Existence of Riemann surfaces through an equilibrium of a tangent holomorphic vector field

Constant Huiszoon

a thesis submitted to the Department of Mathematics
at Utrecht University in partial fulfillment of the requirements for the degree of

Master of Mathematical Sciences

supervisor:	Dr. Heinz Hanßmann
second reader:	Dr. Johan van de Leur

November 1, 2022

Universiteit Utrecht

Abstract

It is shown that every holomorphic vector field that vanishes at a point where its derivative is invertible has a Riemann surface such that the vector field is tangent to the Riemann surface and such that the Riemann surface contains the point. In fact, we give a precise finite lower bound on the number of (germs of) such Riemann surfaces depending on the derivative of the vector field at such a point. We use for this a differentiable version of the Grobman-Hartman theorem. Additionally, we conjecture an upper bound on the number of independent integrals a vector field may have near an equilibrium. In the presentation, a newly developed notional system is used.

Acknowledgments

I would like to thank my supervisor Heinz Hanßmann for helping me finding a good subject, his useful comments on my work, and his time, patience and attention during our regular meetings. And of course I would like to thank my family and friends for their unconditional support.

Contents

CHAPTER 1

Introduction

Given a holomorphic vector field with an equilibrium[1], a natural question to ask is whether there exist invariant submanifolds (or more general, analytic subspaces) that contain the equilibrium. We restrict here to the case of Riemann surfaces. The major result on this so far seems to be that from Camacho and Sad [**CS82**], which states that for a vector field on $\mathbb{C}^2$ there always exists one. In contrast to them, in this thesis we found a result that applies not only to dimension two but to any (finite) dimension. Moreover, we show that there is not only one Riemann surface, but in fact we show that there are at least as many as the number eigenvalues of the derivative of the vector field at the equilibrium that are extremal points of the convex hull of the set of eigenvalues. However, we do have the restriction that the derivative of the vector field at the equilibrium should be invertible. The differences in these kinds of results reflect somewhat the differences in the methods being used. While Camacho and Sad used tools as blow ups of manifolds, and Chern classes, on a differentiable version of the Grobman-Hartman theorem is relied. [2]. This differentiable version is a recent result, and gives criteria under which a real C^∞-vector field with an equilibrium is C^1-equivalent to its linearization at the equilibrium. An outline of the proof is in section 6 of chapter 3.

The original aim was to prove a non-integrability criterion for Hamiltonian systems near equilibria of vector fields. The idea was to apply a theory of Morales-Ruiz and Ramis[**Rui99**]. Before this could be done, it was needed to show the existence of invariant analytic curves through the equilibrium. This approach did not work. Instead, a non-integrability criterion is conjectured, the proof of which the author is still working.

Then, there is another purpose this thesis serves, other than the study of holomorphic vector fields. In the process of writing, a new notional system has been developed. This system has the ability to denote logical statements in a systematic and at the same time human readable way - the motivation for its development. This thesis has been taken as opportunity to both present and test this notational system.

[1] Often called a singularity.

[2] The seemingly incompatible conditions on the eigenvalues of the Grobman-Hartman theorem and invertibility is not a problem

CHAPTER 2

Notation

> A proof should not be great literature; it should be beautiful mathematics. Its beauty lies in its logical structure, not in its prose.
>
> Leslie Lamport

As mentioned in the introduction, we use a new notational system. Here we will point out how it works. We are not giving here any more arguments in favor or against it. Since the reader is not ought to be familiar with it yet, all statements are also written down in the usual prosaic form and the proofs are provided with prosaic proof sketches.

Let A, B be logical statements. From these one can form with the usual operators from propositional logic new statements, e.g., $A \wedge B$ or $A \implies B$ or $A \iff B$. When A and B are long satements, then in the new notation $A \wedge B$ will be written as
2 A
1 $\wedge$
2 B
The numberings at the start of the sentence are called structure numbers, and indicate what the arguments of the operators are. Precisely those statement are the arguments of a binary logical operator that have one lower structure number on either side of the operator. Since
2 A
1 $\wedge$
2 B
is itself a logical statement, it also has its own structure. That is given by the structure number of its operator, in this case 1. In this way one can form any composition of logical statements without having the need for brackets. Let us at look at one more example. $(A \wedge B) \implies C$ can be written as
3 A
2 $\wedge$
3 B
1 $\implies$
2 C
Here $A \wedge B$ has structure 2, hence $A \wedge B$ is the first argument of $\implies$, and similarly for C.

Since sequences of implications and equivalences such as $(A \implies B) \wedge (B \iff C)$ are common, we write this down in an abbreviated way:
2 A
1 $\implies$
2 B
1 $\iff$
2 C
Here, one notes that is necessary to use read the structure numbers to distinghuish this from $A \implies (B \iff C)$.

We also have a shorthand notation for sequences as $A \wedge B \wedge C \wedge D \wedge E$[1] or $A \vee B \vee C \vee D \vee E$. In case of $\wedge$, we do this by indicating the start of the sequence by $\wedge_*$ and the end by $\wedge^*$, and the arguments again have a structure number 1 higher than their arguments. With this, $A \wedge B \wedge C \wedge D \wedge E$ becomes
1 $\wedge_*$
2 A
2 B
2 C
2 D
2 E
1 $\wedge^*$

Combining this with notion one has also has the meaning of

1 $\implies_*$
2 A
2 B
2 C
1 $\implies^*$

We also allow extensions from this of the following form
1 $A \wedge$
2 B
which means $A \wedge B$. Here $A \wedge$ can be seen as an operator with one argument, which should written after it, since $\wedge$ is on the left of A. This notation can be again extended to quantifications $\forall$ and $\exists$. Namely, let a be a term, and P a predicate symbol, then we could write $(\forall a)(P(a))$ as
1 $\forall a$
2 $P(a)$
Often in mathematics, we show the existence of objects with a desired property and after that we go on with, 'for each object with such properties...'. Something similar here is also done:

[1]These kind of expressions are well defined since the operator here is associative.

3 [some proof of existence]
2 $\implies$
3 $\exists a$: $P(a)$
1 $\wedge$
2 $\forall a$: as such
3 ...
'as such' now refers to $P(a)$. The rule is that when one reads "as such" after a quantification, then one should look up latest occurrence of the quantification of a variable with the same symbol and interpret "as such" as the properties that required given at this latest quantification.

Another typical situation is as follows: Suppose we know A_1, A_2 and A_3 are true and B_1 and B_2 are unknown statements and we know the implications $(A_1 \wedge A_2) \implies B_1$ and $(B_1 \wedge A_3) \implies B_2$, then usually the proof of B_2 is written down as
2 B_2
1 $\Longleftarrow$
4 $A_1 \wedge A_2$
3 $\implies$
4 B_1
2 $\wedge$
4 $B_1 \wedge A_3$
3 $\implies$
4 B_2

or

2 B_2
1 $\Longleftarrow$
2 $B_1 \wedge A_3$
1 $\wedge$
3 B_1
2 $\Longleftarrow$
4 $A_1 \wedge A_2$

Of course, there also the variations with the arrows in the opposite direction. $B_1 \wedge A_3$ is a very long statement, then to improve readability We could "4 $B_1 \wedge A_3$" expand as:
5 B_1
4 $\wedge$
5 A_3

Suppose we know $C \Longleftarrow (A \wedge B)$ and $A \Longleftarrow$ [very long proof of A] and $B \Longleftarrow$ [very long proof of B], then typically the proof of C is written down as
2 C

1 $\Longleftarrow$
3 A
2 $\wedge$
3 B
1 $\Longleftarrow$
4 A
3 $\Longleftarrow$
4 [very long proof of A]
2 $\wedge$
4 B
3 $\Longleftarrow$
4 [very long proof of B]
Then there is one last variation of proof writing which deserves mentioning. This is one frequently used in the proof of main last theorem. Suppose we have $A \Longleftarrow$ [very long proof of A] and $B \Longleftarrow$ [very long proof of B using A]. Then a proof of B can be written as

2 A
1 $\wedge$
3 A
2 $\Longleftarrow$
3 [very long proof of A]
1 $\wedge$
3 A
2 $\Longrightarrow$
4 B
3 $\Longleftarrow$
4 [very long proof of B using A]

Some explanation: A written down first as claim that A is true, then after "1 $\wedge$" A is proven, and after the next 1 $\wedge$ A is assumed, since it is needed in the proof of B. If this proof continued with 1 $\wedge$ then apparently, the assumption of A would not be used after it.

Other comments on used conventions
We always quantify over ZFC-sets (ZF+choice). So when there is written $\forall S$ without other assumptions, then it should be read as 'let S be a ZFC-set'.

In most mathematical texts, a map from a set A to a set B is defined as a triple (A, B, R) where R is a subset of $A \times B$ that needs to have the properties the reader is familiar with. Here, however, R itself would be called a map from A to B. It could be seen as a drawback that in this way the notion of 'the codomain of a map' is lost; instead we can talk about 'a codomain of R' as set that contains the image of R.

CHAPTER 3

The proof

The aim of this chapter is to prove the main theorem 37 and its corollary, which is what we mean with the title of this thesis. Just after the statement of 37, and before its proof, we give a sketch or outline of the whole proof. It is advisable to read this first before going into detail. In the sections before the statement and proof of theorem 37, the necessary definitions and results are collected.

1. Polygons in $\mathbb{C}$

Some basic definitions and results are recalled about convex sets in vector spaces.

Definition 1 (convex)**.**
1 $\forall V$: a vector space over $\mathbb{R}$
2 $\forall C \subset V$
4 C is convex
3 : $\Longleftrightarrow$
4 $\forall x_1, x_2 \in C$
5 $\forall t \in [0,1]$
6 $x_1 t + x_2(1-t) \in C$

Definition 2 (extreme point, ext(Λ))**.**
1 $\forall V$: a vector space over $\mathbb{R}$
1 $\forall \Lambda \subset V$: Λ is finite
3 $\forall \lambda \in \Lambda$
6 λ is an extreme point of Λ in V
5 : $\Longleftrightarrow$
6 $\forall \widetilde{\lambda} : \Lambda \to [0,1]$
8 $\sum_{\mu \in \Lambda} \widetilde{\lambda}(\mu) = 1$ and $\sum_{\lambda \in \mu} \widetilde{\lambda}(\mu)\mu = \lambda$
7 $\implies$ $\widetilde{\lambda}(\lambda) = 1$
4 $\wedge$
5 ext(Λ, V) or for short ext(Λ) is the set of extreme points of Λ in V

Definition 3 (convex hull, co(Λ))**.**
1 $\forall V$: a vector space over $\mathbb{R}$
2 $\forall \Lambda \subset V$
3 $\forall \lambda \in V$
6 $\lambda \in$ the convex hull of Λ
5 : $\Longleftrightarrow$

6 $\forall \Lambda' \subset \Lambda : \Lambda'$ is finite
7 $\exists \widetilde{\lambda} : \Lambda' \to [0, 1]$
8 $\sum_{\mu \in \Lambda'} \widetilde{\lambda}(\mu) = 1$ and $\sum_{\lambda' \in \mu} \widetilde{\lambda}(\mu)\mu = \lambda$
4 $\wedge$ co(Λ) = the convex hull of Λ

Proposition 4.
1 $\forall V$: a normed vector space over $\mathbb{R}$
2 $\forall \Lambda \subset V : \Lambda$ is finite
5 $P_{4,a}(\Lambda) :\iff$ co(Λ) is convex and compact wrt the norm topology
4 $\wedge$
5 $P_{4,b}(\Lambda) :\iff$ ext Λ = ext co Λ
3 $\implies$
4 $P_{4,a}(\Lambda) \wedge P_{4,b}(\Lambda)$

Prosaic form of the statement. Let V a normed vector space over $\mathbb{R}$ and let $\Lambda \subset V$ such that Λ is finite. Then co(Λ) is convex and compact wrt the norm topology Λ = ext co Λ

Proof.
See [**Con07**].

This definition is made to reference to the result of 6 in a precise manner.

Definition 5.
1 $\forall \Lambda \subset \mathbb{C} : \Lambda$ is finite
2 $\forall \lambda \in \Lambda$
3 λ is an extreme point of Λ in $\mathbb{C} \implies$
5 $P_6(\Lambda, \lambda)$
4 $:\iff$
5 $\exists \alpha \in \mathbb{C}$: $|\alpha| = 1$
6 $\forall \mu \in \Lambda \backslash \{\lambda\}$
7 re $\alpha\mu <$ re $\alpha\lambda$

The following statement looks innocent, but its proof is quite some work.

Proposition 6.
1 $\forall \Lambda \subset \mathbb{C} : \Lambda$ is finite
2 $\forall \lambda \in \Lambda$
3 λ is an extreme point of Λ in $\mathbb{C} \implies$
4 $P_6(\Lambda, \lambda)$

Prosaic form of the statement. Let $\Lambda \subset \mathbb{C}$ such that Λ is finite; let $\lambda \in \Lambda$ and λ is an extreme point of Λ in $\mathbb{C}$. Then $P_6(\Lambda, \lambda)$ is true.

In the proof, the first 5 lines of the statement of the proposition are taken as assumptions.

Proof sketch. First we show there does not exist a line through λ that intersects the convex hull of Λ at two other points than λ on both sides of λ in this line by contradiction (P(not both)). Then we show that there exists a line through λ that does not intersect the convex hull at any other point than λ (P(nonintersecting line)) by combining P(not both) and that the set of angles for which there exists a nonintersecting half line is compact and connected. This latter fact is achieved by showing that we only we have to look at the angles for which there exists a half line that intersects the convex hull of Λ is the same as of $\Lambda\backslash\{\lambda\}$, and the latter is compact and connected in $\mathbb{C}\backslash\{\lambda\}$ - in contrast to the former. After this, we take a complex number $\widetilde{\alpha}$ that on multiplying makes the non-intersecting line vertical in the complex plane. Then by showing that $\widetilde{\alpha}(\Lambda\backslash\{\lambda\})$ is connected, it should be at either the left or the right side of this line. If on the right side then we take $\widetilde{\alpha} = \alpha$, otherwise $\widetilde{\alpha} = -\alpha$.

Proof.
2 $P(\text{not both}) :\Longleftrightarrow$
3 $\nexists \varphi \in \mathbb{R}$
4 $\exists t_1, t_2 \in \mathbb{R}_{>0}$
5 $\lambda + t_1 e^{i\varphi} \in \text{co}(\Lambda) \wedge \lambda + t_2 e^{i\varphi + i\pi} \in \text{co}(\Lambda)$
1 $\Longrightarrow$
4 $P(\text{not both})$
3 $\Longleftarrow$
4 $\exists \varphi \in \mathbb{R}$
5 $\exists t_1, t_2 \in \mathbb{R}_{>0}$
7 $\lambda + t_1 e^{i\varphi} \in \text{co}(\Lambda) \wedge \lambda + t_2 e^{i\varphi + i\pi} \in \text{co}(\Lambda)$
6 $\Longrightarrow$
8 $\frac{t_2}{t_1+t_2} \in (0,1) \iff \frac{t_2}{t_1+t_2} < \frac{t_1+t_2}{t_1+t_2} = 1$
7 $\wedge$
8 $\frac{t_2}{t_1+t_2}(\lambda + t_1 e^{i\varphi}) + (1 - \frac{t_2}{t_1+t_2})(\lambda + t_2 e^{i\varphi + i\pi}) = \lambda$
6 $\Longrightarrow$
7 $\lambda \notin \text{ext co }(\Lambda) \wedge P_{4,b}$
6 $\Longrightarrow$
7 $\lambda \notin \text{ext }(\Lambda) \Longrightarrow \lightning$
2 $\wedge$
4 $\widetilde{e} := \{(\tilde{\varphi}, \{te^{i\varphi} : t \in \mathbb{R}_{>0}\}) : \tilde{\varphi} \in \mathbb{R}/2\pi\mathbb{Z} \ \wedge \ \varphi \in \mathbb{R} \ \wedge \ [\varphi] = \tilde{\varphi}\}$
3 $\Longrightarrow$
5 $\widetilde{e}$ is a map $\mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}\backslash\{0\}/\mathbb{R}_{>0}$
4 $\wedge$
6 $S := \widetilde{e}^{-1}([\text{co }(\Lambda) - \lambda]\backslash\{0\}/\mathbb{R}_{>0})$
5 $\Longrightarrow$
7 $P(\text{characterization of } S) :\Longleftrightarrow$
9 $\forall \varphi \in \mathbb{R}$
10 $\exists t \in \mathbb{R}_{>0}$
11 $\lambda + te^{i\varphi} \in \text{co}(\Lambda)$

8 $\iff$

9 $[\varphi] \in S$

6 $\implies$

8 P(characterization of S) $\impliedby$

9 $\forall \varphi \in \mathbb{R}$

10 $\exists t \in \mathbb{R}_{>0}$

11 $\lambda + te^{i\varphi} \in \text{co } (\Lambda)\backslash\{\lambda\}$

10 $\iff$

11 $te^{i\varphi} \in [\text{co } (\Lambda) - \lambda]\backslash\{0\}$

10 $\iff$

11 $[\varphi] \in \widetilde{e}^{-1}([\text{co } (\Lambda) - \lambda]\backslash\{0\}/\mathbb{R}_{>0})$

10 $\iff$

11 $[\varphi] \in S$

7 $\wedge$

The next step is to show that S is compact and connected.

9 S is compact and connected

8 $\impliedby$

10 $\widetilde{e}^{-1}(\text{co } ((\Lambda - \lambda)\backslash\{0\})/\mathbb{R}_{>0})$ is compact and connected $\wedge$ $\widetilde{e}^{-1}([\text{co } (\Lambda) - \lambda]\backslash\{0\}/\mathbb{R}_{>0}) = \widetilde{e}^{-1}(\text{co } ((\Lambda - \lambda)\backslash\{0\})/\mathbb{R}_{>0})$

9 $\wedge$

11 $\widetilde{e}^{-1}(\text{co } ((\Lambda - \lambda)\backslash\{0\})/\mathbb{R}_{>0})$ is compact and connected

10 $\impliedby$

12 $P_{4,a} \implies$ co $((\Lambda - \lambda)\backslash\{0\})$ is compact and connected $\implies$ co $((\Lambda - \lambda)\backslash\{0\})/\mathbb{R}_{>0}$ is compact and connected

11 $\wedge$

13 co $((\Lambda - \lambda)\backslash\{0\})/\mathbb{R}_{>0}$ is compact and connected $\wedge$ $\widetilde{e}$ is a homeomorphism

12 $\implies$

13 $\widetilde{e}^{-1}(\text{co } ((\Lambda - \lambda)\backslash\{0\})/\mathbb{R}_{>0})$ is compact and connected

9 $\wedge$

10 $[\text{co } (\Lambda) - \lambda]\backslash\{0\}/\mathbb{R}_{>0} = \text{co } ((\Lambda - \lambda)\backslash\{0\})/\mathbb{R}_{>0}$

11 $\impliedby$

10 $[\text{co } (\Lambda) - \lambda]\backslash\{0\}/\mathbb{R}_{>0} = \text{co } ((\Lambda - \lambda)\backslash\{0\})\backslash\{0\}/\mathbb{R}_{>0} \;\wedge\; 0 \notin \text{co } ((\Lambda - \lambda)\backslash\{0\})/\mathbb{R}_{>0}$

11 $\impliedby$

12 $\wedge_*$

13 $[\text{co } (\Lambda) - \lambda]\backslash\{0\}/\mathbb{R}_{>0} \subset \text{co } ((\Lambda - \lambda)\backslash\{0\})\backslash\{0\}/\mathbb{R}_{>0}$

13 $[\text{co } (\Lambda) - \lambda]\backslash\{0\}/\mathbb{R}_{>0} \supset \text{co } ((\Lambda - \lambda)\backslash\{0\})\backslash\{0\}/\mathbb{R}_{>0}$

13 $0 \notin \text{co } ((\Lambda - \lambda)\backslash\{0\})/\mathbb{R}_{>0}$

12 $\wedge^*$

9 $\wedge$

11 $[\text{co } (\Lambda) - \lambda]\backslash\{0\}/\mathbb{R}_{>0} \subset \text{co } ((\Lambda - \lambda)\backslash\{0\})\backslash\{0\}/\mathbb{R}_{>0}$

10 $\impliedby$

11 $\forall x \in \mathbb{C}$

12 $\forall t \in \mathbb{R}_{>0}$: $xt \in [\text{co } (\Lambda) - \lambda]\backslash\{0\}$

13 $\forall \widetilde{\lambda_1} : \Lambda \to [0,1]$: $\sum_{\mu\in\Lambda} \widetilde{\lambda_1}(\mu) = 1$
14 $\implies$ *
15 $xt = \sum_{\mu\in\Lambda}(\widetilde{\lambda_1}(\mu)\mu) - \lambda$
15 $xt = \sum_{\mu\in\Lambda} \widetilde{\lambda_1}(\mu)(\mu - \lambda)$
15 $xt = \sum_{\mu\in\Lambda\setminus\{\lambda\}} \widetilde{\lambda_1}(\mu)(\mu - \lambda)$
15 $\frac{xt}{\sum_{\mu\in\Lambda\setminus\{\lambda\}} \widetilde{\lambda_1}(\mu)} = \frac{1}{\sum_{\mu\in\Lambda\setminus\{\lambda\}} \widetilde{\lambda_1}(\mu)} \sum_{\mu\in\Lambda\setminus\{\lambda\}} \widetilde{\lambda_1}(\mu)(\mu - \lambda)$
19 $t' := \frac{t}{\sum_{\mu\in\Lambda\setminus\{\lambda\}} \widetilde{\lambda_1}(\mu)}$
16 $\wedge$
17 $\lambda_2 := \{(\mu, \lambda_1(\mu + \lambda)) : \mu \in (\Lambda - \lambda)\setminus\{0\}\}$
15 $\implies$
16 $xt' = \sum_{\mu\in(\Lambda-\lambda)\setminus\{0\}} \widetilde{\lambda_2}(\mu)\mu$ and $\sum_{\mu\in(\Lambda-\lambda)\setminus\{0\}} \widetilde{\lambda_2}(\mu) = 1$ and $xt' \neq 0$
15 $\implies$
16 $[x] \in \text{co } ((\Lambda - \lambda)\setminus\{0\})\setminus\{0\}/\mathbb{R}_{>0}$
14 $\implies$ *
9 $\wedge$
11 $[\text{co } (\Lambda) - \lambda]\setminus\{0\}/\mathbb{R}_{>0} \supset \text{co } ((\Lambda - \lambda)\setminus\{0\})\setminus\{0\}/\mathbb{R}_{>0}$
10 $\impliedby$
11 $\forall x \in \mathbb{C}$
12 $\forall t \in \mathbb{R}_{>0}$: $xt \in \text{co } ((\Lambda - \lambda)\setminus\{0\})\setminus\{0\}$
13 $\forall \widetilde{\lambda}_1 : (\Lambda - \lambda)\setminus\{0\} \to [0,1]$: $\sum_{\mu\in\Lambda} \widetilde{\lambda}_1(\mu) = 1$
15 $xt = \sum_{\mu\in(\Lambda-\lambda)\setminus\{0\}}(\widetilde{\lambda_1}(\mu)\mu)$
14 $\implies$
16 $\widetilde{\lambda}_2 := \{(\lambda, 0)\} \cup \{(\mu, \widetilde{\lambda}_1(\mu - \lambda)) : \mu \in \Lambda\setminus\{\lambda\}\}$
15 $\implies$ $xt = \sum_{\mu\in\Lambda} \widetilde{\lambda_2}(\mu)\mu - \lambda$ and $\sum_{\mu\in\Lambda} \widetilde{\lambda_2}(\mu) = 1$ and $xt \neq 0$
15 $\implies$ $[x] \in [\text{co } (\Lambda) - \lambda]\setminus\{0\}/\mathbb{R}_{>0}$
9 $\wedge$
11 $0 \notin \text{co } ((\Lambda - \lambda)\setminus\{0\})/\mathbb{R}_{>0}$

This proof is done by defining a statement P, then showing it is true, and then showing that $0 \in \text{co } ((\Lambda - \lambda)\setminus\{0\})$ *causes the negation of P, which results in contradiction.*

10 $\impliedby$
12 $P :\iff$
13 $\forall \widetilde{\lambda_1}$: $(\Lambda - \lambda) \to [0,1]$: $\sum_{\mu\in\Lambda-\lambda} \widetilde{\lambda_1}(\mu) = 1$ and $\sum_{\mu\in\Lambda-\lambda} \widetilde{\lambda_1}(\mu)\mu = 0$
14 $\widetilde{\lambda_1}(0) = 1$
11 $\implies$
14 $\lambda \in \text{ext } (\Lambda)$
13 $\implies$
14 $0 \in \text{ext } (\Lambda - \lambda)$
13 $\implies$ P
12 $\wedge$
14 $0 \in \text{co } ((\Lambda - \lambda)\setminus\{0\})$
13 $\implies$

15 $0 \in \text{co}\ ((\Lambda - \lambda)\backslash\{0\}) \implies$

16 $\exists\widetilde{\lambda_2}$: $(\Lambda - \lambda)\backslash\{0\} \to [0,1]$: $\sum_{\mu\in(\Lambda-\lambda)\backslash\{0\}} \widetilde{\lambda_2}(\mu) = 1$ and $\sum_{\mu\in(\Lambda-\lambda)\backslash\{0\}} \widetilde{\lambda_2}(\mu)\mu = 0$

14 $\wedge$

15 $\forall\widetilde{\lambda_2}$: $(\Lambda - \lambda)\backslash\{0\} \to [0,1]$: $\sum_{\mu\in(\Lambda-\lambda)\backslash\{0\}} \widetilde{\lambda_2}(\mu) = 1$ and $\sum_{\mu\in(\Lambda-\lambda)\backslash\{0\}} \widetilde{\lambda_2}(\mu)\mu = 0$

16 $\widetilde{\lambda_1} := \{0,0\} \cup \{(\mu,\lambda_2) : \mu \in (\Lambda-\lambda)\backslash\{0\}\} \implies$

17 $\widetilde{\lambda_1} : \Lambda - \lambda \to [0,1]$ and $\sum_{\mu\in\Lambda-\lambda} \widetilde{\lambda_1}(\mu) = 1$ and $\sum_{\mu\in\Lambda-\lambda} \widetilde{\lambda_1}(\mu)\mu = 0$ and $\widetilde{\lambda_1}(0) \neq 1$

13 $\implies \neg P$

Now we have proven that S is compact and connected. We continue with showing there exists a line through λ that does intersect co (Λ) *at no other point.*

7 $\wedge$

9 P(nonintersecting line) $:\iff$

10 $\exists\varphi \in \mathbb{R}$

11 $\forall t \in \mathbb{R}$

12 $\lambda + te^{i\varphi} \notin \text{co}\ (\Lambda)\backslash\{\lambda\}$

8 $\implies$

11 P(characterization of S) $\wedge$ $(\exists\forall\neg \Leftrightarrow\ \neg\forall\exists)$

10 $\implies$

12 P(nonintersecting line)

11 $\iff$

12 $\exists\varphi \in \mathbb{R} : [\varphi] \in S^C$ and $[\varphi + \pi] \in S^C$

9 $\wedge$

11 S is compact and connected

10 $\implies$

11 $\exists\varphi_1, \varphi_2 \in \mathbb{R}$: $\varphi_1 \leq \varphi_2$ and $S = [\varphi_1, \varphi_2]/2\pi\mathbb{Z}$

9 $\wedge$

10 $\forall\varphi_1, \varphi_2$: as such

12 P(not both) $\wedge$ P(characterization of S) $\implies \varphi_2 < \varphi_1 + \pi$

11 $\wedge$

12 $S^C = (\varphi_2, \varphi_1 + 2\pi)/2\pi\mathbb{Z}$

11 $\wedge$

13 $\varphi_1 + \pi > \varphi_2\ \wedge\ S^C = (\varphi_2, \varphi_1 + 2\pi)/2\pi\mathbb{Z}$

12 $\implies$

14 $\forall\varphi \in (\varphi_2, \varphi_1 + \pi)$

$[\varphi] \in S^C\ \wedge [\varphi + \pi] \in S^C$

13 $\wedge$

14 $\exists\varphi \in (\varphi_2, \varphi_1 + \pi)$

9 $\wedge$

12 $\exists\varphi_1, \varphi_2, \varphi$: as such

11 $\wedge$

12 $\forall\varphi_1, \varphi_2, \varphi$: as such

13 $[\varphi] \in S^C$ and $[\varphi + \pi] \in S^C$

10 $\implies$

11 $\exists\varphi \in \mathbb{R} : [\varphi] \in S^C$ and $[\varphi + \pi] \in S^C$

Remember that we have already proven this equivalence.

