Master's Thesis

Regular Poisson Manifolds of Compact Type

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Contents

Introduction

Poisson manifolds were first introduced in 1977 by Lichnerowicz [Lic77], and further studied by Alan Weinstein in his seminal papers [Wei83] and [Wei87]. A Poisson structure on a manifold is a generalization of a symplectic structure such that one can still capture all essentials of Hamiltonian mechanics. Whereas the existence of a symplectic structure on a closed manifold can be a delicate issue, any manifold admits a Poisson structure. Moreover, Poisson structures can have very distinct behaviour locally around a point. This is in sharp contrast to symplectic structures, which all look the same in Darboux charts. A Poisson structure induces a singular foliation of the manifold by symplectic leaves.

In general, not a lot can be said about Poisson manifolds, so we often restrict to a specific class of Poisson manifolds. The class of Poisson manifolds that we will focus on are the Poisson manifolds of compact type. The name is inspired from Lie theory, where one says that a Lie algebra is of compact type if it is integrated by a compact Lie group. We could naively ask the same for the Lie algebra of smooth functions of a Poisson manifold, but since this is an infinite dimensional vector space, this is not the way go. Therefore, we consider a generalization of Lie groups and Lie algebras, which are Lie groupoids and Lie algebroids. There is a similar correspondence between Lie groupoids and Lie algebroids, where a Lie algebroid is viewed as the infinitesimal counterpart of a Lie groupoid. Any Poisson manifold gives rise to a Lie algebroid, called the cotangent algebroid and the Lie groupoids integrating the Lie algebroids that come from Poisson manifolds carry a multiplicative symplectic form. Therefore, they are called symplectic groupoids. One may then wonder whether the cotangent algebroid integrates to a "compact-like" symplectic groupoid.

Contrary to Lie groups and Lie algebras, there are different notions of compactness for Lie groupoids. For instance, we call a Lie groupoid

- compact if the space of arrows is compact,
- source-proper if the source map is proper,
- proper if the anchor map is proper.

Instead of asking that the cotangent algebroid integrates to some symplectic groupoid, one can demand that there is a source 1-connected symplectic integration. For instance, a proper Poisson manifold is a Poisson manifold whose whose Lie algebroid integrates to a proper symplectic groupoid, whereas a strong proper Poison manifold is a Poisson manifold whose Lie algebroid integrates to a source 1-connected, proper symplectic groupoid. In total, this leads to 6 different notions of compactness for a Poisson manifold, of which strong compactness is the strongest. Poisson manifold of compact type are interesting, because of the following inexhaustive list of properties that they satisfy: [CFT15]

1. The Poisson cohomology admits a Hodge decomposition and Poincaré duality holds.
2. There are natural operations such as Hamiltonian quotients, gauge equivalences, etc. that preserve the comapctness type.
3. The leaf spaces are integral affine orbifolds.
4. The leafwise symplectic form varies linearly in cohomology.

This puts Poisson manifolds of compact type in a prominent position in Poisson geometry.

Producing examples of non-symplectic Poisson manifolds of strong compact type is a non-trivial task. The first example was produced by Martinez-Torres in [Tor13], following up on Kotschick's construction of free circle actions on a symplectic manifold with contractible orbits [Kot05]. In this thesis, we focused on the interaction between the linear variation of the leafwise symplectic form and the integral affine structure on the leaf space of a *simple* Poisson manifold. Usually, a Poisson manifold is called simple if the leaf space is a smooth manifold. A sufficient condition for this is that the Poisson manifold has compact and 1-connected symplectic leaves. In particular, we found a generic construction of Poisson manifolds of strong compact type in Chapter 4, and we show in Chapter 5 that the example by Martinez-Torres is a special example of this.

Informally, the construction scheme is as follows. Essential are the monodromy groups of a Poisson manifold, which we will be recalled in Chapter 2. The monodromy groups of an integrable Poisson manifold form discrete additive subgroups of the cotangent bundle of the leaf space. It turns out that a Poisson manifold is of so-called "strong **s**-proper" type if the symplectic leaves are compact and the monodromy groups are of full rank. By studying the linear variation of the symplectic form on the leaves, we can find suitable Poisson submanifolds whose monodromy groups are full rank. After taking a suitable Poisson quotient, we obtain a Poisson manifold of strong compact type.

Organization of this thesis

In Chapter 1, we introduce the basic and relevant concepts in Poisson geometry. In particular, we discuss Poisson bivectors, the symplectic foliation and Poisson quotients. We illustrate all notions with a couple of relevant examples for the rest of the thesis.

In Chapter 2, we define Lie groupoids and Lie algebroids, and discuss their basic properties. We describe the correspondence between Lie groupoids and Lie algebroids, and give the necessary and sufficient condition for Lie algebroids to be integrable. Then, we make the connection to Poisson geometry, and describe how Poisson manifolds give rise to symplectic groupoids. We also touch upon the integrability problem in Poisson geometry of finding symplectic groupoids integrating the Poisson manifold. The main result of this chapter is to understand the following theorem.

Theorem (Theorem 2.46, [CFM21])**.** *Let $(\mathscr{G},\Omega) \rightrightarrows M$ be a symplectic groupoid. There exists a unique Poisson structure π on M such that $\mathbf{t} : (\mathscr{G},\Omega) \to (M,\pi)$ is a Poisson map. Moreover,*

1. *The connected components of the orbits of $\mathscr{G}$ are precisely the symplectic leaves of (M,π).*
2. *There is a canonical Lie algebroid isomorphism*
$$\sigma_\Omega : Lie(\mathscr{G}) \to T^*M, \qquad \alpha \mapsto -\mathbf{u}^*(\iota_\alpha \Omega).$$
In particular, $\pi^\sharp \circ \sigma_\Omega^{-1} = \rho$, where ρ is the anchor of $Lie(\mathscr{G})$.

Then, we move our discussion to the various compactness types of Poisson manifolds in Chapter 3. We highlight each notion of compactness by clarifying examples, and we describe how compactness types behave when we take Poisson submanifolds, or Poisson quotients. The main result of this chapter is the following theorem.

Theorem (Theorem 3.17, [CFT15])**.** *Let (M,ω,μ) be a free Quasi-Hamiltonian S^1-space, and suppose μ has 1-connected fibers. The reduced Poisson space $(M/S^1,\pi_{red})$ is of strong compact type.*

In chapter 4, we describe a general scheme to construct Poisson manifolds of strong compact type. The key ingredient is the relationship between integral affine structures on the leaf space, and the linear variation of the leafwise symplectic form. The main result in this chapter is the following theorem.

Theorem (Theorem 4.37)**.** *Let (M,π) be a simple Poisson manifold with regular variation, with leaf space B. Let $\mathscr{K}$ be the kernel of the linear variation, and let $F \subseteq B$ be a submanifold such that $TF \cap \mathscr{K} = \{0\}$. Then $M_F = p^{-1}(F)$ is a simple saturated Poisson submanifold of M, whose linear variation is injective. The monodromy groups of M_F satisfy*

$$\mathscr{N}^\vee(M_F,\pi_F) = TF \cap \mathscr{N}^\vee(M,\pi).$$

Moreover, M_F is integrable if and only if $\mathscr{N}^\vee(M_F,\pi)$ defines a smooth lattice in TF, and if it this is the case, M_F is of strong **s***-proper type, with integral affine structure on F determined by $\mathscr{N}^\vee(M_F,\pi_F)$.*

In Chapter 5, we follow the construction of a Poisson manifold of strong compact type, by Martinez-Torres [Tor13], and show that it is an example of the construction scheme in Chapter 4. The Poisson manifold that we construct arises from the theory of complex K3 surfaces, so a majority of this chapter discusses the theory of K3 surfaces and deformations. We end with a short outlook and conjectures on possible generalizations of this approach. In particular, we discuss Enriques surfaces and the Hilbert scheme of n points on a K3 surface.

1 Poisson Geometry

In this chapter, we introduce Poisson manifolds and we discuss various ways to characterise a Poisson structure: as Lie brackets on $C^\infty(M)$, as bivectors and as a skew-symmetric bundle map $T^*M \to TM$. Then, we will discuss important notions such as regularity, the symplectic foliation, the isotropy algebra of a Poisson manifold, Poisson quotients and Poisson submanifolds. All of these concepts will be illustrated by a couple of important examples of Poisson manifolds that appear in this thesis. For this chapter, we mainly follow [CFM21].

1.1 Poisson Brackets and Bivectors

1.1.1 The Classical Definition

The classical definition of a Poisson manifold is that of endowing the algebra of smooth functions on a manifold with a special kind of Lie bracket.

Definition 1.1. A Poisson structure on a smooth manifold M is a Lie bracket on the algebra of smooth functions $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, satisfying the Leibniz identity

$$\{fg,h\} = f\{g,h\} + \{f,h\}g$$

for all $f,g,h \in C^\infty(M)$. We call the pair $(M,\{\cdot,\cdot\})$ a Poisson manifold. A Poisson map between two Poisson manifolds $(M_1,\{\cdot,\cdot\}_1)$ and $(M_2,\{\cdot,\cdot\}_2)$ is a smooth map $\phi : M_1 \to M_2$ satisfying

$$\{\phi^* f, \phi^* g\}_1 = \phi^*\{f,g\}_2$$

for all $f,g \in C^\infty(M_2)$

The Leibniz identity implies that for any $H \in C^\infty(M)$, the operation $\{H,\cdot\}$ is a derivation of $C^\infty(M)$. Therefore, it corresponds to a vector field X_H defined by

$$\mathscr{L}_{X_H}(g) = \{H,g\}.$$

The vector field X_H is called the *Hamiltonian vector field* of H.

Another application of the Leibniz identity is that for any open set $U \subseteq M$ of a Poisson manifold, there exists a unique Poisson bracket $\{\cdot,\cdot\}_U$ on U, satisfying $\{f|_U, g|_U\}_U = \{f,g\}|_U$. In particular, we can restrict to a coordinate chart $(U, x^1, \cdots x^n)$ and express the Poisson bracket in local coordinates. In order to find this expression, note that the Hamiltonian vector field of a function $f \in C^\infty(U)$ must be of the form

$$X_f = \sum_{j=1}^n X_f^j \frac{\partial}{\partial x^j}.$$

The Leibniz identity dictates that $X_{gf}^j = fX_g^j + gX_f^j$, so the map $f \mapsto X_f^j$ is again a derivation for each j. The upshot is that the Hamiltonian vector field can be written as

$$X_f = \sum_{i,j=1}^n \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

for some smooth functions $\pi^{ij} \in C^\infty(U)$. This implies that in a coordinate chart U, the Poisson bracket is of the form

$$\{f,g\}|_U = \sum_{i,j=1}^{n} \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}. \tag{1}$$

The functions π^{ij} are called the structure functions of the Poisson bracket.

Remark 1.2. The structure functions satisfy $\pi^{ij} = \{x^i, x^j\}$. By the skew-symmetry of the Poisson bracket, they define a skew-symmetric matrix, and the Jacobi identity translates to

$$\sum_{\ell=1}^{n} \pi^{\ell k} \frac{\partial \pi_{ij}}{\partial x^\ell} + \pi^{\ell j} \frac{\partial \pi^{ki}}{\partial x^\ell} + \pi^{\ell i} \frac{\pi^{jk}}{\partial x^\ell} = 0. \tag{2}$$

The flow of Hamiltonian vector fields X_H defines an equivalence relation on a Poisson manifold $(M, \{\cdot,\cdot\})$ as follows.

Lemma 1.3. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. The relation*

$$x \sim y \iff \exists f_1, \cdots, f_k \in C^\infty(M) \colon \phi^1_{f_1} \circ \cdots \circ \phi^1_{f_k}(x) = y.$$

is an equivalence relation.

Proof. Reflexivity follows from taking $f = 0$. Transitivity is immediate. For symmetry, it suffices to observe that $\phi^1_{-X_f} = \phi^{-1}_{X_f} = (\phi^1_{X_f})^{-1}$, provided that the flow of X_f at $t = 1$ exists. □

The equivalence classes of $\sim$ are called the *orbits* of $(M, \{\cdot,\cdot\})$. It turns out that that the partition into orbits defines a symplectic foliation of M, see Section 1.2.

Example 1.4. We discuss some important examples of Poisson manifolds that appear throughout this thesis.

1. On $\mathbb{R}^n$, we have constant Poisson structures

   $$\{f,g\} = \sum_{i,j=1}^{n} c^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},$$

   where c^{ij} are constants satisfying $c^{ij} = -c^{ji}$. The orbits of this Poisson structures are all translations of the orbit W containing the origin, where W is the subspace spanned by the vectors

   $$v^i = \left(c^{i1}, \cdots, c^{in}\right).$$

2. Every symplectic manifold (M, ω) has a natural Poisson structure, defined by

   $$\{f,g\} = -\omega(X_f, X_g),$$

   where X_f and X_g are the Hamiltonian vector fields of the symplectic structure. That is, they are the unique vector fields satisfying

   $$\iota_{X_f}\omega = df, \qquad \iota_{X_g}\omega = dg.$$

These Poisson structures are called non-degenerate, and the Poisson structure can be seen as an "inverse" to the symplectic form. The orbits are precisely the connected components of M. Note that the notion of Hamiltonian vector fields for Poisson manifold and symplectic manifolds coincide.

3. A linear Poisson bracket on a vector space V is a Poisson bracket $\{\cdot,\cdot\}$ such that the bracket of two linear functions is again a linear function. There is a one-to-one correspondence

$$\{\text{Linear Poisson structures on } V=\mathfrak{g}^*\} \leftrightarrow \{\text{Lie algebra structures on } V^*=\mathfrak{g}\}$$

determined by the property that $\{\mathrm{ev}_u,\mathrm{ev}_v\}=\mathrm{ev}_{[u,v]}$ for all $u,v\in\mathfrak{g}$, where $\mathrm{ev}:\mathfrak{g}\to C^\infty(\mathfrak{g}^*)$ is the evaluation map: $\mathrm{ev}_u(\xi)=\langle\xi,u\rangle$. This tells that, given a linear Poisson structure on $\mathfrak{g}^*$, the corresponding Lie algebra structure on $\mathfrak{g}$ is obtained by restricting the Poisson bracket to linear functions. Conversely, given a Lie algebra structure on $\mathfrak{g}$, there is a unique linear Poisson structure on $\mathfrak{g}^*$, given by

$$\{f,g\}(\xi)=\langle[(df)_\xi,(dg)_\xi],\xi\rangle, \qquad \xi\in\mathfrak{g}^*.$$

To write this correspondence in coordinates, choose a basis $\{e^i\}$ of $\mathfrak{g}$, and denote by $\{x^i\}$ the induced linear coordinates on $\mathfrak{g}^*$. Let c_k^{ij} be the structure constants of $\mathfrak{g}$ with respect to the chosen basis. The structure functions of the linear Poisson structures are

$$\pi^{ij}=\{x^i,x^j\}=\sum_k c_k^{ij}x^k.$$

The orbits of the linear Poisson structure can be described using any connected Lie group G with Lie algebra $\mathfrak{g}$, as the coadjoint orbits of G.

1.1.2 Poisson Bivectors

The local expression of a Poisson structure in (1) suggests that a Poisson bracket can be encoded by an expression of the form

$$\pi=\sum_{i<j}^n \pi^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j} \tag{3}$$

Such an expression is an example of a bivector field on M, i.e. a section of Λ^2TM. Thus, an alternative way of encoding a Poisson structure is by using a particular type of bivector π. This description has the advantage that it captures the definition of a Poisson structure in terms of tensors, and this happens to be more convenient in practical situations.

Bivector fields are a particular case of multivector fields. To understand the relation between Poisson structures and bivectors, it is best to start with a discussion of the calculus of multivector fields. Essentially, a multivector field is the covariant analogue of a differential form: a section of Λ^kTM. We denote these sections by $\mathfrak{X}^k(M)$. For $k=0$, these are just the smooth functions on M and for $k=1$, we retrieve the standard notion of vector fields. Just as with differential forms, there are two ways to look at multivector fields $\theta\in\mathfrak{X}^k(M)$. We can either interpret it as an $\mathbb{R}$-multilinear, skew-symmetric map

$$\theta_x:\underbrace{T_x^*M\times\cdots\times T_x^*M}_{k\,\text{times}}\to\mathbb{R},$$

or as a $C^\infty(M)$-multilinear, skew-symmetric map

$$\theta : \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k\,\text{times}} \to C^\infty(M).$$

Using the second point of view, we have a wedge product $\wedge : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l}(M)$, given by

$$(\alpha \wedge \beta)(X_1, \cdots, X_{k+l}) = \sum_{\sigma \in S_{k,l}} (-1)^\sigma \alpha(X_{\sigma(1)}, \cdots, X_{\sigma(k)}) \beta(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)}), \tag{4}$$

where the sum is over all (k,l)-shuffles. The wedge product is graded commutative, and associative:

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha, \qquad \alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma,$$

so one obtains a graded commutative algebra structure on $\mathfrak{X}^\bullet(M) = \bigoplus_{k \geq 0} \mathfrak{X}^k(M)$.

In a local coordinate chart $(U, x^1, \cdots, x^n)$, a multivector fields $\theta \in \mathfrak{X}^k(M)$ is of the form

$$\theta|_U = \sum_{i_1 < \cdots < i_k} \theta^{i_1 \cdots i_k} \frac{\partial}{x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}},$$

where $\theta^{i_1 \cdots i_k} \in C^\infty(U)$ are uniquely determined, and skew-symmetric. To state the correspondence between Poisson structures and bivector fields, we need yet another viewpoint on multivector fields: as multiderivations. Recall that any ordinary vector field $X \in \mathfrak{X}(M)$ corresponds to a derivation $\mathscr{L}_X : C^\infty(M) \to C^\infty(M)$. A similar correspondence exists for multivector fields and multiderivations. A multiderivation of degree k is an $\mathbb{R}$-multilinear, alternating map

$$\underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{k\,\text{times}} \to C^\infty(M),$$

which is a derivation in each argument. Each multivector field $\theta \in \mathfrak{X}^k(M)$ gives rise to a multiderivation $\mathscr{L}_\theta$, defined by

$$\mathscr{L}_\theta(f_1, \cdots, f_k) = \theta(df_1, \cdots, df_k).$$

Proposition 1.5. *The association $\theta \mapsto \mathscr{L}_\theta$ defines a one-to-one correspondence between $\mathfrak{X}^k(M)$ and multiderivations of degree k.*

Proof. The proof is similar to the case $k = 1$, which can be found in any introductory text on differential geometry. See for instance [Lee03]. □

In particular, biderivations will play a special role. They are maps denoted by

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

which are skew-symmetric, $\mathbb{R}$-bilinear and satisfy the Leibniz identity. Thus, all properties of a Poisson bracket hold, except for the Jacobi identity. According to the proposition, a biderivation $\{\cdot, \cdot\}$ corresponds to a bivector field π via

$$\pi(df, dg) = \{f, g\}. \tag{5}$$

In order to write the Jacobi identity for π, we generalize the ordinary Lie bracket of vector fields to the Schouten-Nijenhuis bracket of multivector fields.

Definition 1.6. The Schouten-Nijenhuis bracket of two multivector fields $\theta \in \mathfrak{X}^{k+1}(M)$ and $\xi \in \mathfrak{X}^{l+1}(M)$ is the unique multivector field $[\theta,\xi] \in \mathfrak{X}^{k+l+1}(M)$ satisfying

$$\mathscr{L}_{[\theta,\xi]} = \mathscr{L}_\theta \circ \mathscr{L}_\xi - (-1)^{kl} \mathscr{L}_\xi \circ \mathscr{L}_\theta,$$

where

$$(\mathscr{L}_\theta \circ \mathscr{L}_\xi)(f_1,\cdots,f_{k+l-1}) = \sum_{\sigma \in S_{k,l+1}} (-1)^\sigma \mathscr{L}_\theta \left(f_{\sigma(1)},\cdots,f_{\sigma(k-1)},\mathscr{L}_\xi(f_{\sigma(k)},\cdots,f_{\sigma(k+l-1)})\right).$$

Note that the composition of two multiderivations is not again a multiderivation, but only the graded commutator of multiderivations will again be a multiderivation.

For a biderivation $\{\cdot,\cdot\}$ with associated bivector field π, a simple computation shows

$$\mathscr{L}_{[\pi,\pi]}(f,g,h) = 2(\mathscr{L}_\pi \circ \mathscr{L}_\pi)(f,g,h) = 2(\{f\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\}). \tag{6}$$

Thus, the Jacobi identity for $\{\cdot,\cdot\}$ is equivalent to $[\pi,\pi]=0$. Therefore, we have the following result.

Theorem 1.7. *On any manifold M, there is a one-to-one correspondence*

$$\left\{\text{Poisson brackets } \{\cdot,\cdot\} \text{ on } M\right\} \longleftrightarrow \left\{\text{Bivector fields } \pi \text{ satisfying } [\pi,\pi]=0\right\} \tag{7}$$

From now on, we will refer to a Poisson manifold by specifying its associated bivector π, and denote Poisson manifolds by the pair (M,π).

There is yet another useful way to look at Poisson manifolds. Any bivector field $\pi \in \mathfrak{X}^2(M)$ induces a bundle map

$$\pi^\sharp : T^*M \to TM, \qquad \alpha \mapsto \pi^\sharp(\alpha) = \pi(\alpha,\cdot).$$

Then, bivector fields on M are in one-to-one correspondence with bundle maps $\pi^\sharp : T^*M \to TM$ satisfying

$$(\pi^\sharp)^* = -\pi^\sharp.$$

We will denote the map between sections $\pi^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ by the same symbol. We can express the condition that $[\pi,\pi]=0$ in terms of $\pi^\sharp$ using the following bracket $[\cdot,\cdot]_\pi$ on $\Omega^1(M)$:

$$[\alpha,\beta]_\pi = \mathscr{L}_{\pi^\sharp(\alpha)}(\beta) - \mathscr{L}_{\pi^\sharp(\beta)}(\alpha) - d(\pi(\alpha,\beta)).$$

This bracket satisfies the following properties, listed in the proposition below.

Proposition 1.8 ([CFM21])**.** *Let $\pi \in \mathfrak{X}^2(M)$ be a bivector field with associated biderivation $\{\cdot,\cdot\}$ The bracket $[\cdot,\cdot]_\pi$ is the unique bracket satisfying*

1. *On exact 1-forms, it is given by*

$$[df,dg]_\pi = d\{f,g\}$$

   for all $f,g \in C^\infty(M)$.

2. *It satisfies the Leibniz identity*

$$[\alpha,f\beta]_\pi = f[\alpha,\beta]_\pi + \mathscr{L}_{\pi^\sharp(\alpha)}(f)\beta$$

for all $\alpha, \beta \in \Omega^1(M)$ *and all* $f \in C^\infty(M)$.

Moreover, the following are equivalent.

- $[\pi, \pi] = 0$
- $\pi^\sharp : (\Omega^1(M), [\cdot,\cdot]_\pi) \to (\mathfrak{X}(M), [\cdot,\cdot])$ *is bracket preserving.*
- $[\cdot,\cdot]_\pi$ *satisfies the Jacobi identity.*

Remark 1.9. The bracket $[\cdot,\cdot]_\pi$ will play an important role in the next chapter. It gives the triple $(T^*M, \pi^\sharp, [\cdot,\cdot]_\pi)$ the structure of a Lie algebroid.

Thus, a Poisson structure is the same as a skew-symmetric, bracket-preserving map $\pi^\sharp : T^*M \to TM$.

Example 1.10. Let (M, ω) be a symplectic manifold, and denote by $\omega^\flat : TM \to T^*M$ the isomorphism $X \mapsto \iota_X \omega$. The associated Poisson structure $\pi^\sharp : T^*M \to TM$ is precisely $\left(\omega^\flat\right)^{-1}$.

Example 1.11. Let $\mathfrak{g}$ be a Lie algebra, and consider the linear Poisson structure on its dual $(\mathfrak{g}^*, \pi_\mathfrak{g})$. The cotangent bundle of $\mathfrak{g}^*$ can be identified with $\mathfrak{g}^* \times \mathfrak{g}$, and the space of sections can be identified with $C^\infty(\mathfrak{g}^*, \mathfrak{g})$. The linear Poisson structure corresponds to the bundle map $\pi^\sharp_{\mathfrak{g},\xi} : \mathfrak{g} \to \mathfrak{g}^*$, $v \mapsto \mathrm{ad}^*_v(\xi) = \langle \xi, [v, \cdot] \rangle$. Consequently, the bracket $[\cdot,\cdot]_\pi$ on $C^\infty(\mathfrak{g}^*, \mathfrak{g})$ becomes $[f,g]_\pi(\xi) = [f(\xi), g(\xi)]$.

If (M, π) is a Poisson manifold, the map $\pi^\sharp$ allows us to write the Hamiltonian vector fields as

$$X_H = \pi^\sharp(dH), \qquad H \in C^\infty(M).$$

In particular, we see that

$$\mathrm{im}(\pi^\sharp_x) = \{X_{H,x} \colon H \in C^\infty(M)\} \subseteq T_xM.$$

In general, $\mathrm{im}(\pi^\sharp)$ is not of constant rank, so it forms a *singular* distribution of TM. We will pay more attention to Poisson maifolds for which this distribution is smooth in Section 1.3. Since $\pi^\sharp$ preserves the bracket, we have

$$[\pi^\sharp(\alpha), \pi^\sharp(\beta)] = \pi^\sharp([\alpha, \beta]_\pi) \in \mathrm{im}(\pi^\sharp),$$

so this distribution is *involutive*. We can also write the condition that a map is Poisson in terms of $\pi^\sharp$. Finally, note that a map $\phi : (M_1, \pi_1) \to (M_2, \pi_2)$ is Poisson if and only if the following square commutes,

$$\begin{tikzcd} TM_1 \arrow[r, "d\phi"] & TM_2 \\ T^*M_1 \arrow[u, "\pi^\sharp_{M_1}"] & T^*M_2 \arrow[l, "(d\phi)^*"] \arrow[u, "\pi^\sharp_{M_2}"'] \end{tikzcd} \tag{8}$$

or, equivalently, if $\phi^* : (\Omega^1(M_2), [\cdot,\cdot]_{\pi_{M_2}}) \to (\Omega^1(M_2), [\cdot,\cdot]_{\pi_{M_1}})$ is bracket-preserving.

1.2 The Symplectic Foliation of a Poisson Manifold

In this section, we describe the symplectic foliation of a Poisson manifold (M, π). This is the partition of M into the orbits from the equivalence relation defined in Lemma 1.3. It turns out each of the orbits is equipped with a symplectic form, compatible with the Poisson structure.

Theorem 1.12. *Every orbit S of the equivalence relation in Lemma 1.3 has a unique smooth structure for which the inclusion is an immersion. The tangent space of S consists of all Hamiltonian directions*

$$T_xS = \mathrm{im}(\pi_x^\sharp),$$

and S has a symplectic structure ω_S, defined by

$$\omega_{S,x}(\pi_x^\sharp\alpha, \pi_x^\sharp\beta) = -\pi_x(\alpha,\beta). \tag{9}$$

For a complete proof of this theorem, we refer to [CFM21].

The definition of symplectic leaves using orbits is not very useful to compute them. Fortunately, the following proposition helps to classify the symplectic leaves in terms of the tangent spaces.

Proposition 1.13. *Let $\mathscr{S}$ be a collection of connected, immersed submanifolds of (M,π) such that $T_xS = \mathrm{im}(\pi_x^\sharp)$ for each $S \in \mathscr{S}$. Then $\mathscr{S}$ is the symplectic foliation of (M,π).*

Proof. See [CFM21]. □

All symplectic leaves of (M,π) form a foliation of M.

Definition 1.14. The symplectic foliation $\mathscr{F}_\pi$ of (M,π) is the collection

$$\{(S,\omega_S) \colon S \text{ is a symplectic leaf of } M\}.$$

The term symplectic foliation that we have introduced is synonymous for "collection of symplectic leaves". In general, the symplectic leaves can have different dimensions, and form a so-called singular foliation. For more on symplectic foliations, see the appendix.

Definition 1.15. The isotropy Lie algebra at $x \in M$ is $(\mathfrak{g}_x, [\cdot,\cdot]_{\mathfrak{g}_x}$, where $\mathfrak{g}_x = \ker(\pi_x^\sharp)$ and where $[\cdot,\cdot]_{\mathfrak{g}_x}$ is the unique bracket determined by the property that

$$[\alpha(x),\beta(x)]_{\mathfrak{g}_x} = [\alpha,\beta]_\pi|_x.$$

We prove in Proposition 2.21 in the more general framework of Lie algberoids that this is well-defined.

1.2.1 Regular Poisson Manifolds

An important subclass of Poisson manifolds are *regular* Poisson manifolds, which are those Poisson manifolds for which the dimension of the symplectic leaves is constant. In other words, regular Poisson manifolds are those Poisson manifolds for which $\mathrm{im}(\pi_x^\sharp)$ is always of constant rank. Regular Poisson manifolds are a convenient class since $\mathscr{F}_\pi = \mathrm{im}(\pi^\sharp)$ actually forms a regular foliation of M. We denote the normal bundle of this foliation by $\nu(\mathscr{F}_\pi) = TM/\mathscr{F}_\pi$, and the conormal bundle $\nu^*(\mathscr{F}_\pi)$ can be identified with $\ker(\pi^\sharp)$. The symplectic structures on the leaves glue to a foliated symplectic form $\omega_{\mathscr{F}_\pi} \in \Gamma(\Lambda^2 T^*\mathscr{F}_\pi)$, given by

$$\omega_{\mathscr{F}_\pi}(\pi^\sharp(\alpha), \pi^\sharp(\beta)) = -\pi(\alpha, \beta). \tag{10}$$

This is well-defined, because $\pi(\alpha,\beta) = \pi(\alpha',\beta')$ if $\pi^\sharp(\alpha) = \pi^\sharp(\alpha')$ and $\pi^\sharp(\beta) = \pi^\sharp(\beta')$. Moreover, it is indeed a foliated symplectic form, since it is non-degenerate and leafewise closed. In fact, any symplectic foliation gives rise to a regular Poisson structure, defined by (10). Indeed, this is a Poisson structure since

$$d_{\mathscr{F}}\omega_{\mathscr{F}_\pi}(\pi^\sharp(\alpha), \pi^\sharp(\beta), \pi^\sharp(\gamma)) = -[\pi,\pi](\alpha,\beta,\gamma).$$

Therefore, we conclude

Theorem 1.16. *Equation* (10) *defines a one-to-one correspondence*

$$\{\text{Regular Poisson structures } \pi \text{ on } M\} \longleftrightarrow \{\text{Regular symplectic foliations } (\mathscr{F}_\pi, \omega_{\mathscr{F}_\pi}) \text{ on } M\}$$

The importance of this theorem is that it gives us a different method to construct regular Poisson manifolds, namely by specifying a regular symplectic foliation. In practical situations, it is sometimes easier to specify a regular symplectic foliation. One instance of this phenomenon occurs in Chapter 5, when we construct a Poisson manifold of strong compact type.

We end with a discussion of the isotropy algebras $\mathfrak{g}_x = \ker(\pi^\sharp_x)$ of regular Poisson manifold. Of course, all of them have the same dimension, but it turns out that they are all abelian.

Proposition 1.17. *Let (M,π) be a regular Poisson manifold. The isotropy algebra $\mathfrak{g}_x$ is abelian for all $x \in M$.*

Proof. Let $x \in M$. By Weinstein's splitting theorem, (see Theorem 3.2 in [CFM21]), there exists a coordinate chart $(U, p^1, \cdots, p^n, q_1, \cdots, q_n, y^1, \cdots, y^s)$ centered at x such that

$$\pi_U = \sum_{i=1}^{n} \frac{\partial}{\partial p^i} \wedge \frac{\partial}{\partial q^i}.$$

It follows that

$$\mathfrak{g}_x = \mathrm{Span}_{\mathbb{R}}\{(dy^1)_x, \cdots (dy^s)_x\},$$

and the Lie bracket is given by

$$[(dy^i)_x, (dy^j)_x] = [dy^i, dy^j]_\pi|_x = d\{y^i, y^j\}_x = d\left(\pi(dy^i, dy^j)\right)_x = 0,$$

and this concludes the proof. □

Thus, the isotropy algebras of a regular Poisson manifolds form a bundle of abelian Lie algebras $(\nu^*(\mathscr{F}), +)$.

1.3 Quotients and Poisson Submanifolds

In this section, we describe two procedures to construct new Poisson manifolds from known Poisson manifolds, which are Poisson quotients and Poisson submanifolds.

1.3.1 Poisson Quotients

A Poisson quotient is the quotient of a Poisson manifold by a free and proper Poisson action of a Lie group. The resulting quotient will naturally carry a Poisson structure.

Definition 1.18. A Poisson action of a Lie group G on a Poisson manifold (M,π) is a smooth action such that the action map $\Phi_g : M \to M$ is a Poisson map for all $g \in G$. The triple (M,π,G) is called a Poisson G-space.

Note that if S is a symplectic leaf of a Poisson G-space (M,π,G), $S' = \Phi_g(S)$ is also a symplectic leaf of M, and $\Phi_g : S \to S'$ is a symplectomorphism. Whenever a Lie group action is free and proper, the orbit space M/G has a unique smooth structure for which the quotient map $p : M \to M/G$ is a submersion.

Theorem 1.19. *Let (M,π,G) be a free and proper Poisson G-space. There exists a unique Poisson structure $\pi_{M/G}$ on M/G for which $p : M \to M/G$ is a Poisson map.*

Proof. The proof follows from the observation that the quotient map induces an isomorphism between $C^\infty(M/G)$ and the space $C^\infty(M)^G$ of smooth G-invariant functions on M. Since the action of G on M is Poisson, the Poisson bracket of two G-invariant functions is again G-invariant, so we obtain the Poisson structure on M/G. □

The quotient procedure for Poisson manifolds that is outlined here is rather simple, but can have drastic consequences. For instance, non-degenerate Poisson structures can be taken to degenerate ones, and vice versa. It can therefore be hard to describe the symplectic foliation of a Poisson quotient. In some situations, for example if the symplectic leaves are mapped to themselves by the action maps, one may hope that the symplectic foliation of the orbit space has a simple description. For Poisson G-spaces, this is the case if the infinitesimal vector fields for the action are Hamiltonian.

If (M,π,G) is a Poisson G-space, a linear map $\hat{\mu} : \mathfrak{g} \to C^\infty(M), \xi \mapsto \mu_\xi$ is the same as a map $\mu : M \to \mathfrak{g}^*$, by setting $\langle \mu(x), \xi \rangle = \hat{\mu}_\xi(x)$.

Definition 1.20. Let (M,π,G) be a Poisson G-space. A moment map is a smooth map $\mu : M \to \mathfrak{g}^*$ such that

$$X_\xi = -\pi^\sharp\left(d\mu_\xi\right),$$

where X_ξ is the infinitesimal vector field

$$\left(X_\xi\right)_x = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)\cdot x.$$

Since (M,π) is Poisson manifold, the space $C^\infty(M)$ has the structure of a Lie algebra so it is natural to require that $\hat{\mu}$ is a Lie algebra homomorphism. This is equivalent to saying that the moment map μ should be a Poisson map, since

$$\begin{aligned}\{\hat{\mu}(\xi_1),\hat{\mu}(\xi_2)\}_M(x) - \hat{\mu}([\xi_1,\xi_2])(x) &= \{\xi_1\circ\mu,\xi_2\circ\mu\}_M(x) - \langle[\xi_1,\xi_2],\mu(x)\rangle \\ &= \{\xi_1\circ\mu,\xi_2\circ\mu\}_M(x) - \{\xi_1,\xi_2\}_{\mathfrak{g}^*}(\mu(x)).\end{aligned}$$

Given a moment map $\mu : M \to \mathfrak{g}^*$ for a Poisson G-space (M,π,G), we say that μ is *G-equivariant* if $\mu(g\cdot x) = \mathrm{Ad}_g^*\mu(x)$.

Proposition 1.21. *Let (M,π,G) be a Poisson G-space, with G connected. A moment map $\mu : M \to \mathfrak{g}^*$ is G-equivariant if and only if it is a Poisson map.*

Proof. Suppose $\mu : M \to \mathfrak{g}^*$ is G-equivariant, and let $g = \exp(t\xi_1) \in G$. Then we have for each $\xi_2 \in \mathfrak{g}$ that

$$\langle \mu(\exp(t\xi_1)\cdot x), \xi_2\rangle = \langle \mu(x), \mathrm{Ad}_{\exp(-t\xi_1)}(\xi_2)\rangle$$

Differentiating this relation with respect to t at $t = 0$ gives

$$\langle d\mu_x(X_{\xi_1,x}), \xi_2\rangle = -\langle \mu(x), [\xi_1,\xi_2]\rangle = -\{\xi_1,\xi_2\}_{\mathfrak{g}^*}(\mu(x)),$$

which can be rewritten using the moment map condition as

$$-\langle d\mu_x \pi_x^\sharp(d\mu_\xi)_x, \xi_2\rangle = -\{\xi_1\circ\mu, \xi_2\circ\mu\}_M(x),$$

so μ is a Poisson map. The same procedure can be shown to prove the reverse implication. $\square$

Definition 1.22. A Poisson action of a connected Lie group G on a Poisson manifold (M,π) is called Hamiltonian if it admits a G-equivariant moment map $\mu : M \to \mathfrak{g}^*$. The quadruple (M,π,G,μ) is called a Hamiltonian G-space.

For Hamiltonian G-spaces, we have the following reduction theorem, which can be seen as the Poisson variant of the well-known Marsden-Weinstein theorem.

Theorem 1.23 ([FM14])**.** *Let (M,π,G,μ) be a proper, free Hamiltonian G-space. Then $0 \in \mathfrak{g}^*$ is a regular value of μ, and*

$$M//G = \mu^{-1}(0)/G$$

is a Poisson submanifold of M/G. If π is non-degenerate, then so is $\pi_{M//G}$, and the connected components of $M//G$ are the symplectic leaves of M/G.

For a proof of this theorem, we refer to [FM14].

