Universiteit Utrecht

Faculteit Bètawetenschappen

Master's thesis

Chiral Matter in F-Theory Compactifications

Christoph Sieling

First Supervisor : Prof. Dr. Thomas Grimm
Institute for Theoretical Physics
Second Supervisor : Dr. Umut Gürsoy
Institute for Theoretical Physics

Abstract

String theory provides a framework to construct quantum theories of gravity that are UV-complete. It is a key interest of string phenomenology to realize verifiable models of particle physics such as extensions of the Standard Model in this framework. F-theory provides a technique to translate the gauge theory of type IIB superstring theory on orientifolds into geometric quantities, strictly speaking, into singularities of an elliptically fibered Calabi-Yau fourfold. We review a new method to compute the number of chiral particles, called the chiral index, in terms of a background flux and the base manifold in a general way. This method utilizes the intersection theory of resolved elliptic fourfolds, which is deeply connected to the gauge theory and the chiral index. This intersection theory also appears in the log-monodromy matrices associated with the singularities at large complex-structure in the moduli space of the mirror manifold of the resolved elliptic fourfold. Our goal is to examine how much information about the chiral index can be extracted from this large complex-structure regime.

Contents

1 Introduction

Classical physics provides theories that are reasonably good in our everyday life. But when we go to very small or large scales, or doing very precise measurements, classical results become inaccurate and new phenomena occur which cannot be explained classically. For very small scales, classical physics can be replaced by quantum physics, whose predictions fit enormously well with the experiments in particle physics. Quantum physics is also consistent with the results on the everyday scale, because in the limit $\hbar \to 0$ we obtain the classical theory. In a similar fashion, general relativity successfully replaced classical physics on large scales and its corrections to the classical theory are hardly seen in everyday life.

It is a natural to ask for a unifying theory of quantum gravity. This would be a theory which is supposed to coincide with a theory of general relativity on large scales and quantum theory on small scales. String theory appears to be a promising framework to derive candidates for such a unifying theory. The superstring theories, that we want to consider, demand ten spacetime dimensions. In order to fit the theory with our observation of four spacetime dimensions, we compactify the theory on a six dimensional manifold, which is a complex threefold.

In this thesis we focus on the quantum physics of some of these string theories. This is a part of string phenomenology [1]. In particular, we will investigate quantum field theories in the framework of type IIB string theory.

Currently, the most successful quantum field theory of particle physics is the Standard Model, which has a gauge symmetry of $SU(3) \times SU(2) \times U(1)$. But for now, we can just conduct particle physics experiments below a certain energy level, which is currently ~ 13.6 TeV. So it might be possible to turn out that the Standard Model is just a part of another quantum theory. The Georgi–Glashow model is a famous candidate of such a theory and it has a gauge symmetry of $SU(5)$. We take a closer look at this model throughout this thesis.

A crucial aspect in the Standard Model is the existence of chiral matter. These are particles, where only the left or right handed field transforms in certain gauge representations. We are in particular interested in the number of chiral particles, called the chiral index, in type IIB string theories.

Our main framework for analyzing type IIB string theory in regard to their particle physics implications is F-theory [2, 3]. We can roughly think of the idea of F-theory as translating the physical quantities of type IIB string theory into geometrical quantities. F-theory has been successfully used for constructing string theories that comprise the Standard Model of particle physics [4, 5]. The Georgi–Glashow model has also been studied in the F-theory context [6, 7]. Using the mathematical tools of F-theory, it is rather easy to model more gauge theories in this context.

For string phenomenology, not only the gauge group is important but also the matter content, especially the chiral matter. Computing the chiral index in F-theory turns out to be rather complicated, but it has been done for some concrete examples of base threefolds (see for example [8]).

In [9], Jefferson, Taylor and Turner describe a novel approach for the computation of the chiral index, which works for arbitrary base threefolds. We will review the entire derivation of this approach for non-abelian gauge algebras.

The chiral index depends on a background flux G_4. This background flux has to satisfy a number of consistency conditions. We write the G_4 flux in terms of the remaining degrees of freedom in the canonical basis for the $SU(5)$ model. In [9], the basis is given by the remaining degrees of freedom, but for phenomenological reasons it is useful to have a description in the canonical basis. To the author's knowledge, this has not been done before.

It turns out that lots of the geometric information needed for the chiral index lies in the intersection theory of a smooth Calabi-Yau fourfold in terms of quadruple intersection numbers. These intersection numbers also appear in the context of singularities in complex-structure moduli space [10].
A useful framework for studying these singularities is asymptotic Hodge theory, which provides generalizations of Hodge theory. It has been successfully applied to the distance conjecture in the swampland program [11, 12]. The appearance of the quadruple intersection numbers in the context of asymptotic Hodge theory gives rise to the question of a deeper relation with the chiral index. In this thesis we set the stage for further investigations on the connections between the chiral index and asymptotic Hodge theory. In particular, we investigate to which extent the chiral index is based on the quadruple intersection number and discuss information about the intersection numbers extracted from Hodge-Deligne diamonds, which are used to determine the kind of singularity in the moduli space.

The structure of the thesis is as follows: In section 2, we give an overview of string theory and F-theory. We also show how we can realize F-theory in the dual M-theory picture and how we can translate some of the physics of type IIB superstring theory into geometrical quantities. In section 3, we deepen this focusing purely on the mathematical side. We see how gauge algebras and their representation can be associated to singularities in an elliptic fibration. Moreover, we introduce the crucial tool of computing quadruple intersection numbers in terms of triple intersection numbers. Section 4 is dedicated to show how these gauge algebras and the representations are realized in the physical theory. We also discuss how chiral matter and the chiral index are realized from the geometric quantities. In section 5, we review the novel approach of [9] to compute the chiral matter spectrum by matching the Chern-Simons action of the M-theory side with a contribution coming from a one loop correction. In section 6, we review the basics of asmyptotic Hodge theory and investigate possible connections to the chiral matter spectrum. Finally, we conclude what we did and discuss possible further directions.

2 Introduction to F-theory

The goal of string theory is to formulate a UV-complete theory of quantum gravity that is consistent with our observable universe as we know it for low energies. The idea is that all the known fields of elementary particles come from one-dimensional objections called strings, whose size is in order of the Planck length ℓ_p. In order to preserve Poincare symmetry when quantizing the theory, we have to consider additional dimensions if we want the spacetime to contain our observable three space and one time dimension. The compactification of the additional dimensions opens up the aspect of compactifications in string theory, which has become one of the most challenging aspects in the field. Two of the most famous theories of massless string modes in ten dimensions have $\mathcal{N} = 2$ spacetime supersymmetry and are called type IIA and type IIB superstring theory. Especially dealing with D-branes, which are objects where the strings can end, is highly complicated if one does it directly. The idea of F-theory is therefore, to encode the gauge symmetries arising from D-branes and the axion dilaton in geometric quantities. One can think of F-theory as a translation of physics into algebraic geometry.

In this chapter, we want to give a brief overview over this translation - starting with a brief review of type IIB superstring theory in section 2.1 and how it can be obtained from M-theory on a elliptically fibered fourfold in section 2.2. The elliptic fibration will be discussed in more detail in section 2.3. The main sources for this chapter can be found in the reviews [13], [14].

2.1 Superstring Theory

Review of Type IIB Superstring Theory

We will start by briefly reviewing some basic superstring theories. More details can be found in [15], [16].

In ten spacetime dimensions, there are only five known superstring theories. We will mostly be interested in the theories with $\mathcal{N} = 2$ supersymmetry called type IIA and type IIB. We will focus on the latter one.

The type IIB superstring theory contains closed and open strings. The string dynamics can be decomposed into oscillation modes, which can give us the fields we are interested in. We are only interested in the massless modes, because the mass of the fields scales with ℓ_p^{-1}, which is way too heavy to detect. These effective theories at the energy scale much lower than the string energy scale are called supergravity theories and are of interest for us in order to make contact with the known physics.

Depending on the boundary conditions, we can distinguish the fields in the theory between R-R and NS-NS sectors. The mixed sectors R-NS and NS-R correspond to fermions.

The theory is called superstring theory, because it admits a special kind of symmetry, called supersymmetry. Supersymmetry is a local symmetry, where the local parameter is a spinor with spin 1/2. In the type IIB theory, we can have two supersymmetry generators, thus we say it has $\mathcal{N} = 2$ supersymmetry. One can very roughly think about supersymmetry as the shift of a boson by a fermion and the other way around. Therefore, we may consider only the bosonic part, because, due to supersymmetry, the fermionic part can be obtained as well.

The NS-NS sector comprises the fields

- the dilaton Φ, whose expectation value $\langle\Phi\rangle$ appears as well in the string coupling constant $g_s = e^{\langle\Phi\rangle}$,
- the metric tensor g
- and the two-form B_2.

In the R-R sector, we have the p-form fields C_p for $p = 0, 2, 4$. It is common to introduce the axion-dilaton as $\tau := C_0 + ie^{-\phi}$ and the field strengh $H_3 = \mathrm{d}\, B_2$. By introducing in addition the notation

$F_5 = \mathrm{d}\, C_4 - \frac{1}{2} C_2 \wedge \mathrm{d}\, B_2 + \frac{1}{2} B_2 \wedge \mathrm{d}\, C_2$ and $G_3 = \mathrm{d}\, C_2 - \tau\, \mathrm{d}\, B_2$, the ten dimensional action in the long wavelength limit reads [13]

$$\frac{1}{2\pi} S_{IIB} = \int \mathrm{d}^{10} x \sqrt{-g} \left(R - \frac{\partial_\mu \tau \partial^\mu \overline{\tau}}{2 \operatorname{Im} \tau^2} - \frac{1}{2} \frac{|G_3|^2}{\operatorname{Im} \tau} - \frac{1}{4} |F_5|^2 \right) + \frac{1}{4i} \int \frac{1}{\operatorname{Im} \tau} C_4 + G_3 \wedge \overline{G}_3, \tag{2.1}$$

where $|F_p|^2 = \frac{1}{p!} F_{\mu_1 \ldots \mu_p} F^{\mu_1 \ldots \mu_p}$. The condition $F_5 = * F_5$ needs to be supplemented on the level of the equations of motions.

Some Dualities

The action in equation (2.1), exhibits a $SL(2, \mathbb{Z})$-symmetry[1][17]. The group $SL(2, \mathbb{Z})$ consists of matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with}\, a, b, c, d \in \mathbb{Z} \ \text{ and}\, ad - bc \neq 0. \tag{2.2}$$

The group acts on the fields by

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \qquad \text{and} \qquad \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \mapsto M \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}. \tag{2.3}$$

The other fields transform trivially under the group action.
As mentioned above, there are five superstring theories in ten dimensions. They are not completely isolated theories but rather connected by a web of dualities. A duality is a identification of two theories, which means that it is possible to rewrite one theory into the other.
One of these dualities, called T-duality, relates the type IIA and type IIB superstring theories. More precisely, if we consider type IIA superstring theory on a circle S^1_B with radius r_B, we can rewrite the theory to obtain type IIB superstring theory on a circle with radius $\tilde{r}_B = \alpha' r_B^{-1}$ [16], where α' is called the Regge slope parameter.

Compactifications

There are a couple of issues with the presented type IIB theory. First, it is a ten dimensional theory, while our universe has only four observable dimensions. Secondly, we would like to obtain models with $\mathcal{N} = 1$ supersymmetry, which are also called minimal supersymmetric. The reason to demand $\mathcal{N} = 1$ are mainly of phemenological type. For instance the minimal supersymmetric Standard Model is proposed to solve the hierarchy problem amongst many other things [18].
Moreover, the Einstein-gravity within the theory is made on top of a Minkowski space, which means that without any masses, the spacetime would be flat. In our current universe, an acceleration of space has been observed, which suggest a De Sitter spacetime background instead of a Minkowski background. This issue will not be addressed further here.

In order to obtain a theory with four observable ($\sim$ large) spacetime dimensions, we consider the remaining six spacetime dimensions to be compact, i.e. they form a compact manifold B_3. The usual ansatz to think about the ten dimensional spacetime $\mathcal{M}$ is as the product space $\mathcal{M} = \mathbb{M}_{1,3} \times B_3$.
In order to preserve supersymmetry, the manifold, on which we compactify, needs to allow for covariant

[1]The action is even invariant under $SL(2, \mathbb{R})$ on the classical level, but gets broken to $SL(2, \mathbb{Z})$ because of D(-1)-instanton effects[13].

constant spinors. In order to satisfy this flatness condition, we consider the manifold to be a Calabi-Yau manifold. The size of these manifolds has to be very small and by compactifying the fields, massive fields occur with masses that scale with the inverse size of the manifold. These fields need to be integrated out [2].
The geometry of a manifold is given by the metric. In a theory of gravity, this metric is a dynamical quantity and hence, if we compactify a gravity theory on a compact manifold, the geometry of the manifold might vary over the non-compact spacetime. The geometry of a manifold can be roughly understood as its 'size' and 'shape'. For a fixed topology the geometry is given by a bunch of parameters. These parameters then have to be viewed as dynamical fields and are called moduli. The space of all these moduli is called the moduli space. The moduli space can be decomposed in the Kähler moduli space and the complex structure moduli space, which are roughly generalizations of 'size' and 'shape'.
By compactifying type IIB (or type IIA) on a Calabi-Yau threefold (i.e. three complex dimensions), we get a four dimensional theory with $\mathcal{N} = 2$ supersymmetry generators. To obtain the desired $\mathcal{N} = 1$ theory, we perform a orientifold projection.
A string is paramertized by a coordinate σ. The map $\sigma \mapsto -\sigma$ reverses the orientation of the string. Type IIB superstring theory is symmetric under this change of the orientation (it corresponds to the $\mathbb{Z}/2\mathbb{Z}$-part of the $SL(2,\mathbb{Z})$ symmetry above). Therefore we can divide out the orientation of the oriented strings in the type IIB theories. The resulting theory has $\mathcal{N} = 1$ supersymmetry (see [15] 17.2 and 15.4, [14] page 9). The orientifold projection also acts on spacetime itself and after the orientifold projection, so-called orientifold planes (for example O7 branes) at the fixed points of the action can occur.

2.2 F-theory via M-theory

M-theory is a conjectured theory whose low energy effective action is supposed to be the unique eleven dimensional supergravity theory [20]. Although a full, concrete description is still missing, there are many hints for its existence. For instance, the 10d supergravity theory arising from type IIA superstring theory can be obtained by compactifying the 11d supergravity theory on a circle. The radius of the circle can be identified with the string coupling. Therefore we reach the perturbative regime in the type IIA supergravity theory by shrinking the circle to zero size. The idea of M-theory is that this also holds in the UV-completion, which means that M-theory is a UV-complete theory that, compactified on a circle, gives the type IIA superstring theory. Since M-theory is a spacetime supersymmetric theory in eleven dimensions with $\mathcal{N} = 1$ supersymmetry, we can examine the possible states, called BPS states. These are the so-called M2-branes and M5-branes, which are dynamical two and five dimensional objects. Especially the M2-branes will be of great interest later.
Since a UV-description of M-theory is still missing, we will work with the eleven dimensional supergravity description instead. The bosinic part of the action reads [21]

$$S_{11} = \frac{1}{2}\int_{\mathbb{R}^{1,10}} \sqrt{-g}R - \frac{1}{4}\int_{\mathbb{R}^{1,10}} G_4 \wedge * G_4 \underbrace{-\frac{1}{12}\int_{\mathbb{R}^{1,10}} C_3 \wedge G_4 \wedge G_4}_{=:S^{11}_{CS}}, \tag{2.4}$$

where R is the scalar curvature and $G_4 = \mathrm{d}\, C_3$ is the field strength associated to the three form C_3. We denote the latter part of the action as S^{11}_{CS} for later purpose. The correction in higher orders of the Planck length also includes a coupling of the M2-branes to the C_3 field via [13]

$$S_{M2} = 2\pi \int_{M2} \sqrt{-g} + 2\pi \int_{M2} C_3. \tag{2.5}$$

[2] In most cases, this is consistent with setting the massive fields to zero [19].

Next we consider M-theory compactified on a Torus $\mathbb{T} = S^1_A \times S^1_B$ consisting of two circles with radii r_A and r_B. Now we can run through a chain of dualities. First, we know that M-theory on the S^1_A circle gives us type IIA string theory. In the shrinking limit of the circle S^1_A, we reach the pertubative regime of type IIA, which is still compactified of the circle S^1_B. Using T-duality, we can identify this theory with the type IIB string theory compactified on a circle $\tilde{S}^1_B$ which has the radius $\tilde{r}_B = \frac{\ell_s^2}{r_B}$ with ℓ_s^2 the string strength. Thus, in the limit $r_B \to 0$, we obtain Type IIB string theory in the decompactification limit. Therefore, M-theory compactified on a torus can be identified with type IIB string theory in the limit where the torus shrinks to zero, called the F-theory limit.
Later, in section 4, we will see how the C_3 and G_4 form fields and the M2-brane enter in the type IIB description under this chain of dualities.
This chain of dualities still works when promoted fiberwise to a compactified spacetime i.e. we consider M-theory on a six dimensional compact Kähler manifold B_3 with a fibration whose fibers consists of a torus. More precisely, the eleven dimensional spacetime takes the form $\mathbb{R}^{1,2} \times X_4$, where X_4 is a torus fibration. Moreover, we want X_4 to be an elliptically fibered Calabi-Yau fourfold, which implies in addition the existence of a rational section. The Calabi-Yau condition is necessary for preserving spacetime supersymmetry. A careful analysis shows, that the shrinking limit of the elliptic fibration gives us the $\mathcal{N} = 1$ type IIB compactified on B_3 after the orientifold projection [22, 23].
This construction gives rise to the idea of F-theory [2], [3] as a twelve dimensional theory, which on a torus gives $\mathcal{N} = 1$ type IIB on orientifolds and fits in the picture

$$\begin{array}{ccc} \text{M-theory on } \mathcal{M} \times S^1_A \times S^1_B & \xrightarrow{R_A \to 0} & \text{type IIA on } \mathcal{M} \times S^1_B \\ \Big\downarrow & & \Big\downarrow {\scriptstyle R_B \to R_B^{-1} \text{ and } R_B \to 0} \\ \text{F-theory on } \mathcal{M}' \times \mathbb{T} & \xrightarrow{vol(\mathbb{T}) \to 0} & \text{type IIB decompactified.} \end{array} \tag{2.6}$$

We call the shrinking limit, in which the theory can be identified with four dimensional type IIB theory, the F-theory limit. Since no twelve dimensional supergravity description with a single graviton exists [24], we rather think of F-theory as M-theory on a elliptically fibered Calabi-Yau fourfold.

It is convenient to write the torus in the fiber as a quotient of the complex plane by a lattice Λ:

$$\mathbb{T} \sim E_\tau := \mathbb{C}/\Lambda = \{w \in \mathbb{C} : w \sim w + (n + \tau m)\},\ n, m \in \mathbb{Z}. \tag{2.7}$$

Here, τ is called Teichmüller parameter and it is an element of the upper half plane $\mathbb{H}$. In the elliptic fibration, τ can vary over the entire base manifold. Following this field through the chain of dualities (see (2.6)), we can identify the complex sturcture of the elliptic fibration with the axion dilaton in the type IIB theory in the F-theory limit.
From equation (2.7) follows that τ is invariant under the transformation

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \text{ with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{2.8}$$

This was expected, because τ has to satisfy this $SL(2, \mathbb{Z})$ invariance in the F-theory limit as we have seen in the previous subsection. For a given matrix $M \in SL(2, \mathbb{Z})$, the corresponding transformation of τ is the same as the transformation induced by $-M$. By dividing this $\mathbb{Z}_2$ subgroup out of $SL(2, \mathbb{Z})$, we are left with the modular group $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$. Since the torus is the same for τ transformed under this modular group, we demand that the τ lives in the fundamental domain of the modular group $PSL(2,\mathbb{Z})$, that is

$$\tau \in \{z \in \mathbb{H}, |\mathrm{Re}(z)| \leqslant \frac{1}{2} \text{ and } |z| > 1\}. \tag{2.9}$$

If a M2 brane wraps around the torus in the fiber, it gives rise to a $[p,q]$-string in the F-theory limit. A $[p,q]$-string is a BPS bound state in the type IIB theory [25]. The numbers p and q are determined by how often the M2 brane wrapped the circle S^1_A and S^2_B respectively. The p,q-string will then give rise to physical particles. We will come back to this in section 4.1.

2.3 Elliptic Fibrations

A key aspect of F-theory compactification on a Calabi-Yau fourfold is the elliptic fibration. In this subsection, we will examine this further and introduce an alternative formulation using the theory of elliptic curves, which is better to handle.

2.3.1 The Weierstrass Model

At first, we show how the torus given in (2.7) can be written as a elliptic curve. Therefore, we define an elliptic curve[3] to be the vanishing locus of

$$P_W := y^2 - (x^3 + fxz^4 + gz^6) \tag{2.10}$$

in the weighted projective space $\mathbb{P}_{2,3,1}$. This weighted projective space can be described by homogeneous coordinates $[x,y,z]$ with the property that $[x,y,z] = [\lambda^2 x, \lambda^3 y, \lambda z]$ for all $\lambda \in \mathbb{C}^*$. The complex numbers f and g determine the elliptic curve.
Writing the torus fiber in F-theory in terms of equation (2.10) is called the Weierstrass model. It forms a hypersurface in a projective space, which means that the elliptic fibration is a hypersurface in a $\mathbb{P}_{2,3,1}$ fibration over a threefold base. For more details on the geometrical aspects, see appendix A.
Let us define the discriminat of an elliptic curve by

$$\Delta := 4f^3(\tau) + 27g^2(\tau). \tag{2.11}$$

This quantity of an elliptic curve will be useful later.

Next it is shown that the Weierstrass model is really equivalent to the torus $\mathbb{E}_\tau$ given in equation (2.7) by following the line of arguments from [13].
We start with a torus $\mathbb{E}_\tau$ with Teichmüller parameter τ. We define the meromorphic function (also called Weierstrass function)

$$\wp_\tau(w) := \frac{1}{w^2} + \sum_{(n,m)\in\mathbb{Z}^2\setminus(0,0)} \left(\frac{1}{(w+n+\tau m)^2} - \frac{1}{(n-m\tau)^2} \right). \tag{2.12}$$

This function is invariant under a shift of the argument by elements of the lattice $\mathbb{Z} + \tau\mathbb{Z}$ and therefore $\wp_\tau$ is well defined on the torus $\mathbb{E}_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. It has double poles on the lattice and satisfies the differential

[3] There are many equivalent ways to define or represent an elliptic curve. For instance one can directly define an elliptic curve to be a torus with a marked point.

equation

$$(\wp_\tau(w)')^2 = 4\wp_\tau(w)^3 - g_2(\tau)\wp_\tau(w) - g_3(\tau), \tag{2.13}$$

where g_2 and g_3 are the Eisenstein series

$$g_2(\tau) = 60 \sum_{\mathbb{Z}^2\backslash(0,0)} (m+n\tau)^{-4} \tag{2.14a}$$

$$g_3(\tau) = 140 \sum_{\mathbb{Z}^2\backslash(0,0)} (m+n\tau)^{-6}. \tag{2.14b}$$

Now we can define a map from the torus to the projective space by

$$\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \to \mathbb{P}_{2,3,1} \tag{2.15}$$

$$w \mapsto [4^{\frac{2}{3}}\wp_\tau(w) : 2\wp'_\tau(w) : 1] \tag{2.16}$$

Because $\wp$ satisfies equation (2.13), we see that this map really takes values only in the hypersurface given by the Weierstrass equation (2.10) with coefficients

$$f(\tau) := -4^{\frac{1}{3}} g_2(\tau), \qquad g(\tau) = -4g_3(\tau). \tag{2.17}$$

Now we consider the other direction. For the Weierstass model above with $f(\tau)$ and $g(\tau)$, we define the j-invariant (also called Jacobi j-function) as

$$j(\tau) = 4\frac{24^3 f(\tau)^3}{\Delta}, \tag{2.18}$$

where Δ is the discriminant of the Weierstrass model defined in (2.11). This is a modular function, i.e. it is invariant under the action of the modular group on the argument as defined in (2.8). Moreover, it is a bijection from the fundamental domain of $SL(2,\mathbb{Z})$ to the complex plane [26]. One can write down the inverse of the j-invariant, which gives us the Teichmüller parameter depending on f and g back. This yields the inverse of (2.17).

As mentioned above, we have a Weierstrass equation in every point of our base manifold. Thus, the coefficient f and g have to be sections of line bundles. By the identification of equations (2.14) and (2.15), the transition behaviour of this bundle depends on τ. Since we want to consider 7-branes in the type IIB theory as well, τ is rather a section of the line bundle $\mathcal{L}$ than a globally defined function. As explained above, the section τ will correspond to the axion-dilaton in the type IIB theory.
The Weierstrass coeffecients f and g are sections in powers of the line bundle $\mathcal{L}$ over the base manifold

$$f \in \Gamma(B_3, \mathcal{L}^{-4}), \qquad g \in \Gamma(B_3, \mathcal{L}^{-6}). \tag{2.19}$$

The Einstein equations imply a relation between the curvature of the manifold B_3 and the dilaton. Together with the requirement to preserve supersymmetry, this forces

$$c_1(B_3) = c_1(\mathcal{L}). \tag{2.20}$$

This implies that we can identify $\mathcal{L}$ with the anticanonical bundle K_B^{-1}[4], which means for the divisor classes

$$[\mathcal{L}] = -K_B. \tag{2.21}$$

Throughout the entire thesis, we tend to write line bundles over a manifold X as divisors, which are elements in the Chern group $\mathrm{CH}^1(X)$. For details, we refer the reader to appendix A.3.
Note that the base manifold is only required to be a compact Kähler manifold instead of a Calabi-Yau manifold and therefore, K_B does not need to be trivial in contrast to K_Y.

2.3.2 Singularities and D-branes

Next, we want to consider D-branes in our type IIB model and how they are encoded in singularities of the elliptic fibration in F-theory.
A hypersurface is singular at the point $\mathbf{x}$ if $p(\mathbf{x}) = 0$ and $\nabla p \mid_{\mathbf{x}} = 0$. For the fiber in the elliptic fibration this is satisfied, if the discriminant Δ, which is defined in equation (2.11), vanishes

$$\Delta(f, g) = 4f^3 + 27g^2 = 0. \tag{2.22}$$

By looking again at the j-invariant introduced in equation (2.18), we see that it has poles where the fiber is singular. The j-invariant can be expanded

$$j(\tau) = e^{-2\pi i\tau} + 744 + 196884e^{2\pi i\tau} + \mathcal{O}(e^{2\pi i\tau}). \tag{2.23}$$

So the axion dilaton has to go to $i\infty$ (see [13], section 3.3) for $\Delta \to 0$ because τ is an element of the upper half plane. This corresponds to the coupling constant $g_s = e^{\phi} = \mathrm{Im}(\tau)^{-1} \to 0$, which represents the behaviour of the coupling constant when approaching a 7-branes. This is of course not a complete proof but by examining the monodromies around the normal directions of the branes, it is possible to prove that 7-branes are located at the positions on B_3 where the fibers are singular. Stacks of multiple 7-branes are indicated by the severity of the singularity. We will come to this in chapter 4.

[4] In the end we are just interested in the divisor class, which determines the line bundles only up to isomorphisms. For details about the definition of the (anti-) canonical bundle, see appendix A.3.

3 Geometry of Singularities in F-theory

We have seen in the previous section that we can detect 7-branes and even stacks of 7 branes in the type IIB superstring theory by the singularities of the elliptic fibers in F-theory. Since stacks of 7-branes give rise to gauge theories, this will be of interest for the rest of this thesis. Due to the presence of orientifold seven planes, we can get multiple different gauge theories.

In this section we focus purely on the geometric side, saving the physics for section 4. We start this section with the classification of singularities by Kodaira and Néron and their relation to Lie algebras. In subsection (3.3) we resolve the singularities and discuss how the information about the corresponding Lie algebra is still preserved in the resolved geometry. The following subsections 3.4 and 3.5 deal with the intersection theory of divisors on the resolved space and finally, subsection 3.6 discusses singularities in codimension two and their relation to representation theory.

We will assume the functions f and g in the Weierstrass model to be maximally generic if not indicated differently. This means that we can ignore pathological examples.

3.1 Kodaira Classification

From now on, we assume the base three-fold to be a toric variety[5] (for the basics of toric varieties, we refer to the appendix A). The equation $\Delta = 0$ cuts out a hypersurface of the base manifold B_3. The polynomial Δ might factorize $\Delta = \prod_i \Delta_i$ such that the hypersurface is in fact a union of irreducible hypersurfaces

$$\{\Delta = 0\} = \bigcup_s \{\Delta_s = 0\}. \tag{3.1}$$

The corresponding divisor class $[\{\Delta = 0\}] = \Sigma = \sum_s \Sigma_s$ is called singular divisor.

We consider a local coordinate w, whose vanishing locus corresponds locally to an irreducible part of the singular divisor

$$[\{w = 0\}] = \Sigma_s. \tag{3.2}$$

This means that locally, we can factorize the discriminant in a way

$$\Delta = w^{\mathrm{ord}(\Delta)} \Delta', \tag{3.3}$$

such that $\Delta' \mid_{w=0} \neq 0$ and $\mathrm{ord}(\Delta)$ is defined by this condition. The order of f and g are defined in the same way.

Note that we have $\mathrm{ord}(\Delta) \geqslant \min(3\,\mathrm{ord}(f), 2\,\mathrm{ord}(g))$, but it is not an equality, because we have for example $f = -3\left(\frac{g}{2}\right)^{\frac{2}{3}}$, which makes the discriminant vanishing although $f \neq 0 \neq g$. The singular elliptic curve that arises from this choice of f is called *nodal curve*. Singularities of this kind are called Kodaira-type I_1. Writing the vanishing orders in a triple, we can identity

$$(\mathrm{ord}(f), \mathrm{ord}(g), \mathrm{ord}(\Delta)) = (0, 0, 1) \qquad \text{Kodaira-type } \mathrm{I}_1. \tag{3.4}$$

If f and g both vanish, the discriminant vanishes as well and the singular fiber is called a *cuspidal curve*. Singularities with order $(\mathrm{ord}(f), \mathrm{ord}(g), \mathrm{ord}(\Delta)) = (\geqslant 1, 1, 2)$ are known as Kodaira-type II_2.