10 $\iff$

11 P(nonintersecting line)

Now we use that P(nonintersecting line) *is true to just do what is promised in the proof sketch: "we take a complex number* $\widetilde{\alpha}$ *that on multiplying makes the nonintersecting line vertical in the complex plane. Then by showing* $\widetilde{\alpha}(\Lambda\backslash\{\lambda\})$ *is connected, it should be at either the left or the right side of this line. If on the right side then we take* $\widetilde{\alpha} = \alpha$*, otherwise* $\widetilde{\alpha} = -\alpha$*"*

9 $\wedge$

10 P(nonintersecting line) $\wedge$

11 $\forall \varphi \in \mathbb{R}$

13 $\forall t \in \mathbb{R}$

14 $\lambda + te^{i\varphi} \notin \text{co }(\Lambda)\backslash\{\lambda\}$

12 $\implies$

15 $\widetilde{\alpha} := ie^{-i\varphi}$

14 $\wedge$

15 $l := \{\lambda + te^{i\varphi} : t \in \mathbb{R}_{>0}\}$

13 $\implies$

15 $\Longleftarrow$ *

16 $(\widetilde{\alpha}l)^C$ is the disjoint union of $\{x \in \mathbb{C} : \text{re}(x) > \text{re}(\widetilde{\alpha}\lambda)\}$ and $\{x \in \mathbb{C} : \text{re}(x) < \text{re}(\widetilde{\alpha}\lambda)\}$

16 $\widetilde{\alpha}l = \{x \in \mathbb{C} : \text{re}(x) = \text{re}(\widetilde{\alpha}\lambda)\}$

16 $\widetilde{\alpha}e^{i\varphi} = i$

15 $\Longleftarrow$ *

14 $\wedge$

15 $\Longleftarrow$ *

16 $\widetilde{\alpha}(\text{co }(\Lambda)\backslash\{\lambda\})$ is connected

16 co $(\Lambda)\backslash\{\lambda\}$ is connected $\wedge$ multiplying on $\mathbb{C}$ by a nonzero complex number is a homeomorphism

16 co $(\Lambda)\backslash\{\lambda\}$ is connected

16 co $(\Lambda)\backslash\{\lambda\}$ is convex

16 $\forall x, y \in \text{co}(\Lambda)\backslash\{\lambda\}$

17 $\forall t \in [0, 1]$

20 $\text{co}(\Lambda)$ is convex and $x, y \in \text{co}(\Lambda)$

19 $\implies$

20 $xt + (1 - t)y \in \text{co}(\Lambda)$

19 $\implies$

20 $xt + (1 - t)y \in \text{co}(\Lambda)\backslash\{\lambda\} \vee xt + (1 - t)y = \lambda$

18 $\wedge$

19 $P_{4,b} \implies \lambda \in \text{ext co}(\Lambda)$

18 $\wedge$

20 $xt + (1 - t)y = \lambda \ \wedge \ \lambda \in \text{ext co}(\Lambda)$

19 $\implies$

20 $x = \lambda \vee y = \lambda$

15 $\Longleftarrow$ *
14 $\wedge$
15 $\widetilde{\alpha}(\text{co }(\Lambda)\backslash\{\lambda\}) \subset (\widetilde{\alpha}l)^C \Longleftarrow \text{ co }(\Lambda)\backslash\{\lambda\} \cap l = \emptyset$
14 $\wedge$
16 $\wedge_*$
17 $\widetilde{\alpha}(\text{co }(\Lambda)\backslash\{\lambda\}) \subset (\widetilde{\alpha}l)^C$
17 $(\widetilde{\alpha}l)^C = \{x \in \mathbb{C} : \text{re}(x) > \text{re}(\widetilde{\alpha}\lambda)\} \sqcup \{x \in \mathbb{C} : \text{re}(x) < \text{re}(\widetilde{\alpha}\lambda)\}$
17 $\widetilde{\alpha}(\text{co }(\Lambda)\backslash\{\lambda\})$ is connected
16 $\wedge^*$
15 $\Longrightarrow$
16 $\widetilde{\alpha}(\text{co }(\Lambda)\backslash\{\lambda\}) \subset \{x \in \mathbb{C} : \text{re}(x) > \text{re}(\widetilde{\alpha}\lambda)\}$ or $\widetilde{\alpha}(\text{co }(\Lambda)\backslash\{\lambda\}) \subset \{x \in \mathbb{C} : \text{re}(x) < \text{re}(\widetilde{\alpha}\lambda)\}$
15 $\Longrightarrow$
17 $\alpha := \widetilde{\alpha}/|\widetilde{\alpha}|$
18 $\forall \mu \in \Lambda\backslash\{\lambda\}$
19 re $\alpha\mu <$ re $\alpha\lambda$ and $|\alpha| = 1$
16 $\vee$
17 $\alpha := -\widetilde{\alpha}/|\widetilde{\alpha}|$
18 $\forall \mu \in \Lambda\backslash\{\lambda\}$
19 re $\alpha\mu <$ re $\alpha\lambda$ and $|\alpha| = 1$

2. Integral curves, flows and limits

The following shorthand notation is useful now.

Definition 7.
1 $\forall V$: a vector space over $\mathbb{C}$ or $\mathbb{R}$
2 $V_{\mathbb{R}}$ is the underlying real vector space of V, i.e. we restrict the field of scalars to $\mathbb{R}$

To be able to distinguish clearly between complex differentiable and real differentiable, we use the following notation.

Definition 8.
1 $\forall V_1, V_2$: vector spaces over $\mathbb{C}$
2 $\forall U$: an open subset of V_1
3 $\forall f$
5 $f \in C^1(U|_{\mathbb{R}}, V_{1,\mathbb{R}}) \iff f$ is real differentiable map from U to V_2 with respect to the vector space structures of $V_{1,\mathbb{R}}$ and $V_{2,\mathbb{R}}$
4 $\wedge$
5 $f \in \mathrm{Hol}(U, V) \iff f$ is a holomorphic map from U to V

Apart from distinguishing between the complex and real case, these definitions are also here to emphasize that the domain of an integral curve needs to be connected.

Definition 9 (real integral curve)**.**
1 $\forall V$: a vector space over $\mathbb{R}$ or $\mathbb{C}$
2 $\forall U \subset V$: U is open
3 $\forall \xi \in C^1(U_{\mathbb{R}}, V_{\mathbb{R}})$
4 $\forall \gamma$
6 γ is a real integral curve of ξ
5 $\iff$
6 $\exists I$: an open interval of $\mathbb{R}$
8 $\gamma \in C^1(I, U_{\mathbb{R}})$
7 $\wedge$
8 $D\gamma = \xi \circ \gamma$

Definition 10 (complex integral curve)**.**
1 $\forall V$: a vector space over $\mathbb{C}$
2 $\forall U \subset V$: U is open
3 $\forall \xi \in \mathrm{Hol}(U, V)$
4 $\forall \gamma$
6 γ is a complex integral curve of ξ
5 $\iff$
6 $\exists \tilde{U}$: an open connected set of $\mathbb{C}$
8 $\gamma \in \mathrm{Hol}(\tilde{U}, V)$
7 $\wedge$
8 $D\gamma(-)(1) = \xi \circ \gamma$

This proposition is a not so surprising result, and the proof is trivial if use the 'right' definition of real differentiability. The point of this proposition that a real differentiable function that commutes with multiplication with i at only one point behaves for some limits to this point the same as functions that are holomorphic on a neighborhood of this point.

Proposition 11.
1 $\forall V$: a vector space over $\mathbb{C}$
2 $\forall U \subset V$: U is a neighborhood of 0
3 $\forall f \in C^1(U_\mathbb{R}, V_\mathbb{R})$: $f(0) = 0$
4 $\forall \gamma$
7 $\exists x_0 \in \mathbb{R}$
8 $\gamma \in C((-\infty, x_0), \mathbb{C}\backslash\{0\})$
6 $\wedge$
7 $\lim_{t\to-\infty} \gamma(t) = 0$
5 $\implies$
7 $\forall v \in V$
8 $Df(0)(iv) = iDf(0)(v)$
6 $\implies$
7 $\lim_{t\to-\infty} \frac{1}{\gamma(t)} f(\gamma(t)v) = Df(0)(v)$

Prosaic form of the statement. Let V is a finite dimensional vector space over $\mathbb{C}$; U is a neighborhood of 0; $f \in C^1(U_\mathbb{R}, V_\mathbb{R})$ such that $f(0) = 0$; let γ be such that there exists a $x_0 \in \mathbb{R}$ $\gamma \in C((-\infty, x_0), \mathbb{C}\backslash\{0\})$ and $\lim_{t\to-\infty} \gamma(t) = 0$, then, if for each $v \in V$ such that $Df(0)(iv) = iDf(0)(v)$ holds, then $\lim_{t\to-\infty} \frac{1}{\gamma(t)} f(\gamma(t)v) = Df(0)(v)$.

In the proof, the lines of the statement of the proposition up to "5 $\implies$ " are used as assumptions.

Proof sketch. Only elementary analysis is used.

Proof.
1 $\forall || - ||$: a norm on V
4 f is continuously differentiable
3 $\implies$
4 $\exists h : U \to V$
6 $\lim_{x\to0} h(x) = 0$
5 $\wedge$
6 $\forall x \in U$
7 $f(x) = Df(0)(x) + ||x||h(x)$
2 $\wedge$
3 $\forall h$: as such
5 $\forall t \in \mathrm{dom}(\gamma)$: $\gamma(t) \in \mathrm{dom}(f)$
6 $\implies_*$
7 $f(\gamma(t)v) = Df(0)(\gamma(t)v) + ||\gamma(t)v||h(\gamma(t)v)$
7 $\frac{1}{\gamma(t)} f(\gamma(t)v) - \frac{1}{\gamma(t)} Df(0)(\gamma(t)v) = \frac{1}{\gamma(t)} ||\gamma(t)v||h(\gamma(t)v)$

7 $\frac{1}{\gamma(t)}f(\gamma(t)v) - Df(0)(v) = \frac{1}{\gamma(t)}||\gamma(t)v||h(\gamma(t)v)$
7 $||\frac{1}{\gamma(t)}f(\gamma(t)v) - Df(0)(v)|| = ||v||\ ||h(\gamma(t)v)||$
6 $\implies$ *
4 $\wedge$
7 $\forall t \in \mathrm{dom}(\gamma)\colon\ \gamma(t) \in \mathrm{dom}(f)$
8 $||\frac{1}{\gamma(t)}f(\gamma(t)v) - Df(0)(v)|| = ||v||\ ||h(\gamma(t)v)||$
6 $\wedge$
7 $\lim_{t\to-\infty} ||v||\ ||h(\gamma(t)v)|| = 0$
5 $\implies$
7 $\lim_{t\to-\infty} ||\frac{1}{\gamma(t)}f(\gamma(t)v) - Df(0)(v)|| = 0$
6 $\implies$
7 $\lim_{t\to-\infty} \frac{1}{\gamma(t)}f(\gamma(t)v) = Df(0)(v)$

The following definition makes the proof of the next proposition more readable.

Definition 12.
1 $\forall n, m \in \mathbb{N}\colon\ n > m$
2 $\{n, ..., m\} := \emptyset$

This is another proposition concerning limits.

Proposition 13.
1 $\forall t_0 \in \mathbb{R}$
2 $\forall n \in \mathbb{N}$
3 $\forall c : \{1, ..., n\} \to \mathbb{C}$
4 $\forall f : \{1, ..., n\} \to ((-\infty, t_0) \to \mathbb{C})$
5 $\forall C \in \mathbb{C}$
8 $\lim_{t\to-\infty} \sum_{m=1}^{n} c(m)f(m)(t) = C \ \wedge\ \lim_{t\to-\infty} f(1)(t) \neq 0$
7 $\wedge$
8 $\forall m \in \{1, ..., n-1\}$
9 $\lim_{t\to-\infty} \frac{f(m)(t)}{f(m+1)(t)} = 0$
6 $\implies$
8 $c(1) = \frac{C}{\lim_{t\to-\infty} f(1)(t)}$
7 $\wedge$
8 $\forall m \in \{2, ..., n\}$
9 $c(m) = 0$

Prosaic form of the statement. Let $t_0 \in \mathbb{R}$; $n \in \mathbb{N}$; $c : \{1, ..., n\} \to \mathbb{C}$; $f : \{1, ..., n\} \to ((-\infty, t_0) \to \mathbb{C})$; $C \in \mathbb{C}$. Then if $\lim_{t\to-\infty} \sum_{m=1}^{n} c(m)f(m)(t) = C$ and $\lim_{t\to-\infty} f(1)(t) \neq 0$ and for each $m \in \{1, ..., n-1\}$ $\lim_{t\to-\infty} \frac{f(m)(t)}{f(m+1)(t)} = 0$ hold, then $c(1) = \frac{C}{\lim_{t\to-\infty} f(1)(t)}$ holds and for each $m \in \{2, ..., n\}$ $c(m) = 0$ holds.

The first 5 lines of the statement of the proposition are taken as assumptions in the proof.

In the proof, the lines up to "6 $\implies$ " are taken as assumptions.

Proof sketch. First $(\forall m \in \{2, ..., n\})(c(m) = 0)$ is shown, from which $c(1) = \frac{C}{\lim_{t\to-\infty} f(1)(t)}$ easily follows. $(\forall m \in \{2, ..., n\})(c(m) = 0)$ is shown using induction, and the induction base and step are proven by first showing $(\forall m_1, m_2 \in \{1, ..., n\})(m_2 > m_1$
9 $\lim_{t\to-\infty} \frac{f(m_1)(t)}{f(m_2)(t)} = 0 \ \wedge \ \lim_{t\to-\infty} \frac{1}{f(m_2)(t)} = 0)$.

Proof.
3 $(\forall m \in \{2, ..., n\})(c(m) = 0)$
2 $\wedge$
4 $(\forall m \in \{2, ..., n\})(c(m) = 0)$
3 $\implies$
4 $c(1) = \frac{C}{\lim_{t\to-\infty} f(1)(t)}$
1 $\impliedby$
4 $(\forall m \in \{2, ..., n\})(c(m) = 0)$
3 $\impliedby$
5 induction
4 $\wedge$
This second argument of the "4 $\wedge$" contains both the base case as the induction step. For the base case, note that if $m = n$, then $\{m + 1, ..., n\} = \emptyset$.
5 $\forall m \in \{2, ..., n\}$
7 $(\forall k \in \{m + 1, ..., n\})(c(k) = 0)$
6 $\implies$
7 $(\forall k \in \{m, ..., n\})(c(k) = 0)$
4 $\wedge$
6 $\forall m \in \{2, ..., n\}$
8 $(\forall k \in \{m + 1, ..., n\})(c(k) = 0)$
7 $\implies$
8 $(\forall k \in \{m, ..., n\})(c(k) = 0)$
5 $\impliedby$
7 $\forall m_1, m_2 \in \{1, ..., n\}\colon\ m_2 > m_1$
8 $\lim_{t\to-\infty} \frac{f(m_1)(t)}{f(m_2)(t)} = 0 \ \wedge \ \lim_{t\to-\infty} \frac{1}{f(m_2)(t)} = 0$
6 $\wedge$
8 $\forall m_1, m_2 \in \{1, ..., n\}\colon\ m_2 > m_1$
9 $\lim_{t\to-\infty} \frac{f(m_1)(t)}{f(m_2)(t)} = 0 \ \wedge \ \lim_{t\to-\infty} \frac{1}{f(m_2)(t)} = 0$
7 $\implies$
8 $\forall m \in \{2, ..., n\}$
10 $(\forall k \in \{m + 1, ..., n\})(c(k) = 0)$
9 $\implies$
10 $(\forall k \in \{m, ..., n\})(c(k) = 0)$
5 $\impliedby$
8 $\forall m_1, m_2 \in \{1, ..., n\}\colon\ m_2 > m_1$
9 $\lim_{t\to-\infty} \frac{f(m_1)(t)}{f(m_2)(t)} = 0 \ \wedge \ \lim_{t\to-\infty} \frac{1}{f(m_2)(t)} = 0$

7 $\Longleftarrow$

11 $(\forall m \in \{2, ..., n\})(\lim_{t\to-\infty} \frac{f(m)(t)}{f(m+1)(t)} = 0)$

10 $\wedge$

11 $(\forall m_1, m_2 \in \{1, ..., n\}\colon\ m_2 > m_1)(\lim_{t\to-\infty} \frac{f(m_1)(t)}{f(m_2)(t)} = \lim_{t\to-\infty} \prod_{m=m_1}^{m_2-1} \frac{f(m)(t)}{f(m+1)(t)})$

9 $\Longrightarrow$

10 $(\forall m_1, m_2 \in \{1, ..., n\}\colon\ m_2 > m_1)(\lim_{t\to-\infty} \frac{f(m_1)(t)}{f(m_2)(t)} = 0)$

9 $\Longrightarrow$

10 $\lim_{t\to-\infty} \frac{f(1)(t)}{f(m_2)(t)} = 0$

8 $\wedge$

10 $\lim_{t\to-\infty} \frac{f(1)(t)}{f(m_2)(t)} = 0 \ \wedge\ \lim_{t\to-\infty} f(1)(t) \neq 0$

9 $\Longrightarrow$

10 $\lim_{t\to-\infty} \frac{1}{f(m_2)(t)} = 0$

6 $\wedge$

9 $\forall m_1, m_2 \in \{1, ..., n\}\colon\ m_2 > m_1$

10 $\lim_{t\to-\infty} \frac{f(m_1)(t)}{f(m_2)(t)} = 0 \ \wedge\ \lim_{t\to-\infty} \frac{1}{f(m_2)(t)} = 0$

8 $\Longrightarrow$

9 $\forall m \in \{2, ..., n\}$

11 $(\forall k \in \{m+1, ..., n\})(c(k) = 0)$

10 $\Longrightarrow$

11 $(\forall k \in \{m, ..., n\})(c(k) = 0)$

7 $\Longleftarrow$

8 $\forall m \in \{2, ..., n\}$

12 $(\forall k \in \{m+1, ..., n\})(c(k) = 0)$

11 $\wedge$

12 $\lim_{t\to-\infty} \sum_{m'=1}^{n} c(m')f(m')(t) = C$

10 $\Longrightarrow$

11 $\lim_{t\to-\infty} \sum_{m'=1}^{m} c(m')f(m')(t) = C$

10 $\Longrightarrow$

11 $\lim_{t\to-\infty} \frac{1}{f(m)(t)} \lim_{t\to-\infty} \sum_{m'=1}^{m} c(m')f(m')(t) = \lim_{t\to-\infty} \frac{1}{f(m)(t)} \cdot C$

9 $\wedge$

12 $\lim_{t\to-\infty} \frac{1}{f(m)(t)} \lim_{t\to-\infty} \sum_{m'=1}^{m} c(m')f(m')(t) = \lim_{t\to-\infty} \frac{1}{f(m)(t)} \cdot C$

11 $\wedge$

12 $\forall m_1, m_2 \in \{1, ..., n\}\colon\ m_2 > m_1$

13 $\lim_{t\to-\infty} \frac{f(m_1)(t)}{f(m_2)(t)} = 0 \ \wedge\ \lim_{t\to-\infty} \frac{1}{f(m_2)(t)} = 0$

10 $\Longrightarrow$

11 $c(m) = 0$

2 $\wedge$

5 $(\forall m \in \{2, ..., n\})(c(m) = 0)$

4 $\Longrightarrow$

5 $c(1) = \frac{C}{\lim_{t\to-\infty} f(1)(t)}$

3 $\Longleftarrow$

6 $(\forall m \in \{2, ..., n\})(c(m) = 0)$

5 $\wedge$

6 $\lim_{t\to-\infty}\sum_{m'=1}^{n} c(m')f(m')(t) = C$
4 $\implies$
5 $\lim_{t\to-\infty} c(1)f(1)(t) = C$
4 $\implies$
5 $c(1) = \frac{C}{\lim_{t\to-\infty} f(1)(t)}$

Definition 14 (maximal real integral curve)**.**
1 $\forall V, U$: V is a finite dimensional vector space over $\mathbb{R}$ and $U \subset V$ and is open
2 $\xi \in C^1(U, V)$
3 $\forall\gamma$
5 γ is a maximal real integral curve of ξ
4 : $\iff$
6 γ is a real integral curve of ξ
5 $\wedge$
6 $\nexists\tilde{\gamma}$: $\tilde{\gamma}$ is a real integral curve of ξ $\wedge$ $\tilde{\gamma} \supsetneq \gamma$

Theorem 15 ((Part of) fundamental theorem on flows)**.**
1 $\forall V, U$: V is a finite dimensional vector space over $\mathbb{R}$ and $U \subset V$ and is open
2 $\xi \in C^1(U, V)$
4 $\exists!\mathcal{D}_\xi \subset \mathbb{R}\times U$: $\mathcal{D}_\xi$ is open
5 $\exists! F_\xi \in C^1(\mathcal{D}_\xi, U)$
6 $\forall x \in U$: $\{(t, F_\xi(t, x)) : t \in \mathrm{pr}_1(\mathcal{D}_\xi \cap \mathbb{R}\times\{x\})\}$ is a maximal real integral curve of ξ and at 0 this integral curve is x

Prosaic form of the statement. Let V is a finite dimensional vector space over $\mathbb{R}$ and $U \subset V$ an open set and let $\xi \in C^1(U, V)$. Then There exists a unique $\mathcal{D}_\xi \subset \mathbb{R}\times U$ that is open and a unique $F_\xi \in C^1(\mathcal{D}_\xi, U)$ such that for each $x \in U$ $\{(t, F_\xi(t, x)) : t \in \mathrm{pr}_1(\mathcal{D}_\xi \cap \mathbb{R}\times\{x\})\}$ is a maximal real integral curve of ξ and at 0 this integral curve is x.

Proof.
See [**Lee12**], 9.12.

Definition 16 (maximal flow domain, maximal flow, $\mathcal{D}_\xi$, F_ξ)**.**
1 $\forall V, U$: V is a finite dimensional vector space over $\mathbb{R}$ and $U \subset V$ and is open
2 $\xi \in C^1(U, V)$
4 $\mathcal{D}_\xi$ and F_ξ are as in theorem 15
3 $\wedge$
4 the maximal flow domain of ξ is $\mathcal{D}_\xi$, and the maximal flow of ξ is F_ξ

From a maximal flow, all other integral curves can be obtained.

Proposition 17.
1 $\forall V$: a vector space over $\mathbb{R}$ or $\mathbb{C}$
2 $\forall U \subset V$: U is open
3 $\forall \xi \in C^1(U_\mathbb{R}, V_\mathbb{R})$
4 $\forall I$: an open interval of $\mathbb{R}$

5 $\forall \gamma : I \to U$
6 γ is a real integral curve of ξ
5 $\iff$
6 $\forall t' \in I$
7 $\gamma = \{(t, F_\xi(t - t', \gamma(t'))) : t \in I\}$

Prosaic form of the statement. Let V a vector space over $\mathbb{R}$ or $\mathbb{C}$; $U \subset V$ such that U is open; $\xi \in C^1(U_\mathbb{R}, V_\mathbb{R})$; I an open interval of $\mathbb{R}$; γ a map $I \to U$. Then γ is a real integral curve of ξ if and only if for each $t' \in I$ $\gamma = \{(t, F_\xi(t - t', \gamma(t'))) : t \in I\}$ holds.

The first 5 lines of the statement of the proposition are taken as assumptions in the proof.

Proof sketch. Two proof is done in two independent parts, which are written down before "1 $\Longleftarrow$ ", and which are the $\Longrightarrow$ and the $\Longleftarrow$ -implications. After "1 $\Longleftarrow$ " they are proven straightforwardly.