In the case that G preserves the symplectic leaves of M, the symplectic leaves of M are Hamiltonian G-spaces and we infer from Theorem 1.23 that the symplectic foliation of M/G consists of the connected components of the symplectic reductions $S//G$, where S is a symplectic leaf of M.

Another extreme case is a Poisson G-space (M,π,G) such that the stabilizer of each leaf is trivial, i.e. $S\cap g\cdot S = \emptyset$ for all $g \neq e$ and all symplectic leaves S. In this case it turns out that the symplectic foliation of M/G has the same symplectic leaves as M. To see this, we will make a first investigation of the leaf space of a Poisson manifold. We will study leaf spaces in greater detail in Chapter 4.

Let (M,π,G) be a Poisson G-space such that the stabilizer of each leaf is trivial. The leaf space is defined to be $B = M/\mathscr{F}_\pi$, endowed with the quotient topology. In general, this can be a very wild space, but it will be a smooth manifold if the leaves of M are compact and 1-connected. The fibers of the quotient map $p : M \to B$ are precisely the symplectic leaves of M. Since the G-action takes symplectic leaves to symplectic leaves, we have an induced, free G-action on B. The leaf space of M/G can naturally be identified with B/G, and we have a quotient map $\bar{p} : M/G \to B/G$, whose fibers are precisely the symplectic leaves of M/G. Next, remark that the map

$$\begin{aligned} p^{-1}(b) &\to \bar{p}^{-1}(\bar{b}) \\ x &\mapsto [x] \end{aligned}$$

is a diffeomorphism since G acts freely on B, and it preserves the symplectic forms on the fibers by definition of the quotient Poisson structure. Thus, the symplectic leaves of M are the same as the leaves of M/G.

1.3.2 Poisson Submanifolds

If (M, π) is a Poisson manifold, there are several ways that the Poisson tensor can interact with submanifolds $N \subseteq M$. In this section, we will focus on Poisson submanifolds.

Definition 1.24. Let (M, π) be a Poisson manifold. A Poisson submanifold is a Poisson submanifold (N, π_N), together with an injective immersion $i : N \to M$ that is Poisson.

The notion of a submanifold in Poisson geometry is rather strict and as we shall see, the only Poisson submanifolds of a non-degenerate Poisson manifold are the open subsets.The following proposition helps us to determine which submanifolds of a Poisson manifold are Poisson submanifolds.

Proposition 1.25 ([CFM21])**.** *Let (M, π_M) be a Poisson manifold, and let $\iota : N \hookrightarrow M$ be an immersed submanifold. There exists at most one Poisson structure π_N on N such that (N, π_N) is a Poisson submanifold of (M, π_M). This happens if and only if the following equivalent conditions hold*

1. $\mathrm{im}(\pi^\sharp_{M,x}) \subseteq T_xN$ *for all* $x \in N$.
2. $\pi^\sharp_M(TN^\circ) = 0$.
3. *Every Hamiltonian vector field X_H is tangent to N.*

In particular, a Poisson submanifold $N \hookrightarrow M$ intersects each symplectic leaf S of (M, π) in an open subset of S. The connected components of $N \cap S$ are the symplectic leaves of (N, π_N).

Proof. If $\iota : (N, \pi_N) \to (M, \pi_M)$ is a Poisson submanifold, we have that

$$(d\iota_x)^* \pi^\sharp_{N,x}(d\iota_x) = \pi^\sharp_{M,x} \qquad \forall x \in N.$$

Since $(d\iota_x)$ is injective, it follows that π_N is uniquely determined. It also shows that 1 must hold for a Poisson submanifold.

Next, let $\iota : N \to M$ be a submanifold such that $\mathrm{im}(\pi^\sharp_{M,x}) \subseteq (d\iota_x)T_xN$. It suffices to check that $\pi^\sharp_{M,x}(\alpha) = 0$ for $\alpha \in (T_xN)^\circ$. This follows from the skew-symmetry of $\pi^\sharp_M$, as for any $b \in T_xM$, one has

$$\langle \pi^\sharp_{M,x}(\alpha), \beta \rangle = -\langle \alpha, \pi^\sharp_{M,x}(\beta) \rangle = 0.$$

The smoothness and skew-symmetry of π_N are now immediate. What is left to show is the Jacobi identity for π_N. This follows from the Jacobi identity for π. Since ι is a Poisson map, we have by the diagram in Equation

(8) that
$$(d\iota)_x[\pi_N,\pi_N]_x = [\pi_M,\pi_M]_{\iota(x)} = 0,$$
and the result follows because ι is an immersion.

Next, we show the equivalence of 1 and 2. Observe that for $\alpha \in (T_xN)^\circ$ and $\beta \in T_x^*M$, one has
$$\langle \pi^\sharp_{M,x}(\alpha),\beta\rangle = -\langle \alpha, \pi^\sharp_{M,x}(\beta)\rangle = 0 \qquad \Longleftrightarrow \qquad \mathrm{im}(\pi^\sharp_{M,x}) \subseteq T_xN.$$
The equivalence between 1 and 3 is immediate. □

An important class of Poisson submanifolds are the saturated Poisson submanifolds, which are defined as follows.

Definition 1.26. A Poisson submanifold $N \subseteq (M,\pi)$ is called saturated if it is a union of symplectic leaves of M.

Let us now discuss examples of Poisson submanifolds of the main examples of Poisson manifolds that we have been carrying through this chapter.

Example 1.27. Any symplectic leaf is a Poisson submanifold. Symplectic leaves are by definition the minimal saturated submanifolds.

Example 1.28. If (M,ω) is a connected symplectic manifold, there is only one symplectic leaf: M itself. Therefore, the only Poisson submanifolds of (M,ω) are the open subsets of M.

Example 1.29. Let $\mathfrak{g}$ be a Lie algebra and consider the Linear Poisson structure $(\mathfrak{g}^*,\pi_{\mathfrak{g}^*})$ on its dual. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a linear subspace, $\mathfrak{h}^\circ \subset \mathfrak{g}^*$ is a Poisson submanifold if and only if $\mathfrak{h}$ is a Lie algebra ideal. This can be seen using condition 2 in Proposition 1.25. We compute that
$$\begin{aligned}\pi^\sharp_{\mathfrak{g}^*,\xi}\left((T_\xi\mathfrak{h}^\circ)^\circ\right) &= \left\{\pi^\sharp_{\mathfrak{g},\xi}(u) \colon u \in \mathfrak{g}\,\text{vanishing on}\,\mathfrak{h}^\circ\right\}\\ &= \{\mathrm{ad}^*_u(\xi) \colon u \in \mathfrak{h}\},\end{aligned}$$
hence condition 2 in Proposition 1.25 holds if and only if $\xi([\mathfrak{h},\mathfrak{g}]) = 0$ for all $\xi \in \mathfrak{h}^\circ$, i.e. if $[\mathfrak{h},\mathfrak{g}] = \mathfrak{h}$. Moreover, $\mathfrak{h}^\circ$ is a union of symplectic leaves, hence saturated. Under the canonical isomorphism $\mathfrak{h}^\circ \simeq (\mathfrak{g}/\mathfrak{h})^*$, the induced Poisson structure on $\mathfrak{h}^\circ$ coincides with the linear Poisson structure on $(\mathfrak{g}/\mathfrak{h})^*$.

In the next chapter, we will study another principal piece of data associated to a Poisson manifold: a symplectic groupoid.

2 Lie groupoids and Lie algebroids

In this chapter, we describe Lie groupoids and Lie algebroids, which are a natural generalization of Lie groups and Lie algebras. We will see that it is possible to extend the correspondence between Lie groups and Lie algebras to Lie groupoids and Lie algebroids, and describe how a Lie algebroid is the infinitesimal counterpart of a Lie groupoid. This also rises the question whether any Lie algebroid can be integrated to a Lie groupoid, and we will discuss the results on integrability for Lie algebroids. Moreover, it turns out that every Poisson manifold gives rise to a Lie algebroid, whose global counterpart is a *symplectic groupoid*. For the results on Lie groupoids and Lie algebroids, we refer to [CFM21] and for the results on integrability, we mainly refer to [CF02] and [CF01].

2.1 Lie groupoids

We start with a discussion of Lie groupoids. A groupoid is a generalization of a group, in the sense that not all elements can be multiplied. A Lie groupoid is the natural generalization of a Lie group, in the sense that not all elements can be multiplied and that all the structure maps should be smooth. In this section, we make the discussion above exact, and discuss basic examples of Lie groupoids.

2.1.1 Basic Definitions

Definition 2.1. A groupoid is a category whose objects form a set and where each arrow is invertible.

This definition of a groupoid is very concise, but in practice, one should think of a groupoid as follows. A groupoid $\mathscr{G} \rightrightarrows M$ consists of a set of objects M, a set of arrows $\mathscr{G}$, and a collection $\{\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{m}, i\}$ of structure maps. Every arrow $g \in \mathscr{G}$ can be thought of as an arrow $y \xleftarrow{g} x$ starting at the *source* $\mathbf{s}(g) = x$ and ending at the *target* $\mathbf{t}(g) = y$, and this defines the *source* and *target* maps $\mathbf{s}, \mathbf{t} : \mathscr{G} \to M$. Unlike elements in a group, not all elements in a groupoid can be composed. A pair of arrows (g,h) is composable precisely when the source of g coincides with the target of h. We denote by

$$\mathscr{G}^{(2)} = \{(g,h) \in \mathscr{G} \times \mathscr{G} \colon \mathbf{s}(g) = \mathbf{t}(h)\}.$$

the set of all composable arrows. We have a multiplication map $\mathbf{m} : \mathscr{G}^{(2)} \to \mathscr{G}$, and we often denote $\mathbf{m}(g,h) = gh$. We require that $\mathbf{s}(gh) = \mathbf{s}(h)$ and $\mathbf{t}(gh) = \mathbf{t}(g)$. Graphically, this can be depicted as

$$\mathbf{m}(z \xleftarrow{g} y \xleftarrow{h} x) = z \xleftarrow{gh} x.$$

The composition map is required to be associative. That is, $(gh)k = g(hk)$ for all pairs $(g,h),(h,k) \in \mathscr{G}^{(2)}$.

Another vast difference between groups and groupoids is the set of units. Whereas a group has a single unit, a groupoid has precisely one unit element 1_x for each $x \in M$. The unit is required to satisfy $\mathbf{s}(1_x) = \mathbf{t}(1_x) = x$, and $g1_{\mathbf{s}(g)} = g = 1_{\mathbf{t}(g)}g$. This defines the unit map $\mathbf{u} : M \to \mathscr{G}$ by $u(x) = 1_x$.

Finally, every $g \in \mathscr{G}$ is supposed to have an inverse $g^{-1} \in \mathscr{G}$. That is, for every arrow $y \xleftarrow{g} x$, there exists a (necessarily unique) arrow $x \xleftarrow{g^{-1}} y$ such that

$$g^{-1}g = 1_x, \quad gg^{-1} = 1_y.$$

Remark 2.2. Note that the structure maps satisfy the relations $i^2 = \mathrm{Id}$, $\mathbf{s} \circ i = \mathbf{t}$, $\mathbf{s} \circ \mathbf{u} = \mathrm{Id}_M$ and $\mathbf{t} \circ \mathbf{u} = \mathrm{Id}_M$. In particular, i is bijective, $\mathbf{s}$ and $\mathbf{t}$ are surjective and $\mathbf{u}$ is injective.

If $\mathscr{G} \rightrightarrows M$ is a groupoid, the subset of arrows that start and end at x forms a group

$$\mathscr{G}_x = \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x),$$

called the isotropy group of $\mathscr{G}$ at x. Furthermore, $\mathscr{G}$ defines an equivalence relation on M by

$$x \sim y \iff \exists g \in \mathscr{G} \text{ such that } \mathbf{s}(g) = x \text{ and } \mathbf{t}(g) = y.$$

The equivalence classes of this equivalence relation are called the orbits of $\mathscr{G}$. Explicitly, the orbit $\mathscr{O}_x$ through x is the set $\mathbf{t}(\mathbf{s}^{-1}(x))$. We denote by $M/\mathscr{G}$ the space of orbits. Just as with groups, we can talk about left or right multiplication by an arrow $g \in \mathscr{G}$. However, we can only compose arrows if the source and target maps agree so we have to restrict ourselves to $\mathbf{s}$- and $\mathbf{t}$-fibers. To be precise, we fix $g \in \mathscr{G}$ and write $\mathbf{s}(g) = x$ and $\mathbf{t}(g) = y$. Then, left and right multiplication by g define bijections

$$L_g : \mathbf{t}^{-1}(x) \to \mathbf{t}^{-1}(y), \quad R_g : \mathbf{s}^{-1}(y) \to \mathbf{s}^{-1}(x). \tag{11}$$

Let us now define Lie groupoids.

Definition 2.3. A Lie groupoid is a groupoid $\mathscr{G} \rightrightarrows M$ such that $\mathscr{G}$ and M are smooth manifolds. All structure maps are required to be smooth, and $\mathbf{s}$ and $\mathbf{t}$ are required to be submersions.

Remark 2.4. The condition that $\mathbf{s}$ and $\mathbf{t}$ are submersions ensures that the set $\mathscr{G}^{(2)}$ is a smooth submanifold of $\mathscr{G} \times \mathscr{G}$, so it makes sense for $\mathbf{m}$ to be smooth.

Throughout the literature, only the source and target fibers of a Lie groupoid are assumed to be Hausdorff. The total space, on the other hand, can be non-Hausdorff. However, we will only encounter Hausdorff groupoids in this thesis and we will therefore make the assumption that the space of arrows is also Hausdorff. We will pay extra attention to ensure that the space of arrows is in fact Hausdorff.

For a Lie groupoid $\mathscr{G} \rightrightarrows M$, it is not difficult to see that left and right multiplication defines a diffeomorphism between the fibers. The following proposition summarizes some basic properties of Lie groupoids.

Proposition 2.5. *Let $\mathscr{G} \to M$ be a Lie groupoid. Then*

1. *The unit map $\mathbf{u} : M \to \mathscr{G}$ is an embedding.*

2. *The isotropy groups $\mathscr{G}_x$ are endowed with a smooth structure that turns them into a Lie group.*

3. *For every $x \in M$, the orbit $\mathscr{O}_x$ admits a unique smooth structure that turns it into an immersed submanifold of M. Moreover,*
$$\mathbf{t} : \mathbf{s}^{-1}(x) \to \mathscr{O}_x$$
defines a principal $\mathscr{G}_x$-bundle.

Proof. The first statement is not difficult to see. For the second and third statement, we refer to [MM03]. $\square$

There is a natural notion of morphism between groupoids. The smooth version between Lie groupoids is as follows.

Definition 2.6. Let $\mathscr{G} \to M$ and $\mathscr{H} \to N$ be Lie groupoids. A Lie groupoid morphism is a pair of maps $\Phi : \mathscr{G} \to \mathscr{H}$ and $\phi : M \to N$ with commuting source and target maps

$$\begin{tikzcd} \mathscr{G} \arrow[r, "\Phi"] \arrow[d, shift right, "\mathbf{s}"'] \arrow[d, shift left, "\mathbf{t}"] & \mathscr{H} \arrow[d, shift right, "\mathbf{s}"'] \arrow[d, shift left, "\mathbf{t}"] \\ M \arrow[r, "\phi"] & N \end{tikzcd}$$

and compatible multiplications $\Phi(gh) = \Phi(g)\Phi(h)$ for all $(g,h) \in \mathscr{G}^{(2)}$.

If $\Phi : \mathscr{G} \to \mathscr{H}$ is a Lie groupoid morphism, it induces

1. A smooth map between the source fibers $\mathbf{s}^{-1}(x) \to \mathbf{s}^{-1}(\phi(x))$ and the target fibers $\mathbf{t}^{-1}(x) \to \mathbf{t}^{-1}(\phi(x))$.
2. A Lie group homomorphism between the isotropy groups $\mathscr{G}_x \to \mathscr{H}_{\phi(x)}$
3. A smooth map between the orbits $\mathscr{O}_x \to \mathscr{O}_{\phi(x)}$.

Often, we wish to discuss Lie groupoids whose $\mathbf{s}$-fibers are connected. We call those Lie groupoids *source-connected*. If the $\mathbf{s}$-fibers are 1-connected, we call the Lie groupoid *source-1-connected*.

2.1.2 Examples

We will give some important examples of Lie groupoids.

Example 2.7. The simplest example is when $M = \{*\}$. Then, Lie groupoids over M are just ordinary Lie groups. Of course, source-(1-)connectedness is equivalent to (1-)connectedness of the Lie group itself.

Example 2.8. A more interesting example is that of a bundle of Lie groups. Bundles of Lie groups are Lie groupoids $\mathscr{G} \underset{\mathbf{s}}{\overset{\mathbf{t}}{\to}} M$ where the source and target map coincide. An example of this arises from the theory of vector bundles. A vector bundle $E \to M$ can be interpreted as a Lie groupoid, where the source and target maps are the projections to the base and where composition is the fiberwise addition of vectors.

Example 2.9. If M is a manifold, we can form the pair groupoid $M \times M \rightrightarrows M$, where we think of a pair (y,x) as an arrow from x to y. We set $\mathbf{s} = \mathrm{pr}_2$, $\mathbf{t} = \mathrm{pr}_1$ and $\mathbf{m}((x,y),(y,z)) = (y,z)$, and this gives $M \times M \rightrightarrows M$ the structure of a Lie groupoid over M. The isotropy groups are the points (x,x), and there is only one orbit, which is all of M. It is source-1-connected if and only if M is.

A more interesting example of a Lie groupoid is the fundamental groupoid $\Pi_1(M)$.

Example 2.10. If M is a manifold, we can form the fundamental groupoid $\Pi_1(M)$, whose space of arrows consists of path homotopy classes of smooth paths in M. For a homotopy class $[\gamma]$, we set $\mathbf{s}([\gamma]) = \gamma(0)$ and $\mathbf{t}([\gamma]) = \gamma(1)$. This is well-defined, because path homotopies leave the start and end points fixed. The space $\Pi_1(M)$ can be given a smooth structure such that $\mathbf{s}$ and $\mathbf{t}$ are smooth submersions. The multiplication map $\mathbf{m}$

is the concatenation of paths. For $x \in M$, we define the unit 1_x to be the class of the constant path at x. The orbits are the path-components of M, and the isotropy groups are precisely the fundamental groups $\pi_1(M,x)$. The principal $\pi_1(M,x)$-bundle $\mathbf{t} : \mathbf{s}^{-1}(x) \to \mathscr{O}_x$ is a realization of the universal cover for $\mathscr{O}_x$, so the fundamental groupoid is always source-1-connected.

When M is 1-connected, there is only one homotopy class of paths between each two points in M, so the fundamental groupoid coincides with the pair groupoid of M.

Another interesting example is provided by action groupoids.

Example 2.11. Let G be a Lie group acting smoothly on a manifold M. We define the action groupoid $G \ltimes M \rightrightarrows M$, whose space of arrows is $G \times M$, and where $\mathbf{s}(g,x) = x$, $\mathbf{t}(g,x) = gx$. We define multiplication by $\mathbf{m}((h,gx),(g,x)) = (hg,x)$. For this groupoid, we have that

1. Each $\mathbf{s}$-fiber is diffeomorphic to G, so source-1-connectedness of the action groupoid is equivalent to 1-connectedness of G.

2. The isotropy group at x is the isotropy group of the action:
$$G_x = \{g \in G\colon\, gx = x\}.$$

3. The orbit through x coincides with the orbit through x of the G-action
$$\mathscr{O}_x = \{gx\colon\, g \in G\}.$$

We end this section with the definition of a Lie groupoid action.

Let $\mathscr{G} \rightrightarrows M$ be a groupoid and let $\mu : S \to M$ be a smooth map. We set
$$\mathscr{G} \times_M S = \{(g,p) \in \mathscr{G} \times S\colon\, \mathbf{s}(g) = \mu(p)\} \subseteq \mathscr{G} \times S.$$
This is a smooth submanifold of $\mathscr{G} \times S$, since $\mathbf{s}$ is a submersion.

Definition 2.12. An action of $\mathscr{G} \rightrightarrows M$ on $\mu : S \to M$ is a smooth map
$$\mathbf{A} : \mathscr{G} \times_M S \to S, \quad (g,p) \mapsto g \cdot p,$$
satisfying

1. $1_{\mu(p)} \cdot p = p$.

2. $g \cdot (h \cdot p) = (gh) \cdot p$ for all $(g,h) \in \mathscr{G}^{(2)}$.

3. $\mu(g \cdot p) = \mathbf{t}(g)$

An action groupoid $G \ltimes M$ and a groupoid action are related as follows.

Example 2.13. If G is a Lie group acting smoothly on a manifold M, an action of the action groupoid $G \ltimes M$ on a smooth map $\mu : S \to M$ is the same as a smooth action of G on S together with a G-equivariant map $\mu : S \to M$.

2.1.3 Normal Lie subgroupoids and Short Exact Sequences

Normal Lie subgroupoids of a Lie groupoid play the role of normal Lie subgroups, in the sense that one can take quotients by normal Lie subgroupoids and form a new Lie groupoid. In this section, we study the relation between normal Lie subgroupoids and short exact sequence of Lie groupoids.

Definition 2.14. Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid. A Lie subgroupoid is a Lie groupoid $\mathscr{H} \rightrightarrows N$ together with a Lie groupoid morphism $j : \mathscr{H} \to \mathscr{G}$ which is an injective immersion. We call a Lie subgroupoid

1. Wide if $N = M$,
2. Embedded if j is an embedding,
3. Closed if $j(\mathscr{H})$ is closed in $\mathscr{G}$.

Next, we define normal Lie subgroupoids.

Definition 2.15. A normal Lie subgroupoid $\mathscr{H}$ of a Lie groupoid $\mathscr{G}$ is a wide, embedded, closed Lie subgroupoid, such that

1. For all $g \in \mathscr{G}$, one has
$$g\mathscr{H}_{\mathbf{s}(g)}g^{-1} \subseteq \mathscr{H}_{\mathbf{t}(g)}.$$
2. The restriction of $\mathbf{s}$ and $\mathbf{t}$ to $\mathscr{H}$ coincide.

An important group of examples of normal Lie subgroupoids comes form kernels of Lie groupoid morphisms.

Example 2.16. Let $\Phi : \mathscr{G} \to \mathscr{H}$ be a Lie groupoid morphism that is a surjective submersion. Then

$$\ker(\Phi) = \{g \in \mathscr{G} : \Phi(g) = 1_{\phi(\mathbf{t}(g))}\}$$

is a normal Lie subgroupoid of $\mathscr{G}$.

If $\mathscr{H} \subseteq \mathscr{G}$ is a normal Lie subgroupoid, it acts freely and properly on the map $\mathbf{s} : \mathscr{G} \to M$ by $h \cdot g = gh^{-1}$. The quotient $\mathscr{G}/\mathscr{H}$ is naturally a Lie groupoid for which the sequence

$$1 \to \mathscr{H} \to \mathscr{G} \to \mathscr{G}/\mathscr{H} \to 1$$

is a short exact sequence of Lie groupoids. The following can be interpreted as the first isomorphism theorem for Lie groupoids.

Theorem 2.17. *Let $\mathscr{K}$ be a Lie groupoid fitting in a short exact sequence of groupoids over M*

$$1 \to \mathscr{K} \xrightarrow{j} \mathscr{G} \xrightarrow{q} \mathscr{H} \to 1$$

Then $\mathscr{K}$ is a normal Lie subgroupoid of $\mathscr{G}$ isomorphic to $\ker(q)$ *and q descends to an isomorphism $\mathscr{G}/\mathscr{K} \to \mathscr{H}$.*

2.2 Lie algebroids

In this section, we will describe Lie algebroids, and explain how to Lie algebroids can be seen as the infinitesimal counterparts of Lie groupoids.

Definition 2.18. A Lie algebroid over M is a triple $(A, \rho, [\cdot,\cdot]_A)$, consisting of a vector bundle $A \to M$, a vector bundle map $\rho : A \to TM$, called the anchor and a Lie bracket $[\cdot,\cdot]_A$, on the space of sections $\Gamma(A)$ satisfying the Leibniz identity

$$[\alpha, f\beta]_A = f[\alpha,\beta] + \mathscr{L}_{\rho(\alpha)}(f)\beta.$$

The following might have been expected to be part of the definition, but it turns out it is actually a consequence of the other properties.

Proposition 2.19. *Let $(A, \rho, [\cdot,\cdot]_A)$ be a Lie algebroid. The anchor is a Lie algebra map, i.e. satisfies*

$$[\rho(\alpha), \rho(\beta)] = \rho([\alpha,\beta]_A)$$

for all $\alpha, \beta \in \Gamma(A)$.

Proof. The key ingredient in the proof is that the Jacobi identity for $[\cdot,\cdot]_A$ is equivalent to ρ preserving the bracket. To see this, consider the Jacobiator

$$J(\alpha,\beta,\gamma) = [[\alpha,\beta]_A,\gamma]_A + [[\beta,\gamma]_A,\alpha]_A + [[\gamma,\alpha]_A,\beta]_A.$$

We have for $\alpha,\beta,\gamma \in \Gamma(A)$ and $f \in C^\infty(M)$ that

$$J(\alpha,\beta,f\gamma) - fJ(\alpha,\beta,\gamma) = \mathscr{L}_{\rho[\alpha,\beta]_A - [\rho(\alpha),\rho(\beta)]}(f)\gamma.$$

Since $[\cdot,\cdot]_A$ satisfies the Jacobi identity, the left-hand is zero, so the result follows. □

The notion of a Lie algebroid morphism is subtle, since the vector bundles involved can live over different bases. For Lie algebroids over the same base, the definition is clear.

Definition 2.20. Let $(A, \rho_A, [\cdot,\cdot]_A)$ and $(B, \rho_B, [\cdot,\cdot]_B)$ be Lie algebroids over B. A Lie algebroid morphism $\Phi : A \to B$ is a vector bundle map intertwining the anchors

$$\begin{tikzcd} A \arrow[rr, "\Phi"] \arrow[dr, "\rho_A"'] & & B \arrow[dl, "\rho_B"] \\ & M & \end{tikzcd}$$

and preserving the brackets.

For Lie algebroids over different bases, the definition changes slightly, see [Mei17]. We will not use Lie algebroids over different bases.

For a general Lie algebroid $(A, \rho, [\cdot,\cdot]_A)$, we define at each $x \in M$ a Lie algebra, called the isotropy Lie algera of A at x:

$$\mathfrak{g}_x(A) = \ker(\rho_x : A_x \to T_xM).$$

The isotropy Lie algebra carries a Lie bracket, induced by $[\cdot,\cdot]_A$.

Proposition 2.21. *For every $x \in M$, there exists a Lie bracket $[\cdot,\cdot]_{\mathfrak{g}_x}$, uniquely determined by*

$$[\alpha(x),\beta(x)]_{\mathfrak{g}_x} = [\alpha,\beta]_A(x)$$

for all $\alpha,\beta \in \Gamma(A)$ such that $\alpha(x),\beta(x) \in \mathfrak{g}_x(A)$.

Proof. Observe that for $\beta \in \Gamma(A)$ with $\beta(x) \in \mathfrak{g}_x(A)$, one has that $[\alpha, f\beta]_A(x) = f(x)[\alpha,\beta]_A(x)$. It follows by a standard argument in differential geometry that the bracket is well-defined, see for instance Paragraph 10.7 in [Tu17]. $\square$

Let us now discuss some examples of Lie algebroids.

Example 2.22. A Lie algebroid over a point is the same as a Lie algebra, the anchor is the zero map. More general, a Lie algebroid for which the anchor $\rho : A \to TM$ is zero, is a bundle of Lie algebras. The isotropy Lie algebras of this algebroid are the fibers A_x.

If $(A,\rho,[\cdot,\cdot]_A)$ is a Lie algebroid, the isotropy algebras $\ker(\rho_x)$ form a bundle of Lie algebras if and only if $\ker(\rho_x)$ is of constant rank. In that case, we have a short exact sequence of Lie algebroids

$$0 \to \ker(\rho) \to A \to \mathrm{Im}(\rho) \to 0 \tag{12}$$

Example 2.23. Quite trivially, $TM \to M$ is a Lie algebroid, where the anchor is the identity. More general, any regular flotation on M forms an involutive distribution $\mathscr{F} \subseteq TM$. This is a Lie algebroid, and the anchor is the inclusion $\mathscr{F} \to TM$.

Probably the most important example for us is the cotangent algebroid for a Poisson manifold.

Example 2.24. If (M,π) is a Poisson manifold, we define the cotangent algebroid as the triple $(T^*M,\pi^\sharp,[\cdot,\cdot]_\pi)$, where the bundle map $\pi^\sharp : T^*M \to TM$ is the anchor and with Lie bracket $[\cdot,\cdot]_\pi$. We have seen in Proposition 1.8 that the Jacobi identity for $[\cdot,\cdot]_\pi$ is equivalent to π being a Poisson structure. Thus, we only have to verify the Leibniz identity. For this, we get

$$\begin{aligned}
[\alpha, f\beta]_\pi &= \mathscr{L}_{\pi^\sharp(\alpha)}(f\beta) - \mathscr{L}_{\pi^\sharp(f\beta)}(a) - d\left(\pi(\alpha, f\beta)\right) \\
&= \mathscr{L}_{\pi^\sharp(\alpha)}(f)\beta + f\mathscr{L}_{\pi^\sharp(a)}\beta - f\mathscr{L}_{\pi^\sharp(\beta)}\alpha - df \wedge \iota_{\pi^\sharp(\beta)}\alpha - df \wedge \pi(\alpha,\beta)) - fd(\pi(\alpha,\beta)) \\
&= f\mathscr{L}_{\pi^\sharp(\alpha)}\beta - f\mathscr{L}_{\pi^\sharp(\beta)}\alpha + \mathscr{L}_{\pi^\sharp(\alpha)}(f)\beta - df \wedge \pi(\beta,\alpha) - df \wedge \pi(\alpha,\beta) - fd(\pi(\alpha,\beta)) \\
&= f[\alpha,\beta]_\pi + \mathscr{L}_{\pi^\sharp(\alpha)}(f)\beta.
\end{aligned}$$

The isotropy Lie algebras are precisely $\ker(\pi_x^\sharp) = \nu_x^*(S)$, where S is the symplectic leaf through x. These form a bundle of Lie algebras if and only if (M,π) is a regular Poisson manifold and in that case we have a short exact sequence of Lie algebroids

$$0 \to \nu^*(\mathscr{F}) \to T^*M \xrightarrow{\pi^\sharp} \mathscr{F} \to 0 \tag{13}$$

where $\mathscr{F}$ is the symplectic foliation.

2.2.1 The Lie algebroid of a Lie groupoid

In this section, we describe the Lie algebroid of a Lie groupoid. In particular, we will describe the "Lie-functor" which associates a Lie algebroid to a Lie groupoid, and we will discuss important examples.

Inspired by the Lie group-Lie algebra correspondence, we start with the definition of right-invariant vector fields.

Definition 2.25. Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid. A vector field $X \in \mathfrak{X}(\mathscr{G})$ is called right-invariant if

1. X is tangent to the **s**-fibers.
2. $(dR_g)_h X_h = X_{hg}$ for all $(h,g) \in \mathscr{G}^{(2)}$, where R_g is right multiplication by g, defined in Equation (11).

We denote by $\mathfrak{X}_{\mathrm{inv}}(\mathscr{G})$ the space of all right-invariant vector fields on $\mathscr{G}$.

Next, observe that for any $X \in \mathfrak{X}_{\mathrm{inv}}(\mathscr{G})$, the restriction $\alpha = X|_{\mathbf{u}(M)}$ defines a section of the vector bundle

$$A = \mathbf{u}^* \ker(d\mathbf{s}).$$

Conversely, any section $\alpha \in \Gamma(A)$ produces a right-invariant vector field $\vec{\alpha}$ by

$$\vec{\alpha}_g = (dR_g)_{1_{\mathbf{t}(g)}}(\alpha_{\mathbf{t}(g)}).$$

It follows that right-invariant vector fields on $\mathscr{G}$ are in one-to-one correspondence with sections of A. The Lie bracket of two right-invariant vector fields is again a right-invariant vector field. This allows us to define a Lie bracket $[\cdot,\cdot]_A$ by

$$\overrightarrow{[\alpha,\beta]_A} = [\vec{\alpha},\vec{\beta}].$$

Definition 2.26. The Lie algebroid of a Lie groupoid $\mathscr{G} \rightrightarrows M$ is the vector bundle

$$A = \mathbf{u}^* \ker(d\mathbf{s}) \to M,$$

with anchor map $\rho : A \to TM$ given by $\rho_x = (d\mathbf{t})_{1_x} : \ker(d\mathbf{s})_{1_x} \to T_x M$ and Lie bracket $[\cdot,\cdot]_A$.

Proposition 2.27. *The triple $(A, \rho, [\cdot,\cdot]_A)$ is a Lie algebroid.*

Proof. All that is left to show is the Leibniz identity. For this, it is enough to show that

$$\overrightarrow{[\alpha, f\beta]_A} = \overrightarrow{f[\alpha,\beta]_A} + \overrightarrow{\mathscr{L}_{\rho(\alpha)}(f)\beta}.$$

To show this, note that

$$\overrightarrow{f\alpha}_g = (dR_g)_{1_{\mathbf{t}(g)}}\left(f(\mathbf{t}(g))\alpha_{1_{\mathbf{t}(g)}}\right) = f(\mathbf{t}(g))(dR_g)_{1_{\mathbf{t}(g)}}\alpha_{1_{\mathbf{t}(g)}} = (\mathbf{t}^* f)(g) \cdot \vec{\alpha}_g$$

so

$$\overrightarrow{f\alpha} = \mathbf{t}^* f \overrightarrow{\alpha}.$$

Therefore, we have

$$\overrightarrow{[\alpha, f\beta]_A} = [\vec{\alpha}, \overrightarrow{f\beta}] = [\vec{\alpha}, \mathbf{t}^* f\vec{\beta}] = \mathbf{t}^* f[\vec{\alpha}, \vec{\beta}] + \mathscr{L}_{\vec{\alpha}}(\mathbf{t}^* f)\vec{\beta} = \mathbf{t}^* f\overrightarrow{[\alpha, \beta]_A} + \mathscr{L}_{\vec{\alpha}}(\mathbf{t}^* f)\vec{\beta}.$$

Next, remark that

$$\begin{aligned}
\mathscr{L}_{\vec{\alpha}}(\mathbf{t}^* f)(g) &= d(f \circ \mathbf{t})_g(\vec{\alpha}_g) \\
&= df_{\mathbf{t}(g)} d\mathbf{t}_g (dR_g)_{1_{\mathbf{t}(g)}} \alpha_{\mathbf{t}(g)} \\
&= df_{\mathbf{t}(g)} d\mathbf{t}_{1_{\mathbf{t}(g)}} \alpha_{\mathbf{t}(g)} \\
&= \mathbf{t}^*(\mathscr{L}_{\rho(\alpha)}(f))(g),
\end{aligned}$$

so we deduce that

$$\overrightarrow{[\alpha, f\beta]_A} = [\vec{\alpha}, \overrightarrow{f\beta}] = \mathbf{t}^* f\overrightarrow{[\alpha, \beta]_A} + \mathbf{t}^*(\mathscr{L}_{\rho(\alpha)}(f))\vec{\beta} = \overrightarrow{f[\alpha, \beta]_A} + \overrightarrow{\mathscr{L}_{\rho(\alpha)}(f)\beta},$$

and this concludes the proof. □

If $\mathscr{G} \rightrightarrows M$ is a Lie groupoid, we denote by $\mathrm{Lie}(\mathscr{G})$ its Lie algebroid. This describes what the Lie-functor does to Lie groupoids. Of course, we should also describe its action on Lie groupoid morphisms. Let $(\Phi, \phi) : (\mathscr{G}_1, M_1) \to (\mathscr{G}_2, M_2)$ be a morphism of Lie groupoids. Then Φ maps $\mathbf{s}$-fibers to $\mathbf{s}$-fibers, so we can define the bundle map

$$\begin{tikzcd}
\mathrm{Lie}(\mathscr{G}_1) \arrow[r, "\Phi_*"] \arrow[d] & \mathrm{Lie}(\mathscr{G}_2) \arrow[d] \\
M_1 \arrow[r, "\phi"] & M_2
\end{tikzcd} \qquad \Phi_* = d\Phi|_{\mathbf{u}^* \ker(d\mathbf{s})},$$

It is shown in [CFM21] that this indeed defines a morphism of Lie algebroids over different bases. This gives a description of the Lie functor. The following proposition is straightforward.

Proposition 2.28. *Let $\mathscr{G} \to M$ be a Lie groupoid with Lie algebroid $A = Lie(\mathscr{G})$. Then the isotropy Lie algebras* $\ker(d\mathbf{t})_{1_x}$ *coincide with the Lie algebra of the isotropy groups $\mathscr{G}_x$ for all $x \in M$.*

Let us now give some examples.

Example 2.29. If $\mathscr{G} \rightrightarrows M$ is a bundle of Lie groups, $\mathrm{Lie}(\mathscr{G})$ is a bundle of Lie algebras. Since $\mathbf{s} = \mathbf{t}$, the anchor $d\mathbf{t}$ vanishes on $\ker(d\mathbf{s})$.

Example 2.30. The Lie algebroid of the pair groupoid $M \times M$ and of the fundamental groupoid $\Pi_1(M)$ is just TM.

An important observation is that we can pass to the source-connected part of any Lie groupoid, without losing information about the Lie algebroid.

Proposition 2.31. *Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid, with possibly disconnected* $\mathbf{s}$*-fibers. Let $\mathscr{G}^\circ$ be the union of all the connected components of the* $\mathbf{s}$*-fibers containing the units. Then $\mathscr{G}^\circ \rightrightarrows M$ is an open Lie subgroupoid of M. In particular, $Lie(\mathscr{G}) = Lie(\mathscr{G}^\circ)$.*

Proof. For a proof, see [Mac05]. □

2.3 Integration of Lie algebroids

Now that we have a way to associate a Lie algebroid to a Lie groupoid, the natural question that arises is whether any Lie algebroid comes from a Lie groupoid. This is well-known to be true for Lie groups and Lie algebras, in the finite dimensional case, but it turns out that not every Lie algebroid comes from a Lie groupoid. In this section, we discuss Lie's theorems for Lie algebroids and compare them to Lie's theorems for Lie groups and Lie algebras. Then, we describe a necessary and sufficient condition for a Lie algebroid to be integrable. In this section, we have collected results from [CFM21] and [CF01].