These two types of singular fibers do not lead to a singular fourfold, but if we consider higher vanishing orders the Calabi-Yau fourfold becomes singular itself. The singularities that do not exceed the order

[5] Moreover, it is reasonable to exclude Fano threefolds due to the lack of the decoupling limit [27].

$(\mathrm{ord}(f), \mathrm{ord}(g), \mathrm{ord}(\Delta)) = (4, 6, 12)$ have been classified by Kodaira and Néron [28], [29] as Type I_0^*, $\mathrm{I}_{m\geqslant 0}$, II(*), III(*) and IV*. In the subsection 3.3, we will see how Lie algebras can be associated to the types of singularities. The classification can be found in table 1.

	$\mathrm{ord}(f)$	$\mathrm{ord}(g)$	$\mathrm{ord}(\Delta)$	Lie algebra $\mathfrak{g}$
I_0	$\geqslant 0$	$\geqslant 0$	0	–
I_1	0	$\geqslant 0$	1	–
I_2	0	$\geqslant 0$	2	$\mathfrak{su}(2)$
I_m	0	0	m	$\mathfrak{sp}([\frac{m}{2}])$ or $\mathfrak{su}(m)$
II	$\geqslant 1$	$\geqslant 1$	2	–
III	1	$\geqslant 2$	3	$\mathfrak{su}(2)$
IV	$\geqslant 2$	2	4	$\mathfrak{sp}(1)$ or $\mathfrak{su}(3)$
I_0^*	$\geqslant 2$	$\geqslant 3$	6	$\mathfrak{g}_2$ or $\mathfrak{so}(7)$ or $\mathfrak{so}(8)$
I_{2n-5}^* $n \geqslant 3$	2	3	$2n+1$	$\mathfrak{so}(4n-3)$ or $\mathfrak{so}(4n-2)$
I_{2n-4}^* $n \geqslant 3$	2	3	$2n+2$	$\mathfrak{so}(4n-1)$ or $\mathfrak{so}(4n)$
IV*	$\geqslant 3$	4	8	$\mathfrak{f}_4$ or $\mathfrak{e}_6$
III*	3	$\geqslant 5$	9	$\mathfrak{e}_7$
II*	$\geqslant 4$	5	10	$\mathfrak{e}_8$
non-min.	$\geqslant 4$	$\geqslant 6$	$\geqslant 12$	–

Table 1: Kodaira-Tate classification of singularities with orders of f, g and the discriminant Δ as well as the corresponding gauge algebra $\mathfrak{g}$ [30].

3.2 The Tate Model

Some of the singularities mentioned above exhibit the property that the fiber picks up a monodromy when going around the singularity. This global monodromy can factorize and lead to the realization of non-simply laced gauge algebras [31]. At this point it is not clear how the gauge algebras are physically realized, because it will be discussed in section 4, but for the sake of completeness, we want to use this as a motivation to reformulate the Weierstrass model.

There are different mathematical descriptions of the Tate model, but we will follow the one in [32]. The Weierstrass model can locally be reformulated into the so-called Tate form

$$P_T = zy^2 + a_1xyz + a_3yz^2 - \left(x^3 + a_2x^2z + a_4xz^2 + a_6z^3\right). \tag{3.5}$$

The roots of P_T determine the hypersurface in the projective fibration, in which the elliptic curves lives. This projective fibration is obtained by the projectivization of the bundle

$$V = \mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3, \tag{3.6}$$

where $\mathcal{L}$ is the line bundle introduced in section 2.3.1. This means that the ambient space of the elliptically fibered Calabi-Yau fourfold is a fibration

$$\mathbf{Y}_4 := \mathbb{P}(V) \xrightarrow{\tilde{\pi}} B_3. \tag{3.7}$$

Let $K_{\mathbb{P}} \to \mathbb{P}(V)$ be the the canonical line bundle and $\mathbf{H}$ the corresponding divisors class, which we call hyperplane class . The homogeneous coordinates and the coefficient of the Tate model (3.5) are sections of line bundles over $\mathbb{P}(V)$

$$\begin{cases} x \text{ is a section of } K_{\mathbb{P}} \otimes \tilde{\pi}^* \mathcal{L}^2 \\ y \text{ is a section of } K_{\mathbb{P}} \otimes \tilde{\pi}^* \mathcal{L}^3 \\ z \text{ is a section of } K_{\mathbb{P}} \\ a_i \text{ is a section of } \tilde{\pi}^* \mathcal{L}^i. \end{cases} \tag{3.8}$$

Moreover, P_T is a section in $K_{\mathbb{P}}^3 \otimes \tilde{\pi}^* \mathcal{L}^i$. The hypersurface $P_T = 0$ cuts out the elliptially fibered Calabi-Yau manifold Y_4.
In order to model a gauge theory, we can tune these parameters i.e. set $a_n = w^m a_{n,m}$ where w is a coordinate on the base, which gives a singular locus at $w = 0$. Using the Calabi-Yau condition (2.21), the associated divisor class for $a_{n,m}$ is $-nK_B - m\Sigma_s$. The Tate algorithm [33] provides the gauge algebra for a given factorization of the coefficients. This can be found in table 2.

The rank of the gauge group that we model can model is bounded. If the singularity is too strong, i.e. the rank of the gauge group is too large, the fourfold the supersymmetry breaks down [34]. To be precise, the resolution procedure that we introduce in the next chapter spoils the Calabi-Yau condition.

fiber type	gauge group G	a_1	a_2	a_3	a_4	a_6	Δ
I_0	-	0	0	0	0	0	0
I_1	-	0	0	1	1	1	1
I_2	SU(2)	0	0	1	1	2	2
I_3^{ns}	unconven.	0	0	2	2	3	3
I_3^{s}	unconven.	0	1	1	2	3	3
I_{2k}^{ns}	Sp(k)	0	0	k	k	2k	2k
I_{2k}^{s}	SU(2k)	0	1	k	k	2k	2k
I_{2k+1}^{ns}	unconven.	0	0	0	k+1	2k+1	2k+1
I_{2k+1}^{s}	SU(2k+1)	0	1	k	k+1	2k+1	2k+1
II	-	1	1	1	1	1	2
III	SU(2)	1	1	1	1	2	3
IV^{ns}	unconven.	1	1	1	2	2	4
IV^{s}	SU(3)	1	1	1	2	3	4
I_0^{*ns}	G_2	1	1	2	2	3	6
I_0^{*ss}	SO(7)	1	1	2	2	4	6
I_0^{*s}	SO(8)	1	1	2	2	4	6
I_1^{*ns}	SO(9)	1	1	2	3	4	7
I_1^{*s}	SO(10)	1	1	2	3	5	7
I_2^{*ns}	SO(11)	1	1	3	3	5	8
I_2^{*s}	SO(12)	1	1	3	3	5	8
I_{2k-3}^{*ns}	SO(4k+1)	1	1	k	k+1	2k	2k+3
I_{2k-3}^{*s}	SO(4k+2)	1	1	k	k+1	2k+1	2k+3
I_{2k-2}^{*ns}	SO(4k+3)	1	1	k+1	k+1	2k+1	2k+4
I_{2k-3}^{*s}	SO(4k+4)	1	1	k+1	k+1	2k+1	2k+4
IV^{*ns}	F_4	1	2	2	3	4	8
IV^{*s}	E_6	1	2	2	3	5	8
III*	E_7	1	2	3	3	5	9
II*	E_8	1	2	3	4	5	10
non-min	-	1	2	3	4	6	12

Table 2: F-theory Tate algorithm taken from [35], based on [33]. By tuning the coefficents of the Tate model in equation (3.5) in terms of the local coordinate w i.e. $a_n = w^m a_{n,m}$ with in the given orders, the gauge group G is associated to the locus $\{w = 0\}$. The superscripts s, ns, ss correspond to 'split', 'non-split' and 'semi-split' and indicates if the global monodromy of the singularity factorizes.

3.3 Resolution of Singularities

In order to create realistic gauge models, we have to consider singularities of the fibers, which are so severe, that the entire Calabi-Yau Y_4 fourfold becomes singular. There are plenty of inconveniences when working on singular Calabi-Yau manifolds. For instance, we have to distinguish between different kinds of divisors (Cartier and Picard). It will be convenient to resolve the singularities. This means, we construct a smooth Calabi-Yau fourfold $\hat{Y}_4$ which, away from the singularities, exactly looks like Y_4. We want the canonical class not to get affected by the resolution, i.e. we demand the resolution to be crepant. This preserves the Calabi-Yau property.

The resolution procedure is not unique. There are different algorithms in the literature (for example [36], [35], [37]). The resolution of singularities is just supposed to serve as a tool to handle the singular fourfold. The physical results should not depend on the choice of resolution. The question which intermediate objects still depend on the choice of resolution is an open question [9].
The procedure, which we consider is a blowup procedure. Depending on the severity of the singularity, the resolution procedure consists of multiple analog steps. The number of steps is equal to the rank of the gauge algebra. In each step a new coordinate with new scaling relations in the toric description is introduced. The intersection behaviour of the loci of the new coordinates restricted to the fibers takes the form of the extended Dynkin diagram of the corresponding gauge algebra. Therefore, the resolved fourfold is deeply connected to the gauge theory corresponding to the singularity, that we want to resolve. Thus we do not loose information about the singularity by resolving it.
Now we will describe the procedure in more detail. The ith step of the resolution procedure consists of finding a so-called center of the blowup $\{g_{i,1}, g_{i,2}, \dots, g_{i,n_i}\}$. These are given by the coordinates (or expressions) which vanish at the singularity. We introduce the new coordinate e_i and change the blowup centers by

$$g_{i,k} \mapsto g'_{i,k}, \tag{3.9}$$

where $g'_{i,k}$ is defined to satisfy

$$e_i g_{i,k} = g'_{i,k}. \tag{3.10}$$

In the Tate polynomial, which should not change, this corresponds to the replacement $g_{i,k} \to e_i g'_{i,k}$. The new toric coordinate comes with the new scaling behaviour that is determined by the requirement that $g'_{i,k} = g_{i,k} e_i$ should be invariant under the scaling i.e. $(g_{i,1}, g_{i,2}, \dots, g_{i,n_i}, e_i) \sim (\lambda g_{i,1}, \lambda g_{i,2}, \dots, \lambda g_{i,n_i}, \lambda^{-1} e_i)$. This ensures that the resolved manifold coincides with the singular manifold away from the singularity. The blowup center determines the blowup.
We will focus on singularities of Type I_N^s in order to model gauge theories with gauge group $SU(N)$ (c.f. 2). The resolution for other gauge groups can be found for instance in [38, table 6].
For $SU(N)$ gauge groups, the resolution procedure can be computed as follows. We start with the singular fourfold Y_4. The first blowup center is given by $\{x, y, w\}$, where x, y are toric coordinates in the fiber and w is a local base coordinate such that the singularity is at $w = 0$. In the Tate equation, we replace

$$x \to x' e_1 \tag{3.11}$$

$$y \to y' e_1 \tag{3.12}$$

$$w \to w' e_1. \tag{3.13}$$

The second step is determined by the blowup center $\{y', e_1\}$ and the third step is determined by the center $\{x'', e_2\}$. Moreover, for $i > 1$ the ith step, the blowup center is $\{y^{(i)}, e_{i-1}\}$ for even i and $\{x^{(i)}, e_{i-1}\}$ for odd. Each blowup step gives a new divisor class $E_i \in \mathrm{CH}^1(\hat{Y}_4)$, called exeptional divisor, as well as $\mathbf{E}_i \in \mathrm{CH}^1(\hat{\mathbf{Y}}_4)$ for the ambient space $\hat{\mathbf{Y}}_4$, both represented by the loci $\{e_i = 0\}$. Note that the divisor classes of of $x, y = 0$ change with each blowup step. The divisor classes after the ith step in the ambient space $\mathbf{Y}_4$ can be expressed as

$$[\{x^{(i)}=0\}] = \mathbf{H} - 2\mathbf{K} - \mathbf{E}_1 - \sum_{j=1}^{\frac{i-3}{2}} \mathbf{E}_{2j} \tag{3.14}$$

$$[\{y^{(i)}=0\}] = \mathbf{H} - 3\mathbf{K} - \mathbf{E}_1 - \sum_{j=1}^{\frac{i-2}{2}} \mathbf{E}_{2j+1}, \tag{3.15}$$

where we already imposed the Calabi-Yau condition (2.21) and introduced $\mathbf{K} := -\tilde{\pi}^*(\mathcal{L})$. The class $\mathbf{H}$ is the hyperplane class introduced in section 3.2. Written in terms of divisors, the blowup centers for the ith step reads [32, A.29]

$$\{\mathbf{H} - 2\mathbf{K}, \mathbf{H} - 3\mathbf{K}, \tilde{\pi}^*\Sigma_s\} \qquad \text{if } i = 1 \tag{3.16}$$

$$\left\{\mathbf{H} - 3\mathbf{K} - \mathbf{E}_1 - \sum_{j=1}^{\frac{i-2}{2}} \mathbf{E}_{2j}, \mathbf{E}_{i-1}\right\} \qquad \text{if } i \text{ even} \tag{3.17}$$

$$\left\{\mathbf{H} - 2\mathbf{K} - \mathbf{E}_1 - \sum_{j=1}^{\frac{i-3}{2}} \mathbf{E}_{2j+1}, \mathbf{E}_{i-1}\right\} \qquad \text{if } i \geqslant 3 \text{ odd.} \tag{3.18}$$

In the new coordinates, the Tate polynomial can factorize. We divide out the powers of e_i that factorize. This changes the divisor class of the fourfold $\hat{Y}_4$ as hypersurface in the five dimensional ambient space to

$$\hat{Y}_4 = 3\mathbf{H} + 6\mathbf{K} - 2\mathbf{E}_1 - \sum_{i=2}^{N-1} \mathbf{E}_i \tag{3.19}$$

in the I_N^s case.

More geometrically, the singular fibers correspond to pinched tori. The resolution procedure cuts out this singularity and inserts new $\mathbb{P}^1$ curves as schematically given in figure 1. Each resolution step adds one $\mathbb{P}^1$ to the singular fiber[6]. The resolved fiber can still be singular, since the procedure stops when the Calabi-Yau manifold is smooth as a fourfold.

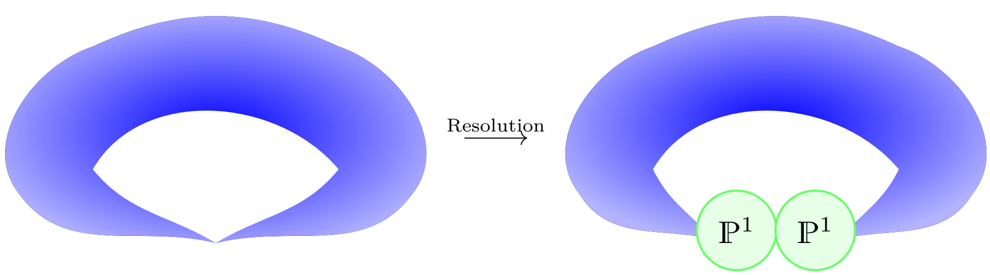


Figure 1: Schematic picture of the resolution of a singular fiber with I_3^s- singularity, corresponding to the gauge group $SU(3)$.

[6]This is true for generic fibers over the singular loci. At certain codimension two loci, the fiber structure might change as it will be dicussed in section 3.6.

For step i in the resolution procedure, the curve $\mathbb{P}^1_i$ is defined by the intersection

$$\mathbb{P}^1_i = \{e_i = 0\} \cap \{P_T = 0\} \cap \{y_a = 0\} \cap \{y_b = 0\}, \tag{3.20}$$

where P_T is the Tate polynomial (3.5) and y_a and y_b are local coordinates of the base manifold B_3 that are neither w nor another singular loci [36].

Example: $SU(2)$

Let us illustrate this by the example of the gauge group $SU(2)$ following [32, section 3.2.1]. Using the Tate algorithm in table 2, a singularity corresponding to the gauge group $SU(2)$ can be modelled by

$$y^2 z + a_1 xyz + a_{3,1} wyz^2 - \left(x^3 + a_{2,1} wx^2 z + a_{4,1} wxz^2 + a_{6,2} w^2 z^3\right). \tag{3.21}$$

At the singular loci $\{w = 0\}$, the equation reads

$$y^2 z + a_1 xyz + x^3 = 0. \tag{3.22}$$

We can see that it is singular at $[x, y, z] = [0, 0, 1]$. We want to blowup along the center $\{x, y, w\}$. Therefore, we introduce the new coordinate e_1 and replace our old coordinates with $x \mapsto \tilde{x}, y \mapsto \tilde{y}, w \mapsto \tilde{w}$, where $\tilde{x}, \tilde{y}$ and $\tilde{w}$ are such that

$$x = e_1 \tilde{x}, \qquad y = e_1 \tilde{y}, \qquad \text{and} \qquad w = e_1 \tilde{w}. \tag{3.23}$$

Now the Tate equation reads

$$\begin{aligned} &e_1^2 \tilde{y}^2 z + e_1^2 a_1 \tilde{x}\tilde{y}z + e_1^2 a_{3,1} \tilde{w}\tilde{x}z^2 - \left(e_1^3 \tilde{x}^3 + e_1^2 a_{2,1} \tilde{w}\tilde{x}^2 z + e_1^2 a_{4,1} \tilde{w}\tilde{x}z^2 + e_1^2 a_{6,2} \tilde{w}^2 z^3\right) = 0 \\ = &e_1^2 \left(\tilde{y}^2 z + a_1 \tilde{x}\tilde{y}z + a_{3,1} \tilde{w}\tilde{x}z^2 - \left(e_1 \tilde{x}^3 + a_{2,1} \tilde{w}\tilde{x}^2 z + a_{4,1} \tilde{w}\tilde{x}z^2 + a_{6,2} \tilde{w}^2 z^3\right)\right) = 0. \end{aligned} \tag{3.24}$$

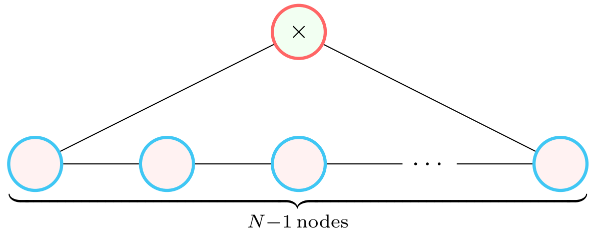
We can see that for $(x, y, w) = (0, 0, 0)$, the new coordinate e_1 must not vanish, which cures the singularity. Moreover, the fiber gets the additional surface $\{e_1 = 0\}$, which at $\{\tilde{w} = 0\}$ gives the $\mathbb{P}^1_1$ curve $[x, y, w, e_1] = [\lambda\tilde{x}, \lambda\tilde{y}, 0, 0]$ at the location of $[0, 0, 0, 1]$. Note that, away from $(x, y, w) = (0, 0, 0)$, we can use the scaling relation to set $e_1 = 1$ if $e_1 \neq 0$, so the resolved manifold does not differ from the singular one away from the singularity. The $\mathbb{P}^1_1$ at $e_1 = 0$ is only connected to the rest via the singularity.
The intersection between the curve $\mathbb{P}^1_1$ and the rest of the original fiber is given by the solution of the equation

$$\tilde{y}^2 z + a_1 \tilde{x}\tilde{y}z = 0. \tag{3.25}$$

Therefore, the curve $\mathbb{P}^1_1$ intersects the original fiber exactly in the two points given by $\tilde{y} = 0$ and $\tilde{y} + a_1 \tilde{x}\tilde{y}$. It is important to note that the class of the singular divisor changes by going from $w \mapsto \tilde{w}$ as

$$[\{w = 0\}] \mapsto [\{w = 0\}] - E_1. \tag{3.26}$$

As mentioned above, for higher gauge groups, more $\mathbb{P}^1_i$s have to be placed in the fiber. We will adjust the index i because the order of the $\mathbb{P}^1_i$s in the diagram does not necessarily coincides with the order in which of the steps of the resolution procedure they were introduced [36]. The intersecting structure of the $\mathbb{P}^1_i$s forms the extended Dynkin diagram of the Lie algebra $\mathfrak{su}(N)$ associated to the resolved singularity. In the $SU(N)$ case, this can be displayed as



where the red node represents the original fiber. The cross denotes the zero section, given by $z = 0$.
Note that we look at generic fibers on the singular loci. As we will see later, over curves of higher codimensions, some $\mathbb{P}^1$s can split further.
Let D_i denote the codimension-one surface that swipes out the curve $\mathbb{P}^1_i$. They can be written in terms of the resolution divisors as [32, A.27]

$$D_i = \begin{cases} E_{2i-1} - E_{2i} & \text{if } i < \lfloor \frac{N}{2} \rfloor \\ E_{N-1} & \text{if } i = N, \\ E_{2N-2i} - E_{2N-2i+1} & \text{if } i < \lceil \frac{N}{2} \rceil \end{cases} . \tag{3.27}$$

These divisors are called Cartan divisors for reasons that should become clear later.
The intersection properties of the $\mathbb{P}^1_i$s among each other is captured by the Dynkin diagram and pass over to the intersection numbers between the Cartan divisors D_i and the curves $\mathbb{P}^1_j$ via

$$D_i \cdot \mathbb{P}^1_j = -C_{ij}, \tag{3.28}$$

where C_{ij} is the Cartan matrix (see appendix B) for the corresponding Lie algebra. This is primarily for combinatorial reasons, but we can use this coincidence to associate geometrical objects with objects in the theory of Lie algebras.
The interpretation goes as

divisor/curve class	Lie algebraic analog
Cartan divisor D_i	co-root T_i
minus resolution curve $-\mathbb{P}^1_j$	simple roots e_{α_j}

.

For the definitions on the Lie algebra side, the reader is referred to appendix B. The roots of a Lie algebra are the weight of the adjoint representation on itself. So we associate a positive linear combination of $\mathbb{P}^1_i$s with the weights of the adjoint representation. Will will come back to this in section 3.6, but first we examine what this resolution procedure means for the intersection theory of the fourfold.

3.4 Intersection numbers

In this subsection, we want to focus on the intersection properties of the divisor classes of the resolved Calabi-Yau fourfold. For details about divisor classes, the reader is referred to appendix A. The intersection product we want to consider is the intersection product in the Chow ring and can be computed in multiple ways. Formally, the quadruple intersection numbers of four divisors D_I, D_J, D_K, D_L in a smooth Calabi-Yau fourfold can be computed as

$$K_{IJKL} := \int_{\hat{Y}_4} \omega_I \wedge \omega_J \wedge \omega_K \wedge \omega_L \tag{3.29}$$

and for triple intersection numbers

$$K^B_{\alpha\beta\gamma} := \int_{B_3} \omega^B_\alpha \wedge \omega^B_\beta \wedge \omega^B_\gamma . \tag{3.30}$$

Here ω_I denotes the $(1,1)$-form that is Poincaré dual to the divisors D_I and ω^B_α the dual of D^B_α respectively, where the divisors of the base threefold are denoted by D^B_α.
If the representatives D_I, D_J, D_K, D_L intersect in points, the intersection product is nothing but

$$K_{IJKL} = \# D_I \cap D_J \cap D_K \cap D_L. \tag{3.31}$$

One can as well consider just the intersection of two codimension-one divisors which gives then a codimension two surface, if they intersect transversal. This gives the multiplication in the Chow ring. By abuse of notation, we do not differ between divisors and their divisor classes in the Chow ring $CH^\bullet(\hat{Y}_4)$ (or $CH^\bullet(B_3)$ respectively).
There are three different kinds of divisor classes in the resolved fourfolds we consider:

- The embedding of the base B_3 in the fourfold via the zero section gives a divisor class that is denoted as D_0 and is called zero divisor. It is defined by $\{z = 0\}$.
- The divisors of the base D^B_α give rise to divisors of the fourfold by their pullback $D_\alpha := \pi^*(D^B_\alpha)$, where π denotes the projection of the elliptic fibration into the base.
- Finally, D_i denotes the Cartan divisors that arise from the resolution procedure and have been introduced in the previous subsection.

We write the divisors classes of the resolved fourfold as D_I with capitalized Latin letters $I = (0, \alpha, i)$. Together with the D_0 divisor, we get

$$h^{1,1}(\hat{Y}_4, \mathbb{Z}) = 1 + h^{1,1}(B_3) + \text{rk}(\mathfrak{g}), \tag{3.32}$$

which is a special case of the Shioda-Tate-Wazir theorem [39, 40].

Recall that we assumed in the definition of the elliptic fibration the existence of a globally defined rational section, that we called the zero section. In fact, there can be additional globally defined rational sections. There exists a natural group law on elliptic curves that we can promote to a group law on all globally defined rational section. This group is called the Mordell-Weil group [41]. The rank of the Mordel-Weil group MW is the minimal number of generators of its non-torsion part.
Due to the additional rational sections, we get new divisors and the Shioda-Tate-Wazir formula now reads

$$h^{1,1}(\hat{Y}_4, \mathbb{Z}) = 1 + h^{1,1}(B_3) + \text{rk}(\mathfrak{g}) + \text{rk}(MW). \tag{3.33}$$

In this thesis, we will only consider trivial Mordell-Weil groups.

In the following we will summarize some known constrains on the quadruple intersection numbers.
Our first observation is that the zero divisor, given by $\{z = 0\}$, never intersects with the exceptional, and

hence with the Cartan divisors. Thus we have

$$K_{0iIJ} = 0 \qquad \text{for all } i, I, J \tag{3.34}$$

An important property of the divisor D_0 is that [42]

$$D_0 \cdot D_0 = K \cdot D_0, \tag{3.35}$$

where K is the pullback of the canonical divisor of the base $\pi^*(K_B)$. Since the zero divisor is a copy of the base, the intersection with divisors D_α can be reduced to intersection on the base.
In particular, this implies [42, 9]

$$\begin{aligned} K_{0000} &= K^\alpha K^\beta K^\gamma K^B_{\alpha\beta\gamma} \\ K_{000\alpha} &= K^\beta K^\gamma K^B_{\alpha\beta\gamma} \\ K_{00\alpha\beta} &= K^\gamma K_{\alpha\beta\gamma} \\ K_{0\alpha\beta\gamma} &= K^B_{\alpha\beta\gamma}, \end{aligned} \tag{3.36}$$

where we used the notation $K_B = K^\alpha D_\alpha$.
Moreover, we have [42]

$$K_{\alpha\beta\gamma\delta} = 0. \tag{3.37}$$

Let us briefly mention a useful fact about triple intersection numbers of a threefold. For a threefold B_3 we have

$$\dim(\text{span}(D^B_\alpha)) = h^{1,1}(B_3) = h^{2,2}(B_3) = \dim(\text{span}(D^B_\alpha D_\beta)). \tag{3.38}$$

Moreover, we can choose a basis for the space $CH^2(B_3)$ also indexed by small Greek letters denoted S^B_γ such that [9]

$$D^B_\alpha \cdot S^B_\beta = \delta_{\alpha\beta}. \tag{3.39}$$

The fact that the intersection product provides a non-degenerate pairing, is a non trivial statement and goes back to the hard Lefschitz theorem [43].

Let us now focus on the intersections between the pullback of base divisors and Cartan divisors. For one Cartan divisor, it turns out that according to [44, Appendix A.8]

$$K_{i\alpha\beta\gamma} = 0. \tag{3.40}$$

Now we consider two Cartan divisors D_i and D_j, arising from the resolution of singularities on the irreducible singular loci Σ_s and $\Sigma_{s'}$, both lying in B_3. We allow them to be the same. The intersection numbers with the divisors D_α and D_β turn out to be

$$K_{ij\alpha\beta} = -\Sigma_s \cdot D^B_\alpha \cdot D^B_\beta \, \delta_{ss'} C^s_{ij}, \tag{3.41}$$

where C^s_{ij} is the Cartan matrix for the Lie algebra corresponding to the singularity at Σ_s and $\delta_{ss'}$ denotes the Kronecker delta. Using the notation $\Sigma_s = \Sigma^\alpha_s D^B_\alpha$, this reads

$$K_{ij\alpha\beta} = -\Sigma^\gamma_s C_{ij} K_{0\alpha\beta\gamma}. \tag{3.42}$$

This could have been expected by the identification of the Cartan divisors with the co-roots of the Lie algebra (see Appendix B). Since different singular loci Σ_s, $\Sigma_{s'}$ do not seem to interact with each other, we are going to focus only one singular Σ loci dropping the index.

Finally, for intersection numbers involving three and four Cartan divisors, there is no known general formula. Instead, the following subsection is concerned with an algorithm to determine these intersection numbers in terms of triple intersection numbers of the base threefold.

3.5 Pushforward Technique

In this subsection, we present an algorithm to compute quadruple intersection numbers in terms of triple intersection numbers of the base. The algorithm is based on [38] and [32, section 3.2.1].

From the analysis of the previous subsection, we see that the only non-trivial cases occur only if Cartan indices and base indices are involved.

We start by looking closer at the projection $\pi : \hat{Y}_4 \to B_3$. Note that $\hat{Y}_4$ is not necessary a elliptic fibration anymore, because due to resolution procedure, the adjusted Tate from does not resemble an elliptic curve anymore. Therefore, we have to define π as the concatenation

$$\pi = \pi \circ f_1 \circ \cdots \circ f_N, \tag{3.43}$$

where $\pi : Y_4 \to B_3$ is the canonical projection of the elliptic fibration and f_i denotes the blowdown which is the reverse of the ith blowup step. The goal is to push the divisor classes forward to the base along these maps.

One of your main tools to deal with pushforwards of intersection products is the following theorem:

Let $f : M \to N$ be a proper morphism and C be a class in $CH^\bullet(N)$ and D a class in $CH^\bullet(M)$. Then the intersection products satisfy the projection formula [45]

$$f_*(f^*(C) \cdot_N D) = C \cdot_M f_* D. \tag{3.44}$$

This allows us to reduce the computation of the pushforwards for the blowdowns of the intersection product to powers of Cartan divisors $D_i \in \mathrm{CH}^1(\hat{Y}_4)$. The Cartan divisors can be written in terms of resolution divisors according to equation (3.27)[7]. Moreover, we want to consider the divisors in the ambient space that we discussed in section 3.3. We can relate the resolution divisor $E_i \in \mathrm{CH}^1(\hat{Y}_4)$ with the resolution divisor in the ambient space $\hat{\mathbf{Y}}_4$ via

$$E_i = \mathbf{E}_i \cdot \hat{Y}_4, \tag{3.45}$$

where we view $\hat{Y}_4$ as a divisor in the five dimensional ambient space $\hat{\mathbf{Y}}_4$. We can write the class $\hat{Y}_4$ in terms of the classes $\mathbf{K}$, $\mathbf{H}$ and $\mathbf{E}_i$ as in equation (3.19).

