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Masters Thesis of Theoretical Physics

On Collective Phenomena Of One And Two Dimensional Fermionic Systems

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«Դու չեն հոգնի, դու կը քայլես, կը բարձրանաս դու սարն ի վեր:
Դրա համար ոգի է պէտք, դրա համար պէտք չեն թեւեր:»
-Պարոյր Սեւակ

I would like to thank everyone who were there for me throughout my years in the Netherlands. My parents who did everything they could and more in helping me get a higher education. I would also like to thank my friends, old and new, who were there for me when I needed them. I would also like to thank Professor Stoof for guiding me at every step throughout my thesis, the insights that he has shared with me are priceless and I will carry them with me throughout my entire academic career.

Abstract

We begin by investigating the physics one-dimensional interacting fermions. We lay the foundation of previous methods which were used in understanding these types of one-dimensional interacting systems as well as give a qualitative analysis of the system itself. We then proceed to use the Hubbard-Stratonovich transformation in order to study the same system of interacting fermions in one-dimension with the hopes of getting similar results to previous methods. After which, we then tried to generalize our approach to higher dimensions where we realized that using the Hubbard Stratonovich transformation had proven useful in studying one-dimensional fermionic systems, as well as, two-dimensional, anisotropic, interacting fermionic systems.

Contents

Chapter 1

Introduction

Ever since 1986, one of the most exciting experimental discovery in condensed matter physics, was the first ever observation of superconductivity at high-temperatures. Which occurred in a layered copper-oxide as mentioned the paper by Bednorz et al.[2]. Since then, other copper-oxide compounds which are identified by a layered structure of CuO_2 planes, have been seen to exhibit increasingly higher critical temperatures. This in fact, is well above the expected temperature for materials to exhibit superconductivity as per the Bardeen–Cooper–Schrieffer[BCS] theory which uses phonon-exchange as the pairing mechanism[1], which is considered to be the mainstream physics that describes the basic properties of superconductors. A huge amount of effort has been put by both the experimental and theoretical community of physicists in order to better understand the mystery behind the weird behavior of copper-oxides-cuprates [8], which has currently no explanation within the Fermi-liquid model. It is an ongoing endeavour and it encompasses the challenges that physics of strongly interacting materials pose. We can see that the peculiarities do not only lie within the superconducting phase of the material. They lie in a phase which is known as the strange metal regime. As we ought to guess, the said phase is characterized by a non-Fermi liquid behavior which is shown from an anomalous temperature behavior of the Hall angle [4] and by an anomalous T proportional resistivity of the material [5, 3], even if super-conductivity is suppressed via a magnetic field[11].

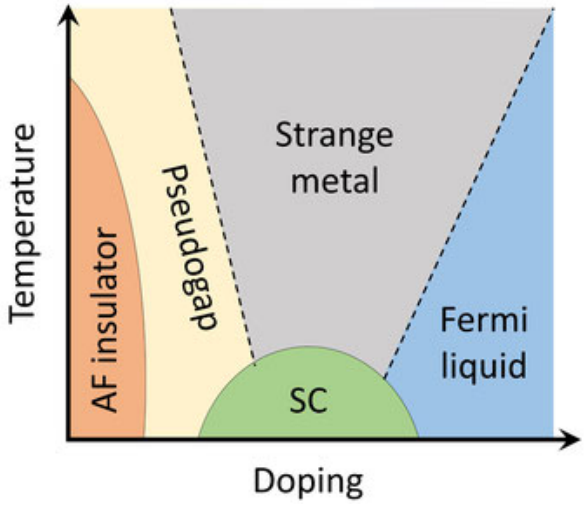

Figure 1.1: Strange metal phase diagram, Temperature versus Doping

We can see that there have been multiple attempts to model such high critical-temperature cuprates, these are, for instance, the $t-J$ models which base themselves on the physics of the Mott insulator in the underdoped region of the cuperate diagram[16, 20], the marginal Fermi liquid for describing the optimally doped strange metal [15, 13] and stripe phases within high temperature superconductors[26]. One technique that stands out is gauge/gravity duality [6] which, is also called, AdS/CFT. This approach relates the response of a strongly interacting system to a higher-dimensional gravitational theory. Although this method originates in high-energy physics and cosmology [18], it has proven itself to be a powerful tool in describing strongly correlated condensed-matter systems, as well as being able to describe properties which have been observed within copper-oxides, via angle resolve photoemission spectroscopy(ARPES) measurements [21]. In addition we can see that, angle-resolved photo-emission spectroscopy measurements point to possible explanation of the phenomenology of the strange metal phase of the cuprates, specifically in the presence of quantum critical phase which is to be treated as local in space. This notion is well captured by the holographic approach of a strongly interacting fermionic system [6] which itself reproduces the marginal Fermi-liquid results near the Fermi surface. As we move away from the Fermi surface, these results retrieved from holography stop giving completely featureless momentum scaling of said Fermi system and

optimal doping, in fact they begin to predict momentum-dependent scaling exponents [6].

These results represent a deviation from the current accepted power-law-liquid (PLL) model as described in [21] of a momentum-independent self energy, with an imaginary part that obeys $\Sigma_{PL}(\omega) \propto (\omega^2)^{\alpha}$. Where in this case α is to be considered as a scaling exponent which is increased with doping, from an initial point $\alpha = \frac{1}{2}$ at optimal doping towards the Fermi-liquid value, but evidently never reaching it[23]. The analysis of ARPES data was done on each momentum distribution curve, that measures the spectral function as a function of momentum with a fixed energy $\hbar\omega$. It is known that the PLL close to the Fermi surface predicts a Lorentzian lineshape for the distribution peaks as

$$A(k;\omega) = \frac{W(\omega)}{\pi} \frac{\Gamma(\omega)/2}{(k - k_*(\omega))^2 + (\Gamma(\omega)/2)^2}, \tag{1.1}$$

where $W(\omega)$ is the intensity and $\Gamma(\omega) = 2\Sigma_{PLL}(\omega)/v_F + G_0(\omega)$ is the full-width at half maximum with $G_0(\omega)$ describing contributions other than the contributions to the electron self energy to the width in the data, e.g., due to phonons, impurities and instrument sensitivity. In this case we see v_F is the renormalized Fermi velocity and $k_*(\omega) = k_F + \omega/v_F$ where k_F is to be considered as the Fermi wave number. A frequency dependent electronic self energy is common for a large class of models which are proposed by gauge/gravity duality. It is important to mention that the momentum dependence is in fact located in the scaling exponent of the self energy. As mentioned by S. Smit et al.[22] we can say that

$$\Sigma(k,\omega) \propto \omega(-\omega^2)^{\nu_k - 1/2}. \tag{1.2}$$

It is worthwhile to also mention that $\omega = \omega + i0$. In their paper, S. Smit et al.[23], they explicitly show that the self-energy, has a branch cut everywhere on the real axis.

$$\Sigma(k,\omega) \propto \omega(-\omega^2)^{\alpha(1-(k-k_F)/k_F)}. \tag{1.3}$$

They also explain how this momentum dependence is peculiar, and reduces to the PLL for that sharp distribution of peaks near the Fermi surface. But our

goal is to stay away from AdS/CFT, such that we develop a more "condensed-matter" approach to study these types of complicated systems, the cuperates. In order to do so we begin by studying systems of one dimension, then generalize our approach to higher dimensions.

1.1 One-Dimension

Problems of strongly correlated fermions in one-dimensional systems has attracted the attention of many condensed-matter physicists. After Tomonaga [25] who was the first to come up with the soluble model in 1950, more people such as Luttinger [17] made great advancements in the field. Even though the behavior of one-dimensional, highly correlated electrons were more or less understood near the end of the 1970's, Haldane's famous paper[10] where he developed the essentials of the bosonization technique, was a remarkable breakthrough. Because this technique involved the one-to-one correspondence of the fermionic annihilation and creation operators with bosonic ones. We investigate the results of another method which "bosonizes" the problem. This method is called the Hubbard-Stratonovich transformation, which can be used to derive the Hartree-Fock theory by using a diagramtic and perturbative approach, or even a zero-temperature variational method. Our main goal is to show how we can use the Hubbard-Stratonovich method to derive results which should be in accordance to results derived via standard bosonization techniques, like the ones of M. Khodas et al.[14], where they evaluate the Green's function of interacting electrons in one dimension. Where the spectral function, which corresponds to the interacting Green's function of their system, displays a power-law singularity on the hole mass-shell, the same way as a Luttinger liquid would.

Luttinger liquids, or non-Fermi liquid theory has been a really useful model in describing one-dimensional quantum fluids in the limit of low energies. This model bases itself on the linearization of the dispersion relation of the particles near the Fermi points of the material. It is an interesting fact that we are referring to points! This is true because we are working with a one dimensional system. It is evident that these methods could and are applied to spectrum of 1D fluid systems, such as 1D spin liquids as well as electrons that exist within quantum wires and ultracold atoms which are confined in one-dimensional traps. In this thesis our aim is to show that another way to

achieve existing results in one dimension there exists a method which is also a great candidate in predicting these results in higher dimensions as well.

1.2 Two-Dimensions

Since our ultimate goal is to have a method that would be able to describe cuprates, in Chapter 3, we would like to jump to two dimension and try to apply our results to anisotropic 2-D systems. In this part of the thesis we will be applying the Hubbard-Stratonovich method to study a 2-D square lattice system with nearest-neighbor hopping. Experimental results show that in hole and electron doped cuprates the polarization operator which is represented via a Lindhard function, which gives rise to plasmonic modes. Plasmons are considered to be found everywhere in condensed matter, which is mostly due to the electric charge of the electron. Although these types of plasmon modes are found using holography [7] which begin to provide a simplified approach to test experimental data, our aim in the third chapter of our thesis is clear, to try and "find" these plasmons using the Hubbard-Stratonovich transformation technique which we will use for the one-dimensional systems in Chapter 2.

Chapter 2

Theoretical Framework Of One-Dimensional Systems

The problem of strongly correlated electrons in one-dimensional systems has attracted the attention of theoretical physicists for some time now. After Tomonaga's [25] seminal paper where he suggested the first soluble model for such systems another paper, by Luttinger, Mattis and Lieb,[17] followed on the proposed model of Tomonaga with considerable contributions. The most famous example of a paper trying to develop a comprehensive model was written by Haldane in 1981 where the author developed the fundamentals of what we have now come to call, bosonisation. Known as the operator approach, the author rigorously showed how to get fermionic, anti-commuting operator out of commuting, bosonic ones. In this chapter I will briefly give an example of bosonisation, then I will talk about treating a Luttinger-Tomonaga liquid using the operator formalism which will be mostly based on the approach done by M. Khodas et al. [14]

2.1 Luttinger-Liquid Theory: Green's Function of Interacting One-Dimensional Fermionic Systems

In order to study fermions in condensed matter, the notion of quasiparticles is used, which are elementary excitations which in turn behave as free quantum particles with a given energy spectrum. This spectrum in turn depends on microscopic interactions. Low-energy aspects of the electronic structure of metals in turn, are accurately described by the theory of the Fermi liquids [19], since its elementary excitations are similar to free fermions. A nice way to view quasiparticles and their interactions is to assume that they are states that are evolving from free fermions when their interactions are turned on adiabatically. It is important to note that quasiparticles are defined by their momenta, specifically speaking their dispersion relation $\xi(\mathbf{k})$ is momentum dependent and differs from the free fermion case. An electron, for example, can easily tunnel into a metal by *disguising* itself as a quasiparticle. In the process of doing so, an electron entering a Fermi-liquid, creates a single quasiparticle state with momentum $\mathbf{k}$ and its corresponding single particle energy $\xi(\mathbf{k})$. In the case of an electron tunneling out of the one dimensional Fermi system it creates a *hole* which also bears a well defined corresponding energy. In both cases the tunneling probability of an electron entering or exiting a low dimensional Fermi system is directly given by the imaginary part of its two point function $\langle T\{\hat{\psi}^\dagger(x,\tau),\hat{\psi}(x',\tau')\}\rangle$, where $\langle\hat{\psi}(x',\tau')\hat{\psi}^\dagger(x,\tau)\rangle$ accounts for the probability of finding a particle at x' and τ' that was initially added to the system at position and time x and τ and $\langle\hat{\psi}^\dagger(x,\tau)\hat{\psi}(x',\tau')\rangle$ accounts for a hole in the opposite direction. The imaginary part of the two-point function is the spectral function, $A(k,\epsilon)\propto\delta(\epsilon-\xi(\mathbf{k}))$. In order to account for residual interactions between quasiparticles a perturbative expansion is necessary which in turn results in the broadening of its width, to a Lorentzian, with a width proportional to its relaxation rate. In the case where there is no momentum conservation the electron tunneling out of the Fermi system leaves behind a superposition of states where these states have the same energy ϵ. In momentum space, the points where the constant-energy is defined is, $\xi(\mathbf{k})=\epsilon$. This relation determines the density of states our system[19],

$$\nu(\epsilon)\propto\int d\mathbf{k}A(\mathbf{k},\epsilon). \tag{2.1}$$

Similarly to free fermions, the tunneling density of states at the Fermi-level in the Fermi liquid is finite. We can see that the both the free fermion and an elementary excitation in a many-body system is marked by the narrow width of $A(\mathbf{k}, \epsilon)$ where the spectral weight is mainly concentrated around the quasiparticle energy. This treatment sometimes works for higher dimensions but it fails in the case of weakly interacting one-dimensional fermions. The lowest order correction for the fermionic spectral function pushes the spectral weight away from the mass shell, $|\epsilon - \xi(k)| \ll |\xi(k)|$. A magic bullet in attempting to resolve the many difficulties of one-dimensional quantum systems was proposed by Tomonaga [25]. It is assumed that replacing the standard parabolic dispersion to a linear one,

$$\xi(k) = \pm v_F(k \mp k_F), \tag{2.2}$$

would simplify the many-particle system dynamics. It is also important to note that the upper signs corresponds to right moving particles and the lower signs for left moving ones and k_F is the Fermi momentum. For free fermions the corresponding single particle energy with a linear spectrum is $E = v_F k$, where k itself is the total momentum of the excitation. Tomonaga encoded these excitations into non-interacting bosons[25]. The dispersion of these bosonic fields are nothing more than acoustic phonons. In Chapter 3 we will be attempting to retrieve theses phonon dispersions for one dimension! Our method, in turn, is the Hubbard-Stratonivich transformation and Hartree approximation of the quartic weak interactions between the fermions.