2.3.1 Lie's Theorems for Lie Algebroids

Definition 2.32. Let $A \to M$ be a Lie algebroid. We say that A is integrable if there some, not necessarily Hausdorff, Lie groupoid $\mathscr{G} \rightrightarrows M$ such that $A \cong \mathrm{Lie}(\mathscr{G})$. We say that $\mathscr{G}$ integrates A.

This strategy of inverting the Lie-functor is well-known in the theory of Lie groups and in fact, many results concerning the integration of Lie algebroids carry over to Lie groupoids. The integration part in Lie theorems for Lie algebras assumes 1-connectedness of the Lie groups, which, in the context of Lie groupoids, should be replaced by 1-connectedness of the $\mathbf{s}$-fibers. The first two theorems of Lie have natural analogues for Lie groupoids.

Theorem 2.33 (Lie I for Lie algebroids, [CFM21])**.** *Let A be an integrable Lie algebroid. There exists a unique (up to isomorphism) Lie groupoid with 1-connected* $\mathbf{s}$*-fibers integrating A, which is possibly non-Hausdorff.*

Theorem 2.34 (Lie II for Lie algebroids, [CFM21])**.** *Let* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *be Lie groupoids with Lie algebroids* A_1 *and* A_2*. If* $\mathscr{G}_1$ *is* $\mathbf{s}$*-1-connected, any Lie algebroid morphism* $\phi : A_1 \to A_2$ *can be integrated to a unique Lie groupoid morphism* $\Phi : \mathscr{G}_1 \to \mathscr{G}_2$*.*

A proof of these theorems can be found in [MM03].

Lie's third theorem for Lie algebras is more delicate and harder to prove.

Theorem 2.35 (Lie III for Lie algebras, [Hal10])**.** *Any finite dimensional Lie algebra is integrable.*

As already stated, not every Lie algebroid comes from a Lie groupoid, so there is no simple analogue of Lie III to Lie algebroids. Fortunately, the failure of integrability is well understood and there are several equivalent integrability criteria. For any Lie algebroid $A \to M$, there is always a natural candidate groupoid: the Weinstein groupoid $\mathscr{G}(A)$. If this groupoid can be given a smooth structure, it is a Lie groupoid with 1-connected $\mathbf{s}$-fibers integrating A. Thus, the integrability of A is equivalent to the smoothness of $\mathscr{G}(A)$.

2.3.2 Weinstein Groupoid

In this section, we describe the Weinstein groupoid $\mathscr{G}(A)$ and make the integrability criteria explicit using monodromy groups. The idea is to construct a special set of paths $a : [0,1] \to A$, known as A-paths, and define a suitable notion of homotopy between A-paths.

Definition 2.36. Let $(A \xrightarrow{\pi} M, \rho, [\cdot,\cdot]_A)$ be a Lie algebroid. An A-path is a path $a : I \to A$ such that

$$\rho(a(t)) = \frac{d\gamma_a}{dt},$$

where $\gamma_a = \pi \circ a$ is the base path of the A-path.

We denote the space of all C^1 A-paths a with C^2 base path γ_a by $\mathscr{P}(A)$. This has the structure of an infinite dimensional Banach manifold.

Remark 2.37. There is a natural interpretation of A-paths as generalized derivatives. For any path $\gamma : I \to M$, there is a path $d\gamma : I \to TM$ which is the derivative of γ. Since a Lie algebroid $A \to M$ can be thought of as a replacement of the tangent bundle, we can think of an A-path as a pair (a, γ_a), where a is the A-derivative of γ_a, related to the derivative of γ_a through the anchor.

We will not go into the detail about the A-homotopy, since the definition is rather involved, see [CF01]. The basic idea is that it is just a collection of paths $a_\varepsilon(t) I \times I \to A$ such that the base paths γ_ε have fixed start and end points, plus some extra information capturing the Lie algebroid structure. If two A-paths a and a' are A-homotopic, we write $a \sim a'$. The notion of an A-homotopy induces an equivalence relation on $\mathscr{P}(A)$, inducing a foliation whose leaves have codimension dim(A), and we denote by $\mathscr{G}(A) = \mathscr{P}(A)/\sim_A$ the leaf space. This is a topological groupoid with 1-connected $\mathbf{s}$-fibers.

Since the base paths of A-homotopic A-paths have the same start and end points, we define the source and target maps as

$$\mathbf{s} : \mathscr{G}(A) \to M, \mathbf{s}([a]) = \gamma_a(0), \quad \mathbf{t} : \mathscr{G}(A) \to M, \mathbf{t}([a]) = \gamma_a(1).$$

The multiplication map is slightly more subtle. For A-paths a and b such that $\gamma_a(0) = \gamma_b(1)$, we wish to define the composition of paths as

$$(a \bullet b)(t) = \begin{cases} 2b(2t) & t \leq \frac{1}{2} \\ 2a(2t-1) & t \geq \frac{1}{2} \end{cases}$$

This is independent of the chosen representative of the A-homotopy class of a and b. Of course, the problem is that this path need not be smooth at $t = \frac{1}{2}$. To solve this problem, we choose a smooth reparametrization $\tau : I \to I$ such that $\tau^{(n)}(0) = \tau^{(n)}(1) = 0$ for all $n \geq 1$. We define the reparametrized path a^τ by $a^\tau(t) = \tau'(t) a(\tau(t))$, and one can show that this is A-homotopic to a. Thus, we just choose smooth representatives for which the composition above forms a smooth path. One can show that this is well-defined, since the composition is independent of the chosen representative.

Finally, we define the units by $\mathbf{u}(x) = 0_x \in A_x$, and inversion maps the path $(t \mapsto a(t))$ to the path $(t \mapsto -a(1-t))$. This gives $\mathscr{G}(A)$ the structure of a topological groupoid. The $\mathbf{s}$-fibers are isomorphic to the path space of M, which we know is 1-connected.

Theorem 2.38 ([CF01]). *A Lie algebroid $A \to M$ is integrable if and only if and only if the smooth structure on $\mathscr{P}(A)$ descends to the leaf space $\mathscr{G}(A) = \mathscr{P}(A)/\sim_A$. If this is the case, $\mathscr{G}(A) \rightrightarrows A$ is a source-1-connected Lie groupoid integrating A.*

As we announced before, the smoothness of $\mathscr{G}(A)$ can be rephrased by a simpler, necessary and sufficient condition. This uses the monodromy groups, which are defined below.

Definition 2.39. Let $A \to M$ be a Lie algebroid, and let $x \in M$. The monodromy group of A at x is defined to be the following subgroup from the isotropy algebra.

$$\mathcal{N}_x(A) = \{v \in Z(\mathfrak{g}_x) \colon v \text{ and } 0_x \text{ are } A\text{-homotopic}\}.$$

When (M,π) is a regular Poisson manifold, we will derive a simpler expression for the monodromy groups in Section 2.5.

Now, we can state the conditions for integrability of a Lie algebroid $A \to M$ in terms of these monodromy groups. Fix a norm d on A and define the function $r : M \to [0,\infty]$ by $r(x) = d(0_x, \mathcal{N}_x(A) \setminus \{0_x\})$, with the convention that $d(0_x, \emptyset) = \infty$.

Definition 2.40. The monodromy groups $\mathcal{N}_x(A)$ are said to be locally uniformly discrete if the following two conditions hold.

1. The monodromy group $\mathcal{N}_x(A)$ is discrete, i.e. $r(x) > 0$, for all $x \in M$.
2. For all $x \in M$, one has $\liminf_{y \to x} r(y) > 0$.

The following theorem is what becomes of Lie III for Lie algebroids.

Theorem 2.41 ([CF01])**.** *Let $A \to M$ be a Lie algebroid. Then, the following are equivalent.*

- *The Weinstein groupoid $\mathcal{G}(A)$ is smooth.*
- *The monodromy groups $\mathcal{N}_x(A)$ are locally uniformly discrete.*
- *The Lie algebroid A is integrable.*

2.4 Integration of Poisson manifolds

In this section, we take a closer look at integrations of the cotangent algebroid $(T^*M, \pi^\sharp)$ of a Poisson manifold. By Theorem 2.41, we already have a good understanding of what it means for the Poisson manifold (M,π) to be integrable: we look for a Lie groupoid integrating T^*M, and the Weinstein groupoid is the natural candidate. It turns out that the Weinstein groupoid of a Poisson manifold comes with more structure, namely that of a multiplicative symplectic form, and is therefore called a symplectic groupoid. Moreover, it turns out that any symplectic groupoid determines a Poisson structure on the base. This leads to the interesting integration problem in Poisson geometry: given a Poisson manifold, is there a symplectic groupoid $(\mathcal{G}, \Omega) \rightrightarrows (M,\pi)$ inducing π? In this section we will properly state and discuss the integration problem in Poisson geometry.

We start with the definition of symplectic groupoids.

Definition 2.42. Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid. A differential form $\omega \in \Omega^k(\mathscr{G})$ is called multiplicative if

$$\mathbf{m}^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega,$$

where $\mathrm{pr}_1, \mathrm{pr}_2 : \mathscr{G}^{(2)} \to \mathscr{G}$ are the restriction of the projections $\mathrm{pr}_1, \mathrm{pr}_2 : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$. We say that $(\mathscr{G}, \omega) \rightrightarrows M$ is a symplectic groupoid if ω is a multiplicative, symplectic form.

Example 2.43. We discuss some basic examples of the symplectic groupoids of Poisson manifolds that appear in this thesis.

1. The cotangent bundle $T^*M \rightrightarrows M$, equipped with the canonical symplectic form, is a symplectic groupoid. This is a groupoid integrating the zero Poisson structure on M.

2. Let G be a Lie group and let $\Sigma = G \ltimes \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$ be action groupoid associated to the coadjoint representation. This is a groupoid integrating the linear Poisson structure $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$, Under the isomorphism $l_*T^*G \simeq G \ltimes \mathfrak{g}^*$ induced by left translation, the pushforward $\Omega = l_*\Omega_{\mathrm{can}}$ is a symplectic multipliative form on Σ.

3. Let (M, ω) be a symplectic manifold. The pair groupoid $\Sigma = M \times M \rightrightarrows M$. The form $\Omega = \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega$ is clearly symplectic, and multiplicative, hence $(\Sigma, \Omega) \to M$ is a symplectic groupoid, which integrates the non-degenerate Poisson manifold (M, ω).

The following proposition collects together several basic facts about symplectic groupoids.

Proposition 2.44 ([CFM21]). *Let $(\mathscr{G}, \Omega) \rightrightarrows M$ be a symplectic groupoid.*

1. *The* $\mathbf{s}$*- and* $\mathbf{t}$*-fibers are symplectic orthogonal:*

$$\ker(d\mathbf{s}) = \ker(d\mathbf{t})^{\perp_\Omega}.$$

2. *The unit section $\mathbf{u} : M \to \mathscr{G}$ is a Lagrangian embedding.*

3. *The inversion map $\iota : \mathscr{G} \to \mathscr{G}$ is anti-symplectic: $\iota^*\Omega = -\Omega$.*

We infer from part 2 of this proposition that $\dim(\mathscr{G}) = 2\dim(M)$.

All examples in Example 2.43 have the property that the target map is a Poisson map. This is no coincidence, and the following theorem explains the connection between symplectic groupoid and Poisson structures.

Theorem 2.45 ([CFM21]). *Let $(\mathscr{G},\Omega) \rightrightarrows M$ be a symplectic groupoid. There exists a unique Poisson structure π on M such that $\mathbf{t} : (\mathscr{G},\pi_\Omega) \to (M,\pi)$ is a Poisson map, where $\pi_\Omega^\sharp = (\Omega^\flat)^{-1}$. Moreover,*

1. *The connected components of the orbits of $\mathscr{G}$ are precisely the symplectic leaves of (M,π).*

2. *There is a canonical Lie algebroid isomorphism*

$$\sigma_\Omega : Lie(\mathscr{G}) \to T^*M, \qquad \alpha \mapsto -\mathbf{u}^*(\iota_\alpha \Omega).$$

In particular, $\pi^\sharp \circ \sigma_\Omega^{-1} = \rho$, where ρ is the anchor of $Lie(\mathscr{G})$.

The integration problem, mentioned at the start of this section, will can now be split up into two parts.

1. Is there a Lie groupoid $\mathscr{G} \rightrightarrows M$ integrating T^*M?

2. If $\mathscr{G} \rightrightarrows M$ is a Lie groupoid whose Lie algebroid is isomorphic to T^*M, does there exist a multiplicative symplectic form Ω on $\mathscr{G}$ inducing π?

The first question can be answered using the integrability criteria in the previous section. The second question was answered by Mackenzie and Xu.

Theorem 2.46 ([CFM21]). *Let (M,π) be a Poisson manifold and let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid with 1-connected $\mathbf{s}$-fibers integrating T^*M, and choose an isomorphism $\sigma : Lie(\mathscr{G}) \to T^*M$. Then there exists a unique multiplicative symplectic form Ω on $\mathscr{G}$ such that $\sigma_\Omega = \sigma$,*

Thus, we conclude that if (M,π) is an integrable Poisson manifold, the Weinstein groupoid $\Sigma(M,\pi) = \mathscr{G}(T^*M)$ carries a multiplicative symplectic form Ω such that $(\Sigma(M,\pi),\Omega) \rightrightarrows (M,\pi)$ is a symplectic integration. We can interpret the symplectic groupoid as a desingularization of the Poisson structure. In later chapters, we will be heavily interested in the topological properties of the symplectic groupoid $\Sigma(M,\pi)$. We end this chapter with a description of the monodromy groups for regular Poisson manifolds.

2.5 Regular Poisson manifolds

In this section, we will revisit the monodromy groups for a regular Poisson manifold, and simplify the expressions in Section 2.3.2, by integrating a certain curvature form over 2-cycles. This description for the monodromy groups of a regular Poisson manifold allows for a quick computations of the monodromy groups of regular Poisson submanifolds. We will take a rather unorthodox approach by relating the curvature form to the transversal derivative, and to prove properties about this curvature form using statements about the transversal derivative. The advantage of this is that the transversal derivative reappears intensely in 4.2 by any means, and introducing the transversal derivative at this stage means that we do not repeatedly present similar proofs.

2.5.1 The Transversal Derivative

Let (M,π) be a regular Poisson manifold and let $\mathscr{F} = \mathrm{im}(\pi^\sharp)$ be the symplectic foliation. Recall from Example 2.24 that we have a short exact sequence of Lie algebroids

$$0 \to \nu^*(\mathscr{F}) \to T^*M \xrightarrow{\pi^\sharp} \mathscr{F} \to 0. \tag{14}$$

Viewing this short exact sequence just as a short exact sequence of vector bundles, there exists a splitting $\tau : \mathscr{F} \to T^*M$. Such a splitting does not have to be a splitting of the short exact sequence as Lie algebroids, since τ might not preserve the bracket. The curvature of τ is a measure to what extent τ is a splitting of the short exact sequence (14) as Lie algebroids.

Definition 2.47. Let (M,π) be a regular Poisson manifold and let τ be a splitting of (14) as vector bundles. We define the curvature of τ as the form $\Omega_\tau \in \Omega^2(\mathscr{F}, \nu^*(\mathscr{F}))$ given by by

$$\Omega_\tau(X,Y) = [\tau(X), \tau(Y)]_\pi - \tau[X,Y]$$

the curvature of τ.

This is well-defined, because Ω_τ is indeed skew-symmetric and $C^\infty(M)$-linear, as the following proposition shows.

Proposition 2.48. *If τ is a splitting of* (14)*, then $\Omega_\tau \in \Omega^2(\mathscr{F}, \nu^*(\mathscr{F}))$.*

Proof. The skew-symmetry of Ω_τ is clear. For $f \in C^\infty(M)$, we compute that

$$\begin{aligned}
\Omega_\tau(fX,Y) &= [f\tau(X), \tau(Y)]_\pi - \tau[fX,Y] \\
&= f[\tau(X), \tau(Y)]_\pi - \mathscr{L}_{\pi^\sharp(\tau(Y))}(f)X - f\tau[X,Y] + \mathscr{L}_Y(f)X \\
&= f[\tau(X), \tau(Y)]_\pi - f\tau[X,Y] \\
&= f\Omega_\tau(X,Y),
\end{aligned}$$

since $\pi^\sharp \circ \tau = \mathrm{Id}$, and thus we see that Ω_τ is $C^\infty(M)$-bilinear.

Finally, we see that Ω_τ indeed takes value in $\nu^*(\mathscr{F})$, since

$$\begin{aligned}
\pi^\sharp(\Omega_\tau(X,Y)) &= \pi^\sharp([\tau(X), \tau(Y)]_\pi) - \pi^\sharp\tau[X,Y] \\
&= [\pi^\sharp\tau(X), \pi^\sharp\tau(Y)] - [X,Y] \\
&= [X,Y] - [X,Y] \\
&= 0,
\end{aligned}$$

so we are done. □

It follows directly from the definition of Ω_τ that τ defines a splitting of (14) as Lie algebroids if and only if $\Omega_\tau = 0$.

Now, we will give a description of the transversal derivative of a manifold M with a regular foliation $\mathscr{F}$, and relate this to Ω_τ in the specific case that $\mathscr{F}$ is the symplectic foliation of a regular Poisson manifold. The transversal derivative is defined in terms of the dual Bott connection on the conormal bundle to the foliation, which is a special case of an $\mathscr{F}$-connection on T^*M.

Definition 2.49. Let $E \to M$ be a smooth vector bundle and let $F \subseteq TM$ be a smooth subbundle. An F-connection on E is an $\mathbb{R}$-bilinear operation

$$\nabla : \Gamma(F) \times \Gamma(E) \to \Gamma(E)$$

such that for all $f \in C^\infty(M)$, $X \in \Gamma(F)$ and $s \in \Gamma(E)$, one has

1. $\nabla_{fX}(s) = f\nabla_X(s)$
2. $\nabla_X(fs) = f\nabla_X(s) + \mathscr{L}_X(f)s$

Note that a TM-connection on E is just an affine connection on E. If $F \subseteq TM$ is a smooth subbundle, we can always restrict an affine connection on E to an F-connection on E, and any F-connection can be extended to an affine connection.

Example 2.50. Let M be a smooth manifold and let $\mathscr{F}$ be a regular foliation on M. The Bott connection is the $\mathscr{F}$-connection on the normal bundle $\nu(\mathscr{F})$ to the foliation, defined by

$$\nabla : \Gamma(\mathscr{F}) \times \Gamma(\nu(\mathscr{F})) \to \Gamma(\nu(\mathscr{F})), \qquad \nabla_X(\overline{Y}) = \overline{[X,Y]}, \tag{15}$$

were $[X,Y]$ is the Lie bracket of vector fields. To check that this is well-defined, let Y' be another representative of $\overline{Y}$. Then the difference $Y - Y' \in \mathscr{F}$, and thus $[X, Y - Y'] \in \mathscr{F}$, since $\mathscr{F}$ is involutive. It follows that $\overline{[X,Y]} = \overline{[X,Y']}$.

To check that this is indeed a $\mathscr{F}$-connection on $\nu(\mathscr{F})$, we remark that for $f \in C^\infty(M)$, one has

$$\begin{aligned}
\nabla_X(f\overline{Y}) &= \overline{[X, fY]} = \overline{f[X,Y]} + \overline{\mathscr{L}_X(f)Y} = f\overline{[X,Y]} + \mathscr{L}_X(f)\overline{Y} = f\nabla_X(Y) + \mathscr{L}_X\overline{Y}, \\
\nabla_{fX}(\overline{Y}) &= \overline{[fX, Y]} = \overline{f[X,Y]} - \overline{\mathscr{L}_Y X} = f\overline{[X,Y]} - \mathscr{L}_Y(f)\overline{X} = f\overline{[X,Y]} = f\nabla_X(\overline{Y}),
\end{aligned}$$

where we used that $\overline{X} = 0$. This shows that the Bott connection indeed defines a $\mathscr{F}$-connection on $\nu(\mathscr{F})$.

Recall that a connection ∇ on a vector bundle $E \to M$ induces a connection ∇^* on the dual bundle $E^* \to M$, by requiring the following Leibniz identity to hold:

$$\mathscr{L}_X(\alpha(s)) = \langle \alpha, \nabla_X(s) \rangle + \langle \nabla^*_X(\alpha), s \rangle.$$

The same applies to any F-connection, and thus we obtain the dual Bott connection ∇^* as an $\mathscr{F}$-connection on the conormal bundle $\nu^*(\mathscr{F})$. Identifying the space of sections $\Gamma(\nu^*(\mathscr{F}))$ with the 1-forms on M that vanish on $\mathscr{F}$, a quick computations shows that for $\omega \in \Gamma(\nu^*(\mathscr{F})), Y \in \Gamma(TM)$:

$$(\nabla^*_X \omega)(Y) = \mathscr{L}_X(\omega(Y)) - \omega([X,Y]) = (\mathscr{L}_X \omega)(Y), \tag{16}$$

so the dual Bott connection $\nabla^* : \Gamma(\mathscr{F}) \times \Gamma(\nu^*(\mathscr{F})) \to \Gamma(\nu^*(\mathscr{F}))$ is simply given by

$$\nabla^*_X \omega = \mathscr{L}_X \omega. \tag{17}$$

We denote by $\Omega^\bullet(\mathscr{F}, \nu^*(\mathscr{F}))$ the space of foliated differential forms with values in the conormal bundle. The dual Bott connection gives rise to a map $d_{\mathscr{F}}^{\nabla^*} : \Omega^p(\mathscr{F}, \nu^*(\mathscr{F})) \to \Omega^{p+1}(\mathscr{F}, \nu^*(\mathscr{F}))$ given by

$$\begin{aligned}(d_{\mathscr{F}}^{\nabla^*}\omega)(X_0,\cdots,X_p) = &\sum_{i=0}^{p}(-1)^i\nabla^*_{X_i}\left(\omega(X_0,\cdots,\hat{X}_i,\cdots,X_p)\right) \\ &+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_p). \end{aligned} \tag{18}$$

It squares to zero since the (dual) Bott connection is flat. The associated cohomology groups, called the foliated cohomology groups with coefficients in $\nu^*(\mathscr{F})$, are denoted by $H^\bullet(\mathscr{F}, \nu^*(\mathscr{F}))$.

If we start with a $d_{\mathscr{F}}$-closed foliated 2-form $\omega_{\mathscr{F}} \in \Omega^2(\mathscr{F})$, we can extend it to a 2-form $\omega \in \Omega^2(M)$, by choosing an explicit splitting $TM \cong \mathscr{F} \oplus \nu(\mathscr{F})$. Note that ω need not be closed, but we can construct a $d_{\mathscr{F}}^{\nabla^*}$-closed 2-form out of $d\omega$ as follows. Since $d_{\mathscr{F}}\omega_{\mathscr{F}} = 0$, the foliated differential form

$$\theta : (X,Y) \mapsto d\omega(X,Y,-)$$

vanishes on $\mathscr{F}$, so it defines an element $\theta \in \Omega^2(\mathscr{F}, \nu^*(\mathscr{F}))$.

Lemma 2.51. *The form $\theta \in \Omega^2(\mathscr{F}, \nu^*(\mathscr{F}))$ is closed, and the class $[\theta] \in H^2(\mathscr{F}, \nu^*(\mathscr{F}))$ is independent of the chosen extension ω.*

Proof. The proof of this lemma consists of two simple computations. For the first one, let $X_0, X_1, X_2 \in \Gamma(\mathscr{F})$ and let $N \in \Gamma(TM)$. Since

$$(\mathscr{L}_X\omega)(Y) = \mathscr{L}_X(\omega(Y)) - \omega([X,Y]),$$

we have that

$$\begin{aligned}\left(d_{\mathscr{F}}^{\nabla^*}\theta\right)(X_0,X_1,X_2)(N) &= \mathscr{L}_{X_0}(\theta(X_1,X_2))(N) - \mathscr{L}_{X_1}(\theta(X_0,X_2))(N) + \mathscr{L}_{X_2}(\theta(X_0,X_2))(N) \\ &\quad - d\omega([X_0,X_1],X_2,N) + d\omega([X_0,X_2],X_1,N) - d\omega([X_1,X_2],X_0,N) \\ &= \mathscr{L}_{X_0}(d\omega(X_1,X_2,N)) - \mathscr{L}_{X_1}(d\omega(X_0,X_2,N)) + \mathscr{L}_{X_2}(d\omega(X_0,X_1,N)) \\ &\quad - d\omega(X_1,X_2,[X_0,N]) + d\omega(X_0,X_2,[X_1,N]) - d\omega(X_0,X_1,[X_2,N] \\ &\quad - d\omega([X_0,X_1],X_2,N) + d\omega([X_0,X_2],X_1,N) - d\omega([X_1,X_2],X_0,N) \\ &= d^2\omega(X_0,X_1,X_2,N) = 0,\end{aligned}$$

where we used in the final line that $\mathscr{L}_N(d\omega(X_0,X_1,X_2)) = 0$ since $\omega_{\mathscr{F}}$ is $d_{\mathscr{F}}$-closed.

For the second statement, suppose ω' is another extension of $\omega_{\mathscr{F}}$, giving rise to a closed form $\theta' \in \Omega^2(\mathscr{F}, \nu^*(\mathscr{F}))$. Denote $\alpha = \omega - \omega'$, and remark that $\alpha(X,Y) = 0$ for $X, Y \in \Gamma(\mathscr{F})$, so we can interpret α as an element in $\Omega^1(\mathscr{F}, \nu^*(\mathscr{F}))$. We have by a similar type of computation that

$$\begin{aligned}(\theta - \theta')(X,Y)(N) &= (d\omega - d\omega')(X,Y,N) \\ &= \mathscr{L}_X\left((\omega-\omega')(Y,N)\right) - \mathscr{L}_Y\left((\omega-\omega')(X,N)\right) \\ &\quad - (\omega-\omega')([X,Y],N) + (\omega-\omega')([X,N],Y) - (\omega-\omega')([Y,N],X) \\ &= (\mathscr{L}_X\alpha(Y))(N) - (\mathscr{L}_Y\alpha(X))(N) - \alpha([X,Y])(N) \\ &= (d_{\mathscr{F}}^{\nabla^*}\alpha)(X,Y)(N),\end{aligned}$$

so $[\theta] = [\theta']$, and this proves the lemma. $\square$

Definition 2.52. Let $\omega_{\mathscr{F}} \in \Omega^2(\mathscr{F})$ be a $d_{\mathscr{F}}$-closed 2-form, and let $\theta \in \Omega^2(\mathscr{F}, \nu^*(\mathscr{F}))$ be as above. We define the transversal derivative as the map

$$d_\nu : H^2(\mathscr{F}) \to H^2(\mathscr{F}, \nu^*(\mathscr{F})), \qquad d_\nu[\omega_{\mathscr{F}}] = [\theta],$$

and we call $[\theta]$ the normal variation of $\omega_{\mathscr{F}}$

Proposition 2.53. *The transversal derivative is well-defined, i.e. does not depend on the chosen representative of* $[\omega_{\mathscr{F}}]$.

Proof. Let $\omega'_{\mathscr{F}} = \omega_{\mathscr{F}} + d_{\mathscr{F}}\alpha$ for some $\alpha \in \Omega^1(\mathscr{F})$, and choose an extension $\omega \in \Omega^2(M)$ of $\omega_{\mathscr{F}}$ and an extension $A \in \Omega^1(M)$ of α. Then dA is an extension of $d_{\mathscr{F}}\alpha$, so $\omega' = \omega + dA$ is an extension of $\omega'_{\mathscr{F}}$. It follows that $d\omega' = d\omega$, so d_ν is well-defined. $\square$

In particular, the transversal derivative measures how $\omega_{\mathscr{F}}$ changes in the normal direction, as the following example aims to makes clear.

Example 2.54. We consider a simple foliation $M = S \times \mathbb{R}^q$. We endow each leaf $S_v = S \times \{v\}$, with a closed 2-form ω_v that vary smoothly, and thus glue together to a $d_{\mathscr{F}}$-closed foliated 2-form $\omega_{\mathscr{F}}$. There exists a natural decomposition $TM = TS \oplus T\mathbb{R}^q$, hence a natural a way to extend foliated 2-form $\omega_{\mathscr{F}}$ to the 2-form $\tilde{\omega} \in \Omega^2(M)$ which vanishes on $T\mathbb{R}^q$. It follows that for $Z \in T\mathbb{R}^q = \nu(\mathscr{F})$, one has

$$\theta(X,Y)(Z) = d\tilde{\omega}(X,Y,Z) = L_Z \omega_{\mathscr{F}}(X,Y),$$

so the variation of the foliated form in the transversal derivative coincides with the directional derivative in the normal direction.

Foliated cohomology groups can be tedious to compute, and adding coefficients does not simplify that task. Therefore, computing the transversal derivative of the foliated symplectic form on a regular Poisson manifold can turn out to be very challenging. Hence, we wish to relate it to an object that we are more acquainted with. It turns out that this object is the curvature Ω_τ of a splitting of the short exact sequence of vector bundles

$$\begin{tikzcd} 0 \arrow[r] & \nu^*(\mathscr{F}) \arrow[r] & T^*M \arrow[r, "\pi^\sharp"] & \mathscr{F} \arrow[r] \arrow[l, bend left, dotted, "\tau"] & 0 \end{tikzcd}.$$

The relationship between the transversal derivative and Ω_τ arises from the following observation.

Proposition 2.55. *Let (M,π) be a regular Poisson manifold, with foliated symplectic form $\omega_{\mathscr{F}}$. Then there exists a one-to-one correspondence*

$$\{\textit{Splittings } \tau : \mathscr{F} \to T^*M \textit{ of } (14)\} \xrightarrow{1-1} \{\textit{Extensions } \omega \in \Omega^2(M) \textit{ of } \omega_{\mathscr{F}} \textit{ such that } TM = \mathscr{F} \oplus \ker(\omega^\flat)\}$$

Proof. We will explain the maps in both directions, and it will be immediate they are inverses. Given a splitting

$\tau : \mathscr{F} \to T^*M$, we define $\omega \in \Omega^2(M)$ by $\iota_X \omega = \tau(X)$ for $X \in \mathscr{F}$, and $\iota_V \omega = 0$ otherwise. Then

$$\begin{aligned}\omega(\pi^\sharp(\alpha), \pi^\sharp(\beta)) &= \langle \iota_{\pi^\sharp(\alpha)} \omega, \pi^\sharp(\beta) \rangle \\ &= \langle \tau(\pi^\sharp(\alpha)), \pi^\sharp(\beta) \rangle \\ &= -\langle \pi^\sharp(\alpha), \beta \rangle \\ &= \omega_{\mathscr{F}}(\pi^\sharp(\alpha), \pi^\sharp(\beta)),\end{aligned}$$

so ω is an extension of $\omega_{\mathscr{F}}$, and this defines the map to the right.

For the map to the left, we define τ by $\tau(X) = \iota_X \omega$.Then τ defines a splitting of (14) since for $X = \pi^\sharp(\alpha) \in \mathscr{F}$, one has

$$\begin{aligned}\langle \beta, \pi^\sharp(\tau(X)) \rangle &= \langle \beta, \pi^\sharp(\iota_{\pi^\sharp(\alpha)} \omega) \rangle \\ &= -\langle \iota_{\pi^\sharp(\alpha)} \omega, \pi^\sharp(\beta) \rangle \\ &= -\omega(\pi^\sharp(\alpha), \pi^\sharp(\beta)) \rangle \\ &= \pi(\alpha, \beta) \\ &= \langle \beta, X \rangle.\end{aligned}$$

It is immediate the maps are each others inverses. □

Let (M, π) is a regular Poisson manifold, with a splitting $\tau : \mathscr{F} \to T^*M$. By the previous proposition, this gives rise to an extension ω of $\omega_{\mathscr{F}}$. The curvature Ω_τ, see Definition 2.47, of this splitting and ω are related by

$$\begin{aligned}\Omega_\tau(X,Y)(Z) &= \langle [\tau(X), \tau(Y)]_\pi, Z \rangle - \langle \tau[X,Y], Z \rangle \\ &= \langle \mathscr{L}_X(\tau(Y))), Z \rangle - \langle \mathscr{L}_Y(\tau(X)), Z \rangle - \mathscr{L}_Z(\pi(\tau(X), \tau(Y))) - \omega([X,Y],Z) \\ &= \mathscr{L}_X(\langle \tau(Y), Z \rangle) - \langle \tau(Y), [X,Z] \rangle - \mathscr{L}_Y(\langle \tau(X), Z \rangle) \\ &+ \langle \tau(X), [Y,Z] \rangle + \mathscr{L}_Z(\omega(X,Y)) - \omega([X,Y],Z) \\ &= \mathscr{L}_X \omega(Y,Z) - \mathscr{L}_Y \omega(X,Z) + \mathscr{L}_Z \omega(X,Y) \\ &- \omega(Y,[X,Z]) + \omega(X,[Y,Z]) - \omega([X,Y],Z) \\ &= d\omega(X,Y,Z).\end{aligned} \tag{19}$$

The following theorem collects together the previous results.

Theorem 2.56. *Let (M, π) be a regular Poisson manifold and choose a splitting τ of* (14)*. Then the curvature form $\Omega_\tau \in \Omega^2(\mathscr{F}, \nu^*(\mathscr{F}))$ is $d_{\mathscr{F}}^{\nabla^*}$-closed, and $d_\nu[\omega_{\mathscr{F}}] = [\Omega_\tau]$. In particular, $[\Omega_\tau]$ is independent of the chosen splitting τ.*

Proof. It follows by Lemma 2.51 and Equation (19) that Ω_τ is closed, and by the definition of the transversal derivative, we have that $d_\nu[\omega_{\mathscr{F}}] = [\Omega_\tau]$. Since $d_\nu[\omega_{\mathscr{F}}]$ is independent of the chosen splitting, so is $[\Omega_\tau]$. □

2.5.2 Monodromy groups of Regular Poisson manifolds

In this section, we will describe how the monodromy groups of the cotangent algebroid of a regular Poisson manifold can be heavily simplified. If (M,π) is a regular Poisson manifold, we will refer to the monodromy groups of the cotangent algebroid of M as the monodromy groups of (M,π), and denote them by $\mathcal{N}_x(M,\pi)$ for $x \in M$.

To obtain a better understanding of the monodromy groups of a Poisson manifold, we take a closer look at the Poisson homotopy groups. If (M,π) is an integrable Poisson manifold, the Poisson homotopy groups $\Sigma_x(M,\pi)$ are the isotropy groups of the Weinstein groupoid. Thhey are in particular Lie groups with Lie algebras $\mathfrak{g}_x = \ker(\pi_x^\sharp)$ and the target map yields a principal $\Sigma_x(M,\pi)$-bundle $\mathbf{s}^{-1}(x) \to S$, where S is the symplectic leaf through x. Since $\mathbf{s}^{-1}(x)$ is 1-connected, the long exact sequence of homotopy groups takes the form

$$\cdots \to \pi_2(S,x) \xrightarrow{\partial_x} \pi_1(\Sigma_x(M,\pi)) \to 1 \to \pi_1(S,x) \to \pi_0(\Sigma_x(M,\pi)) \to 1.$$

In particular, we deduce the following corollary.

Corollary 2.57. *Let (M,π) be an integrable Poisson manifold with 1-connected symplectic leaves. Then the Poisson homotopy groups of (M,π) are connected.*

Denote by $\Pi(\mathfrak{g}_x)$ the unique 1-connected Lie group integrating $\mathfrak{g}_x$. We have a canonical homomorphism $q_x : \Pi(\mathfrak{g}_x) \to \Sigma_x(M,\pi)^\circ$ induced by the inclusion of $\mathfrak{g}_x \to T^*M$. The kernel of q_x is a discrete subgroup contained in $Z(\Pi(\mathfrak{g}_x))$, and

$$\Sigma_x(M,\pi)^\circ \cong \Pi(\mathfrak{g}_x)/\ker(q_x).$$

We deduce from basic Lie theory that $\ker(q_x) = \pi_1(\Sigma_x(M,\pi))$. In particular, we deduce that we have an exact sequence

$$\pi_2(S,x) \xrightarrow{\partial_x}\!\!\to \Pi(\mathfrak{g}_x) \xrightarrow{q_x} \Sigma_x(M,\pi) \xrightarrow{p_x} \pi_1(S,x) \to 1. \tag{20}$$

So far, we have assumed that (M,π) is integrable, but parts of this discussion also makes sense for general Poisson manifolds (M,π); there is still an exact sequence:

Proposition 2.58 ([CFM21], Proposition 14.63)**.** *Let (M,π) be a Poisson manifold. Then there exists an exact sequence*

$$\pi_2(S,x) \xrightarrow{\partial_x} \Pi(\mathfrak{g}_x) \xrightarrow{q_x} \Sigma_x(M,\pi) \xrightarrow{p_x} \pi_1(S,x) \to 1. \tag{21}$$

We will describe the maps involved in more detail. The map $p_x : \Sigma_x(M,\pi) \to \pi_1(S,x)$ sends the class of a cotangent path to the homotopy class of its underlying base path. The map q_x is induced by the inclusion $\mathfrak{g}_x \to T^*M$, and the map $\partial_x : \pi_2(S,x) \to \Pi(\mathfrak{g}_x)$ sends the class of $\sigma : [0,1]^2 \to S$ to the class of a path $a : [0,1] \to \mathfrak{g}_x$, which is cotangent path-homotopic to 0_x via a cotangent path homotopy covering σ. These maps are well-defined, and the sequence is exact. From this description, we infer that $\mathcal{N}_x(M,\pi) = \mathrm{im}(\partial_x) = \ker(q_x)$.