Since the divisors $\mathbf{D}_\alpha$ are defined as pullbacks of base divisors, we have $\pi_* \mathbf{D}_\alpha = D^B_\alpha$. It behaves similarly with resolution divisors that are pulled back to a further resolved space.

[7] This equation just holds for the $\mathfrak{su}(N)$ case, but similar formulas can be derived for other Lie algebras [9, 46].

Now we can use the following statement [32]:

Let f denote the blowup $X \to \tilde{X}$ along the blowup center, which corresponds to the classes $\{\mathbf{g}_1, \ldots, \mathbf{g}_k\}$. Let $\mathbf{E}$ be the resolution divisor. Then we have

$$f_*\mathbf{E}^n = \sum_{m=1}^{k} \mathbf{g}_k \prod_{l=1,\, l\neq m}^{k} \frac{\mathbf{g}_m}{\mathbf{g}_m - \mathbf{g}_l}. \tag{3.46}$$

Now we can combine the previous two statements and make use of the linearity of the pullback as a map between Chow rings to compute quadruple intersection numbers involving Cartan divisors in terms of a pushforward of π of intersection numbers involving five divisor from the classes $\mathbf{H}, \mathbf{K}, \{\mathbf{D}_\alpha\}$. Therefore, only the pushforward along π still has to be computed. The pushforward along π of products of divisors in the ambient space is rather defined by the pushforwards of $\varpi : \mathbf{Y}_4 \to B_3$ (see equation (3.7)). Since $\mathbf{K}$ and $\{\mathbf{D}_\alpha\}$ are pullbacks of base divisor classes, we are only left with the pushforward of ϖ of powers of the hyperplane class $\mathbf{H}$. These can be computed using [38]

$$\varpi_*(1) = 0, \quad \varpi_*(\mathbf{H}) = 0 \quad \varpi_*(\mathbf{H}^{i+2}) = [2(-2)^i - 3(-3)^i]K_B. \tag{3.47}$$

Note that the power of the divisor gets reduced by 2 because we have $\varpi : \mathbf{Y}_4 \to B_3$ from the five dimensional ambient space into the threefold base. This was the last step. This procedure allows us to compute all quadruple intersection numbers in terms of triple intersection numbers of the base.

$SU(2)$ Example

We illustrate this procedure with the example of a $SU(2)$ singularity on the singular divisor $\Sigma \in \mathrm{CH}^1(B_3)$ [9, appendix E.2]. There is only one resolution divisor E_1 and we have $D_1 = E_1 = \mathbf{E}_1 \cdot \hat{\mathbf{Y}}_4$. Written in terms of divisor classes, the blowup center in the ambient space is

$$\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\} = \{x, y, w\} = \{\mathbf{H} - 2\mathbf{K}, \mathbf{H} - 3\mathbf{K}, \varpi^*\Sigma\}. \tag{3.48}$$

Using $\hat{\mathbf{Y}}_4 = 3\mathbf{H} - 6\mathbf{K} - 2\mathbf{E}_1$, we compute

$$\begin{aligned}
K_{1111} &= \pi_*(\mathbf{E}_1^4 \cdot \hat{\mathbf{Y}}_4) \\
&= \varpi_* \circ f_{1*}(\mathbf{E}_1^4 \cdot \hat{\mathbf{Y}}_4) \\
&= \varpi_*((3\mathbf{H} - 6\mathbf{K}) f_{1*}(\mathbf{E}_1^4)) - 2 f_{1*}(\mathbf{E}_1^5)) \\
&= \varpi_* \left((3\mathbf{H} - 6\mathbf{K}) \sum_{i=1}^{3} \mathbf{g}_i^4 \prod_{i\neq j} \frac{\mathbf{g}_j}{\mathbf{g}_j - \mathbf{g}_i} - 2 \sum_{i=1}^{3} \mathbf{g}_i^5 \prod_{i\neq j} \frac{\mathbf{g}_j}{\mathbf{g}_j - \mathbf{g}_i} \right) \\
&= \varpi_* \left((3\mathbf{H} - 6\mathbf{K}) \mathbf{g}_1\mathbf{g}_2\mathbf{g}_3(\mathbf{g}_1 + \mathbf{g}_2 + \mathbf{g}_3) - \mathbf{g}_1\mathbf{g}_2\mathbf{g}_3 \left(\sum_i \mathbf{g}_i^2 + \mathbf{g}_1\mathbf{g}_2 + \mathbf{g}_2\mathbf{g}_3 + \mathbf{g}_1\mathbf{g}_3 \right) \right) \\
&= \varpi_*((\mathbf{H} - 3\mathbf{K})(\mathbf{H} - 2\mathbf{K})\varpi^*\Sigma(3\mathbf{HK} - 8\mathbf{K}^2 - \mathbf{H}\varpi^*\Sigma + 4\mathbf{KS} - 2\varpi^*\Sigma^2)) \\
&= (3K_B - \Sigma)\Sigma \underbrace{\varpi_*(\mathbf{H}^3)}_{=5K_B} + S(-23K_B^2 + 9K_B S - 2S^2) \underbrace{\varpi_*(\mathbf{H}^2)}_{1} \\
&= 2\Sigma \cdot (2K_B\Sigma - 4K_B^2 - \Sigma^2).
\end{aligned} \tag{3.49}$$

In the same way, we find

$$K_{111\alpha} = 4K_B \cdot \Sigma - 2\Sigma^2. \tag{3.50}$$

Using this pushforward technique, we are able to compute all quadruple intersection numbers for the $SU(5)$ case. They can be found in the appendix C.

3.6 Codimension-Two Singularities and Representations

We have seen how singular divisors in codimension-one can be associated to non-ablian Lie groups. This subsection will discuss how codimension two singularities can be associated to representations. How these representations can be realized as representations of physical states, will be discussed in the next section 4.
The Tate polynomial can factorize over certain codimension two loci, which are intersections of the singular loci with a different loci called the enhancement loci[8]. This leads to a splitting of some of the $\mathbb{P}^1$ resolution curves. In this case, the Dynkin diagram becomes larger and the Lie algebra enhances. The analysis of the representations via gauge enhancements is known as the Katz-Vafa method [47]. We exclude the possibility of self intersections of the singular divisor and assume that the singular loci intersects with the enhancement loci only once.
Recall that the $\mathbb{P}^1_i$ curve classes can form the weights of the adjoint representation of the Lie algebra $\mathfrak{g}$ over generic points in the codimension-one loci. With a change of the weight and with additional weights, this corresponds to different representations. We illustrate this with an example.

SU(5) example

The Tate equation for the resolved $SU(5)$ model reads

$$e_4e_2y^2z + a_1xyz + e_4e_2e_1a_{3,2}w^2yz^2 - \left(e_4e_3^2e_1x^3 + e_3e_1a_{2,1}wxz + e_3e_2e_1a_{4,3}xz^2 + e_3e_2^2e_1^3a_{6,5}w^5z^3\right) = 0. \tag{3.51}$$

At the intersection of the loci $\{w = 0\}$ and $\{a_1 = 0\}$, the curve

$$\mathbb{P}^1_3 = \{e_3 = 0\} \cap \{e_4e_2y^2z + a_1xyz + e_4e_2e_1a_{3,2}w^2yz^2\} \cap \{y_b\} \cap \{y_a\} \tag{3.52}$$

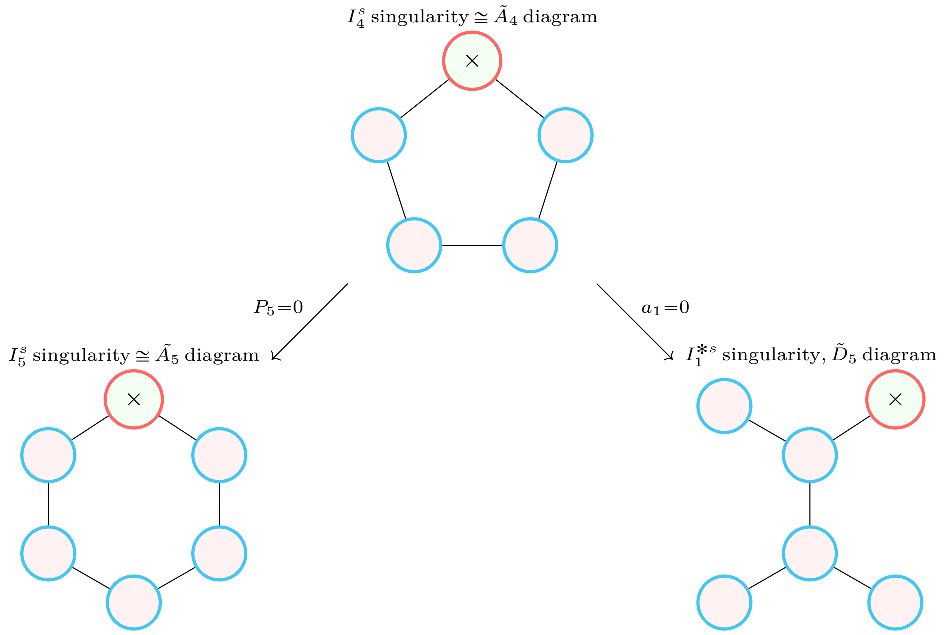
splits into

$$\begin{aligned} &\{e_3 = 0\} \cap \{w = 0\} \cap \{e_4e_2y^2z = 0\} \cap \{y_a\} \\ =& \left(\{e_3 = 0\} \cap \{w = 0\} \cap \{e_2y^2z = 0\} \cap \{y_a\}\right) \cup \left(\{e_3 = 0\} \cap \{w = 0\} \cap \{e_4y^2z = 0\} \cap \{y_a\}\right). \end{aligned} \tag{3.53}$$

The splitting of the $\mathbb{P}^1$ curves leads to a change in the extended Dynkin diagram that is formed by the intersections of the $\mathbb{P}^1$ curves. In the $SU(5)$ model, the relevant enhancements are over the $\{a_1 = 0\}$ and the $\{P_5 = 0\}$ loci with $P_5 = a_{2,1}a_{3,2}^2 - a_1a_{3,2}a_{4,3} + a_1^2a_{6,5}$. The enhancement over these codimension two loci can

[8] This can also be another singular loci.

be displayed as [35]



The tilde indicates the extended Dynkin diagram. Over the loci $P_5 = 0$, the singularity enhances $I_s^5 \to I_6^s$ at this loci. This implies the gauge enhancement $A_4 \to A_5$. In particular, we have the adjoint representation of $\mathfrak{su}(6)$ which breaks as [48][9]

$$\mathbf{Adj}_{\mathfrak{su}(6)} \to \mathbf{Adj}_{\mathfrak{su}(5)} + \mathbf{5} + \overline{\mathbf{5}} + \mathbf{1}. \tag{3.54}$$

From this, we obtain the fundamental representation $\mathbf{5}$ of $SU(5)$ and its conjugate.
In the same way, the enhancement $A_4 \to D_5$ is realized over the enhancement loci $\{a_1 = 0\}$ and corresponds to the 2-index antisymmetric representation denoted by $\mathbf{10}$.

For $N \geqslant 3$, the only representations that can occur for the $SU(N)$ model are the fundamental representation $\mathbf{N}$ and the 2-index antisymmetric $\frac{1}{2}\mathbf{N}(\mathbf{N}-1)$, their complex conjugates $\overline{\mathbf{N}}$ and $\overline{\frac{1}{2}\mathbf{N}(\mathbf{N}-1)}$ as well as the adjoint and the trivial representations [37][10]. This method for obtaining the representations is kown as the Katz-Vafa method [47].
The representations arise over the matter curves $w = 0 = P_N$ for $\mathbf{N}$ and $w = a_1 = 0$ for $\frac{1}{2}\mathbf{N}(\mathbf{N}-1)$ with [9, Appendix F]

$$P_N = \begin{cases} -a_{4,\lfloor \frac{N}{2} \rfloor}^2 + \mathcal{O}(a_1)\,, & N \text{ even} \\ a_{2,1} a_{3,\lfloor \frac{N}{2} \rfloor}^2 + \mathcal{O}(a_1)\,, & N \text{ odd} \end{cases}. \tag{3.55}$$

This can be used to compute the classes for the matter curves to be

$$C_{\mathbf{N}} = \begin{cases} \Sigma \cdot (-8K_B - N\Sigma)\,, & N \neq 3 \\ \Sigma \cdot (-9K_B - 3\Sigma)\,, & N = 3 \end{cases}, \qquad C_{\frac{1}{2}\mathbf{N}(\mathbf{N}-1)} = \Sigma \cdot (-K_B). \tag{3.56}$$

[9]It is not really a breaking in a physical sense, but should rather illustrate the underlying combinatorics.
[10]This is restricted to fibers that can be described by the Tate-model with general coeffients. Under certain conditions and in different models, other representations may occur as examined in [49], [50]. Moreover, higher rank enhancements may occur for $N \geqslant 6$ [48], but they are not considered in this thesis.

4 Physical Realization and the Chiral Index

In the previous section we saw how non-abelian Lie algebras can be associated to codimension-one loci of the elliptic fibration where the fiber degenerates. Moreover, one can associate representations to codimension two loci where the gauge algebra enhances. These associations are until now purely mathematical and can be traced back to combinatorial reasoning.

The first parts 4.1 and 4.2 of this section are aimed to show the physical realization of a gauge theory. This means, there are physical states transforming under the representations of the Lie algebra. In 4.3, we investigate the chiral spectrum using the geometrical description of the previous section. A crucial role in the discussion of chiral matter in F-theory compactifications is played by the G_4 field strength. In the last part 4.4, we list constrains on this G_4 flux.

4.1 Physical Realization of Gauge Symmmetries

At first, let us recall that a non-abelian Lie algebra $\mathfrak{g}$ in F-theory is modelled by a singular Calabi-Yau fourfold. We have already mentioned in the section 2.2 that, due to the monodromy behaviour, the singularities indicate that stacks of 7-branes are wrapping the singular divisor. Thus we get a physical realization of the gauge theory.

When resolving the singularities, the gauge theory breaks into the abelian gauge theory $U(1)^{\mathrm{rk}(\mathfrak{g})}$ with the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ as the Lie algebra. One can think of this as the $\mathbb{P}^1_i$s are separating the 7-branes.

We want to take a different route in understanding the physical consequences, namely, via the M-theory picture introduced in section 2.2.

Our starting point is the effective action of M-theory compactified on a smooth Calabi-Yau fourfold. This has been intensively studied [51, 52, 44]. The three-form field C_3 in the eleven dimensional effective action (2.4) , can be expanded

$$C_3 = A^I \wedge \omega_I + \dots, \tag{4.1}$$

where $\{\omega_I\}_{I=0,\alpha,i}$ denotes the basis of $(1,1)$ forms that is Poincare dual to the divisor D_I. The $\mathrm{rk}(\mathfrak{g}) = N-1$ vector fields A^i in the M-theory survive the chain of dualities to the F-theory side as the massless gauge fields for the Cartan part $U(1)^{\mathrm{rk}(\mathfrak{g})}$ of the gauge algebra.

Another ingredient of M-theory are M2-branes, which can be defined as a black brane solution to the equations of motion of 11-dimensional supergravity. The M2-brane couples to the C_3 fields via the contribution

$$2\pi \int_{\mathrm{M2}} C_3 \tag{4.2}$$

in the 11-dimensional supergravity action. As briefly mentioned in section 2.2, a M2-brane that wraps p times around S^1_A circle and q times around S^1_B circle of the torus in the fiber, becomes a $[p,q]$-string in the F-theory limit. When a M2-brane wraps a holomorphic curve C, it gives rise to a massive particle state with mass [53]

$$|m(C)| \simeq \mathrm{vol}(C). \tag{4.3}$$

If the M2-brane wraps the entire fiber, we can see that it is not charged under base divisors. In the resolved fiber, the M2-brane can wrap multiple $\mathbb{P}^1$s in the chain of $\mathbb{P}^1$s that forms the Dynkin diagram. Lets focus on the $SU(N)$ example, where we have $\mathrm{rk}(\mathfrak{su}(N)) = N-1$. Taking into account the possible orientations, the M2 branes gives in total rise to

$$2\sum_{k=1}^{N-1} k = N^2 - N \tag{4.4}$$

massive states along the brane. Together with the $N-1$ vector fields, this would reassemble the $\mathfrak{su}(N)$ generators, but the mass of this particles causes the breaking of the non-abelian gauge symmetry into the abelian Cartan factors. This is called the Coulomb branch. The volume of the $\mathbb{P}^1_i$s is called the Coulomb branch parameter ξ^i and will be part of the vector multiplet (ξ^i, A_i).
The symmetry gets restored in singular limit where $\mathbb{P}^1_i$s shrink to zero size and the singularity in the fourfold is restored.
These states coming from the M2-branes will be interpreted as W-bosons in the singular limit $\text{vol}(\mathbb{P}^1_i) \to 0$ [54] and together with the abelian gauge factors, we obtain the generators for the entire $\mathfrak{su}(n)$ gauge algebra, transforming in the adjoint representation.
If we would consider a non-trivial Mordell-Weil group, additional global $U(1)$ gauge factors would arise and survive the shrinking limit.

A more geometric feature of the shrinking $\mathbb{P}^1$ is that the Kähler form J of $\hat{Y}_4$ can be expanded as [44]

$$J = R\omega_0 + v^\alpha \omega_\alpha + \zeta^i \omega_i. \tag{4.5}$$

The possible Kähler forms span a cone, called the Kähler cone and ξ^i are parameters in this cone. Thus, the shrinking limit of the $\mathbb{P}^1$s correponds to going to the boundary of the cone.
The remaining vector fields A_0 and A_α appear in terms of the vector mulitplets (R, A_0) and (v^α, A_α) in the 3D supergravity theory of the compactified, effective M-theory. The scalar R measures precisely the volume of a generic (non-degenerate) fiber. We will come back to this when matching the vector multiplet with the vector multiplets on the F-theory side in section 5.2.

4.2 Matter in F-theory

Our goal is to create a model with fermions that are charged under a non-trivial representation, which is not the adjoint representation. M2-branes wrapping around curves in the fiber, give rise to the BPS-states that correspond to chiral multiplets in the $\mathcal{N} = 1$ supergravity theory in four dimensions [13]. Let us first consider the situation over a generic point on the singular loci. Using equation (4.2) the $U(1)$ charge of the M2-brane along a holomorphic curve C is [13]

$$q_i = \int_C w_i = [C] \cdot D_i, \tag{4.6}$$

if the holomorphic curve is the $\mathbb{P}^1_i$, this is the charge with which the M2-brane couples to the vector field A_i. Over a generic fiber, the charges correspond to minus the simple roots. Wrapping the M2-brane along the curve with the opposite orientation gives the simple roots. This is equivalent to replacing the M2-brane with an anti-M2 brane. By wrapping multiple fibers we obtain the entire root system, which give the weights of the representation and hence the chiral multiplet transforms in the adjoint representation as well as mentioned in the previous subsection. On Type IIB orientifolds, chiral multiplets in the adjoint representation correspond to D7-brane moduli [54], [55].
If we move to a codimension two loci on the fully resolved Calabi-Yau fourfold, the curve structure of the fiber enhances because of the splitting of $\mathbb{P}^1_i$ curves. Since the charge of a M2-brane wrapping these curves

still corresponds to the weights of a representation, we obtain a chiral multiplet that transforms under that representation.
From looking at the type IIB picture, one might wonder why we can get chiral matter over the enhancement loci, because, for instance in the $SU(5)$-model, $a_1 = 0$ is not a singular loci wrapped by D7-branes. In fact, the intersection $\{w = 0\} \cap \{a_1 = 0\}$ is the same intersection as $\{w = 0\} \cap \{a_1^2 + 4a_2 = 0\}$ and the latter loci defines the location of the O7 brane [56]. Therefore, the enhancement loci is exactly at the intersection of the stack of D7 branes and the O7 plane. This can be seen by a careful analysis of the Sen limit [22].
The key question is now, how many chiral and anti-chiral multiplets exist and especially, how many chiral particles exist in a given setting. This question will be tackled in the next subsection.

4.3 Chiral Matter

Before we continue with the discussion of chiral matter in the F-theory setting, we take a step back and discuss chiral matter in general.
Our goal is to construct a theory that is consistent with our current experimental observations, which are all at very low energy compared to the inverse Planck length. Therefore the key ingredients of the (minimal supersymmetric) Standard Model should be present in the low energy effective description of our model.

4.3.1 Chiral Matter in Particle Physics

In the mathematical description of particle physics, a fermion in four spacetime dimensions is described by a Dirac spinor Ψ. It can be written in terms of two Weyl spinor ψ_+ and ψ_-

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{4.7}$$

A Weyl spinor is a spinor with a definite chirality [57]. The chirality in four spacetime dimensions is given as the eigenvalue of the operator

$$\overline{\gamma} = i\gamma_0\gamma_1\gamma_2\gamma_3, \tag{4.8}$$

written in terms of Dirac matrices γ_a.
For the decomposition of the Dirac spinor (4.7), the Weyl spinor ψ_+ has the chirality $+1$ and ψ_- has -1. Weyl spinors appear in the $\mathcal{N} = 1$ supergravity theory in terms of chiral (and anti-chiral) multiplets.
Under a gauge transformation, the states of a left handed particle have the opposite charge as the states of the right handed anti-particle and the state of a right handed particle have the opposite charge as the states of the left handed anti-particle, which can be summarized as [58, section 21.3]

$$q_{f_L} = -q_{\overline{f}_R} \tag{4.9}$$

$$q_{f_R} = -q_{\overline{f}_L}. \tag{4.10}$$

If the left and the right handed fermion do not have the same charges, they are called chiral. In the Standard Model, all fermions are chiral. This is the case if the charges of the left handed particles are not opposite to the charges of the left handed anti-particles. If we look at a particular representation, not having the desired charge means not transforming under the representation at all since the representation is defined by the charges.
We are interested in the chiral index $\chi_{\mathbf{R}}$, that is the number of chiral fermions transforming in the rep-

resentation **R**. Since the anti-particles transform under the conjugate representation $\overline{\mathbf{R}}$ the chiral index reads

$$\chi_{\mathbf{R}} = n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}, \tag{4.11}$$

where $n_{\mathbf{R}}$ denotes the number of Weyl- spinors transforming in the representation **R** and $n_{\overline{\mathbf{R}}}$ for the conjugate representation respectively.

4.3.2 The Chiral Index in F-theory

Lets go back to our F-theory model. It turns out that it is possible to count chiral and anti-chiral multiplets on the Type IIB side by the dimensions of the cohomology groups [13, section 4.6], (see also [59], [60])

$$\text{chiral:}\, H^1(\Sigma_k, \mathcal{O}) \oplus H^0(\Sigma_k, \mathcal{O} \otimes K_{\Sigma_k}) \tag{4.12}$$

$$\text{anti-chiral:}\, H^2(\Sigma_k, \mathcal{O}) \oplus H^1(\Sigma_k, \mathcal{O} \otimes K_{\Sigma_k}), \tag{4.13}$$

with K_{Σ_k} being the canonical bundle over the singular divisor Σ_k. If we consider a trivial gauge background, i.e.

$$G_4 = dC_3 = 0, \tag{4.14}$$

then $\mathcal{O}$ is a trivial bundle and it can be shown that

$$\dim(H^1(\Sigma_k, \mathcal{O})) = H^1(\Sigma_k, \mathcal{O} \otimes K_{\Sigma_k}) \tag{4.15}$$

$$\dim(H^2(\Sigma_k, \mathcal{O})) = H^0(\Sigma, \mathcal{O} \otimes K_{\Sigma_k}), \tag{4.16}$$

which leads to a vanishing chiral index

$$\chi = \dim(H^1(\Sigma, \mathcal{O}) \oplus H^0(\Sigma, \mathcal{O} \otimes K_\Sigma)) - \dim(H^2(\Sigma, \mathcal{O}) \oplus H^1(\Sigma, \mathcal{O} \otimes K_\Sigma)) = 0. \tag{4.17}$$

Therefore, we need to consider a non-vanishing gauge background, which leads to a twist of the cohomology groups in (4.12), such that the chiral index does not vanish. The gauge background in F-theory has its dual in the G_4 flux in the M-theory picture. This G_4 flux is a key ingredient, because the chiral index depends on its choice. We will not go into the details of the twist, but refer to the review [13, section 9].
It turns out in the end that the chiral index for a representation **R** depending on the G_4 flux is given by [61]

$$\chi_{\mathbf{R}} = \int_{S_{\mathbf{R}}} G_4. \tag{4.18}$$

The surface $S_{\mathbf{R}}$ is called the matter surface. It can be constructed by fibering the $\mathbb{P}^1$s over the codimension two curve that corresponds to the representation **R**.
The computation of the chiral index has been done for many examples of toric base manifolds, especially for the $SU(5)$ model e.g. [8, 6, 7].
Before we go into the computation of the chiral index in section 5, we discuss some conditions on G_4 that allow us to proceed.

4.4 Flux conditions

As mentioned before, the fourform flux G_4 in the M-theory is an important object for the investigation of the chiral spectrum in F-theory compactifications. It defines a class in the fourth cohomology group $H^4(\hat{Y}_4, \mathbb{R})$. This cohomology group can be decomposed into the three subgroups [62, 63] [11]

$$H^4(\hat{Y}_4, \mathbb{R}) = H^4_{\text{vert}}(\hat{Y}_4, \mathbb{R}) \oplus H^4_{\text{hor}}(\hat{Y}_{2,2}, \mathbb{R}) \oplus H^{2,2}_{\text{rem}}(\hat{Y}_4, \mathbb{R}). \tag{4.19}$$

The vertical part $H^{2,2}_{\text{vert}}(\hat{Y}_4, \mathbb{R})$ is spanned by wedge products of $(1,1)$ forms $\langle H^{1,1}(\hat{Y}_4, \mathbb{R}) \wedge H^{1,1}(\hat{Y}_4, \mathbb{R})\rangle$. The horizontal part is spanned by all contributions of unique harmonic $(4,0)$ form Ω_4 obtained by varying the complex structure. The last part $H^{2,2}_{\text{rem}}(\hat{Y}_4, \mathbb{R})$ comprises all remaining $(1,1)$ forms.
For simplicity, we assume that the G_4 flux does only lie in $H^{2,2}_{\text{vert}}(\hat{Y}_4, \mathbb{R})$. We will drop the subscript vert in the following. Note that its Poincare dual space is spanned by

$$D_{IJ} := D_I \cdot D_J. \tag{4.20}$$

The vertical flux allows us to write the Poincare dual of G_4 as

$$PD[G_4] = PD[G_4]^{IJ} D_{IJ} = \Theta_{KL} K^{IJKL} D_{IJ}, \tag{4.21}$$

where we define

$$\Theta_{IJ} = \int_{\hat{Y}_4} G_4 \wedge w_I \wedge w_J. \tag{4.22}$$

Under the assumption[12] that the Poincare dual of the class $[S_{\mathbf{R}}]$ also lies in vertical part of the cohomology, equation (4.18) implies that the chiral index depends linear on Θ_{IJ}

$$\chi_{\mathbf{R}} = x^{IJ}_{\mathbf{R}} \Theta_{IJ}, \tag{4.23}$$

where $x^{IJ}_{\mathbf{R}} D_{IJ} = PD[S_{\mathbf{R}}]$.
In the next section we will prove even prove the stronger statement

$$\Theta_{IJ} = \chi_{\mathbf{R}} (y_{\mathbf{R}})_{IJ} \tag{4.24}$$

under the assumption that this matter surface has components in D_{IJ}[13]. The knowledge of $(y_{\mathbf{R}})_{IJ}$ will then allow us to compute the chiral spectrum.

Flux Consistency Conditions

The assumption that the G_4-flux lies in the vertical cohomology is useful for the computation. Other restrictions on G_4 have to be imposed for the sake of consistency of the theory. We will briefly enumerate these conditions.
The shifted quantization condition [64] states that the G_4 flux has to satisfy

$$G_4 - \frac{c_2(\hat{Y}_4)}{2} \in H^4(\hat{Y}_4, \mathbb{Z}). \tag{4.25}$$

[11] In the literature, this decomposition is usually done over the complex numbers, but the real decomposition can be achieved by the intersection with $H^4(\hat{Y}_4, \mathbb{R})$
[12] In [61], there is a counterexample to this assumption.
[13] See [9, Appendix G] for examples, where this assumption does not hold.

Since $c_2(\hat{Y}_4)$ is an integer, this implies $G_4 \in H^4(\hat{Y}_4, \mathbb{Z}/2)$. For simplicity, we restrict ourselves to the cases where $c_2(\hat{Y}_4)$ is even, such that we can assume $G_4 \in H^{2,2}_{\text{vert}}(\hat{Y}_4, \mathbb{Z}) := H^4(\hat{Y}_4, \mathbb{Z}) \cap H^{2,2}_{\text{vert}}(\hat{Y}_4, \mathbb{R})$.
In order to preserve supersymmetry, the G_4 flux has to satisfy the primitivity condition [65]

$$J \wedge G_4 = 0 \tag{4.26}$$

for any suitable Kähler form J. Moreover, the G_4 flux must satisfy

$$\int_{D_{\alpha\beta}} G_4 = \int_{D_{\alpha 0}} G_4 = 0 \qquad \text{for all } \alpha, \beta \tag{4.27}$$

in order to not violate Lorentz invariance.
Preserving the gauge symmetry requires in addition that [66][42]

$$\int_{D_{\alpha i}} G_4 = 0 \qquad \text{for all } \alpha, i. \tag{4.28}$$

For the sake of completeness, let us also mention the M2-brane tadpole cancellation condition, which relates the number of M2-branes N_{M2} and the Euler characteristic $\chi(\hat{Y}_4)$ with the G_4 flux [64]

$$N_{\text{M2}} = \frac{\chi(\hat{Y}_4)}{24} - \frac{1}{2} \int_{\hat{Y}_4} G_4 \wedge G_4. \tag{4.29}$$

This condition can be interpreted as a bound on the maximal self-interactions of the divisor class dual to G_4.
Let us define the quantity

$$\Theta_{IJ} = \int_{\hat{Y}_4} G_4 \wedge w_I \wedge w_j. \tag{4.30}$$

This will be crucial in the following section. In this notation, the above conditions read

$$\Theta_{\alpha I} = 0 \tag{4.31}$$

and we are left with Θ_{00} and Θ_{ij} encoding the remaining degrees of freedom. Later in equation (5.53), we will see that $\Theta_{00} = 0$ follows as well. We will therefore focus on Θ_{ij}.

