Author:
Iris R. van Gemeren
Utrecht University

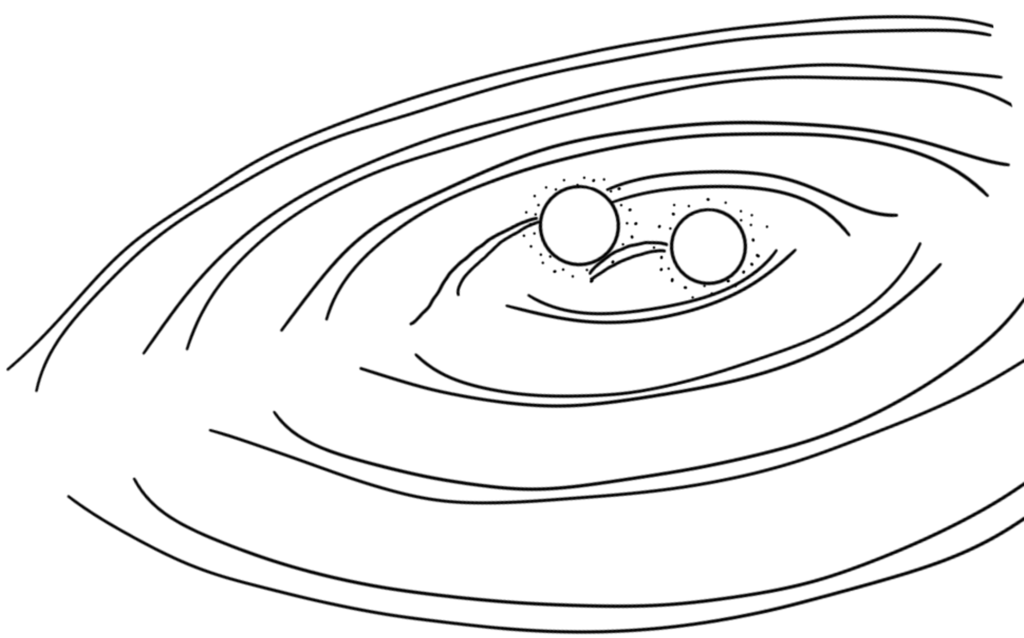
First supervisor:
Dr. Tanja P. Hinderer
Utrecht University

Second supervisor:
Dr. Tomislav Prokopec
Utrecht University

Black holes letting their hair down;

Incorporating tidal effects in the gravitational wave signature of scalarized black holes in quadratic gravity

Master Thesis
Institute for Theoretical Physics
10-07-2022



Abstract

As we enter the era of increasingly accurate gravitational wave observatories and even new detectors in the pipeline, the modelling of gravitational wave emission by black hole binary systems is of great importance. In general a gravity theory can be tested with the gravitational wave detections because the data analysis relies on cross correlating the detector output with the theoretical model. With the upcoming high-accuracy experiments, one expects to be able to distinguish possible corrections to General Relativity (GR) in the strong gravity regime, aiding in the search for an accurate gravity model including a quantum description. We study the gravitational wave emission from the inspiral of a black hole binary in quadratic modified gravity. This class of theories is a promising beyond GR model which can be seen as a higher curvature extension that makes the theory renormalizable. Specifically, we study a candidate theory within this class, namely scalar Gauss Bonnet (sGB) gravity. sGB introduces on top of the Hilbert Einstein action, a topological invariant quadratic curvature term and a free scalar field, leading to the possibility of having black hole solutions with non zero scalar hair. This results in additional scalar radiation from a black hole binary system. For modelling the gravitational wave signal in this theory the inclusion of the curvature corrections and the scalar radiation are required.

We re-derive the results from recent literature of the two body Lagrangian up to first order in the Post Newtonian (PN) expansion in sGB gravity and reconstruct the scalar waveform amplitude and phase evolution. Several typographical and algebraic errors from previous calculations are identified and resolved. For the first time we include tidal effects induced by the scalar field around the black holes in the modelling of the binary system and gravitational radiation. We find that the tidally induced corrections contribute at the same PN order and scale the same with distance and frequency as the sGB correction to the gravitational wave phase. Finally, we investigate the dependency of the sGB correction and tidally induced correction on the coupling constant and physical properties of the binary and find that the tidal effects dominate over the sGB corrections for large separations of the black holes.

Contents

1 Introduction

It starts to become a bit of a cliché to open a study on gravitational waves with the first gravitational wave detection in 2015 by the LIGO-VIRGO detectors[1], but things become a cliche for good reasons. This historical moment verified what Einsteins theory of General Relativity, our current theory for gravity, already predicted approximately hundred years before. Massive objects like black holes or neutron stars in an accelerating asymmetrical motion produce ripples in the fabric of spacetime, what was then named to be gravitational waves (GWs). Spectacular indeed, as well for astronomy as it opened a whole new way to gather information about objects in the universe[2, 3].
Now around seven years after the first detection, many events of gravitational waves coming from black hole and neutron star binaries have been measured[4, 5, 6]. There are even new detectors in the pipeline, for example the more accurate Einstein telescope [7] or the space based telescope LISA [8] probing a different frequency range. Rapid progress is also made on the theory side. One of the many scientific opportunities with GWs is testing the accuracy of models for gravity. As the measurement of GWs was a real breakthrough for Einsteins theory of General Relativity, the theory is not flawless. On many scales this theory works very well, for example on the scales of the solar system, GR passes the tests with high accuracy [9]. However in more extreme situations (high energy) as for example the gravity inside a black hole, or on very small scales when quantum effects come into play, GR cannot describe gravity accurately [10]. Already quite soon after Einstein had proposed his theory, the search for alternatives or alternations began, resulting in a zoo of possible modified versions of a theory of gravity[11]. Ideally an alternative theory would preserve the successes of GR on scales such as the solar system but would have some improvements on the small and strong gravity scales so it can be combined to include quantum effects.

In this thesis we dive into this world of modified gravity to see how one could use GWs to test these theories in the strong gravity regime. For this we focus on a class of modified gravity theories called quadratic gravity which are extensions of GR with quadratic terms in the spacetime curvature [12]. This extension is quite natural as it would be an order higher in the expansion of the curvature. Even more appealing is that these quadratic terms make it possible to quantize the theory which allows for including the description of quantum effects [13]. This is related to the motivation for the inclusion of higher order curvature terms from the low energy limit of string theory[14]. Within this subclass of quadratic gravity theories we will focus on scalar Gauss Bonnet (sGB) gravity. This theory includes a quadratic curvature correction to the GR framework and this curvature term coupled to a new introduced scalar field. A distinct feature in this theory is the possibility for black hole solutions to have non zero scalar field, resulting in black holes with scalar hair[15]. The main focus of this thesis work lies on producing accurate templates of the phase of the GWs coming from a coalescing black hole binary, including the interesting effect of the scalar hair solution of the black holes. These templates can be used to compare with the theoretical wave form phase in GR to see if there are differences in an event in strong gravity. If these differences are there, which is already shown in [16, 17]. In future research these accurate templates can be compared with data from gravitational waves to constrain the modified theory and to see if it would be a more accurate description of gravity on those scales.

Most previous work on constraining gravity theories focuses on tests of GR and constraining parameters [9, 18, 19, 20, 21, 22], for example leaving the coefficients describing corrections to Newtonian gravity on the waveform general and phenomenologically constrain their value with the data. Then one can compare these constraints on the waveform with the corrections that would come from GR. This is called a Parametrized Post Newtonian (PPN) method. In this way one can search for alterations to GR on the waveform but interpretation for the underlying gravitational theory is limited. In contrast constructing whole templates gives much more constraining power then most previously conducted tests. However this is a more time consuming task and therefore it is more interesting to build the framework that covers a broader class of modified gravity theories, which in this case is done for the class of quadratic gravity theories.

In this thesis we focus on a system of two non spinning black holes rotating around each other which would produce GWs in the detectable range [1]. The GWs coming from a merging event of two black holes have different stages. At the beginning of the event the black holes are rotating around each other; this is called the inspiral. When the black holes are so close together they are within the radius of the Innermost Stable Circular Orbit (ISCO)[2] they actually collide and we enter the merger stage. After the merger there is the ringdown as shown Fig. 1. The inspiral can be modelled well analytically, while for the merger one needs for example numerical tools. Therefore we focus in this thesis on the inspiral stage.

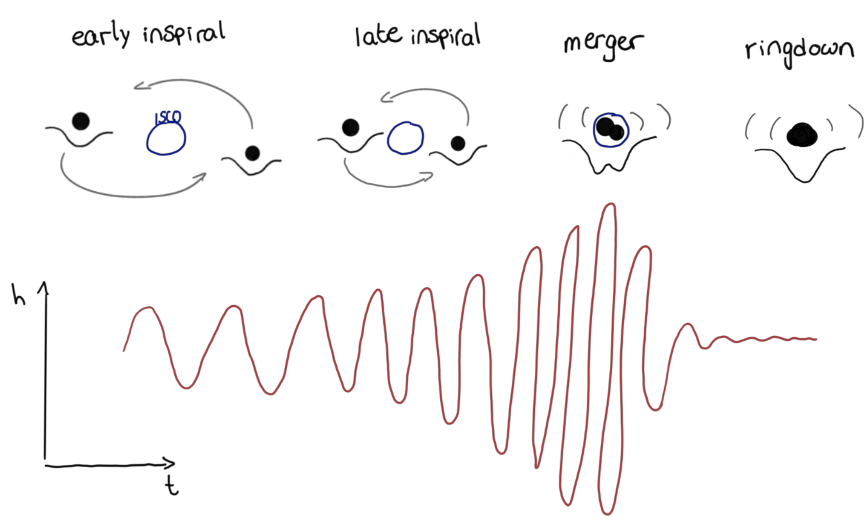


Figure 1: Sketch of the coalescence event of two black holes in a binary system and its gravitational wave signal in time.

[1]This is approximately $10 - 10^4$Hz for the ground based detectors and about $0.1 \times 10^{-3} - 1$Hz for future space based detector LISA[23]

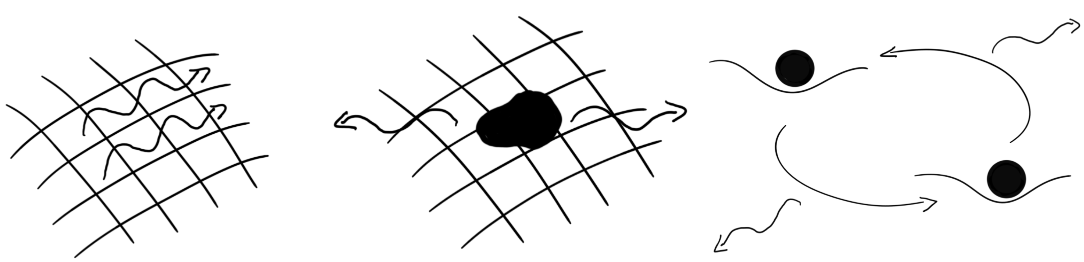
[2]the smallest distance to a black hole at which a stable circular orbit exist, this is in general taken as a proxy for the end of the inspiral and start of the merger stage.

Recently the dynamics of a binary black hole system and the waveform templates in sGB were studied in [24, 16, 17]. Solving the set of coupled non linear equations of motion for the spacetime and scalar field in this theory is very complicated and to analytically solve them requires establishing an approximation scheme. A widely used method for these kind of calculations is the Post Newtonian (PN) approximation where one expands in corrections to the Newtonain gravity limit. This approximation can be used in a region near the GWs source for example to calculate binary dynamics of the system, at larger distances different methods are required. In the references above the authors constructed the Lagrangian of the binary dynamics and the waveforms in sGB for the first time to 1st order in this PN expansion. In this thesis we will re-derive and check their results. On top of that we will include the description of tidal effects coming from the non zero scalar field around the black holes that can exists in vacuum solutions for sGB gravity as opposed to GR. These tidal effects influence the binary dynamics of the system and induce an additional dipole moment to the multipole expansion of the scalar field itself which contributes to the radiation of the scalar field. To include these effects in the calculation of the orbital dynamics and radiation the tidal deformability parameter needs to be obtained. This parameter characterises the sensitivity of the scalar field to tidally deform in the presence of the scalar field of the companion black hole. It is defined as the ratio between the induced scalar dipole moment and the scalar tidal field of the companion black hole. By zooming in on the scalar field solution of one black hole to first order in the perturbation of the scalar tidal field, we can derive the expression for the induced dipole moment and tidal field by looking at the asymptotic multipole expansion of the scalar field solution, resulting in the tidal deformability parameter.

This thesis is organised as follows; we start with technical background chapters on GWs in section 2, modified gravity in section 3 and tidal effects in section 4. Then we continue with the main part of the report starting with binary dynamics analysis in section 5 which includes a study of the binding energy and the effects of the tidal terms in sGB gravity. After that we derive the tidal deformability parameter in section 6. Finally we calculate the phase evolution in section 7 and analyse the dependency of the tidally induced phase corrections compared to the sGB terms and GR phasing and their dependence on the coupling and properties of the binary system. We find that these corrections have opposite dependencies on the coupling and mass ratios of the black holes in stages of the very early inspiral in which the scalar radiation contributions dominate and later in the inspiral. Furthermore for large separations during the early inspiral the tidal corrections to the GW phase actually dominate over the sGB corrections. Lastly we finish with a discussion and conclusion in section 8 and outlook for future research in section 9. The more lengthy and technical calculations can be found in the appendix starting in section A, as well as an overview of the results from GR, we use in this analysis, a description of the Mathematica package we used for the calculations and a discussion on similar analysis for another quadratic gravity theory called dynamical Chern Simons gravity.

2 Gravitational waves

We start with a description on how one can derive the most important features of GWs starting from the equations of motion coming from GR, to give an intuitive picture of the concept of GWs. A short note on detection is given as well. Then we will discuss how one can calculate the wave function of the GWs. An important point to note is that in principle the calculations done on GWs always start from the equations of motion in GR which one solves for a perturbation in the spacetime fabric. However for solving these equations analytically one has to implement approximations as these equations are non linear. Therefore these calculations can be done on very different levels of complexity depending on which approximations can be made, which depends again on the system that generates the waves and/or at what distance from the source you are solving the equations. Therefore we specify explicitly for the calculations which assumptions and approximations we make.
We will start looking at the general features of gravitational waves keeping it unspecified what kind of source would have generated these waves and assuming that the background spacetime is flat, see Fig. 2a, 2b. Then we will shift to the more astrophysical accurate system of two black holes rotating around each other, as shown in Fig. 2c for which we have to reconsider the approximation scheme.
This whole discussion is taking place in a GR context, no modified gravity considered for now. The first part is to give some intuition on gravitational waves. In the last subsection 2.4 we will look at gravitational waves in the strong gravity regime which explains the tools we will also need for calculations of the binary dynamics and gravitational wave phase in the core parts of this report in sections 5, 7.
The information in this section is largely based on Ch 1, 3 and 5 of the book by Maggiore[25].



(a) (b) (c)

Figure 2: Different systems to consider for calculating the gravitational waveforms. Considering a) vacuum solutions with a flat spacetime background, b) solutions for a generic source with a flat spacetime background and c) a gravitationally bound system of two black holes.

2.1 Gravitational waves in linearized theory

As GWs are the ripples in our spacetime, we can start our discussion with looking at the action describing this spacetime. In GR this action is given by

$$S = S_{HE} + S_M = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}R + S_M, \tag{2.1}$$

with S_{HE} the Hilbert Einstein action and S_M the action of the applicable matter distribution. In GR we have the metric $g_{\mu\nu}$ and its determinant g describing the properties of the spacetime and the Riemann tensor $R_{\mu\nu\alpha\beta}$, Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R describing the geometric properties as curvature of the spacetime [3]. Varying the action with respect to the metric $g_{\mu\nu}$ results in the Einstein field equations which are basically the equations of motion of spacetime

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{2.2}$$

Here the energy momentum tensor is defined as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}}. \tag{2.3}$$

These equations describe how the geometry of the spacetime defined by the terms on the LHS are related to the distribution of matter described by the energy momentum tensor.
To see explicitly how linearized GWs arise from the solutions to the Einstein field equations we express the equations in terms of perturbations in the spacetime metric. With this approach we split up the contribution of the rapidly varying part of the metric and the underlying flat spacetime in respectively the wave part and background, see Fig. 2a. Here the perturbations in the metric are regarded as the GWs. Furthermore we can simplify the equations a lot by making use of the symmetries we have in GR. We can define the perturbations to the spacetime metric $h_{\mu\nu}$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{2.4}$$

Furthermore we assume that these perturbations are small $|h_{\mu\nu}| \ll 1$ and that the background metric is flat which is the weak field approximation. If we substitute this metric in the field equations and expand to linear order in $h_{\mu\nu}$, we describe GWs in the so called *linearized theory*. In this framework indices can be raised or lowered with the flat space Minkowski metric $\eta_{\mu\nu}$.
Thus when expanding the Ricci tensor and scalar up to linear order in the metric perturbations[4], substituting in the Einstein equations and making use of the following definition

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{2.5}$$

with the trace $h = \eta^{\mu\nu}h_{\mu\nu}$, this results in the linearized Einstein equations

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \partial^\rho\partial_\mu\bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu}. \tag{2.6}$$

We can do one more simplification using the symmetries of GR. As the Einstein equations are in a covariant, tensorial form, the equations are invariant under all coordinate transformations, as tensors are coordinate independent objects. The system is therefore invariant

[3] see D for the definitions of these expressions defined within the GR framework
[4] see Appendix D for these expanded terms

under translations and Lorentz transformations. In addition, the theory is invariant under active coordinate transformations; transformations for which the transformed coordinate is invertible, differentiable with a differentiable inverse. This transformation can be performed locally and is a local symmetry property. This local symmetry is called the gauge freedom in GR. If one chooses a certain reference frame to express the coordinates in, this gauge freedom gets fixed. However after choosing a set of coordinates there is still a small invariance left; the invariance under infinitesimal local transformations. This freedom gets fixed by the following gauge constraint[5]

$$\partial^\nu \bar{h}_{\mu\nu} = 0, \tag{2.7}$$

which is called the *Lorentz gauge*. Using this constraint the linearized field equations reduce to the wave equation

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}.} \tag{2.8}$$

The calculation of the waveforms of gravitational waves and the effects they have in linear theory comes down to solving this wave equation.

2.1.1 Propagation of gravitational waves in vacuum

As a first study we can look at how the waves propagate on a flat spacetime background, see Fig. 2a. We can then set the source term $T_{\mu\nu}$ in the wave equation to zero as we assume empty space. The wave equation then becomes

$$\Box \bar{h}_{\mu\nu} = 0. \tag{2.9}$$

From this we can already see that since $\Box = -\left(1/c^2\right)\partial_0^2 + \nabla^2$ the propagation speed of gravitational waves is the speed of light c.

Outside the source we can make even more use of the gauge freedom as the Lorentz gauge still does not fix the gauge entirely. This comes from the transformation of $\partial^\nu \bar{h}_{\mu\nu}$, the Lorentz gauge condition is not violated when also applying another infinitesimal coordinate

[5] Deriving this gauge condition works as follows. In general the Einstein equations are invariant under the active coordinate transformation $x^\mu \to x'^\mu(x)$ also called a diffeomorphism. Under this transformation the metric transforms as $g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$. Choosing a set of coordinates in principle fixes this gauge freedom but one can check that the equations are then still invariant under infinitesimal transformations $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$ with $|\partial_\mu \xi_\nu|$ being of similar order as the metric perturbations . Under this transformation the perturbations transform to lowest order as $h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$ using the transformation rule of the metric $g_{\mu\nu}$. Then with $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$ this transformation for $\bar{h}_{\mu\nu}$ is given by $\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho)$. Then taking the partial derivative, the second and third term cancel and one has $\partial^\nu \bar{h}_{\mu\nu} \to \left(\partial^\nu \bar{h}_{\mu\nu}\right)' = \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu$. Now because of the freedom to choose ξ we can always make sure $\left(\partial^\nu \bar{h}_{\mu\nu}\right)' = 0$ by letting $\Box \xi_\mu = \partial^\nu \bar{h}_{\mu\nu}$. This always allows for a solution for ξ as the d'Alembertian is invertible using a Greens function.

transformation.[6] This freedom together with the Lorentz gauge can be used to write the metric in the *transverse-traceless (TT) gauge*:

$$h^{0\mu} = 0, \quad h^i_i = 0, \quad \partial^j h_{ij} = 0 \tag{2.10}$$

We will from now on use the superscript TT to denote the metric written in the TT gauge. To see the consequences of this gauge it is instructive to look at possible solutions of Eq. (2.9). A set of solutions to the homogeneous wave equation are plane waves

$$\bar{h}^{TT}_{\mu\nu} = A_{\mu\nu}(\mathbf{k})e^{ik^\alpha x_\alpha}, \tag{2.11}$$

with $k^\mu = (w/c, \mathbf{k})$, $\mathbf{k}$ the wave vector, $A_{\mu\nu}$ the polarization tensor and $w/c = |\mathbf{k}|$ with w the angular frequency of the wave. At the end of the calculation we take the spatial part of the expression to obtain the physical modes.
From the first TT gauge condition we also have that all temporal components are 0. From the traceless condition follows that $\bar{h}_{\mu\nu} = h_{\mu\nu}$, hence

$$h^{TT}_{ij} = A_{ij}(\mathbf{k})e^{ik^\alpha x_\alpha}. \tag{2.12}$$

From the transverse gauge condition we derive

$$\partial^j h_{ij} = ik^j A_{ij} e^{ik^\alpha x_\alpha} = k^j A_{ij} = 0. \tag{2.13}$$

If we now choose the propagation direction along the z axis, then the only non zero component of the wavevector is k^3 and from Eq. (2.13) we have $A_{i3} = 0$. As the solution should be traceless we also have that $A_{11} = -A_{22}$. As the metric tensor and thus the perturbations are symmetric in its indices it follows that $A_{12} = -A_{21}$. Then taking the real part we obtain the solution

$$h^{\mathrm{TT}}_{ij}(t,z) = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos[\omega(t - z/c)]. \tag{2.14}$$

Or by convention the components of the polarization tensor are called h_+ for A_{11}, plus polarization and $h_\times$ for A_{12}

$$h^{\mathrm{TT}}_{ij}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos[\omega(t - z/c)]. \tag{2.15}$$

So we see that plane waves in vacuum have only two degrees of freedom, the TT and Lorentz gauge reduce the degrees of freedom from 10 to 2 [7].

[6] On top of the transformation of $\partial^\nu \bar{h}_{\mu\nu}$ in the previous footnote we can do another infinitesimal transformation. If we then set $\Box\xi_\mu = 0$ nothing changes for the condition $\partial^\nu \bar{h}_{\mu\nu} = 0$. If $\Box\xi_\mu = 0$ then also $\Box\xi_{\mu\nu} = \Box(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho) = 0$ as the flat space d'Alembertian commutes with the partial derivatives. Then from the transformation in footnote 3: $\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho)$ which in vacuum satisfies $\Box\bar{h}_{\mu\nu} = 0$, it holds that we can choose four independent functions ξ_μ which satisfy $\Box\xi_{\mu\nu} = 0$ to subtract from the 6 components of $\bar{h}_{\mu\nu}$. Then ξ^0 can be chosen so the trace of $\bar{h}_{\mu\nu}$ vanishes making $\bar{h}_{\mu\nu} = h_{\mu\nu}$ and the other ξ^i to set $h^{0i} = 0$. Then together with the Lorentz gauge one derives the TT gauge.

[7] The Einstein equations consists of 4x4=16 equations but as the equation is symmetric under exchange of indices this reduces to 10

In general a symmetric tensor, and thus a general plane wave solution $h_{\mu\nu}$ can always be put in the TT gauge using

$$h_{ij}^{TT} = \Lambda_{ij,kl} h_{kl}, \tag{2.16}$$

where $\Lambda_{ij,kl}$ is the TT projection operator. To construct it explicitly, we first define the projector

$$P_{ij}(\mathbf{n}) = \delta_{ij} - n_i n_j,$$

being a symmetric and transverse tensor with trace 2. Here $\mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|}$ is a unit vector along the propagation direction of the GWs. With this projector we can construct the TT projector operator

$$\Lambda_{ij,kl}(\mathbf{n}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}, \tag{2.17}$$

which is transverse on all indices, traceless with respect to i,j and k,l and symmetric under the exchange of ij with kl. In full this projector is given by

$$\begin{aligned}\Lambda_{ij,kl}(\mathbf{n}) =& \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} \\ &+ \frac{1}{2} n_k n_l \delta_{ij} + \frac{1}{2} n_i n_j \delta_{kl} + \frac{1}{2} n_i n_j n_k n_l.\end{aligned} \tag{2.18}$$

If we would now like to look at the physical effects of a passing gravitational wave, we can not look at the movement of one single test particle as that would only tell us about the coordinate values that may change. As in GR we have the freedom to change coordinates we can always find coordinates in which the particle appears not to move, in fact it turns out that these are the transverse traceless coordinates. We can however look at the relative motion of particles. Relative motion in GR is described by the geodesic deviation equation which describes how the spacetime trajectories, geodesics, of two particles influence each other

$$\frac{D^2\xi}{d\tau^2} = -R^{\mu}_{\nu\rho\sigma} \xi^{\sigma} \frac{dx^{\nu}}{d\tau} \frac{dx^{\rho}}{d\tau}, \tag{2.19}$$

with the covariant derivative given by

$$\frac{D^2\xi}{d\tau^2} = \frac{d\xi}{d\tau} + \Gamma^{\mu}_{\nu\rho} \xi^{\nu} \frac{dx^{\rho}}{d\tau}. \tag{2.20}$$

If we assume slowly moving particles then we can express the four velocity $\frac{dx^{\nu}}{d\tau}$ as a unit vector in time plus corrections first order in $h_{\mu\nu}^{TT}$ but as $R^{\mu}_{\nu\rho\sigma}$ is already first order we can neglect the corrections to the four velocity

$$\frac{dx^{\rho}}{d\tau} = (1, 0, 0, 0). \tag{2.21}$$

Then only $R_{\mu 00\sigma}$ is relevant and is given[8] in the TT gauge

$$R_{\mu 00\sigma} = \frac{1}{2} \partial_0 \partial_0 h_{\mu 0}^{TT}. \tag{2.22}$$

[8]for the Riemann tensor in terms of the metric perturbations see the GR recap in Appendix D

As for slowly moving particles to lowest order the proper time τ approximates the time coordinate t we can write the geodesic deviation in lowest order as

$$\frac{\partial^2}{dt^2}\xi^\mu = \frac{1}{2}\xi^\sigma \frac{\partial^2}{\partial t^2} h_\sigma^{TT\mu}. \tag{2.23}$$

If we look at the indices of this equation, we can see that for a wave travelling in the z direction, only the ξ^1 and ξ^2 components will change, the third component is 0. Thus the relative distance between the test particles is only changed by the wave in the directions perpendicular to the propagation direction, similar to waves in electromagnetism.
Using Eq. (2.15), considering the polarization directions separately, so first set $h_x = 0$

$$\frac{\partial^2}{dt^2}\xi^1 = \frac{1}{2}\xi^1 \frac{\partial^2}{\partial t^2} h_+ \cos[\omega(t - z/c)], \tag{2.24}$$

$$\frac{\partial^2}{dt^2}\xi^2 = -\frac{1}{2}\xi^2 \frac{\partial^2}{\partial t^2} h_+ \cos[\omega(t - z/c)], \tag{2.25}$$

which to lowest order is given by

$$\xi^1 = \xi^1(0) + \xi^1(0)\frac{1}{2}h_+ cos[\omega(t - z/c)], \tag{2.26}$$

$$\xi^2 = \xi^2(0) - \xi^2(0)\frac{1}{2}h_+ cos[\omega(t - z/c)]. \tag{2.27}$$

Thus particles with a separation along the x axis oscillate in the same x direction, same for those in the y direction. If we consider particles intially at rest in a circle in the x,y plane, when a wave passes they will oscillate back and forth forming a plus shape. See Fig. 3. In the case were $h_\times = 0$ we have

$$\xi^1 = \xi^1(0) + \xi^2(0)\frac{1}{2}h_\times cos[\omega(t - z/c)], \tag{2.28}$$

$$\xi^2 = \xi^2(0) - \xi^1(0)\frac{1}{2}h_\times cos[\omega(t - z/c)]. \tag{2.29}$$

Then the circle of particles is moving back and forth in a x shaped manner see Fig. 3, hence the notation of h_+ and $h_\times$.

So essentially as a GW passes, it stretches and squeezes spacetime itself. In the TT gauge the coordinates change similarly to the stretching and squeezing therefore in these coordinates the particles remain stationary. However the proper distance between the particles does change.

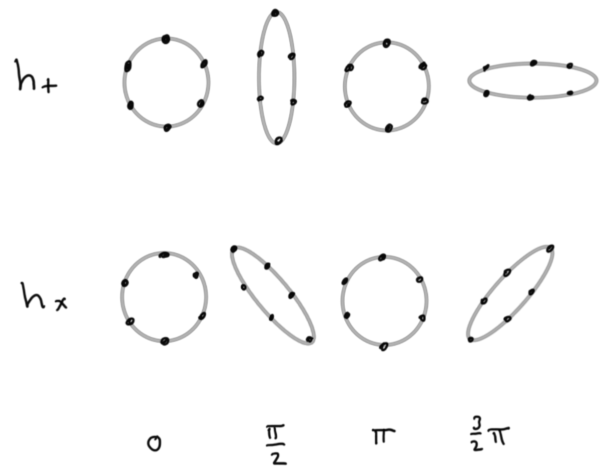


Figure 3: Circle of particles moving back and forth in plus and cross shaped manner when a gravitational wave passes.

2.2 Generation of gravitational waves in linearized theory

In the previous section we solved the wave equation assuming a flat background spacetime and vacuum which is a valid assumption at a position in space far away from any source. However if we want to study the generation of gravitational waves we have to introduce a source, hence a non zero energy momentum tensor. We will still keep it general what kind of source this would be, see Fig. 2b. In this case we will see that in linearized theory we can expand the formulas for gravitational wave production in terms of the velocity of the source divided by the speed of light $\frac{v}{c}$. For sources whose system is determined by non gravitational forces, this expansion is valid and can be separated from the weak field expansion. However in the case of gravitationally bound systems the velocity and gravitational field expansion are coupled and we have to use a different approximation scheme. We will see more about this in the next section 2.4.

We start again with now the sources wave equation

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}. \tag{2.30}$$

As the RHS does not depend on $\bar{h}_{\mu\nu}$ this equation is linear and can be solved with a Greens function. The appropriate solution is the retarded Greens function Eq. (C.10a) as it depends on the past lightcone, respecting causal relations. Then the solution becomes

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{\mu\nu} \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}' \right). \tag{2.31}$$

The solution depends on the retarded time which describes that the gravitational wave at a certain time t was sourced at time $t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$ with $\frac{|\mathbf{x}-\mathbf{x}'|}{c}$ the distance from point x to the source point x' divided over the speed of light, so the time it takes for the wave to travel to point x.

For a point outside the source we can write this too in the TT gauge using the TT projector from Eq. (2.17)

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{4G}{c^4}\Lambda_{ij,kl}(\mathbf{n})\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}T_{kl}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right). \tag{2.32}$$

We will specialize to $\mathbf{n} = \hat{\mathbf{x}}$ and $|\mathbf{x}| = r$. We are interested in the GWs far away from the source, for example at a detector on earth. Then if d is the radius of the source, the distance to the detector is much larger than the radius of the source $r \gg d$, see Fig. 4. We can Taylor expand $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ around small $|\mathbf{x}'|$

$$|\mathbf{x}-\mathbf{x}'| = r - \mathbf{x}'\cdot\mathbf{n} + \dots . \tag{2.33}$$

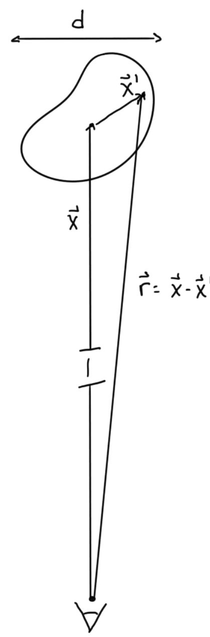


Figure 4: location source relative to the observer

Then we can expand to $O(1/r)$ and take the fraction out of the integral

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\mathbf{n})\int d^3x' T'_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\mathbf{n}}{c}, \mathbf{x}'\right). \tag{2.34}$$

We assume non relativistic sources; the velocities inside the source are small compared to c and hence we can expand in terms of $\frac{v}{c} \ll 1$ as a multipole expansion.
To see how we can expand the energy momentum tensor term it is convenient to look for a moment at its Fourier transform

$$T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\mathbf{n}}{c}, \mathbf{x}'\right) = \int \frac{d^4k}{(2\pi)^4}\tilde{T}_{kl}(\omega,\mathbf{k})e^{-i\omega(t-r/c+\mathbf{x}'\cdot\mathbf{n}/c)+i\mathbf{k}\cdot\mathbf{x}'}. \tag{2.35}$$

Expanding the exponential again for small $|\mathbf{x}'|$

$$
\begin{aligned}
e^{-i\omega(t-r/c+\mathbf{x}'\cdot\mathbf{n}/c)} =& e^{-i\omega(t-r/c)} \\
& \times \left[1 - i\frac{\omega}{c}x'^i n^i + \frac{1}{2}\left(-i\frac{\omega}{c}\right)^2 x'^i x'^j n^i n^j + \dots\right] .
\end{aligned}
\tag{2.36}
$$

This is the same as writing the energy momentum tensor in the expansion

$$
\begin{aligned}
T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}'\cdot\mathbf{n}}{c}, \mathbf{x}'\right) \simeq& T_{kl}\left(t - \frac{r}{c}, \mathbf{x}'\right) \\
& + \frac{x'^i n^i}{c}\partial_t T_{kl} + \frac{1}{2c^2}x'^i x'^j n^i n^j \partial_t^2 T_{kl} + \dots ,
\end{aligned}
\tag{2.37}
$$

with the derivatives evaluated at $(t - r/c, x')$.

We define the following moments

$$
\begin{aligned}
S^{ij}(t) &= \int d^3x T^{ij}(t, \mathbf{x}), \\
S^{ij,k}(t) &= \int d^3x T^{ij}(t, \mathbf{x}) x^k, \\
S^{ij,kl}(t) &= \int d^3x T^{ij}(t, \mathbf{x}) x^k x^l,
\end{aligned}
\tag{2.38}
$$

which for the higher orders contain higher products of x. The expression for h_{ij}^{TT} becomes

$$
\begin{aligned}
h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) =& \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\mathbf{n}) \\
& \times \left[S^{kl} + \frac{1}{c}n_m \dot{S}^{kl,m} + \frac{1}{2c^2}n_m n_p \ddot{S}^{kl,mp} + \dots\right]_{\text{ret}} .
\end{aligned}
\tag{2.39}
$$

The notation of the indices of the moments S are such that the first indices represent the indices of the energy momentum tensor and after the comma the indices of the powers of $\mathbf{x}$. This means that the moments are symmetric in the indices before or after the comma but not by interchanging the two sets. The subscript "ret" denotes evaluation of the derivatives in retarded time. These are the first moments of the full multipole expansion of h_{ij}^{TT}. As the derivatives to x^i in Eq. (2.39) give factors of v and the multipoles are multiplied with factors of $\frac{1}{c}$ this expansion is done in $\frac{v}{c} \ll 1$.

To get a more physical intuition of this expression it is more insightful to re-express the moments. Therefore we define the following mass multipole moments of the energy density T^{00}

$$
\begin{aligned}
m &= \frac{1}{c^2}\int d^3x T^{00}(t,\mathbf{x}), \\
m^i &= \frac{1}{c^2}\int d^3x T^{00}(t,\mathbf{x})x^i, \\
m^{ij} &= \frac{1}{c^2}\int d^3x T^{00}(t,\mathbf{x})x^i x^j,
\end{aligned} \tag{2.40}
$$

and so forth for higher order of x^i. Similar for the following momentum multipole moments of the momentum density T^{0i}

$$
\begin{aligned}
P^i &= \frac{1}{c}\int d^3x T^{0i}(t,\mathbf{x}), \\
P^{i,j} &= \frac{1}{c}\int d^3x T^{0i}(t,\mathbf{x})x^j, \\
P^{i,jk} &= \frac{1}{c}\int d^3x T^{0i}(t,\mathbf{x})x^j x^k \,.
\end{aligned} \tag{2.41}
$$

In linearized gravity the Lorentz gauge Eq. (2.7) together with the wave equation Eq. (2.30) give the following simplification of the conservation of the energy momentum tensor

$$
\partial^\nu T_{\mu\nu} = 0. \tag{2.42}
$$

Using its zero component we can derive

$$
\partial_0 T^{00} = -\partial_i T^{0i}. \tag{2.43}
$$

With this we can rewrite the derivative of the mass monopole as

$$
c\dot{m} = \int_V d^3x \partial_0 T^{00} = -\int_V d^3x \partial_i T^{0i} = -\int_{\partial V} dS^i T^{0i} = 0, \tag{2.44}
$$

with V some volume larger than the source and the energy momentum tensor vanishing on its boundary as the tensor is zero outside the source. This vanishing $\dot{m}$ denotes that in linearized theory there is conservation of mass. However in a physical system GWs would radiate away mass/energy, but in linearized gravity one neglects the back action of the GWs on the source.
Similarly we can write

$$
c\dot{m}^i = \int_V d^3x x^i \partial_0 T^{00} = -\int_V d^3x x^i \partial_j T^{0j} = \int_V d^3x \left(\partial_j x^i\right) T^{0j} = \int_V d^3x \delta^i_j T^{0j} = cP^i. \tag{2.45}
$$

Continuing one can derive the following results

$$
\begin{aligned}
\dot{m}^{ij} &= P^{i,j} + P^{j,i}, \quad \dot{P}^i = 0, \\
\dot{P}^{i,j} &= S^{ij},
\end{aligned} \tag{2.46}
$$

with $\dot{P}^i = 0$ the conservation of momentum.
Using these identities we can re-express the lowest order moment of S. Using the expressions for $\dot{m}^{ij}$ and $\dot{P}^{i,j}$ gives

$$
S^{ij} = \frac{1}{2}\ddot{m}^{ij}. \tag{2.47}
$$

Substituting back in the multipole expansion for h_{ij}^{TT}, to lowest order we have

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\mathbf{n})\ddot{m}^{kl}(t-r/c). \tag{2.48}$$

We can re-write this formula in a more commonly used form. First we can write m^{kl} in terms of its irreducible representations in the following way, decomposing in its trace and traceless parts

$$m^{kl} = \left(m^{kl} - \frac{1}{3}\delta^{kl}m_{ii}\right) + \frac{1}{3}\delta^{kl}m_{ii}. \tag{2.49}$$

The last term can be neglected as the TT projector in h_{ij}^{TT} selects only the traceless part. Now we define the (reduced)[9] quadrupole moment

$$Q^{ij} = m^{ij} - \frac{1}{3}\delta^{ij}m_{kk} = \int d^3x\rho(t,\mathbf{x})\left(x^ix^j - \frac{1}{3}r^2\delta^{ij}\right), \tag{2.50}$$

using $\rho = \frac{1}{c^2}T^{00}$ which to lowest order is the mass density. This then gives for the waveform:

$$\boxed{\begin{aligned} h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) &= \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\mathbf{n})\ddot{Q}_{kl}(t-r) \\ &\equiv \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{\mathrm{TT}}(t-r/c). \end{aligned}} \tag{2.51}$$

Which is well known as the *quadrupole formula* describing the generation of gravitational waves till lowest order, first derived by Einstein [26, 27]. This formula tells us what sources we can expect to generate GWs [28]. The sources do need to have a time varying quadrupole moment for the above formula to be non trivial. This means that we need a change in the distribution of matter around its centre of mass. So for example a spinning object such as a black hole is a symmetric system and even though it is spinning it does not change the distribution. But a black hole or neutron star binary does and would therefore radiate. In general objects accelerating in a non-spherical manner could generate GWs. One even expects that similar to the CMB (Cosmic Microwave Background) in the early universe quantum fluctuations have generated a stochastic GW background [29]. However as we would like to detect the waves, requiring a large enough amplitude, large and varying quadrupole momenta are required which brings us back to the black hole and neutron star binaries.

This formula also highlights the difference in behaviour of GWs versus electromagnetic waves for which one can apply the same procedure with a multipole expansion to approximate the solution to the wave equation. For electromagnetic radiation the monopole moment is zero, as we also have charge conservation, but the dipole moment is not. Therefore electromagnetic radiation can be generated by moving a charge up and down, but GWs can not be generated by moving a mass back and forth. From this we can also discuss another point of confusion, because both gravitational and electromagnetic radiation scale

[9] In general the quadrupole moment is defined as $I_{ij} = \frac{1}{c^2}\int x^ix^jT^{00}d^3x$ thus m^{ij}, the reduced quadrupole moment is the traceless part of this.

as $1/r$ with respect to the amplitude. The energy of the waves scales with the amplitude squared thus scales as $1/r^2$. When we measure electromagnetic radiation, we measure its energy as it arrives at the measurement device. Hence this radiation has fallen of as $1/r^2$ with respect to the source following the inverse square law. However when we detect the GWs the signal has fallen of as $1/r$ with respect to the source. This is because when we measure GWs, we do not measure the energy but the squeezing and squishing of spacetime which is related to the amplitude as we have seen in 2.1.1. This is fortunate as the signal we detect dampens less quickly than the electromagnetic counterpart.

2.3 Detection of gravitational waves

The property of the GWs of stretching and squeezing spacetime as we saw in 2.1.1 is used in the detection of the waves. The GWs detectors called interferometers, they measure the waves in in the following way, also shown in Fig. 5. The detectors have two very long perpendicular arms. A beam of light is split, one part moving into one arm and the other part in the other arm. The two light beams are therefore in phase. The light beams are bouncing back and forth in the arms between two mirrors and finally arrive together at the detector. The detector measures the phase of the two beams. As the two beams are in phase there is destructive interference when there are no perturbations
When a GW passes the relative length of the arms changes and as light always moves with the same velocity, this results in a change in the travel time leading to a shift in phase. The light beams are not in phase anymore and the detector can measure the interference pattern.

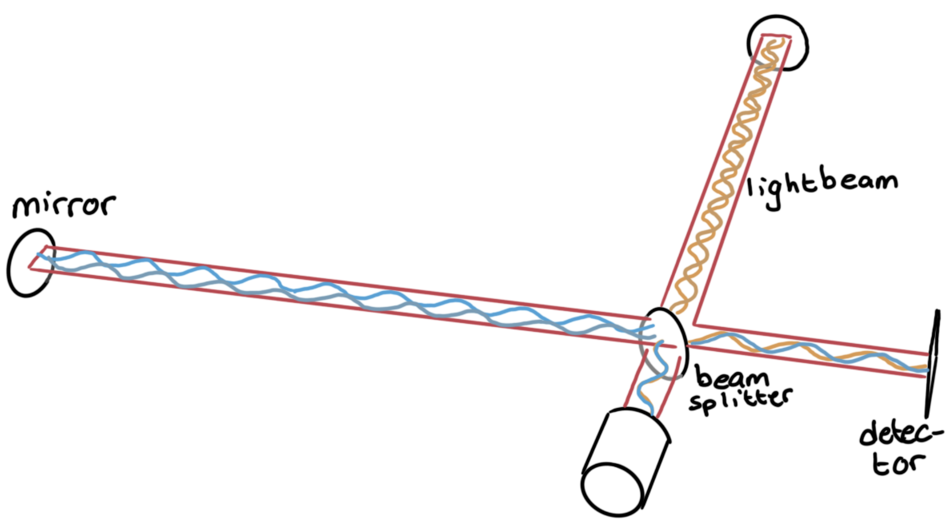


Figure 5: Sketch of the set up of a GW interferometer. The lightbeams are split in the two perpendicular directions and move up and down between the mirrors. When a GW passes a small phase shift is measured at the detector.

The shift that the detectors have to pick up in the signal of the laser is unbelievably tiny. To make an estimate of the order of magnitude we assume Newtonian mechanics for the orbital motion of two black holes rotating around their centre of mass. We paraphrase the calculation done in [30][p.305-307]. For a bound binary system, the force of gravity equals the centrifugal force on circular orbits. We take R as the radius of the system, then we have for the force balance

$$\frac{GM^2}{(2R)^2} = \frac{Mv^2}{R} \rightarrow v = \sqrt{\frac{GM}{4R}}. \tag{2.52}$$

Together with the orbital period $T = \frac{2\pi R}{v}$ this gives as an estimate for the frequency

$$f = \sqrt{\frac{GM}{16\pi^2 R^3}} \sim \sqrt{\frac{c^2 Rs}{R^3}}, \tag{2.53}$$

with $Rs = \frac{2GM}{c^2}$ the Schwarzschild radius of a black hole. Substituting the coordinates and Newtonian approximation for this circular motion in the quadrupole formula Eq. (2.51) results in an estimate for the GWs amplitude. Then if we take a typical system of a black hole binary with both black holes having a mass of 10 solar masses and at approximately 100MPc distance, the frequency and amplitude are given by

$$f \sim 10^2 Hz, \quad h \sim 10^{-21} \tag{2.54}$$

So the detectors have to pick up a change in length of the order 10^{-21}. Therefore making the arms as long as possible results in a larger relative length change between the two sides of the arm. By using optical cavities to let the laser bounce up and down between the mirrors the change in distance is accumulated, making it a bit easier to measure.

There are now multiple GW detectors based on interferometry in different locations. The current ones are LIGO in the US, Virgo in Italy, GEO in Germany and KAGRA in Japan. The most advanced detectors are the advanced LIGO and Virgo, of which LIGO has two detectors in Washington and Louisiana consisting of intereferometers with arms of 4km. It was the LIGO detector that made the first gravitational wave detection facilitated by shared efforts on the data analysis of both the Virgo and Ligo collaborations[1]. Having multiple detectors to measure an GW event is important. As the perpendicular arms of the different detectors have different orientations, one can compare the measurements of the different detectors to determine the direction and orientation in the sky the GW came from.

The two new detectors that are in the pipeline and attract most of the attention are the Einstein telescope (ET) and the LISA detector. The ET will be an underground based detector which will have a wider sensitive frequency bandwidth and will be able to detect smaller strains than the current LIGO/VIRGO detectors. The telescope will be build in the Netherlands or in Italy. The decision for the location will probably be made next year. However as the project is not funded yet, there is no set date for when it will start operating the estimate now is approximately mid-2030s.
The LISA telescope is an even more ambitious plan to construct a detector in space led by

ESA with contributions from NASA and is actually well underway to be launched around 2037. The idea is to place three detectors in a triangle formation in a heliocentric orbit around the sun at approximately 50 million km behind the earth. The detectors will be separated at around 2.5 million km and hence can detect much smaller frequencies. This is also illustrated in Fig. 6. Having the possibility to detect other frequencies is interesting for detecting GWs from other sources than neutron star and black hole binaries and to detect different classes and mass scales of the black hole binaries.

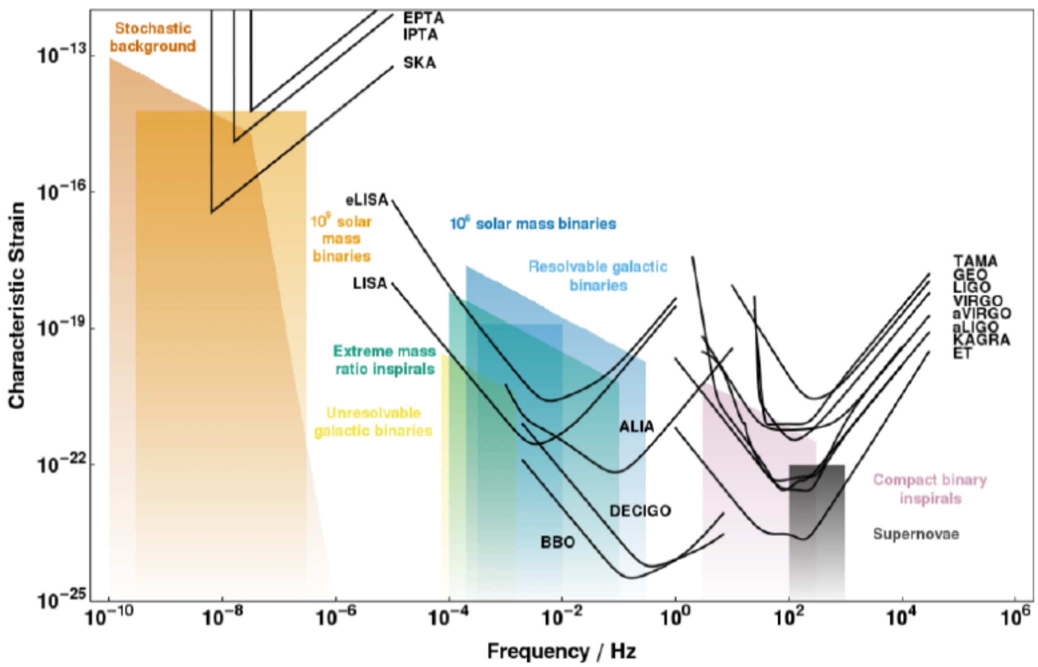


Figure 6: Sensitivity curves of the GW detectors with on the vertical axis the strain and on the horizontal axis the frequency. We focus on the aLIGO/aVIRGO, ET and LISA curves. The colored surfaces show the related GW sources in the corresponding sensitivity regimes. Source:[31].

From Eq. (2.23) we have seen that the effect of GWs on test particles, and therefore on the detector, is given by the waveform $h_{\mu\nu}$ which depends again on the amplitude and the waveform phase. For data analysis most of the information of the GWs is coming from the waveform phase[32]. During a binary black hole coalescence event the first stage is the inspiral part. As the binary system loses energy in the form of GWs the relative distance between the black holes becomes smaller. From Eq.2.52 follows that the orbital velocity will increase which leads to an increase of the frequency as well according to Eq. (2.53). During the inspiral stage the frequency increases slowly. At the end of the inspiral and just before the merger, the freqency goes up in a peak, as is also depicted in Fig. 1. The first part of the inspiral of stellar mass black hole binaries can not be detected by the current GW detectors as the frequency of the waves is too low. For frequencies lower than 10Hz the seismic vibrations on the ground make the GWs indistinguishable from the noise. Generally when the frequency of the waves reaches above the 10Hz to approximately 500Hz. This maximum frequency is roughly the estimate for the frequency at which large mass stellar black hole binaries merge. The GWs from stellar mass binaries contain around thousands of cycles that can be detected by the current ground based detectors [25]. Therefore any information that is related to the phase of the GWs is accumulated over all these cycles. This is very valuable for determining the mass of the BHs which influences the phase but also in our case the additional curvature and scalar field terms of sGB gravity alter the waveform phase. Therefore modelling the inspiral part of the event is not only convenient as it can be done analytically but also has an accumulated effect of the differences of the GB gravity relative to GR. As thus most information is encoded in the phase we focus in this thesis on calculating the phase evolution in section 7.

2.4 Departure from linearized theory

2.4.1 The Relaxed Einstein Equations

In the strong gravity regime, for example in a black hole binary system, we can not assume that our background metric is flat and that the perturbations to the metric are small. Therefore we can not reduce the Einstein equation into the form of Eq. (2.8). However it is still possible to cast the Einstein equations in a wave equation form by expressing the terms in the *gothic metric*

$$\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}. \tag{2.55}$$

With this metric we can define the field

$$h^{\alpha\beta} \equiv (-g)^{1/2}g^{\alpha\beta} - \eta^{\alpha\beta}. \tag{2.56}$$

This definition is exact so we do not assume $h^{\alpha\beta}$ to be a small perturbation.
One can re-express the Einstein equations from Eq. (2.2) in terms of the gothic metric and $h^{\alpha\beta}$, together with imposing the *deDonder/Harmonic* gauge

$$\partial_\beta \mathfrak{g}^{\alpha\beta} = 0 \rightarrow \partial_\beta h^{\alpha\beta} = 0. \tag{2.57}$$

This gauge is the generalization of the Lorentz gauge from Eq. (2.7) to curved spacetime. Then the exact Einstein equations are recast in the form

$$\boxed{\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}\tau^{\alpha\beta}}, \tag{2.58}$$

with $\Box = -\partial^2/\partial t^2 + \nabla^2$ the *flat*space d'Alembertian [33] and

$$\tau^{\alpha\beta} = (-g)T^{\alpha\beta} + \frac{c^4}{16\pi G}\Lambda^{\alpha\beta}. \tag{2.59}$$

Here is $T^{\alpha\beta}$ the energy momentum tensor and the nonlinear field contributions are contained in $\Lambda^{\alpha\beta}$ wich is given by

$$\Lambda^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)t^{\alpha\beta}_{LL} + \left(\partial_\nu \mathrm{h}^{\alpha\mu}\partial_\mu \mathrm{h}^{\beta\nu} - \mathrm{h}^{\mu\nu}\partial_\mu\partial_\nu \mathrm{h}^{\alpha\beta}\right). \tag{2.60}$$

Where $t_{LL}^{\alpha\beta}$ the Landau Lifshitz energy momentum pseudo tensor

$$\begin{aligned}\frac{16\pi G}{c^4}(-g)t_{LL}^{\alpha\beta} =& g_{\lambda\mu}g^{\nu\rho}\partial_\nu h^{\alpha\lambda}\partial_\rho h^{\beta\mu} + \frac{1}{2}g_{\lambda\mu}g^{\alpha\beta}\partial_\rho h^{\lambda\nu}\partial_\nu \mathrm{h}^{\rho\mu} \\ &- g_{\mu\nu}\left(g^{\lambda\alpha}\partial_\rho h^{\beta\nu} + g^{\lambda\beta}\partial_\rho h^{\alpha\nu}\right)\partial_\lambda h^{\rho\mu} \\ &+ \frac{1}{8}\left(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu}\right)\left(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau}\right)\partial_\lambda h^{\nu\tau}\partial_\mu h^{\rho\sigma}.\end{aligned} \tag{2.61}$$

The good news is that even without an approximation we can recast the Einstein equations in a wave equation form, for which we have methods to compute solutions. This recasted equation is called the *relaxed Einstein equation*. This name is coming from the fact that Eq. (2.58) is equal to the Einstein equations only together with the Harmonic gauge condition from Eq. (2.57) which we imposed. However one can independently solve Eq. (2.58) without considering the Harmonic gauge condition. Then it would be less constrained than the Einstein equations, therefore named "relaxed".
The bad news is that the RHS of Eq. (2.58) is highly nonlinear in $h^{\alpha\beta}$, therefore the equation is a nonlinear differential equation. To write Eq. (2.58) in an integral form, we can use a retarded Greens function from Eq. (C.10a), resulting in

$$\begin{aligned}h_{\alpha\beta}(t,\mathbf{x}) &= -\frac{4G}{c^4}\int d^4x' \frac{\tau^{\alpha\beta}\left(t',\mathbf{x}'\right)\delta\left(t'-t+\left|\mathbf{x}-\mathbf{x}'\right|/c\right)}{\left|\mathbf{x}-\mathbf{x}'\right|} \\ &= -\frac{4G}{c^4}\int \frac{d^3x'}{\left|\mathbf{x}-\mathbf{x}'\right|}\tau^{\alpha\beta}\left(t-\left|\mathbf{x}-\mathbf{x}'\right|/c,\mathbf{x}'\right),\end{aligned} \tag{2.62}$$

which is an integral differential equation for $h^{\alpha\beta}$. The effect of the field $h^{\alpha\beta}$ also turning up on the RHS of Eq. (2.58) shows that GWs themselves are a source of GWs. For any realistic source it is not possible to find an exact solution, therefore we need an approximation method. However there is no perturbation scheme that is valid covering the entire spacetime. Instead one has to use different approximations in different regimes depending on the distance to the source and match the different expansions at the boundaries of the valid regions. For the generation of GWs, one can use the so called Post Newtonian expansion, which we will discuss below. However this method breaks down at large distances from the source.
However in general when one solves the equation iteratively to sufficiently high order the computed GWs will cause more GWs at higher orders. Fortunately the expansions we will do in this thesis do not reach high enough order for these effects to come into play but it is an important feature from the non linear nature of gravity.

2.4.2 Methods for solving the Relaxed Einstein Equations

We will give a qualitative overview on solving the differential equation Eq. (2.58) in a perturbative approach, taking into account the different approximation methods at different scales. These methods can be divided in two approaches, one constructed by the group of Blanchet and Damour[34, 35] and the other approach by the group of Will, Wiseman and Pati[33]. In principle the methods are equivalent. We will use the latter method in our calculation in section 7.

The method from Blanchet and Damour splits the calculation into different domains related to the important length scales. These scales are the size of the black hole system d and the distance $\mathcal{R}$ to the source within the Post Newtonian (PN) approximation is valid. We will see later in subsection 2.4.3 what exactly defines this distance. For non relativistic sources the distance for which the PN expansion is valid is much larger than the system size $\mathcal{R} \gg d$, hence distance $\mathcal{R}$ lies outside of the source. For the region from inside the source out to distances $\mathcal{R}$, the field equations can be solved perturbatively in orders deviating from Newtonian gravity. Outside of the source, the energy momentum tensor is zero and the only contributions from the source term $\tau^{\alpha\beta}$ comes from the gravitational field. For sources where the gravitational field inside is not too large, the field outside the source becomes flat fast. Therefore for these sources practically up to the system size d, one can use a Post Minkowskian (PM) approximation in which the field equations can be solved order by order in terms of deviations from flat space. Thus in the region $0 < r < \mathcal{R}$ the equations can be solved order by order with an PN approximation and in the region $d < r < \infty$ one can use the PM approximation, see Fig. 7 for the length scales.

In the overlap region $d < r < \mathcal{R}$ the two expansions need to be matched with a method called matched asymptotic expansion. The general idea of a matched asymptotic expansion is that one has an expansion that is valid for some inner region and an expansion that is valid for an outer region. In the region where the regions of validity overlap the expansions should be equal, hence the outer limit of the inner expansion can be equaled to the inner limit of the outer expansion. This can be solved as the solution in the overlap region. For the composite expansion that would be valid over the whole domain, one can add the solution of the inner and outer expansion and subtract the solution of the overlap region.

Here we will only provide a rough sketch of this approach, since applying this method requires highly nontrivial details for the PM, PN and matched asymptotic expansion. For example within these approximation methods, to solve the wave equation Eq. (2.58) for the different orders of the approximation, one can use a multipole expansion, similar as in electrodynamics. A more detailed description is given in [25][Ch5.3], [36] and on the matched asymptotic expansion for GWs [37].

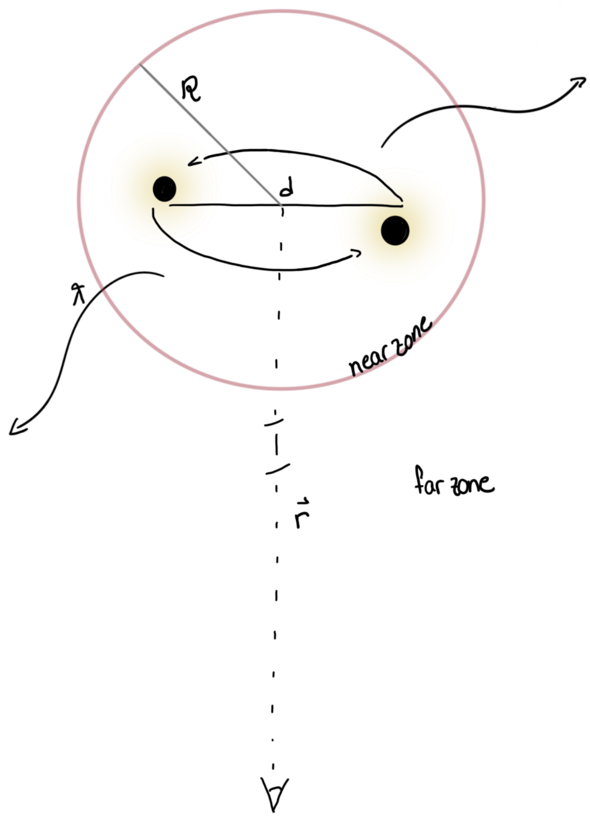


Figure 7: Near zone and far zone regions

In this thesis we will use the approach of Will, Wiseman and Pati to solve Eq. (2.58), which is named the "Direct Integration of the Relaxed Einstein equation" (*DIRE*) approach. As a first step one splits the integral domain of Eq. (2.62) into two parts. From the dirac delta in the integral can be concluded that to calculate the waveform at position $P(ct, \mathbf{x})$, one has to integrate over the past light cone of this point in space. As in our case we are interested in the waveforms that the detector will measure, our point will lie very far away from the actual GW source; the black hole binary. One can define the following regions. We set the the centre of mass of the binary system at the origin of the coordinate system. The system has size d and one defines the position from a field point $\mathbf{x}$ relative to the position of the source $\mathbf{x}'$ as $R = |\mathbf{x} - \mathbf{x}'|$. As the characteristic wavelength is given by

$$\lambda = \frac{v}{c} d, \tag{2.63}$$

we define the zone in which the PN expansion is valid (called near zone) as the worldtube $R < \lambda$. The region outside the near zone is the far zone. The integration is then split over the hypersurface that is given by the part of the past lightcone that intersects with the near zone, called $\mathcal{N}$ and over the part of the lightcone in the far zone called $\mathcal{C} - \mathcal{N}$ also shown in Fig. 8. In principle you can put your field point in the near zone or the far zone, thus resulting in four different integrals with the splitted integration domain. However as we are interested in the GWs at the detector, the field point x lies in the far zone. Adding the results of the integrations over both domains together to get the total result, cancels the dependence on the boundary R. That this happens was first shown up to 2 PN order by Ref. [38] and later via induction for all orders by Ref. [33].