For a regular Poisson manifold, all isotropy Lie algebras are abelian, so $\Pi(\mathfrak{g}_x) = \nu_x^*(S)$. A fortiori, the Poisson homotopy groups must be abelian, q_x is just the exponential map, and the monodromy groups $\mathcal{N}_x(M,\pi)$ are the image of the monodormy operator $\partial_x : \pi_2(S,x) \to \nu_x^*(S)$. The monodromy operator can be related to the transversal derivatve as follows. If $\tau : \mathscr{F} \to T^*M$ is a splitting of (14), we can restrict the curvature form to S, to obtain a class $[\Omega_\tau|_S] \in H^2(S, \nu^*(S))$. Integrating this form over a sphere $\sigma : S^2 \to S$ yields the description of the monodromy operator.

Proposition 2.59 ([CF02])**.** *The monodromy operator ∂_x for a regular Poisson manifold can be described as*

$$\partial_x[\sigma] = \int_\sigma \Omega_\tau|_S.$$

Let us explain what we mean by an integral of a form with coefficients in a vector bundle. Suppose $\omega \in \Omega^2(M,E)$ is a 2-form with coefficients in some flat vector bundle $E \to M$. Integrating ω over $\gamma : S^2 \to M$ means that we integrate the pullback form $\gamma^*\omega \in \Omega^2(S^2, \gamma^*E)$ over S^2. Here, γ^*E is a flat vector bundle over S^2. Since S^2 is 1-connected, this bundle can be trivialized by parallel transport, and this gives a way of integrating differential forms with coefficients in a flat vector bundle over 1-connected cycles. In the case of a regular Poisson manifold, the dual Bott connection is a flat connection on $\nu^*(S) \to S$, and we can trivialize the pullback bundle by parallel transport $\sigma^*\nu^*(S) = S^2 \times \nu_x^*(S)$. The integration coincides with the natural pairing between homology and cohomology. In particular, the expression for ∂_x does not depend on the chosen splitting τ.

We will give two applications of this description of the monodromy groups, that simplify computations in many situations. The first one is that we describe how the monodromy groups can be described by computing the variation of symplectic area. The second one is the computation of monodromy groups of submanifolds of a regular Poisson manifold.

In practical situations, it is often more convenient to compute the monodromy operator of Proposition 2.59 in terms of the variation of the symplectic area. The variation of symplectic area is the defined as follows. Let $[\sigma] \in \pi_2(S,x)$, and pick a smooth representative $\sigma : (S^2, p_N) \to (S,x)$, mapping the north pole of S^2 to x. The symplectic area of σ is the quantity given by

$$A_\omega(\sigma) = \int_{S^2} \sigma^*\omega,$$

where ω is the symplectic form on S. A deformation of σ is a family σ_t with $t \in (-\varepsilon, \varepsilon)$ and $\sigma_0 = \sigma$ such that the image of each σ_t is contained in a symplectic leaf. Given a sphere $\sigma : (S^2, p_N) \to (S,x)$ and a normal vector $v \in \nu_x(S)$, one can find a deformation $\sigma_t : (S^2, p_N) \to (S, x_t)$, where $\sigma_0 = \sigma$ and $t \mapsto x_t$ is a path with $[\dot{x}(0)] = v$. The variation of the symplectic area is the group homomorphism

$$A'_x : \pi_2(S,x) \to \nu_x^*(S), \qquad \langle A'_x[\sigma], v\rangle = \left.\frac{d}{dt}\right|_{t=0} \int_{S^2} \sigma_t^*\omega_t,$$

where ω_t is the symplectic form on the leaf through x_t. If we foliate $S^2 \times I$ by spheres $S^2 \times \{t\}$, we obtain that

$$H^2(\mathscr{F}) \cong C^\infty(I), \qquad H^2(\mathscr{F}, \nu) \cong \Omega^1(I),$$

and d_ν is just the de Rham differential. Interpreting a deformation σ_t as a foliated map $S^2 \times I \to M$, the functoriality of d_ν with respect to foliated maps implies that

$$\frac{d}{dt}\int_{S^2} \sigma_t^*\omega_t = \left\langle \int_{\sigma_t} \Omega_\tau, \frac{d}{dt}\sigma_t(p_N) \right\rangle.$$

Therefore, the variation of the symplectic area coincides with the monodromy operator in Proposition 2.59. This is in many situations the most convenient method to compute the monodromy groups of a regular Poisson manifolds.

Example 2.60. Let $\omega \in \Omega^2(S^2)$ be the standard area form. On the manifold $M = S^2 \times S^2 \times \mathbb{R}^2$, consider a regular Poisson structure with leaves

$$S_y = S^2 \times S^2 \times \mathbb{R}^2, \quad y \in \mathbb{R}^2$$

and foliated symplectic form

$$\omega_{(y_1,y_2)} = f(y_1)\mathrm{pr}_1^*\omega + g(y_2)\mathrm{pr}_2^*\omega,$$

where $f, g \in C^\infty(M)$ are positive smooth functions. For $v = (v_1, v_2) \in \mathbb{R}^2$, the variation of symplectic area is given by

$$\langle A_y'([\sigma]), v\rangle = f'(y_1)v_1 \int_\sigma \mathrm{pr}_1^*\omega + g'(y_2)v_2 \int_\sigma \mathrm{pr}_2^*\omega.$$

Using that $\pi_2(S^2 \times S^2 \times \mathbb{R}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$, we infer that the monodromy groups are given by

$$\mathcal{N}_y(M,\pi) = f'(y_1)\mathbb{Z} \oplus g'(y_2)\mathbb{Z}.$$

Depending on the value of f' and g', the monodromy groups are uniformally discrete, so this is a simple way to produce non-integrable Poisson manifolds.

2.5.3 Monodromy groups of Poisson Submanifolds

In general, the integrability of Poisson submanifolds of a given Poisson manifold can be a delicate issue. For instance, it is well-established that the sphere $S_{\mathfrak{g}^*}$ inside the dual of a Lie algebra $\mathfrak{g}^*$ (different from $\mathfrak{su}(2)^*$) is never integrable. Even if we assume that (M,π) is regular and that N is a saturated Poisson submanifold, integrability of N is not guaranteed. To check that N is integrable, we have to determine the monodromy groups of N, and see if they are uniformally discrete. The goal is to express them in terms of the monodromy groups of (M,π). We will assume that (M,π) regular since we can describe the monodromy groups as the image of ∂_x. Moreover, we assume for simplicity that N is saturated.

Consider a splitting $\tau_M : \mathscr{F}_M \to T^*M$ of the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \nu^*(\mathscr{F}_M) \arrow[r] & T^*M \arrow[r, "\pi_M^\sharp"] & \mathscr{F}_M \arrow[r] \arrow[l, bend left, dotted, "\tau_M"] & 0. \end{tikzcd}$$

This induces a splitting $\tau_N : \mathscr{F}_N \to T^*N$ of the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \nu^*(\mathscr{F}_N) \arrow[r] & T^*N \arrow[r, "\pi_N^\sharp"] & \mathscr{F}_N \arrow[r] \arrow[l, bend left, dotted, "\tau_N"] & 0, \end{tikzcd}$$

by setting $\tau_N(X) = \tau_M(X) \circ d\iota$. Since N is a Poisson submanifold, $\iota : N \to M$ is a Poisson map. Therefore

$$\pi_N^\sharp(\tau_N(X)) = \pi_N^\sharp(\tau_M(X) \circ d\iota) = \pi_M^\sharp(\tau_M(X)) = X,$$

so τ_N is a splitting. The curvatures of the splittings are related by

$$\begin{aligned}\Omega_{\tau_N}(X,Y) &= [\tau_N(X),\tau_N(Y)]_{\pi_N} - \tau_N([X,Y]) \\ &= [\tau_M(X)\circ d\iota, \tau_M(Y)\circ d\iota]_{\pi_N} - \tau_M([X,Y])\circ d\iota \\ &= [\tau_M(X),\tau_M(Y)]_{\pi_M}\circ d\iota - \tau_M([X,Y])\circ d\iota \\ &= \Omega_{\tau_M}(X,Y)\circ d\iota,\end{aligned}$$

where we used in the third line that ι is Poisson. Using Proposition 2.59, we deduce that

$$\partial_x^N([\sigma]) = \int_\sigma \Omega_{\tau_N} = (d\iota_x)^* \int_\sigma \Omega_{\tau_M} = (d\iota_x)^* \partial_x^M([\sigma]).$$

We conclude that

$$\mathcal{N}_x(N,\pi_N) = \mathrm{im}(\partial_x^N) = (d\iota_x)^* \mathrm{im}(\partial_x^M) = (d\iota_x)^* \mathcal{N}_x(M,\pi_M),$$

which gives us an explicit expression for the monodromy groups of N.

In the next chapter, we will investigate the topology of symplectic integrations of Poisson manifolds in greater detail. This gives rise to Poisson manifolds of compact type, which are a natural generalization of Lie algebras of compact type.

3 Poisson Manifolds of Compact Type

In the previous chapters, we have described all the basic concepts in this thesis. Now, we can describe the objects we are truly interested in: Poisson manfolds of compact type. These are the natural generalization of compact Lie algebras, in the sense that they are integrated by a groupoid which has some compactness type. We will see that the notion of compactness for a Poisson manifold is more subtle than for a Lie algebra, as there are six different notions. In this chapter, we will follow [CFT15] and [CFM16].

3.1 Compactness types of Lie groupoids and Poisson manifolds

There are multiple notions of compactness for Lie groupoids. Before we give the definitions of these, let us recall that a map between topological space $f : X \to Y$ is called proper if inverse images of compact subsets of Y are compact in X. If Y is locally compact and Hausdorff, this is equivalent to saying that f is a closed map with compact fibers.

Definition 3.1. Let $\mathcal{G} \rightrightarrows M$ be an $\mathbf{s}$-connected Lie groupoid. We call $\mathcal{G}$

1. proper if the map $(\mathbf{s},\mathbf{t}) : \mathcal{G} \to M \times M$ is proper,
2. $\mathbf{s}$-proper if the map $\mathbf{s} : \mathcal{G} \to M$ is proper,
3. compact if it is compact as a topological space.

Clearly, any compact Lie groupoid is $\mathbf{s}$-proper, and any $\mathbf{s}$-proper Lie groupoid is proper. It turns out that $\mathbf{s}$-properness of a Lie groupoid is completely governed by the topology of the fibers of $\mathbf{s}$.

Lemma 3.2. *Let $\mathcal{G} \rightrightarrows M$ be a source-connected Lie groupoid such that all fibers $\mathbf{s}^{-1}(x)$ are compact. Then $\mathbf{s}$ is proper.*

Proof. The proof uses that open maps between metrizable spaces $f : X \to Y$ have the following property: For every $x \in X, y \in Y$ and sequence $y_n \in Y$ converging to y, there exists a sequence $x_n \in X$ such that $f(x_n) = y_n$ and x_n converges to x.

We will prove the lemma by contradiction. Let $\mathcal{G} \rightrightarrows M$ be as in the statement of the lemma, and assume that $\mathbf{s}$ is not proper. Then there exists a sequence $y_n \in M$ converging to some $y \in M$ and a sequence $x_n \in \mathcal{G}$ such that each $x_n \in \mathbf{s}^{-1}(y_n)$ and x_n diverges to infinity, in the sense that for any compact $K \subseteq \mathcal{G}$, all but finitely x_n lie outside K.

Since $\mathbf{s}$ is surjective, there exists $x \in \mathbf{s}^{-1}(y)$. Since $\mathbf{s}$ is a submersion, it is an open map, and therefore there exists a sequence $z_n \in \mathbf{s}^{-1}(y_n)$ converging to x. Let K be a compact whose interior contains all points $\mathbf{s}^{-1}(y) \cup \{z_n : n \geq N\}$ for some $N \in \mathbb{N}$. This exists, since $\mathbf{s}^{-1}(y)$ is compact and $\mathcal{G}$ is locally compact. By increasing N, we can make sure that $x_n \notin K$ for all $n \geq N$. Since $\mathbf{s}^{-1}(x_n)$ is connected for each n, there exits a sequence $w_n \in \mathbf{s}^{-1}(y_n) \cap \partial K$. But ∂K is compact, so after passing to a subsequence, w_n converges to some $w \in \partial K$. By continuity of $\mathbf{s}$, $\mathbf{s}(w) = y$, so $w \in \partial K \cap \mathbf{s}^{-1}(y)$, which is absurd since $\mathbf{s}^{-1}(y) \subseteq \mathrm{Int}(K)$. $\square$

There are situations in which some of the compactness notions coincide.

Proposition 3.3. *Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid with connected* $\mathbf{s}$*-fibers.*

1. *If all orbits of $\mathscr{G}$ are compact, then $\mathscr{G}$ is proper if and only if it is* $\mathbf{s}$*-proper.*

2. *If M is compact, all notions coincide.*

Proof. For 1, we only have to show that $\mathscr{G}$ is $\mathbf{s}$-proper if it is proper and has compact orbits. By Lemma 3.2, it suffices to show that the $\mathbf{s}$-fibers are compact. Recall from Proposition 2.5 that for any $x \in M$, we have a principal $\mathscr{G}_x$-bundle $\mathbf{t} : \mathbf{s}^{-1}(x) \to \mathscr{O}_x$. By assumption, $\mathscr{O}_x$ and $\mathscr{G}_x = (\mathbf{s},\mathbf{t})^{-1}(x,x)$ are compact. Therefore, $\mathbf{s}^{-1}(x)$ is compact.

For 2, suppose that M is compact. If $\mathbf{s}$ is proper, $\mathscr{G} = \mathbf{s}^{-1}(M)$ is compact, so $\mathbf{s}$-properness is equivalent to compactness. To complete the proof, we have to show that properness implies $\mathbf{s}$-properness, which we will do by showing that the orbits are compact. For $x \in M$, the fiber $\mathbf{s}^{-1}(x) \subseteq \mathscr{G}$ is obviously closed. Since the map $(\mathbf{s},\mathbf{t})$ is proper, it is in particular closed, so $(\mathbf{s},\mathbf{t})(\mathbf{s}^{-1}(x)) = \{x\} \times \mathscr{O}_x \subseteq M \times M$ is closed. This implies that $\mathscr{O}_x$ is closed. Since M is compact, $\mathscr{O}_x$ is also compact and this concludes the proof of the proposition. $\square$

Example 3.4. All three notions of compactness coincide when we consider a Lie groupoid over a point (i.e. a Lie group). The difference between the three compactness notions is best illustrated by considering the action groupoid $G \ltimes M$ associated to the action of a Lie groupoid G on a manifold M. Then $G \ltimes M$ is:

- of compact type if and only if G and M are both compact,
- of $\mathbf{s}$-proper type if and only if G is compact,
- of proper type if the action of G on M is proper.

Example 3.5. Let M be a connected manifold. For the pair groupoid $M \times M \rightrightarrows M$, the map $(\mathbf{s},\mathbf{t}) : M \times M \to M \times M$ is always a homeomorphism, hence proper. Thus, the pair groupoid is always proper, and it is $\mathbf{s}$-proper if and only if it is compact, which happens if and only if M is compact.

It is clear how to generalize the notions of compactness to Poisson manifolds. However, there are three more different compactness notions that we can consider. The analogy with Lie algebras is that even though a Lie algebra can be integrated by a compact Lie group, the unique 1-connected integration need not be compact. The same holds for Poisson manifolds, where we replace the unique 1-connected Lie group by the Weinstein groupoid.

Definition 3.6. Let $\mathscr{C} \in \{\text{proper, } s\text{-proper, compact}\}$. A Poisson manifold is said to be

- Of $\mathscr{C}$-type if there exists an $\mathbf{s}$-connected integration of (M,π) of $\mathscr{C}$-type
- Of strong $\mathscr{C}$-type if the Weinstein groupoid of (M,π) is smooth and of $\mathscr{C}$-type.

Poisson manifolds of (strong) $\mathscr{C}$-type will be referred to as PM(S)CTs. Remark that by Proposition 3.3, properness and $\mathbf{s}$-properness of Poisson manifolds are equivalent if the symplectic leaves are compact, since the symplectic leaves are the orbits of a symplectic integration.

Remark 3.7. The six different compactness type of Poisson manifolds are related as follows

$$\begin{array}{ccccc} \text{strong compact} & \Longrightarrow & \text{strong } \mathbf{s}\text{-proper} & \Longrightarrow & \text{strong proper} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{compact} & \Longrightarrow & \mathbf{s}\text{-proper} & \Longrightarrow & \text{proper} \end{array}$$

When all the symplectic leaves are compact, this becomes

$$\begin{array}{ccccc} \text{strong compact} & \Longrightarrow & \text{strong } \mathbf{s}\text{-proper} & \Longleftrightarrow & \text{strong proper} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{compact} & \Longrightarrow & \mathbf{s}\text{-proper} & \Longleftrightarrow & \text{proper} \end{array},$$

and when M is compact, it becomes

$$\begin{array}{ccccc} \text{strong compact} & \Longleftrightarrow & \text{strong } \mathbf{s}\text{-proper} & \Longleftrightarrow & \text{strong proper} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{compact} & \Longleftrightarrow & \mathbf{s}\text{-proper} & \Longleftrightarrow & \text{proper} \end{array},$$

We will now discuss compactness type of several Poisson structures that we are familiar with.

Example 3.8. The Weinstein groupoid of the zero Poisson structure on M is $(T^*M, \omega_{\mathrm{can}})$. This is of never of strong proper type. However, $(M,0)$ may be of proper type. This happens precisely when there exists an *integral affine structure* on M. We will discuss integral affine structures in more detail in Chapter 4.

Example 3.9. Let $\mathfrak{g}$ be a Lie algebra and consider the linear Poisson structure on its dual $\mathfrak{g}^*$. Let G be the 1-connected integration of $\mathfrak{g}$. The action groupoid $G \ltimes \mathfrak{g}^*$ is a symplectic integration of $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$, and the source fibers coincide with $\mathfrak{g}^*$. Therefore, $G \ltimes \mathfrak{g}^*$ is isomorphic to the Weinstein groupoid. Thus, we see that $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ is strong proper if and only if it is strong **s**-proper. This happens if and only if there exists a 1-connected, compact Lie group G integrating $\mathfrak{g}$, i.e. if and only if $\mathfrak{g}$ is compact and semisimple. It is never of strong compact type.

It is clear that strong compactness is the strongest compactness type a Poisson manifold can have. The majority of this thesis is focused on constructing examples of strong compact Poisson manifolds. As of now, there is only one simple class of examples of strong compact Poisson manifolds.

Example 3.10. Let (M, ω) be a connected symplectic manifold. Then $\omega^\flat : T^*M \to TM$ is a Lie algebroid isomorphism, and $\Pi_1(M) \rightrightarrows M$ is a Hausdorff, 1-connected integration of (M, ω). Moreover,

$$\Omega = \mathbf{t}^*\omega - \mathbf{s}^*\omega$$

is a multiplicative symplectic form on $\Pi_1(M)$. Recall that $\Pi_1(M)$ is source 1-connected, so $(\Pi_1(M), \Omega)$ is the Weinstein groupoid of (M, ω). It is proper if $\pi_1(M)$ is finite, **s**-proper if $\pi_1(M)$ is finite and M has compact path components, and compact if $\pi_1(M)$ is finite and M is compact.

Thus, we see that compact symplectic manifolds (M, ω) with finite $\pi_1(M)$ form a class of examples of strong compact Poisson manifolds. Examples of these are $(S^2, \omega_{\mathrm{std}})$ or $(\mathbb{C}P^n, \omega_{\mathrm{FS}})$.

3.2 Submanifolds of PMCTs

In this section, we explain what happens to the PMCT property when we pass to Poisson submanifolds. We follow the arguments outlined in [Fer06] and [CFT15]. Integrability and compactness Poisson submanifolds are subtle issues. Already in the case of $(S^2, \omega_{\text{std}})$, which is of strong compact type, removing a point gives an open symplectic submanifold, which is not strong compact. It is still of strong proper type since its fundamental group vanishes, but that also breaks down when we remove another point. This is because the fundamental group of $S^2 \setminus \{P, S\}$ is isomorphic to $\mathbb{Z}$, hence not finite. The result follows from Example 3.10.

For a general Poisson submanifold N of (M, π), the restricted subbundle $T^*_N M \to N$ is a Lie subalgebroid of the cotangent algebroid. If (M, π) is integrable, then so is $T^*_N M$, and is integrated by the Lie subgroupoid $\Sigma_N(M) \subseteq \Sigma(M)$:

$$\Sigma_N(M) = \{[a] \in \Sigma(M) \colon a : I \to T^*_N M\}.$$

If Ω is the symplectic form on $\Sigma(M)$, the pair $(\Sigma_N(M), i^*\Omega)$ is a coisotropic submanifold of rank $2\dim(N)$. For coisotropic manifolds of this form, $\ker(i^*\Omega^\flat)$ forms an involutive distrubtion of $T\Sigma_N(M)$. This follows from the fact that $i^*\Omega$ is closed, together with the Koszul formula for the exterior derivative. If the leaf space $\Sigma_N(M)/\ker(i^*\Omega^\flat)$ is smooth, it is a well-known result that there exists a unique symplectic form $\bar{\Omega}$ on it, completely determined by the property $p^*\bar{\Omega} = i^*\Omega^\flat$. Moreover, the quotient $\Sigma_N(M)/\ker(i^*\Omega^\flat)$ is always a topological groupoid, because Ω is multiplicative. It is, as a topological groupoid, isomorphic to $\Sigma(N)$. Thus, if N and M are both integrable, we have an isomorphism

$$(\Sigma(N), \Omega_N) \cong (\Sigma_N(M)/\ker(i^*\Omega^\flat), \bar{\Omega}).$$

Remark that the restriction groupoid $\Sigma(M)|_N$ is another integration of $T^*_N M$, but restricting $\Sigma(M)$ to N may destroy 1-connectedness of the **s**-fibers. This problem does not occur when N is saturated by symplectic leaves, and we must have that $\Sigma(M)|_N \simeq \Sigma_N(M)$. Thus, the previous discussion implies

$$\Sigma(N) = \Sigma(M)|_N/\ker(\Omega^\flat),$$

and it follows that N and M have the same strong compactness type.

3.3 Quotients of PMCTs and Hamiltonian reduction

In this section, we investigate the effect of Lie group quotients on PMCTs. We mainly follow [FOR07] and [CFT15].

We fix a free and proper Poisson G-space (M, π, G), and we assume that the Poisson manifold (M, π) is integrable. As a consequence of Lie's second theorem for Lie algebroids, Theorem 2.34, any Poisson diffeomorphism $\phi : M \to M$ lifts to a symplectomorphism $\Phi : \Sigma(M) \to \Sigma(M)$ covering ϕ. Here, we can apply Lie's second theorem since $\Sigma(M)$ is source-1-connected. Explicitly, Φ is given by

$$\Phi : \Sigma(M) \to \Sigma(M), \quad [a] \mapsto [(d\phi^{-1})^* \circ a].$$

Thus, the action of G on M lifts to a symplectic action on $\Sigma(M)$ by the lifts $\Phi_g : \Sigma(M) \to \Sigma(M)$ of the Poisson diffeomorphism $\phi_g : M \to M$. The following proposition shows that the lifted action is also free and proper.

Proposition 3.11 ([FOR07]). *Let $\mathscr{G} \rightrightarrows M$ be a Lie groupoid, and let G be a Lie group acting smoothly on $\mathscr{G} \rightrightarrows M$ by Lie groupoid automorphisms. Then*

1. *The G-action on $\mathscr{G}$ is free if and only if the G-action on M is free.*

2. *The G-action on $\mathscr{G}$ is proper if and only if the G-action on M is proper.*

Proof. If the action on $\mathscr{G}$ is free, so is the action on M, since M embeds equivariantly in G via the unit $\mathbf{u}$. For the converse, we note that $\mathbf{s}$ is G-equivariant. Therefore, $\mathrm{Stab}_{\mathscr{G}}(x) \subseteq \mathrm{Stab}_M(\mathbf{s}(x))$, so if G acts freely on M, it must act freely on $\mathscr{G}$.

Similarly, the second statement is trivially true in one direction. For the other direction, suppose G acts properly on M. We want to show that the map

$$G \times \mathscr{G} \to \mathscr{G} \times \mathscr{G}, \qquad (g,x) \mapsto (x,gx)$$

is proper. To see this, we choose sequences g_k in $\mathscr{G}$ and x_k in M such that g_k converges to $g \in \mathscr{G}$ and $g_k x_k$ converges to some $y \in M$. We have to show that g_k contains a convergent subsequence. We set $m_k = \mathbf{s}(x_k)$, $m = \mathbf{s}(x)$ and $n = \mathbf{s}(y)$. Obviously, m_k converges to m, and $g_k m_k$ converges to n. Since the action of G on M is proper, g_k has the required convergent subsequence. □

It turns out that the G-action on $\Sigma(M)$ is always Hamiltonian, with moment map $J : \Sigma(M) \to \mathfrak{g}^*$ defined as follows. Given a vector field $X \in \mathfrak{X}(M)$, we can integrate it over a cotangent path $a : I \to T^*M$ by setting

$$\int_a X = \int_0^1 \langle X_{\gamma_a(t)}, a(t) \rangle \, dt.$$

For Hamiltonian vector fields, the value of this integral only depends on the begin and end point of γ_a:

$$\int_a X_H = H(\gamma_a(1)) - H(\gamma_a(0)), \quad H \in C^\infty(M).$$

A basic property of this type of integral, proven in [CF02], is the invariance of the integration of Poisson vector fields under cotangent path homotopies. If X is a Poisson vector field, we obtain a well-defined map

$$c_X : \Sigma(M) \to \mathbb{R}, \quad c_X([a]) = \int_a X.$$

The addivitity of the integral with respect to composition of cotangent paths implies that

$$c_X([a_1] \cdot [a_2]) = c_x([a_1]) + c_x([a_2]).$$

We define the moment map for this action by $J : \Sigma(M) \to \mathfrak{g}^*$ by

$$\langle J([a]), \xi \rangle = c_{X_\xi}([a]),$$

where X_ξ is the infinitesimal vector field of M, which is Poisson. It is shown in [FOR07] that this is indeed a moment map, which is a groupoid cocycle:

$$J(g_1 g_2) = J(g_1) + J(g_2), \qquad \forall (g_1, g_2) \in \Sigma(M)^{(2)}.$$

Moreover, this cocycle is exact, i.e. is of the form $J = \mu \circ \mathbf{s} - \mu \circ \mathbf{t}$, if and only if (M, π, G, μ) is a Hamiltonian G-space.

We can now perform Marsden-Weinstein reduction of $\Sigma(M)$. If 0 is a regular value of J, then $J^{-1}(0)$ is a Lie subgroupoid of $\Sigma(M)$.The symplectic quotient

$$(\Sigma(M)//G, \bar{\Omega}) = (J^{-1}(0)/G, \bar{\Omega})$$

is naturally a symplectic groupoid. Moreover, since the map $v \mapsto \iota_v\Omega$ induces an isomorphism between $\mathrm{Lie}(\Sigma(M))$ and T^*M, the map $v \mapsto \iota_v\bar{\Omega}$ induces an isomorphism between $\mathrm{Lie}(\Sigma(M)//G)$ and $T^*(M/G)$. Therefore, we conclude:

Corollary 3.12. *Let (M, π, G) be a free and proper Poisson G-space. Then M/G is an integrable Poisson manifold and the redued symplectic groupoid*

$$\Sigma(M)//G = J^{-1}(0)/G$$

is a symplectic groupoid integrating M/G.

In general, it is not true that $\Sigma(M)//G$ is not source-1-connected, so $\Sigma(M)//G$ is not always equal to $\Sigma(M/G)$. However, when G is a discrete Lie group, we do have that $J^{-1}(0) = \Sigma(M)$. By the same argument as in Section 1.3.2, the source fibers of $\Sigma(M)$ and $\Sigma(M)/G$ of a Poisson G-space (M, π, G) are isomorphic. Thus if G is discrete, it is true that $\Sigma(M/G) = \Sigma(M)/G$. Therefore, we can conclude the following results

Theorem 3.13. *Let G be a Lie group, and let (M, π, G) be a free and proper Poisson G-space. Then*

1. *If M is of strong* **s***-proper or strong compact type, then M/G is of s-proper or compact type.*
2. *If M is strong* **s***-proper or strong compact and moreover G is discrete, then so is M/G.*

Note that strong properness does not pass to quotients by discrete Lie groups. A simple example of this phenomenon is when we form the symplectic torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by acting on symplectic $\mathbb{R}^2$ by $\mathbb{Z}^2$. Using Example 3.10, we deduce that symplectic $\mathbb{R}^2$ is strong proper, but that the symplectic torus is not. A simple way to see this is that $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$, which is not finite, and the statement follows by Example 3.10

3.3.1 (Quasi-)Hamiltonian Reduction

In the previous section, we have seen that Poisson quotients by discrete Lie group actions preserves strong **s**-properness and strong compactness. Obviously, quotients by discrete Lie groups are not very interesting, so we need to work in a more restrictive setting to guarantee that strong compactness types are preserved. It turns out that the suitable candidates are Poisson manifolds that are reductions of free and proper Hamiltonian G-spaces (M, ω, G, μ). These are symplectic manifolds (M, ω), together with a free and proper Hamiltonian action by G, and a G-equivariant moment map $\mu : M \to \mathfrak{g}^*$. The quotient space $(Q = M/G, \pi)$ is naturally a Poisson manifold, where the symplectic leaves are the connected components of the symplectic quotients

$$\mu^{-1}(\xi)/G_\xi \simeq \mu^{-1}(O_\xi)/G \subseteq M/G,$$

where O_ξ is the coadjoint orbit through $\xi \in \mathfrak{g}^*$ and G_ξ is the stabilizer of ξ. Provided that G is compact, the Poisson manifold (Q, π) is naturally of some compactness type.

Proposition 3.14 ([CFT15])**.** *Let (M, ω, G, μ) be a Hamiltonian G-space, and assume G is compact and acts freely. Assume furthermore that M is connected, and that the fibers of μ are connected (or 1-connected). Then*

1. (Q,π_{red}) *is proper (or strong-proper).*

2. (Q,π_{red}) *is* **s**-*proper (or strong-***s***-proper) if* μ *is proper.*

3. (Q,π_{red}) *is of compact type (or of strong compact) if and only if M is compact and G is finite (hence when Q is symplectic).*

Proof. Since the action is free, the moment map is a submersion, hence the submersion groupoid $M\times_\mu M \rightrightarrows M$ is a smooth subgroupoid of the pair groupoid. The form $\Omega = \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega$ is a closed, multiplicative 2-form on $M\times_\mu M$, with kernel precisely the orbits of the diagonal action of G on $M\times_\mu M$. This action is free and proper, and by groupoid automorphisms. Moreover, Ω is basic with respect to this diagonal action, so descends to a symplectic form on the quotient. The upshot is that the quotient

$$\Sigma = (M\times_\mu M)/G \rightrightarrows M/G = Q$$

is a symplectic groupoid, where the symplectic form Ω is the unique form satisfying $q^*\Omega = \mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega$, where $q : M\times_\mu M \to (M\times_\mu M)/G$ is the quotient map.

This symplectic groupoid is an integration of (Q,π_{red}). Note that we have a commuting diagram,

$$\begin{array}{ccc} (M\times_\mu M,\Omega) & \xrightarrow{\mathrm{pr}_1} & (M,\omega) \\ \downarrow & & \downarrow \\ (\Sigma,\Omega) & \xrightarrow{\mathbf{t}} & (Q,\pi_{\mathrm{red}}) \end{array},$$

so $\mathbf{t}$ is a forward Dirac map (see Chapter 7 in [CFM21]). In particular, it is a Poisson map so π_{red} is the Poisson structure on Q induced by Ω.

The **s**-fibers of Σ coincide with the fibers of μ. If M is connected, and the fibers of μ are connected (or 1-connected), it follows that the **s**-fibers of Σ are connected (or 1-connected) so item 1 and 2 follow. Item 3 follows from the fact that Hamiltonian actions of non-finite Lie groups on compact manifolds are never free, because otherwise it maps the compact manifold M onto on $\mathfrak{g}^*$, which is absurd. □

Because of item 3 of this proposition, reduction of Hamiltonian spaces is not very suitable for producing Poisson manifolds of strong compact type. The previous discussion can be generalized using a notion of S^1-valued momentum maps.

Definition 3.15. A Quasi-Hamiltonian S^1-space is a symplectic manifold (M,ω), together with an S^1-action and an equivariant S^1-valued momentum map, i.e. S^1-equivariant map $\mu : Q \to S^1$, satisfying

$$\iota_X\omega = \mu^*(d\theta),$$

where X is the infinitesimal generator of the action and $d\theta$ the standard volume form on S^1.

If (M,ω,μ) is a free quasi-Hamiltonian S^1-space, the same arguments as in the previous proposition show that $\Sigma = (M\times_\mu M)/S^1 \rightrightarrows M/S^1$ is a symplectic groupoid integrating the Poisson manifold $(M/S^1,\pi_{\mathrm{red}})$. The **s**-fibers of Σ and the fibers of μ coincide, so we conclude:

Theorem 3.16. *Let (M, ω, μ) be a compact, free Quasi-Hamiltonian S^1-space, and suppose μ has 1-connected fibers. The reduced Poisson space $(M/S^1, \pi_{red})$ is of strong compact type.*

Thus, the question of finding Poisson manifolds of strong compact type can be reduced to the study of free Quasi-Hamiltonian S^1-spaces, whose momentum map has 1-connected fibers. Such a space has been constructed by [Kot05], and the explicit construction relies heavily on the geometrical properties of K3 surfaces. We will go over this construction in a Poisson setting in Chapter 5, following the work of Martinez-Torres [Tor13], which also fills in some of the details overlooked at in [Kot05]. In the next chapter, we will describe a generic construction of Poisson manifolds of strong compact type. We will show that the example by Martinez-Torres fits in that framework.

4 Constructing PMCTs using Integral Affine Structures and the Linear Variation

In this chapter, we will discuss another procedure that allows us to construct Poisson manifolds of strong **s**-proper type and to perform a reduction to obtain a Poisson manifold of strong compact type. The PMSCTs that we construct will appear as appropriate Poisson submanifolds of regular Poisson manifolds M whose leaves are compact and 1-connected. The advantage of working with these Poisson manifolds is that the leaf space $B = M/\mathscr{F}$ of M has a smooth structure such that the map $p : M \to B$ is a submersion. The approach we are following is built upon two properties any PMCT (M,π) with a smooth leaf space B share.

1. The leaf space of any PMCT of M has an integral affine structure, that is induced by a proper symplectic integration $(\mathscr{G},\Omega)$ of (M,π).

2. Any $b \in B$ corresponds to a symplectic leaf (S_b,ω_b), and the cohomology groups $H^2(S_b)$ yield a vector bundle $\mathscr{H}^2 \to B$, which carries a flat connection ∇^{GM} that is introduced in Section 4.1.2. The foliated symplectic form gives a section ϖ of $\mathscr{H}^2$. Then, the linear variation $\mathrm{var}^{\mathrm{lin}}_{\varpi} : TB \to \mathscr{H}^2$, $v \mapsto \nabla^{\mathrm{GM}}_v(\varpi)$ is injective if (M,π) is strong **s**-proper.

These two features will be elaborated in the first sections of this chapter, and we will describe how we can apply them to exhibit Poisson manifolds of the aforementioned compactness types.

4.1 Integral Affine Geometry

In this first section, we describe the notion of integral affine structures on a manifold, and we start off with a discussion of integral affine vector spaces.

4.1.1 Integral affine vector spaces

Throughout this section, V will always assumed to be a real, finite-dimensional vector space.

Definition 4.1. A lattice Λ inside V is a discrete, additive subgroup of V. Every lattice gives rise to a dual lattice

$$\Lambda^\vee = \{\alpha \in V^* \colon \alpha(\Lambda) \subseteq \mathbb{Z}\}$$

in V^*.

If Λ is a lattice in V, the *rank* of Λ is the dimension of the $\mathbb{R}$-linear subspace that it spans in V. The lattice is of *full rank* if its rank is equal to the dimension of V.

Definition 4.2. An integral affine vector space is a pair (V,Λ), where V is a real, finite-dimensional vector space and where Λ is a full rank lattice in V. A morphism of integral affine vector spaces $\phi : (V,\Lambda) \to (V',\Lambda')$ is a linear map $\phi : V \to V'$ such that $\phi(\Lambda) \subseteq \Lambda'$.

We say that a set $B = \{b_1,\cdots,b_n\} \subseteq V$ is a basis for the lattice Λ if it is linearly independent over the integers and the integral span is equal to Λ. The number of elements in B is precisely equal to the rank of the lattice.

Similar to vector spaces, any lattice admits a basis. This means that every integral affine vector space is isomorphic to $(\mathbb{R}^n, \mathbb{Z}^n)$ for some integer n.

If (V, Λ) is an integral affine vector space, we can consider a discrete subgroup $\Lambda' \subseteq \Lambda$, which is again a lattice in V. A natural question we can consider is whether there is a basis for Λ' that extends to a basis for Λ. Whereas this is obvious for vector spaces, this is no longer true for lattices. For instance, $2\mathbb{Z} \subseteq \mathbb{Z}$ is a sublattice, but there is no basis for $2\mathbb{Z}$ that extends to a basis for $\mathbb{Z}$.

Definition 4.3. We call a sublattice $\Lambda' \subseteq \Lambda$ primitive if any basis for Λ' extends to a basis for Λ.

It turns out that it is easy to characterise the primitive sublattices of an integral affine vector space (V, Λ).

Proposition 4.4. *Let (V, Λ) be an integral affine vector space and let $\Lambda' \subseteq \Lambda$ be a sublattice. Setting $V' = span_{\mathbb{R}}(\Lambda')$, the following are equivalent:*

1. *Λ' is primitive.*

2. *$V' \cap \Lambda = \Lambda'$.*

4.1.2 Integral affine vector bundles

In this section, we generalize the notions of integral affine vector spaces to lattices in vector bundles. We let $E \to B$ be a smooth, real vector bundle of rank q.

A lattice Λ inside E is a subbundle

$$\Lambda = \bigcup_{b \in B} \Lambda_b \subseteq E$$

consisting of full rank lattices $\Lambda_b \subseteq E_b$. It is called smooth if around each $b_0 \in B$, we can find an open neighbourhood U_0 together with smooth local sections $\lambda_i \; U_0 \to E$ such that

$$\Lambda_b = \bigoplus_{i=1}^{k} \mathbb{Z}\lambda_i(b).$$

Definition 4.5. An integral affine vector bundle is a pair (E, Λ), where $E \to B$ is a vector bundle and Λ is a smooth lattice in E.