The evaluation of the propagator of a fermion along with its spectral function $A(k, \epsilon)$ is rather complicated. At low energies and long-wavelengths the excitations of non-interacting fermions are described by holes and particles with respective momenta k in the region of the Fermi points, $\pm k_F$. Let's study the single-particle energy of right-moving particle near the Fermi point. We have

$$\xi(k) = v_F(k - k_F) + \frac{(k - k_F)^2}{2m^*}, \tag{2.3}$$

it is also the lowest-order expansion of the non-linear dispersion relation. For the case of free fermions where we have Galilean symmetry, m^* is to be considered as the bare mass of the fermion. So the quadratic term in this case could be scaled as $\xi^2/(m^* v_F)^2$. It was pointed out by Haldane[10]

that the non-linear spectrum does not affect the long-range behavior of the fermionic two-point function. This tranlates to the fact that we can map gapless excitation spectra of one-dimensional systems at low energies to the Luttinger-liquid theory. The Luttinger-liquid theory is the phenomenological description of excitations at the long wavelength limit in one dimensional systems[10]. After using this model on a real one-dimensional system the corresponding linear spectrum is of the form, $\omega(k) = v|k|$, this linear spectrum is associated to low-energy excitations of density waves[12], these excitations, which propagate all along the axis, have fixed velocities $\pm v$. Meaning a density perturbation which begins at some point x propagates without changing its shape to $x \pm vt$.

2.1.1 Qualitative Analysis

Before moving on to analyse the perturbative aspect of our one-dimensional system, I would like to take the time to give a qualitative analysis of the system we are trying to study. Primarily, I would like to discuss the kinematic edges of the single-particle energy in phase-space which is a two-dimensional plane (p, ϵ). Most importantly give a qualitative framework in understanding the spectral function. We proceed to define the spectral function using its Lehmann representation,

$$A(k,\epsilon) \propto \sum_{|f\rangle} |\langle f| \psi_R^\dagger(k) |0\rangle|^2 \delta(k - k_{|f\rangle} + k_{|0\rangle}) \delta(\epsilon - E_{|f\rangle} + E_{|0\rangle}). \tag{2.4}$$

Another way of trying to understand equation 2.4, is by viewing it as the probability of tunneling of a particle which, with respect to the origin, has momentum $p_F + k$ and a corresponding energy eingenvalue which is $\epsilon_F + \epsilon$, where ϵ and k are arbitrary values. An important note is that we only consider right moving particles (their momenta are defined as $p_F + k > 0$). We would also like mention that $|f\rangle$ and $|0\rangle$ represent respectively, a final state with corresponding energy $E_{|f\rangle}$ momentum $k_{|f\rangle}$ and an initial ground state of the system with corresponding energy $E_{|0\rangle}$ and momentum $k_{|0\rangle}$. Due to conservation of momentum the particle that is to be added must have k as momentum, such that we can write, $|f\rangle = \psi_R^\dagger(k)|0\rangle$. We can still find regions where the spectral function has trivial values, these regions are due to kinematic constraints. In order to have a straightforward and less complicated analysis we focus on the low energy domain where $|\epsilon| \ll \epsilon_F$ and

consequently momenta which range between $-k_F < k < k_F$. We can define some of these final states via the following equations

$$|\psi_f\rangle = \psi_R^\dagger(k_1)\psi_R^\dagger(k_2)\psi_R(k_3)\,|0\rangle\,,\ \text{such that: } k_1 > 0, k_2 > 0, k_3 < 0, \quad (2.5)$$

$$|\psi_f\rangle = \psi_R^\dagger(k_1)\psi_L^\dagger(k_2)\psi_L(k_3)\,|0\rangle\,,\ \text{such that: } k_1 > 0, k_2 < 0, k_3 > 0. \quad (2.6)$$

We see that equation 2.5 gives us the following energy difference with respect to the ground state energy,

$$E_{|f\rangle} - E_{|0\rangle} = v(k_1 + k_2 - k_3) + \frac{1}{2m}(k_1^2 + k_2^2 - k_3^2). \quad (2.7)$$

If we also take into account the conservation of momentum at $k_1+k_2-k_3 = k$, we get,

$$E_{|f\rangle} - E_{|0\rangle} = vk + \frac{1}{2m}\Big[k_1^2 + k_2^2 - (k_1 + k_2 - k)^2\Big]. \quad (2.8)$$

These constraints on momenta, k_1, k_2 and k_3 guarantee that we always have $k > 0$. So the smallest possible value of the energy need to jump from the ground state to an excited eigenstate for any given k should be when $k_1 = k_2 = 0$ and $k_3 = -k$. We proceed to get a better understanding of equation 2.6, making similar considerations for it we get,

$$E_{|f\rangle} - E_{|0\rangle} = -vk + vk_1 - \frac{k^2}{2m} + \frac{k}{m}(k_1 + k_2) - \frac{k_1 k_2}{m}. \quad (2.9)$$

In this case, we see that for $k < 0$ we get the lowest energy when $k_1 = k_2 = 0$ and $k_3 = -k$. We make a remarkable observation from the analysis we just made that all the lowest energy configurations happen when all the particles are at the Fermi level along with a single hole which is contributing the most to the momentum. So we can see that the states 2.5 and 2.6 correspond to a kinematic boundary $\tilde{\xi}_R(k)$ of the spectral function of a ($\epsilon > 0$). We proceed to combine equations 2.8 and 2.9 which, at $k_1 = k_2 = 0$ yields us

$$\tilde{\xi}_R(k) = v|k| - \frac{k^2}{2m}. \quad (2.10)$$

We would like to focus on the hole part of the spectrum, so we start by writing down the corresponding spectral function in Lehmann representation,

$$A(k,\epsilon) \propto \sum_{|f\rangle} |\langle f| \psi_R(k) |0\rangle|^2 \delta(k + P_{|f\rangle} - P_{|0\rangle}) \delta(\epsilon + E_{|f\rangle} - E_{|0\rangle}). \tag{2.11}$$

We can see that the momentum of the end state is relative to $-p_F$. The matrix elements of Eq. 2.11 are finite only when we have a single hole excitation with $k < 0$. Considering the state,

$$|f\rangle = \psi_R(k_1)\psi_L(k_2)\psi_L^\dagger(k_3) |0\rangle . \tag{2.12}$$

In order to understand what the largest possible value for ϵ is for a given k, we would like to look at the most stable energy configuration final state $E_{|f\rangle}$ in Eq.2.11, The energy spectrum can be written as,

$$E_{|f\rangle} - E_{|0\rangle} = v(-k_1 + k_2 - k_3) - \frac{1}{2m}(k_1^2 + k_2^2 - k_3^2). \tag{2.13}$$

Fermionic Algebra

Seeing that, $\psi(k)$ and $\psi^\dagger(k)$ are the fermionic annihilation and creation operators, respectively, and $\xi(k)$ is the single-particle energy. These operators obey the following algebra,

$$\begin{aligned} &\{\psi^\dagger(k), \psi(k')\} = \delta(k - k'), \\ &\{\psi^\dagger(k), \psi^\dagger(k')\} = \{\psi(k), \psi(k')\} = 0. \end{aligned} \tag{2.14}$$

We define the inverse Fourier transform of these field operators such that,

$$\psi(x) = \frac{1}{\sqrt{N}} \int \frac{dk}{2\pi} e^{ikx} \psi(k), \tag{2.15}$$

and the corresponding Fourier transform,

$$\psi(k) = \frac{1}{\sqrt{N}} \int \frac{dx}{2\pi} e^{-ikx} \psi(x). \tag{2.16}$$

Where k is to be considered as the momentum of the fermion and N the number of lattice sites in our system. It is interesting to see that the field

operator corresponding to creation is just the Hermitian conjugate of the operator corresponding to annihilation. We then proceed by defining the fermionic density operator in momentum space,

$$\rho_R(k) = \int \frac{dk'}{2\pi} \psi_R^\dagger(k+k')\psi_R(k'). \tag{2.17}$$

These operators in some literature, are also called, shifting operators. For example in Eq.2.17 we can consider that the operator that is defined shifts a right moving fermion by a momentum k. Now is a good time to remind ourselves that for low temperatures, all the states below the Fermi points are filled, meaning all the states with $|k| < k_F$. It is interesting for us to notice that,

$$\rho_R^\dagger(k) = \rho_R(-k), \tag{2.18}$$

meaning, that the Hermitian conjugate of the operator itself serves a means to shift the system by a momentum $-k$. This type of operator behavior is expected for bosonic operators! Let us consider the following commutation relation,

$$[\rho_R(-k), \rho(k')] = [\rho_R^\dagger(k), \rho_R(k')] = \frac{kN}{2\pi}\delta(k-k'). \tag{2.19}$$

We can see that there is a subtle result of having $k = k'$, which gives us the possibility of having an infinite number of states in the Fermi sea. We would also like to focus on the commutation relation between the density operator with the linearized non-interacting Hamiltonian,

$$[H, \rho_R(k)] = \int \frac{dk'dk''}{(2\pi)^2} v_F k' [\psi_R^\dagger(k')\psi_R(k'), \psi_R^\dagger(k''+k)\psi_R(k'')] = v_F k \rho_R(k). \tag{2.20}$$

The qualitative results of having an infinite number of particles in the Fermi-sea, along with the fact that these shifting/density operators follow a typical bosonic algebra, will gives us two creation and annihilation renormalized bosonic operators,

$$\begin{aligned} b_R^\dagger(k) &= i\sqrt{\frac{2\pi}{kN}}\rho_R(k), \\ b_R(k) &= -i\sqrt{\frac{2\pi}{kN}}\rho_R(-k). \end{aligned} \tag{2.21}$$

As for the left-moving bosons,

$$\begin{aligned} b_L^\dagger(k) &= -i\sqrt{\frac{2\pi}{kN}}\rho_L(-k), \\ b_L(k) &= i\sqrt{\frac{2\pi}{kN}}\rho_L(k). \end{aligned} \tag{2.22}$$

It is straight forward for us to see that the following commutations pop-up naturally,

$$\begin{aligned} [b_R(k), b_R^\dagger(k')] &= \delta(k-k'), \\ [H, b_R^\dagger(k)] &= v_F k b_R^\dagger(k). \end{aligned} \tag{2.23}$$

Using the results from equations, 2.23 and 2.22 we can define a non-interacting bosonic Hamiltonian,

$$H = \int \frac{dk}{2\pi} v_F k \Big(b_R^\dagger(k) b_R(k) + b_L^\dagger(k) b_L(k) \Big) + \frac{\pi v_F}{N}(n_R^2 + n_L^2), \tag{2.24}$$

where the second term of the right-hand side accounts for zero-modes. The integrand is where the physics actually is. These are in fact the terms that describe the excitations.

Expressing Fermions As Bosons

Taking all the results in the previous section, we define the denistiy operator in position space for the right and left movers,

$$\rho_{R,L}(x) = \psi_{R,L}^\dagger(x)\psi_{R,L}(x). \tag{2.25}$$

Averaging over the the fermionic density operator for the right movers around the ground state, we get,

$$\langle \rho_R(x) \rangle = \int \frac{dk dk'}{(2\pi)^2} e^{ix(k'-k)} \langle \psi_R^\dagger(k)\psi_R(k') \rangle = \infty. \tag{2.26}$$

This problem of infinity, arises because the electronic states in the Luttinger model start from $-\infty$ and extend into 0, ergo we have an infinite amount of particles. This can be solved by taking the normal ordering, which means we rearrange our operators in such a way that we take out the ground state,

$$: \rho_R(x) := \psi_R^\dagger(x)\psi_R(x) - \langle \rho_R(x) \rangle. \tag{2.27}$$

This is accomplished by moving our fermion operators to the right ($k > 0$) and to the left ($k \leq 0$). In this case we have a new definition for our density operator,

$$\int_0^l dx e^{ikx} : \rho_R(x) := \int \frac{dk'}{(2\pi)} : \psi_R^\dagger(k+k')\psi_R(k') := \rho_R(k), \tag{2.28}$$

but only for $k \neq 0$ and when $k = 0$, the integral equalizes to n_R. Taking the inverse Fourier transform, we have the following expression for the fermionic density operator, which is in terms of the bosonic creation and annihilation operators,

$$: \psi_R^\dagger(x)\psi_R(x) := \frac{1}{L}\int \frac{dk}{2\pi}\sqrt{k}\Big(ib_R(k)e^{i\frac{2\pi}{L}kx} - ib_R^\dagger e^{-i\frac{2\pi}{L}kx}\Big) + \frac{n_R}{L}. \tag{2.29}$$

Where L is the wire length. As for the left movers the exponents need to have opposite signs. Now, we can define a typical bosonic field operator for a right-moving particle, such that,

$$\varphi_R(x) = \varphi_R(0) + \frac{n_R x}{L} + \int \frac{dk}{2\pi}\frac{1}{\sqrt{4\pi k}}\Big(e^{i\frac{2\pi k}{L}x}b_R(k) + e^{-i\frac{2\pi k}{L}x}b_R^\dagger(k)\Big). \tag{2.30}$$

Using Eq.2.29 and Eq.2.30, we get a one to one correspondence of the bosonic and fermionic fied operators,

$$: \rho_{R,L}(x) := \frac{1}{\sqrt{\pi}}\partial_x\varphi_{R,L}(x). \tag{2.31}$$

This one-to-one correspondence which is shown in equation 2.31, is important for the next section of this chapter, where we will be defining the Hamiltonian of one dimensional quantum wire with length L, and we will use these bosonic operators to describe the chiral fermions near the Fermi level.