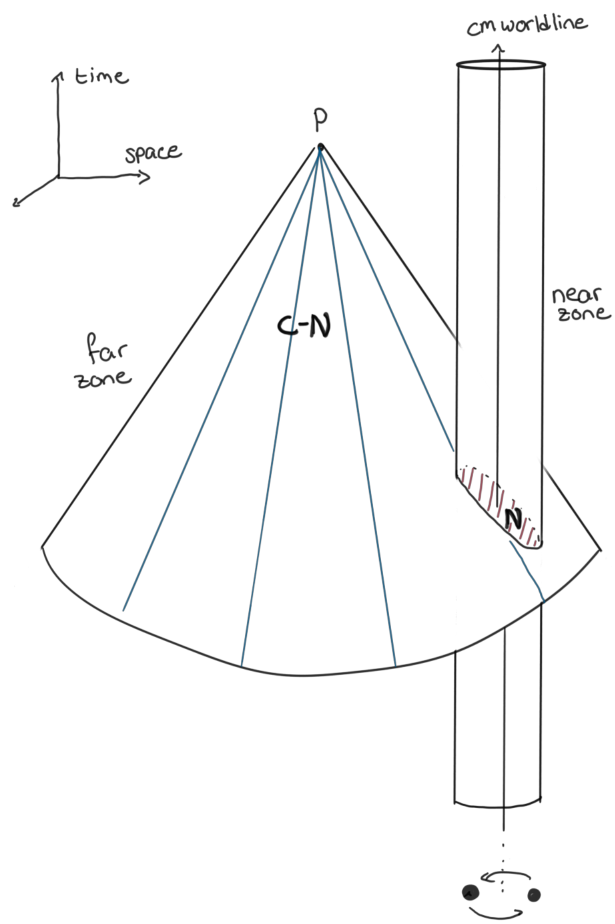


Figure 8: The past lightcone of field point P intersects with the near zone world tube. Figure based in [17].

For the calculation in the near zone, one can express the integral in terms in a multipole expansion of the source term $\tau^{\alpha\beta}$. Then because we are in the near zone we can expand the source term up to 1PN order and solve the integral differential equation perturbatively.

For the far zone calculation, one does the same only the source term does not contain the energy momentum contribution any more as the far zone does not contain the black hole binary itself. However $\tau^{\alpha\beta}$ is not zero. This is because the source in the relaxed Einstein equation Eq. (2.58) depends also on the metric and metric perturbations itself, which are not zero in the far zone.

2.4.3 The Post Newtonian approximation

As discussed before, to solve the relaxed Einstein equations with the DIRE approach, one can expand the source terms with the so called Post Newtonian expansion. We also use this approximation for the calculation of the 2 body Lagragian in section 5.

Assuming linearized gravity, we assumed a flat background metric. Then we can assume that the sources contribute negligibly to the curvature of spacetime. To calculate the GWs, one can expand in $\frac{v}{c}$ separately from the corrections to spacetime. However this is only a valid approximation if the system is governed by non gravitational forces. But as we are interested in the GWs coming from black hole binaries which are gravitationally bound, the approximation from before is no longer entirely accurate.
For self gravitating systems holds

$$\frac{v^2}{c^2} \sim \frac{R_s}{d}, \tag{2.64}$$

with R_s the Schwarzschild radius and d the typical system size. This relation shows that the velocity of the source is coupled to the curvatuve of spacetime, given by $\frac{R_s}{d}$ which measures the strength of the gravitational field near the source. This relation is a consequence of the virial theorem: kinetic energy of a stable system of discrete objects in a bound potential is related to

$$< T > \sim < V_{tot} >, \tag{2.65}$$

with $T \propto mv^2$ and $V_{tot} \propto \frac{-GMm}{r} \propto \frac{R_s}{d}$ for gravitational systems, thus relating the velocity expansion to the strength of gravity.

For describing these systems we need to go beyond a flat background spacetime, which can be described by Newtonian gravity. Considering gravitationally bound (semi) relativistic systems one has to go a step further to the post Newtonian regime. As can be seen in Fig. 9 this PN [39] description is valid in the region where $\frac{R_s}{d}$ and $\frac{v^2}{c^2}$ are comparable and not too close to 1. This is the case for slowly moving sources in a weakly gravitationally bound system. In these situations we have the following small parameter to expand in

$$\varepsilon_{PN} = \frac{v^2}{c^2} \sim \frac{R_s}{d}. \tag{2.66}$$

In practice one often expands in factors of $\frac{1}{c^2}$ which leads to the same results as using the formal expansion parameter ε_{PN}.
The PN expansion is only valid during the inspiral of the coalescence event. When the black holes get too close, the gravitational fields that the black holes move in, becomes very

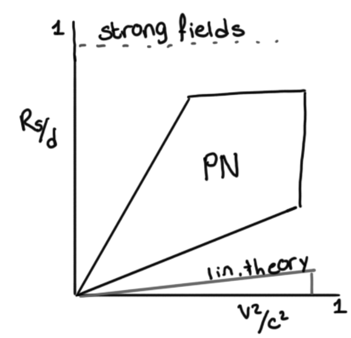


Figure 9: Regimes of validity for the different perturbation methods in the parameter space of the field strength $\frac{R_s}{d}$ and the velocity. Figure based on [25].

strong and the velocities are high. Still the PN expansion works to describe the motion approximately even in this regime where it is not entirely valid. This has been compared with numerical simulations [40]. When the actual merger happens the expansion does break down.

One note on the regime of validity of this expansion. As already mentioned in the previous part we can define the following zones based on the hierarchy of length scales of our system, see also Fig. 7:

- the near zone, the distance to the source is much smaller than the typical reduced wavelength Eq. (2.63) of the GWs: $r \ll \lambda$; retardation effects are negligible and in this region the PN approximation works well.
- far(wave)zone, the distance to the source is much larger than the typical reduced wavelength of the gravitational source: $r \gg \lambda$, PN approximation breaks down.
- intermediate region, is between those two regions: $r \sim \lambda$

That the PN expansion is only valid in the near zone can be reasoned in the following way. We discuss a source moving non-relativistically, therefore the time derivatives are of order v smaller than the spatial derivatives

$$\frac{\partial}{\partial t} \sim \frac{\partial}{\partial x^i}\frac{\partial x^i}{\partial t} \sim v\frac{\partial}{\partial x^i}. \tag{2.67}$$

Because of this relation between the time and spatial derivatives the retardation effects (effects on quantities that are expressed in the retarded time $t - \frac{r}{c}$) are small corrections $\frac{r}{c} \ll t$. This has the following consequence: retarded functions such as the the source terms in Eq. (2.62) can be expanded for small retardation

$$F(t - \frac{r}{c}) \sim F(t) - \frac{r}{c}\dot{F}(t) + \frac{r^2}{2c^2}\ddot{F}(t) + \ldots . \tag{2.68}$$

Each time derivative scales as a factor $1/t$, which is of the order of the frequency and therefore of order of the orbital frequency w. We can write the reduced wavelenght from Eq. (2.63) as

$$\frac{\omega}{c} = \frac{1}{\lambda}, \tag{2.69}$$

using that the orbital angular frequency is half the gravitational wave angular frequency. Thus the expansion is in $\frac{r}{\lambda}$. Therefore PN expansion and the expansion of the retarded function is valid in near zone with $r \ll \lambda$. More intuitively can be seen from above expansion that r should not be too large otherwise this expansion becomes divergent, the "not too large" relative to the scales in the system is then quantified with $r \ll \lambda$.

To implement the PN expansion one usually starts from an ansatz for the expansion of the metric up to a certain PN order. By substituting this expanded metric back into the Einstein equations and energy momentum tensor and expanding and equating the equations order by order, one can solve for the metric components.

To be able to write the metric one PN order higher than Newtonian order, we first need to know the metric in the Newtonian limit. We can begin with the geodesic equation describing the path of a test particle through spacetime [10], writing this equations in the Newtonian limit, it should reduce to the Newtonian equation of motion. Comparing the geodesic equation in this limit with this equation of motion, we can derive what the metric components are in the Newtonian limit. The geodesic equations in GR are given by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0. \tag{2.70}$$

The second derivative of x^μ to the proper time can be rewritten with respect to the coordinate time t using the following relation derived from the acceleration with respect to the coordinate time

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= \left(\frac{dt}{d\tau}\right)^{-1}\frac{d}{d\tau}\left[\left(\frac{dt}{d\tau}\right)^{-1}\frac{dx^i}{d\tau}\right] \\ &= \left(\frac{dt}{d\tau}\right)^{-2}\frac{d^2x^i}{d\tau^2} - \left(\frac{dt}{d\tau}\right)^{-3}\frac{d^2t}{d\tau^2}\frac{dx^i}{d\tau}.\end{aligned} \tag{2.71}$$

With the last term in the second line, we subtract zero as $\frac{d^2t}{dt^2} = 0$[11]. This seems to only makes life a lot more complicated but in writing it this way, we use the geodesic equation with $\mu = i$ and $\mu = 0$ to substitute $\frac{d^2x^i}{d\tau^2}$ and $\frac{d^2t}{d\tau^2}$ in

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= -\Gamma^i_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} + \Gamma^0_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt}\frac{dx^i}{dt} \\ &= -c^2\Gamma^i_{00} - 2\Gamma^i_{0j}c\frac{dx^j}{dt} - \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + \left[c^2\Gamma^0_{00} + 2\Gamma^0_{0j}c\frac{dx^j}{dt} + \Gamma^0_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\right]\frac{dx^i}{dt}.\end{aligned} \tag{2.72}$$

In the Newtonian limit we assume low velocity, weak gravity and a static field. Low velocity means $\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$ thus we can neglect the spatial derivatives with respect to the time derivatives.Then Eq. (2.72) becomes

$$\frac{d^2x^i}{dt^2} = -c^2\Gamma^i_{00}. \tag{2.73}$$

For a static field the time derivative to the metric is zero, $\frac{\partial g_{\mu\nu}}{\partial x^0} = 0$. Then the zero component of the Christoffel symbol reduces to

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{\lambda 0}}{\partial x^0} + \frac{\partial g_{0\lambda}}{\partial x^0} - \partial_\lambda g_{00}\right) = -\frac{1}{2}g^{\mu\lambda}\frac{\partial g_{00}}{\partial x^\lambda}. \tag{2.74}$$

Then in weak gravity we can expand the metric in the Minkowski metric plus a small perturbation as in Eq. (2.4). Therefore to first order in the perturbations we have

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\lambda}\frac{\partial h_{00}}{\partial x^\lambda} \rightarrow \Gamma^i_{00} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i} \tag{2.75}$$

[10] See Appendix (D)

[11] Or in the famous words of S. Wepster, we apply the method of "creatief niks doen".

Then the acceleration comes

$$\frac{d^2x^i}{dt^2} = \frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i}. \tag{2.76}$$

Comparing this with the Newtonian equation of motion

$$\mathbf{a} = \frac{d^2x^i}{dt^2} = -\nabla U, \tag{2.77}$$

with U the Newtonian potential. We have

$$h_{00} = -2\frac{U}{c^2}, \quad g_{00} = -1 + 2\frac{U}{c^2}. \tag{2.78}$$

Therefore in the Newtonian limit we recover for all the metric components

$$\begin{aligned} g_{00} &= -1 + 2\frac{U}{c^2} \\ g_{0i} &= 0 \\ g_{ij} &= \delta_{ij} \end{aligned} \tag{2.79}$$

How to go PN orders beyond this? The counting can be inferred from the action

$$\begin{aligned} S &= -mc^2 \int dt \sqrt{-g_{\mu\nu}\frac{dx_A^\mu}{dt}\frac{dx_A^\nu}{dt}} \\ &= -mc^2 \int dt \sqrt{-g_{00} - 2g_{0i}\frac{v_A^i}{c} - g_{ij}\frac{v_A^i v_A^j}{c^2}}. \end{aligned} \tag{2.80}$$

As g_{ij} is multiplied with PN factor $\frac{v^2}{c^2}$ this components will be one PN order lower than g_{00} which is not multiplied with any factor of $\frac{v^2}{c^2}$. This corresponds to one factor lower than g_{00}. As g_{0i} is multiplied with $\frac{v}{c}$, this component should be expanded only in odd factors, which correspond to half PN orders. Because of this multiplication, g_{0i} counts one factor $\frac{v}{c}$ below g_{00}. Therefore the metric components one PN order higher will be

$$\begin{aligned} g_{00} &= -1 + 2\frac{U}{c^2} - 2\frac{U^2}{c^4} \\ g_{0i} &= 0 - 4\frac{g_i}{c^3} \\ g_{ij} &= \delta_{ij} + 2\delta_{ij}\frac{U}{c^2}. \end{aligned} \tag{2.81}$$

Or expressed differently

$$\begin{aligned} g_{00} &= e^{\frac{-2U}{c^2}} + \mathcal{O}\left(\frac{1}{c^6}\right) \\ g_{0i} &= 0 - 4\frac{g_i}{c^3} + \mathcal{O}\left(\frac{1}{c^5}\right) \\ g_{ij} &= \delta_{ij}e^{\frac{2U}{c^2}} + \mathcal{O}\left(\frac{1}{c^4}\right). \end{aligned} \tag{2.82}$$

The fact that the potential U can be taken the same for both the time and purely spatial components and why one can write the potential of g_{0i} with only one spatial index, can be derived starting with taking a general expression for these higher order metric components and solving the Einstein equations order by order. This is done in Appendix E.

This metric expansion will be the starting point for applying a PN expansion in the subsequent sections. In this thesis we expand up to 1PN order. Going to higher orders will become very cumbersome very quickly. Also when expanding to higher orders beyond Newtonian gravity the nonlinearity of GR comes into play. GWs will backreact on the matter sources beyond a certain order of the expansion, influencing the equations of motion. Also the gravitational field itself is source for GWs, but then GWs at higher order become also source of GWs. When working to 1PN order we do not need to worry about these effects as they come into play at 2.5PN in GR.

3 Modified gravity

Thus far we have studied GWs describing gravity with GR. However in this thesis we are interested in new phenomena that arise for GWs when gravity differs from GR. Therefore we will first look in general at why it would be interesting to modify the theory of gravity and which modifications are proposed. Then we give a qualitative overview of the theories that is studies in this thesis; scalar Gauss Bonnet gravity.

3.1 Motivation to modify the theory of gravity

A valid question to pose is 'Why do we need a modification of the current theory of gravity?'. Here we would like to give some motivation. Our current theory of gravity is Einstein's theory of General relativity which describes gravity as the effect due to curvature in the fabric of spacetime (see appendix D for a short recap of the GR formalism). Even after more than a century, it still holds up against many experimental test and is the best description of how gravity works on macroscopic scales. For example, GR predicted very precisely events in our solar system as the precession of the perihelion of Mercury's orbit and the deflection of lightrays because of the gravitational field of the sun. Also the direct measurements of the predicted GWs[1] and black holes[41] are a major accomplishment in the direction of GR (for an overview see [9]).

However on small scales when high energies and elementary particles become important, GR does not hold up. Modelling gravity on those scales becomes important when describing the insides of black holes or the beginning of the universe. The most important problem is that when one writes this description of gravity in a quantum field theory[42] format, which is now our current model to describe the physics on smallest scales with the standard model, the theory becomes nonrenormalizable[10]. This means that the infinities that arise in the quantities can not be compensated by including counterterms to cancel them. When a theory is nonrenormalizable it is not an accurate description on those scales. The goal of modified theories of gravity is to solve the problems of GR on small scales but also to reduce to GR on the macroscopic scales on which it is well tested.
Other open questions that can not be answered with GR are for example the nature of dark energy and dark matter, the matter-antimatter asymmetry in the early universe and the existence of singularities in a blackhole. One of the possible answers to those questions is that the description of gravity with GR is incomplete. There is quite some motivation to look beyond the theory of GR for possible improvements. Since the formulation of GR, a wide variety of possible modified gravity theories has been explored[11].

3.2 How to modify the theory of gravity?

In 1971 David Lovelock[43] showed that when starting from an action which only contains second order derivatives in the spacetime metric, the Einstein field equations can be the only possible equations of motion, later named Lovelock's *theorem*. In the same spirit he formulated the now so called Lovelock's *gravity*; the most general description of gravity which coincides with GR in three and four dimensions.

Lovelock's *theorem* has some implications[44] for if we want to modify GR and hence the Einstein field equations. If one wants to have another outcome for the equations of motion, there are five possibilities:

(i) Include higher order derivatives of the metric

(ii) Introduce other fields next to the metric tensor

(iii) Another number of spacetime dimensions

(iv) Introduce non-locality

(v) Introduce emergence; the equations of motion are not derived from the action.

Modified gravity theories can be categorised by these options and sometimes they use multiple, see Fig.3. However the fifth one is often not considered as it would undermine the basis of all field theories.

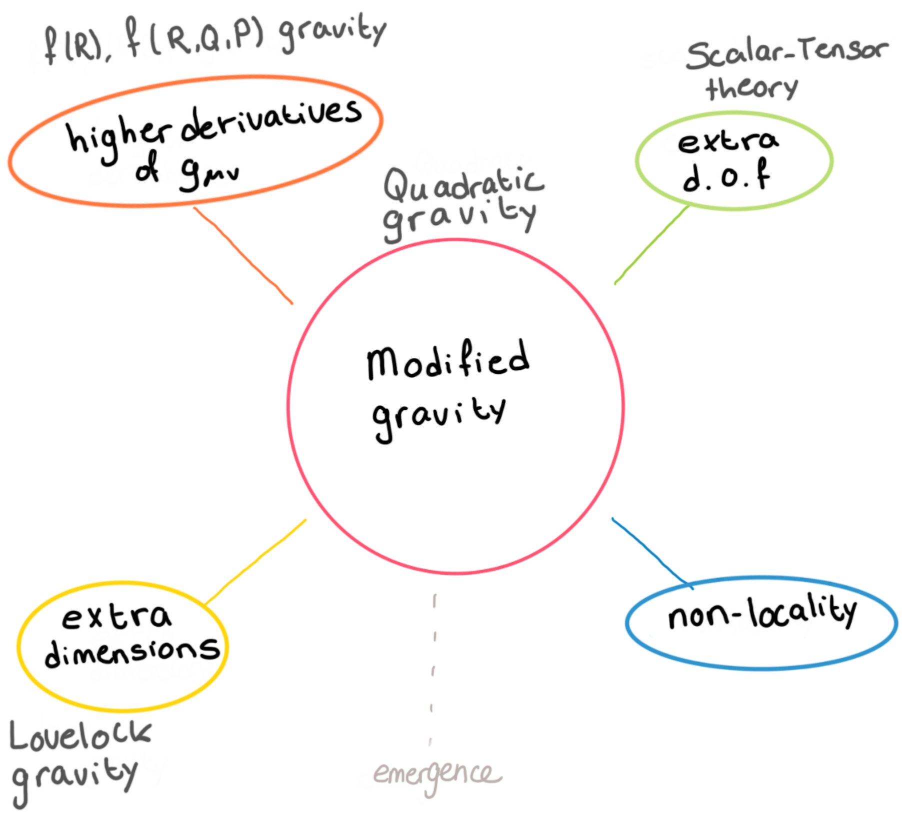


Figure 10: Violations of Lovelocks theorem

Item (i) can be achieved by adding extra scalar curvature terms to the Hilbert Einstein action. This is because the curvature terms depend on derivatives of the metric, hence higher powers of these terms introduce higher order derivatives. Instead of only the Ricci scalar R one could also add terms as R^n

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}f(R), \tag{3.1}$$

with $f(R) = R + \gamma R^n$. This would be a quite natural extension as the curvature in the Universe is quite small. On solar system scales then only the leading order term R would play a role and GR would be the accurate approximation in this limit. The higher order terms would only become important in strong gravity, thus strong curvature regimes. One can also think about adding other curvature scalars as contractions of the Ricci tensor and Riemann tensor like $Q = R_{\mu\nu}R^{\mu\nu}$ or $P = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. The theories which add extra powers of the Ricci scalar are called $f(R)$ theories and theories that add also other invariants are called $f(R,Q,P)$ theories. These theories modify GR only in the pure gravitational sector. The downside of these theories is that when introducing higher order derivative terms in the metric in most cases one also introduces ghost instabilities. That higher order derivatives in the action and in the EOMs lead to instabilities was proved by Ostrogradsky [45] by showing that the corresponding Hamiltonian becomes unbounded from below[12]. Ghosts are fields with a negative energy or norm which indicates an instability in the theory. Because of this negativity the vacuum is unstable as creating pairs of fields with positive and thus with negative energy can be a process of zero net energy. Therefore this will happen infinitely, which can not be physical. One can deal with these instabilities by introducing a cut off that suppresses the ghost fields at the scales at which the theory is valid. There are specific combinations of higher order curvature terms that do not introduce ghost fields as we will discuss in the case of scalar Gauss Bonnet gravity below.

Item (ii), introducing other fields is interesting as this is also motivated from different directions. For example a possible candidate for a quantum gravity theory is string theory. However string theory does not describe gravity as GR. This can be reasoned from the fundamental vibration modes of loops of the fundamental strings[30][p299-300]. Two of the modes have the same polarization as that of gravitational waves and represent the states of a massless spin 2 particle, the graviton. But the strings also have another vibration mode corresponding to a spin 0 particle, or a scalar. This latter mode is an indication that string theory rather predicts a modified gravity theory of GR with an extra scalar field degree of freedom. There is also motivation of the addition of a scalar field from cosmology. To explain the acceleration of the expansion of our universe the quintessence field is introduced which is a very light scalar field. A well known class of theories that introduce such an extra scalar field is scalar-tensor modified gravity theories (tensor refers to the metric tensor).

[12]One can show this in the general case as is also given in Appendix C of [46]. But as an instructive example one can also see it from the classical example of a higher derivative oscillator Lagrangian, paraphrasing this reference: $\mathcal{L} = \frac{1}{2}\ddot{q}^2 - \frac{1}{2}\left(m_1^2 + m_2^2\right)\dot{q}^2 + \frac{1}{2}m_1^2m_2^2q^2$. The Legendre transformation for the Hamiltonian from the Lagrangian for higher order derivatives generalizes to $\mathcal{H} = \sum_{a=1}^{N} P_a q^{(a)} - \mathcal{L}$ with momenta $P_a = \sum_{i=a}^{N}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{i-a}\frac{\partial\mathcal{L}}{\partial q^{(i)}}$. So in our case we have momenta $P_1 = -\dddot{q} - \left(m_1^2 + m_2^2\right)\dot{q}$ and $P_2 = \ddot{q}$ and Hamiltonian $\mathcal{H} = P_1\dot{q} + P_2^2 - \mathcal{L}_{\mathrm{PU}}\left(q,\dot{q},P_2\right) = P_1\dot{q} + \frac{1}{2}P_2^2 + \frac{1}{2}\left(m_1^2 + m_2^2\right)\dot{q}^2 - \frac{1}{2}m_1^2m_2^2q^2$. Momentum P1 linear in the Hamiltonian which makes the Hamiltonian unbounded.

In this thesis we only work with modified gravity theories which break Lovelocks theorem by introducing item (i) and (ii). However as (super) string theory is one of the possible candidates for a quantum gravity model, it is interesting to also look at higher dimensional gravity theories. These theories assume that the dimensions can be compactified at solar system scales to the 4D case but can open up on other scales. Introducing non locality is motivated by string theory as it also has non local aspects[47].

From this general overview we can conclude that there are many modified gravity theories, how to make a choice which one to focus on? First of all, the theory has to reduce to GR on the macroscopic scales and therefore has to pass the tests that are already done for GR on those scales. Therefore most of the theories named above add terms with a coupling term to the Hilbert Einstein action of GR so it can reduce to GR again at the right scales. Secondly the theory should be free from mathematical instabilities, as for example the Ostrogradski instabilities described above.
Furthermore the goal of a modified gravity is that it can explain (some of the) questions that are still open in GR. It is therefore interesting if the theory originates from a quantum mechanical description of gravity so it can be used to describe gravity on the smallest scales, for example fundamental quantum gravity candidates as string theory and quantum loop gravity.
Lastly the modified gravity theories should be tested in the regime where they differ from GR, the regime of small scales or high energies. An excellent probe for this goal are gravitational waves as they are generated by the extreme event of a merger of a black hole binary and therefore in a system with high energies and strong gravity. For these events one expects the modified gravity theory to differ from GR and hence predict a difference in the produced GWs [48].

3.3 Scalar Gauss Bonnet gravity

In this thesis we focus on a specific class of modified gravity theories called scalar Gauss Bonnet gravity. These theories are an extension to GR which introduces an extra scalar field φ which is nonminimally coupled to a quadratic curvature term. This quadratic curvature term is the Gauss Bonnet invariant

$$R^2_{\rm GB} = R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{3.2}$$

with coupling parameter α. The total sGB action is given by

$$S = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}[R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \alpha f(\varphi)R^2_{GB}]. \tag{3.3}$$

Thus the Hilbert Einstein action gets extended by a kinetic term of the scalar field and the non minimally coupled quadratic curvature term to a coupling function depending on the scalar field $f(\varphi)$. Different choices for the coupling functions lead to different 'flavours' of sGB gravity, which we will discuss below.
This theory therefore avoids Lovelocks theorem by introducing another field which is

this scalar field, and includes higher order derivatives terms of the metric by including a quadratic curvature term (the Riemann tensor already contains second order derivatives of the metric). Note that the scalar field is part of the gravitational sector and not a matter field.
Including a quadratic curvature term is a natural extension of GR as it can be regarded as the lowest order in a series expansion in the curvature. It can therefore be seen as an effective theory only up to the second-lowest order in curvature. Another attractive feature of quadratic curvature terms is that they make the theory renormalizable [13], as these terms have the same form as the one loop divergent terms that shows up when one tries to quantize gravity and therefore can act as a counterterm [11]. Regarding the ghost instabilities that higher order derivative terms can introduce, the particular combination of the quadratic curvature terms of the Gauss Bonnet invariant are such that in the equations of motion, the higher order derivatives are canceled. Therefore the theory is still free[13] of Ostrogradsky instabilities [49].
Also it was shown that the theory in the weak coupling limit is mathematically well posed [50]. And the theory can be derived from Lovelock gravity, the most general theory for gravity which results in second order equations of motion [51], [52].
On top of all this, the GB term also turns up in the low energy limit in the effective action of the bosonic sector in heterotic string theory [14] [55], and in the low energy expansion of supersymmetric string theory [56] and is therefore a candidate for a gravity theory with quantum corrections.

There are different kinds of sGB gravity theories depending on the form of the coupling function between the scalar field and the GB invariant. Different choices for the coupling function also influence the possible black hole solutions in the theory and what constraints there are for the coupling constant α. One can divide the choices for the coupling functions into two types, categorized according to the way in which the resulting theory can violate the no-hair theorem [57][58][59]. This theorem states that black holes can be described solely by their external observables, namely their mass, charge and angular momentum. All the information about "hair", the matter that formed the black hole or is falling into it disappears behind the horizon.

[13]At first sight this seems to have something to do with the GB term to be a topological invariant term which in 4D is a surface term which would vanish in the integral of the action. However as we have a scalar field coupled to this term it is a bit more subtle. As explained in [49] the GB invariant coupled to a general field can still have higher order derivatives in the field equations because with the equation of motion of this extra field, this field can be written as a function of the GB invariant and be substitute in the field equations resulting in higher order derivatives. This can be solved by adding a kinetic term of this field to the action (which is the case in sGB gravity). Then with the equations of motion from that action the metric and (scalar)field can be uniquely determined and the equations on motion do not contain higher than second order derivatives, as explained more explicitly in the beginning of section B of [49].

[14]Ref. [14] was the first to seriously study if string theory could be a theory for unifying all interactions. They did a small Regge slope expansion, carrying this expansion in their low energy field limit to second order in metric derivative terms and carried further this would give higher derivative terms, leading with quadratic curvature terms. This was indeed the case in the low energy limit of heterotic super string theory[53]. This quadratic term consisting of a contraction of two Riemann tensors is most conveniently written as the GB invariant, the other terms can be neglected as they can be made vanishing by a redefinition of the field and hence carry no physical meaning [54]

Much recent interests has focused on finding black hole solutions for which this no hair theorem *is* violated as this would be different from GR and would leave an imprint on the GW signal. This violation can happen in two different ways. In general one can look at the equation of motion of the scalar field which is

$$\Box\varphi = -\frac{1}{4}\alpha f'(\varphi)R^2_{GB}. \tag{3.4}$$

(We will derive this in section 5). The GB invariant contains the scalar invariant $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ named the Kretschmann scalar. This scalar is generally non zero for black hole solutions. Consequently a non-zero derivative of the coupling function results in a non-trivial solution for Eq. (3.4). It can be that the first derivative of the coupling function never vanishes. In this case the non zero scalar solution is present in the theory. We call solutions corresponding to a non vanishing derivative of the coupling function, solutions of type I. It can also be that the derivative of the coupling function does vanish for some values of the scalar field, in this case a non trivial scalar solution can happen spontaneously. We call solutions of this type, type II.

3.3.1 Compact object in type I sGB theory

Coupling functions of type I are for example a shift symmetric coupling function $f = 2\varphi$ and a dilatonic one $f \approx e^{2\varphi}$. The naming of these theories are respectively shift symmetric scalar Gauss Bonnet gravity (ssGB) and Einstein dilation scalar Gauss Bonnet gravity (EdGB). sGB gravity with a dilatonic coupling function is explicitly motivated by string theory. In the low energy limit after compactification to 4D in heterotic string theory, the extra low energy degrees of freedom appear in the effective action as a dilaton field with a GB coupling [60]. The coupling constant is then proportional to the Regge slope. Black hole solutions in this theory have nontrivial scalar hair, which modifies the gravitational mass of the black hole.

The black hole solutions of these type I coupling functions, or more specific for shift symmetric sGB [61] and dilatonic sGB[62, 63, 15] always have a non vanishing scalar field. Therefore the black hole solutions of GR acquire corrections [64, 65, 66]. The evolution of the scalar field in this type is also studies in dynamical collapse scenarios [67, 68, 69], in these systems the scalar field eventually relaxes to the static configurations again. One important thing to note is that in this type of sGB theory neutron stars can not scalarize.

The argument of a vanishing scalar field around a neutron star is based on the following reasoning[70, 71]. The scalar charge is defined as the lowest order term in a multipole expansion of the asymptotic limit of the scalar field (we will do this expansion explicitly in section 6). The asymptotic limit is the limit of large distances r from the source where the scalar field reduces to

$$\varphi \sim \frac{\mu}{r} + \mathcal{O}(r^{-2}), \tag{3.5}$$

with μ in this case the scalar charge. The equation of motion of the scalar field is given by Eq. (3.4). On the RHS of this equation is the derivative of coupling function but in the asymptotic expansion the derivative reduces to a constant. R^2_{GB} is a topological invariant which in 4D is a surface term. Performing the integral over both sides of the EOM makes the RHS vanish. Then as for an isolated compact object one is interested in stationairy solutions, the box operator reduces to only spatial derivatives. Using stokes theorem gives

$$\int \sqrt{-g}\nabla\varphi = \int \sqrt{-g}\left(\partial_i\varphi\right) n^i dS = \int \sqrt{-g}\left(\partial_r\varphi\right) dS = 0, \tag{3.6}$$

with n^i a radial unit vector and the integration is done over a two sphere at infinity. After substituting Eq. (5.81), the integral has the unique solution of $\mu = 0$, hence a vanishing scalar charge. The question now is why does this argument not hold for black holes? As stated in [71] this argument holds for compact object without a horizon. If one does that calculation of the scalar charge explicitly for neutron stars and black holes, the difference lies with that for neutron stars one has to match the solution of the GB invariant inside the star, depending on its equation of state, with the solution outside the neutron star which corresponds to the Schwarzschild solution. For black holes, everything inside the horizon is shielded and to first approximation at large distances the black hole can be described as a point particle with a mass that dependents on the scalar field (we will say more about this in section 5.1.1), so no matching has to be done. The integral in the argument above for black holes will therefore be over all of space (instead of only outside the compact object) with a localised contribution from a point particle description with a scalar field dependent mass and therefore does not vanish [24]. We will actually calculate the scalar charge and dipole moment of the scalar field explicitly in section 6 .

3.3.2 Compact object in type II sGB theory

Coupling functions of type II are for example quadratic functions $f \approx \varphi^2$ or Gaussian functions $f \approx e^{\varphi^2}$. In these models the well known GR BH solutions as Kerr and Schwarzschild exist in certain limits, but can have configurations with a nontrivial scalar field for specific scalar field bands. This is known as spontaneous scalarization[72, 73, 74, 75, 76, 77]. This spontaneous scalarization is allowed for both black holes and neutron stars [78]. In a dynamical system as for example a black hole collapse, it is possible for the black holes to have a dynamical spontaneous scalarized scalar field [79, 80, 81] and end up with a scalar field in the remnant. Or they descalarize, ending up with no scalar field in the final configuration[82].

3.3.3 Constraints on the coupling

Tests to constrain the value of the coupling constant so far are only done for coupling functions of type I, more specifically for a dilatonic coupling function in EdGB theory. The first constraints are coming from the fact that the theory should recover the results on solar system scales from GR. Basically the theory should reduce to GR on those scales. The constraints for the coupling constant of EdGB gravity coming from the solar system test of Shapiro time delay (gravitational time delay) measured by the Cassini probe is given by $\sqrt{\alpha} < 8.9 \times 10^6$ km [83]. If the coupling constant would be greater than this value, the differences with GR would be too large at the solar system scale. However there are now stronger constraints coming from low mass x-ray binary observations [84] and from Bayesian parameter estimation with GW detections [85]. Current bounds from GWs are now set to be $\sqrt{\alpha} \lesssim 1.7$ km [86] [87], improved recently by [88] to $\sqrt{\alpha} \lesssim 1.18 - 1.33$ km.

In this thesis we intend to keep the coupling function general and specify particular couplings only for analysis parts. However we do assume in general that the black holes we consider have a non trivial scalar field. This results has the consequence that an inspiralling black hole binary radiates not only quadrupole radiation corresponding with GWs but also dipolar radiation coming from the scalar field. The waveform templates and waveform phase evolution of these two forms of radiation were analyzed for the first time by [16][17]. On top of that, the scalar field leads to the interesting result of scalar tidal effect as we will discuss further in section 4.

4 Tidal effects

In this thesis we will not only look at the waveforms and phase of the scalar field radiation coming from a scalarized black hole binary in sGB gravity but on top of that we will include tidal effects in the framework as well.

When thinking about tidal effects the first thing that comes to mind is often the tidal effects here on earth from the gravitational pull from the moon (and to some lesser extend from the sun). Because the gravitational field scales with $1/r^2$, the side of the earth facing the moon experiences a stronger gravitational field than the other side. Or to be more specific, to lowest order the side of the earth closest to the moon experiences a higher acceleration towards the moon than the opposite side of the earth. Also the upper and lower side have a component of their acceleration inwards resulting in the gravitational quadrupolar shape bulge of water we experience as the tides in the sea, see Fig. 11.

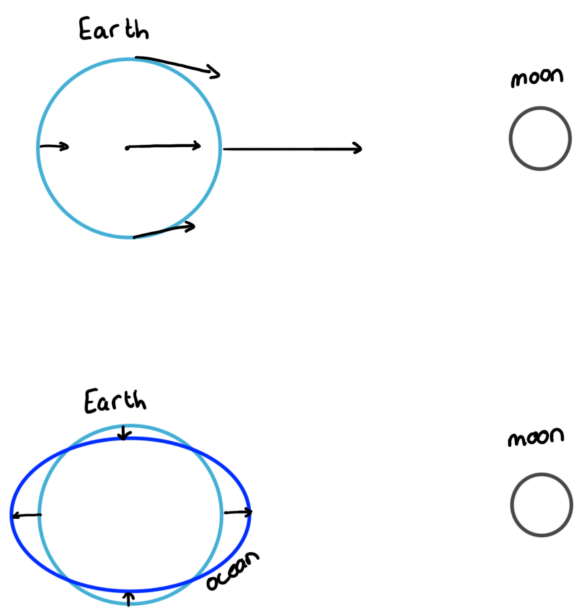


Figure 11: Tidal forces and tidal bulge in earth moon system as often explained in Newtonian gravity

In this thesis we will actually not focus on gravitational tidal effects but on tidal effects coming purely from the scalar fields around black holes in sGB gravity. However as this is less intuitive, it is still interesting to first take a look at gravitational tidal effects as a lot of the processes can be taken over to the scalar field case.

4.1 Gravitational tidal effects

To start, it is interesting to find out why the gravitational tides are quadrupolar effects. In Newtonian mechanics one can describe tidal effects as follows. The Newtonian gravitational potential is given by the solution to the Poisson equation

$$\nabla^2 U_A = -4\pi\rho_A \rightarrow U_A(t,\mathbf{x}) = \int d^3\mathbf{x}'\rho_A\left(t,\mathbf{x}'\right)\frac{1}{|\mathbf{x}-\mathbf{x}'|}, \tag{4.1}$$

with U_A the gravitational potential of body A and ρ_A the mass density. For positions far away from the source $\mathbf{x} > \mathbf{x}'$, one can Taylor expand around $\mathbf{x}'^i = 0$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{n^i x'^i}{r^2} + \frac{3}{2}\frac{n^i n^j - \delta^{ij}/3}{r^3}x'^i x'^j + \ldots, \tag{4.2}$$

with $r = |\mathbf{x}-\mathbf{x}'|$ and $n^i = \frac{\mathbf{x}-\mathbf{x}'}{r}$ the unit vector. We will use the notation $n^i n^j - \delta^{ij}/3 = n^{<i}n^{j>}$ with $<>$ denoting symmetric trace free tensors (STF tensors). These tensors are symmetric under exchange of any two indices and vanish when taking to trace to any index. One can use a basis from STF tensors to construct these multipole expansions, which is equivalent to using a basis of spherical harmonics [89]. Substituting Eq. (4.2) in Eq.(4.1) gives

$$U_A = \frac{1}{r}\int \rho_A(t,\mathbf{x}')d^3\mathbf{x}' + \frac{n^i}{r^2}\int \rho_A(t,\mathbf{x}')x'^i d^3\mathbf{x}' - \frac{3}{2}\frac{n^{<i}n^{j>}}{r^3}\int \rho_A(t,\mathbf{x}')x'^{<i}x'^{j>}d^3\mathbf{x}' + \ldots. \tag{4.3}$$

then similar to Eq. (2.40) we define the multipole moments

$$m_A = \int_A d^3x\rho_A(t,x), \quad Q_A^{ij} = \int_A d^3x\rho_A(t,x)x'^{<i}x'^{j>}, \tag{4.4}$$

the monopole moment giving the mass of the body and the quadrupole moment Q^{ij}. The dipole moment

$$Q_A^i = \int \rho_A(t,\mathbf{x}')x'^i d^3\mathbf{x}' \tag{4.5}$$

is proportional to the centre of mass as this is defined as

$$x_{CM} = \frac{1}{m_A}\int d^3\mathbf{x}\rho_A(t,\mathbf{x})x^i. \tag{4.6}$$

By setting our origin $\mathbf{x}'^i$ in the centre of mass, this dipole term vanishes. This results in the expansion of Eq.(4.3)

$$U_A = \frac{m_A}{r} + \frac{3}{2}\frac{1}{r^3}Q^{ij}n^{<i}n^{j>} + \ldots. \tag{4.7}$$

Now we can assume that our body A is in the external gravitational field of a companion body with the distance between the bodies much larger than their characteristic size. The potential that is felt by body A because of external sources can be Taylor expanded around the origin as well

$$U_A^{ext}(t,\mathbf{x}) = U_A^{ext}(t,\mathbf{x}) + \frac{\partial U_A^{ext}}{\partial x^i}\,|_O\, x^i - \frac{1}{2}\mathcal{E}_{ij}x^i x^j + \mathcal{O}\left(r^2\right), \tag{4.8}$$

with

$$\mathcal{E}_{ij} = -\frac{\partial^2 U_A^{ext}}{\partial x^i \partial x^j} |_O, \tag{4.9}$$

the tidal field. Without loss of generality we can set the constant value in Eq. (4.8) to zero and if our origin is set to the centre of mass, the dipole term vanishes as before. Furthermore as in the origin the external field does not have a source (source of this potential is the companion body but we are considering the field at body A), the Poisson equation for this external potential is $\nabla^2 U_A^{ext} = \delta^{ij}\mathcal{E}_{ij} = 0$. This makes the tidal field trace free and as it is already symmetric in its indices we can write

$$U_A^{ext} = -\frac{1}{2}\mathcal{E}_{ij}x^{<i}x^{j>} = -\frac{1}{2}\mathcal{E}_{ij}n^{<i}n^{j>}r^2. \tag{4.10}$$

Thus the total potential is given by

$$U_A = \frac{m_A}{r} + \frac{3}{2}\frac{1}{r^3}Q_{A,ij}n^{<i}n^{j>} + \mathcal{O}\left(\frac{1}{r^4}\right) - \frac{1}{2}\mathcal{E}_{ij}r^2 n^{<i}n^{j>} + O\left(r^3\right). \tag{4.11}$$

Next, we construct the Lagrangian of the binary system from $\mathcal{L} = T - V$ with $T = T_A + T_B$ the total kinetic energy and $V = V_A + V_B$ the total potential energy. Writing the energies in terms of the motion of the centre of mass z_A and the internal part as done in [90]

$$T_A = \frac{1}{2}\int_A d^3\mathbf{x}\rho_A \dot{\mathbf{z}}_A^2 + T_A^{\text{int}}, \quad V_A = \frac{1}{2}\int_A d^3\mathbf{x}\rho_A U_{ext} + V_{\text{int}} = \frac{1}{2}\int_A d^3\mathbf{x}(\frac{1}{2}Q^{ij}\mathcal{E}_{ij}U_{ext}) + ... + V_{\text{int}}. \tag{4.12}$$

We can assume for body A that companion B is a point mass and vice versa. The linear order contribution, if both bodies would be extended objects, can be extracted from adding both contributions. Then moving to the CM frame of the binary system gives

$$T = \frac{1}{2}\mu v^2 + T_{\text{int}}, \quad V = -\frac{\mu m}{r} + \frac{1}{2}Q_{ij}\mathcal{E}_{ij} + ... + V_{\text{int}}, \tag{4.13}$$

with the total mass $m = m_A + m_B$, the reduced mass $\mu = m_A m_B/m$, the separation $\mathbf{r} = \mathbf{z_A} - \mathbf{z_B}$ with magnitude r and relative velocity $v^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$. We can construct the Lagrangian and thus the action

$$S = S_{\text{orbit}} + \int dt \left[-\frac{1}{2}Q_{ij}\mathcal{E}_{ij} + ... + \mathcal{L}^{\text{int}}\right], \tag{4.14}$$

with $S_{\text{orbit}} = \int dt \mathcal{L}_{\text{orbit}}$, $\mathcal{L}_{\text{orbit}} = (\mu/2)v^2 + \mu M/r$ and $\mathcal{L}^{int}$ describing the internal dynamics. In the case of a spherical symmetric body in a binary system, the multipole moments of the potential are induced by the tidal field of the other body [91]. For a neutron star the Lagrangian for the dynamics of these induced moments is given by a tidally driven harmonic oscillator [92]

$$\mathcal{L}^{\text{int}} = \frac{1}{2}\left[\frac{1}{2\lambda w_0^2}\dot{Q}_{ij}\dot{Q}^{ij} - \frac{1}{2\lambda}Q_{ij}Q^{ij} + ...\right], \tag{4.15}$$

with w_0 the oscillation mode frequencies and λ the tidal deformability parameter. The tidal deformability parameter is often considered in the adiabatic limit; assuming the internal timescale being much faster than the time scale of the variations in the tidal field. Subsituting the Lagrangian Eq. (4.15) back into the action Eq. (4.14) and varying with respect to the quadrupole moment gives the following equation of motion for the quadrupole

$$\ddot{Q}^{ij} + \omega_2^2 Q^{ij} = -\lambda \omega_2^2 \mathcal{E}^{ij}. \tag{4.16}$$

In the adiabatic limit for which the second derivative becomes negligible, we have the solution

$$Q_{ij} = -\lambda \mathcal{E}_{ij}. \tag{4.17}$$

Showing that indeed the quadrupole moment is induced by the quadrupole tidal field. The tidal deformability parameter λ describes the strength of the response to the tidal field. Following the same calculation with general expansion of the potential up to general order in the multipoles gives this relation for general order multipoles, each with a different λ. Usually these higher multipoles are subdominant over the quadrupole contribution.

However what you might have noticed that everything up to now was done for Newtonian gravity. In GR, similar to expanding around the CM in Newtonian dynamics, one can define the gravitational potential around a reference centre of mass worldline [93]. At the scales of large distances outside the body compared to the size of the object but small compared to the radius of curvature of the gravitational field of the companion, the multipole moments can be determined by expanding the asymptotic metric in a local asymptotic rest frame [90]. The lowest order in this expansion corresponds to a point particle description. For a non rotating black hole this would mean that it is only described by its mass. This expansion method is called skeletonization. We discuss skeletonization again in section 5.1.1 specific for scalarized black holes in sGB gravity in which case the scalar field dependency can be described as funcion of the mass.
As an example of the expansion, the 00 component of the metric in Schwarzschild coordinates for a black hole in an external tidal field of its companion is given by

$$g_{00} = -1 + \frac{2m}{r} + \frac{3}{r^3} Q_{ij} n^{<i} n^{j>} + O\left(\frac{1}{r^3}\right) - \mathcal{E}_{ij} r^2 n^{<i} n^{j>} + O\left(r^3\right), \tag{4.18}$$

which is the same expansion as for our Newtonian potential above.
To derive the Lagrangian in GR we can go back to the action in Eq. (4.14) and use the minimal coupling principle [30](see also Appendix D) and put in in the covariant form. The corresponding Lagrangian is then given by[94],

$$\mathcal{L} = \frac{z}{4\lambda}\left[\frac{1}{z^2\omega_f^2}\frac{DQ_{\mu\nu}}{d\sigma}\frac{DQ^{\mu\nu}}{d\sigma} - Q_{\mu\nu}Q^{\mu\nu}\right] - \frac{z}{2}E_{\mu\nu}Q^{\mu\nu}, \tag{4.19}$$

with D a covariant derivative, $z = \sqrt{-u^\mu u_\mu}$ ensuring invariance under different parametrizations, $u^\mu = \dot{y}^\mu$ with wordline $y^\mu(\sigma)$ and σ the worldline parameter.
In the case of GR, the tidal effects are now described by the curvature of spacetime and the tidal field in this case is defined as projections of the curvature tensor on the worldline, expressed as projections of the Weyl tensor corresponding to the companion body. The Weyl tensor is the trace free part of the riemann curvature tensor.

$$\mathcal{E}_{\mu\nu} = z^{-2} C_{\mu\alpha\nu\beta} u^\alpha u^\beta \tag{4.20}$$

As this expression is symmetric under exchange of indices, the tidal field is an STF tensor. In the Newtonian limit the Weyl tensor is related to the gravitational potential and the tidal field reduces to Eq. (4.9).
Starting from the relativistic action Eq. (4.19), one can derive that in the adiabatic limit the relation Eq. (4.17) still holds in GR but the spatial indices replaced with spacetime indices. What is also different from the Newtonian case is that in GR there is an additional tidal field due to frame dragging effects. This field is called the "gravitomagnetic" tidal field which induces current multipole moments. This effect does not play a further role in the scope of this thesis.

4.2 Scalar tidal effects

In our situation we are considering two black holes which do not contain water as on the earth, or at all consists of matter that can deform as for neutron stars. Hence these gravitational tidal effects do not deform the black holes. What is possible is that in sGB, black holes can have a non zero scalar field. The gradient in the scalar field of the companion black hole can exert tidal effects on the scalar field of the other black hole in a similar way as for gravitational tidal effects. Therefore we can use many of the expressions from the previous section for the scalar field case. However one important difference is that, when expanding the scalar field as a Taylor expansion around the origin, the dipole moment does not vanish as we will see explicitly in section 6. Therefore the lowest order tidal effects in the case of the scalar field are coming from dipolar effects. Also the Lagrangian for these effects can again be described as a tidally driven harmonic oscillator and thus with Eq. (4.19) for the scalar dipole moment $Q_\mu^{(s)}$. We denote scalar field related terms with the superscript (s). As we saw before, in the adiabatic limit one can derive the relation between the tidal field and the in this case now the dipole moment

$$Q_\mu^{(s)} = -\lambda_{(s)}\mathcal{E}_\mu^{(s)}, \tag{4.21}$$

with λ_s the scalar tidal deformability parameter. The scalar tidal effects are generated by the gradient in the scalar field of the companion, similarly as in the Newtonian case for the Newtonian gravitational potential. The scalar tidal field is therefore expressed as

$$\mathcal{E}_\mu^{(s)} \equiv \partial_\mu\varphi. \tag{4.22}$$

Also when plugging Eq. (4.21) back in the Lagrangian one gets the Lagrangian in the adiabatic limit for the scalar tidal effects

$$\mathcal{L}_{tid} = \frac{\lambda_{(s)}}{2}\mathcal{E}_\mu^{(s)}\mathcal{E}^\mu_{(s)}. \tag{4.23}$$

Together with Eq. (4.22) we have the action describing the scalar tidal effects of body A in the tidal field of B and vice versa

$$S_{tid} = -\sum_{A\neq B}\frac{1}{2}\lambda_A^{(s)}\int \mathrm{d}s_B c\left(g^{\mu\nu}\right)_B\left(\partial_\mu\varphi\right)_B\left(\partial_\nu\varphi\right)_B, \tag{4.24}$$

with

$$ds_B = \sqrt{-g_{\alpha\beta} dx_B^\alpha dx_B^\beta} = \sqrt{-g_{\alpha\beta} \frac{dx_B^\alpha}{dt} \frac{dx_B^\beta}{dt}} dt, \tag{4.25}$$

defined as the evolution along the wordline. This is the action we can use to incorporate the scalar tidal effects in the orbital dynamics of the BH system in section 5. We will also include the tidally induced dipole moment Q_μ in the expansion of the scalar field in section 7, to incorporate the effects on the scalar waveform. To actually analyse these effects we need to know the expression for the scalar tidal deformability parameter which we calculate in section 6.

An interesting thing to note is that we derived this action now specifically describing the scalar tidal effects. These effects fall under a broader category of finite size effects; effects coming from higher order terms of the expansion in the derivatives of the scalar field and the metric along the worldline. In [95] they show that when starting from the most general action, considering second order terms in the expansion, up to two derivatives to the scalar field and metric, only the scalar tidal field terms are nonvanishing after making use of the gauge symmetries of the system. These are exactly the tidal terms we described in this section.

5 Binary dynamics

After our general discussion on GWs, modified gravity and tidal effects we can now specify to the calculations on gravitational radiation from a black hole binary system in sGB gravity. For this we start with a calculation of the binary dynamics; the Lagrangian and energy of the inspiralling black hole system. The changes in the dynamics during the inspiral carry through in the gravitational radiation that is sent into space, therefore we need to look at the system itself first.
We start with the sGB, matter and tidal action and use the variational principle to derive from it the equations of motion. Then we expand these equations of motion for small tidal effects and in the PN expansion. We solve the equations order by order up to linear order in λ_s and to 1PN. With these solutions we can construct the metric and substitute in the action to derive the Lagrangian. Subsequently the binding energy up to 1PN can be derived from the Lagrangain. The binding energy as a function of the frequency for circular orbits is a gauge invariant quantity, for which we analyse the effects of including the scalar tidal effects on the system.

5.1 The action

The total action of a binary black hole system is described by the gravitational action of sGB gravity[16, 17], the matter action and the tidal action

$$S = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}[R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \alpha f(\varphi)R^2_{GB}] + S_m + S_{tidal}. \tag{5.1}$$

Where we have the Ricci scalar R, the metric $g_{\mu\nu}$, the scalar field φ which is non minimally coupled via a coupling function to the quadratic curvature term given by the GB invariant Eq. (3.2). We will use a different way of writing this invariant term

$$R^2_{GB} = {}^*R^*_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{5.2}$$

The double dual of the Riemann tensor is defined as

$${}^*R^*_{c\mu\nu d} = \frac{1}{4}\epsilon^{abef}R_{efgh}\epsilon^{ghcd}, \tag{5.3}$$

with ϵ^{ghcd} being the anti symmetric Levi Civita tensor. We will discuss the matter action of Eq. (5.1) below and the tidal action is given by Eq. (4.24).

In this calculation we keep the coupling function general but as discussed in section 3.3 there are different types of functions. However as we are interested in the effect of including tidal terms because of the scalar hair on the black holes, we assume a coupling function for which the scalar hair does not vanish and scalar tidal effects are present. We specify specific coupling functions for the analysis in sections 5.8, and 7.5.

5.1.1 Skeletonization and the matter action

The system we are considering consists two non spinning black holes. What kind of black holes are considered keeps being general in this calculation, however for the analysis part we assume stellar mass black holes. These are black holes formed from gravitational collapse of a star. The masses of these black holes typically range from order 5 to tens of solar masses[15]. Black holes are vacuum solutions of the Einstein equations (see Appendix D) and are not made of any matter, only purely described by the curvature of space time. Still we would like to describe them with a "matter" action, how is this valid?
Hints are given by the no hair theorem described in section 3.3. The black holes are solely described by their mass, charge and angular momentum. As we are talking about non spinning chargeless black holes, we only have to care about the mass. This is similar to electrons[16] which we can describe well with a point particle description. Furthermore the external gravitational field of a black hole is the same as would be for a point particle of similar mass. Lastly for our calculations we only consider the inspiral part of the coalescence, when the black holes are far away from each other, relative to the scales of the system one can therefore treat them approximately as point particles.
The formal route to this approximation is that similar as in Newtonian mechanics, one can do a expansion around the centre of mass, which in GR would be considered a CM wordline, as a multipole expansion. Elaborate but dense details on this can be found in the analysis of W.G. Dixon [93]. Then one could argue that the lowest order, corresponding to a point particle approximation, is accurate enough. The procedure of reducing the description of the black holes to an effective point particle description is called *skeletonization*, see Fig. 12. The action of a point particle is given by $S_{pp} = -c\int ds\, m,$
with m the mass of the particle.

In our modified gravity context however we assume the black holes also have scalar hair, violating the no hair theorem. We therefore need to implement this scalar field in the skeletonization description. In 1975 Eardley [97] was the first to generalize this description for modified gravity, more specific for scalar tensor theory. In [98] it was effectively implemented for Einstein-Maxwell dilation theories and by among others the same author also for sGB gravity [24]. In this description the scalar field can be implemented by letting the mass depend on the scalar field. For this the mass function $m(\varphi)$ is introduced. The idea behind this is that if one expands the metric in a PN expansion and looks at the 00 component, to lowest order the Poisson equation from Newtonain gravity should be recovered Eq. (4.1) (as we saw too in the end of section 2.4.3 for GR). However if one does this for scalar tensor theory one recovers Eq. poissoneq with an altered effective Newtonian constant G_{eff}, which depends on the scalar field. The value of the G_{eff} can therefore vary in this theory. According to the equivalence principle, the gravitational mass should be equal to the inertial mass. However if G_{eff} can vary with the scalar field then the inertial mass m will too. Therefore making the mass a function of the scalar field can describe the contribution of the scalar field in the skeletonization procedure. A caveat is that in sGB

[15] As opposed to the hypothesized primordial black holes that are formed from dark matter overdensities in the early universe [96]; these black holes can have a smaller masses.

[16] In the case of electrons charge does matter of course.

gravity we do recover the Poisson equation to lowest order. Still the scalar field is part of the gravitational sector and hence plays a role in the gravitational mass, which becomes a scalar-field-dependent mass function.

To sum up, on the orbital scale we describe the black holes in the binary system as two point particles with their mass depending on the scalar field. They form the "matter" in this description given by the point particle action. For now we will keep the number of bodies in the system general but when calculating the Lagrangian we specify to the two bodies in the binary. Thus the matter action becomes

$$S_m = -c \sum_A \int ds_A m_A(\varphi), \tag{5.4}$$

with ds_A defined as Eq. (4.25).

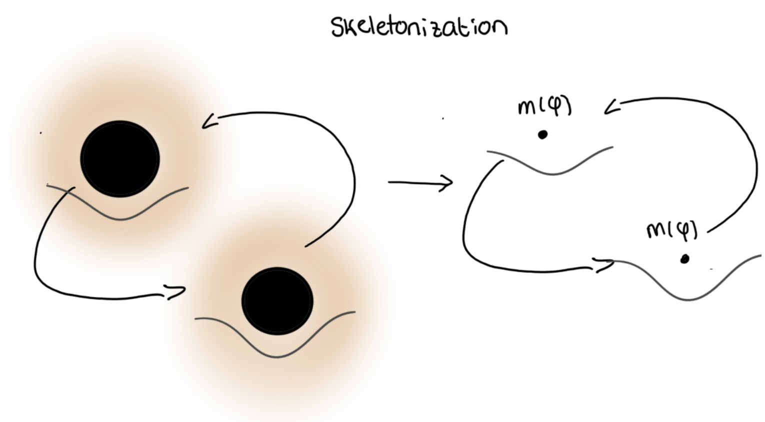


Figure 12: Sketch of skeletonization

5.2 The field equations

To get the equations of motion, we vary the total action Eq. (5.1) with respect to the metric and the scalar field. The variation to the metric, after quite some rewriting of the variation of the GB invariant [17] results in

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{c^4}{16\pi G}\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 2\nabla_\mu\varphi\nabla_\nu\varphi + g_{\mu\nu}\nabla_\rho\varphi\nabla^\rho\varphi + \alpha\epsilon^{\lambda\omega}_{\rho\mu}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\nu\alpha}\nabla^\alpha\nabla^\rho f(\varphi)\right]$$
$$- \frac{\delta S_m}{\delta g^{\mu\nu}} - \frac{\delta S_{tidal}}{\delta g^{\mu\nu}} = 0. \tag{5.5}$$

[17] For explicitly varying the curvature terms in the GB invariant, one can use the expressions in Appendix D. Doing this gives a whole string of terms which can conveniently be rewritten in the form of the term proportional to the coupling constant in Eq. (5.5). In Appendix F we link to a mathematica notebook by L. Stein which describes this rewriting.

Thus for the field equations we have

$$\underbrace{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R}_{GR} = 2\partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi - \alpha\epsilon_{\rho\mu}{}^{\lambda\omega}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}{}_{\nu\alpha}\nabla^\alpha\nabla^\rho f(\varphi) + \underbrace{\frac{16\pi G}{c^4}\frac{1}{\sqrt{-g}}}_{GR}(\frac{\delta S_m}{\delta g^{\mu\nu}} + \frac{\delta S_{tidal}}{\delta g^{\mu\nu}}). \tag{5.6}$$

We see that sGB is indeed a modified gravity theory, it modifies the Einstein equations with extra terms: respectively the kinetic terms in the scalar field, a term proportional to the coupling constant which in further reference we will call the GB term and the last term includes the tidal effects. Here we also see in practice that the GB invariant term cancels higher derivative terms of the metric and thus only second order derivative terms to the metric in the equations of motion.

It is more convenient to have this expression in the trace reversed form. Therefore we first take the trace by multiplying with $g^{\mu\nu}$ resulting in an expression for R, after relabelling its indices we substitute this expression back into Eq. (5.6) which gives

$$R_{\mu\nu} = 2\nabla_\nu\varphi\nabla_\mu\varphi + \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\alpha\epsilon^{\lambda\omega}_{\rho\alpha}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\beta\xi}\nabla^\xi\nabla^\rho f(\varphi) - \alpha\epsilon^{\lambda\omega}_{\rho\mu}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\nu\xi}\nabla^\xi\nabla^\rho f(\varphi) + \frac{8\pi G}{c^4}\left(T^m_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^m\right) + \frac{8\pi G}{c^4}\left(T^{tidal}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{tidal}\right), \tag{5.7}$$

with

$$T^m_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}, \quad T^{tid}_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{tidal}}{\delta g^{\mu\nu}}, \tag{5.8a}$$

and

$$T^m = g^{\alpha\beta}T^m_{\alpha\beta}, \quad T^{tid} = g^{\alpha\beta}T^{tid}_{\alpha\beta}. \tag{5.9a}$$

The energy momentum tensor and its trace corresponding to the matter and tidal action.

We checked that this is consistent with equation D.1a in [24] by rewriting their GB term in the field equations.
From variation with respect to the scalar field follows

$$\Box\varphi = -\frac{1}{4}\alpha f'(\varphi)R^2_{GB} - \frac{4\pi G}{c^4}(\bar{\delta}S_m + \bar{\delta}S_{tidal}), \tag{5.10}$$

with

$$\bar{\delta}S_m = \frac{1}{\sqrt{-g}}\frac{\delta S_m}{\delta\varphi}, \quad \bar{\delta}S_{tidal} = \frac{1}{\sqrt{-g}}\frac{\delta S_m}{\delta\varphi}, \tag{5.11a}$$

and $\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ the d'Alembertian operator.

To actually vary the matter and tidal action in Eq. (5.8) and Eq. (5.11) we substitute in these definitions the actions Eq. (5.4) and Eq. (4.24). The energy momentum tensor describes the matter distribution. As we describe our black holes as point particles, we want to localize the energy momentum tensor on the positions of the black holes. Therefore we introduce a delta function to localize the integrals in the actions on the location $x_A(t)$ of the black hole. We put the actual variation to the metric of this terms in the appendix A as the expressions are quite long. Here we give the results. With Eq. (A.1) and Eq. (A.2) energy momentum tensors from Eq. (5.8) become

$$T^m_{\mu\nu} = c\sum_A m_A(\varphi)\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\frac{\frac{dx^A_\mu}{dt}\frac{dx^A_\nu}{dt}}{\sqrt{gg_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}}, \tag{5.12a}$$

$$T^{tidal}_{\mu\nu} = c\sum_A \lambda^{(s)}_A\left[\partial_\mu\varphi\partial_\nu\varphi\sqrt{\frac{g_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}{g}} + g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi\frac{\frac{dx^A_\mu}{dt}\frac{dx^A_\nu}{dt}}{2\sqrt{gg_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}}\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)). \tag{5.12b}$$

Then for the expressions in Eq. (5.11), with Eq. (A.3) and Eq. (A.4) we have

$$\bar{\delta}S_m = -c\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\frac{dm_A(\varphi)}{d\varphi}\sqrt{\frac{g_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}{g}}, \tag{5.13a}$$

$$\bar{\delta}S_{tidal} = \sum_{A\neq B}\lambda^{(s)}_B\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))c\left[g^{\mu\nu}\partial_\mu\partial_\nu\varphi\sqrt{\frac{g_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}{g}} + \frac{\partial_\nu(\sqrt{-g_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}g^{\mu\nu})}{\sqrt{-g}}\partial_\mu\varphi\right]. \tag{5.13b}$$

Next we will expand both sides of the equations of motion from Eq. (5.7) and Eq. (5.10) for weak gravitational field and non relativistic velocities in a Post Newtonian expansion. Solving the equations to linear order and up to 1PN.

5.3 Post Newtonian expansion of the sGB field equations

To solve the equations of motion in a similar way as discussed in section 2.4 for GR, in a gravitationally bound system we need to go beyond linearized gravity and use a PN expansion.