Proof.
4 γ is a real integral curve of ξ
3 $\Longrightarrow$
4 $\forall t' \in I$
5 $\gamma = \{(t, F_\xi(t - t', \gamma(t'))) : t \in I\}$
2 $\wedge$
4 γ is a real integral curve of ξ
3 $\Longleftarrow$
4 $\forall t' \in I$
5 $\gamma = \{(t, F_\xi(t - t', \gamma(t'))) : t \in I\}$
1 $\Longleftarrow$
5 γ is a real integral curve of ξ
4 $\Longrightarrow$
5 $\forall t' \in I$
6 $\gamma = \{(t, F_\xi(t - t', \gamma(t'))) : t \in I\}$
3 $\Longleftarrow$
7 $D\{(t, \gamma(t+t')) : t \in (I - t')\} = D\{(t, D\gamma(t+t') \cdot D\{(t'', t'+t'') : t'' \in (I - t')\}) : t \in (I - t')\}$
6 $\wedge$
7 γ is a real integral curve of ξ
6 $\wedge$
7 $D\{(t'', t' + t'') : t'' \in (I - t')\} = 1$
5 $\Longrightarrow$
6 $D\{(t, \gamma(t + t')) : t \in (I - t')\} = D\{(t, \xi(\gamma(t + t'))) : t \in (I - t')\}$
5 $\Longrightarrow$
6 $\{(t, \gamma(t + t')) : t \in (I - t')\}$ is a real integral curve of ξ
4 $\wedge$
7 $\{(t, \gamma(t + t')) : t \in (I - t')\}$ is a real integral curve of ξ

6 $\wedge$
7 $F_\xi(-,\gamma(t'))$ is a real integral curve of ξ
6 $\wedge$
7 $\{(t,\gamma(t+t')) : t \in (I-t')\}(0) = F_\xi(0,\gamma(t'))$
6 $\wedge$
7 uniqueness of solutions of ODEs
5 $\Longrightarrow$
7 $\{(t,\gamma(t+t')) : t \in (I-t')\}|_{(I-t')\cap \mathrm{pr}_1((\mathbb{R}\times\{x\})\cap\mathcal{D}_\xi)} = F_\xi(-,\gamma(t'))|_{(I-t')\cap \mathrm{pr}_1((\mathbb{R}\times\{x\})\cap\mathcal{D}_\xi)}$
6 $\wedge$
7 $\{(t,\gamma(t+t')) : t \in (I-t')\}$ is a real integral curve of ξ
6 $\wedge$
7 $F_\xi(-,\gamma(t'))$ is a real integral curve of ξ
5 $\Longrightarrow$
6 $\{(t,\gamma(t+t')) : t \in (I-t')\} \cup F_\xi(-,\gamma(t'))$ is a real integral curve of ξ
4 $\wedge$
7 $\{(t,\gamma(t+t')) : t \in (I-t')\} \cup F_\xi(-,\gamma(t'))$ is a real integral curve of ξ
6 $\wedge$
7 $F_\xi(-,\gamma(t'))$ is a maximal real integral curve of ξ
5 $\Longrightarrow$
6 $\neg$
7 $\{(t,\gamma(t+t')) : t \in (I-t')\} \cup F_\xi(-,\gamma(t')) \supsetneq F_\xi(-,\gamma(t'))$
4 $\wedge$
7 $\neg$
8 $\{(t,\gamma(t+t')) : t \in (I-t')\} \cup F_\xi(-,\gamma(t')) \supsetneq F_\xi(-,\gamma(t'))$
6 $\wedge$
7 $\{(t,\gamma(t+t')) : t \in (I-t')\} \cup F_\xi(-,\gamma(t')) \supset F_\xi(-,\gamma(t'))$
5 $\Longrightarrow$
6 $\{(t,\gamma(t+t')) : t \in (I-t')\} = F_\xi(-,\gamma(t'))$
5 $\Longrightarrow$
6 $\{(t,\gamma(t)) : t \in I\} = \{(t, F_\xi(t-t',\gamma(t'))) : t \in I\}$
2 $\wedge$
5 $\forall t' \in I$
6 $\gamma = \{(t, F_\xi(t-t',\gamma(t'))) : t \in I\}$
4 $\Longrightarrow$
5 γ is a real integral curve of ξ
3 $\Longleftarrow$
6 $D\{(t, F_\xi(t-t',\gamma(t'))) : t \in I\} = \{(t, D_1F_\xi(t-t',\gamma(t')) \cdot D\{(t'', t''-t') : t'' \in I\}(t)) : t \in I\}$
5 $\wedge$
6 $\forall t \in I$
7 $D\{(t'', t''-t') : t'' \in I\}(t) = 1$
5 $\wedge$
6 $\forall x \in U$
7 $F_\xi(-,x)$ is a real integral curve of ξ

4 $\implies$
5 $D\{(t, F_\xi(t - t', \gamma(t'))) : t \in I\} = \{(t, \xi(F_\xi(t - t', \gamma(t')))) : t \in I\}$
4 $\implies$
5 $\{(t, F_\xi(t - t', \gamma(t'))) : t \in I\}$ is a real integral curve of ξ

Readers familiar with the definition of a Lie bracket of vector fields can skip this definition.

Definition 18 (Lie bracket, $[\xi, \eta]$)**.**
1 $\forall V$: V a vector space over $\mathbb{C}$ or $\mathbb{R}$
2 $\forall U \subset V$: U is a neighborhood of 0
3 $\forall \xi, \eta \in C^1(U_\mathbb{R}, V)$
5 $[\xi, \eta] = \{(\ x,\ D\eta(x)(\xi(x)) - D\xi(x)(\eta(x))\ ),\ x \in U\}$
4 $\wedge$
5 The Lie bracket of ξ and η is $[\xi, \eta]$

Theorem 19 (commuting flows $\iff$ commuting vector fields)**.**
1 $\forall V$: V a vector space over $\mathbb{C}$ or $\mathbb{R}$
2 $\forall U \subset V$: U is a neighborhood of 0
3 $\forall \xi, \eta \in C^1(U_\mathbb{R}, V)$
5 $[\xi, \eta] = 0$
4 $\iff$
5 $\forall t, s, x$: $(s, x) \in \mathcal{D}_\eta$
7 $\forall s' \in [0, s]$
8 $(t, F_\eta(s', x)) \in \mathcal{D}_\xi$
6 $\implies$
8 $(t, x) \in \mathcal{D}_\xi$ and $(s, F_\xi(t, x)) \in \mathcal{D}_\eta$
7 $\wedge$
8 $F_\xi(t, F_\eta(s, x)) = F_\eta(s, F_\xi(t, x))$

Prosaic form of the statement. Let V be a vector space over $\mathbb{C}$ or $\mathbb{R}$; U a neighborhood of 0; $\xi, \eta \in C^1(U_\mathbb{R}, V)$. Then $[\xi, \eta] = 0$ if and if only if for each $(s, x) \in \mathcal{D}_\eta$ such that for each $s' \in [0, s]$ $(t, F_\eta(s', x)) \in \mathcal{D}_\xi$, then both $(t, x) \in \mathcal{D}_\xi$ and $(s, F_\xi(t, x)) \in \mathcal{D}_\eta$ and $F_\xi(t, F_\eta(s, x)) = F_\eta(s, F_\xi(t, x))$ hold.
Proof.
See [**Lee12**], 9.44.

Proposition 20 (Lie brackets and holomorphic maps)**.**
1 $\forall V$: V a vector space over $\mathbb{C}$
2 $\forall U \subset V$: U is a neighborhood of 0
3 $\forall \xi \in \mathrm{Hol}(U, V)$
4 $\forall \alpha \in \mathbb{C}$
5 $[\alpha\xi, \xi] = 0$

Prosaic form of the statement. Let V a vector space over $\mathbb{C}$; U is a neighborhood of 0; $\xi \in \mathrm{Hol}(U, V)$; $\alpha \in \mathbb{C}$. Then $[\alpha\xi, \xi] = 0$.
In the proof the first 4 lines of the statement of the proposition are taken as assumptions.

Proof.
1 $\forall x \in U$
2 $[\alpha\xi, \xi](x) = D(\alpha\xi)(x)(\xi(x)) - D\xi(x)((\alpha\xi)(x)) = (D\alpha \circ D\xi)(x)(\xi(x)) - \alpha D\xi(x)(\xi(x)) =$
$= \alpha D\xi(x)(\xi(x)) - \alpha D\xi(x)(\xi(x)) = 0$

3. Periodicity

Definition 21 (T-periodic)**.**
1 $\forall V: V$ is a finite dimensional vector space over $\mathbb{C}$ or $\mathbb{R}$
2 $\forall U \subset \mathbb{C}$
3 $\forall \gamma : U \to V$
4 $\forall T \in \mathbb{C}$
6 γ is T-periodic
5 $\iff$:
6 $\forall k \in \mathbb{Z}$
7 $\forall t \in U$: $t + kT \in U$
8 $\gamma(t) = \gamma(t + kT)$

A non-injective real integral curve is periodic.

Proposition 22.
1 $\forall V, U : V$ is a finite dimensional vector space over $\mathbb{C}$ or $\mathbb{R}$ and $U \subset V$ and is open
2 $\forall \xi \in C^1(U_{\mathbb{R}}, V_{\mathbb{R}})$
3 $\forall T \in \mathbb{R}\backslash\{0\}$
4 $\forall x \in U$
6 $(T, x) \in \mathcal{D}_\xi \ \wedge \ F_\xi(T, x) = x$
5 $\implies$
6 $\mathcal{D}_\xi \supset \mathbb{R} \times \{x\} \ \wedge \ \{(t, F_\xi(t, x)) : (t, x) \in \mathcal{D}_\xi\}$ is T-periodic

Prosaic form of the statement. Let V is a finite dimensional vector space over $\mathbb{C}$ or $\mathbb{R}$ and $U \subset V$ and is open; $\xi \in C^1(U_{\mathbb{R}}, V_{\mathbb{R}})$; $T \in \mathbb{R}\backslash\{0\}$ and $x \in U$ such that $(T, x) \in \mathcal{D}_\xi$ and $F_\xi(T, x) = x$ then $\mathcal{D}_\xi \supset \mathbb{R} \times \{x\}$ and $\{(t, F_\xi(t, x)) : (t, x) \in \mathcal{D}_\xi\}$ is T-periodic

In the proof of the statement of the proposition, everything of the statement up to "5 $\implies$ " is taken as assumption.

Proof sketch. First γ is defined as the maximal integral curve that starts at x, and $\tilde{\gamma}$. Then it is shown that $\tilde{\gamma}$ is a T-periodic real integral curve. Since it also has domain $\mathbb{R}$ and γ is 'maximal', they should be equal, whence γ is also T-periodic and has domain $\mathbb{R}$.

Proof.
3 $\gamma := \{(t, F_\xi(t, x)) : (t, x) \in \mathcal{D}_\xi\}$
2 $\wedge$
3 $\tilde{\gamma} := \{(t, F_\xi(t + kT, x)) : t + kT \in [0, T] \ \wedge \ k \in \mathbb{Z} \ \wedge \ t \in \mathbb{R}\}$
1 $\implies$
4 $\tilde{\gamma}$ is a T-periodic real integral curve of ξ
3 $\impliedby$
6 $\tilde{\gamma}$ is a T-periodic real map from $\mathbb{R}$
5 $\impliedby$
6 $\forall t, k, k_1, k_2$: $t \in \mathbb{R} \ \wedge \ k_1, k_2 \in \mathbb{Z}$

9 $t + k_1T \in [0,T] \ \wedge \ t + kT + k_2T \in [0,T]$
8 $\implies$
9 $t + k_1T = t + kT + k_2T \ \vee \ t + k_1T, t + kT + k_2T \in \{0,T\}$
7 $\wedge$
8 $t + k_1T = t + kT + k_2T \implies \gamma(t + k_1T) = \gamma(t + kT + k_2T)$
7 $\wedge$
9 $t + k_1T, t + kT + k_2T \in \{0,T\} \ \wedge \ \gamma(0) = \gamma(T)$
8 $\implies \gamma(t + k_1T) = \gamma(t + kT + k_2T)$
4 $\wedge$
6 $\forall k \in \mathbb{Z}$
7 $\forall t \in [0,T]$
8 $\tilde{\gamma}(t + kT) = \gamma(t)$
5 $\impliedby$
6 $t + kT - (kT) \in [0,T]$
4 $\wedge$
6 $\tilde{\gamma}$ is continuous
5 $\impliedby$
9 $\forall k \in \mathbb{Z}$
10 $\forall t \in [0,T]$
11 $\tilde{\gamma}(t + kT) = \gamma(t)$
8 $\wedge$
9 γ is continuous
7 $\implies$
8 $\forall k \in \mathbb{Z}$
9 $\tilde{\gamma}|_{[Tk,T(k+1)]}$ is continuous
6 $\wedge$
9 $\forall k \in \mathbb{Z}$
10 $\tilde{\gamma}|_{[Tk,T(k+1)]}$ is continuous $\wedge$ $[Tk, T(k+1)]$ is closed
8 $\wedge$
9 $\mathbb{R} = \bigcup_{k\in\mathbb{Z}}[Tk, T(k+1)]$
7 $\implies$
8 $\tilde{\gamma}$ is continuous
4 $\wedge$
6 $\tilde{\gamma}$ is differentiable and a real integral curve of ξ
5 $\impliedby$
9 $\forall k \in \mathbb{Z}$
10 $\forall t \in [0,T]$
11 $\tilde{\gamma}(t + kT) = \gamma(t)$
8 $\wedge$
9 γ is C^1 and $D\gamma = \xi \circ \gamma$
7 $\implies$
8 $\forall k \in \mathbb{Z}$
9 $\tilde{\gamma}|_{(Tk,T(k+1))}$ is C^1 and $D\tilde{\gamma}|_{(Tk,T(k+1))} = \xi \circ \tilde{\gamma}|_{(Tk,T(k+1))}$

6 $\wedge$
9 $\forall k \in \mathbb{Z}$
10 $\tilde{\gamma}|_{(Tk,T(k+1))}$ is C^1 and $D\tilde{\gamma}|_{(Tk,T(k+1))} = \xi \circ \tilde{\gamma}|_{(Tk,T(k+1))}$
8 $\wedge$
9 ξ and $\tilde{\gamma}$ are continuous
7 $\implies$
8 $\tilde{\gamma}$ is C^1 and $D\tilde{\gamma} = \xi \circ \tilde{\gamma}$
2 $\wedge$
4 $\wedge_*$
5 $\gamma(0) = x = \tilde{\gamma}(0)$
5 γ and $\tilde{\gamma}$ are real integral curves ξ
5 uniqueness of integral curves of ODEs
4 $\wedge^*$
3 $\implies$
4 $\gamma|_{\mathrm{dom}(\gamma)\cap\mathrm{dom}(\tilde{\gamma})} = \tilde{\gamma}|_{\mathrm{dom}(\gamma)\cap\mathrm{dom}(\tilde{\gamma})}$
2 $\wedge$
4 $\mathrm{dom}(\tilde{\gamma}) = \mathbb{R} \;\wedge\; \mathrm{dom}(\gamma) \subset \mathbb{R}$
3 $\implies$
4 $\mathrm{dom}(\gamma) \cap \mathrm{dom}(\tilde{\gamma}) = \mathrm{dom}(\gamma)$
2 $\wedge$
4 $\gamma|_{\mathrm{dom}(\gamma)\cap\mathrm{dom}(\tilde{\gamma})} = \tilde{\gamma}|_{\mathrm{dom}(\gamma)\cap\mathrm{dom}(\tilde{\gamma})} \;\wedge\; \mathrm{dom}(\gamma) \cap \mathrm{dom}(\tilde{\gamma}) = \mathrm{dom}(\gamma)$
3 $\implies$
4 $\tilde{\gamma} \supset \gamma$
2 $\wedge$
4 γ is a maximal real integral curve of ξ $\wedge$ $\tilde{\gamma}$ is a real integral curve of ξ
3 $\implies$
$\neg$ *is used to denote negation*
4 $\neg\ \tilde{\gamma} \supsetneq \gamma$
2 $\wedge$
4 $\tilde{\gamma} \supset \gamma \;\wedge\; \neg\ \tilde{\gamma} \supsetneq \gamma$
3 $\implies$
4 $\tilde{\gamma} = \gamma$
2 $\wedge$
4 $\tilde{\gamma}$ is T-periodic $\wedge$ $\mathrm{dom}(\tilde{\gamma}) = \mathbb{R} \;\wedge\; \tilde{\gamma} = \gamma$
3 $\implies$
4 γ is T-periodic $\wedge$ $\mathrm{dom}(\gamma) = \mathbb{R}$

Proposition 23 (pullback of periodic function)**.**
1 $\forall V$: V is a finite dimensional vector space over $\mathbb{C}$
2 $\forall U \subset \mathbb{C}$: open
3 $\forall T \in \mathbb{C}\backslash\{0\}$
4 $\forall f \in \mathrm{Hol}(U,V)$: f is T-periodic
5 $\tilde{f} := \{(z, f(w)) : w \in U \;\wedge\; z = \exp(\frac{2\pi i w}{T})\} \implies$
7 $\tilde{f} \in \mathrm{Hol}(\exp(U\frac{2\pi i}{T}), V)$

6 $\wedge$
7 $\forall w \in U$
8 $z := \exp(\frac{2\pi i w}{T}) \implies Df(w) = \frac{2\pi i z}{T} D\tilde{f}(z)$

Prosaic form of the statement. Let V is a finite dimensional vector space over $\mathbb{C}$; $U \subset \mathbb{C}$ open; $T \in \mathbb{C}\backslash\{0\}$; $f \in \mathrm{Hol}(U,V)$ such that f is T-periodic, and let $\tilde{f} := \{(z, f(w)) : w \in U \ \wedge \ z = \exp(\frac{2\pi i w}{T})\}$. Then $\tilde{f} \in \mathrm{Hol}(\exp(U\frac{2\pi i}{T}), V)$ and $(\forall w \in U)(z := \exp(\frac{2\pi i w}{T}) \implies Df(w) = \frac{2\pi i z}{T} D\tilde{f}(z))$.

In the proof of the statement of the proposition, the first 5 lines are taken as assumptions.

Proof sketch. Up to "1 $\Longleftarrow$ " the steps in which the proof is done are written down, namely, showing $\tilde{f}$ is a map, them that this implies it is a holomorphic map, and that from this follows that the derivative identity - at the last two lines of the proposition - follows. After "1 $\Longleftarrow$ ", these three steps are shown to be true.

Proof.
3 $\tilde{f}$ is a map from $\exp(U\frac{2\pi i}{T})$
2 $\wedge$
3 $\tilde{f}$ is a map from $\exp(U\frac{2\pi i}{T}) \implies \tilde{f} \in \mathrm{Hol}(\exp(U\frac{2\pi i}{T}), V)$
2 $\wedge$
3 $\tilde{f} \in \mathrm{Hol}(\exp(U\frac{2\pi i}{T}), V) \implies$
4 $\forall w \in U$
5 $z := \exp(\frac{2\pi i w}{T}) \implies Df(w) = \frac{2\pi i z}{T} D\tilde{f}(z)$
1 $\Longleftarrow$
4 $\tilde{f}$ is a map from $\exp(U\frac{2\pi i}{T})$
3 $\Longleftarrow$
4 $\forall z \in \exp(U\frac{2\pi i}{T})$
5 $\forall v_1, v_2 \in V$
6 $(z, v_1), (z, v_2) \in \tilde{f} \implies v_1 = v_2$
3 $\Longleftarrow$
4 $\forall z \in \exp(U\frac{2\pi i}{T})$
5 $\forall v_1, v_2 \in V$
6 $(z, v_1), (z, v_2) \in \tilde{f} \implies$
8 $\exists w_1, w_2 \in U$: $z = \exp(\frac{2\pi i w_1}{T}) \ \wedge \ z = \exp(\frac{2\pi i w_2}{T})$
9 $v_1 = f(w_1) \ \wedge \ v_2 = f(w_2)$
7 $\wedge$
8 $\forall w_1, w_2$: as such
10 $\exp(\frac{2\pi i w_1}{T}) = \exp(\frac{2\pi i w_2}{T}) \implies$
11 $\exists k \in \mathbb{Z}$: $\frac{2\pi i w_1}{T} + k \cdot 2\pi i = \frac{2\pi i w_2}{T}$
9 $\wedge$
10 $\forall k$: as such
12 $w_1 + kT = w_2$

11 $\wedge$
13 $w_1 + kT = w_2 \;\wedge\; f$ is T-periodic
12 $\implies$
13 $f(w_1) = f(w_2)$
12 $\implies$
13 $v_1 = v_2$
2 $\wedge$
4 $\tilde{f}$ is a map from $\exp(U\frac{2\pi i}{T}) \implies \tilde{f} \in \mathrm{Hol}(\exp(U\frac{2\pi i}{T}), V)$
3 $\Longleftarrow$
4 $\forall z \in \exp(\frac{2\pi i w}{T})$
5 $\exists U_1 \subset \exp(\frac{2\pi i w}{T})$: U_1 is open in $\mathbb{C} \;\wedge\; z \in U_1$
6 $\tilde{f} \in \mathrm{Hol}(U_1, V)$
3 $\Longleftarrow$
4 $\forall z \in \exp(\frac{2\pi i w}{T})$
5 $\forall w \in U : z = \exp(\frac{2\pi i w}{T})$
8 $D\{(u, \exp(\frac{2\pi i u}{T})) : u \in U\}(w) \neq 0 \wedge$ inverse function theorem for the holomorphic case
7 $\implies$
8 $\exists U_2 \subset U$: open in $\mathbb{C} \;\wedge\; w \in U_2$
9 $\exp(\frac{2\pi i}{T}U_2)$ is open $\;\wedge\; \{(u, \exp(\frac{2\pi i u}{T})) : u \in U_2\}$ has a holomorphic inverse (w.r.t. its image)
6 $\wedge$
7 $\forall U_2$: as such
9 $\tilde{f}|_{\exp(\frac{2\pi i}{T}U_2)} = f \circ \{(u, \exp(\frac{2\pi i u}{T})) : u \in U_2\}^{-1} \;\wedge\; f \circ (\exp|_{U_2})^{-1} \in \mathrm{Hol}(U_2, V)$
10 $\implies \tilde{f}|_{\exp(\frac{2\pi i}{T}U_2)} \in \mathrm{Hol}(\exp(\frac{2\pi i}{T}U_2), V)$
8 $\wedge$
9 $w \in U_2 \implies z \in \exp(\frac{2\pi i}{T}U_2)$
2 $\wedge$
4 $\tilde{f} \in \mathrm{Hol}(\exp(U\frac{2\pi i}{T}), V) \implies$
5 $\forall w \in U$
6 $z := \exp(\frac{2\pi i w}{T}) \implies Df(w) = \frac{2\pi i z}{T} D\tilde{f}(z)$
3 $\Longleftarrow$
5 $\tilde{f} \circ \exp|_{\exp(U\frac{2\pi i}{T})} = f$
4 $\wedge$
6 $\tilde{f} \circ \exp|_{\exp(U\frac{2\pi i}{T})} = f \;\wedge\; \tilde{f} \in \mathrm{Hol}(\exp(U\frac{2\pi i}{T}), V)$
5 $\implies$
6 $\{(w, \exp(\frac{2\pi i w}{T}) \cdot D\tilde{f}(\exp(\frac{2\pi i w}{T}))) : w \in U\} = \{(w, Df(w)) : w \in U\}$

4. Flow of a linear vector field

The definition below gives a shorthand notation for stating whether a vector field is near some equilibrium equivalent to its linear part - in some specific way.

Definition 24.
1 $\forall V, U : V$ is a finite dimensional vector space over $\mathbb{R}$ and U a neighborhood of 0
2 $\forall \xi \in C^\infty(U, V) : \xi(0) = 0$
4 $P_{24}(V, U, \xi)$
3 $:\iff$
4 $\exists U_1, U_2 \subset U$: neighborhoods of 0
5 $\exists \varphi \in \mathrm{Diff}^1(U_1, U_2)$: $\varphi(0) = 0$ and $D\varphi(0) = \mathrm{id}_V$
6 $\forall a, b \in [-\infty, \infty]$: $a < b$
7 $\forall \gamma \in C^1((a, b), U_1)$: $D\gamma = D\xi(0) \circ \gamma$
8 $D(\varphi \circ \gamma) = \xi \circ \varphi \circ \gamma$

The main and most non-elementary tool we use, is the differentiable version of the Grobman-Hartman theorem.

Theorem 25 (Differentiable version Grobman-Hartman theorem for vector fields)**.**
1 $\forall V, U : V$ is a finite dimensional vector space over $\mathbb{R}$ and U a neighborhood of 0
2 $\forall \xi \in C^\infty(U, V) : \xi(0) = 0$
4 Each eigenvalue of $\mathrm{id}_\mathbb{C} \otimes_\mathbb{R} D\xi(0)$ has nonzero real part (i.e. ξ is hyperbolic at 0)
3 $\implies$
4 $P_{24}(V, U, \xi)$

Prosaic form of the statement. Let V is a finite dimensional vector space over $\mathbb{R}$ and U a neighborhood of 0; $\xi \in C^\infty(U, V)$ such that $\xi(0) = 0$. If each eigenvalue of $\mathrm{id}_\mathbb{C} \otimes_\mathbb{R} D\xi(0)$ has nonzero real part (i.e. ξ is hyperbolic at 0) then $P_{24}(V, U, \xi)$ holds.

Proof sketch.
In [**GHR03**], it is proven that a C^∞-map on a vector space that has a hyperbolic fixed point is C^1 equivalent to its linear part. One can use the existence of this 'conjugating' diffeomorphism to produce φ, in exactly the same way as is done in [**PM82**]. Here, it is done for the topological (not differentiable) case, but in this process (i.e. going from the case fixed points of maps to case the equilibria of vector fields) differentiability is preserved.

This definition gives us a concise way to choose generalized eigenvectors for a linear endomorphism.

Definition 26 (eigensystem)**.**
1 $\forall V$: a finite dimensional vector space over $\mathbb{C}$
2 $\forall A \in \mathrm{L}(V, V)$
3 $\forall \lambda_E, E, v_E, v_E^*$
5 $(\lambda_E, E, v_E, v_E^*)$ is an eigensystem for A

4 : $\iff$
5 $\wedge_*$
6 $\lambda_E = \{(W, \text{the eigenvalue of } A|_W)\colon W$ a generalized eigenspace of $A\}$
6 $E = \{(W, n)\colon W$ a generalized eigenspace of A, $n \in \mathbb{N}_{\geq 1}$ and $n \leq \dim(W)\}$
6 $v_E : E \to$ generalized eigenvectors of A
6 $\forall W, m : (W, m) \in E$
8 $Av_E(W, 1) = \lambda_E(W)v_E(W, 1)$
7 $\wedge$
8 $Av_E(W, m+1) = \lambda_E(W)v_E(W, m+1) + v_E(W, m)$
6 $v_E^* : E \to$ generalized eigenvectors of A
6 $\forall W, W', m, m' : (W, m), (W', m') \in E$
8 $(W, m) \neq (W', m') \implies v_E^*(W, n)(v(W', n')) = 0$
7 $\wedge$
8 $(W, m) = (W', m') \implies v_E^*(W, n)(v(W', n')) = 1$
5 $\wedge^*$

Proposition 27 (existence eigensystem)**.**
1 $\forall V$: a finite dimensional vector space over $\mathbb{C}$
2 $\forall A \in \mathrm{L}(V, V)$
3 $\exists \lambda_E, E, v_E, v_E^*$
4 $(\lambda_E, E, v_E, v_E^*)$ is an eigensystem for A

Prosaic form of the statement. Let V a finite dimensional vector space over $\mathbb{C}$; $A \in \mathrm{L}(V, V)$; then there exists an eigensystem $(\lambda_E, E, v_E, v_E^*)$ for A.