2.1.2 Defining The Hamiltonian For Typical Interacting Spinless Interacting Fermions: The Luttinger Liquid

We would like to study the Luttinger liquid near the Fermi energy, we can focus on the slow moving fermions in the latter part of this thesis, I will define

my Hamiltonian for the Fermions which are interacting via weak Coulomb repulsion. We see that,

$$H = \sum_{\nu\in\{R,L\}} \int dx \xi_\nu(k)\psi^\dagger_\nu(x)\psi_\nu(x) + \frac{1}{L}\int dx dx' \psi^\dagger_R(x)\psi_R(x)V(x-x')\psi^\dagger_L(x')\psi_L(x') + \frac{1}{2L}\sum_{\nu\in\{R,L\}}\int dx dx' \psi^\dagger_\nu(x)\psi_\nu(x)V(x-x')\psi^\dagger_\nu(x')\psi_\nu(x'), \tag{2.32}$$

where $\xi(k)$ is the single-particle energy. We would like to look at this Hamiltonian in Fourier space in order to get the momentum conservations and have a better understanding of the system at hand. Taking the Fourier components of all the terms in the Hamiltonian, we get,

$$\begin{aligned} H = &\sum_{\nu\in\{R,L\}}\sum_{k,k'}\int dx \xi_\nu(k)\psi^\dagger_\nu(k)e^{\mp ikx}\psi_\nu(k')e^{\pm ik'x} \\ &+ \frac{1}{L}\sum_{k,k',k'',k'''}\sum_{q\neq 0}\int dx dx' \psi^\dagger_R(k)e^{-ikx}\psi_R(k')e^{ik'x}V(q)e^{iq(x-x')} \\ &\times \psi^\dagger_L(k'')e^{ik''x'}\psi_L(k''')e^{-ik'''x'} \\ &+ \frac{1}{2L}\sum_{\nu\in\{R,L\}}\sum_{k,k',k'',k'''}\sum_{q\neq 0}\int dx dx' \psi^\dagger_\nu(k)e^{\mp ikx}\psi_\nu(k')e^{\pm ik'x} \\ &\times V(q)e^{iq(x-x')}\psi^\dagger_\nu(k'')e^{\mp ik''x'}\psi_\nu(k''')e^{\pm ik'''x'} \\ &= \sum_{\nu\in\{R,L\}}\sum_k \xi_\nu(k)\psi^\dagger_\nu(k)\psi_\nu(k) + \frac{1}{2L}\sum_{q\neq 0}\Big(2V(q)\rho_R(q)\rho_L(-q) \\ &+ \sum_{\nu\in\{R,L\}} V(q)\rho_\nu(q)\rho_\nu(-q)\Big). \end{aligned} \tag{2.33}$$

In equation 2.33 we have written the Hamiltonian of the system in density formalism in order to utilise the bosonic operators which were defined in the previous section.

Adding Impurities To The Hamiltonian And Bosonizing The Fermi-Level.

A typical way of trying to understand our system is to use a technique called bosonization where we express our fermionic density operators in terms of

bosonic operators. Our total Hamiltonian will be composed of interactions between R and R, R and L and also $R-d$ and $L-d$ subbands. Our complete Hamiltonian will have the form,

$$H = H_d + H_{R,L} + H_{d,R/L}. \tag{2.34}$$

It is also important to mention that particle-boson interactions typically have small momenta transfer, which means we can approximate our interactions into screened Coulomb potentials which are momentum independent, or "constant". Accounting for the effects of the deep, slow moving excitations interacting with the Fermi-level particles are crucial for the evaluation of the Green's function of the d subband. The interaction between the R and L subbands does not contribute to the logarithmic approximations of the Green's function, it is safe to neglect these terms for the rest of the thesis.[14] We have,

$$\begin{aligned} H_{R,L} &= \sum_{\nu=r,l} \sum_{k} \xi_\nu(k) \psi^\dagger_\nu(k) \psi_\nu(k), \\ H_{d,R/L} &= \frac{V}{L} \sum_{|q|<\lambda k} \rho_d(q) \rho_R(-q) + \frac{V}{L} \sum_{|q|<\lambda k^2/mv} \rho_d(q) \rho_L(-q). \end{aligned} \tag{2.35}$$

Because of the strict wavelengths of density fluctuations it is safe to say that the R and L subbands can be linearized[14]. Since both these subbands are also contained near the Fermi level we can write their density or "shifting" operators in terms of bosonic field operators. So we get the following form for $H_{R,L}$ and $H_{d,R/L}$,

$$\begin{aligned} H_{R,L} &= \frac{1}{4\pi L} \sum_{|q|<\lambda k} v q^2 |\varphi_R(q)|^2 + \frac{1}{4\pi L} \sum_{|q|<\lambda k^2/mv} v q^2 |\varphi_L(q)|^2, \\ H_{d,R/L} &= -i \frac{V}{2\pi L} \sum_{|q|<\lambda k} q \rho^d_q \varphi_R(-q) - i \frac{V}{2\pi L} \sum_{|q|<\lambda k^2/mv} q \rho_d(q) \varphi_L(-q). \end{aligned} \tag{2.36}$$

Notice that the upper cut-off for the R subband and $d-R$ interactions is λk and the cut-off for the L subband and $d-L$ interactions is $\frac{\lambda k^2}{mv}$. This is due to a step similar to the renormalization group (RG) procedure done by Haldane[9]. Where the method is based on the work of Khodas et al. [14], after taking out the relatively high-energy states $\lambda k < |p - p_F|$ and $\frac{\lambda k^2}{mv} < |p + p_F|$ where they define $\lambda \ll 1$, using RG transformation they reduce the energy bands of the system into three strips, where $|p - p_F| < \lambda k$ and $|p + p_F| < \frac{\lambda k^2}{mv}$ are the R band and L band respectively.

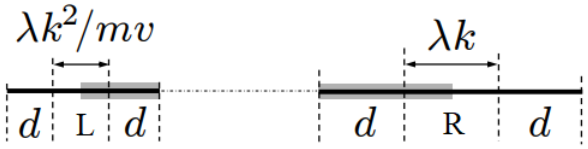

Figure 2.1: The separation of momenta that describes the Hamiltonian 2.34 and is the result of the RG transformation procedure done by Khodas et al. [14].

Behavior Of $G_d(k|x,t)$

Now that we have a complete understanding of which interactions are important for the evaluation of our Green's function which describes the deep, slow moving excitation, we can start with the evaluation of $G_d(k|x,t)$, All second order approximation of the self-energy can be disregarded since they are small[14], hence, we would like to focus on the deep particle-hole alone. Neglecting any interactions with the Fermi/level the Green's function satisfies the following equation,

$$\Big\{i\hbar\partial_t - \xi(k) + i\hbar v_d\partial_x\Big\}G_d(k|x,t) = -\hbar\delta(x)\delta(t). \tag{2.37}$$

It is important to note that the linearization of the d subband spectrum stems from the fact that particle-boson interactions are associated with small transfer momenta, this also extends to the fact that we have taken our interaction potentials to be constants. Neglecting vertex corrections we can see that the effect of bosonic field is merely to induce a slowly varying potential in space-time where the d-fermions move. We express the potential as $\phi(x,t)$ and our new interacting Green's function obeys the following equation.

$$\Big\{i\hbar\partial_t - \xi(k) + i\hbar v_d\partial_x + \phi(x,t)\Big\}G_d^{\phi}(k|x,t) = -\hbar\delta(x)\delta(t), \tag{2.38}$$

Where the slow varying potential $\phi(x,t)$ can be expressed as a linear combination of the bosonic operators, $\partial_x\varphi_R$ and $\partial_x\varphi_L$.

2.2 Conclusion

In the next chapter, we will be attempting to reproduce Eq.2.38, using the Hubbard-Stratonovich transformation, which is dubbed to be a more elegant method of introducing a slow varying potential to our system. We will then apply the same methodology for a two-dimensional system that carries a four-fold symmetry, our hope will be to find plasmonic behavior for the singularities of the Coulomb propagator.

Chapter 3

Interactions In One Dimension and The Hubbard Stratonovich Transformation

In this chapter we will introduce the functional-integral formulation of quantum field theory. We will do this in order to treat many-body systems in one dimension. We will be tackling the interacting Green's function of the system that was discussed in Chapter 2. Since in general it is not an easy task to determine the interacting Green's function exactly, we begin by cancelling the quartic, interaction terms of the action and add a new field, which will become the field $\phi(x,t)$ which was discussed in Chapter 2, then via variational methods we will determine the averaged (with respect to the $\phi(x,t)$ fields) Green's function of the slow moving fermions. In this section we will be adopting a different approach in trying to comprehensively describe quantum wires with weak interactions. In Chapter 2 we had used second quantization and Haldane's[8] bosonization approach in order describe the interactions in terms of non-interacting bosons. We find that the method that we will employ in this chapter is a more subtle and straightforward way of trying to understand these one-dimensional systems. Our hope is to expand and try to tackle two-dimensional square lattice systems in order to be able to describe strange metals. The method we will be using is the Hubbard Stratonovich transformation where later on we will also approximate our quantum field theory to the Hartree theory. Before that, we would like to discuss which

part of the propagator we would like to study. As previously stated it is our intention to study the behavior of the a slow moving hole in a one dimensional quantum wire, which means

$$\begin{aligned}&\langle T\{\hat{\psi}_d(x,\tau)\hat{\psi}_d^\dagger(x',\tau')\}\rangle \\ &= \theta(\tau-\tau')\langle\hat{\psi}_d(x,\tau)\hat{\psi}_d^\dagger(x',\tau')\rangle + \theta(\tau'-\tau)\langle\hat{\psi}_d^\dagger(x',\tau')\hat{\psi}_d(x,\tau)\rangle.\end{aligned} \tag{3.1}$$

It is important to note that, $\hat{\psi}_d$ and $\hat{\psi}_d^\dagger$ are the annihilation and creation operators respectively[24]. So in our case we would like to study the second term of the right hand side, which is considered to be the $\langle\psi_d^\dagger(x',\tau')\psi_d(x,\tau)\rangle$ two-point function of a hole propagating.

3.1 Non-Interacting Green's Function Of Dirac Fermions

Before we tackle the interaction, as in the previous chapter, we would like to first understand the behavior of the two-point function without any interactions. This would gives a good understanding of the bare aspects of our system, where later on we can introdcuce the quartic interacting terms in our action which will in turn be problems when integrating over the fermionic fields but that is why we will be using a transformation that will render our integral into a simple gaussian with respect to those fermionic fields. Firstly, since we are still working with fermions we would like to using Grassmann field ψ whose algebra is mention in Appendix B. In order to better understand our system at the Fermi/level which are located at $\pm p_F$ we then, in an approximate manner, split up our Grassmann field into 3 parts,

$$\begin{aligned}\psi(x) \simeq &\int_{-\Lambda}^{\Lambda}\frac{dp}{2\pi}e^{i(p+p_F)x}\psi(p+p_F) + \int_{-\Lambda}^{\Lambda}\frac{dp}{2\pi}e^{i(p-p_F)x}\psi(p-p_F) \\ &+ \int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}e^{i(p+k)x}\psi(p+k).\end{aligned} \tag{3.2}$$

Where it is straightforward to see that each range of momenta are associated with a given chiral fermion, $\psi_R(p) = \psi(p+p_F)$, $\psi_L(p) = \psi(p-p_F)$ and $\psi_d(k) = \psi(p+k)$. This in turn gives us,

$$\psi(x) \simeq \psi_R(x)e^{ip_Fx} + \psi_L(x)e^{-ip_Fx} + \psi_d(x)e^{ikx} \tag{3.3}$$

We now move on to a much more important point. That is linearizing the spectrum in order to better understnad our system and try to replicate the results using Hubbard-Stratonovich transformations. Here we can assume that the free fermionic case has low-energy excitations and in 1 dimension instead of having a Fermi surface they have 2 Fermi points at $\pm p_F$.

3.1.1 Linearizing The Spectrum

We first write down the standard action for a non-interacting, free fermionic system with a parabolic approximation of its dispersion.

$$S = \int_0^L dx \int_0^{\hbar\beta} d\tau \psi^*(x,\tau)\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu\right)\psi(x,\tau), \tag{3.4}$$

It is important to note that the length of the wire is L. We now would like to write our field as the sum of three different grassmann fields, we will be using the same definition, but in this case we will be studying our right and left moving particles at the Fermi levels so we can write our fields such as: $\psi_R(p_F)$ and $\psi_L(p_F)$, the action then takes the form of,

$$\begin{aligned}
&S_0[\psi_R^*, \psi_R, \psi_L^*, \psi_L, \psi_d^*, \psi_d] \\
&= \int_0^L dx \int_0^{\hbar\beta} d\tau \psi_R^*(x,\tau) e^{-ip_F x}\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu\right)\psi_R(x,\tau)e^{ip_F x} \\
&+ \int_0^L dx \int_0^{\hbar\beta} d\tau \psi_L^*(x,\tau) e^{ip_F x}\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu\right)\psi_L(x,\tau)e^{-ip_F x} \\
&+ \int_0^L dx \int_0^{\hbar\beta} d\tau \psi_d^*(x,\tau) e^{-ikx}\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu\right)\psi_d(x,\tau)e^{ikx}
\end{aligned}$$

Taking the term which corresponds to $\psi_d(k)$ and $\psi_d^*(k)$, we can see that we can re-write the dispersion relation by simply applying the derivative on the

fields such that,

$$
\begin{aligned}
&S_0[\psi_d^*,\psi_d] \\
&= \int_0^L dx \int_0^{\hbar\beta} d\tau \psi_d^*(x,\tau) e^{-ikx} \left\{ \hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu \right\} \psi_d(x,\tau) e^{ikx} \\
&= \int_0^L dx \int_0^{\hbar\beta} d\tau \psi_d^*(x,\tau) e^{-ikx} \left\{ \hbar \frac{\partial}{\partial \tau} e^{ikx} - \frac{\hbar^2}{2m}\partial_x \Big(ike^{ikx} + e^{ikx}\partial_x \Big) - \mu e^{ikx} \right\} \psi_d(x,\tau) \\
&= \int_0^L dx \int_0^{\hbar\beta} d\tau \psi_d^*(x,\tau) e^{-ikx} \left\{ \hbar \partial_\tau e^{ikx} - \frac{\hbar^2}{2m}\Big(i2ke^{ikx}\partial_x - k^2 e^{ikx} \Big) - \mu e^{ikx} \right\} \psi_d(x,\tau) \\
&= \int_0^L dx \int_0^{\hbar\beta} d\tau \psi_d^*(x,\tau) e^{-ikx} \left\{ \hbar \partial_\tau + \Big(\frac{\hbar^2}{2m} k^2 + i\frac{\hbar^2}{m} k\partial_x \Big) - \mu \right\} \psi_d(x,\tau) e^{ikx} \\
&= \int_0^L dx \int_0^{\hbar\beta} d\tau \psi_d^*(x,\tau) \left\{ \hbar \partial_\tau + \Big(\xi(k) - i\hbar v_d \partial_x \Big) - \mu \right\} \psi_d(x,\tau),
\end{aligned}
\tag{3.5}
$$

where in the last line we took the limit of the momentum to be $k \to 0$, which implies that $e^{\pm ikx} \to 1$.