5 Approach via Chern-Simons Term

The only way to compute the chiral index is until now is via the matter surface (4.18). Since this surface is difficult to compute and its calculation appears to need concrete knowledge about the base threefold, we discuss a different way to compute the chiral index based on [9]. The idea is to match a Chern-Simons action on the M-theory side with a one loop contribution of the type IIB side.
At first, we focus on the M-theory side in 5.1 and in section 5.2 we find the corresponding contributions on the Type IIB side. In section 5.3 we will discuss how to use the previous discussion for the computation of the chiral index. This allows us to determine the chiral index in terms of the Chern-Simons terms for the $SU(5)$ model. In the end, we want to obtain the chiral index for a representation only depending on the G_4 flux and the base threefold B. Following [9], we complete this last step in section 5.4 and give the concrete chiral index in a canonical basis for the G_4 flux.

5.1 Effective Chern-Simons action in M-theory

At first, we look again at the effective action of the eleven dimensional supergravity theory (2.4). It comprises the part

$$S_{SC} \sim \int_{\mathbb{R}^{1,10}} C_3 \wedge G_4 \wedge G_4. \tag{5.1}$$

We want to compactify the action (5.1) on the smooth Calabi-Yau fourfold $\hat{Y}_4$. Using the expansion (4.1), we get [44, p. 5.12]

$$\begin{aligned} S_{SC} \sim -\frac{1}{12}\int_{\mathbb{R}^{1,2}}\int_{\hat{Y}_4} C_3 \wedge G_4 \wedge G_4 &= -\frac{1}{2}\int_{\mathbb{R}^{1,2}} A_I \wedge dA_J \underbrace{\int_{\hat{Y}_4} G_4 \wedge w_I \wedge w_J}_{=:\Theta_{IJ}} + \dots \\ &= -\frac{1}{2}\Theta_{IJ}\int_{\mathbb{R}^{1,2}} A_I \wedge dA_J. \end{aligned} \tag{5.2}$$

The term Θ_{IJ} has already been defined in (4.30).
The expansion of the three form and the compactification of the eight dimensions give rise to other contributions, but we are only interested in the Chern-Simons actions for the 3D vector fields A_I. It comes with the factors Θ_{IJ} which have been introduced in (4.30). We want to follow these Chern-Simons terms via the chain of dualities (2.6). If we do not take the F-theory limit of the shrinking fibers yet, the effective M-theory description is dual to effective type IIB string theory compactified on a circle.
One often performs a change of basis of the divisors

$$D_{\hat{0}} := D_0 - \frac{1}{2}K^\alpha D_\alpha \tag{5.3}$$

for the sake of convenience in the uplift to the Type IIB side [67]. The coefficents K^α are defined by $K_B = K^\alpha D^B_\alpha$. We will not do this change of basis, because we are only interested in the purely Cartan indexed part of the Chern-Simons terms Θ_{ij}, as we saw at the end of section 4.4. The change of basis becomes more relevant when including a non-trivial Mordell-Weil group [42].

Recall from sections 4.2 and 4.1 that we work over the smooth i.e. resolved Calabi-Yau fourfold. This implies that we are in the Coulomb branch, where the gauge group is broken to its Cartan factors with the abelian gauge fields A_i. This also means that the Weyl fermions acquire a mass that coincides with the Coulomb branch parameter which is nothing but the size of the blowup $\mathbb{P}^1_i$s.

For the very small fiber volume that we consider, the masses of the fermions are very large. So in order to investigate the low energy effective action on the Type IIB side for the matching with M-theory, we have to integrate out these heavy fermions.

5.2 Effective Fermion Action Three Dimensional Supergravity

In this subsection, we will discuss the low energy effective action of type IIB superstring theory after the orientifold projection in order to find the match with the effective M-theory Chern-Simons terms. This low energy effective action is described by the corresponding supergravity action. We have seen the bosonic part in section 2.1, but now, we are interested in the fermionic part.
Following [42], the kinetic part for the Weyl spinor Ψ of a chiral multiplet in four dimensions reads

$$\mathcal{L}_{f,Weyl} = -\frac{i}{\kappa_4^2}\overline{\Psi}^f \sigma^A e_A^M \left(\partial_M + iq_i^f A_M^i\right)\Psi^f, \tag{5.4}$$

where κ_4^2 denotes the four dimensional Plank mass, $\sigma = (-\mathbb{1}, \sigma^a)$ are the Pauli matrices and e_A^M is the four dimensional tetrad. The index M indicates the local coordinate, while A indicates coordinates of the four dimensional Minkovski space. The vector fields A_i are the gauge fields for the gauge group $U(1)^{\mathrm{rk}(\mathfrak{g})}$, which lie in the Coloumb branch.

5.2.1 Kaluza-Klein Reduction

Now we want the compactify this on a circle S^1 using Kaluza-Klein reduction. Here $M, N = 0\dots 3$ denote four dimensional local coordinates, while μ, ν denote three dimensional local coordinates. We compactify the direction $M = 3$ on a circle with radius r, that is assumed to be constant. The compactified coordinate is denoted as $x^3 =: y$ for better distinction. The tetrad in three dimensions comes with a tilde. The ansatz for the metric takes the form [42]

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + r^2 A_\mu^0 A_\nu^0 & -r^2 A_\mu^0 \\ -r^2 A_\nu^0 & r^2 \end{pmatrix}. \tag{5.5}$$

Equation (5.5) implies for the tetrad

$$e^a = \tilde{e}_\mu^a dx^\mu \qquad\qquad e^3 = r(dy - A_\mu^0 dx^\mu), \tag{5.6}$$

with $e_M^A = (e_M^a, e_M^3)$. For the inverse tetrad appearing in (5.4) follows

$$e_A = (e_a, e_3) = \left(\tilde{e}_a^\nu \frac{\partial}{\partial x} + \tilde{e}_a^\nu A_\nu^0 \frac{\partial}{\partial y}, \frac{1}{r}\frac{\partial}{\partial y}\right). \tag{5.7}$$

The vector field A_μ^0 is exactly the vector field that we already know from the effective M-theory discussion 4.1 as well as the gauge fields A^i of the gauge group $U(1)^{\mathrm{rk}(\mathfrak{g})}$. The expectation value of the scalar R will be identified with $\langle R\rangle = r^{-2}$ [68] and the expectation value of the scalar accompanying the vector fields A^i can be identified as the Coulomb branch parameters $\langle \xi^i \rangle$. These scalar fields become Wilson line degrees of freedoms in the F-theory lift, which can be computed by the holonomy around S^1 of a vector field A^i [42]

$$\zeta^i(x^\mu) = \int_{S^1} A^i(x^\mu, y). \tag{5.8}$$

The vector multiplets (v^α, A_α) lift to chiral multiplets in four dimensions [44].
The vector fields A^i in the Kaluza-Klein ansatz take the form

$$A^i(x^\mu, y) = A^i_\mu dx^\mu + \zeta r(dy - A^0_\mu) = A^i_\mu dx^\mu + \zeta e^3 \tag{5.9}$$

and the Kaluza-Klein ansatz for the Weyl spinors that are charged under the $U(1)$ factors take the form

$$\Psi^f(x^\mu, y) = \sum_{n=-\infty}^{\infty} \Psi^f_n(x^\mu) e^{iyn}. \tag{5.10}$$

Inserting this in the Lagrangian (5.4) gives

$$\begin{aligned}\mathcal{L}^4_{f,KK} &= \sum_{n,m=-\infty}^{\infty} \overline{\Psi}^f_m e^{-iym} \left(\sigma^a e^\mu_a(\partial_\mu + irq_i(A^i - ir\zeta^i A^0_\mu)) + \sigma^a e^\nu_a A^0_\nu \left(\frac{\partial}{\partial y} + irq_i\zeta^i\right) + \sigma^3 \frac{1}{r}\left(\frac{\partial}{\partial y} + irq_i\zeta^i\right)\right) \Psi^f_n e^{iyn} \\ &= \sum_{n,m=-\infty}^{\infty} \overline{\Psi}^f_m e^{-iy(m-n)} \left(\sigma^a e^\mu_a(\partial_\mu + irq_iA^i + iA^0_\mu n) + \sigma^3 \frac{i}{r}\left(n + rq_i\zeta^i\right)\right) \Psi^f_n.\end{aligned} \tag{5.11}$$

In order to obtain the three dimensional theory, we have to integrate over the circle coordinate y, which gives

$$\mathcal{L}^3_{KK} = -\frac{i}{\kappa_4^2} \sum_{n=-\infty}^{\infty} \overline{\Psi}^f_n(x^\mu)\sigma^a \tilde{e}^\mu_a \left(\partial_\mu + iq_iA^i_\mu + inA^0_\mu\right) \Psi^f_n + i\overline{\Psi}^f_n \sigma^3 \left(\frac{n}{r} + q_i\zeta^i\right) \Psi^f_n. \tag{5.12}$$

We can read of the charges $q_0 = n$, q_i and the mass

$$M^f_n = \frac{n}{r} + q^f_i \langle \zeta^i \rangle. \tag{5.13}$$

Only the zero modes become massless in the limit of vanishing Coulomb branch parameter. But as long as we are still in the Coulomb branch, all modes are heavy and they all have to get integrated out to obtain the low energy effective action for the abelian gauge fields A^i.

5.2.2 Computation of the Loop Diagram

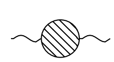
We will not compute the entire low energy effective action, but rather ask, which Chern-Simons-like contributions can there be. We want to compute the factor Θ^F_{ij} in the Chern-Simons term [8]

$$S^F_{SC} = \frac{1}{2}\Theta^F_{ij} \int_{\mathbb{R}^{1,2}} A^i \wedge dA^j \tag{5.14}$$

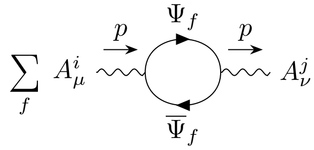
that appears in the effective action of effective type IIB string theory on a circle. Locally, in momentum space, this action reads

$$S^F_{SC} = \frac{1}{2}\Theta^F_{ij} \int_{\mathbb{R}^{1,2}} A^i_\mu \left(\epsilon^{\mu\nu\rho} p_\nu\right) A^j_\rho \, dk^\mu dk^\nu dk^\rho. \tag{5.15}$$

A given Θ_{ij} should correspond to Feynman diagrams of the form

A^i_μ  A^j_ν .

Since there are no propagators for A^i in the action available, the blob can only be a single fermion loop



We have to check that the contribution of this diagram takes the desired form (5.15). In the 3D theory given by the Lagrangian (5.12), we can compute this diagram to be

$$
\begin{aligned}
&\sum_f \sum_{n=-\infty}^{\infty} \sigma^a \int d^3k\, Tr\left[\frac{1}{\kappa_4^4} q_i^f q_j^f \tilde{e}_a^\mu \sigma^b \tilde{e}_b^\nu \frac{1}{i\not{k} + M_n^f}\frac{1}{\not{p} - i\not{k} + M_n^f}\right] \\
=&\sum_f \sum_{n=-\infty}^{\infty} \int d^3k\, Tr\left[\frac{1}{\kappa_4^4} q_i^f q_j^f \sigma^a \tilde{e}_a^\mu \sigma^b \tilde{e}_b^\nu \frac{i\not{k} - M_n^f}{k^2 + (M_n^f)^2}\frac{i\not{p} - i\not{k} - M_n^f}{(p-k)^2 + (M_n^f)^2}\right] \\
=&\sum_f \sum_{n=-\infty}^{\infty} \int d^3k\, \frac{1}{\kappa_4^4} q_i^f q_j^f \tilde{e}_a^\mu \tilde{e}_b^\nu \frac{-Tr[\sigma^a\sigma^b\not{k}\not{p}] + Tr[\sigma^a\sigma^b](M_n^f)^2 + Tr[\sigma^a\sigma^b\not{k}\not{k}] - iM_n^f Tr[\sigma^a\sigma^b\not{p}]}{(k^2 + (M_n^f)^2)\cdot((p-k)^2 + (M_n^f)^2)}.
\end{aligned}
\tag{5.16}
$$

We used the slashed notation for curved spacetime $\not{v} = \sigma^a \tilde{e}_a^\mu v_\mu$. The first term in the nominator of the last line in equation (5.16) is anti-symmetric in k, but the denonimator is symmetric, thus this term vanishes in the integral. By looking again at the Chern-Simons term (5.15), we see that it is parity violating i.e. antisymmetric in p^ν. Since we want to match this with a contribution coming from the loop diagram, we are only interested in the parity violation part of (5.16), which has $-iM^f Tr[\sigma^a\sigma^b\not{p}]$ in the nominator. But there are still non-parity-violating parts in it. In order to really match the contribution of the form of (5.15), we expand around $p^\rho = 0$ and consider only the linear part [53], because only this has the correct shape. We get

$$
\begin{aligned}
&\sum_f \sum_{n=-\infty}^{\infty} \int d^3k\, \frac{1}{\kappa_4^4} q_i^f q_j^f \tilde{e}_a^\mu \tilde{e}_b^\nu \frac{-iM_n^f Tr[\sigma^a\sigma^b\not{p}]}{(k^2 + (M_n^f)^2)\cdot(k^2 + (M_n^f)^2)} \\
=2&\sum_f \sum_{n=-\infty}^{\infty} \int d^3k\, \frac{1}{\kappa_4^4} q_i^f q_j^f \frac{\tilde{e}_a^\mu \tilde{e}_b^\nu M_n^f p_\rho e_c^\rho \epsilon^{abc}}{(k^2 + (M_n^f)^2)^2}.
\end{aligned}
\tag{5.17}
$$

In order to match with (5.15), we can consider our expression locally. Locally, the spacetime manifold looks like $\mathbb{M}^{1,2}$ and therefore, the coordinates can be chosen such that the tetrad $\tilde{e}_a^\mu$ is a Kronecker delta. Thus we are left with the expression

$$
\begin{aligned}
&2\frac{1}{\kappa_4^4}\epsilon^{\mu\nu\rho}p_\rho \sum_f q_i^f q_j^f \sum_{n=-\infty}^{\infty} M_n^f \int d^3k\, \frac{1}{(k^2 + (M_n^f)^2)^2} \\
=&2\frac{1}{\kappa_4^4}\epsilon^{\mu\nu\rho}p_\rho \sum_f q_i^f q_j^f \sum_{n=-\infty}^{\infty} M_n^f \int_0^\infty dk\, \frac{2\pi^2 k^2}{(k^2 + (M_n^f)^2)^2} \\
=&2\frac{1}{\kappa_4^4}\epsilon^{\mu\nu\rho}p_\rho \sum_f q_i^f q_j^f \sum_{n=-\infty}^{\infty} M_n^f \frac{\pi^2}{2|M_n^f|} \\
=&\frac{2\pi^2}{\kappa_4^4}\epsilon^{\mu\nu\rho}p_\rho \sum_f q_i^f q_j^f \sum_{n=-\infty}^{\infty} \mathrm{sign}(M_n^f).
\end{aligned}
\tag{5.18}
$$

We go to natural units, such that $1 = \frac{2\pi}{\kappa_4^2}$. Finally, comparing with (5.15), we get [8, 69]

$$\Theta_{ij}^F = \frac{1}{2}\sum_f q_i^f q_j^f \sum_{n=-\infty}^{\infty} \mathrm{sign}(M_n^f) = \frac{1}{2}\sum_f q_i^f q_j^f \sum_{n=-\infty}^{\infty} \mathrm{sign}\left(\frac{n}{r} + q_i^f \zeta^i\right). \tag{5.19}$$

We will consider the KK-radius to be very small, even smaller than the Coulomb branch parameters, i.e. $r^{-1} \gg q_i \zeta^i$ as in [42]. For $n \neq 0$, the term $\frac{n}{r}$ is always dominant in the argument of the sign. Since for every $n \in \mathbb{Z}\backslash 0$, there is a number with opposite sign, the signs in the sum (5.19) cancel each other and we are left with the contribution at $n = 0$, that is

$$\Theta_{ij}^F = \frac{1}{2}\sum_f q_i^f q_j^f \mathrm{sign}(q_i^f \zeta^i). \tag{5.20}$$

At next, we want to take a closer look at the remaining sum, that runs over the fermions. For a gauge G with representation ρ, the fermions arise in G-multiplet with size $\dim(\rho)$. A particle in this multiplet is associated with a weight $\mathbf{w} = (\mathbf{w}_1 \dots \mathbf{w}_{\mathrm{rk}(\mathfrak{g})})$. The $U(1)$ charge of this particle under the Cartan generator with index i is precisely w_i. This can be used to rewrite the sum over the fermions in (5.20) to

$$\sum_f \simeq \sum_{\mathbf{R}} n_{\mathbf{R}} \sum_{\mathbf{w} \in W(\mathbf{R})}, \tag{5.21}$$

where $n_{\mathbf{R}}$ denotes the number of fermions transforming under the representation $\mathbf{R}$ and $\mathbf{w} \in W(\mathbf{R})$ runs over the weights $\mathbf{w}$ of the representation $\mathbf{R}$.
A more physical explanation of this is that the fermions arise from M2 branes wrapping $\mathbb{P}_i^1$s according to a weight $\mathbf{w} = (\mathbf{w}_1 \dots \mathbf{w}_{\mathrm{rk}(\mathfrak{g})})$. The curve consisting of these $\mathbb{P}_i^1$s is denoted $\mathcal{C}_{\mathbf{w}}$ [8]. The $U(1)$ charges appearing in (5.19) are computed by equation (4.6) with $C = \mathcal{C}_{\mathbf{w}}$. The resulting $U(1)_i$ charge is nothing but the weight component $\mathbf{w}_i$. This can be seen by writing the weight as a linear combination of simple roots, where the coefficents are the Dynkin labels.
The Chern-Simons term now reads

$$\Theta_{ij}^F = \frac{1}{2}\sum_{\mathbf{R}} n_{\mathbf{R}} \sum_{\mathbf{w} \in W(\mathbf{R})} \mathbf{w}_i \mathbf{w}_j \mathrm{sign}(\mathbf{w}_k \zeta^k). \tag{5.22}$$

Since the conjugate representation corresponds to weight with opposite signs, only chiral fermions contribute in equation (5.22) and we have

$$\Theta_{ij}^F = \frac{1}{2}\sum_{\mathbf{R}} \chi_{\mathbf{R}} \sum_{\mathbf{w} \in W(\mathbf{R})} \mathbf{w}_i \mathbf{w}_j \mathrm{sign}(\mathbf{w}_k \zeta^k) \tag{5.23}$$

with the chiral index $\chi_{\mathbf{R}}$.
We already identified the gauge fields A^i of the M-theory and type IIB side. Since the theories are dual, their contribution in the low energy effective theory has to match, i.e.

$$-\Theta_{ij} = \Theta_{ij}^F. \tag{5.24}$$

such that we get

$$\Theta_{ij} = -\frac{1}{2}\sum_{\mathbf{R}} \chi_{\mathbf{R}} \sum_{\mathbf{w} \in W(\mathbf{R})} \mathbf{w}_i \mathbf{w}_j \mathrm{sign}(\mathbf{w}_k \zeta^k). \tag{5.25}$$

We have to keep in mind that Θ_{ij} and Θ_{ji} represent the same degree of freedom since there is only one vertical cohomology class $PD(D_iD_j) = PD(D_jD_i)$.
This result is remarkable, because it gives a connection between the Chern-Simons terms, which depends only on the G_4-flux and the intersections number, on the one side and a formula including the chiral index on the other side. In the following the task will be to extract the chiral index of equation (5.25). We will closely follow the discussion in [60].

5.3 Computation of the Chiral Index

In order to extract the chiral index from equation (5.25), we need to compute the sign function in a first step and in a second step, we either need to solve the system of linear equations or we need to get the chiral index out of the sum over the representations. The weights are determined by the representation and can be read from tables such as 5 or 6 in the $SU(5)$ case.

5.3.1 Computation of the Sign Function

At first we look at the sign-functions. The idea is to consider the same gauge theory but in six non-compact spacetime dimensions. This is achieved by compactifying M-theory on an elliptically fibered Calabi-Yau threefold. The Chern-Simons terms in this theory have three Cartan indices and are given by triple intersection numbers. We assume that the hypermultiplet representations characterizing the gauge sector of a 6D supergravity theory can be recovered from a 5D KK theory, at least for representations corresponding to local matter in the F-theory geometry . This means we can write [70]

$$D_i^{5D} \cdot D_j^{5D} \cdot D_k^{5D} = K_{IJK}^{5D} = k_{ijk}^{5D} = -\frac{1}{2}\sum_f \text{sign}(q_l^f \zeta^l) q_i^f q_j^f q_k^f = -\frac{1}{2}\sum_{\mathbf{R}} n_{\mathbf{R}} \sum_{\mathbf{w}\in W(\mathbf{R})} \text{sign}(\mathbf{w}_l \zeta^l)\mathbf{w}_i \mathbf{w}_j \mathbf{w}_k. \quad (5.26)$$

The triple intersection numbers of the resolved elliptically fibered Calabi-Yau threefold on the left side can be associated to quadruple intersection numbers of the resolved elliptically fibered fourfold of the four dimensional theory via

$$K_{IJK} = W_{IJK} \qquad \text{with} \qquad K_{IJK\alpha} = W_{IJK} \cdot D_\alpha^B. \quad (5.27)$$

Using the pushforward technique from section 3.5, we can compute the most left term of equation (5.26).
On a Calabi-Yau threefold, the codimension two loci giving rise to matter are just points. The number of points gives the number of multiplets $n_{\mathbf{R}}$. When going back from 5D to 3D, the left hand side is $\text{CH}^2(B_3)$ valued and the number $n_{\mathbf{R}}$ becomes the class of the matter curves. For $SU(N)$ gauge theores, these classes can be computed by (3.56).
Now we have enough information to solve equation (5.26) for the sign function $\text{sign}(w_l q^l)$, which is the same as in the 3D case.
We will illustrate this procedure in the example of the gauge group $SU(2)$.

SU(2) Example

Let us examine this procedure of finding the signs for the familiar $SU(2)$ example [9, section 6.3].
In the $SU(2)$-model, there is only one Cartan devisor D_1. The triple intersection number are computed using the pushforward technique (3.50)

$$W_{111} = 4\Sigma \cdot K_B - 2\Sigma^2. \quad (5.28)$$

The $SU(2)$ model has the two matter representations **2** and **3** [71] . They have the weights $\{\mathbf{w}^{\mathbf{2},1}, \mathbf{w}^{\mathbf{2},2}\}$ and $\{\mathbf{w}^{\mathbf{3},1}, \mathbf{w}^{\mathbf{3},2}, \mathbf{w}^{\mathbf{3},3}\}$, which are

$$\mathbf{w}^{\mathbf{2},1} = -1 \qquad \mathbf{w}^{\mathbf{2},2} = 1 \tag{5.29}$$

$$\mathbf{w}^{\mathbf{3},1} = -2 \qquad \mathbf{w}^{\mathbf{3},2} = 0 \qquad \mathbf{w}^{\mathbf{3},3} = 2. \tag{5.30}$$

We can ignore $\mathbf{w}^{\mathbf{3},2}$, because its sign function vanishes. The number of matter points of the representation **2** are according to (3.56)

$$C_{\mathbf{2}} = \Sigma(-8K_B - 2\Sigma) \tag{5.31}$$

and the number of matter points of the representation **3** are [9]

$$C_{\mathbf{3}} = \frac{\Sigma(\Sigma + K_B)}{2}. \tag{5.32}$$

Plugging this into equation (5.26) gives

$$\begin{aligned}
4K_B\Sigma - 2\Sigma^2 &\overset{!}{=} -\frac{1}{2}C_{\mathbf{2}}(\mathrm{sign}(\mathbf{w}^{\mathbf{2},1}\zeta)(\mathbf{w}^{\mathbf{2},1})^3 + \mathrm{sign}(\mathbf{w}^{\mathbf{2},2}\zeta)(\mathbf{w}^{\mathbf{2},2})^3) + C_{\mathbf{3}}(\mathrm{sign}(\mathbf{w}^{\mathbf{3},1}\zeta)(\mathbf{w}^{\mathbf{3},1})^3 + \mathrm{sign}(\mathbf{w}^{\mathbf{2},3}\zeta)(\mathbf{w}^{\mathbf{3},3})^3) \\
&= -\frac{1}{2}\left[\Sigma(-8K_B - 2\Sigma)\right](-\mathrm{sign}(\mathbf{w}^{\mathbf{2},1}\zeta) + \mathrm{sign}(\mathbf{w}^{\mathbf{2},2}\zeta) + \left[\frac{\Sigma(\Sigma + K_B)}{2}\right](-8\,\mathrm{sign}(\mathbf{w}^{\mathbf{3},3}\zeta) + 8\,\mathrm{sign}(\mathbf{w}^{\mathbf{3},3}\zeta)) \\
&= 2K_B\Sigma\underbrace{(-2\,\mathrm{sign}(\mathbf{w}^{\mathbf{2},1}\zeta) + 2\,\mathrm{sign}(\mathbf{w}^{\mathbf{2},2}\zeta) - \mathrm{sign}(\mathbf{w}^{\mathbf{1},3}\zeta) + \mathrm{sign}(\mathbf{w}^{\mathbf{3},3}\zeta))}_{\overset{!}{=}2} \\
&\quad - \Sigma^2\underbrace{(\mathrm{sign}(\mathbf{w}^{\mathbf{2},1}\zeta) - \mathrm{sign}(\mathbf{w}^{\mathbf{2},2}\zeta) + 2\mathrm{sign}(\mathbf{w}^{\mathbf{3},1}\zeta) - 2\mathrm{sign}(\mathbf{w}^{\mathbf{3},3}\zeta))}_{\overset{!}{=}2}.
\end{aligned} \tag{5.33}$$

Using $\mathrm{sign}(w^{\mathbf{2},1}\zeta) = -\mathrm{sign}(w^{\mathbf{2},2}\zeta)$ and $\mathrm{sign}(w^{\mathbf{3},1}\zeta) = -\mathrm{sign}(w^{\mathbf{3},3}\zeta)$, we obtain the sign functions

$$\mathrm{sign}(\mathbf{w}^{\mathbf{2},1}\zeta) = -1 \tag{5.34}$$

$$\mathrm{sign}(\mathbf{w}^{\mathbf{2},2}\zeta) = 1 \tag{5.35}$$

$$\mathrm{sign}(\mathbf{w}^{\mathbf{3},1}\zeta) = -1 \tag{5.36}$$

$$\mathrm{sign}(\mathbf{w}^{\mathbf{3},3}\zeta) = 1. \tag{5.37}$$

In a similar fashion, the sign functions for the $SU(5)$ representations **5** and **10** can be computed. They can be found in the tables 5 and 6.

With the information about the sign function, equation (5.25) gives a set of linear equation, whose solution is the chiral index for all representations involved. We can simplify the task of solving this set of equations by restricting the chiral index by a consistency condition, which we will encounter in the next subsection.

5.3.2 Chiral Anomaly Cancellation

In order for a quantum theory to be consistent, the demanded symmetries have to be persevered while quantizing. In the quantisation process, anomalies can occur that spoil the symmetry. The requirement of the cancellation of these anomalies are additional consistency conditions to the theory.
We are in particular interested in the cancellation of anomalies in the four-dimensional gauge theory of chiral

fermions in the F-theory setting using the Green-Schwarz method [42]. In terms of intersections of divisors and holomorphic curves, the anomaly cancellation conditions can be translated into

$$\frac{1}{3}\sum_{S_{\mathbf{R}}}\sum_{c\subset S_{\mathbf{R}}} S_{\mathbf{R}}\cdot[G_4](c\cdot D_{(i})(c\cdot D_j)(c\cdot D_{k)}) = \frac{1}{2}[G_4]\cdot D_{(i}\cdot\pi_*(D_j\cdot D_{k)}) \tag{5.38}$$

$$\frac{1}{3}\sum_{S_{\mathbf{R}}}\sum_{c\subset S_{\mathbf{R}}} S_{\mathbf{R}}\cdot[G_4])(c\cdot D_i) = [G_4]\cdot\pi^* K_B\cdot D_i, \tag{5.39}$$

where $S_{\mathbf{R}}$ is the class of the matter surface introduced in section 4.3.2. The holomorphic curves $c\subset S_{\mathbf{R}}$ are the holomorphic curves in the fiber over the codimension two loci that arise from the resolution procedure i.e. they are (possible split components of) the resolution $\mathbb{P}^1$-curves over the matter curve. The intersection of these curves with the cartan divisors give the weight vectors of the representation as in 4.6. For details, the reader is referred to [42].
Using equation (4.18), we can write the intersection product between the class of the G_4 flux and the matter surface as the chiral index. The pushforward $\pi_*(D_jD_k)$ can be computed to be $-\Sigma^\alpha C_{kl}D^B_\alpha$ because the pairing $D^B_\alpha\cdot(D^B_\gamma D^B_\delta)$ is non-degenerate (see section 3.4) and

$$-C_{jk}\Sigma^\alpha D^B_\alpha\cdot D^B_\gamma D^B_\delta = K_{jk\gamma\beta} = \pi_*(D_iD_jD_\gamma D_\delta) \overset{(3.44)}{=} (\pi_*(D_jD_k))^\alpha D^B_\alpha\cdot(D^B_\gamma D^B_\delta). \tag{5.40}$$

Moreover, we can write the right hand side in terms of the Chern-Simons term using the definition (4.30) to

$$\frac{1}{3}\sum_{\mathbf{R}}\chi_{\mathbf{R}}\sum_{\mathbf{w}_{\mathbf{R}}}\mathbf{w}_{(i}\mathbf{w}_j\mathbf{w}_{k)} = -\frac{1}{2}\Theta_{\alpha(i}C_{jk)}\Sigma^\alpha \tag{5.41}$$

$$\frac{1}{3}\sum_{\mathbf{R}}\chi_{\mathbf{R}}\sum_{w\in\mathbf{R}}\mathbf{w}_i = K^\alpha\Theta_{i\alpha}. \tag{5.42}$$

Considering a $SU(5)$ model with the representations **5** and **10**, whose weights are given in the tables 5 and 6, we see that the left hand side of the second equation (5.42) is trivial. The vanishing of the right hand side is implied by the condition (4.28).
Applying these flux conditions (4.28) on the term to the right hand side of equation (5.41), we see that the right hand side vanishes.
Examining the first equation for $i=1, j=2, k=2$, we get the condition

$$\chi_{\mathbf{5}}+\chi_{\mathbf{10}} = 0, \tag{5.43}$$

where we used that $\chi_{\overline{\mathbf{R}}} = -\chi_{\mathbf{R}}$. The result of equation (5.43) is crucial, because it allows us to express the two chiral indices as each other such that only one independent chiral index family is left.
Now all ingredient are collected to express the chiral index in terms of the Chern-Simons term Θ_{ij}.

