pi^4}{\beta^4} \right)^{-\frac{1}{2}} \end{aligned} \quad (3.6)$$

and

$$ds_{\text{out}}^2 = -\frac{l^4}{\epsilon^4} dy^+ dy^- = \frac{\beta^2 l^4}{4\pi^2 \epsilon^4} \frac{1}{w^+ w^-} dw^+ dw^- = \Omega_{\text{out}}^{-2} dw^+ dw^-, \quad \Omega_{\text{out}} = \frac{2\pi \epsilon^2}{\beta l^2} \sqrt{w^+ w^-}. \quad (3.7)$$

The dilaton inside the physical boundary can be expressed as

$$\phi = \frac{l^3}{r(\sigma(w^+, w^-))^3}. \quad (3.8)$$

Note that  $r(\sigma(w^+, w^-))$  is a composite function of  $w^+$  and  $w^-$  by combining the implicit function (2.71) (now  $\epsilon = 0$ ) and  $\sigma = \frac{\beta}{4\pi} \log |w^+ w^-|$ , namely the inverse function of

$$|w^+ w^-| = e^{-2 \left[ \arctan \left( \frac{\pi}{\beta} r(\sigma(w^+, w^-)) \right) + \operatorname{arctanh} \left( \frac{\pi}{\beta} r(\sigma(w^+, w^-)) \right) \right]}. \quad (3.9)$$

We are ready to give the computation of the true quantum gravity fine-grained entropy of the region  $B = [0, b]$  in the bath at time  $t = u = 0$ , which is tethered to the boundary of black hole on the left end, see Figure 12. As

in JT gravity, the physical meaning of the entropy of the interval  $B$  outside the eternal Narayan black hole is the true quantum gravity fine-grained entropy of one side (right side) of the eternal Narayan black hole at  $u = 0$ . Actually, the calculation involves an interval  $B' = [-a, b]$  at  $t = u = 0$  sticking into the eternal black hole, with  $a, b > 0$ , which will be upheld by the analysis in section 4.

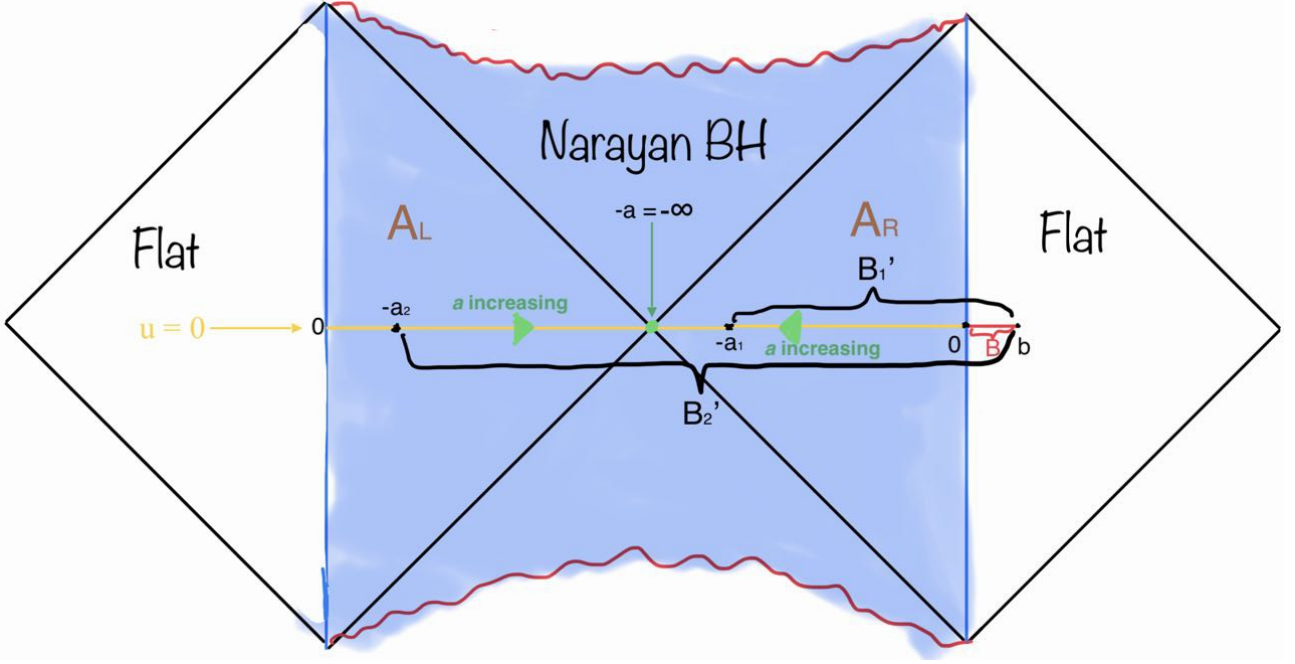


Figure 12: The single interval configuration in Lorentzian signature. We are calculating the true quantum gravity fine-grained entropy of the interval  $B = [0, b]$  (in red). This calculation, however, involves extremizing the generalized entropy of the interval  $B' = [-a, b]$  whose left end point  $\sigma = -a$  can move left and right freely on the part of time reflection surface  $u = 0$  in the eternal Narayan black hole (painted in blue). Note that, according to our choice of Rindler coordinates,  $a \in (0, \infty)$  in both  $A_L$  and  $A_R$  and  $a$  increases in value when it moves closer to the bifurcation surface (throat) where  $a = \infty$ . Here two possible typical configurations are shown:  $B'_1 = [-a_1, b]$  whose left end point lies in  $A_R$  and  $B'_2 = [-a_2, b]$  whose left end point lies in  $A_L$ .

Note that we assume  $c \gg 1$  so that we can ignore the contribution to the total fine-grained entropy and the total stress-energy tensor from gravitons. Therefore, quantum fluctuation from the on-shell solution, i.e. the saddle point in gravitational path integral can thus be ignored. For  $B'$ , the generalized entropy is the sum of the Bekenstein-Hawking term evaluated at the left end point of  $B'$  and the von Neumann entropy of the semi-classical state of the CFT matter on  $B'$

$$\begin{aligned}
S_{\text{gen}}(B') &= S_{\text{grav}}(-a) + S_{vN, \text{CFT}}([-a, b]) \\
&= \frac{1}{4G_2} [\phi_0 + \phi(-a)] + \frac{c}{6} \log \left[ \frac{[w^+(b) - w^+(-a)][w^-(b) - w^-(-a)]}{\epsilon_{-a, UV} \epsilon_{b, UV} \Omega_{\text{in}}(-a) \Omega_{\text{out}}(b)} \right] \\
&= \frac{1}{4G_2} \left( \phi_0 + \frac{l^3}{r(\sqrt{w^+ w^-} = e^{-\frac{2\pi}{\beta} a})^3} \right) + \frac{c}{6} \log \left\{ \frac{[e^{\frac{2\pi}{\beta} b} \mp e^{-\frac{2\pi}{\beta} a}] [e^{-\frac{2\pi}{\beta} (-b)} \mp e^{-\frac{2\pi}{\beta} a}]}{\left( \frac{2\pi}{\beta l^2} e^{\frac{2\pi}{\beta} b} \right) \left( \frac{2\pi \epsilon^2}{\beta l^2} e^{-\frac{2\pi}{\beta} a} \right) \left[ \frac{1}{r(\sqrt{w^+ w^-} = e^{-\frac{2\pi}{\beta} a})^4} - \frac{\pi^4}{\beta^4} \right]^{-\frac{1}{2}}} \right\} \\
&= \frac{1}{4G_2} \left( \phi_0 + \frac{l^3}{r(\sqrt{w^+ w^-} = e^{-\frac{2\pi}{\beta} a})^3} \right) + \frac{c}{6} \log \left( \cosh \frac{2\pi}{\beta} (a+b) \mp 1 \right) + \frac{c}{12} \log \left[ \frac{1}{r(\sqrt{w^+ w^-} = e^{-\frac{2\pi}{\beta} a})^4} - \frac{\pi^4}{\beta^4} \right] \\
&+ \frac{c}{6} \log \frac{\beta^2 l^6}{2\pi^2}, \quad \epsilon = 0,
\end{aligned} \tag{3.10}$$

where the minus sign in the  $\mp$  on the third and the fourth line corresponds to the left end of the interval  $\sigma = -a$  lying in the right asymptotic region  $A_R$  and the plus sign corresponds to  $\sigma = -a$  lying in the left asymptotic region  $A_L$ . We also absorbed  $\epsilon$  and the UV divergence  $\epsilon_{-a, UV}$  into the Newton constant  $G_2$  and dropped  $\epsilon_{b, UV}$

since it is constant and thus does not feed into the extremization of  $S_{\text{gen}}(B')$  below.

$S_{\text{gen}}(B')$  possibly can have more than one stationary points on the time reflection Cauchy slice  $u = 0$ , which is different from the situation in JT. The quantum extremal surface (QES) of  $S_{\text{gen}}(B')$ , by its definition, lies at the stationary point of  $S_{\text{gen}}(B')$  where the value of  $S_{\text{gen}}(B')$  there is smaller than that at any other stationary point of  $S_{\text{gen}}(B')$ .

The extremization goes as

$$\begin{aligned}
0 &= \partial_a S_{\text{gen}}(B') \equiv \partial_a S_{\text{gen}}([-a, b]) \\
&= \frac{3l^3}{4G_2 r^4} \frac{dr}{d\sigma} \Big|_{\sigma=-a} + \frac{c}{6} \cdot \frac{\sinh \frac{2\pi}{\beta}(a+b) \cdot \frac{2\pi}{\beta}}{\cosh \frac{2\pi}{\beta}(a+b) \mp 1} - \frac{c}{12} \cdot \frac{r^4 r_0^4}{r_0^4 - r^4} \cdot \left(-\frac{4}{r^5}\right) \cdot \frac{dr}{d\sigma} \Big|_{\sigma=-a} \\
&= \frac{c}{3} \left( \frac{\pi}{\beta} \cdot \frac{\sinh \frac{2\pi}{\beta}(a+b)}{\left[\cosh \frac{2\pi}{\beta}(a+b) \mp 1\right]} - \frac{1}{r(\sigma=-a)} \right) - \frac{3\pi^4 l^3}{4G_2 \beta^4} \left( \frac{\beta^4}{\pi^4 r(\sigma=-a)^4} - 1 \right).
\end{aligned} \tag{3.11}$$

As a result, the location of the QES,  $\sigma = -a$ , implicitly satisfies following condition

$$\begin{cases} \coth \frac{\pi}{\beta}(a+b) - \frac{\beta}{\pi r(\sigma=-a)} = \frac{9\pi^3 l^3}{4cG_2 \beta^3} \left[ \left( \frac{\beta}{\pi r(\sigma=-a)} \right)^4 - 1 \right], & \text{if } \sigma = -a \text{ is in } A_R, \\ \tanh \frac{\pi}{\beta}(a+b) - \frac{\beta}{\pi r(\sigma=-a)} = \frac{9\pi^3 l^3}{4cG_2 \beta^3} \left[ \left( \frac{\beta}{\pi r(\sigma=-a)} \right)^4 - 1 \right], & \text{if } \sigma = -a \text{ is in } A_L. \end{cases} \tag{3.12}$$

Alternatively, the above equations for "a" can be transformed into equations for  $\frac{\beta}{\pi r}$  by use of (2.71)

$$\begin{cases} \coth \left( \frac{1}{2} \arctan \frac{1}{x} + \frac{1}{2} \operatorname{arctanh} \frac{1}{x} + \frac{\pi b}{\beta} \right) - x - \frac{9\pi^3 l^3}{4cG_2 \beta^3} (x^4 - 1) = 0, & \text{if } \sigma = -a \text{ is in } A_R, \\ \tanh \left( \frac{1}{2} \arctan \frac{1}{x} + \frac{1}{2} \operatorname{arctanh} \frac{1}{x} + \frac{\pi b}{\beta} \right) - x - \frac{9\pi^3 l^3}{4cG_2 \beta^3} (x^4 - 1) = 0, & \text{if } \sigma = -a \text{ is in } A_L, \end{cases} \quad x = \frac{\beta}{\pi r} \in [1, +\infty). \tag{3.13}$$

It is easy to see  $x = 1 \Leftrightarrow r = \frac{\beta}{\pi} \Leftrightarrow a = +\infty$  is a solution to both equations in (3.13), therefore the bifurcation sphere of the event horizon (or throat) is always a stationary point of  $S_{\text{gen}}(B')$ . Since we require

$$\frac{9\pi^3 l^3}{4cG_2 \beta^3} = \frac{9l^3}{4cG_2 r_0^3} > 0, \quad \frac{\pi b}{\beta} = \frac{b}{r_0} > 0 \tag{3.14}$$

there is no stationary point of  $S_{\text{gen}}(B')$  in the left asymptotic region  $A_L$ , as expected from the range of the function  $\tanh$ . From numerical analysis, the right derivative of the function

$$\coth \left( \frac{1}{2} \arctan \frac{1}{x} + \frac{1}{2} \operatorname{arctanh} \frac{1}{x} + \frac{\pi b}{\beta} \right) - x - \frac{9\pi^3 l^3}{4cG_2 \beta^3} (x^4 - 1)$$

at  $x = 1$  is always positive infinite, so there is always another stationary point of  $S_{\text{gen}}(B')$  in the right asymptotic region  $A_R$  whose location depends on the values of the dimensionless parameters involved, namely  $\frac{9\pi^3 l^3}{4cG_2 \beta^3} = \frac{9l^3}{4cG_2 r_0^3}$  and  $\frac{\pi b}{\beta} = \frac{b}{r_0}$ . Further numerical analysis shows, unless  $\frac{9\pi^3 l^3}{4cG_2 \beta^3} = \frac{9l^3}{4cG_2 r_0^3}$  and  $\frac{\pi b}{\beta} = \frac{b}{r_0}$  both take very small values ( $\ll 1$ ), the other stationary point is extremely close to the throat  $x = 1 \Leftrightarrow r = \frac{\beta}{\pi} \Leftrightarrow a = +\infty$  (see Figures 13, 14 and Figures 15, 16 on the next page), meaning that the QES always exists and it lies in  $A_R$  — almost exactly at the bifurcation sphere (throat) of the event horizon when at least one of  $\frac{9\pi^3 l^3}{4cG_2 \beta^3} = \frac{9l^3}{4cG_2 r_0^3}$  and  $\frac{\pi b}{\beta} = \frac{b}{r_0}$  does not take very small value ( $\ll 1$ ).

In the general case where at least one of  $\frac{9\pi^3 l^3}{4cG_2 \beta^3} = \frac{9l^3}{4cG_2 r_0^3}$  and  $\frac{\pi b}{\beta} = \frac{b}{r_0}$  does not take very small value ( $\ll 1$ ), the QES lies extremely close to the bifurcation surface (throat), we have

$$S(B) \approx S_{\text{gen}}([-\infty, b]) = \frac{1}{4G_2} \left( \phi_0 + \frac{\pi^3 l^3}{\beta^3} \right) + \frac{\pi c}{3\beta} \cdot b + \frac{c}{3} \log \frac{\beta l^3}{\pi} + \frac{\pi c}{24} - \frac{c}{12} \log 2. \tag{3.15}$$

We see, from (3.15) and (2.84), that the true entropy of one side of the black hole  $S(B)$  is dominated by the Bekenstein-Hawking term associated with classical gravitational entropy of the one-side black hole if  $b$  is small ( $b \ll 1$ ). Also, due to the existence of a time-like Killing vector outside the event horizon of eternal Narayan black hole, this one-side black hole configuration is invariant under time translation and the general QES at  $u \neq 0$  is related by a time translation.

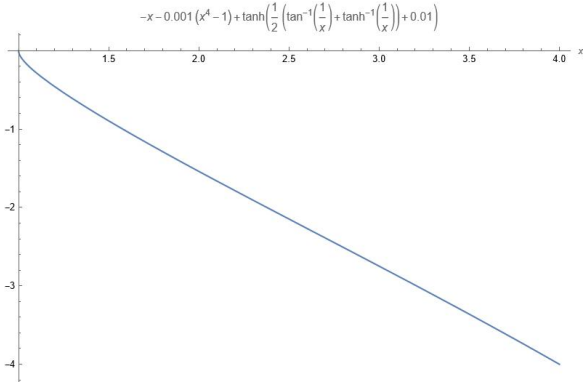


Figure 13: Plotting  $\frac{c}{3}\partial_a S_{\text{gen}}(B')$  against  $x = \frac{\beta}{\pi r}$  if the left end point of the interval  $B'$ ,  $\sigma = -a$ , lies in  $A_L$  and when  $\frac{9\pi^3 l^3}{4cG_2\beta^3}$  takes the very small value of 0.001 and  $\frac{\pi b}{\beta}$  takes the very small value of 0.01. As expected from the range of  $\tanh$ , there is no stationary point of  $S_{\text{gen}}(B')$  in  $A_L$ .