For the PN expansion the small parameter is given by $\varepsilon_{PN} = \frac{R_s}{d} \approx \frac{v^2}{c^2}$. In practice this comes down to tracking the factors of $1/c^2$, hence in our calculation we expand explicitly in these factors. Then we solve the differential equations perturbatively per order in the expansion.

5.3.1 PN expansion of the fields

Before we can expand the expressions in the field equations in Eq. (5.7) and Eq. (5.10), we first expand the fields U and g_i, which were defined in the PN expansion of the metric in Eq. (2.81) and the scalar field.

The PN expansion contains quite some subtleties which can make tracking the order right be a daunting task. It turns out[18] that the most consistent way of tracking the PN orders and solving order per order happens when tracking and expanding in factors of $1/c^2$. Therefore we omit the parameter ε_{PN} and just track the factors of $1/c^2$. When solving the equations per expansion order we link back to actual PN orders. Expanding to 1PN corresponds to expanding to $1/c^4$, see Eq. (2.81).

For the U and g_i fields this results in

$$\frac{U}{c^2} = \frac{U^{(0)}}{c^2} + \frac{U^{(1)}}{c^4} + \mathcal{O}(1/c^6), \tag{5.14}$$

$$\frac{g_i}{c^3} = \frac{g_i^{(0)}}{c^3} + \mathcal{O}(1/c^5). \tag{5.15}$$

For the scalar field we use a slightly different approach, combining the convention from [24], defining the scalar field as a background field plus perturbations

$$\varphi = \varphi_0 + \delta\varphi, \tag{5.16}$$

for which we PN expand the perturbations. In the expansions the scalar field dependent mass $m(\varphi)$ and the coupling function $f(\varphi)$ are expanded around this background value φ_0. For the PN expansion of the perturbations we redefine the field

$$\delta\varphi = \frac{\varphi_c}{c^2}, \tag{5.17}$$

including the explicit $1/c^2$. It turns out that this is more convenient to work with as the U and g_i field are also defined with this explicit factor of $1/c^2$, and as combinations of these fields arise during the calculation, they are in this way treated at equal footing. During the calculation we will therefore express everything in terms of φ_c and when we look at the solutions for the EOMs we will translate back to the actual scalar field solution. The expansion of this redefined scalar field perturbations is given by

$$\frac{\varphi_c}{c^2} = \frac{\varphi_c^{(0)}}{c^2} + \frac{\varphi_c^{(1)}}{c^4} + \mathcal{O}(1/c^6). \tag{5.18}$$

Now we can expand the expressions in the field equations and define the EOMs per expansion order, matching the same factors of $1/c^2$.

5.3.2 Expanding the scalar field equation

We start with the PN expansion of the scalar field equation, which in the redefined field φ_c is given by

[18] We explain the reasons after this calculation, see 5.4.4

$$\Box\frac{\varphi_c}{c^2} = -\frac{1}{4}\alpha f'(\varphi)R^2_{GB} - \frac{4\pi G}{c^4}(\bar{\delta}S_m + \bar{\delta}S_{tid}). \tag{5.19}$$

Here on the LHS we only include the scalar field perturbations Eq. (5.17) as the background field is unperturbed.

We look at the expansions of the terms on the RHS of Eq. (5.19). We have the expansion of the coupling function using Eq. (5.16) and Eq. (5.18)

$$f'(\varphi) = f'(\varphi_0) + f''(\varphi_0)\frac{\varphi_c^{(0)}}{c^2} + f''(\varphi_0)\frac{\varphi_c^{(0)}}{c^4} + \mathcal{O}(\frac{1}{c^6}). \tag{5.20}$$

We expand the other quantities in the EOM in the same way. For the box operator this results in

$$\Box = \Box^{(0)} + \frac{1}{c^2}\Box^{(1)} + \frac{1}{c^4}\Box^{(1)} + \mathcal{O}(\frac{1}{c^6}), \tag{5.21}$$

where the (0) order term corresponds to the flat space d'Alembertian $\Box_\eta$.

For the expansion of the expressions R^2_{GB} , $\bar{\delta}S_m$ and $\bar{\delta}S_{tidal}$ we substitute the PN expansion of the metric Eq. (2.81) and the expansion of the scalar field φ_c from Eq. (5.18) and gravitational fields U Eq. (5.14) and g_i Eq.(5.15)) and expand in $1/c^2$. We have to pay attention to the prefactor $4\pi G/c^4$ in Eq. (5.19), as this already carries the factor of $1/c^4$. Again writing out explicitly these expansions becomes a bit cumbersome so we show this in the Appendix A.

The expressions concerning the matter action Eq. (5.4) and Eq. (5.13a) contain the mass function depending on the scalar field which is therefore also expanded in the following way [24]

$$\begin{aligned} m_A(\varphi) &= m_A(\varphi_0) + m'_A(\varphi_0)\frac{\varphi_c^{(0)}}{c^2} + \frac{1}{2}m''{}_A(\varphi_0)\frac{\varphi_c^{(1)}}{c^4} + \mathcal{O}(1/c^6) \\ &= m_A^0\left[1 + \alpha_A^0\frac{\varphi_c^{(0)}}{c^2} + \frac{1}{2}\left(\alpha_A^{0\,2} + \beta_A^0\right)\frac{\varphi_c^{(1)}}{c^4}\right] + \mathcal{O}\left(1/c^6\right), \end{aligned} \tag{5.22}$$

with $m_A^0 = m_A(\varphi_0)$ and

$$\begin{aligned} \alpha_A(\varphi) &\equiv \frac{d\ln m_A(\varphi)}{d(\varphi)} \\ \beta_A(\varphi) &\equiv \frac{d\alpha_A(\varphi)}{d(\varphi)}, \end{aligned} \tag{5.23}$$

with $\alpha_A^0 = \alpha_A(\varphi_0)$ called the *scalar charge*, measuring the strength of the coupling of the physical mass to the back ground scalar field. In the same manner we define $\beta_A^0 = \beta_A(\varphi_0)$. For the expansion of the GB invariant we used the Mathematica package xAct (see Appendix F) to calculate the curvature expressions and to substitute the expanded metric components from Eq. (2.81). Looking at the lowest order terms in $\frac{1}{c^2}$ results in

$$R^2_{GB} = 8\left((\partial_i\partial_j\frac{U^{(0)}}{c^2})(\partial_i\partial_j\frac{U^{(0)}}{c^2}) - \triangle\frac{U^{(0)}}{c^2}\triangle\frac{U^{(0)}}{c^2}\right) + \mathcal{O}(1/c^6), \tag{5.24}$$

and thus this term contributes $R_{GB}^{2\,(1)}$ at order $\mathcal{O}(1/c^4)$, $R_{GB}^{2\,(0)}$ vanishes.

The explicit expansion of $\bar{\delta}S_m$ in Eq. (A.20) results in

$$\begin{aligned}\bar{\delta}S_m = -\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\,\big[c^2\alpha_A^0 m_A^0 + (m_A^0((\alpha_A^0)^2+\beta_A^0)\varphi_c^{(0)} \\ -\alpha_A^0 m_A^0 U^{(0)} - \frac{1}{2}\alpha_A^0 m_A^0 v_A^2)\Big] + \mathcal{O}(1/c^2),\end{aligned} \tag{5.25}$$

giving a leading order contribution of $\bar{\delta}S_m^{(-1)}$ corresponding to order $\mathcal{O}(c^2)$.

The explicit expansion of $\bar{\delta}S_{tid}$ in Eq. (A.21) then gives

$$\bar{\delta}S_{tid} = \sum_A \lambda_A^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\Box\varphi_c^{(0)} + \mathcal{O}(1/c^2). \tag{5.26}$$

Thus resulting in only a contributing of $\bar{\delta}S_{tid}^{(0)}$ corresponding to order $\mathcal{O}(c^0)$.

Substituting these expansions for R_{GB}^2 , $\bar{\delta}S_m$, $\bar{\delta}S_{tidal}$ back in Eq. (5.19) for the different expansion orders, only looking at the equations with matching orders in the expansion parameters as we found above

$$\mathcal{O}(1/c^2)\quad \Box_\eta\frac{\varphi_c^{(0)}}{c^2} = -\frac{4\pi G}{c^4}(\bar{\delta}S_m^{(-1)}), \tag{5.27a}$$

$$\mathcal{O}(1/c^4)\quad \Box_\eta\frac{\varphi_c^{(1)}}{c^4} = -\frac{1}{4}\alpha f'\left(\frac{\varphi_c^{(0)}}{c^2}\right)R_{GB}^{2\,(1)} - \frac{4\pi G}{c^4}(\bar{\delta}S_m^{(0)} + \bar{\delta}S_{tid}^{(0)}). \tag{5.27b}$$

Substituting the expansions

$$\mathcal{O}(1/c^2)\quad \Box_\eta\frac{\varphi_c^{(0)}}{c^2} = \frac{4\pi G}{c^4}(\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))c^2\alpha_A^0 m_A^0), \tag{5.28a}$$

$$\begin{aligned}\mathcal{O}(1/c^4)\quad \Box_\eta\frac{\varphi_c^{(1)}}{c^4} = &-2\alpha f'\left(\frac{\varphi_c^{(0)}}{c^2}\right)\left((\partial_i\partial_j\frac{U^{(0)}}{c^2})(\partial_i\partial_j\frac{U^{(0)}}{c^2}) - \triangle\frac{U^{(0)}}{c^2}\triangle\frac{U^{(0)}}{c^2}\right) - \frac{4\pi G}{c^4} \\ &\left(-\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\left[(m_A^0((\alpha_A^0)^2+\beta_A^0)\varphi_c^{(0)} - \alpha_A^0 m_A^0 U^{(0)} - \frac{1}{2}\alpha_A^0 m_A^0 v_A^2)\right]\right. \\ &\left.+\sum_A \lambda_A^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\Box\varphi_c^{(0)}\right).\end{aligned} \tag{5.28b}$$

Next, we do the same for the gravitational equations of motion.

5.3.3 Expanding the R_{00} and R_{0i} components

Starting from the gravitational equation of motion

$$\begin{aligned} R_{\mu\nu} = 2\nabla_\nu\varphi\nabla_\mu\varphi + \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\alpha\epsilon^{\lambda\omega}_{\rho\alpha}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\beta\xi}\nabla^\xi\nabla^\rho f(\varphi) - \alpha\epsilon^{\lambda\omega}_{\rho\mu}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\nu\xi}\nabla^\xi\nabla^\rho f(\varphi) \\ + \frac{8\pi G}{c^4}\left(T^m_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^m\right) + \frac{8\pi G}{c^4}\left(T^{tidal}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{tidal}\right). \end{aligned} \tag{5.29}$$

We can split this in the different components. From solving these equations to 1PN we recover the dynamics of the metric, more specifically the fields we defined in our metric PN expansion U and g_i. It turns out that we only need the 00 component equation and $0i$ component equation to solve for these fields. This can be seen from expanding R_{00} and R_{0i} in terms of $1/c^2$ by substituting the PN expansion of the metric Eq. (2.81). This can be done with the mathematica package xAct (see Appendix F), which results in

$$\begin{aligned} R_{00} &= -3\partial_0\partial_0\frac{U}{c^2} - \partial_i\partial_i\frac{U}{c^2} - 4\partial_0\partial_i\frac{g_i}{c^3}, \\ R_{0i} &= -2\partial_i\partial_0\frac{U}{c^2} - 2(\partial_k\partial_i\frac{g_k}{c^3} - \partial_k\partial_k\frac{g_i}{c^3}). \end{aligned} \tag{5.30}$$

Using the harmonic gauge $\partial_\beta\sqrt{-g}g^{\alpha\beta} = 0$ which in the PN expansion becomes

$$\partial_0\frac{U}{c^2} = -\partial_k\frac{g_k}{c^3}, \tag{5.31}$$

results in

$$\begin{aligned} R_{00} &= \partial_0\partial_0\frac{U}{c^2} - \partial_i\partial_i\frac{U}{c^2} = -\Box_\eta\frac{U}{c^2} + \mathcal{O}(1/c^4), \\ R_{0i} &= 2\partial_k\partial_k\frac{g_i}{c^3} = 2\triangle\frac{g_i}{c^3} + \mathcal{O}(1/c^5). \end{aligned} \tag{5.32}$$

Hence solving the equations of motion for R_{00} and R_{0i} gives the solutions for the fields U and g_i.
Starting with the 00 component substituting Eq. (5.32)

$$\begin{aligned} R_{00} = -\Box_\eta\frac{U}{c^2} = 2\partial_0\frac{\varphi_c}{c^2}\partial_0\frac{\varphi_c}{c^2} + \frac{1}{2}g_{00}g^{\alpha\beta}\alpha\epsilon^{\lambda\omega}_{\rho\alpha}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\beta\xi}\nabla^\xi\nabla^\rho f(\varphi) - \alpha\epsilon^{\lambda\omega}_{\rho0}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{0\xi}\nabla^\xi\nabla^\rho f\left(\frac{\varphi_c}{c^2}\right) \\ + \frac{8\pi G}{c^4}\left(T^m_{00} - \frac{1}{2}g_{00}T^m\right) + \frac{8\pi G}{c^4}\left(T^{tid}_{00} - \frac{1}{2}g_{00}T^{tid}\right). \end{aligned} \tag{5.33}$$

We can analyse the RHS to see in what orders the expansion is going to be. In the first term we again immediately substituted the scalar field perturbations Eq. (5.17) as the derivatives acting on the background field vanish. This first term can be neglected as it would be at least of order $1/c^6$ (additional factor of $1/c^2$ comes from the two time derivatives) which is too high for a 1PN expansion.

We first look at the expansion in $1/c^2$ of the terms in the RHS of Eq. (5.33). Starting with the second and third term using for the derivatives on the coupling function the chain rule, followed by the lowest order expression in Eq. (5.20)

$$\begin{aligned}&\frac{1}{2}g_{00}g^{\alpha\beta}\alpha\epsilon^{\lambda\omega}_{\rho\alpha}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\beta\xi}\nabla^{\xi}\nabla^{\rho}f(\varphi)-\alpha\epsilon^{\lambda\omega}_{\rho 0}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{0\xi}\nabla^{\xi}\nabla^{\rho}f(\varphi)=\\&\left(\frac{1}{2}g_{00}g^{\alpha\beta}\alpha\epsilon^{\lambda\omega}_{\rho\alpha}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\beta\xi}-\alpha\epsilon^{\lambda\omega}_{\rho 0}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{0\xi}\right)\left(f''(\varphi_0)\partial^{\xi}\frac{\varphi_c^{(0)}}{c^2}\partial^{\rho}\frac{\varphi_c^{(0)}}{c^2}+f'(\varphi_0)\partial^{\xi}\partial^{\rho}\frac{\varphi_c^{(0)}}{c^2}\right).\end{aligned} \tag{5.34}$$

Now if ξ and ρ are both 0 the Levi Cevita tensor vanishes which leads to a trivial result. If one of the indices is temporal, the partial derivatives on the scalar field would give a factor of $1/c$ resulting in an odd power of c which does not match with the orders related to the expansion of U in Eq. (5.14) on the LHS of Eq. (5.33). Hence we consider both indices spatial. We expand the term inside the first set of brackets in Mathematica with the xAct package, substituting the PN expanded metric components in the Riemann tensor. At the lowest order in $1/c^2$ this gives

$$\frac{1}{2}g_{00}g^{\alpha\beta}\alpha\epsilon^{\lambda\omega}_{i\alpha}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\beta j}-\alpha\epsilon^{\lambda\omega}_{i0}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{0j}=-4\alpha(\delta_{ij}\partial_i\partial^i\frac{U^{(0)}}{c^2}-\partial_i\partial_j\frac{U^{(0)}}{c^2})+\mathcal{O}(1/c^2). \tag{5.35}$$

Thus in total the only part of this expression that contributes to the differential equations with the right order of $1/c^2$ is

$$\begin{aligned}&\frac{1}{2}g_{00}g^{\alpha\beta}\alpha\epsilon^{\lambda\omega}_{\rho\alpha}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\beta\xi}\nabla^{\xi}\nabla^{\rho}f(\varphi)-\alpha\epsilon^{\lambda\omega}_{\rho 0}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{0\xi}\nabla^{\xi}\nabla^{\rho}f(\varphi)=\\&4\alpha(\delta_{ij}\partial_i\partial^i\frac{U^{(0,0)}}{c^2}-\partial_i\partial_j\frac{U^{(0,0)}}{c^2})f'(\varphi_{c_0})\partial^{\xi}\partial^{\rho}\frac{\varphi_c^{(00)}}{c^2}.\end{aligned} \tag{5.36}$$

For the last two terms in Eq. (5.33) we have the following term regarding the EM tensors: $T_{00}-\frac{1}{2}Tg_{00}$. This can be rewritten using the PN expansion of the metric components Eq. (2.81) only to order c^0

$$\begin{aligned}T_{00}-\frac{1}{2}Tg_{00}&=g_{0\mu}g_{0\nu}T^{\mu\nu}-\frac{1}{2}Tg_{0\mu}g_{0\nu}g^{\mu\nu}=g_{00}g_{00}T^{00}-\frac{1}{2}g_{00}g_{00}g^{00}g_{\rho\sigma}T^{\rho\sigma}=T^{00}-\frac{1}{2}T^{00}+\frac{1}{2}T^{ii}\\&=\frac{1}{2}(T^{00}+T^{ii}).\end{aligned} \tag{5.37}$$

Then we are interested in the expansion in $1/c^2$ of T^{00} and T^{ii}. We write the actual expansions again in the Appendix A and give here the results. For the matter energy momentum tensors we have expanding as Eq. (A.22) and Eq. (A.23)

$$T^m_{00}=\sum_A(m_A^0c^2+m_A^0\alpha_A^0\varphi_c^{(0)}-m_A^0U^{(0)}+m_A^0\frac{1}{2}v_A^2)\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))+\mathcal{O}(1/c^2) \tag{5.38a}$$

$$T^m_{ii}=\sum_A m_A^0\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))v_A^2+\mathcal{O}(1/c^2). \tag{5.38b}$$

For the tidal energy momentum tensors using Eq. (A.24) and Eq. (A.25) we have

$$T_{00}^{tid} = \sum_A \lambda_A^{(s)} \left[c^2 \partial_0 \frac{\varphi_c^{(0)}}{c^2} \partial_0 \frac{\varphi_c^{(0)}}{c^2} + \frac{1}{2} c^2 g^{\rho\sigma} \partial_\rho \frac{\varphi_c^{(0)}}{c^2} \partial_\sigma \frac{\varphi_c^{(0)}}{c^2} \right] \delta^{(3)}(\mathbf{x} - \mathbf{x_A}(t)) + \mathcal{O}(1/c^4) \quad (5.39a)$$

$$T_{ii}^{tid} = \sum_A \lambda_A^{(s)} \left[c^2 \partial_i \frac{\varphi_c^{(0)}}{c^2} \partial_i \frac{\varphi_c^{(0)}}{c^2} \right] \delta^{(3)}(\mathbf{x} - \mathbf{x_A}(t)) + \mathcal{O}(1/c^4). \quad (5.39b)$$

The lowest order contribution to T_{00} starts at $\mathcal{O}(1/c^2)$ since expanding $g^{\rho\sigma}\partial_\rho \frac{\varphi_c^{(0)}}{c^2} \partial_\sigma \frac{\varphi_c^{(0)}}{c^2}$ leads to terms at this and higher orders.

Thus the differential equations from Eq. (5.33) after fully performing the PN expansions, turn into the following system of equations per order in the expansion

$$\mathcal{O}(1/c^2) \qquad -\Box_\eta \frac{U^{(0)}}{c^2} = \frac{4\pi G}{c^4} \left(\sum_A \delta^{(3)}(\mathbf{x} - \mathbf{x_A}(t)) m_A^0 c^2 \right) \quad (5.40a)$$

$$\mathcal{O}(1/c^4) \qquad -\Box_\eta \frac{U^{(1)}}{c^4} = \frac{4\pi G}{c^4} \left(\sum_A (m_A^0 \alpha_A^0 \varphi_c^{(0)} - m_A^0 U^{(0)} + m_A^0 \frac{3}{2} v_A^2) \delta^{(3)}(\mathbf{x} - \mathbf{x_A}(t)) \right) + 4\alpha(\delta_{ij}\partial_i\partial^i \frac{U^{(0)}}{c^2} - \partial_i\partial_j \frac{U^{(0)}}{c^2}) f'(\varphi_0) \partial^\xi \partial^\rho \frac{\varphi_c^{(0)}}{c^2}. \quad (5.40b)$$

(5.40c)

Next, we consider the expansions of the $0i$ component of the field equations given by

$$R_{0i} = 2\Delta \frac{g_i}{c^3} = 2\partial_0 \frac{\varphi_c^{(0)}}{c^2} \partial_i \frac{\varphi_c^{(0)}}{c^2} + \left(\frac{1}{2} g_{0i} g^{\alpha\beta} \alpha \epsilon_{\rho\alpha}^{\lambda\omega} R_{\lambda\omega\sigma\varepsilon} \epsilon_{\beta\xi}^{\sigma\varepsilon} - \alpha \epsilon_{\rho 0}^{\lambda\omega} R_{\lambda\omega\sigma\varepsilon} \epsilon_{i\xi}^{\sigma\varepsilon} \right) \nabla^\xi \nabla^\rho f(\varphi) + \frac{8\pi G}{c^4} \left(T_{0i}^m - \frac{1}{2} g_{0i} T^m \right) + \frac{8\pi G}{c^4} \left(T_{0i}^{tid} - \frac{1}{2} g_{0i} T^{tid} \right). \quad (5.41)$$

As in the PN expansion of the metric Eq. (2.81), we saw that g_i/c^3 is already at 1PN order we do not need to expand further. Analysing again the RHS to see in what orders the expansion is going to be, we see that the first term can be neglected again as it is of order $1/c^5$ and the LHS is of order $1/c^3$.
Again we look at the expansion in $1/c^2$ for the terms on the RHS of Eq. (5.41). Starting with the second term in the expression, rewriting the derivatives of the coupling function in the same way as in Eq. (5.36)

$$(\frac{1}{2} g_{0i} g^{\alpha\beta} \alpha \epsilon_{\rho\alpha}^{\lambda\omega} R_{\lambda\omega\sigma\varepsilon} \epsilon_{\beta\xi}^{\sigma\varepsilon} - \alpha \epsilon_{\rho 0}^{\lambda\omega} R_{\lambda\omega\sigma\varepsilon} \epsilon_{i\xi}^{\sigma\varepsilon}) \nabla^\xi \nabla^\rho f(\varphi) = (\frac{1}{2} g_{0i} g^{\alpha\beta} \alpha \epsilon_{\rho\alpha}^{\lambda\omega} R_{\lambda\omega\sigma\varepsilon} \epsilon_{\beta\xi}^{\sigma\varepsilon} - \alpha \epsilon_{\rho 0}^{\lambda\omega} R_{\lambda\omega\sigma\varepsilon} \epsilon_{i\xi}^{\sigma\varepsilon}) \left(f''(\varphi_0) \partial^\xi \frac{\varphi_c^{(0)}}{c^2} \partial^\rho \frac{\varphi_c^{(0)}}{c^2} + f'(\varphi_0) \partial^\xi \partial^\rho \frac{\varphi_c^{(0)}}{c^2} \right). \quad (5.42)$$

The term proportional to the first derivative of the coupling function is of order $\mathcal{O}(1/c^2)$ if ξ and ρ are spatial or of order $\mathcal{O}(1/c^3)$ if one of them is 0 and the other spatial. The term proportional to the second derivative of the coupling function is at least order of $\mathcal{O}(1/c^4)$ which is in both cases of higher order than we are considering for Eq. (5.41). We can calculate the expansion of both index combinations in Mathematica with xAct which gives

$$\frac{1}{2}g_{0i}g^{\alpha\beta}\alpha\epsilon^{\lambda\omega}_{j\alpha}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\beta k}-\alpha\epsilon^{\lambda\omega}_{j0}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{ik}=4\alpha(-2\partial_k\partial_i\frac{g_j}{c^3}+2\partial_k\partial_j\frac{g_i}{c^3}-\partial^i\partial_j\frac{g_i}{c^3}\delta_{ik}+2\partial^i\partial_i\frac{g_j}{c^3}\delta_{ik} + \partial^j\partial_i\frac{g_j}{c^3}\delta_{jk}-2\partial^j\partial_j\frac{g_i}{c^3}\delta_{jk})+\mathcal{O}(1/c^5) \tag{5.43a}$$

$$\frac{1}{2}g_{0i}g^{\alpha\beta}\alpha\epsilon^{\lambda\omega}_{j\alpha}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{\beta 0}-\alpha\epsilon^{\lambda\omega}_{j0}R_{\lambda\omega\sigma\varepsilon}\epsilon^{\sigma\varepsilon}_{i0}=-4\alpha(-\delta_{ij}\partial_i\partial^i\frac{U^{(0)}}{c^2}+\partial_i\partial_j\frac{U^{(0)}}{c^2})+\mathcal{O}(1/c^4). \tag{5.43b}$$

These terms are multiplied with the coupling function first derivative term respectively of orders $\mathcal{O}(1/c^2)$ and $\mathcal{O}(1/c^3)$. Thus both terms are of order $\mathcal{O}(1/c^5)$ and can be neglected.

For the last two terms in Eq. (5.41) we can again rewrite the energy momentum tensor term inside the brackets as follows, using the PN expanded metric components Eq. (2.81) of order c^0

$$T_{0i}-\frac{1}{2}Tg_{0i}=g_{00}g_{ii}T^{0i}-\frac{1}{2}g_{00}g_{ii}g^{0i}g_{\rho\sigma}T^{\rho\sigma}=T^{0i}. \tag{5.44}$$

Hence we are interested in the expansion of T^{0i}, which can again be found in Appendix A. For T^m_{0i} we have with Eq. (A.26)

$$T^m_{0i}=c\sum_A m^0_A v^i_A\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))+\mathcal{O}(1/c). \tag{5.45}$$

For T^{tid}_{0i} with Eq. (A.27) this results in

$$T^{tid}_{0i}=\sum_A\lambda^{(s)}_A\left[c^2\partial_0\frac{\varphi^{(0)}_c}{c^2}\partial_i\frac{\varphi^{(0)}_c}{c^2}+\frac{1}{2}cg^{\rho\sigma}v^A_i\partial_\rho\frac{\varphi^{(0)}_c}{c^2}\partial_\sigma\frac{\varphi^{(0)}_c}{c^2}\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))+\mathcal{O}(1/c^4). \tag{5.46}$$

We see that the lowest order here is of higher order than we are considering and thus this term does not contribute.

Thus the differential equation from Eq. (5.41) with matching orders of $1/c$ results in

$$\mathcal{O}(1/c^3)\quad 2\Delta\frac{g^{(0)}_i}{c^3}=-\frac{8\pi G}{c^3}\sum_A m^0_A v^i_A\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)). \tag{5.47}$$

5.4 Solving the differential equations

To sum up we have the following differential equations resulting from the field equations Eq. (5.7) and Eq. (5.10) per specific order in the PN expansion. For the scalar field:

$$\mathcal{O}(1/c^2) \quad \Box_\eta \frac{\varphi_c^{(0)}}{c^2} = \frac{4\pi G}{c^4}(\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))c^2\alpha_A^0 m_A^0), \tag{5.48a}$$

$$\begin{aligned}\mathcal{O}(1/c^4) \quad \Box_\eta \frac{\varphi_c^{(1)}}{c^4} = & -2\alpha f'\left(\frac{\varphi_c^{(0)}}{c^2}\right)\left((\partial_i\partial_j\frac{U^{(0)}}{c^2})(\partial_i\partial_j\frac{U^{(0)}}{c^2}) - \triangle\frac{U^{(0)}}{c^2}\triangle\frac{U^{(0)}}{c^2}\right) - \frac{4\pi G}{c^4} \\ & \left(-\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\left[(m_A^0((\alpha_A^0)^2+\beta_A^0)\varphi_c^{(0)} - \alpha_A^0 m_A^0 U^{(0)} - \frac{1}{2}\alpha_A^0 m_A^0 v_A^2)\right]\right. \\ & \left. + \sum_A \lambda_A^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\Box\varphi_c^{(0)}\right).\end{aligned} \tag{5.48b}$$

For the U potential:

$$\mathcal{O}(1/c^2) \quad -\Box_\eta \frac{U^{(0)}}{c^2} = \frac{4\pi G}{c^4}\left(\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))m_A^0 c^2\right) \tag{5.49a}$$

$$\begin{aligned}\mathcal{O}(1/c^4) \quad -\Box_\eta \frac{U^{(1)}}{c^4} = & \frac{4\pi G}{c^4}\left(\sum_A (m_A^0\alpha_A^0\varphi_c^{(0)} - m_A^0 U^{(0)} + m_A^0\frac{3}{2}v_A^2)\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\right) \\ & + 4\alpha(\delta_{ij}\partial_i\partial^i\frac{U^{(0)}}{c^2} - \partial_i\partial_j\frac{U^{(0)}}{c^2})f'(\varphi_0)\partial^\xi\partial^\rho\frac{\varphi_c^{(0)}}{c^2}.\end{aligned} \tag{5.49b}$$

(5.49c)

For the g_i potential

$$\mathcal{O}(1/c^3) \quad \Delta\frac{g_i^{(0)}}{c^3} = -\frac{4\pi G}{c^3}\sum_A m_A^0 v_A^i \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)). \tag{5.50}$$

We now discuss how to relate the various terms to specific PN orders. The lowest order terms with a factor $1/c^2$ correspond to 0PN. One order in factors of c higher then corresponds to 1PN, thus $1/c^4$. The lowest order of the g_i solution carries a factor of $1/c^3$ and we saw in the PN expansion of the metric Eq. (2.81) that this corresponds to 1PN.

In the next sections we solve these differential equations per PN order. On top of that we consider the equations in zeroth and first order in $\lambda^{(s)}$, the tidal deformability parameter, as a separate expansion in the tidal effects.

5.4.1 Solving for leading PN order

First we look at the equations at 0PN

$$\Box_\eta U^{(0)} = -\frac{4\pi G}{c^2}\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))m_A^0 c^2, \tag{5.51a}$$

$$\Box_\eta \varphi_c^{(0)} = 4\pi G(\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\alpha_A^0 m_A^0). \tag{5.51b}$$

To solve for the fields we can make use of the inverse of the box operator which is given by a Greens function, see Appendix C. Here we will use the half retarded half advanced[19] Greens function solution Eq. (C.11). Then the solution to Eq. (5.51a) becomes

$$\begin{aligned} U^{(0)} &= G\sum_A \int d^4x' \left(\frac{\delta\left(t-t'\right)}{|\mathbf{x}-\mathbf{x}'|} + \frac{|\mathbf{x}-\mathbf{x}'|}{2}\frac{1}{c^2}\partial_t^2\delta\left(t-t'\right)\right) m_A^0\delta^3(\mathbf{x}'-\mathbf{x}_A(t)) \\ &= G\sum_A m_A\left(\frac{1}{|\mathbf{x}-\mathbf{x}_A(t)|} + \frac{1}{c^2}\partial_t^2\int dt'\frac{|\mathbf{x}-\mathbf{x}_A\left(t'\right)|}{2}\delta\left(t-t'\right)\right) \\ &= G\sum_A m_A\left(\frac{1}{|\mathbf{x}-\mathbf{x}_A(t)|} + \frac{1}{c^2}\partial_t^2\frac{|\mathbf{x}-\mathbf{x}_A(t)|}{2}\right). \end{aligned} \tag{5.52}$$

We define $\mathbf{r}_A = \mathbf{x}_A(t) - \mathbf{x}$, $r_A = |\mathbf{r}_A|$. We can rewrite $\frac{1}{c}\partial_t r_A = \frac{1}{c}(\mathbf{n}_A\cdot\mathbf{v}_A)$ with $\mathbf{n}_A = \mathbf{r}_A/r_A$ the directional unit vector, as

$$\begin{aligned} \frac{1}{c^2}\partial_t^2 r_A &= \frac{1}{c^2}\partial_t\left(\mathbf{n}_A\cdot\mathbf{v}_A\right) = \frac{1}{c^2}\frac{r_A\left(\mathbf{r}_A\cdot\mathbf{a}_A+\mathbf{v}_A^2\right) - \left(\mathbf{r}_A\cdot\mathbf{v}_A\right)\left(\mathbf{n}_A\cdot\mathbf{v}_A\right)}{r_A^2} \\ &= \frac{1}{c^2}\frac{1}{r_A}\left[\mathbf{v}_A^2 + \mathbf{r}_A\cdot\mathbf{a}_A - \left(\mathbf{n}_A\cdot\mathbf{v}_A\right)^2\right], \end{aligned} \tag{5.53}$$

with $\mathbf{a}_A = \partial_t\mathbf{v}_A$ the relative acceleration.

And in the solution for $U^{(0)}$ Eq. (5.52) we can substitute

$$\frac{1}{r_A} + \frac{1}{c^2}\partial_t^2\frac{r_A}{2} = \frac{1}{r_A}\left[1+\frac{1}{c^2}\left(\frac{1}{2}\mathbf{v}_A^2 - \frac{1}{2}\left(\mathbf{n}_A\cdot\mathbf{v}_A\right)^2\right)\right] + \frac{1}{2c^2}\mathbf{n}_A\cdot\mathbf{a}_A \equiv \frac{1}{\tilde{\rho}_A} \tag{5.54}$$

After substitution this results in

$$U^{(0)} = G\sum_A \frac{m_A^0}{\tilde{\rho}_A}. \tag{5.55}$$

[19]In general the retarded solution is seen as the most physical solution. This is because the solution of the equation can be seen as an initial value problem with its evolution depending on sources in the past, respecting causal structure. However here we chose the solution resulting from a linear combination of the retarded and advanced Greens functions. This means that the solution is given half by sources in the past and half by the same sources in the future. The reason for selecting this solution is, if the solution is preferred to be energy conserving and therefore to be symmetric in time, which is the case here.

Using the same method for Eq. (5.51b) gives

$$\varphi_c^{(0)} = -G\sum_A \frac{m_A^0\alpha_A^0}{\tilde{\rho}_A}. \tag{5.56}$$

This were the solutions for 0PN, now we look at 1PN.

5.4.2 Solving for 1PN order corrections

We start with the solution for the g_i potential which is in the form of a Poisson equation

$$\Delta g_i^{(0,0)} = -4\pi G\sum_A m_A^0 v_A^i \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)). \tag{5.57}$$

Which can be solved directly with a Greens function Eq. (C.4)

$$g_i^{(0,0)} = G\sum_A \frac{m_A^0 v_i^A}{r_A}. \tag{5.58}$$

Secondly we look at the equations for U and ϕ_c at 1PN to zeroth order in $\lambda^{(s)}$.

$$\begin{aligned}
\Box_\eta U^{(1)} =& -4\pi G\sum_A (m_A^0\alpha_A^0\varphi_c^{(0)} - m_A^0 U^{(0)} + m_A^0\frac{3}{2}v_A^2)\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)) \qquad (5.59a)\\
&-4\alpha f'(\varphi_0)(\triangle U^{(0)}\triangle\varphi_c^{(0)} - (\partial_i\partial_j U^{(0)})(\partial_i\partial_j\delta\varphi_c^{(0)})),\\
\Box_\eta \varphi_c^{(1)} =& \; 4\pi G(\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))m_A^0((\alpha_A^0)^2+\beta_A^0)\varphi_c^{(0)} \qquad (5.59b)\\
&-\alpha_A^0 m_A^0 U^{(0)} - \frac{1}{2}\alpha_A^0 m_A^0 v_A^2)\\
&-2\alpha f'(\varphi_0)\left(\partial_i\partial_j U^{(0)}\partial_i\partial_j U^{(0)} - \triangle U^{(0)}\triangle U^{(0)}\right)\\
=& 4\pi G(\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))m_A^0((\alpha_A^0)^2+\beta_A^0)\varphi_c^{(0)}\\
&-\alpha_A^0 m_A^0 U^{(0)} - \frac{1}{2}\alpha_A^0 m_A^0 v_A^2)\\
&+2\alpha f'(\varphi_0)\left(\triangle U^{(0)}\triangle U^{(0)} - \partial_i\partial_j U^{(0)}\partial_i\partial_j U^{(0)}\right).
\end{aligned}$$

Substituting the 0PN solutions for $U^{(0)}$ from Eq. (5.55) and for $\varphi_c^{(0)}$ from Eq. (5.56), taking the lowest order term from $1/\tilde{\rho}_A$ within these solutions

$$\begin{aligned}\Box_\eta U^{(1)} =& -4\pi G\sum_A(m_A^0\frac{3}{2}v_A^2 - G\sum_{B\neq A}\frac{m_B^0}{r_B}(1+\alpha_B^0\alpha_A^0))\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)) \\ & -4\alpha f'(\varphi_0)G^2\sum_{B,A}m_A^0 m_B^0\alpha_A^0(\partial_A^i\partial_{A,i}\partial_B^j\partial_{B,j}-\partial_{A,i}\partial_{B,i}\partial_{A,j}\partial_{B,j})\frac{1}{r_A}\frac{1}{r_B} \\ &= -4\pi G\sum_A(m_A^0\frac{3}{2}v_A^2 - G\sum_{B\neq A}\frac{m_B^0}{r_B}(1+\alpha_B^0\alpha_A^0))\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)) \\ & -4\alpha f'(\varphi_0)G^2\sum_{B,A}m_A^0 m_B^0\alpha_A^0\triangle h_{AB},\end{aligned} \tag{5.60a}$$

$$\begin{aligned}\Box_\eta\varphi_c^{(1)} =& 4\pi G(\sum_A\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\alpha_A^0 m_A^0(-\frac{1}{2}v_A^2 - G\sum_{B\neq A}\frac{m_B^0}{r_B}(1+\alpha_B^0\alpha_A^0-\frac{\beta_A^0\alpha_B^0}{\alpha_A^0})) \\ & +2\alpha f'(\varphi_0)G^2\sum_{B,A}m_A^0 m_B^0\alpha_A^0\triangle h_{AB},\end{aligned} \tag{5.60b}$$

using the definition [24]

$$\triangle h_{AB} = (\partial_A^i\partial_{A,i}\partial_B^j\partial_{B,j}-\partial_{A,i}\partial_{B,i}\partial_{A,j}\partial_{B,j})\frac{1}{r_A}\frac{1}{r_B}. \tag{5.61}$$

Applying the Greens function Eq. (C.11) with Eq. (5.54)

$$U^{(1)} = G\sum_A\frac{m_A^0}{\tilde{\rho}_A}(\frac{3}{2}v_A^2 - G\sum_{B\neq A}\frac{m_B^0}{r_B}(1+\alpha_B^0\alpha_A^0)) - 4\alpha f'(\varphi_0)G^2\sum_{B,A}m_A^0 m_B^0\alpha_A^0 h_{AB}(\mathbf{x}) \tag{5.62a}$$

$$\varphi_c^{(1)} = G\sum_A\frac{m_A^0\alpha_A^0}{\tilde{\rho}_A}(\frac{1}{2}v_A^2 + G\sum_{B\neq A}\frac{m_B^0}{r_B}(1+\alpha_B^0\alpha_A^0-\frac{\beta_A^0\alpha_B^0}{\alpha_A^0})) + 2\alpha f'(\varphi_0)G^2\sum_{B,A}m_A^0 m_B^0\alpha_A^0 h_{AB}(\mathbf{x}) \tag{5.62b}$$

Next, we consider separately at the equation first order in $\lambda^{(s)}$, substituting the 0PN solution Eq. (5.56)

$$\begin{aligned}\Box_\eta\varphi_c^{(1)} &= 4\pi G(\sum_A\lambda_A^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\Box\varphi_c^{(0)}) \\ &= 16\pi^2 G^2(\sum_A\lambda_A^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))(\sum_{B\neq A}\delta^{(3)}(\mathbf{x}-\mathbf{x_B}(t))\alpha_B^0 m_B^0)\end{aligned} \tag{5.63}$$

When the term with the product of delta functions is convoluted with the Greens function, this term will always give zero if $x_A \neq x_B$, which is the case for two black holes which are not at the same position. Thus it does not contribute to our solution of the scalar field up to 1PN.

5.4.3 Solutions for the potentials and scalar field up to 1PN

Substituting the solutions obtained from the order by oder expansion discussed above gives the following fields

$$\begin{aligned}\frac{U}{c^2} &= \varepsilon_c^1\frac{U^{(0)}}{c^2} + \frac{U^{(1)}}{c^4}\\ &= \frac{G}{c^4}\sum_A \frac{m_A^0}{\tilde{\rho}_A}(c^2 + \frac{3}{2}v_A^2 - G\sum_{B\neq A}\frac{m_B^0}{r_B}(1+\alpha_B^0\alpha_A^0)) - 4\alpha\frac{f'(\varphi_0)}{c^4}G^2\sum_{B,A} m_A^0 m_B^0 \alpha_A^0 h_{AB}(\mathbf{x}),\end{aligned} \tag{5.64}$$

$$\frac{g_i}{c^3} = \frac{g_i^{(0)}}{c^3} = G\sum_A \frac{m_A^0 v_i^A}{r_A}, \tag{5.65}$$

$$\begin{aligned}\varphi &= \varphi_0 + \frac{\varphi_c}{c^2} = \varphi_0 + \varepsilon_c^1\frac{\varphi_c^{(0)}}{c^2} + \varepsilon_c^2\frac{\varphi_c^{(1)}}{c^4}\\ &= \varphi_0 - \frac{G}{c^4}\sum_A \frac{m_A^0\alpha_A^0}{\tilde{\rho}_A}(c^2 - \frac{1}{2}v_A^2 - G\sum_{B\neq A}\frac{m_B^0}{r_B}(1+\alpha_B^0\alpha_A^0 - \frac{\beta_A^0\alpha_B^0}{\alpha_A^0}))\\ &+ 2\alpha\frac{f'(\varphi_0)}{c^4}G^2\sum_{B,A} m_A^0 m_B^0\alpha_A^0 h_{AB}(\mathbf{x}).\end{aligned} \tag{5.66}$$

These solutions correspond to the 1PN near zone fields calculated in [24].

5.4.4 Motivation expansion method and generalization for tidal perturbations

Before we continue with the calculation of the 2 body Lagrangian we first elaborate on the choice for performing the PN expansion by explicitly tracking factors of $1/c^2$ as this is not necessarily convention.

This choice solves the following ambiguities that arise otherwise. In the expansion of the terms in the EOMs, the terms proportional to U and the scalar field in the expressions should be treated as first order, so when solving the EOMS order by order, you can plug in the zeroth order solutions in these terms. Furthermore, it is the contribution -1 from the PN metric expansion Eq. (2.81) that results in the 0th order equation of motion. There is therefore some conflict, the two terms in g_{00} up to 1PN are treated as two different orders, although it is defined as a whole as 0PN.
This can be resolved by saying that U at lowest order inherently carries a 1PN order, which is done in [24]. The confusing thing about this is that the GB term proportional to α in for example the equation of motion of R_{00} from Eq. (5.33) is proportional to $\triangle U\triangle\varphi - \partial_{ij}U\partial_{ij}\varphi$ at lowest order. This product of the two fields which both carry inherently an order of 1PN would make it total a second order term, however it is still part of the 1PN solution.

In this way it also seems odd to talk about the 0PN solution for U as the U in this case would carry the 1PN order. The nice thing about including the factors of c and tracking them is that the prefactor of the terms linear in the fields carries a factor of $1/c^2$ and the GB term does not. As the two fields at lowest order carry a factor of $1/c^2$, this guarantees that the factors of c in all these terms are equal and can all be treated as 1PN. Also automatically you treat the -1 and $2U/c^2$ in g_{00} as different orders.

When considering the solution for the scalar field to linear order in the tidal deformability in Eq. (5.63) we found that to 1PN order this contribution vanishes. This is a consequence of the skeletonization approach we discussed in section 5.1.1. The tidal effects to this order are decoupled from the near zone fields and need to be included separately, as we will see in the next section when calculating the Lagrangian. It would have been interesting to derive this from scratch using a more general expansion framework. This can be done by expanding the fields first for small tidal perturbations with ε_{tid} followed by the PN expansion, e.g. for the scalar field

$$\varphi = \varphi^{(0)} + \varepsilon_{tid}\varphi^{(1)} + \mathcal{O}(\varepsilon_{tid}^2), \tag{5.67}$$

with $\varphi^{(k)} \propto \lambda_{(s)}^k$. Each of the tidal expansion coefficients $\varphi^{(0)}$ and $\varphi^{(1)}$ is further expanded in a PN approximation to 1PN order. For $\varphi^{(0)}$ this is

$$\varphi^{(0)} = \varphi^{(0,0)} + \varepsilon_{PN}^1\varphi^{(0,1)} + \mathcal{O}(\varepsilon_{PN}^2), \tag{5.68}$$

with $\varphi^{(k,l)} \propto \lambda_s^k(\frac{v^2}{c^2})^l$. By substituting these expansions of the fields in the expressions of the EOMs Eq. (5.19), Eq. (5.33) and Eq. (5.41), one can again solve the differential equations per expansion order using similar methods as in previous section. This time additional terms of the tidal expansion of the fields would arise. To 1PN order this still should result in a vanishing contribution as this is a property of the skeletonization, but we leave it for future work to show this explicitly.

5.5 The 2 body Lagrangian up to leading PN order

With the expressions for the fields up to 1PN we can calculate important quantities such as the acceleration and binding energy regarding the dynamics of the binary system. Up till now we kept the number of bodies general by expressing everything in a sum over the particles A. Now we specify to a two body system and we start by calculating the Lagrangian.

For the Lagrangian of particle A in the field of a point particle B we assume: the mass of the test particle to be zero $m_A = 0$ and $r_A = r_B = |\mathbf{x}_A - \mathbf{x}_B| \equiv r$ (and consequently $\mathbf{n}_A = -\mathbf{n}_B = (\mathbf{r}_A - \mathbf{r}_B)/r \equiv \mathbf{n}$). The Lagrangian is given by

$$\mathcal{L}_A = \frac{dS_A}{dt} + \frac{dS_{tidal}}{dt}. \tag{5.69}$$

With the action expanded up to 1PN for the metric components using Eq. (2.81)

$$\begin{aligned} S_A &= -m_A(\varphi)c^2 \int dt \sqrt{-g_{\mu\nu}\frac{dx_A^\mu}{dt}\frac{dx_A^\nu}{dt}} \\ &\approx -m_A(\varphi)c^2 \int dt \sqrt{1 - \frac{2U}{c^2} + \frac{8g_i^{(0)}}{c^3}\frac{v_a^i}{c} - \frac{v_a^2}{c^2} - \frac{2U^{(0)}}{c^2}\frac{v_a^2}{c^2}} \\ &\approx -m_A(\varphi)c^2 \int dt (1 - \frac{U}{c^2} + \frac{4g_i^{(0)}}{c^3}\frac{v_a^i}{c} - \frac{1}{2}\frac{v_a^2}{c^2} + \frac{1}{2}\frac{(U^{(0)})^2}{c^4} - \frac{3}{2}\frac{U^{(0)}v_a^2}{c^4} - \frac{1}{8}\frac{v_a^4}{c^4}). \end{aligned} \tag{5.70}$$

In the latter equation we Taylor expanded the squareroot. Then we have, using the expansion of $m(\varphi)$ Eq. (5.22)

$$\begin{aligned} \frac{dS_A}{dt} = -m_A^0 c^2 \Bigg[& 1 + \alpha_A^0 \frac{\varphi_c}{c^2} + \frac{1}{2}\left(\alpha_A^{0\,2} + \beta_A^0\right)\frac{(\varphi_c^{(0)})^2}{c^4} - \frac{U}{c^2} + \frac{4g_i^{(0)}}{c^3}\frac{v_a^i}{c} \\ & -\frac{1}{2}\frac{v_a^2}{c^2} + \frac{1}{2}\frac{(U^{(0)})^2}{c^4}] - \frac{3}{2}\frac{U^{(0)}v_a^2}{c^4} - \frac{1}{8}\frac{v_a^4}{c^4} - \alpha_A^0\frac{U^{(0)}\varphi_c^{(0)}}{c^4} - \alpha_A^0\frac{v_a^2\varphi_c^{(0)}}{c^4}\Bigg]. \end{aligned} \tag{5.71}$$

Furthermore in the two body case we can rewrite the expression of $\tilde{\rho}$ as follows

$$\begin{aligned} & \frac{d}{dt}\left(\mathbf{n}\cdot\mathbf{v}_A\right) = \frac{1}{r}\left[\mathbf{v}\cdot\mathbf{v}_A - \left(\mathbf{n}\cdot\mathbf{v}_A\right)\left(\mathbf{n}\cdot\mathbf{v}\right)\right] + \mathbf{n}\cdot\mathbf{a}_A \\ \text{So, } & \frac{1}{\tilde{\rho}_A} \equiv \frac{1}{r_A}\left[1 + \frac{1}{2c^2}(\mathbf{v}_A^2 - (\mathbf{n}_A\cdot\mathbf{v}_A))^2\right] + \frac{1}{2c^2}\mathbf{n}_A\cdot\mathbf{a}_A \\ & = \frac{1}{r}\left[1 + \frac{1}{2c^2}(\mathbf{v}_A\cdot\mathbf{v}_B - \left(\mathbf{n}\cdot\mathbf{v}_A\right)\left(\mathbf{n}\cdot\mathbf{v}_B\right))\right] + \frac{1}{2c^2}\frac{d}{dt}\left(\mathbf{n}\cdot\mathbf{v}_A\right). \end{aligned} \tag{5.72}$$

In the integration the total derivative term will be integrated out. Therefore we have $1/\tilde{\rho}_A = 1/\tilde{\rho}_B = 1/\tilde{\rho}$.

Then writing down the different terms for particle A in the field of B that contribute to the action Eq. (5.71) up to 1PN contains using above expression for $\tilde{\rho}$ Eq. (5.72)

$$\frac{U}{c^2} = \frac{Gm_B^0}{c^4 r}(c^2 + \frac{1}{2}(\mathbf{v}_A\cdot\mathbf{v}_B) - \frac{1}{2}(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B) - \frac{3}{2}v_B^2) - 4\alpha\frac{f'(\varphi_0)}{c^4}G^2(m_B^0)^2\alpha_B^0 h_{BB}(x), \tag{5.73a}$$

$$\frac{\varphi_c}{c^2} = \frac{-Gm_B^0\alpha_B^0}{c^4 r}(c^2 - \frac{1}{2}(\mathbf{v}_A\cdot\mathbf{v}_B) - \frac{1}{2}(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B) - \frac{1}{2}v_B^2) + \frac{2\alpha f'(\varphi_0)G^2(m_B^0)^2}{c^4}h_{BB}(x). \tag{5.73b}$$

And for the order $\mathcal{O}(1/c^4)$ terms

$$\frac{\varphi_c^{(00)2}}{c^4} = \frac{1}{c^4}\frac{G^2(m_B^0\alpha_B^0)^2}{r^2}, \tag{5.74a}$$

$$\frac{4g_i^{(0,0)}v_A^i}{c^4} = \frac{4}{c^4}\frac{Gm_B^0\mathbf{v}_B\cdot\mathbf{v}_A}{r}, \tag{5.74b}$$

$$\frac{1}{2}\frac{(U^{(0,0)})^2}{c^4} = \frac{1}{2c^4}\frac{G^2(m_B^0)^2}{r^2}, \tag{5.74c}$$

$$-\frac{3}{2}\frac{U^{(0,0)}v_A^2}{c^4} = -\frac{3}{2}\frac{1}{c^4}G\frac{m_B^0v_A^2}{r}, \tag{5.74d}$$

$$-\alpha_A^0\frac{U^{(0,0)}\varphi_c^{(00)}}{c^4} = \frac{\alpha_A^0}{c^4}\frac{G^2(m_B^0)^2\alpha_B^0}{r^2}, \tag{5.74e}$$

$$-\alpha_A^0\frac{v_A^2\varphi_c^{(00)}}{c^4} = \alpha_A^0v_A^2G\frac{m_B^0\alpha_B^0}{rc^4}. \tag{5.74f}$$

Furthermore we use for the Greens function as shown in [24](correcting a minus sign typo)

$$h_{BB} = -\frac{1}{2r^4}. \tag{5.75}$$

Substituting this all in Eq. (5.71) gives

$$\begin{aligned}\frac{dS_A}{dt} = -m_Ac^2\Bigg[&1 - \frac{1}{2}\frac{\mathbf{v}_A^2}{c^2} - \frac{1}{8}\frac{\mathbf{v}_A^4}{c^4} - \frac{m_B^0G}{rc^4}\Bigg(c^2 + \frac{3}{2}v_B^2 + \frac{1}{2}(\mathbf{v}_A\cdot\mathbf{v}_B) - \frac{1}{2}(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)\\ &+ \alpha_A^0\alpha_B^0\left(c^2 - \frac{1}{2}\mathbf{v}_B^2 + \frac{1}{2}(\mathbf{v}_A\cdot\mathbf{v}_B) - \frac{1}{2}(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)\right) - 4(\mathbf{v}_A\cdot\mathbf{v}_B) + \frac{3}{2}v_A^2 - \frac{1}{2}\alpha_A^0\alpha_B^0v_A^2\Bigg)\\ &+ \frac{G^2(m_B^0)^2}{c^4r^2}\left(\frac{1}{2} + \frac{1}{2}((\alpha_A^0)^2 + \beta_A^0)(\alpha_B^0)^2 + \alpha_A^0\alpha_B^0\right) - (\alpha_A^0 + 2\alpha_B^0)\frac{\alpha f'(\varphi_0)(m_B^0)^2G^2}{c^4r^4}\Bigg].\end{aligned} \tag{5.76}$$

For the action regarding the tidal Eq. (4.24) we have using Eq. (2.81) and Eq. (5.18)

$$\begin{aligned}\frac{dS_{tidal}}{dt} &= -\frac{1}{2}c\lambda_B^{(s)}\sqrt{-g_{\alpha\beta}\frac{dx_B^\alpha}{dt}\frac{dx_B^\beta}{dt}}g^{\mu\nu}\partial_\mu\frac{\varphi_c^{(0)}}{c^2}\partial_\nu\frac{\varphi_c^{(0)}}{c^2}\\ &= -\frac{1}{2}c\lambda_B^{(s)}\sqrt{c^2(1 - \frac{U^{(0)}}{c^2} - \frac{\mathbf{v}_B^2}{2c^2})}[-e^{\frac{2U}{c^2}}(\partial_0\frac{\varphi_c^{(0)}}{c^2})^2 + e^{\frac{-2U}{c^2}}(\nabla\frac{\varphi_c^{(0)}}{c^2})^2]\\ &= -\frac{1}{2}\lambda_B^{(s)}c^2(\nabla\frac{\varphi_c^{(0)}}{c^2})^2,\end{aligned} \tag{5.77}$$

to lowest order. Using our lowest order solution for $\frac{\varphi_c}{c^2}$ from Eq. (5.56) this results in

$$(\nabla\frac{\varphi_c}{c^2})^2 = G^2\frac{(m_A^0\alpha_A^0)^2}{c^4r^4}. \tag{5.78}$$

Combining Eq. (5.69), Eq. (5.76) and Eq. (5.77) we derive for the the Lagrangian for particle A

$$\begin{aligned}\mathcal{L}_A = &-m_A^0c^2 + \frac{1}{2}m_A^0v_A^2 + \frac{1}{8c^2}m_A^0v_A^4 + \frac{Gm_A^0m_B^0}{c^2r}(c^2 + \frac{3}{2}v_B^2 + \frac{1}{2}(\mathbf{v}_A\cdot\mathbf{v}_B) - \frac{1}{2}(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)\\ &+ \alpha_A^0\alpha_B^0(c^2 - \frac{1}{2}\mathbf{v}_B^2 + \frac{1}{2}(\mathbf{v}_A\cdot\mathbf{v}_B) - \frac{1}{2}(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)) - 4(\mathbf{v}_A\cdot\mathbf{v}_B) + \frac{3}{2}v_A^2 - \frac{1}{2}\alpha_A^0\alpha_B^0v_A^2)\\ &- \frac{G^2m_A^0(m_B^0)^2}{c^2r^2}(\frac{1}{2} + \frac{1}{2}((\alpha_A^0)^2 + \beta_A^0)(\alpha_B^0)^2 + \alpha_A^0\alpha_B^0) + \frac{\alpha f'(\varphi_0)G^2m_A^0(m_B^0)^2}{c^2r^4}(\alpha_A^0 + 2\alpha_B^0)\\ &- \frac{1}{2}\lambda_A^{(s)}\frac{G^2(m_B^0\alpha_B^0)^2}{c^2r^4}\end{aligned} \tag{5.79}$$

For constructing total Lagrangian of the two body system we add the contribution from body B, which has the same functional form as Eq. (5.5) but with indices A and B interchanged. We introduce the the expressions

$$\begin{aligned}\bar{\alpha} &= 1 + \alpha_A^0\alpha_B^0, & \bar{\gamma} &= -\frac{2\alpha_A^0\alpha_B^0}{1+\alpha_A^0\alpha_B^0},\\ \bar{\beta}_A &= \frac{\beta_A^0(\alpha_B^0)^2}{2(1+\alpha_A^0\alpha_B^0)^2}, & \bar{\delta}_A &= \frac{\alpha_B^0 + 2\alpha_A^0}{(1+\alpha_A^0\alpha_B^0)^2},\\ m &= m_A + m_B, & \mu &= \frac{m_Am_B}{m}.\end{aligned} \tag{5.80}$$

with for the scalar charges to lowest order, the solution is calculated in [24], given by

$$\begin{aligned}\alpha_A^0 &= -\frac{\alpha f'(\varphi_0)c^4}{2G^2(m_A^0)^2},\\ \beta_A^0 &= -\frac{\alpha f''(\varphi_0)c^4}{2G^2(m_A^0)^2}.\end{aligned} \tag{5.81}$$

And for the terms proportional to the tidal deformability $\lambda^{(s)}$ we define

$$\zeta \equiv \lambda_A^{(s)}\frac{m_B^0\alpha_B^{0\,2}}{\bar{\alpha}^2m_A^0} + \lambda_B^{(s)}\frac{m_A^0\alpha_A^{0\,2}}{\bar{\alpha}^2m_B^0}, \tag{5.82}$$

which can be used to rewrite the term in proportional to the tidal effects, which we will name $\mathcal{L}_{tid}$

$$\begin{aligned}\mathcal{L}_{tid} &= \frac{1}{2}\frac{G^2}{c^2}(\lambda_A^{(s)}\frac{(m_B^0\alpha_B^0)^2}{r^4} + \lambda_B^{(s)}\frac{(m_A^0\alpha_A^0)^2}{r^4})\\ &= \frac{1}{2}\frac{G^2\bar{\alpha}^2m_A^0m_B^0}{c^2r^4}(\lambda_A^{(s)}\frac{m_B^0(\alpha_B^0)^2}{\bar{\alpha}^2m_A^0} + \lambda_B^{(s)}\frac{m_A^0(\alpha_A^0)^2}{\bar{\alpha}^2m_B^0})\\ &= \frac{1}{2}\frac{G^2\bar{\alpha}^2\mu m}{r^4c^2}\zeta.\end{aligned} \tag{5.83}$$

Our total two body Lagriangian is given by

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{orbit} + \mathcal{L}_{tid} \\
&= -m_A^0 c^2 - m_B^0 c^2 + \frac{1}{2} m_A^0 v_A^2 + \frac{1}{2} m_B^0 v_B^2 + \frac{\bar{\alpha} G m_A^0 m_B^0}{r} + \frac{1}{8c^2} m_A^0 v_A^4 + \frac{1}{8c^2} m_B^0 v_B^4 \\
&+ \frac{\bar{\alpha} G m_A^0 m_B^0}{2c^2 r}(3(v_A^2 + v_B^2) - 7(\mathbf{v}_A \cdot \mathbf{v}_B) - (\mathbf{n} \cdot \mathbf{v}_A)(\mathbf{n} \cdot \mathbf{v}_B) + 2\bar{\gamma}(\mathbf{v}_A - \mathbf{v}_B)^2) \\
&- \frac{\bar{\alpha}^2 G^2 m_A^0 m_B^0}{2r^2 c^2}(m_A^0(1 + 2\bar{\beta}_B) + m_B^2(1 + 2\bar{\beta}_A)) + \frac{\alpha f'(\varphi_0) \bar{\alpha}^2 G^2 m_A^0 m_B^0}{c^2 r^4}(m_A^0 \bar{\delta}_A + m_B^0 \bar{\delta}_B) \\
&- \frac{1}{2} \frac{G^2 \bar{\alpha}^2 \mu m}{r^4 c^2} \zeta.
\end{aligned}
\tag{5.84}
$$

In the first two and a half lines this Lagrangian contains 0PN and 1PN order GR terms for $\bar{\alpha} = 1$ concerning the kinetic energy terms and gravitational energy terms. With the $\mathcal{O}(c^2) + \mathcal{O}(c^0)$ order terms being the Newtonian; 0PN contributions and the $\mathcal{O}(1/c^2)$ order terms, the 1PN GR corrections. The term proportional to the coupling constant is the purely GB related term and thus comes in at 1PN. The last term proportional to ζ is due to the inclusion of the tidal action and also comes in at 1PN. We also note that the scaling with r of the GB and tidal term is the same, however, they contribute with a different sign so their effects are opposite.

This result up to the tidal terms corresponds to the result of the two body Lagrangian in [24]. In [17] the GB term has a minus sign as this expression was calculated with a typo in the Greens function from [24], which we corrected in Eq. (5.75).

