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ENTROPY OF DE SITTER SPACETIME

Analysis of Replica Wormholes

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Abstract

Taking into account quantum effects, the cosmological horizon in de Sitter spacetime is seen to radiate by a static observer, just like the event horizon of the Schwarzschild black hole. A temperature and thermodynamic entropy is then associated to both. Regarding both universes as quantum systems, we can also consider the von Neumann entropy of their subregions. A side-by-side comparison between these spacetimes is made on their thermodynamic properties, quantum entropy and the application of replica wormholes to the von Neumann entropy of their radiation. For the black hole it is known a classical replica derivation of its von Neumann entropy results in exactly its thermodynamic entropy. We use the same method for the static patch in de Sitter spacetime and also find its thermodynamic entropy. Recently it was shown a semi-classical replica method can reproduce the island formula for radiation from an eternal black hole by including replica wormholes. Again, we apply the same calculation to de Sitter spacetime in the Bunch-Davies vacuum and recover the island formula.

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Chapter 1

Introduction

In 1975 Stephen Hawking studied the Schwarzschild black hole in a semi-classical approximation, and found a static observer standing outside of it would see radiation being created by the black hole [1]. Even before this discovery, the link between black holes and thermodynamics had been established by writing the dynamic relation between the black hole mass and its horizon area in complete analogy to the first law of thermodynamics [2]. The other three laws were also used to represent properties of a black hole. It was proposed that the black hole temperature is proportional to its surface gravity, and it would have an entropy proportional to the area of the event horizon $\mathcal{A}$. Hawking's calculation did indeed confirm this relation for the temperature and fixed the proportionality constant for the entropy, called the Bekenstein-Hawking entropy:

$$S_{BH} = \frac{\mathcal{A} k_B c^3}{4G\hbar} \,.$$

As this thermodynamic entropy arises from taking into account the quantum aspects of a black hole, we are moving into the realm of quantum gravity. General relativity is non-renormalizable, so the usual way of quantizing a theory does not work. The main problem lies in the fact that the metric is now also a dynamical field, and not just a fixed description of the background spacetime. Many paths to quantum gravity have been tried and are still being worked on (see Chapter 14 of Wald [3] for a short overview). At this time string theory is the most promising candidate in reconciling these two theories, and the Bekenstein-Hawking entropy has turned up in it as a counting of black hole microstates [4].

For working purposes, however, a semi-classical approximation is usually taken where the background spacetime is fixed (but can be curved) and a quantum field theory is then defined on it. Already at the first order of this approximation Hawking radiation appears. The next order would be the backreaction of the quantum fields on the background metric, as their presence would change it via the Einstein field equations.

Now the customary definition of entropy for a quantum subsystem is the von Neumann entropy (or entanglement entropy when the total state is a pure state), and Srednicki showed it should also be proportional to the area of the boundary of the subsystem [5]. This has a striking similarity to the Bekenstein-Hawking entropy. In addition, the interpretation of the von Neumann entropy as being a measure of how much is unknown about a system suits the black hole.

This lead Ryu and Takayanagi to propose a holographic derivation of the von Neumann entropy [6], where it is proportional to the area of some minimal surface. Holography describes the duality of a quantum gravity theory in a bulk spacetime to a quantum field theory on its boundary and has been very useful in the study of black hole quantum properties. The proposal by Ryu and Takayanagi has been proven and extended to a covariant expression containing the first semi-classical correction in the quantum extremal surface formulation [7][8]. Considering a radiating and evaporation black hole, this formulation has been found to successfully describe the unitary evolution of the black hole von Neumann entropy in time [9].

Special interest goes out to the von Neumann entropy of the radiation and whether it evolves unitarily. This is the core problem in the Black Hole Information Paradox and is made apparent in the Page curve describing this entropy [10]. The unitary evolution is described by the quantum extremal surface formalism when allowing for disconnected surfaces. This is also called the island formula, and will be a central topic in this text.