SU(5) Model

In the case of the $SU(5)$ model that we like to consider, we evaluate the relation between the Chern-Simons term and the one-loop contribution in equation (5.25) using the sign functions computed in the previous subsection and the weights of the representations which are all given in the tables 5 and 6 to find the non-vanishing components of the Chern-Simons contribution to be [9, section 6.4.1]

$$\chi_{\mathbf{5}} = -\chi_{\mathbf{10}} = -\Theta_{22} = \Theta_{24} = \Theta_{33} = -\Theta_{34}. \tag{5.44}$$

5.4 Reduced Intersection Pairing

Our goal is to express the chiral index for a given gauge group in terms of the background flux and the information about the base manifold. From the previous discussion, we can already express the chiral index in terms of the Cherm-Simons term Θ_{IJ}[14]. According to the definition of this Chern-Simons term 4.30 depending on the background flux $[G_4] = \phi^{IJ} D_I D_J$ in the bases $D_I D_J$, the Chern-Simons term reads

$$\Theta_{IJ} = \phi^{KL} K_{IJKL}, \tag{5.45}$$

with the quadruple intersection numbers that have already been computed in table 3. In the basis $D_I D_J$, the quadruple intersection numbers can be interpreted as a matrix, called the intersection matrix[15] M. In order to express the chiral index in terms of the flux input ϕ^{IJ}, we have to solve the set of linear equations (5.45).

In principle, we can just use the pushforeward technique to express the quadruple intersection numbers in terms of triple intersection numbers but in addition we would like to impose the consistency constrains at the same time.

The idea proposed in [9], is to reduce the intersection matrix, i.e restrict it to a subspace. That means we change the base $D_I D_J$ to a possible smaller basis without losing information. There are two approaches to do so. The first one employs the flux consistency conditions (4.31) and investigates the resulting restriction on the flux input ϕ^{KL}. The other approach involves finding nullvectors of the intersection matrix and deleting these redundancies from the basis.

We pursue the first one and start with employing the condition $\Theta_{\alpha I} = 0$ which is a redundancy of the size of the subspace spanned by $\{D_I D_\alpha\}$. This information can be used to express all components $\phi^{I\alpha}$ in terms of the components $\phi^{\hat{J}\hat{K}}$ with $\hat{J} \neq \alpha \neq \hat{K}$. We will keep this notation $\hat{I} = (0, i)$ [16]. Let us first sketch the general idea. We can order the basis like $\Theta_{IJ} = (\Theta_{I\alpha}, \Theta_{\hat{K}\hat{L}})$ and write equation (5.45) as

$$\begin{pmatrix} \Theta_{I\alpha} \\ \Theta_{\hat{K}\hat{L}} \end{pmatrix} = \underbrace{\begin{pmatrix} A & Q \\ Q^t & B \end{pmatrix}}_{\text{intersection}-\text{matrix}\, M} \begin{pmatrix} \phi^{I\alpha} \\ \phi^{\hat{K}\hat{L}} \end{pmatrix} \overset{(4.31)}{=} \begin{pmatrix} 0 \\ \Theta_{\hat{K}\hat{L}} \end{pmatrix} \tag{5.46}$$

We perform the change of basis

$$\begin{pmatrix} \phi^{I\alpha} \\ \phi^{\hat{K}\hat{L}} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\phi}^{I\alpha} \\ \tilde{\phi}^{\hat{K}\hat{L}} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & A^{-1}Q \\ 0 & 1 \end{pmatrix}}_{=:U} \begin{pmatrix} \phi^{I\alpha} \\ \phi^{\hat{K}\hat{L}} \end{pmatrix} \tag{5.47}$$

such that the condition (4.31) reads $\tilde{\phi}^{I\alpha} = 0$. Note that this is rather a sketch of the idea. At this point we do not know if the inverse of A does exists. So we can rather think of A^{-1} as a pseudo-inverse.

Since the intersection matrix has to be viewed as a bilinear form, the matrix in the new basis reads

$$U^{-t} M U^{-1} = \begin{pmatrix} A & 0 \\ 0 & B - Q^t A^{-1} Q \end{pmatrix}. \tag{5.48}$$

After imposing the condition $\tilde{\phi}^{I\alpha} = 0$, we can reduce the matrix to $\tilde{M} = B - Q^t A Q$. This is the intersection matrix that we want to find.

[14] That fact that we actually found a dependence only on Θ_{ij} and Θ_{00} as shown in section 4.4 will be used later.
[15] We have to keep in the back of our mind that not every entry represents a degree of freedom.
[16] These parameters are called distinctive in [9].

Since we have

$$\begin{pmatrix} \Theta^{I\alpha} \\ \Theta^{\hat{K}\hat{L}} \end{pmatrix} = \begin{pmatrix} A & 0 \\ Q^t & \tilde{M} \end{pmatrix} \begin{pmatrix} 0 \\ \phi^{\hat{K}\hat{L}} \end{pmatrix}, \tag{5.49}$$

we can find the matrix $\tilde{M}$ by expressing $\Theta^{\hat{K}\hat{L}}$ in terms of $\phi^{\hat{K}\hat{L}}$.

Let us do this in more detail following [9]. At first we express Θ_{IJ} in equation (5.45) in terms of different parts of the basis $D_I D_J$. We can write

$$\Theta_{IJ} = \phi^{\hat{K}\hat{L}} K_{IJ\hat{K}\hat{L}} + \phi^{\gamma 0} K_{\gamma 0 IJ} + \phi^{\gamma\delta} K_{\gamma\delta IJ} + \phi^{\gamma k} K_{\gamma k IJ}. \tag{5.50}$$

Imposing the condition (4.31), we get

$$0 = \Theta_{\alpha 0} = \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\alpha 0} + \phi^{\gamma 0} K_{\gamma 0\alpha 0} + \phi^{\gamma\delta} K_{\gamma\delta\alpha 0} + \phi^{\gamma k} K_{\gamma k\alpha 0} = \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\alpha 0} + \phi^{\gamma 0} K_{\gamma 0\alpha 0} + \phi^{\gamma\delta} K_{\gamma\delta\alpha 0} \tag{5.51a}$$

$$0 = \Theta_{\alpha\beta} = \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\alpha\beta} + \phi^{\gamma 0} K_{\gamma 0\alpha\beta} + \phi^{\gamma\delta} K_{\gamma\delta\alpha\beta} + \phi^{\gamma k} K_{\gamma k\alpha\beta} = \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\alpha\beta} + \phi^{\gamma 0} K_{\gamma 0\alpha\beta} \tag{5.51b}$$

$$0 = \Theta_{\alpha i} = \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\alpha i} + \phi^{\gamma 0} K_{\gamma 0\alpha i} + \phi^{\gamma\delta} K_{\gamma\delta\alpha i} + \phi^{\gamma k} K_{\gamma k\alpha i} = \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\alpha i} + \phi^{\gamma k} K_{\gamma k\alpha i}. \tag{5.51c}$$

The third equality sign in the equations (5.51) uses the properties of the quadruple intersections numbers discussed in section 3.4. We want to express the terms involving $\phi^{\alpha I} K_{\alpha I\hat{F}\hat{G}}$ in terms of $\phi^{\hat{K}\hat{L}}$. From the equations 5.51 we get the relations

$$\begin{aligned} \phi^{\alpha 0} K_{\alpha 000} &= K^\gamma K^\delta \phi^{\alpha 0} K_{\alpha 0\gamma\delta} \\ &\overset{(5.51b)}{=} -K^\gamma K^\delta \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\gamma\delta} \end{aligned} \tag{5.52a}$$

$$\begin{aligned} \phi^{\alpha\beta} K_{\alpha\beta 00} &\overset{(5.51a)}{=} -(\phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\alpha\gamma} + \phi^{\gamma 0} K_{\delta 0\gamma 0}) K^\gamma \\ &\overset{(5.51b)}{=} (-\phi^{\hat{K}\hat{L}} K_{\gamma 0\hat{K}\hat{L}} + K^\delta \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\gamma\delta}) K^\gamma \end{aligned} \tag{5.52b}$$

$$\begin{aligned} \phi^{\alpha\beta} K_{\alpha\beta ik} &= -C_{ik} \Sigma^\delta \phi^{\alpha\beta} K_{\alpha\beta\delta 0} \\ &= (-\phi^{\hat{K}\hat{L}} K_{\alpha 0} + K^\delta \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\alpha\delta}) \Sigma^\alpha (-C_{ik}) \end{aligned}, \tag{5.52c}$$

where C_{ik} is the Cartan matrix and Σ^α is defined via $\Sigma = \Sigma^\alpha D^B_\alpha$. We have not expressed $\phi^{\alpha i}$ yet since it is more difficult. We will circumvent this problem later.
At next, we look at $\Theta^{\hat{K}\hat{L}}$ in terms of $\phi^{\hat{K}\hat{L}}$. Note that Θ_{i0} always vanishes. For Θ_{00} we compute

$$\begin{aligned} \Theta_{00} &= \phi^{\alpha 0} K_{\alpha 000} + \phi^{\alpha\beta} K_{\alpha\beta 00} + \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}00} \\ &= -K^\gamma K^\delta \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\gamma\delta} + \phi^{00} K_{0000} - K^\alpha \phi^{\hat{K}\hat{L}} + K^\alpha K^\delta \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\alpha\delta} \\ &= 0. \end{aligned} \tag{5.53}$$

This implies that only Θ_{ij} is non-trivial. When expressing the remaining Θ_{ij} in terms of $\phi^{\hat{K}\hat{L}}$, we reach the point where we need to deal with $\phi^{k\gamma} K_{ijk\gamma}$, which is nontrivial. We compute

$$-\phi^{kl} K_{kl\alpha i} (C^{-1})^{ij} W_{fg|j} \overset{5.51c}{=} \phi^{\gamma k} W_{\gamma k\alpha i} (C^{ij}) W_{fg|j} = \phi^{k\gamma} K_{fgk\gamma}. \tag{5.54}$$

The expression $W_{fg|i}$ is defined by $K_{fgi\alpha} = D^B_\alpha \Sigma W_{fg|i}$. Recall, that Σ is the class of the singular divisor in the base threefold. By carefully looking at the pushforward procedure 3.5, especially at equation (3.46) for the last blow-down step, we see that this definition is well defined.
We can use equation (5.54) for the remaining Chern-Simons part

$$\begin{aligned}\Theta_{ij} &= \phi^{\alpha\beta} K_{\alpha\beta ij} + \phi^{\alpha k} K_{\alpha kij} + \phi^{kl} K_{ijkl} && (5.55a)\\ &= (-\phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\alpha 0} + K^\delta \phi^{\hat{K}\hat{L}} K_{\hat{K}\hat{L}\alpha\delta})\Sigma^\alpha C_{ij} && (5.55b)\\ &= K^\delta \phi^{kl} K_{\alpha\delta kl})\Sigma^\alpha C_{ij} + \phi^{k\alpha} K_{ijk\alpha} && (5.55c)\\ &\overset{(5.54),(3.41)}{=} \phi^{kl} \left[K_{ijkl} + K\Sigma^2 C_{ij} C_{kl} - W_{klf} (C^{-1})^{fg} W_{ij|g} \right]. && (5.55d)\end{aligned}$$

From the last line (5.55d), we can read off the reduced intersection matrix

$$\tilde{M}_{(ij)(kl)} = K_{ijkl} + K\Sigma^2 C_{ij} C_{kl} - W_{klf} (C^{-1})^{fg} W_{ij|g}, \tag{5.56}$$

which coincides with the result of [9, equation C.13].
This completes the story of the chiral index, because we can write the chiral index for a given gauge algebra just in terms of the triple intersection numbers using the pushforward technique 3.5 and the G_4 flux input written in a new basis, such that it is consistent with the consistency conditions (4.31).

Let us apply this to the Georgi–Glashow $SU(5)$ model.

SU(5) Grand Unified Theory Model

Now we have all ingredients to express the chiral index of the $SU(5)$ theory in terms of triple intersection numbers and the flux-input $\tilde{\phi}^{kl}$, such that the flux is consistent with all the consistency conditions. We can compute the reduced intersection matrix (5.56) concretely using the quadruple intersection numbers of table 3 and 4. We have to combine this with the relation between the chiral matter index and the Chern-Simons term given in equation (5.44) to compute[17]

$$\begin{aligned}\chi_{\mathbf{5}} = -\chi_{\mathbf{10}} = -\Theta_{22} &= -\tilde{\phi}^{22}\tilde{K}_{2222} + -\tilde{\phi}^{24}\tilde{K}_{2422} + -\tilde{\phi}^{33}\tilde{K}_{3322} - \tilde{\phi}^{34}\tilde{K}_{3422} && (5.57)\\ &= (-\tilde{\phi}^{22} + \tilde{\phi}^{24} + \tilde{\phi}^{33} - \tilde{\phi}^{34})\frac{1}{5} K_B \Sigma (6K_B + 5\Sigma) && (5.58)\\ &= (-\tilde{\phi}^{22} + \tilde{\phi}^{24} + \tilde{\phi}^{33} - \tilde{\phi}^{34})\frac{1}{5} \Sigma \cdot [a_1] \cdot [a_{6,5}]. && (5.59)\end{aligned}$$

All other ϕ^{kl} have to be zero.

This closes the discussion of the dependency of the chiral index on the flux input, except for the question, how the input in the usual basis $D_I D_J$ would look like. We know that $\phi^{\hat{K}\hat{L}} = \tilde{\phi}^{\hat{K}\hat{L}}$. It remains to derive the components $\phi^{I\alpha}$ from the given components above. They are determined by the conditions (5.52). From the first one (5.52a), we see

$$\phi^{0\gamma} = -\phi^{00} K^\gamma - \phi^{ij} C_{ij} \Sigma^\gamma. \tag{5.60}$$

[17] We use different labels for the Cartan divisors compared to [9].

We can use this together with the second condition (5.52b) to determine the components involving two Greek letters to be

$$\begin{aligned} \phi^{\gamma\beta} D^B_\alpha D^B_\beta D^B_\gamma &= D^B_\alpha K^2 \phi^{00} - D^B_\alpha D^B_\beta D^B_\gamma K^\gamma \phi^{0\beta} \\ &\overset{(5.60)}{=} D^B_\alpha K^2 \phi^{00} - D^B_\alpha D^B_\beta D^B_\gamma K^\gamma (-\phi^{00} - \phi^{ij} C_{ij} \Sigma^\beta) \\ &= D^B_\alpha D^B_\beta D^B_\gamma K^\gamma \Sigma^\beta \phi^{ij} C_{ij} \\ \Rightarrow \phi^{\gamma\beta} &= K^\beta \Sigma^\gamma C_{ij} \phi^{ij}. \end{aligned} \tag{5.61}$$

The remaining components of the form $\phi^{j\beta}$ are much harder to compute. We have to solve

$$\phi^{j\beta} K_{i\alpha j\beta} = \phi^{kl} K_{i\alpha kl} \tag{5.62}$$

$$\Leftrightarrow \phi^{j\beta} D^B_\alpha D^B_\beta \Sigma = -\phi^{kl} (C^{-1})^{ij} K_{i\alpha kl}. \tag{5.63}$$

There is no way that the author is familiar with, how to do this in general. But in the $SU(5)$ model, we know every single component of $K_{i\alpha kl}$ by table 4 and which of the ϕ^{kl} components are non-zero by (5.57). Using this information, we obtain the missing components

$$\phi^{i\beta} = \frac{1}{5} \begin{pmatrix} -(8\phi^{22} + 2(\phi^{24} + \phi^{33}) + 3\phi^{34}) K^\beta - 5(2\phi^{22} + \phi^{33}) \Sigma^\beta \\ (-6\phi^{22} + \phi^{24} - 4\phi^{33} - 6\phi^{34}) K^\beta - 5(3\phi^{22} + 2\phi^{33}) \Sigma^\beta \\ (11\phi^{22} - \phi^{24} + 9\phi^{33} - 4\phi^{34}) K^\beta - 5\phi^{33} \Sigma^\beta \\ (8\phi^{22} + 2(\phi^{24} + \phi^{33}) + 3\phi^{34}) K^\beta + 5(\phi^{34} - \phi^{33}) \Sigma^\beta \end{pmatrix}. \tag{5.64}$$

The flux input $\phi^{i\beta}$ also has to satisfy the integrality condition (4.25), which imposes contrains on the flux input ϕ^{ij}. In [9] it was argued that this ensures the chiral index to be an integer despite the fraction in (5.59).

To conclude, we derived an algorithm that allows us to compute the chiral index $\chi_{\mathbf{R}}(\tilde{\phi}^{ij}, K^B_{\alpha\beta\gamma})$ in terms of the flux input $\tilde{\phi}^{ij}$ in the basis described above and the triple intersection numbers $K^B_{\alpha\beta\gamma}$ of the base threefold. For the $SU(5)$ model, we even derived $\chi_{\mathbf{R}}(\phi^{ij}, K^B_{\alpha\beta\gamma})$ in the canonical basis $[G_4] = \phi^{ij} D_{ij}$. The procedure above suggests that with a complete knowledge of the intersection numbers, that can be gained with the pushforward technique, it might be possible to write $\chi_{\mathbf{R}}(\phi^{ij}, K^B_{\alpha\beta\gamma})$ in the canonical flux basis for arbitrary gauge groups.

Let us take one step back and recall how we obtained the chiral index in terms of the Chern-Simons term $\chi_{\mathbf{R}}(\Theta_{ij})$. The key observation was the matching of the M-theory Chern-Simons action with the corresponding contribution on the type IIB side, which included the chiral index (5.25). The resources used to evaluate $\chi_{\mathbf{R}}$ in terms of Θ_{ij} are the weights of representation w_i, the values of the sign functions of the fermion masses $\text{sign}(M^f) = \text{sign}(\mathbf{w}_k \zeta^k)$ and the anomaly cancellation. The weights are purely Lie algebraic information, determined by the representation. The possible representations are determined by the gauge enhancement over the codimension two loci. If we restrict ourselves to $\mathfrak{su}(N)_{n\geqslant 3}$ Lie algebras and assume only the Tate model with general coefficients, we only have $\mathbf{N}$ and $\frac{1}{2}\mathbf{N}(\mathbf{N}-1)$ as representations [37]. The gauge algebra can be recovered from the Cartan matrix and the Cartan matrix in encoded in the intersection numbers via $K_{ij\alpha\beta} = -C_{ij}\Sigma^\gamma K^B_{\alpha\beta\gamma}$.

On the other side, the sign function was determined by equation (5.26) using again the weights as well as the intersection numbers involving three Cartan indices and the classes of the matter curves. The first piece of information is as well encoded in the quadruple intersection numbers of the resolved fourfold.

The anomaly cancellation was used to reduce the question of determining two chiral indices to the question of one independent chiral index. The entire algorithm presented here is not necessarily suitable for multiple independent chiral indices, if we do not want to explicitely solve the system of linear equations (5.25). It has been shown for certain models [9] that the chiral anomaly cancellation reduces the chiral index to one independend family.

Finally, the translation of the Chern-Simons term Θ_{ij} in the flux input ϕ^{ij} seems to be computable, at least into the basis $\tilde{\phi}^{ij}$, only with the quadruple intersection numbers as input.

To summarize, under the certain conditions mentioned previously, the chiral index can be fully determined using the quadruple intersection numbers of the resolved fourfold. The authors of [9] even found, that in all the models they studied, with the exception of $SO(11)$, that the number of independent chiral families matches with the rank of the reduced intersection matrix $\tilde{K}_{(\hat{I}\hat{J}),(\hat{K}\hat{L})}$.

The quadruple intersection numbers appear to be the gate to the chiral matter spectrum. We want to use the quadruple intersection numbers as a connection point to asymptotic Hodge theory, that will be introduced the following section. The same quadruple intersection numbers will appear in that theory as well, which suggest a deeper connection between the asymptotic Hodge theory and the chiral index.

6 Asymptotic Hodge Theory

We briefly mentioned the moduli space already in section 2.1. One can roughly think about the moduli space as the space of all deformations of the geometry of a Calabi-Yau manifold. Asymptotic Hodge theory has been sucessfully used to classify singularities in complex structure moduli space. The reason, why we are interested in this theory here, is that in a certain setting it comprises the quadruple intersection numbers. Technically speaking, the quadruple intersection numbers of a smooth Calabi-Yau fourfold appear in the log monodromy matrix associated to the singularities in the complex structure moduli space of the mirror manifold of that smooth, resolved Calabi-Yau fourfold. In particular, the space of all deformations of the complex-structure of a complex manifold is called the complex structure moduli space. This is the motivation to investigate the possibility of combining the previous story about the chiral index with the asmyptotic Hodge theory.
We start reviewing mixed Hodge structures in 6.1 and how they can be used to classify singularities in the moduli space of a Calabi-Yau manifold in 6.2. In the latter section we also see that quadruple intersections are arising from certain singularities. In section 6.3, we set the stage for investigations on possible connections between the chiral index and singularities in moduli space. We first extract some information about the intersection numbers from the classification of singularities and later discuss the significance of quadruple intersection numbers in the derivation of the chiral index.

6.1 Mixed Hodge structures

Before we review mixed Hodge structures, we start with the special case of a pure Hodge structure primarily following [72]. This might be known from the Hodge decomposition of the cohomology space of forms on a manifold, but we start slightly more general.

Pure Hodge Structure

A pure Hodge structure of weight $n \geqslant 0$ on the rational vector space V is given by the decomposition

$$V_{\mathbb{C}} := V \otimes \mathbb{C} = \bigoplus_{p+q=n} V_{\mathbb{C}}^{p,q} \quad \text{with} \quad \overline{V^{p,q}} = V^{q,p} \text{ and } p, q \in \mathbb{N}. \tag{6.1}$$

An equivalent definition of a pure Hodge structure is via the finite, decreasing Hodge filtration

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 = V_{\mathbb{C}} \quad \text{such that } V_{\mathbb{C}} = F^k \oplus \overline{F^{n+1-k}}. \tag{6.2}$$

These equivalent definitions are related by

$$F^k = \bigoplus_{p \geqslant k} V^{p,n-p} \quad \text{and} \quad V^{p,q} = F^p \cap \overline{F^q}. \tag{6.3}$$

At next we want to equip the Hodge structure with an additional structure called a polarization, which is given by the bilinear form Q on $V_{\mathbb{C}}$. A Hodge structure is called $Q-$polarized if the Hodge-Riemann relations

$$Q(V^{p,q}, V^{r,s}) = 0 \quad \text{for} \quad (p,q) \neq (r,s) \tag{6.4}$$

$$i^{p-q} Q(v, \overline{v}) > 0 \quad \text{for all} \quad 0 \neq v \in V^{p,q} \tag{6.5}$$

are satisfied.

Mixed Hodge Structure

A mixed Hodge structure can be defined by two filtrations W_ℓ and F^k, where W_ℓ is a decreasing filtration of V and F^k is an increasing filtration of $V_\mathbb{C}$ such that it induces a Hodge structure on the graded quotients

$$W_\ell^{gr} := W_\ell / W_{\ell-1} \tag{6.6}$$

with weight ℓ.

Let us illustrate this definition with an example. Let V be the sum of all rational cohomology spaces of a Kähler manifold Y

$$V = H(Y, \mathbb{Q}) = \bigoplus_n H^n(Y, \mathbb{Q}) \tag{6.7}$$

By defining the two filtrations

$$W_\ell = \bigoplus_{n \leqslant \ell} H^n(X, \mathbb{Q}) \tag{6.8}$$

$$F^k = \bigoplus_{p \geqslant k} H^{p,\bullet}(Y, \mathbb{C}) = \bigoplus_j \bigoplus_{p \geqslant k} H^{p,j}(Y, \mathbb{C}), \tag{6.9}$$

V is equipped with a mixed Hodge structure. The graded quotient of the increasing filtration is given by

$$W_\ell^{gr} = W_\ell / W_{\ell-1} = H^\ell(Y, \mathbb{Q}). \tag{6.10}$$

The decreasing filtration F^k defines a pure Hodge structure on the quotient W_ℓ^{gr} with weight ℓ via the Hodge filtration

$$F_\ell^k = \bigoplus_{p \geqslant k} H^{p,\ell-p}(Y, \mathbb{C}). \tag{6.11}$$

The condition (6.2) is satisfied via

$$H^\ell(Y, \mathbb{C}) = \left(\bigoplus_{p \geqslant k} H^{p,\ell-p}\right) \bigoplus \left(\bigoplus_{q<k} H^{q,\ell-q}\right) = \left(\bigoplus_{p \geqslant k} H^{p,\ell-p}\right) \bigoplus \left(\bigoplus_{-((\ell-p)+1) \geqslant -k} H^{\ell-p,p}\right) = F_\ell^k \oplus \overline{F^{\ell+1-k}}. \tag{6.12}$$

At next we want to discuss polarized, mixed Hodge structures. Before we come to the definition of a polarized mixed Hodge structure, let us mention that an increasing filtration W_ℓ with $W_{2D} = V$ can be described by a nilpotent endomophism $N : V \to V$ that satisfies the conditions

$$N(W_\ell) \subset W_{\ell-2} \tag{6.13}$$

and that N^ℓ induces an isomorphism $W_{D+\ell}^{gr} \to W_{D-\ell}^{gr}$ for all $0 \leqslant \ell \leqslant D$ [73, lemma 6.4], [12]. The increasing filtration can be computed inductively for a given N and is denoted by $W(N)$.
A Q-polarized mixed Hodge structure on V is a mixed Hodge structure given by the two filtrations (W, F), where $W_{2D} = V$, and a set of nilpotent, commuting endomorphisms $\{N_1 \dots N_r\}$ on V satisfying the conditions

- For all $i = 1 \dots r$ holds $N_i^{D+1} = 0$ and $W = W(N_i)$.

- F is Q-isotropic, i.e. $Q(F^p, F^q) = 0$ for all $p + q = 2D + 1$.
- $N_i F^p \subset F^{p-1}$
- F induces a Hodge structure on the primitive space

$$P(N_i)_\ell := \ker(N_i^{\ell+1} : W^{gr}_{D+\ell} \to W^{gr}_{D-\ell-2}) \tag{6.14}$$

which is polarized by $Q^N(\cdot, \cdot) := Q(\cdot, N^\ell \cdot)$.

Hodge Deligne Splitting

There is another way to encode a mixed Hodge structure, namely via the Deligne splitting, which is a decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}, \tag{6.15}$$

If we define the subspaces $I^{p,q}$ as

$$I^{p,q} = F^p \cap W_{p+q} \otimes \mathbb{C} \cap \left(\overline{F^q} \cap W_{p+q} \otimes \mathbb{C} + \overline{\sum_{j \geqslant 0} F^{q-1-j} \cap W_{p+q-2-j} \otimes \mathbb{C}} \right) \tag{6.16}$$

it relates to the filtrations via

$$F^p = \bigoplus_s \bigoplus_{p \geqslant r} I^{r,s}, \quad W_\ell \otimes \mathbb{C} = \bigoplus_{p+q \leqslant \ell} I^{p,q}. \tag{6.17}$$

This also implies that the complexified, graded quotient $W^{gr}_\ell \otimes \mathbb{C}$ admits the decomposition in terms of the $I^{p,q}$ splitting

$$W^{gr}_\ell \otimes \mathbb{C} = \oplus_{p+q=\ell} I^{p,q}. \tag{6.18}$$

We call a mixed Hodge structure $\mathbb{R}$-split, if $\overline{I^{p,q}} = I^{q,p}$ holds.
In the example above, we can compute that $I^{p,q} = H^{p,q}$ and hence this mixed Hodge structure is indeed $\mathbb{R}$-split.