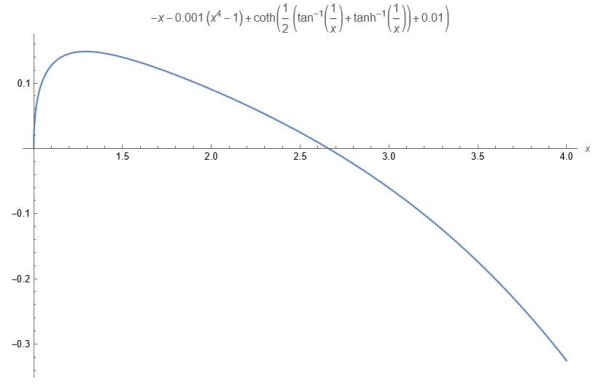


Figure 14: Plotting  $\frac{c}{3}\partial_a S_{\text{gen}}(B')$  against  $x = \frac{\beta}{\pi r}$  if the left end point of the interval  $B'$ ,  $\sigma = -a$ , lies in  $A_R$  and when  $\frac{9\pi^3 l^3}{4cG_2\beta^3}$  takes the very small value of 0.001 and  $\frac{\pi b}{\beta}$  takes the very small value of 0.01. There is another stationary point of  $S_{\text{gen}}(B')$  other than the bifurcation surface (throat) at  $x = \frac{\beta}{\pi r} \approx 2.6559$ , showing a local minimum of  $S_{\text{gen}}(B')$  there. Since  $\partial_a S_{\text{gen}}(B')$  is always negative when  $x > 1 \Leftrightarrow a < \infty \Leftrightarrow r > \frac{\beta}{\pi}$  in  $A_L$ , and,  $\partial_a S_{\text{gen}}(B')$  is positive in  $A_R$  in the vicinity of the bifurcation surface at  $x = 1 \Leftrightarrow a = \infty \Leftrightarrow r = \frac{\beta}{\pi}$  as shown in this Figure, we see the bifurcation surface of the event horizon is not an extremum (Figure 12 is helpful to see this). As a result, the QES lies at the local minimum  $x = \frac{\beta}{\pi r} \approx 2.6559$ .

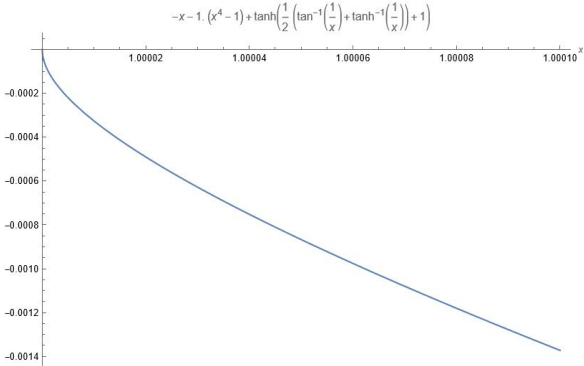


Figure 15: Plotting  $\frac{c}{3}\partial_a S_{\text{gen}}(B')$  against  $x = \frac{\beta}{\pi r}$  if the left end point of the interval  $B'$ ,  $\sigma = -a$ , lies in  $A_L$  and when  $\frac{9\pi^3 l^3}{4cG_2\beta^3}$  and  $\frac{\pi b}{\beta}$  both take the value 1. Still, as expected from the range of  $\tanh$ , there is no stationary point of  $S_{\text{gen}}(B')$  in  $A_L$ .

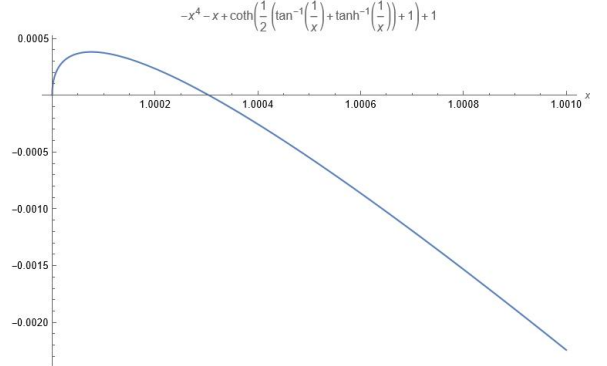


Figure 16: Plotting  $\frac{c}{3}\partial_a S_{\text{gen}}(B')$  against  $x = \frac{\beta}{\pi r}$  if the left end point of the interval  $B'$ ,  $\sigma = -a$ , lies in  $A_R$  and when  $\frac{9\pi^3 l^3}{4cG_2\beta^3}$  and  $\frac{\pi b}{\beta}$  both take the value 1. There is another stationary point other than the bifurcation surface (throat) at  $x = \frac{\beta}{\pi r} \approx 1.0003$ , showing a local minimum of  $S_{\text{gen}}(B')$  there, which is still very close to the bifurcation surface (throat) of the event horizon. By the same reason mentioned in the caption of Figure 14, we see the bifurcation surface of the event horizon is not an extremum. As a result, the QES lies at the local minimum  $x = \frac{\beta}{\pi r} \approx 1.0003$ , though it is very close to the bifurcation surface.

### 3.2 Two intervals in the Narayan eternal black hole

We now move on to formally set up our version of the black hole information paradox in the eternal Narayan black hole configuration constructed in subsection 2.3. For the initial state, we choose a pure thermofield double state for the black hole plus radiation. This initial state can be prepared by an Euclidean gravitational path integral which will be explored in section 4, while the corresponding Lorentzian geometry in this setup is shown in Figure 17.

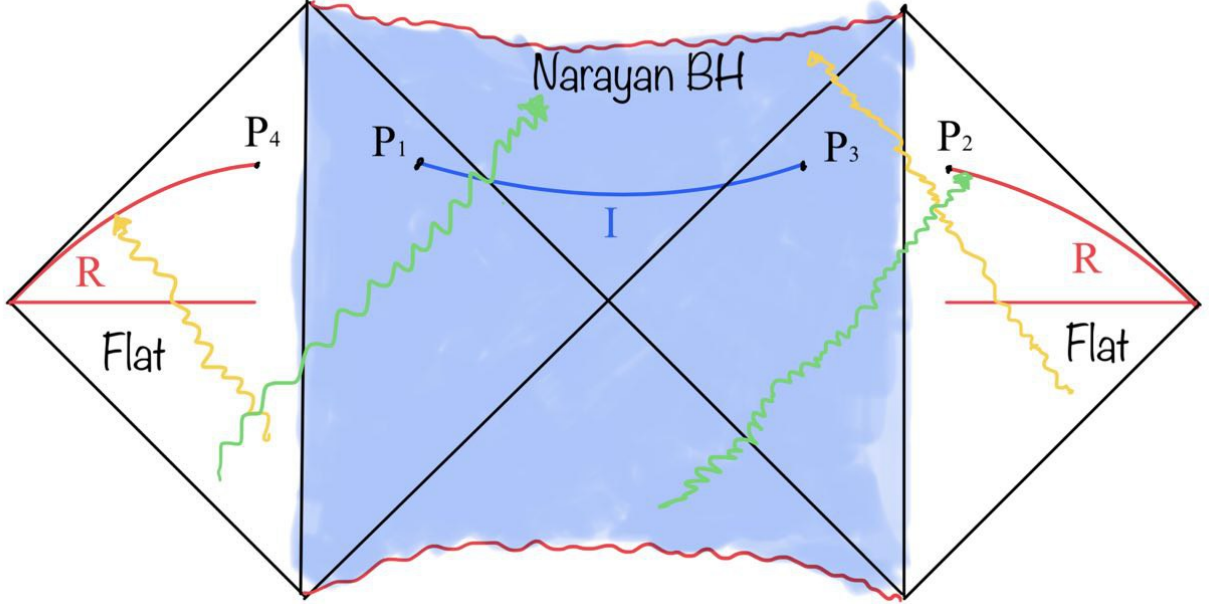


Figure 17: The two interval configuration in Lorentzian signature. Here we visualize two instants in time:  $u = 0$  and late time. The two particles of the same color are entangled as a Hawking pair.

In Figure 17, the Rindler  $(u, \sigma)$  coordinates of the points are

$$P_1 = (-u_{-a}, -a), \quad P_2 = (u_b, b), \quad P_3 = (u_{-a}, -a), \quad P_4 = (-u_b, b). \quad (3.16)$$

The radiation region  $R$  is

$$R = [P_4, i_L^0] \cup [P_2, i_R^0], \quad (3.17)$$

and the island region  $I$  is

$$I = [P_1, P_3]. \quad (3.18)$$

The CFT state on the full Cauchy slice is pure, so we have

$$S_{\text{CFT}}(I \cup R) = S_{\text{CFT}}([P_4, P_1] \cup [P_3, P_2]). \quad (3.19)$$

which clearly depends on the CFT itself. For simplicity, we assume the CFT matter in our model is  $c$  free Dirac fermions [40]. The entanglement entropy of the two intervals combined  $([P_4, P_1] \cup [P_3, P_2])$  on the metric  $ds^2 = \Omega^{-2} dw^+ dw^-$  is

$$S_{\text{fermions}} = \frac{c}{6} \log \left[ \frac{|x_{21} x_{32} x_{34} x_{41}|^2}{|x_{24} x_{31}|^2 \Omega_1 \Omega_2 \Omega_3 \Omega_4} \right], \quad (3.20)$$

where all the UV divergences are dropped.

Plugging in all the configuration data, (3.20) gives the following entanglement entropy in the limit  $\epsilon \rightarrow 0$

$$S_{\text{fermions}}(I \cup R) = \frac{c}{6} \log \left\{ l^8 \left[ \left( \frac{\beta}{\pi r(\sigma = -a)} \right)^4 - 1 \right] \times e^{\frac{-4\pi}{\beta}(a+2b-u_b)} \left( e^{\frac{2\pi}{\beta}(-a+u_{-a})} + e^{\frac{2\pi}{\beta}(b-u_b)} \right)^2 \right. \\ \left. \times \frac{\left( -1 + e^{\frac{2\pi}{\beta}(a+b+u_{-a}-u_b)} \right)^2 \left( -1 + e^{\frac{2\pi}{\beta}(a+b-u_{-a}+u_b)} \right)^2 \left( 1 + e^{\frac{2\pi}{\beta}(a+b+u_{-a}+u_b)} \right)^2}{16 \left( 1 + e^{\frac{4\pi}{\beta}u_{-a}} \right)^2 \left( 1 + e^{\frac{4\pi}{\beta}u_b} \right)^2} \right\}. \quad (3.21)$$

The generalized entropy of the radiation region  $R$  plus the island region  $I$  is thus

$$S_{\text{gen}}(I \cup R) = 2(\phi_0 + \phi(-a)) + S_{\text{fermions}}(I \cup R) = 2\left(\phi_0 + \frac{l^3}{r(\sigma = -a)^3}\right) + S_{\text{fermions}}(I \cup R) \quad (3.22)$$

Next, we run an analysis of the generalized entropy of the radiation part in both scenarios where an island is either absent or present.

**When island is absent** We first calculate the generalized entropy in the simpler scenario where there is no island, which corresponds to the Hawking saddle where there is no conical singularities,

$$\begin{aligned} S_{\text{gen}}^{\text{no island}} &= S_{\text{gen}}(R) = S_{\text{fermions}}(R) = S_{\text{fermions}}([P_2, P_4]) \\ &= \frac{c}{6} \log \left[ \frac{[w^+(P_2) - w^+(P_4)][w^-(P_2) - w^-(P_4)]}{\epsilon_{P_2, UV} \epsilon_{P_4, UV} \Omega_{\text{out}}(P_2) \Omega_{\text{out}}(P_4)} \right] \\ &= \frac{c}{6} \log \left[ \frac{\left( e^{\frac{2\pi}{\beta}(u_b+b)} + e^{\frac{2\pi}{\beta}(-u_b+b)} \right) \left( e^{-\frac{2\pi}{\beta}(u_b-b)} + e^{-\frac{2\pi}{\beta}(-u_b-b)} \right)}{\left( \frac{2\pi}{\beta l^2} \right)^2 e^{\frac{4\pi b}{\beta}}} \right] \\ &= \frac{c}{3} \log \left[ \frac{\beta l^2}{\pi} \cosh \left( \frac{2\pi}{\beta} u_b \right) \right], \quad \epsilon = 0. \end{aligned} \quad (3.23)$$

Obviously, the generalized entropy in this scenario is automatically an extreme value since there is no QES at all to extremalize with.

We start to collect Hawking radiation in region  $R$  in Figure 17 from  $u_b = 0$ . The radiation region  $R$ , as a function of  $u_b$ , moves upwards in both the baths to the left and right of the black hole. According to (3.23), the generalized entropy of the radiation without an island, now equal to the von Neumann entropy of the matter, grows monotonically with the time  $u_b$ . Especially, at late times, the generalized entropy of the radiation (3.23) can be approximated as

$$S_{\text{gen}}^{\text{no island}} = S_{\text{gen}}(R) \approx \frac{c}{3} \cdot \frac{2\pi}{\beta} \cdot u_b, \quad u_b \gg 0, \quad (3.24)$$

showing a linear growth with  $u_b$ , see Figure 18. This growth of radiation entropy, matching Hawking's original calculation, can be explained as follows. Initially, the radiation modes on the left and the right are entangled and in pairs. As time moves on, in more and more pairs of entangled modes, only one mode reaches and registers itself in the radiation region  $R$  while the other falls into the black hole, therefore the entanglement entropy of the radiation continues to increase. However, if this growth continued forever, the entanglement entropy of the radiation would exceed the thermodynamic/coarse-grained entropy of the two-side eternal Narayan black hole, which is equal to double the Bekenstein-Hawking entropy of the one-side eternal Narayan black hole, causing a contradiction and posing our version of black hole information paradox. It calls for some large non-perturbative corrections to show up as early as around the page time, whose representation in Lorentzian geometry is discussed as follows.

**When island is present** We turn into the more interesting scenario where there is an island corresponding to the replica wormhole saddle in Euclidean path integral. According to (3.21) and (3.22), when  $u_{-a} = u_b = 0$  we have in the limit  $\epsilon \rightarrow 0$

$$\begin{aligned} S_{\text{gen}}^{\text{island}} &= S_{\text{gen}}(I \cup R) \\ &= 2\left(\phi_0 + \frac{l^3}{r(\sigma = -a)^3}\right) + \frac{c}{6} \log \left\{ \frac{1}{16} l^8 \sinh \frac{2\pi}{\beta} (a+b)^4 \left[ \left( \frac{\beta}{\pi r(\sigma = -a)} \right)^4 - 1 \right] \right\} \end{aligned} \quad (3.25)$$

Extremalizing the generalized entropy gives

$$\begin{aligned} 0 &= \partial_a S_{\text{gen}}^{\text{island}} = S_{\text{gen}}(I \cup R) = \partial_a S_{\text{gen}}([P_4, P_1] \cup [P_3, P_2]) \\ &= \frac{6l^3}{r(\sigma = -a)^4} \frac{dr}{d\sigma} \Big|_{\sigma=-a} + \frac{4\pi c}{3\beta} \coth \frac{2\pi}{\beta} (a+b) + \frac{2c}{3} \cdot \frac{\left( \frac{\beta}{\pi r(\sigma=-a)} \right)^4}{\left( \frac{\beta}{\pi r(\sigma=-a)} \right)^4 - 1} \cdot \frac{1}{r(\sigma = -a)} \frac{dr}{d\sigma} \Big|_{\sigma=-a} \end{aligned} \quad (3.26)$$

$$= \left[ \frac{6l^3}{r(\sigma = -a)^4} + \frac{2c}{3r(\sigma = -a)} \cdot \frac{\left(\frac{\beta}{\pi r(\sigma = -a)}\right)^4}{\left(\frac{\beta}{\pi r(\sigma = -a)}\right)^4 - 1} \right] \left( -1 + \frac{\pi r(\sigma = -a)^4}{\beta} \right) \Big|_{\sigma = -a} + \frac{4\pi c}{3\beta} \coth \frac{2\pi}{\beta}(a+b).$$

Consequently, the location  $\sigma = -a$  of the QES's satisfies the following identity,

$$\frac{6}{l} \left[ \left( \frac{l}{r(\sigma = -a)} \right)^4 - \left( \frac{\pi l}{\beta} \right)^4 \right] + \frac{2c}{3r(\sigma = -a)} = \frac{4\pi c}{3\beta} \coth \frac{2\pi}{\beta}(a+b). \quad (3.27)$$

Note again that  $r$  and  $\sigma$  is related by (2.71) while  $\epsilon = 0$ , and that as  $a$  goes up from 0 to  $\infty$ ,  $r(\sigma = -a)$  goes up, monotonically, from 0 to  $\frac{\beta}{\pi}$ .

There **always exist** solutions to (3.27) at  $u_{-a} = u_b = 0$ . Here is why : As  $a$  goes up from 0 to  $\infty$ , the LHS of (3.27) decreases monotonically from  $\infty$  to  $\frac{2\pi c}{3\beta}$  and the RHS of (3.27) goes monotonically down from  $\frac{4\pi c}{3\beta} \coth \frac{2\pi}{\beta} b = \frac{4\pi c}{3\beta} \left( \frac{\beta}{2\pi b} + \frac{2\pi b}{3\beta} + \dots \right) < \infty$ , when  $b$  is small but not zero, to  $\frac{4\pi c}{3\beta} > \frac{2\pi c}{3\beta}$ . It is natural to infer that the end points of the island are located somewhere very near the event horizon ( $a = \infty, r = r_0 = \frac{\beta}{\pi}$ ). The fact that the solutions  $\sigma = -a$  to (3.27) at  $u_{-a} = u_b = 0$  always exist means, even if the replica saddle is not in dominance so early, there is **always an island solution in existence** at  $u_{-a} = u_b = 0$  irrelevant of the value of the parameters in (3.27) so long as  $b > 0$ , which is different from the situation in JT gravity where the existence of the island at  $u = 0$  depends on the value of parameters  $b$  and the boundary condition for the dilaton  $\phi$  [11].