Instead of assuming a thermodynamic law, the Bekenstein-Hawking entropy was also derived using the relation between a Euclidean gravity theory and a statistical thermodynamic theory [11]. Another way is to use Euclidean path integrals in combination with a technique called the replica trick [7]. This method can also be applied to calculate the von Neumann entropy of a subsystem. In 2019, two groups even showed that the replica method can reproduce the island formula and performed explicit calculations for the eternal black hole [12][13].

Very quickly after the finding of Hawking radiation, the same phenomenon was shown to appear at other types of horizons [14]. Here, we want to study the cosmological horizon, which arises in the expanding de Sitter spacetime for a static observer. They will observe thermal radiation coming from this horizon at a temperature proportional to the surface gravity. The horizon dynamics have also been formulated in thermodynamic-like laws and the associated thermodynamic entropy is completely similar to the Bekenstein-Hawking entropy:

$$S_{dS} = \frac{\mathcal{A} k_B c^3}{4 G \hbar} ,$$

with $\mathcal{A}$ the area of the cosmological horizon. These similarities between the event horizon of a black hole and the cosmological horizon in de Sitter spacetime motivate us to study how far the resemblance reaches and to discuss the differences of de Sitter in comparison.

The semi-classical properties of de Sitter have been derived as arising in quantum field theory on curved spacetime, and using the relation between Euclidean gravity and thermodynamics [14]. Also the island formula has been applied to de Sitter spacetime, although it does not yet have a complete holographic dual description [15][16]. Our goal is, then, to analyse the replica method in deriving the island formula in de Sitter spacetime. A successful formulation will give more justification to its direct use.

The text is structured as follows: in chapter 2 first the necessary definitions are introduced to discuss quantum information, then the Schwarzschild black hole and de Sitter spacetime are described as solutions to classical general relativity and with their semi-classical thermodynamic laws. In addition, the black hole information paradox and possible “cosmological information paradox" are discussed. In chapter 3 we introduce the Euclidean path integral formulation of quantum field theory and the basics of holography, then we go over the steps taken to formulate the quantum extremal surface prescription of von Neumann entropy and in particular the island formula. In the final chapter 4 we first re-derive the Bekenstein-Hawking entropy using a classical replica method, and perform the same calculation for de Sitter spacetime. We find at the classical level their von Neumann entropies are exactly the thermodynamic entropies. Then, we outline the semi-classical replica method used to derive the island formula for the eternal Schwarzschild black hole and discuss its analogy in de Sitter. We make a direct comparison to the quantum extremal surface derivation by Watse Sybesma [15] and find the descriptions agree.

Chapter 2

Preliminaries

2.1 Quantum Information

In this section we introduce the necessary axioms, definitions and theorems to talk about quantum systems. The following is heavily based on lecture notes by Michael Walter and Maris Ozols [17].

To be able to describe a quantum system mathematically, we introduce the following axiom:

Axiom *To every quantum system, we associate a Hilbert space $\mathcal{H}$.*

A Hilbert space is a complex vector space with an inner product $\langle\phi|\psi\rangle$, where $|\phi\rangle$ is a vector in $\mathcal{H}$, and $\langle\psi|$ a dual vector in $\mathcal{H}^*$. For our purposes we only consider finite dimensional Hilbert spaces. The inner product has the following properties:

1. The inner product associates a complex number to each pair of elements in $\mathcal{H}$.
2. The inner product of a pair of elements is the same as the complex conjugate of the inner product of the swapped elements: $\langle\phi|\psi\rangle = \overline{\langle\psi|\phi\rangle}$.
3. The inner product is anti-linear in its first argument: $\langle a\phi + b\chi|\psi\rangle = \overline{a}\,\langle\phi|\psi\rangle + \overline{b}\,\langle\chi|\psi\rangle$.
4. The inner product of an element with itself is positive definite:

$$\begin{aligned} \langle\phi|\phi\rangle &> 0 \qquad \text{if } \phi \neq 0\,, \\ \langle\phi|\phi\rangle &= 0 \qquad \text{if } \phi = 0\,. \end{aligned}$$