6.2 Singularities in Moduli Space

We want to use the concept of mixed Hodge structures in order to investigate what happens to a Calabi-Yau manifold X with complex dimension D, if we approach a boundary of the complex structure moduli space $\mathcal{M}_{cs}$. A useful object for the study of Calabi-Yau manifolds is the period vector Π which is defined as the coefficients of the unique $(D, 0)$-form Ω in a fixed basis of the middle cohomology $H^D(X, \mathbb{C})$.
The period vector varies over the moduli space and actually diverges when approaching a singularity in moduli space. This means that the middle cohomology and even the entire Hodge decomposition is not valid at these singularities. The behaviour of the period vector near the singularity still gives lots of information about the singularity itself though.
Let us consider a singular point P in the moduli space. It may be part of a larger singularity which is locally of dimension $\dim(\mathcal{M}_{cs}) - n_p$. We fix a local coordinate system $z_1 \dots z_{n_P}$ such that $z_1 = \dots z_{n_P} = 0$ gives this larger singularity in which P lies. This means that P lies on the intersection of the singular locis $\{z_j = 0\}$

in the local picture. When going around one of these singular locis (symbolized by $z \to e^{2\pi i}z$), the period vector picks ups a monodromy

$$\Pi(z_j) \to \Pi(e^{2\pi i}z_j) = T_j\Pi. \tag{6.19}$$

It turns out that monodromy matrices T_j commute and are quasi-unipotent [12], i.e. there exist $m_i, n_i \in \mathbb{N}$ such that

$$(T_j^{m_j} - \mathbb{1})^{n_j} = 0. \tag{6.20}$$

From this we can conclude the nilpotency of the log-monodromy matrices which are defined by

$$N_j := \frac{1}{m_j}\ln(T_j^{m_j}), \tag{6.21}$$

where m_j are the minimal numbers satisfying equation (6.20).
In the limit of the singular loci $\{z_j = 0\}$, the Hodge filtration F^p approaches the limit

$$F_0^p := \lim_{z_j\to 0}\exp\left[-\frac{1}{2\pi i}\ln(z_j)N_j\right]F^p. \tag{6.22}$$

The filtration F_0^p, called the limiting Hodge filtration, form no longer a polarized Hodge filtration, because the bilinear form degenerates. But instead, together with the information of the matrix N_j, $(W(N_j), F_0, N_j)$ forms a polarized mixed Hodge structure. This is a consequence of Schmids Sl_2-orbit theorem [73]. This polarized mixed Hodge structure contains information about the singularity. It is a useful to use the Deligne splitting $I^{p,q}$ of this Hodge structure. It can be shown that by a change of basis of $H^D(X,\mathbb{C})$, the Deligne splitting can be made $\mathbb{R}$-split. The complex dimension of the spaces $I^{p,q}$ is denoted by $i^{p,q}$ and can be arranged in a diagram called the Hodge-Deligne diamond. For a fourfold, this is depicted in figure 2. These numbers are related to the Hodge numbers by [12]

$$\sum_{q=0}^{D} i^{p,q} = h^{p,D-p}. \tag{6.23}$$

Moreover, they have the properties

$$0 \leqslant i^{p,q} = i^{q,p} = i^{D-p,D-q} \qquad \text{for all}\, p,q \tag{6.24}$$
$$i^{p,q} \geqslant i^{p-1,q-1} \qquad \text{for}\, p+q \leqslant D. \tag{6.25}$$

In addition, we like to mention the fact [12]

$$N_j I^{p,q} \subset I^{p-1,q-1}. \tag{6.26}$$

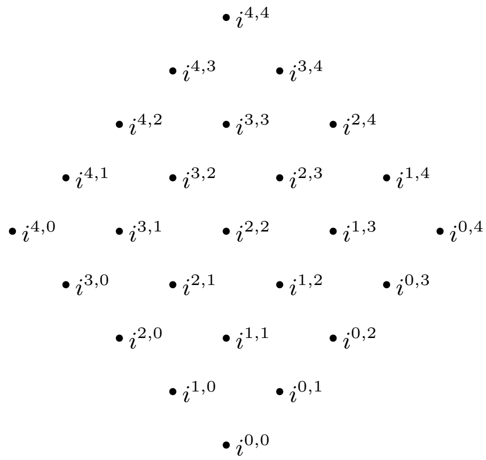


Figure 2: Arrangement of the Deligne splitting $i^{p,q} = \dim(I^{p,q})$ for Calabi-Yau fourfolds.

Note that, due to $h^{D,0}(X, \mathbb{C}) = 1$, we always have $\sum_{q=0}^{D} i^{p,q} = 1$ and thus only for one $i^{D,q} = 1$ and the other terms vanish. We can assign classes labeled by Latin numerals I, II, III, IV, V to Hodge-Deligne diamonds associated to the number $q = 0, 1, 2, 3, 4$ where $i^{D,q} = 1$.

If we approach the point P which lies at the intersection of multiple singular loci in moduli space

$$P \in \bigcap_j \{z_j = 0\}, \tag{6.27}$$

the limiting Hodge filtration F_0^p is obtained by

$$F_0^p = \lim_{z_j \to 0} \exp\left[-\frac{1}{2\pi i} \sum_{j=1}^{n_P} \ln(z_j) N_j\right] F^p. \tag{6.28}$$

This limiting Hodge filtration, together with the matrix $\sum_{j=1}^{n_p} N_j$ gives rise to another Hodge-Deligne diamond. This also implies that at the intersection of two singular loci, the mixed Hodge-structure might change. For higher codimension singularities, this can be realized as a network of Hodge-Deligne diamonds. Given the type of the Hodge-Deligne diamond, the Hodge-Deligne diamond can be fully computed by the rank of powers of the corresponding nilpotent log-monodromy matrix $N = \sum_{j \in J \subseteq \mathbb{N}_{\leqslant n_p}} N_j$. The reason for this lies in the condition, that the log monodromy matrices induce an isomorphism

$$N^\ell : \bigoplus_{p+q=D+\ell} I^{p,q} \overset{\cong}{\to} \bigoplus_{p+q=D-\ell} I^{p,q}. \tag{6.29}$$

We used that, due to equation (6.6), we can write the quotient (6.17) as

$$W_\ell^{gr} \otimes \mathbb{C} = \oplus_{p+q=\ell} I^{p,q}. \tag{6.30}$$

Moreover, we can see that if $N_A^4 \neq 0$, the Hodge-Deligne diamond has to be of type V and if $N_A^3 \neq 0$, it has be at least of rank IV with $\mathrm{rk}(N_A^3) = 2$.
This is the first step of a classification of Hodge-Deligne diagrams. A classification of Hodge-Deligne diagrams also means a classification of the polarized, mixed Hodge structures of $H^D(X, \mathbb{C})$ and therefore a classification of the singularities in the complex-structure moduli space.

In the following we want to focus on a specific kind of singularities in the complex structure moduli space of smooth Calabi-Yau fourfolds X, namely the large complex-structure regime. The complex structure moduli space has dimension $h^{3,1}(X)$ and its coordinates are complex structure moduli t^j. We can define the singularity in the moduli space of X by the limits [11]

$$t^j = x^j + iy^j = \frac{1}{2\pi i}\log(z^j) \to i\infty. \tag{6.31}$$

We want to make contact with the F-theory setting, as discussed previously. Therefore, we consider again the smooth Calabi-Yau fourfold $\hat{Y}_4$, which is the resolved fourfold of a singular elliptically fibered fourfold Y_4 over a base threefold B_3. The fourfold $\hat{Y}_4$ has $h^{1,1}(\hat{Y})$ Kähler moduli. Let X denote the mirror fourfold of $\hat{Y}_4$. We will not review mirror symmetry but only refer to the reviews [74, 75]. A key feature of mirror symmetry is, that the complex-structure moduli space and the Kähler moduli space are exchanged under the mirror map. Thus X has $h^{1,1}(\hat{Y}_4)$ complex-structure moduli.
We assume the Kähler cone to be simplicial, which implies that there are $h^{1,1}(\hat{Y}_4)$ generators of the Kähler cone. Let D_Ω be a basis of the Kähler cone (in the following indexed by capital Greek letters). Finding this basis in terms of our known basis D_I is in general not an easy task. The Kähler cone is dual to the Mori cone of effective holomorphic curves. There is exists an algorithm [76, 77] to determine the Mori cone and hence the Kähler cone. There is no general formula $D_\Omega(D_I)$ known.
The space $H^{2,2}(\hat{Y}_4)$ is spanned by $D_{\Lambda\Sigma} = D_\Lambda \cdot D_\Sigma$, but this basis might be degenerate. Let H_μ denote a non-degenerate basis. So there exists a linear map ζ such that

$$D_{\Lambda\Sigma} = \zeta^\mu_{\Lambda\Sigma} H_\mu. \tag{6.32}$$

With this, we can reproduce the quadruple intersection numbers as

$$K_{\Lambda\Sigma\Gamma\Delta} = \zeta^\mu_{\Lambda\Sigma} H_\mu \cdot H_\nu \zeta^\nu_{\Gamma\Delta} = \zeta^\mu_{\Lambda\Sigma}\eta_{\mu\nu}\zeta^\nu_{\Gamma\delta} \tag{6.33}$$

with $\eta_{\mu\nu} = H_\mu \cdot H_\nu$.
We introduced this notation because in [10] it has been found that the log-monodromy matrix N_Ω with D_Ω being a generator of the Kähler cone, for the singularity $t_\Omega \to i\infty$ in the complex-structure moduli space of the mirror fourfold X of $\hat{Y}_4$ takes the form

$$N_\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\delta_{\Omega\Lambda'} & 0 & 0 & 0 & 0 \\ 0 & -\eta_{\mu'\rho}\zeta^\rho_{\Omega\Sigma} & 0 & 0 & 0 \\ 0 & 0 & -\zeta^\mu_{\Omega\Delta'} & 0 & 0 \\ 0 & 0 & 0 & -\delta_{\Omega\Delta} & 0 \end{pmatrix}. \tag{6.34}$$

We denote the row labels with a prime and the column labels without a prime. The matrix N_Ω acts on vectors in the basis $(1, D_\Sigma, H_\mu, D_\Delta, 1)^T$.

We see that the quadruple intersection numbers of $\hat{Y}_4$ appear in the log monodromy matrices N_Ω. This is exactly the connection point to the F-theory and chiral matter picture described in the previous section. We will exploit this connection further in the next subsection.

6.3 Applications to F-theory

In this subsection we want to deepen the discussion about the connection of large complex-structure singularities, preferable in terms of Hodge-Deligne diamonds, and the chiral index. The connection point is, as explained earlier, the quadruple intersection numbers appearing in the log-monodromy matrices N_Ω given in (6.34) and in the computation of the chiral index depending on the flux input.
There are now two approaches that we want to follow: At first we want to use our knowledge about the intersection numbers from previous sections to gain information about the Hodge-Deligne diamonds. We do this to investigate how much information about the intersection numbers we can extract for a given network of singularities. Similar analysis has been done for elliptically fibered threefolds in [78]. The second approach is to start with the chiral index and decompose it in the terms of the information that it consists of. Finally we discuss how far these loose ends can be tied together.

6.3.1 Hodge-Deligne Diamonds from Quadruple Intersection Numbers

We start with the Hodge-Deligne diamond that we try to determine computing the ranks of powers of the log-monodromy matrices N_Ω in equation (6.34).
We compute

$$N_\Omega^2 = \begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ \eta_{\mu'\rho}\zeta^\rho_{\Omega\Omega} & 0 & 0 & & \\ 0 & K_{\Omega\Sigma\Omega\Delta'} & 0 & 0 & \\ 0 & 0 & \zeta^\mu_{\Omega\Omega} & 0 & 0 \end{pmatrix}, \tag{6.35a}$$

$$N_\Omega^3 = \begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ 0 & 0 & 0 & & \\ -K_{\Omega\Omega\Omega\Delta'} & 0 & 0 & 0 & \\ 0 & -K_{\Omega\Omega\Omega\Sigma} & 0 & 0 & 0 \end{pmatrix}, \tag{6.35b}$$

$$N_\Omega^4 = \begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \\ K_{\Omega\Omega\Omega\Omega} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.35c}$$

We sum over indices that appear double, except of Ω.

Even if the generators of the Kähler cone are known $D_\Lambda(D_I)$, it is still difficult to determine the rank of the log-monodromy matrix N_Ω due to the presence of the ζ-term depending on the basis of $H^{\text{vert}}_{2,2}(\hat{Y})$.
The set of formal products $\{D_\Lambda D_\Sigma\}$ might have more elements than the dimension of $H^{\text{vert}}_{2,2}$ and therefore it degenerates as a basis. Let V denote the space[18] spanned by $D_\Lambda D_\Sigma$ as independent symbols. We can interpret ζ as the projection in V on $H^{\text{vert}}_{2,2}$. According to [9], the reduced intersection matrix $M_{(\Lambda\Sigma)(\Gamma\Delta)}$, which is defined by dividing out the nullspace of the intersection matrix, gives a non-degenerate intersection

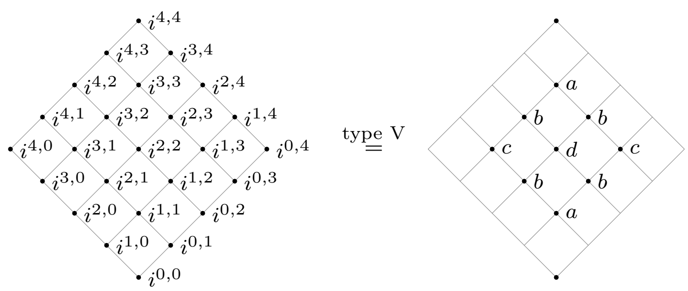
[18]actually, it is a lattice, since we are looking at $H^{\text{vert}}_{2,2}(\hat{Y}_4, \mathbb{Z})$.

pairing on $H^{\text{vert}}_{2,2}$.
This means that $\eta_{\mu\nu}$ is non-degenerate as a bilinear form due to (6.33) and we have $\text{rk}(\eta_{\mu\nu}) = \text{rk}(M_{(\Lambda\Sigma)(\Gamma\Delta)})$. Although $M_{(\Lambda\Sigma)(\Gamma\Delta)}$ is not necessarily a projection on $H^{\text{vert}}_{2,2}$, it is surjective on that space and zero everywhere else. Therefore, $\zeta^{\mu}_{\Lambda\Sigma}$ can be replaced by $\delta^{(\Gamma\Delta)}_{(\Lambda\Sigma)} M_{(\Gamma\Delta)(\Pi\Psi)}$[19] without a change of the rank. Doing this replacement, we get the matrix $\tilde{N}_\Omega$ with $\text{rk}(\tilde{N}_\Omega) = \text{rk}(N_\Omega)$ and

$$\tilde{N}_\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\delta_{\Omega\Lambda'} & 0 & 0 & 0 & 0 \\ 0 & -M_{(\Gamma'\Delta')\Omega\Sigma} & 0 & 0 & 0 \\ 0 & 0 & -M_{\Delta'\Omega(\Lambda\Sigma)} & 0 & 0 \\ 0 & 0 & 0 & -\delta_{\Omega\Delta} & 0 \end{pmatrix}. \tag{6.36}$$

The ranks of all powers of the log-monodromy matrices can now be determined by the quadruple intersection numbers of the generators of the Kähler cone $K_{\Lambda\Sigma\Gamma\Delta}$.

Next we briefly touch how the ranks of N^k_Ω can be used to determine the diamond.
If $N^4_\Omega \neq 0$, it follows from equation (6.26) that the corresponding Hodge-Deligne diamond has to be of type V. Utilizing the relation (6.24) of the $i^{p,q}$ entries, a Hodge-Deligne diamond of type V takes the form as depicted in 6.37.



(6.37)

A dot without additional label represents a one.
We can use equation (6.23) to determine two of the four numbers in (6.37) in terms of the Hodge numbers $h^{1,3}(X)$ and $h^{2,2}(X)$, i.e.

$$c = h^{1,3}(X) - a - b \tag{6.38}$$
$$d = h^{2,2}(X) - 2b. \tag{6.39}$$

Only a, b are unknown yet but they will be related to the rank of N_Ω and N^2_Ω in the following.
We start analyzing the isomorphisms (6.29) for $\ell = 2$. According to equation (6.26), the operator N^2_Ω maps the one-dimensional space $I^{4,4}$ to the d dimensional space $I^{2,2}$ and from this space into the one dimensional space $I^{0,0}$ which cannot be the zero map. That first map and the second map have both a one-dimensional image. In addition we know that N^2_Ω induces an isomorphism between the a-dimensional spaces $I^{3,3}$ and $I^{1,1}$. Therefore we have

$$\text{rk}(N^2_\Omega) = a + 2. \tag{6.40}$$

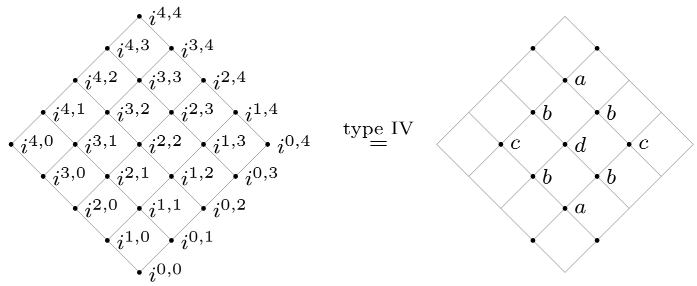
[19] In practice, we have to be a bit careful here, because $(\Lambda\Sigma)$ and $(\Sigma\Lambda)$ denote the same index.

In the same way, we can analyze the rank of N_Ω to be

$$\mathrm{rk}(N_\Omega) = 2a + 2b + 2, \tag{6.41}$$

because N_Ω induces an isomorphism on $I^{2,3} \to I^{1,3}$ and their complex conjugates.

In an analog way, if $K_{\Omega\Omega\Omega\Omega} = 0$ and $K_{\Omega\Omega\Omega\Sigma} \neq 0$, we are in type IV. A Hodge-Delinge diamond of type IV takes the form



(6.42)

Using again the isomorphism in equation (6.29), we get the relations

$$\mathrm{rk}(N_\Omega) = 4 + 2a + 2b \tag{6.43a}$$

$$\mathrm{rk}(N_\Omega^2) = 4 + a \tag{6.43b}$$

$$\mathrm{rk}(N_\Omega^3) = 2 \tag{6.43c}$$

$$\mathrm{rk}(N_\Omega^4) = 0 \tag{6.43d}$$

where a and b are the unknown components in the interior of the Hodge-Deligne diamond 6.42.

At next, we assume that the Kähler cone is generated by $D_\Lambda = D_{\tilde{I}}$ with $I = (\tilde{0}, \alpha, i)$[20]. The component $D_{\tilde{0}}$ is defined by the shift

$$D_{\tilde{0}} := D_0 - K. \tag{6.44}$$

[20] At the first look, this is a rather restrictive assumption. For a smooth elliptic fibration, i.e without any blow-ups, the Kähler cone is generated by $D_{\tilde{0}}, D_\alpha$, where D_α are choosen such that D_α^B generate the Kähler cone of the base [79]. The key question, how the Kähler cone changes under blow-ups, remains.

Viewed as a matrix $M_{(\tilde{I}\tilde{J})(\tilde{K}\tilde{L})} = K_{\tilde{I}\tilde{J}\tilde{K}\tilde{L}}$, the intersection numbers in the basis (6.44) take the form

$$\begin{array}{c|cccccc}
 & \tilde{0}\tilde{0} & \tilde{0}\alpha & \alpha\beta & \tilde{0}i & \alpha i & ij \\
\hline
\tilde{0}\tilde{0} & -K^3 & K^2 D_\alpha & -K D_\alpha D_{\alpha'} & 0 & 0 & K^\gamma K^\delta K_{\gamma\delta ij} \\
\tilde{0}\alpha' & K^2 D_{\alpha'} & -K D_\alpha D_{\alpha'} & D_{\alpha'} D_\alpha D_\beta & 0 & 0 & -K^\gamma K_{\alpha'\gamma ij} \\
\alpha'\beta' & -K D_{\alpha'} D_{\beta'} & D_\alpha D_{\alpha'} D_\beta & 0 & 0 & 0 & K_{\alpha'\beta' ij} \\
\tilde{0}i' & 0 & 0 & 0 & K^\gamma K^\delta K_{\gamma\delta i' i} & -K^\gamma K_{\gamma\alpha i i'} & -K^\gamma K_{\gamma i j i'} \\
i'\alpha' & 0 & 0 & 0 & -K^\gamma K_{\alpha\gamma i i'} & K_{\alpha\alpha' i i'} & K_{\alpha' i' i j} \\
i'j' & K^\gamma K^\delta K_{\gamma\delta i' j'} & -K^\gamma K_{\gamma i i' j'} & K_{\alpha\beta i' j'} & -K^\gamma K_{\gamma i i' j'} & K_{\alpha i i' j'} & K_{i j i' j'}
\end{array} \tag{6.45}$$

We see that the first column and the first row are linear combinations of the second line and the second row respectively.

We start with a systematic analysis of the singularities, beginning with $\Omega = \tilde{0}$.
Our first observation is that the diagram corresponding to the singularity $t_{\tilde{0}} \to i\infty$ is of type V, if $K^3 \neq 0$, because $K_{\tilde{0}\tilde{0}\tilde{0}\tilde{0}} = -K^3$ [21].

We now start computing the rank of $N_{\tilde{0}}^2$, which is given in (6.35a). The components involving $\zeta_{\tilde{0}}^\mu$ must be non-vanishing, because otherwise $K_{\tilde{I}\tilde{J}\tilde{0}\tilde{0}} = 0$ for all $\tilde{I}, \tilde{J}$, which is not true. Therefore we are left with the rank of $K_{\tilde{0}\tilde{K}\tilde{0}\tilde{L}'}$.
From (6.45), we can extract the relevant matrix

$$K_{\tilde{0}\tilde{J}\tilde{0}\tilde{L}'} = \begin{pmatrix} -K^3 & K^2 D_\alpha & 0 \\ K^2 D_{\alpha'} & -K D_\alpha D_\alpha & 0 \\ 0 & 0 & K^\gamma K^\delta K_{\gamma\delta i' i} \end{pmatrix}, \tag{6.46}$$

which has rank $\mathrm{rk}(\mathfrak{g}) + \mathrm{rk}(K D_\alpha D_{\alpha'})$. With equation (6.41) we get

$$a_{\tilde{0}} = \mathrm{rk}(\mathfrak{g}) + \mathrm{rk}(K D_\alpha D_{\alpha'}). \tag{6.47}$$

Next we focus on the rank of $N_{\tilde{0}}$ using the auxiliary matrix $\tilde{N}$ defined in (6.36). For $\Omega = \tilde{0}$, we have to compute the rank of $M_{L'A(IJ)} = K_{L'AIJ}$, which can be thought of as a linear map $H_{2,2}^{\mathrm{vert}} \to H_{1,1}$. Looking at (6.45), we can delete the $D_{\tilde{0}} D_{\tilde{0}}$-components for the computation of the rank because they are a linear combination of the others. The components of the remaining matrix are

$$\begin{array}{c|ccc}
 & \alpha\beta & \alpha i & ij \\
\hline
\tilde{0}\alpha' & D_{\alpha'} D_\alpha D_\beta & 0 & -K^\gamma K_{\alpha'\gamma ij} \\
\tilde{0}i' & 0 & -K^\gamma K_{\gamma\alpha i i'} & -K^\gamma K_{\gamma i j i'}
\end{array} \quad .$$

Since the intersection pairing on the base $(D_\alpha D_\beta) \cdot D_{\alpha'}$ is non-degenerate and the Cartan matrix is as well, we obtain the rank $h^{1,1}(B_3) + \mathrm{rk}(\mathfrak{g})$.
Using the fact that $h^{1,3}(X) = h^{1,1}(\hat{Y}_4) = 1 + h^{1,1}(B_3) + \mathrm{rk}(\mathfrak{g})$, we can specify the values of a, b, c and d in

[21] We first shift the basis back to our old basis (6.44) and then we can apply equations (3.36).

the Hodge-Deline diamond (6.37) to be

$$a_{\tilde{0}} = \text{rk}(\mathfrak{g}) + \text{rk}(KD_\alpha D_{\alpha'}) \tag{6.48}$$
$$b_{\tilde{0}} = h^{1,1}(B_3) - \text{rk}(KD_\alpha D_{\alpha'}) \tag{6.49}$$
$$c_{\tilde{0}} = 1 \tag{6.50}$$
$$d_{\tilde{0}} = h^{2,2}(\hat{Y}_4) - 2h^{1,1}(B_3) + 2\text{rk}(KD_\alpha D_{\alpha'}). \tag{6.51}$$

At next we want focus on $\Omega = i$. Due to the complexity of the quadruple intersection numbers involving three and four Cartan indices, we have to presume more assumptions and general results are rather speculative.
At first, we see that we are in type V again, if $K_{iiii} \neq 0$. This holds for the $SU(5)$ case (see table 3), if the emerging combinations of the base divisors K and Σ are non-vanishing. We suggest that this holds in general, but a proof is still missing. We continue assuming being in type V. Under this assumption, the rank of N_i^2 is

$$\text{rk}(N_i^2) = 2 + \text{rk}\begin{pmatrix} K_{\alpha\alpha' ii} & K_{\alpha' jii} \\ K_{j\alpha' ii} & K_{j' jii} \end{pmatrix}. \tag{6.52}$$

For the $SU(5)$ model with generic base intersection, we can compute that $\text{rk}(M_{ii(j')(j)}) = 4 = \text{rk}(\mathfrak{su}_5)$ for all $i = 1 \ldots 4$. The complete determination is difficult, because it involves detailed knowledge about the intersection ring of the base.
We face similar problem in the computation of $\text{rk}(N_i)$. There we can use the trick of replacing N_i with $\tilde{N}_i$ given in (6.36), but at this stage, we cannot conclude any further statement.

Finally, we take a look at $\Omega = \alpha$. As already, mentioned, we have $N_\alpha^4 = 0$ for all α. Under the assumption that $(D_\alpha^B)^3 \neq 0$, we are in type IV because $\text{rk}(N_\alpha^3) = 2$. According to equation (6.43), we have

$$(\text{rk}(N_\alpha), \text{rk}(N_\alpha^2), \text{rk}(N_\alpha^3), \text{rk}(N_\alpha)^4) = (4 + 2a_\alpha + 2b_\alpha, 4 + a_\alpha, 2, 0). \tag{6.53}$$

The rank of N_α^2 can be computed to be $4 + \text{rk}(\mathfrak{g})$ if $(D_\alpha^B)^2 \neq 0$. We use again $\tilde{N}_\alpha$ instead of N_α and compute

$$\text{rk}(N_\alpha) = 2 + 2(1 + \text{rk}(\mathfrak{g}) + \text{rk}(D_\alpha D_{\beta'} D_\beta)). \tag{6.54}$$

In summary, the Hodge-Deligne diamond for $\Omega = \alpha$ is determined by

$$a_\alpha = \text{rk}(\mathfrak{g}) \tag{6.55}$$
$$b_\alpha = \text{rk}(D_\alpha D_\beta D_{\beta'}) \tag{6.56}$$

and c and d are determined by the Hodge numbers as discussed above.

We can conclude that for this Kähler cone the computation of the Hodge-Deligne diamonds depends on the information about the base threefold. But we can look at this the other way around: Given all the Hodge-Deligne diamonds, we can extract information about the base. In particular, we can obtain the rank of $D_\alpha^B D_\beta^B D_{\beta'}^B$ for fixed α and $KD_\beta^B D_{\beta'}^B$. Due to the missing insights on a general structure of the quadruple intersection number involving four Cartan indices, the information that can be gained from the large complex structure regime might be even richer.

6.3.2 The Chiral Index

We shift our focus now to the chiral index and discuss possible connections to the previous story. As already indicated at the end of section (5.4), for a given gauge theory, the only ingredient of the chiral index, next to information regarding the representation theory, is the geometry in terms of the quadruple intersection numbers. Let us revisit this in more detail. We start with the question, to which extent the chiral index can be written in terms of the quadruple intersection numbers.

We can view $\chi_{\mathbf{R}}$ as a vector indexed by $\mathbf{R}$. The chiral anomaly cancellation can be viewed as the projection denoted $P_{\mathfrak{g}}$ on a subspace. Rewriting equation (5.41), this subspace is the kernel of the map

$$\chi \mapsto \sum_{\mathbf{R}} \left(\sum_{\mathbf{w_R}} \mathbf{w}_{(i} \mathbf{w}_j \mathbf{w}_{k)} \right) \chi_{\mathbf{R}}. \tag{6.57}$$

The dimension of this kernel gives the number of independent chiral matter families. It is one for the $SU(5)$ case in a generic Tate model. A chiral index vector satisfying the anomaly cancellation condition satisfies $P_{\mathfrak{g}}\chi = \chi$.

Recall that a key intermediate step in the derivation of the chiral index in terms of the Chern-Simons terms $\chi_{\mathbf{R}}(\Theta_{ij})$ was the matching with the loop contribution (5.25). We can write this as

$$2\Theta_{ij} = -\sum_{\mathbf{R}} \left(\sum_{\mathbf{w_R}} \mathbf{w}_i \mathbf{w}_j \mathrm{sign}(\mathbf{w}_k \zeta^k) \right) \chi_{\mathbf{R}} = -\sum_{\mathbf{R}} \underbrace{\left(\sum_{\mathbf{w_R}} \mathbf{w}_i \mathbf{w}_j \mathrm{sign}(\mathbf{w}_k \zeta^k) \right) P_{\mathfrak{g}}}_{=:\mathcal{A}^{\mathbf{R}}_{ij}} \chi_{\mathbf{R}} = -(\mathcal{A}\chi)_{ij} \tag{6.58}$$

The (pseudo-)inverse for $\mathcal{A}$ is not necessarily unique. One solution is given by the matter surface, if this lies in the vertical cohomology (see equation (4.18)). If there is only one chiral index family, another possibility is

$$P_{\mathfrak{g}}\chi = -2 \frac{\sum_{i,j} \left[\Theta_{ij} \left(\sum_{\mathbf{w_R}} \mathbf{w}_i \mathbf{w}_j \mathrm{sign}(\mathbf{w}_k \zeta^k) \right) P_{\mathfrak{g}} \right]}{\sum_{i,j} \left[\left(\sum_{\mathbf{w_R}} \mathbf{w}_i \mathbf{w}_j \mathrm{sign}(\mathbf{w}_k \zeta^k) \right) P_{\mathfrak{g}} \left(\sum_{\mathbf{w_R}} \mathbf{w}_i \mathbf{w}_j \mathrm{sign}(\mathbf{w}_k \zeta^k) \right) \right]}. \tag{6.59}$$

Note that $\left(\sum_{\mathbf{w_R}} \mathbf{w}_i \mathbf{w}_j \mathrm{sign}(\mathbf{w}_k \zeta^k) \right) P_{\mathfrak{g}}$ is a number for given i and j.

It remains to discuss the sign-function as it has been done in section 5.3.1. We can rewrite the crucial equation (5.26) to

$$\sum_{\mathbf{R},\mathbf{w_R}} \underbrace{(C_{\mathbf{R}} \mathbf{w}_i \mathbf{w}_j \mathbf{w}_k)}_{=:\mathcal{V}_{\mathbf{R},\mathbf{w_R},(ijk)}} \mathrm{sign}(\mathbf{w}_{\mathbf{R},l} \zeta^l) = W_{ijk}. \tag{6.60}$$

It has been argued in [9], that one can determine all the sign-functions with this equation. Therefore, we can symbolically write a (pseudo-) inverse for $\mathcal{V}$ such that

$$\mathrm{sign}(\mathbf{w}_{\mathbf{R},l} \zeta^l) = (\mathcal{V}^{-1})^{(ijk)}_{\mathbf{w_R}} W_{ijk}. \tag{6.61}$$

Note that $\mathcal{V}^{-1}_{\mathbf{w_R}}$ is in the dual of the Chow group $\mathrm{CH}^2(B_3)$.