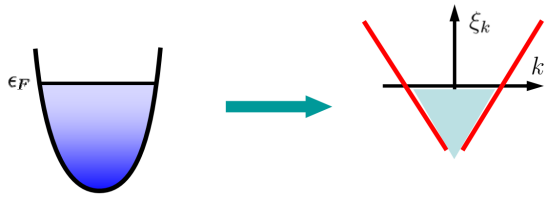

Figure 3.1: Graphical representation of linearizing the spectrum

Taking the time-dependent non interacting term for the slow moving fields in the action we have,

$$
S_0[\psi_d^*,\psi_d] = \int d\tau dx \psi_d^*(x,\tau) \left\{ \hbar \partial_\tau + \Big(\xi(k) - i\hbar v_d \partial_x \Big) - \mu \right\} \psi_d(x,\tau). \tag{3.6}
$$

Seeing that this in fact requires a lot of book-keeping we proceed to define the following notation associated to the algebra of Grassmann variables and their respective path integrals.

$$
S_0[\psi_d^*,\psi_d] = (\psi_d | \hbar \partial_\tau + \Big(\xi(k) - i\hbar v_d \partial_x \Big) - \mu | \psi_d). \tag{3.7}
$$

3.1.2 Computing The Non-Interacting Green's Function For The Slow Moving Fermions

Seeing the form of the action that describes the slow-moving fermions it is straightforward to see that the corresponding Green's function in position space satisfies the following equation,

$$\Big\{\hbar\partial_\tau + \Big(\xi(k) - i\hbar v_d\partial_x\Big) - \mu\Big\}G_d(k;x,\tau) = -\hbar\delta(x)\delta(\tau), \tag{3.8}$$

where it is important to note the momentum dependence comes from linearizing the spectrum and not any Fourier transformations. We proceed to do a wick rotation in order to get this same equality but in real time, it reads,

$$\left(i\hbar\frac{\partial}{\partial t} + iv_d\hbar\frac{\partial}{\partial x} - \xi_R(k) + \mu\right)G_d(k|x,t) = \hbar\delta(x)\delta(t). \tag{3.9}$$

We can actually solve this Green's function, after Wick rotating and getting our results in real time we get

$$G(k,\omega) = \frac{1}{\omega + v_d k - \left(\frac{\xi_R(k)-\mu}{\hbar}\right)}. \tag{3.10}$$

It is straightforward to take the inverse Fourier transform of Eq.3.10, we get,

$$\begin{aligned}\langle\psi_d^*(x,\tau)\psi_d(0,0)\rangle = G_d(k|x,t) &= \lim_{\epsilon\to 0^+}\int dk\int d\omega\frac{e^{ikx-i\omega^+ t}}{\omega + kv_d - \left(\frac{\xi_R(k)-\mu}{\hbar}\right) + i\epsilon} \\ &= \int dk - i\theta(t)e^{ikx-ikv_d}e^{i\left(\frac{\xi_R(k)-\mu}{\hbar}\right)t} \\ &= -i\theta(t)\delta(x-v_d t)e^{i\left(\frac{\xi_R(k)-\mu}{\hbar}\right)t}.\end{aligned} \tag{3.11}$$

As we can clearly see that in this section we have been able to derive $G_d(k|x,t)$ from Chapter 2. Which although describes how our slow moving fermions propagate in our system, but it does not give us a good picture of how it behaves while interacting with fermions which are located at Fermi level. In Section 3.2 we would like to study these fermionic fields when they are

interacting via, a Coulomb potential. Since the interaction terms are of fourth-order we will also be adding a new field, via the Hubbard-Stratonovich transformation which will cancel all quartic terms and leave us with a simple Gaussian integral over the fermionic fields.

3.2 Interacting Green's Function Via Coulomb Potential

In the previous section, we derived the non-interacting Green's function for a slow moving fermion using the path integral formalism for quantum field theory. In this section we will be using a technique which is called the Hubbard Stratonovich transformation which will help us in deriving the interacting Green's function. We are able to use perturbation theory in order to expand the interacting terms and use Feynman diagrams in order to better understand the interactions, the lowest order contributions are then modified via Hartree diagrams which give us a self-consistent Hartree theory. We can also minimize our newly found effective action in order to get the Hartree theory. In our case, we can see that we have three different fermionic field to account for, which are in fact the right-moving and left-moving fields which are located close to the Fermi level, and we are also taking into account the slow moving field which has a single particle energy lower than the Fermi energy.

$$
\begin{aligned}
&Z[J, J^*] \\
&= \int d[\psi_R^*]d[\psi_R]d[\psi_L^*]d[\psi_L]d[\psi_d^*]d[\psi_d] \exp\Bigg\{ -\frac{1}{\hbar}\Big[(\psi_R|\hbar\partial_\tau - i\hbar v_F\partial_x|\psi_R) \\
&+ (\psi_L|\hbar\partial_\tau + i\hbar v_F\partial_x|\psi_L) + (\psi_d|\hbar\partial_\tau + \Big(\xi(k) - i\hbar v_d\partial_x\Big) - \mu|\psi_d) \\
&+ \hbar(\psi_d|J) + \hbar(J|\psi_d) + \frac{1}{2}(|\psi_R|^2|V||\psi_R|^2) + \frac{1}{2}(|\psi_L|^2|V||\psi_L|^2) \\
&+ \frac{1}{2}(|\psi_d|^2|V||\psi_d|^2) + (|\psi_R|^2|V||\psi_L|^2) + (|\psi_R|^2|V||\psi_d|^2) + (|\psi_L|^2|V||\psi_d|^2)\Big]\Bigg\}.
\end{aligned}
\tag{3.12}
$$

We would now like to study this fourth-order interaction between the slow moving and Fermi level fermions via a collective Coulomb field, such as in [14]. In order to do that we would like to multiply our path integral with a real filed which is serves to describe said interaction and we define the following integral,

$$1 = \int d[\phi] \exp\Big[\frac{1}{2\hbar}(\phi - V\sum_{\mu}|\psi_{\mu}|^2|V^{-1}|\phi - V\sum_{\nu}|\psi_{\nu}|^2)\Big]$$
$$= \int d[\phi] \exp\Big[\frac{1}{2\hbar}(\phi|V^{-1}|\phi) - 2(\phi|\sum_{\mu}|\psi_{\mu}|^2) + (\sum_{\mu}|\psi_{\mu}|^2|V|\sum_{\nu}|\psi_{\nu}|^2)\Big].$$

Such that $\{\mu, \nu\} \in \{R, L, d\}$. So when multiplying this unity to our path integral the quartic interaction terms ought to cancel, leaving us with a Gaussian equation with respect to the Coulomb field. We would also like to mention that the Coulomb field is meant to replace the density field of all three fermion fields. We then re-write Eq.3.12,

$$\begin{aligned} Z[J, J^*] = \int d[\psi_R^*]d[\psi_R]d[\psi_L^*]d[\psi_L]d[\psi_d^*]d[\psi_d] \int d[\phi] \\ \times \exp\Big\{ -\frac{1}{\hbar}\Big[(\psi_R|\hbar\partial_\tau - i\hbar v_F\partial_x|\psi_R) + (\psi_L|\hbar\partial_\tau + i\hbar v_F\partial_x|\psi_L) \\ + (\psi_d|\hbar\partial_\tau + \Big(\xi(k) - i\hbar v_d\partial_x\Big) - \mu|\psi_d) + \hbar(\psi_d|J) + \hbar(J|\psi_d) \\ + \frac{1}{2}(|\psi_R|^2|V||\psi_R|^2) + \frac{1}{2}(|\psi_L|^2|V||\psi_L|^2) + \frac{1}{2}(|\psi_d|^2|V||\psi_d|^2) \\ + (|\psi_R|^2|V||\psi_L|^2) + (|\psi_R|^2|V||\psi_d|^2) + (|\psi_L|^2|V||\psi_d|^2)\Big]\Big\} \\ \times \exp\left\{\frac{1}{2\hbar}\Big[(\phi|V^{-1}|\phi) - 2(\phi|\sum_{\mu}|\psi_{\mu}|^2) + (\sum_{\mu}|\psi_{\mu}|^2|V|\sum_{\nu}|\psi_{\nu}|^2)\Big]\right\}. \end{aligned}$$

We can now group the linear terms of ϕ with terms which account for the

non-interacting Green's function which were treated in the previous section.

$$
\begin{aligned}
&Z[J,J^*] \\
&= \int d[\psi_R^*]d[\psi_R]d[\psi_L^*]d[\psi_L]d[\psi_d^*]d[\psi_d]\int d[\phi] \\
&\times \exp\left\{-\frac{1}{\hbar}\Big[(\psi_R|\hbar\partial_\tau - i\hbar v_d\partial_x + \phi(x,\tau)|\psi_R) + (\psi_L|\hbar\partial_\tau + i\hbar v_F\partial_x + \phi(x,\tau)|\psi_L)\Big]\right\} \\
&\times \exp\left\{-\frac{1}{\hbar}\Big[(\psi_d|\hbar\partial_\tau + \Big(\xi(k) - i\hbar v_d\partial_x\Big) - \mu + \phi(x,\tau)|\psi_d) + (J|\psi_d) + (\psi_d|J)\Big]\right\} \\
&\times \exp\left\{\frac{1}{2\hbar}(\phi|V^{-1}|\phi)\right\}.
\end{aligned}
\tag{3.13}
$$

It is important to see that after collecting all the quadratic terms in the fermionic fields we can identify the self-energy correction, $\Sigma_\phi = \Sigma(x,x';\tau,\tau'|\phi)$ for the sake of simplicity we would like to set the x' and τ' to 0. So taking the terms which are quadratic in the fermionic fields we get the following interacting Green's function,

$$
\sum_{\nu\in\{R,L,d\}}(\psi_\nu| - G^{-1}_{\nu,}(x,\tau;\phi)|\psi_\nu) = \sum_{\nu\in\{R,L,d\}}\Big((\psi_\nu| - G^{-1}_{\nu,0}|\psi_\nu) + (\phi|\psi^*_\nu\psi_\nu)\Big).
\tag{3.14}
$$

We can re-write the second term of the right-hand side of Eq. 3.14 such as,

$$
\begin{aligned}
&\Big(\phi|\sum_{\nu\in\{R,L,d\}}\psi^*_\nu\psi_\nu\Big) \\
&= \int dx\int d\tau d\tau'\phi(x,\tau)\sum_{\nu\in\{R,L,d\}}|\psi_\nu(x,\tau)|^2 \\
&= \int dxdx'\int d\tau\phi(x,\tau)\delta(x-x')\delta(\tau-\tau')\sum_{\nu\in\{R,L,d\}}\psi^*_\nu(x',\tau')\psi_\nu(x',\tau') \\
&= \int dxdx'\int d\tau d\tau'\sum_{\nu\in\{R,L,d\}}\psi^*_\nu(x',\tau')\phi(x,\tau)\delta(x-x')\delta(\tau-\tau')\psi_\nu(x'\tau'),
\end{aligned}
$$

where it is evident to see that $\hbar\Sigma(x,\tau;\phi) = \phi(x,\tau)\delta(x)\delta(\tau)$, and it is diagonal in the chirality of the fermions.

Integrating Over The Fermionic Fields

We would also like to integrate over the $\psi_L(x,\tau)$, $\psi_R(x,\tau)$ and slow moving, $\psi_d(x,\tau)$. It is important to note that the currents associated to the right and left moving fermions have been set to 0, since we are only interested of the two point correlation function of the slow-moving, deep fermions, $\langle T[\psi_d(x,\tau)\psi_d^*(x',\tau')]\rangle$, It is important to note that we will be studying the advanced two-point correlation function which is of the form, $\langle\psi_d^*(x',\tau')\psi_d(x,\tau)\rangle$. We can begin by writing our generating function with respect to our Hubbard-Stratonovich field and the slow-moving fermionic fields,

$$\begin{aligned} Z[J,J^*] = \int d[\psi_d^*]d[\psi_d]\int d[\phi]\exp\Big\{ -\Big[\log\Big(\hbar\partial_\tau + i\hbar v_F\partial_x + \phi(x,\tau)\Big)\Big]\Big\} \\ \times\exp\Big\{ -Tr\Big[\log\Big(\hbar\partial_\tau - i\hbar v_F\partial_x + \phi(x,\tau)\Big)\Big]\Big\} \\ \times\exp\Big\{ -\frac{1}{\hbar}\Big[(\psi_d|\hbar\partial_\tau + \Big(\xi(k) - i\hbar v_d\partial_x\Big) + \phi(x,t)|\psi_d)\Big\} \\ \times\exp\Big\{(J|\psi_d) + (\psi_d|J) - \frac{1}{2}(\phi|V^{-1}|\phi)\Big]\Big\}. \end{aligned} \tag{3.15}$$

In order to retrieve the Green's function corresponding to the slow-moving $\psi_d(x,\tau)$ and $\psi_d^*(x,\tau)$, we ought to complete the square in the action such that,

$$\begin{aligned} Z[J,J^*] = \int d[\psi_d]d[\psi_d^*]\int d[\phi]\exp\Big\{ -\mathrm{Tr}\Big[\log\Big(\hbar\partial_\tau + i\hbar v_F\partial_x + \phi(x,\tau)\Big)\Big]\Big\} \\ \times\exp\Big\{ -\mathrm{Tr}\Big[\log\Big(\hbar\partial_\tau - i\hbar v_F\partial_x + \phi(x,\tau)\Big)\Big]\Big\} \\ \times\exp\Big\{ -\frac{1}{\hbar}\Big[(\psi_d + G_dJ| - G_d^{-1}(x,\tau;x',\tau')|\psi_d + G_dJ)\Big\} \\ \times\exp\Big\{ -(J|G_d(x,\tau;x',\tau')|J) - \frac{1}{2}(\phi|V^{-1}|\phi)\Big]\Big\}, \end{aligned} \tag{3.16}$$