Intuitively, it could help to view $|\phi\rangle$ as a column vector, and $\langle\phi|$ as its corresponding row vector obtained by taking the conjugate transpose. With this inner product, we can define a norm on the Hilbert space as $\|\phi\| := \sqrt{\langle\phi|\phi\rangle}$. A unit vector has a norm equal to unity. Now we can define the orthonormal basis of the Hilbert space to be the collection of vectors $\{|e_i\rangle\}$ obeying $\langle e_i|e_j\rangle = 0$ if $i \neq j$, and $\langle e_i|e_j\rangle = 1$ if $i = j$. This means the identity operator can be written as

$$I = \sum_i |e_i\rangle\langle e_i|\ , \tag{2.1}$$

for any orthonormal basis $\{|e_i\rangle\}$. The trace of an operator, or matrix, M is

$$\mathrm{tr}[M] = \sum_i \langle e_i| M |e_i\rangle \ . \tag{2.2}$$

To be more specific, an operator M on $\mathcal{H}$ is a linear operator that transforms a vector in $\mathcal{H}$ into another vector in $\mathcal{H}$ or another Hilbert space $\mathcal{K}$. That is, $M \in \mathrm{L}(\mathcal{H}, \mathcal{K})$ or $M \in \mathrm{L}(\mathcal{H})$, with

$$\mathrm{L}(\mathcal{H}, \mathcal{K}) := \{A : \mathcal{H} \to \mathcal{K} \text{ linear}\} , \quad \mathrm{L}(\mathcal{H}) := \mathrm{L}(\mathcal{H}, \mathcal{H}) := \{A : \mathcal{H} \to \mathcal{H} \text{ linear}\} . \tag{2.3}$$

Every operator $M \in \mathrm{L}(\mathcal{H}, \mathcal{K})$ has an adjoint $M^\dagger \in \mathrm{L}(\mathcal{K}, \mathcal{H})$ defined by

$$\langle \phi| M^\dagger |\psi\rangle = \overline{\langle \phi| M |\psi\rangle} \qquad \forall \, |\phi\rangle \in \mathcal{H} \, , |\psi\rangle \in \mathcal{K} \, . \tag{2.4}$$

Choosing a specific orthonormal basis, you can write the operator M as a matrix. Its adjoint is then the conjugate transpose with respect to this basis: $M^\dagger = \overline{A^T} = (\overline{A})^T$. We call an operator $A \in \mathrm{L}(\mathcal{H})$ Hermitian when $A = A^\dagger$. Hermitian operators have real eigenvalues and their eigenvectors form an orthonormal basis of the Hilbert space. This means we can always write the operator in its eigendecomposition:

$$A = \sum_{i=1}^{d} a_i |\phi_i\rangle\langle\phi_i| \ , \tag{2.5}$$

where $d = \dim(\mathcal{H})$, $a_1, \dots, a_d \in \mathbb{R}$ are its eigenvalues, and $|\phi_1\rangle , \dots , |\phi_d\rangle$ is an orthonormal basis of $\mathcal{H}$, such that for all $1 \leq i \leq d$ we have $A |\phi_i\rangle = a_i |\phi_i\rangle$. This means each $|\phi_i\rangle$ is an eigenvector of A.