Inserting this in equation (6.59) and using equation (5.45) gives

$$\chi_{\mathbf{R}} = 2 \frac{\sum_{i,j} \left[\phi^{KL} K_{ijKL} \left(\sum_{\mathbf{w_R}} \mathbf{w}_i \mathbf{w}_j (\mathcal{V}^{-1})^{(klm)}_{\mathbf{w_R}} W_{klm} \right) P_{\mathfrak{g}} \right]}{\sum_{i,j} \left[\left(\sum_{\mathbf{w_R}} \mathbf{w}_i \mathbf{w}_j (\mathcal{V}^{-1})^{(klm)}_{\mathbf{w_R}} W_{klm} \right) P_{\mathfrak{g}} \left(\sum_{\mathbf{w_R}} \mathbf{w}_i \mathbf{w}_j (\mathcal{V}^{-1})^{(klm)}_{\mathbf{w_R}} W_{klm} \right) \right]}. \tag{6.62}$$

The next step towards the asymptotic Hodge theory involves the replacement of the quadruple intersection numbers with the log-monodromy matrices. It is not obvious how to do this. We postpone further discussion and speculations to the conclusion.

Considering again the $SU(5)$ model, we can use our knowledge about weights and matter curves and hence about the chiral index. It is possible to combine our result on the chiral matter index in equation (5.44) and use the log monodromy matrices of equation (6.34) to express the chiral index in terms of the log monodromy matrices

$$\chi_{\mathbf{5}} = -\Theta_{33} = -\phi^{IJ} K_{IJ33} = (0,0,\phi^{IJ},0,0) N_3 (0, \delta_{3J} 0, 0, 0)^T. \tag{6.63}$$

This shows that with sufficient additional information in form of equation (5.44), it is possible extract the chiral index from a singularity in the mirror moduli space.

As described in section 5, the chiral index for the $SU(5)$ model can be determined by equation (5.59) in terms of triple intersection numbers instead of quadruple intersection numbers

If we view the chiral index as an assignment of a number to a given consistent flux imput, the only unknown numbers in the formula (5.59) are $K_B^2\Sigma$ and $K_B\Sigma^2$. This reduces the necessary information coming from the singularity to two integers.
Information about the base manifold is needed in order to compute these numbers, but also the choice of Σ in the base threefold has to be determined. The quadruple intersection numbers contain this information.
In details, we have

$$K_B^2\Sigma = -\frac{1}{4} K^\alpha K^\beta K_{\alpha\beta ij} (C^{-1})^{ij}). \tag{6.64}$$

The singular divisor Σ can be fully determined from

$$D_\alpha^B D_\beta^B \Sigma = -\frac{1}{4} K_{\alpha\beta ij} (C^{-1})^{ij},$$

if the isomorphism $H^{1,1}(B_3) \to H^{2,2}(B_3)$ is known. This isomorphism is determined by the triple intersection numbers of the base divisors. These intersection numbers are encoded by

$$D_\alpha^B D_\beta^B D_\gamma^B = K_{0\alpha\beta\gamma}.$$

Moreover, the coefficients K^α of the canonical divisor can be determined by $K_{00\beta\gamma} = K^\alpha K_{0\alpha\beta\gamma}$.
The chiral index in the $SU(5)$-model is therefore completely determined by the quadruple intersection numbers $K_{0\alpha\beta\gamma}, K_{00\alpha\beta}$ and $K_{\alpha\beta ij}$, which does not involve the more complicated quadruple intersection number $K_{\alpha ijk}$ and K_{ijkl}.

7 Conclusion

In this work, we discussed crucial aspects about chiral matter in F-theory compactifications. We reviewed the approach described in [9] of computing the chiral index in 4D F-theory without further knowledge about the threefold base in terms of a vertical G_4 flux background and triple intersections of the threefold base. A key aspect involves utilizing the duality with M-theory in 3D. In particular, we matched the Chern-Simons term in the 3D effective M-theory action to the corresponding contribution on the type IIB side, which captures the chiral index. In this analysis, we considered only fluxes that lie in the vertical part of the cohomology.
For concreteness, we focused on the $SU(5)$ model because it provides a famous candidate for a grand unified extension of the Standard Model (called Georgi–Glashow model) and it is the smallest $SU(N)$-model admitting chiral matter. Various consistency conditions restrict the G_4 flux. We provide a full description of the consistent G_4-flux in the canonical basis for the $SU(5)$ model. To the authors knowledge, this has not been done before.
Another focus of this work lies in the investigation of the connection between the chiral index and asymptotic Hodge theory. At the first view, these two topics do not seem related, but the motivation to study a possible connection lies in the fact that the computation of the chiral index is heavily based on the geometry of the resolved fourfold in terms of quadruple intersection numbers and intersections numbers are appearing in the log-monodromy matrices of singularities in complex-structure moduli space of the mirror manifold of the resolved fourfold $\hat{Y}_4$. It remains an issue that the quadruple intersection numbers appearing in the log monodromy matrix are written in terms of a divisor basis, that is also the basis of the Kähler cone. Knowledge about this basis in terms of the usual basis $D_I = (D_0, D_\alpha, D_i)$ is inevitable to build the connection.
The strategy was investigating the connection between the chiral index and the quadruple intersection numbers and investigating the connection between the quadruple intersection numbers and the singularities.
Indeed, the information about the relation between the chiral index and the Chern-Simons terms can be used to express the latter one with the log-monodromy matrices and the flux input as done in equation (6.63) for a simple basis of the Kähler cone, but this formula does not provide any insights in a deeper connection as anticipated.
There are multiple issues with extracting the chiral index as a dual element of the vertical cohomology using the knowledge about the quadruple intersection numbers. The missing information is the representation theory. The representation theory is determined by the codimension two singular loci of the elliptic fibration and does not enter the quadruple intersection numbers directly. Since the chiral index arises in a sum over all possible representations, knowledge over all possible representations is required. The weights enter the expression (6.62) directly and in the projection $P_{\mathfrak{g}}$ encoding the anomaly cancellation. Not only the weights of the representations enter the chiral index, but also enters the curve class of the codimension two loci the formula in (6.60).
Another ingredient of the chiral index are the 5D triple intersection numbers W_{ijk}. Recall that they were only an auxiliary quantity in the computation of the sign-function in section 5.3.1. They can indeed be computed from the quadruple intersection numbers via $K_{ijk\alpha} = W_{ijk} \cdot D^B_\alpha$. Therefore, just by looking at the information content, it seems reasonable to suggest that the chiral index can be determined only by the quadruple intersection numbers, if information about the entire representation theory of the gauge group in terms of matter curve classes and the weights is provided.
At this point we can take a step back and recall that the underlying idea of F-theory is to encode information such as the chiral index in the geometric quantities. Indeed, the weights are completely determined by the fiber enhancement over the codimension two matter curves.
On the other hand, the gauge algebra itself is encoded in the intersection numbers via the Cartan matrix and for many gauge algebras, the rank-one gauge enhancements are already well known. Moreover, the classes of

the matter curves are also known for many models. For the $SU(N)$ models, equation (3.56) provides a full answer for $N \geqslant 3$.

In this work we also started from the other side, guided by the question how much information can be extracted about the intersection theory of a resolved fourfold from given Hodge-Deligne diamonds for the large complex-structure singularities. We assumed the simple basis $\{D_{\tilde{0},D_\alpha,D_i}\}$ as basis for the Kähler cone due to the lack of a general formula for a resolved Calabi-Yau fourfold. This choice is motivated by the fact that $\{D_{\tilde{0}}, D_\alpha\}$ provides a basis for the Kähler cone of a smooth elliptic fibration [79]. For proper statements and concrete examples, one could verify the choice of the basis by appropriate algorithms.
We introduced a novel technique to compute the ranks of the log-monodromy matrices only by ranks of matrices of quadruple intersection numbers. Under the assumption that $K^3 \neq 0$ and $D_\alpha^3 \neq 0$, we determined the rank of $KD_\beta D_{\beta'}$ and $D_\alpha D_\beta D_{\beta'}$ from the Hodge-Deligne diamonds for $t_{\tilde{0}} \to i\infty$ and $t_\alpha \to i\infty$ respectively. The case of $t_i \to \infty$ is more difficult. Due to the complexity and amount of non-trivial information contained in quadruple intersection numbers involving multiple Cartan indices, we suspect that lots of non-trivial information can be gained from these singularities. With our current techniques, we could not access these information.
We only considered one-moduli singularities. One might gain more information considering higher codimension singularities in the moduli space. In fact, the large complex structure regime provides an entire network of singularities, which potentially encodes more information. The information encoded in the network of singularities appears to be rich and especially the unknown territory involving Cartan divisors might give various insights in future research.

The key question is whether the information gained from the singularities is sufficient to determine to chiral index using provided information about the representation theory. In this thesis, we only set the stage for further research in this direction. Any attempt of answering this question at this point is just speculation. One problem lies in the fact that the log-monodromy matrices do comprise the intersection numbers, but the Hodge-Deligne diamonds are just based on their ranks. The formulation for chiral index as we reviewed in this thesis on the other hand requires more detailed insights of the intersection numbers. In fact, we showed that for the $SU(5)$-model, the only intersection numbers needed are $K_{00\alpha\beta}, K_{0\alpha\beta\gamma}$ and $K_{\alpha\beta ij}$.
These numbers are sufficient because they can be use to determine the two triple intersection numbers $K_B^2\Sigma$ and $K_B\Sigma^2$ and we found in (5.59) for the $SU(5)$ model that the chiral index as a map from the flux input to integers only depend on the two numbers $K_B^2\Sigma$ and $\Sigma^2 K_B$. These numbers are determined by the singular divisor and the base geometry. We have seen that this information can be obtained from the quadruple intersection numbers, therefore is a more relevant question to only ask for these two numbers from the network of singularities instead of more information about quadruple intersection numbers.

Throughout the entire thesis, we considered only non-abelian gauge theories. It is a natural question to ask for the impact of global Mordell-Weil $U(1)$ gauge factors. Moreover, other assumptions made here can yield further research directions, such as including higher rank enhancements.

As a final note, we want to mention that in [9] it was argued that since the number of independent components of Θ_{IJ} for consistent flux inputs equals the number of independent chiral index families, the rank of the reduced intersection matrix equals that number as well. Therefore we speculate that it might be possible to find the number of independent chiral index families in the Hodge-Delinge diamonds of the large complex-structure regime.

Acknowledgements

First of all, I would like to thank Thomas Grimm for the opportunity to work on these interesting topics and for his supervision of my project. Secondly, I would like to thank Jeroen Monee for helping me with the many questions I had and giving valuable feedback.
Furthermore, I am in dept to Mick van Vliet, Sam Woldringh and Thomas van Vuren. I really benefitted from our discussions talks and feedback. It was a great pleasure to work with you. Moreover, I would like to thank Maarten Rottier, Artim Bassant, Robin van Bijleveld and Lorenz Schlechter.

A Algebraic and Toric Geometry

In this section, we will review some basics of toric geometry that are used throughout this thesis. We start with the description of toric varieties, which are geometric spaces that admit a faithful action of an algebraic torus, such that there is a single maximal torus orbit. Toric varieties are natural generalization of projective spaces and provide a rich class of concrete descriptions of compact Kähler manifolds[22].
Later we discuss divisor classes and how they can be computed in toric description. We mainly follow [14, 80, 81, 82].

A.1 Toric Varieties

A general[23] toric variety with $d = n - r$ dimensions over the complex numbers can be described as the quotient

$$V_\Delta = (\mathbb{C}^n \backslash Z_\Delta)/T_\Delta, \tag{A.1}$$

where $T_\Delta = (\mathbb{C}^r)^* \times G$ for a finite group G. For pedagogical reasons, we restrict ourselves to trivial groups G. The set $Z_\Delta \subset \mathbb{C}$ will be discussed later. The coordinates $x_1, \ldots, x_n$ in $\mathbb{C}^n$ are called homogeneous coordinates of V_Δ. From this description, one obtains the projective space by setting $r = 1$ and $Z = \{0\} \subset \mathbb{C}^{d+1}$. With the action defined by

$$g(\lambda) : (x_1, \ldots, x_{d+1}) \mapsto (\lambda x_1, \ldots, \lambda x_{d+1}), \tag{A.2}$$

equation (A.1) is nothing but the definition of the complex projective space in d dimensions

$$\mathbb{CP}^d = \frac{\mathbb{C}^{d+1}\backslash\{0\}}{\mathbb{C}^*}. \tag{A.3}$$

A key ingredient of the definition (A.1) is the action of T_Δ. The action can be decomposed in $g(\lambda_1, \ldots, \lambda_r) = g_1(\lambda_1) \cdot \cdots \cdot g_r(\lambda_r)$ with

$$g_a(\lambda_a) : (x_1, \ldots, x_n) \mapsto ((\lambda_a^{Q_a^1})x_1, \ldots, (\lambda_a^{Q_a^n})x_n) \quad \text{for all}\, a = 1 \ldots r \,\text{and}\, \lambda \neq 0. \tag{A.4}$$

Hence the action of $(\mathbb{C}^r)^*$ is determined by a $n \times r$ integer matrix with entries Q_a^i. But this description is not unique. Any non-degenerate r linear combinations of the integer vectors Q^i will give the same toric variety, since the variety just depends on the r subspace spanned by the vectors Q^i in $\mathbb{R}^n$.
Another important example is the weighted projective space $\mathbb{P}_{2,3,1}$, which is defined by $Q = (2, 3, 1)$ and $T_\Delta = \{0\}$.

We want to encode the toric variety in a different mathematical structure, namely a fan of strongly convex rational polyhedral cones Δ in $\mathbf{N} \otimes_\mathbb{Z} \mathbb{R}$ for a lattice $\mathbf{N} \sim \mathbb{Z}^d$. In order to make sense of this definition, we want to take a step back and introduce some underlying concepts first.
A rational polyhedral cone is the set [83]

$$\sigma = \left\{ \sum_a v_a \alpha_a, \, \alpha_a \geqslant 0 \right\} \tag{A.5}$$

[22] Hypersurefaces of toric varieties yield a richt class of Calabi-Yau manifolds, but there cannot be a Calabi-Yau manifold in a purely toric description as we will see later.

[23] We consider only simplicial toric varieties to avoid subtleties with taking the quotiens, see [80, footnote 5]

generated by a finite set of vectors v_a in $\mathbf{N}$. The finiteness of generators is equivalent to the attribute 'polyhedral' and the attribute 'rational' comes from the condition that the vectors v_a lie in the lattice $\mathbf{N}$. The cone is called strongly convex if $\sigma \cap (-\sigma) = \{0\}$.
A fan Δ of strongly convex rational polyhedral cones is a collection of this kind of cones with the properties that

- all faces of cones in Δ are cones and are also in Δ,
- any intersection of two cones is a face of both cones.

In order to match this description with the toric variety defined in the quotient (2.7), there have to be exactly n one-dimensional cones. These n cones have integral generators denotes by $(v_a)_{a=1\ldots n}$. Written as a $(d \times n)$ matrix with the vectors v_i as rows (called the weight matrix), the kernel of this matrix gives the subspace that defines the action of T_δ in the quotient picture. This means in particular that the scalings Q_i^a in (A.4) are defined by

$$\sum_{a=1}^{n} v_a Q_i^a = 0 \quad \text{for all}\, i = 1 \ldots r = n - d. \tag{A.6}$$

This definition is not unique as mentioned above.
The set F_Δ in the quotient description (2.7) is defined by the union of hyperplane intersections

$$F_\Delta = \bigcap_{i \in I} \{x_i = 0\} \tag{A.7}$$

for all index sets $I \subset \{1 \ldots n\}$, such that $\{v_i\}_{i \in I}$ are not contained in any cone of Δ.
The advantage of the fan-description is that certain conditions on the fans lead to specific classes of varieties. In particular

- The fan Δ is called complete if all the cones in Δ cover $\mathbf{N} \times \mathbb{R}$.
  The variety V_Δ is compact if and only if Δ is complete.
- A cone spanned by vectors $w_1 \ldots w_k \in \mathbb{Z}^d$ is called regular if the coordinates in each vector w_i are coprime (w_i is then called primitive) and if there are primitive vectors $w_{k+1} \ldots w_d$ such that $\det(w_1 \ldots w_n) = \pm 1$. If all cones in a fan Δ are regular, the fan is called regular and this happens if and only if the toric variety V_Δ is smooth.

For compact varieties one can think of the fan as a polytop, such that the corners generate the one dimensional cones and all faces and edges generate higher cones. This picture is often used for toric varieties in the physics literature.

A.2 Divisors

In this subsection we want to introduce the notion of a divisor and important equivalence classes of divisors. One can think of a divisor as the formal sum of codimension-one subvarieties in a variety $\mathbf{X}$. In general, we like to distinguish between two different notions of divisor, namely the Cartier and Weil divisors.

A Weil divisor is an element of the free abelian group $\mathrm{Div}(\mathbf{X})$ of closed integral subvarieties, i.e. it is the formal sum of irreducible hypersurfaces Y_i with integer coefficients[24]

$$D = \sum_i a_i Y_i. \tag{A.8}$$

A Weil divisor is called effective if all the coefficients are non-negative.

A Cartier divisor is defined by a set of pairs

$$(U_\alpha, f_\alpha), \tag{A.9}$$

where U_α is an open cover of $\mathbf{X}$ and f_α a rational function up to multiplication by a function that has neither zeros nor poles. The functions f_α have to satisfy the condition that f_α/f_β has neither zero nor poles on the open set $U_\alpha \cap U_\beta$.
On the set of Cartier divisors, we define the addition

$$(U_\alpha, f_\alpha) + (U_\alpha, g_\alpha) = (U_\alpha, f_\alpha g_\alpha). \tag{A.10}$$

If the functions f_α are given by the restriction of a globally defined function $f_\alpha = f\mid_{U_\alpha}$, then the corresponding Cartier divisor is called principal. If two Cartier divisors differ by a principal divisor, they are called rational equivalent. If we apply the above group structure to the Cartier divisors modulo rational equivalence, the resulting group is called the Picard group $\mathrm{Pic}(\mathbf{X})$.
By definition, a Cartier divisor D also defines a holomorphic line bundle on $\mathbf{X}$ that we denote $\mathcal{O}(D)$. Actually, a different definition of the Picard group is the group of all isomorphism classes of line bundles with the tensor product. The line bundle corresponding to a principal divisor is the trivial line bundle.

To a principal Cartier divisor with defining function f we can associate the principal Weil divisor by

$$D = \sum_i \mathrm{ord}_{Y_i}(f) Y_i. \tag{A.11}$$

With this notion, we can define the divisor class group $\mathrm{Cl}(\mathbf{X})$ to be the abelian group of Weil divisors modulo principal Weil divisors.
In the case of a smooth variety $\mathbf{X}$, Cartier divisors and Weil divisors are the same and we have $\mathrm{Cl}(X) = \mathrm{Pic}(X)$. This is one of the reasons why we prefer to work on smooth varieties. We use the term divisor for both, the subvariety and the equivalence class, depending on the context [84].
For a smooth, simply connected, compact Kähler manifold, we even have that [85, proposition 3.3.2]

$$\mathrm{Pic}(X) = H^{1,1}(X, \mathbb{Z}) = H^{1,1}(\mathbf{X}, \mathbb{C}) \cap H^2(\mathbf{X}, \mathbb{Z}). \tag{A.12}$$

This identification can be made concrete by [85, Proposition 4.4.13]

$$c_1(\mathcal{O}(D)) = [D] \in H^{1,1}(X, \mathbb{Z}). \tag{A.13}$$

This is exaclty the Poincaré dual of the divisor D, also denoted as $PD[D]$.

Let us generalize the notion of divisors and rational equivalence for higher codimension subvarieties. We call two complex codimension p-cycles Z_1 and Z_2 in $\mathbf{X}$ rational equivalent if there exists a cycle V on $\mathbf{X} \times \mathbb{P}^1$ such that $V \cap (X \times \{t_1\}) - V \cap (X \times \{t_2\}) = Z_1 - Z_2$. The free group of this codimension p-cyles modulo

[24] This is a slightly simplified definition because we restrict ourselves to complex varieties instead of schemes.

rational equivalence is the chow group $\mathrm{CH}^p(\mathbf{X})$.
We can define the Chow ring

$$\mathrm{CH}(\mathbf{X}) = \bigoplus_{k=0} \mathrm{CH}^k(\mathbf{X}) \tag{A.14}$$

to be the graded ring of all chow groups and $\mathrm{CH}(\mathbf{X})^0 := \mathbb{Z}$ equipped with the intersection product as multiplication. The intersection product is a associative and commutative map $\mathrm{CH}^a(\mathbf{X}) \times \mathrm{CH}^b(\mathbf{X}) \to \mathrm{CH}^{a+b}(\mathbf{X})$. As the name suggests, for transversal intersection, this product counts the intersection points $\# \bigcap_i D_i$.
For a compact, toric variety, the ring structure is determined by the intersection structure of the divisors [80]. Thus it is sufficient for our purposes to concern ourselves only with the intersection product of divisors. For more details, we refer to [45].
On a smooth compact Kähler manifold $\mathbf{X}$ with n complex dimensions, the intersection product of n divisors $D_1, \ldots, D_n$ is given by

$$D_1 \cdot \ldots \cdot D_n = \int_{\mathbf{X}} PD[D_1] \wedge \cdots \wedge PD[D_n] \in \mathbb{Z}, \tag{A.15}$$

where $PD[D_i]$ is the Poincare dual class in $H^{1,1}(X, \mathbb{C})$.

A.3 Divisors in Toric Varieties

At next, we want to give a concrete description of divisors, following [14]. We have seen above that we can describe divisors by roots of locally defined, rational functions with transition functions. Globally defined functions correspond to trivial divisors. In a toric variety, a function can be defined globally if it is invariant under the action of T_Δ.
Since the addition of divisors (A.10) corresponds to the multiplication of the functions, all divisors can be generated by classes represented by

$$D_i := \{x_i = 0\} \tag{A.16}$$

for $\{x_i\}$ being the homogeneous coordinates of the toric variety. These divisors are not necessarily independent.
For example $\mathbb{P}^n$, all the functions x_i/x_j are invariant under the rescaling action and therefore

$$D_i - D_j = 0 \qquad \text{for all } i, j = 1 \ldots n \tag{A.17}$$

and hence all the divisors are same and there exists only one divisor in $\mathbb{P}^n$.
For general toric varieties we can see that the polynomial

$$\prod_{a=1}^{n} x^{\langle m, v_a \rangle} \tag{A.18}$$

is invariant under the action (A.4) for all $m \in \mathbf{M} \sim \mathbb{Z}^d$. The lattice $\mathbf{M}$ is the dual lattice to $\mathbf{N}$. This invariance translates in the linear relations for the divisor classes

$$\sum_{a=1}^{n} \langle m, v_a \rangle D_a = 0 \qquad \text{for all } m \in \mathbf{M}. \tag{A.19}$$

Since the dual lattice $\mathbf{M}$ is generated by d vectors, the linear relations (A.19) leaves exactly $r = n - d$ independent (also called irreducible) divisors.
There are additional, non-linear relations on the divisors or, to be more precise, on the Chow ring. These relations are of the form

$$\prod_{a \in A} D_a = 0 \tag{A.20}$$

and form an ideal in the Chow-ring that is known as the Stanlay Reisner ideal.

An important divisor class is the canonical class, that is the class in $\text{Pic}(X)$ associated to the canonical line bundle. The canonical line bundle of a manifold in n dimensions is defined as the nth exterior power of the cotangent bundle

$$K_X := \det(\Omega_X) = \Omega_X^n. \tag{A.21}$$

We call the divisor class corresponding to the canonical line bundle the canonical class (throughout the thesis, it is denoted by K_X or simply K as well, if the context is clear). In a basis of irreducible divisors $\{D_\alpha\}$, the canonical class can be written as a linear combination

$$K = K^\alpha D_\alpha. \tag{A.22}$$

For a compact Kähler manifold X, it holds [86]

$$-c_1(K_X) = c_1(X) := c_1(TX) \tag{A.23}$$

and therefore, the divisor class that corresponds to the first chern class of the manifold $c_1(X)$ is the negative canonical class.
Moreover, the divisor corresponding to the first chern class of a toric variety X with homogeneous coordinates $\{x_i\}$ can be written as [14]

$$[c_1(X)] = \sum_i D_i. \tag{A.24}$$

B Lie Algebras

B.1 Root System

A Lie algebra $\mathfrak{g}$ is a vector space over a field $\mathbb{K}$ with a bilinear map (called the Lie bracket) $[\cdot, \cdot] \to \mathbb{K}$ such that the following identities hold:

$$[a, b] = -[b, a] \qquad \text{antisymmetric} \tag{B.1}$$

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \qquad \text{Jacobi Identity} \tag{B.2}$$

Lie algebras are of interest for us because they appear as the tangent bundle of a Lie group and therefore describe infinitesimal symmetry transformations.
A Lie algebra is called simple if it does not contain any non-trivial proper ideal. Lie algebras consisting of direct products of simple algebras are called semi-simple. In the following, we consider all Lie algebras to be semi-simple. In addition we will only consider finite dimensional Lie algebras. Important examples of Lie

algebras are finite dimensional algebras equipped with the Lie bracket $[a, b] = ab - ba$.
The Cartan algebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is the largest abelian Lie algebra contained in $\mathfrak{g}$. The dimension of the Cartan algebra is called the rank of $\mathfrak{g}$. We call $\alpha \in \mathfrak{h}^*$ a root if there exists $X_\alpha \in \mathfrak{g}$ such that

$$[H, X_\alpha] = \alpha(H) X_\alpha \tag{B.3}$$

for all $H \in \mathfrak{h}$. Let us denote the set of roots by Δ. Let $\Delta^+ \subseteq \Delta$ be the set of positive roots. It is defined by the properties

- for a root α, either α or $-\alpha$ is in Δ^+
- if α and β are positive and $\alpha + \beta$ is a root, then $\alpha + \beta$ is positive.

Positive roots that cannot be written as the sum of positive roots are called simple. The set of simple roots is a key quantity of a semi-simple Lie algebra and they are used for their classification. There are $\mathrm{rk}(\mathfrak{g})$ simple roots for the Lie algebra $\mathfrak{g}$.
For every semi-simple Lie algebra, we can associate a graph, called the Dynkin diagram. Given a set of simple roots S, we can construct the Dynkin diagram as follows:
The nodes of the Dynkin diagram is formed by the simple roots and the number of edges between α and β is given by $\frac{1}{4}\left(\frac{\langle \alpha, \beta \rangle}{||\alpha|| ||\beta||}\right)^2$. For double and triple edges, we mark a direction towards the shorter simple root. The resulting Dynkin diagam is independent of the choice of the simple roots.
Using the Dynkin diagrams we can classify Lie algebras. If there are no double or triple edges in a Dynkin diagram, the diagram is called simply laced. The corresponding Lie algebras belong to the classes A, D, E. We say these Lie algebras belong to the ADE-classification. For these Lie algebras, we define the Cartan matrix by

$$C_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{||\alpha_i||}, \tag{B.4}$$

where $\{\alpha_i\}_{i=1 \ldots \mathrm{rk}(\mathfrak{g})}$ are all simple roots.
We can find the basis $\{T_i\}_{i=1 \ldots \mathfrak{g}}$ of the Cartan subalgebra $\mathfrak{h}$, such that

$$[T_i, X_{\alpha_j}] = C_{ij} X_{\alpha_j}. \tag{B.5}$$

The generators T_i are called co-roots.

B.2 Representations

Let V be a vector space. A representation of the semi-simple Lie algebra $\mathfrak{g}$ is a linear map $\rho : \mathfrak{g} \to \mathrm{End}(V)$, such that $\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$.
Let $\mathbf{w}$ be a vector with $\mathrm{rk}(\mathfrak{g})$ components. If there is a vector $v_{\mathbf{w}}$ such that $H_i v_{\mathbf{w}} = \mathbf{w}_i v_{\mathbf{w}}$ for all $H_i \in \mathfrak{h}$, then $\mathbf{w}$ is called weight (vector).
A representation is fully determined by the highest weight, because all other weights can be obtained by subtracting suitable positive linear combinations of simple roots.
For the $SU(5)$ example the weights of the fundamental representation **5** and the two index antisymmetric representation **10** are given in table 5 and 6.

C Intersection Numbers for the $SU(5)$ Model

We provide all quadruple intersection numbers of divisor classes of the resolved Calabi-Yau fourfold, where the resolved singularity corresponds to the gauge group $SU(5)$, in terms of triple intersection numbers of the base B_3. According to section 3.4, we know

$$K_{0000} = K^3 \tag{C.1}$$

$$K_{000\alpha} = D_\alpha K^2 \tag{C.2}$$

$$K_{00\alpha\beta} = K D_\alpha D_\beta \tag{C.3}$$

$$K_{0\alpha\beta\gamma} = D_\alpha D_\beta D_\gamma . \tag{C.4}$$

Despite these, the only non-trivial quadruple intersection numbers are those containing four, three or two Cartan indices. We compute them using the technique of section 3.5. The calculation was done using Mathematica.

The quadruple intersection numbers involving four Cartan indices can be found in table 3.

The quadruple intersection numbers involving three Cartan indices can be found in table 4.

The quadruple intersection numbers involving two Cartan indices are according to 3.4

$$K_{ij\alpha\beta} = -C_{ij}\Sigma D_\alpha D_\beta = -\Sigma D_\alpha D_\beta \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} , \tag{C.5}$$

where C_{ij} denotes the Cartan matrix of $SU(5)$. This can be verified using the pushforward technique of section 3.5.