Remark 4.6. By viewing E as a Lie groupoid over M, an integral affine structure Λ on E is a normal Lie subgroupoid with the properties that each isotropy group Λ_x is a full rank lattice in E_x.

Every integral affine vector bundle (E, Λ) naturally carries a connection ∇, known as the Gauss-Manin connection, uniquely defined by declaring all (local) sections of Λ to be flat.

Proposition 4.7. *The Gauss-Manin connection is always flat.*

Proof. Let $s : B \to E$ be a section, and let $X, Y \in \mathfrak{X}(B)$. We have an open cover $\{U_i\}$ of B such that for every U in this cover, we can write

$$s|_U = \sum_{i=1}^{k} f_i \lambda_i$$

where the $f_i : U \to \mathbb{R}$ are smooth functions and where the λ_i are smooth setions taking values in $\Lambda|_U$. We can naturally restrict the connection ∇ to a connection ∇^U on $E|_U \to U$, which satisifies $\nabla^U_{X|_U}(s|_U) = \nabla_X(s)|_U$. Thus, ∇^U still has the property that local sections of $\Lambda|_U$ are flat.

$$\begin{aligned} \left(\nabla_X \nabla_Y s - \nabla_Y \nabla_X - \nabla_{[X,Y]} s\right)\big|_U &= \nabla_{X|_U} \nabla_{Y|_U}(s|_U) - \nabla_{Y|_U} \nabla_{X|_U}(s|_U) - \nabla_{[X|_U, Y|_U]}(s|_U) \\ &= \sum_{i=1}^{k} (L_{X|_U} L_{Y|_U} - L_{Y|_U} L_{X|_U} - L_{[X|_U, Y|_U]}) f_i \lambda_i \\ &= 0, \end{aligned}$$

so the Gauss-Manin connection is indeed flat. □

The Gauss-Manin connection will reappear in Section 4.2.

Since every integral affine vector bundle has to carry a flat connection, there is a topological obstruction to having an integral affine structure. By Proposition 4.7, any integral affine vector bundle must carry a flat connection. This gives rise to a firm restriction on on which vector bundles are integral affine. For instance, the tangent bundle TS^2 of the 2-sphere cannot have an integral affine structure, since the first Chern class of TS^2 is non-zero. Following this argument, the only connected oriented surface whose tangent bundle has an integral affine structure is the torus.

4.1.3 Integral affine manifolds

If B is a manifold, with an integral affine structure (T^*B, Λ) on the cotangent bundle, we remark that Λ can be interpreted as a submanifold of T^*B of dimension equal to $\dim(B) = \frac{1}{2}\dim(T^*B)$. Since the cotangent bundle is always a symplectic manifold, one might wonder when Λ is a Lagrangian submanifold of T^*B. It turns out that this is precisely the case when B is an integral affine manifold, to be defined below.

Let $\mathrm{GL}_{\mathbb{Z}}(\mathbb{R}^q)$ consist of all invertible linear transformations $\mathbb{R}^q \to \mathbb{R}^q$ for which both A and A^{-1} have integer coefficients.

Definition 4.8. We denote by $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^q) = \mathrm{GL}_{\mathbb{Z}}(\mathbb{R}^q) \ltimes \mathbb{R}^q$ the group of integral affine transformations, i.e. transformations of the form

$$\mathbb{R}^q \to \mathbb{R}^q, \qquad v \mapsto Av + w \tag{22}$$

for $A \in \mathrm{GL}_{\mathbb{Z}}(\mathbb{R}^q)$ and $w \in \mathbb{R}^q$.

Definition 4.9. An integral affine atlas on a manifold B is an atlas $\mathscr{A} = \{(U_i, \chi_i)\}_{i\in I}$ for which each transition map $\chi_i \circ \chi_j^{-1}$ is (the restrictions of) an integral affine transformation. We call an element $(U_i, \chi_i) \in \mathscr{A}$ an integral affine chart for B. An integral affine structure on B is a maximal integral affine atlas, and we call $(B, \mathscr{A})$ an integral affine manifold.

The following theorem states the correspondence between integral affine structures on B and Lagrangian lattices in T^*B.

Theorem 4.10. *Let B be a q-dimensional manifold. There exists a one-to-one correspondence between*

1. *Integral affine structures $\mathscr{A}$ on B,*
2. *Smooth Lagrangian lattices Λ inside T^*B,*
3. *Smooth lattices $\Lambda^\vee$ inside TB such that all local vector fields with values in $\Lambda^\vee$ commute.*

In this correspondence, Λ and $\Lambda^\vee$ correspond to the lattices

$$\Lambda_b = \bigoplus \mathbb{Z} dx^i|_b, \qquad \Lambda_b^\vee = \bigoplus_{i=1}^{q} \mathbb{Z} \frac{\partial}{\partial x^i}\bigg|_b, \tag{23}$$

where $(x^1, \cdots, x^q)$ is an integral affine chart of B.

Proof. First, we note that the representations of Λ and $\Lambda^\vee$ are independent of the chosen integral affine chart, because we have for any two integral affine chart (U, χ) and (V, ψ) that

$$\mathrm{Jac}(\chi \circ \psi^{-1}) \in \mathrm{GL}_{\mathbb{Z}}(\mathbb{R}^q).$$

We first prove equivalence of 1 and 2. Let $\mathscr{A}$ be an integral affine atlas on B, giving rise to the lattice Λ. By construction, Λ is a smooth lattice in T^*B. To show that it is Lagrangian inside T^*B, it suffices to show that all local sections are Lagrangian (i.e. that they have Lagrangian images inside T^*B), since each local section is a diffeomorphism onto an open inside Λ. A local section α of T^*B is Lagrangian if and only if $\alpha^*\Omega_{\text{can}} = d\alpha = 0$, by the defining property of the canonical symplectic form on T^*B. Note that any local section of Λ is of the form $\alpha = \sum_{i=1}^q n_i dx^i$ for some $n_i \in \mathbb{Z}$, and is thus closed. Therefore, Λ is indeed a smooth Lagrangian lattice inside $(T^*B, \Omega_{\text{can}})$.

Next, let Λ inside $(T^*B, \Omega_{\text{can}})$ be a smooth Lagrangian lattice. Let $b_0 \in B$ and let U be an open neighbourhood around b_0 with a local frame $\{\lambda_1, \cdots, \lambda_q\}$ such that for all $b \in U$:

$$\Lambda_b = \bigoplus_{i=1}^{q} \mathbb{Z}\lambda_i(b).$$

Since the lattice is Lagrangian, each λ_i has to be a closed 1-form on U. By shrinking the open neighbourhood further, we may assume that each $\lambda_i = d\chi_i$, for smooth functions $\chi_i : U \to \mathbb{R}$. This gives us a smooth map $\chi = (\chi_1, \cdots, \chi_q) : U \to \mathbb{R}^q$ with the property that all $d\chi_i|_b$ for $b \in U$ are linearly independent. This means in particular that $(d\chi)_{b_0}$ is invertible, so we can apply the inverse function theorem to shrink U even further to obtain a chart (U, χ) around b_0. Doing this for all $b_0 \in B$ gives rise to an atlas $\mathscr{A}$ of B with the property that all charts (U,ϕ) in $\mathscr{A}$ satisfy that for any $b \in U$, one has

$$\Lambda_b = \bigoplus_{i=1}^{q} \mathbb{Z} d\phi_i(b).$$

If (U,χ) and (V,ψ) are two of such charts, we may assume that the intersection $U \cap V$ is connected. Since we must have that

$$\Lambda_b = \bigoplus_{i=1}^{q} \mathbb{Z} d\psi_i(b) = \bigoplus_{i=1}^{q} \mathbb{Z} d\chi_i(b),$$

we must have that

$$\mathrm{Jac}(\chi \circ \psi^{-1})(x) \in \mathrm{GL}_{\mathbb{Z}}(\mathbb{R}^q) \qquad \forall x \in \psi(U \cap V).$$

Now $\mathrm{GL}_{\mathbb{Z}}(\mathbb{R}^q)$ is discrete and $\psi(U \cap V)$ is connected so it follows that $\mathrm{Jac}(\chi \circ \psi^{-1})(x) = A$ for all x. It follows that $\chi \circ \psi^{-1}(v) = Av + w$ for some $w \in \mathbb{R}^q$, thus the transition maps are integral affine transformations. Letting $\mathscr{A}'$ be the maximal atlas containing $\mathscr{A}$, we constructed a map from the smooth, Lagrangian lattices in T^*B to the set of integral affine structures on B, and these are easily verified to be inverses.

Finally, we prove the equivalence of 2 and 3. This follows from the following observation: If $\alpha : U \to \Lambda|_U$ is a section of a smooth lattice Λ inside T^*B, we must have that for any local vector field $X : U \to \Lambda^\vee|_U$ that $\alpha(X) \in \mathbb{Z}$ is constant. Therefore, we have for any two local vector fields $X, Y : U \to \Lambda^\vee$ that

$$d\alpha(X,Y) = L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha[X,Y] = -\alpha[X,Y].$$

Thus, if X and Y commute, $d\alpha = 0$ and any local section is closed, so Λ is Lagrangian. On the other hand, if $d\alpha = 0$ for all local sections α, we have that $\alpha[X,Y] = 0$ for all local sections α, which implies that X and Y commute, and this proves the theorem. □

From now on, we denote integral affine manifolds by (B,Λ), where $\Lambda \subseteq T^*B$ is a smooth, Lagrangian lattice.

Definition 4.11. A morphism $\phi : (B,\Lambda) \to (B,\Lambda')$ of integral affine manifolds is a smooth map $\phi : B \to B'$ with the property that for each $q \in B$, the map

$$(d\phi_q)^* : (T^*_{\phi(q)}B, \Lambda_{\phi(q)}) \to (T^*_q B, \Lambda_q)$$

is a morphism of integral affine vector spaces.

Remark 4.12. The condition in Definition 4.11 is equivalent to saying that the map $\phi : B \to B'$ on each connected integral affine coordinate domain is (the restriction of) an integral affine map.

Definition 4.13. Let (B,Λ) be an integral affine manifold. We say that (B',Λ') is an integral affine submanifold if B' is a submanifold of B and $\Lambda'^\vee \subseteq \Lambda^\vee$. We call it primitve if furthermore, see Proposition 4.4

$$\Lambda'^\vee = TB' \cap \Lambda^\vee.$$

Remark 4.14. The condition that (B',Λ') is an integral affine submanifold of (B,Λ) is equivalent to saying that the inclusion $B' \to B$ is an integral affine map.

Let us now discuss some examples of integral affine structures, and see a first glimpse of their relation to PMCTs.

Example 4.15. If (V,Λ) is an integral affine vector space, there is naturally an integral affine structure $V \times \Lambda^* \subseteq V \times V^* = T^*V$.

Example 4.16. An important class of integral affine manifolds are the so-called complete integral affine manifolds. Let $\Gamma \subseteq \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^q)$ be a discrete subgroup, and assume that it acts freely and properly on $\mathbb{R}^q$. The smooth manifold $B = \mathbb{R}^q/\Gamma$ carries an integral affine structure. To see this, note that the projection $\mathbb{R}^q \to B$ is a local diffeomorphism, so local sections $B \to \mathbb{R}^q$ may serve as charts for B. This implies that the transition maps are in Γ, meaning they define an integral affine structure on B. Important examples of these include the subgroup $\Gamma \subseteq \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^q)$ generated by $x \mapsto x+1$ which endows $S^1 = \mathbb{R}/\Gamma$ with an integral affine structure.

The final example requires more introduction. Given a smooth lattice Λ inside T^*M, one can form the quotient $\mathscr{T}_\Lambda = T^*M/\Lambda$. This will be a torus bundle over M, whose fibers are the tori T^*_xM/Λ_x. By a similar argument as in Theorem 4.10, the symplectic form Ω_{can} on T^*M descends to a symplectic form Ω_Λ on $\mathscr{T}_\Lambda$ if and only if Λ is an integral affine structure on M, and we call $(\mathscr{T}_\Lambda, \Omega_\Lambda)$ a symplectic torus bundle over M. It turns out that every symplectic torus bundle over M is of this form, see Proposition 3.1.6 in [CFM16].

Theorem 4.17. *Proper symplectic integrations of $(M,0)$ are in one-to-one correspondence to integral affine structures on M.*

Proof. By the discussion above, it suffices to show that proper symplectic integrations of $(M,0)$ are in one-to-one correspondence with symplectic torus bundles over. M.

By definition, a symplectic torus bundle $(\mathscr{T}_\Lambda,)$ is a symplectic groupoid over M whose target and source map coincide, and whose isotropy groups are tori. In particular, $(\mathscr{T}_\Lambda, \Omega_\Lambda)$ induces a Poisson structure on M for which the source map is both Poisson and anti-Poisson, so this Poisson structure must be the zero Poisson structure. For $x \in M$, the fiber $\mathbf{s}^{-1}(x)$ is a torus, so they are compact, and by Lemma 3.2, the $\mathbf{s}$-map is proper. It follows that $(\mathscr{T}_\Lambda, \Omega_\Lambda)$ is a proper integration of $(M,0)$.

Conversely, suppose that $(\mathscr{G},\Omega) \rightrightarrows (M,0)$ is a proper integration. Both the $\mathbf{s}$ and $\mathbf{t}$-fibers form a foliation of $\mathscr{G}$. The target map $\mathbf{t} : (\mathscr{G},\Omega) \to (M,0)$ is a symplectic realization. Since symplectic realizations of the zero Poisson structure have coisotropic fibers, we have by Proposition 2.44 that $\ker(d\mathbf{s}) = \ker(d\mathbf{t})^\Omega = \ker(d\mathbf{t})$. Since the $\mathbf{s}$ and $\mathbf{t}$-fibers are connected, they must be the same, hence $\mathbf{s} = \mathbf{t}$. Since the zero Poisson structure is regular, the isotropy algebras must be abelian and the same must a fortiori hold for the isotropy groups. Since $\mathscr{G}$ is proper, the isotropy groups are compact, so they must be tori. We conclude that $(\mathscr{G},\Omega)$ is a symplectic torus bundle. $\square$

4.1.4 Transverse Integral Structures

We have seen in the previous section that an integral affine structure gives rise to a Lagrangian lattice in the cotangent bundle. In this section, we will be discussing a similar, but more general notion, namely that of a transverse integral affine structure on a (regularly) foliated manifold $(B,\mathscr{K})$. This will again be a Lagrangian lattice, but not in the full cotangent bundle of B but taking values in the conormal bundle $\nu^*(\mathscr{K})$ of a foliation. This is to be thought of as an integral affine structure on the leaf space $B/\mathscr{K}$, but this maybe non-smooth, or even very pathological.

To describe this structure in greater detail, recall that a regular foliation $(B,\mathscr{K})$ of codimension q has a foliation atlas $\{(U_i,\chi_i)\colon i\in I\}$, where the U_i are opens cover of M, together with submersions $\chi_i : U_i \to \mathbb{R}^q$, whose fibers are the plaques of $\mathscr{K}$. For an introduction to foliations and plaques, we refer to [Ban06].

Definition 4.18 ([CFM16])**.** A transverse integral affine structure on a regular foliation $(B,\mathscr{K})$ is a choice of maximal foliation atlas, with the property that each transition function $\chi_i \circ \chi_j^{-1} : \chi_j(U_i \cap U_j) \to \chi_i(U_i \cap U_j)$ is the restriction of an integral affine transformation.

Just as with integral affine structures, it is easier to work with lattices in the cotangent bundle. There is an analogue of Theorem 4.10 where we have replace the cotangent bundle by the conormal bundle, which we state as follows.

Theorem 4.19 ([CFM16])**.** *Let $(B,\mathscr{K})$ be a regular foliation. There is a one-to-one correspondence between*

1. *Transverse integral affine structures on $(B,\mathscr{K})$.*
2. *Smooth Lagrangian lattices Λ inside $\nu^*(\mathscr{K})$.*

In this correspondence, Λ is given by

$$\Lambda_x = \bigoplus_{i=1}^{q} \mathbb{Z} dx^i|_x,$$

where $(x^1,\cdots,x^q)$ is a transverse integral affine coordinate system.

We will be mostly interested in transverse integral affine structures on *simple* foliations, i.e. foliations for which the leaf space $B/\mathscr{K}$ is a smooth manifold.

Example 4.20. If the case of a simple foliaton $(B,\mathscr{K})$, the leaf space $B/\mathscr{K}$ is a smooth manifold and its cotangent bundle is isomorphic to the conormal bundle $\nu^*(\mathscr{K})$ via the isomorphism $(dp)^*_x : T^*_{p(x)}(B/\mathscr{K}) \xrightarrow{\simeq} \nu^*_x(\mathscr{K})$. Through this isomorphism, there is a one-to-one correspondence between integral affine structures on $B/\mathscr{K}$, and transverse integral affine structures on $(B,\mathscr{K})$.

We are now ready to describe the transverse integral affine associated to any PMCT. Let $(\mathscr{G},\Omega)$ be a proper integration of a Poisson manifold (M,π). Then, the isotropy groups $\mathscr{G}_x$ are compact, and the isotropy Lie algebras $\mathfrak{g}_x$ correspond to the isotropy Lie algebras $\nu^*_x(\mathscr{F}) = \ker(\pi^\sharp_x)$, under the isomorphism from Theorem 2.45. Therefore, they must be abelian and

$$\Lambda_{\mathscr{G},x} = \ker(\exp_{\mathfrak{g}} : \mathfrak{g}_x \to \mathscr{G}_x) \tag{24}$$

defines a lattice inside $\mathfrak{g}_x$, or equivalently, in $\nu^*_x(\mathscr{F})$. This defines a bundle of lattices inside $\nu^*(\mathscr{F})$

$$\Lambda_{\mathscr{G}} = \bigcup_{x\in M} \Lambda_{\mathscr{G},x}.$$

The following theorem is the promised result from the introduction of this chapter.

Theorem 4.21. *Each proper integration $(\mathscr{G},\Omega)$ of a regular Poisson manifold (M,π) induces a transverse integral affine structure $\Lambda_{\mathscr{G}}$ on $(M,\mathscr{F})$. Moreover, we always have that $\mathscr{N}(M,\pi)\subseteq\Lambda_{\mathscr{G}}$, and $\mathscr{N}(M,\pi)=\Lambda_{\Sigma(M)}$ if and only if (M,π) is strong proper, where $\mathscr{N}(M,\pi)$ are the monodromy groups of a regular Poisson manifold, defined in 2.39.*

The proof of this theorem uses the following proposition.

Proposition 4.22. *Let $(\mathscr{G},\Omega)$ be a symplectic groupoid and let $\vec{\alpha}$ be a right-invariant vector field on $\mathscr{G}$, corresponding to a section α of Lie$(\mathscr{G})$. Then we have that*

$$\iota_{\vec{\alpha}}\Omega=\mathbf{t}^*\sigma_\Omega(\alpha)$$

Proof. Let $v\in T_g\mathscr{G}$. We need the following identities

$$v=\frac{d}{dt}\bigg|_{t=0}\gamma_v=\frac{d}{dt}\bigg|_{t=0}\mathbf{m}(\mathbf{u}(\mathbf{t}(\gamma_v)),\gamma_v)=d\mathbf{m}_{(1_{\mathbf{t}(g)},g)}\left(d\mathbf{u}_{1_{\mathbf{t}(g)}}d\mathbf{t}_g v,v\right)$$
$$\vec{\alpha}_g=(dR_g)_{1_{\mathbf{t}(g)}}\alpha_{1_{\mathbf{t}(g)}}=\frac{d}{dt}\bigg|_{t=0}\gamma_\alpha\cdot g=d\mathbf{m}_{(1_{\mathbf{t}(g)},g)}\left(\alpha_{1_{\mathbf{t}(g)}},0\right).$$

Using the multiplicativity of Ω, we deduce that

$$\begin{aligned}\Omega_g(v,\vec{\alpha}_g)&=(\mathbf{m}^*\Omega)_{(\mathbf{1}_{\mathbf{t}(\mathbf{g})},g)}\left(\left((d\mathbf{u})_{1_{\mathbf{t}(g)}}d\mathbf{t}_g v,v\right),\left(\alpha_{1_{\mathbf{t}(g)}},0\right)\right)\\&=(\mathrm{pr}_1^*\Omega)_{(1_{\mathbf{t}(g)},g)}\left(\left((d\mathbf{u})_{1_{\mathbf{t}(g)}}d\mathbf{t}_g v,v\right),\left(\alpha_{1_{\mathbf{t}(g)}},0\right)\right)+(\mathrm{pr}_2^*\Omega)_{(1_{\mathbf{t}(g)},g)}\left(\left((d\mathbf{u})_{1_{\mathbf{t}(g)}}d\mathbf{t}_g v,v\right),\left(\alpha_{1_{\mathbf{t}(g)}},0\right)\right)\\&=\Omega_{1_{\mathbf{t}(g)}}\left((d\mathbf{u})_{1_{\mathbf{t}(g)}}d\mathbf{t}_g v,\alpha_{1_{\mathbf{t}(g)}}\right)+\Omega_g(v,0)\\&=\Omega_{1_{\mathbf{t}(g)}}\left((d\mathbf{u})_{1_{\mathbf{t}(g)}}d\mathbf{t}_g v,\alpha_{1_{\mathbf{t}(g)}}\right)\\&=-\sigma_\Omega(\alpha_{1_{\mathbf{t}(g)}})(d\mathbf{t}_g)v.\end{aligned}$$

The statement follows immediately. □

We can now prove Theorem 4.21. The proof fills in the details of the proof of Theorem 3.3.1 in [CFM16].

Proof. We follow the argument exposed in [CFM16], and fill in some details. Let $(\mathscr{G},\Omega)$ and $\Lambda_{\mathscr{G}}$ be as above. We show that the bundle of lattices is smooth, which we will do by giving a different description for the individual lattices $\Lambda_{\mathscr{G},x}$. Each $\alpha_x\in\nu_x^*(\mathscr{F})$ corresponds to a unique right-invariant vector field $\vec{\alpha}$ on $\mathscr{G}_x$. We denote by $\mathscr{G}_x^\circ$ the connected component containing 1_x, and restrict $\vec{\alpha}$ to a vector field on $\mathscr{G}_x^\circ$. The bundle of connected components of the isotropy groups

$$\mathscr{G}_M=\bigcup_{x\in M}\mathscr{G}_x^\circ$$

is a bundle of Lie groups over M, and hence a Lie subgroupoid of $\mathscr{G}$. There is an action of the bundle of abelian Lie groups $(\nu^*(\mathscr{F}),+)\rightrightarrows M$ on $\mathscr{G}_M\rightrightarrows M$ by

$$\theta:\nu^*(\mathscr{F})\times_M\mathscr{G}_M\to\mathscr{G}_M,\qquad\theta(\alpha,g)=\phi^1_{\vec{\alpha}}(g),$$

where $\phi^1_{\vec{\alpha}}$ denotes the flow at $t=1$ of $\vec{\alpha}$. Indeed, this is a groupoid action, because we have that

1. $\mathbf{s}(\alpha_x\cdot g)=\mathbf{s}\circ\phi^1_{\vec{\alpha}}(g)=\mathbf{s}(g)=\mathbf{t}(g)$, since $\mathbf{s}\circ\phi^1_{\vec{\alpha}}=\mathbf{s}$,

2. $0 \cdot g = \phi_0^1(g) = g$,

3. For $\alpha_x, \beta_x \in \nu_x^*(\mathscr{F})$, we have

$$\beta_x \cdot (\alpha_x \cdot g) = \beta_x \cdot \phi^1_{\vec{\alpha}}(g) = \phi^1_{\vec{\beta}}\phi^1_{\vec{\alpha}}(g) = \phi^1_{\vec{\alpha}+\vec{\beta}}(g) = (\alpha_x + \beta_x) \cdot g,$$

where we used in the third equality that the $[\vec{\alpha}, \vec{\beta}] = 0$ since $\nu^*(\mathscr{F})$ is abelian.

Since $\vec{\alpha}$ is tangent to the isotropy groups, we have by definition of the exponential map for Lie groups that $\exp_{\mathfrak{g}_x}(\alpha) = \phi^1_{\vec{\alpha}}(1_x)$. This allows us to identify

$$\Lambda_{\mathscr{G},x} = \{\alpha \in \nu_x^*(\mathscr{F}) \colon \phi^1_{\vec{\alpha}} = \mathrm{Id}\}.$$

We should show that $\Lambda_{\mathscr{G}}$ forms a smooth lattice. For that, we need to show that for each $\alpha_{x_0} \in \Lambda_{\mathscr{G},x_0}$, there is an open neighbourhood U_0 of x_0 together with a local section λ such that $\lambda(x) \in \Lambda_{\mathscr{G},x}$ for all $x \in U_0$. In other words, we should have that

$$\phi^1_{\overrightarrow{\lambda(x)}}(1_x) = 1_x \text{for all } x \in U. \tag{25}$$

We define the map

$$F_{\mathbf{u}} : \nu^*(\mathscr{F}) \to \mathscr{G}_M, \qquad F_{\mathbf{u}}(\xi_x) = \phi^1_{\vec{\xi}}(1_x).$$

This is a bundle map, covering the identity. Moreover, the map $\xi_x \mapsto \vec{\xi}$ is injective, so the action is locally free. Therefore, $f_{\mathbf{u}}$ is fiberwise a local diffeomorphism, and thus $F_{\mathbf{u}}$ is a local diffeomorphism. This implies that there exists a local section λ satisfying (25), and this proves that the lattice is smooth.

We are left to show that this lattice is Lagrangian. By the characterisation of transverse integral affine structures in Theorem 4.21, it suffices to show that any local section $\alpha : U \to \Lambda_{\mathscr{G}}|_U$ is closed. Using Proposition 4.22, we have for a local section $\alpha : U \to \Lambda_{\mathscr{G}}|_U$ that

$$\begin{aligned}
\mathbf{t}^* d\alpha &= \int_0^1 \mathbf{t}^* d\alpha \, d\tau \\
&= \int_0^1 \left(\phi^\tau_{\vec{\alpha}}\right)^* \mathbf{t}^* d\alpha \, dt && \text{(Since } \mathbf{t} \circ \phi^\tau_{\vec{\alpha}} = \mathbf{t}\text{)} \\
&= \int_0^1 \left(\phi^\tau_{\vec{\alpha}}\right)^* d\iota_{\vec{\alpha}} \Omega \, d\tau \\
&= \int_0^1 \left(\phi^\tau_{\vec{\alpha}}\right)^* \mathscr{L}_{\vec{\alpha}} \Omega \, d\tau \\
&= \int_0^1 \frac{d}{d\tau} \left(\phi^\tau_{\vec{\alpha}}\right)^* \Omega \, d\tau \\
&= \left(\phi^1_{\vec{\alpha}}\right)^* \Omega - \Omega \\
&= 0.
\end{aligned}$$

Since $\mathbf{t}$ is a submersion, $d\alpha = 0$, hence the lattice is Lagrangian. To prove the remaining statements of the theorem, recall from 2.58 that the monodromy groups can be described as the image of the map ∂_{mon} in the exact sequence

$$\pi_2(S,x) \xrightarrow{\partial_x} \nu_x^*(S) \xrightarrow{\exp_{\mathfrak{g}_x}} \Sigma_x(M,\pi) \xrightarrow{p} \pi_1(S,x) \to 0.$$

Exactness implies that $\mathrm{im}(\partial_x) = \ker(\exp_{\mathfrak{g}_x})$. Starting with $\alpha \in \mathscr{N}_x(M,\pi)$, one has that the right-invariant vector field $\vec{\alpha}$ on $\Sigma(M)$ satisfies $\phi^1_{\vec{\alpha}} = \mathrm{Id}$. There is always a covering projection $\Phi : \Sigma(M) \to \mathscr{G}$, such that $\vec{\alpha}$ is mapped to a right-invariant vector field $\vec{\alpha}'$ on $\mathscr{G}$, which must also satisfy $\phi^1_{\vec{\alpha}'} = \mathrm{Id}$, so $\mathscr{N}_x(M,\pi) \subseteq \Lambda_{\mathscr{G},x}$.

If (M,π) is strong proper, we have by definition $\Lambda_{\Sigma(M,\pi),x} = \ker(\exp_{\mathfrak{g}_x}) = \mathcal{N}_x(M,\pi)$. On the other hand, suppose that $\mathcal{N}(M,\pi) = \Lambda_{\Sigma(M,\pi)}$. Since the monodromy groups of (M,π) form a lattice, $\Sigma(M,\pi)$ is proper in this case. This concludes the proof. $\square$

4.2 Linear Variation of the Symplectic Forms

In this section, we focus on the other property of a PMCT, which is that of full linear variation of the symplectic forms on the symplectic leaves. Recall that a regular Poisson manifold (M,π) can equivalently be described by a symplectic foliation $(\mathscr{F},\omega_{\mathscr{F}})$. The aim of this section is to study how the symplectic form changes in the normal direction $\nu(\mathscr{F}) = TM/\mathscr{F}$. We will give two different, but in the end equivalent, descriptions of the linear variation of the foliated symplectic form. We start with probably the most intuitive description of the two, which we have already seen in Section 2.5.1 and is that of the transversal derivative d_ν. Let us briefly recall the construction of d_ν. Consider a regular foliation $\mathscr{F}$ of a manifold M, and consider a $d_{\mathscr{F}}$-closed 2-form $\omega_{\mathscr{F}}$. If we choose any extension $\omega \in \Omega^2(M)$ of $\omega_{\mathscr{F}}$, the form

$$\theta : (X,Y) \mapsto d\omega(X,Y,-) \in \Omega^2(\mathscr{F},\nu^*(\mathscr{F}))$$

is $d^*_{\mathscr{F}}$-closed, and its cohomology class is independent of the chosen extension ω. This defines the transversal derivative

$$d_\nu : H^2(\mathscr{F}) \to H^2(\mathscr{F},\nu^*(\mathscr{F})), \qquad [\omega_{\mathscr{F}}] \mapsto [\theta].$$

Definition 4.23. The linear variation of a $d_{\mathscr{F}}$-closed 2-form $\omega_{\mathscr{F}}$ is defined by $d_\nu[\omega_{\mathscr{F}}]$.

We have argued in Example 2.54 why the transveral derivative of the foliated symplectic form can be interpreted as the normal variation of this form. Also, we have shown that

$$d_\nu[\omega_{\mathscr{F}}] = [\Omega_\tau],$$

where $\tau : \mathscr{F} \to T^*M$ is a splitting of the short exact sequence associated to the cotangent algebroid of a Poisson manifold.

Next, we present a second description of the linear variation of the foliated symplectic form, in the special case that the symplectic leaves form a simple foliation. The standing assumption is that (M,π) is a regular (connected) Poisson manifold with *compact* and *1-connected* symplectic leaves. Under these assumptions, the leaf space of $B = M/\mathscr{F}$ of M is a smooth manifold such that the map $p : M \to B$ is a submersion [CFM16]. We will call these Poisson manifolds *simple Poisson manifolds* since the symplectic foliation is simple. For such simple Poisson manifolds, we define a (set-theoretical) real vector bundle $\mathscr{H}^2 \to B$, whose fibers are the degree 2 cohomology groups of the leaves: $\mathscr{H}^2_b = H^2(S_b,\mathbb{R})$. To see that this is indeed a smooth vector bundle, we need to have trivializing opens, and these can be obtained from local trivializations of p. For local trivializations opens of p, we use Ehresmann's lemma.

Theorem 4.24 (Ehresmann's lemma, [VS02])**.** *Let $p : X \to Y$ be a surjective submersion that is proper. Then p is a fiber bundle.*

By the same argument as in Lemma 3.2, the quotient map $p : M \to B$ is proper if its fibers are compact. Thus, we have the following lemma.

Lemma 4.25. *If (M,π) is a simple Poisson manifold, the quotient map $p: M \to B$ is a fiber bundle.*

The local trivializations of p induce local trivializations of $\mathscr{H}^2 \to B$ in the following sense. If $U \subseteq B$ is an open trivialization such that $p|_U : U \times S \to U$, we have that that $\mathscr{H}^2|_U \cong U \times H^2(S,\mathbb{R})$. This endows $\mathscr{H}^2 \to B$ with the structure of a smooth vector bundle. By replacing the coefficient group by $\mathbb{Z}$, we obtain a smooth lattice $\mathscr{H}^2_{\mathbb{Z}} \to B$ inside $\mathscr{H}^2$, so we can endow $\mathscr{H}^2 \to B$ with the Gauss-Manin connection ∇. There is a rich interplay between the integral affine geometry of the integral affine vector bundle $(\mathscr{H}^2, \mathscr{H}^2_{\mathbb{Z}})$ and the integral affine geometry of the leaf space B, which is depicted through the linear variation map. To define the linear variation map, we use that the foliated symplectic form $\omega_{\mathscr{F}}$ gives a canonical section $\varpi : B \to \mathscr{H}^2$ by $\varpi(b) = [\omega|_{S_b}] \in H^2(S_b,\mathbb{R})$.

Definition 4.26. The linear variation is defined as the bundle map

$$\mathrm{var}^{\mathrm{lin}}_{\varpi} : TB \to \mathscr{H}^2, \qquad v \mapsto \mathrm{var}^{\mathrm{lin}}_{\varpi}(v) = \nabla_v \varpi,$$

where ∇ is the Gauss-Manin connection, defined in Section 4.1.2.

We can relate the map $\mathrm{var}^{\mathrm{lin}}_{\varpi}$ to the curvature form Ω_τ from the previous section, as follows. Under the isomorphism $(dp)_x : \nu_x(\mathscr{F}) \xrightarrow{\simeq} T_{p(x)}B$, elements in $H^2(\mathscr{F}, \nu^*(\mathscr{F}))$ correspond to elements in $\Omega^1(B,\mathscr{H}^2)$ under the isomorphism

$$H^2(\mathscr{F}, \nu^*(\mathscr{F})) \ni [\alpha] \mapsto (v_b \mapsto [\langle v_b, \alpha\rangle]) \in \Omega^1(B,\mathscr{H}^2).$$

Proposition 4.27 ([CFM16]). *Under the previous identifications, we have that the linear variation map var^{lin}_{ϖ} satisfies*

1. $[\Omega_\tau] = var^{lin}_{\varpi} \in \Omega^1(B,\mathscr{H}^2)$*, where* $\tau : \mathscr{F} \to T^*M$ *is a splitting of* (14).
2. *For* $v \in T_{p(x)}B$ *and* $\sigma : (S^2, p_N) \to (S,x)$*, one has*

$$\int_\sigma var^{lin}_{\varpi}(v) = \langle \partial_x([\sigma]), v\rangle,$$

where S is the symplectic leaf through $x \in M$.

Proof. Part 2 follows from 1, together with the interpretation of monodromy in terms of curvature from Section 2.5.1.

To prove part 1, we note that right from the definitions, we have a commuting square

$$\begin{tikzcd} H^2(\mathscr{F}) \arrow[r, "\simeq"] \arrow[d, "d_\nu"] & \Gamma(\mathscr{H}^2) \arrow[d, "d_\nabla"] \\ H^2(\mathscr{F}, \nu^*(\mathscr{F})) \arrow[r, "\simeq"] & \Omega^1(B,\mathscr{H}^2) \end{tikzcd}.$$

To see this, choose a splitting $TM \cong \nu(\mathscr{F}) \oplus \mathscr{F}$, which gives a way to extend foliated forms $\omega_{\mathscr{F}} \in \Omega^\bullet(\mathscr{F})$ to forms $\omega \in \Omega^\bullet(M)$, satisfying $\iota_V \omega = 0$ for all $V \in \Gamma(\nu(\mathscr{F}))$. Now, consider the following claim that is proven in [CFM16].

- Let $V \in \Gamma(\nu(\mathscr{F}))$ be a vector field defined on an open around S_b whose restriction to S_b projects to a vector $v_b \in T_bB$. For any section $\overline{\eta} \in \Gamma(\mathscr{H}^2)$ represented by a 2-form $\eta \in \Omega^2(M)$ satisfying $\iota_V\eta = 0$, one has

$$\nabla_{v_b}\overline{\eta} = [(\mathscr{L}_V\eta)|_{S_b}] \in H^2(S_b,\mathbb{R}). \tag{26}$$

The claim follows from the following observation. We can lift vector fields on B to vector fields on M, using the splitting of TM. The flow of this lifted vector field is a 1-parameter group of diffeomorphisms of the fibers, which preserves the integral cohomology of the fibers. Since the Gauss-Manin connection is defined as the unique connection which vanishes on sections of the integral cohomology lattice, (26) follows.

From this claim, we deduce the commutativity of the diagram. Let $v_b \in T_bB$ and let $V \in \Gamma(\nu(\mathscr{F}))$ be a vector field defined on an open around S_b projection to v_b. Let $\eta_{\mathscr{F}}$ be a foliated 2-form representing $\overline{\eta} \in \Gamma(\mathscr{H}^2)$. Since we have a splitting of TM, we have a natural extension of $\eta_{\mathscr{F}}$ to $\eta \in \Omega^2(M)$ with $\iota_V\eta = 0$. Then, we have for $X,Y \in \mathscr{F}$ that

$$d\eta(X,Y,V) = \mathscr{L}_V(\eta(X,Y)) + \eta([X,V],Y) - \eta([Y,V],X) = (\mathscr{L}_V\eta)(X,Y).$$

It follows from the definition of the transversal derivative d_ν and the claim that the diagram commutes.

From the commutativity of the diagram, we deduce the statement. The canonical section ϖ corresponds to the class $[\omega_{\mathscr{F}}] \in H^2(\mathscr{F})$. By Theorem 2.56, $d_\nu[\omega_{\mathscr{F}}] = [\Omega_\tau]$, which by commutativity of the diagram corresponds to $\nabla\varpi$, and completes the proof. □

Recall from Example 4.20 that a proper groupoid integrating a simple Poisson manifold induces an integral affine structure on the leaf space. The advantage of this new description of the linear variation is that interacts nicely with the integral affine geometry on the leaf space. Using the linear variation map, the monodromy groups can be expressed in terms of the integral affine bundle $(\mathscr{H}^2, \mathscr{H}^2_{\mathbb{Z}})$.