By acting with a function on a Hermitian operator we can construct a new operator. This is defined by acting with the function on the eigenvalues, while keeping the eigenvectors the same. More formally, let $f : D \to \mathbb{R}$ be an arbitrary function with $D \subseteq \mathbb{R}$. For any Hermitian operator A with eigendecomposition (2.5) and all eigenvalues $a_i \in D$, we define

$$f(A) := \sum_{i=1}^{d} f(a_i) |\phi_i\rangle\langle\phi_i| \ . \tag{2.6}$$

A special subset of Hermitian operators are the positive semidefinite (PSD) operators, which we will denote by

$$\mathrm{PSD}(\mathcal{H}) = \{A \in \mathrm{L}(\mathcal{H}) : \ A \text{ positive semidefinite}\} . \tag{2.7}$$

Besides being Hermitian, they have the additional property of having nonnegative eigenvalues. As in (2.5), we can write them in their eigendecomposition with $a_i \geq 0$. A positive definite (PD) operator has only positive eigenvalues $a_i > 0$, so these are invertible. Using PSD operators we can describe the specific states a quantum system can be in:

Definition *A state, quantum state, density operator or density matrix ρ is a positive semidefinite operator with* $\mathrm{tr}[\rho] = 1$.

The set of states on a Hilbert space $\mathcal{H}$ is

$$\mathrm{D}(\mathcal{H}) = \{\rho \in \mathrm{PSD}(\mathcal{H}) : \ \mathrm{tr}[\rho] = 1\} . \tag{2.8}$$

Axiom *The state space of a quantum system with Hilbert space $\mathcal{H}$ is $D(\mathcal{H})$.*

One might be more familiar with quantum states being described by unit vectors. And indeed, $\rho = |\psi\rangle\langle\psi|$ is a quantum state for any unit vector $|\psi\rangle \in \mathcal{H}$. All states of this form are called pure states; they have only one nonzero eigenvalue, that is automatically equal to unity. There are, however, also quantum states that are not pure. These are called mixed.

So far we considered situations being described by a single Hilbert space, now we will extend this to composite systems consisting of two or more subsystems:

Axiom *For a quantum system composed of n subsystems with Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$, the overall Hilbert space is given by their tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$.*

This overall Hilbert space has a convenient product basis constructed from the basisvectors of $\mathcal{H}_1, \ldots, \mathcal{H}_n$. Let B_i be the set of basisvectors of $\mathcal{H}_i$, then the basis B of $\mathcal{H}$ consists of all vectors $|b_i\rangle \otimes \ldots \otimes |b_n\rangle$, with $|b_i\rangle \in B_i$.

One possible type of state on such a joint system is the product state. For $\rho_i \in \mathrm{D}(\mathcal{H}_i)$ with $i = 1, \ldots, n$, the state $\rho = \rho_i \otimes \ldots \otimes \rho_n$ is a product state $\rho \in \mathrm{D}(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n)$. When a state is not a product state, it is called correlated.

Here we used the tensor product of operators, which is defined by the tensor product of vectors. Take $M \in \mathrm{L}(\mathcal{H}_1, \mathcal{K}_1)$ and $N \in \mathrm{L}(\mathcal{H}_2, \mathcal{K}_2)$, then their tensor product $M \otimes N$ is a linear operator in $\mathrm{L}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$ defined as

$$(M \otimes N)(|\psi\rangle \otimes |\phi\rangle) \equiv M|\psi\rangle \otimes N|\phi\rangle \qquad \forall\, |\psi\rangle \in \mathcal{H}_1\,, |\phi\rangle \in \mathcal{H}_2\,. \tag{2.9}$$

Describing their tensor product in the product basis, we get the following matrix elements:

$$\langle a, b| M \otimes N |c, d\rangle = (\langle a| \otimes \langle b|)(M \otimes N)(|c\rangle \otimes |d\rangle) = \langle a| M |c\rangle \langle b| N |d\rangle\ , \tag{2.10}$$

and we recognize this is just the Kronecker product of matrices.

Now we know how to construct a state in a composite system from the states of the subsystems, we also want a way to distill the state describing a subsystem from the total state. We can do so with the partial trace. For every operator $M_{AB} \in \mathrm{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the partial trace over B is the linear map $\mathrm{tr}_B : \mathrm{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathrm{L}(\mathcal{H}_A)$ defined as

$$\mathrm{tr}_B[M_{AB}] := \sum_b (I_A \otimes \langle b|) M_{AB} (I_A \otimes |b\rangle)\,, \tag{2.11}$$

where $\{|b\rangle\}$ is an arbitrary orthonormal basis of $\mathcal{H}_B$.