(n_1, n_2, n_3, n_4)	$D_1^{n_1} \cdot D_2^{n_2} \cdot D_3^{n_3} \cdot D_4^{n_4}$
$(4,0,0,0)$	$-2\Sigma(7K^2 + 10K\Sigma + 4\Sigma^2)$
$(0,4,0,0)$	$-\Sigma(16K^2 + 29K\Sigma + 14\Sigma^2)$
$(0,0,4,0)$	$-\Sigma(8K^2 + 9K\Sigma + 4\Sigma^2)$
$(0,0,0,4)$	$-2\Sigma(3K^2 + 6K\Sigma + 4\Sigma^2)$
$(3,1,0,0)$	$\Sigma(3K + 2\Sigma)^2$
$(3,0,1,0)$	0
$(3,0,0,1)$	$4K\Sigma(K + \Sigma)$
$(1,3,0,0)$	$\Sigma(2K + \Sigma)^2$
$(0,3,1,0)$	$\Sigma(K + \Sigma)(13K + 9\Sigma)$
$(0,3,0,1)$	$-K\Sigma^2$
$(1,0,3,0)$	0
$(0,1,3,0)$	$\Sigma(3K + 2\Sigma)^2$
$(0,0,3,1)$	$\Sigma(-K^2 + K\Sigma + \Sigma^2)$
$(1,0,0,3)$	0
$(0,1,0,3)$	$K\Sigma(K + 2\Sigma)$
$(0,0,1,3)$	$4\Sigma(K + \Sigma)^2$
$(2,2,0,0)$	$-\Sigma(2K + \Sigma)(3K + 2\Sigma)$
$(2,0,2,0)$	$K\Sigma^2$
$(2,0,0,2)$	$-\Sigma(K + \Sigma)^2$
$(0,2,2,0)$	$-\Sigma(3K + 2\Sigma)(4K + 3\Sigma)$
$(0,2,0,2)$	$K\Sigma^2$
$(0,0,2,2)$	$-\Sigma(K + \Sigma)^2$
$(2,1,1,0)$	0
$(2,1,0,1)$	$-K\Sigma(3K + 2\Sigma)$
$(2,0,1,1)$	0
$(1,2,1,0)$	0
$(1,2,0,1)$	$K\Sigma(2K + \Sigma)$
$(0,2,1,1)$	$-K\Sigma(K + \Sigma)$
$(1,1,2,0)$	0
$(1,0,2,1)$	0
$(0,1,2,1)$	$K\Sigma(3K + 2\Sigma)$
$(1,1,0,2)$	$K\Sigma^2$
$(1,0,1,2)$	0
$(0,1,1,2)$	$-2K\Sigma(K + \Sigma)$
$(1,1,1,1)$	0

Table 3: Quadruple intersection numbers of the form $D_1^{n_1} \cdot D_2^{n_2} \cdot D_3^{n_3} \cdot D_4^{n_4}$ involving four Cartan divisors in terms of triple intersections of the base divisors K_B and Σ, where $K = K^\alpha D_\alpha$ is the canonical class and Σ is the singular divisor. Computation is done via section 3.5 and Mathematica.

(n_1, n_2, n_3, n_4)	$D_1^{n_1} \cdot D_2^{n_2} \cdot D_3^{n_3} \cdot D_4^{n_4} \cdot D_\alpha$
$(3,0,0,0)$	$-4\Sigma(K+\Sigma)D_\alpha$
$(0,3,0,0)$	$-\Sigma(3K+4\Sigma)D_\alpha$
$(0,0,3,0)$	$\Sigma(4K+\Sigma)D_\alpha$
$(0,0,0,3)$	$-2\Sigma(K+2\Sigma)D_\alpha$
$(2,1,0,0)$	$\Sigma(3K+2\Sigma)D_\alpha$
$(2,0,1,0)$	0
$(2,0,0,1)$	$2K\Sigma D_\alpha$
$(1,2,0,0)$	$-\Sigma(2K+\Sigma)D_\alpha$
$(0,2,1,0)$	$\Sigma(4K+3\Sigma)D_\alpha$
$(0,2,0,1)$	$K\Sigma D_\alpha$
$(1,0,2,0)$	0
$(0,1,2,0)$	$-\Sigma(3K+2\Sigma)D_\alpha$
$(0,0,2,1)$	$-\Sigma(K+\Sigma)D_\alpha$
$(1,0,0,2)$	0
$(0,1,0,2)$	$K\Sigma D_\alpha$
$(0,0,1,2)$	$2\Sigma(K+\Sigma)D_\alpha$
$(1,1,1,0)$	0
$(1,1,0,1)$	$-K\Sigma D_\alpha$
$(1,0,1,1)$	0
$(0,1,1,1)$	$-K\Sigma D_\alpha$

Table 4: Quadruple intersection numbers of the form $D_1^{n_1} \cdot D_2^{n_2} \cdot D_3^{n_3} \cdot D_4^{n_4} \cdot D_\alpha$ involving three Cartan divisors in terms of triple intersections of the base divisors D_α, K_B and Σ, where $K = K^\beta D_\beta$ is the canonical class and Σ is the singular divisor. Computation is done via section 3.5 and Mathematica.

The weights and the sign functions for the $SU(5)$ representations **5** and **10** can be found in table 5 and 6.

$\text{sign}(w_k \zeta^k)$	w_1	w_2	w_3	w_4
$+$	1	0	0	0
$+$	-1	1	0	0
$+$	0	-1	1	0
$-$	0	0	-1	1
$-$	0	0	0	-1

Table 5: Weights w_i of the fundamental representation **5** of the algebra $\mathfrak{su}_5$ and the sign function $\text{sign}(w_k \zeta^k)$ that can be computed by the method described in 5.3.1, [9, table 3]

$\text{sign}(w_k\zeta^k)$	w_1	w_2	w_3	w_4
$+$	0	1	0	0
$+$	1	-1	0	0
$+$	-1	0	1	0
$+$	1	0	-1	1
$+$	-1	1	-1	1
$-$	1	0	0	-1
$-$	-1	0	0	-1
$-$	0	-1	0	1
$-$	0	-1	1	-1
$-$	0	0	-1	0

Table 6: Weights w_i of the fundamental representation $\mathbf{5}$ of the algebra $\mathfrak{su}_5$ and the sign function $\text{sign}(w_k\zeta^k)$ that can be computed by the method described in 5.3.1, [9, table 3]

References

[1] Mirjam Cvetic et al. *Snowmass White Paper: String Theory and Particle Physics.* 2022. DOI: 10.48550/ARXIV.2204.01742. arXiv: 2204.01742. URL: https://arxiv.org/abs/2204.01742.

[2] David R. Morrison and Cumrun Vafa. "Compactifications of F-theory on Calabi-Yau threefolds. (I)". In: *Nuclear Physics B* 473.1-2 (Aug. 1996), pp. 74–92. DOI: 10.1016/0550-3213(96)00242-8. arXiv: hep-th/9602114. URL: https://doi.org/10.1016%2F0550-3213%2896%2900242-8.

[3] Cumrun Vafa. "Evidence for F-theory". In: *Nuclear Physics B* 469.3 (June 1996), pp. 403–415. DOI: 10.1016/0550-3213(96)00172-1. arXiv: hep-th/9602022. URL: https://doi.org/10.1016%2F0550-3213%2896%2900172-1.

[4] Mirjam Cvetič et al. "Three-family particle physics models from global F-theory compactifications". In: *Journal of High Energy Physics* 2015.8 (Aug. 2015). DOI: 10.1007/jhep08(2015)087. arXiv: 1503.02068. URL: https://doi.org/10.1007%2Fjhep08%282015%29087.

[5] Mirjam Cvetič et al. "Quadrillion F-Theory Compactifications with the Exact Chiral Spectrum of the Standard Model". In: *Physical Review Letters* 123.10 (Sept. 2019). DOI: 10.1103/physrevlett.123.101601. arXiv: 1903.00009. URL: https://doi.org/10.1103%2Fphysrevlett.123.101601.

[6] Joseph Marsano, Natalia Saulina, and Sakura Schäfer-Nameki. "A note on G-fluxes for F-theory model building". In: *Journal of High Energy Physics* 2010.11 (Jan. 2010). DOI: 10.1007/jhep11(2010)088. arXiv: 1006.0483. URL: https://doi.org/10.1007%2Fjhep11%282010%29088.

[7] Yu-Chieh Chung. "On global flipped SU(5) GUTs in F-theory". In: *Journal of High Energy Physics* 2011.3 (Mar. 2011). DOI: 10.1007/jhep03(2011)126. arXiv: hep-th/1008.2506. URL: https://doi.org/10.1007%2Fjhep03%282011%29126.

[8] Thomas W. Grimm and Hirotaka Hayashi. "F-theory fluxes, chirality and Chern-Simons theories". In: *Journal of High Energy Physics* 2012.3 (Mar. 2012). DOI: 10.1007/jhep03(2012)027. arXiv: 1111.1232. URL: https://doi.org/10.1007%2Fjhep03%282012%29027.

[9] Patrick Jefferson, Washington Taylor, and Andrew P. Turner. *Chiral matter multiplicities and resolution-independent structure in 4D F-theory models.* 2021. arXiv: 2108.07810 [hep-th].

[10] Thomas W. Grimm, Erik Plauschinn, and Damian van de Heisteeg. "Moduli stabilization in asymptotic flux compactifications". In: *Journal of High Energy Physics* 2022.3 (Mar. 2022). DOI: 10.1007/jhep03(2022)117. arXiv: 2110.05511. URL: https://doi.org/10.1007%2Fjhep03%282022%29117.

[11] Thomas W. Grimm, Chongchuo Li, and Eran Palti. "Infinite distance networks in field space and charge orbits". In: *Journal of High Energy Physics* 2019.3 (Mar. 2019). DOI: 10.1007/jhep03(2019)016. arXiv: 1811.02571. URL: https://doi.org/10.1007%2Fjhep03%282019%29016.

[12] Thomas W. Grimm, Eran Palti, and Irene Valenzuela. "Infinite distances in field space and massless towers of states". In: *Journal of High Energy Physics* 2018.8 (Aug. 2018). DOI: 10.1007/jhep08(2018)143. arXiv: 1802.08264. URL: https://doi.org/10.1007%2Fjhep08%282018%29143.

[13] Timo Weigand. *TASI Lectures on F-theory.* 2018. arXiv: 1806.01854 [hep-th].

[14] Frederik Denef. *Les Houches Lectures on Constructing String Vacua.* 2008. DOI: 10.48550/ARXIV.0803.1194. URL: https://arxiv.org/abs/0803.1194.

[15] Stefan Theisen Ralph Blumenhagen Dieter Lüst. *Basic Concepts of String Theory.* Springer, 2013. ISBN: 978-3-642-29497-6.

[16] J. H. Schwarz K. Becker M. Becker. *STRING THEORY AND M-THEORY. A Modern Introduction*. Cambridge University Press, 2007. ISBN: 978-0-511-25653-0.

[17] John H. Schwarz. "An SL(2, Z) multiplet of type IIB superstrings". In: *Physics Letters B* 360.1-2 (Oct. 1995), pp. 13–18. DOI: 10.1016/0370-2693(95)01138-g. arXiv: hep-th/9508143. URL: https://doi.org/10.1016%2F0370-2693%2895%2901138-g.

[18] Ian J R Aitchison. *Supersymmetry and the MSSM: An Elementary Introduction*. 2005. DOI: 10.48550/ARXIV.HEP-PH/0505105. arXiv: hep-th/0505105. URL: https://arxiv.org/abs/hep-ph/0505105.

[19] M.J. Duff, B.E.W. Nilsson, and C.N. Pope. "Kaluza-Klein supergravity". In: *Physics Reports* 130.1 (1986), pp. 1–142. ISSN: 0370-1573. DOI: https://doi.org/10.1016/0370-1573(86)90163-8. URL: https://www.sciencedirect.com/science/article/pii/0370157386901638.

[20] Petr Hořava and Edward Witten. "Heterotic and Type I string dynamics from eleven dimensions". In: *Nuclear Physics B* 460.3 (Feb. 1996), pp. 506–524. DOI: 10.1016/0550-3213(95)00621-4. arXiv: hep-th/9510209. URL: https://doi.org/10.1016%2F0550-3213%2895%2900621-4.

[21] M.J. Duff, James T. Liu, and R. Minasian. "Eleven-dimensional origin of string/string duality: a one-loop test". In: *Nuclear Physics B* 452.1 (1995), pp. 261–282. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(95)00368-3. arXiv: hep-th/9506126. URL: https://www.sciencedirect.com/science/article/pii/0550321395003683.

[22] Ashoke Sen. "Orientifold limit of F-theory vacua". In: *Physical Review D* 55.12 (June 1997), R7345–R7349. DOI: 10.1103/physrevd.55.r7345. arXiv: hep-th/9702165. URL: https://doi.org/10.1103%2Fphysrevd.55.r7345.

[23] Ashoke Sen. "Orientifold limit of F-theory vacua". In: *Nuclear Physics B - Proceedings Supplements* 68.1-3 (Nov. 1998), pp. 92–98. DOI: 10.1016/s0920-5632(98)00143-1. arXiv: hep-th/9709159. URL: https://doi.org/10.1016%2Fs0920-5632%2898%2900143-1.

[24] W Nahm. "Supersymmetries and their representations". In: *Nucl. Phys. B* 135 (July 1977), 149–166. 26 p. DOI: 10.1016/0550-3213(78)90218-3. URL: http://cds.cern.ch/record/132743.

[25] John H. Schwarz. "An SL(2, Z) multiplet of type IIB superstrings". In: *Physics Letters B* 360.1-2 (Oct. 1995), pp. 13–18. DOI: 10.1016/0370-2693(95)01138-g. arXiv: hep-th/9508143. URL: https://doi.org/10.1016%2F0370-2693%2895%2901138-g.

[26] David Singerman Gareth A Jones. *Complex Functions. An algebraic and geometric viewpoint*. Press Syndicale of Ihe University of Camhridge, 1987. ISBN: 0-52l-30893-3.

[27] Clay Cordova. "Decoupling Gravity in F-Theory". In: (2009). DOI: 10.48550/ARXIV.0910.2955. arXiv: 0910.2955. URL: https://arxiv.org/abs/0910.2955.

[28] Kunihiko Kodaira. "On compact analytic surfaces II". In: *Annals of Mathematics* 77 (1963), pp. 563–626.

[29] André Néron. "Modèles minimaux des variétés abéliennes sur les corps locaux et globaux". fr. In: *Publications Mathématiques de l'IHÉS* 21 (1964), pp. 5–128. URL: http://www.numdam.org/item/PMIHES_1964__21__5_0/.

[30] Antonella Grassi and David R. Morrison. *Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds*. 2011. DOI: 10.48550/ARXIV.1109.0042. arXiv: hep-th/1109.0042. URL: https://arxiv.org/abs/1109.0042.

[31] Antonella Grassi et al. "Non-simply-laced symmetry algebras in F-theory on singular spaces". In: *Journal of High Energy Physics* 2018.9 (Sept. 2018). DOI: 10.1007/jhep09(2018)129. arXiv: 1805.06949. URL: https://doi.org/10.1007%2Fjhep09%282018%29129.

[32] Mboyo Esole and Shu-Heng Shao. *M-theory on Elliptic Calabi-Yau Threefolds and 6d Anomalies*. 2015. DOI: 10.48550/ARXIV.1504.01387. URL: https://arxiv.org/abs/1504.01387.

[33] J. Tate. "Algorithm for determining the type of a singular fiber in an elliptic pencil". In: *Modular Functions of One Variable IV*. Ed. by Bryan J. Birch and Willem Kuyk. Berlin, Heidelberg: Springer Berlin Heidelberg, 1975, pp. 33–52. ISBN: 978-3-540-37588-3.

[34] David R. Morrison and Cumrun Vafa. "Compactifications of F-theory on Calabi-Yau threefolds (II)". In: *Nuclear Physics B* 476.3 (Sept. 1996), pp. 437–469. DOI: 10.1016/0550-3213(96)00369-0. arXiv: hep-th/9603161. URL: https://doi.org/10.1016%2F0550-3213%2896%2900369-0.

[35] Mboyo Esole and Shing-Tung Yau. "Small resolutions of SU(5)-models in F-theory". In: *Adv. Theor. Math. Phys.* 17.6 (2013), pp. 1195–1253. DOI: 10.4310/ATMP.2013.v17.n6.a1. arXiv: 1107.0733 [hep-th].

[36] Sven Krause, Christoph Mayrhofer, and Timo Weigand. "-flux, chiral matter and singularity resolution in F-theory compactifications". In: *Nuclear Physics B* 858.1 (May 2012), pp. 1–47. DOI: 10.1016/j.nuclphysb.2011.12.013. arXiv: 1109.3454. URL: https://doi.org/10.1016%2Fj.nuclphysb.2011.12.013.

[37] Mboyo Esole, Shu-Heng Shao, and Shing-Tung Yau. "Singularities and Gauge Theory Phases II". In: *Adv. Theor. Math. Phys.* 20 (2016), pp. 683–749. DOI: 10.4310/ATMP.2016.v20.n4.a2. arXiv: 1407.1867 [hep-th].

[38] Mboyo Esole, Patrick Jefferson, and Monica Jinwoo Kang. "Euler Characteristics of Crepant Resolutions of Weierstrass Models". In: *Communications in Mathematical Physics* 371.1 (Sept. 2019), pp. 99–144. DOI: 10.1007/s00220-019-03517-1. arXiv: 1703.00905. URL: https://doi.org/10.1007%2Fs00220-019-03517-1.

[39] Tetsuji SHIODA. "On elliptic modular surfaces". In: *Journal of the Mathematical Society of Japan* 24.1 (1972), pp. 20 –59. DOI: 10.2969/jmsj/02410020. URL: https://doi.org/10.2969/jmsj/02410020.

[40] Rania Wazir. "Arithmetic on elliptic threefolds". In: *Compositio Mathematica* 140.3 (2004), pp. 567–580. DOI: 10.1112/S0010437X03000381.

[41] Matthias Schuett and Tetsuji Shioda. *Elliptic Surfaces*. 2009. DOI: 10.48550/ARXIV.0907.0298. arXiv: 0907.0298. URL: https://arxiv.org/abs/0907.0298.

[42] Mirjam Cvetič, Thomas W. Grimm, and Denis Klevers. "Anomaly cancellation and abelian gauge symmetries in F-theory". In: *Journal of High Energy Physics* 2013.2 (Feb. 2013). ISSN: 1029-8479. DOI: 10.1007/jhep02(2013)101. arXiv: 1210.6034. URL: http://dx.doi.org/10.1007/JHEP02(2013)101.

[43] Joseph Harris Phillip Griffiths. *Principles of algebraic geometry*. Wiley classics library, 1994. ISBN: 0-471-05059-8.

[44] Thomas W. Grimm. "The N=1 effective action of F-theory compactifications". In: *Nuclear Physics B* 845.1 (Apr. 2011), pp. 48–92. DOI: 10.1016/j.nuclphysb.2010.11.018. arXiv: 1008.4133. URL: https://doi.org/10.1016%2Fj.nuclphysb.2010.11.018.

[45] William Fulton. *Intersection Theory*. Springer, 1998. ISBN: 3-540-62046-X.

[46] Patrick Jefferson and Andrew P. Turner. *Generating functions for intersection products of divisors in resolved F-theory models.* 2022. DOI: 10.48550/ARXIV.2206.11527. URL: https://arxiv.org/abs/2206.11527.

[47] Sheldon Katz and Cumrun Vafa. "Matter from geometry". In: *Nuclear Physics B* 497.1-2 (July 1997), pp. 146–154. DOI: 10.1016/s0550-3213(97)00280-0. arXiv: hep-th/9606086. URL: https://doi.org/10.1016%2Fs0550-3213%2897%2900280-0.

[48] Lara B. Anderson et al. "Matter in transition". In: *Journal of High Energy Physics* 2016.4 (Apr. 2016), pp. 1–104. DOI: 10.1007/jhep04(2016)080. arXiv: 1512.05791. URL: https://doi.org/10.1007%2Fjhep04%282016%29080.

[49] David R. Morrison and Washington Taylor. "Matter and singularities". In: *Journal of High Energy Physics* 2012.1 (Jan. 2012). DOI: 10.1007/jhep01(2012)022. arXiv: 1106.3536. URL: https://doi.org/10.1007%2Fjhep01%282012%29022.

[50] Jonathan Mboyo Esole, James Fullwood, and Shing-Tung Yau. "D5 elliptic fibrations: Non-Kodaira fibers and new orientifold limits of F-theory". In: *Communications in Number Theory and Physics* 9 (Oct. 2011). DOI: 10.4310/CNTP.2015.v9.n3.a4. arXiv: 1110.6177. URL: https://arxiv.org/abs/1110.6177.

[51] Kenneth Intriligator et al. "Conifold Transitions in M-theory on Calabi-Yau Fourfolds with Background Fluxes". In: *Adv. Theor. Math. Phys.* 17.3 (2013), pp. 601–699. DOI: 10.4310/ATMP.2013.v17.n3.a2. arXiv: 1203.6662 [hep-th].

[52] Katrin Becker and Melanie Becker. "M-theory on eight-manifolds". In: *Nuclear Physics B* 477.1 (Oct. 1996), pp. 155–167. DOI: 10.1016/0550-3213(96)00367-7. arXiv: hep-th/9605053. URL: https://doi.org/10.1016%2F0550-3213%2896%2900367-7.

[53] Edward Witten. "Phase transitions in M-theory and F-theory". In: *Nuclear Physics B* 471.1-2 (July 1996), pp. 195–216. DOI: 10.1016/0550-3213(96)00212-x. arXiv: hep-th/9603150. URL: https://doi.org/10.1016%2F0550-3213%2896%2900212-x.

[54] Timo Weigand. "Lectures on F-theory compactifications and model building". In: *Classical and Quantum Gravity* 27.21 (Oct. 2010), p. 214004. DOI: 10.1088/0264-9381/27/21/214004. URL: https://doi.org/10.1088%2F0264-9381%2F27%2F21%2F214004.

[55] Hans Jockers and Jan Louis. "The effective action of D7-branes in Calabi–Yau orientifolds". In: *Nuclear Physics B* 705.1-2 (Jan. 2005), pp. 167–211. DOI: 10.1016/j.nuclphysb.2004.11.009. arXiv: hep-th/0409098. URL: https://doi.org/10.1016%2Fj.nuclphysb.2004.11.009.

[56] Sven Krause, Christoph Mayrhofer, and Timo Weigand. "Gauge fluxes in F-theory and type IIB orientifolds". In: *Journal of High Energy Physics* 2012.8 (Aug. 2012). DOI: 10.1007/jhep08(2012)119. arXiv: 1202.3138. URL: https://doi.org/10.1007%2Fjhep08%282012%29119.

[57] Yoshiaki Tanii. *Introduction to Supergravity.* Springer, 2014. ISBN: 978-4-431-54828-7.

[58] Barton Zwiebach. *A First Course in String Theory.* 2nd ed. Cambridge University Press, 2009. ISBN: 978-0-511-47932-8.

[59] Jacques Distler and Brian Greene. "Aspects of (2, 0) string compactifications". In: *Nuclear Physics B* 304 (1988), pp. 1–62. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(88)90619-0. URL: https://www.sciencedirect.com/science/article/pii/0550321388906190.

[60] Sheldon Katz and Eric Sharpe. “D-branes, open string vertex operators, and Ext groups”. In: *Advances in Theoretical and Mathematical Physics* 6.6 (2002), pp. 979–1030. ISSN: 1095-0753. DOI: 10.4310/atmp.2002.v6.n6.a1. arXiv: hep-th/0208104. URL: http://dx.doi.org/10.4310/ATMP.2002.v6.n6.a1.

[61] Andreas P. Braun, Andres Collinucci, and Roberto Valandro. “G-flux in F-theory and algebraic cycles”. In: *Nucl. Phys. B* 856 (2012), pp. 129–179. DOI: 10.1016/j.nuclphysb.2011.10.034. arXiv: 1107.5337 [hep-th].

[62] A. P. Braun and T. Watari. “The vertical, the horizontal and the rest: anatomy of the middle cohomology of Calabi-Yau fourfolds and F-theory applications”. In: *Journal of High Energy Physics* 2015.1 (Jan. 2015). DOI: 10.1007/jhep01(2015)047. arXiv: 1408.6167. URL: https://doi.org/10.1007%2Fjhep01%282015%29047.

[63] Brian R. Greene, David R. Morrison, and M. Ronen Plesser. “Mirror manifolds in higher dimension”. In: *Communications in Mathematical Physics* 173.3 (Nov. 1995), pp. 559–597. DOI: 10.1007/bf02101657. arXiv: hep-th/9402119. URL: https://doi.org/10.1007%2Fbf02101657.

[64] Edward Witten. “On flux quantization in M-theory and the effective action”. In: *Journal of Geometry and Physics* 22.1 (Apr. 1997), pp. 1–13. DOI: 10.1016/s0393-0440(96)00042-3. arXiv: hep-th/9609122. URL: https://doi.org/10.1016%2Fs0393-0440%2896%2900042-3.

[65] Keshav Dasgupta, Govindan Rajesh, and Savdeep Sethi. “M-theory, orientifolds and G-flux”. In: *Journal of High Energy Physics* 1999.08 (Aug. 1999), pp. 023–023. DOI: 10.1088/1126-6708/1999/08/023. arXiv: hep-th/9908088. URL: https://doi.org/10.1088%2F1126-6708%2F1999%2F08%2F023.

[66] Ron Donagi and Martijn Wijnholt. “Model building with F-theory”. In: *Advances in Theoretical and Mathematical Physics* 15 (Oct. 2011). DOI: 10.4310/ATMP.2011.v15.n5.a2. arXiv: 0802.2969. URL: https://arxiv.org/abs/0802.2969.

[67] Federico Bonetti and Thomas W. Grimm. “Six-dimensional (1,0) effective action of F-theory via M-theory on Calabi-Yau threefolds”. In: *Journal of High Energy Physics* 2012.5 (May 2012). DOI: 10.1007/jhep05(2012)019. arXiv: 1112.1082. URL: https://doi.org/10.1007%2Fjhep05%282012%29019.

[68] Thomas W. Grimm and Raffaele Savelli. “Gravitational instantons and fluxes from M/F-theory on Calabi-Yau fourfolds”. In: *Physical Review D* 85.2 (Jan. 2012). DOI: 10.1103/physrevd.85.026003. arXiv: 1109.3191. URL: https://doi.org/10.1103%2Fphysrevd.85.026003.

[69] O. Aharony et al. “Aspects of N = 2 supersymmetric gauge theories in three dimensions”. In: *Nuclear Physics B* 499.1–2 (Aug. 1997), pp. 67–99. ISSN: 0550-3213. DOI: 10.1016/s0550-3213(97)00323-4. arXiv: hep-th/9703110. URL: http://dx.doi.org/10.1016/S0550-3213(97)00323-4.

[70] Thomas W. Grimm, Andreas Kapfer, and Jan Keitel. *Effective action of 6D F-Theory with U(1) factors: Rational sections make Chern-Simons terms jump*. 2013. DOI: 10.48550/ARXIV.1305.1929. URL: https://arxiv.org/abs/1305.1929.

[71] Mboyo Esole, Shu-Heng Shao, and Shing-Tung Yau. *Singularities and Gauge Theory Phases*. 2014. DOI: 10.48550/ARXIV.1402.6331. URL: https://arxiv.org/abs/1402.6331.

[72] C. Robles. *Degenerations of Hodge structure*. 2016. DOI: 10.48550/ARXIV.1607.00933. URL: https://arxiv.org/abs/1607.00933.

[73] Wilfried Schmid. “Variation of Hodge Structure: The Singularities of the Period Mapping.” In: *Inventiones mathematicae* 22 (1973), pp. 211–320. URL: http://eudml.org/doc/142246.

[74] P. Berglund and S. Katz. *Mirror Symmetry Constructions: A Review*. 1994. DOI: 10.48550/ARXIV.HEP-TH/9406008. URL: https://arxiv.org/abs/hep-th/9406008.

[75] Babak Haghighat. *Mirror Symmetry and Modularity*. 2017. DOI: 10.48550/ARXIV.1712.00601. URL: https://arxiv.org/abs/1712.00601.

[76] Per Berglund, Sheldon Katz, and Albrecht Klemm. "Mirror symmetry and the moduli space for generic hypersurfaces in toric varieties". In: *Nuclear Physics B* 456.1-2 (Dec. 1995), pp. 153–204. DOI: 10.1016/0550-3213(95)00434-2. arXiv: hep-th/9506091. URL: https://doi.org/10.1016%2F0550-3213%2895%2900434-2.

[77] Victor V. Batyrev. "Quantum cohomology rings of toric manifolds". en. In: *Journées de géométrie algébrique d'Orsay - Juillet 1992*. Astérisque 218. Société mathématique de France, 1993. URL: http://www.numdam.org/item/AST_1993__218__9_0/.

[78] Pierre Corvilain, Thomas W. Grimm, and Irene Valenzuela. "The Swampland Distance Conjecture for Kähler moduli". In: *Journal of High Energy Physics* 2019.8 (Aug. 2019). DOI: 10.1007/jhep08(2019)075. arXiv: 1812.07548. URL: https://doi.org/10.1007%2Fjhep08%282019%29075.

[79] Cesar Fierro Cota, Albrecht Klemm, and Thorsten Schimannek. *Modular Amplitudes and Flux-Superpotentials on elliptic Calabi-Yau fourfolds*. 2017. DOI: 10.48550/ARXIV.1709.02820. URL: https://arxiv.org/abs/1709.02820.

[80] David R. Morrison and M.Ronen Plesser. "Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties". In: *Nuclear Physics B* 440.1-2 (Apr. 1995), pp. 279–354. DOI: 10.1016/0550-3213(95)00061-v. arXiv: hep-th/9412236. URL: https://doi.org/10.1016%2F0550-3213%2895%2900061-v.

[81] Volker Braun. "Toric elliptic fibrations and F-theory compactifications". In: *Journal of High Energy Physics* 2013.1 (Jan. 2013). DOI: 10.1007/jhep01(2013)016. arXiv: 1110.4883. URL: https://doi.org/10.1007%2Fjhep01%282013%29016.

[82] Johanna Knapp and Maximilian Kreuzer. "Toric Methods in F-Theory Model Building". In: *Advances in High Energy Physics* 2011 (2011), pp. 1–18. DOI: 10.1155/2011/513436. arXiv: hep-th/1103.3358. URL: https://doi.org/10.1155%2F2011%2F513436.

[83] Cyril Closset. *Toric geometry and local Calabi-Yau varieties: An introduction to toric geometry (for physicists)*. 2009. DOI: 10.48550/ARXIV.0901.3695. arXiv: 0901.3695. URL: https://arxiv.org/abs/0901.3695.

[84] Robin Hartshorne. *Algebraic Geometry*. Springer New York, NY. ISBN: 978-1-4757-3849-0. DOI: https://doi.org/10.1007/978-1-4757-3849-0.

[85] Daniel Huybrechts. *Complex Geometry*. Springer, 2005. ISBN: 3-540-21290-6.

[86] Werner Ballmann. *Lectures on Kähler Manifolds*. European Mathematical Society, 2006. ISBN: 978-3037190258.