Where it is clear for us to see that:

$$-\frac{1}{\hbar}G_d^{-1}(x,\tau;x',\tau'|\phi) = \Big\{\hbar\partial_\tau + \Big(\xi(k) - i\hbar v_d\partial_x\Big) + \phi(x,\tau)\Big\}\delta(x-x')\delta(\tau-\tau') \tag{3.17}$$

Taking the functional derivative of the path integral with respect to the current J and J^* respectively. Seeing that the functional integral has become quadratic in terms of the fermionic fields we can integrate them out of the integral and get the following,

$$\begin{aligned}\langle G_d(k|x,\tau)\rangle_\phi = \int d[\phi]\Big\{\hbar\partial_\tau + \Big(\xi(k) - i\hbar v_d\partial_x\Big) + \phi(x,\tau)\Big\}^{-1} \\ \times \exp\Big\{ -\mathrm{Tr}\Big(\log\Big[\hbar\partial_\tau + \xi(k) - i\hbar v_d\partial_x + \phi(x,t)\Big]\Big)\Big\} \\ \times \exp\Big\{ -\mathrm{Tr}\Big(\log\Big[\hbar\partial_\tau + i\hbar v_F\partial_x + \phi(x,\tau)\Big]\Big)\Big\} \\ \times \exp\Big\{ -\mathrm{Tr}\Big(\log\Big[\hbar\partial_\tau - i\hbar v_F\partial_x + \phi(x,\tau)\Big]\Big) - \frac{1}{2}(\phi|V^{-1}|\phi)\Big\}.\end{aligned} \tag{3.18}$$

The term in the exponent is now called the effective action which is solely dependent on the Hubbard-Stratonovich field, $\phi(x,\tau)$, we now can write the effective action as,

$$\begin{aligned}S_{\text{eff}}[\phi] = -\hbar\mathrm{Tr}\Big(\log\Big[\hbar\partial_\tau + \xi(k) - i\hbar v_d\partial_x + \phi(x,t)\Big]\Big) \\ - \hbar\mathrm{Tr}\Big(\log\Big[\hbar\partial_\tau + i\hbar v_F\partial_x + \phi(x,\tau)\Big]\Big) \\ - \hbar\mathrm{Tr}\Big(\log\Big[\hbar\partial_\tau - i\hbar v_F\partial_x + \phi(x,\tau)\Big]\Big) + \frac{1}{2\hbar}(\phi|V^{-1}|\phi).\end{aligned} \tag{3.19}$$

Minimizing The Effective Action

We now proceed to minimize the effective action by putting $\phi(x,\tau) = \langle\phi(x,\tau)\rangle + \phi'(x,\tau)$ where we know the averaged ϕ term minimizes the effective action. We then get the basis-independent formulation that gives us,

$$G^{-1} = G_0^{-1} - \frac{\langle\phi\rangle}{\hbar} - \frac{\phi'}{\hbar}. \tag{3.20}$$

This is actually a shift in our integration variables we can also proceed to set, $\frac{\delta S_{\text{eff}}[\phi]}{\delta\phi(x,\tau)} = 0$, we can also neglect any fluctuations, we then have the Hartree treatment of our system, which in one equation can be written as $Z \sim e^{-\frac{S_{\text{eff}}[\phi]}{\hbar}}$. All of this, in a basis independent notation, can be translated to,

$$G^{-1} = G^{H^{-1}} - \Sigma' = G^{H^{-1}}(1 - G^H\Sigma'), \tag{3.21}$$

where it is straightforward to see, $G^{H^{-1}} = G_0^{-1} = \frac{\langle\phi\rangle}{\hbar}$ as well as $\hbar\Sigma'(x - x', \tau - \tau') = \phi'(x,\tau)\delta(x - x')\delta(\tau - \tau')$.

Here we would like to study the minimized effective action in order to try and cancel out all the linear terms in ϕ'. It is important to mention that for the continuation of the chapter we would be excluding the term which accounts for the deep slow moving fermions described by the $\psi_d^*(x,\tau)$ and $\psi_d(x,\tau)$ fields. So we're left with,

$$S^{\text{eff}}[\phi] = -\hbar\text{Tr}(\log(-G_R^{-1})) - \hbar\text{Tr}(\log(-G_L^{-1})) + \frac{1}{2\hbar}(\phi|V^{-1}|\phi). \tag{3.22}$$

Where we can see that $G_R^{-1} = G_{R_0}^{-1} - \frac{\langle\phi\rangle}{\hbar} - \frac{\phi'}{\hbar}$, here we find it convenient to use the following basis independent notation:

$$G_R^{-1} = G_{R_0}^{-1} - \Sigma' = G_R^{H-1}(1 - G^H\Sigma). \tag{3.23}$$

We can now use the logarithmic expansion in order to get the following equation

$$\log(1 - G_R^H\Sigma') = -G_R^H\Sigma' - \frac{1}{2}(G_R^H\Sigma'G_R^H\Sigma') + ... \tag{3.24}$$

We can now proceed to set all linear terms to zero such that,

$$-\hbar\text{Tr}[-G_R^H\Sigma'] - (\phi'|V^{H-1}|\langle\phi\rangle) = 0 \tag{3.25}$$

By setting these linear terms to zero we actually assume that $\langle\phi\rangle$ is the minimum and that the effective action actually contributes to the Hartree selfenergy.

We now proceed to apply this logarithimic expansion to the effective action in ϕ':

$$S^{\text{eff}}[\phi'] = \frac{1}{2}(\phi'|V^{-1}|\phi') + \frac{\hbar}{2}\text{Tr}[G_R^H\Sigma' G_R^H\Sigma'] + \frac{\hbar}{2}\text{Tr}[G_L^H\Sigma' G_L^H\Sigma'] + ... \quad (3.26)$$

We can now see that this quadratic term in full, we will only work on the right handed electrons because applying this same method to the left handed electrons should be trivial and straightforward.

$$\begin{aligned}
&\text{Tr}[G_R^H\Sigma' G_R^H\Sigma'] \\
&= \int d\tau d\tau' d\tau'' d\tau''' \int dx dx' dx'' dx''' G_R^H(x,\tau;x',\tau') \\
&\times \Sigma'(x',\tau';x'',\tau'')G_R^H(x'',\tau'';x''',\tau''')\Sigma'(x''',\tau''';x,\tau) \\
&= \int_0^{\hbar\beta} d\tau \int dx \int_0^{\hbar\beta} d\tau' \int dx' G_R^H(x,\tau,x',\tau')\frac{\phi'(x',\tau')}{\hbar} G_R^H(x',\tau',x,\tau)\frac{\phi'(x,\tau)}{\hbar},
\end{aligned} \quad (3.27)$$

where we can clearly see that $\hbar\Sigma'(x,\tau,x',\tau';\phi') = \phi'(x,\tau)\delta(x-x')\delta(\tau-\tau')$.

We briefly remind ourselves that,

$$G_R^{H-1}(x,\tau;x',\tau') = -\frac{1}{\hbar}\bigg(\hbar\partial_\tau - i\hbar v_F\partial_x\bigg)\delta(x-x')\delta(\tau-\tau'). \quad (3.28)$$

where in Fourier space we can write it as,

$$G_R^H(q,i\omega_n) = \frac{-\hbar}{-i\hbar\omega_n + \hbar v_F q} \quad (3.29)$$

We now proceed to calculate the quadratic term of the effective action post Fourier transform.

$$\begin{aligned}
S^{\text{eff}}[\phi'] = \sum_{q,n}\phi'(q,i\omega_n)\bigg\{&\frac{1}{V_k} + \int\frac{dq}{2\pi}G_R^H(q',i\omega_{n'})G_R^H(q+q',i\omega_n+i\omega_{n'}) \\
&+ \int\frac{dq}{2\pi}G_L^H(q',i\omega_{n'})G_L^H(q+q',i\omega_n+i\omega_{n'})\bigg\}\phi'(q,i\omega_n).
\end{aligned} \quad (3.30)$$

In this case we would like to study the polarization operators for both the right movers and left movers which are located at the Fermi points k_F and $-k_F$ respectively. It is important to note that the polarization operators have corresponding diagrams which are known to be bubble diagrams. The analytic expressions of these bubble diagrams are well known. We will proceed to derive them from the generic Lindhard function, taking the long-wavelength limit and taking into account we get the following.

$$\Pi_R[q, i\omega_n] = \frac{1}{\hbar\beta}\int \frac{dq'}{2\pi}\frac{-\hbar}{-i\hbar\omega_n + \hbar v_F q}\Big\{G_R^H(q'+q, i\omega_n + i\omega_{n'}) - G_R^H(q', i\omega_{n'})\Big\}. \tag{3.31}$$

Taking the first term of the right hand side of Eq. 3.31 we compute,

$$\begin{aligned}\Pi_R^{(1)}(q, i\omega_n) &= \left(\frac{1}{i\omega_n - v_F q}\right)\sum_{n'}\int\frac{dq'}{2\pi}\frac{-\hbar}{-i\hbar(\omega_n+\omega_{n'}) + \hbar v_F(q+q')} \\ &= \left(\frac{1}{i\omega_n - v_F q}\right)\int\frac{dq'}{2\pi}\sum_{n'}\frac{1}{i(\omega_n+\omega_{n'}) - v_F(q+q')} \\ &= \left(\frac{1}{i\omega_n - v_F q}\right)\int\frac{dq'}{2\pi}N_{FD}(q+q').\end{aligned} \tag{3.32}$$

And here the same can be applied to the second term and we would get the following equation.

$$\Pi_R[q, i\omega_n] = \left(\frac{1}{i\omega_n - v_F q}\right)\int\frac{dq'}{2\pi}\left[\frac{1}{e^{\beta\left(v_F(q+q')\right)}+1} - \frac{1}{e^{\beta\left(v_F q'\right)}+1}\right], \tag{3.33}$$

and for the left moving particles we ought to get,

$$\Pi_L[q, i\omega_n] = \left(\frac{1}{i\omega_n + v_F q}\right)\int\frac{dq'}{2\pi}\left[\frac{1}{e^{\beta\left(-v_F(q+q')\right)}+1} - \frac{1}{e^{\beta\left(-v_F q'\right)}+1}\right]. \tag{3.34}$$

We can now see that these Π_R and Π_L are to be summed such that we get the following relations,

$$\Pi(q,\omega_n) = \Pi_R(q,\omega_n) + \Pi_L(q,\omega_n) = \nu \frac{v_F^2 q^2}{\omega_n^2 + v_F^2 q^2}. \tag{3.35}$$

Where $\nu = \frac{1}{\pi v_F}$. We can set the renormalized velocity to be the following,

$$v = v_F\left(\sqrt{1 + \frac{V_0}{\pi v_F}}\right). \tag{3.36}$$

So we can finally get the following propagator for the screened Coulomb

$$\langle \phi(q, i\omega_n)\phi(-q, -i\omega_n)\rangle = V_0 \frac{\omega_n^2 + v_F^2 q^2}{\omega_n^2 + v^2 q^2}. \tag{3.37}$$

which satisfies the following equation:

$$\Big[i\omega_n - v_d q\Big]\theta(q, i\omega_n) = \phi(q, i\omega_n). \tag{3.38}$$

We can now see that the formal solution for the deep green's function can be expressed as.

$$\langle G_d(k|x,\tau)\rangle_\phi = -G_d(k|x,\tau)\exp\Big[\frac{1}{2}\langle\theta(x,\tau) - \theta(0,0)\rangle^2\Big]. \tag{3.39}$$

Our Hubbard Stratonovich gave us the same exact Green's function for the ϕ field we get the following explicit equation for the Green's function for the deep field. Without loss of generality and for the sake of simplicity we will be studying the averaged propagator of a slow moving hole which starts at the origin of the wire,

$$
\begin{aligned}
&\langle G_d(k|x,\tau)\rangle_\phi \\
&= G_d(k|x,\tau)\exp\left[\int \frac{dq}{(2\pi)}\sum_{\omega_n}\frac{\langle\phi\phi\rangle}{(-i\omega_n+v_dq)^2}\Big(1-\cos(qx-\omega_n\tau)\Big)\right] \\
&= G_d(k|x,\tau)\exp\left[\int \frac{dq}{(2\pi)}\sum_{\omega_n}\frac{V_0}{(-i\omega_n+v_dq)^2}\frac{\omega_n^2+v_F^2q^2}{\omega_n^2+v^2q^2}\Big(1-\cos(qx-\omega_n\tau)\Big)\right] \\
&= G_d(k|x,\tau) \\
&\times\exp\left[\int \frac{dq}{2\pi}\sum_{\omega_n}\frac{V_0}{(-i\omega_n+v_dq)^2}\right. \\
&\left.\times\left(1+\frac{v_F^2/vq}{-i\omega_n+vq}+\frac{v_F^2/vq}{i\omega_n+vq}\right)\Big(1-\cos(qx-\omega_n\tau)\Big)\right].
\end{aligned}
\tag{3.40}
$$

Before proceeding to solve this integral and sum we would like to take the zero-temperature limit such that $T\to 0$ where our Matsurbara frequencies become continuous.

$$\sum_{n\in\mathbb{Z}} i\omega_n \to \int \frac{id\omega}{(2\pi)}.$$

We first proceed to find the useful poles and they are, $i\omega = \pm vq$ where in this case v is the renormalized velocity.

$$
\begin{aligned}
\langle G_d(k|x,\tau)\rangle_\phi &= G_d(k|x,\tau) \\
&\times\exp\left[\int \frac{dqd(i\omega)}{(2\pi)^2}\frac{V_0}{(-i\omega+v_dq)^2}\Big(1-\cos(qx-\omega\tau)\Big)\right. \\
&+\int \frac{dqd(i\omega)}{(2\pi)^2}\frac{V_0}{(-i\omega+v_dq)^2}\frac{\alpha(k)q}{-i\omega+vq}\Big(1-\cos(qx-\omega\tau)\Big) \\
&\left.+\int \frac{dqd(i\omega)}{(2\pi)^2}\frac{V_0}{(-i\omega+v_dq)^2}\frac{\alpha(k)q}{i\omega+vq}\Big(1-\cos(qx-\omega\tau)\Big)\right],
\end{aligned}
\tag{3.41}
$$

where $\alpha = v_F\frac{k_F}{k}$, where $k = mv$. We continue by using the residue theorem

for the integral over the frequency.