Applying the partial trace to a state results in a state again, which completely describes the subsystem independently of the system that is traced out. Thus this is a good method to find the state of a subsystem, also called the reduced state.

Definition *For a state ρ_{AB} on system AB, its reduced state on subsystem A is defined by $\rho_A := \mathrm{tr}_B[\rho_{AB}]$. Similarly, the reduced state on subsystem B is $\rho_B := \mathrm{tr}_A[\rho_{AB}]$.*

This definition can be naturally extended to states on three or more subsystems. Before, we mentioned that states that are not product states are correlated. These correlations come in two types: classical or quantum (also called entanglement). To distinguish between the two we introduce separable and entangled states.

A state $\rho_{AB} \in \mathrm{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is called separable or unentangled, if it is a convex combination of product states:

$$\rho_{AB} = \sum_{i \in I} p_i \, \rho_{A,i} \otimes \rho_{B,i} \, , \tag{2.12}$$

with $(p_i)_{i \in I}$ a probability distribution and $\rho_{A,i} \in \mathrm{D}(\mathcal{H}_A)$, $\rho_{B,i} \in \mathrm{D}(\mathcal{H}_B)$ quantum states. In this construction only classical correlations are used. When a state is not separable it is called entangled, and can not be constructed using classical correlations only. Note that all product states are separable.

An interesting entangled state is the maximally entangled state. This is a pure state on two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of the same dimension d constructed like

$$\rho_{AB} = |\Phi_{AB}\rangle\langle\Phi_{AB}| \, , \qquad |\Phi_{AB}\rangle \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |e_i\rangle \otimes |f_i\rangle \, , \tag{2.13}$$

where $\{|e_i\rangle\}_{i=1}^d$ and $\{|e_i\rangle\}_{i=1}^d$ are orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Its reduced states ρ_A and ρ_B are maximally mixed, which means they are of the form I/d, with I the identity matrix on their corresponding Hilbert space.

Another interesting state is the thermofield double state [18], constructed from the eigenstates and -values of a Hamiltonian H on two copies of the same Hilbert space:

$$\rho_{TFD} = \frac{1}{\mathcal{Z}} \sum_{m,n} e^{-\beta(E_m+E_n)/2} \, |m\rangle \, \langle n|_A \otimes |m\rangle \, \langle n|_{A'} \, , \tag{2.14}$$

where $H\,|n\rangle = E_n\,|n\rangle$, $\mathcal{Z} = \sum_n e^{-\beta E_n}$ is the normalization factor. The overall state is pure, while the reduced states are thermal states:

$$\begin{aligned} \rho_A &= \frac{1}{\mathcal{Z}} \sum_n e^{-\beta E_n} \, |n\rangle \, \langle n|_A \\ &= \frac{1}{\mathcal{Z}} e^{-\beta H} \equiv \rho_{th} \, . \end{aligned} \tag{2.15}$$

They describe a state in statistical thermodynamics using a canonical ensemble at inverse temperature $T = 1/\beta$, and we recognize $\mathcal{Z}$ as its partition function.

In the following chapters the quantity of interest is the entropy, which we can define for quantum systems in the following way:

Definition *The von Neumann entropy of a state* ρ *is* $S_{vN}(\rho) = -\operatorname{tr}[\rho \log(\rho)]$.