5.6 Relative acceleration

Now we will look into calculating several properties of the dynamics. We start with the relative acceleration of the two bodies.
One can derive the relative acceleration from the Euler Lagrange equations

$$
\frac{1}{m_A^0} \frac{\partial \mathcal{L}}{\partial x_A} - \frac{1}{m_B^0} \frac{\partial \mathcal{L}}{\partial x_B} = \frac{1}{m_A^0} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \mathbf{v_A}} - \frac{1}{m_B^0} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \mathbf{v}_B}. \tag{5.85}
$$

Calculating these derivatives is a straightforward but a bit tedious because of the long expressions. Therefore this part is moved to Appendix B. Here we will look only at the results for the acceleration

$$
\begin{aligned}
\mathbf{a} = &-\frac{G\bar{\alpha} m \mathbf{n}}{r^2} + \frac{G\bar{\alpha} m}{c^2 r^2} \left\{ \mathbf{n} \left[\frac{3}{2} \eta \dot{r}^2 - (1 + 3\eta + \bar{\gamma}) \mathbf{v}^2 \right] + 2\mathbf{v} \dot{r} [2 - \eta + \bar{\gamma}] \right. \\
&\left. + \frac{2G\bar{\alpha} m \mathbf{n}}{r} \left[2 + \eta + \bar{\gamma} + \beta_+ - \frac{\Delta m}{m} \beta_- - \frac{2\alpha f'(\varphi_0)}{\bar{\alpha}^{3/2} r^2} \left(3S_+ + \frac{\Delta m}{m} S_- \right) + \frac{\zeta}{m r^2} \right] \right\},
\end{aligned}
\tag{5.86}
$$

using the definitions

$$\mathcal{S}_\pm \equiv \frac{\alpha_A^0 \pm \alpha_B^0}{2\sqrt{\bar\alpha}}, \qquad \beta_\pm \equiv \frac{\bar\beta_A \pm \bar\beta_B}{2}, \\ \eta \equiv \frac{m_A^0 m_B^0}{m^2}, \qquad \Delta m \equiv m_A^0 - m_B^0. \tag{5.87}$$

Now to simplify this expression we make the assumption that the orbits are approximately circular. For a later moment in the inspiral this is a valid approximation, non circular motions get removed from the orbits by dissipative effects of the GWs [25]. During the inspiral the relative distance shrinks much more slowly than the orbital period and one can in general assume it to be constant over an orbit. This assumption is defined as

$$\dot r = \mathbf{n}\cdot\mathbf{v} = 0 \text{ and } \ddot r = \frac{1}{r}\left[\mathbf{a}\cdot\mathbf{r} + \mathbf{v}^2 - (\mathbf{n}\cdot\mathbf{v})^2\right] = 0 \\ \to \mathbf{v}^2 = -\mathbf{a}\cdot\mathbf{r} \approx \frac{G\bar\alpha m}{r} + \mathcal{O}(1/c^2). \tag{5.88}$$

The relative acceleration becomes

$$\mathbf{a} = -\frac{G\bar\alpha m\mathbf{n}}{r^2} + \frac{G\bar\alpha m}{c^2 r^2}\left\{\mathbf{n}\left[-(1+3\eta+\bar\gamma)\frac{G\bar\alpha m}{r}\right] \right.\\ \left. + \frac{2G\bar\alpha m\mathbf{n}}{r}\left[2+\eta+\bar\gamma+\beta_+ - \frac{\Delta m}{m}\beta_- - \frac{2\alpha f'(\varphi_0)}{\bar\alpha^{3/2}r^2}\left(3S_+ + \frac{\Delta m}{m}S_-\right) + \frac{\zeta}{mr^2}\right]\right\}. \tag{5.89}$$

Then using $w^2 = \frac{v^2}{r^2} = -\frac{\mathbf{a}\cdot\mathbf{r}}{r^2}$ we have for the angular frequency

$$w^2 = \frac{G\bar\alpha m}{r^3}\left\{1 - \frac{G\bar\alpha m}{rc^2}\left[3-\eta+\bar\gamma+2\beta_+ - 2\frac{\Delta m}{m}\beta_- - \frac{4\alpha f'(\varphi_0)}{\bar\alpha^{3/2}r^2}\left(3S_+ + \frac{\Delta m}{m}S_-\right) + \frac{2\zeta}{mr^2}\right]\right\}. \tag{5.90}$$

As the information about the binary system comes from GWs, it is most convenient to express for example the binding energy or the phase in terms of the angular frequency, which we can deduce from the waves instead of the distance r. Also is the distance r a gauge dependent quantity. For this we can use above relation for the angular frequency which we can invert perturbatively in r in the following way.

We want to invert this expansion up to 1PN order, or factor $1/c^2$. Say that we have the following ansatz for $r(w)$ up to 1PN

$$r = r_0\left(1 + \frac{r_1}{c^2}\right). \tag{5.91}$$

Substituting this ansatz in Eq. (5.90) and expanding up to $1/c^2$ gives

$$w^2 = \frac{G\bar\alpha m}{r_0^3} + \frac{1}{c^2}\left(-\frac{3G\bar\alpha m r_1}{r_0^3} - \frac{G\bar\alpha m}{r_0^4}\left[3-\eta+\bar\gamma+2\beta_+ - 2\frac{\Delta m}{m}\beta_- \right.\right.\\ \left.\left. - \frac{4\alpha f'(\varphi_0)}{\bar\alpha^{3/2}r_0^2}\left(3S_+ + \frac{\Delta m}{m}S_-\right) + \frac{2\zeta}{mr_0^2}\right]. \tag{5.92}$$

Equating the lowest order terms results in

$$w^2 = \frac{G\bar{\alpha}m}{r_0^3} \to r_0 = \left(\frac{G\bar{\alpha}m}{w^2}\right)^{1/3}. \tag{5.93}$$

Equating the second order terms gives

$$0 = \frac{1}{c^2}\left(-\frac{3G\bar{\alpha}mr_1}{r_0^3} - \frac{G\bar{\alpha}m}{r_0^4}\left[3 - \eta + \bar{\gamma} + 2\beta_+ - 2\frac{\Delta m}{m}\beta_- - \frac{4\alpha f'(\varphi_0)}{\bar{\alpha}^{3/2}r_0^2}\left(3S_+ + \frac{\Delta m}{m}S_-\right) + \frac{2\zeta}{mr_0^2}\right]\right). \tag{5.94}$$

Solving this for r_1

$$r_1 = \frac{-G\bar{\alpha}m}{3r_0}\left[3 - \eta + \bar{\gamma} + 2\beta_+ - 2\frac{\Delta m}{m}\beta_- - \frac{4\alpha f'(\varphi_0)}{\bar{\alpha}^{3/2}r_0^2}\left(3S_+ + \frac{\Delta m}{m}S_-\right) + \frac{2\zeta}{mr_0^2}\right] \tag{5.95}$$

Resulting in the inverted expression

$$\begin{aligned} r(w) = r_0(1 + \frac{r_1}{c^2}) &= \left(\frac{G\bar{\alpha}m}{w^2}\right)^{1/3}\left(1 - \frac{(G\bar{\alpha}w)^{2/3}}{3c^2}\left[3 - \eta + \bar{\gamma} + 2\beta_+ - 2\frac{\Delta m}{m}\beta_-\right.\right. \\ &\left.\left. - \frac{4\alpha f'(\varphi_0)\, w^2}{c^2\bar{\alpha}^{3/2}}\left(S_+ + \frac{\Delta m}{3m}S_-\right) - \frac{2\zeta w^2}{3c^2 m}\right]\right. \\ &= (\frac{\bar{\alpha}Gm}{w^2})^{1/3} - \frac{G\bar{\alpha}m}{c^2}(1 + \frac{\bar{\gamma}}{3} - \frac{\eta}{3} + \frac{2\beta_+}{3} - \frac{2\Delta m}{3m}\beta_-) + (\frac{Gmw^4}{\bar{\alpha}^{7/2}})^{1/3}\frac{4\alpha f'[\varphi_0]}{c^2}(S_+ + \frac{\Delta m}{3m}S_-) \\ &- (\frac{G\bar{\alpha}w^4}{m^2})^{1/3}\frac{2\zeta}{3c^2}. \end{aligned} \tag{5.96}$$

5.7 The binding energy

One can derive the energy from the Lagrangian with a Lagrange transformation

$$E = \sum_{i=1}^{n} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}. \tag{5.97}$$

So in our case

$$E = \mathbf{v}_A \frac{\partial \mathcal{L}}{\partial \mathbf{v}_A} + \mathbf{v}_B \frac{\partial \mathcal{L}}{\partial \mathbf{v}_B} - \mathcal{L}. \tag{5.98}$$

Calculating the derivatives to the velocity of Eq. (5.84)

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}_A} = m_A^0 \mathbf{v}_A + \frac{1}{2}m_A^0 \mathbf{v}_A^3 + \frac{G\bar{\alpha}m_A^0 m_B^0}{2c^2 r}\left[6\mathbf{v}_A - 7\mathbf{v}_B - \mathbf{n}\left(\mathbf{n}\cdot\mathbf{v}_B\right) + 4\bar{\gamma}\left(\mathbf{v}_A - \mathbf{v}_B\right)\right], \tag{5.99}$$

and for $\frac{\partial \mathcal{L}}{\partial \mathbf{v}_A}$ the same expression with $A \leftrightarrow B$.

Filling in Eq. (5.98)

$$\begin{aligned}E &= m_A^0c^2 + m_B^0c^2 + \frac{1}{2}m_A^0v_A^2 + \frac{1}{2}m_B^0v_B^2 - \frac{G\bar{\alpha}m_A^0m_B^0}{r} + \frac{3}{8}m_A^0v_A^4 + \frac{3}{8}m_B^0v_B^4 \\ &- \frac{G\bar{\alpha}m_A^0m_B^0}{2c^2r}[-3(v_A^2+v_B^2) + 7(\mathbf{v}_A\cdot\mathbf{v}_B) + (\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B) - 2\bar{\gamma}(\mathbf{v}_A-\mathbf{v}_B)^2] \\ &+ \frac{\bar{\alpha}^2G^2m_A^0m_B^0}{2r^2c^2}(m_A^0(1+2\bar{\beta}_B) + m_B^2(1+2\bar{\beta}_A)) - \frac{\alpha f'(\varphi_0)\bar{\alpha}^2G^2m_A^0m_B^0}{c^2r^4}(m_A^0\bar{\delta}_A + m_B^0\bar{\delta}_B) \\ &+ \frac{1}{2}\frac{G^2\bar{\alpha}^2\mu m}{r^4c^2}\zeta.\end{aligned} \tag{5.100}$$

Now we switch to CM frame, to lowest order approximation[20] the coordinates in this frame are

$$\begin{aligned}\mathbf{x}_A &\approx \tfrac{m_B^0}{m}\mathbf{r}, \quad \mathbf{x}_B \approx -\tfrac{m_A^0}{m}\mathbf{r}, \\ \mathbf{v}_A &\approx \tfrac{m_B^0}{m}\mathbf{v}, \quad \mathbf{v}_B \approx -\tfrac{m_A^0}{m}\mathbf{v}.\end{aligned} \tag{5.101}$$

Thus in the CM frame the expression is given by

$$\begin{aligned}E &= m_A^0c^2 + m_B^0c^2 + \frac{1}{2}\mu v^2 - \frac{G\bar{\alpha}m_A^0m_B^0}{r} + \frac{3}{8}\mu\frac{(m_B^0)^3+(m_A^0)^3}{m^3}v^4 + \frac{G\bar{\alpha}m_A^0m_B^0}{2c^2r}[3\frac{(m_B^0)^2+(m_A^0)^2}{m^2} \\ &+ 7\frac{m_B^0m_A^0}{m^2}v^2 + \frac{m_B^0m_A^0}{m^2}(\mathbf{n}\cdot\mathbf{v})^2 + 2\bar{\gamma}v^2] + \frac{\bar{\alpha}^2G^2m_A^0m_B^0}{2r^2c^2}(m_A^0(1+2\bar{\beta}_B) + m_B^2(1+2\bar{\beta}_A)) \\ &- \frac{\alpha f'(\varphi_0)\bar{\alpha}^2G^2m_A^0m_B^0}{c^2r^4}(m_A^0\bar{\delta}_A + m_B^0\bar{\delta}_B) + \frac{1}{2}\frac{G^2\bar{\alpha}^2\mu m}{r^4c^2}\zeta.\end{aligned} \tag{5.102}$$

Using $\dot{r} = (\mathbf{v}\cdot\mathbf{r})/r = \mathbf{n}\cdot\mathbf{v}$ together with Eq. (5.87) and Eq. (5.80) this reduces to

$$\begin{aligned}E = mc^2 + \mu\Bigg[&\frac{1}{2}v^2 - \frac{G\bar{\alpha}m}{r} + \frac{3}{8c^2}(1-3\eta)v^4 + \frac{G\bar{\alpha}m}{2c^2r}[(3+\eta+2\bar{\gamma})v^2 + \eta\dot{r}^2] \\ &+ \frac{\bar{\alpha}^2G^2m^2}{r^2c^2}(\frac{1}{2} + \beta_+ - \frac{\Delta m}{m}\beta_-) - \frac{\alpha f'(\varphi_0)\sqrt{\bar{\alpha}}G^2m^2}{c^2r^4}\left(3S_+ + \frac{\Delta m}{m}S_-\right) \\ &+ \frac{1}{2}\frac{G^2\bar{\alpha}^2m}{r^4c^2}\zeta\Bigg].\end{aligned} \tag{5.103}$$

Writing this in terms of the angular frequency we can substitute for r the relation Eq. (5.96). Then using the dimensionless PN counting parameter which depends on the angular frequency

$$x = \left(\frac{G\bar{\alpha}m\omega}{c^3}\right)^{2/3}, \tag{5.104}$$

[20] In section 7.2 we discuss the calculation of the CM coordinates up to 1PN. We find that the next terms in Eq. (7.14) and Eq. (7.16) are of order 1PN, as $\mathbf{v}$ in the binding energy always comes as the square or to the fourth power, substituting these higher order terms would give terms of 2PN order or higher which we do not consider here.

the binding energy becomes

$$\boxed{E(x) = -\frac{\mu c^2 x}{2}\left(1 + x\left[-\frac{3}{4} - \frac{\eta}{12} - \frac{2}{3}\bar{\gamma} + \frac{2}{3}\beta_+ - \frac{2}{3}\frac{\Delta m}{m}\beta_- - \frac{10}{3}\frac{c^4\alpha f'(\varphi_0)}{G^2\bar{\alpha}^{7/2)}m^2}x^2\left(3S_+ + \frac{\Delta m}{m}S_-\right) + \frac{5}{3}\frac{c^4\zeta}{G^2m^3\bar{\alpha}^2}x^2\right]\right).} \tag{5.105}$$

This result differs slightly from [17] as the GB term has a minus sign in the brackets compared to the plus in this paper, this was due to the typo in [48] already mentioned when discussing the two body Lagrangian. Furthermore, our result here corrects the numerical prefactor to be $\frac{10}{3}$ instead of $\frac{22}{3}$ as in [17].

5.8 Binding energy analysis

With the expression of the binding energy up to 1PN we can already analyze the effects of the tidal contributions on the dynamics and their dependence on the parameters. The change in binding energy during the coalescence event does work through in the radiation signal, but it is not an observable that is measured. However it is interesting for the comparison with numerical relativity simulations. As the binding energy as function of the frequency is a gauge independent quantity, it is interesting on its own right to study its features, in section 7.5 we will see how the binding energy dependencies work through in the GW phase evolution.

The tidal term ζ depends on the tidal deformability parameter λ_s. In the next section 6 we show that it is given by

$$\lambda^{(s)} = \frac{7}{6}m_{BH}\alpha f''\left(\varphi_\infty\right), \tag{5.106}$$

with m_{BH} the mass of the black hole in the binary, furthermore depending linearly on the coupling constant and the second derivative of the coupling function evaluated at the scalar field at infinity. We set the scalar field at infinity to zero which is equal to the background field $\varphi_\infty = \varphi_0 = 0$. As discussed in section 3.3 there are different types of coupling functions that are commonly used. For including tidal effects, the second derivative of the coupling function evaluated at the scalar field at infinity should not be zero as follows from Eq. (5.106). This is the case for exponential, quadratic and Gaussian coupling functions. Other effects from the scalar field and the GB nonlinear curvature contributions depend on the scalar charge Eq. (5.81), which depends again on the first derivative of the coupling function. Non vanishing first derivative at φ_0 is only the case for linear and exponential coupling. The only choice for which both effects are present simultaneously is an exponential coupling resulting in EdGB with coupling function $f(\varphi) = \frac{1}{4}e^{2\varphi}$.

We are interested in the contribution of the term in Eq. (5.105) proportional to ζ relative to the GB term proportional to α and their total contribution with respect to the GR case.

The binding energy in GR is recovered from Eq. (5.105) for zero coupling

$$E_{GR}(x) = -\frac{\mu c^2 x}{2}\left(1 + x\left[-\frac{3}{4} - \frac{\eta}{12}\right]\right). \tag{5.107}$$

First we compare the tidal and GB term. In Eq. (5.105) they have opposite sign. All the terms are positive except for the terms depending on the scalar charge and its derivative which are defined with a minus sign, see Eq. (5.81). Therefore S_+ and S_- are negative, canceling the minus sign in front of the GB term, while ζ depends only on the square of the scalar charge, hence no sign change occurs in the tidal term. Therefore in the end the contribution of the tidal term and Gauss Bonnet term is both additive (or substractive as the binding energy is defined with an overall minus sign).
From Eq. (5.105) we can also see that both terms have the same scaling with x from Eq. (5.104). Rewriting this PN parameter in terms of the frequency using $w = \pi f$[21] Thus also the same scaling with the orbital frequency. We find the same scaling with total mass as well. We do find a different dependence on the coupling constant α and the mass ratio $q = \frac{m_A}{m_B}$. We plot the contours of the ratio of the term proportional to ζ in Eq. (5.105) and the term proportional to α, showing the dependencies on the mass ratio and coupling constant, see Fig. 13.

[21] For GWs holds $w_{GW} = 2w_{orbital}$ in the adiabatic and quadrupolar approximation.

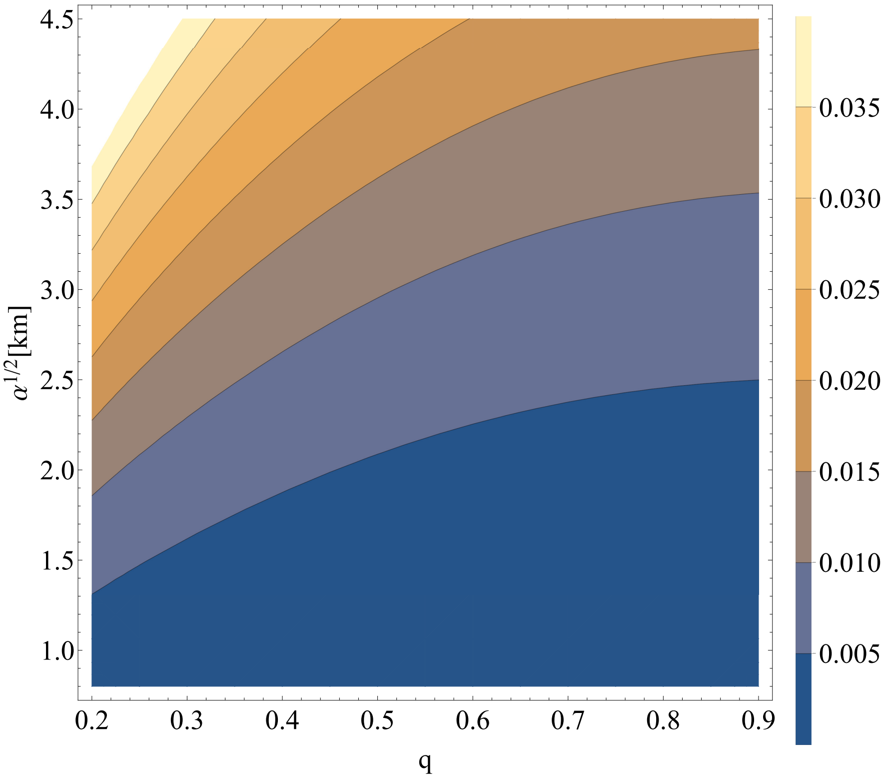


Figure 13: Contour values of the ratio $\frac{E_{binding,\zeta}}{E_{binding,GB}}$ for different values of the coupling constant, here given in the squareroot corresponding to how the bounds on this parameter are set, and the mass ratio $q = \frac{m_A}{m_B}$. Furthermore setting $f[\varphi] = \frac{1}{4}e^{2\varphi}$, $m = 15M_\odot$, at a frequency of $f \approx 586$Hz corresponding to the ISCO frequency[22]

As the values of these contours in Fig. 13 are positive, the sign of the tidal and GB term in the binding energy are indeed the same. From the values we can also see that the tidal term is about a factor $10^{-2} - 10^{-3}$ smaller than the GB term. While the extra contribution from the tidal term to the binding energy is quite small, it can nevertheless be significant in the GW signal as differences are accumulated in the phase during the inspiral. We can see that for a small value of the mass ratio and a larger coupling constant the value of the energy ratio is highest. However our current upperbound for the coupling constant is around ≈ 1km, this in the lower section of this plot, however in this analysis the focus lies at gaining intuition on the parameter dependencies.

Next we analyse the total binding energy, compared to the binding energy in GR, see Fig. 14

[22]This frequency corresponds to the frequency at the radius of the ISCO which is a proxy for the end of the inspiral.

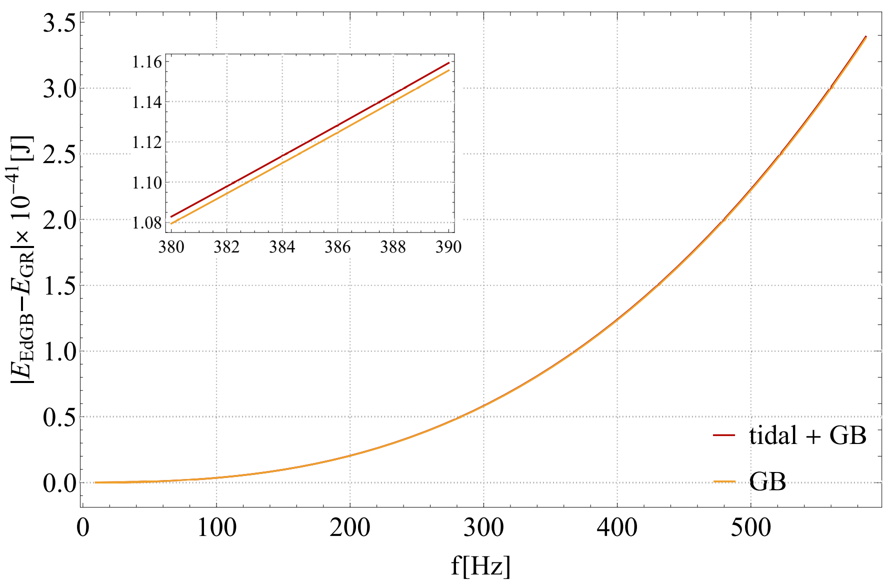


Figure 14: The absolute value of the difference between the binding energy in EdGB gravity and GR as a function of the GW frequency. The red line with tidal contribution and the orange line without. The inset shows a magnified part of frequency range 380-390Hz. Again setting $f[\varphi] = \frac{1}{4}e^{2\varphi}$, $m = 15M_{\odot}$, $q = \frac{1}{2}$ and $\sqrt{\alpha} = 1.7 \times 10^3$km.

In general we see that the difference in binding energies becomes larger for larger frequencies. The difference when including the tidal effects is very small and can only be seen in the inset. If we subtract both lines we find the results shown in Fig. 15 for different mass ratios.

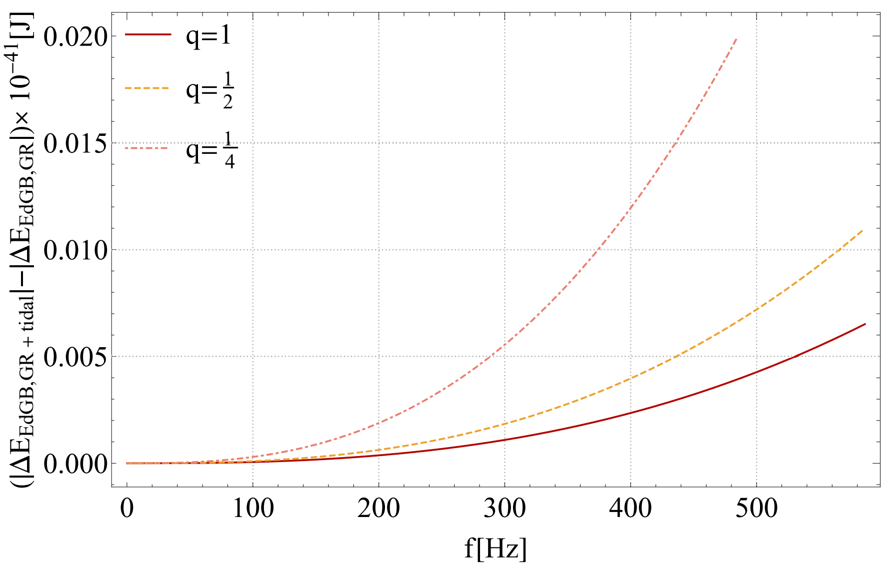


Figure 15: The absolute value of the difference between the binding energy in EdGB gravity with tidal effects and GR minus the difference without tidal effects for different mass ratios. Again setting $f[\varphi] = \frac{1}{4}e^{2\varphi}$, $m = 15M_\odot$, $q = \frac{1}{2}$ and $\sqrt{\alpha} = 1.7 \times 10^3$km.

From this plot we see that the difference with GR by including tidal effects on top of the GB corrections is larger for smaller values of q as corresponding with the results shown in Fig. 13.

From this we can conclude that we see some non trivial effects in the binding energy from including the tidal term. The scaling with the frequency and total mass is degenerate with the scaling of the GB terms but has different dependencies on the coupling constant and mass ratios. Larger α and smaller q result in the largest differences from the GR values. The effects are however very small but we can not conclude from this how significant the effects will be in the gravitational radiation signal, that we will analyse in section 7.5.

6 Tidal deformability calculation

In this part we focus on calculating the strength of the response of our scalar field φ around one of the black holes in the binary to the gradient of the scalar field of the companion black hole; the scalar tidal field. This strength is given by the scalar tidal deformability parameter $\lambda_{(s)}$ as we have seen in section 4. We start with the equation of motion of our scalar field of the scalarized black hole

$$\Box\varphi = \frac{4\pi G}{c^4}\mu_s, \tag{6.1}$$

with the source term given by

$$\mu_s = \frac{-\delta S_m}{\sqrt{-g}\delta\varphi} - \frac{c^4}{16\pi G}\alpha f'(\varphi)R^2_{GB}. \tag{6.2}$$

The wave equations we encountered in linearized gravity in Eq. (2.30) and the gothic formulation in Eq. (2.58) had a flat space d'Alembertian operator and we could therefore use a Greens function to write it in an integral form. In Eq. (6.1) we are not in the gothic formulation and the d'Alembertian is still general. As an instructive example, we do assume we have a flat spacetime background for the following discussion.

In the case of a flat background we can invert the d'Alembertian operator with the retarded Greens function Eq. (C.10a)

$$\varphi = \frac{-G}{c^4}\int \frac{\mu_s\left(\tau, \mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}d^3x', \tag{6.3}$$

where the delta function was already evaluated making τ the retarded time. We can write this in a multipole expansion framework similar as for the Newtonian potential Eq. (4.3) as we are dealing again with a scalar quantity. In this case we keep the expansion to all orders.
If we are in a position far away from the source $\mathbf{x} \gg \mathbf{x}'$, we can do the following Taylor expansion

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{r} + \frac{n^i x^i}{r^2} + \frac{3}{2}\frac{n^i n^j - \delta^{ij}/3}{r^3}x^i x'^j + \dots \\ &= \sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\frac{(2l-1)!!}{l!}n^{<L>}\mathbf{x}'_{<L>},\end{aligned} \tag{6.4}$$

which is the same as Eq. (4.2) but to general order. As $n^i n^j - \frac{\delta^{ij}}{3}$ removes the trace of $n^i n^j$ which is also symmetric in its indices, from these unit vectors the STF part is taken. This can also be done for the higher order terms. The L summation denotes the product corresponding with dimensionality l. Substituting this in Eq. (6.3) gives the asymptotic expansion of the scalar field

$$\varphi = \frac{G}{c^4}\int\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\frac{(2l-1)!!}{l!}n^{<L>}x'_{<L>}\mu_s\left(\tau, \mathbf{x}'\right)d^3x'. \tag{6.5}$$

We define the "mass" monopole and higher order multipoles as

$$\begin{aligned} m^{(s)} &= -\int d^4x'\mu(\tau,\mathbf{x}'), \\ Q_L^{(s)} &= -\int d^4x'\mu(\tau,\mathbf{x}')\mathbf{x}'_{<L>}. \end{aligned} \tag{6.6}$$

Hence our asymptotic scalar field can be written in the multipole expansion

$$\varphi = \frac{G}{c^4}\frac{m^{(s)}}{r} + \frac{G}{c^4}\sum_{l=1}^{\infty}\frac{(2l-1)!!}{l!}\frac{1}{r^{l+1}}n^{<L>}Q^{(s)}_{<L>}. \tag{6.7}$$

Here we come back to the discussion on neutron stars having vanishing scalar hair for type 1 coupling functions in section 3.3.1. The scalar charge is defined as the coefficient scaling as $1/r$, so here this would be $m^{(s)}$. As this is the integral over the source from Eq. (6.2), we see that this integral is over the GB invariant *and* over the matter action corresponding to a point particle with scalar dependent mass varied with respect to the scalar field. Because of this extra integral, the scalar charge for black holes non zero while for neutron stars it vanishes.

We assume the black hole and its companion to be at large distance relative to each other. Then we Taylor expand the scalar field of the companion that is felt by our black hole around the origin

$$\begin{aligned} \varphi_{ext} &= \varphi_{ext}\mid_O + \sum_{l=1}^{\infty}\frac{1}{l!}\mathbf{x}'^{<L>}\frac{\partial\varphi_{ext}}{\partial x'^{<L>}}\mid_O \\ &= \sum_{l=1}^{\infty}\frac{1}{l!}n^{<L>}\mathcal{E}^{(s)}_{<L>}r^l, \end{aligned} \tag{6.8}$$

with $\mathcal{E}$ the scalar tidal field defined in Eq. (4.22). This tidal field influences the multipoles of the scalar field. If we assume we have a scalarized black hole with scalar field background φ_B given by Eq. (6.5) which is placed in the scalar field of an external source the scalar field can be described as follows with the effect of the tidal field on the multipole structure

$$\lim_{r\to\infty}\varphi = \varphi_B + \varepsilon_{tid}\sum_{\ell=1}^{\infty}\left[\frac{(2\ell-1)!!}{\ell!}\frac{Gn^{<L>}Q^s_{<L>}}{c^4r^{\ell+1}} - \frac{1}{\ell!}n^{<L>}r^{\ell}\mathcal{E}^s_{<L>}\right]. \tag{6.9}$$

We express the tidal perturbations with dimensionless parameter ε_{tid}. In this section we will not track explicitly the PN factors or factors of $1/c^2$ as in section 5 as we are not explicitly doing a PN expansion. For lowest order this reduces to the dipole moments

$$\begin{aligned} \lim_{r\to\infty}\varphi &\sim \ldots + \varepsilon_{tid}\left[\frac{Gn^iQ^s_i}{c^2r^2} + O\left(r^{-3}\right) - rn^i\mathcal{E}^s_i + O\left(r^2\right)\right] \\ &\sim \ldots + \varepsilon_{tid}\sum_{m=-1}^{1}Y_{1m}\left[\frac{G\tilde{Q}_{1m}}{c^2r^2} - r\tilde{E}_{1m} + \ldots\right]. \end{aligned} \tag{6.10}$$

In the last line we switched from Cartesian basis with the STF tensors forming the complete basis structure to a spherical harmonic basis using

$$Q_i^s n^i = \sum_{m=-1}^{1} \tilde{Q}_{1m} Y_{1m}, \tag{6.11}$$

which in general holds for STF vectors and tensors contracted with higher powers of n[99]. As we have seen in section 4 in the adiabatic limit the induced dipole moment is related to the tidal field via

$$Q_i^s = -\lambda^s \mathcal{E}_i^s \quad \text{or} \quad \tilde{Q}_i^s = -\lambda^s \tilde{\mathcal{E}}_i^s. \tag{6.12}$$

To derive Eq. (6.10) we did an asymptotic expansion to very large distances. At far distances from the source we can assume spacetime to be flat therefore does this final asymptotic result also hold in the case of curved spacetime in the region near the source.

Now to calculate $\lambda^{(s)}$ we are solving the EOM of the scalar field for static perturbations to the scalar field from the tidal field. We will solve for the dominant contribution of the perturbations up to leading order in ε_{tid}. Then writing this solution in the asymptotic expansion which allows for reading of the expression for the induced dipole moment scaling with $\frac{1}{r^2}$ and the tidal field scaling with r. Using the relation Eq. (6.12) results in the expression for the scalar tidal deformability.

We start again with the action and the EOM in Eq. (5.1), Eq. (5.6) and Eq. (5.10) respectively. This time we do not include the matter related terms in the form of the energy momentum tensor as we are purely interested in the reaction of the scalar field to the tidal field and not in the generation of the field itself.
We would like to solve the EOM of the scalar field perturbatively. In GR the solution of the Einstein equation for a static black hole is given by the Schwarzschild metric

$$ds^2_{\text{Schwarzschild}} = -(1-u)c^2dt^2 + \frac{1}{1-u}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{6.13}$$

with

$$u = \frac{r_S}{r}, \quad r_S = \frac{2Gm}{c^2}. \tag{6.14}$$

Here r_S is the Schwarzschild radius (see Appendix D. We assume a small coupling limit, in which the GB coupling parameter is small. It is more convenient to work with a dimensionless version of the coupling constant by dividing the coupling constant by the characteristic lenght scale in the system, the Schwarzschild radius

$$\hat{\alpha} \equiv \frac{\alpha}{r_S^2} \ll 1 \propto \varepsilon_{\hat{\alpha}}, \tag{6.15}$$

tracking this small coupling parameter with $\varepsilon_{\hat{\alpha}}$. Looking at the scalar field EOM in Eq. (5.10) we can see that the solutions will be of order $\hat{\alpha}$ and thus the dominant tidal response will come from linear order in this parameter. If we look at the gravitational EOM in Eq. (5.6) all the terms on the RHS are of order $\hat{\alpha}^2$ as they contain two derivatives of φ which

is of linear order. Or when applying the chain rule on the derivatives on coupling function, the GB term of order $\hat{\alpha}$ multiplied with two derivatives of the scalar field which is also of order $\hat{\alpha}$. Hence at linear order in $\hat{\alpha}$ the GB terms in the gravitational field equations do not play a role yet and we can take the GR Schwarzschild metric. This metric does not change at this order.
In the Schwarzschild metric one can explicitly calculate the components of the curvature tensors and hence the expression of the GB invariant which is given by

$$R^2_{GB} = \frac{48G^2m^2}{c^4r^6}. \tag{6.16}$$

In this metric the EOM becomes

$$\Box\varphi = -\frac{48G^4m^4\hat{\alpha}}{c^8r^6}f'(\varphi)\varepsilon_{\hat{\alpha}}. \tag{6.17}$$

We expand the scalar field up to linear order in $\hat{\alpha}$

$$\varphi = \varphi^{(0)} + \varepsilon_{\hat{\alpha}}\varphi^{(1)} + O\left(\varepsilon^2_{\hat{\alpha}}\right). \tag{6.18}$$

Then we would also like to study the linear perturbation in the scalar field because of the tidal effects. Thus we expand the scalar field also in a background part denoting with the second superscript being 0 and a static (adiabatic limit) tidal perturbation with second superscript being 1, tracking this expansion with small tidal perturbation parameter ε_{tid}. As we are interested in the dominant effects we expand up to linear order

$$\varphi = \varphi^{(0,0)} + \varepsilon_{tid}\varphi^{(0,1)} + \varepsilon_{\hat{\alpha}}\varphi^{(1,0)} + \varepsilon_{tid}\varepsilon_{\hat{\alpha}}\varphi^{(1,1)} + \mathcal{O}(\varepsilon^2_{tid}\varepsilon^2_{\hat{\alpha}}). \tag{6.19}$$

Similar to the expansion of the scalar field in section 5 we substitute this expansion in the EOM and equate the terms with the same expansion parameters using the Taylor expansion for the coupling function

$$f'(\varphi) = f'(\varphi^{(0,0)}) + f'(\varphi^{(0,0)})\varepsilon_{\hat{\alpha}}\varphi^{(1,0)} + f'(\varphi^{(0,0)})\varepsilon_{tid}\varphi^{(0,1)} + \mathcal{O}(\varepsilon^2_{\hat{\alpha}}\varepsilon^2_{tid}) \tag{6.20}$$

This results in the following equations

$$\begin{aligned} \Box\varphi^{(0,0)} &= 0, \qquad \varepsilon_{\hat{\alpha}}\Box\varphi^{(1,0)} = -\frac{48G^4m^4\hat{\alpha}}{c^8r^6}f'\left(\varphi^{(0,0)}\right)\varepsilon_{\hat{\alpha}} \\ \varepsilon_{tid}\Box\varphi^{(0,1)} &= 0, \qquad \varepsilon_{tid}\varepsilon_{\hat{\alpha}}\Box\varphi^{(1,1)} = -\frac{48G^4m^4\hat{\alpha}}{c^8r^6}f''\left(\varphi^{(0,0)}\right)\varphi^{(0,1)}\varepsilon_{tid}\varepsilon_{\hat{\alpha}}, \end{aligned} \tag{6.21}$$

where the box operator is the d'Alembertian which we can write out in coordinates as we specify our metric being the Schwarzschild metric, as opposed to the flat space d'Alembertian in section 5. As we are interested in the static solutions the operator reduces to the Laplacian in the Schwarzschild metric

$$\begin{aligned} \Box\varphi(r,\theta,\phi) = \Bigg\{ &\frac{1}{r^2}\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}\right] + \left(\frac{2}{r} - \frac{2Gm}{r^2c^2}\right)\frac{\partial}{\partial r} + \\ &\left(1 - \frac{2Gm}{rc^2}\right)\frac{\partial^2}{\partial r^2}\Bigg\}\varphi(r,\theta,\phi). \end{aligned} \tag{6.22}$$

Solutions to these differential equations can be found with the method of separation of variables in a spherical harmonic basis as they are defined as eigenfunctions of the angular part of the laplacian. Therefore we start with the ansatz

$$\varphi = R(r)S(\theta,\phi) = \sum_{\ell m} R^{lm} Y_{lm}(\theta,\phi) = \sum_{\ell m} R^{lm} e^{-im\phi} \mathcal{S}_{\ell m}(\theta), \tag{6.23}$$

with Y_{lm} the spherical harmonics given by

$$Y_{\ell m}(\theta,\phi) = e^{-im\phi}\mathcal{S}_{\ell m}(\theta). \tag{6.24}$$

The θ dependent function has the following properties: $\mathcal{S}_{\ell m}(\theta) = N_{\ell m} P_{\ell m}(\cos\theta)$, with P_{lm} Legendre polynomials and $N^2_{\ell m} = (2\ell+1)(\ell-m)!/[4\pi(\ell+m)!]$, thereby holds the following property for the derivative

$$\frac{\partial^2 \mathcal{S}_{\ell m}}{\partial\theta^2} = -\left[\ell(\ell+1) - \frac{m^2}{\sin^2(\theta)}\right]\mathcal{S}_{\ell m}(\theta) - \frac{\cos\theta}{\sin\theta}\frac{\partial S_{\ell m}}{\partial\theta}. \tag{6.25}$$

These are standard properties of spherical harmonics.

6.1 Solution at $\mathcal{O}(\varepsilon^0_{tid}\varepsilon^0_{\hat{\alpha}})$

Plugging in the ansatz Eq. (6.23) in the 0th order differential equation and taking the derivatives gives

$$-e^{-im\phi}\mathcal{S}_{\ell m}(\theta)\left[(\frac{2Gm}{rc^2} - 1){R^{(0,0)}}''_{lm}(r) + (\frac{2Gm}{r^2c^2} - \frac{2}{r}){R^{(0,0)}}'_{lm}(r) + \frac{l(1+l)}{r^2}R^{(0,0)}_{lm}(r)\right] = 0. \tag{6.26}$$

So we can divide by the ϕ and θ dependent terms and for convenience we rewrite the coordinate r with Eq. (6.14)

$$(1-u){R^{(0,0)}_{B\ell m}}''(u) - {R^{(0,0)}_{\ell m}}'(u) - \frac{\ell(\ell+1)}{u^2}R^{(0,0)}_{\ell m}(u) = 0. \tag{6.27}$$

The solutions to this equation are the hypergeometric functions ${}_2F_1$

$$R^{(0,0)}_{lm} = (-1)^l \frac{1}{u} c_1\, {}_2F_1(-l;-l;-2l;u) + (-1)^{1+l}u^{1+l}c_2\, {}_2F_1(1+l;1+l;2+2l;u). \tag{6.28}$$

As we are not considering any perturbations for this solution we are interested in the ground-state solution for $l=0$. The solution reduces to $R_{00} = c_1 + c_2\log(1-u)$ but as the $\log$ is divergent at the horizon $u=1$, we have to set $c_2 = 0$ to have a well defined solution. Thus the solution for the scalar field becomes to this order

$$\varphi^{(0,0)} = c_1 Y_{00} = c_1\frac{1}{2\sqrt{\pi}} = \varphi_\infty, \tag{6.29}$$

where we chose $c_1 = 2\sqrt{\pi}\varphi_\infty$ absorbing the numerical factors and making sure the solution is equal to the field far from the source of the perturbations.

6.2 Solution at $\mathcal{O}(\varepsilon_{tid}^0\varepsilon_{\hat{\alpha}}^1)$

Substituting the ansatz Eq. (6.23) in the (1,0)th order differential equation and taking the derivatives gives

$$\left(\frac{lS_{lm}(\theta)}{r^2}-\frac{l^2S_{lm}(\theta)}{r^2}\right)R_{lm}^{(1,0)}(r)+\left(-\frac{2GmS_{lm}(\theta)}{c^2r^2}+\frac{2S_{lm}(\theta)}{r}\right)R^{(1,0)\prime}{}_{lm}(r)$$
$$+\left(S_{lm}(\theta)-\frac{2GmS_{lm}(\theta)}{c^2r}\right)R^{(1,0)\prime\prime}{}_{lm}(r)=\frac{48G^4m^4}{c^8r^6}\hat{\alpha}f'(\varphi^{(0,0)}). \tag{6.30}$$

As the RHS has no angular dependence as we can use our previous solution Eq. (6.29) as argument of the coupling function, we can set the indices $l=m=0$ and putting the equation again in terms of u gives

$$(1-u)R_{00}^{(1,0)\prime\prime}(u)-R_{00}^{(1,0)\prime}(u)=-6\sqrt{\pi}u^2\hat{\alpha}f'\left(\varphi_\infty\right). \tag{6.31}$$

Of which the solution is given by

$$R_{00}^{(1,0)}=c_2+2\sqrt{\pi}(u+\frac{1}{2}u^2+\frac{1}{3}u^3)f'(\varphi_\infty)+\log[1-u](-c_1+2\sqrt{\pi}\hat{\alpha}f'(\varphi_\infty)). \tag{6.32}$$

For a regular solution we choose c_1 such that the $\log$ term vanishes. Also at infinity the theory should reduce to GR again and thus should not have a contribution at this order. Therefore we choose $c_2=0$. Then for the total φ at this order we have

$$\varphi^{(1,0)}=R_{00}^{(1,0)}Y_{00}=\hat{\alpha}f'\left(\varphi_\infty\right)\left(u+\frac{1}{2}u^2+\frac{1}{3}u^3\right), \tag{6.33}$$

hereby reproducing the result in [24].

So the total scalar field up to linear order in the coupling is given by

$$\varphi_{background}=\varphi_\infty+\frac{\alpha}{r_S^2}f'\left(\varphi_\infty\right)\left(\frac{r_S}{r}+\frac{1}{2}(\frac{r_S}{r})^2+\frac{1}{3}(\frac{r_S}{r})^3\right), \tag{6.34}$$

without any tidal perturbation. There is no angular dependence and radially the scalar field scales as (assuming the scalar field at infinity to be zero) shown in Fig. 16.

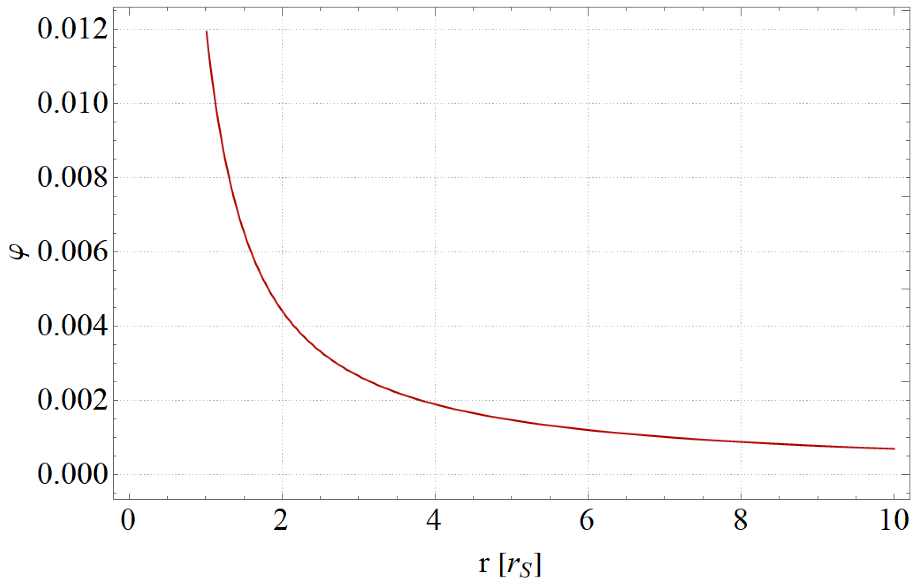


Figure 16: Radial profile of the unperturbed scalar field around a black hole as a function of the radius in units of r_S.

6.3 Solution at $\mathcal{O}(\varepsilon_{tid}^1 \varepsilon_{\hat{\alpha}}^0)$

Substituting in the ansatz Eq. (6.23) in the (0,1)th order differential equation and taking the derivatives gives

$$e^{-im\phi} S_{lm}(\theta) \left(-\frac{l}{r}(1+l) R_{lm}^{(1,0)}(r) + 2\left(1 - \frac{Gm}{c^2 r}\right) R_{lm}^{(1,0)\prime}(r) + \left(r - \frac{2Gm}{c^2}\right) R_{lm}^{(1,0)\prime\prime}(r) \right) = 0. \tag{6.35}$$

Dividing by the angular terms again and expressing the equation in terms of u

$$(1-u) R_{lm}^{(1,0)\prime\prime}(u) - R_{lm}^{(1,0)\prime}(u) - \frac{l(l+1)}{u^2} R_{lm}^{(1,0)}(u) = 0. \tag{6.36}$$

Which again has the solution with the hypergeometric functions ${}_2F_1$ but this time with the argument $1+2l$ compared to Eq. (6.28). As we are interested in the first order tidal effects corresponding to the dipole we can set $l=1$ to obtain explicitly

$$R_{1m}^{01} = \frac{(-2+u)c_1 + 4c_2}{u} - \frac{(-2+u)c_2 \log[1-u]}{u}. \tag{6.37}$$

Setting $c_2 = 0$ to eliminate the $\log$ due to the boundary condition that the solution mucht be regular at the horizon at $u=1$. We rename the constant $c_1 = c_{10}$. So we have for φ

$$\varphi^{(1,0)} = \sum_{m=-1}^{1} R_{1m}^{(0,1)} Y_{1m} = \sum_{m=-1}^{1} c_1 (1 - \frac{2}{u}) Y_{1m}. \tag{6.38}$$

6.4 Solution at $\mathcal{O}(\varepsilon_{tid}^1 \varepsilon_{\hat{\alpha}}^1)$

Next substituting the ansatz Eq. (6.23) in the (1,1)th order differential equation and taking the derivatives gives

$$\begin{aligned} &e^{-im\phi} S_{lm}(\theta) \left(-\frac{l(1+l)}{r^2} R_{lm}^{(1,1)}(r) + 2(\frac{1}{r} - \frac{Gm}{c^2 r^2}) R_{lm}^{(1,1)\prime}(r) + (1 - \frac{2Gm}{c^2 r}) R_{lm}^{(1,1)\prime\prime}(r) \right) \\ &= -\frac{48 G^4 m^4 \hat{\alpha}}{c^8 r^6} f''(\varphi_\infty) e^{-im\phi} S_{lm}(\theta) R_{lm}^{(0,1)}. \end{aligned} \tag{6.39}$$

Dividing by the angular terms and rewriting everything in terms of u

$$(1-u) R_{1m}^{11\prime\prime}(u) - R_{1m}^{11\prime}(u) - \frac{2}{u^2} R_{1m}^{11} = -3u^2 f''\left(\varphi_\infty\right) R_{1m}^{01}(u). \tag{6.40}$$

Substituting the previous solution Eq. (6.37) and solving gives

$$R_{1m}^{(1,1)} = -\frac{(-2+u) log[1-u](c_2 - 7c_{10} f''(\varphi_\infty)}{u} + \frac{6(-2+u)c_1 + 24c_2 + c_1 u(84 - 7u^2 + 2u^3) f''(\varphi_\infty)}{6u} \tag{6.41}$$

We take the constant c_2 again so the $\log$ vanishes. This results in

$$R_{1m}^{11} = [c_1 - 14c_{10} f''\left(\varphi_\infty\right)] \left(1 - \frac{2}{u}\right) + \frac{1}{3} c_{10} f''\left(\varphi_\infty\right) \left(-\frac{7}{2}u^2 + u^3\right). \tag{6.42}$$

Here the first term has the same structure as the $\mathcal{O}(\varepsilon_{tid}^1 \varepsilon_{\hat{\alpha}}^0)$ solution Eq. (6.38) corresponding to an external field. We are interested in the tidal field of order $\varepsilon_{\hat{\alpha}}^1$, therefore we set this first term to zero fixing c_1.

$$\varphi^{(1,1)} = \sum_{m=-1}^{1} Y_{1m} \frac{1}{3} c_1 f''\left(\varphi_\infty\right) \left(-\frac{7}{2}u^2 + u^3\right). \tag{6.43}$$

Thus the total scalar field solution for first order in the perturbation is given by

$$\varphi^{(1)} = \sum_m \left\{ c_{10} \left(1 - \frac{2r}{r_S}\right) + \frac{1}{6} \hat{\alpha} f''\left(\varphi_\infty\right) c_{10} (-7 + 2\frac{r_S}{r}) \right\} Y_{1m} + O\left(\hat{\alpha}^2\right) \tag{6.44}$$

So the perturbation in the scalar field is proportional to $l = 1$ spherical harmonics corresponding to the dipole. Plotting the value of the perturbed scalar field squared as radius (taking the real part of the spherical harmonics) results in a rescaled dipole spherical harmonic shape as shown in Fig.17.

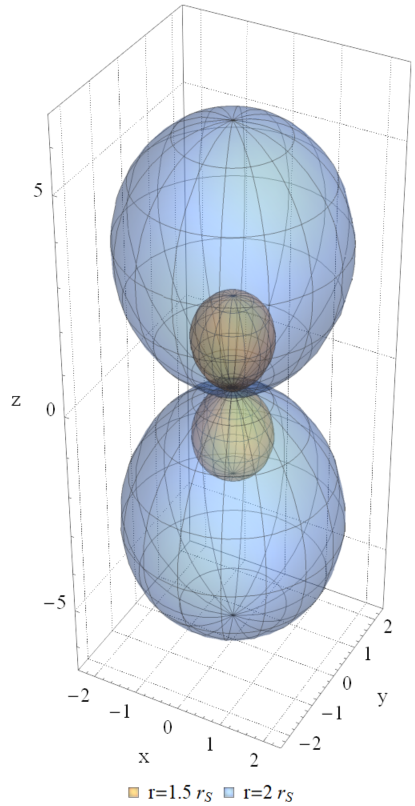


Figure 17: The value of the scalar field squared as radii varying over the angles θ: $[0, \pi]$, ϕ: $[0, 2\pi]$. The blue surface corresponds to $r = 2r_S$ and orange surface to $r = 1.5r_S$.

Thus we have some symmetric field in the ϕ direction. For angles in θ ranging from 0 to $\frac{1}{2}\pi$ we have a positive values scalar field and for angles ranging from $\frac{1}{2}\pi$ to π a negative value. As the radius is larger for larger distances r we have that the perturbations in the scalar field become larger in magnitude further away from the black hole.

It is interesting to compare this to gravitational tidal effects in GR which perturb the gravitational field in a quarupolar manner, hence resulting in the field being proportional to the $l = 2$ spherical harmonic depicted in Fig. 18.

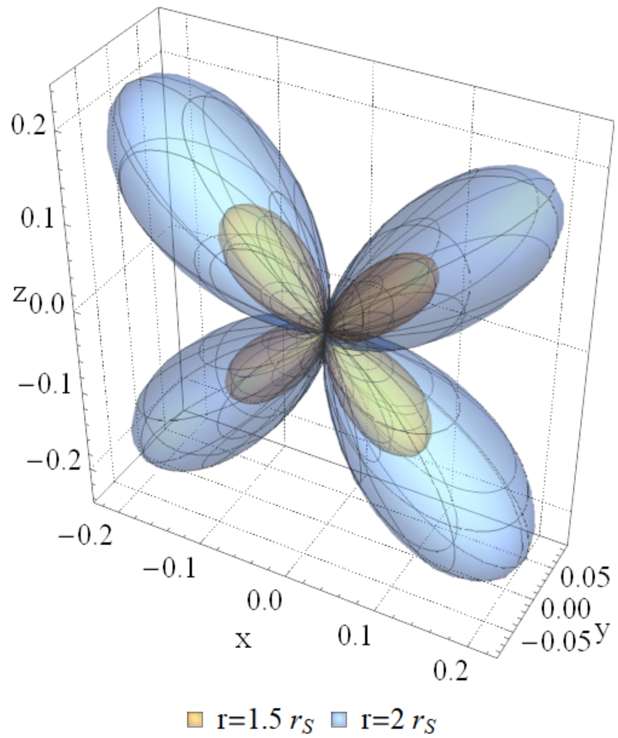


Figure 18: Real part of spherical harmonic $\sum_{m=-2}^{2} Y_2^m$ squared varying over the angles θ: $[0, \pi]$, ϕ :$[0, 2\pi]$. The blue surface corresponding to $r = 2r_S$ and orange surface to $r = 1.5r_S$.

Fig. 19 shows a sketch of the scalar fields around the black hole based on the solutions we found above.

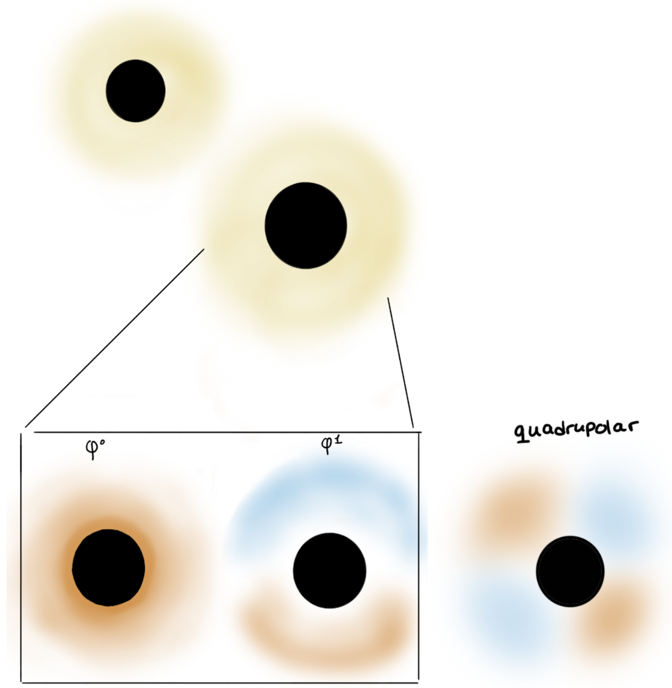


Figure 19: Sketch of the unperturbed and perturbed scalar fields around a black hole. Blue color corresponds to negative values and orange colors to positive values. For comparison a quadrupolar scaling scalar field is also shown.

6.5 Extracting the tidal deformability parameter

To read off the dipole moment and tidal field from our solution, we perform an asymptotic expansion of our scalar field solution as in Eq. (6.10). As we are solely interested in the induced dipole and tidal field terms we only have to look at the linear perturbation part of the solution given by Eq. (6.44). Expanding for $r \to \infty$ or $u \to 0$ gives

$$\lim_{u\to 0} \varphi^{(1)} = \sum_{m=-1}^{1} Y_{1m}(\theta,\varphi) c_{10} \left[-\frac{7\hat{\alpha} u^2 f''(\varphi_\infty)}{6} - \frac{2c^{10}}{u} \right] + \dots . \tag{6.45}$$

Or in terms of r

$$\lim_{r\to\infty} \varphi^{(1)} = \sum_{m=-1}^{1} Y_{1m}(\theta,\varphi) c_{10} \left[-\frac{14\hat{\alpha} G^2 m^2 f''(\varphi_\infty)}{3c^4} \frac{1}{r^2} + \frac{c^2}{Gm} r \right] + \dots . \tag{6.46}$$

Comparing with Eq. (6.10) we can read off

$$\tilde{Q}^{(s)}_{1m} = -c_{10}\frac{14\hat{\alpha}Gm^2 f''\left(\varphi_\infty\right)}{3c^2}, \quad \tilde{E}^{(s)}_{1m} = \frac{c^2}{Gm}c_{10}. \tag{6.47}$$

Thus the tidal deformability parameter is given by

$$\lambda^{(s)}_{\ell=1} = -\frac{\tilde{Q}^{(s)}_{1m}}{\tilde{E}^{(s)}_{1m}} = \frac{14}{3}\hat{\alpha}\frac{G^2m^3}{c^4}f''\left(\varphi_\infty\right), \tag{6.48}$$

$$\boxed{\lambda^{(s)}_{\ell=1} = \frac{7}{6}m\alpha f''\left(\varphi_\infty\right).} \tag{6.49}$$

This tells us that the sensitivity of the scalar field of a black hole in sGB gravity depends linearly on its mass, coupling and second derivative of the coupling function.

7 Scalar waveform and phase evolution

Thus far we only looked at the dynamics of the black hole binary system and calculated the fields up to 1PN in the near zone, but we not yet calculated its radiation. Ultimately we are interested in the effect of the GB and tidal terms on the GW signal from these systems. As we have seen in section 2, one can calculate the gravitational waveforms by solving the Einstein field equations for a perturbation in the gravitational field. As we are considering a system of gravitationally bound black holes we have to look further than linearized gravity and use the relaxed Einstein equations as our starting point. However in the case of sGB gravity as opposed to GR, we also have a scalar field which can radiate too. We solve the EOMS up to 1PN with the DIRE approach as already qualitatively discussed in section 2.4.2. This calculation in sGB gravity is previously in [16] [17] and we rederive the scalar field results from scratch, correcting some algebraic errors. Up to the GB related terms, the the expression should be consistent with the results in the same calculation in scalar tensor theory done in [100][101][102][92].
In this thesis we are interested in the effect of including tidal terms in the calculation of the waveform compared to the 1PN GR and 1PN GB terms. As shown in section 5 the tidal terms do not contribute up to 1PN to the solutions for the near zone fields. However at 1PN there is a non-vanishing tidally induced scalar dipole moment as shown in section 6.5. In the calculation of the the scalar waveform, expanding the field in multipole moments, the tidal term is included in the dipole moment. As the tidal terms do not play a role in the metric perturbation calculation at the orders we are considering, we will use the results from [17] in this regard and only calculate explicitly the scalar waveform including the tidal terms. Using our results for the scalar waves together with the solutions for the tensor waves from [17], we derive the phasing terms of the radiation using energy balance: the change in binding energy which we calculated in section 5.7 should be equal to the rate of change of the radiation, a.k.a the energy flux out of the binary. As the binding energy also has a tidal contribution, the phasing terms have this contribution as well.
With these phasing terms we conduct an analysis of the expressions to quantify the effect of the tidal terms compared to the GB and GR terms, and study the dependencies on the parameters.

7.1 Waveform calculation

As discussed in section 2.4.1 the Einstein equations can be written in a wave equation form named the relaxed Einstein equations, using the gothic metric from Eq. (2.55). Together with the harmonic gauge condition from Eq. (2.57) and the definition of the field $h^{\alpha\beta}$ from Eq. (2.56), the sGB field equations from Eq. (5.7) and Eq. (5.10) can be written in the relaxed form as well.

This results in [17]

$$
\begin{aligned}
\Box_\eta h^{\alpha\beta} &= \frac{16\pi G}{c^4}\mu^{\alpha\beta}, \\
\mu^{\alpha\beta} &= (-g)T_m^{\alpha\beta} + \frac{c^4}{16\pi G}\left(\Lambda_{GB}^{\alpha\beta} + \Lambda_{GR}^{\alpha\beta}\right), \\
\Lambda_{GR}^{\alpha\beta} &= \frac{16\pi G}{c^4}(-g)t_{LL}^{\alpha\beta} + h^{\alpha\nu}_{,\mu}h^{\beta\mu}_{,\nu} - h^{\mu\nu}h^{\alpha\beta}_{,\mu\nu}, \\
\Lambda_{GB}^{\alpha\beta} &= -8\alpha(-g)\left({}^*\hat{R}^{*c\alpha\beta d}f(\varphi)_{,cd}\right) + 4\varphi_{,c}\varphi_{,d}\left(\mathfrak{g}^{\alpha c}\mathfrak{g}^{\beta d} - \frac{1}{2}\mathfrak{g}^{\alpha\beta}\mathfrak{g}^{cd}\right).
\end{aligned}
\tag{7.1}
$$

Here we denote expressions written in terms of the gothic metric with $\hat{}$. An extra Λ_{GB} is present as opposed to the GR case Eq. (2.59). For the scalar field this becomes

$$
\begin{aligned}
\Box_\eta \varphi &= \frac{4\pi G}{c^4}\mu_s \\
\mu_s &= -\frac{\delta S_m}{\delta\varphi\sqrt{-g}} - \frac{c^4}{16\pi G}\alpha f'(\varphi)\hat{\mathcal{R}}^2_{GB}.
\end{aligned}
\tag{7.2}
$$

This equation still has the same form as the original equation of motion from Eq. (5.7) but the expression of the GB invariant is now in terms of the gothic metric.

Here we will focus on the calculation of the waveform and energy loss regarding the scalar field, as the tidal terms only play a role here. We use the results of the calculation for the metric perturbations in [17]. The calculation of the latter is similar to the scalar field case, however the scalar field calculation is a bit less involved as it is a scalar instead of a tensor.