In the proof, the first 2 lines are taken as assumptions.

Proof sketch. E are λ_E defined in the only way possible. For the existence of v_E, Jordan normal form theorem is used. By this same theorem, the image of such a v_E forms a basis for V, from which the existence of v_E^* follows.

Proof.
3 $E := \{(W, n)\colon W$ a generalized eigenspace of A, $n \in \mathbb{N}_{\geq 1}$ and $n \leq \dim(W)\}$
2 $\wedge$
3 $\lambda_E := \{(W, \text{the eigenvalue of } A|_W)\colon W$ a generalized eigenspace of $A\}$
1 $\implies$
4 Jordan normal form theorem
3 $\implies$
4 $\exists v_E : E \to$ generalized eigenvectors of A
4 $\forall W, m : (W, m) \in E$
6 $Av_E(W, 1) = \lambda_E(W)v_E(W, 1)$
5 $\wedge$
6 $Av_E(W, m+1) = \lambda_E(W)v_E(W, m+1) + v_E(W, m)$
2 $\wedge$

4 Jordan normal form theorem
3 $\implies$
4 $\forall v_E$: as such
5 $\{v_E(W,m)\}_{(W,m)\in E}$ is a basis of V
3 $\implies$
4 $\forall v_E$: as such
5 $\exists v_E^* : E \to$ generalized eigenvectors of A
6 $\forall W, W', m, m' : (W,m), (W',m') \in E$
8 $(W,m) \neq (W',m') \implies v_E^*(W,n)(v(W',n')) = 0$
7 $\wedge$
8 $(W,m) = (W',m') \implies v_E^*(W,n)(v(W',n')) = 1$

The following lemma gives an expression for solutions of linear ODEs with constant coefficients in terms of generalized eigenvectors, exponentials and polynomials. It was harder to find a reference, than to figure out the proof itself.[1]

Lemma 28 (solutions of linear system of differential equations with constant coefficients)**.**
1 $\forall V$: a finite dimensional vector space over $\mathbb{C}$
2 $\forall A \in \mathrm{L}(V,V)$
3 $\forall I$: an open interval of $\mathbb{R}$
4 $\forall \gamma : I \to V$: $D\gamma = A \circ \gamma$
5 $\forall \lambda_E, E, v_E, v_E^*$: $(\lambda_E, E, v_E, v_E^*)$ is an eigensystem for A
6 $\gamma \in \mathrm{span}_{\mathbb{C}}\{\{t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!} : t \in I\}, (W,n) \in E\}$

Prosaic form of the statement. V is a finite dimensional vector space over $\mathbb{C}$; $A \in \mathrm{L}(V,V)$; I an open interval of $\mathbb{R}$; $\gamma : I \to V$ such that $D\gamma = A \circ \gamma$; $(\lambda_E, E, v_E, v_E^*)$ is an eigensystem for A. Then $\gamma \in \mathrm{span}_{\mathbb{C}}\{\{t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!} : t \in I\}, (W,n) \in E\}$.

In the proof of the statement of the lemma, the first 5 lines are used as assumptions.

Proof sketch. The proof consists of three parts. In the proof, these are the arguments of the two 1$\wedge$-connectives. The first can be seen as a sufficient condition for being a real integral curve of the respective ODE. The second part shows that the real integral curves found in this way span V for every instant. The second part uses the first part. The third and last part uses the two previous parts and uniqueness of solutions of ODEs and thus that the 'sufficient condition' is also necessary.

Proof.
3 $\forall \tilde{\gamma} \in \mathrm{span}_{\mathbb{C}}\{\{(t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!}) : t \in \mathbb{R}\} : (W,n) \in E\}$
4 $D\tilde{\gamma} = A \circ \tilde{\gamma}$
2 $\impliedby$
5 $\forall W, n$: $(W,n) \in E$

[1]Allegedly, the reader can find also a proof in the first edition of Smale's and Hirsch's *Differential equations, dynamical systems, and an introduction to chaos.*

6 $\forall \tilde{\gamma} \in \{\{(t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!}) : t \in \mathbb{R}\} : (W,n) \in E\}$
7 $v^*(W,n)(D\tilde{\gamma}) = v^*(W,n)(A \circ \tilde{\gamma})$
4 $\Longleftarrow$
5 $\forall W_1, n_1, W_2, n_2$: $(W_1,n_1),(W_2,n_2) \in E$
7 $(W_1 \neq W_2 \vee n_1 > n_2) \implies$
9 $v^*(W_1,n_1)(D\{(t, \sum_{n'=1}^{n_2} e^{\lambda(W_2)t} v(W_2,n') \frac{t^{n_2-n'}}{(n_2-n')!}) : t \in \mathbb{R}\}) = 0$
8 $\wedge$
9 $v^*(W_1,n_1)(A\{(t, \sum_{n'=1}^{n_2} e^{\lambda(W_2)t} v(W_2,n') \frac{t^{n_2-n'}}{(n_2-n')!}) : t \in \mathbb{R}\}) = 0$
6 $\wedge$
7 $(W_1 = W_2 \wedge n_1 \leq n_2) \implies$
9 $v^*(W_1,n_1)D\{(t, \sum_{n'=1}^{n_2} e^{\lambda(W_2)t} v(W_2,n') \frac{t^{n_2-n'}}{(n_2-n')!}) : t \in \mathbb{R}\} = D\{(t, e^{\lambda(W_1)t} \frac{t^{n_2-n_1}}{(n_2-n_1)!}) : t \in \mathbb{R}\} =$

$$\begin{cases} n_1 = n_2 : & \lambda(W_2) e^{\lambda(W_2)t} \\ n_1 \neq n_2 : & \lambda(W_2) e^{\lambda(W_2)t} \frac{t^{n_2-n_1}}{(n_2-n_1)!} + e^{\lambda(W_2)t} \frac{t^{n_2-n_1-1}}{(n_2-n_1-1)!} \end{cases}$$

8 $\wedge$
9 $v^*(W_1,n_1)(A\{(t, \sum_{n'=1}^{n_2} e^{\lambda(W_2)t} v(W_2,n') \frac{t^{n_2-n'}}{(n_2-n')!}) : t \in \mathbb{R}\}) =$
$v^*(W_1,n_1)(\{(t,$
$\sum_{n'=1}^{n_2} \lambda(W_2) e^{\lambda(W_2)t} v(W_2,n') \frac{t^{n_2-n'}}{(n_2-n')!} + \sum_{n'=2}^{n_2} e^{\lambda(W_2)t} v(W_2,n'-1) \frac{t^{n_2-n'}}{(n_2-n')!}) \ : t \in \mathbb{R}\}) =$

$$\begin{cases} n_1 = n_2 : & \lambda(W_2) e^{\lambda(W_2)t} \\ n_1 \neq n_2 : & \lambda(W_2) e^{\lambda(W_2)t} \frac{t^{n_2-n_1}}{(n_2-n_1)!} + e^{\lambda(W_2)t} \frac{t^{n_2-n_1-1}}{(n_2-n_1-1)!} \end{cases}$$

3 $\wedge$
6 $\forall W, n$: $(W,n) \in E$
7 $\forall \tilde{\gamma} \in \{\{(t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!}) : t \in \mathbb{R}\} : (W,n) \in E\}$
8 $v^*(W,n)(D\tilde{\gamma}) = v^*(W,n)(A \circ \tilde{\gamma})$
5 $\wedge$
6 $\{v^*(W,n) : (W,n) \in E\}$ is a basis of V^* $\Longleftarrow$ Jordan normal form theorem
4 $\implies$
5 $\forall \tilde{\gamma} \in \{\{(t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!}) : t \in \mathbb{R}\} : (W,n) \in E\}$
6 $D\tilde{\gamma} = A \circ \tilde{\gamma}$
3 $\wedge$
6 $\forall \tilde{\gamma} \in \{\{(t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!}) : t \in \mathbb{R}\} : (W,n) \in E\}$
7 $D\tilde{\gamma} = A \circ \tilde{\gamma}$
5 $\wedge$
6 the set of solutions is a vector space over $\mathbb{C}$ $\Longleftarrow$ the ODE is $\mathbb{C}$-linear
4 $\implies$
5 $\forall \tilde{\gamma} \in \operatorname{span}_{\mathbb{C}}\{\{(t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!}) : t \in \mathbb{R}\} : (W,n) \in E\}$
6 $D\tilde{\gamma} = A \circ \tilde{\gamma}$
1 $\wedge$
3 $\forall t_0 \in \mathbb{R}$

4 $V = \mathrm{span}_{\mathbb{C}}\{\sum_{n'=1}^{n} e^{\lambda(W)t_0} v(W,n') \frac{t_0^{n-n'}}{(n-n')!} : (W,n) \in E\}$
2 $\Longleftarrow$
3 $S := \{\{(t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!}) : t \in \mathbb{R}\},$
W a generalized eigenspace of A, $n \in \mathbb{N}_{\geq 1}$ and $n \leq \dim(W)\} \implies$
4 $\forall t_0 \in I$
6 $|\{\gamma_S(t_0) : \gamma_S \in S\}| = \dim V$
5 $\wedge$
6 $\{\gamma_S(t_0) : \gamma_S \in S\}$ is linear independent $\Longleftarrow$
7 $\forall c : S \to \mathbb{C}$: $\sum_{\gamma_S \in S} c(\gamma_S)\gamma_S(t_0) = 0$
10 $\wedge_*$
11 $\sum_{\gamma_S \in S} c(\gamma_S)\gamma_S(t_0) = 0$
11 0 is an equilibrium for A as vector field
11 $\sum_{\gamma_S \in S} c(\gamma_S)\gamma_S$ is a real integral curve of A as vector field
10 $\wedge^*$
9 $\implies$
10 $\sum_{\gamma_S \in S} c(\gamma_S)\gamma_S = 0$
9 $\implies$
10 $\sum_{\gamma_S \in S} c(\gamma_S)\gamma_S(0) = 0$
8 $\wedge$
10 $\sum_{\gamma_S \in S} c(\gamma_S)\gamma_S(0) = 0 \;\; \wedge \;\; \{\gamma_S(0) : \gamma_S \in S\}$ is a set of independent generalized eigenvectors
9 $\implies$
10 $\forall \gamma_S \in S$: $c(\gamma_S) = 0$
1 $\wedge$
4 $\forall \tilde{\gamma} \in \mathrm{span}_{\mathbb{C}}\{\{(t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!}) : t \in \mathbb{R}\} : (W,n) \in E\}$
5 $D\tilde{\gamma} = A \circ \tilde{\gamma}$
3 $\wedge$
4 $\forall t_0 \in \mathbb{R}$
5 $V = \mathrm{span}_{\mathbb{C}}\{\sum_{n'=1}^{n} e^{\lambda(W)t_0} v(W,n') \frac{t_0^{n-n'}}{(n-n')!} : (W,n) \in E\}$
2 $\implies$
3 $\forall t_0 \in I$
4 $\exists \tilde{\gamma} \in \mathrm{span}_{\mathbb{C}}\{\{(t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!}) : t \in \mathbb{R}\} : (W,n) \in E\}$
5 $\tilde{\gamma}(t_0) = \gamma(t_0) \;\; \wedge \;\; D\tilde{\gamma} = A \circ \tilde{\gamma}$
1 $\wedge$
4 $\forall t_0 \in I$
5 $\exists \tilde{\gamma} \in \mathrm{span}_{\mathbb{C}}\{\{(t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W,n') \frac{t^{n-n'}}{(n-n')!}) : t \in \mathbb{R}\} : (W,n) \in E\}$
6 $\tilde{\gamma}(t_0) = \gamma(t_0) \;\; \wedge \;\; D\tilde{\gamma} = A \circ \tilde{\gamma}$
3 $\wedge$
4 γ is a real integral curve of $D\tilde{\gamma} = A \circ \tilde{\gamma}$
3 $\wedge$
4 uniqueness of solutions of ODEs
2 $\implies$

3 $\exists\tilde{\gamma} \in \text{span}_{\mathbb{C}}\{\{(t, \sum_{n'=1}^{n} e^{\lambda(W)t} v(W, n') \frac{t^{n-n'}}{(n-n')!}) : t \in \mathbb{R}\} : (W, n) \in E\}$
4 $\tilde{\gamma}|_I = \gamma$

One could describe the following proposition in fancy way as 'taking (space) derivative and taking flow of a vector vector field commute'.

Proposition 29.
1 $\forall V$: a vector space over $\mathbb{C}$ or $\mathbb{R}$
2 $\forall U \subset V$: a neighborhood of 0
3 $\forall \xi \in C^1(U_{\mathbb{R}}, V_{\mathbb{R}})$: $\xi(0) = 0$
4 $\forall t \in \mathbb{R}$
6 $D_2F_\xi(t, 0)$ and $F_{D\xi(0)}(t, -)$ are defined
5 $\wedge$
6 $D_2F_\xi(t, 0) = F_{D\xi(0)}(t, -)$

Prosaic form of the statement. Let V a vector space over $\mathbb{C}$ or $\mathbb{R}$; U a neighborhood of 0 in V; $\xi \in C^1(U_{\mathbb{R}}, V_{\mathbb{R}})$ such that $\xi(0) = 0$. Then for each $t \in \mathbb{R}$ $D_2F_\xi(t, 0)$ and $F_{D\xi(0)}(t, -)$ are defined and $D_2F_\xi(t, 0) = F_{D\xi(0)}(t, -)$.

In the proof, the first three lines of the statement of the proposition are used as assumptions.

Proof sketch. First the statement about the domain is proved using the previous proposition. The tools used for proving the equality are commutation of derivatives and the fundamental theorem of calculus.

Proof.
3 proposition 28
2 $\implies$
3 $\forall t \in \mathbb{R}$
4 $F_{D\xi(0)}(t, -)$ is defined
1 $\wedge$
2 proposition 15 $\implies$ $\mathcal{D}_\xi$ is open
1 $\wedge$
3 $\mathbb{R} \times \{0\} \subset \mathcal{D}_\xi$ and $\mathcal{D}_\xi$ is open
2 $\implies$
3 $\forall t \in \mathbb{R}$
4 $D_2F_\xi(t, 0)$ is defined
1 $\wedge$
3 $D_2F_\xi(t, 0) = F_{D\xi(0)}(t, -)$
2 $\impliedby$
3 $\wedge_*$
4 $\forall t \in \mathbb{R}$
5 $F_{D\xi(0)}(t, -) = F_{D\xi(0)}(0, -) + \int_0^t D_1F_{D\xi(0)}(t', -)dt'$ and $D_2F_\xi(t, 0) = D_2F_\xi(0, 0) + \int_0^t D_1D_2F_\xi(t', 0)dt'$

4 $F_{D\xi(0)}(0,-) = D_2F_\xi(0,0)$
4 $\forall t \in \mathbb{R}$
5 $D_1F_{D\xi(0)}(t',-) = D_1D_2F_\xi(t',0)$
3 $\wedge^*$
2 $\Longleftarrow$
3 $\wedge_*$
5 $\forall t \in \mathbb{R}$
6 $F_{D\xi(0)}(t,-) = F_{D\xi(0)}(0,-)+\int_0^t D_1F_{D\xi(0)}(t',-)dt'$ and $D_2F_\xi(t,0) = D_2F_\xi(0,0)+\int_0^t D_1D_2F_\xi(t',0)dt'$
4 $\Longleftarrow$
5 fundamental theorem of calculus
5 $F_{D\xi(0)}(0,-) = D_2F_\xi(0,0)$
4 $\Longleftarrow$
6 definition 16 $\implies F_{D\xi(0)}(0,0) = \mathrm{id}_V$
5 $\wedge$
6 definition 16 $\implies (\forall x \in U)(F_\xi(0,x) = x) \implies D_2F_\xi(0,0) = \mathrm{id}_V$
5 $\forall t \in \mathbb{R}$
6 $D_1F_{D\xi(0)}(t,-) = D_1D_2F_\xi(t,0)$
4 $\Longleftarrow$
5 $\forall t \in \mathbb{R}$
7 $D_1F_{D\xi(0)}(t,-) = D\xi(0)$
6 $\wedge$
7 $D_1D_2F_\xi(t,0) = D_2D_1F_\xi(t,0) = D_2\{((t,x),\xi(x)),(t,x) \in \mathcal{D}_\xi\}(t,0) = D\xi(0)$
3 $\wedge^*$

This notation is convenient in the proof of the proposition after it.

Definition 30 (im and re on $\mathbb{C} \otimes_\mathbb{R} V$)**.**
1 $\forall V$: a vector space over $\mathbb{C}$ or $\mathbb{R}$
3 $\mathrm{im}, \mathrm{re} \in \mathrm{L}((\mathbb{C} \otimes_\mathbb{R} V)_\mathbb{R}, V_\mathbb{R})$
2 $\wedge$
3 $\forall \alpha \in \mathbb{C}$
4 $\forall v \in V$
5 $\mathrm{im}(\alpha \otimes v) = \mathrm{im}(\alpha)v$ and $\mathrm{re}(\alpha \otimes v) = \mathrm{re}(\alpha)v$

When a complex linear map is tensored over $\mathbb{C}$ with the identity, the spectrum does not change. However, when the tensoring is done over $\mathbb{R}$ it can change, and this proposition gives a restriction on this change.

Proposition 31.
1 $\forall V$: a vector space over $\mathbb{C}$
2 $\forall A \in \mathrm{L}(V,V)$
3 $\forall \lambda \in \mathbb{C}$
5 $\ker(\mathrm{id}_\mathbb{C} \otimes_\mathbb{R} A - \lambda \otimes_\mathbb{R} \mathrm{id}_V) \neq \{0\}$
4 $\implies$
5 $\ker(A - \lambda\ \mathrm{id}_V) \neq \{0\}$ or $\ker(A - \bar{\lambda}\ \mathrm{id}_V) \neq \{0\}$

Prosaic form of the statement. Let V be a vector space over $\mathbb{C}$; $A \in \mathrm{L}(V, V)$ such that $\ker(\mathrm{id}_{\mathbb{C}} \otimes_{\mathbb{R}} A - \lambda \otimes_{\mathbb{R}} \mathrm{id}_V) \neq \{0\}$. Then $\ker(A - \lambda\ \mathrm{id}_V) \neq \{0\}$ or $\ker(A - \bar{\lambda}\ \mathrm{id}_V) \neq \{0\}$.

In the proof, the first four lines are taken as assumptions.

Proof sketch.
For each $v \in \ker(\mathrm{id}_{\mathbb{C}} \otimes_{\mathbb{R}} A - \lambda \otimes_{\mathbb{R}} \mathrm{id}_V) \backslash \{0\}$ it shown that this gives rise to an element in $\ker(A - \lambda\ \mathrm{id}_V) \neq \{0\}$ or $\ker(A - \bar{\lambda}\ \mathrm{id}_V) \neq \{0\}$. First some equations are proven. With this, the cases that $\{\mathrm{re}(v), \mathrm{im}(v)\}$ is independent and dependent are treated separately. In the dependent case, the proof branches in four different cases in total.

Proof.
1 $\forall v \in \ker(\mathrm{id}_{\mathbb{C}} \otimes_{\mathbb{R}} A - \lambda \otimes_{\mathbb{R}} \mathrm{id}_V) \backslash \{0\}$
2 $\forall u, w \in V\colon\ \mathrm{re}(v) = u\ \wedge\ \mathrm{im}(v) = w$
5 $P(\text{equations})$
4 $: \Longleftrightarrow$
5 $Au = \mathrm{re}(\lambda)u - \mathrm{im}(\lambda)w\ \wedge\ Aw = \mathrm{im}(\lambda)u + \mathrm{re}(\lambda)w$
3 $\Longrightarrow$
6 $P(\text{equations})$
5 $\Longleftarrow$
7 $(\mathrm{id}_{\mathbb{C}} \otimes_{\mathbb{R}} A - \lambda \otimes_{\mathbb{R}} \mathrm{id}_V)v = 0$
6 $\Longrightarrow$
7 $\mathrm{im}((\mathrm{id}_{\mathbb{C}} \otimes_{\mathbb{R}} A - \lambda \otimes_{\mathbb{R}} \mathrm{id}_V)v) = 0$ and $\mathrm{re}((\mathrm{id}_{\mathbb{C}} \otimes_{\mathbb{R}} A - \lambda \otimes_{\mathbb{R}} \mathrm{id}_V)v) = 0$
6 $\Longrightarrow$
7 $Au = \mathrm{re}(\lambda)u - \mathrm{im}(\lambda)w\ \wedge\ Aw = \mathrm{im}(\lambda)u + \mathrm{re}(\lambda)w$
4 $\wedge$
5 $P(\text{equations}) \Longrightarrow$
8 $\{u, w\}$ linear dependent $\Longrightarrow \ker(A - \lambda\ \mathrm{id}_V) \neq \{0\}$ or $\ker(A - \bar{\lambda}\ \mathrm{id}_V) \neq \{0\}$
7 $\wedge$
8 $\{u, w\}$ linear independent $\Longrightarrow \ker(A - \lambda\ \mathrm{id}_V) \neq \{0\}$
6 $\Longleftarrow$
9 $\{u, w\}$ linear dependent $\Longrightarrow \ker(A - \lambda\ \mathrm{id}_V) \neq \{0\}$ or $\ker(A - \bar{\lambda}\ \mathrm{id}_V) \neq \{0\}$
8 $\Longleftarrow$
10 $w = 0 \vee w \neq 0$
9 $\wedge$
10 $w = 0 \Longrightarrow$
13 $v \neq 0\ \wedge\ w = 0$
12 $\Longrightarrow$
13 $u \neq 0$
11 $\wedge$
12 $P(\text{equations}) \Longrightarrow$
13 $Au = \mathrm{re}(\lambda)u\ \wedge\ 0 = \mathrm{im}(\lambda)u$
11 $\wedge$

13 $Au = \mathrm{re}(\lambda)u \ \wedge \ 0 = \mathrm{im}(\lambda)u \ \wedge \ u \neq 0$
12 $\implies$
13 $Au = \lambda u \ \wedge \ u \neq 0$
9 $\wedge$
10 $w \neq 0 \implies$
12 $u = iw \ \vee \ u = -iw \ \vee \ \mathrm{im}(\lambda) = 0$
11 $\wedge$
13 $u = iw \ \vee \ u = -iw \ \vee \ \mathrm{im}(\lambda) = 0$
12 $\impliedby$
15 $\{u, w\}$ linear dependent $\wedge \ w \neq 0$
14 $\implies$
15 $\exists \beta \in \mathbb{C}\colon \ u = \beta w$
13 $\wedge$
14 $\forall \beta \in \mathbb{C}\colon \ u = \beta w$
16 $P(\text{equations})$
15 $\implies$
16 $\beta A w = (\beta \mathrm{re}(\lambda) - \mathrm{im}(\lambda))w \ \wedge \ \beta A w = (\beta^2 \mathrm{im}(\lambda) + \beta \mathrm{re}(\lambda))w$
15 $\implies$
16 $\beta \mathrm{re}(\lambda) - \mathrm{im}(\lambda) = \beta^2 \mathrm{im}(\lambda) + \beta \mathrm{re}(\lambda)$
15 $\implies$
16 $\beta = i \ \vee \ \beta = -i \ \vee \ \mathrm{im}(\lambda) = 0$
11 $\wedge$
13 $u = iw \ \wedge \ P(\text{equations})$
12 $\implies$
13 $Aw = \lambda w$
12 $\implies$
13 $w \in \ker(A - \lambda \ \mathrm{id}_V) \backslash \{0\}$
11 $\wedge$
13 $u = -iw \ \wedge \ P(\text{equations})$
12 $\implies$
13 $Aw = \bar{\lambda} w$
12 $\implies$
13 $w \in \ker(A - \bar{\lambda} \ \mathrm{id}_V) \backslash \{0\}$
11 $\wedge$
13 $\mathrm{im}(\lambda) = 0 \ \wedge \ P(\text{equations})$
12 $\implies$
13 $Au = \mathrm{re}(\lambda)u = \lambda u$
12 $\implies$
13 $u \in \ker(A - \lambda \ \mathrm{id}_V) \backslash \{0\}$
7 $\wedge$
9 $\{u, w\}$ linear independent $\implies \ker(A - \lambda \ \mathrm{id}_V) \neq \{0\}$
8 $\impliedby$
10 $\{u, v\}$ linear independent $\wedge \ P(\text{equations})$

9 $\implies$
10 $\forall \mu \in \mathbb{C}$
12 $\det(A|_{\mathrm{span}(u,v)} - \mu \cdot \mathrm{id}_{\mathrm{span}(u,v)}) = (\mathrm{re}(\lambda) - \mu)^2 - \mathrm{im}(\lambda)^2$
11 $\wedge$
13 $\det(A|_{\mathrm{span}(u,v)} - \mu \cdot \mathrm{id}_{\mathrm{span}(u,v)}) = (\mathrm{re}(\lambda) - \mu)^2 - \mathrm{im}(\lambda)^2$
12 $\implies$
13 $\mu = \lambda \implies \det(A|_{\mathrm{span}(u,v)} - \mu \cdot \mathrm{id}_{\mathrm{span}(u,v)}) = 0$
12 $\implies$
13 $\ker(A - \lambda\ \mathrm{id}_V) \neq \{0\}$

5. Some miscellaneous topology

Proposition 32.
1 $\forall X, Y, \tau_X, \tau_Y$: (X, τ_X) and (Y, τ_Y) are topological spaces
2 $\forall K \subset X$: K is τ_X-compact
3 $\forall p \in Y$
4 $\forall U_1 \subset X \times Y$: open in the product topology of τ_X and τ_Y and $U_1 \supset K \times \{p\}$
5 $\exists U_2 : U_2 \in \tau_Y \ \wedge \ p \in U_2 \ \wedge \ K \times U_2 \subset U_1$

Proof sketch. The first four lines in the statement of the proposition are taken as assumptions, and then a U_2 is constructed.

Prosaic form of the statement. Let (X, τ_X) and (Y, τ_Y) are topological spaces; $K \subset X$ and K is τ_X-compact; $p \in X$; $U_1 \subset X \times Y$: open in the product topology of τ_X and τ_Y and $U_1 \supset K \times \{p\}$. Then there exists U_2 such that $U_2 \in \tau_Y \ \wedge \ p \in U_2 \ \wedge \ K \times U_2 \subset U_1$.