The logarithm is to base 2 (i.e. $\log 2 = 1$), since the information is measured in bits. Recall that when $\rho = \sum_i \lambda_i \, |\phi_i\rangle\langle\phi_i|$ is its eigendecomposition, $\log(\rho)$ is defined as $\log(\rho) = \sum_i \log(\lambda_i) \, |\phi_i\rangle\langle\phi_i|$. The von Neumann entropy can then also be calculated in terms of its eigenvalues as $S_{vN}(\rho) = -\sum_i \lambda_i \log(\lambda_i)$. Since the function $f(x) = x \log x$ can be continuously extended to $x = 0$, the von Neumann entropy is well defined for all states $\rho \in \mathrm{D}(\mathcal{H})$.

Some properties of the von Neumann entropy are:

1. *Nonnegativity* $S_{vN}(\rho) \geq 0$, and $S_{vN}(\rho) = 0$ if and only if ρ is pure.
2. *Upper bound* $S_{vN}(\rho) \leq \log(\dim \mathcal{H})$, and $S_{vN}(\rho) = \log(\dim \mathcal{H})$ if and only if ρ is maximally mixed (i.e. $\rho = I/\dim \mathcal{H}$).
3. *Invariance under isometries* $S_{vN}(\rho) = S_{vN}(V \rho V^\dagger)$ for any isometry V.
4. *Continuity*

An operator $V \in \mathrm{L}(\mathcal{H}, \mathcal{K})$ is an isometry if $V^\dagger V = I_\mathcal{H}$. When $\dim \mathcal{H} = \dim \mathcal{K}$, any isometry U is a unitary and is also satisfies $UU^\dagger = I_\mathcal{K}$.

To talk about entanglement one always has to refer to two or more subsystems, so we will go over what this means for the von Neumann entropy. For convenience we will use the following notation: $S_{vN}(AB)_\rho = S_{vN}(\rho_{AB})$ or even leave out the subscript when it is clear to what state we refer. The entropies of subsystems are related to the entropy of the overall system by the following inequalities:

1. *ρ_{AB} is pure* $S_{vN}(AB) = 0$ and $S_{vN}(A) = S_{vN}(B)$, which we call the entanglement entropy.
2. *ρ_{AB} is a product state* $S_{vN}(AB) = S_{vN}(A) + S_{vN}(B)$.
3. *Subadditivity* $S_{vN}(A) + S_{vN}(B) \geq S_{vN}(AB)$.
4. *No monotonicity* $S_{vN}(AB) \ngeq S_{vN}(A)$ and $S_{vN}(AB) \ngeq S_{vN}(B)$.
5. *Araki-Lieb or triangle inequality* $S_{vN}(AB) \geq |S_{vN}(A) - S_{vN}(B)|$.
6. *Strong subadditivity* $S_{vN}(AC) + S_{vN}(BC) \geq S_{vN}(ABC) + S_{vN}(C)$.
7. *Weak monotonicity* $S_{vN}(AC) + S_{vN}(BC) \geq S_{vN}(A) + S_{vN}(B)$.

The von Neumann entropy can be interpreted as a measure of how much is unknown about a system. This is reflected by the minimal entropy of a pure state, and the maximal entropy of a maximally mixed state. Or from the upper bound of the von Neumann entropy we see it counts the number of dimensions of the corresponding Hilbert space: $\dim \mathcal{H} = 2^{S_{vN}}$. In statistical mechanics entropy is a counting of the number of microstates associated with a particular macrostate: $S_{st} = k_B \ln(\text{number of microstates})$ [19], similar to the counting of Hilbert space dimensions. In thermodynamics it is usually seen as a measure of uncertainty (Gibbs' entropy: $S_{th} = -k_B \sum_i P_i \ln P_i$, with P_i the probability of the system being in a certain macrostate), or disorder ("entropy of mixing"). If we re-scale $k_B = 1$ and use a logarithm to base e instead of 2, we see the von Neumann entropy of a classical state and Gibbs' entropy match in definition.