Using a retarded Greens function C.10a[23], we can write the wave equations as the integrals

$$
h^{\alpha\beta}(t,\mathbf{x}) = -\frac{4G}{c^4}\int d^4x' \frac{\mu^{\alpha\beta}(t',\mathbf{x}')\,\delta\left(t'-t+|\mathbf{x}-\mathbf{x}'|/c\right)}{|\mathbf{x}-\mathbf{x}'|}, \tag{7.3}
$$

$$
\varphi(x) = -\frac{G}{c^4}\int d^4x' \frac{\mu_s(t',\mathbf{x}')\,\delta\left(t'-t+|\mathbf{x}-\mathbf{x}'|/c\right)}{|\mathbf{x}-\mathbf{x}'|}. \tag{7.4}
$$

To solve for the scalar waveform φ up to 1PN we make use of the DIRE approach described in section 2.4.2. This comes down to splitting the integration domain, which is the past lightcone, into the contribution from the near zone crossing the past lightcone and the domain outside in the far zone as illustrated in Fig.8. We first focus on the near zone contribution and we will argue that the far zone contribution is higher PN order than we are considering here.

7.2 Near zone contribution to the scalar waveform

For the scalar waveform we start with Eq. (7.4). Since we are in the near zone (defined in section 2.4.3) the integration variable $\mathbf{x}'$ satisfies $|\mathbf{x}'| < \lambda$ while for our field point at the

[23]Here we choose again the retarded Greens function instead of the half retarded, half advanced Greens function in the calculation in section 5. The retarded Greens function depends on the sources in the past, as is the case here for the waveforms sources by the binary system.

detector, we have $R = |\mathbf{x} - \mathbf{x}'| > \lambda$. Therefore we can expand the integration variable $\mathbf{x}'$ in the $|\mathbf{x} - \mathbf{x}'|$ terms as an expansion in $\mathbf{x}'/R$ to express the field in the form

$$\begin{aligned}\varphi(x) &= \sum_{l=0}^{\infty} \delta\varphi_l(x) \\ &= -\frac{G}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left(\frac{1}{R} I_s^L(\tau) \right),\end{aligned} \tag{7.5}$$

with

$$I_s^L(\tau) = \int_{\mathcal{M}} d^3\mathbf{x}' \mu_s(\tau, \mathbf{x}') \mathbf{x}'^L. \tag{7.6}$$

Here I_s^L are the scalar multipole moments and $\delta\varphi_l(x)$ the scalar field moments. Furthermore we use the retarded time is $\tau = t - R/c$ and $\mathcal{M}$ is the hypersurface cut out by the intersection of the near zone with the constant time hypersurface $t_{\mathcal{M}} = \tau$. As the region $\mathcal{M}$ is bounded, the integral is convergent.

For GWs we are only interested in the spatial part and use the fact that

$$\partial_i I_s(\tau) = \frac{\partial \tau}{\partial x^i} \frac{dI_s}{d\tau} = -\frac{n^i}{c} \frac{dI_s}{d\tau} = -\frac{n^i}{c} \frac{dI_s}{dt}, \tag{7.7}$$

with n^i the unit vector in the observational direction. Finally we have for the scalar field moments

$$\delta\varphi_l(x) = \frac{G}{Rc^4} \frac{n_L}{l!} \left(\frac{\partial}{c\partial t} \right)^l I_s^L + \mathcal{O}\left(R^{-2}\right). \tag{7.8}$$

from Eq.(7.5) follows that summing these moments gives the total scalar waveform. Thus to construct the scalar waveform we are interested in the scalar multipole moments I_s^L and its derivatives.

In section 5 we showed that the 1PN expansion of the scalar field φ is given by (leaving out the tidal terms as we saw they do not contribute) Eq. (5.66) or here written down slightly rewritten

$$\begin{aligned}\Box_\eta \varphi = &-2\alpha \frac{1}{c^4} f'(\varphi_0) \left((\partial_i \partial_j U^{(0)})(\partial_i \partial_j U^{(0)}) - \triangle U^{(0)} \triangle U^{(0)} \right) \\ &+ \frac{4\pi G}{c^4} \sum_A \delta^{(3)}(\mathbf{x} - \mathbf{x_A}(t)) \left[m_A^0 \alpha_A^0 \left(c^2 + (\alpha_A^0 + \frac{\beta_A^0}{\alpha_A^0}) \varphi_c^{(0)} - U^{(0)} - \frac{1}{2} v_A^2 \right) \right] + (A \leftrightarrow B).\end{aligned} \tag{7.9}$$

Now for the multipole moments we need the 1PN expansion of the source term μ_s. To get this expression we factor out the prefactor $\frac{4\pi G}{c^4}$ corresponding to Eq. (7.2). Hence the 1PN expansion of the source term is given by

$$\begin{aligned}\mu_s = &-\frac{\alpha}{2\pi G} f'(\varphi_0) \left((\partial_i \partial_j U^{(0)})(\partial_i \partial_j U^{(0)}) - \triangle U^{(0)} \triangle U^{(0)} \right) \\ &+ \sum_A \delta^{(3)}(\mathbf{x} - \mathbf{x_A}(t)) \left[m_A^0 \alpha_A^0 \left(c^2 + (\alpha_A^0 + \frac{\beta_A^0}{\alpha_A^0}) \varphi_c^{(0)} - U^{(0)} - \frac{1}{2} v_A^2 \right) \right].\end{aligned} \tag{7.10}$$

Note however that we did not write the R^2_{GB}, corresponding to the first term in Eq. (7.10), in terms of the gothic metric. The expression for R^2_{GB} in terms of the gothic metric as given in [17], have the same dependencies on the potentials as in Eq. (7.10). In this reference is also shown that the integral over this term in the expression for the multipole moments Eq. (7.6) is zero, hence this contribution is vanishing[24]. Therefore we will not explicitly write R^2_{GB} in terms of the gothic metric, but the (very long) expression can be found in Appendix A of [17].

Thus the source term up till 1PN is given by

$$\mu_s = \sum_A \delta^{(3)}(\mathbf{x} - \mathbf{x_A}(t)) m^0_A c^2 \alpha^0_A (1 - \frac{v^2_A}{2c^2} - \frac{U^{(0)}}{c^2} + (\alpha^0_A + \frac{\beta^0_A}{\alpha^0_A})\frac{\varphi^{(0)}_c}{c^2}). \tag{7.11}$$

The multipole moments are computed from Eq. (7.11) and Eq. (7.6)

$$\begin{aligned} I_s &= m^0_A c^2 \alpha^0_A \left\{ 1 - \frac{v^2_A}{2c^2} - \frac{G m^0_B \alpha_{AB}}{r c^2} + \mathcal{O}\left(c^{-4}\right) \right\} + (A \leftrightarrow B), \\ I^i_s &= x^i_A m^0_A c^2 \alpha^0_A \left\{ 1 - \frac{v^2_A}{2c^2} - \frac{G m^0_B \alpha_{AB}}{r c^2} + \mathcal{O}\left(c^{-4}\right) \right\} + (A \leftrightarrow B), \\ I^{ij}_s &= x^{ij}_A m^0_A c^2 \alpha^0_A \left\{ 1 + \mathcal{O}\left(c^{-2}\right) \right\} + (A \leftrightarrow B), \\ I^{ijk}_s &= x^{ijk}_A m^0_A c^2 \alpha^0_A \left\{ 1 + \mathcal{O}\left(c^{-2}\right) \right\} + (A \leftrightarrow B), \end{aligned} \tag{7.12}$$

using the lowest order solution of potential U from Eq. (5.55) and φ_0 from Eq. (5.56) and $\alpha_{AB} = \left(1 + \alpha_A \alpha_B + \beta_A \alpha_B \alpha_A^{-1}\right)$. To continue we need to cast these expressions in the centre of mass frame of our binary system and calculate the appropriate derivatives according to Eq. (7.8). The induced scalar tidal dipole moment adds linearly to the orbital dipole from Eq.(7.12).

The reason why we have expanded the multipoles to the specific orders given in Eq. (7.12) is because the quadrupole and octupole get an extra factor of $1/c^2$ because of the time derivatives taken in Eq. (7.8).

The PN order counting will work for this calculation a bit different than in the section before. As gravitational radiation is a purely GR phenomemon, we do not have a Newtonian order result anymore. It is convention to take the GR result as 0PN. This is given by the order of the quadruoplar formula. We saw in Eq. (2.51) this expression is proportional to $1/c^4$, which thus correspond to 0PN in this context. As the lowest order terms in the source of the scalar field are proportional to c^2, multiplied with $1/c^4$ in the formula of the scalar field moments Eq. (7.8), this will be of order $1/c^2$ and thus relative -1PN order.

[24]One can show that after integrating by parts the contributing terms either depend on $\mathcal{R}$ or are proportional to $\nabla^2 U = -4\pi \sum_A m_A \delta^3(\mathbf{x} - \mathbf{x}')$. The former means that the expression depends on the term splitting the DIRE integral contributions. The answers should not depend on this boundary and therefore will vanish. The latter gives terms that are proportional to $\delta^3(\mathbf{x}_A - \mathbf{x}_B)$ which is always vanishes during the inspiral as $\mathbf{x}_A \neq \mathbf{x}_B$.

With these scalar multipole moments we can calculate the scalar field multipole moments Eq. (7.8) by first transforming these expressions to the CM frame and taking the appropriate time derivatives.

The CM frame coordinates coordinates are given by the generalised version of Eq. (4.6)

$$X^i_{CM} = \frac{1}{m}\int \mu^{00} r^i d^3\mathbf{r}, \tag{7.13}$$

with μ^{00} the component of the gravitational source term in Eq. (7.1) which can be expanded to $1/c^2$. The integral is done in [17] which gives for the positions

$$\begin{aligned} \mathbf{x}_A &= \left[\frac{m_B}{m} + \frac{\mu\Delta m}{2m^2c^2}\left(v^2 - \frac{Gm\bar{\alpha}}{r}\right)\right]\mathbf{r} + \delta + \mathcal{O}\left(c^{-3}\right) \\ \mathbf{x}_B &= \mathbf{x}_A \,\text{with}\, m_B \to -m_A, \end{aligned} \tag{7.14}$$

with

$$\delta = -2\eta\left(\frac{Gm\bar{\alpha}}{rc^2}\frac{\alpha f'(\varphi_0)}{\sqrt{\bar{\alpha}}r^2}\mathcal{S}_+\right)\mathbf{r}. \tag{7.15}$$

And for the velocities, differentiating the expressions above to time

$$\begin{aligned} \mathbf{v}_A &= \frac{m_B}{m}\mathbf{v} + \frac{\mu\Delta m}{2m^2c^2}\left[\left(v^2 - \frac{Gm\bar{\alpha}}{r}\right)\mathbf{v} - \frac{Gm\bar{\alpha}}{r^2}\dot{r}\mathbf{r}\right] + \dot{\delta} + \mathcal{O}\left(c^{-3}\right), \\ \mathbf{v}_B &= \mathbf{v}_A \,\text{with}\, m_B \to -m_A, \end{aligned} \tag{7.16}$$

with

$$\dot{\delta} = 2\eta\frac{Gm\bar{\alpha}}{rc^2}\frac{\alpha f'(\varphi_0)}{\sqrt{\bar{\alpha}}r^2}\mathcal{S}_+(3\dot{r}\mathbf{r} - \mathbf{v}). \tag{7.17}$$

We also recall the earlier defined expressions Eq. (5.80) and Eq. (5.87) in which we will express the equations again. This makes the expressions easier to compare with other literature. In literature for scalar tensor theory, they use the terms $\mathcal{S}_\pm, \beta_\pm, \bar{\gamma}$ as well and although the definitions for these terms are different as it is a different theory, they play the same 'role'. Therefore the dependencies of the terms in our expressions excluded from the GB and tidal terms, can be directly compared to the results for scalar tensor theory in [100][101][102].

By doing the differentiation in Eq. (7.8) we make use of the following identities

$$\begin{aligned} &\frac{\partial \mathbf{r}}{\partial t} = \dot{r}, && \frac{\partial \mathbf{v}}{\partial t} = \mathbf{a}, \\ &\frac{\partial v}{\partial t} = \frac{\mathbf{v}\cdot\mathbf{a}}{v}, && \frac{\partial \mathbf{n}}{\partial t} = 0, \\ &\mathbf{r}\cdot\mathbf{v} = \dot{r}r, \\ &\ddot{r} = \frac{\dot{r}^2}{r} + \frac{v^2}{r} + \frac{\mathbf{r}\cdot\mathbf{a}}{r}, \end{aligned} \tag{7.18}$$

and the relative acceleration $\mathbf{a}$ given by Eq. (5.86).

7.2.1 Monopole moment

Starting with the monopole moment I_s in Eq. (7.12). Substituting the CM expression for v_A and v_B and expanding to order $\mathcal{O}(c^0)$ gives

$$\begin{aligned} I_{s,CM} =& m\bar{\alpha}^{1/2}\left(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) \\ &+ \mu\bar{\alpha}^{1/2}\left(-\frac{1}{2}\left(\mathcal{S}_+ - \frac{\Delta m}{m}\mathcal{S}_-\right)v^2 + \left[-2\mathcal{S}_+ + \frac{8}{\bar{\gamma}}\left(\mathcal{S}_+\bar{\beta}_+ + \mathcal{S}_-\bar{\beta}_-\right)\right]\frac{G\bar{\alpha}m}{r}\right). \end{aligned} \tag{7.19}$$

Then substituting in Eq. (7.8) results in

$$\begin{aligned} \delta\varphi_0 =& \frac{Gm\sqrt{\bar{\alpha}}}{Rc^2}\left(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) \\ &+ \frac{G\mu\sqrt{\bar{\alpha}}}{Rc^4}\left\{-\frac{v^2}{2}\left(\mathcal{S}_+ - \frac{\Delta m}{m}\mathcal{S}_-\right) + \left(\frac{8}{\bar{\gamma}}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) - \mathcal{S}_+\right)\frac{G\bar{\alpha}m}{r}\right\} \end{aligned} \tag{7.20}$$

Thus as opposed to GR we have a non-vanishing monopole moment. However only the parts depending on v and r contribute to the scalar radiation.

7.2.2 Dipole moment

Continuing for the dipole moment I_s^i from Eq. (7.12). To this expression we add the contribution of the induced scalar tidal dipole moment. This dipole moment is given in section 4.2, here we slightly change the notation and as before we are only interested in the spatial part

$$Q_i^{tid} = -\lambda^{(s)}\partial_i\varphi. \tag{7.21}$$

We found the lowest order solution of the scalar field to be Eq. (5.56). Substituting in the induced dipole moment results in

$$\begin{aligned} Q_i^{tid} &= \frac{\lambda_A^{(s)}}{c^2}\partial_i\frac{Gm_A^0\alpha_A^0}{r} + \frac{\lambda_B^{(s)}}{c^2}\partial_i\frac{Gm_B^0\alpha_B^0}{r} \\ &= \bar{\zeta}\frac{r_i}{c^2r^3}, \end{aligned} \tag{7.22}$$

with

$$\bar{\zeta} = \lambda_A Gm_B^0\alpha_B^0 + \lambda_B Gm_A\alpha_A^0. \tag{7.23}$$

Note that this tidal term is slightly different from the tidal term that contributed to the binding energy Eq. (5.82), among others does Eq. (7.23) depend linearly instead of quadratic on the scalar charge.

Then the scalar dipole moment in relative coordinates including the tidal term is given by

$$
\begin{aligned}
I^i_{s,CM} = \mu\bar{\alpha}^{1/2}c^2 \Bigg\{ & 2\mathcal{S}_- r^i + \left(\frac{\Delta m}{m}\mathcal{S}_+ - \eta\mathcal{S}_-\right)\frac{v^2}{c^2}r^i \\
& + \left[\frac{1}{2}\frac{\Delta m}{m}\mathcal{S}_+ + \left(-\frac{3}{2} + 2\eta\right)\mathcal{S}_- - \frac{4}{\bar{\gamma}}\frac{\delta m}{m}\left(\mathcal{S}_+\bar{\beta}_+ + \mathcal{S}_-\bar{\beta}_-\right) + \frac{4}{\bar{\gamma}}\left(\mathcal{S}_-\bar{\beta}_+ + \mathcal{S}_+\bar{\beta}_-\right)\right]\frac{G\alpha m}{c^2 r}r^i \\
& - \frac{2Gm\bar{\alpha}}{r}\frac{r^i}{r^2}\frac{\alpha f'[\varphi_0]}{\sqrt{\bar{\alpha}}c^2}S_+\left(\frac{\Delta m}{m}S_- + S_+\right) + \frac{r^i\bar{\zeta}}{c^2\sqrt{\bar{\alpha}}r^3\mu}\Bigg\}.
\end{aligned}
\tag{7.24}
$$

Taking one time derivative and substituting in Eq. (7.8) gives

$$
\begin{aligned}
\delta\varphi_1 = \frac{G\mu\sqrt{\bar{\alpha}}}{Rc^3}\Bigg\{ & (\mathbf{n}\cdot\mathbf{v})\Bigg[2\mathcal{S}_- + \frac{v^2}{c^2}\left(\frac{\Delta m}{m}\mathcal{S}_+ - \eta\mathcal{S}_-\right) \\
& + \frac{G\bar{\alpha}m}{rc^2}\Bigg(\frac{\Delta m}{2m}\mathcal{S}_+ + \left(2\eta - \frac{3}{2}\right)\mathcal{S}_- - \frac{4}{\bar{\gamma}}\frac{\Delta m}{m}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) + \frac{4}{\bar{\gamma}}\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right) \\
& - 2\mathcal{S}_+\left(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right)\frac{\alpha f'(\varphi_0)}{\sqrt{\bar{\alpha}}r^2} + \frac{\bar{\zeta}}{\bar{\alpha}^{3/2}Gm^2\eta r^2}\Bigg)\Bigg] + \frac{G\bar{\alpha}m}{c^2r^2}\dot{r}(\mathbf{n}\cdot\mathbf{r})\Bigg[-\frac{5}{2}\frac{\Delta m}{m}\mathcal{S}_+ + \frac{3}{2}\mathcal{S}_- \\
& + \frac{4}{\bar{\gamma}}\frac{\Delta m}{m}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) - \frac{4}{\bar{\gamma}}\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right) + 6\mathcal{S}_+\left(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right)\frac{\alpha f'(\varphi_0)}{\sqrt{\bar{\alpha}}r^2} \\
& - \frac{3\bar{\zeta}}{\bar{\alpha}^{3/2}Gm^2\eta r^2}\Bigg]\Bigg\}.
\end{aligned}
\tag{7.25}
$$

Hence in this theory when moving to the centre of mass frame the dipole does not vanish compared with GR, see section 4. As we will see in the total waveform, the first term of the dipole moment sources the lowest order contribution of the scalar radiation.

7.2.3 Quadrupole moment

Next we look at the quadrupole moment I_s^{ij} in Eq. (7.12). In relative coordinates it becomes

$$
I^{ij}_{s,CM} = \mu\sqrt{\bar{\alpha}}c^2(\mathcal{S}_+ - \frac{\Delta m}{m}\mathcal{S}_-)r^i r^j. \tag{7.26}
$$

Taking two time derivatives and substituting in Eq. (7.8) gives

$$
\delta\varphi_2 = \frac{G\mu\sqrt{\bar{\alpha}}}{Rc^4}\left(\mathcal{S}_+ - \frac{\Delta m}{m}\mathcal{S}_-\right)\left\{(\mathbf{n}\cdot\mathbf{v})^2 - \frac{G\bar{\alpha}m}{r}\left(\frac{\mathbf{n}\cdot\mathbf{r}}{r}\right)^2\right\} \tag{7.27}
$$

7.2.4 Octupole moment

Lastly for the octupole moment I_s^{ijk} in Eq. (7.12), we write the expression in relative coordinates

$$
I^{ijk}_{s,CM} = \mu\sqrt{\bar{\alpha}}c^2(-\frac{\Delta m}{m}\mathcal{S}_+ + (1-2\eta)\mathcal{S}_-)r^i r^j r^k. \tag{7.28}
$$

Taking three time derivatives and substituting again in Eq. (7.8) gives

$$\delta\varphi_3 = \frac{G\mu\sqrt{\bar{\alpha}}}{Rc^5}\left((1-2\eta)\mathcal{S}_- - \frac{\Delta m}{m}\mathcal{S}_+\right)\left(\frac{3}{2}\frac{G\bar{\alpha}m}{r^4}\dot{r}(\mathbf{n}\cdot\mathbf{r})^3 - \frac{7}{2}\frac{G\bar{\alpha}m}{r^3}(\mathbf{n}\cdot\mathbf{v})(\mathbf{n}\cdot\mathbf{r})^2 + (\mathbf{n}\cdot\mathbf{v})^3\right). \tag{7.29}$$

7.2.5 Far zone contribution

As described in appendix C of [17] the far zone contributions to the tensor and scalar waveforms are beyond 1PN order. We will make an argument why this is the case for the scalar waveform. In the far zone the integration domain lies outside of the source so does not contain any matter contribution, see Fig.8. The source μ_s therefore only contains terms related to the fields. In the far zone these fields have contributions coming from the near zone and the far zone, which add to give the total field contribution in this region. This time we can not just substitute the lowest order field solutions we found in section 5 in our expression for μ_s as these fields are only calculated in the near zone. The near zone contributions to these field can be calculated with Eq. (7.5) and similar expression for the metric fields, which gives the same results for the fields as described above and in [17] for the metric perturbation field.

The far zone contributions to the field can be calculated from the far zone field integrals [100], given by Eq. (7.3) and Eq. (7.4) which in the far zone can not be expanded in $|x'|/R$ as in the near zone. However one can do a change of coordinates to retarded time. The integral domain in this case is over the far zone excluding the near zone. However these contributions to the waveforms can only come from backreaction effects of the GWs as in this region there is no matter source. Backreaction effects come in at higher PN order which makes the far zone contributions to the field also higher PN order than we consider. This is shown explicitly for scalar tensor theory in [100, 101].
To actually calculate the waveform contribution in the far zone one can use these far zone integrals again over the source terms without the matter part, substituting the solutions for the fields, which thus contain only a contribution from the near zone up till the PN order we are considering. However in the source term of the scalar field without the matter in Eq. (7.2), the only contribution is coming from R^2_{GB} which we have seen has a vanishing contribution when integrated over up to 1PN in the expansion. So in the orders we are considering we have no far zone contribution to the waveforms for the scalar field.
Similar argument is given for the far zone part of the tensor waveform which is also higher order, see appendix C of [17].

7.2.6 The scalar waveform to relative 0.5PN order

The total scalar waveform is given by the sum of the near zone scalar field moments

$$\begin{aligned}
\varphi &= \delta\varphi_0 + \delta\varphi_1 + \delta\varphi_2 + \delta\varphi_3 \\
&= \frac{Gm\sqrt{\bar{\alpha}}}{Rc^2}\left(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) \\
&+ \frac{G\mu\sqrt{\bar{\alpha}}}{Rc^4}\left\{-\frac{v^2}{2}\left(\mathcal{S}_+ - \frac{\Delta m}{m}\mathcal{S}_-\right) + \left(\frac{8}{\bar{\gamma}}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) - \mathcal{S}_+\right)\frac{G\bar{\alpha}m}{r}\right\} \\
&\frac{G\mu\sqrt{\bar{\alpha}}}{Rc^3}\left\{(\hat{\mathbf{n}}\cdot\mathbf{v})\left[2\mathcal{S}_- + \frac{v^2}{c^2}\left(\frac{\Delta m}{m}\mathcal{S}_+ - \eta\mathcal{S}_-\right)\right.\right. \\
&+ \frac{G\bar{\alpha}m}{rc^2}\left(\frac{\Delta m}{2m}\mathcal{S}_+ + \left(2\eta - \frac{3}{2}\right)\mathcal{S}_- - \frac{4}{\bar{\gamma}}\frac{\Delta m}{m}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) + \frac{4}{\bar{\gamma}}\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right)\right. \\
&\left.\left. -2\mathcal{S}_+\left(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right)\frac{\alpha f'\left(\varphi_0\right)}{\sqrt{\bar{\alpha}}r^2} + \frac{\bar{\zeta}}{\bar{\alpha}^{3/2}Gm^2\eta r^2}\right)\right] + \frac{G\bar{\alpha}m}{c^2r^2}\dot{r}(\mathbf{n}\cdot\mathbf{r})\left[-\frac{5}{2}\frac{\Delta m}{m}\mathcal{S}_+ + \frac{3}{2}\mathcal{S}_-\right. \\
&\left.\left. +\frac{4}{\bar{\gamma}}\frac{\Delta m}{m}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) - \frac{4}{\bar{\gamma}}\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right) + 6\mathcal{S}_+\left(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right)\frac{\alpha f'\left(\varphi_0\right)}{\sqrt{\bar{\alpha}}r^2} - \frac{3\bar{\zeta}}{\bar{\alpha}^{3/2}Gm^2\eta r^2}\right]\right\} \\
&\frac{G\mu\sqrt{\bar{\alpha}}}{Rc^4}\left(\mathcal{S}_+ - \frac{\Delta m}{m}\mathcal{S}_-\right)\left\{(\mathbf{n}\cdot\mathbf{v})^2 - \frac{G\bar{\alpha}m}{r}\left(\frac{\mathbf{n}\cdot\mathbf{r}}{r}\right)^2\right\} \\
&\frac{G\mu\sqrt{\bar{\alpha}}}{Rc^5}\left((1-2\eta)\mathcal{S}_- - \frac{\Delta m}{m}\mathcal{S}_+\right)\left(\frac{3}{2}\frac{G\bar{\alpha}m}{r^4}\dot{r}(\mathbf{n}\cdot\mathbf{r})^3 - \frac{7}{2}\frac{G\bar{\alpha}m}{r^3}(\mathbf{n}\cdot\mathbf{v})(\mathbf{n}\cdot\mathbf{r})^2 + (\mathbf{n}\cdot\mathbf{v})^3\right).
\end{aligned} \tag{7.30}$$

Sorting this per PN order, leaving out the non radiative part of the monopole moment leads to

$$\varphi = \frac{G\mu\sqrt{\bar{\alpha}}}{Rc^3}\left\{P^{-1/2}\tilde{\varphi} + \frac{1}{c}\tilde{\varphi} + \frac{1}{c^2}P^{1/2}\tilde{\varphi} + \mathcal{O}\left(c^{-3}\right)\right\}, \tag{7.31}$$

$$P^{-1/2}\tilde{\varphi} = 2\mathcal{S}_-(\mathbf{n}\cdot\mathbf{v}), \tag{7.32}$$

$$\begin{aligned}
\tilde{\varphi} = &\left(\mathcal{S}_+ - \frac{\Delta m}{m}\mathcal{S}_-\right)\left[-\frac{G\bar{\alpha}m}{r}\left(\frac{\mathbf{n}\cdot\mathbf{r}}{r}\right)^2 + (\mathbf{n}\cdot\mathbf{v})^2 - \frac{1}{2}v^2\right] \\
&+ \frac{G\bar{\alpha}m}{r}\left[-2\mathcal{S}_+ + \frac{8}{\bar{\gamma}}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right)\right],
\end{aligned} \tag{7.33}$$

$$
\begin{aligned}
P^{1/2}\tilde{\varphi} = &\left(-\frac{\Delta m}{m}\mathcal{S}_+ + (1-2\eta)\mathcal{S}_-\right)\left[\frac{3}{2}\frac{G\bar{\alpha}m}{r^4}\dot{r}(\mathbf{n}\cdot\mathbf{r})^3 - \frac{7}{2}\frac{G\bar{\alpha}m}{r^3}(\mathbf{n}\cdot\mathbf{v})(\mathbf{n}\cdot\mathbf{r})^2 + (\mathbf{n}\cdot\mathbf{v})^3\right] \\
&+ (\mathbf{n}\cdot\mathbf{v})\left\{\left(\frac{\Delta m}{m}\mathcal{S}_+ - \eta\mathcal{S}_-\right)v^2 + \frac{G\bar{\alpha}m}{r}\left[\frac{1}{2}\frac{\Delta m}{m}\mathcal{S}_+ + \left(2\eta - \frac{3}{2}\right)\mathcal{S}_- \right.\right. \\
&\left.\left. - \frac{4}{\bar{\gamma}}\frac{\Delta m}{m}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) + \frac{4}{\bar{\gamma}}\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right)\right]\right\} \\
&+ \frac{G\bar{\alpha}m}{r^2}\dot{r}(\mathbf{n}\cdot\mathbf{r})\left[\frac{3}{2}\mathcal{S}_- - \frac{5}{2}\frac{\Delta m}{m}\mathcal{S}_+ + \frac{4}{\bar{\gamma}}\frac{\Delta m}{m}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) - \frac{4}{\bar{\gamma}}\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right)\right] \\
&+ 2\frac{G\bar{\alpha}m}{r}\frac{\alpha f'(\varphi_0)}{\sqrt{\bar{\alpha}}r^2}\mathcal{S}_+\left(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right)\left[3\frac{\dot{r}}{r}(\mathbf{n}\cdot\mathbf{r}) - (\mathbf{n}\cdot\mathbf{v})\right] \\
&- \frac{\bar{\zeta}}{\sqrt{\bar{\alpha}}\mu r^3}\left[3\frac{\dot{r}}{r}(\mathbf{n}\cdot\mathbf{r}) - (\mathbf{n}\cdot\mathbf{v})\right].
\end{aligned}
\tag{7.34}
$$

Where the first term corresponds to -0.5PN, the second term to 0PN and the third term to 0.5PN. The -0.5PN term comes from the dipole moment, the 0PN term from the quadrupole and radiative monopole parts and the 0.5PN term comes from the octupole moment and the other terms in the dipole moment, also including the GB (terms proportional to α directly) and tidal terms. This result is consistent with [17] except for an overall factor of 2. This factor is present in [101] but should be absent in this derivation as the prefactor of our EOM is proportional to 4π and in the cited paper 8π is used.

7.3 The energy loss

With the scalar waveform we can construct the expression for the energy loss related to the rate of change in the waveform. The energy of a wave is in general given by the square of the amplitude. The energy loss would then result from its time derivative. More specifically:

$$
\dot{E}_s = \frac{1}{4\pi}\frac{c^3R^2}{G}\oint \dot{\varphi}^2\, d^2\Omega. \tag{7.35}
$$

The prefactor of this formula is different from that in [17]. The correct prefactor is given by [70] and included in Eq. (7.35). Note that in the differentiation of φ, the relative acceleration appears when differentiating the velocity. For the lowest order term in the scalar waveform in Eq. (7.31) the higher order terms of the relative acceleration contribute. To calculate the term inside the integral of Eq. (7.35), we differentiate ant take the square of the scalar waveform Eq. (7.31). Only terms up to a total factor of $1/c^5$ are kept, corresponding to a relative 0.5PN order. This results in the terms

$$
\dot{\varphi}^2 = P^{-1/2}\dot{\tilde{\varphi}}^2 + \dot{\tilde{\varphi}}^2 + P^{-1/2}\dot{\tilde{\varphi}}\dot{\tilde{\varphi}} + P^{-1/2}\dot{\tilde{\varphi}}P^{1/2}\dot{\tilde{\varphi}}. \tag{7.36}
$$

The integration is done over the directional unit vectors $\mathbf{n}$ in the scalar waveform by using the following identities[99]

$$\begin{aligned} &\int n_{k_1\ldots k_m} d^2\Omega = 0 \quad (m \text{ odd }), \\ &\int n_{k_1\ldots k_m} d^2\Omega = [4\pi/(m+1)!!] \times \left[\delta_{k_1k_2}\ldots\delta_{k_{m-1}k_m} + \text{ distinct permutations }\right] (m \text{ even }). \end{aligned} \tag{7.37}$$

Thus odd products of $\mathbf{n}$ are zero, therefore the product $P^{-\dot{1}/2}\tilde{\varphi}\dot{\tilde{\varphi}}$ vanishes. Up to the required order for the even products, we only have products of 2 and 4 unit vectors

$$\begin{aligned} &\int n_j n_k d^2\Omega = \frac{4\pi}{3}\delta_{jk}, \\ &\int n_j n_k n_n n_p d^2\Omega = \frac{4\pi}{15}(\delta^{jk}\delta^{np} + \delta^{jn}\delta^{kp} + \delta^{jp}\delta^{kn}). \end{aligned} \tag{7.38}$$

This results in the energy loss expression

$$\boxed{\begin{aligned} \dot{E}_S = &\frac{\eta^2}{G\bar{\alpha}c^3}\left(\frac{G\bar{\alpha}m}{r}\right)^4 \left[\frac{4}{3}\mathcal{S}_-^2 + \frac{8}{15c^2}\left(\frac{G\overline{\alpha}m}{r}\left[\left(-23+\eta-10\bar{\gamma}-10\beta_+ + 10\frac{\Delta m}{m}\beta_-\right)\mathcal{S}_-^2\right.\right.\right. \\ &\left.-2\frac{\Delta m}{m}\mathcal{S}_+\mathcal{S}_-\right] + v^2\left[2\mathcal{S}_+^2 + 2\frac{\Delta m}{m}\mathcal{S}_+\mathcal{S}_- + (6-\eta+5\bar{\gamma})\mathcal{S}_-^2 - \frac{10}{\bar{\gamma}}\frac{\Delta m}{m}\mathcal{S}_-\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right)\right. \\ &\left.+\frac{10}{\bar{\gamma}}\mathcal{S}_-\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right)\right] + \dot{r}^2\left[\frac{23}{2}\mathcal{S}_+^2 - 8\frac{\Delta m}{m}\mathcal{S}_+\mathcal{S}_- + \left(9\eta - \frac{37}{2} - 10\bar{\gamma}\right)\mathcal{S}_-^2 - \frac{80}{\bar{\gamma}}\mathcal{S}_+\left(\mathcal{S}_+\beta_+\right.\right. \\ &\left.\left.\left.+\mathcal{S}_-\beta_-\right) + \frac{30}{\bar{\gamma}}\frac{\Delta m}{m}\mathcal{S}_-\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) - \frac{10}{\bar{\gamma}}\mathcal{S}_-\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right) + \frac{120}{\bar{\gamma}^2}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right)^2\right]\right) \\ &-\frac{4}{c^2}\left(\frac{\alpha f'(\varphi_0)\mathcal{S}_-\mathcal{S}_+}{\sqrt{\alpha}r^2}\right)\left(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right)\left[-3\dot{r}^2 + v^2 - \frac{2G\bar{\alpha}m}{3r}\right] \\ &-\frac{16}{c^2}\left(\frac{\alpha f'(\varphi_0)\mathcal{S}_-^2}{\alpha^{3/2}r^2}\right)(3S_+ + \frac{\Delta m}{m}S_-)(-\frac{2Gm\bar{\alpha}}{3r}) \\ &\left.+\frac{2\bar{\zeta}S_-}{Gm\bar{\mu}\alpha^{3/2}c^2r^2}[-3\dot{r}^2 + v^2 - \frac{2Gm\bar{\alpha}}{3r}] + \frac{8\zeta S_-^2}{mc^2r^2}(-\frac{2Gm\bar{\alpha}}{3r}) + \mathcal{O}\left(c^{-3}\right)\right]. \end{aligned}} \tag{7.39}$$

Note that the two different tidal contributions arise, namely the tidal term related to the orbital dynamics ζ in Eq. (5.82) and the tidal term resulting from the induced scalar dipole moment $\bar{\zeta}$ in Eq. (7.23).

We have an extra GB term compared to [17] coming from the product $P^{-1/2}\dot{\tilde{\varphi}}P^{1/2}\dot{\tilde{\varphi}}$ in which $P^{-1/2}\dot{\tilde{\varphi}}$ contains a GB term coming from the higher order terms in the relative acceleration.

7.4 Gravitational wave phase evolution

For the calculation of the phase we also need the tensor energy loss from [17]. This energy loss is calculated along the same lines as the scalar energy loss. The gravitational waveform is constructed from the spatial components of the perturbation in the metric making use of a multipole moment construction in the near zone. Differentiating and taking the square of the waveform, after the integration of the angular integrals results in

$$
\begin{aligned}
\dot{E}_T = &\frac{8}{15}\frac{\eta^2}{G\bar{\alpha}^2 c^5}\left(\frac{G\bar{\alpha}m}{r}\right)^4 \Big\{ \left(12v^2 - 11\dot{r}^2\right) \\
&+ \frac{1}{28c^2}\Bigg[-16\left(170 - 10\eta + 63\bar{\gamma} + 84\beta_+ - 84\frac{\Delta m}{m}\beta_-\right)v^2\frac{G\bar{\alpha}m}{r} \\
&+ (785 - 852\eta + 336\bar{\gamma})v^4 - 2(1487 - 1392\eta + 616\bar{\gamma})v^2\dot{r}^2 + 3(687 - 620\eta + 280\bar{\gamma})\dot{r}^4 \\
&+8\left(367 - 15\eta + 140\bar{\gamma} + 168\beta_+ - 168\frac{\Delta m}{m}\beta_-\right)\dot{r}^2\frac{G\bar{\alpha}m}{r} + 16(1-4\eta)\left(\frac{G\bar{\alpha}m}{r}\right)^2\Bigg] \\
&+ \frac{3f'\left(\varphi_0\right)\alpha}{4\sqrt{\alpha}r^2c^2}\Bigg[-4(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-)\left(-4v^2\left(18\tilde{E} + 13\frac{G\bar{\alpha}m}{r} - 45\dot{r}^2\right)\right. \\
&\left.-\dot{r}^2\left(108\tilde{E} + 85\frac{G\bar{\alpha}m}{r} - 150\dot{r}^2\right) + 54\dot{r}^4\right) + \frac{G\alpha m}{r}\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right)\left(32v^2 + 56\dot{r}^2\right)\Bigg] \\
&+ \frac{45f'\left(\varphi_0\right)\alpha}{8\sqrt{\bar{\alpha}}r^2c^2}\left[\mathcal{S}_+(1+2\eta) + \mathcal{S}_-\frac{\Delta m}{m}\right]\Bigg[-\frac{4}{7}v^2\left(22\tilde{E} + 77\frac{G\alpha m}{r} - \frac{199}{3}\dot{r}^2 + \frac{18}{5}v^2\right) \\
&+\frac{24}{5}\dot{r}^4 - \frac{16}{5}v^2\dot{r}^2 - \frac{\dot{r}^2}{7}\left(992\tilde{E} + 737\frac{G\bar{\alpha}m}{r} - \frac{2404}{3}\dot{r}^2 + \frac{8}{5}v^2\right)\Bigg] + \mathcal{O}\left(c^{-3}\right)\Big\}.
\end{aligned}
\tag{7.40}
$$

We have now the expressions for different energies in the system, namely the binding energy in Eq. (5.105) of the two black holes orbiting each other and the energy losses from the gravitational and scalar radiation out of the system given by Eq. (7.40), Eq. (7.39). In general we assume adiabatic motion during the inspiral. The circular orbits gradually have a shrinking radius, hence the change of the orbital velocity over the period of the orbit is much smaller than 1, or related to the angular velocity this becomes $\dot{w}/w^2 \ll 1$. In this limit the source from the energy loss/flux comes from the change in binding energy:

$$\frac{dE(x)}{dt} = -F(x), \tag{7.41}$$

with $F = \dot{E}_S + \dot{E}_T$ the scalar and tensor energy flux and x given by Eq. (5.104). Using that the angular velocity equals the change in the phase angle: $\dot{\phi}_{phase} = w$ with ϕ_{phase} the phase angle in radians, the energy balance can be rewritten as

$$\frac{d\phi_{phase}}{dt} - \frac{c^3x^3}{G\bar{\alpha}m} = 0, \quad \frac{dx}{dt} + \frac{F(x)}{E'(x)} = 0. \tag{7.42}$$

Thus we are interested in the quantities in terms of x, which we can rewrite using Eq. (5.96) and Eq. (5.104).

For the scalar energy loss from Eq.(7.39) this becomes

$$\dot{E}_S(x) = S4x^4c^5 + (S5 + S5_{GB}x^2c^4 + S5_{tidal}x^2c^4)x^5c^5, \tag{7.43}$$

$$\begin{aligned}
S4 &= \frac{4\eta^2 S_-^2}{3\bar{\alpha}G}, \\
S5 &= \left(\frac{8\eta^2 S_-}{45\bar{\alpha}G}\right)\left(-\frac{30\Delta m(S_-\beta_- + S_+\beta_+)}{\bar{\gamma}m} + \frac{30(S_-\beta_+ + S_+\beta_-)}{\bar{\gamma}} + 10\frac{\Delta m}{m}S_-\beta_-\right. \\
&\left. - S_-(5\bar{\gamma} + 10\beta_+ + 10\eta + 21) + \frac{6S_+^2}{S_-}\right), \\
S5_{GB} &= \left(\frac{4\alpha f'[\varphi_0]\eta^2 S_-}{3\bar{\alpha}^{7/2}G^3m^2}\right)\left(\frac{8S_-}{3\bar{\alpha}}(3S_+ + \frac{\Delta m}{m}S_-) - S_+(S_+ + \frac{\Delta m}{m}S_-)\right), \\
S5_{tidal} &= \left(\frac{2\eta S_-}{3G^3\bar{\alpha}m^3}\right)\left(\frac{\bar{\zeta}}{G\bar{\alpha}^{3/2}m} - \frac{8\eta S_-\zeta}{3}\right).
\end{aligned} \tag{7.44}$$

And for the tensor energy loss from Eq. (7.40) this is given by

$$\dot{E}_T(x) = T5x^5c^5 + (T6 + T6_{GB}x^2c^4 + T6_{tidal}x^2c^4)x^6c^5, \tag{7.45}$$

$$\begin{aligned}
T5 &= \frac{32\eta^2}{5\bar{\alpha}^2G}, \\
T6 &= \left(\frac{2\eta^2}{105\bar{\alpha}^2G}\right)\left(-1247 - 448\bar{\gamma} + 896\frac{\Delta m}{m}\beta_- - 896\beta_+ - 980\eta\right), \\
T6_{GB} &= \left(\frac{128\alpha f'[\varphi_0]\eta^2}{5\bar{\alpha}^{9/2}G^3m^2}\right)\left(-\frac{2\Delta mS_-}{3\bar{\alpha}m} - \frac{2S_+}{\bar{\alpha}} - \frac{233\Delta mS_-}{56m} + \frac{\Delta mS_+}{m} - \frac{261\eta S_+}{28} + S_- - \frac{177S_+}{56}\right), \\
T6_{tidal} &= \frac{128\eta^2\zeta}{15G^3\bar{\alpha}^4m^3}.
\end{aligned} \tag{7.46}$$

Lastly the derivative of the binding energy from Eq. (5.105) with respect to x

$$E'(x) = -\frac{\mu c^2}{2}(1 + E_2'x), \tag{7.47}$$

$$E_2' = -\frac{3}{2} - \frac{\eta}{6} - \frac{4\bar{\gamma}}{3} + \frac{4}{3}\left(\beta_+ - \frac{\Delta m}{m}\beta_-\right) - \frac{40c^4}{3G^2}\frac{\alpha f'(\varphi_0)}{m^2\bar{\alpha}^{7/2}}x^2\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) + \frac{20c^4}{3\bar{\alpha}^2G^2m^3}\zeta x^2. \tag{7.48}$$

Then the total energy flux is given by the sum of the scalar and tensor energy loss. Sorting per PN order this gives

$$F_{total} = F_{-1,S} + F_{0,S} + F_{0,T} + F_{1,T}, \tag{7.49}$$

with the scalar terms from Eq. (7.43) given by

$$F_{-1,S} = \frac{4\eta^2 S_-^2}{3\bar{\alpha}G}c^5x^4, \tag{7.50}$$

$$F_{0,S} = \dot{E}_S(x) - F_{-1,S}. \tag{7.51}$$

And the tensor terms from Eq. (7.40) given by

$$F_{0,T} = \frac{32\eta^2}{5\bar{\alpha}^2 G} c^5 x^5, \tag{7.52}$$

$$F_{1,T} = \dot{E}_T(x) - F_{0,T}. \tag{7.53}$$

There are different methods [103] for treating the ratio of the flux and the derivative of the binding energy in Eq. (7.42) depending in which way one expands the ratio. We follow here the same approximation method as in [17], called Taylor T4. In this approximant one expands the whole ratio to the desired Post Newtonian order.
To expand the ratio of the flux over the derivative of the binding energy we split the calculation in two regimes in which different terms dominate. The first is the regime where the scalar dipole part dominates, corresponding to the -1PN term, this is called the dipolar driven regime (DD). The other is the regime for which the 0PN order tensor flux dominates, this is the quadrupolar driven regime (QD).

We will see in a moment that in the quadrupole driven regime, the dipole term in the ratio scales as $5\bar{\alpha}S_-^2/24x$ compared to the quadrupole term. Therefore for frequencies for which this term becomes dominant instead of the quadrupole term. For these frequencies the dipolar driven regime holds. This is thus given by

$$x_{DD} \ll \frac{5\bar{\alpha}S_-^2}{24}, \quad f_{GW}^{DD} \ll \left(\frac{5}{24}\right)^{3/2} \frac{c^3 \mathcal{S}_-^2 \sqrt{\bar{\alpha}}}{\pi G m}, \tag{7.54}$$

using $x = (\frac{G\bar{\alpha}m\pi f}{c^3})^{2/3}$.

Starting with the dipolar driven regime, here the $F_{-1,S}$ term dominates, hence we factor this term out of the total flux.

$$F^{DD} = \frac{4\eta^2 S_-^2}{3\bar{\alpha}G} c^5 x^4 (1 + f_2^{DD} x + ...), \tag{7.55}$$

with

$$\begin{aligned} f_2^{DD} = &\frac{24}{5\bar{\alpha}S_-^2} + \frac{4S_+^2}{5S_-^2} - \frac{4\beta_+}{3} + \frac{4\beta_-\Delta m}{3m} - \frac{14}{5} - \frac{4\eta}{3} - \frac{2\bar{\gamma}}{3} + \frac{4\beta_- S_+}{\bar{\gamma}S_-} - \frac{4\beta_+\Delta m S_+}{\bar{\gamma}m S_-} + \frac{4\beta_+}{\bar{\gamma}} - \frac{4\beta_-\Delta m}{\bar{\gamma}m} \\ &+ \frac{\alpha f'[\varphi_0] c^4 x^2}{\bar{\alpha}^{5/2} G^2 m^2 S_-} \left(\frac{8S_-}{3\bar{\alpha}} (3S_+ + \frac{\Delta m}{m} S_-) - S_+ (S_+ + \frac{\Delta m}{m} S_-) \right) \\ &+ \frac{\bar{\zeta} c^4 x^2}{2\bar{\alpha}^{7/2} \mu G^3 m^3 S_-} - \frac{4^4 x^2}{3\bar{\alpha}^2 G^2 m^3}. \end{aligned} \tag{7.56}$$

Then the ratio of Eq. (7.55) and Eq. (7.47) becomes, expanding to 1PN corresponding to x^5

$$\frac{F^{DD}(x)}{E'(x)} = -\frac{8\eta c^3 \mathcal{S}_-^2 x^4}{3G\bar{\alpha}m} \left[1 + \left(f_2^{DD} - E_2'\right) x + ...\right]. \tag{7.57}$$

We do the same for the quardupole driven regime. So first factoring the quadrupole term in the total flux.

$$F^{QD} = \frac{32\eta^2}{5\bar{\alpha}^2 G} c^5 x^5 (\xi + f_{2,T}^{nd} x + ...) + \frac{4\eta^2 S_-^2}{3\bar{\alpha} G} c^5 x^4. \tag{7.58}$$

with

$$\xi = \frac{5\bar{\alpha} S_-^2}{24} f_2^{DD}, \tag{7.59}$$

$$\begin{aligned} f_{2,T}^{nd} = &-\frac{8\beta_+}{3} - \frac{4\bar{\gamma}}{3} - \frac{35\eta}{12} + \frac{8\beta_- \Delta m}{3m} - \frac{1247}{336} \\ &- \frac{4\alpha f'[\varphi_0] c^4 x^2}{\bar{\alpha}^{5/2} G^2 m^2} \left(\frac{2}{3\bar{\alpha}} (3S_+ + \frac{\Delta m}{m} S_-) - (\frac{\Delta m}{m} S_+ + S_-) + \frac{1}{56} (177 S_+ + 233 \frac{\Delta m}{m} S_-) \right. \\ &\left. + \frac{261}{28} \eta S_+ - S_- \right) + \frac{4\zeta c^4 x^2}{3\bar{\alpha}^2 G^2 m^3}. \end{aligned} \tag{7.60}$$

Next, we expand the ratio of Eq. (7.58) and Eq. (7.47) up to 1PN

$$\frac{\mathcal{F}^{QD}(x)}{E'(x)} = -\frac{64\eta c^3 x^5}{5Gm\bar{\alpha}^2} \left[\left(\xi - E_2' \frac{5\bar{\alpha}\mathcal{S}_-^2}{24} \right) + \left(f_{2,T}^{nd} - \xi E_2' \right) x + \frac{5\bar{\alpha}\mathcal{S}_-^2}{24} x^{-1} \right]. \tag{7.61}$$

One can now substitute these ratios in Eq. (7.42) and solve the differential equations for x and the phase angle φ numerically for the different regimes. We compute examples using as the initial condition for x its value at the frequency for entering the LIGO/VIRGO sensitivity band ≈ 10Hz. Solving for x and hence the frequency shows nicely the frequency evolution during the inspiral and the change in the proxy for the merger time[25] relative to GR, see Fig. 20.

[25]The PN expansion should break down before the merger so this divergence behaviour does not exactly mimic the merger, however the PN expansion for comparable black hole masses works better than expected outside its validity range [40].

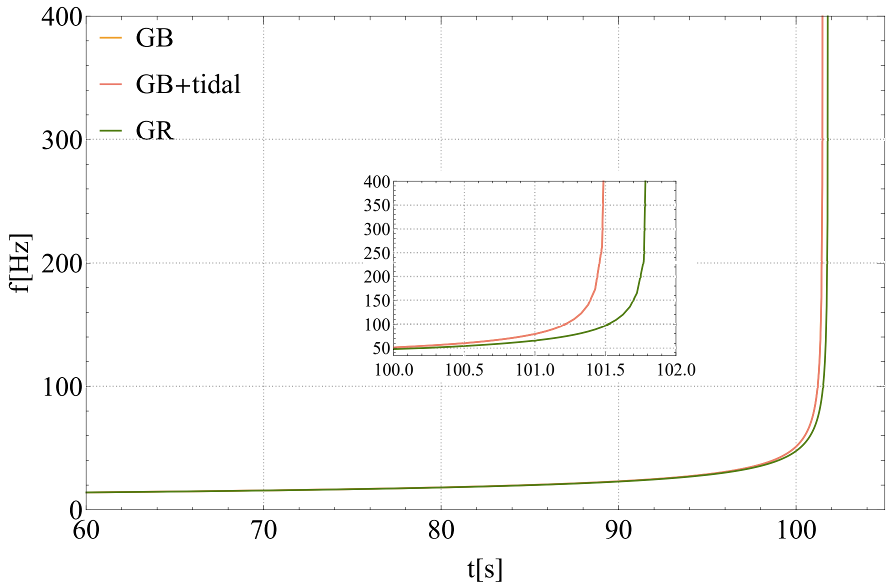


Figure 20: Frequency evolution in time during the quadrupolar driven phase with and without tidal contributions, which overlap in this figure, and compared with the 1PN GR frequency. The moment when the frequency diverges is a proxy for the merger time. We chose here $f[\varphi_0] = \frac{1}{4}e^{2\varphi}$, $\sqrt{\bar{\alpha}} = 1.7$km, $m = 15M_\odot$ and $q = 0.25$.

In the plot we can already see a difference in merging time between the GR frequency evolution and the GB and tidal case. The difference without and with tidal terms is not visible in this plot. Besides considering the time-domain phasing, it is also useful to study the phase evolution in the frequency domain, as the GW data is also studied in this domain. For this we need to do a Fourier transform. In Fourier space we will use the so called Taylor T2 approximant for expanding the expressions [103].

Similar to what we have seen in section 2 one can write the gravitational waveform in the exponential form $h(t) = A(t)e^{-i\phi_{phase}(t)}$. Its Fourier transform is given by

$$\tilde{h}(f) = \int dt A(t) e^{i(2\pi f t - \phi_{phase}(t))}. \tag{7.62}$$

During the inspiral we can assume the stationary phase approximation (SPA)[104]. In this approximation is assumed that the amplitude of the signal changes much more slowly than the phase, which is valid to assume during the inspiral. In that case the exponential osciallates in the integral while the amplitude slowly varies. The postive and negative values in the oscillation cancel each other and the main contribution of the integral is given by the points where the derivative of the phase is $2\pi f$ so the exponential has a stationary

point. Thus we define the stationary point t_f as the point where $\dot{\phi}_{phase}(t_f) = 2\pi f$.
Then we can Taylor expand the exponent around this stationary point t_f

$$2\pi f t - \phi_{phase}(t) \simeq 2\pi f t_f - \phi_{phase}(t_f) - \frac{1}{2}\ddot{\phi}_{phase}(t_f)(t - t_f)^2 + \dots . \tag{7.63}$$

Substituting this expansion back in the Fourier integral results in

$$\begin{aligned} \tilde{h}(f) &= A(t_f) e^{i(2\pi f t_f - \phi_{phase}(t_f))} \int dt e^{-i\frac{1}{2}\ddot{\phi}_{phase}(t_f)(t-t_f)^2} \\ &= A(t_f) e^{i(2\pi f t_f - \phi_{phase}(t_f))} \left(\frac{2}{\ddot{\phi}_{phase}(t_f)} \right)^{1/2} \int dx e^{-ix^2} \\ &= A(t_f) \left(\frac{2\pi}{\ddot{\phi}_{phase}(t_f)} \right)^{1/2} e^{-i(\psi_f(t_f) + \pi/4)}. \end{aligned} \tag{7.64}$$

We assume the amplitude varies slowly around the stationary point and in the last equality we use $\int_{-\infty}^{\infty} dx e^{-ix^2} = \sqrt{\pi} e^{-i\pi/4}$, the known fresnel integral.
Then the Fourier phase angle is given by $\psi_f(t) \equiv 2\phi_{phase}(t) - 2\pi f t$

We can calculate the expressions for the time evolution and orbital phase with Eq. (7.42) rewritten in terms of an integral and using Eq. (5.104) with $w = \pi f$ and $f = 1/t$

$$\begin{aligned} t(\bar{v}) &= t(\bar{v}_{ref}) + \int_{\bar{v}}^{\bar{v}_{ref}} \frac{E'(\bar{v})}{\mathcal{F}(\bar{v})} d\bar{v}, \\ \phi_{phase}(\bar{v}) &= \phi_{phase}(\bar{v}_{ref}) + \frac{c^3}{G\bar{\alpha}m} \int_{\bar{v}}^{\bar{v}_{ref}} \bar{v}^3 \frac{E'(\bar{v})}{\mathcal{F}(\bar{v})} d\bar{v}. \end{aligned} \tag{7.65}$$

Here we defined the new variable $\bar{v}$ as

$$\bar{v} = x^{1/2} = \left(G\bar{\alpha}m\omega/c^3 \right)^{1/3} = \left(\frac{G\bar{\alpha}m\pi f}{c^3}\right)^{1/3}, \tag{7.66}$$

and $\bar{v}_{ref}$ is the variable at some chosen reference point.

Then the Fourier phase angle is given by

$$\psi_f(t_f) = 2 \left(\phi_{phase}(v) - \frac{c^3}{G\bar{\alpha}m} \bar{v}^3 t(\bar{v}) \right) \Bigg|_{\bar{v}=\bar{v}_f}, \tag{7.67}$$

written in terms of the frequency $\bar{v}_f \equiv (\pi G m \bar{\alpha} f / c^3)^{1/3}$ at the stationary point.
Note that the ratio of the derivative of the binding energy and the flux is reversed in Eq. (7.65) compared to Eq. (7.42). Also note that the derivative is now taken with respect to the new variable $\bar{v}$ in the binding energy.

We start again with expanding this ratio in the dipolar driven regime. We express the binding energy Eq. (5.105) and the dipolar driven flux Eq. (7.55) in terms of the new variable $\bar{v} = x^{1/2}$ and take the derivative of the binding energy to $\bar{v}$. Then we expand the ratio up till first order in $x = \bar{v}^2$ which corresponds to

$$\frac{E'(\bar{v})}{\mathcal{F}^{DD}(\bar{v})} = \frac{-3\bar{\alpha}Gm}{4\eta S_-^2 c^3 \bar{v}^7}(1 + (E_0' - f_2^{DD})\bar{v}^2), \tag{7.68}$$

$$E_0' = -\frac{3}{2} - \frac{\eta}{6} - \frac{4\bar{\gamma}}{3} + \frac{4}{3}\left(\beta_+ - \frac{\Delta m}{m}\beta_-\right) - \frac{40c^4}{3G^2}\frac{\alpha f'(\varphi_0)}{m^2\bar{\alpha}^{7/2}}\bar{v}^4\left(3\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right) + \frac{20c^4}{3\bar{\alpha}^2 G^2 m^3}\zeta\bar{v}^4, \tag{7.69}$$

$$\begin{aligned} f_2^{DD} = {} & \frac{24}{5\bar{\alpha}S_-^2} + \frac{4S_+^2}{5S_-^2} - \frac{4\beta_+}{3} + \frac{4\beta_-\Delta m}{3m} - \frac{14}{5} - \frac{4\eta}{3} - \frac{2\bar{\gamma}}{3} + \frac{4\beta_- S_+}{\bar{\gamma}S_-} - \frac{4\beta_+\Delta m S_+}{\bar{\gamma}mS_-} + \frac{4\beta_+}{\bar{\gamma}} - \frac{4\beta_-\Delta m}{\bar{\gamma}m} \\ & + \frac{\alpha f'[\varphi_0]c^4\bar{v}^4}{\bar{\alpha}^{5/2}G^2m^2S_-}\left(\frac{8S_-}{3\bar{\alpha}}(3S_+ + \frac{\Delta m}{m}S_-) - S_+(S_+ + \frac{\Delta m}{m}S_-)\right) \\ & + \frac{\bar{\zeta}c^4\bar{v}^4}{2\bar{\alpha}^{7/2}\mu G^3 m^3 S_-} - \frac{4^4\bar{v}^4}{3\bar{\alpha}^2G^2m^3}. \end{aligned} \tag{7.70}$$

Substituting this expansion in Eq. (7.65) and integrating gives the time and orbital phase. Substituting these results in Eq. (7.67) results in the Fourier phase evolution in the dipolar driven regime, which results in

$$\begin{aligned} \psi(t_f) = {} & -\frac{1}{4\eta\mathcal{S}_-^2\bar{v}_f^3}\Bigg[1 + \frac{9}{2}\rho^{DD}\bar{v}_f^2 + (\rho_{GB}^{DD} + \rho_{tid}^{DD})\log(\bar{v}_f)\,\bar{v}_f^6 + \left(\frac{\bar{v}_f}{\bar{v}_{ref}}\right)^6\Bigg(1 + \frac{1}{3}\rho^{DD}\bar{v}_{ref}^2 \\ & - (\rho_{GB}^{DD} + \rho_{tid}^{DD})\bar{v}_{ref}^6\log(\bar{v}_{ref}) - 3(\rho_{GB}^{DD} + \rho_{tid}^{DD})\bar{v}_{ref}^6 \\ & -2\left(\frac{\bar{v}_f}{\bar{v}_{ref}}\right)^3\left(1 + \frac{3}{2}\rho^{DD}\bar{v}_{ref}^2 - 6(\rho_{GB}^{DD} + \rho_{tid}^{DD})\bar{v}_{ref}^6\right)\Bigg] + \varphi(\bar{v}_{ref}) - 2\pi f t(\bar{v}_{ref}), \end{aligned} \tag{7.71}$$

$$\begin{aligned} \rho^{DD} = {} & -\frac{108}{5\bar{\alpha}S_-^2} - \frac{18}{\bar{\gamma}}\left(\beta_+\frac{\Delta m}{m}\beta_-\right) + \frac{18}{\bar{\gamma}}\frac{S_+}{S_-}(\frac{\Delta m}{m}\beta_+ - \beta_-) - 3\bar{\gamma} + \frac{21\eta}{4} + 12(\beta_+ - \frac{\Delta m}{m}\beta_-), \\ & - \frac{18S_+^2}{5S_-^2} + \frac{117}{20} \\ \rho_{GB}^{DD} = {} & \frac{c^4}{G^2}\frac{6\alpha f'(\varphi_0)}{m^2\bar{\alpha}^{5/2}S_-}\left[\frac{-16S_-}{\bar{\alpha}}(\frac{2\Delta m}{3m}S_- + 3S_+) + S_+(\frac{\Delta m}{m}S_- + S_+)\right], \\ \rho_{tid}^{DD} = {} & \frac{3c^4}{G^2\bar{\alpha}^2m^3\eta}\left[-\frac{\bar{\zeta}}{\bar{\alpha}^{3/2}GmS_-} + 16\eta\zeta\right]. \end{aligned} \tag{7.72}$$

Note that the GB and tidal related terms always come in the combination of $\rho_{GB} + \rho_{tid}$ hence scaling in a degenerate way with the frequency. Also as both $\bar{\zeta}$ and ζ are part of ρ_{tid}^{DD}, both the binary dynamics tidal contribution and the tidal contribution from the induced dipole moment are present in this regime.