Meanwhile, at any given time  $u_{-a}$  and  $u_b$  the extremity condition  $\partial_a S_{\text{gen}}^{\text{island}} = 0$  leads to the following identity satisfied by  $\sigma = -a$

$$\frac{6}{l} \left[ \left( \frac{l}{r(\sigma = -a)} \right)^4 - \left( \frac{\pi l}{\beta} \right)^4 \right] + \frac{2c}{3r(\sigma = -a)} = \frac{\pi c}{3\beta} \left[ \coth \frac{\pi}{\beta}(a+b+u_{-a}-u_b) + \coth \frac{\pi}{\beta}(a+b-u_{-a}+u_b) + \tanh \frac{\pi}{\beta}(a+b-u_{-a}-u_b) + \tanh \frac{\pi}{\beta}(a+b+u_{-a}+u_b) \right]. \quad (3.28)$$

Following the same reasoning as under (3.27) for  $u_{-a} = u_b = 0$ , one can easily show there always exists an island at any timelike separated  $u_{-a}$  and  $u_b$  prior to extremalizing along temporal direction  $u$ . Therefore for any given  $u_b$ , in principle, one can locate the islands by Maximin algorithm [41] [42] — extremalizing the generalized entropy in temporal coordinate  $u_{-a}$  after solving (3.28) which extremalizes the generalized entropy in  $a$  at every  $u_{-a}$ . Furthermore, at late times, it is valid that the entanglement entropy can be approximated by doubling the modified single-interval answer, where we restore time dependence in (3.10), in the OPE limit, namely

$$S_{\text{fermions}}(I \cup R) \approx 2S_{\text{fermions}}([P_1, P_2]) = \frac{c}{3} \log \left[ \cosh \frac{2\pi}{\beta}(a+b) - \cosh \frac{2\pi}{\beta}(u_b - u_{-a}) \right] + (\text{terms independent of the temporal coordinate } u), \quad (3.29)$$

we see the extremalization in temporal direction further mandates that  $u_{-a} = u_b$  at the end points of the island at late times.

According to the QES prescription, the true fine-grained full entropy for the radiation in two-integral configuration is

$$S_{\text{Rad}} = \min \left\{ S_{\text{gen}}^{\text{no island}}, S_{\text{gen}}^{\text{island}} \right\}. \quad (3.30)$$

As we have seen in (3.24), the entropy in the absence of island grows linearly with time at late times, hence island not only always exists but dominates the entropy at late times. This suggests that we should regard the island, which is inside the black hole, as a subsystem of the outgoing Hawking radiation at late times. This amazing new insight leads to the observation that the entanglement across the event horizon of the entangled pairs no longer contributes to the entanglement entropy of outgoing radiation, for the two quanta in each of those entangled pairs now belong to the same system and thus purify each other. As a result, a unitary Page curve for the fine-grained entropy of the radiation is produced as shown in Figure 18. At the same time, because the CFT state on the full Cauchy slice at any given time is pure, the fine-grained matter entropy of the radiation is always equal to that of the two-side black hole at any given time, plus the Bekenstein-Hawking term is always

the same in both fine-grained full entropy of the radiation and of the black hole, we conclude the fine-grained full entropy of the two-side eternal Narayan black hole is equal to that of the radiation at any given time,

$$S_{\text{BH, two-side}} = S_{\text{Rad}}. \quad (3.31)$$

More specifically, we are essentially extremizing two always equal quantities, namely (see Figure 17)

$$S_{\text{gen, BH}}^{\text{trivial}} \equiv S_{\text{gen}}([P_4, P_2]) = S_{\text{gen}}(R) \equiv S_{\text{gen, rad}}^{\text{no island}} \quad (3.32)$$

before the Page time and

$$S_{\text{gen, BH}}^{\text{disconnected}} \equiv S_{\text{gen}}([P_4, P_1] \cup [P_3, P_2]) = S_{\text{gen}}(I \cup R) \equiv S_{\text{gen, rad}}^{\text{island}} \quad (3.33)$$

since the Page time, over the locations of the same set of points (no points to extremize over before the Page time, extremizing over the locations of  $P_1$  and  $P_3$  since the Page time), according to the QES prescription.

Therefore, the fine-grained full entropy of the two-side eternal Narayan black hole follows the same Page curve as the true fine-grained entropy of the radiation does in Figure 18, which shows the eternal Narayan black hole and the radiation remain perfectly are entangled all the time and thus unitarity is preserved.

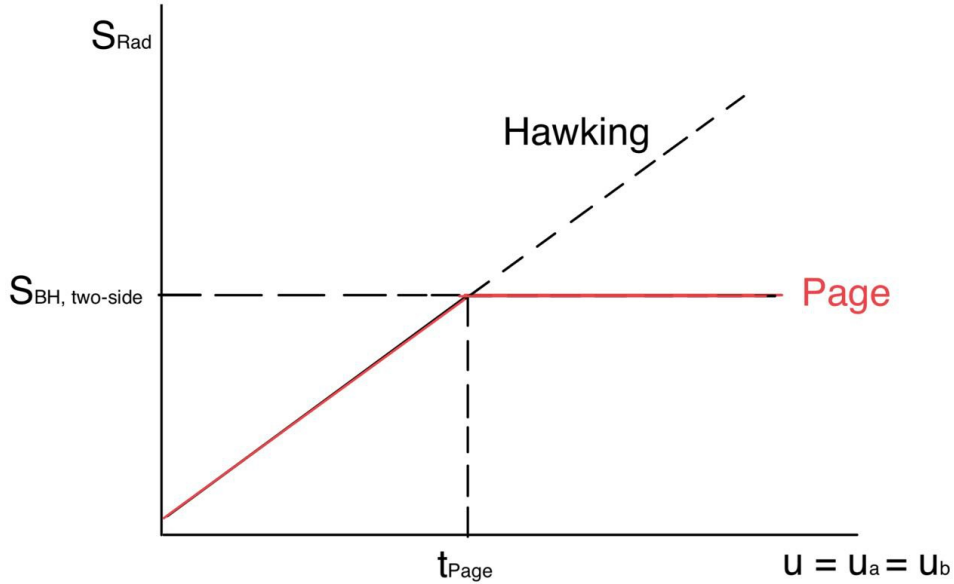


Figure 18: Page curve for the fine-grained entropy of the radiation in our model. The oblique dash line represents the linear growth originally derived by Hawking while the horizontal dash line represents the thermodynamic/coarse-grained entropy of the two-side eternal Narayan black hole. The page curve for our model traces along the minimum of the two. The true fine-grained entanglement entropy of the two-side eternal Narayan black hole also follows the same Page curve.

**Our new version of black hole information problem in the eternal Narayan black holes is thus resolved.**



## 4 The ‘‘Replica geometry’’ of eternal black holes in Narayan theories

In this section we try to build the replica geometry of eternal Narayan black holes on the basis of a hypothesis and then attempt to recover the extremity condition of the QES in one-interval case constructed in subsection 3.1 by taking the limit  $n \rightarrow 1$ . In subsection 4.1, we first specify more details about applying the replica trick to the theory at issue, that is, the Narayan theory plus a CFT. Then, in subsection 4.2, given the modes of boundary particle, we discuss the ‘‘conformal welding problem’’ of joining the gravitational region and the flat region by finding holomorphic maps of both inside and outside the boundary onto the complex  $z$  plane that are compatible at the boundary, following [11]. We also make a hypothesis of the replica geometry of the covering manifold and derive the replica equation of motion for the boundary mode in the orbifold, generalizing (2.58). We then make an attempt to extract the extremity condition for the single interval case from this replica equation of motion. Although this effort proves to be only partially successful in the end, the same method/algorithm deriving the bulk action with conical singularities can be used in more general settings. Furthermore, there is still more interesting stuff worth being discussed in the wake of our failure of recovering the extremity condition in the one-interval case (3.12), which will be further explored in section 5.

### 4.1 Applying the replica trick to and setting up conformal boundary problem in Narayan theory

As specified in subsection 2.3, the same CFT theory lives in the Narayan black hole where the gravity is dynamic and flat baths where the gravity is non-dynamic, we impose transparent boundary conditions at the physical boundary. We thus have the following Euclidean full action on any smooth geometry

$$I_{\text{total}} = -\frac{\phi_0}{16\pi G_2} \left( \int_{M_2} R + 2 \int_{\partial M_2} K \right) - \frac{1}{16\pi G_2} \left[ \int_{M_2} (\phi R + 12l^{-2}\phi^{\frac{1}{3}}) + 2 \int_{\partial M_2} \phi K - \frac{6}{l} \int_{\partial M_2} \phi^{\frac{2}{3}} \right] + \log Z_{\text{CFT}}[g]. \quad (4.1)$$

Instead of working on a smooth geometry on the covering manifold  $\tilde{\mathcal{M}}_n$ , as mentioned earlier, we choose to quotient  $\tilde{\mathcal{M}}_n$  by  $Z_n$  and work on the orbifold  $\mathcal{M}_n = \tilde{\mathcal{M}}_n/Z_n$  for our convenience. As a result, the total gravitational action on  $\mathcal{M}_n$  contains extra terms involving cosmic branes

$$\begin{aligned} I_{\text{grav}}[\mathcal{M}_n] &= \frac{1}{n} I_{\text{grav}}[\tilde{\mathcal{M}}_n] \\ &= -\frac{\phi_0}{16\pi G_2} \left( \int_{\mathcal{M}_n} R + 2 \int_{\partial \mathcal{M}_n} K \right) - \frac{1}{16\pi G_2} \left[ \int_{\mathcal{M}_n} (\phi R + 12l^{-2}\phi^{\frac{1}{3}}) + 2 \int_{\partial \mathcal{M}_n} \phi K - \frac{6}{l} \int_{\partial \mathcal{M}_n} \phi^{\frac{2}{3}} \right] \\ &\quad + \frac{1}{4G_2} \left( 1 - \frac{1}{n} \right) \sum_{\alpha, \text{Cosmic Branes}} [\phi_0 + \phi(w_\alpha)]. \end{aligned} \quad (4.2)$$

In some senses, the above action can be regarded as a new gravity theory and we then add  $n$  copies of CFT on the orbifold  $\mathcal{M}_n$ . In addition, we place twist fields at the position of the cosmic branes,  $w_\alpha$ .

We define an interior complex coordinate  $w$  so that the metric on  $\mathcal{M}_n$  in the gravitational region inside the boundary can be written in conformal gauge

$$ds_{\text{in}}^2 = e^{2\rho} dw d\bar{w}, \quad |w| \leq 1 \quad (4.3)$$

where the boundary between the gravitational and non-gravitational region lies at  $|w| = 1$ ,  $w = e^{i\frac{2\pi}{\beta}\theta}$ . There are conical singularities at certain  $w_\alpha$  with an opening angle  $2\pi/n$ . By varying the action (4.2) with respect to the dilaton, the function  $\rho$  in the conformal factor in (4.3) is subject to the following dilaton equation of motion

$$-4\partial_w \partial_{\bar{w}} \rho + 2l^{-2} e^{2\rho} \phi^{-\frac{2}{3}} = 4\pi \left( 1 - \frac{1}{n} \right) \sum_{\alpha} \delta^{(2)}(w - w_\alpha). \quad (4.4)$$

The above equation is coupled with the Einstein equation derived by varying (4.2) with respect the metric

$$\partial_w^2 \phi - 2\partial_w \phi \partial_w \rho = -8\pi G_2 \langle T_{ww}^{\text{CFT}} \rangle, \quad (4.5)$$

$$\partial_w \partial_{\bar{w}} \phi - 3l^{-2} \phi^{\frac{1}{3}} e^{2\rho} = 8\pi G_2 \langle T_{w\bar{w}}^{\text{CFT}} \rangle, \quad (4.6)$$

$$\partial_{\bar{w}}^2 \phi - 2\partial_{\bar{w}} \phi \partial_{\bar{w}} \rho = -8\pi G_2 \langle T_{\bar{w}\bar{w}}^{\text{CFT}} \rangle. \quad (4.7)$$

Once we impose the above Einstein equations and dilaton EOM, the cosmic brane action terms in (4.2) are cancelled by the contribution from the singular Ricci curvature scalar which is, in turn, caused by the singular

part in  $\rho$ .

Now that we have set up the coordinate system in the gravitational region, we need to do the same for the flat space outside the boundary. We consider a finite temperature configuration where  $\tau \sim \tau + \beta$ . We define the coordinate  $v = e^{\frac{2\pi}{\beta}y}$ ,  $y = \sigma + i\tau$  in the flat space such that

$$ds_{\text{out}}^2 = \frac{l^4}{\epsilon^4} dy d\bar{y}, \quad |v| \geq 1. \quad (4.8)$$

At the boundary  $|w| = |v| = 1$ , we have  $w = e^{i\frac{2\pi}{\beta}\theta(\tau)}$  and  $v = e^{i\frac{2\pi}{\beta}\tau}$ . Much noteworthy here is that, different from the  $n = 1$  original solution where  $\theta(\tau) \equiv \tau$  when we take  $\epsilon$  to 0 after finishing boundary renormalization, we generally can **NOT** directly extend the coordinate  $v$  on the outside to a holomorphic coordinate inside the boundary. Nevertheless, we can find another coordinate  $z$  so that there are holomorphic maps from both the interior  $|w| \leq 1$  and the exterior  $|v| \geq 1$  to the complex  $z$  plane, see Figure 5. The problem of finding this coordinate map  $z$  covering the whole orbifold  $\mathcal{M}_n$  can be framed as the ‘‘conformal welding problem’’, following [11]. For given data of  $\theta(\tau)$  at the boundary, we need to find two functions  $G$  and  $F$  such that

$$\begin{aligned} z &= G(w), \quad \text{for } |w| \leq 1 \\ z &= F(v), \quad \text{for } |v| \geq 1 \\ G(e^{i\frac{2\pi}{\beta}\theta(\tau)}) &= F(e^{i\frac{2\pi}{\beta}\tau}), \quad \text{for } |w| = |v| = 1. \end{aligned} \quad (4.9)$$

We require the functions  $G$  and  $F$  are holomorphic in their respective domains but they do not have to be holomorphic at the boundary.  $G$  and  $F$  depend non-locally on  $\theta(\tau)$  at the boundary and, respectively, they map the inside disk  $|w| \leq 1$  and the outside disk  $|v| \geq 1$  to the inside and outside of some irregular region on the complex  $z$  plane. Put into the context of replica geometry of Narayan black holes,  $\theta(\tau)$  represents the boundary mode in the Narayan theory, in parallel with that in the nearly- $AdS_2$  gravity [28] [29] [30]. Note again that, in the original  $n = 1$  theory developed in subsection 2.3, the boundary mode  $\theta(\tau) = \tau$  when  $\epsilon = 0$  such that the physical boundary  $\sigma = -\epsilon$  is identical with the true boundary  $\sigma = 0$  after the action is renormalized.

## 4.2 A partially successful attempt to construct replica geometry for one-interval case in eternal Narayan black holes

For a given boundary mode  $\theta(\tau)$ , we have already defined the conformal welding problem of matching the inside and outside of the boundary in a compatible way at the boundary. We then need to deal with the problem of deriving the equation of motion for the  $\theta(\tau)$  on the orbifold  $\mathcal{M}_n$ . It is important to notice that the on-shell gravitational action for  $\mathcal{M}_n$  is no longer the Euclidean version of that in (2.55) for the  $n = 1$  original solution, namely

$$-\frac{l^3}{16\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \left( \frac{3\pi^4\theta'(\tau)}{\beta^4} + \frac{21\theta''(\tau)^4}{32\theta'(\tau)^7} - \frac{3\theta'''(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} \right). \quad (4.10)$$

since we need to take into account the conical singularities on  $\mathcal{M}_n$ . When  $n \sim 1$ , these conical singularities lead to small deviations in the interior metric from the  $n = 1$  Narayan black hole metric below

$$ds_{\text{in},\mathcal{M}_1}^2 = e^{2\rho_0} dw d\bar{w} = \frac{\beta^2 l^4}{4\pi^2} \left( \frac{1}{r(|w|)^4} - \frac{\pi^4}{\beta^4} \right) \frac{1}{|w|^2} dw d\bar{w} \quad (4.11)$$

where we define  $w = e^{\frac{2\pi}{\beta}(\gamma+i\theta)}$  and  $y$  was defined in (2.76),  $r(|w|)$  is the inverse function of

$$|w| = e^{-[\arctan \frac{\pi}{\beta} r(|w|) + \operatorname{arctanh} \frac{\pi}{\beta} r(|w|)]}. \quad (4.12)$$

Therefore, in order to figure out the exact form of the metric perturbation when  $n \sim 1$  in the case of a single interval in eternal Narayan black hole studied in section 3.1, we need to seek the geometry of the replica wormhole on the  $n$ -th-covering manifold,  $\tilde{\mathcal{M}}_n$ , which in turn is the  $n$ -fold cover of the orbifold  $\mathcal{M}_n$  where there are branch points at  $-a$  and  $b$  and a branch cut in between, see Figure 19. Obviously, due to the existence of the branch points and the branch cut, it is more convenient to introduce different coordinates on the inside and outside just like the initial setup in the conformal welding problem in the previous section. Likewise, we define  $w = e^{\frac{2\pi}{\beta}(\gamma+i\theta)}$ ,  $|w| < 1$  for the interior and  $v = e^{\frac{2\pi}{\beta}y} = e^{\frac{2\pi}{\beta}(\sigma+i\tau)}$ ,  $|v| > 1$  for the exterior, on the basis of (4.9). We write the coordinates of the branch points as

$$w = A = e^{-\frac{2\pi}{\beta}a}, \quad v = B = e^{\frac{2\pi}{\beta}b}. \quad (4.13)$$

following the convention in [11].

We now make the following important hypothesis, which is in part inspired by  $AdS_2$  geometry in JT gravity studied in [11], that the leading saddle of the gravitational action on  $\tilde{\mathcal{M}}_n$ ,  $I_{\text{grav}}[\tilde{\mathcal{M}}_n]$ , is still an Euclidean Narayan black hole of the same temperature  $\beta$  which is totally smooth on  $\tilde{\mathcal{M}}_n$  when we fill with it the inside of the boundary on  $\tilde{\mathcal{M}}_n$ . In other words, the geometry of the so-called "replica wormhole" in one-interval case is nothing but a Narayan black hole of the temperature  $\beta$  but is "n times as large as" the original Narayan black hole solution on  $\mathcal{M}_1$ .