$$\langle G_d(k|x,\tau)\rangle_\phi = G_d(k|x,\tau)\exp\Bigg[-\frac{\alpha(k)V_0}{(v_d-v)^2}\int_0^{\frac{\lambda k^2}{mv}}\frac{dq}{q}\Big(1-\cos((x+v\tau)q)\Big) + \frac{\alpha(k)V_0}{(v_d+v)^2}\int_0^{\lambda k}\frac{dq}{q}\Big(1-\cos((x-v\tau)q)\Big)\Bigg]. \tag{3.42}$$

As per section 2.1.2 and Figure 2.1, it is important to see that we have set cutoffs for our momenta which are $|q| < \lambda k^2/mv$ for the $L-d$ interactions and $|q| < \lambda k$ for the $R-d$ interactions. It is also important to note that because the scalar field θ only enters when combined with $\delta(x-v_d\tau)$ the double-pole at $v_d q$ does not show up in Eq. 3.42. For the long-wavelength/low-momentum, spectrum we solely get the logarithmic contributions. Which is similar to the results discussed in [14]. Using Mathematica we get,

$$\langle G_d(k|x,\tau)\rangle_\phi = G_d(k|x,\tau)\exp\Bigg\{-\frac{\alpha(k)V_0}{(v_d+v)^2}\log\left[\frac{\lambda k^2}{mv}(x+v\tau)\right] - \frac{\alpha(k)V_0}{(v_d-v)^2}\log\Big[\lambda k(x-v\tau)\Big]\Bigg\}. \tag{3.43}$$

3.3 Conclusion

After applying the Hubbard-Stratonovich transformation to the system extensively discussed in Chapter 2, we evaluated the Green's function by also accounting for the slow varying potential $\phi(x,t)$ by minimizing it and applying the RPA method in order to deduce the its own averaged propagator. The next step is to try and generalize this method of introducing a new field in order to kill off the quartic terms in the action and leave us solely with a Gaussian integrals over the interacting fields which can easily be calculated.

Chapter 4

$\omega \sim \sqrt{q}$ Singularities In 2+1 Dimensions Using Hubbard Stratonovich Transformation

Newly found properties of two-dimensional systems such as the high-T_C superconducting phenomena of cuprates, is a really good candidate to investigate in this final chapter of our thesis. After computing the Green's function for the slow moving excitations we move on and try to understand what our technique has to say for systems that are anistoropic in two-dimensions. In this chapter we will be studying a four-fold symmetric lattice with a near-neighbor, tight-binding dispersion.

4.1 The Tight-Binding Theory

Materials that are mostly composed of closed-shell atomic or ionic lattices, can not be treated by the parabolic approximation, or better yet the free electron dispersion, instead it is more appropriate to use the tight-binding model. We begin by taking an element with one atom place in each unit-cell, where each atom is allowed one valence orbital to be defined as $\phi_n(r)$. We take the a linear combination of a state such that it obeys Bloch's theorem,

$$\psi_{n\mathbf{k}}(\mathbf{r}+\mathbf{R}) = e^{i\mathbf{k}\mathbf{R}}\psi_{n\mathbf{k}}(\mathbf{r}), \tag{4.1}$$

where $\mathbf{R}$ is a translation vector, such that $\mathbf{R} \in \mathbb{R}$.

4.1.1 Bloch's Theorem

The theorem states that solutions to the Schrödinger equation in a periodic potential take the form of a plane wave modulated by a periodic function. Taking a linear combination of the orbitals we get the following equation,

$$\psi_{n\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{N}} \sum_{\mathbf{R}} e^{i\mathbf{kR}} \phi_n(\mathbf{r} - \mathbf{R}). \tag{4.2}$$

Where N is the number of lattice sites and the factor $\frac{1}{\sqrt{N}}$ is for normalizing the state. Assume that $\mathbf{R'} \in \mathbb{R}$ we write,

$$\psi_{n\mathbf{k}}(\mathbf{r} - \mathbf{R}') = \frac{1}{\sqrt{N}} \sum_{\mathbf{R}} e^{i\mathbf{kR}} \phi_n(\mathbf{r} - (\mathbf{R} - \mathbf{R'})), \tag{4.3}$$

we define $\mathbf{R} - \mathbf{R}' = \mathbf{R}''$ such that we re-write Equation 4.3 as,

$$\begin{aligned} \psi_{n\mathbf{k}}(\mathbf{r} + \mathbf{R}') &= \frac{1}{\sqrt{N}} \sum_{\mathbf{R}''} e^{i\mathbf{k}(\mathbf{R}'+\mathbf{R}'')} \phi_n(\mathbf{r} - \mathbf{R}'') \\ &= e^{i\mathbf{kR}'} \frac{1}{\sqrt{N}} \sum_{\mathbf{R}''} e^{i\mathbf{kR}''} \phi_n(\mathbf{r} - \mathbf{R}'') \\ &= e^{i\mathbf{kR}'} \psi_{n\mathbf{k}}(\mathbf{r}) \end{aligned} \tag{4.4}$$

4.1.2 s-band in a 2D lattice

We define the disperion relation to be,

$$\begin{aligned} \epsilon(\mathbf{k}) &= \int d\mathbf{r} \psi^*_{\mathbf{k}}(\mathbf{r}) H \psi_{\mathbf{k}}(\mathbf{r}) \\ &= \frac{1}{N} \sum_{\mathbf{R}} \sum_{\mathbf{R}'} e^{i\mathbf{k}(\mathbf{R}'-\mathbf{R})} \int d\mathbf{r} \phi_s(\mathbf{r} - \mathbf{R}) H \phi_s(\mathbf{r\text{-}R'}) \\ &= \frac{1}{N} \sum_{\mathbf{R}} \sum_{\mathbf{R}'} e^{i\mathbf{k}(\mathbf{R}-\mathbf{R}')} \int d\mathbf{x} \phi_s(\mathbf{x}) H(\mathbf{x}) \phi_s(\mathbf{r} - \mathbf{R}'), \end{aligned} \tag{4.5}$$

where $\mathbf{x} = \mathbf{r} - \mathbf{R}$ and H is unchanged due to periodic boundary conditions (i.e. $H(\mathbf{r\text{-}R}) = H(\mathbf{r})$). Reminding ourselves of $\mathbf{R} - \mathbf{R}' = \mathbf{R}''$, we get,

$$\begin{aligned} \epsilon(\mathbf{k}) &= \frac{1}{N} \sum_{\mathbf{R}} \sum_{\mathbf{R}''} e^{i\mathbf{kR}''} \int d\mathbf{x} \phi_s(\mathbf{x}) H \phi_s(\mathbf{r} - \mathbf{R}'') \\ &= \sum_{\mathbf{R}''} e^{i\mathbf{kR}''} \int d\mathbf{x} \phi_s(\mathbf{x}) H \phi_s(\mathbf{x} - \mathbf{R}''). \end{aligned} \tag{4.6}$$

The factor N comes from summing over $\mathbf{R}$. The orbitals are tightly localized, meaning that they are large for $|\mathbf{r}| \ll 1$ and decay rapidly away from $\mathbf{r} = \mathbf{0}$. Meaning that if the translation vector $\mathbf{R}''$ is large, then

$$\int d\mathbf{x}\phi_s(\mathbf{x})H\phi_s(\mathbf{x}-\mathbf{R}'') \approx 0. \tag{4.7}$$

Setting $\mathbf{R}''$ to be not large we get,

$$\begin{aligned}\epsilon(\mathbf{k}) &= \sum_{\mathbf{R}''} e^{i\mathbf{k}\mathbf{R}''} \int d\mathbf{x}\phi_s(\mathbf{x})H\phi_s(\mathbf{x}-\mathbf{R}'') \\ &= -\gamma(\cos(k_x)+\cos(k_y)).\end{aligned} \tag{4.8}$$

Where $-\gamma = \int d\mathbf{x}\phi_s(\mathbf{x})H\phi_s(\mathbf{x}-\mathbf{R}'')$. We have absorbed the lattice constant a in the momentum to make $k = (k_x, k_y)$ a dimensionless entity.

4.2 Interacting Green's function.

Therefore, we can define a non-interacting action,

$$S_0[\psi^*,\psi] = \int dxdy \int d\tau\psi^*(x,y;\tau)\left[\hbar\frac{\partial}{\partial\tau} + \epsilon(k_x,k_y) - \mu\right]\psi(x,y;\tau). \tag{4.9}$$

For simplicity we can use the following notation:

$$S_0[\psi^*,\psi] = (\psi|\hbar\partial_\tau + \xi(k_x,k_y)|\psi).$$

Where $\xi(k_x,k_y) = \epsilon(k_x,k_y) - \mu$.

$$\left(\hbar\frac{\partial}{\partial\tau} + \xi(k_x,k_y)\right)G_k(x_1-x_2;y_1-y_2;\tau-\tau') = \delta(x_1-x_2)\delta(y_1-y_2)\delta(\tau-\tau'). \tag{4.10}$$

Here we can safely see that our non-interacting green's function for the deep electronic fields can be written as. We will now proceed to apply Hubbard-Stratonovich transformation to density fields such that,

$$\begin{aligned}Z[J,J^*] = \int d[\psi^*]d[\psi]\exp\left\{-\frac{1}{\hbar}\int dx\psi^*(x,\tau)(\hbar\partial_\tau + \xi(k_x,k_y))\psi(x,\tau)\right\} \\ \times\exp\left\{\frac{1}{2\hbar}\int dxdx'\psi^*(x,\tau)\psi^*(x',\tau)V(x-x')\psi(x,\tau)\psi(x',\tau)\right\},\end{aligned} \tag{4.11}$$

we now move on to apply a Hubbard-Stratonovich transformation to our generating functional such that,

$$\begin{aligned} 1 &= \int d[\phi] \exp\Big[\frac{1}{2\hbar}(\phi - V|\psi|^2|V^{-1}|\phi - V|\psi|^2)\Big] \\ &= \int d[\phi] \exp\Big[\frac{1}{2\hbar}(\phi|V^{-1}|\phi) - 2(\phi||\psi|^2) + (|\psi|^2|V||\psi|^2)\Big]. \end{aligned}$$

Where $V = V(x - x')$. We multiply the unity that was previously defined to the path integral of our system.

$$\begin{aligned} Z[J, J^*] = &\int d[\psi^*]d[\psi] \exp\Bigg\{ -\frac{1}{\hbar}\Bigg[\int dx d\tau \psi^*(x,\tau)\Big(\hbar\partial_\tau + \xi(k_x, k_y)\Big)\psi(x,\tau) \\ &+ \frac{1}{2}\int dx dx' d\tau d\tau' \psi^*(x,\tau)\psi^*(x',\tau')V\psi(x,\tau)\psi(x',\tau')\Bigg]\Bigg\} \\ &\times \int d[\phi] \exp\left\{ \frac{1}{2\hbar}(\phi|V_0^{-1}|\phi) - 2(\phi||\psi|^2) + (|\psi|^2|V||\psi|^2)\right\} \\ &= \int d[\psi^*]d[\psi] \int d[\phi] \exp\Bigg[-\frac{1}{\hbar}\Big[(\psi|\Big(\hbar\partial_\tau + \xi(k_x,k_y) + \phi(x,y,\tau)\Big)|\psi) \\ &+ (J|\psi) + (\psi|J) - \frac{1}{2}(\phi|V^{-1}|\phi)\Big]\Bigg]. \end{aligned} \tag{4.12}$$

After this we integrate the fermions out of the action and we finally get,

$$Z[J, J^*] = \int d[\phi] \exp\left\{ -\hbar Tr \log[\hbar\partial_\tau + \xi(k_x,k_y) + \phi(x,y,\tau)] - \frac{1}{2}(\phi|V^{-1}|\phi)\right\}. \tag{4.13}$$

Where it is clear for us to see that

$$-\frac{1}{\hbar}G^{-1}(x_1, y_1, \tau, |\phi) = \Big\{\hbar\partial_\tau + \epsilon(k_x,k_y) - \mu + \phi(x_1,y_1,\tau)\Big\}\delta(x_1 - x_2)\delta(y_1 - y_2)\delta(\tau - \tau'). \tag{4.14}$$

Here we can see that the effective action can be written as,

$$S_{\text{eff}}[\phi] = -\hbar \text{Tr}\Bigg(\log\Big[\hbar\partial_\tau + \xi(k_x, k_y) + \phi(x, y, \tau)\Big]\Bigg)) + \frac{1}{2\hbar}(\phi|V^{-1}|\phi). \quad (4.15)$$

We now proceed to minimize the effective action by putting $\phi(x, y, \tau) = \langle\phi(x, y, \tau)\rangle + \phi'(x, y, \tau)$ where we know the averaged ϕ term minimizes the effective action. In a basis independent notation, can be translated to,

$$G^{-1} = G^{H^{-1}} - \Sigma' = G^{H^{-1}}(1 - G^H\Sigma'), \quad (4.16)$$

where it is straightforward to see, $G^{H^{-1}} = G_0^{-1} = \frac{\langle\phi\rangle}{\hbar}$ and that $\hbar\Sigma(x_1 - y_1, x_2 - y_2; \tau - \tau')' = \phi'(x_1, x_2; \tau)\delta(x_1 - y_1)\delta(x_2 - y_2)\delta(\tau - \tau')$.

Now the main part to study is the minimized effective action. Where for the sake of simpler book-keeping we neglect the deep field term.

$$S_{\text{eff}}[\phi] = -\hbar \text{Tr}(\log(-G^{-1})) + \frac{1}{2\hbar}(\phi|V^{-1}|\phi), \quad (4.17)$$

where we can see that $G^{-1} = G_0^{-1} - \frac{\langle\phi\rangle}{\hbar} - \frac{\phi'}{\hbar}$ and this leads to the following conclusion,

$$G^{-1} = G_0^{-1} - \Sigma' = G^{H-1}(1 - G^H\Sigma). \quad (4.18)$$

We can now use the logarithmic expansion in order to get the following equation,

$$\log(1 - G^H\Sigma') = -G^H\Sigma' - \frac{1}{2}(G^H\Sigma' G^H\Sigma'). \quad (4.19)$$

We can now proceed to set all linear terms to zero such that.

$$-\hbar \text{Tr}[-G^H\Sigma'] - (\phi'|V^{H-1}|\langle\phi\rangle) = 0. \quad (4.20)$$

By setting these linear terms to zero we actually assume that $\langle\phi\rangle$ is the minimum and that the effective action actually contributes to the Hartree selfenergy.