Proof.
4 $\{U_3 \times U_4 : U_3 \in \tau_X \ \wedge \ U_4 \in \tau_Y\}$ is a basis for the product topology of τ_X and τ_Y
3 $\wedge$
4 U_1 is open in the product topology of τ_X and τ_Y
3 $\wedge$
4 $K \times \{p\} \subset U_2$
2 $\implies$
3 $\exists \mathcal{U}$: a map $K \to \tau_X \times \tau_Y$
4 $\forall x \in K$
6 $\mathcal{U}(x)[1] \times \mathcal{U}(x)[2] \subset U_1$
5 $\wedge$
6 $x \in \mathcal{U}(x)[1] \ \wedge \ p \in \mathcal{U}(x)[2]$
1 $\wedge$
2 $\forall \mathcal{U}$: as such
5 K is τ_X-compact
4 $\implies$
5 $\exists K' \subset K$: K' is finite
6 $\bigcup_{x \in K'} \mathcal{U}(x)[1] \supset K$
3 $\wedge$
4 $\forall K'$: as such
5 $U_2 := (\bigcap_{x \in K'} \mathcal{U}(x)[2]) \implies$
6 $U_2 \in \tau_Y \ \wedge \ p \in U_2 \ \wedge \ K \times U_2 \subset U_1$
5 $\impliedby$
7 $U_2 \in \tau_Y \impliedby U_2$ is a finite intersection of τ_Y-opens
6 $\wedge$
7 $p \in U_2 \impliedby$
8 $\forall x \in K$

9 $p \in \mathcal{U}(x)[2]$
6 $\wedge$
7 $K \times U_2 \subset U_1 \Longleftarrow$
8 $\forall x, y$: $(x, y) \in K \times U_2$
10 $\exists x' : x' \in K' \wedge x \in \mathcal{U}(x')[1]$
9 $\wedge$
10 $\forall x'$: as such
11 $(x, y) \in \mathcal{U}(x')[1] \times \mathcal{U}(x')[2] \subset U \Longrightarrow$
12 $(x, y) \in U$

The following three definitions are to handle the situation that there appear multiple possible topologies. These definitions are needed in the proof of theorem 37, but they are referred to from the place where they are used.

Definition 33.
1 $\forall V$: a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$
2 τ_V^{norm} := the norm topology on V

Definition 34.
1 $\forall X, \tau$: (X, τ) a topological space
2 $\forall Y \subset X$
3 $\tau|_Y$ is subspace topology on Y from X

Definition 35.
1 $\forall X, \tau$: (X, τ) a topological space
2 $\forall x_{\mathbb{N}}$: a map $\mathbb{N} \to X$ and $x_{\mathbb{N}}$ converges with respect to τ
3 $\lim^{\tau}_{n\to\infty} x_{\mathbb{N}}$:= the limit of $x_{\mathbb{N}}$ with respect to τ

6. The statement and proof of the main result

This section starts with one small definition, and has as core the proof of the main theorem. At the end there is also a corollary, which is states what is meant by the title of this thesis. Also there is a remark on normal modes of Hamiltonian systems.

This is more precise notation for the complement of a set with respect to a set that contains it. It is used in the beginning of the proof of theorem 37.

Definition 36 ($S_2^{C(S_1)}$)**.**
1 $\forall S_1$
2 $\forall S_2 \subset S_1$
3 $S_2^{C(S_1)} := S_1 \backslash S_2$

Theorem 37.
1 $\forall V$: V is a finite dimensional vector space over $\mathbb{C}$
2 $\forall U \subset V$: U is a neighborhood of 0 in V
3 $\forall \xi \in \mathrm{Hol}(U, V)$: $\xi(0) = 0$ and $D\xi(0)$ is invertible
4 $\forall \lambda$: λ is an extreme point of the union of $\{0\}$ and the eigenvalues of $D\xi(0)$
5 $\forall v \in \ker(D\xi(0) - \lambda \cdot \mathrm{id}_V)\backslash\{0\}$
6 $\exists j$: j is a holomorphic injective immersion of a disc centered at 0 in $\mathbb{C}$ to U
8 $\exists \gamma$: a complex integral curve of ξ
9 $\mathrm{im}(j) = \mathrm{im}(\gamma) \cup \{0\} \ \wedge \ j(0) = 0 \ \wedge \ Dj(0)(1) = v$
7 $\wedge$
8 $\forall z \in \mathrm{dom}(j)$
9 $\lambda z Dj(z)(1) = \xi(j(z))$

Prosaic form of the statement. Let V be a finite dimensional vector space over $\mathbb{C}$; U is a neighborhood of 0 in V; $\xi \in \mathrm{Hol}(U, V)$ such that $\xi(0) = 0$ and $D\xi(0)$ is invertible; λ is an extreme point of the union of $\{0\}$ and the eigenvalues of $D\xi(0)$; $v \in \ker(D\xi(0) - \lambda \cdot \mathrm{id}_V)\backslash\{0\}$. Then there exists j and γ such that j is a holomorphic injective immersion of a disc centered at 0 in $\mathbb{C}$ to U and γ is complex integral curve of ξ and the idenities $\mathrm{im}(j) = \mathrm{im}(\gamma) \cup \{0\} \wedge j(0) = 0 \wedge Dj(0)(1) = v$ and $(\forall z \in \mathrm{dom}(j))(\lambda z Dj(z)(1) = \xi(j(z)))$ hold.

In the proof, the first 5 lines of the statement of the theorem are used as assumptions.

Proof sketch. (This is an outline, within the proof more specific and detailed comments are also given.) First it is shown that there exists an $\alpha \in \mathbb{C}$ such that $\alpha\lambda$ has a positive real part strictly greater than the other eigenvalues of $\alpha D\xi(0)$, and such that the $\alpha\xi$ satisfies the condition of the Grobman-Hartman theorem. Then for each $s \in \mathbb{R}$, an open interval is defined, and is shown to be unbounded in the negative direction and open. After that, for each $s \in \mathbb{R}$, γ_s is defined as some map from I_s to V and shown to be a real integral curve of $\alpha\xi$. Then, it is shown that γ_0 and $\gamma_{2\pi}$ agree on their common domain. To proof this statement, we look at their "linearized version" obtained by applying the diffeomorphism from the differentiable Grobman-Hartman theorem. From this linearized version we can compute a certain limit by looking at the "infinitesimal flow" of $\frac{i}{\lambda}\xi$ near 0. Using the lemma on solutions of linear (time independent) ODEs, and that α is chosen such that $\alpha\lambda$ has a positive real part strictly greater than the other eigenvalues of $\alpha D\xi(0)$, and this already computed limit, it is shown that γ_0 and $\gamma_{2\pi}$ agree on their common domain. After that, γ is defined and by "pasting together" γ_s for each s. Periodicity of γ_s follows easily from the fact that γ_0 and $\gamma_{2\pi}$ agree on their common domain and this also has an implication for the domain of γ since real periodic integrals can be extended to the whole real number line. The fact that for each s γ_s is a real integral curve is used another time, and now to show that γ is a complex integral curve (this includes by the definition made in this thesis that γ is holomorphic). Using holomorphy and periodicity of γ we can pullback γ to a holomorphic map that is called $\tilde{\gamma}$. $\tilde{\gamma}$ is shown to be defined on a punctured disc, which is quite easy, and shown that it can be continuously be extended to its center, which is more work, although the arguments are all elementary. Being holomorphic on a punctured disc and continuously extendable on a disc implies that this extension is holomorphic on the disc, and we call this extension j. Then using holomorphy of j the identity $Dj(0)(1) = v$ is proven. $(\forall z \in \mathrm{dom}(j))(\lambda z Dj(z)(1) = \xi(j(z)))$ has a very short proof. At last we show that j is an injective immersion: We could at this point also take a new j restricted to a smaller disc and simply using $Dj(0)(1) = v$ and the rank theorem from analysis, but it pleased the author to show that this restriction is not necessary, i.e. that the "original" j is also an injective immersion.

Proof.
Here we define some objects, among which are A_1 and A_2. After these definitions we show that $\exists\alpha \in A_1 \cap A_2$.
2 $\Lambda :=$ (the set of eigenvalues of $D\xi(0)$)
1 $\implies$
4 $\forall\mu \in \Lambda \cup \{0\}$
5 $A_{1,\mu} := \{\alpha \in \mathbb{C} : |\alpha| = 1 \land \mathrm{re}(\alpha\mu) < \mathrm{re}(\alpha\lambda)\}$
3 $\land$
4 $A_1 := \bigcap_{\mu\in\Lambda\cup\{0\}} A_{1,\mu}$
3 $\land$
4 $\forall\mu \in \Lambda$
5 $A_{2,\mu} := \{\alpha \in \mathbb{C} : |\alpha| = 1 \land \mathrm{re}(\alpha\mu) \neq 0\}$
3 $\land$
4 $A_2 := \bigcap_{\mu\in\Lambda} A_{2,\mu}$
2 $\implies$
5 $\exists\alpha \in A_1 \cap A_2$
4 $\impliedby$
5 $A_1 \cap A_2 \neq \emptyset$
4 $\impliedby$
5 A_1 is open in $\{\alpha \in \mathbb{C} : |\alpha| = 1\}$ $\land$ $A_1 \neq \emptyset$ $\land$ A_2 is dense in $\{\alpha \in \mathbb{C} : |\alpha| = 1\}$

4 $\Longleftarrow$
7 A_1 is open
6 $\Longleftarrow$
9 $\forall \mu \in \Lambda \cup \{0\}$
10 $\{(\alpha, \mathrm{re}(\alpha\mu - \alpha\lambda)) : \alpha \in \mathbb{C}$ and $|\alpha| = 1\}$ is a continuous map
8 $\Longrightarrow$
9 $\forall \mu \in \Lambda \cup \{0\}$
10 $A_{1,\mu}$ is open in $\{\alpha \in \mathbb{C} : |\alpha| = 1\}$
7 $\wedge$
10 $\forall \mu \in \Lambda \cup \{0\}$
11 $A_{1,\mu}$ is open in $\{\alpha \in \mathbb{C} : |\alpha| = 1\}$
9 $\wedge$
10 $\Lambda \cup \{0\}$ is finite
8 $\Longrightarrow$
9 $\bigcap_{\mu \in \Lambda \cup \{0\}} A_{1,\mu}$ is open in $\{\alpha \in \mathbb{C} : |\alpha| = 1\}$
8 $\Longrightarrow$
9 A_1 is open
5 $\wedge$
6 $A_1 \neq \emptyset \;\Longleftarrow\; P_6(\Lambda \cup \{0\}, \lambda) \;\Longleftarrow\;$ (proposition 6 $\;\wedge\; \lambda \in \mathrm{ext}(\Lambda \cup \{0\})$)
5 $\wedge$
7 A_2 is dense
6 $\Longleftarrow$
8 $\forall \mu \in \Lambda$
9 $\forall \alpha \in \mathbb{C}$: $|\alpha| = 1$
The upper C with an argument is defined in 36.
11 $\alpha \in A_{2,\mu}^{C(\{\alpha' \in \mathbb{C} : |\alpha'| = 1\})}$
10 $\Longleftrightarrow$
11 $\mathrm{re}(\alpha\mu) = 0$
10 $\Longleftrightarrow$
11 $\alpha\mu = i|\alpha||\mu| \;\vee\; \alpha\mu = -i|\alpha||\mu|$
10 $\Longleftrightarrow$
11 $\alpha = i\frac{\mu}{|\mu|} \;\vee\; \alpha = -i\frac{\mu}{|\mu|}$
7 $\wedge$
12 $\forall \mu \in \Lambda$
13 $\forall \alpha \in \mathbb{C}$: $|\alpha| = 1$
15 $\alpha \in A_{2,\mu}^{C(\{\alpha' \in \mathbb{C} : |\alpha'| = 1\})}$
14 $\Longleftrightarrow$
15 $\alpha = i\frac{\mu}{|\mu|} \;\vee\; \alpha = -i\frac{\mu}{|\mu|}$
11 $\Longrightarrow$
12 $\forall \mu \in \Lambda$
13 $A_{2,\mu}^{C(\{\alpha \in \mathbb{C} : |\alpha| = 1\})}$ is finite
11 $\Longrightarrow$

12 $\bigcup\limits_{\mu\in\Lambda} A_{2,\mu}^{C(\{\alpha\in\mathbb{C}:|\alpha|=1\})}$ is finite
11 $\implies$
12 $(\bigcap\limits_{\mu\in\Lambda} A_{2,\mu})^{C(\{\alpha\in\mathbb{C}:|\alpha|=1\})}$ is finite
11 $\implies$
12 A_2 is dense in $\{\alpha \in \mathbb{C} : |\alpha| = 1\}$
3 $\wedge$
4 $\forall \alpha \in A_1 \cap A_2$
7 $D(\alpha\xi)(0)$ has no eigenvalues with zero real part $\wedge$ proposition 31
6 $\implies$
7 $\mathrm{id}_{\mathbb{C}} \otimes_{\mathbb{R}} D(\alpha\xi)(0)$ has no eigenvalues with zero real part
5 $\wedge$
7 $\mathrm{id}_{\mathbb{C}} \otimes_{\mathbb{R}} D(\alpha\xi)(0)$ has no eigenvalues with zero real part and theorem 25
6 $\implies$
7 $P_{24}(V_{\mathbb{R}}, U, \alpha\xi)$
6 $\implies$
7 $\exists U_1, U_2 \subset U$: neighborhoods of 0
8 $\exists \varphi \in \mathrm{Diff}^1(U_1, U_2)$: $\varphi(0) = 0$ and $D\varphi(0) = \mathrm{id}_V$
9 $\forall a, b \in [-\infty, \infty]$: $a < b$
10 $\forall \gamma \in C^1((a, b), U_1)$: $D\gamma = D(\alpha\xi)(0) \circ \gamma$
11 $D(\varphi \circ \gamma) = (\alpha\xi) \circ \varphi \circ \gamma$
5 $\wedge$
6 $\forall U_1, U_2, \varphi$: as such
Now define for each $s \in \mathbb{R}$, and show that is a nonempty open interval that is unbounded in the negative direction, which is important because we will use as the domain of a real integral curve. For the definition of $\mathcal{D}_{\frac{i}{\lambda}\xi}$ see definition 16
8 $\forall s \in \mathbb{R}$
9 $I_s := \big(-\infty,\ \sup\{t_s : (\forall t \in \mathbb{R}_{\le t_s})(ve^{\lambda\alpha t} \in U_1 \ \wedge\ (s, \varphi(ve^{\lambda\alpha t})) \in \mathcal{D}_{\frac{i}{\lambda}\xi})\} \ \big)$
7 $\implies$
10 $\forall s \in \mathbb{R}$
11 $I_s = (-\infty, \sup(I_s)) \ \wedge\ \sup(I_s) > -\infty$
9 $\impliedby$
11 $I_s = (-\infty, \sup(I_s)) \ \impliedby$
12 $I_s = \big(-\infty,\ \sup\{t_s : (\forall t \in \mathbb{R}_{\le t_s})(ve^{\lambda\alpha t} \in U_1 \ \wedge\ (s, \varphi(ve^{\lambda\alpha t})) \in \mathcal{D}_{\frac{i}{\lambda}\xi})\} \ \big)$
10 $\wedge$
11 $\sup(I_s) > -\infty \ \impliedby$
14 $\lim_{t\to-\infty} ve^{\lambda\alpha t} = 0 \ \wedge\ U_1$ is a neighborhood of 0 in V
13 $\implies$
14 $\exists t' \in \mathbb{R}$
15 $\forall t \in \mathbb{R} : t < t'$
16 $ve^{\lambda\alpha t} \in U_1$
12 $\wedge$
13 $\forall t'$: as such

14 $\forall s \in \mathbb{R}$
16 $\frac{i}{\lambda}\xi(0) = 0 \implies \mathbb{R} \times \{0\} \subset \mathcal{D}_{\frac{i}{\lambda}\xi} \implies \{s\} \times \{0\} \subset \mathcal{D}_{\frac{i}{\lambda}\xi}$
15 $\wedge$
17 $\lim\limits_{t<t' \wedge t \to -\infty} (s, \varphi(ve^{\lambda\alpha t})) = (s, 0) \ \wedge \ \mathcal{D}_{\frac{i}{\lambda}\xi}$ is open $\wedge \ \{s\} \times \{0\} \subset \mathcal{D}_{\frac{i}{\lambda}\xi}$
16 $\implies$
17 $\exists t_s \in \mathbb{R}$
18 $\forall t \in \mathbb{R} : t < t_s$
19 $ve^{\lambda\alpha t} \in U_1 \ \wedge \ (s, \varphi(ve^{\lambda\alpha t})) \in \mathcal{D}_{\frac{i}{\lambda}\xi}$
8 $\wedge$
10 $\forall s \in \mathbb{R}$
11 $I_s = (-\infty, \sup(I_s)) \ \wedge \ \sup(I_s) > -\infty$
9 $\implies$

Now we define for every s a map γ_s, and show that it is a real integral curve. The idea of the proof is: The assumptions on φ imply γ_0 is a real integral curve, and this and the commutation of the flows of $\alpha\xi$ and $\frac{i}{\lambda}\xi$ that for every s γ_s is a real integral curve. For the definition of $F_{\frac{i}{\lambda}\xi}$ see definition 16

11 $\forall s \in \mathbb{R}$
12 $\gamma_s := \{(t, F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t}))) : t \in I_s\}$
10 $\implies$
12 $(\forall s \in \mathbb{R})(\gamma_s$ is a real integral curve of $\alpha\xi)$
11 $\wedge$
13 $(\forall s \in \mathbb{R})(\gamma_s$ is a real integral curve of $\alpha\xi)$
12 $\impliedby$
13 $\forall s \in \mathbb{R}$
14 $\forall t' \in I_s$
17 $\gamma_s = F_{\frac{i}{\lambda}\xi}(s, -) \circ \gamma_0$
16 $\wedge$
17 $\gamma_s = F_{\frac{i}{\lambda}\xi}(s, -) \circ \gamma_0 \implies \gamma_s = \{(t, F_{\frac{i}{\lambda}\xi}(s, (F_{\alpha\xi}(t - t', \gamma_0(t'))))) : t \in I_s\}$
16 $\wedge$
17 $\gamma_s = \{(t, F_{\frac{i}{\lambda}\xi}(s, (F_{\alpha\xi}(t-t', \gamma_0(t'))))) : t \in I_s\} \implies \gamma_s = \{(t, F_{\alpha\xi}(t-t', (F_{\frac{i}{\lambda}\xi}(s, \gamma_0(t'))))) : t \in I_s\}$
16 $\wedge$
18 $\gamma_s = \{(t, F_{\alpha\xi}(t - t', (F_{\frac{i}{\lambda}\xi}(s, \gamma_0(t'))))) : t \in I_s\} \implies \gamma_s$ is a real integral curve of $\alpha\xi$
15 $\impliedby$
17 $\gamma_s = F_{\frac{i}{\lambda}\xi}(s, -) \circ \gamma_0$
16 $\wedge$
18 $\gamma_s = F_{\frac{i}{\lambda}\xi}(s, -) \circ \gamma_0 \implies \gamma_s = \{(t, F_{\frac{i}{\lambda}\xi}(s, (F_{\alpha\xi}(t - t', \gamma_0(t'))))) : t \in I_s\}$
17 $\impliedby$
21 $\{(t, ve^{\lambda\alpha t}) : t \in I_0\}$ is a real integral curve of $\alpha D\xi(0)$
20 $\wedge$
21 $\forall a, b \in [-\infty, \infty]\colon \ a < b$
22 $\forall \gamma \in C^1((a, b), U_1)\colon \ D\gamma = D(\alpha\xi)(0) \circ \gamma$

23 $D(\varphi \circ \gamma) = (\alpha\xi) \circ \varphi \circ \gamma$

19 $\Longrightarrow$

20 γ_0 is a real integral curve of $\alpha\xi$

18 $\wedge$

20 γ_0 is a real integral curve of $\alpha\xi$ and proposition 17

19 $\Longrightarrow$

20 $\gamma_0 = \{(t, F_{\alpha\xi}(t - t', \gamma_0(t'))) : t \in I_s\}$

18 $\wedge$

20 $\gamma_0 = \{(t, F_{\alpha\xi}(t - t', \gamma_0(t'))) : t \in I_s\} \ \wedge \ \gamma_s = F_{\frac{i}{\lambda}\xi}(s, -) \circ \gamma_0$

19 $\Longrightarrow$

20 $\gamma_s = \{(t, F_{\frac{i}{\lambda}\xi}(s, (F_{\alpha\xi}(t - t', \gamma_0(t'))))) : t \in I_s\}$

16 $\wedge$

18 $\gamma_s = \{(t, F_{\frac{i}{\lambda}\xi}(s, (F_{\alpha\xi}(t{-}t', \gamma_0(t'))))) : t \in I_s\} \implies \gamma_s = \{(t, F_{\alpha\xi}(t{-}t', (F_{\frac{i}{\lambda}\xi}(s, \gamma_0(t'))))) : t \in I_s\}$

17 $\Longleftarrow$

20 ξ is holomorphic and proposition 20

19 $\Longrightarrow$ $\frac{i}{\lambda}\xi$ and $\alpha\xi$ commute

18 $\wedge$

20 $\frac{i}{\lambda}\xi$ and $\alpha\xi$ commute and proposition 19

19 $\Longrightarrow$

20 $F_{\frac{i}{\lambda}\xi}$ and $F_{\alpha\xi}$ commute

19 $\Longrightarrow$

20 $\{(t, F_{\frac{i}{\lambda}\xi}(s, (F_{\alpha\xi}(t - t', \gamma_0(t'))))) : t \in I_s\} = \{(t, F_{\alpha\xi}(t - t', (F_{\frac{i}{\lambda}\xi}(s, \gamma_0(t'))))) : t \in I_s\}$

18 $\wedge$

20 $\{(t, F_{\frac{i}{\lambda}\xi}(s, (F_{\alpha\xi}(t - t', \gamma_0(t'))))) : t \in I_s\} = \{(t, F_{\alpha\xi}(t - t', (F_{\frac{i}{\lambda}\xi}(s, \gamma_0(t'))))) : t \in I_s\} \ \wedge$ $\gamma_s = \{(t, F_{\frac{i}{\lambda}\xi}(s, (F_{\alpha\xi}(t - t', \gamma_0(t'))))) : t \in I_s\}$

19 $\Longrightarrow$

20 $\gamma_s = \{(t, F_{\alpha\xi}(t - t', (F_{\frac{i}{\lambda}\xi}(s, \gamma_0(t'))))) : t \in I_s\}$

16 $\wedge$

18 $\gamma_s = \{(t, F_{\alpha\xi}(t - t', (F_{\frac{i}{\lambda}\xi}(s, \gamma_0(t'))))) : t \in I_s\} \implies \gamma_s$ is a real integral curve of $\alpha\xi$

17 $\Longleftarrow$

19 proposition 17

18 $\Longrightarrow$

19 $\{(t, F_{\alpha\xi}(t - t', (F_{\frac{i}{\lambda}\xi}(s, \gamma_0(t'))))) : t \in I_s\}$ is a real integral curve of $\alpha\xi$

17 $\wedge$

19 $\{(t, F_{\alpha\xi}(t - t', (F_{\frac{i}{\lambda}\xi}(s, \gamma_0(t'))))) : t \in I_s\}$ is a real integral curve of $\alpha\xi$ and $\gamma_s = \{(t, F_{\alpha\xi}(t - t', (F_{\frac{i}{\lambda}\xi}(s, \gamma_0(t'))))) : t \in I_s\}$

18 $\Longrightarrow$

19 γ_s is a real integral curve of $\alpha\xi$

Now we have proven the stament "$(\forall s \in \mathbb{R})(\gamma_s$ is a real integral curve of $\alpha\xi)$*", we use it to proof* $\gamma_{2\pi} = \gamma_0|_{I_{2\pi}}$*. The proof is lengthy and the idea as follows: for different values of s, the* γ_s *are by definition related by the flow of* $\frac{i}{\lambda}\xi$*. This is implies that for different* *values of s* $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(\gamma_s(t))$ *are related by the infinitesimal flow of* $\frac{i}{\lambda}\xi$*, which for* $s = 2\pi$ *fixes* v*. Hence* $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v$*.* $\varphi^{-1}\circ\gamma_{2\pi}$ *is a real integral curve of the linear ODE induced by* $D(\alpha\xi)(0)$ *and by assumption on* φ *and because* $\gamma_{2\pi}$ *is a real integral curve. Using* $\alpha \in A_2$*, this "limit property" and being solution of the linear ODE induced by* $D(\alpha\xi)(0)$ *uniquely determine* $\varphi^{-1}\circ\gamma_{2\pi}$*, and this yields that* $\gamma_{2\pi}$ *and* γ_0 *are equal on their common domain.*

11 $\wedge$

13 $(\forall s \in \mathbb{R})(\gamma_s$ is a real integral curve of $\alpha\xi)$

12 $\implies$

14 $\gamma_{2\pi} = \gamma_0|_{I_{2\pi}}$

13 $\wedge$

15 $\gamma_{2\pi} = \gamma_0|_{I_{2\pi}}$

14 $\impliedby$

16 $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v$

15 $\wedge$

17 $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v$

16 $\impliedby$

18 $DF_{\frac{i}{\lambda}\xi}(2\pi,-)(0)$ is $\mathbb{C}$ linear

17 $\wedge$

18 $DF_{\frac{i}{\lambda}\xi}(2\pi,-)(0)$ is $\mathbb{C}$ linear $\implies \lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(F_{\frac{i}{\lambda}\xi}(2\pi,\varphi(ve^{\lambda\alpha t})) = DF_{\frac{i}{\lambda}\xi}(2\pi,v)(0)$

17 $\wedge$

18 $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(F_{\frac{i}{\lambda}\xi}(2\pi,\varphi(ve^{\lambda\alpha t})) = DF_{\frac{i}{\lambda}\xi}(2\pi,v)(0) \implies$

19 $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v$

16 $\impliedby$

19 $DF_{\frac{i}{\lambda}\xi}(2\pi,-)(0)$ is $\mathbb{C}$ linear

18 $\impliedby$

19 $\frac{i}{\lambda}\xi$ is holomorphic $\implies F_{\frac{i}{\lambda}\xi}(2\pi,-)$ is holomorphic $\implies DF_{\frac{i}{\lambda}}(2\pi,-)(0)$ is $\mathbb{C}$

17 $\wedge$

19 $DF_{\frac{i}{\lambda}\xi}(2\pi,-)(0)$ is $\mathbb{C}$ linear $\implies \lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(F_{\frac{i}{\lambda}\xi}(2\pi,\varphi(ve^{\lambda\alpha t})) = DF_{\frac{i}{\lambda}\xi}(2\pi,v)(0)$

18 $\impliedby$

20 $\lim_{t\to-\infty} e^{\alpha\lambda t} = 0 \impliedby \alpha \in A_1$

19 $\wedge$

21 $\wedge_*$

22 $\{(t, e^{\alpha\lambda t}) : t \in I_{2\pi}\} \in C^1(I_{2\pi}, \mathbb{C}\backslash\{0\})$ and $\lim_{t\to-\infty} e^{\alpha\lambda t} = 0$