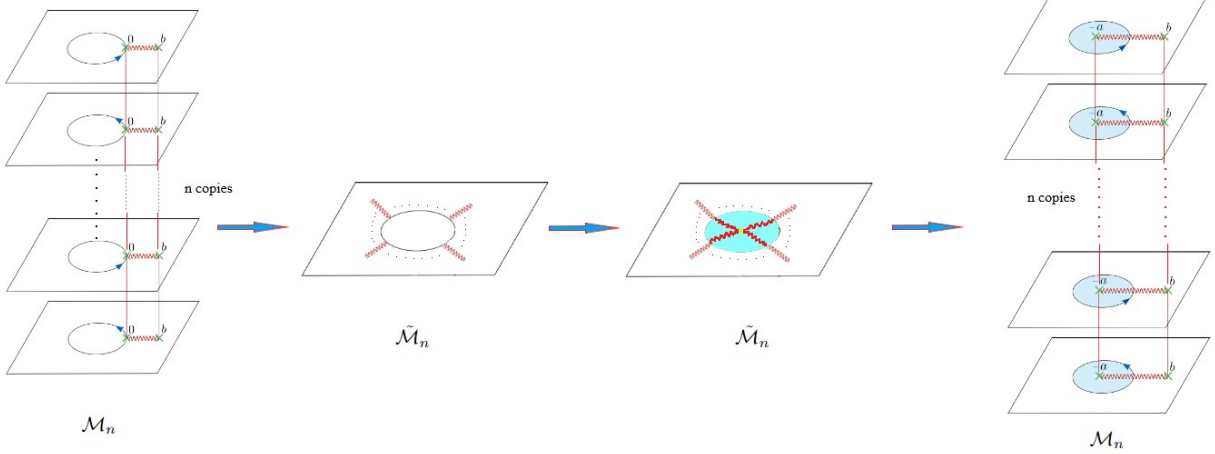


Figure 19: From left to right: We have  $n$  copies of  $\mathcal{M}_n$ . On each copy, there is a flat space field theory outside the boundary shown as a unit circle. In the one-interval scenario, we want to calculate the true fine-grained entropy of an interval in the exterior with one of its end tethered to the boundary. The disk is hollow in  $\mathcal{M}_n$  and will be filled in with gravitational configurations subject to the boundary conditions on the unit circle. We then glue together these  $n$  copies along the  $n$  intervals, each now shown as a branch cut on each  $\mathcal{M}_n$ . The blue arrows on each  $\mathcal{M}_n$  indicate how to go around the branch cuts. The  $n$  boundaries on  $n$  copies of  $\mathcal{M}_n$  are now connected into a single long circle  $n$  times as long as the original one on the covering manifold  $\tilde{\mathcal{M}}_n$ . According to the hypothesis we made below (4.13), we fill in the same standard Narayan black hole metric of inverse temperature  $\beta$  in the interior of this elongated boundary on  $\tilde{\mathcal{M}}_n$  which is perfectly smooth and has a topology of disk. Then, moving to the farthest right, the same replica wormhole geometry on  $\tilde{\mathcal{M}}_n$  we just filled in can also be described by  $n$  copies of, again, orbifold  $\mathcal{M}_n$ . However, the point at the center of replica geometry on  $\tilde{\mathcal{M}}_n$  is mapped to a branch point inside the boundary on every copy of  $\mathcal{M}_n$ , which now stands for a conical singularity with an opening angle of  $2\pi/n$ . This makes the branch cut on each  $\mathcal{M}_n$  extend into the interior, anchoring one of its end point to the conical singularity. This figure is based on figure 11 in [11].

We then use the following map from  $\mathcal{M}_n$  to  $\tilde{\mathcal{M}}_n$  to uniformize  $n$  copies of the inner disk

$$\tilde{w} = \left( \frac{w - A}{1 - Aw} \right)^{1/n} \quad (4.14)$$

which maps the boundary in each copy of  $\mathcal{M}_n$  to the elongated boundary in  $\tilde{\mathcal{M}}_n$  and the branch point at  $A$  in each copy of  $\mathcal{M}_n$  to  $0$  in the unit disk in  $\tilde{\mathcal{M}}_n$  where the metric is smooth there. We define a new auxiliary coordinate  $w_n$  on the unit disk in  $\tilde{\mathcal{M}}_n$

$$\tilde{w} = \frac{w_n - A}{1 - Aw_n}, \quad \text{or} \quad w_n = \frac{\tilde{w} + A}{1 + A\tilde{w}} = \frac{\left( \frac{w - A}{1 - Aw} \right)^{1/n} + A}{1 + A \left( \frac{w - A}{1 - Aw} \right)^{1/n}}, \quad (4.15)$$

we see  $w_n$  becomes  $w$  when  $n$  goes to 1. Then, the Euclidean Narayan black hole metric on  $\tilde{\mathcal{M}}_n$ , the hypothesized

leading saddle point of  $I_{\text{grav}}[\tilde{\mathcal{M}}_n]$ , takes the following form in  $\{\tilde{w}, \bar{\tilde{w}}\}$  coordinates

$$\begin{aligned} ds_{\text{in}, \tilde{\mathcal{M}}_n}^2 &= \frac{\beta^2 l^4}{4\pi^2} \left( \frac{1}{r_n (|w_n|)^4} - \frac{\pi^4}{\beta^4} \right) \frac{1}{|w_n|^2} dw_n d\bar{w}_n \\ &= \frac{\beta^2 l^4}{4\pi^2} \left( \frac{1}{r_n \left( \left| \frac{\tilde{w}+A}{1+A\tilde{w}} \right| \right)^4} - \frac{\pi^4}{\beta^4} \right) \frac{(1-A^2)^2}{\left| \frac{\tilde{w}+A}{1+A\tilde{w}} \right|^2 |1+A\tilde{w}|^4} d\tilde{w} d\bar{\tilde{w}} \end{aligned} \quad (4.16)$$

where, similar to (4.12),

$$|w_n| = \left| \frac{\tilde{w}+A}{1+A\tilde{w}} \right| = e^{-(\arctan \frac{\pi}{\beta} r_n + \text{arctanh} \frac{\pi}{\beta} r_n)} \quad (4.17)$$

and  $ds_{\text{in}, \tilde{\mathcal{M}}_n}^2 = ds_{\text{in}, \mathcal{M}_1}^2$  when  $n = 1$ , as expected.

Going back onto  $\mathcal{M}_n$ , from a combination of (4.11), (4.14), (4.16) and the previous definition  $w = e^{\frac{2\pi}{\beta}(\gamma+i\theta)}$  we have when  $n \sim 1$

$$ds_{\text{in}, \mathcal{M}_n}^2 = e^{2\rho} dw d\bar{w} = e^{2\delta\rho} ds_{\text{in}, \mathcal{M}_1}^2, \quad \rho = \rho_0 + \delta\rho, \quad (4.18)$$

where  $\delta\rho$  goes as (we shall always only keep the  $o(n-1)$  order for our purpose)

$$\delta\rho = U_1(\theta) + \gamma^2 U_2(\theta) + \gamma^4 U_3(\theta) + o(\gamma^6), \quad \text{as } |w| \rightarrow 1 (\gamma \rightarrow 0) \quad (4.19)$$

and

$$U_1(\theta) = (n-1) \left[ 1 + \frac{A e^{-\frac{2i\pi\theta}{\beta}} \left( 1 - e^{\frac{4i\pi\theta}{\beta}} \right) \log \left( \frac{e^{\frac{2i\pi\theta}{\beta}} - A}{1 - A e^{\frac{2i\pi\theta}{\beta}}} \right)}{1 - A^2} \right], \quad (4.20)$$

$$\begin{aligned} U_2(\theta) &= \frac{(n-1) A \pi^2 e^{-\frac{2i\pi\theta}{\beta}}}{3(A^2-1)\beta^2 \left( A - e^{\frac{2i\pi\theta}{\beta}} \right)^2 \left( -1 + A e^{\frac{2i\pi\theta}{\beta}} \right)^2} \times \\ &\left[ -2(A^2-1) e^{\frac{2i\pi\theta}{\beta}} \left( -2A^2 e^{\frac{2i\pi\theta}{\beta}} \left( 1 + e^{\frac{4i\pi\theta}{\beta}} \right) + A \left( 6e^{\frac{4i\pi\theta}{\beta}} + e^{\frac{8i\pi\theta}{\beta}} + 1 \right) - 2e^{\frac{2i\pi\theta}{\beta}} \left( 1 + e^{\frac{4i\pi\theta}{\beta}} \right) \right) \right. \\ &\left. - 2 \left( -1 + e^{\frac{4i\pi\theta}{\beta}} \right) \left( A^2 \left( -e^{\frac{2i\pi\theta}{\beta}} \right) + A e^{\frac{4i\pi\theta}{\beta}} + A - e^{\frac{2i\pi\theta}{\beta}} \right)^2 \log \left( \frac{A - e^{\frac{2i\pi\theta}{\beta}}}{-1 + A e^{\frac{2i\pi\theta}{\beta}}} \right) \right], \end{aligned} \quad (4.21)$$

$$U_3(\theta) = \frac{2(n-1)\pi^4 (A^2-1)^2 e^{\frac{4i\pi\theta}{\beta}} \left( A^4 e^{\frac{4i\pi\theta}{\beta}} + A^2 \left( -12e^{\frac{4i\pi\theta}{\beta}} + 5e^{\frac{8i\pi\theta}{\beta}} + 5 \right) + e^{\frac{4i\pi\theta}{\beta}} \right)}{5\beta^4 \left( A - e^{\frac{2i\pi\theta}{\beta}} \right)^4 \left( -1 + A e^{\frac{2i\pi\theta}{\beta}} \right)^4}. \quad (4.22)$$

Clearly, the metric perturbation functions  $U_1$ ,  $U_2$  and  $U_3$ , are all of order 1 in  $(n-1)$  and depend on  $A$ , the location of the conical singularity in  $\mathcal{M}_n$ , and thus also on the moduli of the Riemann surface.

The on-shell gravitational action on  $\mathcal{M}_n$ , after a long derivation in Appendix B, is then (B.32)

$$\begin{aligned} I_{\text{grav}}[\mathcal{M}_n] &= -\frac{l^3}{16\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \left[ \left( \frac{3\pi^4 \theta'(\tau)}{\beta^4} + \frac{21\theta''(\tau)^4}{32\theta'(\tau)^7} - \frac{3\theta'''(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} \right) \right. \\ &+ \left( \frac{11\pi^4 \theta'(\tau) U_1(\theta(\tau))}{3\beta^4} + \theta'(\tau) U_2''(\theta(\tau)) + \frac{5\theta''(\tau) U_2'(\theta(\tau))}{2\theta'(\tau)} + \frac{27\theta''(\tau)^2 U_2(\theta(\tau))}{4\theta'(\tau)^3} - \frac{50}{3} \theta'(\tau) U_3(\theta(\tau)) \right. \\ &- \frac{\theta''(\tau) U_1'''(\theta(\tau))}{6\theta'(\tau)} - \frac{3\theta''(\tau)^2 U_1''(\theta(\tau))}{8\theta'(\tau)^3} + \frac{\theta''(\tau)^3 U_1'(\theta(\tau))}{\theta'(\tau)^5} + \frac{63\theta''(\tau)^4 U_1(\theta(\tau))}{32\theta'(\tau)^7} - \frac{3\theta'''(\tau) U_2(\theta(\tau))}{\theta'(\tau)^2} \\ &\left. \left. - \frac{1}{12} \theta'(\tau) U_1^{(4)}(\theta(\tau)) - \frac{3\theta'''(\tau)\theta''(\tau) U_1'(\theta(\tau))}{4\theta'(\tau)^4} - \frac{9\theta'''(\tau)\theta''(\tau)^2 U_1(\theta(\tau))}{8\theta'(\tau)^6} \right) \right]. \end{aligned} \quad (4.23)$$

By varying (4.23) with respect to  $\theta(\tau)$  and following the same variation of the CFT part in Appendix A of [11] where they did a variation of the exterior metric localized at the boundary in lockstep with the variation of  $\theta(\tau)$

maintaining the same boundary condition, we get the following equation of motion for the boundary mode  $\theta(\tau)$

$$\begin{aligned}
& \frac{l^3}{16\pi G_2} \left[ \left( \frac{3\pi^4\theta''(\tau)}{\beta^4} - \frac{147\theta''(\tau)^5}{32\theta'(\tau)^8} - \frac{3\theta^{(4)}(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} + \frac{39\theta'''(\tau)\theta''(\tau)^3}{8\theta'(\tau)^7} - \frac{3\theta'''(\tau)^2\theta''(\tau)}{4\theta'(\tau)^6} \right) \right. \\
& + \frac{11\pi^4\theta'(\tau)^2U_1'(\theta(\tau))}{3\beta^4} + \frac{11\pi^4\theta''(\tau)U_1(\theta(\tau))}{3\beta^4} + \frac{7}{2}\theta''(\tau)U_2''(\theta(\tau)) - \frac{50}{3}\theta'(\tau)^2U_3'(\theta(\tau)) - \frac{50}{3}\theta''(\tau)U_3(\theta(\tau)) \\
& \frac{17\theta''(\tau)^3U_1''(\theta(\tau))}{8\theta'(\tau)^4} - \frac{97\theta''(\tau)^4U_1'(\theta(\tau))}{32\theta'(\tau)^6} - \frac{441\theta''(\tau)^5U_1(\theta(\tau))}{32\theta'(\tau)^8} + \frac{17\theta''(\tau)^2U_2'(\theta(\tau))}{4\theta'(\tau)^2} - \frac{81\theta''(\tau)^3U_2(\theta(\tau))}{4\theta'(\tau)^4} \\
& - \frac{5\theta''(\tau)^2U_1'''(\theta(\tau))}{24\theta'(\tau)^2} - \frac{3\theta'''(\tau)\theta''(\tau)U_1''(\theta(\tau))}{2\theta'(\tau)^3} + \theta'(\tau)^2U_2'''(\theta(\tau)) - \frac{\theta'''(\tau)U_2'(\theta(\tau))}{2\theta'(\tau)} + \frac{39\theta'''(\tau)\theta''(\tau)U_2(\theta(\tau))}{2\theta'(\tau)^3} \\
& - \frac{\theta'''(\tau)U_1''(\theta(\tau))}{6\theta'(\tau)} - \frac{3\theta'''(\tau)^2U_1'(\theta(\tau))}{4\theta'(\tau)^4} + \frac{39\theta'''(\tau)\theta''(\tau)^2U_1'(\theta(\tau))}{8\theta'(\tau)^5} + \frac{117\theta'''(\tau)\theta''(\tau)^3U_1(\theta(\tau))}{8\theta'(\tau)^7} - \frac{1}{4}\theta''(\tau)U_1^{(4)}(\theta(\tau)) \\
& \left. - \frac{3\theta^{(4)}(\tau)\theta''(\tau)U_1'(\theta(\tau))}{4\theta'(\tau)^4} + \frac{9\theta'''(\tau)^2\theta''(\tau)U_1(\theta(\tau))}{4\theta'(\tau)^6} - \frac{3\theta^{(4)}(\tau)U_2(\theta(\tau))}{\theta'(\tau)^2} - \frac{1}{12}\theta'(\tau)^2U_1^{(5)}(\theta(\tau)) - \frac{9\theta^{(4)}(\tau)\theta''(\tau)^2U_1(\theta(\tau))}{8\theta'(\tau)^6} \right] \\
& = i \left[ T_{yy} \left( e^{i\frac{2\pi}{\beta}\tau} \right) - T_{\bar{y}\bar{y}} \left( e^{i\frac{2\pi}{\beta}\tau} \right) \right] = T_{\tau\sigma}.
\end{aligned} \tag{4.24}$$

The physical stress-energy tensor on the RHS of (4.24) can be calculated using the conformal anomaly in the setting of conformal welding problem. Note that there is an ambiguity under  $SL(2, C)$  transformation of  $z$  for both  $G$  and  $F$  maps that we can use to map the twist operator at  $w = A$  to  $z = 0$  and the twist operator at  $v = B$  to  $z = \infty$ . So for any original maps  $G$  and  $F$  to the complex  $z$  plane, we can define a new  $G'$  and a new  $F'$  such that

$$G'(w) = \frac{G(w) - A}{B - G(w)}, \quad F'(v) = \frac{F(v) - A}{B - F(v)}. \tag{4.25}$$

From now on, we assume  $G$  and  $F$  by default map  $w = A$  to  $z = 0$  and  $v = B$  to  $z = \infty$ . Provided this choice of maps  $G$  and  $F$ , we have on and outside the boundary

$$T_{yy} \left( e^{i\frac{2\pi}{\beta}y} \right) = \left( \frac{dF \left( e^{i\frac{2\pi}{\beta}y} \right)}{dy} \right)^2 T_{zz} \left( F \left( e^{i\frac{2\pi}{\beta}y} \right) \right) - \frac{c}{24\pi} \left\{ F \left( e^{i\frac{2\pi}{\beta}y} \right), y \right\} \tag{4.26}$$

and a similar expression for  $T_{\bar{y}\bar{y}}$ . Even though the  $z$  coordinate covers the entire complex plane in a holomorphic way in the sense of conformal welding problem, see the discussion following (4.9), there are still twist operators at the branch points  $z = 0$  and  $z = \infty$  and a branch cut between them. So we further define the standard mapping  $\tilde{z} = z^{\frac{1}{n}}$  to remove them. The vacuum expectation value of the stress-energy tensor vanishes on the  $\tilde{z}$  plane due to conformal symmetry, so back on the  $z$ -plane we have

$$T_{zz}(z) = -\frac{c}{24\pi} \{z^{\frac{1}{n}}, z\} = -\frac{c}{48\pi} \left( 1 - \frac{1}{n^2} \right) \frac{1}{z^2}. \tag{4.27}$$

By inverting the conformal welding map to the  $v$ -plane, we finally obtain the expression for  $T_{yy}$

$$T_{yy}(y) = \frac{4\pi^2}{\beta^2} e^{\frac{4\pi}{\beta}y} \left[ F'(v)^2 T_{zz}(F(v)) - \frac{c}{24\pi} \{F, v\} + \frac{c}{48\pi} \right] \tag{4.28}$$

and a similar one for  $T_{\bar{y}\bar{y}}$  by taking the complex conjugation of (4.28).