For the quadrupolar driven regime we would like to do the same calculation. However the quadrupole flux Eq. (7.58) contains the dipole term which is very small in this regime. Therefore we split the contribution of the flux in the Fourier domain in a part not containing dipole terms and a part only containing the dipole terms. The latter will be subdominant in this regime. As the dipole term scales with S_-, we can use this to track the small terms in the flux:

$$\begin{aligned} \mathcal{F}^{\mathbf{QD}} &= \mathcal{F}_{\text{non-dip}} + \mathcal{F}_{\text{dip}} \,, \\ \mathcal{F}_{\text{non-dip}} &\equiv \lim_{\mathcal{S}_- \to 0} \mathcal{F}, \quad \mathcal{F}_{\text{dip}} \equiv \mathcal{F} - \mathcal{F}_{\text{non-dip}} \,. \end{aligned} \tag{7.73}$$

The ratio of the binding energy and flux in Eq. (7.65) can be expanded as

$$\frac{E'(\bar{v})}{\mathcal{F}(\bar{v})} \simeq \frac{E'(\bar{v})}{\mathcal{F}_{\text{non-dip}}(\bar{v})} \left(1 - \frac{\mathcal{F}_{\text{dip}}(\bar{v})}{\mathcal{F}_{\text{non-dip}}(\bar{v})}\right). \tag{7.74}$$

We rewrite the quadrupole flux in Eq. (7.58) in terms of $\bar{v}$ and split the two contributions according to Eq. (7.73)

$$\begin{aligned} \mathcal{F}_{\text{non-dip}}(\bar{v}) &= \frac{32\eta^2\bar{\xi}c^5}{5G\bar{\alpha}^2}\bar{v}^{10}\left[1 + f_2^{nd}\bar{v}^2 + \mathcal{O}\left(c^{-3}\right)\right], \\ \mathcal{F}_{\text{dip}}(\bar{v}) &= \frac{4\mathcal{S}_-^2\eta^2c^5}{3G\bar{\alpha}}\bar{v}^8\left[1 + f_2^d\bar{v}^2 + \mathcal{O}\left(c^{-3}\right)\right], \end{aligned} \tag{7.75}$$

with $\bar{\xi} = \left(1 + \mathcal{S}_+^2\bar{\alpha}/6\right)$. The coefficients given by the coefficients from Eq. (7.58) altered to this splitting procedure in the following way

$$\begin{aligned} f_2^{nd} &= \frac{f_{2,T}^{nd}}{\bar{\xi}}(S_- \to 0, x \to \bar{v}^2), \\ f^d &= f_2^{DD}(\frac{1}{S_-^2} \to 0, x \to \bar{v}^2). \end{aligned} \tag{7.76}$$

The ratio is then given by

$$\begin{aligned} \frac{E'(\bar{v})}{\mathcal{F}(\bar{v})} \simeq &-\frac{5Gm\bar{\alpha}^2}{32c^3\eta\bar{\xi}\bar{v}^9}\left[1 + \left(E_0' - f_2^{nd}\right)\bar{v}^2 + \mathcal{O}\left(c^{-4}\right)\right] \\ &+ \frac{25Gm\bar{\alpha}^3\mathcal{S}_-^2}{768c^3\bar{\xi}^2\eta\bar{v}^{11}}\left[1 + \left(E_0' - 2f_2^{nd} + f^d\right)\bar{v}^2 + \mathcal{O}\left(c^{-4}\right)\right]. \end{aligned} \tag{7.77}$$

We substitute this expansion in Eq. (7.65) and we integrate, which gives the time and orbital phase. Substituting these expressions in Eq. (7.67) results in the Fourier phase related to the dipolar and non dipolar terms. In total, they form the Fourier phase angle evolution in the quadrupole regime:

$$\psi_{\text{non-dip}}\,(t_f) = -\frac{6\bar{\alpha}}{256\eta\bar{\xi}\bar{v}_f^5}\left[1+\frac{20}{9}\rho^{nd}\bar{v}_f^2 - 20(\rho^{nd}_{GB}+\rho^{nd}_{tid})\bar{v}_f^6\right.$$
$$\left.+\frac{5}{3}\left(\frac{\bar{v}_f}{\bar{v}_{ref}}\right)^8\left(1+\frac{4}{3}\rho^{nd}\bar{v}_{ref}^2 - 4(\rho^{nd}_{GB}+\rho^{nd}_{tid})\bar{v}_{ref}^6\right) - \frac{8}{3}\left(\frac{\bar{v}_f}{\bar{v}_{ref}}\right)^5\left(1+\frac{5}{3}\rho^{nd}\bar{v}_{ref}^2 + 5(\rho^{nd}_{GB}+\rho^{nd}_{tid})\bar{v}_{ref}^6\right)\right],$$
$$\psi_{\text{dip}}\,(t_f) = \frac{10\mathcal{S}_-^2\bar{\alpha}^2}{3584\eta\bar{\xi}^2\bar{v}_f^7}\left[1+\frac{7}{4}\rho^{d}\bar{v}_f^2 + \frac{70}{4}(\rho^{d}_{GB}+\rho^{d}_{tid})\bar{v}_f^6\right.$$
$$\left.+\frac{7}{3}\left(\frac{\bar{v}_f}{\bar{v}_{ref}}\right)^{10}\left(1+\frac{5}{4}\rho^{d}\bar{v}_{ref}^2 + \frac{5}{2}(\rho^{d}_{GB}+\rho^{d}_{tid})\bar{v}_{ref}^6\right) + \frac{10}{3}\left(\frac{\bar{v}_f}{\bar{v}_{ref}}\right)^7\left(1+\frac{7}{5}\rho^{d}\bar{v}_{ref}^2 + 7(\rho^{d}_{GB}+\rho^{d}_{tid})\bar{v}_{ref}^6\right)\right] \tag{7.78}$$

Here the coefficients are given by

$$\begin{aligned}
\rho^{nd} =& \frac{1247}{336\bar{\xi}} - \frac{3}{2} + \frac{4}{3}(\bar{\xi}-1)\bar{\gamma} + (\frac{4}{3}+\frac{8}{3\bar{\xi}})(\beta_+ - \frac{\Delta m}{m}\beta_-),\\
\rho^{nd}_{GB} =& \frac{c^4}{G^2}\frac{f'(\varphi_0)\,\alpha}{m^2\bar{\alpha}^{5/2}}\left[\frac{40}{3\bar{\alpha}}(3S_+ + \frac{\Delta m}{m}S_-) + \frac{S_+}{\bar{\xi}}(-177-261\eta+\frac{4\Delta m}{m}) - \frac{8S_+}{\bar{\alpha}\bar{\xi}}\right],\\
\rho^{nd}_{tid} =& \frac{c^4\zeta}{G^2\bar{\alpha}^2m^3}\left[\frac{-20}{3}+\frac{4}{3\bar{\xi}}\right]\\
\rho^{d} =& \frac{1247}{168\bar{\xi}} - \frac{43}{10} + (\frac{35}{6\bar{\xi}}-\frac{3}{2})\eta + (\frac{8}{3\bar{\xi}}-2)\bar{\gamma}\\
&+\frac{4}{\bar{\gamma}}(\beta_+ - \frac{\Delta m}{m}\beta_-) + \frac{4S_+}{\bar{\gamma}S_-}(\beta_- - \frac{\Delta m}{m}\beta_+) + \frac{16}{3\bar{\xi}}(\beta_+ - \frac{\Delta m}{m}\beta_-),\\
\rho^{d}_{GB} =& \frac{c^4}{G^2}\frac{f'(\varphi_0)\,\alpha}{m^2\bar{\alpha}^{5/2}S_-}\left[-S_+(S_+ + \frac{\Delta m}{m}S_-) + \frac{8S_-}{3\bar{\alpha}}(3S_+ + \frac{\Delta m}{m}S_-) + \frac{S_-S_+}{\bar{\xi}}(\frac{177}{7} - \frac{8\Delta m}{m} + \frac{16}{\bar{\alpha}} + \frac{522\eta}{7})\right],\\
\rho^{d}_{tid} =& \frac{c^4}{G^2\bar{\alpha}^2m^3}\left[\frac{\bar{\zeta}}{2\bar{\alpha}^{3/2}G\mu S_-} + \frac{16\zeta}{3}\right].
\end{aligned} \tag{7.79}$$

Again we have the degenerate scaling with the frequency of the GB and tidal terms. In this regime the non dipolar part dominates over the dipolar part. The dominating tidal term in this regime is therefore ρ^{nd}_{tid} which only contains a term proportional to ζ. This contribution comes from the tidal contribution in the binary dynamics.

7.5 Analysis of the phasing terms

We are interested in the contribution of the tidal term with respect to the GB term and the GR result to the phase. If we look at expressions for the phases in fourier space Eq. (7.71), Eq. (7.78) we see that the contribution of the GB and tidal term always come in this combination $\rho_{GB} + \rho_{tid}$ and they scale therefore degenerate with respect to the frequency.

We begin by extracting the terms from the phase expression which are proportional to the tidal terms $\bar{\zeta}$ and ζ and the GB term proportional to α. We can compare there relative contributions. Studying the frequency dependence of these two contributions is trivial as they have the same scaling. We look at the frequency dependence when we look at the entire phase expression. We are especially interested in configurations where the two contributions have the same sign as the total contribution depends on $\rho_{GB}^{DD} + \rho_{GB}^{DD}$ which is enhanced instead of canceled for same signs.

7.5.1 Analysis of the DD phase

We start by analysing the phasing in the dipolar driven regimen given in Eq. (7.71). We consider this expression in the frequency range of 0.1×10^{-3}Hz to an upper bound of f^{DD} from Eq. (7.54), which depends on the choice of coupling constant and mass. The lower bound corresponds to the LISA lower bound frequency. In general for stellar mass black hole binaries and choice of the coupling constant is consistent with recent emperical constrants [105] (also discussed in section 3.3.3. For these choices the upperbound frequency f^{DD} is below 10Hz, which is the lower bound frequency for LIGO/VIRGO. However it is within the LISA frequency band of $0.1 \times 10^{-3}\text{Hz} - 1\text{Hz}$. We compare the contribution of ρ_{GB}^{DD} with the contribution of ρ_{tid}^{DD}.
As before, we have that the scalar charge and its derivative are defined with a minus sign in Eq.(5.81), making $S_\pm$ negative for mass ratios $q < 1$ which we in general consider. For mass ratios larger than 1, one can swap the role of body A and B in the system and again do the analysis for $q < 1$. The expression of ρ_{GB}^{DD} is therefore negative and thus the GB contribution to the phase makes the phase smaller relative to GR, which corresponds to the findings in [16, 17]. We have seen that ζ in Eq. (5.82) is positive as it depends on the square of the scalar charge. The tidal term related to $\bar{\zeta}$ from Eq. (7.23) is negative as it depends linearly on the scalar charge, divided by negative S_- and having an overall minus sign. Thus ρ_{tid}^{DD} consists of a positive and a negative term, making the sign of the overall term depend on the mass ratio and total mass.
For the analysis below we use the following properties

$$\begin{aligned}
&f[\varphi] = \frac{1}{4}e^{2\varphi}, && m = 15M_\odot, \\
&f_{min,DD} = 0.1 \times 10^{-3}\text{Hz}, && f_{max,DD} = \left(\frac{5}{24}\right)^{3/2} \frac{c^3 \mathcal{S}_-^3 \sqrt{\bar{\alpha}}}{\pi G m}\text{Hz}, \\
&f_{min,QD} = 10\text{Hz}, && f_{max,DD} = 586\text{Hz},
\end{aligned} \tag{7.80}$$

and for $\sqrt{\alpha}$ we use the current constraints from literature corresponding to EdGB gravity as described in section 3.3.3: $\sqrt{\alpha} = 1.18$km and $\sqrt{\alpha} = 1.7$km .

When we study the dependency of the tidal phase contributions and GB phase contributions with the total mass we find that they scale the same. Therefore we look at the dependence of the ratio of the tidal and GB terms with respect to the mass ratio, see Fig. 21.

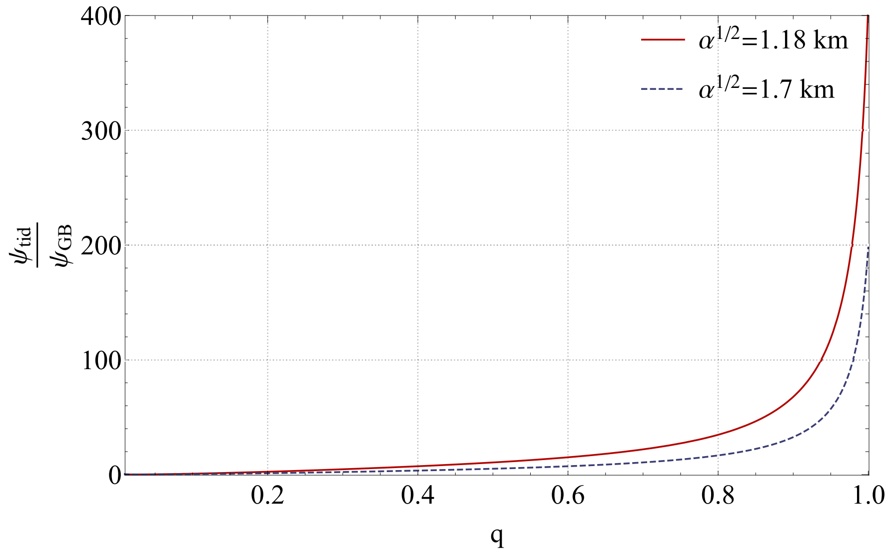


Figure 21: The ratio of the tidal phase contributions over the GB phase contributions versus the mass ratio $q = \frac{m_A}{m_B}$. The red line corresponding to the coupling constant $\sqrt{\alpha} = 1.18$km and the purple dashed line to $\sqrt{\alpha} = 1.7$km. We use the properties in Eq.(7.80).

As the ratio is positive, the two contributions have the same sign for all mass ratios. The ratio is of order 10^2 therefore the tidal contribution to the phase in the dipolar driven regime is a factor 100 larger than the GB contribution in this frequency range. This plot also shows that the ratio becomes smaller for a larger coupling constant value. This implies that we find the largest difference in the phase occurs for mass ratios close to 1 and a smaller values of the coupling constant.

The fact that the ratio of the tidal and GB contributions is larger for smaller coupling constats is interesting, as one would generally expect corrections to GR to scale positively with the coupling constant. The dependency of the terms on the coupling constant is not so easy to track explicitly, as the quanitites $S_\pm$, $\beta_\pm$, $\bar{\gamma}$, $\bar{\alpha}$, ζ and $\bar{\zeta}$ depend on the scalar charge, which in turn depends on the coupling constant. To see where this behaviour comes from we plot explicitly the dependency on the coupling constant of the tidal and GB phasing terms and the ratio shown in Fig. 22.

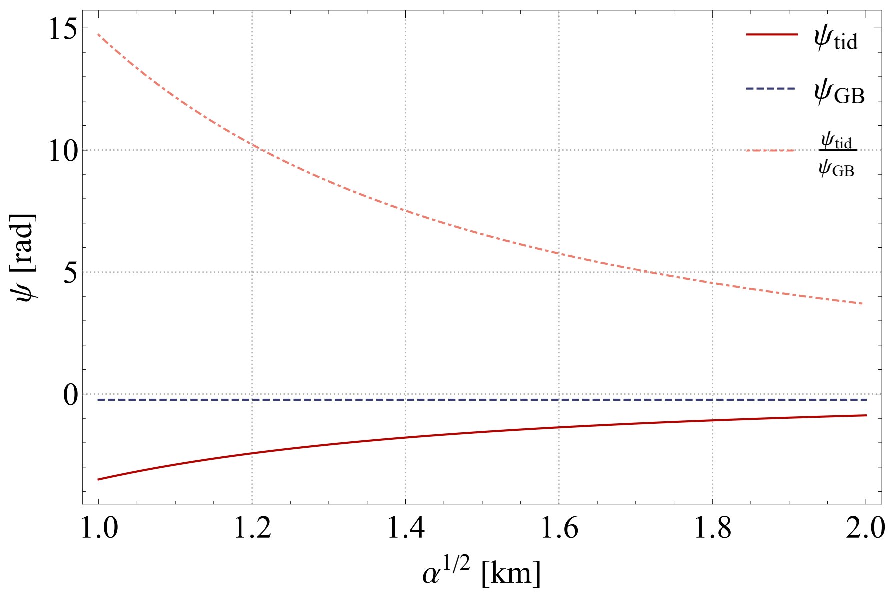
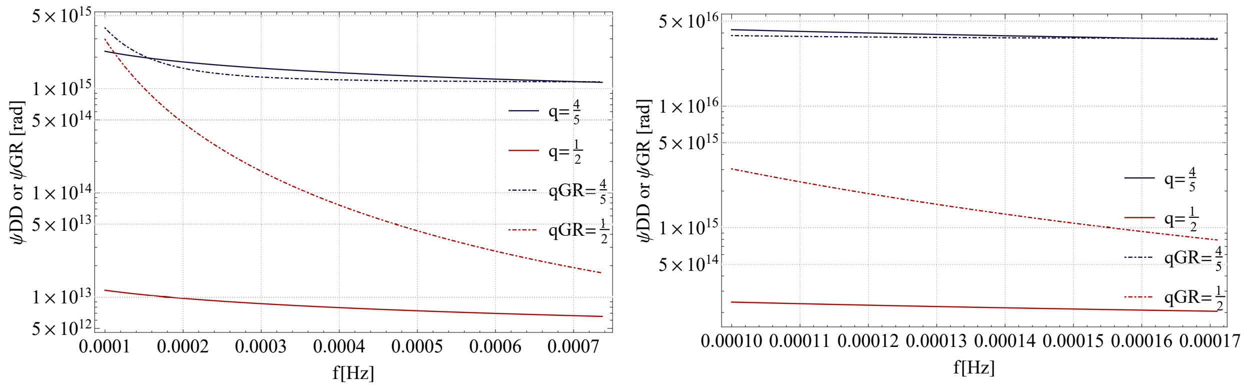


Figure 22: The tidal and GB contributions of the phase and the ratio of the tidal phase contributions over the GB phase contributions as functions of the coupling constant. Furthermore we use Eq. (7.80) and $q = 0.5$.

We see that the contribution of the tidal terms becomes less negative for a larger coupling constant, the GB contribution stays nearly constant at -0.23rad. Therefore in total the contribution to the phasing of the GB and tidal terms scales negatively with the coupling constant and we thus expect an enlarged difference with the GR phasing in this regime for a small coupling constant.

We compare the total dipolar driven phase with the 1PN GR phase (which can be found in [25] but can also be recovered from setting $\alpha = 0$ and hence also the scalar charge to zero), see Fig. 23. We match the integration constants such that the GR phase matches with the DD phase at the end of the frequency validity regime. This is done because there is no GR equivalent of dipole radiation, only quadrupole radiation. For frequencies larger than the DD validity frequency the quadrupole term becomes dominant and can hence be compared with the GR case.

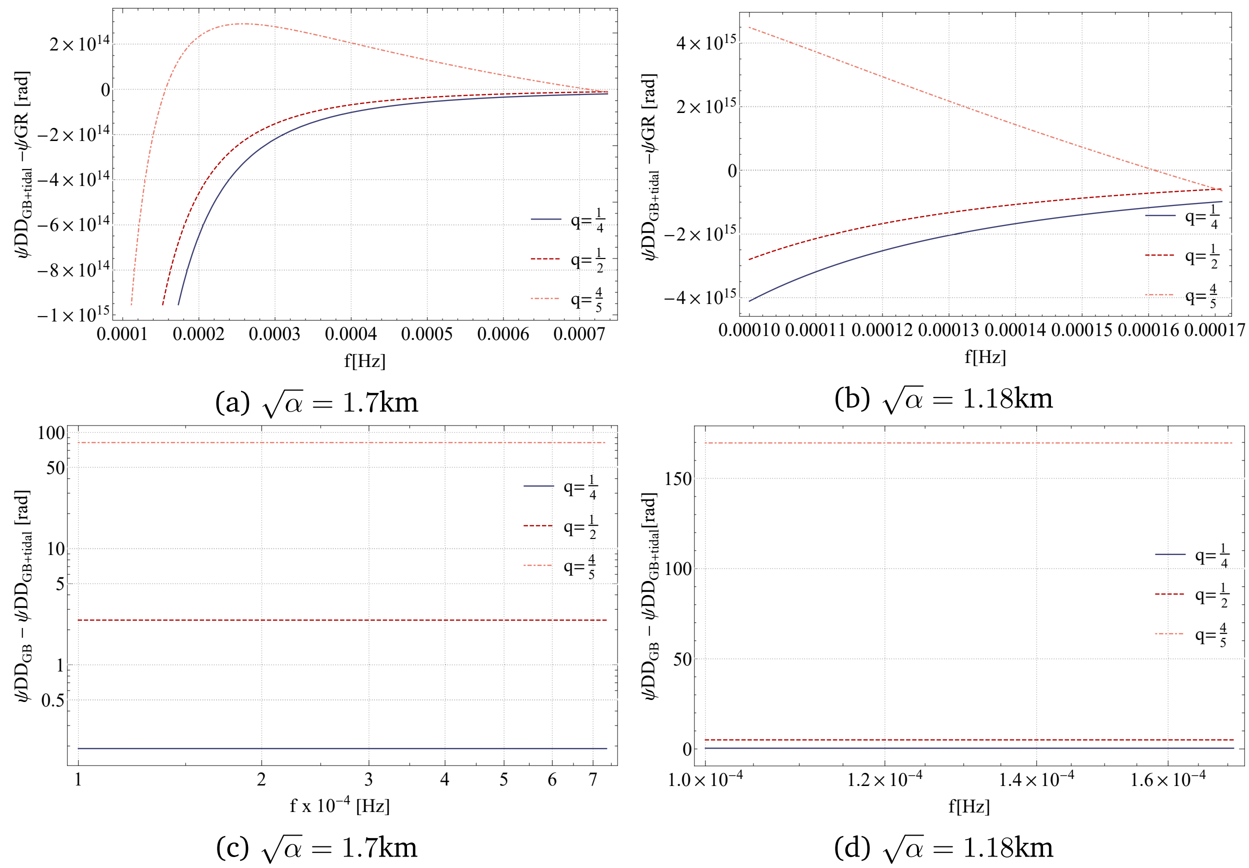


(a) $\sqrt{\alpha} = 1.7\text{km}$ (b) $\sqrt{\alpha} = 1.18\text{km}$

Figure 23: Log linear plot of total phase in dipolar driven regime including tidal contributions for two different coupling constants. The GR phase is also included and matched with the DD phase for the upper bound frequency. The solid lines correspond to the EdGB DD result and the dot-dashed line to GR. We use Eq. (7.80).

From the total dipolar driven phase evolution we see that for a mass ratio closer to 1 the phase is two orders of magnitude greater. We also see that for a large mass ratio and larger coupling constant the GR phase is first larger and for higher frequencies smaller than the GB phase, while for smaller mass ratios the GR phase is always larger.

To see the differences more explicitly we plot the difference of the total dipolar driven phase with the 1PN GR phase for different mass ratios and two coupling constants, and the differences between the dipolar driven phase with and without tidal terms shown in Fig. 24.
What we see from the difference with GR plots is that for a smaller coupling constant the GB + tidal contributions are always larger than the GR phase for a mass ratio near 1, while for a larger coupling constant for small frequencies the GR phase is first larger and becomes smaller for larger frequencies.
For the difference between the phase with and without tidal terms we see that it is enlarged for mass ratio close to 1 and for a smaller coupling constant. Furthermore, the difference is positive, hence in the dipolar driven regime the inclusion of the tidal contribution makes the phase smaller. This is because both the tidal and GB contributions to the phase carry a minus sign.



(a) $\sqrt{\alpha} = 1.7$km

(b) $\sqrt{\alpha} = 1.18$km

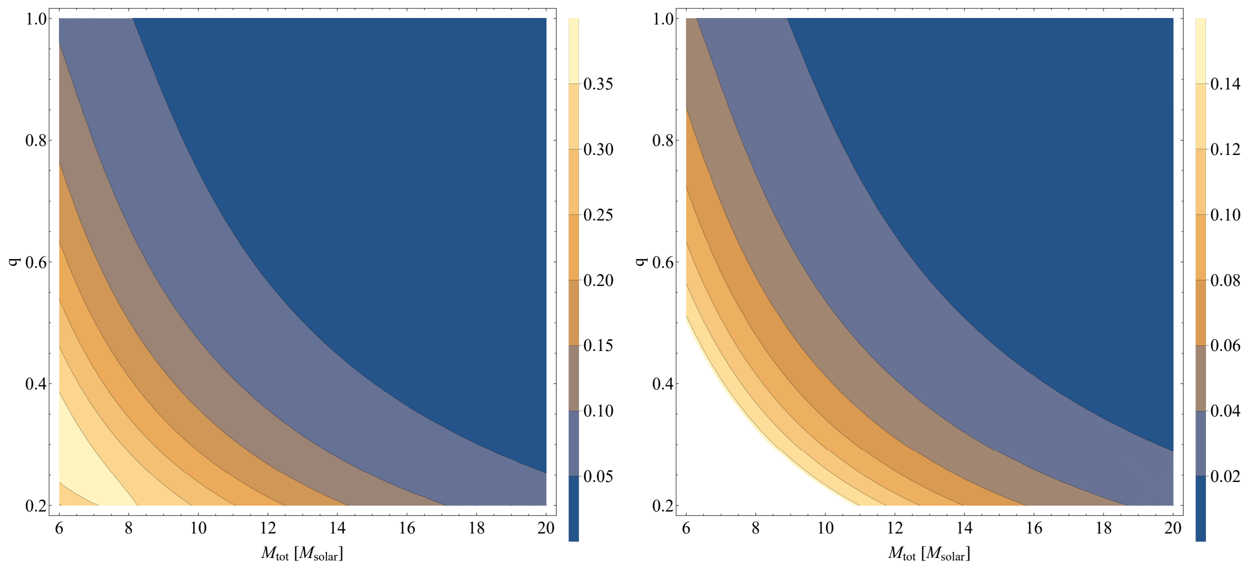
(c) $\sqrt{\alpha} = 1.7$km

(d) $\sqrt{\alpha} = 1.18$km

Figure 24: The top two plots showing the total phase in the dipolar driven regime and the 1PN GR phase for a mass ratio of $1/4$, $1/2$ and $4/5$ for two different values of the coupling constant. The bottom two plots showing explicitly the difference between the dipolar driven phase with and without tidal terms for mass ratios of $1/4$, $1/2$ and 1 for two different coupling constants. We use Eq. (7.80).

7.5.2 Analysis of the QD phase

We repeat the same analysis for the quadrupolar driven expression Eq. (7.78). We consider this regime for a frequency range of LIGO/VIRGO $10 - 586$Hz with the upperbound being the ISCO estimate. Here we do find that the scaling of the tidal contribution and GB contribution with the total mass is different. We therefore study a contourplot, varying both the total mass and mass ratio as shown in Fig. 25.



(a) $\sqrt{\alpha} = 1.7$km (b) $\sqrt{\alpha} = 1.18$km

Figure 25: Contour of the ratio of the tidal phase contributions over the GB phase contributions varying the mass ratio $q = \frac{m_A}{m_B}$ and total mass for two different coupling constants. We use Eq. (7.80).

Again we find the ratio to be positive, hence same sign contributions of the GB and tidal terms. In the quadrupole regime the contribution of the tidal terms is largest for small mass ratio and total mass, opposite to the dipolar driven case. Similarly, the decrease in the ratio of the tidal to GB terms for smaller coupling is also opposite to the trend in the DD regime.

To see where this behaviour with respect to the coupling constant comes from we plot again the contribution to the phase of the tidal terms, GB term and the ratio versus the coupling constant, see Fig. 26.

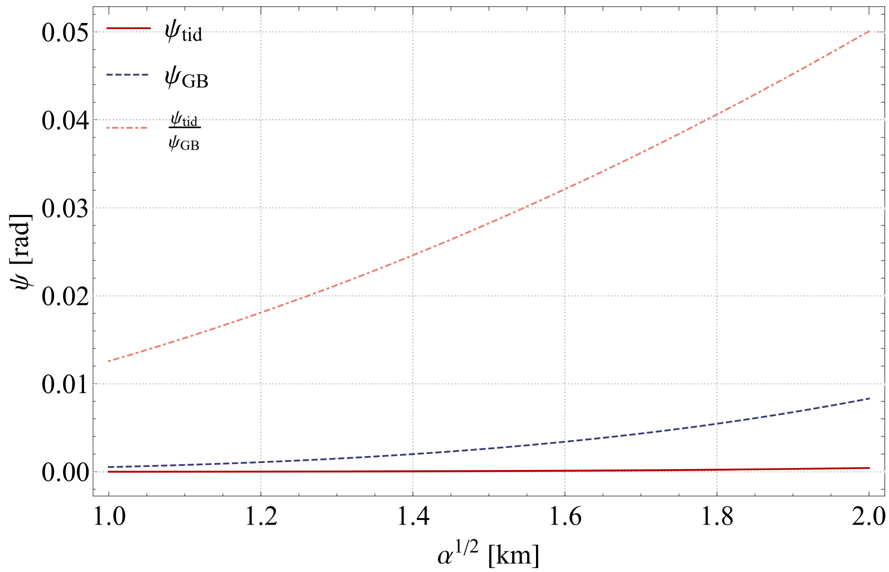
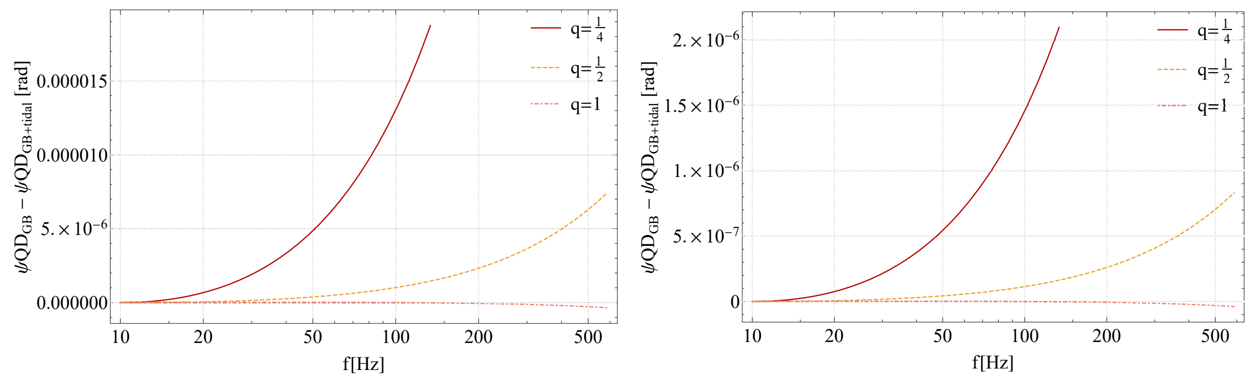


Figure 26: The tidal and GB contributions of the phase and the ratio of the tidal phase contributions over the GB phase contributions against the coupling constant. We use Eq. (7.80) and $q = 0.5$.

This time both contributions are positive and the tidal contribution scales less quickly with the coupling constant than the GB contribution. The magnitude of the contributions is much smaller than in the dipolar driven regime. If we compare the expression of the quadrupolar driven phase Eq. (7.78) with the dipolar driven phase Eq. (7.71) one sees that the dipolar part of the QD phase has similar scaling as the DD phase. However the dipolar part is very small in this regime and the non dipolar part dominates which has the dependency on the coupling constant we see above in the plot.

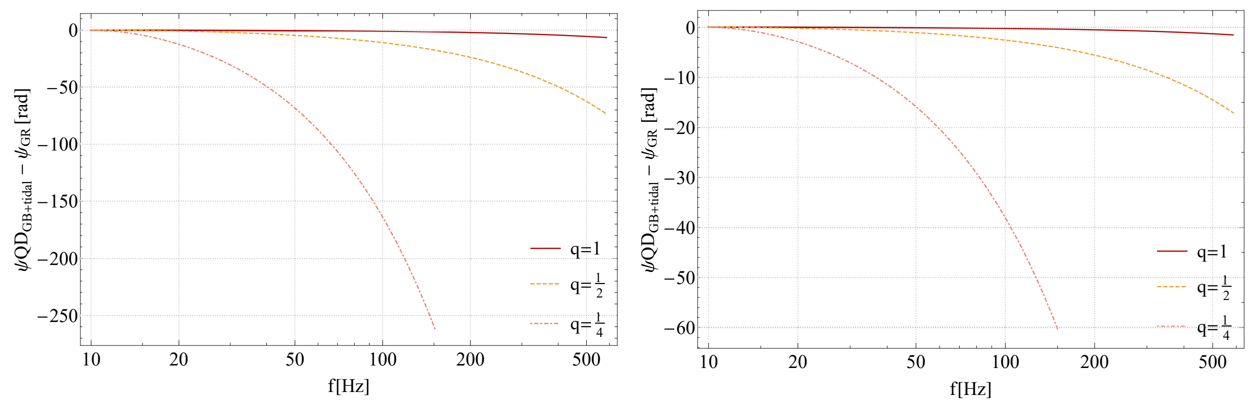
If we now look at the difference between the total quadrupole phase with and without tidal contribution, we indeed see this behaviour shown in Fig. 27.
For a smaller mass ratio the difference is enlarged and is of an order of magnitude of $10^{-5}/10^{-6}$. The sign of the difference, opposite to the dipolar driven case, is negative meaning that the contribution of the tidal terms to the phase makes the total quadrupolar driven phase larger.

Next, we look at the total quadrupolar phase evolution with respect to the phase in GR. We do not show the quadrupolar phase without tidal effects as the difference is to small to be visible in these plots, see Fig. 28.
We see that the difference with GR is largest for small mass ratios and is of order 10^3 for the larger coupling constant and of order 10^2 for the smaller coupling constant.



(a) $\sqrt{\alpha} = 1.7$km (b) $\sqrt{\alpha} = 1.18$km

Figure 27: Difference between the quadrupolar phase without tidal contribution and with tidal contribution for two different values of the coupling constant. We use Eq. (7.80).



(a) $\sqrt{\alpha} = 1.7$km (b) $\sqrt{\alpha} = 1.18$km

Figure 28: The difference between the quadrupole driven phase and the 1PN GR phase for mass ratios of $1/4$, $1/2$ and 1 for two different coupling constants. We use Eq. (7.80).

The features that the tidal contribution scales negatively in the DD regime and that its contribution is so much larger than in the quadrupole driven regime is interesting. The tidal contribution to the consists of the sum of the tidal term proportional to ζ which comes from the orbital contribution of the tidal effects introduced in section 5 and the term proportional to $\bar{\zeta}$ coming from the induced dipole moment which we included in the multipole expansion of the scalar field. The dipolar driven regime is valid for low frequencies during the early inspiral. As the black holes are here far apart the dominating contribution is not coming from curvature related terms as the GB terms, but from the scalar field. As the movement is here also very slow the contributions of the binary dynamics are also not dominant. Therefore in this regime the dominating tidal contribution comes from the induced scalar dipole moment, which has a negative scaling with the coupling constant. The QD regime is valid for higher frequencies for which the binary dynamics related tidal contribution becomes the dominant contribution, this term has a positive scaling with the coupling constant. The latter correspondsto the scalings we also found in the binding energy analysis in section 5.8.

7.5.3 Analysis for multiband detection

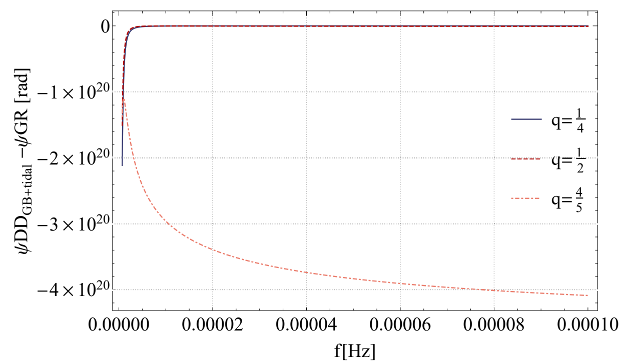
In previous analysis we studied the phasing in the quadrupolar and dipolar driven regimes seperately. We focussed on a binary system with a total mass of $15M_{\odot}$ corresponding to stellar black holes in the light mass range and making the comparison possible to the previous results in [17] which studied the same system. It is interesting to study this system to analyse the dependencies of the phasing terms, but to say something about detectability of the differences requires considering more massive stellar black holes in the dipolar driven regime.

As the frequency regime lies within the LISA frequency band it would be possible to detect the dipolar driven radiation with this space based telescope. Although stellar mass black holes are not the main target of the LISA telescope, it can still measure its radiation corresponding to different stages in the events [106, 107]. An even more promising tool would be to make use of multiband detection, meaning that radiation from the same event is first measured by LISA from the very early inspiral stages and later picked up by ground based detectors as aLIGO/VIRGO or ET [108, 109, 110, 111]. For example it has been shown that the GWs from event GW150914 of a black hole binary with the black hole masses $\approx 30M_{solar}$ detected by LIGO was strong enough to be also measured with LISA [112]. As the strain of the gravitational waves scales with the masses of the black holes (see section 2), it has been shown that for black hole masses comparable to the GW150914 binary or larger the radiation will be suitable for multiband detection [108]. Therefore to have an estimate of the detectibility of the differences in phase regarding the possibility for multiband detection we repeat the analysis for a system of $m = 60M_{\odot}$ and $q = 0.5$.

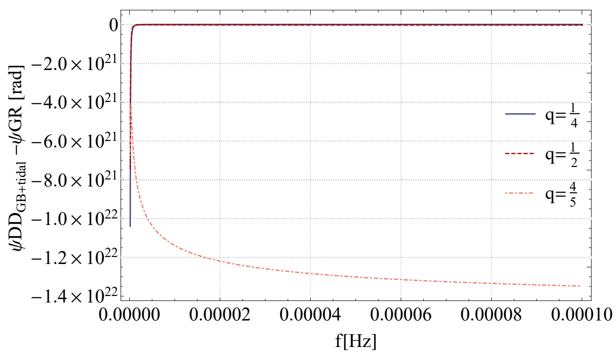
The ISCO frequency scales as $f_{ISCO} \propto 1/m$[25], thus for this new system the ISCO frequency is lowered from 586Hz to 146Hz[26]. The merger happens for lower frequencies, but still in the LIGO/VIRGO frequency range. This frequency will be our upperbound frequency in the QD regime for these systems.

We first look at the DD regime results for the phase, focussing on the total phase differences with respect to GR and with and without tidal terms shown in Fig. 29.

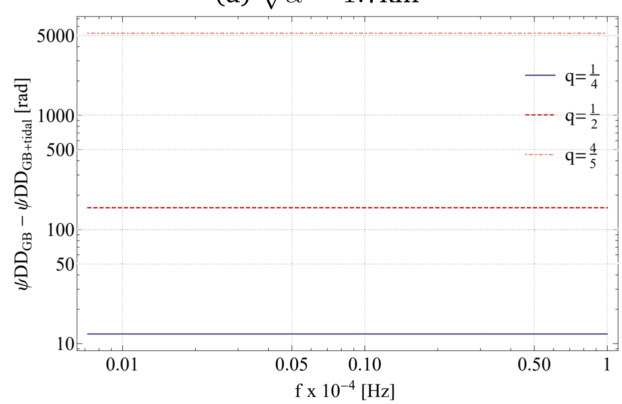
[26] Here we use the ISCO frequency estimation from [17]



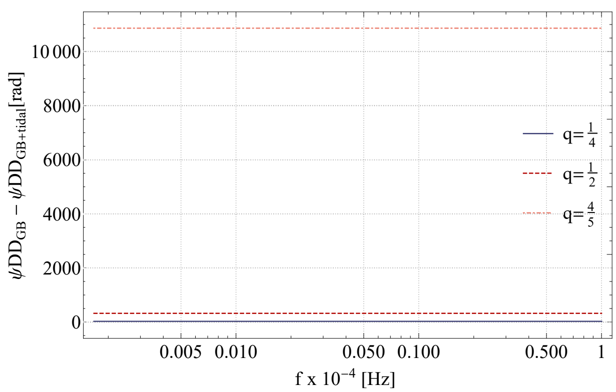
(a) $\sqrt{\alpha} = 1.7$km



(b) $\sqrt{\alpha} = 1.18$km



(c) $\sqrt{\alpha} = 1.7$km

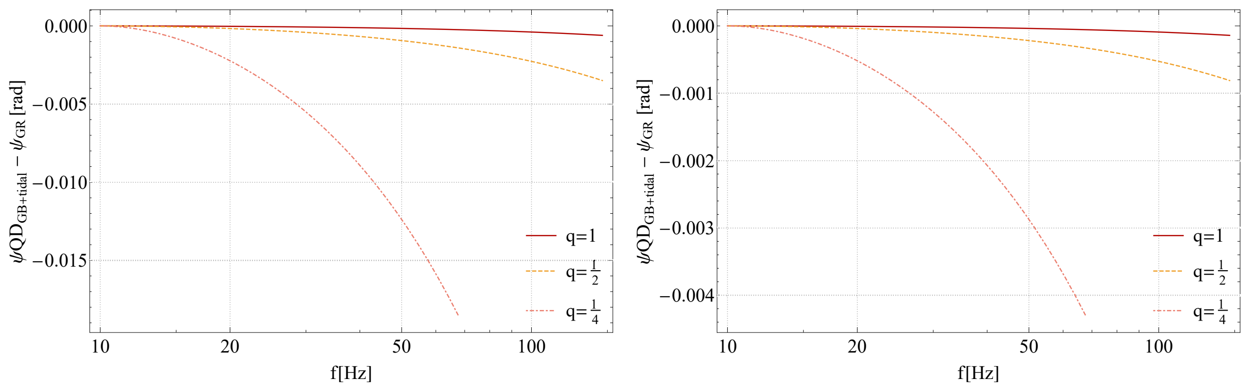


(d) $\sqrt{\alpha} = 1.18$km

Figure 29: The top two plots show the total phase in the dipolar driven regime and the 1PN GR phase for a mass ratio of $1/4$, $1/2$ and $4/5$ for two different values of the coupling constant. The bottom two plots show explicitly the difference between the dipolar driven phase with and without tidal terms for mass ratios of $1/4$, $1/2$ and 1 for two different coupling constants.

We see that for a total mass of $60M_\odot$ the difference of the phase with respect to GR is only negative and the magnitude of the difference is enlarged compared to Fig. 24. Also the contribution of the tidal terms is enlarged compared to the $15M_\odot$ case as can be seen from the bottom plots.

For the QD regime we only plot the difference in phase with the GR as the differences between the phase with and without tidal contributions becomes too small in the case of larger black holes masses. The difference is shown in Fig. 30.



(a) $\sqrt{\alpha} = 1.7$km (b) $\sqrt{\alpha} = 1.18$km

Figure 30: The top two plots show the total phase in the quadrupolar driven regime and the 1PN GR phase for a mass ratio of $1/4$, $1/2$ and $4/5$ for two different values of the coupling constant. The bottom two plots show explicitly the difference between the quadrupolar driven phase with and without tidal terms for mass ratios of $1/4$, $1/2$ and 1 for two different coupling constants.

In the quadrupolar regime the magnitude of the differences is a factor of 10^5 smaller compared to Fig. 28.

A rough estimate for detectability is a difference in phase of about $\mathcal{O}(1)$ rad. All the differences we analysed for a system of $m = 15M_\odot$ in the previous section are above this order in the phase for both the QD and DD regimes, except for the phase difference between the tidal and non tidal included QD phase. But the differences with GR fall in the detectable range. However such a binary system will not be detectable with LISA as the signal would be too weak for such a small masses. If we analyse a more massive system that could be used for multiband detection $m = 60M_\odot$, we find the differences in the DD regime are enlarged and still much above the detectability estimate. However the differences with the GR phase in the QD regime for such a system are probably too small for detection with our current detectors. However it can probably be detected with the higher accuracy of ET.

8 Discussion and conclusion

In this thesis we calculated the near zone gravitational and scalar fields up to 1PN order and the waveform phase evolution in quadratic gravity, specifically in scalar Gauss Bonnet gravity.
For the near zone fields up to 1PN we recovered the results from [24] and we found that the tidal terms do not play a role in these fields up to the order we are considering. They are introduced separately in the two body Lagrangian and contribute at the same 1PN order to the Lagrangian, relative acceleration and binding energy as the GB term and also scale in the same way with the relative separation. Compared to the result for the binding energy in [17] we corrected the prefactor of the GB term in this expression.

After a first analysis of the contribution of the tidal terms in the binding energy, which turned out to be small, but it has the same sign as the GB related terms, we continued to zoom in on the scalar field around one black hole in the binary system specifically. By analysing the scalar field around this black hole and the first order perturbations from the scalar field of the companion, we found that the scalar tidal effects induce a scalar tidal dipole moment as opposed to a quadrupolar effect for gravitational tidal effects. From the asymptotic expansion of the linear perturbation in the scalar field we recovered the tidal deformability parameter for these scalar tidal effects.

With the 1PN near zone fields and tidal deformability parameter at hand, we continued with the calculation of the scalar waveform and the gravitational wave phase using the DIRE approach. Here we corrected some numerical factors and an additional GB term in the scalar waveform compared to [17]. The tidally induced dipole moment adds linearly to the scalar field dipole moment and again has the same PN order and scaling with the relative distance as the GB term. However this tidal term has a different dependency on the total mass/mass ratio and coupling constant as the tidal term related to the orbital dynamics.
From the scalar en tensor waveforms, we constructed the phase evolution in the time and frequency domains. Splitting the regimes in a dipolar dominated frequency range and a quadrupolar dominated frequency range. From the expressions of the phase in the Fourier domain we could already see that the GB and tidal contribution have a degenerate scaling with the frequency.

From our analysis of the phasing terms we found that in the DD regime, the contribution of the tidal and GB terms is around three orders of magintude larger than in the quadrupolar driven regime. In both regimes for the sign of both contributions are equal although in the DD regime they are both negative and in the QD regime they are positive. The scaling of the tidal and GB contributions to the DD phase is the same for the frequency and total mass, while the ratio between the two contributions is largest for mass ratios closer to 1. As the tidal contribution becomes less negative for a larger coupling constant, while the GB term stays constant, the total contribution in this regime becomes smaller for a larger coupling constant, which is an interesting result. This is due to the fact that in the DD regime the dipolar induced tidal contribution dominates from the orbital dynamics contribution.

The situation is reversed in the QD regime.
In the QD regime we found that the ratio between the tidal and GB contributions was largest for small mass ratios and total mass, which is the opposite as in the DD case. Also as the contribution of the tidal and GB terms both increase with a larger coupling constant the total difference with GR in this regime does become larger with a larger coupling constant.

We repeated the analysis for a system of larger black hole masses that could be detected with the LISA telescope and can possibly be used for multiband detection. We found that the phase differences in the DD regime are enlarged compared to the smaller black hole mass system. In the QD regime the differences are smaller. As a first rough estimate for the possibility for detection, the phase differences with GR in QD regime for a $m = 15M_{\odot}$ system is in detectable range, the DD phase differences with GR as well. However for these masses the signal can not be measured with LISA. For a $m = 60M_{\odot}$ system the phase differences with GR in de DD regime are detectable but as the differences in the QD regime are smaller for this system and they are out of the sensitivity range for current detectors. There are however possibilities regarding the ET for detecting these differences. This possibility for multiband detection is an opportunity for resolving the degeneracy of the scaling with the frequency for the tidal and GB contributions. One can use their opposite scaling behaviour with the mass ratio and coupling constant in the two different regimes to mitigate the degeneracy in the two contributions.

All in all our work resulted in an interesting first analysis of the inclusion of scalar tidal effects in the GW signatures in sGB gravity, as we found interesting dependencies of the tidal terms with respect to the coupling and even a dominant behaviour in the DD regime. This shows that these tidal effects play a non-negligible role in the GW analysis for the kind of systems we considered.

9 Outlook

Lastly we have a short discussion on the possibilities for future research related to our findings. One of the most straightforward ways to continue for future research would be to extend this calculation to higher PN orders. In that case back reaction effects of the GWs come into play and the far zone contribution enters. Also higher order tidal effects than the dipolar contribution can be included which requires the extension of the calculation of the tidal deformability parameter to higher orders.
Also already up to 1PN the contribution of the tensor flux in the phasing is till 1PN but the scalar flux only till 0.5PN, so the extension of the scalar field up to 1PN would already be interesting.

To make our current analysis more complete, we can add sGB corrections to the Schwarzschild background we are using in the calculation of the scalar tidal deformability parameter. However as these corrections come in at second order in the coupling constant[113], the calculation of the tidal deformability also needs to be extended up to higher order in the coupling constant.

Furthermore as we find that as the tidal contributions are not negligible and even dominating in the DD regime, there is a bias on the parameter estimation in studies that derive bounds on sGB, as this effect is not included before. For example there are probably stronger constraints on the coupling constant when taking into account the scalar tidal effects. It would be interesting to reconsider these results including this effect.

Lastly doing the same analysis for other quadratic gravity theories is an interesting contribution. We started looking into this analysis for dynamical Chern Simons gravity as described in Appendix G. However as we found the dCS contributions relative to GR to enter at higher PN orders, we left the full analysis for future work.

10 Acknowledgements

Here a word of thanks to the people involved in the makings of this thesis, as I am really grateful for their contributions. First of all I would like to thank my supervisor Tanja Hinderer for all our weekly meetings, valuable feedback, answers to (not so smart) questions, kind words and motivation. I cannot believe the level of detail and time she invested for feedback on this report and I am really grateful for the possibility of doing this thesis under her supervison. Secondly, I would like to thank Banafsheh Shiralilou for the collaboration on a lot of the calculations and the idea for setting up a missing minus sign and factors of two finding business. Furthermore was the GW group at ITP an inspiring environment and the discussions in our meetings lead to quite some sections in this report. Lastly I would like to thank Casper, Thijs, Justus, Franca, Olaf, Lucas, Margot and Salma for keeping me sane during the stressful times and in general for being the greatest studygroup and friends one can hope for, providing a lot of inspirational quotes, fun lunchbreaks, outlet for stressful rants and even a physics discussion once in a while.

References

[1] P. Abbott *et al.*, “Observation of gravitational waves from a binary black hole merger,” *Phys. Rev. Lett.*, vol. 116, p. 061102, Feb 2016.

[2] D. Blair *et al.*, “Gravitational wave astronomy: the current status,” *Science China Physics, Mechanics, and Astronomy*, vol. 58, p. 5748, Dec. 2015.

[3] M. Bailes *et al.*, “Gravitational-wave physics and astronomy in the 2020s and 2030s,” *Nature Reviews Physics*, vol. 3, pp. 344–366, Apr. 2021.

[4] P. B. Abbott *et al.*, “GWTC-1: A gravitational-wave transient catalog of compact binary mergers observed by LIGO and virgo during the first and second observing runs,” *Physical Review X*, vol. 9, sep 2019.

[5] R. Abbott *et al.*, “GWTC-2: Compact binary coalescences observed by LIGO and virgo during the first half of the third observing run,” *Physical Review X*, vol. 11, jun 2021.

[6] The LIGO Scientific Collaboration, The Virgo Collaboration, The KAGRA Collaboration, *et al.*, “Gwtc-3: Compact binary coalescences observed by ligo and virgo during the second part of the third observing run,” 2021.

[7] M. Maggiore *et al.*, “Science Case for the Einstein Telescope,” *JCAP*, vol. 03, p. 050, 2020.

[8] P. Amaro-Seoane *et al.*, “Laser interferometer space antenna,” 2017.

[9] C. M. Will, “The Confrontation between General Relativity and Experiment,” *Living Rev. Rel.*, vol. 17, p. 4, 2014.

[10] A. Shomer, “A Pedagogical explanation for the non-renormalizability of gravity,” 9 2007.

[11] A. Y. Petrov, “Introduction to Modified Gravity,” SpringerBriefs in Physics, Springer, 5 2020.

[12] A. Salvio, “Quadratic Gravity,” *Front. in Phys.*, vol. 6, p. 77, 2018.

[13] K. S. Stelle, “Renormalization of higher-derivative quantum gravity,” *Phys. Rev. D*, vol. 16, pp. 953–969, Aug 1977.

[14] J. Scherk and J. H. Schwarz, “Dual Models for Nonhadrons,” *Nucl. Phys. B*, vol. 81, pp. 118–144, 1974.

[15] P. Pani and V. Cardoso, “Are black holes in alternative theories serious astrophysical candidates? the case for einstein-dilaton-gauss-bonnet black holes,” *Physical Review D*, vol. 79, apr 2009.

[16] B. Shiralilou, T. Hinderer, S. Nissanke, N. Ortiz, and H. Witek, "Nonlinear curvature effects in gravitational waves from inspiralling black hole binaries," *Phys. Rev. D*, vol. 103, no. 12, p. L121503, 2021.

[17] B. Shiralilou, T. Hinderer, S. Nissanke, N. Ortiz, and H. Witek, "Post-Newtonian Gravitational and Scalar Waves in Scalar-Gauss-Bonnet Gravity," 5 2021.

[18] B. Abbott *et al.*, "Tests of general relativity with the binary black hole signals from the LIGO-virgo catalog GWTC-1," *Physical Review D*, vol. 100, nov 2019.

[19] R. Abbott *et al.*, "Tests of general relativity with binary black holes from the second LIGO-virgo gravitational-wave transient catalog," *Physical Review D*, vol. 103, jun 2021.

[20] B. Abbott *et al.*, "Tests of general relativity with GW150914," *Physical Review Letters*, vol. 116, may 2016.

[21] N. Yunes, K. Yagi, and F. Pretorius, "Theoretical physics implications of the binary black-hole mergers GW150914 and GW151226," *Physical Review D*, vol. 94, oct 2016.

[22] M. Isi, M. Giesler, W. M. Farr, M. A. Scheel, and S. A. Teukolsky, "Testing the no-hair theorem with GW150914," *Physical Review Letters*, vol. 123, sep 2019.

[23] M. Pitkin, S. Reid, S. Rowan, and J. Hough, "Gravitational wave detection by interferometry (ground and space)," *Living Reviews in Relativity*, vol. 14, jul 2011.

[24] F.-L. Julié and E. Berti, "Post-Newtonian dynamics and black hole thermodynamics in Einstein-scalar-Gauss-Bonnet gravity," , vol. 100, p. 104061, Nov. 2019.

[25] M. Maggiore, *Gravitational Waves. Vol. 1: Theory and Experiments*. Oxford Master Series in Physics, Oxford University Press, 2007.

[26] A. Einstein, "Näherungsweise Integration der Feldgleichungen der Gravitation," *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin*, pp. 688–696, Jan. 1916.

[27] A. Einstein, "Über Gravitationswellen," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, vol. 1918, pp. 154–167, 1918.

[28] B. S. Sathyaprakash, "Gravitational waves and astrophysical sources," *Comptes Rendus Physique*, vol. 14, pp. 272–287, Apr. 2013.

[29] B. Allen, "Stochastic gravity-wave background in inflationary-universe models," *Phys. Rev. D*, vol. 37, pp. 2078–2085, Apr 1988.

[30] S. M. Carroll, *Spacetime and Geometry*. Cambridge University Press, 7 2019.

[31] C. J. Moore, R. H. Cole, and C. P. L. Berry, "Gravitational-wave sensitivity curves," *Classical and Quantum Gravity*, vol. 32, p. 015014, dec 2014.

[32] C. Cutler and É . E. Flanagan, “Gravitational waves from merging compact binaries: How accurately can one extract the binary’s parameters from the inspiral waveform?,” *Physical Review D*, vol. 49, pp. 2658–2697, mar 1994.

[33] M. E. Pati and C. M. Will, “Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations: Foundations,” , vol. 62, p. 124015, Dec. 2000.

[34] L. Blanchet and T. Damour, “Radiative gravitational fields in general relativity I. general structure of the field outside the source,” *Phil. Trans. Roy. Soc. Lond. A*, vol. 320, pp. 379–430, 1986.

[35] L. Blanchet, “Radiative Gravitation Fields in General Relativity II. Asymptotic Behaviour at Future Null Infinity,” *Proceedings of the Royal Society of London Series A*, vol. 409, pp. 383–399, Feb. 1987.

[36] L. Blanchet, “Gravitational radiation from post-newtonian sources and inspiralling compact binaries,” *Living Reviews in Relativity*, vol. 17, feb 2014.

[37] W. L. Burke, “Gravitational radiation damping of slowly moving systems calculated using matched asymptotic expansions,” *J. Math. Phys.*, vol. 12, pp. 401–418, 1971.

[38] C. M. Will and A. G. Wiseman, “Gravitational radiation from compact binary systems: Gravitational waveforms and energy loss to second post-newtonian order,” *Physical Review D*, vol. 54, pp. 4813–4848, oct 1996.

[39] L. Blanchet and T. Damour, “Post-newtonian generation of gravitational waves,” *Annales de l’I.H.P. Physique théorique*, vol. 50, no. 4, pp. 377–408, 1989.

[40] C. M. Will, “On the unreasonable effectiveness of the post-Newtonian approximation in gravitational physics,” *Proc. Nat. Acad. Sci.*, vol. 108, p. 5938, 2011.

[41] E. H. T. Collaboration *et al.*, “First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole,” , vol. 875, p. L1, Apr. 2019.

[42] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley, 1995.

[43] D. Lovelock, “The Einstein tensor and its generalizations,” *J. Math. Phys.*, vol. 12, pp. 498–501, 1971.

[44] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, “Modified Gravity and Cosmology,” *Phys. Rept.*, vol. 513, pp. 1–189, 2012.

[45] M. Ostrogradsky, “Mémoires sur les équations différentielles, relatives au problème des isopérimètres,”

[46] A. Joyce, B. Jain, J. Khoury, and M. Trodden, “Beyond the Cosmological Standard Model,” *Phys. Rept.*, vol. 568, pp. 1–98, 2015.

[47] G. Calcagni and L. Modesto, "Nonlocality in string theory," *Journal of Physics A: Mathematical and Theoretical*, vol. 47, p. 355402, aug 2014.

[48] E. Berti, K. Yagi, and N. Yunes, "Extreme Gravity Tests with Gravitational Waves from Compact Binary Coalescences: (I) Inspiral-Merger," *Gen. Rel. Grav.*, vol. 50, no. 4, p. 46, 2018.

[49] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, "Ghost-free Gauss-Bonnet Theories of Gravity," *Phys. Rev. D*, vol. 99, no. 4, p. 044050, 2019.

[50] A. D. Kovács and H. S. Reall, "Well-posed formulation of Lovelock and Horndeski theories," *Phys. Rev. D*, vol. 101, no. 12, p. 124003, 2020.

[51] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, "General second order scalar-tensor theory, self tuning, and the Fab Four," *Phys. Rev. Lett.*, vol. 108, p. 051101, 2012.

[52] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, "Self-tuning and the derivation of a class of scalar-tensor theories," *Physical Review D*, vol. 85, May 2012.

[53] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "Heterotic String Theory. 1. The Free Heterotic String," *Nucl. Phys. B*, vol. 256, p. 253, 1985.

[54] B. Zwiebach, "Curvature Squared Terms and String Theories," *Phys. Lett. B*, vol. 156, pp. 315–317, 1985.

[55] D. J. Gross and J. H. Sloan, "The quartic effective action for the heterotic string," *Nuclear Physics B*, vol. 291, pp. 41–89, 1987.

[56] D. G. Boulware and S. Deser, "String-generated gravity models," *Phys. Rev. Lett.*, vol. 55, pp. 2656–2660, Dec 1985.

[57] W. Israel, "Event horizons in static electrovac space-times," *Commun. Math. Phys.*, vol. 8, pp. 245–260, 1968.

[58] B. Carter, "Axisymmetric Black Hole Has Only Two Degrees of Freedom," , vol. 26, pp. 331–333, Feb. 1971.

[59] J. D. Bekenstein, "Nonexistence of baryon number for static black holes," *Phys. Rev. D*, vol. 5, pp. 1239–1246, Mar 1972.

[60] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, and E. Winstanley, "Dilatonic black holes in higher curvature string gravity," *Physical Review D*, vol. 54, pp. 5049–5058, oct 1996.

[61] T. P. Sotiriou and S.-Y. Zhou, "Black hole hair in generalized scalar-tensor gravity: An explicit example," *Physical Review D*, vol. 90, dec 2014.

[62] S. Mignemi and N. R. Stewart, "Charged black holes in effective string theory," *Physical Review D*, vol. 47, pp. 5259–5269, jun 1993.

[63] J. L. Ripley and F. Pretorius, "Gravitational collapse in einstein dilaton-gauss–bonnet gravity," *Classical and Quantum Gravity*, vol. 36, p. 134001, jun 2019.

[64] B. Kleihaus, J. Kunz, S. Mojica, and E. Radu, "Spinning black holes in einstein–gauss-bonnet–dilaton theory: Nonperturbative solutions," *Physical Review D*, vol. 93, feb 2016.

[65] B. Kleihaus, J. Kunz, and E. Radu, "Rotating black holes in dilatonic einstein-gauss-bonnet theory," *Physical Review Letters*, vol. 106, apr 2011.

[66] N. Yunes and L. C. Stein, "Nonspinning black holes in alternative theories of gravity," *Physical Review D*, vol. 83, may 2011.

[67] R. Benkel, T. P. Sotiriou, and H. Witek, "Black hole hair formation in shift-symmetric generalised scalar-tensor gravity," *Classical and Quantum Gravity*, vol. 34, p. 064001, feb 2017.

[68] R. Benkel, T. P. Sotiriou, and H. Witek, "Dynamical scalar hair formation around a schwarzschild black hole," *Physical Review D*, vol. 94, dec 2016.

[69] J. L. Ripley and F. Pretorius, "Scalarized black hole dynamics in einstein-dilaton-gauss-bonnet gravity," *Physical Review D*, vol. 101, feb 2020.

[70] K. Yagi, L. C. Stein, N. Yunes, and T. Tanaka, "Post-Newtonian, Quasi-Circular Binary Inspirals in Quadratic Modified Gravity," *Phys. Rev. D*, vol. 85, p. 064022, 2012. [Erratum: Phys.Rev.D 93, 029902 (2016)].

[71] K. Yagi, L. C. Stein, and N. Yunes, "Challenging the presence of scalar charge and dipolar radiation in binary pulsars," *Physical Review D*, vol. 93, jan 2016.

[72] D. D. Doneva and S. S. Yazadjiev, "New gauss-bonnet black holes with curvature-induced scalarization in extended scalar-tensor theories," *Physical Review Letters*, vol. 120, mar 2018.

[73] E. Berti, L. G. Collodel, B. Kleihaus, and J. Kunz, "Spin-induced black hole scalarization in einstein-scalar-gauss-bonnet theory," *Physical Review Letters*, vol. 126, jan 2021.

[74] D. D. Doneva, L. G. Collodel, C. J. Krüger, and S. S. Yazadjiev, "Black hole scalarization induced by the spin: –2+1 time evolution," *Physical Review D*, vol. 102, nov 2020.

[75] L. G. Collodel, B. Kleihaus, J. Kunz, and E. Berti, "Spinning and excited black holes in einstein-scalar-gauss–bonnet theory," *Classical and Quantum Gravity*, vol. 37, p. 075018, mar 2020.

[76] C. A. Herdeiro, E. Radu, H. O. Silva, T. P. Sotiriou, and N. Yunes, "Spin-induced scalarized black holes," *Physical Review Letters*, vol. 126, jan 2021.

[77] A. Dima, E. Barausse, N. Franchini, and T. P. Sotiriou, “Spin-induced black hole spontaneous scalarization,” *Physical Review Letters*, vol. 125, dec 2020.