22 $DF_{\frac{i}{\lambda}\xi}(2\pi,-)(0)$ is $\mathbb{C}$ linear

22 $D\varphi(0) = \mathrm{id}_V$

22 proposition 11

21 $\wedge^*$

20 $\implies$

21 $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(F_{\frac{i}{\lambda}\xi}(2\pi,\varphi(ve^{\lambda\alpha t})) = DF_{\frac{i}{\lambda}\xi}(2\pi,v)(0)$

17 $\wedge$

19 $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(F_{\frac{i}{\lambda}\xi}(2\pi,\varphi(ve^{\lambda\alpha t})) = DF_{\frac{i}{\lambda}\xi}(2\pi,v)(0) \implies$

20 $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v$
18 $\Longleftarrow$
22 $D(\{(t, ve^{it}) : t \in \mathbb{R}\}) = i\{(t, ve^{it}) : t \in \mathbb{R}\}$
21 $\wedge$
22 $\forall t \in \mathbb{R}$
23 $D(\frac{i}{\lambda}\xi)(0)(ve^{it}) = ive^{it}$
20 $\Longrightarrow$
21 $D(\{(t, ve^{it}) : t \in \mathbb{R}\}) = D(\frac{i}{\lambda}\xi)(0)(\{(t, ve^{it}) : t \in \mathbb{R}\})$
20 $\Longrightarrow$
21 $\forall t \in \mathbb{R}$
22 $F_{D\frac{i}{\lambda}\xi(0)}(t, v) = ve^{it}$
20 $\Longrightarrow$
21 $F_{D\frac{i}{\lambda}\xi(0)}(2\pi, v) = v$
19 $\wedge$
21 $F_{D\frac{i}{\lambda}\xi(0)}(2\pi, v) = v$ and proposition 29
20 $\Longrightarrow$
21 $DF_{\frac{i}{\lambda}\xi}(2\pi, v)(0) = v$
19 $\wedge$
21 $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(F_{\frac{i}{\lambda}\xi}(2\pi, \varphi(ve^{\lambda\alpha t})) = DF_{\frac{i}{\lambda}\xi}(2\pi, v)(0)$ and $DF_{\frac{i}{\lambda}\xi}(2\pi, v)(0) = v$
20 $\Longrightarrow$
21 $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(F_{\frac{i}{\lambda}\xi}(2\pi, \varphi(ve^{\lambda\alpha t})) = v$
20 $\Longrightarrow$
21 $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v$
15 $\wedge$
17 $\lim_{t\to-\infty} e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v$
16 $\Longrightarrow$
18 $\tilde{I}_{2\pi} := (-\infty, \sup\{t \in I_{2\pi} : (\forall t' \in \mathbb{R}_{<t})(\gamma_{2\pi}(t') \in U_2)\})$
17 $\Longrightarrow$
19 $(\forall t \in \tilde{I}_{2\pi})(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v)$
18 $\wedge$
20 $(\forall t \in \tilde{I}_{2\pi})(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v)$
19 $\Longleftarrow$
Now an eigensystem (see definition 26) is picked. This is to be able write down solutions of a linear ODE nicely.
22 proposition 27
21 $\Longrightarrow$
22 $\exists \lambda_E, E, v_E, v_E^*$: $(\lambda_E, E, v_E, v_E^*)$ is an eigensystem for $D\xi(0)$
20 $\wedge$
21 $\forall \lambda_E, E, v_E, v_E^*$: as such
23 $(\forall t \in \tilde{I}_{2\pi})(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v)$
22 $\Longleftarrow$
24 $\{v_E^*(W, n) : (W, n) \in E\}$ is a basis of V^* $\Longleftarrow$ Jordan normal form theorem
23 $\wedge$

24 $\forall W, n\colon (W, n) \in E$
25 $(\forall t \in \tilde{I}_{2\pi})(v_E^*(W, n)(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t))) = v_E^*(W, n)(v))$
23 $\wedge$
25 $\forall W, n\colon (W, n) \in E$
26 $(\forall t \in \tilde{I}_{2\pi})(v_E^*(W, n)(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t))) = v_E^*(W, n)(v))$
24 $\Longleftarrow$
25 $\forall W, n\colon (W, n) \in E$
27 $W_v :=$ the generalized eigenspace of $D\xi(0)$ containing v
26 $\Longrightarrow$
28 $(W, n) \neq (W_v, 1) \implies v_E^*(W, n)(v) = 0$
27 $\wedge$
29 $(W, n) \neq (W_v, 1) \implies v_E^*(W, n)(v) = 0$
28 $\Longleftarrow$
31 $v_E(W_v, 1) \neq 0 \;\wedge\; v_E(W_v, 1), v \in \{$eigenvectors of $D\xi(0)|_{W_v}\} \;\wedge$
$\{$eigenvectors of $D\xi(0)|_{W_v}\}$ is a 1-dimensional subspace of V
30 $\Longrightarrow$
31 $\exists b \in \mathbb{C}\backslash\{0\}\colon\; v = bv_E(W_v, 1)$
29 $\wedge$
30 $\forall b$: as such
32 $(W, n) \neq (W_v, 1)$
31 $\Longrightarrow$
32 $v_E^*(W, n)(v) = v_E^*(W, n)(bv_E(W_v, 1)) = bv_E^*(W, n)(v_E(W_v, 1)) = b \cdot 0 = 0$
27 $\wedge$
29 $(W, n) \neq (W_v, 1) \implies v_E^*(W, n)(v) = 0$
28 $\Longrightarrow$
32 $\forall a, b \in [-\infty, \infty]\colon\; a < b$
33 $\forall \gamma \in C^1((a, b), U_1)\colon\; D\gamma = D(\alpha\xi)(0) \circ \gamma$
34 $D(\varphi \circ \gamma) = (\alpha\xi) \circ \varphi \circ \gamma$
31 $\wedge$
32 $\gamma_{2\pi}$ is a real integral curve of $\alpha\xi$
30 $\Longrightarrow$
31 $\varphi^{-1} \circ (\gamma_{2\pi}|_{\tilde{I}_{2\pi}})$ is a real integral curve of $D(\alpha\xi)(0)$
29 $\wedge$
31 proposition 28 $\;\wedge\;$ $\varphi^{-1} \circ (\gamma_{2\pi}|_{\tilde{I}_{2\pi}})$ is a real integral curve of $D(\alpha\xi)(0)$ $\;\wedge$
$(\alpha\lambda_E, E, v_E, v_E^*)$ is an eigensystem for $D(\alpha\xi)(0)$
30 $\Longrightarrow$
31 $\exists c\colon \{1, ..., n\} \to \mathbb{C}$
32 $\forall t \in \tilde{I}_{2\pi}$
33 $v_E^*(W, n)(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t))) = e^{-\alpha(\lambda - \lambda(W))t} \sum_{m=1}^{\dim(W)-n+1} c(m)t^{m-1}$
29 $\wedge$
30 $\forall c$: as such
33 $\lim_{t \in \tilde{I}_{2\pi};\; t \to -\infty} e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v$
32 $\Longrightarrow$

33 $\lim_{t\in\tilde{I}_{2\pi};\ t\to-\infty} v_E^*(W,n)(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t))) = v_E^*(W,n)(v)$
32 $\implies$
33 $\lim_{t\in\tilde{I}_{2\pi};\ t\to-\infty} e^{-\alpha(\lambda-\lambda(W))t}\sum_{m=1}^{\dim(W)-n+1} c(m)t^{m-1} = v_E^*(W,n)(v)$
32 $\wedge$
33 $\wedge_*$
34 $\lim_{t\in\tilde{I}_{2\pi};\ t\to-\infty} e^{-\alpha(\lambda-\lambda(W))t}\sum_{m=1}^{\dim(W)-n+1} c(m)t^{m-1} = v_E^*(W,n)(v)$
34 $\lim_{t\in\tilde{I}_{2\pi};\ t\to-\infty} e^{-\alpha(\lambda-\lambda(W))t} \neq 0 \impliedby \mathrm{re}(\alpha\lambda) \geq \mathrm{re}(\alpha\lambda(W)) \impliedby \alpha\in A_1$
34 $\forall m\in\mathbb{N}_{\geq 2}$
35 $\lim_{t\in\tilde{I}_{2\pi};\ t\to-\infty} \frac{e^{-\alpha(\lambda-\lambda(W))t}t^m}{e^{-\alpha(\lambda-\lambda(W))t}t^{m+1}} = \lim_{t\in\tilde{I}_{2\pi};\ t\to-\infty} t^m/t^{m+1} = 0$
34 proposition 13
33 $\wedge^*$
32 $\implies$
34 $c(1) = v_E^*(W,n)(v)/(\lim_{t\in\tilde{I}_{2\pi};\ t\to-\infty} e^{-\alpha(\lambda-\lambda(W))t})$
33 $\wedge$
34 $\forall m\in\{2,...,\dim(W)-n+1\}$
35 $c(m) = 0$
32 $\implies$
36 $(W,n)\neq(W_v,1) \implies v_E^*(W,n)(v) = 0$
35 $\wedge$
36 $c(1) = v_E^*(W,n)(v)/(\lim_{t\in\tilde{I}_{2\pi};\ t\to-\infty} e^{-\alpha(\lambda-\lambda(W))t})$
34 $\implies$
36 $(W,n)=(W_v,1) \implies c(1) = v_E^*(W,n)(v)$
35 $\wedge$
36 $(W,n)\neq(W_v,1) \implies c(1) = 0$
33 $\wedge$
35 $\wedge_*$
36 $\forall t\in\tilde{I}_{2\pi}$
37 $v_E^*(W,n)(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t))) = e^{-\alpha(\lambda-\lambda(W))t}\sum_{m=1}^{\dim(W)-n+1} c(m)t^{m-1}$
36 $(W,n)=(W_v,1) \implies c(1) = v_E^*(W,n)(v)$
36 $(W,n)\neq(W_v,1) \implies c(1) = 0$
36 $\forall m\in\{2,...,\dim(W)-n+1\}$
37 $c(m) = 0$
35 $\wedge^*$
34 $\implies$
35 $\forall t\in\tilde{I}_{2\pi}$
37 $(W,n)=(W_v,1) \implies v_E^*(W,n)(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t))) = v_E^*(W,n)(v)$
36 $\wedge$
37 $(W,n)\neq(W_v,1) \implies v_E^*(W,n)(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t))) = 0$
33 $\wedge$
34 $\forall t\in\tilde{I}_{2\pi}$
36 $\wedge_*$
37 $(W,n)=(W_v,1) \implies v_E^*(W,n)(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t))) = v_E^*(W,n)(v)$
37 $(W,n)\neq(W_v,1) \implies v_E^*(W,n)(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t))) = 0$

37 $(W, n) \neq (W_v, 1) \implies v_E^*(W, n)(v) = 0$
36 $\wedge^*$
35 $\implies$
36 $v_E^*(W, n)(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t))) = v_E^*(W, n)(v)$
18 $\wedge$
20 $(\forall t \in \tilde{I}_{2\pi})(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v)$
Now this is done, it only has to be shown that they agree on their whole common domain, for which uniqueness of solutions of ODEs is used.
19 $\implies$
21 $\tilde{I}_{2\pi} \neq \emptyset$
20 $\wedge$
22 $\tilde{I}_{2\pi} \neq \emptyset$
21 $\impliedby$
22 $\wedge_*$
23 $\tilde{I}_{2\pi} = (-\infty, \sup\{t \in I_{2\pi} : (\forall t' \in \mathbb{R}_{<t})(\gamma_{2\pi}(t') \in U_2)\})$
23 $\lim_{t \in I_{2\pi};\ t \to -\infty} \gamma_{2\pi}(t) = 0$
23 U_2 is a neighborhood of 0 in V
23 $I_{2\pi} = (-\infty, \sup(I_{2\pi}))$
22 $\wedge^*$
20 $\wedge$
22 $\tilde{I}_{2\pi} \neq \emptyset$
21 $\implies$
24 $(\forall t \in \tilde{I}_{2\pi})(e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t)) = v)$
23 $\implies$
24 $\{(t, e^{-\alpha\lambda t}\varphi^{-1}(\gamma_{2\pi}(t))) : t \in \tilde{I}_{2\pi}\} = \{(t, v) : t \in \tilde{I}_{2\pi}\}$
23 $\implies$
24 $\{(t, \varphi^{-1}(\gamma_{2\pi}(t))) : t \in \tilde{I}_{2\pi}\} = \{(t, ve^{\alpha\lambda t}) : t \in \tilde{I}_{2\pi}\}$
23 $\implies$
24 $\{(t, \gamma_{2\pi}(t)) : t \in \tilde{I}_{2\pi}\} = \{(t, \varphi(ve^{\alpha\lambda t})) : t \in \tilde{I}_{2\pi}\}$
23 $\implies$
24 $\gamma_{2\pi}|_{\tilde{I}_{2\pi}} = \gamma_0|_{\tilde{I}_{2\pi}}$
22 $\wedge$
24 $\wedge_*$
25 $\gamma_{2\pi}|_{\tilde{I}_{2\pi}} = \gamma_0|_{\tilde{I}_{2\pi}}$
25 uniqueness of solutions of ODEs
25 $\gamma_{2\pi}$ and $\gamma_0|_{I_{2\pi}}$ are real integral curves of $\alpha\xi$
25 $\tilde{I}_{2\pi} \neq \emptyset$
25 $\tilde{I}_{2\pi} \subset I_{2\pi}$
24 $\wedge^*$
23 $\implies$
24 $\gamma_{2\pi} = \gamma_0|_{I_{2\pi}}$
13 $\wedge$
15 $\gamma_{2\pi} = \gamma_0|_{I_{2\pi}}$

14 $\implies$
Here γ is defined. In the rest a the proof it will be shown that satisfies the conditions on the γ from the statement of the theorem.
16 $\gamma := \{(\alpha t + \frac{i}{\lambda}s, F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t}))) : (s, \varphi(ve^{\lambda\alpha t})) \in \mathcal{D}_{\frac{i}{\lambda}\xi}$ and $t \in I_{2\pi}\}$
15 $\implies$
17 $\wedge_*$
18 γ is a complex integral curve of ξ
18 γ is $\frac{2\pi i}{\lambda}$-periodic
18 $\mathrm{dom}(\gamma) = \{\alpha t + \frac{i}{\lambda}s : s \in \mathbb{R} \ \wedge \ t \in I_{2\pi}\}$
17 $\wedge^*$
16 $\wedge$
18 $\wedge_*$
19 γ is a complex integral curve of ξ
19 γ is $\frac{2\pi i}{\lambda}$-periodic
19 $\mathrm{dom}(\gamma) = \{\alpha t + \frac{i}{\lambda}s : s \in \mathbb{R} \ \wedge \ t \in I_{2\pi}\}$
18 $\wedge^*$
17 $\impliedby$
What happens now is that first "α and $\frac{i}{\lambda}$ are linear independent over $\mathbb{R}$ " *is proven and then* "γ is a map".
19 α and $\frac{i}{\lambda}$ are linear independent over $\mathbb{R}$ $\wedge$ γ is a map
18 $\wedge$
20 α and $\frac{i}{\lambda}$ are linear independent over $\mathbb{R}$ $\wedge$ γ is a map
19 $\impliedby$
21 α and $\frac{i}{\lambda}$ are linear independent over $\mathbb{R}$
20 $\wedge$
22 α and $\frac{i}{\lambda}$ are linear independent over $\mathbb{R}$
21 $\impliedby$
22 $\forall c_1, c_2 \in \mathbb{R}$: $c_1\alpha = c_2\frac{i}{\lambda}$
23 $c_1\alpha\lambda = ic_2 \implies$
25 $c_1 \neq 0 \implies \mathrm{re}(\alpha\lambda) = 0 \implies (\neg)(\alpha \in A_1) \implies$ ↯
24 $\wedge$
25 $c_1 = 0 \implies c_2 = 0$
20 $\wedge$
22 α and $\frac{i}{\lambda}$ are linear independent over $\mathbb{R}$
21 $\implies$
23 γ is a map
22 $\impliedby$
23 $\forall z, v_1, v_2$: $(z, v_1) \in \gamma \ \wedge \ (z, v_2) \in \gamma$
24 $v_1 = v_2$
22 $\impliedby$
23 $\forall z, v_1, v_2$: $(z, v_1) \in \gamma \ \wedge \ (z, v_2) \in \gamma$
25 $\exists t_1, s_1, t_2, s_2$
27 $z = \alpha t_1 + \frac{i}{\lambda}s_1$ and $(s_1, \varphi(ve^{\lambda\alpha t_1})) \in \mathcal{D}_{\frac{i}{\lambda}\xi}$ and $t_1 \in I_{2\pi}$ and $v_1 = F_{\frac{i}{\lambda}\xi}(s_1, \varphi(ve^{\lambda\alpha t_1}))$

26 $\wedge$
27 $z = \alpha t_2 + \frac{i}{\lambda} s_2$ and $(s_2, \varphi(ve^{\lambda\alpha t_2})) \in \mathcal{D}_{\frac{i}{\lambda}\xi}$ and $t_2 \in I_{2\pi}$ and $v_2 = F_{\frac{i}{\lambda}\xi}(s_2, \varphi(ve^{\lambda\alpha t_2}))$
24 $\wedge$
25 $\forall t_1, s_1, t_2, s_2$: as such
28 α and $\frac{i}{\lambda}$ are linear independent over $\mathbb{R}$ $\wedge$ $z = \alpha t_1 + \frac{i}{\lambda} s_1$ $\wedge$ $z = \alpha t_2 + \frac{i}{\lambda} s_2$
27 $\implies$
28 $t_1 = t_2$ $\wedge$ $s_1 = s_2$
26 $\wedge$
28 $t_1 = t_2$ $\wedge$ $s_1 = s_2$ $\wedge$ $v_1 = F_{\frac{i}{\lambda}\xi}(s_1, \varphi(ve^{\lambda\alpha t_1}))$ $\wedge$ $v_2 = F_{\frac{i}{\lambda}\xi}(s_2, \varphi(ve^{\lambda\alpha t_2}))$
27 $\implies$
28 $v_1 = v_2$
18 $\wedge$
20 α and $\frac{i}{\lambda}$ are linear independent over $\mathbb{R}$ $\wedge$ γ is a map
19 $\implies$
21 $\mathrm{dom}(\gamma) = \{\alpha t + \frac{i}{\lambda} s : s \in \mathbb{R} \wedge t \in I_{2\pi}\}$ $\wedge$ γ is $\frac{2\pi i}{\lambda}$-periodic
20 $\wedge$
22 $\mathrm{dom}(\gamma) = \{\alpha t + \frac{i}{\lambda} s : s \in \mathbb{R} \wedge t \in I_{2\pi}\}$ $\wedge$ γ is $\frac{2\pi i}{\lambda}$-periodic
21 $\Longleftarrow$
23 $\forall t \in I_{2\pi}$
25 $\gamma_0|_{I_{2\pi}} = \gamma_{2\pi} \implies \gamma_0(t) = \gamma_{2\pi}(t)$
24 $\wedge$
26 $\gamma_0(t) = \gamma_{2\pi}(t)$ $\wedge$ $\gamma_0(t) = \gamma(\alpha t)$ $\wedge$ $\gamma_{2\pi}(t) = \gamma(\alpha t + \frac{2\pi i}{\lambda})$
25 $\implies \gamma(\alpha t) = \gamma(\alpha t + \frac{2\pi i}{\lambda}) \implies \varphi(ve^{\lambda\alpha t}) = F_{\frac{i}{\lambda}\xi}(2\pi, \varphi(ve^{\lambda\alpha t}))$
24 $\wedge$
27 $\varphi(ve^{\lambda\alpha t}) = F_{\frac{i}{\lambda}\xi}(2\pi, \varphi(ve^{\lambda\alpha t}))$
26 $\wedge$
27 proposition 22
25 $\implies$
26 $\mathcal{D}_{\frac{i}{\lambda}\xi} \supset \mathbb{R} \times \{\varphi(ve^{\lambda\alpha t})\}$ $\wedge$ $\{(s, F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t}))) : (s, \varphi(ve^{\lambda\alpha t})) \in \mathcal{D}_{\frac{i}{\lambda}\xi}\}$ is 2π-periodic
20 $\wedge$
23 $\mathrm{dom}(\gamma) = \{\alpha t + \frac{i}{\lambda} s : s \in \mathbb{R} \wedge t \in I_{2\pi}\}$
22 $\wedge$
23 α and $\frac{i}{\lambda}$ are linear independent over $\mathbb{R}$
21 $\implies$
22 $\mathrm{dom}(\gamma)$ is open and connected in $\mathbb{C}$
21 $\implies$
23 γ is a complex integral curve of ξ
22 $\Longleftarrow$
24 $\mathrm{dom}(\gamma)$ is open and connected in $\mathbb{C}$
23 $\wedge$

The intuition of the proof of the following statement is that it is first proven that γ satisfies the complex ODE corresponding to ξ in directions of α and $\frac{i}{\lambda}$ and then the rest follows by $\mathbb{R}$ linearity.

24 γ is holomorphic and $D\gamma(-)(1) = (\xi \circ \gamma)(-)$
22 $\wedge$
24 γ is holomorphic and $D\gamma(-)(1) = (\xi \circ \gamma)(-)$
23 $\Longleftarrow$
24 $(\forall s' \in \mathbb{R})(\forall t' \in I_{2\pi})$
26 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\alpha) = \alpha\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
25 $\wedge$
27 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\alpha) = \alpha\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
26 $\Longleftarrow$
29 $\gamma \circ \{(t, \alpha t + \frac{i}{\lambda}s') : t \in I_{2\pi}\} = \{(t, \gamma(\alpha t + \frac{i}{\lambda}s')) : t \in I_{2\pi}\} = \gamma_{s'}$
28 $\Longrightarrow$
29 $D\gamma(\alpha t' + \frac{i}{\lambda}s') \circ D\{(t, \alpha t + \frac{i}{\lambda}s') : t \in I_{2\pi}\} = D\gamma_{s'}(t')$
28 $\Longrightarrow$
29 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\alpha) = D\gamma_{s'}(t')$
27 $\wedge$
30 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\alpha) = D\gamma_{s'}(t')$
29 $\wedge$
30 $D\gamma_{s'}(t') = \alpha\xi(\gamma_{s'}(t')) \Longleftarrow \gamma_{s'}$ is a real integral curve of ξ
28 $\Longrightarrow$
29 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\alpha) = \alpha\xi(\gamma_{s'}(t'))$
28 $\Longrightarrow$
29 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\alpha) = \alpha\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
25 $\wedge$
27 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\alpha) = \alpha\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
26 $\Longrightarrow$
28 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\frac{i}{\lambda}) = \frac{i}{\lambda}\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
27 $\wedge$
29 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\frac{i}{\lambda}) = \frac{i}{\lambda}\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
28 $\Longleftarrow$
31 $\gamma \circ \{(s, \alpha t' + \frac{i}{\lambda}s) : s \in \mathbb{R}\} = \{(s, \gamma(\alpha t' + \frac{i}{\lambda}s)) : s \in \mathbb{R}\} = \{(s, F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t'}))) : s \in \mathbb{R}\}$
28 $\Longrightarrow$
31 $D\gamma(\alpha t' + \frac{i}{\lambda}s') \circ D\{(s, \alpha t' + \frac{i}{\lambda}s) : s \in \mathbb{R}\}(s') = D\{(s, F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t'}))) : s \in \mathbb{R}\}(s')$
30 $\Longrightarrow$
31 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\frac{i}{\lambda}) = D\{(s, F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t'}))) : s \in \mathbb{R}\}(s')$
29 $\wedge$
32 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\frac{i}{\lambda}) = D\{(s, F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t'}))) : s \in \mathbb{R}\}(s')$
31 $\wedge$
32 $D\{(s, F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t'}))) : s \in \mathbb{R}\}(s') = \frac{i}{\lambda}\xi(F_{\frac{i}{\lambda}\xi}(s', \varphi(ve^{\lambda\alpha t'}))) \Longleftarrow$ definition flow (16)
31 $\wedge$
32 $\frac{i}{\lambda}\xi(F_{\frac{i}{\lambda}\xi}(s', \varphi(ve^{\lambda\alpha t'}))) = \frac{i}{\lambda}\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
30 $\Longrightarrow$
31 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\frac{i}{\lambda}) = \frac{i}{\lambda}\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
27 $\wedge$

29 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\frac{i}{\lambda}) = \frac{i}{\lambda}\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
28 $\implies$
30 γ is holomorphic and $D\gamma(-)(1) = (\xi \circ \gamma)(-)$
29 $\impliedby$
30 $(\forall z \in \mathbb{C})(D\gamma(\alpha t' + \frac{i}{\lambda}s')(z) = z\xi(\gamma(\alpha t' + \frac{i}{\lambda}s')))$
29 $\impliedby$
30 $\forall z \in \mathbb{C}$
33 α and $\frac{i}{\lambda}$ are linear independent over $\mathbb{R}$
32 $\implies$
33 $\exists c_1, c_2 \in \mathbb{R}$: $z = c_1\alpha + c_2\frac{i}{\lambda}$
31 $\wedge$
32 $\forall c_1, c_2$: as such
34 $\wedge_*$
35 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(z) = D\gamma(\alpha t' + \frac{i}{\lambda}s')(c_1\alpha + c_2\frac{i}{\lambda})$
35 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\frac{i}{\lambda}) = \frac{i}{\lambda}\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
35 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(\alpha) = \alpha\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
35 $D\gamma(\alpha t' + \frac{i}{\lambda}s')$ is $\mathbb{R}$-linear
34 $\wedge^*$
33 $\implies$
34 $D\gamma(\alpha t' + \frac{i}{\lambda}s')(z) = z\xi(\gamma(\alpha t' + \frac{i}{\lambda}s'))$
16 $\wedge$
18 $\wedge_*$
19 γ is a complex integral curve of ξ
19 γ is $\frac{2\pi i}{\lambda}$-periodic
19 $\mathrm{dom}(\gamma) = \{\alpha t + \frac{i}{\lambda}s : s \in \mathbb{R} \ \wedge \ t \in I_{2\pi}\}$
18 $\wedge^*$
17 $\implies$
20 $\tilde{\gamma} := \{(z, \gamma(w)) : w \in \mathrm{dom}(\gamma) \ \wedge \ z = \exp(\lambda w)\}$
19 $\wedge$
20 $j := \tilde{\gamma} \cup \{(0,0) \in \mathbb{C} \times V\}$
18 $\implies$
20 proposition 23 $\wedge$ γ is $\frac{2\pi i}{\lambda}$-periodic $\wedge$ γ is holomorphic
19 $\implies$
20 $\tilde{\gamma} \in \mathrm{Hol}(\exp(\lambda \cdot \mathrm{dom}(\gamma)), V)$
19 $\implies$