Proposition 4.28. *Let (M,π) be a simple Poisson manifold then $\mathscr{N}_x^\vee(M,\pi) = (var^{lin}_{\varpi})^{-1}(H^2(S_x,\mathbb{Z}))$, where S_x is the symplectic leaf through x.*

Proof. The proof is a direct computation. Fix $b \in B$ and $v_b \in T_b^*B$. Choose $x \in p^{-1}(b)$. Since the symplectic leaves are compact, we have by Proposition 4.27 that

$$\begin{aligned} v_b \in \mathscr{N}_x^\vee(M,\pi) &\iff \lambda(v_b) \in \mathbb{Z} \text{ for all } \lambda \in \mathscr{N}_x(M,\pi) \\ &\iff \langle \partial_x([\sigma]), v\rangle \in \mathbb{Z} \text{ for all } [\sigma] \in \pi_2(S_x,x) \\ &\iff \int_\sigma \mathrm{var}^{\mathrm{lin}}_{\varpi}(v) \in \mathbb{Z}, \text{ for all } [\sigma] \in \pi_2(S_x,x) \\ &\iff \mathrm{var}^{\mathrm{lin}}_{\varpi}(v_b) \in H^2(S_x,\mathbb{Z}). \end{aligned} \tag{27}$$

In the last implication, we used that the leaves are simply-connected and passed through the Hurewicz isomorphism. It follows that

$$\mathscr{N}_x^\vee(M,\pi) = (\mathrm{var}^{\mathrm{lin}}_{\varpi})^{-1}\left(H^2(S_x,\mathbb{Z})\right).$$

□

Note that not every saturated Poisson submanifold (i.e. a union of symplectic leaves) of a simple Poisson manifold (M,π) is again simple, because the leaf space will only be a subset of the leaf space B, but need not be a submanifold of B. However, if $F \subseteq B$ is a submanifold, then $M_F = p^{-1}(F)$ is a simple, saturated submanifold of M with leaf space F. We can express the linear variation of M_F in terms of the linear variation of M, as follows.

We have that the bundle $\mathscr{H}^2_{M_F} \to F$ of second cohomology groups of the symplectic leaves of M_F is just the restriction of the bundle $\mathscr{H}^2 \to B$ to F. Let $\varpi^M : B \to \mathscr{H}^2$ and $\varpi^{M_F} : F \to \mathscr{H}^2_{M_F}$ be the sections describing the cohomology classes of the symplectic form on the leaves. There is a commuting square

$$\begin{tikzcd} \mathscr{H}^2_{M_F} \arrow[r] \arrow[d] & \mathscr{H}^2 \arrow[d] \\ F \arrow[r, "\iota"] & B \end{tikzcd},$$

and we have that $\varpi^{M_F} = \iota^*\varpi^M$. The Gauss-Manin connection ∇_{M_F} on $\mathscr{H}^2_{M_F} \to F$ is the pullback connection $\iota^*\nabla$, where ∇ is the Gauss-Manin connection on $\mathscr{H}^2 \to B$. For $v_b \in T_bF \subseteq T_bB$, we have that

$$\mathrm{var}^{\mathrm{lin}}_{\varpi^{M_F}}(v_b) = \nabla^{M_F}_{v_b}\varpi = \iota^*\nabla_{v_b}\left(\iota^*\varpi^M\right) = \nabla_{v_b}\varpi^M = \mathrm{var}^{\mathrm{lin}}_{\varpi^M}(v_b).$$

Thus, we deduce that $\mathrm{var}^{\mathrm{lin}}_{\varpi^{M_F}} = \mathrm{var}^{\mathrm{lin}}_{\varpi^M}|_{TF}$. The consequence is that we have a simple way to compute the monodromy groups of M_F.

Corollary 4.29. *Let (M,π) be a simple Poisson manifold with leaf space B, $F \subseteq B$ a submanifold and $M_F = p^{-1}(N)$. Then*

$$\mathscr{N}^\vee_x(M_F,\pi) = T_xF \cap \mathscr{N}^\vee_x(M,\pi)$$

In the sequel, we will be looking for appropriate submanifolds $F \subseteq B$ of the leaf space such that $M_F = p^{-1}(F)$ is a strong **s**-proper submanifold of (M,π). The following proposition gives a first glimpse of what kind of submanifolds we should be looking for.

Proposition 4.30. *Let (M,π) be a simple, integrable Poisson manifold. Then M is of strong* **s***-proper type if and only if the linear variation is injective.*

Proof. If (M,π) is a simple strong **s**-proper Poisson manifold, the monodromy groups form a lattice of full rank. By Proposition 4.27, we have that

$$\ker(\mathrm{var}^{\mathrm{lin}}_{\varpi})_b = \{v \in T_bB\colon \langle v,\alpha\rangle = 0,\ \alpha \in N_x(M,\pi)\} = N_x(M,\pi)^\circ = 0, \tag{28}$$

hence $\mathrm{var}^{\mathrm{lin}}_{\varpi}$ is injective.

Next, suppose that the linear variation map is injective. If M is integrable, the monodromy groups of M are all discrete. By (28), the rank of the annihilators of the monodromy groups is zero, which means that the rank of the monodromy groups is everywhere maximal, so (M,π) is of strong **s**-proper type. □

Remark 4.31. It is not true that every simple Poisson manifold with injective linear variation is integrable. For this, we present an example. Consider the regular Poisson manifold $M = S^2 \times S^2 \times (0,\infty)^2$, with symplectic leaves $S_y = S^2 \times S^2$, where the symplectic form is simply $\omega_y = y_1\mathrm{pr}_1^*\omega + y_2\mathrm{pr}_2^*\omega_2$, where ω_i denotes the standard symplectic form on S^2. The linear variation of this Poisson manifold is the bundle map

$$\mathrm{var}^{\mathrm{lin}}_{\varpi} : \mathbb{R}^2 \to \mathscr{H}^2, \qquad \mathrm{var}^{\mathrm{lin}}_{\varpi}(v_1,v_2) = v_1\omega_1 + v_2\omega_2,$$

which is clearly injective. Using Proposition 4.28, the monodromy groups are $\mathcal{N}_y(M,\pi) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Inside M, we have the Poisson submanifold $N = S^2 \times S^2 \times \{y = \sqrt{q}x\colon q > 0, \text{irrational}\}$. This is a Poisson submanifold of M, so its linear variation is still injective, and given by the map

$$\mathrm{var}^{\mathrm{lin}}_{\varpi^N}(v_1, \sqrt{q}v_1) = v_1\omega_1 + qv_1\omega_1.$$

In particular, we see that

$$\mathcal{N}^{\vee}_{(x,\sqrt{q}x)}(N,\pi_N) = \{\lambda \in \mathbb{R}\colon (\lambda, q\lambda) \in \mathbb{Z}^{\oplus 2}\} = 0.$$

From this, we conclude that N is not integrable.

4.3 Constructing strong s-proper Poisson manifolds

In this section, we use Corollary 4.30 to describe a procedure that can be used to produce strong **s**-proper Poisson manifolds. Moreover, we will describe how the leaf space of any integrable, simple Poisson manifold carries a transverse integral affine structure (under mild assumptions on $\mathrm{var}^{\mathrm{lin}}_{\varpi}$), generalizing Theorem 4.21.

Lemma 4.32. *Let (M,π) be a simple Poisson manifold with leaf space B such that $\mathscr{K} = \ker(var^{lin}_{\varpi}) \subseteq TB$ is of constant rank. Then $\mathscr{K}$ defines a regular foliation on B.*

Proof. This follows from the flatness of the Gauss-Manin connection. For any $X \in \Gamma(\mathscr{K})$, we have that $\mathrm{var}^{\mathrm{lin}}_{\varpi}(X) = \nabla_X\varpi = 0$. Therefore, we have for $X,Y \in \Gamma(\mathscr{K})$ that

$$\mathrm{var}^{\mathrm{lin}}_{\varpi}([X,Y]) = \nabla_{[X,Y]}\varpi = \nabla_X\nabla_Y\varpi - \nabla_Y\nabla_X\varpi = \nabla_X(0) - \nabla_Y(0) = 0,$$

so $[X,Y] \in \Gamma(\mathscr{K})$. □

Motivated by Lemma 4.32, we set the following definition.

Definition 4.33. Let (M,π) be a simple Poisson manifold. We say that (M,π) has regular variation if $\ker(\mathrm{var}^{\mathrm{lin}}_{\varpi}) \subseteq TB$ is of constant rank.

A big class of simple Poisson manifold with regular variations come from simple PMCTs.

Theorem 4.34. *Let (M,π) be a simple Poisson manifold with a proper symplectic integration $(\mathscr{G},\Omega)$. Then (M,π) has regular variation. However, it is not true that any simple Poisson manifold with regular variation is a PMCT.*

Proof. If $(\mathscr{G},\Omega)$ is a proper integration of (M,π), we know from Theorem 4.21 that there exists a transverse integral affine structure $\Lambda_{\mathscr{G}}$ on $(M,\mathscr{F})$, hence an integral affine structure on the leaf space B. This determines a flat connection ∇^B on $TB \to B$, and therefore we have an action of the fundamental groupoid $\Pi_1(B) \rightrightarrows B$ on both TB and $\mathscr{H}^2$ by parallel transport. Explicitly, this means the following.

Let $v \in T_bB$ and let $[\gamma] \in \Pi_1(B)$ with $\mathbf{s}([\gamma]) = b$ be represented by a smooth path $\gamma\colon I \to B$. Let $u : I \to TB$ with $u(0) = v$ be the unique parallel path covering γ, hence $[\gamma]\cdot v = u(1)$. It follows that $\mathrm{var}^{\mathrm{lin}}_{\varpi}([\gamma]\cdot v) = \mathrm{var}^{\mathrm{lin}}_{\varpi}(u(1)) =$

$\nabla_{u(1)}\varpi$. At the same time, we have that $\alpha : I \to \mathscr{H}^2, t \mapsto \nabla_{u(t)}\varpi$ is a parallel path, starting at $\mathrm{var}^{\mathrm{lin}}_{\varpi}(v)$ covering γ. Thus,

$$[\gamma] \cdot \mathrm{var}^{\mathrm{lin}}_{\varpi}(v) = \alpha(1) = \nabla_{u(1)}\varpi = \mathrm{var}^{\mathrm{lin}}_{\varpi}([\gamma] \cdot v).$$

Therefore, we conclude that $\mathrm{var}^{\mathrm{lin}}_{\varpi}$ is $\Pi_1(B)$-equivariant, from which it follows that $\mathscr{K}$ is regular.

Not every simple Poisson manifold with regular variation is a PMCT. We know from Theorem 4.17 that proper integrations of a Poisson manifold $(M,0)$ with the zero Poisson structure are in one-to-one correspondence with integral affine structures on M. The Poisson manifold $(S^2,0)$ is simple, and has regular variation since the linear variation is just zero. However, it has no proper integration since S^2 has no integral affine structure. $\square$

Simple Poisson manifolds of regular variation are a special class of simple Poisson manifolds for which the monodromy groups form a transverse integral affine structure on $(B,\mathscr{K})$.

Theorem 4.35. *Let (M,π) be an integrable, simple Poisson manifold with regular variation. Then the monodromy groups $\mathscr{N}_x(M,\pi)$ form a transverse integral affine structure on $(B,\mathscr{K})$.*

Proof. Since M is integrable, the monodromy groups $\mathscr{N}_x(M,\pi)$ form discrete subgroups of T^*_bB for each $b \in B$, so they form a lattice inside $\mathscr{N}_x^{\mathbb{R}} := \mathrm{span}_{\mathbb{R}}(\mathscr{N}_x(M,\pi))$. By (28), we know that $\mathscr{N}^{\mathbb{R}} = \mathscr{K}^\circ$, so the monodromy groups form a lattice inside $\nu^*(\mathscr{K})$. By Theorem 4.19, we have to show that these lattices vary smoothly, and are Lagrangian.

Since $\mathscr{N}_x^\vee(M,\pi) = \left(\mathrm{var}^{\mathrm{lin}}_{\varpi}\right)^{-1}\left(H^2(S_x,\mathbb{Z})\right)$, we can equivalently write the monodromy groups as

$$\mathscr{N}(M,\pi) = \left\{\xi \in T^*B \colon \xi(v) \in \mathbb{Z} \text{ for all } v \in \left(\mathrm{var}^{\mathrm{lin}}_{\varpi}\right)^{-1}\left(\mathscr{H}^2_{\mathbb{Z}}\right)\right\}.$$

Next, let us fix $b \in B$. Since $\mathscr{K}$ is of constant rank, it is a subbundle of TB and hence there exist local sections $\{s_1,\cdots,s_k\}$, defined on an open U around b, of $\mathscr{K}$. The monodromy groups form a smoothly varying full rank lattice in $\nu(\mathscr{K})$. Thus, we have local sections $\{s_{k+1},\cdots,s_n\}$, defined over an open V around b, of $\mathscr{N}(M,\pi)$. The combination of these two frames forms a local frame $\{s_1,\cdots,s_n\}$ over $U \cap V$ of TB.

Let $\sigma : U \cap V \to \mathscr{N}(M,\pi)$ be a local section. Since $\mathscr{N}^{\mathbb{R}} = \mathscr{K}^\circ$, note that $\sigma(v) = 0$ for all $v \in \mathscr{K}$ Next, let V,W be two vector fields defined on $U \cap V$. We expand them as

$$V = \sum_{i=1}^n \alpha_i s_i, \qquad W = \sum_{j=1}^n \beta_j s_j$$

Then we have that

$$\begin{aligned} d\sigma(V,W) &= \mathscr{L}_V\sigma(W) - \mathscr{L}_W\sigma(V) - \sigma[V,W] \\ &= \sum_{i,j=1}^n (\alpha_i d\beta_j(s_i)m_j - \beta_j d\alpha_i(s_j)m_i) - \sigma[V,W] \\ &= -\alpha_i\beta_j\sigma[s_i,s_j]. \end{aligned}$$

The flatness of the Gauss-Manin connection implies not only that $\mathscr{K}$ is involutive, but also that $[s_i,s_j] \in \mathscr{K}$, since

$$\mathrm{var}^{\mathrm{lin}}_{\varpi}[s_i,s_j] = \nabla_{[s_i,s_j]}\varpi = \nabla_{s_i}\nabla_{s_j}\varpi - \nabla_{s_j}\nabla_{s_i}\varpi = 0,$$

where we used that $\nabla_{s_i}\varpi \in \Gamma(\mathscr{H}^2_{\mathbb{Z}})$, so acting another time with the Gauss-Manin connection yields 0. It follows that $d\sigma = 0$, hence the lattice is Lagrangian. $\square$

Remark 4.36. [Comparison to Theorem 4.21] If $(\mathscr{G},\Omega)$ is a proper integration of a simple Poisson manifold (M,π), we know that there exists a transverse integral affine structure $\Lambda_{\mathscr{G}}$ on $(M,\mathscr{F})$, containing the bundle of the monodromy grous. Projecting to the leaf space, $\Lambda_{\mathscr{G}}$ forms an integral affine structure on B, and the monodromy bundle $\mathscr{N}(M,\pi)$ form a Lagrangian lattice inside $\nu^*(\mathscr{K})$, i.e. a transverse integral affine structure on $(B,\mathscr{K})$. If (M,π) is strong $\mathbf{s}$-proper, $\mathscr{K}=0$ and the integral affine structure on the leaf space coincides again with the monodromy lattices.

We are now ready to state the main result of this chapter, which collects together the previous results.

Theorem 4.37. *Let (M,π) be a simple Poisson manifold with regular variation, with leaf space B.*

1. *If $F \subseteq B$ is a submanifold such that $TF \cap \mathscr{K} = \{0\}$. Then $M_F = p^{-1}(F)$ is a simple saturated Poisson submanifold of M, whose linear variation is injective.*

2. *The monodromy groups of M_F satisfy*

$$\mathscr{N}^{\vee}(M_F,\pi_F) = TF \cap \mathscr{N}^{\vee}(M,\pi). \tag{29}$$

3. *M_F is integrable if and only if $\mathscr{N}^{\vee}(M_F,\pi)$ defines a smooth lattice in TF, and if it this is the case, M_F is of strong* $\mathbf{s}$*-proper type, with integral affine structure on F determined by* (29).

By Corollary 4.29, $\mathscr{N}^{\vee}(M_F,\pi_F)$ is a quantity that is straightforward to compute. We are just one step away of constructing a strong compact Poisson submanifold, and this step is performing a symplectic reduction of an integrable Poisson manifold M_F as in Theorem 4.37 by a discrete Poisson action. If G is a discrete group acting freely and properly on M_F through Poisson maps, the quotient is still strong $\mathbf{s}$-proper by Theorem 3.13. Thus, if M_F/G becomes compact, it is automatically of strong compact type. In the next chapter, we will build the example of a Poisson manifold of strong compact type, and show that it fits in the framework of Theorem 4.37.

5 K3-surfaces and Examples of PMSCTs

In this chapter, we will explicitly produce an example of a Poisson manifold of strong compact type, which is not a compact symplectic manifold with finite fundamental group, cf. Example 3.10. We will mainly be following the idea originally put forward by Kotschick, exposed in [Kot05], in a Poisson context. The adaptation of this idea to a Poisson setting and the example of PMSCT are originally due to [Tor13], and further worked out in [Zwa21]. This construction relies on deep properties of the moduli space of marked K3 surfaces. We will briefly summarize what makes K3 surfaces so exceptional, and highlight some of the results that appear in this chapter. Since K3 surfaces are compact Kähler manifolds, we have the famous Hodge Decomposition Theorem that relates the Dolbeault cohomology groups to the de Rham cohomology groups. The de Rham cohomology groups of K3 surfaces are well-known, and the the second cohomology groups is exceptionally rich. Then, there is the strong Torelli theorem, that allows us to produce biholomorphisms out of certain maps between the second cohomology groups of the K3's. Another property that makes K3 surfaces suitable for our discussion is that a deformation of a K3 surface is again a K3 surface. This will allow us to construct a Poisson manifold, with leaf space is the moduli space of a K3 surface and whose symplectic leaves are agian K3 surfaces. In the first sections of this chapter, we mainly collect some facts on K3 surfaces from [BPV84] and [Huy16], and omit the proofs that go beyond the scope of this thesis. In the second part of this chapter, we will discuss the example by Martinez-Torres.

5.1 Differential forms on Complex Manifolds

We will start with a discussion of differential forms on complex manifolds, to set conventions. The following proposition is well-known and will be used throughout this chapter.

Proposition 5.1. *Let X be a smooth manifold. A splitting of the complexified tangent bundle $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$ with $\overline{T^{0,1}X} = T^{1,0}X$ is the same as an almost complex structure $J : TX \to TX$. Moreover, J is integrable if and only if $T^{1,0}X$ is involutive.*

The bundles $T^{1,0}X$ and $T^{0,1}X$ are called the holomorphic and anti-holomorphic tangent bundles, respectively.

Starting with a complex manifold, we can dualize the splitting

$$T^*_{\mathbb{C}}X = (T^{1,0}X)^* \oplus (T^{0,1}X)^*$$

and take wedge products:

$$\Lambda^k T^*_{\mathbb{C}}X = \bigoplus_{p+q=k} \Lambda^p (T^{1,0}X)^* \otimes \Lambda^q (T^{0,1}X)^*.$$

Consequently, we get a decomposition of the differential forms

$$\Omega^k(X,\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(X),$$

where the space $\Omega^{p,q}(X)$ is called the space of (p,q)-forms. The top exterior power of the dual of the holomorphic bundle $K_X = \Lambda^n(T^{1,0}X)^*$ is a line bundle, known as the canonical bundle of X. Note that this bundle is trivial if and only if there exists a nowhere vanishing holomorphic n-form on X. This form is necessarily unique, up to scaling.

If X is a complex manifold, the exterior derivative splits as $d = \partial + \overline{\partial}$, where

$$\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \qquad \overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X).$$

Since $d^2 = 0$, it follows that

$$\partial^2 = 0, \qquad \partial\overline{\partial} + \overline{\partial}\partial = 0, \qquad \overline{\partial}^2 = 0.$$

Since $\overline{\partial}^2 = 0$, we can associate a cohomology theory to the cochain complex

$$\cdots \xrightarrow{\overline{\partial}} \Omega^{p,q-1}(X) \xrightarrow{\overline{\partial}} \Omega^{p,q}(X) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}(X) \xrightarrow{\overline{\partial}} \cdots,$$

called Dolbeault cohomology. Thus, the Dolbeault cohomology groups are defined as

$$H^{p,q}(X) = \frac{\ker(\overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X))}{\mathrm{im}(\overline{\partial} : \Omega^{p,q-1}(X) \to \Omega^{p,q}(X))}.$$

Immediately, the Dolbeault cohomology groups satisfy

$$H^{p,0}(X) = \Omega^p(X,\mathbb{C}), \qquad \overline{H^{p,q}(X)} = H^{q,p}(X). \tag{30}$$

Moreover, there is Dolbeault's theorem saying that

$$H^{p,q}(X) = H^q(X, \Omega^p),$$

where the right-hand side should be interpreted as sheaf cohomology.

There is no topological analogue of the Dolbeault cohomology groups of a complex manifold, in contrast to de Rham cohomology, and they are therefore hard to compute and can be rather wild. The situation is slightly better when X is a compact Kähler manifold.

5.1.1 Kähler manifolds

In this section, we will discuss an imoprtant class of complex manifolds, known as Kähler manifolds. We will first discuss some basic results in complex geometry.

Lemma 5.2. *Let (X,J) be a complex manifold and let g be a Riemannian metric on X. There exists a Riemannian metric on X that is J-orthogonal.*

Proof. Let g be a Riemannian metric on X. Define another metric $\tilde{g}$ on X by $\tilde{g}(X,Y) = g(X,Y) + g(JX,JY)$. It is straightforward that

$$\tilde{g}(JX,JY) = g(JX,JY) + g(J^2X,J^2Y) = g(X,Y) + g(JX,JY) = \tilde{g}(X,Y).$$

□

Whenever we have fixed an orthogonal metric g on a complex manifold (X,J), we can define a non-degenerate (1,1)-form ω, whose real part is

$$\omega(X,Y) = g(JX,Y).$$

Indeed, ω is skew-symmetric since

$$\begin{aligned}\omega(X,Y) &= g(JX,Y)\\ &= -g(X,JY)\\ &= -g(JY,X)\\ &= -\omega(Y,X),\end{aligned}$$

and it satisfies

$$\omega(X,JY) = g(JX,JY) = g(X,Y).$$

Since g is a metric, ω is always non-degenerate. If it is also closed, it is also a symplectic form. Moreover, it is compatible with the complex and the Riemannian structure on X. This brings us to the definition of a Kähler manifold.

Definition 5.3. A Kähler manifold is a complex manifold (X,J), together with a J-orthogonal Riemannian metric g such that the 2-form ω defined by $\omega(X,Y) = g(JX,Y)$ is closed, hence symplectic.

Remark 5.4. There are several equivalent definitions around of a Kähler manifold. This definition is often referred to as the " complex viewpoint" in literature. The other common definition, referred to as the "symplectic viewpoint" starts with a complex manifold (X,J) and a symplectic form ω such that $g(X,Y) = \omega(X,JY)$ defines a Riemannian metric. It is easy to see these definitions are equivalent.

Example 5.5. Important examples of Kähler manifolds are (open subsets of) $\mathbb{C}^n$, and $\mathbb{C}P^n$ with the Fubini-Study form. In particular, any complex submanifold of a Kähler manifold is again Kähler.

The Dolbeault cohomology groups of compact Kähler manifolds are better behaved, since they can be related to the de Rham cohomology groups, by the Hodge Decomposition theorem.

Theorem 5.6 (Hodge Decomposition Theorem, [Huy05])**.** *Let X be a compact Kähler manifold. We have a decomposition of the de Rham cohomology of X in terms of the Dolbeault cohomology groups*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X). \tag{31}$$

Thus, we already see that the Dolbeault cohomology groups of compact Kähler manifolds are always finite dimensional. We set the Hodge numbers $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X)$. The following properties of the Hodge numbers are immediate.

Proposition 5.7. *The Hodge numbers on an n-dimensional compact Kähler manifold satisfy*

1. $h^{p,q} = h^{q,p}$.
2. $b_k(X) = \sum\limits_{p+q=k} h^{p,q}$, *where b_k is the k-th Betti number of X.*
3. $h^{p,q} = h^{n-p,n-q}$.

Proof. Item 1 follows directly from (30), item 2 direct from the Hodge Decomposition Theorem and for item 3, we refer to [GH94]. $\square$

A simple consequence is that the odd Betti numbers of a compact Kähler manifold must always be even. This can for instance be used to prove that the complex manifold $S^3 \times S^1$ has no Kähler structure. There exists an example very interesting compact Kähler manifold with remarkable properties, which is that of a K3 surface. They will be the main topic of the next chapter.

5.2 K3-surfaces

In this section, we recall the basic definitions and properties of K3 surfaces, with a special focus on their topological properties.

Definition 5.8. A K3 surface is a 1-connected, compact, complex surface (i.e. 2-dimensional complex manifold) with trivial canonical bundle.

Whereas the definition of a K3 surface is rather simple, it turns out they are remarkably interesting. The triviality of the canonical bundle implies that there exists a holomorphic, nowhere vanishing 2-form σ which is unique up to scaling. Since a K3 surface is compact, this 2-form must be constant. Thus, the first property we deduce is that if X is a K3 surface, we have $H^{0,2}(X) \cong H^{2,0}(X) \cong \mathbb{C}\sigma$. Note that Definition 5.8 does not include the condition that a K3 surface must be Kähler. However, it follows from the Enriques-Kodaira classification of compact complex surfaces that a K3 surface is always Kähler, since the first Betti number vanishes.

Theorem 5.9 ([BPV84])**.** *A compact, complex surface X is Kähler if and only if $b_1(X)$ is even.*

The proof of this theorem goes far beyond the scope of this thesis, and uses the full Enriques-Kodaira classification of compact complex surfaces. It can for instance be found in [BPV84]. Another remarkable fact of K3 surfaces, is that the underlying smooth manifolds of all K3 surfaces are diffeomorphic as smooth manifolds. There is one simplest model of a K3 surface, which is the Fermat quartic.

Example 5.10 ([Huy16])**.** The underlying manifold of a K3 surface is always diffeomorphic to the Fermat quartic $X \subseteq \mathbb{C}P^3$ is the curve

$$X = \left\{[X_0 : X_1 : X_2 : X_3] \in \mathbb{C}P^3 \colon X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0\right\}.$$

Thus, what really tells K3 surfaces apart is the complex structure on them. The following proposition allows us to classify complex structures on a K3 surface in terms of certain 2-forms.

Proposition 5.11. *Let X_0 be the Fermat quartic. There is a one-to-one correspondence*

$$\{\textit{Complex structures on } X_0\} \overset{1-1}{\longleftrightarrow} \left\{\sigma \in \Omega^2(X_0,\mathbb{C}) \colon d\sigma = 0, \sigma \wedge \sigma = 0, \int_{X_0} \sigma \wedge \overline{\sigma} > 0\right\} / \mathbb{C}^* \qquad (32)$$

Proof. Starting with a complex structure J on X_0, we define the map to the right by sending (X_0, J) to its unique (up to scaling) nowhere vanishing, holomorphic 2-form. This form satisfies the following properties.

1. $\partial\sigma = 0$, since $\Omega^{3,0}(X) = 0$. Moreover, σ is holomorphic so $\overline{\partial}\sigma = 0$ and therefore, $d\sigma = 0$.

2. $\sigma \wedge \sigma = 0$, since $\Omega^{4,0}(X) = 0$.

3. $\sigma \wedge \overline{\sigma}$ is nowhere vanishing, and of top degree. Therefore it is a volume form and

$$\int_{X_0} \sigma \wedge \overline{\sigma} > 0.$$

We conclude that this map to the right is well-defined.

For the map in the other direction, we invoke Proposition 5.1. Let $\sigma \in \Omega^2(X_0, \mathbb{C})$ such that $d\sigma = 0$, $\sigma \wedge \sigma = 0$ and $\sigma \wedge \overline{\sigma}$ is a volume form. We view σ a map $T_{\mathbb{C}}X \to \Omega^1(X_0, \mathbb{C})$, and define $T^{0,1}X_0 = \ker(\sigma)$. The complex dimension is even since σ is skew-symmetric. The condition that $\sigma \wedge \sigma = 0$ implies that σ is degenerate, and σ is non-zero since $\sigma \wedge \overline{\sigma}$ is a volume form. Also, the latter fact implies that $T^{0,1}X \cap \overline{T^{0,1}X} = \{0\}$, so we have an almost complex structure on X_0. This comes from a complex structure if and only if it is involutive. Since $d\sigma = 0$, involutivity follows directly from Koszul's formula.

To see that these maps are each others inverse, observe that a σ on the right becomes the unique nowhere vanishing holomorphic vanishing 2-form on the K3 surface. Thus, it gets mapped to itself again. If we start with a complex structure on a K3 surface, with antiholomorphic tangent bundle $T^{0,1}X$, the complex structure induced by the nowhere vanishing holomorphic 2-form will also have antiholomorphic tangent bundle $T^{0,1}X$. Since the integration of almost complex structures is unique, the two complex structures have to agree. □

5.2.1 Cohomology Groups of the K3 Surface and the Intersection Form

We will now determine the cohomology of the K3 surface X. In this section, we follow [Sal99]. Since a K3 surface is connected, we know that $H^0(X, \mathbb{Z}) \cong H^4(X, \mathbb{Z}) \cong \mathbb{Z}$. Moreover, a K3 surface is 1-connected, so $H^1(X, \mathbb{Z}) = 0$. By Poincaré Duality, we must also have that $H^3(X, \mathbb{Z}) = 0$, so $H^2(X, \mathbb{Z})$ is torsion-free. We compute $H^2(X, \mathbb{Z})$, by explicitly realizing X as the Fermat quartic. Then, we use the following strategy to determine the second Chern class of X, which is due to [Sal99].

Step 1: The first step is to realize the canonical generator h of $H^2(\mathbb{C}P^3, \mathbb{Z})$ as $c_1(\gamma^*)$, where γ is the tautological line bundle over $\mathbb{C}P^3$.

Step 2: Let $\underline{\mathbb{C}}$ be the trivial complex line bundle over $\mathbb{C}P^3$. Since $T_\ell \mathbb{C}P^3 = \mathrm{Hom}(\ell, \ell^\perp)$ and $\mathrm{Hom}(\ell, \ell) = \mathbb{C}$, we have an isomorphism

$$T\mathbb{C}P^3 \oplus \underline{\mathbb{C}} = \gamma^{*\bigoplus 4}.$$

Step 3: The normal bundle ν_X of X is a complex line bundle over X, and satisfies $c_1(\nu_X) = 4\iota^* h$, where $\iota : X \to \mathbb{C}P^3$ is the inclusion. To see this, let $\Sigma \subseteq X$ be a submanifold such that $\iota(\Sigma) \subseteq \mathbb{C}P^3$ represents a 2-dimensional homology class of degree k. Then, the intersection number $X \cdot \Sigma = 4k$. This intersection number coincides with the number of zeros (counted with multiplicity) of a generic section $s : X \to \nu_X$, restricted to Σ. This means that

$$\langle c_1(\nu_X), [\Sigma] \rangle = 4k = 4\langle \iota^* h, [\Sigma] \rangle.$$

Step 4: The cohomology class $\iota^* h^2 \in H^4(X, \mathbb{Z})$ is given by

$$\langle \iota^* h^2, [X] \rangle = 4.$$

To see this, note that h^2 is a generator of $H^4(\mathbb{C}P^3, \mathbb{Z})$, and it is Poincaré dual is a line. Any such line intersects X in 4 points, counted with multiplicity.

With these preparations done, we choose a splitting $T_X\mathbb{C}P^3 \cong TX \oplus \nu_X$. Taking the total Chern classes yields

$$c(T_X\mathbb{C}P^3) = c(TX)c(\nu_X).$$

Since X is a K3-surface, we have that

$$c_1(TX) = c_1(T^{1,0}X) = -c_1(T^{0,1}X^*) = -c_1(K_X) = 0.$$

Moreover, we have by Step 2, that $c(T_X\mathbb{C}P^3) = (1+\iota^*h)^4$ and by Step 3, $c(\nu_X) = 1+4\iota^*h$. Therefore, we have that

$$(1+\iota^*h)^4 = \big(1+\underbrace{c_1(TX)}_{=0}+c_2(TX)\big)(1+4\iota^*h).$$

Solving for $c_2(TX)$ yields

$$c_2(TX) = 6\iota^*h^2. \tag{33}$$

Knowing $c_2(TX)$ allows us to compute the second Betti number $b_2(X)$ of X. If $\chi(X)$ denotes the Euler characteristic of X, we have by Step 4 that

$$b_2(X) = \chi(X) - 2 = \langle c_2(TX), [X]\rangle - 2 = 24-2 = 22.$$

Therefore, we conclude that $H^2(X,\mathbb{Z}) = \mathbb{Z}^{\oplus 22}$. Moreover, we arrange the Hodge numbers in a so-called Hodge diamond, which takes the form

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

There is more that we can say about the second cohomology groups of a K3 surface. Recall that on any 4-dimensional manifold, the intersection form is a unimodular quadratic form on $H^2(X,\mathbb{Z})$. The goal of the rest of this section is to explain what this means, and to understand the properties of this form on a K3 surface.

Definition 5.12. By a lattice Λ, we mean a finitely generated free abelian group. A symmetric bilinear $Q : \Lambda \times \Lambda \to \mathbb{Z}$ is called unimodular if the induced map

$$\Lambda \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}), \, a \mapsto Q(a, \cdot)$$

is an isomorphism.

Two forms Q_0, Q_1 are called equivalent if there exists an isomorphism $T : \Lambda_0 \to \Lambda_1$ such that $Q_0(v,w) = Q_1(Tv, Tw)$ for all $v, w \in \Lambda_0$.

There are several important invariants of these unimodular forms that in favourable situations completely classify all forms.

Definition 5.13. Let $Q : \Lambda \times \Lambda \to \mathbb{Z}$ be a unimodular symmetric form. We define the rank of Q to be the rank of Λ. Moreover, we call Q

1. *Even* if $Q(a,a) \in 2\mathbb{Z}$ for all $a \in \Lambda$, and *odd* otherwise.

2. Of *signature* (p,q) if there exists an orthonormal basis $(e_1, \cdots, e_p, f_1, \cdots, f_q)$ of Λ such that $Q(e_i, e_j) = -\delta_{ij}, Q(f_i, f_j) = \delta_{ij}$ and $Q(e_i, f_j) = 0$.

The index of Q is defined to be $\tau(Q) = p - q$. We call Q positive definite if $p = \mathrm{Rank}(\Lambda)$, resp. negative definite if $q = \mathrm{Rank}(\Lambda)$, and indefinite otherwise.

Let us discuss some basic examples of forms of symmetric bilinear forms that will appear in the classification of indefinite forms.

Example 5.14.

1. The symmetric bilinear form $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $(m,n) \mapsto mn$ is a unimodular, even, positive definite, symmetric bilinear form represented by the matrix
$$\ell = (1).$$

2. The hyperbolic plane H is the unimodular, even symmetric bilinear form $\mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$ represented by the matrix
$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
It is of signature $(1,1)$.

3. The E_8-lattice is the unimodular, even, positive definite symmetric bilinear form $\mathbb{Z}^8 \times \mathbb{Z}^8 \to \mathbb{Z}$, represented by the matrix
$$E_8 = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

4. The K3-lattice is defined as $L = H^{\oplus 3} \oplus (-E_8)^{\oplus 2}$. It is unimodular, even and of signature (3,19). As the name suggests, this lattice is related to the cohomology of the K3 surface.

The next theorem asserts that indefinite unimodular forms are completely classified by their rank, signature and parity. For a proof, we refer to [Ser73].

Theorem 5.15 (Hasse-Minkowski)**.** *Let Q be a unimodular, indefinite symmetric bilinear form.*

1. *If Q is odd, it can be diagonalized over the integers and is thus isomorphic to*

$$m\ell \oplus n(-\ell), \qquad m,n \geq 1.$$

2. *If Q is even, it is isomorphic to*

$$mE_8 \oplus nH, \qquad m \in \mathbb{Z}, n \geq 0.$$

Let X be a compact, oriented, 4k-dimensional manifold, and assume for simplicity that $H^{2k}(X,\mathbb{Z})$ and $H_{2k}(X,\mathbb{Z})$ are torsion-free, so they are finitely generated, free abelian groups. Then, the cup product is a symmetric, bilinear form

$$\cup : H^{2k}(X,\mathbb{Z}) \times H^{2k}(X,\mathbb{Z}) \to H^{4k}(X,\mathbb{Z}) \cong \mathbb{Z}.$$

The map $a \mapsto a \cup -$ is the composition of the isomorphism $H^{2k}(X,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_{2k}(X,\mathbb{Z}),\mathbb{Z})$ from the Universal Coefficient Theorem, and the isomorphism $\mathrm{Hom}_{\mathbb{Z}}(H_{2k}(X,\mathbb{Z})) \to \mathrm{Hom}_{\mathbb{Z}}(H^{2k}(X,\mathbb{Z}))$ from Poincaré duality. Thus, the cup product endows $H^{2k}(X,\mathbb{Z})$ with a unimodular, symmetric bilinear form. This form is called the *intersection form*, and is an important object in the study of 4-manifolds. The intersection form of a 4-manifolds endows the second cohomology groups with the structure of a lattice. We will determine this lattice structure for a K3 surface.

Let Q_{K3} be the intersection form of a K3 surface. Using the Hirzebruch signature formula, we can compute the signature of Q_{K3} using Hirzebruch signature theorem [Sal99]. Explicitly, we have that

$$\begin{aligned} \sigma(Q_{\mathrm{K3}}) &= \frac{1}{3}\langle c_1(TX)^2 - 2c_2(TX), [X]\rangle \\ &= -\frac{2\langle c_2(TX), [X]\rangle}{3} = -16, \end{aligned} \tag{34}$$

so the intersection form is indefinite, so it is classified along the lines of Theorem 5.15. Therefore, we need to determine whether it is even or odd.