We now proceed to apply this logarithimic expansion to the effective action in ϕ':

$$S^{\text{eff}}[\phi'] = \frac{1}{2}(\phi'|V^{-1}|\phi') + \frac{\hbar}{2}\text{Tr}[G_H\Sigma' G_H\Sigma']. \tag{4.21}$$

We can now see that this quadratic term in full, we will only work on the right handed electrons because applying this same method to the left handed electrons should be trivial and straightforward.

$$\begin{aligned}
\text{Tr}[G^H\Sigma' G^H\Sigma'] &= \int d\tau d\tau' d\tau'' d\tau''' \int dx_1 dx_2 dx_3 dx_4 \int dy_1 dy_2 dy_3 dy_4 \\
&\times G^H(x_1 - x_2, y_1 - y_2; \tau - \tau')\Sigma'(x_2 - x_3, y_2 - y_3, \tau' - \tau'') \\
&\times G^H(x_3 - x_4, y_3 - y_4; \tau'' - \tau''')\Sigma'(x_4 - x_1, y_4 - y_1; \tau''' - \tau) \\
&= \int_0^{\hbar\beta} d\tau d\tau' \int dx_1 dx_2 \int dy_1 dy_2 \\
&\times G^H(x_1 - x_2, y_1, y_2; \tau - \tau')\frac{\phi'(x_2, y_2, \tau')}{\hbar} \\
&\times G^H(x_2 - x_1, y_2 - y_1; \tau' - \tau)\frac{\phi'(x_1, y_1; \tau)}{\hbar},
\end{aligned} \tag{4.22}$$

where we can clearly see that $\Sigma'(x_1 - x_2, y_1 - y_2; \tau - \tau') = \frac{1}{\hbar}\phi'(x_1, \tau)\delta(x_1 - x_2)\delta(y_1 - y_2)\delta(\tau - \tau')$. We briefly remind ourselves that:

$$\begin{aligned}
G^{H-1}(x_1 - x_2, y_1 - y_2; \tau - \tau') = -\frac{1}{\hbar}\Big\{\hbar\partial_\tau + \epsilon(k_x, k_y) - \mu + \langle\phi(x_1, y_1; \tau)\rangle\Big\} \\
\times \delta(x_1 - x_2)\delta(y_1 - y_2)\delta(\tau - \tau').
\end{aligned} \tag{4.23}$$

Due to positive background we get the following Green's function in Fourier space,

$$G_H(k, i\omega_n) = \frac{-\hbar}{-i\hbar\omega_n + \epsilon(k_x, k_y) - \mu}. \tag{4.24}$$

We now proceed in order to determine the polarization operator which is also sometimes called as a linear response function. We can rewrite this linear response function as a difference between.

$$\begin{aligned}\Pi(q,\omega_n) &= \int \frac{d\mathbf{k}}{(2\pi)^2} f(k)\left[\frac{1}{-i\hbar\omega_n - \epsilon_{\mathbf{k}} + \epsilon_{\mathbf{k}-\mathbf{q}}} - \frac{1}{-i\hbar\omega_n + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}}\right] \\ &= \frac{-1}{i\hbar\omega_n}\int \frac{dk_x dk_y}{(2\pi)^2}\left[1 - \left(\frac{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}-\mathbf{q}}}{i\hbar\omega_n}\right) + \left(\frac{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}-\mathbf{q}}}{i\hbar\omega_n}\right)^2 - \left(\frac{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}-\mathbf{q}}}{i\hbar\omega_n}\right)^3\right. \\ &+ \left(\frac{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}-\mathbf{q}}}{i\hbar\omega_n}\right)^4 - 1 + \left(\frac{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}}{i\hbar\omega_n}\right) - \left(\frac{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}}{i\hbar\omega_n}\right)^2 \\ &+ \left.\left(\frac{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}}{i\hbar\omega_n}\right)^3 - \left(\frac{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}}{i\hbar\omega_n}\right)^4\right].\end{aligned} \tag{4.25}$$

We will be focusing on the contributions that actually stand out in terms of four-fold symmetry. We can see that the contributions we get,

$$\begin{aligned}\Pi^{(2)}(\mathbf{q},\omega_n) &= \int \frac{dk_x dk_y}{(2\pi)^2}\left(\frac{\gamma}{\hbar\omega_n}\right)^2\Big(-2q_x^2 + k_x^2 q_x^2 + \frac{q_x^4}{6} + k_y^2 q_y^2 - 2q_y^2 + \frac{q_y^4}{6}\Big) \\ &= \frac{\gamma}{2\pi}\frac{1}{(\hbar\omega_n)^2}\Big(-8\pi q_x^2 + \frac{4\pi^3 q_x^2}{3} + \frac{2\pi q_x^4}{3} - 8\pi q_y^2 + \frac{4\pi^3 q_y^2}{3} + \frac{2\pi q_y^4}{3}\Big).\end{aligned} \tag{4.26}$$

As for the fourth order contributions we get,

$$\begin{aligned}\Pi^{(4)}(\mathbf{q},\omega_n) &= \frac{-4\gamma^3}{(\hbar\omega_n)^4}\Big(-8\pi^3 q_x^4 + 4\pi^5 q_x^4 - \frac{2\pi^7 q_x^4}{3} + \frac{\pi^9 q_x^4}{27} \\ &- 16\pi^3 q_x^2 q_y^2 + 4\pi^5 q_x^2 q_y^2 - \frac{4}{15}\pi^7 q_x^2 q_y^2 + \frac{1}{378}\pi^9 q_x^2 q_y^2 - 8\pi^3 q_y^4 + 4\pi^5 q_y^4 \\ &- \frac{\pi^7 q_y^4}{3} + \frac{\pi^9 q_y^4}{27}\Big) \\ &= \frac{-4\gamma^3}{(\hbar\omega_n)^4}\Big(-8\pi^3(q_x^2+q_y^2)^2 + 4\pi^5(q_x^4 + q_x^2 q_y^2 + q_y^4) \\ &- \frac{2}{15}\pi^7(5q_x^4 + 2q_x^2 q_y^2 + 5q_y^4) + \frac{1}{378}\pi^9(14q_x^4 + q_x^2 q_y^2 + 14q_y^4)\Big).\end{aligned} \tag{4.27}$$

4.3 Plasmonic Singularities of $\langle\phi(\mathbf{q})\phi(-\mathbf{q})\rangle$

We calculate the propagator for the Hubbard Stratonovich field, which can be written in the following form,

$$\langle\phi(\mathbf{q})\phi(-\mathbf{q})\rangle = \frac{1}{\frac{1}{V(q)} + \Pi(\mathbf{q},\omega_n)} = \frac{V(q)z^4}{z^4 + V(q)\chi_1(q^4)z^2 + V(q)\chi_2(q^4)}, \quad (4.28)$$

where we set:

$$\chi_1(q^4) = \frac{\gamma}{2\pi}\Big(- 8\pi q_x^2 + \frac{4\pi^3 q_x^2}{3} + \frac{2\pi q_x^4}{3} - 8\pi q_y^2 + \frac{4\pi^3 q_y^2}{3} + \frac{2\pi q_y^4}{3}\Big),$$
$$\chi_2(q^4) = \frac{-4\gamma^3}{(2\pi)}\Big(- 8\pi^3(q_x^2+q_y^2)^2 + 4\pi^5(q_x^4 + q_x^2q_y^2 + q_y^4)$$
$$- \frac{2}{15}\pi^7(5q_x^4 + 2q_x^2q_y^2 + 5q_y^4) + \frac{1}{378}\pi^9(14q_x^4 + q_x^2q_y^2 + 14q_y^4)\Big).$$

We now proceed to study each pole, meaning we take the poles at the long wavelength limit in order to see how they behave. It is also important to note that in two dimensions the Coulomb potential behaves inversely to the transfer momentum, such that, $V(q) \sim \frac{1}{q}$. So we know that the we have four poles that show up and they are of the form,

$$\omega = \pm\frac{\sqrt{V(q)\chi_1(q^4) - \sqrt{V(q)(-4\chi_2(q^4) + \chi_1^2(q^4))}}}{\sqrt{2}}, \quad (4.29)$$

where for $q \to 0$ these poles behave similar to plasmons, such that $\omega \sim \sqrt{q}$.

4.4 Conclusion

In this chapter we have used the same methods that we had used in one dimension. The calculation of the pole structure of the propagator of the Hubbard Stratonovich field has given us plasmons for the long wavelength limit. This in turn is well expected for two-dimensional anisotropic square lattices where the fermions interact via a Coulomb potential.

Chapter 5

Conclusion and Future Outlook

High-T_C superconductors are an interesting topic to study but new tools are needed which give a physically intuitive approach to better understand them and point experimentalists in the right direction. In chapter 3, we tested to see if the Hubbard Stratonovich transformation is able to reproduce results of other methods in 1D. After getting critical exponents in the Green's function of slow moving excitation in a quantum wire, we generalized our approach to higher dimensions where we study the singularities of $\langle\phi\phi\rangle$ and find plasmonic dispersions. This creates incentive to study the propagation of an excitation in a two-dimensional four-fold symmetric system which is similar to the atomic structure of high-T_C superconductors.

Appendix A

Gaussian Integrals

A.1 Real Variables

The most common type of distribution found in probability theory is the Gaussian distribution.

$$\Phi(x) = \sqrt{\frac{\alpha}{\pi}} \exp\{-\alpha x^2\} \tag{A.1}$$

Integrating over all-space gives us.

$$\int_{\mathbb{R}} dx \exp\{-\alpha x^2\} = \sqrt{\frac{\pi}{\alpha}}$$

In order to determine averages we can use the following formula.

$$\langle x \rangle = \int_{\mathbb{R}} dx x \exp\{-\alpha(x-\kappa)^2\} = \kappa$$

For our cases it is convenient for us to write our parameter as $-\frac{G^{-1}}{2}$ with G<0

$$\int dx \exp\left\{\frac{1}{2} G^{-1} x^2\right\} = \sqrt{-2\pi G} = \sqrt{2\pi} \exp\left\{-\frac{1}{2}\log(-G^{-1})\right\} \tag{A.2}$$

Generating Function

I would like to define now what a generating function could be, taking a general Gaussian equation we add a linear term in x and multiply it by some

variable called J which will later be defined as the current associated with the variable itself such that we get:

$$\begin{aligned} Z(J) &= \int \frac{dx}{\sqrt{2\pi}} \exp\left\{\frac{1}{2}G^{-1}(x-\kappa)^2 + Jx\right\} \\ &= \int \frac{dx}{\sqrt{2\pi}} \exp\left\{\frac{1}{2}G^{-1}(x+GJ)^2 - \frac{1}{2}GJ^2 + J\kappa\right\} \\ &= \exp\left\{-\frac{1}{2}GJ^2 + J\kappa - \frac{1}{2}\log(-G^{-1})\right\} \end{aligned} \tag{A.3}$$

The good thing about this is that the expectation value of x can be easily calculated by the following procedure.

$$\langle x\rangle = \frac{1}{Z(J)}\frac{d}{dJ}Z(J)_{J=0} = \kappa \tag{A.4}$$

so it is straightforward to see that

$$\langle x^2\rangle = \frac{1}{Z(J)}\frac{d^2}{dJ^2}Z(J)_{J=0} = -G + \kappa^2 = -G + \langle x\rangle^2 \tag{A.5}$$

And since we can always apply the following shift to our variables $x \to x + \kappa$ so without loss of generality we set $\kappa = 0$. Some identities that I will add without proof are,

$$\langle x^{2m+1}\rangle = \int dx x^{2m+1} \exp\left\{\frac{1}{2}G^{-1}x^2\right\} = \frac{1}{Z(J)}\frac{d^{2m+1}}{dJ^{2m+1}}Z(J)_{J=0} = 0 \tag{A.6}$$

And for even exponents we get the following result

$$\langle x^{2m}\rangle = (2m-1)!!(-G)^m \tag{A.7}$$

Appendix B

Grassmann Variables

B.1 Algebra

In this section we will be discussing everything we need to know about Grassmann Variables and their importance with our thesis. A Grassmann algebra is defined by a set of Grassmann variables, these are seen to be as a generator algebera, because they span a complex *linear* space by making a linear combination of these variables with complex coefficients. The simplest Grassmann set can be $\{1, \psi\}$. The definition of the Grassmann variable ψ is that the anticommutator becomes.

$$\{\psi, \psi\} = \psi\psi + \psi\psi = 0 \tag{B.1}$$

It is best to think of the elements 1 and ψ as basis vector of a linear space. We can assume that in terms of matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \psi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{B.2}$$

It is important to note that $\phi^2 = 0$ as well as the fact that the most general function of ϕ can be written as

$$F(\psi) = f_1 + f_2\psi \tag{B.3}$$

This can also be generalised to the complex set such that $\{1, \psi, \psi^*, \psi\psi^*\}$, through this set we can see that the algebra satisfies the following the anticommutation relation.

$$\{\psi, \psi^*\} = \psi\psi^* + \psi^*\psi = 0 \tag{B.4}$$

We also remind ourselves of $\psi^2 = \psi^{*2} = 0$.