When the statements up to the first 20 $\impliedby$ *from here are proven, then everything is proven, since "γ is complex integral curve" is already done, and $im(j) = im(\gamma) \cup \{0\}$ and $j(0) = 0$ are trivial from the definitions. After the* 20 $\impliedby$ *the statements are proven in the same order.*

22 $\lim_{z \to 0} \tilde{\gamma}(z) = 0$
21 $\wedge$
22 $\lim_{z \to 0} \tilde{\gamma}(z) = 0 \implies j$ is a holomorphic map from a disc in $\mathbb{C}$ centered at 0 to V
21 $\wedge$

22 j is holomorphic $\implies$ $Dj(0)(1) = v$
21 $\wedge$
22 j is holomorphic $\implies$
23 $\forall z \in \mathrm{dom}(j)$
24 $\lambda z Dj(z)(1) = \xi(j(z))$
21 $\wedge$
22 (j is holomorphic $\wedge$ $Dj(0)(1) = v$) $\implies$ j is an injective immersion
20 $\Longleftarrow$
23 $\lim_{z\to 0} \tilde{\gamma}(z) = 0$
22 $\Longleftarrow$
23 $\forall z_{\mathbb{N}}$: a map $\mathbb{N} \to \exp(\lambda \cdot \mathrm{dom}(\gamma))$ $\wedge$ $\lim_{n\to\infty} z_{\mathbb{N}}(n) = 0$
24 $\lim_{n\to\infty} \tilde{\gamma}(z_{\mathbb{N}}(n)) = 0$
22 $\Longleftarrow$
23 $\forall z_{\mathbb{N}}$: a map $\mathbb{N} \to \exp(\lambda \cdot \mathrm{dom}(\gamma))$ $\wedge$ $\lim_{n\to\infty} z_{\mathbb{N}}(n) = 0$
25 $\exists t_{\mathbb{N}}, s_{\mathbb{N}}$
26 $\forall n \in \mathbb{N}$
27 $\exp(\lambda\alpha t_{\mathbb{N}}(n) + i s_{\mathbb{N}}(n)) = z_{\mathbb{N}}(n)$
24 $\wedge$
25 $\forall t_{\mathbb{N}}, s_{\mathbb{N}}$: as such
28 $\lim_{n\to\infty} \gamma(\alpha t_{\mathbb{N}}(n) + \frac{i}{\lambda} s_{\mathbb{N}}(n)) = 0$
27 $\wedge$
28 $\lim_{n\to\infty} \gamma(\alpha t_{\mathbb{N}}(n) + \frac{i}{\lambda} s_{\mathbb{N}}(n)) = 0 \implies \lim_{n\to\infty} \tilde{\gamma}(z_{\mathbb{N}}(n)) = 0$
26 $\Longleftarrow$
29 $\lim_{n\to\infty} \gamma(\alpha t_{\mathbb{N}}(n) + \frac{i}{\lambda} s_{\mathbb{N}}(n)) = 0$
28 $\Longleftarrow$
31 $X := \{(s, \varphi(v e^{\lambda\alpha t})) : s \in \mathbb{R} \text{ and } t \in I_{2\pi}\} \cup (\mathbb{R} \times \{0\})$
30 $\wedge$
31 $g := F_{\frac{i}{\lambda}\xi}|_X$
29 $\Longrightarrow$
31 $\lim_{n\to\infty} \gamma(\alpha t_{\mathbb{N}}(n) + \frac{i}{\lambda} s_{\mathbb{N}}(n)) = 0$
30 $\Longleftarrow$
31 $\lim_{n\to\infty} g(s_{\mathbb{N}}(n), \varphi(e^{\lambda\alpha t_{\mathbb{N}}(n)})) = 0$
30 $\Longleftarrow$
31 $\forall U_3 \in \tau_V^{\mathrm{norm}}$ $\wedge$ $0 \in U_3$
32 $\exists m \in \mathbb{N}$
33 $\forall n \in \mathbb{N} : n \geq m$
34 $g(s_{\mathbb{N}}(n), \varphi(e^{\lambda\alpha t_{\mathbb{N}}(n)})) \in U_3$
30 $\Longleftarrow$
31 $\forall U_3 \in \tau_V^{\mathrm{norm}}$ $\wedge$ $0 \in U_3$
For the meaning of $\tau_V^{norm}, \tau_V^{norm}|_X$ and $\lim_{n\to\infty}^{\tau_V^{norm}}$, see the definitions 33, 34, 35.
33 $g^{-1}(U_3) \in \tau_{\mathbb{R}\times V}^{\mathrm{norm}}|_X$ $\Longleftarrow$ g is a continuous map $(X, \tau_{\mathbb{R}\times V}^{\mathrm{norm}}|_X) \to (V, \tau_V^{\mathrm{norm}})$ $\Longleftarrow$ $F_{\frac{i}{\lambda}\xi}$ is a continuous map $(X, \tau_{\mathbb{R}\times V}^{\mathrm{norm}}|_{\mathcal{D}_{\frac{i}{\lambda}\xi}}) \to (V, \tau_V^{\mathrm{norm}})$
32 $\wedge$

34 $[0,2\pi] \times \{0\} \subset g^{-1}(U_3) \;\wedge\; g^{-1}(U_3) \in \tau^{\text{norm}}_{\mathbb{R}\times V}|_X \wedge\; [0,2\pi]$ is compact $\wedge$ proposition 32
33 $\implies$
34 $\exists U_4 \in \tau^{\text{norm}}_V|_X$
35 $0 \in U_4 \;\wedge\; [0,2\pi] \times U_4 \subset g^{-1}(U_3)$
32 $\wedge$
33 $\forall U_4$: as such
35 $\lim_{n\to\infty} z_{\mathbb{N}}(n) = 0 \implies \lim_{n\to\infty} |z_{\mathbb{N}}(n)| = 0 \implies \lim_{n\to\infty} |\exp(\lambda\alpha t_{\mathbb{N}}(n) + is_{\mathbb{N}}(n))| = 0$
34 $\wedge$
36 $\lim_{n\to\infty} |\exp(\lambda\alpha t_{\mathbb{N}}(n) + is_{\mathbb{N}}(n))| = 0 \wedge (\forall n \in \mathbb{N})(|\exp(\lambda\alpha t_{\mathbb{N}}(n) + is_{\mathbb{N}}(n))| = |\exp(\lambda\alpha t_{\mathbb{N}}(n))|)$
35 $\implies$
36 $\lim_{n\to\infty} |\exp(\lambda\alpha t_{\mathbb{N}}(n)| = 0$
35 $\implies$
36 $\lim_{n\to\infty} \exp(\lambda\alpha t_{\mathbb{N}}(n)) = 0$
34 $\wedge$
36 $\varphi(0) = 0 \;\wedge\; \varphi$ is continuous $\wedge\; \lim_{n\to\infty} \exp(\lambda\alpha t_{\mathbb{N}}(n)) = 0$
35 $\implies \lim^{\tau_V^{\text{norm}}}_{n\to\infty} \varphi(e^{\lambda\alpha t_{\mathbb{N}}(n)}) = 0$
34 $\wedge$
37 $(\forall n \in \mathbb{N})(\varphi(e^{\lambda\alpha t_{\mathbb{N}}(n)}) \in X) \;\wedge\; \lim^{\tau_V^{\text{norm}}}_{n\to\infty} \varphi(e^{\lambda\alpha t_{\mathbb{N}}(n)}) = 0$
35 $\implies \lim^{\tau_V^{\text{norm}}|_X}_{n\to\infty} \varphi(e^{\lambda\alpha t_{\mathbb{N}}(n)}) = 0$
36 $\wedge$
37 $U_4 \in \tau^{\text{norm}}_V|_X \;\wedge\; 0 \in U_4 \;\wedge\; \lim^{\tau_V^{\text{norm}}|_X}_{n\to\infty} \varphi(e^{\lambda\alpha t_{\mathbb{N}}(n)}) = 0$
35 $\implies$
36 $\exists m \in \mathbb{N}$
37 $\forall n \in \mathbb{N} : n \geq m$
38 $\varphi(e^{\lambda\alpha t_{\mathbb{N}}(n)}) \in U_4$
34 $\wedge$
35 $\forall m$: as such
38 γ has period $\frac{2\pi i}{\lambda} \;\wedge\;$ 0 is an equilibrium of $\xi_{\frac{i}{\lambda}}$
37 $\implies$
38 g is 2π-periodic in its first factor
36 $\wedge$
38 g is 2π-periodic in its first factor $\wedge\; [0,2\pi] \times U_4 \subset g^{-1}(U_3)$
37 $\implies \mathbb{R} \times U_4 \subset g^{-1}(U_3)$
36 $\wedge$
39 $\mathbb{R} \times U_4 \subset g^{-1}(U_3)$
38 $\wedge$
39 $\forall n \in \mathbb{N} : n \geq m$
40 $\varphi(e^{\lambda\alpha t_{\mathbb{N}}(n)}) \in U_4$
37 $\implies$
38 $\forall n \in \mathbb{N} : n \geq m$
39 $g(s_{\mathbb{N}}(n), \varphi(e^{\lambda\alpha t_{\mathbb{N}}(n)})) \in U_3$
27 $\wedge$
29 $\lim_{n\to\infty} \gamma(\alpha t_{\mathbb{N}}(n) + \frac{i}{\lambda} s_{\mathbb{N}}(n)) = 0 \implies \lim_{n\to\infty} \tilde{\gamma}(z_{\mathbb{N}}(n)) = 0$

28 $\Longleftarrow$

30 proposition 23

29 $\Longrightarrow$

30 $\forall n \in \mathbb{N}$

31 $\gamma(\alpha t_{\mathbb{N}}(n) + \frac{i}{\lambda} s_{\mathbb{N}}(n)) = \tilde{\gamma}(e^{\alpha t_{\mathbb{N}}(n) + \frac{i}{\lambda} s_{\mathbb{N}}(n)})$

29 $\Longrightarrow$

30 $\forall n \in \mathbb{N}$

31 $\gamma(\alpha t_{\mathbb{N}}(n) + \frac{i}{\lambda} s_{\mathbb{N}}(n)) = \tilde{\gamma}(z_{\mathbb{N}}(n))$

29 $\Longrightarrow$

30 $\lim_{n\to\infty} \gamma(\alpha t_{\mathbb{N}}(n) + \frac{i}{\lambda} s_{\mathbb{N}}(n)) = \lim_{n\to\infty} \tilde{\gamma}(z_{\mathbb{N}}(n))$

29 $\Longrightarrow$

30 $\lim_{n\to\infty} \tilde{\gamma}(z_{\mathbb{N}}(n)) = 0$

21 $\wedge$

The idea of the following proof is using that a holomorphic map from a punctured disc in $\mathbb{C}$ can be holomorphically extended to the disc if it can be extended continuously. Remember that we have already proven that $\tilde{\gamma}$ is holomorphic (right after its definition).

23 $\lim_{z\to 0} \tilde{\gamma}(z) = 0 \implies j$ is a holomorphic map from a disc in $\mathbb{C}$ centered at 0 to V

22 $\Longleftarrow$

24 $\tilde{\gamma}$ is a map from a punctured disc in $\mathbb{C}$ centered at 0 to V

23 $\wedge$

24 ($\tilde{\gamma}$ is a map from a punctured disc in $\mathbb{C}$ centered at 0 to $V \;\wedge\; \lim_{z\to 0} \tilde{\gamma}(z) = 0) \implies$ j is a holomorphic map from a disc in $\mathbb{C}$ centered at 0 to V

22 $\Longleftarrow$

25 $\tilde{\gamma}$ is a map from a punctured disc in $\mathbb{C}$ centered at 0 to V

24 $\Longleftarrow$

25 $\exp(\lambda \cdot \mathrm{dom}(\gamma))$ is a punctured disc in $\mathbb{C}$ centered at 0

24 $\Longleftarrow$

27 $\mathrm{dom}(\gamma) = \{\alpha t + \frac{i}{\lambda} s : s \in \mathbb{R} \;\wedge\; t \in I_{2\pi}\}$

26 $\Longrightarrow$

27 $\exp(\lambda \cdot \mathrm{dom}(\gamma)) = \{\exp(\mathrm{re}(\alpha\lambda)t + i\mathrm{im}(\alpha\lambda)t + is) : s \in \mathbb{R} \;\wedge\; t \in I_{2\pi}\}$

26 $\Longrightarrow$

27 $\exp(\lambda \cdot \mathrm{dom}(\gamma)) = \{z : |z| = \exp(\mathrm{re}(\alpha\lambda)t) : t \in I_{2\pi}\}$

25 $\wedge$

27 $\exp|_{\mathbb{R}}$ is a monotone increasing map from $(\mathbb{R}, >)$ to $(\mathbb{R}, >) \;\wedge \mathrm{re}\,(\alpha\lambda) > 0$

26 $\Longrightarrow$

27 $\{(t, \exp(\mathrm{re}(\alpha\lambda)t) : t \in I_{2\pi}\}$ is a monotone increasing map from $(I_{2\pi}, >)$ to $(\mathbb{R}, >)$

25 $\wedge$

28 $\exp(\lambda \cdot \mathrm{dom}(\gamma)) = \{z : |z| = \exp(\mathrm{re}(\alpha\lambda)t) : t \in I_{2\pi}\}$

27 $\wedge$

28 $\{(t, \exp(\mathrm{re}(\alpha\lambda)t) : t \in I_{2\pi}\}$ is a monotone increasing map from $(I_{2\pi}, >)$ to $(\mathbb{R}, >)$

27 $\wedge$

28 $I_{2\pi} = (-\infty, \sup I_{2\pi}) \;\wedge\; \sup I_{2\pi} > -\infty$

26 $\Longrightarrow$

27 $\exp(\lambda \cdot \mathrm{dom}(\gamma))$ is a punctured disc in $\mathbb{C}$ centered at 0 of radius $\exp(\mathrm{re}(\alpha\lambda) \cdot \sup I_{2\pi})$
23 $\wedge$
25 ($\tilde{\gamma}$ is a map from punctured a disc in $\mathbb{C}$ centered at 0 to V $\wedge$ $\lim_{z\to 0} \tilde{\gamma}(z) = 0$) $\implies$ j is a holomorphic map from a disc in $\mathbb{C}$ centered at 0 to V
24 $\Longleftarrow$
26 $\wedge_*$
27 $\tilde{\gamma}$ is a map from a punctured disc in $\mathbb{C}$ centered at 0 to V
27 $\tilde{\gamma}$ is holomorphic
27 $\lim_{z\to 0} \tilde{\gamma}(z) = 0$
27 theorem on removable singularities
26 $\wedge^*$
25 $\implies$
26 $\tilde{\gamma} \cup \{0\}$ is a holomorphic map from a disc in $\mathbb{C}$ centered at 0 to V
25 $\implies$
26 j is a holomorphic map from a disc in $\mathbb{C}$ centered at 0 to V
21 $\wedge$
In this proof, $Dj(0)(1)$ is computed by computing $\lim_{z\to 0} Dj(z)(1)$ and using holomorphy of j.
23 j is holomorphic $\implies$ $Dj(0)(1) = v$
22 $\Longleftarrow$
24 $Dj(0)(1) = v$
23 $\Longleftarrow$
25 j is holomorphic
24 $\wedge$
25 $\exists z_{\mathbb{N}}$: a map $\mathbb{N} \to \mathrm{dom}(j)$ $\wedge$ $\lim_{n\to\infty} z_{\mathbb{N}}(n) = 0$ $\wedge$ $\lim_{n\to\infty} Di(z_{\mathbb{N}}(n))(1) = v$
24 $\wedge$
26 $\exists z_{\mathbb{N}}$: a map $\mathbb{N} \to \mathrm{dom}(j)$ $\wedge$ $\lim_{n\to\infty} z_{\mathbb{N}}(n) = 0$ $\wedge$ $\lim_{n\to\infty} Di(z_{\mathbb{N}}(n))(1) = v$
25 $\Longleftarrow$
26 $\exists z_{\mathbb{N}}$: a map $\mathbb{N} \to \mathrm{dom}(\tilde{\gamma})$ $\wedge$ $\lim_{n\to\infty} z_{\mathbb{N}}(n) = 0$ $\wedge$ $\lim_{n\to\infty} D\tilde{\gamma}(z_{\mathbb{N}}(n))(1) = v$
25 $\Longleftarrow$
27 re $(\lambda\alpha) > 0$ $\wedge$ $\mathrm{dom}(\tilde{\gamma}) = \exp(\lambda \cdot \mathrm{dom}(\gamma))$
28 $\implies$
29 $\exists z_{\mathbb{N}}$: a map $\mathbb{N} \to \mathrm{dom}(\tilde{\gamma})$ $\wedge$ $\lim_{n\to\infty} z_{\mathbb{N}}(n) = 0$
30 $\exists t_{\mathbb{N}}$: a map $\mathbb{N} \to \mathbb{R}$ $\wedge$ $(\forall n \in \mathbb{N})(e^{\alpha\lambda t_{\mathbb{N}}(n)} = z_{\mathbb{N}}(n))$
26 $\wedge$
27 $\forall z_{\mathbb{N}}, t_{\mathbb{N}}$: as such
The proof of the following statement has two parts, the statements of which are written down below and its proofs after that.
28 $\lim_{n\to\infty} D\tilde{\gamma}(z_{\mathbb{N}}(n))(1) = v$ $\Longleftarrow$
30 $\forall n \in \mathbb{N}$
31 $D\tilde{\gamma}(z_{\mathbb{N}}(n))(1) = \frac{1}{z_{\mathbb{N}}(n)\lambda\alpha} D\varphi(z_{\mathbb{N}}(n)v)(z_{\mathbb{N}}(n)\lambda\alpha v)$
29 $\wedge$
30 $\lim_{n\to\infty} \frac{1}{z_{\mathbb{N}}(n)\lambda\alpha} D\varphi(z_{\mathbb{N}}(n)v)(z_{\mathbb{N}}(n)\lambda\alpha v) = v$

28 $\Longleftarrow$
31 $\forall n \in \mathbb{N}$
32 $D\tilde{\gamma}(z_{\mathbb{N}}(n))(1) = \frac{1}{z_{\mathbb{N}}(n)\lambda\alpha} D\varphi(z_{\mathbb{N}}(n)v)(z_{\mathbb{N}}(n)\lambda\alpha v)$
30 $\Longleftarrow$
31 $\forall n \in \mathbb{N}$
34 $D\tilde{\gamma}(z_{\mathbb{N}}(n))(1) = \frac{1}{z_{\mathbb{N}}(n)\lambda} D\gamma(\alpha t_{\mathbb{N}}(n))(1)$
33 $\wedge$
34 $D\gamma(\alpha t_{\mathbb{N}}(n))(1) = \frac{1}{\alpha} D\varphi(z_{\mathbb{N}}(n)v)(z_{\mathbb{N}}(n)\lambda\alpha v)$
32 $\Longleftarrow$
35 $D\tilde{\gamma}(z_{\mathbb{N}}(n))(1) = \frac{1}{z_{\mathbb{N}}(n)\lambda} D\gamma(\alpha t_{\mathbb{N}}(n))(1)$
34 $\Longleftarrow$
35 $z_{\mathbb{N}}(n) = e^{\lambda(\alpha t_{\mathbb{N}}(n))}$ $\wedge$ proposition 23
34 $\wedge$
35 $D\gamma(\alpha t_{\mathbb{N}}(n))(1) = \frac{1}{\alpha} D\varphi(z_{\mathbb{N}}(n)v)(z_{\mathbb{N}}(n)\lambda\alpha v)$
34 $\Longleftarrow$
36 $D\gamma(\alpha t_{\mathbb{N}}(n))(1) = \frac{1}{\alpha} D\gamma(\alpha t_{\mathbb{N}}(n))(\alpha)$ $\Longleftarrow$ γ is holomorphic
35 $\wedge$
36 $D\gamma(\alpha t_{\mathbb{N}}(n))(\alpha) = D\varphi(z_{\mathbb{N}}(n)v)(z_{\mathbb{N}}(n)\lambda\alpha v)$
35 $\wedge$
37 $D\gamma(\alpha t_{\mathbb{N}}(n))(\alpha) = D\varphi(z_{\mathbb{N}}(n)v)(z_{\mathbb{N}}(n)\lambda\alpha v)$
36 $\Longleftarrow$
40 $D\gamma(\alpha t_{\mathbb{N}}(n))(\alpha) = D\{((s,t), F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t}))) : t \in I_{2\pi}\}(0,t) \circ D\{((s,t), \alpha t + \frac{i}{\lambda}s) : s,t \in \mathbb{R}\}^{-1}(\alpha)(\alpha t_{\mathbb{N}}(n))$
39 $\wedge$
40 $D\{((s,t), \alpha t + \frac{i}{\lambda}s) : s,t \in \mathbb{R}\}^{-1}(\alpha)(\alpha t_{\mathbb{N}}(n)) = (0,1)$
38 $\Longrightarrow$
39 $D\gamma(\alpha t_{\mathbb{N}}(n))(\alpha) = D\{((s,t), F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t}))) : t \in I_{2\pi}\}(0, t_{\mathbb{N}}(n))(0,1)$
37 $\wedge$
40 $D\gamma(\alpha t_{\mathbb{N}}(n))(\alpha) = D\{((s,t), F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t}))) : t \in I_{2\pi}\}(0, t_{\mathbb{N}}(n))(0,1)$
39 $\wedge$
40 $D\{((s,t), F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t}))) : t \in I_{2\pi}\}(0, t_{\mathbb{N}}(n))(0,1) =$
$= D_2F_{\frac{i}{\lambda}\xi}(0, \varphi(ve^{\lambda\alpha t_{\mathbb{N}}(n)}))(D\varphi(ve^{\lambda\alpha t_{\mathbb{N}}(n)})(\lambda\alpha ve^{\lambda\alpha t_{\mathbb{N}}(n)}))$
38 $\Longrightarrow$
39 $D\gamma(\alpha t_{\mathbb{N}}(n))(\alpha) = D_2F_{\frac{i}{\lambda}\xi}(0, \varphi(ve^{\lambda\alpha t_{\mathbb{N}}(n)}))(D\varphi(ve^{\lambda\alpha t_{\mathbb{N}}(n)})(\lambda\alpha ve^{\lambda\alpha t_{\mathbb{N}}(n)}))$
37 $\wedge$
40 $D\gamma(\alpha t_{\mathbb{N}}(n))(\alpha) = D_2F_{\frac{i}{\lambda}\xi}(0, \varphi(ve^{\lambda\alpha t_{\mathbb{N}}(n)}))(D\varphi(ve^{\lambda\alpha t_{\mathbb{N}}(n)})(\lambda\alpha ve^{\lambda\alpha t_{\mathbb{N}}(n)}))$
39 $\wedge$
40 $D_2F_{\frac{i}{\lambda}\xi}(0, \varphi(ve^{\lambda\alpha t})) = \mathrm{id}_V$ $\Longleftarrow$ $\{F_{\frac{i}{\lambda}\xi}(0,x) : x \in U\} = \mathrm{id}_U$
38 $\Longrightarrow$
39 $D\gamma(\alpha t_{\mathbb{N}}(n))(\alpha) = D\varphi(ve^{\lambda\alpha t_{\mathbb{N}}(n)})(\lambda\alpha ve^{\lambda\alpha t_{\mathbb{N}}(n)})$
38 $\Longrightarrow$
39 $D\gamma(\alpha t_{\mathbb{N}}(n))(\alpha) = D\varphi(z_{\mathbb{N}}(n)v)(z_{\mathbb{N}}(n)\lambda\alpha v)$
29 $\wedge$

31 $\lim_{n\to\infty} \frac{1}{z_{\mathbb{N}}(n)\lambda\alpha} D\varphi(z_{\mathbb{N}}(n)v)(z_{\mathbb{N}}(n)\lambda\alpha v) = v$
30 $\Longleftarrow$
31 $\{(x, ||x||) : x \in V\}$ is a norm on $V \implies$
32 $\forall r \in \mathbb{R}_{>0}$
33 $\exists m \in \mathbb{N}$
34 $\forall n \in \mathbb{N}$: $n \geq m$
35 $||\frac{1}{z_{\mathbb{N}}(m)\lambda\alpha}(D\varphi(z_{\mathbb{N}}(m)v)(z_{\mathbb{N}}(m)\lambda\alpha v)) - v|| < r$
30 $\Longleftarrow$
31 $\{(x, ||x||) : x \in V\}$ is a norm on $V \implies$
32 $\forall r \in \mathbb{R}_{>0}$
35 φ is continuously differentiable $\wedge$ $D\varphi(0) = \mathrm{id}_V$ $\wedge$ $\lim_{n\to\infty} z_{\mathbb{N}}(n) = 0$
34 $\implies$
35 $\exists m \in \mathbb{N}$
36 $\forall n \in \mathbb{N}$: $n \geq m$
37 $||D\varphi(z_{\mathbb{N}}(m)v) - \mathrm{id}_V|| < \frac{r}{||v||}$
33 $\wedge$
34 $\forall m, n$: as such
36 $||\frac{1}{z_{\mathbb{N}}(m)\lambda\alpha}(D\varphi(z_{\mathbb{N}}(m)v)(z_{\mathbb{N}}(m)\lambda\alpha v)) - v|| = ||\frac{1}{z_{\mathbb{N}}(m)\lambda\alpha}(D\varphi(z_{\mathbb{N}}(m)v) - \mathrm{id}_V)(z_{\mathbb{N}}(m)\lambda\alpha v)|| <$ $|\frac{1}{z_{\mathbb{N}}(m)\lambda\alpha}|\frac{r}{||v||}||z_{\mathbb{N}}(m)\lambda\alpha v|| = r$
35 $\implies$ $||\frac{1}{z_{\mathbb{N}}(m)\lambda\alpha}(D\varphi(z_{\mathbb{N}}(m)v)(z_{\mathbb{N}}(m)\lambda\alpha v)) - v|| < r$
21 $\wedge$
23 j is holomorphic $\implies$
24 $\forall z \in \mathrm{dom}(j)$
25 $\lambda z Dj(z)(1) = \xi(j(z))$
22 $\Longleftarrow$
25 $j(0) = 0 \wedge \xi(0) = 0$
24 $\implies$
25 $z = 0 \implies \lambda z Dj(z)(1) = \xi(j(z))$
23 $\wedge$
25 proposition 23 $\wedge$ γ is $\frac{2\pi i}{\lambda}$-periodic $\wedge$ γ is holomorphic
24 $\implies$
25 $\forall z \in \mathrm{dom}(\tilde{\gamma})$
26 $\forall w \in \mathrm{dom}(\gamma)$: $z = e^{\lambda w}$
27 $D\tilde{\gamma}(z) = \lambda z D\gamma(w)$
24 $\implies$
25 $\forall z \in \mathrm{dom}(j)\backslash\{0\}$
26 $\forall w \in \mathrm{dom}(\gamma)$: $z = e^{\lambda w}$
27 $D\tilde{\gamma}(z) = \lambda z D\gamma(w)$
23 $\wedge$
25 $\forall z \in \mathrm{dom}(j)\backslash\{0\}$
26 $\forall w \in \mathrm{dom}(\gamma)$: $z = e^{\lambda w}$
29 $\lambda z D\tilde{\gamma}(z) = D\gamma(w)$ $\wedge$ $Dj(z) = D\tilde{\gamma}(z)$
28 $\implies$