Lemma 5.16. *Let X be a compact, complex surface. For all $\alpha \in H^2(X,\mathbb{Z})$, we have that*

$$\langle c_1(TX), PD(\alpha)\rangle = Q_{K3}(\alpha,\alpha) \mod 2,$$

where $\mathrm{PD}(\alpha)$ denotes the Poincaré dual of α.

Proof. This is Lemma 1.45 in [Sal99]. $\square$

With this lemma, we have immediately that Q_{K3} is even since $c_1(TX) = 0$. Therefore, we have by the classification theorem that Q_{K3} is isomorphic to the K3-lattice $L = H^{\oplus 3} \oplus (-E_8)^{\oplus 2}$. We will denote by $L_{\mathbb{R}} = L \otimes \mathbb{R}$

and $L_{\mathbb{C}} = L \otimes \mathbb{C}$ the models for the real and complex second cohomology groups.

The Kähler cone

In this section, we describe the "Kähler cone" of a K3 surface. This section is based on [Huy16] and [Huy05]. The complex structure on a compact Kähler manifold X gives rise to a symplectic form ω (the Kähler form). The cohomology class of such a form lives in

$$H^{1,1}(X,\mathbb{R}) = H^{1,1}(X) \cap H^2(X,\mathbb{R}),$$

and the set of all Kähler classes is called the *Kähler Cone*, denoted by $\mathscr{K}_X$. Hodge theory allows us to see the Kähler cone as an open, convex cone inside $H^{1,1}(X,\mathbb{R})$. we will now describe this cone for a K3 surface.

A Kähler from ω on a K3 surface satisfies $([\omega],[\omega]) = \int_X \omega \wedge \omega > 0$, so every Kähler class is thus contained in the set

$$\{a \in H^{1,1}(X,\mathbb{R})\colon\, (a,a) > 0\}.$$

This set consists of two connected components, which are mapped to one another by $-\mathrm{Id}$. The connected component containing the Kähler cone is denoted by C_X.

Inside $H^{1,1}(X,\mathbb{R})$, we have the Néron-Severi lattice $\mathrm{NS}(X)$, defined as

$$\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z}).$$

This is precisely the image of the map $c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$, coming from the exponential sheaf sequence $0 \to \underline{\mathbb{Z}} \to \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \to 0$. Inside the Néron-Severi lattice, we have a set of roots Δ_X, which are the elements with self-intersection -2:

$$\Delta_X = \{d \in \mathrm{NS}(X)\colon\, (d,d) = -2\}.$$

The Kähler cone can be described in terms of the roots.

Theorem 5.17. *Let X be a K3 surface. Then the Kähler cone of X is*

$$\mathscr{K}_X = \{a \in C_X\colon\, (a,d) \neq 0 \text{ for all } d \in \Delta_X\}.$$

For a proof, we refer to [BPV84].

5.2.2 The Strong Torelli Theorem

In this section, we will describe another of the remarkable properties of a K3 surface, being the strong Torelli theorem. This theorem allows us to integrate certain isomorphisms between the second cohomology groups of K3 surfaces to biholomorphisms between them. It is precisely this theorem that will allow us perform the reduction argument explained in Chapter 4 to pass from a strong **s**-proper Poisson manifold to a strong compact Poisson manifold.

Definition 5.18. Let X, X' be K3 surfaces. A linear map $\phi : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is called a Hodge isometry if

1. It preserves the intersection form, i.e. it is an isometry,
2. The $\mathbb{C}$-linear extension $\phi_{\mathbb{C}} : H^2(X',\mathbb{C}) \to H^2(X,\mathbb{C})$ preserves the Hodge decomposition.

Moreover, we call ϕ effective if it maps a Kähler class of X' to one of X.

Using that $H^{2,0}(X) \cong \mathbb{C}\sigma$, where σ is a nowhere vanishing holomorphic 2-form, it is straihtforward to see that the Hodge decomposition is preserved if and only if $\phi_{\mathbb{C}}$ maps a nowhere vanishing holomorphic 2-form of X' to one of X. Moreover, it is readily verified that ϕ is effective if and only if it maps the Kähler cone of X' to the Kähler cone of X.

We have now defined the right notions to describe the strong Torelli theorem.

Theorem 5.19 (strong Torelli theorem)**.** *Let X, X' be K3 surfaces and let $\phi : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$ be an effective Hodge isometry between K3 surfaces. Then there exists a unique biholomorphism $f : X \to X'$ such that $\phi = f^*$.*

For a proof of this theorem, we refer to [BPV84].

5.3 Deformations of K3 surfaces and Universal Families

In this section, we will discuss several results concerning families of complex manifolds. In particular we will see that all K3 surfaces are deformation equivalent. The results that we collect in this section will be needed when we are constructing moduli spaces of K3 surfaces. In this section, we follow [BPV84].

Definition 5.20. A *smooth family* of compact complex manifolds is a triple $(\mathscr{X}, p, S)$, with $\mathscr{X}, S$ connected complex manifolds and $p : \mathscr{X} \to S$ a proper holomorphic surjective submersion such that the fibers $\mathscr{X}_s := p^{-1}(s)$ are complex submanifolds of $\mathscr{X}$ for all $s \in S$. A morphism between smooth families is a pair of holomorphic maps $f : S_1 \to S_2$ and $F : \mathscr{X}_1 \to \mathscr{X}_2$ such that the following square commutes

$$\begin{array}{ccc} \mathscr{X}_1 & \xrightarrow{F} & \mathscr{X}_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ S_1 & \xrightarrow{f} & S_2 \end{array}$$

Since the map p in Definition 5.20 is proper, it is a fiber bundle by Ehresmann's lemma, and all fibers will be diffeomorphic. However, they do not have to be biholomorphic.

There is a natural notion of a pullback of these families along maps between bases. Let $(\mathscr{X}, p, S)$ be a smooth family and $f : S' \to S$ a holomorphic map. We can form the fiber product

$$\mathscr{X}' = \mathscr{X} \times_S S' = \{(x, s') \in \mathscr{X} \times S' : \ p(x) = f(s')\}.$$

We define $p' : \mathscr{X}' \to S'$ by $p'(x,s') = s'$, and this turns $(\mathscr{X}', p', S')$ into a smooth family of compact complex manifolds, and we say $\mathscr{X}'$ is the pullback of $\mathscr{X}$ by f. Defining $F : \mathscr{X}' \to \mathscr{X}$ by $F(x,s') = x$, we have a commuting square

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "F"] \arrow[d, "p'"] & \mathscr{X} \arrow[d, "p"] \\ S' \arrow[r, "f"] & S \end{tikzcd},$$

so we always have a morphism $(\mathscr{X}', p', S') \to (\mathscr{X}, p, S)$. The main idea is to view these smooth families as deformations of a specific fiber.

Definition 5.21. Let X be a compact complex manifold. A *smooth deformation* of X is a smooth family $(\mathscr{X}, p, S)$ together with a fixed basepoint $s_0 \in S$ and a biholomorphism $\mathscr{X}_{s_0} \simeq X$. We call it complete if for any other smooth deformation $(\mathscr{X}', p', S' \ni s_0')$, there exists a holomorphic map $f : (S', s_0') \to (S, s_0)$ such that $\mathscr{X}'$ is isomorphic to the pullback of $\mathscr{X}$ by f. Moreover, we call it versal if the derivative $(df)_{s_0'}$ is uniquely determined by $\mathscr{X}'$, and universal if f is uniquely determined by $\mathscr{X}$'.

The following theorem well-known theorem by Kuranishi asserts that versal deformations always exist.

Theorem 5.22. *Let X be a compact, complex surface. Then there exists a versal deformation of X.*

This family is called the Kuranishi family, and its existence is proven in [Col66].

Associated to any deformation $(\mathscr{X}, p, S)$ of a compact complex manifold X is the Kodaira-Spencer map KS, which is defined as follows. Since p is a fiber bundle, the normal bundle of X is trivial. Thus, the normal bundle sequence of X reads

$$0 \to T^{1,0}X \to T^{1,0}_X \mathscr{X} \to X \times T^{1,0}_{s_0} S \to 0.$$

Then, we take the long exact sequence in sheaf cohomology of the sheaves of holomorphic sections of these bundles and define the Kodaira-Spencer map as the connecting homomorphism in the long exact sequence

$$\mathrm{KS} : T^{1,0}_{s_0} S \to H^1(X, T^{1,0}X).$$

Furthermore, there are the following facts on the Kuranishi family, for which we also refer to [Col66].

Proposition 5.23. *Let X be a compact complex surface and let $(\mathscr{X}, p, S)$ be the Kuranishi family from Theorem 5.22. Then*

1. *If the map $s \mapsto \dim_{\mathbb{C}} H^1(\mathscr{X}_s, T^{1,0}\mathscr{X}_s)$ is constant, the deformation is versal for all of its fibers.*
2. *If $H^2(X, T^{1,0}X) = 0$, the Kodaira-Spencer map is an isomorphism.*
3. *If $H^0(X, T^{1,0}X) = 0$, there exists a universal deformation of X. Moreover, it has the property that not only the map between the bases, but also between the total spaces is uniquely determined.*
4. *Every versal deformation is isomorphic to the Kuranishi family.*

If X is a compact, complex surface with trivial canonical bundle, there exists a nowhere vanishing holomorphic 2-form on X, If $(\mathscr{X}, p, S)$ is a deformation of X, then (after possibly shrinking S), we can perturb this 2-form a little bit such that it becomes a nowhere vanishing holomorphic 2-form on $\mathscr{X}_s$, so this surface also has a trivial canonical bundle. The exact details of this argument are found in [BPV84].

Collecting all these facts together, we have the following corollary on deformations of K3 surfaces.

Corollary 5.24. *Let X be a K3 surface. There exists a universal deformation of X, all whose fibers are K3 surfaces and whose associated Kodaira-Spencer map is an isomorphism (for all fibers), so the base space has complex dimension 20.*

Proof. Let $(\mathscr{X},p,S)$ be the Kuranishi family, and we shrink it such that all fibers have trivial canonical bundle. Since the fibers of p have the same topology as X (by Ehresmann's theorem), the fibers of p are 1-connected, so each fiber is a K3 surface. Using Serre Duality and the computation of the Hodge numbers for a K3 surface, we see that

$$\begin{aligned} &h^0(X,T^{1,0}X) = h^{1,0} = 0,\\ &h^2(X,T^{1,0}X) = h^{2,1} = 0,\\ &\dim_{\mathbb{C}} H^1(\mathscr{X}_s,T^{1,0}\mathscr{X}_s) = h^{1,1} = 20. \end{aligned}$$

Thus, the Kuranishi family is a universal deformation, and the Kodaira-Spencer map is an isomorphism for all its fibers. Therefore, $\dim_{\mathbb{C}}(T^{1,0}_{s_0}S) = \dim_{\mathbb{C}}(H^1(X,T^{1,0}X)) = 20$, so the base space has complex dimension 20. □

5.4 The Moduli Space of a K3 surface

In this section, we will investigate the moduli space of complex structures associated to K3 surfaces. It turns out that it is most convenient to work with *marked* K3 surfaces (X,ϕ) instead. Those are K3 surfaces together with a marking $\phi : H^2(X,\mathbb{Z}) \to L$, where L is the K3 lattice from Example 5.14. The moduli space for these markings is smooth, and is closely related to the base of the Kuranishi family, via the period map. It turns out that to get a moduli space which is Hausdorff, we need to refine the moduli space by also taking Kähler classes into account. This will give a new period map, that is a diffeomorphism. In this section, we will mainly follow [BPV84] and [Zom14].

5.4.1 Marked K3 Surfaces and the Period Map

We have seen in Proposition 5.11 that smooth structures on a K3 surface are classified by certain complex-valued 2-forms on the underlying manifold. Motivated by this proposition, we define the period domain as follows.

Definition 5.25. The period domain Ω is defined as

$$\Omega = \{[\sigma] \in \mathbb{P}(L_{\mathbb{C}})\colon (\sigma,\sigma) = 0, (\sigma,\overline{\sigma}) > 0\},$$

where $\mathbb{P}(L_{\mathbb{C}})$ is the projectivization of the complexified K3 lattice.

The following proposition is inspired by [Zom14].

Proposition 5.26. *The period domain has the structure of a smooth manifold, with tangent space*

$$T_{[\sigma]}\Omega = \frac{\{v \in \mathbb{C}^{22} : (\sigma, v) = 0\}}{\mathbb{C}\sigma} = Hom(\mathbb{C}\sigma, \mathbb{C}\sigma^{\perp}/\mathbb{C}\sigma)$$

Proof. Note that Ω is open inside the space

$$\Omega' = \{[\sigma] \in \mathbb{P}(L_{\mathbb{C}}) : (\sigma, \sigma) = 0\}.$$

Define $f : \mathbb{C}^{22} \setminus \{0\} \to \mathbb{C}$ by $f(\alpha) = (\alpha, \alpha)$. For $v \in \mathbb{C}^{22}$, we have

$$(df)_{\alpha}(v) = 2(\alpha, v),$$

so f is a submersion. In particular, $f^{-1}(0)$ is a smooth submanifold of $\mathbb{C}^{22}$. The standard $\mathbb{C}^{\times}$ on $\mathbb{C}^{22} \setminus \{0\}$ is free and proper, and restricts to a free and proper action on $f^{-1}(0)$. Thus,

$$\Omega' = \frac{f^{-1}(0)}{\mathbb{C}^{\times}}$$

is a smooth manifold, with tangent space

$$T_{[\sigma]}\Omega' = \frac{\ker(df)_{\sigma}}{\mathbb{C}\sigma} = \frac{\{v \in \mathbb{C}^{22} : (\sigma, v) = 0\}}{\mathbb{C}\sigma} = \mathrm{Hom}(\mathbb{C}\sigma, \mathbb{C}\sigma^{\perp}/\mathbb{C}\sigma).$$

□

Next, we define the moduli space of marked K3 surfaces is defined as follows.

Definition 5.27. A marked K3 surface is a pair (X, ϕ), where $\phi : H^2(X, \mathbb{Z}) \to L$ is an isometry. Two marked K3 surfaces (X, ϕ) and (X', ϕ') are equivalent if there exists a biholomorphism $f : X \to X'$ such that $f^* = \phi^{-1} \circ \phi'$. We set

$$M_1 = \{\text{Marked K3 surfaces } (X, \phi)\}/\sim.$$

When (X, ϕ) is a marked K3 surface, we can naturally extend the marking ϕ to an isometry $\phi_{\mathbb{R}} : H^2(X, \mathbb{R}) \to L_{\mathbb{R}}$ or $\phi_{\mathbb{C}} : H^2(X, \mathbb{C}) \to L_{\mathbb{C}}$. As of now, M_1 is just a set. To obtain a smooth structure, we transport the smooth structure from the period domain to M_1, by what is called the period map.

Definition 5.28. The period map $\tau : M_1 \to \Omega$ is defined as

$$\tau[(X, \phi)] = [\phi_{\mathbb{C}}(\sigma_X)],$$

where σ_X is a nowhere vanishing holomorphic 2-form on X.

This definition is well-defined. If $\phi : H^2(X, \mathbb{Z}) \to L$ is a marking and if σ_X is a nowhere vanishing holomorphic 2-form, then $[\phi(\sigma_X)] \in \Omega$ since we have by Proposition 5.11 that

$$\big(\phi_{\mathbb{C}}(\sigma_X), \phi_{\mathbb{C}}(\sigma_X)\big) = (\sigma_X, \sigma_X) = 0, \qquad (\phi_{\mathbb{C}}(\sigma_X), \phi_{\mathbb{C}}(\overline{\sigma}_X)) = (\sigma_X, \overline{\sigma}_X) > 0.$$

Next, we note that all nowhere vanishing holomorphic 2-form are a non-zero scalar multiple of one another, so they get mapped to the same line under $\phi_{\mathbb{C}}$. Moreover, the definition of the period map is independent of the chosen marking ϕ. If $[(X, \phi)] = [(X', \phi')]$, there exists a biholomorphism $f : X \to X'$ such that $f^* = \phi^{-1} \circ \phi'$. If

$\sigma_{X'}$ is a nowhere vanishing holomorphic 2-form on X', then $f^*\sigma_{X'}$ is a nowhere vanishing holomorphic 2-form on X. Thus, we have that

$$\tau[(X',\phi')] = [\phi'_{\mathbb{C}}(\sigma_{X'})] = [\phi_{\mathbb{C}}(f^*\sigma_{X'})] = \tau[(X,\phi)].$$

Theorem 5.29. *The period map $\tau : M_1 \to \Omega$ is surjective.*

Proof. A simple proof of this theorem can be found in [Siu81]. □

We may also wonder to what extend the period map is injective, and it turns out that it is not. The reason for this is that if $\tau[(X,\phi)] = \tau[(X',\phi')]$, the composite $\phi^{-1}\circ\phi'$ is a Hodge isometry which need not be effective, so it cannot be integrated to a biholomorphism $X \to X'$. Thus, it is not true that M_1 is simply biholomorphic to Ω. We have to do more work to obtain a smooth structure on M_1.

5.4.2 Universal families over M_1

The basic idea in this section is to put a smooth structure on M_1, by gluing together local deformations of marked K3 surfaces.

Let (X,ϕ) be a marked K3 surface, and let $(\mathscr{X}, p, S \ni s_0)$ be a deformation of X whose fibers are K3 surfaces. Without loss of generality, we assume that S is contractible. The marking ϕ for $X \cong \mathscr{X}_{s_0}$ induces smoothly varying markings $\phi_s : H^2(\mathscr{X}_s,\mathbb{Z}) \to L$ for all $s\in S$, in the sense that they are a trivialization of the bundle

$$\mathscr{H}^2 = \bigcup_{s\in S} H^2(\mathscr{X}_s,\mathbb{R}) \to S\times L_{\mathbb{R}}. \tag{35}$$

For this family, we can define the local period map $\tau^\circ : S \to \Omega$ by

$$\tau^\circ(s) = [\phi_{s,\mathbb{C}}(\sigma_s)],$$

where σ_s is the nowhere vanishing holomorphic 2-form on $\mathscr{X}_s$. It is not difficult to see that

$$\mathscr{H}^{2,0} = \bigcup_{s\in S} H^{2,0}(\mathscr{X}_s)$$

is a holomorphic subbundle of $\mathscr{H}^2$. As a direct consequence, the local period map is holomorphic. It is shown in [BPV84] that $(d\tau^\circ)_{s_0} : T^{1,0}_{s_0}S \to T_{\tau^\circ(s)}\Omega$ is holomorphic. Note that we can make the identification

$$T_{[\sigma]}\Omega = \frac{\{v\in\mathbb{C}^{22} :\ (v,\sigma)=0\}}{\mathbb{C}\sigma} = \mathrm{Hom}(\mathbb{C}\sigma, \mathbb{C}\sigma^\perp/\mathbb{C}\sigma).$$

Since $H^{2,0}(\mathscr{X}_s) = \mathbb{C}\sigma_s$, we note that $\tau^\circ(s) = [H^{2,0}(\mathscr{X}_s)] \in \Omega$. For type reasons, we have that

$$H^{2,0}(\mathscr{X}_s)^\perp = H^{1,1}(\mathscr{X}_s) \oplus H^{2,0}(\mathscr{X}_s),$$

so $T_{\tau^\circ(s)}\Omega \cong \mathrm{Hom}(H^{2,0}(\mathscr{X}_s), H^{1,1}(\mathscr{X}_s))$. The cup product induces an isomorphism

$$H^1(\mathscr{X}_s, T^{1,0}\mathscr{X}_s) \otimes H^0(\mathscr{X}_s, \Omega^2\mathscr{X}_s) \xrightarrow{\simeq} H^1(\mathscr{X}_s, \Omega^1\mathscr{X}_s),$$

which can be dualized to identify

$$H^1(\mathscr{X}_s, T^{1,0}\mathscr{X}_s) \xrightarrow{\simeq} \mathrm{Hom}(H^{2,0}(\mathscr{X}_s), H^{1,1}(\mathscr{X}_s)).$$

Under these identifications, $(d\tau^\circ)_{s_0} : T^{1,0}_{s_0}S \to H^1(\mathscr{X}_s, T^{1,0}\mathscr{X}_s)$ coincides with the Kodaira-Spencer map for the deformation. Thus, we conclude that the local period map is a local biholomorphism if and only if the Kodaira-Spencer map is an isomorphism. Combining this information with Corollary 5.24, we deduce:

Theorem 5.30. *Let (X, ϕ) be a marked K3 surface, and let $(\mathscr{X}, p, S)$ be the Kuranishi family for X. After possibly shrinking the base S, the marking ϕ induces markings on all of the fibers, and the local period map is a holomorphic open embedding.*

Since τ° in the theorem above is an embedding, no $(\mathscr{X}_s, \phi_s)$ and $(\mathscr{X}_{s'}, \phi_{s'})$ (for $s, s' \in S$) are isomorphic.

This theorem allows us to put a holomorphic structure on M_1, as follows. We start with the disjoint union, indexed over marked K3 surfaces, of all bases S as in Theorem 5.30, and we identify elements in the base if the corresponding fibers are isomorphic marked K3 surfaces. This forms M_1, and it shows that every element in $b \in M_1$ has a neighbourhood isomorphic to some basis S. Therefore, M_1 has the structure of a 20-dimensional complex manifold. Moreover, part 3 of Proposition 5.23 allows us to glue the families in the same way as the bases, and no two points in the same fiber come together so that over M_1, we obtained a family of marked K3 surfaces. This family is by construction universal. This proves the following theorem.

Theorem 5.31 ([BPV84]). *The space M_1 has the structure of a 20-dimenisonal complex manifold, that is possibly non-Hausdorff, such that the period map $\tau : M_1 \to \Omega$ is a local biholomorphism with respect to the complex structure on M_1. Moreover, there exists a smooth universal family $\mathscr{U} \to M_1$.*

The universal family $\mathscr{U}$ is such that the fiber over $b \in M_1$ is a marked K3 surface (X_b, ϕ_b) with $[(X_b, \phi_b)] = b$.

Thus, we have parametrized the space of marked K3 surfaces as a (non-Hausdorff) complex manifold, which is related to the period domain via the period map τ, which is a local biholomorphism. To solve the issues of the non-Hausdorfness of the moduli space, and the non-injectivity of the period map, we refine the moduli space to include Kähler classes. Using the strong Torelli theorem, this will allow us to produce stronger statements.

5.4.3 The Refined Moduli Space and the Refined Period Map

The idea here is that instead of marked K3 surfaces (X, ϕ), we look at marked pairs $(X, \phi, [\omega])$, which consist of a marked K3 surface (X, ϕ), with a chosen Kähler class $[\omega]$. The following proposition tells us that the collection of marked pairs can be thought of as the total space of a real-analytic bundle over M_1, hence has a real-analytic structure.

Proposition 5.32 ([BPV84]). *The collection of vector spaces*

$$M_2' = \bigcup_{b \in M_1} H^{1,1}(X_b, \mathbb{R})$$

forms a real analytic subbundle of $\mathscr{H}^2$ in (35) *in which the subset M_2 consisting of all Kähler classes is open.*

By this proposition, the set M_2 is a real-analytic manifold. Its dimension is 60, since the real dimension of M_1, and the spaces $H^{1,1}(X_b,\mathbb{R})$ are 20-dimensional. Elements of M_2 should be thought of as equivalence classes of marked pairs $[(X,\phi)]$, together with a chosen Kähler class $[\omega]$ on X..

We will think of M_2 as the refined moduli space. To define a refined period map with domain M_2, we will also refine the period domain. Recall that any Kähler form ω on a a compact Kähler manifold satisfies $([\omega],[\omega])=0$. Moreover, if σ is a nowhere vanishing holomorphic 2-form, we obviously have $([\omega],[\sigma])=0$ for type reasons. Thus, we define

$$K\Omega = \{(k,[\sigma]) \in L_{\mathbb{R}} \times \Omega \colon (k,k)>0, (k,[\sigma])=0\}.$$

However, this is not good enough yet, since we know from Theorem 5.17 that the Kähler classes have to satisfy more relations. Recall that the roots Δ_X of a K3 surface are the elements in the Neron-Séveri lattice $H^{1,1}(X)\cap H^2(X,\mathbb{Z})$ with self-intersection $(d,d)=-2$. For type reasons, the roots are perpendicular to $H^{2,0}$ and $H^{0,2}$. Thus, it is sensible to define

$$K\Omega^0 = \{(k,[\sigma]) \in K\Omega \colon (k,d)\neq 0, (\sigma,d)=0 \text{ for all } d\in L \text{ with } (d,d)=-2\}.$$

The space $K\Omega^\circ$ is called the refined period domain. We have the following proposition.

Proposition 5.33. *The space $K\Omega$ is a smooth manifold of dimension 60 with*

$$T_{(k,[\sigma])}K\Omega = \frac{\{(v,w)\in\mathbb{R}^{22}\oplus\mathbb{C}^{22} \colon (\sigma,v)+(k,w)=0, (\sigma,w)=0\}}{\mathbb{C}(0,\sigma)}.$$

Moreover, the space $K\Omega^\circ$ is open in $K\Omega$.

Proof. First, we prove that $K\Omega$ is smooth and of dimension 60, and we follow the same strategy as before. We define the map

$$g : L_{\mathbb{R}} \times L_{\mathbb{C}} \setminus \{0\} \to \mathbb{C}^2, \qquad (k,\sigma)\mapsto \big((k,\sigma),(\sigma,\sigma)\big).$$

Then $(dg)_{(k,\sigma)}(v,w) = \big((\sigma,v)+(k,w), 2(\sigma,w)\big)$, so g is a submersion and $g^{-1}(0)$ is in particular a smooth manifold of dimension 62. There is a free and proper $\mathbb{C}^\times$ action on $g^{-1}(0,0)$ by $\lambda\cdot(k,\sigma)=(k,\lambda\sigma)$, so the quotient $K\Omega = g^{-1}(0,0)/\mathbb{C}^\times$ is a 60-dimensional smooth manifold, with

$$T_{(k,[\sigma])}K\Omega = \frac{\ker(dg)_{(k,\sigma)}}{\mathbb{C}(0,\sigma)} = \frac{\{(v,w)\in\mathbb{R}^{22}\oplus\mathbb{C}^{22} \colon (\sigma,v)+(k,w)=0, (\sigma,w)=0\}}{\mathbb{C}(0,\sigma)}.$$

The group group of isometries of L, denoted by $\mathrm{Aut}(L)$, is discrete inside $\mathrm{Aut}(L_{\mathbb{R}})\cong O(3,19)$, which acts properly on $K\Omega$. Thus, $\mathrm{Aut}(L)$ acts properly on $K\Omega$. A fortiori, this holds for subroup $W\subseteq \mathrm{Aut}(L)$, generated by the reflections $s_d : L\to L$ for $d\in L$ with $(d,d)=-2, (d,\sigma)=0$, where as usual

$$s_d(x) = x + (x,d)d$$

is the reflection through the hyperplanes H_d orthogonal to these d. In particular, we obtain $K\Omega^\circ$ from $K\Omega$ by removing these hyperplanes. Since W is discrete and acts properly on $K\Omega$, these hyperplanes form a locally finite collection, hence their union is closed. Indeed, every point in $K\Omega$ has an open neighbourhood U such that $U\cap s_d(U)$ for only finitely many s_d, so only finitely many H_d meet U. ☐

Next, we define the refined period map.

Definition 5.34. The refined period map $\tau' : M_2 \to K\Omega^0$ is defined by

$$\tau'(b,[\omega]) = \big(\phi_{b,\mathbb{C}}([\omega]), \tau(b)\big),$$

where (X_b, ϕ_b) is a marked K3 surface such that $[(X_b, \phi_b)] = b$.

By the description of the Kähler cone, we see that the refined period map indeed takes values in $K\Omega^\circ$. The refined period map is a smooth map, since the markings ϕ_b vary smoothly. Moreover, the refined period map is still surjective, and it happens to be injective as well.

Proposition 5.35. *The refined period map $\tau' : M_2 \to K\Omega^\circ$ is injective.*

Proof. Suppose $\tau'(b,[\omega]) = \tau'(b',[\omega'])$. Then we note that $\phi_{b,\mathbb{C}}^{-1}\left(\phi_{b',\mathbb{C}}[\omega']\right) = [\omega]$, so $\phi_b^{-1} \circ \phi_{b'}$ is now a *effective* Hodge isometry (something we could not realize for the normal period map), and can by the Torelli theorem be integrated to a biholomorphism $f : X_{b'} \to X_b$ such that $f^* = \phi_b^{-1} \circ \phi_{b'}$. This implies that $b = b'$, and therefore $[\omega] = [\omega']$. $\square$

Remark 5.36. In fact, the strong Torelli theorem is equivalent to the injectivity of τ'.

In particular, we deduce that $\tau' : M_2 \to K\Omega^\circ$ is a diffeomorphism, and M_2 is Hausdorff. We can form the commuting square,

$$\begin{array}{ccc} M_2 & \xrightarrow{\tau'} & K\Omega^\circ \\ \big\downarrow p & & \big\downarrow q \\ M_1 & \xrightarrow{\tau} & \Omega \end{array}$$

where p and q are just forgetful maps. We obtain a universal family over M_1, by pulling back the universal family over M_2.

Definition 5.37. We define $K\mathscr{U} = (p \circ \tau'^{-1})^* \mathscr{U}$.

This is a real-analytic family over $K\Omega^\circ$, with the property that the fiber over each $b = (k,[\sigma])$ is a marked pair (X,ϕ) together with a chosen Kähler class $[\omega]$ such that $\phi([\omega]) = k$, and $\phi^{-1}([\sigma])$ is the class of a nowhere vanishing holomorphic 2-form. Thus, we can think of $K\mathscr{U}$ as foliated by K3 surfaces. In the next section, we will choose a foliated symplectic form on $K\mathscr{U}$, thereby turning it into a Poisson manifold.

5.5 Building the Poisson manifold: Yau's theorem and the Poisson action

To equip the fibers of $K\mathscr{U}$ with smoothly varying symplectic forms, we use the celebrated result of Yau that allows us to pick representatives of the Kähler forms of a K3 surface, in a smooth manner. To describe this, we recall the relevant Riemannian geometry.

Definition 5.38. Let (X,g) be a Riemannian manifold with Levi-Civita connection ∇, whose curvature we denote by R. The Ricci tensor is defined as

$$\mathrm{Ric}(X_1,X_2) = \mathrm{Tr}(X_3 \mapsto R(X_3,X_1)X_2).$$

We say that g is an Einstein metric if $\mathrm{Ric} = \alpha g$ for some $\alpha \in \mathbb{R}$. If $\alpha = 0$, we call g Ricci flat.

Remark 5.39. The name Einstein metric comes from the fact that a metric is called Einstein if and only if the metric is a solution to the Einstein equations in a vacuum.

On a Kähler manifold with Kähler metric g, we can define the Ricci form of X as

$$\rho(X_1,X_2) = \mathrm{Ric}(JX_1,X_2),$$

which is a (1,1)-form. By the second Bianchi identity, this form is closed. Moreover, it is shown in [Mor04] that ρ represents the first Chern class of X.

If the metric g is Einstein, and if $\rho = \lambda\omega$ for some $\lambda \in \mathbb{R}$, we call g Kähler-Einstein. A special situation is when $\lambda = 0$, in which we call g Calabi-Yau.

The following result we dicscuss is Yau's theorem, which gives that any form representing the first Chern class comes from a Kähler metric.

Theorem 5.40 (Yau's theorem, [Mor04])**.** *Let X be a compact Kähler manifold Kähler form ω and let ρ be a closed (1,1)-form representing the first Chern class of X. There exists a unique Kähler metric g' on X such that ρ is the Ricci form of g, and whose Kähler form ω' is cohomologous to ω.*

Since the first Chern class of a K3 surface vanishes, we obtain a special variant of Yau's theorem.

Corollary 5.41. *Let X be a K3 surface. For any Kähler class k, there exists a unique Ricci flat Kähler metric whose Kähler form belongs to k.*

The fibers of $K\mathscr{U} \to K\Omega^\circ$ are marked K3 surfaces with a specified Kähler class. By this corollary, we can endow the fibers with symplectic forms, whose class is the specified Kähler class. By construction, the Kähler classes on the fibers vary "smoothly" when moving through $K\Omega^0$, since the markings vary smoothly. Moreover, the metric in Yau's theorem is the solution of a differential equation so that the symplectic forms on the fibers also vary smoothly: they form a regular symplectic foliation of $K\mathscr{U}$, hence a regular Poisson structure. Therefore, we have the following result.

Theorem 5.42. *The family $K\mathscr{U}$ admits a regular Poisson structure whose symplectic leaves are the fibers of the bundle $K\mathscr{U} \to K\Omega^0$, and such that the symplectic form on a fiber (X_b,ϕ_b) over $b = (k,[\sigma])$ is the Kähler form of the unique Ricci flat metric with Kähler class $\phi_b^{-1}(k)$.*

Let $\mathrm{Aut}(L)$ denote the group of isometries of the lattice L. Since this action preserves the roots, there is an induced action on $K\Omega^\circ$ by $\gamma\cdot(k,[\sigma]) = (\gamma\cdot k,[\gamma\cdot\sigma])$. We can use the strong Torelli theorem to obtain an action of $\mathrm{Aut}(L)$ on $K\mathscr{U}$ by Poisson maps.

Theorem 5.43. *There is a smooth, equivariant Poisson action of $Aut(L)$ on $K\mathscr{U} \to K\Omega^0$.*

Proof. Let $b = (k, [\sigma]) \in K\Omega^0$, and set $b' = \gamma \cdot t = (\gamma(k), \gamma([\sigma]) \in K\Omega^0$, with fibers (X_b, ϕ_b, k_b) and $(X_{b'}, \phi_{b'}, k_{b'})$, respectively. Let σ_b and $\sigma_{b'}$ be nowhere vanishing holomorphic 2-forms on X_b and X_b. We observe that

$$\phi_{b'}(k_{b'}) = \gamma(k) = (\gamma \circ \phi_b)(k_b)$$
$$\phi_{b'}(\sigma_{b'}) = \gamma([\sigma]) = (\gamma \circ \phi_b)(\sigma_b).$$

Thus, the composite $\phi_b^{-1} \circ \gamma^{-1} \circ \phi_{b'} : H^2(X_{b'}, \mathbb{Z}) \to H^2(X_b, \mathbb{Z})$ is an effective Hodge isometry. By the strong Torelli theorem, there is a unique biholomorphism $f_\gamma : X_b \to X_{b'}$ integrating it. Thus, we obtain a unique biholomorphism between any two fibers in the orbit of each $\gamma \in \mathrm{Aut}(L)$ in $K\Omega^\circ$. Since $H^0(X, T^{1,0}X) = 0$ for any K3 surface X, there are no global holomorphic vector fields on X, hence the f_γ fit in a real-analytic map $F_\gamma : K\mathscr{U} \to K\mathscr{U}$ (for details, see [Mee11]). By the uniqueness part of Torelli's theorem, we have that $F_{\gamma\circ\gamma'} = F_\gamma \circ F_{\gamma'}$. To check that it acts by Poisson maps, we have to check that the f_γ preserve the symplectic forms on the leaves, i.e. that $f_\gamma^*\omega_{b'} = \omega_b$, where ω_b and ω_b' are the Kähler forms of the Kähler metrics g_b and $g_{b'}$ on X_b and $X_{b'}$. By the uniqueness statement in Corollary 5.41, this follows if $f_\gamma^* g_{X'} = g_X$. To show this, we note that $f_\gamma^* g_{b'}$ is a Ricci-flat metric, with Kähler class $f_\gamma^* k_{b'} = k_b$, so we are done. ☐

5.6 Examples of PMSCTs with leaf space $\mathbb{R}^q/\Gamma$

In this section, we present the promised explicit examples of Poisson manifolds of strong compact type, using the results of Chapter 4 and the results on K3 surfaces we have collected in this chapter. The following proposition lies at the heart of the construction.

Proposition 5.44 ([Zwa21]). *Let $f : \mathbb{R}^q \to K\Omega^0$ be an embedding and let $\Gamma \subseteq Aut(L)$ be a subgroup such that*

1. *The first component is given by*
$$f_1(v) = a + \sum_{i=1}^{q} v_i a_i,$$
where $a \in L_\mathbb{R}$ and $\{a_1, \cdots, a_q\}$ are linearly independent in L.

2. *The action of a subgroup Γ preserves the image of f.*

3. *The induced action on $\mathbb{R}^q$ is free and proper, and by integral affine maps.*

Then $M = f^*K\mathscr{U}/\Gamma$ is a Poisson manifold of strong **s***-proper type with integral affine structure on the leaf space $B = \mathbb{R}^q/\Gamma$ determined by Γ. If $B = \mathbb{R}^q/\Gamma$ is compact, M is of strong compact type.*

Proof. We first remark that $M' = f^*K\mathscr{U}$ is a union of symplectic leaves of $K\mathscr{U}$. This is therefore a simple Poisson manifold, with leaf space $f(\mathbb{R}^q) \cong \mathbb{R}^q$. Condition 2 in the theorem implies that Γ acts on M', and this action is through Poisson maps and recall from Theorem 5.43 that the $p : K\mathscr{U} \to K\Omega^\circ$ is Γ-equivariant. Therefore, Γ-action on M' is also free and proper, so M is a simple Poisson manifold with leaves K3 surfaces and with leaf space $B = \mathbb{R}^q/\Gamma$. We will show that M' is strong **s**-proper. Then, it follows by Theorem 3.13 that M is also strong **s**-proper. Moreover, M is compact, hence of strong compact type, if B is compact.