Putting it all together, the equation of motion when  $n \sim 1$  is

$$\begin{aligned}
& \frac{3l^3}{2cG_2} \left[ \left( \frac{3\pi^4\theta''(\tau)}{\beta^4} - \frac{147\theta''(\tau)^5}{32\theta'(\tau)^8} - \frac{3\theta^{(4)}(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} + \frac{39\theta'''(\tau)\theta''(\tau)^3}{8\theta'(\tau)^7} - \frac{3\theta'''(\tau)^2\theta''(\tau)}{4\theta'(\tau)^6} \right) \right. \\
& + \frac{11\pi^4\theta'(\tau)^2U_1'(\theta(\tau))}{3\beta^4} + \frac{11\pi^4\theta''(\tau)U_1(\theta(\tau))}{3\beta^4} + \frac{7}{2}\theta''(\tau)U_2''(\theta(\tau)) - \frac{50}{3}\theta'(\tau)^2U_3'(\theta(\tau)) - \frac{50}{3}\theta''(\tau)U_3(\theta(\tau)) \\
& \frac{17\theta''(\tau)^3U_1''(\theta(\tau))}{8\theta'(\tau)^4} - \frac{97\theta''(\tau)^4U_1'(\theta(\tau))}{32\theta'(\tau)^6} - \frac{441\theta''(\tau)^5U_1(\theta(\tau))}{32\theta'(\tau)^8} + \frac{17\theta''(\tau)^2U_2'(\theta(\tau))}{4\theta'(\tau)^2} - \frac{81\theta''(\tau)^3U_2(\theta(\tau))}{4\theta'(\tau)^4} \\
& - \frac{5\theta''(\tau)^2U_1'''(\theta(\tau))}{24\theta'(\tau)^2} - \frac{3\theta'''(\tau)\theta''(\tau)U_1''(\theta(\tau))}{2\theta'(\tau)^3} + \theta'(\tau)^2U_2'''(\theta(\tau)) - \frac{\theta'''(\tau)U_2'(\theta(\tau))}{2\theta'(\tau)} + \frac{39\theta'''(\tau)\theta''(\tau)U_2(\theta(\tau))}{2\theta'(\tau)^3} \\
& \left. - \frac{\theta'''(\tau)U_1''(\theta(\tau))}{6\theta'(\tau)} - \frac{3\theta'''(\tau)^2U_1'(\theta(\tau))}{4\theta'(\tau)^4} + \frac{39\theta'''(\tau)\theta''(\tau)^2U_1'(\theta(\tau))}{8\theta'(\tau)^5} + \frac{117\theta'''(\tau)\theta''(\tau)^3U_1(\theta(\tau))}{8\theta'(\tau)^7} \right] \\
& = i \left[ T_{yy} \left( e^{i\frac{2\pi}{\beta}\tau} \right) - T_{\bar{y}\bar{y}} \left( e^{i\frac{2\pi}{\beta}\tau} \right) \right] = T_{\tau\sigma}.
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
& -\frac{1}{4}\theta''(\tau)U_1^{(4)}(\theta(\tau)) - \frac{3\theta^{(4)}(\tau)\theta''(\tau)U_1'(\theta(\tau))}{4\theta'(\tau)^4} + \frac{9\theta'''(\tau)^2\theta''(\tau)U_1(\theta(\tau))}{4\theta'(\tau)^6} - \frac{3\theta^{(4)}(\tau)U_2(\theta(\tau))}{\theta'(\tau)^2} \\
& -\frac{1}{12}\theta'(\tau)^2U_1^{(5)}(\theta(\tau)) - \frac{9\theta^{(4)}(\tau)\theta''(\tau)^2U_1(\theta(\tau))}{8\theta'(\tau)^6} \Big] = i\frac{4\pi^2}{\beta^2}e^{i\frac{4\pi}{\beta}\tau} \left[ -\frac{1}{2}\left(1 - \frac{1}{n^2}\right) \frac{F'\left(e^{i\frac{2\pi}{\beta}\tau}\right)^2}{F\left(e^{i\frac{2\pi}{\beta}\tau}\right)^2} - \{F, e^{i\frac{2\pi}{\beta}\tau}\} \right] + c.c.
\end{aligned}$$

We now have an EOM entirely written on the orbifold  $\mathcal{M}_n$ , when  $n \sim 1$ . Note that  $\theta(\tau + \beta) = \theta(\tau) + \beta$ . The time reflection symmetry of the twist fields insertions allows us to choose a solution satisfying  $\theta(-\tau) = -\theta(\tau)$  which automatically obeys  $\theta(0) = 0$ ,  $\theta''(0) = 0$ , and as a result we should also impose  $\theta(\beta/2) = \beta/2$  and  $\theta''(\beta/2) = 0$ . This EOM can be considerably simplified thanks to perturbative analysis and gauge-fixing, as we show below.

We now expand perturbatively

$$\theta(\tau) = \tau + \delta\theta(\tau), \quad (4.30)$$

hence

$$\theta'(\tau) = 1 + \delta\theta'(\tau), \quad (4.31)$$

where  $\delta\theta(\tau)$  is of the order  $o(n-1)$  and all higher derivatives of  $\theta(\tau)$  with respect to  $\tau$  are at least of the order  $o(n-1)$ .

Keeping all the terms up to the order  $o(n-1)$  and noting that all  $U_i(\theta(\tau))$  (4.20) (4.21)(4.22) and their derivative are already at least of the order  $o(n-1)$ , only a few terms on the LHS of (4.29) survive. We have

$$\begin{aligned}
\text{LHS} &= \frac{3l^3}{2cG_2} \left( \frac{3\pi^4\theta''(\tau)}{\beta^4} + \frac{11\pi^4\theta'(\tau)^2U_1'(\theta(\tau))}{3\beta^4} - \frac{50}{3}\theta'(\tau)^2U_3'(\theta(\tau)) + \theta'(\tau)^2U_2'''(\theta(\tau)) - \frac{1}{12}\theta'(\tau)^2U_1^{(5)}(\theta(\tau)) \right) \\
&= \frac{3l^3}{2cG_2} \left( \frac{3\pi^4}{\beta^4}\delta\theta''(\tau) + \frac{11\pi^4U_1'(\tau)}{3\beta^4} - \frac{50}{3}U_3'(\tau) + U_2'''(\tau) - \frac{1}{12}U_1^{(5)}(\tau) \right) + o(n-1)^2.
\end{aligned} \quad (4.32)$$

Meanwhile, it is shown in Appendix B of [11] that if  $\theta(\tau)$  is expanded in Fourier series

$$\theta(\tau) = \tau + \sum_{m=-\infty}^{\infty} c_m e^{i\frac{2m\pi}{\beta}\tau}, \quad (4.33)$$

then for any given  $\delta\theta(\tau)$ , one can always gauge-fix the ambiguities in the Fourier expansions of both functions  $G$  and  $F$

$$G(w) = w + \sum_{l=0}^{\infty} g_l w^l, \quad F(v) = v + \sum_{l=-\infty}^2 f_l v^l, \quad (4.34)$$

such that there is a unique solution where

$$g_0 = f_1 = f_2 = 0 \quad \text{and} \quad f_l = ic_{l-1} \text{ for } l \leq 0, \quad g_l = -ic_{l-1} \text{ for } l > 0. \quad (4.35)$$

On this account, after a little bit of algebra one can show the second term on the RHS of (4.29) becomes

$$e^{i\frac{4\pi}{\beta}\tau} \{F, e^{i\frac{2\pi}{\beta}\tau}\} = - \left( \delta\theta'(\tau) + \frac{\beta^2}{4\pi^2} \delta\theta'''(\tau) \right)_- = i\frac{2\pi}{\beta} \sum_{l=-\infty}^{-2} l(l^2 - 1)c_l e^{i\frac{2\pi l}{\beta}\tau} \quad (4.36)$$

where the minus subscript indicates that the expression inside the parenthesis is projected onto the modes of negative frequency.

At the same time, note that  $F$  is already multiplied by a factor of  $(n-1)$  in the first term on the RHS of (4.29), we can thus set  $F = \frac{v-A}{B-v}$  per our convention below (4.25), ignoring the effect of conformal welding. We have

$$\frac{1}{2} \left( 1 - \frac{1}{n^2} \right) \frac{F'\left(e^{i\frac{2\pi}{\beta}\tau}\right)^2}{F\left(e^{i\frac{2\pi}{\beta}\tau}\right)^2} = (n-1) \cdot \frac{(A-B)^2}{\left(A - e^{i\frac{2\pi}{\beta}\tau}\right)^2 \left(B - e^{i\frac{2\pi}{\beta}\tau}\right)^2}. \quad (4.37)$$

Ultimately, the perturbed EOM (4.29) can be simplified to

$$\begin{aligned} & \frac{9l^3\pi^4}{2cG_2\beta^4}\delta\theta''(\tau) + \left[ -i \left( \delta\theta'''(\tau) + \frac{4\pi^2}{\beta^2}\delta\theta'(\tau) \right) + c.c. \right] \\ &= -\frac{3l^3}{2cG_2} \left( \frac{11\pi^4 U_1'(\tau)}{3\beta^4} - \frac{50}{3}U_3'(\tau) + U_2'''(\tau) - \frac{1}{12}U_1^{(5)}(\tau) \right) + \frac{4\pi^2(n-1)}{\beta^2} \left[ -\frac{i(A-B)^2 e^{i\frac{4\pi}{\beta}\tau}}{\left(A - e^{i\frac{2\pi}{\beta}\tau}\right)^2 \left(B - e^{i\frac{2\pi}{\beta}\tau}\right)^2} + c.c. \right]. \end{aligned} \quad (4.38)$$

We see, unlike in JT gravity (see (3.27) of [11]) where all the terms on the LHS of the perturbative EOM (4.38) are the functionals of  $\delta\theta'''(\tau) + \frac{4\pi^2}{\beta^2}\delta\theta'(\tau)$ , we actually have a term proportional to  $\delta\theta''(\tau)$  on the LHS here. As a result, when we look at the  $k=1$  Fourier mode of the LHS of (4.38), the second term vanishes because of (4.36) but the Fourier mode of  $\delta\theta''(\tau)$  survives and we can **NOT** extract the extremity condition just from the  $k=1$  Fourier mode of the RHS of (4.38) as in [11]. Actually, we need to directly solve  $\delta\theta(\tau)$  before we Fourier transform (4.38) and then plug it back in an attempt to recover the extremity condition (3.12).

Instead of going through it, we now show, by cutting corners, that the hypothesis about the leading saddle point of the replica geometry on  $\tilde{\mathcal{M}}_n$  is not the complete story because we can only recover from it one of two the stationary point of the  $S_{\text{gen}}(B') \equiv S_{\text{gen}}([-a, b])$  — the one at the bifurcation surface (throat) of the event horizon of an eternal Narayan black hole (located at  $x=1 \Leftrightarrow a=\infty \Leftrightarrow r=\frac{\beta}{\pi}$  in the context of subsection 3.1), not the true QES. So the leading point we hypothesized earlier is actually the subleading saddle point.

As shown in [11], there is actually a shortcut to derive the on-shell replica gravitational action from a given saddle point. We made a hypothesis earlier in this subsection that the leading saddle of the gravitational action on  $\tilde{\mathcal{M}}_n$  is still an Euclidean Narayan black hole of the same temperature  $\beta$  but "n times as large as" the original Narayan black hole solution on  $\mathcal{M}_1$ . As a result, according to the on-shell gravitational action of the original  $n=1$  Narayan black hole (4.10), we can immediately write down the gravitational action on  $\tilde{\mathcal{M}}_n$

$$I_{\text{grav}}[\tilde{\mathcal{M}}_n] = -\frac{l^3}{16\pi G_2} \int_{\partial\tilde{\mathcal{M}}_n} d\tau \left( \frac{3\pi^4\theta_n'(\tau)}{\beta^4} + \frac{21\theta_n''(\tau)^4}{32\theta_n'(\tau)^7} - \frac{3\theta_n'''(\tau)\theta_n''(\tau)^2}{8\theta_n'(\tau)^6} \right) \quad (4.39)$$

where  $\theta_n(\tau)$ , the counterpart of  $\theta(\tau)$  on the boundary of  $\tilde{\mathcal{M}}_n$ , replaces  $\theta(\tau)$  in (4.10). Using the expression of  $\theta_n$  at the boundary of  $\tilde{\mathcal{M}}_n$  (B.28), we can now return to the orbifold  $\mathcal{M}_n$  and obtain the on-shell replica gravitational action on  $\mathcal{M}_n$

$$\begin{aligned} I_{\text{grav}}[\mathcal{M}_n] &= \frac{1}{n} I_{\text{grav}}[\tilde{\mathcal{M}}_n] \\ &= -\frac{l^3}{16\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \left[ \left( \frac{3\pi^4\theta'(\tau)}{\beta^4} + \frac{21\theta''(\tau)^4}{32\theta'(\tau)^7} - \frac{3\theta'''(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} \right) + \left( -\frac{3\pi^4\theta'(\tau)U_1(\theta(\tau))}{\beta^4} + \frac{3\theta''(\tau)^2 U_1''(\theta(\tau))}{8\theta'(\tau)^3} \right. \right. \\ &\quad \left. \left. - \frac{3\theta''(\tau)^3 U_1'(\theta(\tau))}{2\theta'(\tau)^5} + \frac{3\theta'''(\tau)\theta''(\tau)U_1'(\theta(\tau))}{4\theta'(\tau)^4} + \frac{63\theta''(\tau)^4 U_1(\theta(\tau))}{32\theta'(\tau)^7} - \frac{9\theta'''(\tau)\theta''(\tau)^2 U_1(\theta(\tau))}{8\theta'(\tau)^6} \right) \right] \end{aligned} \quad (4.40)$$

whose  $o(n-1)$  order perturbation terms are formally different from those in the same on-shell replica gravitational action on  $\mathcal{M}_n$  we obtained earlier (4.23).

However, the two results (4.23) and (4.40) for the same on-shell replica gravitational action  $I_{\text{grav}}[\mathcal{M}_n]$  derived through different methods but from the same hypothesis — the leading saddle point of  $I_{\text{grav}}[\tilde{\mathcal{M}}_n]$  is still an Euclidean Narayan black hole of the same temperature  $\beta$  — must be equal if the replica metric we used to derive (4.23) and (4.40) is the true on-shell replica metric, meaning not only the replica metric takes the form in (4.18) - (4.22) but  $A = e^{-\frac{2\pi}{\beta}a}$  inside the metric perturbation functions —  $U_i(\theta)$ ,  $i=1, 2, 3$  — indeed takes the right value for the on-shell replica geometry. One important observation is that, if the conical singularity/branch point on  $\mathcal{M}_n$  lies at  $A = e^{-\frac{2\pi}{\beta}a} = 0 \Leftrightarrow a = \infty$  (see 4.13), we have

$$U_1(\theta) = n-1, \quad U_2(\theta) = 0, \quad U_3(\theta) = \frac{2\pi^4}{5\beta^4}(n-1) \quad (4.41)$$

by setting  $A = 0$  in (4.20), (4.21) and (4.22). Plugging (4.41) into (4.23), we get

$$\begin{aligned}
I_{\text{grav}}[\mathcal{M}_n] &= -\frac{l^3}{16\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \left[ \left( \frac{3\pi^4\theta'(\tau)}{\beta^4} + \frac{21\theta''(\tau)^4}{32\theta'(\tau)^7} - \frac{3\theta'''(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} \right) + (n-1) \cdot \left( \frac{11\pi^4\theta'(\tau)}{3\beta^4} \right. \right. \\
&\quad \left. \left. - \frac{50}{3}\theta'(\tau) \cdot \frac{2\pi^4}{5\beta^4} + \frac{63\theta''(\tau)^4}{32\theta'(\tau)^7} - \frac{9\theta'''(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} \right) \right] \\
&= -\frac{l^3}{16\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \left[ \left( \frac{3\pi^4\theta'(\tau)}{\beta^4} + \frac{21\theta''(\tau)^4}{32\theta'(\tau)^7} - \frac{3\theta'''(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} \right) + (n-1) \cdot \left( -\frac{3\pi^4\theta'(\tau)}{\beta^4} \right. \right. \\
&\quad \left. \left. + \frac{63\theta''(\tau)^4}{32\theta'(\tau)^7} - \frac{9\theta'''(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} \right) \right]. \tag{4.42}
\end{aligned}$$

On the other hand, again plugging (4.41) into (4.40), we get the result finally matching that in (4.42)

$$\begin{aligned}
I_{\text{grav}}[\mathcal{M}_n] &= \frac{1}{n} I_{\text{grav}}[\tilde{\mathcal{M}}_n] \\
&= -\frac{l^3}{16\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \left[ \left( \frac{3\pi^4\theta'(\tau)}{\beta^4} + \frac{21\theta''(\tau)^4}{32\theta'(\tau)^7} - \frac{3\theta'''(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} \right) + (n-1) \cdot \left( -\frac{3\pi^4\theta'(\tau)}{\beta^4} \right. \right. \\
&\quad \left. \left. + \frac{63\theta''(\tau)^4}{32\theta'(\tau)^7} - \frac{9\theta'''(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} \right) \right]. \tag{4.43}
\end{aligned}$$

Therefore, we see from the identical results in (4.42) and (4.43) that, even without explicitly solving the EOM of the boundary mode  $\theta(\tau)$  (4.38), the saddle point we chose in our central hypothesis (we filled in an Euclidean Narayan black hole of the same temperature  $\beta$  inside the boundary of covering manifold  $\tilde{\mathcal{M}}_n$ ) mandates the conical singularity/branch point on the orbifold  $\mathcal{M}_n$  be exactly at  $w = A = e^{-\frac{2\pi}{\beta}a} = 0 \Leftrightarrow a = \infty$ . Otherwise, there would be other values of  $A = e^{-\frac{2\pi}{\beta}a}$  where the two results (4.23) and (4.40) for the same on-shell replica gravitational action  $I_{\text{grav}}[\mathcal{M}_n]$  match, which leads to the existence of other conical singularities/branch points in contradiction with the fact that the disk topology of the saddle point we hypothesized can lead to only one conical singularity/branch point on the orbifold  $\mathcal{M}_n$ . The perturbed equation of motion for the boundary mode  $\theta(\tau)$  associated with the hypothesized saddle point is thus further simplified from (4.38) to

$$\frac{9l^3\pi^4}{2cG_2\beta^4} \delta\theta''(\tau) + \left[ -i \left( \delta\theta'''(\tau) + \frac{4\pi^2}{\beta^2} \delta\theta'(\tau) \right)_- + c.c. \right] = \frac{4\pi^2(n-1)}{\beta^2} \left[ -\frac{iB^2}{\left( B - e^{i\frac{2\pi}{\beta}\tau} \right)^2} + c.c. \right] \tag{4.44}$$

after we plugged  $A = 0$  into (4.38), which we will not try to solve in this article. Going back to Lorentzian signature, this conical singularity/branch point translates into  $a = \infty \Leftrightarrow r = \beta/\pi$  at the bifurcation surface (throat) of the event horizon at  $u = 0$ , **so we just recovered one of the two stationary points — the stationary point other than, but generally still very close to, the QES in the one-interval scenario (see the analysis below (3.13)).**

Based on the arguments in subsection 1.2, the results in subsection 3.1 show that there are actually two saddle points of the Euclidean Narayan action leading to two stationary points of the generalized entropy in the one-interval case — the bifurcation surface(throat) of the event horizon and the true QES. The saddle point we earlier hypothesized to be the leading saddle point, namely a standard Euclidean Narayan black hole of the same temperature  $\beta$  but larger, is actually the subleading saddle point. To recover the other stationary point, the true QES, we need to find and fill in the leading saddle point of Euclidean Narayan action. This leading saddle point has inverse temperature  $\beta$  and, at the same time, satisfying the boundary conditions for the metric (2.42) and dilaton (2.47) as the replica wormhole geometry. We have yet found an answer to that so far. It is an interesting open problem and some thoughts are shared in the next section.