[78] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou, and E. Berti, “Spontaneous scalarization of black holes and compact stars from a gauss-bonnet coupling,” *Physical Review Letters*, vol. 120, mar 2018.

[79] L. Annulli, “Close limit approximation for modified gravity: Scalar instabilities in binary black hole spacetimes,” *Physical Review D*, vol. 104, dec 2021.

[80] W. E. East and J. L. Ripley, “Dynamics of spontaneous black hole scalarization and mergers in einstein-scalar-gauss-bonnet gravity,” *Physical Review Letters*, vol. 127, sep 2021.

[81] D. D. Doneva and S. S. Yazadjiev, “Dynamics of the nonrotating and rotating black hole scalarization,” *Physical Review D*, vol. 103, mar 2021.

[82] H. O. Silva, H. Witek, M. Elley, and N. Yunes, “Dynamical descalarization in binary black hole mergers,” *Physical Review Letters*, vol. 127, jul 2021.

[83] B. Bertotti, L. Iess, and P. Tortora, “A test of general relativity using radio links with the Cassini spacecraft,” *Nature*, vol. 425, pp. 374–376, 2003.

[84] K. Yagi, “New constraint on scalar Gauss-Bonnet gravity and a possible explanation for the excess of the orbital decay rate in a low-mass x-ray binary,” , vol. 86, p. 081504, Oct. 2012.

[85] R. Nair, S. Perkins, H. O. Silva, and N. Yunes, “Fundamental physics implications for higher-curvature theories from binary black hole signals in the LIGO-virgo catalog GWTC-1,” *Physical Review Letters*, vol. 123, nov 2019.

[86] S. E. Perkins, R. Nair, H. O. Silva, and N. Yunes, “Improved gravitational-wave constraints on higher-order curvature theories of gravity,” *Physical Review D*, vol. 104, jul 2021.

[87] H.-T. Wang, S.-P. Tang, P.-C. Li, M.-Z. Han, and Y.-Z. Fan, “Tight constraints on einstein-dilation-gauss-bonnet gravity from GW190412 and GW190814,” *Physical Review D*, vol. 104, jul 2021.

[88] Z. Lyu, N. Jiang, and K. Yagi, “Constraints on einstein-dilation-gauss-bonnet gravity from black hole-neutron star gravitational wave events,” *Physical Review D*, vol. 105, mar 2022.

[89] E. Poisson and C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic*. Cambridge University Press, 2014.

[90] T. Hinderer, L. Rezzolla, and L. Baiotti, “Gravitational Waves from Merging Binary Neutron-Star Systems,” in *Astrophysics and Space Science Library* (L. Rezzolla, P. Pizzochero, D. I. Jones, N. Rea, and I. Vidaña, eds.), vol. 457 of *Astrophysics and Space Science Library*, p. 575, Jan. 2018.

[91] M. Shibata, “Effects of tidal resonances in coalescing compact binary systems,” *Prog. Theor. Phys.*, vol. 91, pp. 871–884, 1994.

[92] L. Bernard, “Dipolar tidal effects in scalar-tensor theories,” *Phys. Rev. D*, vol. 101, no. 2, p. 021501, 2020.

[93] W. G. Dixon, “Dynamics of extended bodies in general relativity. I. Momentum and angular momentum,” *Proc. Roy. Soc. Lond. A*, vol. 314, pp. 499–527, 1970.

[94] J. Steinhoff, T. Hinderer, A. Buonanno, and A. Taracchini, “Dynamical tides in general relativity: Effective action and effective-one-body hamiltonian,” *Physical Review D*, vol. 94, nov 2016.

[95] T. Damour and G. Esposito-Farese, “Gravitational wave versus binary - pulsar tests of strong field gravity,” *Phys. Rev. D*, vol. 58, p. 042001, 1998.

[96] A. M. Green and B. J. Kavanagh, “Primordial black holes as a dark matter candidate,” *Journal of Physics G: Nuclear and Particle Physics*, vol. 48, p. 043001, feb 2021.

[97] D. M. Eardley, “Observable effects of a scalar gravitational field in a binary pulsar.,” , vol. 196, pp. L59–L62, Mar. 1975.

[98] F.-L. Julié, “On the motion of hairy black holes in einstein-maxwell-dilaton theories,” *Journal of Cosmology and Astroparticle Physics*, vol. 2018, pp. 026–026, jan 2018.

[99] K. S. Thorne, “Multipole expansions of gravitational radiation,” *Rev. Mod. Phys.*, vol. 52, pp. 299–339, Apr 1980.

[100] R. N. Lang, “Compact binary systems in scalar-tensor gravity. II. tensor gravitational waves to second post-newtonian order,” *Physical Review D*, vol. 89, apr 2014.

[101] R. N. Lang, “Compact binary systems in scalar-tensor gravity. III. scalar waves and energy flux,” *Physical Review D*, vol. 91, apr 2015.

[102] N. Sennett, S. Marsat, and A. Buonanno, “Gravitational waveforms in scalar-tensor gravity at 2pn relative order,” *Physical Review D*, vol. 94, oct 2016.

[103] A. Buonanno, B. R. Iyer, E. Ochsner, Y. Pan, and B. S. Sathyaprakash, “Comparison of post-newtonian templates for compact binary inspiral signals in gravitational-wave detectors,” *Physical Review D*, vol. 80, oct 2009.

[104] S. Isoyama, R. Sturani, and H. Nakano, “Post-Newtonian templates for gravitational waves from compact binary inspirals,” 12 2020.

[105] S. E. Perkins, R. Nair, H. O. Silva, and N. Yunes, “Improved gravitational-wave constraints on higher-order curvature theories of gravity,” *Phys. Rev. D*, vol. 104, p. 024060, Jul 2021.

[106] A. Toubiana, S. Marsat, S. Babak, J. Baker, and T. D. Canton, “Parameter estimation of stellar-mass black hole binaries with LISA,” *Physical Review D*, vol. 102, dec 2020.

[107] A. Toubiana, S. Marsat, S. Babak, E. Barausse, and J. Baker, "Tests of general relativity with stellar-mass black hole binaries observed by LISA," *Physical Review D*, vol. 101, may 2020.

[108] A. Sesana, "Prospects for multiband gravitational-wave astronomy after GW150914," *Physical Review Letters*, vol. 116, jun 2016.

[109] E. Barausse, N. Yunes, and K. Chamberlain, "Theory-agnostic constraints on black-hole dipole radiation with multiband gravitational-wave astrophysics," *Physical Review Letters*, vol. 116, jun 2016.

[110] Z. Carson and K. Yagi, "Multi-band gravitational wave tests of general relativity," *Classical and Quantum Gravity*, vol. 37, p. 02LT01, dec 2019.

[111] G. Gnocchi, A. Maselli, T. Abdelsalhin, N. Giacobbo, and M. Mapelli, "Bounding alternative theories of gravity with multiband GW observations," *Physical Review D*, vol. 100, sep 2019.

[112] S. Vitale, "Multiband gravitational-wave astronomy: Parameter estimation and tests of general relativity with space- and ground-based detectors," *Physical Review Letters*, vol. 117, jul 2016.

[113] A. Maselli, P. Pani, L. Gualtieri, and V. Ferrari, "Rotating black holes in einstein-dilaton-gauss-bonnet gravity with finite coupling," *Phys. Rev. D*, vol. 92, p. 083014, Oct 2015.

[114] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. New York: John Wiley and Sons, 1972.

[115] N. Straumann, *General Relativity*. Graduate Texts in Physics, Dordrecht: Springer, 2013.

[116] T. Damour, M. Soffel, and C. Xu, "General-relativistic celestial mechanics. I. Method and definition of reference systems," , vol. 43, pp. 3273–3307, May 1991.

[117] S. Alexander and N. Yunes, "Chern-Simons modified general relativity," , vol. 480, pp. 1–55, Aug. 2009.

[118] S. Alexander and N. Yunes, "Parametrized post-Newtonian expansion of Chern-Simons gravity," , vol. 75, p. 124022, June 2007.

[119] A. Lue, L. Wang, and M. Kamionkowski, "Cosmological Signature of New Parity-Violating Interactions," , vol. 83, pp. 1506–1509, Aug. 1999.

[120] S. H. Alexander, M. Peskin, and M. M. Sheikh-Jabbari, "Gravi-Leptogenesis: Leptogenesis from Gravity Waves in Pseudo-scalar Driven Inflation Models," *arXiv e-prints*, pp. hep–ph/0701139, Jan. 2007.

[121] K. Yagi, N. Yunes, and T. Tanaka, "Gravitational Waves from Quasicircular Black-Hole Binaries in Dynamical Chern-Simons Gravity," , vol. 109, p. 251105, Dec. 2012.

[122] K. Yagi, N. Yunes, and T. Tanaka, “Slowly Rotating Black Holes in Dynamical Chern-Simons Gravity: Deformation Quadratic in the Spin,” *Phys. Rev. D*, vol. 86, p. 044037, 2012. [Erratum: Phys.Rev.D 89, 049902 (2014)].

[123] L. D. Landau and E. M. Lifshitz, *The classical theory of fields*. 1975.

[124] T. Delsate, D. Hilditch, and H. Witek, “Initial value formulation of dynamical Chern-Simons gravity,” *Phys. Rev. D*, vol. 91, no. 2, p. 024027, 2015.

[125] D. D. Doneva and S. S. Yazadjiev, “Spontaneously scalarized black holes in dynamical Chern-Simons gravity: dynamics and equilibrium solutions,” *Phys. Rev. D*, vol. 103, no. 8, p. 083007, 2021.

[126] C. M. Will and A. G. Wiseman, “Gravitational radiation from compact binary systems: Gravitational wave forms and energy loss to second postNewtonian order,” *Phys. Rev. D*, vol. 54, pp. 4813–4848, 1996.

[127] A. Guarnizo Trilleras, *A Model-Independent Approach to Dark Energy Cosmologies: Current and Future Constraints*. PhD thesis, 01 2016.

[128] L. Hui and A. Nicolis, “No-hair theorem for the galileon,” *Physical Review Letters*, vol. 110, jun 2013.

[129] D. D. Doneva, A. Vañó-Viñuales, and S. S. Yazadjiev, “Dynamical descalarization with a jump during black hole merger,” 2022.

[130] Z. Lyu, N. Jiang, and K. Yagi, “Constraints on einstein-dilation-gauss-bonnet gravity from black hole-neutron star gravitational wave events,” *Physical Review D*, vol. 105, mar 2022.

[131] S. Deser, R. Jackiw, and S. Templeton, “Three-dimensional massive gauge theories,” *Phys. Rev. Lett.*, vol. 48, pp. 975–978, Apr 1982.

[132] R. Jackiw and S.-Y. Pi, “Chern-simons modification of general relativity,” *Physical Review D*, vol. 68, nov 2003.

[133] S. M. Carroll, G. B. Field, and R. Jackiw, “Limits on a lorentz- and parity-violating modification of electrodynamics,” *Phys. Rev. D*, vol. 41, pp. 1231–1240, Feb 1990.

[134] H. Motohashi and T. Suyama, “Black hole perturbation in nondynamical and dynamical chern-simons gravity,” *Phys. Rev. D*, vol. 85, p. 044054, Feb 2012.

[135] S. Dyda, E. E. Flanagan, and M. Kamionkowski, “Vacuum instability in chern-simons gravity,” *Phys. Rev. D*, vol. 86, p. 124031, Dec 2012.

[136] H. O. Silva, A. M. Holgado, A. Cárdenas-Avendaño, and N. Yunes, “Astrophysical and theoretical physics implications from multimessenger neutron star observations,” *Phys. Rev. Lett.*, vol. 126, p. 181101, May 2021.

[137] Y. Ali-Haïmoud and Y. Chen, “Slowly rotating stars and black holes in dynamical chern-simons gravity,” *Physical Review D*, vol. 84, dec 2011.

[138] N. Yunes and F. Pretorius, “Dynamical chern-simons modified gravity: Spinning black holes in the slow-rotation approximation,” *Physical Review D*, vol. 79, apr 2009.

A Explicit expressions from binary dynamics calculation

At the start of section 5 we vary the action with respect to the metric and scalar field to derive the equations of motion. Here we give the variation of the energy momentum tensor related terms Eq. (5.8) and Eq. (5.11). First for the terms of Eq. (5.8)

$$\begin{aligned}
T^{m}_{\mu\nu} &= \frac{-2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(-c\sum_A\int\delta^{(4)}(x-x_A(t))m_A(\varphi)\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}dt\right)\\
&= \frac{-2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(-c\sum_A\int\delta^{(4)}(x-x_A(t))m_A(\varphi)\frac{1}{2}\frac{g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}{\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}}dt\right)\\
&= c\sum_A m_A(\varphi)\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\frac{\frac{dx_\mu^A}{dt}\frac{dx_\nu^A}{dt}}{\sqrt{gg_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}}.
\end{aligned} \tag{A.1}$$

In the second equality we used $\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta}$. Then for the tidal energy momentum tensor we have

$$\begin{aligned}
T^{tidal}_{\mu\nu} &= \frac{-2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(-\frac{1}{2}\sum_{A\neq B}\lambda_B^{(s)}\int c\delta^{(4)}(x-x_A(t))(g^{\mu\nu})_A(\partial_\mu\varphi)_A(\partial_\nu\varphi)_A\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}dt\right)\\
&= \frac{-2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}(-\frac{1}{2}\sum_{A\neq B}\lambda_B^{(s)}\int c\delta^{(4)}(x-x_A(t))\\
&\left[(\delta g^{\mu\nu})_A(\partial_\mu\varphi)_A(\partial_\nu\varphi)_A\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}+(g^{\rho\sigma})_A(\partial_\rho\varphi)_A(\partial_\sigma\varphi)_A\frac{1}{2}\frac{g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}{\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}}\right]dt)\\
&= c\sum_A\lambda_A^{(s)}\left[\partial_\mu\varphi\partial_\nu\varphi\sqrt{\frac{g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}{g}}+g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi\frac{\frac{dx_\mu^A}{dt}\frac{dx_\nu^A}{dt}}{2\sqrt{gg_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}}\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)).
\end{aligned} \tag{A.2}$$

Then for the expressions Eq. (5.11):

$$\begin{aligned}
\triangle S_m &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta\varphi}(-c\sum_A\int\delta^{(4)}(x-x_A(t))m_A(\varphi)\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}dt)\\
&= -c\sum_A\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\frac{dm_A(\varphi)}{d\varphi}\sqrt{\frac{g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}{g}}.
\end{aligned} \tag{A.3}$$

$$\begin{aligned}\triangle S_{tidal} &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta\varphi}(-\frac{1}{2}\sum_{A\neq B}\lambda_B^{(s)}\int\delta^{(4)}(x-x_A(t))cg^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}dt)\\ &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta\varphi}(-\sum_{A\neq B}\lambda_B^{(s)}\int\delta^{(4)}(x-x_A(t))cg^{\mu\nu}\partial_\mu\varphi\partial_\nu\delta\varphi\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}dt)\\ &= \sum_{A\neq B}\lambda_B^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))c\left[g^{\mu\nu}\partial_\mu\partial_\nu\varphi\sqrt{\frac{g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}{g}}+\frac{\partial_\nu(\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}g^{\mu\nu})}{\sqrt{-g}}\partial_\mu\varphi\right].\end{aligned} \tag{A.4}$$

Where in the second to last equality we integrated by parts.

To solve the equations of motion order by order, we need to expand the different terms in the equations in $1/c^2$. This is done by substituting the expansion of the scalar field and the gravitational fields U and g_i in the expressions, together with the PN expansion of the metric

$$\begin{aligned}g_{00} &= -e^{-\frac{2U}{c^2}}+\mathcal{O}(\varepsilon_{PN}^2),\\ g_{0i} &= -\frac{4}{c^3}g_{i,1}+\mathcal{O}(\varepsilon_{PN}^2),\\ g_{ij} &= \delta_{ij}e^{\frac{2U}{c^2}}+\mathcal{O}(\varepsilon_{PN}^2).\end{aligned} \tag{A.5}$$

With the fields having the expansions as discussed in section 5.3.1

$$\frac{U}{c^2}=\frac{U^{(0)}}{c^2}+\frac{U^{(1)}}{c^4}+\mathcal{O}(1/c^6), \tag{A.6}$$

$$\frac{g_i}{c^3}=\frac{g_i^{(0)}}{c^3}+\mathcal{O}(1/c^5). \tag{A.7}$$

For the scalar field we have the in background and perturbations

$$\varphi=\varphi_0+\delta\varphi, \tag{A.8}$$

followed by

$$\delta\varphi=\frac{\varphi_c}{c^2}=\frac{\varphi_c^{(0)}}{c^2}+\frac{\varphi_c^{(1)}}{c^4}+\mathcal{O}(1/c^6). \tag{A.9}$$

Next we expand the expressions turning up in the RHS of the equations of motion, using the expansions of the fields above. Therefore we first need to expand the following combinations:

The following expression needs to be expanded up to $\mathcal{O}(1/c^2)$

$$\begin{aligned} g^{\mu\nu}\partial_\mu\partial_\nu\varphi &= g^{00}\partial_0\partial_0\frac{\varphi_c^{(0)}}{c^2} + g^{0i}\partial_0\partial_i\frac{\varphi_c^{(0)}}{c^2} + g^{ij}\partial_i\partial_j\frac{\varphi_c^{(0)}}{c^2} \\ &= -\partial_0\partial_0\frac{\varphi_c^{(0)}}{c^2} + \delta_{ij}\partial_i\partial_j\frac{\varphi_c^{(0)}}{c^2} = \Box_\eta\frac{\varphi_c^{(0)}}{c^2}. \end{aligned} \tag{A.10}$$

Next we expand the metric determinant. We would like to expand the upcoming expressions up to $\mathcal{O}(1/c^2)$. As the $0i$ component of the metric is already in $1/c^3$ we can neglect this contribution, which makes our metric diagonal. The determinant of a diagonal matrix is given by the product of the diagonal elements

$$g = -e^{-\frac{2U}{c^2}+3\frac{2U}{c^2}} = -e^{\frac{4U}{c^2}} = -1 - \frac{4U^{(0)}}{c^2} + \mathcal{O}\left(1/c^4\right). \tag{A.11}$$

Then we have also some other combinations of terms which we expand in $1/c^2$:

$$\begin{aligned} g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt} &= (-1+2\frac{U^{(0)}}{c^2})\left(c\frac{dt}{dt}\right)^2 + (\delta_{ij}+2\delta_{ij}\frac{U^{(0)}}{c^2})\mathbf{v}_A^2 \\ &= -c^2 + (U^{(0)}+\mathbf{v}_A^2) + \mathcal{O}\left(1/c^2\right), \end{aligned} \tag{A.12}$$

$$\begin{aligned} \frac{1}{\sqrt{g g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}} &= \frac{1}{\sqrt{(-1)(-c^2+2U^{(0)}+v_A^2)}} \\ &= \frac{1}{c}\frac{1}{\sqrt{1+\frac{2U^{(0)}}{c^2}-\frac{v_A^2}{c^2}}} \\ &= \frac{1}{c}(1-\frac{U^{(0)}}{c^2}+\frac{v_A^2}{2c^2}) + \mathcal{O}(1/c^4), \end{aligned} \tag{A.13}$$

$$\begin{aligned} \frac{1}{\sqrt{-g}}\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}} &= \frac{1}{\sqrt{e^{\frac{4U}{c^2}}}}\sqrt{-(-c^2+2U^{(0)}+\mathbf{v}_A^2)} \\ &= \frac{1}{\sqrt{1}}c\sqrt{1-\frac{2U^{(0)}}{c^2}-\frac{v_A^2}{c^2}} = c(1-\frac{U^{(0)}}{c^2}-\frac{v_A^2}{2c^2}) \\ &= c(1-\frac{U^{(0)}}{c^2}-\frac{v_A^2}{2c^2}) + \mathcal{O}(1/c^4), \end{aligned} \tag{A.14}$$

$$
\begin{aligned}
\frac{\partial_\nu\left(\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}g^{\mu\nu}\right)}{\sqrt{-g}} &= (1-\frac{2U^{(0)}}{c^2})\left[g^{\mu\nu}\partial_\nu\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}+\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}\partial_\nu g^{\mu\nu}\right] \\
&= (1-\frac{2U^{(0)}}{c^2})\left[\frac{g^{\mu\nu}}{2\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}}\partial_\nu\left(-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}\right)+\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}\partial_\nu g^{\mu\nu}\right] \\
&= (1-\frac{2U^{(0)}}{c^2})\left[\frac{g^{\mu\nu}(1+\frac{U^{(0)}}{c^2}+\frac{v_A^2}{c^2})}{2c}(-2\partial_\nu U^{(0)}+\frac{4U^{(0)}}{c^2}\partial_\nu U-\frac{2v_A^2}{c^2}\partial_\nu U^{(0)})\right. \\
&\left.+c(1-\frac{U^{(0)}}{c^2}-\frac{v_A^2}{2c^2})\partial_\nu g^{\mu\nu}\right] \\
&= \frac{1}{2}\frac{1}{c}(1-\frac{2U^{(0)}}{c^2})(1+\frac{U^{(0)}}{c^2}+\frac{v_A^2}{c^2})g^{\mu\nu}(-2\partial_\nu U+\frac{4U^{(0)}}{c^2}\partial_\nu U-\frac{2(U^{(0)})^2}{c^2}\partial_\nu U^{(0)}) \\
&+c(1-\frac{2U^{(0)}}{c^2})(1-\frac{U^{(0)}}{c^2}-\frac{v_A^2}{2c^2})\partial_\nu g^{\mu\nu} \\
&= c\partial_\nu g^{\mu\nu}-\frac{3U^{(0)}}{c}\partial_\nu g^{\mu\nu}-\frac{v_A^2}{2c}\partial_\nu g^{\mu\nu}-\frac{1}{c}g^{\mu\nu}\partial_\nu U+\mathcal{O}(1/c^2).
\end{aligned}
\tag{A.15}
$$

Lastly we also expand the mass function

$$
\begin{aligned}
m_A(\varphi) &= m_A\left(\varphi_0\right)+m_A'\left(\varphi_0\right)\frac{\varphi_c^{(0)}}{c^2}+\frac{1}{2}m''{}_A\left(\varphi_0\right)\frac{\varphi_c^{(1)}}{c^4}+\mathcal{O}(1/c^6) \\
&= m_A^0\left[1+\alpha_A^0\frac{\varphi_c^{(0)}}{c^2}+\frac{1}{2}\left(\alpha_A^{0\,2}+\beta_A^0\right)\frac{\varphi_c^{(1)}}{c^4}\right]+\mathcal{O}\left(1/c^6\right),
\end{aligned}
\tag{A.16}
$$

with $m_A^0=m_A\left(\varphi_0\right)$ and

$$
\begin{aligned}
\alpha_A(\varphi) &\equiv \frac{d\ln m_A(\varphi)}{d(\varphi)} \\
\beta_A(\varphi) &\equiv \frac{d\alpha_A(\varphi)}{d(\varphi)},
\end{aligned}
\tag{A.17}
$$

with $\alpha_A^0=\alpha_A(\varphi_0)$ called the *scalar charge* and $\beta_A^0=\beta_A(\varphi_0)$. The scalar charge measures the strength of the coupling of the physical mass to the back ground scalar field. The expression for this scalar charge is calculated in [24] for a small coupling approximation. To first order it is given by

$$
\alpha_A^0=-\frac{\alpha f'(\varphi_0)c^4}{2G^2(m_A^0)^2}. \tag{A.18}
$$

And thus for the β^0 at lowest order

$$
\beta_A^0=-\frac{\alpha f''(\varphi_0)c^4}{2G^2(m_A^0)^2}. \tag{A.19}
$$

Then we can use these expansions to calculate the RHS of the differential equations of the EOMs. Starting with the terms on the RHS of Eq. (5.19):

For the expansion of $\bar{\delta}S_m$ we get using Eq. (A.14) and Eq. (5.22)

$$\begin{aligned}\bar{\delta}S_m &= -c\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\frac{dm_A(\varphi_c)}{d\varphi_c}\sqrt{\frac{g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}{g}} \\ &= -c\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))(0+m_A^0\alpha_A^0+m_A^0((\alpha_A^0)^2+\beta_A^0)\frac{\varphi_c^{(0)}}{c^2})c(1-\frac{U^{(0)}}{c^2}-\frac{v_A^2}{c^2}) \\ &= -\sum_A \delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))c^2\alpha_A^0 m_A^0+(m_A^0((\alpha_A^0)^2+\beta_A^0)\varphi_c^{(0)} \\ &-\alpha_A^0 m_A^0 U^{(0)}-\frac{1}{2}\alpha_A^0 m_A^0 v_A^2)+\mathcal{O}(1/c^2).\end{aligned} \tag{A.20}$$

For the expansion of $\bar{\delta}S_{tid}$ we get using Eq. (A.15), Eq. (A.14) and Eq. (A.10)

$$\begin{aligned}\bar{\delta}S_{tid} &= \sum_A \lambda_A^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))c\left[g^{\mu\nu}\partial_\mu\partial_\nu\varphi\sqrt{\frac{g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}{g}}+\frac{\partial_\nu(\sqrt{-g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}g^{\mu\nu})}{\sqrt{-g}}\partial_\mu\varphi\right] \\ &= \sum_A \lambda_A^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))c[\Box\frac{\varphi_c^{(0)}}{c^2}c(1-\frac{U^{(0)}}{c^2}-\frac{v_A^2}{c^2}) \\ &+(c\partial_\nu g^{\mu\nu}-\frac{2U^{(0)}}{c}\partial_\nu g^{\mu\nu}-\frac{U^{(0)}}{c}\partial_\nu g^{\mu\nu}-\frac{v_A^2}{2c}\partial_\nu g^{\mu\nu}-\frac{1}{c}g^{\mu\nu}\partial_\nu U^{(0)})\partial_\mu\frac{\varphi_c^{(0)}}{c^2}] \\ &= \sum_A \lambda_A^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))c^2[\Box\varphi_c^{(0)}+\partial_\nu g^{\mu\nu}\partial_\mu\delta\varphi] \\ &= \sum_A \lambda_A^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))c^2[\Box\frac{\varphi_c^{(0)}}{c^2}+\partial_i g^{ii}\partial_i\frac{\varphi_c^{(0)}}{c^2}] \\ &= \sum_A \lambda_A^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))c^2[\Box\frac{\varphi_c^{(0)}}{c^2}+\partial_i(1-\frac{2U^{(0)}}{c^2})\partial_i\frac{\varphi_c^{(0)}}{c^2}] \\ &= \sum_A \lambda_A^{(s)}\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\varphi_c^{(0)}+\mathcal{O}(1/c^2).\end{aligned} \tag{A.21}$$

Then for the terms on the RHS of the differential equation of the U field, Eq. (5.33), we have the following expansions for the energy momentum tensor terms. Using Eq. (A.13) and Eq. (5.22)

$$
\begin{aligned}
T^m_{00} &= c\sum_A m_A(\varphi)\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\frac{\frac{dx^A_0}{dt}\frac{dx^A_0}{dt}}{\sqrt{gg_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}} \\
&= c^3\sum_A m^0_A(1+\alpha^0_A\frac{\varphi^{(0)}_c}{c^2})\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\frac{1}{c}(1-\frac{U^{(0)}}{c^2}+\frac{v^2_A}{2c^2}) \\
&= \sum_A(m^0_Ac^2+m^0_A\alpha^0_A\varphi^{(0)}_c-m^0_AU^{(0)}+m^0_A\frac{1}{2}v^2_A)\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))+\mathcal{O}(1/c^2).
\end{aligned}
\tag{A.22}
$$

And for T^m_{ii}

$$
\begin{aligned}
T^m_{ii} &= c\sum_A m_A(\varphi)\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\frac{\frac{dx^A_i}{dt}\frac{dx^A_i}{dt}}{\sqrt{gg_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}} \\
&= c\sum_A m^0_A(1+\alpha^0_A\frac{\varphi^{(0)}_c}{c^2})\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))v^2_A\frac{1}{c}(1-\frac{U^{(0)}}{c^2}+\frac{v^2_A}{2c^2}) \\
&= \sum_A m^0_A\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))v^2_A+\mathcal{O}(1/c^2).
\end{aligned}
\tag{A.23}
$$

For the tidal energy momentum tensor using Eq. (A.13) and Eq. (A.14), we have

$$
\begin{aligned}
T^{tid}_{00} &= c\sum_A\lambda^{(s)}_A\left[\partial_0\frac{\varphi^{(0)}_c}{c^2}\partial_0\frac{\varphi^{(0)}_c}{c^2}\sqrt{\frac{g_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}{g}}+g^{\rho\sigma}\partial_\rho\frac{\varphi^{(0)}_c}{c^2}\partial_\sigma\frac{\varphi^{(0)}_c}{c^2}\frac{\frac{dx^A_0}{dt}\frac{dx^A_0}{dt}}{2\sqrt{gg_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}}\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)) \\
&= \sum_A\lambda^{(s)}_A\left[c^2\partial_0\frac{\varphi^{(0)}_c}{c^2}\partial_0\frac{\varphi^{(0)}_c}{c^2}(1-\frac{U^{(0)}}{c^2}-\frac{v^2_A}{c^2})+c^2g^{\rho\sigma}\partial_\rho\frac{\varphi^{(0)}_c}{c^2}\partial_\sigma\frac{\varphi^{(0)}_c}{c^2}\frac{1}{2}(1-\frac{U^{(0)}}{c^2}+\frac{v^2_A}{2c^2})\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)) \\
&= \sum_A\lambda^{(s)}_A\left[c^2\partial_0\frac{\varphi^{(0)}_c}{c^2}\partial_0\frac{\varphi^{(0)}_c}{c^2}+\frac{1}{2}c^2g^{\rho\sigma}\partial_\rho\frac{\varphi^{(0)}_c}{c^2}\partial_\sigma\frac{\varphi^{(0)}_c}{c^2}\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))+\mathcal{O}(1/c^4).
\end{aligned}
\tag{A.24}
$$

And for T^{tid}_{ii}

$$
\begin{aligned}
T^{tid}_{ii} &= c\sum_A\lambda^{(s)}_A\left[\partial_i\frac{\varphi^{(0)}_c}{c^2}\partial_i\frac{\varphi^{(0)}_c}{c^2}\sqrt{\frac{g_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}{g}}+g^{\rho\sigma}\partial_\rho\frac{\varphi^{(0)}_c}{c^2}\partial_\sigma\frac{\varphi^{(0)}_c}{c^2}\frac{\frac{dx^A_i}{dt}\frac{dx^A_i}{dt}}{2\sqrt{gg_{\alpha\beta}\frac{dx^\alpha_A}{dt}\frac{dx^\beta_A}{dt}}}\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)) \\
&= \sum_A\lambda^{(s)}_A\left[c^2\partial_i\frac{\varphi^{(0)}_c}{c^2}\partial_i\frac{\varphi^{(0)}_c}{c^2}(1-\frac{U^{(0)}}{c^2}-\frac{v^2_A}{c^2})+v^2_Ag^{\rho\sigma}\partial_\rho\frac{\varphi^{(0)}_c}{c^2}\partial_\sigma\frac{\varphi^{(0)}_c}{c^2}\frac{1}{2}(1-\frac{U^{(0)}}{c^2}+\frac{v^2_A}{2c^2})\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)) \\
&= \sum_A\lambda^{(s)}_A\left[c^2\partial_i\frac{\varphi^{(0)}_c}{c^2}\partial_i\frac{\varphi^{(0)}_c}{c^2}\varepsilon^1_c\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))+\mathcal{O}(\frac{1}{c^4}).
\end{aligned}
\tag{A.25}
$$

Lastly for the expansion of the RHS terms in Eq. (5.41) we have for the energy momentum tensor using Eq. (A.13) and Eq. (5.22)

$$\begin{aligned}T^m_{0i} &= c\sum_A m_A(\varphi)\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))\frac{\frac{dx_0^A}{dt}\frac{dx_i^A}{dt}}{\sqrt{gg_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}} \\ &= c^2\sum_A m_A^0(1+\alpha_A^0\frac{\varphi_c^{(0)}}{c^2})\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))v_A^i\frac{1}{c}(1-\frac{U^{(0)}}{c^2}+\frac{v_A^2}{2c^2}) \\ &= c\sum_A m_A^0 v_A^i\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))+\mathcal{O}(1/c).\end{aligned} \tag{A.26}$$

And for the tidal energy momentum tensor using Eq. (A.14) and Eq. (A.13)

$$\begin{aligned}T^{tid}_{0i} &= c\sum_A\lambda_A^{(s)}\left[\partial_0\frac{\varphi_c^{(0)}}{c^2}\partial_i\frac{\varphi_c^{(0)}}{c^2}\sqrt{\frac{g_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}{g}}+g^{\rho\sigma}\partial_\rho\frac{\varphi_c^{(0)}}{c^2}\partial_\sigma\frac{\varphi_c^{(0)}}{c^2}\frac{\frac{dx_0^A}{dt}\frac{dx_i^A}{dt}}{2\sqrt{gg_{\alpha\beta}\frac{dx_A^\alpha}{dt}\frac{dx_A^\beta}{dt}}}\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)) \\ &= \sum_A\lambda_A^{(s)}\left[c^2\partial_0\frac{\varphi_c^{(0)}}{c^2}\partial_i\frac{\varphi_c^{(0)}}{c^2}(1-\frac{U}{c^2}-\frac{v_A^2}{c^2})+cg^{\rho\sigma}v_i^A\partial_\rho\frac{\varphi_c^{(0)}}{c^2}\partial_\sigma\frac{\varphi_c^{(0)}}{c^2}\frac{1}{2}(1-\frac{U^{(0)}}{c^2}+\frac{v_A^2}{2c^2})\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t)) \\ &= \sum_A\lambda_A^{(s)}\left[c^2\partial_0\frac{\varphi_c^{(0)}}{c^2}\partial_i\frac{\varphi_c^{(0)}}{c^2}+\frac{1}{2}cg^{\rho\sigma}v_i^A\partial_\rho\frac{\varphi_c^{(0)}}{c^2}\partial_\sigma\frac{\varphi_c^{(0)}}{c^2}\right]\delta^{(3)}(\mathbf{x}-\mathbf{x_A}(t))+\mathcal{O}(1/c^4).\end{aligned} \tag{A.27}$$

We use these expansions in section 5 to calculate the near zone fields.

B Calculation relative acceleration

One can derive the relative acceleration from the Euler Lagrange equations in relative form

$$\frac{1}{m_A^0}\frac{\partial\mathcal{L}}{\partial x_A}-\frac{1}{m_B^0}\frac{\partial\mathcal{L}}{\partial x_B}=\frac{1}{m_A^0}\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{v_A}}-\frac{1}{m_B^0}\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{v}_B}. \tag{B.1}$$

To calculate these derivatives we can use the following from $\mathbf{r}=\mathbf{x}_A-\mathbf{x}_B$ and $\mathbf{n}=\frac{\mathbf{r}}{r}$

$$\frac{\partial r}{\partial x_A}=\frac{\mathbf{r}}{r}=-\frac{\partial r}{\partial x_B}, \tag{B.2}$$

$$\frac{\partial}{\partial x_A}\frac{1}{r^n}=-\frac{n\mathbf{n}}{r^{n+1}}=-\frac{\partial}{\partial x_B}\frac{1}{r^n}, \tag{B.3}$$

$$\begin{gathered}\frac{\partial}{x_A}(\mathbf{n}\cdot\mathbf{v}_A)=\frac{\partial}{x_A}(\frac{\mathbf{r}}{r})\cdot\mathbf{v}_A+\mathbf{n}\frac{\partial}{\partial x_A}\frac{\partial x_A}{\partial t}=(\frac{1}{r}*1-\frac{1}{r^2}\mathbf{nr})\cdot\mathbf{v_A}= \\ \frac{1}{r}(\mathbf{v}_A-(\mathbf{n}\cdot\mathbf{v}_A)\mathbf{n})=-\frac{\partial}{x_B}(\mathbf{n}\cdot\mathbf{v}_A).\end{gathered} \tag{B.4}$$

Then we calculate the derivatives with the Lagrangian Eq. (5.84). To make this more manageable we split the Lagrangian in three parts:

$$\mathcal{L}_0 = -m_A^0 c^2 - m_B^0 c^2 + \frac{1}{2} m_A^0 v_A^2 + \frac{1}{2} m_B^0 v_B^2 + \frac{G\bar{\alpha} m_A^0 m_B^0}{r} + \frac{1}{8c^2} m_A^0 v_A^4 + \frac{1}{8c^2} m_B^0 v_B^4, \tag{B.5}$$

$$\begin{aligned}\mathcal{L}_1 = &\frac{\bar{\alpha} G m_A^0 m_B^0}{2c^2 r}(3(v_A^2 + v_B^2) - 7(\mathbf{v}_A \cdot \mathbf{v}_B) - (\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B) + 2\bar{\gamma}(\mathbf{v}_A - \mathbf{v}_B)^2) \\ &- \frac{\bar{\alpha}^2 G^2 m_A^0 m_B^0}{2r^2 c^2}(m_A^0(1 + 2\bar{\beta}_B) + m_B^2(1 + 2\bar{\beta}_A)),\end{aligned} \tag{B.6}$$

$$\mathcal{L}_2 = \frac{\alpha f'(\phi_0)\bar{\alpha}^2 G^2 m_A^0 m_B^0}{c^2 r^4}(m_A^0 \bar{\delta}_A + m_B^0 \bar{\delta}'_B) - \frac{1}{2}\lambda_A^{(s)} \frac{G^2 (m_B^0 \alpha_B^0)^2}{c^2 r^4} - \frac{1}{2}\lambda_B^{(s)} \frac{G^2 (m_A^0 \alpha_A^0)^2}{c^2 r^4}. \tag{B.7}$$

B.1 $\mathcal{L}_0$ part

Starting with Eq. (B.5), we first take the derivatives to x

$$\frac{\partial \mathcal{L}_0}{\partial x_A} = -\frac{G\bar{\alpha} m_A^0 m_B^0}{r^2}\mathbf{n} = -\frac{\partial \mathcal{L}_0}{\partial x_B}. \tag{B.8}$$

Next, we take the derivatives to $\mathbf{v}$

$$\begin{aligned}\frac{d}{dt}\frac{\partial \mathcal{L}_0}{\partial \mathbf{v}_A} &= \frac{d}{dt}(m_A^0 \mathbf{v}_A + \frac{1}{2c^2} m_A^0 (\mathbf{v}_A)^2 \mathbf{v}_A) \\ &= m_A^0 \mathbf{a}_A + \frac{1}{c^2} m_A^0 (\mathbf{v}_A \cdot \mathbf{a}_A)\mathbf{v}_A + \frac{1}{2c^2} m_A^0 (\mathbf{v}_A)^2 \mathbf{a}_A = \frac{d}{dt}\frac{\partial \mathcal{L}_0}{\partial \mathbf{v}_B}.\end{aligned} \tag{B.9}$$

Then we substitute these equations into Eq. (B.1)

$$\frac{1}{m_A^0}\frac{\partial \mathcal{L}_0}{\partial x_A} - \frac{1}{m_B^0}\frac{\partial \mathcal{L}_0}{\partial x_B} = -\frac{G\bar{\alpha} m}{r^2}\mathbf{n}, \tag{B.10}$$

$$\begin{aligned}\frac{1}{m_A^0}\frac{d}{dt}\frac{\partial \mathcal{L}_0}{\partial \mathbf{v}_A} - \frac{1}{m_B^0}\frac{d}{dt}\frac{\partial \mathcal{L}_0}{\partial \mathbf{v}_B} =& \mathbf{a}_A - \mathbf{a}_B + \frac{1}{c^2}(\mathbf{v}_A \cdot \mathbf{a}_A)\mathbf{v}_A - \frac{1}{c^2}(\mathbf{v}_B \cdot \mathbf{a}_B)\mathbf{v}_B + \\ &\frac{1}{2c^2}\mathbf{v}_A^2 \mathbf{a}_A - \frac{1}{2c^2}\mathbf{v}_B^2 \mathbf{a}_B.\end{aligned} \tag{B.11}$$

B.2 $\mathcal{L}_1$ part

We perform similar calculation for Eq. (B.6)

$$\begin{aligned}\frac{\partial\mathcal{L}_1}{\partial x_A} &= -\frac{G\bar{\alpha}m_A^0m_B^0}{2r^2c^2}\mathbf{n}(3(\mathbf{v}_A^2+\mathbf{v}_B^2)-7(\mathbf{v}_A\cdot\mathbf{v}_B)-(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)+2\bar{\gamma}(\mathbf{v}_A-\mathbf{v}_B)^2\\ &-\frac{G\bar{\alpha}}{r}(m_A^0(1+2\bar{\beta}_B)+m_B^0(1+2\bar{\beta}_A)))+\frac{G\bar{\alpha}m_A^0m_B^0}{2rc^2}(-\frac{1}{r}(\mathbf{v}_A-(\mathbf{n}\cdot\mathbf{v}_A)\mathbf{n})(\mathbf{n}\cdot\mathbf{v}_B)\\ &-(\mathbf{n}\cdot\mathbf{v}_A)\frac{1}{r}(\mathbf{v}_B-(\mathbf{n}\cdot\mathbf{v}_B)\mathbf{n})+\frac{G\bar{\alpha}\mathbf{n}}{r^2}(m_A^0(1+2\bar{\beta}_B)+m_B^0(1+2\bar{\beta}_A)))\\ &=\frac{G\bar{\alpha}m_A^0m_B^0}{2c^2r^2}\left\{-\mathbf{v}_A\left(\mathbf{n}\cdot\mathbf{v}_B\right)-\mathbf{v}_B\left(\mathbf{n}\cdot\mathbf{v}_A\right)+\left[3\left(\mathbf{n}\cdot\mathbf{v}_A\right)\left(\mathbf{n}\cdot\mathbf{v}_B\right)-3\left(\mathbf{v}_A^2+\mathbf{v}_B^2\right)\right.\right.\\ &\left.\left.+7\left(\mathbf{v}_A\cdot\mathbf{v}_B\right)-2\bar{\gamma}\left(\mathbf{v}_A-\mathbf{v}_B\right)^2+\frac{2G\bar{\alpha}}{r}\left(m_A^0\left(1+2\bar{\beta}_B\right)+m_B^0\left(1+2\bar{\beta}_A\right)\right)\right]\mathbf{n}\right\}=-\frac{\partial\mathcal{L}_1}{\partial x_B}\end{aligned} \tag{B.12}$$

The derivative to $\mathbf{v}$ is given by

$$\frac{\partial\mathcal{L}_1}{\partial\mathbf{v}_A}=\frac{G\bar{\alpha}m_A^0m_B^0}{2c^2r}\left[6\mathbf{v}_A-7\mathbf{v}_B-\mathbf{n}\left(\mathbf{n}\cdot\mathbf{v}_B\right)+4\bar{\gamma}\left(\mathbf{v}_A-\mathbf{v}_B\right)\right], \tag{B.13}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial\mathcal{L}_1}{\partial\mathbf{v}_A} &= -\frac{G\bar{\alpha}m_A^0m_B^0(\mathbf{n}\cdot\mathbf{v})}{2c^2r^2}\left[6\mathbf{v}_A-7\mathbf{v}_B-\mathbf{n}\left(\mathbf{n}\cdot\mathbf{v}_B\right)+4\bar{\gamma}\mathbf{v}\right]\\ &+\frac{G\bar{\alpha}m_A^0m_B^0}{2c^2r}\left(6\mathbf{a}_A-7\mathbf{a}_B-\frac{\mathbf{v}\left(\mathbf{n}\cdot\mathbf{v}_B\right)}{r}-\frac{\mathbf{n}\left(\mathbf{v}\cdot\mathbf{v}_B\right)}{r}-\mathbf{n}\left(\mathbf{n}\cdot\mathbf{a}_B\right)\right.\\ &\left.+\frac{2\mathbf{n}\left(\mathbf{n}\cdot\mathbf{v}_B\right)\left(\mathbf{n}\cdot\mathbf{v}\right)}{r}+4\bar{\gamma}\mathbf{a}\right)\\ &=\frac{G\bar{\alpha}m_A^0m_B^0}{2c^2r}(6\mathbf{a}_A-7\mathbf{a}_B+4\bar{\gamma}\mathbf{a}-\mathbf{n}\left(\mathbf{n}\cdot\mathbf{a}_B\right)\\ &+\frac{1}{r}\left[-\mathbf{v}\left(\mathbf{n}\cdot\mathbf{v}_B\right)-\mathbf{n}\left(\mathbf{v}\cdot\mathbf{v}_B\right)+\left(\mathbf{n}\cdot\mathbf{v}\right)\left(3\mathbf{n}\left(\mathbf{n}\cdot\mathbf{v}_B\right)-6\mathbf{v}_A+7\mathbf{v}_B-4\bar{\gamma}\mathbf{v}\right)\right])\\ &=\frac{d}{dt}\frac{\partial\mathcal{L}_1}{\partial\mathbf{v}_B}(A\leftrightarrow B,\mathbf{v}\to-\mathbf{v},\mathbf{n}\to-\mathbf{n}).\end{aligned} \tag{B.14}$$

Then we substitute this back in Eq. (B.1) resulting in

$$\begin{aligned}\frac{1}{m_A^0}\frac{\partial\mathcal{L}_1}{\partial\mathbf{x}_A}-\frac{1}{m_B^0}\frac{\partial\mathcal{L}_1}{\partial\mathbf{x}_B} &= \frac{G\bar{\alpha}m}{2c^2r^2}\left\{-\mathbf{v}_A\left(\mathbf{n}\cdot\mathbf{v}_B\right)-\mathbf{v}_B\left(\mathbf{n}\cdot\mathbf{v}_A\right)\right.\\ &+\left[3\left(\mathbf{n}\cdot\mathbf{v}_A\right)\left(\mathbf{n}\cdot\mathbf{v}_B\right)-3\left(\mathbf{v}_A^2+\mathbf{v}_B^2\right)+7\left(\mathbf{v}_A\cdot\mathbf{v}_B\right)-2\bar{\gamma}\mathbf{v}^2\right.\\ &\left.\left.+\frac{2G\bar{\alpha}}{r}\left(m_A^0\left(1+2\bar{\beta}_B\right)+m_B^0\left(1+2\bar{\beta}_A\right)\right)\right]\mathbf{n}\right\},\end{aligned} \tag{B.15}$$

$$
\begin{aligned}
\frac{1}{m_A^0}\frac{d}{dt}\frac{\partial \mathcal{L}_1}{\partial \mathbf{v}_A}-\frac{1}{m_B^0}\frac{d}{dt}\frac{\partial L_1}{\partial \mathbf{v}_B}=&\frac{G\bar{\alpha}m_B^0}{2c^2r}\left\{6\mathbf{a}-\mathbf{a}_B+4\bar{\gamma}\mathbf{a}-\mathbf{n}\left(\mathbf{n}\cdot\mathbf{a}_B\right)\right.\\
&\left.+\frac{1}{r}\left[-\mathbf{v}\left(\mathbf{n}\cdot\mathbf{v}_B\right)-\mathbf{n}\left(\mathbf{v}\cdot\mathbf{v}_B\right)+\left(\mathbf{n}\cdot\mathbf{v}\right)\left(3\mathbf{n}\left(\mathbf{n}\cdot\mathbf{v}_B\right)-6\mathbf{v}+\mathbf{v}_B-4\bar{\gamma}\mathbf{v}\right)\right]\right\}\\
&-\frac{G\bar{\alpha}m_A^0}{2c^2r}\left\{-6\mathbf{a}-\mathbf{a}_A-4\bar{\gamma}\mathbf{a}-\mathbf{n}\left(\mathbf{n}\cdot\mathbf{a}_A\right)\right.\\
&\left.+\frac{1}{r}\left[-\mathbf{v}\left(\mathbf{n}\cdot\mathbf{v}_A\right)-\mathbf{n}\left(\mathbf{v}\cdot\mathbf{v}_A\right)+\left(\mathbf{n}\cdot\mathbf{v}\right)\left(3\mathbf{n}\left(\mathbf{n}\cdot\mathbf{v}_A\right)+6\mathbf{v}+\mathbf{v}_A+4\bar{\gamma}\mathbf{v}\right)\right]\right\}.
\end{aligned}
\tag{B.16}
$$

B.3 $\mathcal{L}_2$ part

Then for the last part of the Lagrangian Eq. (B.7) we recover the following expressions

$$
\begin{aligned}
\frac{\partial \mathcal{L}_2}{\partial \mathbf{x}_A}=&-\frac{4\alpha f'\left(\phi_0\right)G^2\bar{\alpha}^2m_A^0m_B^0\mathbf{n}}{c^2r^5}\left(m_A^0\bar{\delta}_A+m_B^0\bar{\delta}_B\right)+2\lambda_A^{(s)}\frac{G^2(m_B^0\alpha_B^0)^2\mathbf{n}}{c^2r^5}+2\lambda_B^{(s)}\frac{G^2(m_A^0\alpha_A^0)^2\mathbf{n}}{c^2r^5}\\
=&-\frac{\partial \mathcal{L}_2}{\partial \mathbf{x}_B},
\end{aligned}
\tag{B.17}
$$

$$
\frac{d}{dt}\frac{\partial \mathcal{L}_2}{\partial \mathbf{v}_A}=0=\frac{d}{dt}\frac{\partial \mathcal{L}_2}{\partial \mathbf{v}_B}. \tag{B.18}
$$

Then we substitute this back in Eq. (B.1) resulting in

$$
\begin{aligned}
\frac{1}{m_A^0}\frac{\partial \mathcal{L}_2}{\partial \mathbf{x}_A}-\frac{1}{m_B^0}\frac{\partial \mathcal{L}_2}{\partial \mathbf{x}_B}=&-\frac{4\alpha f'\left(\phi_0\right)G^2\bar{\alpha}^2m\mathbf{n}}{c^2r^5}\left[m_A^0\bar{\delta}_A+m_B^0\bar{\delta}_B\right]+2\lambda_A^{(s)}\frac{G^2(m_B^0\alpha_B^0)^2\mathbf{n}}{m_A^0c^2r^5}\\
&+2\lambda_B^{(s)}\frac{G^2m_A^0(\alpha_A^0)^2\mathbf{n}}{c^2r^5}+2\lambda_A^{(s)}\frac{G^2m_B^0(\alpha_B^0)^2\mathbf{n}}{c^2r^5}\\
&+2\lambda_B^{(s)}\frac{G^2(m_A^0\alpha_A^0)^2\mathbf{n}}{m_B^0c^2r^5}\\
=&-\frac{4\alpha f'\left(\phi_0\right)G\bar{\alpha}^2m\mathbf{n}}{c^2r^5}\left[m_A^0\bar{\delta}_A+m_B^0\bar{\delta}_B\right]\\
&+\frac{2G^2m\bar{\alpha}^2\mathbf{n}}{c^2r^5}(\lambda_A^{(s)}\frac{m_B^0(\alpha_B^0)^2}{\bar{\alpha}^2m_A^0}+\lambda_B^{(s)}\frac{m_A^0(\alpha_A^0)^2}{\bar{\alpha}^2m_B^0}),
\end{aligned}
\tag{B.19}
$$

with $\zeta=\lambda_A^{(s)}\frac{m_B^0(\alpha_B^0)^2}{\bar{\alpha}^2m_A^0}+\lambda_B^{(s)}\frac{m_A^0(\alpha_A^0)^2}{\bar{\alpha}^2m_B^0}$, and

$$
\frac{1}{m_A^0}\frac{d}{dt}\frac{\partial \mathcal{L}_2}{\partial \mathbf{v}_A}-\frac{1}{m_B^0}\frac{d}{dt}\frac{\partial \mathcal{L}_2}{\partial \mathbf{v}_B}=0. \tag{B.20}
$$

Next, we substitute all these parts together in Eq. (B.1), resulting in

$$\begin{aligned}
&-\frac{G\bar{\alpha}m\mathbf{n}}{r^2}+\frac{G\bar{\alpha}m}{2c^2r^2}\{-\mathbf{v}_A(\mathbf{n}\cdot\mathbf{v}_B)-\mathbf{v}_B(\mathbf{n}\cdot\mathbf{v}_A)+[3(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)\\
&-3\left(\mathbf{v}_A^2+\mathbf{v}_B^2\right)+7(\mathbf{v}_A\cdot\mathbf{v}_B)-2\bar{\gamma}\mathbf{v}^2+\frac{2G\bar{\alpha}}{r}\left(m_A^0\left(1+2\bar{\beta}_B\right)+m_B^0\left(1+2\bar{\beta}_A\right)\right)\Big]\mathbf{n}\Big\}\\
&-\frac{4\alpha f'(\phi_0)G^2\bar{\alpha}^2m\mathbf{n}}{c^2r^5}\left[m_A^0\bar{\delta}_A+m_B^0\bar{\delta}_B\right]+\frac{2G^2m\bar{\alpha}^2\mathbf{n}}{c^2r^5}\xi\\
&=\mathbf{a}+(\mathbf{v}_A\cdot\mathbf{a}_A)\mathbf{v}_A-(\mathbf{v}_B\cdot\mathbf{a}_B)\mathbf{v}_B+\frac{1}{2}(\mathbf{v}_A)^2\mathbf{a}_A-\frac{1}{2}(\mathbf{v}_B)^2\mathbf{a}_B\\
&+\frac{G\bar{\alpha}m_B^0}{2c^2r}\Big\{6\mathbf{a}-\mathbf{a}_B+4\bar{\gamma}\mathbf{a}-\mathbf{n}(\mathbf{n}\cdot\mathbf{a}_B)+\frac{1}{r}[-\mathbf{v}(\mathbf{n}\cdot\mathbf{v}_B)-\mathbf{n}(\mathbf{v}\cdot\mathbf{v}_B)\\
&+(\mathbf{n}\cdot\mathbf{v})(3\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_B)-6\mathbf{v}+\mathbf{v}_B-4\bar{\gamma}\mathbf{v})]\}-\frac{G\bar{\alpha}m_A^0}{2c^2r}\{-6\mathbf{a}-\mathbf{a}_A-4\bar{\gamma}\mathbf{a}-\mathbf{n}(\mathbf{n}\cdot\mathbf{a}_A)\\
&+\frac{1}{r}[-\mathbf{v}(\mathbf{n}\cdot\mathbf{v}_A)-\mathbf{n}(\mathbf{v}\cdot\mathbf{v}_A)+(\mathbf{n}\cdot\mathbf{v})(3\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_A)+6\mathbf{v}+\mathbf{v}_A+4\bar{\gamma}\mathbf{v})]\Big\}.
\end{aligned}\tag{B.21}$$

We rewrite in terms of the relative acceleration

$$\begin{aligned}
\mathbf{a}=&-\frac{G\bar{\alpha}m\mathbf{n}}{r^2}+\frac{G\bar{\alpha}m}{2c^2r^2}\{-\mathbf{v}_A(\mathbf{n}\cdot\mathbf{v}_B)-\mathbf{v}_B(\mathbf{n}\cdot\mathbf{v}_A)+[3(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)\\
&-3\left(\mathbf{v}_A^2+\mathbf{v}_B^2\right)+7(\mathbf{v}_A\cdot\mathbf{v}_B)-2\bar{\gamma}\mathbf{v}^2+\frac{2G\bar{\alpha}}{r}\left(m_A^0\left(1+2\bar{\beta}_B\right)+m_B^0\left(1+2\bar{\beta}_A\right)\right)\Big]\mathbf{n}\Big\}\\
&-\frac{4\alpha f'(\phi_0)G^2\bar{\alpha}^2m\mathbf{n}}{c^2r^5}\left[m_A^0\bar{\delta}_A+m_B^0\bar{\delta}_B\right]+\frac{2G^2m\bar{\alpha}^2\mathbf{n}}{c^2r^5}\xi-\Big((\mathbf{v}_A\cdot\mathbf{a}_A)\mathbf{v}_A-(\mathbf{v}_B\cdot\mathbf{a}_B)\mathbf{v}_B+\frac{1}{2}(\mathbf{v}_A)^2\mathbf{a}_A\\
&-\frac{1}{2}(\mathbf{v}_B)^2\mathbf{a}_B+\frac{G\bar{\alpha}m_B^0}{2c^2r}\Big\{6\mathbf{a}-\mathbf{a}_B+4\bar{\gamma}\mathbf{a}-\mathbf{n}(\mathbf{n}\cdot\mathbf{a}_B)+\frac{1}{r}[-\mathbf{v}(\mathbf{n}\cdot\mathbf{v}_B)-\mathbf{n}(\mathbf{v}\cdot\mathbf{v}_B)\\
&+(\mathbf{n}\cdot\mathbf{v})(3\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_B)-6\mathbf{v}+\mathbf{v}_B-4\bar{\gamma}\mathbf{v})]\}-\frac{G\bar{\alpha}m_A^0}{2c^2r}\{-6\mathbf{a}-\mathbf{a}_A-4\bar{\gamma}\mathbf{a}-\mathbf{n}(\mathbf{n}\cdot\mathbf{a}_A)\\
&+\frac{1}{r}[-\mathbf{v}(\mathbf{n}\cdot\mathbf{v}_A)-\mathbf{n}(\mathbf{v}\cdot\mathbf{v}_A)+(\mathbf{n}\cdot\mathbf{v})(3\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_A)+6\mathbf{v}+\mathbf{v}_A+4\bar{\gamma}\mathbf{v})]\Big\}\Big).
\end{aligned}\tag{B.22}$$

From this we can see that the lowest order correction (Newtonian) is given by $\mathbf{a}=-\frac{G\bar{\alpha}m\mathbf{n}}{r^2}$ and thus also $\mathbf{a}_A=-\frac{G\bar{\alpha}m_B^0\mathbf{n}}{r^2}$ $\mathbf{a}_B=\frac{G\bar{\alpha}m_A^0\mathbf{n}}{r^2}$. Substituting the latter two in the expression for $\mathbf{a}_A$ and $\mathbf{a}_B$ and grouping the terms results in

$$\begin{aligned}
\mathbf{a}=&-\frac{G\bar{\alpha}m\mathbf{n}}{r^2}+\frac{G\bar{\alpha}}{2c^2r^2}\Big[\frac{G\bar{\alpha}\mathbf{n}}{r}[8m^2+4m_B^0m_A^0+4\bar{\gamma}m^2+4m(m_A^0\bar{\beta}_B+m_B^0\bar{\beta}_A)\\
&-\frac{8\alpha f'(\phi_0)m}{r^2}(m_A^0\bar{\delta}_A+m_B^0\bar{\delta}_B)+\frac{4m}{r^2}\xi]+\mathbf{n}[3m(\mathbf{n}\cdot\mathbf{v}_A)(\mathbf{n}\cdot\mathbf{v}_B)\\
&+3(\mathbf{n}\cdot\mathbf{v})\left(\mathbf{n}\cdot\left(m_A^0\mathbf{v}_A-m_B^0\mathbf{v}_B\right)\right)-3m\left(\mathbf{v}_A^2+\mathbf{v}_B^2\right)\\
&+7m(\mathbf{v}_A\cdot\mathbf{v}_B)-2m\bar{\gamma}\mathbf{v}^2-\mathbf{v}\cdot\left(m_A^0\mathbf{v}_A-m_B^0\mathbf{v}_B\right)+m_B^0\mathbf{v}_A^2+m_A^0\mathbf{v}_B^2]\\
&-m\mathbf{v}_A(\mathbf{n}\cdot\mathbf{v}_B)-m\mathbf{v}_B(\mathbf{n}\cdot\mathbf{v}_A)+2m_B^0\mathbf{v}_A(\mathbf{n}\cdot\mathbf{v}_A)+2m_A^0\mathbf{v}_B(\mathbf{n}\cdot\mathbf{v}_B)\\
&+\left(6m\mathbf{v}-m_B^0\mathbf{v}_B+m_A^0\mathbf{v}_A+4\bar{\gamma}m\mathbf{v}\right)(\mathbf{n}\cdot\mathbf{v})-\mathbf{v}\left(\mathbf{n}\cdot\left(m_A^0\mathbf{v}_A-m_B^0\mathbf{v}_B\right)\right)\Big].
\end{aligned}\tag{B.23}$$

Rewriting even further gives

$$
\begin{aligned}
\mathbf{a} = &-\frac{G\bar{\alpha}m\mathbf{n}}{r^2} + \frac{G\bar{\alpha}}{c^2r^2}\left\{\frac{2G\bar{\alpha}\mathbf{n}}{r}[2m^2 + m_B^0 m_A^0 + \bar{\gamma}m^2 + m(m_A^0\bar{\beta}_B + m_B^0\bar{\beta}_A)\right. \\
&-\frac{2\alpha f'(\phi_0)m}{r^2}(m_A^0\bar{\delta}_A + m_B^0\bar{\delta}_B) + \frac{m}{r^2}\xi] \\
&+\mathbf{n}\left[\frac{3}{2}\left(m_A^0\left(\mathbf{n}\cdot\mathbf{v}_A\right)^2 + m_B^0\left(\mathbf{n}\cdot\mathbf{v}_B\right)^2\right) - 2m\mathbf{v}^2 + m_A^0\mathbf{v}_B^2 + m_B^0\mathbf{v}_A^2 - m\bar{\gamma}\mathbf{v}^2\right] \\
&\left.+\mathbf{v}\left[4m(\mathbf{n}\cdot\mathbf{v}) - m_A^0\left(\mathbf{n}\cdot\mathbf{v}_A\right) + m_B^0\left(\mathbf{n}\cdot\mathbf{v}_B\right) + 2\bar{\gamma}m(\mathbf{n}\cdot\mathbf{v})\right]\right\}.
\end{aligned}
\tag{B.24}
$$

We move to the CM frame with the relative coordinates given by

$$
\begin{aligned}
&\mathbf{x}_A \approx \frac{m_B^0}{m}\mathbf{r}, \quad \mathbf{x}_B \approx -\frac{m_A^0}{m}\mathbf{r}, \\
\Rightarrow \quad &\mathbf{v}_A \approx \frac{m_B^0}{m}\mathbf{v}, \quad \mathbf{v}_B \approx -\frac{m_A^0}{m}\mathbf{v}.
\end{aligned}
\tag{B.25}
$$

Substituting these expressions in Eq. (B.24) gives

$$
\begin{aligned}
\mathbf{a} = &-\frac{G\bar{\alpha}m\mathbf{n}}{r^2} + \frac{G\bar{\alpha}}{c^2r^2}\left\{\frac{2G\bar{\alpha}\mathbf{n}}{r}\left[2m^2 + m_A^0 m_B^0 + \bar{\gamma}m^2\right.\right. \\
&\left.+m\left(m_A^0\bar{\beta}_B + m_B^0\bar{\beta}_A\right) - \frac{2\alpha f'\left(\phi_0\right)m}{r^2}\left(m_A^0\bar{\delta}_A + m_B^0\bar{\delta}_B\right) + \frac{m}{r^2}\xi\right] \\
&+\mathbf{n}\left[\frac{3}{2}m\frac{m_A^0 m_B^0}{m^2}(\mathbf{n}\cdot\mathbf{v})^2 + \left(\frac{(m_A^0)^3 + (m_B^0)^3}{m^2} - 2m - m\bar{\gamma}\right)\mathbf{v}^2\right] \\
&\left.+2m\mathbf{v}(\mathbf{n}\cdot\mathbf{v})\left[2 - \frac{m_A^0 m_B^0}{m^2} + \bar{\gamma}\right]\right\}.
\end{aligned}
\tag{B.26}
$$

Then, using the definitions

$$
\begin{aligned}
&\mathcal{S}_\pm \equiv \frac{\alpha_A^0 \pm \alpha_B^0}{2\sqrt{\alpha}}, \qquad \beta_\pm \equiv \frac{\bar{\beta}_A \pm \bar{\beta}_B}{2}, \\
&\eta \equiv \frac{m_A^0 m_B^0}{m^2}, \qquad \Delta m \equiv m_A^0 - m_B^0,
\end{aligned}
\tag{B.27}
$$

and reintroducing $\dot{r} = (\mathbf{n}\cdot\mathbf{v})$, we can rewrite the relative acceleration as

$$
\begin{aligned}
\mathbf{a} = &-\frac{G\bar{\alpha}m\mathbf{n}}{r^2} + \frac{G\bar{\alpha}m}{c^2r^2}\left\{\mathbf{n}\left[\frac{3}{2}\eta\dot{r}^2 - (1 + 3\eta + \bar{\gamma})\mathbf{v}^2\right] + 2\mathbf{v}\dot{r}[2 - \eta + \bar{\gamma}]\right. \\
&\left.+\frac{2G\bar{\alpha}m\mathbf{n}}{r}\left[2 + \eta + \bar{\gamma} + \beta_+ - \frac{\Delta m}{m}\beta_- - \frac{2\alpha f'\left(\phi_0\right)}{\bar{\alpha}^{3/2}r^2}\left(3S_+ + \frac{\Delta m}{m}S_-\right) + \frac{\xi}{mr^2}\right]\right\}.
\end{aligned}
\tag{B.28}
$$

From this expression we continue in section 5.6.