B.2 Differentiation

Let the function $A(\psi^*, \psi) = a_{11} + a_{12}\psi + a_{21}\psi^* + a_{22}\psi^*\psi$ differentiation with Grassmnan variables look like.

$$\frac{\partial}{\partial\psi}A(\psi, \psi^*) = a_{12} - a_{22}\psi^* \tag{B.5}$$

and

$$\frac{\partial}{\partial\psi^*}A(\psi, \psi^*) = a_{21} + a_{22}\psi \tag{B.6}$$

Furthermore we get,

$$-\frac{\partial^2}{\partial\psi\partial\psi^*}A(\psi, \psi^*) = \frac{\partial^2}{\partial\psi^*\partial\psi}A(\psi, \psi^*) = -a_{22} \tag{B.7}$$

Appendix C

Hubbard Stratonovich Transformation

In this section we would like to discuss the power of the HS transformation. We see that we can start with any action such as the one stateed in section []

$$S[\psi^*,\psi] = \int_0^L dx\psi^*(x)\bigg(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-\mu\bigg)\psi(x) + \int_0^L dxdx'\psi^*(x)\psi(x)U(x-x')\psi^*(x')\psi(x')$$

Here $V(x-x')$ is assumed to be the coulomb barrier which is often screened by the media surrounding in this case our one-dimensional system. We can proceed to define the Green function or our two-point function as the product of our fermionic operators averaged over time ordering.

$$G(x,\tau;x',\tau') = \langle\psi^*_\nu(x,\tau)\psi_\nu(x',\tau')\rangle \tag{C.1}$$

So the formal form of the green's function is in terms of a functional integral of coherent states which are mathematically represented as grassmann variables. Which as expressed in the previous section. We finally get the following equation,

$$G(x,x';\tau,\tau') = \frac{\int d[\psi^*]d[\psi]\psi^*(x,\tau)\psi(x',\tau')\exp\Big\{-\frac{S[\psi^*,\psi)]}{\hbar}\Big\}}{\int d[\psi^*]d[\psi]\exp\Big\{-\frac{S[\psi^*,\psi]}{\hbar}\Big\}} \tag{C.2}$$

We proceed to re-write these in terms of a different formalism where:

$$S[\psi^*,\psi] = (\psi|G_0^{-1}|\psi) + (|\psi|^2|V||\psi|^2) \tag{C.3}$$

Where J is the current associated with the fermionic field ψ and the last term of the action is the all-space integral over the quartic interaction term. In order to facilitate our calculations and render the path integral to a gaussian integral we can multiply the path integral with the identity such that:

$$\begin{aligned} 1 &= \int d[\kappa] \exp\Big[\frac{1}{2\hbar}(\kappa - V|\psi|^2|V^{-1}|\kappa - V|\psi|^2)\Big] \\ &= \int d[\kappa] \exp\Big[\frac{1}{2\hbar}(\kappa|V^{-1}|\kappa) - 2(\kappa||\psi|^2) + (|\psi|^2|V||\psi|^2)\Big] \end{aligned} \tag{C.4}$$

Where we can see that the integration measure contains the factor $\exp\{Tr[\log(-V^{-1}/\hbar)]/2\}$ which is then seen to be canceled with the result to of the Gaussian integral. We now proceed to calculate the path integral post multiplication of the identity.

$$\begin{aligned} Z[J,J^*] &= \int d[\psi^*]d[\psi] \int d[\kappa] \exp\left\{ -\frac{1}{\hbar}S[\psi^*,\psi] + (J|\psi) + (\psi|J)\right\} \\ &\qquad \times \exp\left\{\frac{1}{2\hbar}(\kappa - V\psi^*\psi|V^{-1}|\kappa - V\psi^*\psi)\right\} \\ &= \int d[\psi^*]d[\psi] \int d[\kappa] \exp\left\{\frac{1}{2\hbar}(\kappa|V^{-1}|\kappa) + (J|\psi) + (\psi|J)\right\} \times \exp\left\{(\psi|G_0^{-1} - \Sigma|\psi)\right\} \end{aligned} \tag{C.5}$$

Where we define the Hartree channel self energy to be in the form of $\hbar\Sigma(x,\tau;x'\tau'|\kappa) = \delta(\tau-\tau')\delta(x-x')\kappa(x,\tau)$. We can now discuss the resulting functional integral. We see that it has become quadratice in terms of the fermionic fields which implies that the integral itself is a functional gaussian integral. We now write the path integral:

$$Z[J,J^*] = \int d[\kappa] \exp\left\{\frac{1}{2\hbar}(\kappa|V^{-1}|\kappa) + Tr[\log(-G^{-1}) - (J|G|J)]\right\} \tag{C.6}$$

Where, from the Dyson equation we can see that:

$$\begin{aligned} G^{-1}(x,\tau;x',\tau') &= G_0^{-1}(x,\tau;x',\tau') - \Sigma(x,\tau;x',\tau'|\kappa) \\ &= \frac{1}{\hbar}\left\{ -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} - \mu + \kappa(x,\tau)\right\}\delta(x-x')\delta(\tau-\tau') \end{aligned} \tag{C.7}$$

Which when multiplied to the path integral we get a Gaussian integral with respect to the fermionic fields. This then gives us a path integral in terms of the κ field such that we now get. So now the effective action after integrating out the fermionic fields is:

$$S_{eff}[\kappa] = -\hbar \mathrm{Tr}\left[\log\left(\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} - \mu\right)\right] + \frac{1}{2\hbar}(\kappa|V^{-1}|\kappa) \tag{C.8}$$

Here we can see that the effective action is Gaussian in terms of the κ field. the action itself describes.

Appendix D

Powerful Theorems In Complex Analysis

Here we talk about the basics of complex analysis we use this section to better understand the Cauchy-Riemann equations as well as to better understand Grassmann variables and our fermionic path integrals. Assume we have a

function ϕ that maps, $\phi : \mathbb{C} \to \mathbb{C}$. The first theorem that we will be writing insures the differentiability of the function,

Theorem 1 *Let $\phi(u, v) = u(x, y) + iv(x, y)$, if ϕ is complex differentiable, then ϕ must obey the Cauchy-Riemann Equations, $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$.*

The next theorem will help us in solving integrals using the complex plane, meaning that it is crucial to know for the sake of solving for the singularities of the Coulomb propagator.

Theorem 2 *The Residue Theorem states: An analytic function $\phi(z)$ that has a Laurent series ,$\phi(z) = \sum_{n=\infty}^{\infty} a_n(z - z_0)^n$ can be integrated by contour C which encircles the singularity z_0, If the C encloses multiple poles then, $\int_C dz\phi(z) = 2\pi i \sum_n Res\phi(z_n)$.*

Bibliography

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer. "Theory of Superconductivity". In: *Phys. Rev.* 108 (5 Dec. 1957), pp. 1175–1204. DOI: 10.1103/PhysRev.108.1175. URL: https://link.aps.org/doi/10.1103/PhysRev.108.1175.

[2] J. G. Bednorz and K. A. Mller. "Possible highT c superconductivity in the Ba?La?Cu?O system". In: *Zeitschrift fr Physik B Condensed Matter* 64.2 (June 1986), pp. 189–193. DOI: 10.1007/bf01303701. URL: https://doi.org/10.1007/bf01303701.

[3] J. A. N. Bruin et al. "Similarity of Scattering Rates in Metals Showing T-Linear Resistivity". In: *Science* 339.6121 (2013), pp. 804–807. DOI: 10.1126/science.1227612. eprint: https://www.science.org/doi/pdf/10.1126/science.1227612. URL: https://www.science.org/doi/abs/10.1126/science.1227612.

[4] T. R. Chien, Z. Z. Wang, and N. P. Ong. "Effect of Zn impurities on the normal-state Hall angle in single-crystal $YBa_2Cu_{3-x}Zn_xO_{7-\delta}$". In: *Phys. Rev. Lett.* 67 (15 Oct. 1991), pp. 2088–2091. DOI: 10.1103/PhysRevLett.67.2088. URL: https://link.aps.org/doi/10.1103/PhysRevLett.67.2088.

[5] R. A. Cooper et al. "Anomalous Criticality in the Electrical Resistivity of La sub 2– ix/i /sub Sr sub ix/i /sub CuO sub4/sub". In: *Science* 323.5914 (Jan. 2009), pp. 603–607. DOI: 10.1126/science.1165015. URL: https://doi.org/10.1126/science.1165015.

[6] Tom Faulkner et al. "Holographic non-Fermi-liquid fixed points". In: *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 369.1941 (Apr. 2011), pp. 1640–1669. DOI: 10.1098/rsta.2010.0354. URL: https://doi.org/10.1098/rsta.2010.0354.

[7] Ulf Gran, Marcus Tornsö, and Tobias Zingg. "Plasmons in holographic graphene". In: *SciPost Phys.* 8 (2020), p. 093. DOI: 10.21468/SciPostPhys.8.6.093. URL: https://scipost.org/10.21468/SciPostPhys.8.6.093.

[8] Richard L. Greene et al. "The Strange Metal State of the Electron-Doped Cuprates". In: *Annual Review of Condensed Matter Physics* 11.1 (2020), pp. 213–229. DOI: 10.1146/annurev-conmatphys-031119-050558. eprint: https://doi.org/10.1146/annurev-conmatphys-031119-050558. URL: https://doi.org/10.1146/annurev-conmatphys-031119-050558.

[9] F. D. M. Haldane. "Effective Harmonic-Fluid Approach to Low-Energy Properties of One-Dimensional Quantum Fluids". In: *Phys. Rev. Lett.* 47 (25 Dec. 1981), pp. 1840–1843. DOI: 10.1103/PhysRevLett.47.1840. URL: https://link.aps.org/doi/10.1103/PhysRevLett.47.1840.

[10] F. D. M. Haldane. "Luttinger's Theorem and Bosonization of the Fermi Surface". In: (2005). DOI: 10.48550/ARXIV.COND-MAT/0505529. URL: https://arxiv.org/abs/cond-mat/0505529.

[11] N E Hussey, J Buhot, and S Licciardello. "A tale of two metals: contrasting criticalities in the pnictides and hole-doped cuprates*". In: *Reports on Progress in Physics* 81.5 (Mar. 2018), p. 052501. DOI: 10.1088/1361-6633/aaa97c. URL: https://dx.doi.org/10.1088/1361-6633/aaa97c.

[12] Adilet Imambekov, Thomas L. Schmidt, and Leonid I. Glazman. "One-dimensional quantum liquids: Beyond the Luttinger liquid paradigm". In: *Reviews of Modern Physics* 84.3 (Sept. 2012), pp. 1253–1306. DOI: 10.1103/revmodphys.84.1253. URL: https://doi.org/10.11032Frevmodphys.84.1253.

[13] Y. Kakehashi and P. Fulde. "Marginal Fermi Liquid Theory in the Hubbard Model". In: *Phys. Rev. Lett.* 94 (15 Apr. 2005), p. 156401. DOI: 10.1103/PhysRevLett.94.156401. URL: https://link.aps.org/doi/10.1103/PhysRevLett.94.156401.

[14] M. Khodas et al. "Fermi-Luttinger liquid: Spectral function of interacting one-dimensional fermions". In: *Physical Review B* 76.15 (Oct. 2007). DOI: 10.1103/physrevb.76.155402. URL: https://doi.org/10.11032Fphysrevb.76.155402.

[15] J. Kokalj, N. E. Hussey, and Ross H. McKenzie. "Transport properties of the metallic state of overdoped cuprate superconductors from an

anisotropic marginal Fermi liquid model". In: *Phys. Rev. B* 86 (4 July 2012), p. 045132. DOI: 10.1103/PhysRevB.86.045132. URL: https://link.aps.org/doi/10.1103/PhysRevB.86.045132.

[16] Patrick A. Lee, Naoto Nagaosa, and Xiao-Gang Wen. "Doping a Mott insulator: Physics of high-temperature superconductivity". In: *Rev. Mod. Phys.* 78 (1 Jan. 2006), pp. 17–85. DOI: 10.1103/RevModPhys.78.17. URL: https://link.aps.org/doi/10.1103/RevModPhys.78.17.

[17] J. M. Luttinger. "An Exactly Soluble Model of a Many-Fermion System". In: *Journal of Mathematical Physics* 4.9 (Sept. 1963), pp. 1154–1162. DOI: 10.1063/1.1704046. URL: https://doi.org/10.1063/1.1704046.

[18] Juan Maldacena. In: *International Journal of Theoretical Physics* 38.4 (1999), pp. 1113–1133. DOI: 10.1023/a:1026654312961. URL: https://doi.org/10.1023/a:1026654312961.

[19] P. Nozières. *Theory of Interacting Fermi Systems*. Frontiers in physics. W.A. Benjamin, 1964. URL: https://books.google.com.lb/books?id=JQlRAAAAMAAJ.

[20] Masao Ogata and Hidetoshi Fukuyama. "The t–J model for the oxide high-Tc superconductors". In: *Reports on Progress in Physics* 71.3 (Feb. 2008), p. 036501. DOI: 10.1088/0034-4885/71/3/036501. URL: https://dx.doi.org/10.1088/0034-4885/71/3/036501.

[21] T. J. Reber et al. "A unified form of low-energy nodal electronic interactions in hole-doped cuprate superconductors". In: *Nature Communications* 10.1 (Dec. 2019). DOI: 10.1038/s41467-019-13497-4. URL: https://doi.org/10.1038/s41467-019-13497-4.

[22] S. Smit et al. "Momentum-dependent scaling exponents of nodal self-energies measured in strange metal cuprates and modelled using semi-holography". In: (Dec. 2021). arXiv: 2112.06576 [cond-mat.str-el].

[23] S. Smit et al. *Momentum-dependent scaling exponents of nodal self-energies measured in strange metal cuprates and modelled using semi-holography*. 2021. DOI: 10.48550/ARXIV.2112.06576. URL: https://arxiv.org/abs/2112.06576.

[24] H.T.C. Stoof, D.B.M. Dickerscheid, and K. Gubbels. *Ultracold Quantum Fields*. Theoretical and Mathematical Physics. Springer Netherlands, 2008. ISBN: 9781402087639. URL: https://books.google.com.lb/books?id=-9c2Ns2J5P4C.

[25] S.-i. Tomonaga. "Remarks on Bloch's Method of Sound Waves applied to Many-Fermion Problems". In: *Progress of Theoretical Physics* 5.4

(July 1950), pp. 544–569. DOI: 10.1143/ptp/5.4.544. URL: https://doi.org/10.1143/ptp/5.4.544.

[26] J. Zaanen et al. "Dynamical stripe correlations in Cuprate superconductors". In: *Journal of Low Temperature Physics* 105.3-4 (Nov. 1996), pp. 569–579. DOI: 10.1007/bf00768446. URL: https://doi.org/10.1007/bf00768446.