## 5 Conclusion and Discussion

In this article, we first built an eternal black hole on the basis of maximally extended Narayan black hole geometry coupled with two flat baths and then we established a new version of black hole information paradox in the setting of the same CFT living on an eternal vacuum Narayan black hole joined to flat baths on both sides. We derived the extremity conditions and analyzed the existence and the locations of the QES/island solutions in both one-interval and two-interval scenarios and found that there are both similarities and remarkable differences from the JT gravity case. In the two-interval setting which constitutes our version black hole information paradox, we found that, just as in JT gravity, the Hawking saddle is exponentially suppressed in the Euclidean path integral as the time goes on and the entropy continues to grow inside the boundary. Therefore, another saddle of a different topology, in this case the replica wormhole, dominates because the associated generalized entropy finally saturates at that of two copies of single interval entropy at late times and the contribution in the Euclidean path integral stops to diminish for long periods. Although we provided solid justifications for the QES prescription we used to write down the extremity condition and calculate the fined-grained entropy of the radiation and the two-side eternal Narayan black hole, for the one-interval scenario we only partially successfully constructed the replica geometry on the covering manifold (the subleading saddle point), having recovered one of the two stationary point, but did not recover the extremity condition itself and the other stationary point — the true QES. It is intriguing to guess what the other solution of the Narayan action satisfying the boundary conditions and leading to the true QES is. Here we share some thoughts on this open problem.

Fundamentally different from the JT gravity where a constant curvature  $R = -2$  Euclidean  $AdS_2$  saddle point is dictated by the JT action on  $\mathcal{M}_1$  by varying with respect to the dilaton  $\phi$  in the bulk,

$$I_{\text{JT}}[\mathcal{M}_1] = -\frac{1}{16\pi G_2} \left[ \int_{\mathcal{M}_1} \phi(R+2) + 2\phi_b \int_{\partial\mathcal{M}_1} (K+1) \right] + (\text{topological}), \quad (5.1)$$

there are likely more than one classical solutions to the Narayan action on  $\mathcal{M}_1$

$$I_{\text{Narayan}}[\mathcal{M}_1] = -\frac{1}{16\pi G_2} \left[ \int_{\mathcal{M}_1} (\phi R + 12l^{-2}\phi^{-\frac{1}{3}}) + 2 \int_{\partial\mathcal{M}_1} \phi K - \frac{6}{l} \int_{\partial\mathcal{M}_1} \phi^{\frac{2}{3}} \right] + (\text{topological}) \quad (5.2)$$

are of inverse temperature  $\beta$  and also satisfying the boundary conditions for both the metric (2.42) and the dilaton (2.47). To account for the true QES in the one-interval scenario, we need to find a classical solution other than the Narayan black hole satisfying all the boundary conditions and can be shown to have inverse temperature  $\beta$ . On top of it, the on-shell gravitational action of this solution on  $\mathcal{M}_1$ ,  $I_{\text{Narayan, the other solution}}[\mathcal{M}_1]$ , should be no smaller than the on-shell gravitational action of the Narayan black hole solution (2.80) dominating the Narayan black hole saddle so that we still have the correct gravitational entropy for the Narayan black hole (2.82). It is a very interesting open problem to find this solution, as it concerns the evaluation of the saddle points of Euclidean path integral in a theory other than JT gravity, which is obviously more complicated than JT gravity but is relatively simple given that it has a conformally  $AdS_2$  solution.

We now move on to highlight some similarities and differences in regard to the eternal black holes in Narayan theory and JT and some other possibilities in future developments.

In section 2 we imposed a kinetic constraint for the boundary mode (2.46) and constructed a consistent eternal black hole on the basis of it, the same construction can be generalized to Narayan theory with  $d_i > 3$  without fundamental obstacles despite an algebraic mess. However, unlike JT where there is no inconsistency on the boundary conditions to begin with, it is problematic when a black hole joined by two baths is not eternal (say an evaporating black hole) in the Narayan theory and the flux across the boundary is not infinitesimal, say of order 1: if we insist the kinetic constraint (2.46) be imposed: there is no solution to (2.58) satisfying (2.46). Therefore, if we intend to study the non-eternal Narayan black holes, we need to find a new way to join the baths to the black hole, which may involve baths with a curved but non-dynamic metric (possibly conformally flat) or a natural but a different way than (2.41) to extend the Rindler coordinates on flat bath into the Narayan black hole. It is challenging but interesting to design a more general formalism of joining baths to asymptotically conformally  $AdS_2$  geometries as it allows for evaporating black holes which can lead to interesting dynamics, providing more insights to the black hole information paradox in 2D.

In section 3, some interesting results were presented. In one-interval scenario in eternal Narayan black hole, like in the JT construction, the QES always exists and lies somewhere to the right of the bifurcation surface (throat) of the event horizon, albeit generally even closer to the bifurcation surface. In the physically significant two-interval scenario, although at late times a Page curve is produced just as in JT gravity, we found that real solutions to extremity condition for islands always exist, independent on the parameters and even before Page time, which is different than the same scenario in JT.

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## A Calculation of the extrinsic curvature in an $n = 1$ vacuum Narayan black hole

In this appendix we calculate the extrinsic curvature on any curve of constant  $\sigma$  in a vacuum Narayan black hole of inverse temperature  $\beta$ , which will be used in subsection 2.3 to compute the on-shell gravitational action of a vacuum Narayan black hole in Lorentzian signature. (As to Euclidean vacuum Narayan black hole, we get the same result up to several opposite signs.)

For convenience, we choose to set up the calculation in the tortoise coordinates  $\{t, r_*\}$  and name the curve of constant value of  $\sigma$  in question  $\Sigma$ . As the Rindler coordinates  $\{u, \sigma\}$  are extended from the baths to the vacuum Narayan black hole, even though we do not know as of yet the exact form of this coordinate extension,  $\Sigma$  is by definition a  $u$  coordinate line. We adopt the convention that the derivatives with respect to  $u$  is denoted by a prime mark ' and the derivatives with respect to  $\sigma$  is denoted by dot notation and from (2.41) we have one important observation

$$\begin{aligned} t' &= \frac{\partial t}{\partial x^+} \frac{dx^+}{dy^+} \frac{dy^+}{du} + \frac{\partial t}{\partial x^-} \frac{dx^-}{dy^-} \frac{dy^-}{du} = \frac{1}{2} \left( \frac{dx^+}{dy^+} + \frac{dx^-}{dy^-} \right), \\ \dot{t} &= \frac{\partial t}{\partial x^+} \frac{dx^+}{dy^+} \frac{dy^+}{d\sigma} + \frac{\partial t}{\partial x^-} \frac{dx^-}{dy^-} \frac{dy^-}{d\sigma} = \frac{1}{2} \left( \frac{dx^+}{dy^+} - \frac{dx^-}{dy^-} \right), \\ r_*' &= \frac{\partial r_*}{\partial x^+} \frac{dx^+}{dy^+} \frac{dy^+}{du} + \frac{\partial r_*}{\partial x^-} \frac{dx^-}{dy^-} \frac{dy^-}{du} = \frac{1}{2} \left( \frac{dx^+}{dy^+} - \frac{dx^-}{dy^-} \right), \\ \dot{r}_* &= \frac{\partial r_*}{\partial x^+} \frac{dx^+}{dy^+} \frac{dy^+}{d\sigma} + \frac{\partial r_*}{\partial x^-} \frac{dx^-}{dy^-} \frac{dy^-}{d\sigma} = \frac{1}{2} \left( \frac{dx^+}{dy^+} + \frac{dx^-}{dy^-} \right), \end{aligned} \tag{A.1}$$

therefore,

$$t' = \dot{r}_*, \quad \dot{t} = r_*'. \tag{A.2}$$

So now we have the following expression of the tangent vector  $t^a$  and outwardly pointing normal vector  $n^a$  on  $\Sigma$  according to the metric of vacuum Narayan black holes of inverse temperature  $\beta$  (2.24)

$$t^a = \frac{1}{l^2 \sqrt{\left(\frac{1}{r^4} - \frac{\pi^4}{\beta^4}\right) (t'^2 - r_*'^2)}} \left( \frac{\partial}{\partial u} \right)^a = \frac{t'}{l^2 \sqrt{\left(\frac{1}{r^4} - \frac{\pi^4}{\beta^4}\right) (t'^2 - r_*'^2)}} \left( \frac{\partial}{\partial t} \right)^a + \frac{r_*'}{l^2 \sqrt{\left(\frac{1}{r^4} - \frac{\pi^4}{\beta^4}\right) (t'^2 - r_*'^2)}} \left( \frac{\partial}{\partial r_*} \right)^a \tag{A.3}$$

and

$$n^a = -\frac{r_*'}{l^2 \sqrt{\left(\frac{1}{r^4} - \frac{\pi^4}{\beta^4}\right) (t'^2 - r_*'^2)}} \left( \frac{\partial}{\partial t} \right)^a - \frac{t'}{l^2 \sqrt{\left(\frac{1}{r^4} - \frac{\pi^4}{\beta^4}\right) (t'^2 - r_*'^2)}} \left( \frac{\partial}{\partial r_*} \right)^a, \tag{A.4}$$

$$n_a = \frac{l^2 r_*' \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{\sqrt{t'^2 - r_*'^2}} (dt)_a - \frac{l^2 t' \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{\sqrt{t'^2 - r_*'^2}} (dr_*)_a. \tag{A.5}$$

Meanwhile, the inverse metric and the non-zero Christoffel symbols in tortoise coordinates are

$$g^{tt} = -g^{r_* r_*} = -l^{-4} \left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right)^{-1}, \quad g^{tr_*} = g^{r_* t} = 0, \tag{A.6}$$

and

$$\Gamma_{tr_*}^t = \Gamma_{r_* t}^t = \Gamma_{tt}^{r_*} = \Gamma_{r_* r_*}^{r_*} = \frac{2\beta^4}{\pi^4 r^5 - \beta^4 r} * \frac{dr}{dr_*} = -\frac{2}{r}. \tag{A.7}$$

The extrinsic curvature  $K$  with respect to  $n^a$  on  $\Sigma$  is thus

$$\begin{aligned}
K &= g^{\mu\nu} \nabla_\mu n_\nu = g^{\mu\nu} (\partial_\mu n_\nu - \Gamma_{\mu\nu}^\rho n_\rho) = g^{tt} \partial_t n_t + g^{r_* r_*} \partial_{r_*} n_{r_*} \\
&= l^{-4} \left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right)^{-1} \left[ - \left( \frac{\partial u}{\partial t} n'_t + \frac{\partial \sigma}{\partial t} \dot{n}_t \right) + \left( \frac{\partial u}{\partial r_*} n'_{r_*} + \frac{\partial \sigma}{\partial r_*} n_{r_*} \right) \right] \\
&= l^{-4} \left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right)^{-1} \left[ - \frac{t'}{t'^2 - r_*'^2} \cdot \frac{\partial}{\partial u} \left( \frac{l^2 r'_* \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{\sqrt{t'^2 - r_*'^2}} \right) - \frac{-r'_*}{t'^2 - r_*'^2} \cdot \frac{\partial}{\partial \sigma} \left( \frac{l^2 r'_* \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{\sqrt{t'^2 - r_*'^2}} \right) \right] \\
&\quad + l^{-4} \left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right)^{-1} \left[ \frac{-r'_*}{t'^2 - r_*'^2} \cdot \frac{\partial}{\partial u} \left( \frac{-l^2 t' \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{\sqrt{t'^2 - r_*'^2}} \right) + \frac{t'}{t'^2 - r_*'^2} \cdot \frac{\partial}{\partial \sigma} \left( \frac{-l^2 t' \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{\sqrt{t'^2 - r_*'^2}} \right) \right] \\
&= -l^{-2} \left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right)^{-1} \times \\
&\quad \left\{ \frac{t'}{t'^2 - r_*'^2} \cdot \left[ \frac{r''_* \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{\sqrt{t'^2 - r_*'^2}} + \frac{r'_* \cdot \left( -\frac{4}{r^5} \frac{dr}{dr_*} \right) \cdot r'_*}{2 \sqrt{\left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right) (t'^2 - r_*'^2)}} - \frac{r'_* \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{2 (t'^2 - r_*'^2)^{\frac{3}{2}}} \cdot (2t' t'' - 2r'_* r''') \right] \right. \\
&\quad - \frac{r'_*}{t'^2 - r_*'^2} \cdot \left[ \frac{t'' \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{\sqrt{t'^2 - r_*'^2}} + \frac{r'_* \cdot \left( -\frac{4}{r^5} \frac{dr}{dr_*} \right) \cdot t'}{2 \sqrt{\left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right) (t'^2 - r_*'^2)}} - \frac{r'_* \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{2 (t'^2 - r_*'^2)^{\frac{3}{2}}} \cdot (2t' r''_* - 2r'_* t''') \right] \\
&\quad - \frac{r'_*}{t'^2 - r_*'^2} \cdot \left[ \frac{t'' \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{\sqrt{t'^2 - r_*'^2}} + \frac{r'_* \cdot \left( -\frac{4}{r^5} \frac{dr}{dr_*} \right) \cdot t'}{2 \sqrt{\left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right) (t'^2 - r_*'^2)}} - \frac{t' \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{2 (t'^2 - r_*'^2)^{\frac{3}{2}}} \cdot (2t' t'' - 2r'_* r''') \right] \\
&\quad \left. \frac{t'}{t'^2 - r_*'^2} \cdot \left[ \frac{r''_* \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{\sqrt{t'^2 - r_*'^2}} + \frac{t' \cdot \left( -\frac{4}{r^5} \frac{dr}{dr_*} \right) \cdot t'}{2 \sqrt{\left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right) (t'^2 - r_*'^2)}} - \frac{t' \sqrt{\frac{1}{r^4} - \frac{\pi^4}{\beta^4}}}{2 (t'^2 - r_*'^2)^{\frac{3}{2}}} \cdot (2t' r''_* - 2t' r''') \right] \right\} \\
&= -l^{-2} \left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right)^{-\frac{1}{2}} \times \left[ \frac{2 (t' r''_* - t'' r'_* + t' r^{-1} r_*'^2 - t'^3 r^{-1})}{(t'^2 - r_*'^2)^{\frac{3}{2}}} + \frac{r_*'^2 (t' r''_* - t'' r'_*) - t'^2 (t' r''_* - t'' r'_*)}{(t'^2 - r_*'^2)^{\frac{5}{2}}} \right] \\
&= l^{-2} r^{-1} (t'^2 - r_*'^2)^{-\frac{3}{2}} \left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right)^{-\frac{1}{2}} (2t'^3 - 2t' r_*'^2 + t'' r r'_* - t' r r''_*).
\end{aligned} \tag{A.8}$$

In Euclidean signature,  $K$  takes the following form

$$K = l^{-2} r^{-1} (t'^2 + r_*'^2)^{-\frac{3}{2}} \left( \frac{1}{r^4} - \frac{\pi^4}{\beta^4} \right)^{-\frac{1}{2}} (2t'^3 + 2t' r_*'^2 + t' r r''_* - t'' r r'_*). \tag{A.9}$$

## B An attempt to derive the on-shell replica gravitational action

In this appendix we offer a detailed derivation of the  $n \sim 1$  on-shell replica gravitational action based on our hypothesis about the leading saddle point of replica gravitational path integral above Figure 19. Although the results of this appendix only account for the saddle point we hypothesized, the method used here is more general and can be easily applied to other cases without significant changes.