C Greens functions

In the main part of this thesis we often encounter wave equations as they are the differential equations governing the evolution of the GWs. Here we discuss some background on Greens functions which are often used to solve wave like equations.

In the three dimensional case, the Greens function is a common way to solve the Poisson equation

$$\nabla^2 f(\mathbf{x}) = g(\mathbf{x}), \tag{C.1}$$

with the solution

$$f(\mathbf{x}) = \int_{\mathbf{x}'} G\left(\mathbf{x}, \mathbf{x}'\right) g\left(\mathbf{x}'\right) d\mathbf{x}'. \tag{C.2}$$

Here is G the Greens function defined as

$$\nabla^2 G\left(\mathbf{x}, \mathbf{x}'\right) = \delta\left(\mathbf{x} - \mathbf{x}'\right). \tag{C.3}$$

The solution for the Greens function of the equation above is given by

$$G\left(\mathbf{x}, \mathbf{x}'\right) = -\frac{1}{4\pi} \cdot \frac{1}{\left|\mathbf{x} - \mathbf{x}'\right|}. \tag{C.4}$$

In the (flat space) relativistic case, the Greens function is used to solve the wave equation

$$\Box_\eta \phi(\mathbf{x}, t) = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{x}, t) = -\rho(\mathbf{x}, t). \tag{C.5}$$

Then the relativistic Greens function satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G\left(\mathbf{x}, t, \mathbf{x}', t'\right) = \delta\left(\mathbf{x} - \mathbf{x}'\right)\delta\left(t - t'\right). \tag{C.6}$$

To derive the solution for G it is easiest to switch to Fourier space using the identity

$$\delta\left(t - t'\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega. \tag{C.7}$$

Then in Fourier space Eq. (C.6) becomes

$$\left(\nabla^2 + k^2\right) G\left(\mathbf{x}, \mathbf{x}', \omega\right) = \delta\left(\mathbf{x} - \mathbf{x}'\right) e^{i\omega t'}. \tag{C.8}$$

This allows for the following solutions

$$\begin{aligned} G_0\left(\mathbf{x}, \mathbf{x}', \omega\right) &= \frac{-\cos\left(k\left|\mathbf{x} - \mathbf{x}'\right|\right)}{4\pi\left|\mathbf{x} - \mathbf{x}'\right|} e^{i\omega t'}, \\ G_+\left(\mathbf{x}, \mathbf{x}', \omega\right) &= \frac{-e^{+ik\left|\mathbf{x} - \mathbf{x}'\right|}}{4\pi\left|\mathbf{x} - \mathbf{x}'\right|} e^{i\omega t'}, \\ G_-\left(x, x', \omega\right) &= \frac{-e^{-ik\left|x - x'\right|}}{4\pi\left|x - x'\right|} e^{i\omega t'}. \end{aligned} \tag{C.9}$$

Transforming these solutions back to real space gives

$$G_{\pm}\left(\mathbf{x},t,\mathbf{x}',t'\right)=\frac{-\delta\left(\left(t-t'\right)\mp\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{4\pi\left|\mathbf{x}-\mathbf{x}'\right|}, \tag{C.10a}$$

$$G_{0}\left(\mathbf{x},t,\mathbf{x}',t'\right)=\frac{1}{2}\left(G_{+}\left(\mathbf{x},t,\mathbf{x}',t'\right)+G_{-}\left(\mathbf{x},t,\mathbf{x}',t'\right)\right). \tag{C.10b}$$

Where for the first solution, the $+$ stands for the retarded Greens function, the solution in this case depends on the sources in the past. The $-$ stands for the advanced Greens function, which depends on the sources in the future. In general the retarded solution is seen as the most physical solution. In this case the solution of the equation can then be seen as an initial value problem with its evolution depending on sources in the past, respecting causal structure.
The G_0 solution is formed from a linear combination of the retarded and advanced Greens functions. This means that the solution is given half from sources in the past and half from the same sources in the future. A reason for selecting this solution is, if the solution is preferred to be energy conserving and therefore to be symmetric in time. For this reason we select this solution in the calculation in section 5. This solution Eq. (C.10b) can be expanded as

$$\begin{aligned}G_{0}\left(x,x'\right)&=\frac{1}{2}\left[\frac{\delta\left(t-t'-|\mathrm{x}-\mathrm{x}'|\right.}{|\mathrm{x}-\mathrm{x}'|}+\frac{\delta\left(t-t'+|\mathrm{x}-\mathrm{x}'|\right.}{|\mathrm{x}-\mathrm{x}'|}\right]\\&=\frac{\delta\left(t-t'\right)}{|\mathrm{x}-\mathrm{x}'|}+\frac{|\mathrm{x}-\mathrm{x}'|}{2}\partial_t^2\delta\left(t-t'\right)+\cdots.\end{aligned} \tag{C.11}$$

D Expressions from General Relativity

As gravitational waves are a phenomenon fully described by GR, this thesis is heavily relying on its fundamentals. However it is a bit out of the scope to discuss the whole theory here. Instead we mention the most important expressions from GR that we make use of in this thesis and refer for more elaborate study to [30]. This is also the reference on which this discussion is based. In this appendix we set $c=1$.

Einsteins General Relativity describes gravity as the curvature of spacetime. The properties of the spacetime are described by the spacetime metric $g_{\mu\nu}$. This tensor is contained in

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu, \tag{D.1}$$

the invariant spacetime distance element, for which the metric defines the "shortest distance" in spacetime. When spacetime is curved this is no longer a straight line and the metric tells you what it is instead.

The curvature of spacetime is characterised by different curvature tensors and a scalar. We have the Riemann tensor defined as

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{v\sigma} - \partial_{v}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{v\sigma} - \Gamma^{\rho}_{v\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{D.2}$$

with the following symmetry properties

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu},$$
$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu},$$
$$R_{\rho\sigma\mu v} = R_{\mu v\rho\sigma\cdot},$$
$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu},$$
$$R_{\rho\sigma\mu\nu} + R_{\rho\mu v\sigma} + R_{\rho v\sigma\mu} = 0.$$

Hereby is $\Gamma^{\rho}_{v\sigma}$ the Levi-Civita connection defined as

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu}\right). \tag{D.3}$$

Contractions of the Riemann tensor result in the Ricci tensor

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}, \tag{D.4}$$

which is symmetric in its indices. Taking the trace of this tensor results in the Ricci Scalar R.

The path followed by a test particle through spacetime, called a geodesic, is described by the geodesic equation

$$\frac{d^2x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^{\rho}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0. \tag{D.5}$$

For generalizing flat space equations to curved backgrounds, one upgrades the partial derivative to the covariant derivative and writes the equations in a covariant, coordinate invariant form. This is called the minimal coupling principle. The covariant derivative is defined as

$$\nabla_{\mu}V^{\nu} = \partial_{\mu}V^{\nu} + \Gamma^{\nu}_{\mu\lambda}V^{\lambda}, \tag{D.6}$$

$$\nabla_{\mu}\omega_{\nu} = \partial_{\mu}\omega_{\nu} - \Gamma^{\lambda}_{\mu\nu}\omega_{\lambda}. \tag{D.7}$$

Acting on a scalar, covariant derivatives reduce to normal partial derivatives.

The action describing the spacetime is the Hilbert Einstein action

$$S_{HE} = \int \sqrt{-g}Rd^4x. \tag{D.8}$$

Varying this action with respect to the metric results in the Einstein field equations, in the case that there is also a matter action, the field equations are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}, \tag{D.9}$$

which describes how the curvature of spacetime reacts to the presence of matter. Due to the symmetry in its indices, these equations consists of a set of ten equations. Taking the trace of these equations and substituting the solution back into the equation above gives the trace reversed form of the Einstein equations

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right). \tag{D.10}$$

Solving the Einstein equations gives the solution for the spacetime metric. From vacuum solutions one can derive the spacetimes outside black holes. In the case of spherical symmetry the solution is given by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 d\Omega^2, \tag{D.11}$$

which describes the spacetime outside a non spinning black hole. $R_s = 2GM$ is defined as the Schwarzschild radius of the black hole, which defines the event horizon. This is the only black hole solution we consider in this thesis but an example of a more astrophysical solution would be the solution for a spinning black hole given by the Kerr metric.

In the case of linearized gravity, when substituting the metric defined as the Minkowski metric and small perturbations $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the above defined curvature related terms become, after neglecting higher order terms in $h_{\mu\nu}$

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right) \\ &= \frac{1}{2} \eta^{\rho\lambda} \left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu} \right), \end{aligned} \tag{D.12}$$

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda} \partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \eta_{\mu\lambda} \partial_\sigma \Gamma^{\lambda}_{\nu\rho} \\ &= \frac{1}{2} \left(\partial_\rho \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\mu h_{\nu\rho} - \partial_\sigma \partial_\nu h_{\mu\rho} - \partial_\rho \partial_\mu h_{\nu\sigma} \right), \end{aligned} \tag{D.13}$$

$$R_{\mu\nu} = \frac{1}{2} \left(\partial_\sigma \partial_\nu h^{\sigma}_{\mu} + \partial_\sigma \partial_\mu h^{\sigma}_{\nu} - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} \right), \tag{D.14}$$

and lastly for the Ricci scalar

$$R = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h. \tag{D.15}$$

These curvature expressions up to linear order in the metric perturbations are required in our discussion on linearized gravity in section 2.

E Post Newtonain expansions in GR

In section 2.4.3 we gave some qualitative arguments why the PN expanded metric has the form of Eq. (2.81). Here we present a PN expansion of the metric more from first principle and in a slightly different formulation. In the end is shown that it leads to the same metric expansion as Eq. (2.81). This derivation is fully done in GR and we set c=1. This part is largely based on the books by Maggiore [25], Weinberg [114] and Straumann[115].

E.1 The post Newtonian expansion formalism

The procedure of calculating the lowest order PN correction terms of GR is as follows: First we expand the different components of the metric tensor of our spacetime and the energy momentum tensor in the orders corresponding to the small parameter: $\epsilon \sim \sqrt{\frac{R_s}{d}} \sim \frac{v}{c}$. Then we recover the contribution of the Newtonian limit on the metric tensor components. To compute the contribution of the first post Newtonian order (1PN) we have to insert the expansion of the metric tensor and the energy momentum tensor in the Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$ up to the right order. Therefore we first need to calculate the Christoffel symbols to the right order, substituting this in the Ricci tensor, and applying the standard post Newtonian gauge to simplify these expressions. Filling these expansions together with the expansion of the energy momentum tensor in the Einstein equations we obtain differential equations for the metric components in the 1PN orders. These can be solved with the Greens function method introducing four potentials.

Thus we begin by analysing the expansion of the metric tensor components and the EM tensor in orders of ϵ.

In a classical system with conservative forces is invariant under time reversal (when neglecting radiation). The components g_{00}, g_{ij} are even and g_{0i} is odd under time reversal, as the velocity changes sign under reversed time, therefore g_{00}, g_{ij} should only contain even powers of v and g_{0i} only odd powers.

Based on this we have the following ansatz:

$$
\begin{aligned}
&g_{00} = -1 +^{(2)} g_{00} +^{(4)} g_{00} +^{(6)} g_{00} + ... \\
&, g_{0i} =^{(3)} g_{0i} +^{(5)} g_{0i} + ... \\
&, g_{ij} = \delta_{ij} +^{(2)} g_{ij} +^{(4)} g_{ij} + ...
\end{aligned}
\tag{E.1}
$$

Which will be verified later as they lead to consistent solutions of the Einstein equations. In the same way we also expand the energy momentum tensor

$$
\begin{aligned}
T^{00} &= {}^{(0)}T^{00} + {}^{(2)}T^{00} + \dots, \\
T^{0i} &= {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + \dots, \\
T^{ij} &= {}^{(2)}T^{ij} + {}^{(4)}T^{ij} + \dots .
\end{aligned}
\tag{E.2}
$$

An important thing to note is that we discuss a source moving at non-relativistic velocities, therefore the time derivatives are of order v smaller than the spatial derivatives $(\frac{\partial}{\partial t} = \frac{\partial}{\partial x^i}\frac{\partial x^i}{\partial t} = \frac{\partial}{\partial x^i}v)$. So we have for time derivatives and the d'Alembertian

$$
\begin{aligned}
&\frac{\partial}{\partial t} = O(v)\frac{\partial}{\partial x^i}, \\
&-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 = [1 + O(\epsilon^2)]\nabla^2.
\end{aligned}
\tag{E.3}
$$

Now we start analysing the Newtonian limit situation to see how we can get to a higher order of 1PN. We begin with the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0. \tag{E.4}$$

The acceleration term $\frac{d^2x^i}{dt^2}$ can be rewritten as

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= \left(\frac{dt}{d\tau}\right)^{-1}\frac{d}{d\tau}\left[\left(\frac{dt}{d\tau}\right)^{-1}\frac{dx^i}{d\tau}\right] \\ &= \left(\frac{dt}{d\tau}\right)^{-2}\frac{d^2x^i}{d\tau^2} - \left(\frac{dt}{d\tau}\right)^{-3}\frac{d^2t}{d\tau^2}\frac{dx^i}{d\tau}.\end{aligned} \tag{E.5}$$

In which the last term in the second line is creatively subtract zero as $\frac{d^2t}{dt^2} = 0$. This seems to only make life a lot more complicated but in writing it this way we use the geodesic equation with $\mu = i$ and $\mu = 0$ to substitute $\frac{d^2x^i}{d\tau^2}$ and $\frac{d^2t}{d\tau^2}$

$$\begin{aligned}\frac{d^2x^i}{dt^2} &= -\Gamma^i_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} + \Gamma^0_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt}\frac{dx^i}{dt} \\ &= -\Gamma^i_{00} - 2\Gamma^i_{0j}\frac{dx^j}{dt} - \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + \left[\Gamma^0_{00} + 2\Gamma^0_{0j}\frac{dx^j}{dt} + \Gamma^0_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\right]\frac{dx^i}{dt}.\end{aligned} \tag{E.6}$$

In the Newtonian limit we assume low velocity, weak gravity and a static field. Low velocity means $\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$ thus we can neglect the spatial derivatives with respect to the time derivatives.
Then Eq. (E.6) becomes

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00}. \tag{E.7}$$

From a static field follows $\frac{\partial g_{\mu\nu}}{\partial x^0} = 0$ and then $\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}(\frac{\partial g_{\lambda 0}}{\partial x^0} + \frac{\partial g_{0\lambda}}{\partial x^0} - \partial_\lambda g_{00}) = -\frac{1}{2}g^{\mu\lambda}\frac{\partial g_{00}}{\partial x^\lambda}$.
Then in weak gravity we can expand the metric in the Minkowski metric plus perturbations: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. Therefore to first order in ther perturbations we have $\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\lambda}\frac{\partial h_{00}}{\partial x^\lambda}$ thus $\Gamma^i_{00} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i}$. Hence we have

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}. \tag{E.8}$$

Comparing this with the Newtonian equation of motion $\mathbf{a} = -\nabla\phi$, we have $h_{00} = -2\phi$ the potential and $g_{00} = -1 - 2\phi$.Therefore in the Newtonian limit we recover $g_{00} = -1 +^2 g_{00}$, $g_{0i} = 0$, $g_{ij} = \delta_{ij}$.
Now the gravitational potential $\frac{GmM}{r}$ is of order $O\left(\frac{R_s}{d}\right) = O\left(\frac{v^2}{c^2}\right)$ or as we set $c = 1$ of

order v^2. To go one step further to the 1PN order we would have to compute up to order v^4. Looking at Eq. (E.6) we therefore need:

$$\begin{aligned} &\Gamma^i_{00} \text{ to order } v^4, \\ &\Gamma^i_{0j} \text{ to order } v^3, \\ &\Gamma^i_{jk} \text{ to order } v^2, \\ &\Gamma^0_{00} \text{ to order } v^3, \\ &\Gamma^0_{0j} \text{ to order } v^2, \\ &\Gamma^0_{jk} \text{ to order } v. \end{aligned} \tag{E.9}$$

For recovering the expressions of these Christoffel symbols we also need the expansion of the metric components for the high indices which can be derived from $g_{\mu\nu}g^{\lambda\nu} = \delta^\lambda_\mu$

$$\begin{aligned} &g^{i\mu}g_{0\mu} = 0, \\ &g^{0\mu}g_{0\mu} = 1, \\ &g^{i\mu}g_{j\mu} = \delta_{ij}. \end{aligned} \tag{E.10}$$

Which gives results in

$$\begin{aligned} {}^{(2)}g^{00} &= -{}^{(2)}g_{00}, \\ {}^{(2)}g^{ij} &= -{}^{(2)}g_{ij}, \\ {}^{(3)}g^{i0} &= \ {}^{(3)}g_{i0}. \end{aligned} \tag{E.11}$$

Now the Christoffel symbols are given by

$$\Gamma^\mu_{v\lambda} = \frac{1}{2}g^{\mu\rho}\left\{\frac{\partial g_{\rho v}}{\partial x^\lambda} + \frac{\partial g_{\rho\lambda}}{\partial x^v} - \frac{\partial g_{v\lambda}}{\partial x^\rho}\right\}. \tag{E.12}$$

Looking at Eq. (E.9) we know that we will have the following expansions :

Γ^i_{00}, Γ^i_{jk}, Γ^0_{0i} have the expansion $\Gamma^\mu_{\nu\lambda} =^2 \Gamma^\mu_{\nu\lambda} +^4 \Gamma^\mu_{\nu\lambda} + ...$

Γ^i_{0j}, Γ^0_{00}, Γ^0_{ij} have the expansion $\Gamma^\mu_{\nu\lambda} =^3 \Gamma^\mu_{\nu\lambda} +^5 \Gamma^\mu_{\nu\lambda} + ...$

Working out Γ^i_{00} explicitly gives

$$\Gamma^i_{00} = \frac{1}{2} g^{i\rho} \left(\frac{\partial g_{\rho 0}}{\partial x^0} + \frac{\partial g_{\rho 0}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\rho} \right)$$
$$= \frac{1}{2} (\delta_{ij} +^{(2)} g^{i\rho} +^{(4)} g^{i\rho}) \left(\frac{\partial}{\partial x^0} (^{(3)} g_{\rho 0} +^{(5)} g_{0\rho}) + \frac{\partial}{\partial x^0} (^{(3)} g_{\rho 0} +^{(5)} g_{\rho 0}) - \frac{\partial}{\partial x^\rho} (-1 +^{(2)} g_{00} +^{(4)} g_{00}) \right). \tag{E.13}$$

Hence collecting terms to the right order corresponding Eq. (E.9) using Eq. (E.1) and Eq. (E.3) results in

$$\begin{aligned} ^{(2)}\Gamma^i_{00} &= -\frac{1}{2} \frac{\partial\, ^{(2)}g_{00}}{\partial x^i}, \\ ^{(4)}\Gamma^i_{00} &= -\frac{1}{2} \frac{\partial\, ^{(4)}g_{00}}{\partial x^i} + \frac{\partial\, ^{(3)}g_{i0}}{\partial t} + \frac{1}{2}\, ^{(2)}g_{ij} \frac{\partial\, ^{(2)}g_{00}}{\partial x^j}. \end{aligned} \tag{E.14}$$

Applying the same method for the other components of the Christoffel symbols

$$\begin{aligned} ^{(3)}\Gamma^i_{0j} &= \frac{1}{2} \left[\frac{\partial\, ^{(3)}g_{i0}}{\partial x^j} + \frac{\partial\, ^{(2)}g_{ij}}{\partial t} - \frac{\partial\, ^{(3)}g_{j0}}{\partial x^i} \right], \\ ^{(2)}\Gamma^i_{jk} &= \frac{1}{2} \left[\frac{\partial\, ^{(2)}g_{ij}}{\partial x^k} + \frac{\partial\, ^{(2)}g_{ik}}{\partial x^j} - \frac{\partial\, ^{(2)}g_{jk}}{\partial x^i} \right], \\ ^{(3)}\Gamma^0_{00} &= -\frac{1}{2} \frac{\partial\, ^{(2)}g_{00}}{\partial t}, \\ ^{(2)}\Gamma^0_{0i} &= -\frac{1}{2} \frac{\partial\, ^{(2)}g_{00}}{\partial x^i}, \\ ^{(1)}\Gamma^0_{ij} &= 0. \end{aligned} \tag{E.15}$$

To fill in these expressions further we calculate the explicit formulation of the orders of the metric components. Therefore we need to solve the Einstein field equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$. Thus we shall continue deriving the expressions of the right orders for the Ricci tensor.

The Ricci tensor is given by

$$R_{\mu\nu} = \frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\alpha} - \frac{\partial \Gamma^\alpha_{\mu\alpha}}{\partial x^\nu} + \Gamma^\rho_{\mu\nu}\Gamma^\alpha_{\rho\alpha} - \Gamma^\rho_{\mu\alpha}\Gamma^\alpha_{\rho\nu}. \tag{E.16}$$

Then with E.1 we find for the expansion

$$\begin{aligned} R_{00} &=^{(2)} R_{00} +^{(4)} R_{00} + ..., \\ R_{i0} &=^{(3)} R_{i0} +^{(5)} R_{i0} + ..., \\ R_{ij} &=^{(2)} R_{ij} +^{(4)} R_{ij} + ... \end{aligned} \tag{E.17}$$

When writing explicitly the zero component results in

$$R_{00} = \frac{\partial \Gamma^\alpha_{00}}{\partial x^\alpha} - \frac{\partial \Gamma^\alpha_{0\alpha}}{\partial x^0} + \Gamma^\rho_{00}\Gamma^\alpha_{\rho\alpha} - \Gamma^\rho_{0\alpha}\Gamma^\alpha_{\rho 0} \tag{E.18}$$

Using E.1 and Eq. (E.3) we have for the zero components to second and fourth order

$$
\begin{aligned}
{}^{(2)}R_{00} &= \frac{\partial \Gamma^i_{00}}{\partial x^i}, \\
{}^{(4)}R_{00} &= \frac{\partial\, {}^{(4)}\Gamma^i_{00}}{\partial x^i} - \frac{\partial\, {}^{(3)}\Gamma^i_{0i}}{\partial t} +^{(2)} \Gamma^i_{00}{}^{(2)}\Gamma^i_{ij} -^{(2)} \Gamma^0_{0i}{}^{(2)}\Gamma^i_{00}.
\end{aligned}
\tag{E.19}
$$

Similar approach for the other components gives

$$
\begin{aligned}
{}^{(3)}R_{i0} &= \frac{\partial\, {}^{(2)}\Gamma^0_{0i}}{\partial t} + \frac{\partial\, {}^{(3)}\Gamma^j_{0i}}{\partial x^j} - \frac{\partial\, {}^{(3)}\Gamma^0_{00}}{\partial x^i} - \frac{\partial\, {}^{(3)}\Gamma^j_{0j}}{\partial x^i}, \\
{}^{(2)}R_{ij} &= \frac{\partial\, {}^{(2)}\Gamma^k_{ij}}{\partial x^k} - \frac{\partial\, {}^{(2)}\Gamma^0_{i0}}{\partial x^j} - \frac{\partial\, {}^{(2)}\Gamma^k_{ik}}{\partial x^j}.
\end{aligned}
\tag{E.20}
$$

Next, to get the components of the Ricci tensor in terms of the metric components to right order, we substitute the Christoffel symbol components by Eq. (E.14) and Eq. (E.15), which gives

$$
{}^{(2)}R_{00} = -\frac{1}{2}\nabla^{2(2)}g_{00}, \tag{E.21}
$$

$$
\begin{aligned}
{}^{(4)}R_{00} = &-\frac{1}{2}\nabla^{2(4)}g_{00} + \frac{\partial^2\, {}^{(3)}g_{i0}}{\partial t \partial x^i} - \frac{1}{2}\frac{\partial^2\, {}^{(2)}g_{ij}}{\partial t^2} + \frac{1}{2}{}^{(2)}g_{ij}\frac{\partial^2\, {}^{(2)}g_{00}}{\partial x^i \partial x^j} \\
&+ \frac{1}{2}(\frac{\partial\, {}^{(2)}g_{ij}}{\partial x^j})(\frac{\partial\, {}^{(2)}g_{00}}{\partial x^i}) - \frac{1}{4}(\frac{\partial\, {}^{(2)}g_{00}}{\partial x^i})(\frac{\partial\, {}^{(2)}g_{jj}}{\partial x^i}) - \frac{1}{4}(\frac{\partial\, {}^{(2)}g_{00}}{\partial x^i})(\frac{\partial\, {}^{(2)}g_{00}}{\partial x^i}),
\end{aligned}
\tag{E.22}
$$

$$
{}^{(3)}R_{i0} = -\frac{1}{2}\frac{\partial^2\, {}^{(2)}g_{jj}}{\partial x^i \partial t} + \frac{1}{2}\frac{\partial^2\, {}^{(3)}g_{j0}}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial^2\, {}^{(2)}g_{ij}}{\partial x^j \partial t} - \frac{1}{2}\nabla^2\, {}^{(3)}g_{i0}, \tag{E.23}
$$

$$
{}^{(2)}R_{ij} = \frac{1}{2}\frac{\partial^2\, {}^{(2)}g_{00}}{\partial x^i \partial x^j} - \frac{1}{2}\frac{\partial^2\, {}^{(2)}g_{kk}}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial^2\, {}^{(2)}g_{ik}}{\partial x^k \partial x^j} + \frac{1}{2}\frac{\partial^2\, {}^{(2)}g_{kj}}{\partial x^k \partial x^i} - \frac{1}{2}\nabla^2\, {}^{(2)}g_{ij}. \tag{E.24}
$$

We can simplify these expressions by choosing a gauge; the standard post Newtonian gauge:

$$
\begin{aligned}
&\frac{\partial g_{0j}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{jj}}{\partial x^0} = O(c^{-5}), \\
&\frac{\partial g_{ij}}{\partial x^j} - \frac{1}{2}(\frac{\partial (g_{jj} - g_{00})}{\partial x^i} = O(c^{-4}).
\end{aligned}
\tag{E.25}
$$

Which results in

$$
\frac{\partial\, {}^{(3)}g_{0k}}{\partial x^k} - \frac{1}{2}\frac{\partial\, {}^{(2)}g_{kk}}{\partial x^0} = 0, \tag{E.26}
$$

$$\frac{1}{2}\frac{\partial\ ^{(2)}g_{00}}{\partial x^i} + \frac{\partial\ ^{(2)}g_{ij}}{\partial x^j} - \frac{1}{2}\frac{\partial\ ^{(2)}g_{jj}}{\partial x^i} = 0. \tag{E.27}$$

Now we differentiate Eq. (E.27) with respect to x^k which gives

$$\frac{1}{2}\frac{\partial^2\ ^{(2)}g_{00}}{\partial x^i\partial x^k} + \frac{\partial^2\ ^{(2)}g_{ij}}{\partial x^j\partial x^k} - \frac{1}{2}\frac{\partial^2\ ^{(2)}g_{jj}}{\partial x^i\partial x^k} = 0. \tag{E.28}$$

Also interchanging i and k in Eq. (E.28) and adding this new equation and Eq. (E.28) gives

$$\frac{\partial^2\ ^{(2)}g_{00}}{\partial x^i\partial x^k} + \frac{\partial^2\ ^{(2)}g_{ij}}{\partial x^j\partial x^k} + \frac{\partial^2\ ^{(2)}g_{kj}}{\partial x^j\partial x^i} - \frac{\partial^2\ ^{(2)}g_{jj}}{\partial x^i\partial x^k} = 0. \tag{E.29}$$

With this expression Eq. (E.24) simplifies to

$$^{(2)}R_{ij} = -\frac{1}{2}\nabla^2\ ^{(2)}g_{ij}. \tag{E.30}$$

Now again back to the gauge equations, we differentiate Eq. (E.26) to time and replace index k by i, resulting in

$$\frac{\partial^2\ ^{(3)}g_{0i}}{\partial x^i\partial x^0} - \frac{1}{2}\frac{\partial^2\ ^{(2)}g_{ii}}{(\partial x^0)^2} = 0. \tag{E.31}$$

This then simplifies Eq. (E.22) to

$$^{(4)}R_{00} = -\frac{1}{2}\nabla^{2(4)}g_{00}+\frac{1}{2}{}^{(2)}g_{ij}\frac{\partial^2\ ^{(2)}g_{00}}{\partial x^i\partial x^j}+\frac{1}{2}(\frac{\partial\ ^{(2)}g_{ij}}{\partial x^j})(\frac{\partial\ ^{(2)}g_{00}}{\partial x^i})-\frac{1}{4}(\frac{\partial\ ^{(2)}g_{00}}{\partial x^i})(\frac{\partial\ ^{(2)}g_{jj}}{\partial x^i})-\frac{1}{4}(\frac{\partial\ ^{(2)}g_{00}}{\partial x^i})(\frac{\partial\ ^{(2)}g_{00}}{\partial x^i}). \tag{E.32}$$

Using Eq. (E.26) differentiated to x^j to simplify Eq. (E.23) results in

$$^{(3)}R_{i0} = -\frac{1}{4}\frac{\partial^2\ ^{(2)}g_{jj}}{\partial x^i\partial t} + \frac{1}{2}\frac{\partial^2\ ^{(2)}g_{ij}}{\partial x^j\partial t} - \frac{1}{2}\nabla^2\ ^{(3)}g_{i0}. \tag{E.33}$$

With these expressions for the Ricci tensor components to the right order, we only need the energy momentum tensor to the right order to fill in the Einstein equations $R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right)$.
As we saw in the beginning, the energy momentum tensor has the expansion of Eq. (E.2). For the field equations we then need $S_{\mu\nu} := T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\lambda_\lambda$, giving by the expansion

$$\begin{aligned} S_{00} &= {}^{(0)}S_{00} + {}^{(2)}S_{00} + \dots, \\ S_{i0} &= {}^{(1)}S_{i0} + {}^{(3)}S_{i0} + \dots, \\ S_{ij} &= {}^{(0)}S_{ij} + {}^{(2)}S_{ij} + \dots\ . \end{aligned} \tag{E.34}$$

With for the specific order expressions, we substitute in Eq. (E.2) and Eq. (E.1) to the right order results in

$$
\begin{aligned}
{}^{(0)}S_{00} &= \frac{1}{2}{}^{(0)}T^{00}, \\
{}^{(2)}S_{00} &= \frac{1}{2}\left({}^{(2)}T^{00} - 2{}^{(2)}g_{00}{}^{(0)}T^{00} + {}^{(2)}T^{ii}\right), \\
{}^{(1)}S_{0i} &= -{}^{(1)}T^{0i}, \\
{}^{(0)}S_{ij} &= \frac{1}{2}\delta_{ij}{}^{(0)}T^{00}.
\end{aligned} \tag{E.35}
$$

Finally substituting the expansions in the field equations $R_{\mu\nu} = 8\pi G S_{\mu\nu}$ results in

$$\nabla^2\left[{}^{(2)}g_{00}\right] = -8\pi G^{(0)}T^{00}, \tag{E.36}$$

$$
\begin{aligned}
\nabla^2\left[{}^{(4)}g_{00}\right] ={}& {}^{(2)}g_{ij}\frac{\partial^2\, {}^{(2)}g_{00}}{\partial x^i\partial x^j} + \frac{\partial\, {}^{(2)}g_{ij}}{j}\frac{\partial\, {}^{(2)}g_{00}}{\partial x^i} - \frac{1}{2}\frac{\partial\, {}^{(2)}g_{00}}{\partial x^i}\frac{\partial\, {}^{(2)}g_{jj}}{\partial x^i} \\
&- 8\pi G\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii} - 2{}^{(2)}g_{00}{}^{(0)}T^{00}\right\},
\end{aligned} \tag{E.37}
$$

$$\nabla^2\left[{}^{(2)}g_{ij}\right] = -8\pi G\delta_{ij}{}^{(0)}T^{00}, \tag{E.38}$$

$$\nabla^2\left[{}^{(3)}g_{0i}\right] = -\frac{1}{2}\frac{\partial^2\, {}^{(2)}g_{jj}}{\partial x^0\partial x^i} + \frac{\partial^2\, {}^{(2)}g_{ij}}{\partial x^0\partial x^j} + 16\pi G^{(1)}T^{0i}. \tag{E.39}$$

The differential equations of the from above can be solved with the Greens function method, in the same way as solving the Poisson equation. Therefore for Eq. (E.36) and Eq. (E.39) we have to solutions

$${}^{(2)}g_{00} = -2\phi, \quad {}^{(2)}g_{ij} = -2\delta_{ij}\phi. \tag{E.40}$$

Where we introduced the potential ϕ, which is the Newtonian potential

$$\phi = -G\int\frac{T^{00}\left(t,\mathbf{x}'\right)}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3x' \tag{E.41}$$

We substitute this in the remaining Eq. (E.37) and Eq. (E.38) gives

$$\nabla^2\, {}^{(3)}g_{0i} = 16\pi G^{(1)}T^{i0} + \frac{\partial^2\phi}{\partial x^0\partial x^i}, \tag{E.42}$$

$$\nabla^2\, {}^{(4)}g00 = -8\pi G\left({}^{(2)}T^{00} + 4\phi^{(0)}T^{00} + {}^{(2)}T^{ii}\right) + 4\phi\nabla^2\phi - 4(\nabla\phi)^2. \tag{E.43}$$

Now we can use the Poisson equation $\Delta\phi = 4\pi G^{(0)}T^{00}$ and the identity $(\nabla\phi)^2 = \frac{1}{2}\nabla^2\left(\phi^2\right) - \phi\nabla^2\phi$ to rewrite Eq. (E.43) in the following form

$$\nabla^2\left({}^{(4)}g_{00} + 2\phi^2\right) = -8\pi G\left({}^{(2)}T^{00} + {}^{(2)}T^{ii}\right). \tag{E.44}$$

Then we define the potential ψ as

$$^{(4)}g_{00} = -2\phi^2 - 2\psi. \tag{E.45}$$

Hence this potential satisfies the equation

$$\nabla^2\psi = 4\pi G\left({}^{(2)}T^{00} + {}^{(2)}T^{ii}\right). \tag{E.46}$$

Since $^{(4)}g_{00}$ must vanish at infinity, the solution for ψ is again calculated with the Greens function method

$$\psi = -G\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\left({}^{(2)}T^{00}(\mathbf{x}',t) + {}^{(2)}T^{ii}(\mathbf{x}',t)\right). \tag{E.47}$$

We also define the potentials ξ_i and χ as

$$\xi_i(\mathbf{x},t) = -4G\int\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\,{}^{(1)}T^{i0}(\mathbf{x}',t), \tag{E.48}$$

$$\chi(\mathbf{x},t) = -\frac{G}{2}\int|\mathbf{x}-\mathbf{x}'|\,{}^{(0)}T^{00}(\mathbf{x}',t)\,d^3x'. \tag{E.49}$$

They satisfy (which can be verified by using the Greens function method again on these relations) the following equations

$$\nabla^2\xi_i = 16\pi G^{(1)}T^{i0}, \tag{E.50}$$

$$\nabla^2\chi = \phi. \tag{E.51}$$

With this we can write, following from Eq. (E.43)

$$^{(3)}g_{i0} = \xi_i + \frac{\partial^2\chi}{\partial x^i\partial x^0}. \tag{E.52}$$

Thus we have in the first post Newtonian order of the expansion of the components of the metric expressed in terms of the potentials ϕ, ξ_i, χ and ψ

$$\begin{aligned} &{}^{(2)}g_{00} = -2\phi, \quad {}^{(4)}g_{00} = -2\left(\phi^2+\psi\right), \\ &{}^{(2)}g_{ij} = -2\delta_{ij}\phi, \quad {}^{(3)}g_{i0} = \xi_i + \tfrac{\partial\chi}{\partial x^i\partial x^0}. \end{aligned} \tag{E.53}$$

Now we expressed the metric components in terms of the instantaneous potentials ϕ, ψ, ξ_i and χ; their value at a certain time depends on the energy momentum tensor at that same moment. We can also express the components in terms of a retarded potential; then the value at time t depends on the value of the energy momentum tensor at retarded time $t - |\mathbf{x}-\mathbf{x}'|$. This way of expressing the metric components is a useful starting point for more complicated calculations as for higher order post-Newtonian approximations or the post-Newtonian approach in a modified gravity theory.

For g_{00} we can write

$$\begin{aligned} g_{00} &= -1 - 2\phi - 2(\phi^2 + \psi) + O(\epsilon^6) \\ &= -1 - 2(\phi + \psi) - 2\phi^2 + O(\epsilon^6). \end{aligned} \tag{E.54}$$

Note that ψ is of higher order than ϕ, hence we can replace ϕ^2 with $(\phi+\psi)^2$. The additional terms are above 1PN order and will therefore be neglected. Then we introduce the potential

$$V = -(\phi + \psi), \tag{E.55}$$

which is of order v^2. Now we can rewrite g_{00} as

$$\begin{aligned} g_{00} &= -1 + 2V - 2V^2 + O(\epsilon^6) \\ &= -e^{-2V} + O(\epsilon^6). \end{aligned} \tag{E.56}$$

Combining Eq. (E.36), Eq. (E.40) and Eq. (E.46) we have the differential equation

$$\nabla^2(\phi + \psi) = \partial_0^2\phi + 4\pi G \left[{}^{(0)}T^{00} + {}^{(2)}T^{00} + {}^{(2)}T^{ii}\right]. \tag{E.57}$$

As to this order, we can again replace $\partial_0^2\phi$ by $\partial_0^2(\phi + \psi)$ then we can write this as

$$\begin{aligned} \Box V &= -4\pi G \left[{}^{(0)}T^{00} + {}^{(2)}T^{00} + {}^{(2)}T^{ii}\right] \\ &= -4\pi G \left[T^{00} + T^{ii}\right]. \end{aligned} \tag{E.58}$$

With $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ the flat space d'Alembertian and we could replace the orders of the energy momentum tensor components by the whole expressions if we keep in mind to cut them of at 1PN order. Hence we can write

$$\Box V = -4\pi G\sigma, \tag{E.59}$$

with the active gravitational mass density defined as $\sigma := T^{00} + T^{ii}$. So the potential can be written as an retarded integral using the (relativistic) greens function method

$$V(t, \mathbf{x}) = G \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \sigma\left(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}'\right). \tag{E.60}$$

Similar procedure can be done for g_{0i} only in our case this is clear how to do so. However if one works with the Donder gauge condition $\partial_\mu(\sqrt{-g}g^{\mu\nu})$ in harmonic coordinates instead of Eq. (E.25) the expression of the field equation in terms of g_{0i} is

$$\nabla^2\left[{}^{(3)}g_{0i}\right] = 16\pi G^{(1)}T^{0i}, \tag{E.61}$$

with

$$\begin{aligned} {}^{(3)}g_{0i} &= \zeta_i, \\ \zeta_i(t, \mathbf{x}) &= -4G \int \frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} {}^{(1)}T^{0i}\left(t, \mathbf{x}'\right). \end{aligned} \tag{E.62}$$

Then using the active mass current density $\sigma_i = T^{0i}$ and that retardation effects in ζ_i are of higher order we can write, replacing ζ_i as V_i

$$V_i(t,\mathbf{x}) = G\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\sigma_i\left(t - |\mathbf{x}-\mathbf{x}'|/c, \mathbf{x}'\right). \tag{E.63}$$

Therefore in harmonic coordinates at 1PN order we can write the metric components as

$$\begin{aligned} g_{00} &= -e^{-2V} + O(v^6) \\ g_{0i} &= -4V_i + O(v^5) \\ g_{ij} &= \delta_{ij}e^{2V} + O(v^4) \end{aligned} \tag{E.64}$$

When starting from these expansions of the metric, one can obtain the 1PN Lagrangian in a similar way by plugging these components into the field equations to solve them for 1PN orders of V and V_i and use these solutions together with the expression of the energy momentum tensor to the right order to explicitly formulate the action and hence the Lagrangian. In the main part of this thesis (section 5) we use the latter approach with a slight change of convention to use the fields U and g_i respectively instead of V and V_i.

F Mathematica package xact

Mathematica was a widely used tool for most of the calculations presented in this thesis. As for example doing the expansions, differentiations, numerical integration and plotting the figures are standard possibilities in Mathematica, we also made use of the xAct package. Within this package the the GR properties and curvature terms as the Riemann tensor are defined and one can work with index notation in an abstract, tensorial manner without defining coordinates. Also we used subpackages as xpert which allows for varying the field equations with respect to the metric or expanding and varying in a specific coordinate frame and metric with xcoba.

The package, instruction and expanation on all the subpackages can be found on http://www.xact.es/. On https://github.com/xAct-contrib/examples example notebooks can be found for using the packages for GR related problems. Specifically https://github.com/xAct-contrib/examples/blob/master/EDGB-and-DCS-EOMs-and-C-tensors-simplified.nb was used to rewrite the varied R^2_{GB} in the field equations in a less messy form.

G Dynamical Chern Simons gravity

Another theory in the class of quadratic gravity is dynamical Chern Simons gravity. In [131] the three dimensional Chern Simons term

$$CS(\Gamma) = \frac{1}{4\pi^2}\int d^3x \varepsilon^{ijk}\left(\frac{1}{2}{}^3\Gamma^p_{iq}\partial^3_j\Gamma^q_{kp} + \frac{1}{3}{}^3\Gamma^p_{iq}{}^3\Gamma^q_{jr}{}^3\Gamma^r_{kp}\right), \tag{G.1}$$

was introduced as an interesting extension on top of the Hilbert Einstein action in three dimensions as it is a massive gauge invariant quantity. In [132] for the first time they extended the Chen Simons term to four dimensions with an embedding coordinate formed by the divergence of the scalar field and included it to the Hilbert Einstein action in four dimensions. More explicitly, to extend to four dimensions they used the Chern Simons topological current K^μ

$$K^\rho = 2\epsilon^{\rho\mu\nu\lambda}\left(\Gamma_{\mu a}{}^{b}\partial_\nu \Gamma^a_{\lambda b} + \frac{2}{3}\Gamma^b_{\mu a}\Gamma^c_{\nu b}\Gamma^a_{\lambda c}\right), \tag{G.2}$$

and the divergence of this term

$$\partial_\rho K^\rho = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu\gamma\delta}R^{\gamma\delta}_{\alpha\beta} \equiv {}^*RR. \tag{G.3}$$

Which is the 3D CS term with an extra fourth component K^0 including 4D Christoffel symbols and can therefore be included in the 4D Hilbert Einstein action. Introducing an embedding field $\partial_\mu\vartheta$ they added the following term to the action

$$-\int d^4x\frac{1}{2}\partial_\mu\vartheta K^\mu = \int d^4x\frac{1}{4}\vartheta^*RR. \tag{G.4}$$

Using integration by parts assuming the boundary term vanishes as the scalar field at the boundary is zero. Applying this embedding in this way comes from similar analysis of modifying Maxwell theory with a Chern Simons term[133]. Adding this term to the Lagrangian introduces to the inhomogeneous equation of motion the dual term of the electromagnetic field tensor multiplied by the embedding coordinate, similar to the term in .

With the embedding the scalar field in the gravity theory, which could be treated as a constant external field, resulting in non dynamical Chern Simons gravity, or as a dynamical field, including a kinetic term for this field in the action. Both lead to fundamentally different theories. The non dynamical framework there is no physical choice for what the scalar field should be and often a quite ad hoc choice is made that simplifies the formulas. We consider here the dynamical framework for which the scalar field is part of the gravitational sector similar as in sGB gravity. The total action of dynamical Chern Simons is then given by

$$S = \frac{16\pi G}{c^4}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}\beta(g^{ab}\left(\nabla_a\vartheta\right)\left(\nabla_b\vartheta\right) + 2V(\vartheta)) + \alpha\frac{1}{4}f(\vartheta)^*RR\right]. \tag{G.5}$$

With the scalar field ϑ and the scalar field dependent potential $V(\vartheta)$. In string theory context potential becomes nonzero after supersymmetry breaking, but it is therefore only relevant at such scales, can be neglected in semi classical systems [117]. and the quadratic curvature term which goes by the name of the Pontryagin density given by

$${}^*RR := R\tilde{R} = {}^*R^a{}_b{}^{cd}R^b{}_{acd}. \tag{G.6}$$

In which the star stands for the dual

$${}^*R^a_{b\,b}{}^{cd} := \frac{1}{2}\epsilon^{\mathrm{cdef}}R^a_{bef}. \tag{G.7}$$

In this theory generally there are two dimensional coupling constants α and β, in the nondynamical version β is 0, we consider here the dynamical version for which we do have an equation of motion for the scalar field.
Instabilities can arise in a theory due to the appearance of third order or higher order time derivatives in the equations of motion. As opposed to sGB gravity the quadratic curvature term in dCS is not such that it cancels the higher order derivatives to the metric in the equations of motion. When the scalar field does not vanish this theory therefore does contain ghost instabilities[134]. However quadratic gravity theories are often considered as effective theories, begin the lowest order expansion of the higher curvature terms that would be included in high energy regimes. It turns out that the ghost instabilities that appear are in the high frequency regime[135] where the effective theory is no longer valid so they do not have to cause problems.

As the dCS term in the action contains the dual of the Riemann tensor which contains again the Levi Civita tensor, this term is odd under parity transformations. Because of this dCS gravity allows for parity violating solutions. Parity violation is for example seen in weak interactions in the Standard model. When varying the action with respect to the scalar field one finds that the equation of motion for the scalar field is sources by the Pointygarin density, hence the solutions for the scalar field are parity violating in this theory.
A bound on the value of the coupling constant was set by multimessenger neutron star observations being $\alpha^{1/2} = 8.5km$[136] but could not lead to a meaningfull constraint using gravitational wave measurements [105]. In previous literature sCS corrections to the Kerr metric were constructed in a slow rotation approximation[137, 138]. These terms come in at 2 Post Newtonian order for these rotating systems. The corrections due to the scalar field are dipolar and as an effect it weakens frame dragging and shifts the location of the inner most stable circular orbit around the black holes.

The Chern Simons term has motivation in particle physics, string theory and quantum loop gravity. This discussion is based on [117].
In particle physics, the CS invariant term turns up in the gravitational anomaly in the standard model. An anomaly describes a quantum mechanical violation of a classically conserved current. If we have via Noether's theorem the conserved current $\partial_a j^{aA} = 0$, then an anomaly would be quantum correction $\mathcal{A}^{\mathcal{A}}$ with $\partial_a j^{aA} = \mathcal{A}^{\mathcal{A}}$. Global anomalies do not lead to inconsistences although they do have physical consequences. However gauge anomalies are a statement that the quantum theory is quantum mechanically inconsistent. Gauge symmetries can be used to eliminate negative norm states in the quantum theory, but in order to remain unitary, the path integral must also remain gauge invariant. Quantum effects involving gauge interactions with fermions can spoil this gauge invariance and thus lead to a loss of unitarity and render the quantum formulation inconsistent. Therefore, if one is to construct a well-defined unitary quantum theory and if gauge currents are anomalous, these anomalies must be cancelled by counterterms.

An example of a global anomaly in the Standard Model is the violation of the U(1) axial current by a one-loop triangle diagram between fermion loops and the gauge field external legs. This leads to the following famous ABJ anomaly: $\partial^a j^A_a = -\frac{1}{8\pi^2}\epsilon^{abcd}F_{ab}F_{cd}$. The deriva-

tion of the ABJ anomaly also applies for gravitational anomaly. Instead of the field strenght tensor, the Riemann curvature tensor is used, resulting in the gravitational ABJ anomaly: $D^a j_a^A = -\frac{1}{384\pi^2}\frac{1}{2}\epsilon^{abcd}R_{abef}R_{cd}^{ef}$ which contains the Pontryagin density. The gravitational ABJ anomaly can be canceled by adding the appropriate counter term to the action, which turns out to be the CS modification in the HE action.
The CS action can also be induced by other means, for example through Dirac fermions coupling to the gravitational field in radiative fermion loop corrections, or it also arises in Yang-Mills theories and non linearized gravity through proper time method and functional integration.

In heterotic String theory the CS modification to GR arises from the Green Schwarz anomaly cancelling mechanism. The key idea is that a quantum effect due to the gauge field that couples to the string induces a CS term in the effective low energy 4D GR.
CS is extension of GR with addition of a parity violating term. This required by all 4D compactifications of string theory for mathematical consistency because it cancels the Green Schwarz anomaly. The low energy limit of superstring theories are 10 dimensional supergravity theories. As in the particle physics cause, a triangle loop diagram between gravitions and fermions will generate the gravitational anomaly, similarly hexagon loop diagrams generate anomalies in 10 dimensions.
To cancel the anomaly the three form gauge field strength tensor in 10D supergravity is shifted. After compactfying the theory to 4D this shifted term in the action after integration by parts ends up to be the gravitational Pointryagin interaction: $\int d^4x f(\theta) R\Lambda R$.

Quantum loop gavity is an effort towards the quantization of GR through the postulate that spacetime itself is discrete. When analyzing P and CP conservation in loop quantum gravity it leads to CS theory with a constant CS parameter. However this is not yet the dCS term. But when promoting the Barbero Immirzi parameter to a scalar field, the term can be recoverd. This leads to torsion and parity violation when coupled to fermions to the theory. When this torsion is used to construct an effective action they found that one unavoidingly obtains CS modified gravity. One recovers $S_{\mathrm{CS}} = +\alpha\frac{1}{4}\int_{\mathcal{V}} d^4x\sqrt{-g}\vartheta^* RR$ and $S_\vartheta = -\beta\frac{1}{2}\int_{\mathcal{V}} d^4x\sqrt{-g}\left[g^{ab}\left(\nabla_a\vartheta\right)\left(\nabla_b\vartheta\right) + 2V(\vartheta)\right]$ with $\vartheta = \frac{3}{2\kappa}^{1/2}\tilde{\beta}$ with $\tilde{\beta}$ the Barbero Immirzi scalar field and $\beta = -1$, $\alpha = \frac{3}{32\pi^2}\sqrt{3\kappa}$.

G.1 Binary dynamics in dCS

Lastly we will shortly review similar calculation in dCS. At a first instance neglecting the tidal part, we will make a comment on introducing the tidal terms in this theory at the end of this section.

Starting from the action from

$$S = \frac{16\pi G}{c^4}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}\beta(g^{ab}\left(\nabla_a\vartheta\right)\left(\nabla_b\vartheta\right) + 2V(\vartheta)) + \alpha\frac{1}{4}f(\vartheta)^* RR\right]. \tag{G.8}$$

With the Pontryagin density given by

$$^*RR := R\tilde{R} = {}^*R^a{}_b{}^{cd}R^b{}_{acd}, \tag{G.9}$$

with the dual Riemann tensor corresponding to

$$^*R^a_b{}_b{}^{cd} := \frac{1}{2}\epsilon^{\text{cdef}}\, R^a_{bef}. \tag{G.10}$$

Varying with respect to the metric and scalar field results in the equations of motion, rewriting the variation of the Pontryagin density in the form of the C tensor

$$\begin{aligned} G_{\mu\nu} + \alpha C_{\mu\nu} &= \frac{8\pi G}{c^4}T^{mat}_{\mu\nu} + \frac{1}{2}T^{\vartheta}_{\mu\nu} \\ \beta\Box\vartheta &= \frac{\alpha}{4}\frac{df(\vartheta)}{d\vartheta}{}^*RR. \end{aligned} \tag{G.11}$$

Hereby is the energy momentum tensor of the scalar field defined as

$$T^{\vartheta}_{\mu\nu} = \beta\left[(\nabla_\mu\vartheta)(\nabla_\nu\vartheta) - \frac{1}{2}g_{\mu\nu}(\nabla_\rho\vartheta)(\nabla^\rho\vartheta)\right]. \tag{G.12}$$

And we define the following expression as the C-tensor

$$C^{\mu\nu} := v_\lambda\epsilon^{\lambda\rho\epsilon(\mu}\nabla_\epsilon R^{\nu)}{}_\rho + v_\rho{}^*R^{\rho(\mu\nu)\epsilon}, \tag{G.13}$$

with

$$\begin{aligned} v_\mu &:= \nabla_\mu f(\vartheta) = \frac{\partial f(\vartheta)}{\partial\vartheta}\partial_\mu\vartheta, \\ v_{\mu\nu} &:= \nabla_\mu\nabla_\nu f(\vartheta) = \nabla_{(\mu}\nabla_{\nu)}f(\vartheta) = \frac{\partial^2 f(\vartheta)}{\partial\vartheta^2}\partial^\mu\vartheta\partial^\nu\vartheta + \frac{\partial f(\vartheta)}{\partial\vartheta}\partial^\mu\partial^\nu\vartheta. \end{aligned} \tag{G.14}$$

Using that $f(\vartheta)$ is a scalar function. Again it is more convenient to work with the trace reversed version of the EOM. Taking the trace and substituting the expression for R back gives[122]

$$R_{\mu\nu} = -\alpha C_{\mu\nu} + \frac{8\pi G}{c^4}(T^{mat}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{mat}) + \frac{1}{2}\beta\nabla_a\vartheta\nabla_b\vartheta. \tag{G.15}$$

Using that the C tensor is tracefree.

Again we are now interested in the expansions of the EOM terms. We are not considering any tidal terms, so only the expansion in terms of $\frac{1}{c^2}$ is done. We expand the scalar field and U and g_i in the same way as in Eq. (5.14) and Eq. (5.15).

For the scalar field EOM we expand the Pontryagin density in mathematica with the xAct package, plugging in the 1PN expanded metric in the definition of the Riemann tensor. To lowest order this gives

$$\begin{aligned}{}^*RR &= 8\epsilon^{0i\mu\nu}\partial_\mu\partial_0\frac{U^{(0)}}{c^2}\partial_\nu\partial_i\frac{U^{(0)}}{c^2} - 4\epsilon^{ij\rho\lambda}\partial_\rho\partial_i\frac{U^{(0)}}{c^2}\partial_\lambda\partial_j\frac{U^{(0)}}{c^2}\\ &= 8\epsilon^{0ijk}\partial_j\partial_0\frac{U^{(0)}}{c^2}\partial_k\partial_i\frac{U^{(0)}}{c^2} + 8\epsilon^{0ikj}\partial_k\partial_0\frac{U^{(0)}}{c^2}\partial_j\partial_i\frac{U^{(0)}}{c^2}\\ &- 4\epsilon^{ij0k}\partial_0\partial_i\frac{U^{(0)}}{c^2}\partial_k\partial_j\frac{U^{(0)}}{c^2} - 4\epsilon^{ijk0}\partial_k\partial_i\frac{U^{(0)}}{c^2}\partial_0\partial_j\frac{U^{(0)}}{c^2},\end{aligned} \tag{G.16}$$

where we used the property of ϵ being 0 for two the same indices. We can simplify this further by relabelling the double indices k to j and j to k in the second term and i to j and j to i in the last term. Permuting the indices in epsilon so they all get the same order as the first one gives

$${}^*RR = 8\epsilon^{0ijk}\partial_j\partial_0\frac{U^{(0)}}{c^2}\partial_k\partial_i\frac{U^{(0)}}{c^2}. \tag{G.17}$$

The expanded scalar field EOM then becomes

$$\begin{aligned}\Box\vartheta &= \frac{\alpha}{\beta 4}{}^*RR,\\ \Box\vartheta &= \frac{2\alpha}{\beta}\epsilon^{0ijk}\partial_j\partial_0\frac{U^{(0)}}{c^2}\partial_k\partial_i\frac{U^{(0)}}{c^2}.\end{aligned} \tag{G.18}$$

To match up the orders on both sides of the EOM, the lowest order needs to be $\mathcal{O}(c^5)$.
For R_{00} and R_{0i} we have the same expressions as before 5.32.
For the C tensor components in the gravitational EOMs of the 00 component and $0i$ component, we also expand their Riemann tensor terms by substituting the PN expanded metric components with mathematica package xact. For the lowest order terms we then get

$$\begin{aligned}C_{00} &= -2\epsilon_{0\mu\nu i}\frac{\partial f(\vartheta)}{\partial\vartheta}\partial^\mu\vartheta\;\partial^\nu\partial^i\partial_0\frac{U^{(0)}}{c^2} - \epsilon_{0\rho ij}(\frac{\partial^2 f(\vartheta)}{\partial\vartheta^2}\partial^\rho\vartheta\partial^i\vartheta + \frac{\partial f(\vartheta)}{\partial\vartheta}\partial^d\partial^i\vartheta)\partial_j\partial_0\frac{U^{(0)}}{c^2}\\ &+ 2\epsilon_{0\mu i\nu}\frac{\partial f(\vartheta)}{\partial\vartheta}\partial^\mu\vartheta\partial^\nu\partial^j\partial_i\frac{g_j^{(3)}}{c^3} - 2\epsilon^{j}_{0\mu\nu}\frac{\partial f(\vartheta)}{\partial\vartheta}\partial^\mu\vartheta\partial^\nu\partial^i\partial_i\frac{g_j^{(3)}}{c^3},\end{aligned} \tag{G.19}$$

$$\begin{aligned}C_{0i} = \frac{1}{2}\Bigg[&-\epsilon_{0\mu ij}(\frac{\partial^2 f(\vartheta)}{\partial\vartheta^2}\partial^\mu\vartheta\partial^i\vartheta + \frac{\partial f(\vartheta)}{\partial\vartheta}\partial^\mu\partial^i\vartheta)\partial_k\partial_j\frac{U^{(0)}}{c^2} + \epsilon_{0\mu i\nu}\frac{\partial f(\vartheta)}{\partial\vartheta}\partial^\mu\vartheta\partial^\nu\partial_j\partial^j\frac{U}{c^2}\\ &+\epsilon_{\mu 0ij}(\frac{\partial^2 f(\vartheta)}{\partial\vartheta^2}\partial^\mu\vartheta\partial^\nu\vartheta + \frac{\partial f(\vartheta)}{\partial\vartheta}\partial^\mu\partial^\nu\vartheta)\partial_i\partial_\nu\frac{U^{(0)}}{c^2} + \epsilon_{0\mu ij}(\frac{\partial^2 f(\vartheta)}{\partial\vartheta^2}\partial^\mu\vartheta\partial^\nu\vartheta +\\ &\frac{\partial f(\vartheta)}{\partial\vartheta}\partial^\mu\partial^\nu\vartheta)\partial_j\partial_\nu\frac{U^{(0)}}{c^2}\Bigg].\end{aligned} \tag{G.20}$$

These C tensor expressions are already of order $\mathcal{O}(1/c^3)$ respectively $\mathcal{O}(1/c^2)$ and by substituting the lowest order solution of the scalar field this raises to at least order $\mathcal{O}(1/c^7)$ and $\mathcal{O}(1/c^8)$. If we look at the total gravitational EOM Eq. (G.15). For the zeroth order component this would mean that any dCS contribution comes in at at least $\mathcal{O}(1/c^8)$ and for the $0i$ component at order $\mathcal{O}(1/c^7)$. So at 1PN order, corresponding to $\mathcal{O}(1/c^2)$, we do not have any dCS contribution and just recover the GR case.

To get non trivial dCS behaviour we could expand to higher PN orders, but this becomes very cumbersome very quickly so this would be out of the scope of this thesis. Even more so it seems that as the scalar field is sourced by a parity violating quantity, the Pontryagin density, it is only non trivial in parity violating dynamics, aka considering black holes with spin. As all the dCS contributions to the EOM depend on the scalar field this would mean that for static configurations dCS reduces to GR. Therefore another option is to study (slowly) spinning black holes[137, 138]. The energy momentum tensor would then need to include spin corrections. Due to these terms, the near zone scalar field solution already gets contributions at leading PN order[70]. Since the scalar field is sourced by the Pontryagin density which is a parity odd quantity, it therefore had influence on parity odd quantities as the angular momentum of the black holes. In the skeletonization proces you would therefore make the angular momentum a function of the scalar field instead of the mass in sGB gravity.
For this thesis we leave the discussion on dCS gravity up till here and focus on sGB gravity. Nevertheless it is still a worhtly pursuit to calculate the 1PN gravitational and scalar waves in dCS for rotating black holes, possibly including tidal effects too. However this would be a more elaborate study because of the inclusion of spin corrections, therefore we leave it for future research.