First, we are going to determine the linear variation of the symplectic forms on the fibers of $K\mathscr{U}$. Start with a basis $\{e_1, \cdots, e_{22}\}$ of L. Then, this is in paritcular a basis for $L_\mathbb{R}$, and has the property that $c_i(b) = \phi_b^{-1}(e_i) \in$

$H^2(X_b,\mathbb{Z})$ for all $b \in K\Omega^\circ$. A fortiori, they are all linear independent. The section $\varpi : K\Omega^\circ \to \mathscr{H}^2$ is given by

$$\varpi(k,[\sigma]) = \phi_b^{-1}(k) = \sum_{i=1}^{22} k_i c_i(b).$$

It simply follows that the linear variation is given by

$$\mathrm{var}^{\mathrm{lin}}_{\varpi}(X_b) = \nabla_{X_b}\varpi = \sum_{i=1}^{22}(d\mathrm{pr}^1_i)(X_b)c_i(b) \qquad X_b \in T_bK\Omega^\circ, \tag{36}$$

where $\mathrm{pr}^1_i : K\Omega^\circ \to \mathbb{R}$ defines the projection on the i-th coordinate of the first component of $b=(k,[\sigma])$ with respect to the basis $\{e_1\cdots,e_{22}\}$, i.e. $\mathrm{pr}^1_i(k,[\sigma]) = k_i$. Using the description of $T_bK\Omega^\circ$ from Proposition 5.33, equation (36) reduces to

$$\mathrm{var}^{\mathrm{lin}}_{\varpi}(v_b,w_b) = \sum_{i=1}^{22} v_i c_i(b), \qquad (v_b,w_b) \in T_bK\Omega^\circ.$$

This is equal to zero if and only if $v=0$. Thus, we have that $\mathscr{K}_b = \ker(\mathrm{var}^{\mathrm{lin}}_{\varpi})_b$ is given by

$$\mathscr{K}_b = \frac{\{w \in \mathbb{C}^{22} \colon (k,w) = (\sigma,w) = 0\}}{\mathbb{C}\sigma}, \qquad b=(k,[\sigma]).$$

In particular, we see that $\mathscr{K}$ defines a regular foliation of $K\Omega^\circ$ of codimension 22, so $(K\mathscr{U},\pi_{K\mathscr{U}})$ is simple and has regular variation.

We deduce that

$$\mathscr{N}^\vee(K\mathscr{U},\pi_{K\mathscr{U}}) = \left(\mathrm{var}^{\mathrm{lin}}_{\varpi}\right)^{-1}\left(\mathscr{H}^2_{\mathbb{Z}}\right) = \{(v,w) \in \mathbb{R}^{22}\times\mathbb{C}^{22} \in TK\Omega^\circ \colon v_i \in \mathbb{Z}\}.$$

If $\{\alpha_1,\cdots,\alpha_q\}$ is the standard basis for $\mathbb{R}^q$, observe that

$$T_{f(x)}f(\mathbb{R}^q) = (df_x)(\mathbb{R}^q) = \mathrm{span}_{\mathbb{R}}\{(a_j,(df_2)_x(\alpha_j)\} \subseteq T_{f(x)}K\Omega^\circ.$$

Using that $a_j \in \mathbb{Z}^{\oplus 22}$, we deduce that $T_{f(x)}f(\mathbb{R}^q)\cap\mathscr{N}^\vee_{f(x)}(K\mathscr{U},\pi_{K\mathscr{U}})$ is a smooth lattice. By Theorem 4.37, it follows that M' is a strong $\mathbf{s}$-proper Poisson manifold, with integral affine structure on the leaf space $\mathbb{R}^q$ just given by the standard lattice defined by the α_i. In conclusion, M is a strong $\mathbf{s}$-proper Poisson manifold, with integral affine structure on the leaf space $\mathbb{R}^q/\Gamma$ determined by Γ. This concludes the proof. □

By the proposition above, finding examples of PMSCTs has been reduced to two steps: the first is finding an appropriate embedding $f:\mathbb{R}^q \to K\Omega^\circ$, and the second is finding an appropriate subgroup $\Gamma \subseteq \mathrm{Aut}(L)$, acting on $\mathrm{im}(f)$. In [Zwa21], these two steps have been carried out for the cases $q=1$ and $q=2$ (in principle, one could this up to $q=22$), yielding Poisson manifolds of strong compact type, with leaf space a circle or a torus, respectively. We will now discuss these embeddings.

Let $\{u,v\},\{x,y\}$ and $\{z,t\}$ be three copies of the standard basis of H, so $(u,v)=(x,y)=(z,t)=1$, and all other combinations are zero. Next, let $\{e_1,\cdots,e_8\}$ be a set of real numbers such that the set

$$\{1,e_1,\cdots,e_8,e_1^2,e_1e_2,\cdots e_7^2,e_7e_8,e_8^2\}$$

is linear independent over the integers. Such a set always exists, see [Bes40]. Next, we set $e=(e_1,\cdots,e_8)\in(-E_8)_{\mathbb{R}}$, and scale it such that $|(e,e)|\leq 3$. We set $a=(0,e)$ and $b=(e,0)$ as elements in $(-E_8)^{\oplus 2}_{\mathbb{R}}$.

The following example is originally from [Tor13], and the details are from [Zwa21]. Consider the embedding

$$\begin{aligned} f : \mathbb{R} &\to L_{\mathbb{R}} \times \mathbb{P}(L_{\mathbb{C}}) \\ s &\mapsto (2u+v+sy, [x-su+2y+a+i(z+2t+b)]). \end{aligned}$$

We claim that the image of f is contained in $K\Omega^\circ$. Since $K\Omega^\circ$ is an embedded submanifold of $L_{\mathbb{R}} \times \mathbb{P}(L_{\mathbb{C}})$, $f : \mathbb{R} \to K\Omega^\circ$ is an embedding as well.

Let $s \in \mathbb{R}$, set $f_1(s) = 2u+v+sy$, $f_2(s) = x-su+2y+a$ and $f_3(s) = z+2t+b$. Simple arithmetic shows us that

$$\begin{aligned} (f_2(s), f_2(s)) &= (x-su+2y+a), x-su+2y+a) \\ &= 4(x,y) + (a,a) \\ &= 4 + (e,e) \\ &\geq 1 > 0, \\ (f_2(s), f_3(s)) &= (x-su+2y+a, z+2t+b) \\ &= 0, \\ (f_3(s), f_3(s)) &= (z+2t+b, z+2t+b) \\ &= 4(z,t) + (b,b) \\ &\geq 1 > 0. \end{aligned}$$

These computations show that $[f_2(s) + if_3(s)] \in \Omega$. Moreover, we have that

$$\begin{aligned} (f_1(s), f_2(s)) &= (2u+v+sy, x-su+2y+a) = s(y,x) - s(u,v) = 0 \\ (f_1(s), f_3(s)) &= (2u+v+sy, z+2t+b) = 0, \end{aligned}$$

so $f(s) \in K\Omega$. It remains to check that f lands in $K\Omega^\circ$, i.e. we have to check the condition on the roots. We will do this by contradiction, so suppose there exists $d \in L$ with $(d,d) = -2$ and $(d, f_1(s)) = (d, f_2(s)) = (d, f_3(s)) = 0$. Write

$$d = Au + Bv + Cx + Dy + Ez + Ft + d_1 + d_2,$$

for some integers $A, \cdots, F$ and some d_1, d_2 in a copy of $-E_8$. Since $-E_8$ is even and negative definite, we can write $(d_i, d_i) = -2n_i$ for some $n_i \in \mathbb{Z}_{>0}$. Then, these four equations translate into the following set of equations

$$AB + CD + EF = 1 - n_1 - n_2 \tag{37}$$
$$2B + A + sC = 0 \tag{38}$$
$$D - sB + 2C + (d_2, e) = 0 \tag{39}$$
$$F + 2E + (d_1, e) = 0. \tag{40}$$

Note that (d_1, e) is just an integral combination of the numbers $\{e_1, \cdots, e_8\}$ and since $\{1, e_1, \cdots, e_8\}$ are linear independent over the integers, we deduce from Equation (40) that $F + 2E = 0$. Even more, we deduce that the coefficients in front of the e_i should be zero, so in particular $d_1 = n_1 = 0$.

Next we consider two cases: $C = 0$ and $C \neq 0$. If $C = 0$, it follows from Equation (38) that $2B + A = 0$ and Equation (37) becomes

$$2B^2 + 2E^2 = 1 - n_2.$$

It follows that $B = E = 0$ and $n_2 = 1$. This means that $d_2 \neq 0$, but that implies that

$$D + (d_2, e) = 0,$$

which is absurd, by the linear independence of the set $\{1, e_1, \cdots, e_8\}$.

Thus, we assume that $C \neq 0$. From equation (38), we get that

$$s = -\frac{2B + A}{C}.$$

Combining Equation (37) and (39) yields

$$2B^2 + 2C^2 + 2E^2 + C(d_2, e) = 1 - n_2.$$

Again by the linear independence argument, we infer that $d_2 = 0$, hence $n_2 = 0$. Therefore,

$$2B^2 + 2C^2 + 2E^2 = 1,$$

which is absurd, and this proves the claim that f takes values in $K\Omega^\circ$.

Next, we construct a suitable subgroup $\Gamma \subseteq \mathrm{Aut}(L)$. Define $\phi : L \to L$ by $u \mapsto u, v \mapsto v + y, x \mapsto x - u, y \mapsto y$, and as the identity on the other summands of L. It is not difficult to see that ϕ preserves $(\cdot, \cdot)$, since this only has to be checked on the first two copies of H, where the computation is trivial. Then, we note that

$$\begin{aligned} \phi \cdot f(s) &= (\phi(2u + v + sy), [\phi(x - su + 2y + a + i(z + 2t + b)] \\ &= (2u + v + (s+1)y, [x - (s+1)u + 2y + a + i(z + 2t + b)] \\ &= f(s+1). \end{aligned}$$

Thus, the subgroup $\langle \phi \rangle \subseteq \mathrm{Aut}(L)$ preserves the image of f. Moreover, the induced action on $\mathbb{R}$ is just the standard action of $\mathbb{Z}$ on $\mathbb{R}$. All conditions in the proposition are satisfied, so we obtain a Poisson manifold of strong compact type, whose leaf space is S^1 with the standard integral affine structure.

By a similar kind of embedding, and a similar kind of subgroup, we can form a Poisson manifold of strong compact type, whose leaf space is a torus, with standard integral affine structure.

We define

$$\begin{aligned} g : \mathbb{R}^2 &\to L_{\mathbb{R}} \times \mathbb{P}(L_{\mathbb{C}}) \\ (s, r) &\mapsto (2u + v + sy + rt, [x - su + 2y + a + i(z + 2t - ru + b)] \end{aligned}$$

By a similar sequence of computations, one can show that g actually embeds into $K\Omega^\circ$. Moreover, define $\psi : L \to L$ by $u \mapsto u, v \mapsto v + t, z \mapsto z - u, t \mapsto t$ on two copies of H, and by the identity on the other summands of L. Obviously, $\psi \in \mathrm{Aut}(L)$ exactly by the same argument that $\phi \in \mathrm{Aut}(L)$. Moreover, we compute that

$$\begin{aligned}
\phi \cdot g(s,r) &= (\phi(2u+v+sy+rt), [\phi(x-su+2y+a+i(z+2t-ru+b)]) \\
&= (2u+v+(s+1)y+rt, [x-(s+1)u+2y+a+i(z+2t-ru+b)] \\
&= g(s+1,r) \\
\psi \cdot g(s,r) &= (\psi(2u+v+sy+rt), [\psi(x-su+2y+a+i(z+2t-ru+b)]) \\
&= (2u+v+sy+(r+1)t, [x-su+2y+a+i(z+2t-(r+1)u+b)] \\
&= g(s,r+1).
\end{aligned}$$

Thus the action of $\Gamma = \langle \phi, \psi \rangle \subseteq \mathrm{Aut}(L)$ preserves the image of g, and the induced action on $\mathbb{R}^2$ is the standard action of $\mathbb{Z}^2$. Thus, we obtain a Poisson manifold of strong compact type, whose leaf space is the torus with the standard integral affine structure. Using a similar type of embedding $\mathbb{R}^2 \to K\Omega^\circ$, it is shown in [Zwa21] that any strongly integral 2-torus appears as the leaf space of a Poisson manifold of strong compact type.

6 Outlook

In this final chapter, we will discuss the other constructions of PMSCTs that have been tried, but were unsuccessful for reasons that we will see. This chapter serves as an outlook and inspiration for new ideas.

6.1 Products of K3 surfaces

The success of the strategy in Chapter 5 was an accumulation of several specific and strong results on K3 surfaces. Therefore, the simplest generalizations will be those Poisson manifolds, realized as the pullback of a period map of a universal family, whose leaves are algebraic surfaces that share certain properties of K3 surfaces. In this chapter, we discuss three appropriate compact complex manifolds which have a great resemblance with K3 surface. The first of these is the product of two K3 surfaces.

Let X and Y be K3 surfaces. First, we will the Kähler cone of $X \times Y$. By Künneth's theorem, the wedge product

$$H^{p,q}(X) \otimes H^{r,s}(Y) \to H^{p+r,q+s}(X \times Y), \qquad ([\omega_X],[\omega_Y]) \mapsto [\pi_X^*\omega_X \wedge \pi_Y^*\omega_Y]$$

induces an isomoprhism

$$H^{u,v}(X \times Y) \cong \bigoplus_{\substack{p+r=u \\ q+s=v}} \left(H^{p,q}(X) \otimes H^{r,s}(Y)\right),$$

see [GH94]. In particular, we deduce that

$$H^{1,1}(X \times Y) \cong H^{1,1}(X) \oplus H^{1,1}(Y).$$

If ω_1 and ω_2 are Kähler forms on X and Y respectively, the form

$$\omega = \pi_X^*\omega_1 + \pi_Y^*\omega_2$$

defines a Kähler form on $X \times Y$, with respect to the product metric. Thus, the wedge product defines an injection

$$\mathscr{K}_X \times \mathscr{K}_Y \to \mathscr{K}_{X \times Y}. \tag{41}$$

It turns out this map is also onto. If ω is a Kähler form on $X \times Y$, the restriction of ω to each fibers $X \times \{y\}$ and $\{x\} \times Y$ is again a Kähler form, and this defines an inverse $\mathscr{K}_{X \times Y} \to \mathscr{K}_X \times \mathscr{K}_Y$.[1] The Kuranishi family $(\mathscr{X}, p, S)$ for $X \times Y$ has fibers which are products of K3 surfaces. We infer from Künneth's theorem that

$$\begin{aligned} h^0\big(X \times Y, T^{1,0}(X \times Y)\big) &= 0 \\ h^2\big(X \times Y, T^{1,0}(X \times Y)\big) &= 0 \\ \dim_{\mathbb{C}} H^1\big(\mathscr{X}_s, T^{1,0}\mathscr{X}_s\big) &= 40 \end{aligned}$$

As an analogy to Corollary 5.24, we have the following.

Proposition 6.1. *Let X and Y be K3 surfaces. There exists a universal deformation of $X \times Y$, all whose fibers are products of K3 surfaces and whose associated Kodaira-Spencer map is an isomorphism, so the base space has complex dimension 40.*

[1] This argument actually holds for all compact Kähler manifolds X and Y such that at least one of $b_1(X)$ or $b_1(Y)$ is zero

Next, we will discuss the period map and moduli spaces for the products of K3 surfaces. We will be brief here, since the discussion is dual to the discussion in Chapter 5. By Künneth's theorem, the lattice structures on $H^2(X,\mathbb{Z}) \cong L$ and $H^2(Y,\mathbb{Z}) \cong L$ induce a lattice structure on $H^2(X\times Y,\mathbb{Z})$ isomorphic to $L\oplus L$. Therefore, a marking for $X\times Y$ will consist of a pair (ϕ_X,ϕ_Y), where ϕ_X and ϕ_Y are markings for X and Y, respectively. The moduli space of marked products of K3 surfaces splits as the product of the moduli spaces of marked K3 surfaces, and the period map $\tau_{X\times Y}$ is just the product $\tau_X\times\tau_Y$. The refined moduli space of $X\times Y$ splits as the products of the refined moduli spaces, and is thus isomorphic to the product $K\Omega^\circ\times K\Omega^\circ$, via the refined period map. The upshot of this disucssion is that the method in Chapter 5 constructs the product Poisson manifold $K\mathscr{U}\times K\mathscr{U}\to K\Omega^\circ\times K\Omega^\circ$.

The group $\mathrm{Aut}(L)$ of lattice isometries acts diagonally on $K\Omega^\circ\times K\Omega^\circ$, and the same argument as in Theorem 5.43 shows that it acts smoothly and equivariantly by Poisson maps on $K\mathscr{U}\times K\mathscr{U}\to K\Omega^\circ\times K\Omega^\circ$. We can now apply Proposition 5.44 to find PMSCTs, by constructing appropriate embeddings $\mathbb{R}^q\to K\Omega^\circ\times K\Omega^\circ$.

Proposition 6.2. *Let $g:\mathbb{R}^q\to K\Omega^\circ$ and $h:\mathbb{R}^{q'}\to K\Omega^\circ$ be embeddings and write $f=(g,h):\mathbb{R}^q\times\mathbb{R}^{q'}\to K\Omega^\circ\times K\Omega^\circ$. Let $\Gamma_1,\Gamma_2\subseteq Aut(L)$ be subgroups such that*

1. *The first components of g and h are given by*

$$g(x)=a+\sum_{i=1}^{q}a_ix_i,\qquad h(x)=b+\sum_{i=1}^{q'}b_ix_i$$

   where $a,b\in L$ and $\{a_1,\cdots,a_q\}$ and $\{b_1,\cdots,b_{q'}\}$ are both linear independent in L.

2. *The action of Γ_1 preserves* $\mathrm{im}(g)$ *and the action of Γ_2 preserves* $\mathrm{im}(h)$.

3. *The action of $\Gamma=\Gamma_1\times\Gamma_2$ on $\mathbb{R}^q\times\mathbb{R}^{q'}$ is free, proper and by integral affine maps.*

Then $f^*\left(K\mathscr{U}\times K\mathscr{U}\right)/\Gamma\to\mathbb{R}^q\times\mathbb{R}^{q'}/\Gamma$ is a Poisson a manifold of strong **s***-proper type.*

This proposition allows us to construct a greater class of PMSCTs by combining the possible embeddings from Chapter 5.

6.2 Enriques Surfaces

The next compact Kähler manifold that we discuss is the Enriques surface. These are smooth algebraic varieties that share many of the good properties of K3 surfaces, in the sense that K3 surfaces are a 2-1 cover of an Enriques surface. For instance, all Enriques surfaces are deformation equivalent, and the space of $(1,1)$-forms is 10-dimensional. However, we will see they are not 1-connected, so the leaf space B of the Poisson manifold will no longer be a smooth manifold. Instead, it is an orbifold, and $\mathscr{H}^2\to B$ is an orbibundle. The map $\mathrm{var}_{\varpi}^{\mathrm{lin}}$ becomes an orbibundle map. It is explained in [CFM16] how injective of $\mathrm{var}_{\varpi}^{\mathrm{lin}}$ as an orbibundle map leads to the construction of strong **s**-proper Poisson manifolds in a similar fashion as in Chapter 4. Since the second cohomology groups of Enriques surfaces are still sufficiently rich, there is enough room for $\mathrm{var}_{\varpi}^{\mathrm{lin}}$ to be injective.

The definition of an Enriques surface is very similar to that of a K3 surface.

Definition 6.3 ([BPV84]). An Enriques surface Y is a compact, complex surface such that

1. $K_Y \neq 0$, but $K_Y^{\otimes 2} = 0$,
2. $b_1(Y) = 0$,
3. $H^{2,0}(Y) = 0$.

Since $h^{2,0} = 0$, it is not possible to directly define the period map for an Enriques surface and we need the relation between Enriques surfaces and K3 surfaces here. We will be brief, and omit most of the proofs, as they can be found in [BPV84]. We start with the topological and analytical invariants of Enriques surfaces.

Proposition 6.4 ([BPV84]). *Let Y be an Enriques surface. Then*

1. $h^{1,0} = h^{0,1} = h^{2,0} = h^{0,2} = 0$ *and* $h^{1,1} = 10$.
2. $\pi_1(Y) = \mathbb{Z}/2$ *and the universal cover of Y is a K3 surface.*
3. *The intersection form on $H^2(Y,\mathbb{Z})/Torsion$ is isometric to $K = U \oplus -E_8 \cong \mathbb{Z}^{\oplus 10}$.*

If $(\mathscr{X}, p, S)$ is a deformation of an Enriques surface, we have that (after possibly shrinking S) every fiber is again an Enriques surface. Using the results on the Hodge numbers from Proposition 6.4, we have as an analogy to Corollary 5.24:

Corollary 6.5. *Let Y be an Enriques surface. There exists a universal deformation of Y, all whose fibers are Enriques surfaces and whose Kodaira-Spencer map is an isomorphism. Consequently, the base space has complex dimension 10.*

It is not possible to define a period map for Enriques surfaces since $h^{2,0} = 0$. However, a derived period map, using K3 surfaces, is often defined as follows. Recall that $L = (-E_8)^{\oplus 2} \oplus U^{\oplus 3}$ is the K3 lattice from Chapter 5, and define an automorphism $\rho : L \to L$ by

$$\rho(x,y,z_1,z_2,z_3) = (y,x,-z_1,z_3,z_2).$$

If $\pi : X \to Y$ is the universal cover of an Enriques surface Y, there is always a natural involution $\sigma : X \to X$ that switches the sheets of π. The following lemma is important in the definition for a marked Enriques surface.

Lemma 6.6 ([BPV84]). *Let $X \to Y$ be the universal cover of an Enriques surface Y. There exists an marking $\phi : H^2(X,\mathbb{Z}) \to L$ for X such that*

$$\phi \circ \sigma^* = \rho \circ \phi.$$

This lemma motivates the following definition.

Definition 6.7. [BPV84] A marked Enriques surface is a pair (Y, ϕ) where Y is an Enriques surface and where $\phi : H^2(X,\mathbb{Z}) \to L$ is a marking for X such that $\phi \circ \sigma^* = \rho \circ \phi$.

Next, let $\mathfrak{X} \to S$ be the family of Enriques surfaces from Corollary 6.5, with S contractible, and fix $s_0 \in S$. A marking $\phi_{s_0} : H^2(\mathscr{X}_{s_0}, \mathbb{Z}) \to L$ satisfying $\phi_{s_0} \circ \sigma_{s_0}^* = \rho_{s_0} \circ \phi_{s_0}$ induces a markings ϕ_s satisfying $\phi_s \circ \sigma_s^* = \rho_s \circ \phi_s$ for all $s \in S$. This motivates the definition of the period map for an Enriques surface

$$\tilde{\tau} : S \to \Omega, \tilde{\tau}(s) = [\phi_s(\sigma_s)]$$

Since $\phi_s \circ \sigma_s^* = \rho_s \circ \phi_s$, the image of $\tilde{\tau}$ lies inside the set

$$\Omega^- = \{[\omega] \in \Omega \colon \rho(\omega) = -\omega\}$$

The following theorem is a variant of the local Torelli theorem 5.31.

Theorem 6.8. *The Kuranishi family for an Enriques surface Y_0 is universal at all points in a small neighbourhood U around the base point of Y_0 in S. This base is a 10-dimensional complex manifold, and the period map $\tilde{\tau} : S \to \Omega^-$ is a local biholomorphism on U.*

The period map is not a biholomorphism, and Ω^- is not the period domain of the Enriques surface. Analagously to K3 surface, we have to refine the period domain, and this is done as follows.

Let $L^- = \{\ell \in L \colon \rho(\ell) = -\ell\}$, and define

$$\Gamma = \{g|_{L^-} \colon g \in \mathrm{Aut}(L), \rho \circ g = g \circ \rho\}.$$

It turns out that Γ is an arithmetic subgroup of $\mathrm{Aut}(L^{-1} \otimes \mathbb{R})$. It acts properly discontinuous on Ω^-, and the quotient $D = \Omega^-/\Gamma$ is a quasi-projective variety. For $d \in L^-$ with $(d,d) = -2$, let H_d be the set

$$H_d = \{[\omega] \in \Omega \colon (\omega, d) = 0\}.$$

No point in H_d is in the image of the period map of the Enriques surface. The double coset space

$$D^\circ = D \backslash \left(\bigcup_d H_d \right) / \Gamma$$

is also quasi-projective, and is called the period domain of the Enriques surface. The period map takes values in this space. The following theorem, which can be seen as an analogy to the global Torelli theorem, justifies this nomenclature.

Theorem 6.9 (Global Torelli Theorem for Enriques Surfaces, [BPV84])**.** *The isomorphism class of an Enriques surface is completely determined by its period point. Moreover, all points in D° appear as period points of an Enriques surface.*

Next, one has to construct a refined moduli space KD°, which also encodes the Kähler forms of an Enriques surface. Unfortunately, the description of the Kähler cone of an Enriques surface is far less convenient than the description of the Kähler cone of a K3 surface, and therefore this refined moduli space and hence the refined period map will be hard to describe. Yet, we expect that this refined moduli space and the refined period map are rational [Kon94]. Following the same line of arguing as in Section 5.5 and 5.6, we expect the following conjecture to hold.

Theorem 6.10 (Conjecture 1). *There exists a Poisson manifold $p : \mathfrak{X} \to B$ of strong compact type, whose symplectic leaves are Enriques surfaces and whose leaf space is a quasi-projective subvariety of KD°.*

By Theorem 6.8 and 6.9, there exists a universal family $p : M \to KD^{\circ}$ (by the same argument as in the previous chapter), Moreover, the first Chern class of an Enriques surface vanishes so we can use Yau's theorem to endow the fibers of is family with smoothly varying symplectic forms, hence we endow M with a regular Poisson structure. A proof of this conjecture would include a reformulation of Theorem 4.37 in terms of orbifolds and orbifold bundles, a computation of the kernel of the linear variation of the foliated symplectic form and a construction of appropriate subvarieties of KD°.

6.3 Hyperkähler Manifolds and Hilbert Schemes of Points on a K3 Surface

Rather than looking at other complex surfaces with similar properties as K3 surfaces, we can also look for compact Kähler manifolds of higher dimension, which share some of the properties of K3 surfaces. The possibility that we choose to consider are hyperkähler manifolds. They can be thought of as quaternionic analogies of Kähler manifolds. A hyperkähler manifold is a Riemannian manifold of dimension $4k$, equipped with three distinct complex structures I, J, K satisfying $I^2 = J^2 = K^2 = IJK = -1$ such that the metric is Kähler for each structure. Thus, there are also three Kähler forms, one for each complex structure. Any hyperkähler manifold (X, I, J, K), viewed as a complex manifold (X, I), carries a holomorphic symplectic 2-form $\Omega = \omega_J + i\omega_K$ and thus has a trivial canonical bundle. This implies in particular that every hyperkähler is a Calabi-Yau manifold. Calabi-Yau manifolds are very important in string theory, and their theory is therefore well-understood. It might by worthwhile to investigate the connection between Poisson geometry and Calabi-Yau manifolds further [Col92].

Given a compact Kähler manifold with trivial canonical bundle, Yau's theorem provides the existence of a Kähler metric with vanishing Ricci form. A theorem by Bochner shows that any holomorphic form on a compact Kähler manifold with vanishing Ricci tensor is covariantly constant. Thus for every Kähler manifold with holomorphic symplectic form, for instance a K3 surface, an application of these theorems yields a hyperkähler metric on the manifold. It can be shown that the K3 surface is the only non-trivial example of a hyperkähler manifold in dimension 4. In higher dimensions, it has been discovered by Beauville that the Hilbert scheme of k points on a compact hyperkähler 4-manifold is a hyperkähler manifold of dimension $4k$. This gives two classes of hyperkähler manifolds in higher dimensions: Hilbert schemes of points on a K3 surface, and generalized Kummer varieties. We will focus on the first of the two [Deb18].

Let X be a K3 surface, and denote by $X^{[m]}$ the Hilbert scheme of m points on X. This space is 1-connected, and $H^2(X^{[m]}, \mathbb{Z}) \cong \mathbb{Z}^{\oplus 23}$. The lattice structure is given by

$$H^2(X^{[m]}, \mathbb{Z}) = L \oplus (-2(m-1))I_1,$$

where $I_1 = (1)$. A general Torelli theorem for hyperkähler manifolds was found by Verbitsky [Ver09].

Theorem 6.11 ([Ver09]). *Let X and Y be K3 surfaces, and let m be an integer such that $m-1$ is a prime power. Let $\phi : H^2(Y^{[m]}, \mathbb{Z}) \to H^2(X^{[m]}, \mathbb{Z})$ be an effective hodge isometry. Then there exists a unique isomorphism $\sigma : X^{[m]} \to Y^{[m]}$ such that $\sigma^* = \phi$.*

This version of the Torelli theorem is just as strong as the strong Torelli theorem for K3 surfaces. Moreover, we can define a period map τ for $X^{[m]}$. Therefore, we expect the following conjecture to hold:

Theorem 6.12 (Conjecture 2). *For each natural number m, there exists a simple Poisson manifold $p : \mathfrak{X} \to B$ of strong compact type, whose symplectic leaves are the Hilbert schemes of k points on a K3 surface, and whose leaf space is a smooth submanifold of the moduli space of complex structures on the Hilbert scheme of m points on a K3 surface.*

A proof of this theorem includes the characterization of the moduli space of complex structure of $X^{|m|}$, and the construction of a universal family over this moduli space. Then, one has to check that the first Chern class of $X^{|m|}$ vanishes, and that this implies we can endow the fibers of this universal family with a foliated symplectic form. Finally, one has to formulate and proof an appropriate version of Theorem 4.37, and follow the same line of thought as in Chapter 5.

7 Appendix: Symplectic Foliations

In the appendix, we provide background information on symplectic foliations, following [CFM21]. We start with the definition of a foliation

Definition 7.1. A foliation of codimension q on a manifold M is a partition $\mathscr{F}$ of M into immersed, connected submanifolds of codimension q:

$$M = \bigcup_{L \in \mathscr{F}} L,$$

called the leaves of $\mathscr{F}$, satisfying the local triviality property: every $x \in M$ has an open neighbourhood U such that

$$\mathscr{F}|_U = \{L \cap U : L \in \mathscr{F}\}$$

coincides with the fibers of a submersion $f : U \to \mathbb{R}^q$

Due to the local form of a submersion, the local triviality is equivalent to saying that every point $x \in M$ belongs to a foliated chart (U, χ). These are charts (U, χ) where

$$\chi : U \xrightarrow{\simeq} V \times W, \qquad V \subseteq \mathbb{R}^q, W \subseteq \mathbb{R}^p \, V, W \text{ open}$$

such that $\mathscr{F}|_U$ corresponds to the partition $V \times \{w\}$.

This is not the only way to look at foliations on M, we can also view them as involutive rank q distributions of TM. This correspondence is formulated by the Frobenius theorem.

Theorem 7.2 (Frobenius, [CFM21])**.** *For any manifold M, there is a one-to-one correspondence*

$$\{\textit{Foliations on } M \textit{ of codimension } q\} \longleftrightarrow \{\textit{Distrbutions of } TM \textit{ of rank } q\}$$

Proof. In one direction, the correspondence works as follows. Given a foliation $\mathscr{F}$ on M, one obtains an involutive distribution $\mathscr{D} = T\mathscr{F}$, defined as $\mathscr{D}_x = T_x\mathscr{F} = T_xL$, where L is the leaf through x.

For the other direction, we start with an involutive distribution $\mathscr{D}$. To recover a foliation $\mathscr{F}$ such that $\mathscr{D} = T\mathscr{F}$, we define the leaves set-theoretically as follows. For $x, y \in M$, we say that they are in the same leaf if and only if there exists a smooth path $\gamma : I \to M$ such that $\gamma(0) = x, \gamma(1) = y$ and γ is everywhere tangent to $\mathscr{D}$, i.e. $\dot{\gamma}(t) \in \mathscr{D}_{\gamma(t)}$ for all $t \in I$. Then, one needs to show that these sets carry a smooth structure that turns them into an immersed submanifold of M, satisfying the local triviality condition. □

We can consider geometric objects on foliations, such as Riemannian metrics, vector fields, differential forms, etc. We will focus on the differential forms on a foliation $\mathscr{F}$ on M.

Definition 7.3. We define the foliated k-forms $\Omega^k(\mathscr{F})$ as

$$\Omega^k(\mathscr{F}) = \Gamma(\Lambda^k T\mathscr{F}).$$

Naturally, there is a foliated de Rham differential $d_{\mathscr{F}} : \Omega^k(\mathscr{F}) \to \Omega^{k+1}(\mathscr{F})$, which is defined as

$$\begin{aligned} d_{\mathscr{F}}\omega_{\mathscr{F}}(X_0,\cdots,X_k) = &\sum_{i=0}^{k}(-1)^i \mathscr{L}_{X_i}(\omega_{\mathscr{F}}(X_0,\cdots,\hat{X}_i,\cdots,X_k) \\ &+\sum_{i<j}(-1)^{i+j}\omega_{\mathscr{F}}([X_i,X_j],X_0,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_k). \end{aligned}$$

It is readily verified that $d_{\mathscr{F}}^2 = 0$. Thus, it makes sense to define foliated cohomology as

$$H^\bullet(\mathscr{F}) = \frac{\ker(d_{\mathscr{F}})}{\mathrm{im}(d_{\mathscr{F}})}.$$

In general, foliated cohomology groups are infinite dimensional and hard to compute. Yet, they can still contain valuable geometric information. Note that a foliated form $\omega_{\mathscr{F}}$ can be seen as a sequence of smoothly varying forms $\{(L,\omega_L)\colon L \text{a leaf}\}$, and that $\omega_{\mathscr{F}}$is $d_{\mathscr{F}}$ closed if and only if it is leafwise closed for all leaves. However, a foliated form that is leafwise exact does not have to be exact as a foliated form. This is what makes foliated cohomology groups so complicated.

A foliated 2-form $\omega_{\mathscr{F}} \in \Omega^2(\mathscr{F})$ induces a bundle map

$$\omega_{\mathscr{F}}^\flat : T\mathscr{F} \to T^*\mathscr{F}, \qquad X \mapsto \iota_X \omega_{\mathscr{F}}.$$

We call $\omega_{\mathscr{F}}$ non-degenerate if this map is an isomorphism. If it is furthermore $d_{\mathscr{F}}$-closed, we call it a foliated symplectic form. Note, again, that a foliated symplectic form is just a collection of smoothly varying forms on the leaves of the foliation. This brings us to the definition of a symplectic foliation.

Definition 7.4. A symplectic foliation on a manifold M is a pair $(\mathscr{F},\omega_{\mathscr{F}})$ consisting of a foliation $\mathscr{F}$ on M and a foliated symplectic form $\omega_{\mathscr{F}}$.

So far, all foliations we have described have leaves of the same dimension, and are therefore also frequently called regular foliations. In Poisson geometry, the foliations that we encounter need not be regular, so we need to define singular foliations.

The notion of a singular foliation is more subtle than one would expect in first instance. There is still a partition of M into leaves, of varying dimensions, but this does not capture all information. The key idea to define singular foliations is inspired by the Frobenius theorem, and is to characterize singular foliations via vector fields tangent to the leaves.

We call $C^\infty(M)$-submodule $\mathscr{V} \subseteq \mathfrak{X}(M)$ involutive if it is closed under the Lie brakcet of vector fields, i.e. if it is a Lie subalgebra of $\mathfrak{X}(M)$. Moreover, we call $\mathscr{V}$ local if it satisfies one of the following, equivalent, conditions.

1. If $X \in \mathfrak{X}(M)$ is locally in $\mathscr{V}$, then $X \in \mathscr{V}$.

2. If $X \in \mathfrak{X}(M)$ satisfies $fX \in \mathscr{V}$ for all compactly supported $f \in C_c^\infty(M)$, then $X \in \mathscr{V}$.

3. $\mathscr{V}$ is closed under locally finite sums.

There is a one-to-one correspondence between local modules of vector fields $\mathscr{V} \subseteq \mathfrak{X}(M)$ and sheaves of submodules of vector fields $\mathfrak{V} \subseteq \mathfrak{X}_M$, by setting

$$\mathscr{V} = \Gamma(M, \mathfrak{V}), \qquad \Gamma(U, \mathfrak{V}) = \{X \in \mathfrak{X}(U) \colon fX \in \mathscr{V} \text{ for all } f \in C_c^\infty(U)\}.$$

We say that a local submodule $\mathscr{V}$, corresponding to a sheaf $\mathfrak{V}$, is locally finitely generated if each $x \in M$ has an open neighbourhood U and sections $X_1, \cdots, X_k \in \Gamma(U, \mathfrak{V})$ such that

$$\Gamma_c(U, \mathfrak{V}) = C_c^\infty(U)X_1 + \cdots + C_c^\infty(U)X_k.$$

We have now all the tools to define singular foliations.

Definition 7.5. A singular foliation on a manifold M is a local submodule of vector fields $\mathscr{V} \subseteq \mathfrak{X}(M)$ that is involutive and locally finitely generated. The associated singular tangent distribution is defined by

$$T_x\mathscr{V} = \{X_x \colon X \in \mathscr{V}\} \subseteq T_xM.$$

A leaf of $\mathscr{V}$ is a maximal, connected, immersed submanifold $L \subseteq M$ such that $T_xL = T_x\mathscr{V}$ for all $x \in L$.

We have the following Frobenius theorem for singular foliations.

Theorem 7.6 ([CFM21]). *Let $\mathscr{V}$ be a singular foliation on M. Then every $x \in M$ belongs to a unique leaf $L \subseteq M$, and the leaves are initial submanifolds.*

We end with discussing some examples of singular foliations.

Example 7.7. Any regular foliation $\mathscr{F}$ on a manifold M is also a singular foliation. To see this, we define $\mathscr{V} = \mathfrak{X}(\mathscr{F})$, and this is an involutive, locally finite submodule of $\mathfrak{X}(M)$. The leaves of $\mathscr{V}$, seen as singular distribution, coincide with the leaves of $\mathscr{F}$.

For us, the most important example of a singular foliation comes from Poisson geometry.

Example 7.8. Let (M, π) be a Poisson manifold. The local submodule $\mathscr{V} \subseteq \mathfrak{X}(M)$ spanned by the Hamiltonian vector fields X_f defines a singular foliation on M. The leaves are precisely the symplectic leaves of (M, π).

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