The replica gravitational action on the orbifold  $\mathcal{M}_n$  is

$$\begin{aligned} I_{\text{grav}}[\mathcal{M}_n] &= \frac{1}{n} I_{\text{grav}}[\tilde{\mathcal{M}}_n] \\ &= -\frac{\phi_0}{16\pi G_2} \left( \int_{\mathcal{M}_n} R + 2 \int_{\mathcal{M}_n} K \right) - \frac{1}{16\pi G_2} \left[ \int_{\mathcal{M}_n} (\phi R + 12l^{-2}\phi^{\frac{1}{3}}) + 2 \int_{\partial\mathcal{M}_n} \phi K - \frac{6}{l} \int_{\partial\mathcal{M}_n} \phi^{\frac{2}{3}} \right] \\ &+ \frac{1}{4G_2} \left( 1 - \frac{1}{n} \right) \sum_{\alpha, \text{Cosmic Branes}} [\phi_0 + \phi(w_\alpha)], \end{aligned} \quad (\text{B.1})$$

To compute the two boundaries terms in (B.1), we need to calculate the extrinsic curvature  $K$  and the value of the dilaton  $\phi$  at the physical boundary  $\sigma = -\epsilon$ .

We start with the expansion of the replica metric near the boundary, (4.18) and (4.19),

$$\begin{aligned} ds_{\text{in}, \mathcal{M}_n}^2 &= \frac{\beta^2 l^4}{4\pi^2} \left( \frac{1}{r(|w|)^4} - \frac{\pi^4}{\beta^4} \right) \frac{1}{|w|^2} dw d\bar{w} [1 + 2U_1(\theta) + 2\gamma^2 U_2(\theta) + 2\gamma^4 U_3(\theta) + o(\gamma^6) + o(n-1)^2] \\ &= l^4 \left( \frac{1}{r \left( e^{\frac{2\pi}{\beta}\gamma} \right)^4} - \frac{\pi^4}{\beta^4} \right) (d\gamma^2 + d\theta^2) [1 + 2U_1(\theta) + 2\gamma^2 U_2(\theta) + 2\gamma^4 U_3(\theta) + o(\gamma^6) + o(n-1)^2] \end{aligned} \quad (\text{B.2})$$

where  $w = e^{\frac{2\pi}{\beta}(\gamma+i\theta)}$  with  $|w| < 1 \Leftrightarrow \gamma < 0$ ,  $r$  satisfies (4.12) and  $U_i(\theta)$ 's are of the form (4.20) (4.21) and (4.22),  $i = 1, 2, 3$ .

Here the boundary conditions for the metric is still

$$ds_{\text{bd}, \mathcal{M}_n}^2 = \frac{l^4}{\epsilon^4} (d\tau^2 + d\sigma^2) \quad (\text{B.3})$$

which gives rise to

$$\left( \frac{1}{r \left( e^{\frac{2\pi}{\beta}\gamma} \right)^4} - \frac{\pi^4}{\beta^4} \right) (\gamma'^2 + \theta'^2) [1 + 2U_1(\theta) + 2\gamma^2 U_2(\theta) + 2\gamma^4 U_3(\theta) + o(\gamma^6) + o(n-1)^2] = \frac{1}{\epsilon^4} \quad (\text{B.4})$$

in parallel to (2.48) at the physical boundary  $\sigma = -\epsilon$  (we still mark the derivative with respect to the Rindler time  $\tau$  with  $'$ ). In the same spirit of (2.49), we make another ansatz for  $\gamma$  at the boundary

$$\gamma_{\text{bd}}(\tau) = -\epsilon \theta'(\tau)^{\frac{1}{2}} \left( 1 + f_{\gamma 0}(\tau) + \epsilon f_{\gamma 1}(\tau) + \epsilon^2 f_{\gamma 2}(\tau) + \epsilon^3 f_{\gamma 3}(\tau) + \epsilon^4 f_{\gamma 4}(\tau) + o(\epsilon^5) \right) \quad (\text{B.5})$$

and plug it into (B.4). The solution, to the first order of  $(n-1)$ , is

$$f_{\gamma 0}(\tau) = \frac{1}{2} U_1(\theta(\tau)), \quad (\text{B.6})$$

$$f_{\gamma 2}(\tau) = \frac{1}{2} U_2(\theta(\tau)) \theta'(\tau), \quad (\text{B.7})$$

$$f_{\gamma 4}(\tau) = -\frac{\pi^4 \theta'(\tau)^2}{20\beta^4} + \left( -\frac{\pi^4 \theta'(\tau)^2 U_1(\theta(\tau))}{8\beta^4} + \frac{1}{8} \theta''(\tau) U_2'(\theta(\tau)) + \frac{3\theta''(\tau)^2 U_2(\theta(\tau))}{16\theta'(\tau)^2} + \frac{1}{2} \theta'(\tau)^2 U_3(\theta(\tau)) \right), \quad (\text{B.8})$$

$$f_{\gamma 1}(\tau) = f_{\gamma 3}(\tau) = 0, \quad (\text{B.9})$$

where  $U_i'(\theta(\tau))$  means the derivative of  $U_i(\theta(\tau))$  with respect to  $\theta(\tau)$ .

We see from the above equations that the replica wormhole geometry we hypothesized leads to  $o(n-1)$  perturbations in every order of  $\epsilon$  in (B.5), even the zeroth order, 1, receives  $o(n-1)$  perturbation.

We can now proceed to calculate the function  $r$  at the boundary through the following expansion of  $r$  in  $\gamma$  near the boundary

$$r = -\gamma + \frac{\pi^4 \gamma^5}{5\beta^4} + \frac{4\pi^8 r_*^9}{45\beta^8} + \dots, \quad (\text{B.10})$$

which is essentially the same thing as (2.43). After plugging in (B.5), we get

$$r_{\text{bd}}(\tau) = \epsilon \theta'(\tau)^{\frac{1}{2}} \left\{ 1 + \frac{1}{2} U_1(\theta(\tau)) + \frac{1}{2} \epsilon^2 \theta'(\tau) U_2(\theta(\tau)) + \epsilon^4 \left[ -\frac{\pi^4 \theta'(\tau)^2}{4\beta^4} + \left( \frac{1}{2} \theta'(\tau)^2 U_3(\theta(\tau)) - \frac{5\pi^4 \theta'(\tau)^2 U_1(\theta(\tau))}{8\beta^4} \right. \right. \right. \\ \left. \left. \left. + \frac{1}{8} \theta''(\tau) U_2'(\theta(\tau)) + \frac{3\theta''(\tau)^2 U_2(\theta(\tau))}{16\theta'(\tau)^2} \right) \right] \right\} \quad (\text{B.11})$$

which reduces to (2.52) when  $n = 1$  and switched back to Lorentzian signature. We can then plug (B.11) into the metric and compute the tangent vector  $t^\mu$  on and normal vector  $n^\mu$  to the boundary, which sets the conditions to calculate the extrinsic curvature  $K$  at the boundary,

$$K = t^\mu t^\nu \nabla_\mu n_\nu = \epsilon l^{-2} \theta'(\tau)^{\frac{1}{2}} \left\{ 2 - U_1(\theta(\tau)) + \epsilon^2 \left[ -\frac{\theta''(\tau)^2}{\theta'(\tau)^3} + \frac{\theta'''(\tau)}{2\theta'(\tau)^2} + \left( \frac{1}{2} \theta'(\tau) U_1''(\theta(\tau)) + \frac{\theta''(\tau) U_1'(\theta(\tau))}{2\theta'(\tau)} \right. \right. \right. \\ \left. \left. \left. - 3\theta'(\tau) U_2(\theta(\tau)) - \frac{\theta'''(\tau) U_1(\theta(\tau))}{4\theta'(\tau)^2} - \frac{\theta''(\tau)^2 U_1(\theta(\tau))}{2\theta'(\tau)^3} \right) \right] + \epsilon^4 \left[ \frac{\pi^4 \theta'(\tau)^2}{2\beta^4} + \frac{21\theta''(\tau)^4}{64\theta'(\tau)^6} - \frac{3\theta'''(\tau)\theta''(\tau)^2}{16\theta'(\tau)^5} \right. \right. \\ \left. \left. + \left( \frac{3\pi^4 \theta'(\tau)^2 U_1(\theta(\tau))}{4\beta^4} + \frac{1}{2} \theta'(\tau)^2 U_2''(\theta(\tau)) + \frac{7}{4} \theta''(\tau) U_2'(\theta(\tau)) - \frac{3\theta''(\tau)^2 U_2(\theta(\tau))}{8\theta'(\tau)^2} - 5\theta'(\tau)^2 U_3(\theta(\tau)) \right. \right. \right. \\ \left. \left. \left. - \frac{3\theta''(\tau)^2 U_1''(\theta(\tau))}{16\theta'(\tau)^2} + \frac{\theta''(\tau)^3 U_1'(\theta(\tau))}{2\theta'(\tau)^4} - \frac{3\theta'''(\tau)\theta''(\tau) U_1'(\theta(\tau))}{8\theta'(\tau)^3} + \frac{63\theta''(\tau)^4 U_1(\theta(\tau))}{128\theta'(\tau)^6} + \frac{3\theta'''(\tau) U_2(\theta(\tau))}{4\theta'(\tau)} \right. \right. \right. \\ \left. \left. \left. - \frac{9\theta'''(\tau)\theta''(\tau)^2 U_1(\theta(\tau))}{32\theta'(\tau)^5} \right) \right] + o(\epsilon^6) \right\} \quad (\text{B.12})$$

which reduces to (2.54) if we take  $n = 1$  — all  $U_i(\theta(\tau))$ 's thus vanish because they are of the order  $o(n - 1)$  — and switch from Euclidean back to Lorentzian signature.

The dilaton  $\phi$  on  $\tilde{\mathcal{M}}_n$  is an integral part of the leading saddle point solution we hypothesized, together with the replica metric (4.16). Therefore,  $\phi$  should take the following form

$$\phi = \frac{l^3}{r_n^3} \quad (\text{B.13})$$

where  $r_n$  is defined and related to  $w_n$  and  $w$  through relation (4.17) and (4.14).

In order to find the expression of  $r_n$  we need to first relate it to  $r$ , whose boundary expression has already been obtained in (B.11). From the equation that defines  $r_n$  (4.17), (4.14) and  $w = e^{\frac{2\pi}{\beta}(\gamma+i\theta)}$  we have

$$\log |w_n| = \log \left| \frac{\tilde{w} + A}{1 + A\tilde{w}} \right| = \log \left| \frac{\left( \frac{w-A}{1-Aw} \right)^{1/n} + A}{1 + A \left( \frac{w-A}{1-Aw} \right)^{1/n}} \right| = \log \left| \frac{\left( \frac{e^{\frac{2\pi}{\beta}(\gamma+i\theta)} - A}{1 - A e^{\frac{2\pi}{\beta}(\gamma+i\theta)}} \right)^{1/n} + A}{1 + A \left( \frac{e^{\frac{2\pi}{\beta}(\gamma+i\theta)} - A}{1 - A e^{\frac{2\pi}{\beta}(\gamma+i\theta)}} \right)^{1/n}} \right| \\ = - \left( \arctan \frac{\pi r_n}{\beta} + \operatorname{arctanh} \frac{\pi r_n}{\beta} \right). \quad (\text{B.14})$$

Meanwhile from (4.12) we also have

$$\log |w| = \frac{2\pi\gamma}{\beta} = - \left( \arctan \frac{\pi r}{\beta} + \operatorname{arctanh} \frac{\pi r}{\beta} \right). \quad (\text{B.15})$$

It is convenient to expand  $r_n$  as the perturbed ansatz below

$$r_n = r + (n - 1)\delta r \quad (\text{B.16})$$

and plug it into (B.14). Next, by subtracting (B.15) from (B.14) and then expanding both sides in  $(n-1)$ , we get

$$\begin{aligned} \delta r = & \frac{\beta^4 - \pi^4 r^4}{4\pi\beta^3} (A^2 - 1) \left[ \left( -A^2 + Ae^{\frac{2\pi(-\gamma+i\theta)}{\beta}} + Ae^{\frac{2\pi(\gamma-i\theta)}{\beta}} - 1 \right) \log \left( \frac{A - e^{\frac{2\pi(\gamma-i\theta)}{\beta}}}{-1 + Ae^{\frac{2\pi(\gamma-i\theta)}{\beta}}} \right) \right. \\ & \left. + \left( -A^2 + Ae^{\frac{2\pi(\gamma+i\theta)}{\beta}} + Ae^{-\frac{2\pi(\gamma+i\theta)}{\beta}} - 1 \right) \log \left( \frac{A - e^{\frac{2\pi(\gamma+i\theta)}{\beta}}}{-1 + Ae^{\frac{2\pi(\gamma+i\theta)}{\beta}}} \right) \right]. \end{aligned} \quad (\text{B.17})$$

Plugging (B.5) into (B.17), we get a boundary expression of  $\delta r$ . We then plug it and (B.11) into (B.16), we finally have the boundary expression for  $r_n$ ,

$$\begin{aligned} r_{n,\text{bd}}(\tau) = & \epsilon\theta'(\tau)^{\frac{1}{2}} \left\{ 1 - \frac{1}{2}U_1(\theta(\tau)) + \frac{3}{2}\epsilon^2\theta'(\tau)U_2(\theta(\tau)) + \epsilon^4 \left[ -\frac{\pi^4\theta'(\tau)^2}{4\beta^4} + \left( \frac{7\pi^4\theta'(\tau)^2U_1(\theta(\tau))}{24\beta^4} + \frac{5}{6}\theta'(\tau)^2U_3(\theta(\tau)) \right. \right. \right. \\ & \left. \left. + \frac{1}{8}\theta''(\tau)U_2'(\theta(\tau)) + \frac{3\theta''(\tau)^2U_2(\theta(\tau))}{16\theta'(\tau)^2} \right) \right] \right\}. \end{aligned} \quad (\text{B.18})$$

We are then all set to compute  $\phi_{\text{bd}}$  by plugging (B.18) into (B.13), it gives

$$\begin{aligned} \phi_{\text{bd}} = & l^3 \left\{ \frac{1}{\epsilon^3} \left( \frac{1}{\theta'(\tau)^{3/2}} + \frac{3U_1(\theta(\tau))}{2\theta'(\tau)^{3/2}} \right) - \frac{1}{\epsilon} \left( \frac{9U_2(\theta(\tau))}{2\theta'(\tau)^{1/2}} \right) + \epsilon \left[ \frac{3\pi^4\theta'(\tau)^{1/2}}{4\beta^4} + \left( \frac{5\pi^4\theta'(\tau)^{1/2}U_1(\theta(\tau))}{8\beta^4} \right. \right. \\ & \left. \left. - \frac{3\theta''(\tau)U_2'(\theta(\tau))}{8\theta'(\tau)^{3/2}} - \frac{9\theta''(\tau)^2U_2(\theta(\tau))}{16\theta'(\tau)^{7/2}} - \frac{5}{2}\theta'(\tau)^{1/2}U_3(\theta(\tau)) \right) \right] - \epsilon^3 \left( \frac{9\pi^4\theta'(\tau)^{3/2}U_2(\theta(\tau))}{2\beta^4} \right) \right\} \end{aligned} \quad (\text{B.19})$$

and

$$\begin{aligned} \phi_{\text{bd}}^{2/3} = & l^2 \left\{ \frac{1}{\epsilon^2} \left( \frac{1}{\theta'(\tau)} + \frac{U_1(\theta(\tau))}{\theta'(\tau)} \right) - 3U_2(\theta(\tau)) + \epsilon^2 \left[ \frac{\pi^4\theta'(\tau)}{2\beta^4} + \left( \frac{\pi^4\theta'(\tau)U_1(\theta(\tau))}{6\beta^4} - \frac{5}{3}\theta'(\tau)U_3(\theta(\tau)) \right. \right. \right. \\ & \left. \left. - \frac{\theta''(\tau)U_2'(\theta(\tau))}{4\theta'(\tau)} - \frac{3\theta''(\tau)^2U_2(\theta(\tau))}{8\theta'(\tau)^3} \right) \right] \right\}. \end{aligned} \quad (\text{B.20})$$