29 $\lambda z Dj(z) = D\gamma(w)$
27 $\wedge$
29 $\lambda z Dj(z) = D\gamma(w) \ \wedge \ D\gamma(w) = \xi(\gamma(w))$
28 $\implies$
29 $\lambda z Dj(z) = \xi(\gamma(w))$
28 $\implies$
29 $\lambda z Dj(z) = \xi(j(z))$
21 $\wedge$
23 (j is holomorphic $\wedge$ $Dj(0)(1) = v$) $\implies$ j is an injective immersion
Instead of the following proof, the theorem could also have been proven using an j and γ restricted to smaller sets and using the rank theorem and $Dj(0)(1) = v$. However, it seemed nice to show that the restriction is not necessary. Proof outline: first showing that $j^{-1}(\{0\}) = \{0\}$ and j is immersion which is done by first showing that γ is not constant. Then injectivity is done.
22 $\impliedby$
25 j is an immersion
24 $\wedge$
25 $\forall z_1, z_2 \in \mathrm{dom}(j)$: $j(z_1) = j(z_2) \ \wedge \ 0 \in \{z_1, z_2\}$
26 $z_1 = z_2$
23 $\wedge$
26 j is an immersion
25 $\wedge$
26 $\forall z_1, z_2 \in \mathrm{dom}(j)$: $j(z_1) = j(z_2) \ \wedge \ 0 \in \{z_1, z_2\}$
27 $z_1 = z_2$
24 $\impliedby$
26 γ is non-constant
25 $\wedge$
27 γ is non-constant
26 $\impliedby$
27 $\{(\alpha t + \frac{i}{\lambda} s, F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t}))) : s \in \mathbb{R}$ and $t \in I_{2\pi}\}$ is non-constant
26 $\impliedby$
27 $\forall s \in \mathbb{R}$
28 $\{(t, F_{\frac{i}{\lambda}\xi}(s, \varphi(ve^{\lambda\alpha t}))) : t \in I_{2\pi}\}$ is non-constant
26 $\impliedby$
27 $\wedge_*$
28 $|I_{2\pi}| \geq 2 \impliedby (I_{2\pi} = (-\infty, \sup(I_{2\pi})) \ \wedge \ \sup(I_{2\pi}) > -\infty)$
28 $(\forall s \in \mathbb{R})(F_{\frac{i}{\lambda}\xi}(s, -))$ is injective
28 φ is injective
28 $\{(t, ve^{\lambda\alpha t}) : t \in I_{2\pi}\}$ is injective
27 $\wedge^*$
25 $\wedge$
27 γ is non-constant
26 $\implies$

28 γ is a non-constant and γ is a complex integral curve
27 $\Longrightarrow$
28 γ is an immersion
27 $\Longrightarrow$
30 proposition 23 $\wedge$ γ is $\frac{2\pi i}{\lambda}$-periodic $\wedge$ γ is holomorphic
29 $\Longrightarrow$
30 $\forall z \in \mathrm{dom}(j)\backslash\{0\}$
31 $\forall w \in \mathrm{dom}(\gamma)$
32 $\lambda z D j(z)(1) = D\gamma(w)$
28 $\wedge$
32 $\forall z \in \mathrm{dom}(j)\backslash\{0\}$
33 $\forall w \in \mathrm{dom}(\gamma)$
34 $\lambda z D j(z)(1) = D\gamma(w)$
31 $\wedge$
32 γ is an immersion
30 $\Longrightarrow$
31 $\forall z \in \mathrm{dom}(j)\backslash\{0\}$
32 $Dj(z)(1) \neq 0$
28 $\wedge$
31 $\forall z \in \mathrm{dom}(j)\backslash\{0\}$
32 $Dj(z)(1) \neq 0$
30 $\wedge$
31 $Dj(0)(1) = v \neq 0$
29 $\Longrightarrow$
30 j is an immersion
27 $\wedge$
29 γ is a non constant complex integral curve of ξ $\wedge\, \xi(0) = 0$
28 $\Longrightarrow$
29 $0 \notin \mathrm{im}(\gamma)$
28 $\Longrightarrow$
29 $\forall z_1, z_2 \in \mathrm{dom}(j)\colon\ j(z_1) = j(z_2)\ \wedge\ 0 \in \{z_1, z_2\}$
30 $z_1 = z_2$
24 $\wedge$
26 $\forall z_1, z_2 \in \mathrm{dom}(j)\colon\ j(z_1) = j(z_2)\ \wedge\ 0 \in \{z_1, z_2\}$
27 $z_1 = z_2$
25 $\Longrightarrow$

The proof of injectivity is the only thing that remains. By what we have proven so far, we can concentrate on the case that j is equal on two nonzero complex numbers. The main ingredients are proving that j is locally injective near 0, and using that real integral curves are either periodic or injective.

27 j is injective
26 $\Longleftarrow$
28 $\forall z_1, z_2 \in \mathrm{dom}(j)\colon\ j(z_1) = j(z_2)\ \wedge\ 0 \in \{z_1, z_2\}$

29 $z_1 = z_2$
27 $\wedge$
28 $\forall z_1, z_2 \in \mathrm{dom}(j)\colon\ j(z_1) = j(z_2)\ \wedge\ 0 \notin \{z_1, z_2\}$
29 $z_1 = z_2$
27 $\wedge$
29 $\forall z_1, z_2 \in \mathrm{dom}(j)\colon\ j(z_1) = j(z_2)\ \wedge\ 0 \notin \{z_1, z_2\}$
30 $z_1 = z_2$
28 $\Longleftarrow$
29 $\forall z_1, z_2 \in \mathrm{dom}(j)\colon\ j(z_1) = j(z_2)\ \wedge\ 0 \notin \{z_1, z_2\}$
32 $Dj(0)(1) = v\ \wedge\ v \neq 0\ \wedge$ rank theorem[2]
31 $\Longrightarrow$
32 $\exists U_3 \subset \mathrm{dom}(j)$: a neighborhood of 0 in $\mathrm{dom}(j)$
33 $j|_{U_3}$ is injective
30 $\wedge$
31 $\forall U_3$: as such
32 $\forall w_1, w_2\colon\ e^{\lambda w_1} = z_1\ \wedge\ e^{\lambda w_2} = z_2$
36 $\mathrm{re}(\alpha\lambda) > 0\ \Longleftarrow\ \alpha \in A_1$
35 $\Longrightarrow$
36 $\lim_{t\to-\infty} \exp(\alpha\lambda t + \lambda w_1) = 0\ \wedge\ \lim_{t\to-\infty} \exp(\alpha\lambda t + \lambda w_2) = 0$
35 $\Longrightarrow$
36 $\exists t \in \mathbb{R}\colon\ \exp(\alpha\lambda t + \lambda w_1), \exp(\alpha\lambda t + \lambda w_1) \in U_3$
34 $\wedge$
35 $\forall t \in \mathbb{R}$: as such
38 $z_1 = z_2$
37 $\Longleftarrow$
38 $\exp(\lambda w_1) = \exp(\lambda w_2)$
37 $\Longleftarrow$
38 $\exp(\alpha\lambda t + \lambda w_1) = \exp(\alpha\lambda t + \lambda w_2)$
37 $\Longleftarrow$
38 $j(\exp(\alpha\lambda t+\lambda w_1)) = j(\exp(\alpha\lambda t+\lambda w_2)) \wedge j|_{U_3}$ is injective $\wedge\ \exp(\alpha\lambda t+\lambda w_1), \exp(\alpha\lambda t+ \lambda w_1) \in U_3$
36 $\wedge$
38 $j(\exp(\alpha\lambda t + \lambda w_1)) = j(\exp(\alpha\lambda t + \lambda w_2))$
37 $\Longleftarrow$
38 $\gamma(\alpha t + w_1) = \gamma(\alpha t + w_2)$
37 $\Longleftarrow$
38 $\forall w \in \{w_1, w_2\}$
39 $\{(t', \gamma(\alpha t' + w)) : \alpha t' + w \in \mathrm{dom}(\gamma)\} = \{(t', F_{\alpha\xi}(t', j(z_1))) : \alpha t' + w \in \mathrm{dom}(\gamma)\}$
37 $\Longleftarrow$
38 $\forall w \in \{w_1, w_2\}$
41 γ is a complex integral curve of ξ
40 $\Longrightarrow$

[2]The rank theorem from analysis ([**Lee12**]), not from linear algebra.

41 $\{(t', \gamma(\alpha t' + w)) : \alpha t' + w \in \mathrm{dom}(\gamma)\}$ is a real integral curve of $\alpha\xi$
40 $\wedge$
41 $\{(t', \gamma(\alpha t' + w)) : \alpha t' + w \in \mathrm{dom}(\gamma)\}$ is a real integral curve of $\alpha\xi \ \wedge$ prop. 17
40 $\implies$
41 $\{(t', \gamma(\alpha t' + w)) : \alpha t' + w \in \mathrm{dom}(\gamma)\} = \{(t', F_{\alpha\xi}(t', \gamma(w))) : \alpha t' + w \in \mathrm{dom}(\gamma)\}$
39 $\wedge$
41 $\{(t', \gamma(\alpha t' + w)) : \alpha t' + w \in \mathrm{dom}(\gamma)\} = \{(t', F_{\alpha\xi}(t', \gamma(w))) : \alpha t' + w \in \mathrm{dom}(\gamma)\}$
40 $\implies$
43 $\gamma(w_1) = j(z_1) \ \wedge \ \gamma(w_2) = j(z_2) \ \wedge \ j(z_1) = j(z_2)$
42 $\implies$
43 $\gamma(w) = j(z_1)$
41 $\wedge$
44 $\gamma(w) = j(z_1)$
43 $\wedge$
44 $\{(t', \gamma(\alpha t' + w)) : \alpha t' + w \in \mathrm{dom}(\gamma)\} = \{(t', F_{\alpha\xi}(t', \gamma(w))) : \alpha t' + w \in \mathrm{dom}(\gamma)\}$
42 $\implies$
43 $\{(t', \gamma(\alpha t' + w)) : \alpha t' + w \in \mathrm{dom}(\gamma)\} = \{(t', F_{\alpha\xi}(t', j(z_1))) : \alpha t' + w \in \mathrm{dom}(\gamma)\}$

Now, the hard work is done, and what is promised in the title of this thesis, is a corollary of the above result. But to state the corollary, some definitions are needed. The reader is expected to be familiar with the notion of a Riemann surface. While often in notation explicit reference to a maximal holomorphic atlas is suppressed, it seemed necessary to do here. The definitions here are not meant to be new or surprising, but they are included so to make the statements involving them clearer. Only "Riemann subsurface" is new, but it just means a complex submanifold of complex dimension 1.

Definition 38.
1 $\forall V$: a complex vector space of finite dimension
2 $\mathcal{A}_V :=$ (the maximal holomorphic atlas on V generated by id_V)

Definition 39.
1 $\forall V$: a complex vector space of finite dimension
2 $\forall \Sigma, \mathcal{A}$
3 $\forall j \in \mathrm{Hol}((\Sigma, \mathcal{A}), (V, \mathcal{A}_V))$
2 $D(\mathcal{A}, \mathcal{A}_V)(j) :=$ (the derivative of j with respect to the maximal holomorphic atlasses $\mathcal{A}$ and $\mathcal{A}_V$)

Definition 40 (Riemann subsurface)**.**
1 $\forall V$: a complex vector space of finite dimension
2 $\forall \Sigma, \mathcal{A}$
4 $(\Sigma, \mathcal{A})$ is a Riemann subsurface of V
3 $:\iff$
4 $\wedge_*$
5 $(\Sigma, \mathcal{A})$ is Riemann surface
5 $\Sigma \subset V$
5 $\mathrm{id}_\Sigma \in \mathrm{Hol}((\Sigma, \mathcal{A}), (V, \mathcal{A}_V))$

5 $\forall x \in \Sigma$
6 $D(\mathcal{A}, \mathcal{A}_V)(\mathrm{id}_\Sigma)(x) \neq 0$
4 $\wedge^*$

Definition 41 (tangent to)**.**
1 $\forall V$: a complex vector space of finite dimension
2 $\forall \Sigma, \mathcal{A}$: $(\Sigma, \mathcal{A})$ is a Riemann subsurface of V
3 $\forall U \subset V$: U is open in V
4 $\forall \xi : U \to V$
6 ξ is tangent to $(\Sigma, \mathcal{A})$
5 $:\iff$
6 $\forall x \in \Sigma \cap U$
7 $\xi(x) \in \mathrm{im}(D(\mathcal{A}, \mathcal{A}_V)(\mathrm{id}_\Sigma)(x))$

Corollary 42.
1 $\forall V : V$ is a finite dimensional vector space over $\mathbb{C}$
2 $\forall U \subset V$: U is a neighborhood of 0 in V
3 $\forall \xi \in \mathrm{Hol}(U, V)$: $\xi(0) = 0$ and $D\xi(0)$ is invertible
4 $\forall \lambda$: λ is an extreme point of the union of $\{0\}$ and the eigenvalues of $D\xi(0)$
5 $\forall v \in \ker(D\xi(0) - \lambda \cdot \mathrm{id}_V)\backslash\{0\}$
6 $\exists \Sigma, \mathcal{A}$: $(\Sigma, \mathcal{A})$ a Riemann subsurface of V and $\Sigma \subset U$
7 $0 \in \Sigma \ \wedge \ \xi$ is tangent to $(\Sigma, \mathcal{A})$

Prosaic form of the statement. Let V is a finite dimensional vector space over $\mathbb{C}$; U is a neighborhood of 0 in V; $\xi \in \mathrm{Hol}(U, V)$: $\xi(0) = 0$ and $D\xi(0)$ is invertible; λ is an extreme point of the union of $\{0\}$ and the eigenvalues of $D\xi(0)$; $v \in \ker(D\xi(0) - \lambda \cdot \mathrm{id}_V)\backslash\{0\}$. Then there exists a $(\Sigma, \mathcal{A})$ that is a Riemann subsurface of V and such that $\Sigma \subset U$ and such that $0 \in \Sigma$ and ξ is tangent to $(\Sigma, \mathcal{A})$

In the proof, the first 5 lines are taken as assumptions.

Proof sketch. It follows easily from theorem 37. If one takes j as in theorem 37, then one can take $\Sigma := \mathrm{im}(j)$ and take the induced maximal holomorphic atlas from j. The assertions will then easily follow from the conditions on j.

Proof.
3 theorem 37
2 $\implies$
3 $\exists j$: j is a holomorphic injective immersion of a disc in $\mathbb{C}$ to U
4 $\exists \gamma$: a complex integral curve of ξ
6 $\mathrm{im}(j) = \mathrm{im}(\gamma) \cup \{0\} \ \wedge \ j(0) = 0 \ \wedge \ Dj(0)(1) = v$
5 $\wedge$
6 $\forall z \in \mathrm{dom}(j)$
7 $Dj(z)(1) = \lambda z \xi(j(z))$
1 $\wedge$

2 $\forall j, \gamma$: as such
5 $\Sigma := \mathrm{im}(j)$
4 $\wedge$
5 $\mathcal{A}$:=(the maximal holomorphic atlas on $\mathrm{im}(j)$ generated by j^{-1})
3 $\implies$
6 j is a holomorphic injective immersion of a disc in $\mathbb{C}$ to U
5 $\implies$
6 $(\Sigma, \mathcal{A})$ is Riemann subsurface of V
4 $\wedge$
6 $\mathrm{im}(j) = \mathrm{im}(\gamma) \cup \{0\} \;\wedge\; \mathrm{im}(\gamma) \cup \{0\} \subset U$
5 $\implies$
6 $\Sigma \subset U$
4 $\wedge$
5 $\mathrm{im}(j) = \mathrm{im}(\gamma) \cup \{0\} \implies 0 \in \Sigma$
4 $\wedge$
6 $\forall z \in \mathrm{dom}(j)$
7 $\lambda z D j(z)(1) = \xi(j(z))$
5 $\implies$
6 ξ is tangent to $(\Sigma, \mathcal{A})$

Note on normal modes
The informed reader might recall the existence of so called 'normal modes near real elliptic Hamiltonian equilibria. Normal modes are certain periodic orbits near such an equilibrium. One may ask what the relation is between the embedded Riemann surfaces whose existence we just proved, and these normal modes. The first thing to observe is that a normal mode cannot be contained in such a Riemann surface: since we assume analyticity a real periodic curve is of the form $\sum_{n=0}^{\infty}(a_n e^{i\omega n t} + \bar{a}_n e^{-i\omega n t})$. The Riemann surface in consideration, however, tends in some complex direction to the equilibrium, whereas one readily sees that functions of the form $e^{i\omega n t}$ and $e^{-i\omega n t}$ cannot converge simultaneously for $n \geq 1$ and for any complex direction of time. Secondly, as t goes to infinity in the negative imaginary direction, and as long as the expression $\sum_{n=0}^{\infty}(a_n e^{i\omega n t} + \bar{a}_n e^{-i\omega n t})$ converges for finite t, then $\sum_{n=0}^{\infty}(a_n e^{i\omega n t} + \bar{a}_n e^{-i\omega n t}) \to \sum_{n=0}^{\infty} a_n e^{i\omega n t}$. The Riemann surface through the equilibrium has $\sum_{n=0}^{\infty} a'_n e^{i\omega' n t}$ as expansion, so it is possible that in some cases in the complexified space the normal mode is contained in a Riemann surface that tends to a Riemann surface through the equilibrium.

CHAPTER 4

Obstructions for the existence of integrals

The original aim of this thesis was to give a non-integrability[1] condition of Hamiltonian systems near equilibria using the theory Morales-Ruiz and Ramis. To apply this theory to an equilibrium, one needs theorem 37. However the author[2] failed to find any obstructions in this way. On the other hand, he noted that in this local case near an equilibrium, a very similar theory could be built, that would fulfill the original aim. Philosophically, the reason behind this is that this altered theory would only require *formal power series solutions* of the differential equation, whereas for an analytic curve, one would also need convergence. Hence there are more objects that give an obstruction to the existence of integrals in this new theory. Sadly, there was not enough time left to elaborate the whole theory sufficiently, but this will be done later and the author is optimistic about its chances to succeeding. For now, what the author hoped to prove is written down as a conjecture.

Definition 43.
1 $\forall m, n \in \mathbb{N}_{\geq 1}$
2 $I_{m,n} := \{i : \text{a map } \{1, .., m\} \to \{1, ..., n\} \ \wedge \ \sum_{i=1}^{m} i(m) = n\}$

Definition 44.
1 $\forall V$: a complex vector space of finite dimension
2 $\forall U \subset V$: a neighborhood of 0
3 $\forall \xi \in C^{\infty}(U, V)$
4 $\tilde{D}^n \xi := \frac{1}{n!} D^n \xi$

Definition 45 (Taylor independent)**.**
1 $\forall V, W$: a complex vector spaces of finite dimension
2 $\forall U \subset V$: a neighborhood of 0
3 $\forall f \in C^{\infty}(U, W)$
5 f is Taylor independent at 0
4 $:\Longleftrightarrow$
5 $\nexists g \in C^{\infty}(W, \mathbb{C})$
6 $\forall n \in \mathbb{N}_{\geq 1}$
7 $D^n(g \circ f)(0) = 0$

Definition 46 (Taylor integral)**.**
1 $\forall V, W$: a complex vector spaces of finite dimension

[1]Integrability of a Hamiltonian system is roughly the existence of enough integrals that are in 'involution'.
[2]the author := the author of this thesis

2 $\forall U \subset V$: a neighborhood of 0
3 $\forall \xi \in C^\infty(U,V)$
4 $\forall f \in C^\infty(V,W)$
6 f is a Taylor integral of ξ at 0
5 $: \Longleftrightarrow$
6 $\forall n \in \mathbb{N}_{\geq 0}$
7 $D^n(Df(-)(\xi))(0) = 0$

If the reader is familiar with the notion of *functional independence* and of an *integral* then he can convince himself that in the holomorphic case, Taylor integral is the same as integral, and Taylor independent is the same as functionally independent.

Conjecture 47.
1 $\forall V$: a complex vector space of finite dimension
2 $\forall U \subset V$: a neighborhood of 0
3 $\forall \xi \in C^\infty(U,V)$: $\xi(0) = 0 \ \wedge \ D\xi(0)$ is invertible
4 $\forall \lambda$: an eigenvalue of $D\xi(0)$
6 $\nexists k \in \mathbb{N}_{\geq 2}$
7 $k\lambda$ is an eigenvalue of $D\xi(0)$
5 $\Longrightarrow$
6 $\forall v \in V$: $D\xi(0)v = \lambda v$
8 $\gamma := \{(1,v)\} \cup \{(n, (n\lambda - D\xi(0))^{-1}(\sum_{m=2}^{n} \sum_{i \in I_{m,n}} \tilde{D}^m \xi(0)(\gamma(i(1)), ..., \gamma(i(m))))) : n \in \mathbb{N}_{\geq 2}\}$
7 $\Longrightarrow$
9 $\forall \mu \in \mathbb{C}$
10 $\forall w \in V$
13 $\mu w = D\xi(0)w$
12 $\wedge$
13 $\exists m \in \mathbb{N}_{\geq 1}$
14 $\mu + m\lambda$ is an eigenvalue of $D\xi(0)$
11 $\Longrightarrow$
13 $m_\mu := \min\{m \in \mathbb{N}_{\geq 1} : \mu + m\lambda$ is an eigenvalue of $D\xi(0)\}$
12 $\wedge$
13 $y_w := \{(1,w)\} \cup \{(n, (\mu + (n-1)\lambda - D\xi(0))^{-1}(\sum_{m=2}^{n} \sum_{i \in I_{m,n}} \tilde{D}^m \xi(0)(y_w(i(1)), \gamma(i(2)), ..., \gamma(i(m))))) : n \in \{1, ..., m_\mu\}\}$
8 $\Longrightarrow$
9 $\forall W$: a complex vector space
10 $\forall f \in C^\infty(U,W)$: f is Taylor independent at 0 and a Taylor integral of ξ at 0
11 $\dim W \leq$
$|\{(w,\mu) \in V \times \mathbb{C} : \mu w = D\xi(0)w \ \wedge \ (\exists m)(\mu + m\lambda$ is an eigenvalue of $D\xi(0)) \ \wedge \ y(m_\mu) \notin \operatorname{im}\,(\mu + m_\mu\lambda - D\xi(0))\}|$

Prosaic form of the statement. Let V be a complex vector space of finite dimension; $U \subset V$: a neighborhood of 0; $\xi \in C^\infty(U,V)$ such that $\xi(0) = 0$ and $D\xi(0)$ is invertible; λ an eigenvalue of $D\xi(0)$ such that there does not exist a $k \in \mathbb{N}_{\geq 2}$ such that $k\lambda$ is an

eigenvalue of $D\xi(0)$; let $v \in V$ such that $D\xi(0)v = \lambda v$. Define γ inductively by $\gamma := \{(1,v)\} \cup \{(n,(n\lambda - D\xi(0))^{-1}(\sum_{m=2}^{n}\sum_{i\in I_{m,n}} \tilde{D}^m\xi(0)(\gamma(i(1)),...,\gamma(i(m))))) : n \in \mathbb{N}_{\geq 2}\}$. Then for each $\mu \in \mathbb{C}$ and $w \in V$ such that $\mu w = D\xi(0)w$ and there exists$m \in \mathbb{N}_{\geq 1}$ such that $\mu + m\lambda$ is an eigenvalue of $D\xi(0)$, define $m_\mu := \min\{m \in \mathbb{N}_{\geq 1} : \mu + m\lambda$ is an eigenvalue of $D\xi(0)\}$ and y_w inductively by $y_w := \{(1,w)\} \cup \{(n,(\mu + (n-1)\lambda - D\xi(0))^{-1}(\sum_{m=2}^{n}\sum_{i\in I_{m,n}} \tilde{D}^m\xi(0)(y_w(i(1)),\gamma(i(2)),...,\gamma(i(m)))) : n \in \{1,...,m_\mu\}\}$. Then for each complex vector space W and $f \in C^\infty(U,W)$: f is Taylor independent at 0 and a Taylor integral of ξ at 0, the inequality $\dim W \leq$
$|\{(w,\mu) \in V \times \mathbb{C} : \mu w = D\xi(0)w \ \wedge \ (\exists m)(\mu + m\lambda$ is an eigenvalue of $D\xi(0)) \ \wedge \ y(m_\mu) \notin$ im $(\mu + m_\mu\lambda - D\xi(0))\}|$ holds.

Given a linear map of which the eigenvalues obey suitable resonance relations, then the above conjecture would give an integer k and a map from vector fields to a vector space, that is polynomial, and depends only on the first k derivatives of the vector field. If the map applied to this polynomial mapping is nonzero, then this would give an upper bound on the number of independent integrals. For example, in case that the eigenvalues are all integers and one eigenvalue is equal to one, and the rest is negative, then the conjecture gives an obstruction to more than one independent integrals. Another example: if the set of eigenvalues is $\{-3,-2,-1,1,2,3\}$, then the upper bound on independent integrals is for some vector fields 2.

Bibliography

[Con07] Conway.
A Course in Functional Analysis.
Springer New York, NY, 2 edition, 2007.

[CS82] Cesar Camacho and Paulo Sad.
Invariant varieties through singularities of holomorphic vector fields.
Annals of Mathematics, 115(3):579–595, 1982.

[GHR03] Misha Guysinsky, Boris Hasselblatt, and Victoria Rayskin.
Differentiability of the hartman–grobman linearization.
Discrete & Continuous Dynamical Systems, 5(2):979–984, 2003.

[Lee12] John Lee.
Introduction to Smooth Manifolds.
Springer New York, NY, 2 edition, 2012.

[PM82] Jacob Palis and Welington Melo.
Geometric Theory of Dynamical Systems.
Springer New York, NY, 1 edition, 1982.

[Rui99] Juan J. Morales Ruiz.
Differential Galois Theory and Non-Integrability of Hamiltonian Systems.
Birkhäuser Basel, 1 edition, 1999.