The two boundary terms in the replica gravitational action (B.1) are thus

$$\begin{aligned} I_{\text{grav, bd, 1}}[\mathcal{M}_n] = & \frac{1}{16\pi G_2} \cdot 2 \int_{\partial\mathcal{M}_n} \phi K = \frac{1}{8\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \cdot \frac{l^2}{\epsilon^2} \cdot \phi K \\ = & \frac{l^3}{16\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \left\{ \frac{1}{\epsilon^4} \left( \frac{4}{\theta'(\tau)} + \frac{4U_1(\theta(\tau))}{\theta'(\tau)} \right) + \frac{1}{\epsilon^2} \left[ \frac{\theta'''(\tau)}{\theta'(\tau)^3} - \frac{2\theta''(\tau)^2}{\theta'(\tau)^4} + \left( U_1''(\theta(\tau)) + \frac{\theta''(\tau)U_1'(\theta(\tau))}{\theta'(\tau)^2} \right. \right. \right. \\ & \left. \left. - \frac{4\theta''(\tau)^2U_1(\theta(\tau))}{\theta'(\tau)^4} - 24U_2(\theta(\tau)) + \frac{2\theta'''(\tau)U_1(\theta(\tau))}{\theta'(\tau)^3} \right) \right] + \left( \frac{4\pi^4\theta'(\tau)U_1(\theta(\tau))}{\beta^4} + \theta'(\tau)U_2''(\theta(\tau)) - 20\theta'(\tau)U_3(\theta(\tau)) \right. \\ & \left. - \frac{3\theta''(\tau)^2U_1'(\theta(\tau))}{8\theta'(\tau)^3} + \frac{\theta''(\tau)^3U_1'(\theta(\tau))}{\theta'(\tau)^5} + \frac{63\theta''(\tau)^4U_1(\theta(\tau))}{32\theta'(\tau)^7} + \frac{2\theta''(\tau)U_2'(\theta(\tau))}{\theta'(\tau)} + \frac{6\theta''(\tau)^2U_2(\theta(\tau))}{\theta'(\tau)^3} \right. \\ & \left. \left. - \frac{3\theta'''(\tau)\theta''(\tau)U_1'(\theta(\tau))}{4\theta'(\tau)^4} - \frac{9\theta'''(\tau)\theta''(\tau)^2U_1(\theta(\tau))}{8\theta'(\tau)^6} - \frac{3\theta''(\tau)U_2(\theta(\tau))}{\theta'(\tau)^2} \right) \right\} \end{aligned} \quad (\text{B.21})$$

and

$$\begin{aligned} I_{\text{grav, bd, 2}}[\mathcal{M}_n] = & -\frac{1}{16\pi G_2} \cdot \frac{6}{l} \int_{\partial\mathcal{M}_n} \phi^{\frac{2}{3}} = -\frac{3}{8\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \cdot \frac{l}{\epsilon^2} \cdot \phi^{\frac{2}{3}} \\ = & \frac{l^3}{16\pi G_2} \int_{\partial\mathcal{M}_n} d\tau \left\{ -\frac{1}{\epsilon^4} \left( \frac{6}{\theta'(\tau)} + \frac{6U_1(\theta(\tau))}{\theta'(\tau)} \right) + \frac{1}{\epsilon^2} \cdot 18U_2(\theta(\tau)) + \left[ -\frac{3\pi^4\theta'(\tau)}{\beta^4} + \left( -\frac{\pi^4\theta'(\tau)U_1(\theta(\tau))}{\beta^4} \right. \right. \right. \\ & \left. \left. + 10\theta'(\tau)U_3(\theta(\tau)) + \frac{3\theta''(\tau)U_2'(\theta(\tau))}{2\theta'(\tau)} + \frac{9\theta''(\tau)^2U_2(\theta(\tau))}{4\theta'(\tau)^3} \right) \right] \right\}. \end{aligned} \quad (\text{B.22})$$

The remaining challenge that we have never been confronted with in JT gravity — the problem of evaluating the bulk term in action (B.1) which initially involves conical singularities and cosmic brane terms cancelling them — can be circumvented by converting it to another equivalent evaluation much easier to tackle, namely that we directly evaluate the bulk term on  $\tilde{\mathcal{M}}_n$  and then divide it by  $n$ , altogether doing away with the singularities from the very beginning of our calculation. One important note before we start off — though we can cut corners tremendously by skillfully switching between two pictures, the covering manifold  $\tilde{\mathcal{M}}_n$  and the orbifold  $\mathcal{M}_n$  of the replica geometry, we must take heed of exactly which boundary time we are integrating over on  $\tilde{\mathcal{M}}_n$ .

Following the definition  $w = e^{\frac{2\pi}{\beta}(\gamma+i\theta)}$ , we define "space" coordinate  $\gamma_n$  and "time" coordinate  $\theta_n$  for the auxiliary coordinate  $w_n$  defined on  $\tilde{\mathcal{M}}_n$  by

$$w_n = e^{\frac{2\pi}{\beta}(\gamma_n+i\theta_n)}. \quad (\text{B.23})$$

The metric (4.16) on  $\tilde{\mathcal{M}}_n$  can thus be alternatively written as

$$ds_{\text{in},\tilde{\mathcal{M}}_n}^2 = \frac{\beta^2 l^4}{4\pi^2} \left( \frac{1}{r_n^4} - \frac{\pi^4}{\beta^4} \right) (d\gamma_n^2 + d\theta_n^2) = \frac{l^4}{r_n^4} \left[ \left( 1 - \frac{\pi^4 r_n^4}{\beta^4} \right) d\theta_n^2 + \left( 1 - \frac{\pi^4 r_n^4}{\beta^4} \right)^{-1} dr_n^2 \right] \quad (\text{B.24})$$

where  $r_n$  and  $\gamma_n$  are related by

$$\gamma_n = -\frac{\beta}{2\pi} \left( \arctan \frac{\pi r_n}{\beta} + \operatorname{arctanh} \frac{\pi r_n}{\beta} \right) \quad (\text{B.25})$$

in parallel with (2.23) according to (4.17) and (B.23).

Next up, here comes the more intricate part - what is the relationship between the two boundary times —  $\theta_n$  and  $\theta$  — on the boundary? To answer it, we first obtain  $\gamma_n$  at the boundary through  $r_{n,\text{bd}}(\tau)$  (B.18) and asymptotic expansion of (B.25) near/at the boundary, then we extract  $\theta_{n,\text{bd}}$  as a function of  $\theta$  from (B.23),(4.15) and  $w = e^{\frac{2\pi}{\beta}(\gamma+i\theta)}$  combined. Since the geometry on  $\tilde{\mathcal{M}}_n$  is the same Narayan black hole of inverse temperature  $\beta$ , we have from (B.25) and (B.18)

$$\begin{aligned} \gamma_{n,\text{bd}}(\tau) &= - \left( r_{n,\text{bd}}(\tau) + \frac{\pi^4 r_{n,\text{bd}}(\tau)^5}{5\beta^4} \right) \\ &= -\epsilon \theta'(\tau)^{\frac{1}{2}} \left\{ 1 - \frac{1}{2} U_1(\theta(\tau)) + \frac{3}{2} \epsilon^2 U_2(\theta(\tau)) \theta'(\tau) + \epsilon^4 \left[ -\frac{\pi^4 \theta'(\tau)^2}{20\beta^4} + \left( -\frac{5\pi^4 \theta'(\tau)^{5/2} U_1(\theta(\tau))}{24\beta^4} \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{1}{8} \theta'(\tau)^{\frac{1}{2}} \theta''(\tau) U_2'(\theta(\tau)) + \frac{3\theta''(\tau)^2 U_2(\theta(\tau))}{16\theta'(\tau)^{3/2}} + \frac{5}{6} \theta'(\tau)^{5/2} U_3(\theta(\tau)) \right) \right] \right\}. \end{aligned} \quad (\text{B.26})$$

Then we have from (4.15) and  $w = e^{\frac{2\pi}{\beta}(\gamma+i\theta)}$

$$\theta_n = \frac{1}{i} \left( \frac{\beta}{2\pi} \log w_n - \gamma_n \right) = -i \left[ \frac{\beta}{2\pi} \log \left( \frac{\left( \frac{e^{\frac{2\pi}{\beta}(\gamma+i\theta)} - A}{1 - A e^{\frac{2\pi}{\beta}(\gamma+i\theta)}} \right)^{1/n} + A}{1 + A \left( \frac{e^{\frac{2\pi}{\beta}(\gamma+i\theta)} - A}{1 - A e^{\frac{2\pi}{\beta}(\gamma+i\theta)}} \right)^{1/n}} \right) - \gamma_n \right]. \quad (\text{B.27})$$

We then plug in  $\gamma$  at the boundary (B.5) and  $\gamma_n$  at the boundary (B.26) and expand to the first order of  $(n-1)$ , it gives  $\theta_{n,\text{bd}}(\tau)$  as a function of the boundary mode  $\theta(\tau)$

$$\begin{aligned} \theta_{n,\text{bd}}(\tau) &= \theta(\tau) + (n-1) \cdot \frac{i\beta e^{-\frac{2i\pi\theta(\tau)}{\beta}} \left( A^2 \left( -e^{\frac{2i\pi\theta(\tau)}{\beta}} \right) + A e^{\frac{4i\pi\theta(\tau)}{\beta}} + A - e^{\frac{2i\pi\theta(\tau)}{\beta}} \right) \log \left( \frac{A - e^{\frac{2i\pi\theta(\tau)}{\beta}}}{-1 + A e^{\frac{2i\pi\theta(\tau)}{\beta}}} \right)}{2\pi(A^2 - 1)} \\ &\quad + \frac{1}{2} \epsilon^2 \theta'(\tau) U_1'(\theta(\tau)) - \frac{1}{24} \epsilon^4 \theta'(\tau)^2 U_1'''(\theta(\tau)) \end{aligned} \quad (\text{B.28})$$

where the  $(n-1)$  perturbation to the zeroth order in  $\epsilon$  satisfies

$$(n-1) \cdot \frac{i\beta e^{-\frac{2i\pi\theta(\tau)}{\beta}} \left( A^2 \left( -e^{\frac{2i\pi\theta(\tau)}{\beta}} \right) + A e^{\frac{4i\pi\theta(\tau)}{\beta}} + A - e^{\frac{2i\pi\theta(\tau)}{\beta}} \right) \log \left( \frac{A - e^{\frac{2i\pi\theta(\tau)}{\beta}}}{-1 + A e^{\frac{2i\pi\theta(\tau)}{\beta}}} \right)}{2\pi(A^2 - 1)} = - \int d\theta(\tau) U_1(\theta(\tau)). \quad (\text{B.29})$$



Finally, we can calculate the following Jacobian at the  $\tilde{\mathcal{M}}_n$  boundary

$$\frac{d\theta_{n,\text{bd}}(\tau)}{d\theta(\tau)} = 1 + \left[ -U_1(\theta(\tau)) + \epsilon^2 \left( \frac{1}{2} \theta'(\tau) U_1''(\theta(\tau)) + \frac{\theta''(\tau) U_1'(\theta(\tau))}{2\theta'(\tau)} \right) + \epsilon^4 \left( -\frac{1}{24} \theta'(\tau)^2 U_1^{(4)}(\theta(\tau)) - \frac{1}{12} \theta''(\tau) U_1'''(\theta(\tau)) \right) \right] \quad (\text{B.30})$$

This will be exploited below to convert the integral on the boundary of  $\tilde{\mathcal{M}}_n$  to the integrals on  $n$  copies of  $\mathcal{M}_n$ .

Putting together all we have derived above, the bulk term in  $I_{\text{grav}}$  on  $\mathcal{M}_n$  is

$$\begin{aligned} I_{\text{grav, bulk}}[\mathcal{M}_n] &= - \left[ (\text{topological on } \mathcal{M}_n) + \frac{1}{16\pi G_2} \int_{\mathcal{M}_n} (\phi R + 12l^{-2} \phi^{\frac{3}{2}}) - \frac{1}{4G_2} \left( 1 - \frac{1}{n} \right) \sum_{\alpha, \text{Cosmic Branes}} [\phi_0 + \phi(w_\alpha)] \right] \\ &= \frac{1}{n} I_{\text{grav, bulk}}[\tilde{\mathcal{M}}_n] = \frac{1}{n} \left[ (\text{topological on } \tilde{\mathcal{M}}_n) + \frac{1}{16\pi G_2} \int_{\tilde{\mathcal{M}}_n} (\phi R + 12l^{-2} \phi^{\frac{3}{2}}) \right] \\ &= \frac{1}{16\pi n G_2} \int_{\tilde{\mathcal{M}}_n} d\theta_n \int_{r_{n,\text{bd}}(\tau)}^{r_{n,\text{horizon}}=\frac{\beta}{\pi}} dr_n \left( \frac{l^4}{r_n^4} \right) \cdot \left[ \frac{l^3}{r_n^3} * \frac{-4r_n^2}{l^4} + 12l^{-2} \left( \frac{l^3}{r_n^3} \right)^{\frac{1}{3}} \right] \\ &= \frac{l}{2\pi n G_2} \int_{\tilde{\mathcal{M}}_n} d\theta_n \int_{r_{n,\text{bd}}(\tau)}^{r_{n,\text{horizon}}=\frac{\beta}{\pi}} dr_n \frac{1}{r_n^5} = -\frac{l}{8\pi n G_2} \int_{\tilde{\mathcal{M}}_n} d\theta_n \left( \frac{1}{r_n^4} \right) \Big|_{r_{n,\text{bd}}(\tau)}^{\frac{\beta}{\pi}} \\ &= -\frac{l}{8\pi n G_2} \int_0^\beta d\theta_n \frac{\pi^4}{\beta^4} + \frac{l}{8\pi n G_2} \int_{\partial \tilde{\mathcal{M}}_n} d\theta_n \left( \frac{1}{r_n^4} \right) = -\frac{\pi^3 l}{8n G_2 \beta^3} + \frac{l}{8\pi n G_2} \int_{\partial \tilde{\mathcal{M}}_n} d\theta_n \left( \frac{1}{r_n^4} \right) \\ &= \frac{l}{8\pi G_2} \cdot \frac{1}{n} \cdot n \int_{\partial \mathcal{M}_n} d\tau \frac{d\theta_{n,\text{bd}}}{d\theta} \theta'(\tau) \frac{1}{r_{n,\text{bd}}(\tau)^4} \\ &= \frac{l^3}{16\pi G_2} \int_{\partial \mathcal{M}_n} d\tau \left\{ \frac{1}{\epsilon^4} \left( \frac{2}{\theta'(\tau)} + \frac{2U_1(\theta(\tau))}{\theta'(\tau)} \right) + \frac{1}{\epsilon^2} \left( U_1''(\theta(\tau)) + \frac{\theta''(\tau) U_1'(\theta(\tau))}{\theta'(\tau)^2} - 12U_2(\theta(\tau)) \right) \right. \\ &\quad + \left[ \frac{2\pi^4 \theta'(\tau)}{\beta^4} + \left( \frac{2\pi^4 \theta'(\tau) U_1(\theta(\tau))}{3\beta^4} - \frac{\theta''(\tau) U_2(\theta(\tau))}{\theta'(\tau)} - \frac{3\theta''(\tau)^2 U_2(\theta(\tau))}{2\theta'(\tau)^3} - \frac{20}{3} \theta'(\tau) U_3(\theta(\tau)) \right. \right. \\ &\quad \left. \left. - \frac{1}{12} \theta'(\tau) U_1^{(4)}(\theta(\tau)) - \frac{\theta''(\tau) U_1'''(\theta(\tau))}{6\theta'(\tau)} \right) \right] \left. \right\} \quad (\text{B.31}) \end{aligned}$$

where the topological bulk term in the second line and the contribution to the definite integral over  $r_n$  from  $r_{n,\text{horizon}} = \frac{\beta}{\pi}$  in the fifth line are dropped because these two terms are merely constants and thus totally irrelevant to the boundary modes  $\theta(\tau)$ . Also it is notable that we made use of the Jacobian (B.30) on the sixth line to convert the integral over  $\theta(\tau)$  on  $\tilde{\mathcal{M}}_n$  boundary to the integral over  $\tau$  on  $n$  copies of  $\mathcal{M}_n$  boundary, hence the factor of  $n$  appears in the sixth line, cancelling the factor  $\frac{1}{n}$  to give the final answer where  $n$  is absent.

In the end, we add up the bulk (B.31) and boundary parts (B.21) (B.22) of the on-shell replica gravitational action on  $\mathcal{M}_n$  (B.1) and get

$$\begin{aligned} I_{\text{grav}}[\mathcal{M}_n] &= I_{\text{grav, bulk}}[\mathcal{M}_n] + I_{\text{grav, bd, 1}}[\mathcal{M}_n] + I_{\text{grav, bd, 2}}[\mathcal{M}_n] \\ &= -\frac{l^3}{16\pi G_2} \int_{\partial \mathcal{M}_n} d\tau \left\{ \frac{1}{\epsilon^2} \left[ \left( \frac{2\theta''(\tau)^2}{\theta'(\tau)^4} - \frac{\theta'''(\tau)}{\theta'(\tau)^3} \right) + \left( -2U_1''(\theta(\tau)) - \frac{2\theta''(\tau) U_1'(\theta(\tau))}{\theta'(\tau)^2} - \frac{2\theta'''(\tau) U_1(\theta(\tau))}{\theta'(\tau)^3} \right. \right. \right. \\ &\quad \left. \left. + \frac{4\theta''(\tau)^2 U_1(\theta(\tau))}{\theta'(\tau)^4} + 18U_2(\theta(\tau)) \right) \right] + \left[ \left( \frac{3\pi^4 \theta'(\tau)}{\beta^4} + \frac{21\theta''(\tau)^4}{32\theta'(\tau)^7} - \frac{3\theta'''(\tau)\theta''(\tau)^2}{8\theta'(\tau)^6} \right) \right. \\ &\quad + \left( \frac{11\pi^4 \theta'(\tau) U_1(\theta(\tau))}{3\beta^4} + \theta'(\tau) U_2''(\theta(\tau)) + \frac{5\theta''(\tau) U_2'(\theta(\tau))}{2\theta'(\tau)} + \frac{27\theta''(\tau)^2 U_2(\theta(\tau))}{4\theta'(\tau)^3} - \frac{50}{3} \theta'(\tau) U_3(\theta(\tau)) \right) \\ &\quad - \frac{\theta''(\tau) U_1'''(\theta(\tau))}{6\theta'(\tau)} - \frac{3\theta''(\tau)^2 U_1''(\theta(\tau))}{8\theta'(\tau)^3} + \frac{\theta''(\tau)^3 U_1'(\theta(\tau))}{\theta'(\tau)^5} + \frac{63\theta''(\tau)^4 U_1(\theta(\tau))}{32\theta'(\tau)^7} - \frac{3\theta'''(\tau) U_2(\theta(\tau))}{\theta'(\tau)^2} \\ &\quad \left. \left. - \frac{1}{12} \theta'(\tau) U_1^{(4)}(\theta(\tau)) - \frac{3\theta'''(\tau)\theta''(\tau) U_1'(\theta(\tau))}{4\theta'(\tau)^4} - \frac{9\theta'''(\tau)\theta''(\tau)^2 U_1(\theta(\tau))}{8\theta'(\tau)^6} \right) \right] \left. \right\} \quad (\text{B.32}) \end{aligned}$$

where there is no  $1/\epsilon^4$  divergence. Although there is still a  $1/\epsilon^2$  divergence, it can be cancelled by the Euclidean version of the same counterterm (2.56), of course now evaluated in replica geometry, just like in (2.55). So we can safely throw it off at the end of the day.

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