This thermodynamic entropy is a specific type of coarse-grained entropy [20], which is linked to the von Neumann entropy of a general state, also called fine-grained or quantum entropy. The coarse-grained entropy of a state ρ is defined as

$$S_{cg}(\rho) = \max_{\tilde{\rho}\in \mathrm{D}(\mathcal{H})} \left[-\operatorname{tr}[\tilde{\rho}\ln\tilde{\rho}]\right] , \tag{2.16}$$

where $\tilde{\rho}$ describes the same statistics as state $\rho \in \mathrm{D}(\mathcal{H})$ for a number of observables O_i we are interested in: $\operatorname{tr}[\tilde{\rho}\, O_i] = \operatorname{tr}[\rho\, O_i]$. Working, for instance, in the canonical ensemble, these observables are the number of particles, volume and temperature of the system. The thermal state (2.15) satisfies the maximum $S_{cg}(\rho) = S_{vN}(\rho_{th})$, and we can check this gives the thermodynamic entropy[1]:

$$\begin{aligned}
S_{vN}(\rho_{th}) &= -\operatorname{tr}\left[\frac{1}{\mathcal{Z}}\sum_n e^{-\beta E_n}\,|n\rangle\langle n|\,\ln\left(\frac{1}{\mathcal{Z}}\sum_m e^{-\beta E_m}\,|m\rangle\langle m|\right)\right] \\
&= -\operatorname{tr}\left[\frac{1}{\mathcal{Z}}\sum_n e^{-\beta E_n}\,\ln\left(\frac{1}{\mathcal{Z}}\,e^{-\beta E_n}\right)|n\rangle\langle n|\right] \\
&= \sum_n \frac{1}{\mathcal{Z}}e^{-\beta E_n}\,\ln\left(\frac{1}{\mathcal{Z}}\,e^{-\beta E_n}\right) \\
&= \sum_n P_n \ln(P_n) = S_{th} ,
\end{aligned} \tag{2.17}$$

where $P_n = \frac{1}{\mathcal{Z}}e^{-\beta E_n}$ is the probability the system has energy E_n. In contrast to the von Neumann entropy, which is invariant under unitary time transformations, the coarse-grained entropy obeys the second law of thermodynamics: $dS_{cg} \geq 0$. From its definition we also see

$$S_{vN}(\rho) \leq S_{cg}(\rho) \qquad \forall\rho \in \mathrm{D}(\mathcal{H}) , \tag{2.18}$$

where equality is included since we can always consider ρ itself as one of the options for $\tilde{\rho}$.

Determining the von Neumann entropy of a system is quite hard as its definition includes the logarithm of a density matrix. An easier way to obtain this quantity is to consider the Rényi-n entropy [21] in the limit of $n \to 1$. For a quantum state $\rho \in D(\mathcal{H})$, the Rényi entropy is defined as

$$S^{(n)}(\rho) = \frac{1}{1-n}\ln\left(\operatorname{tr}\left[\rho^n\right]\right) , \qquad n \in \mathbb{Z}_+ . \tag{2.19}$$

When we extend this definition to $n \in \mathbb{R}_+$, we can use l'Hôpital's rule to take the limit:

$$\begin{aligned}
\lim_{n\to 1} S^{(n)}(\rho) &= \lim_{n\to 1}\frac{\partial_n \ln\left(\operatorname{tr}\left[\rho^n\right]\right)}{\partial_n(1-n)} \\
&= -\lim_{n\to 1}\frac{1}{\operatorname{tr}[\rho^n]}\,\partial_n\left(\operatorname{tr}\left[\rho^n\right]\right) \\
&= -\lim_{n\to 1}\partial_n\left(\operatorname{tr}\left[e^{n\ln\rho}\right]\right) \\
&= -\lim_{n\to 1}\operatorname{tr}[\rho^n\ln\rho] \\
&= -\operatorname{tr}[\rho\ln\rho] \equiv S_{vN}(\rho) ,
\end{aligned} \tag{2.20}$$

[1] Note that we changed the base of the logarithm relative to its definition. This will just scale the entropy by a factor $1/\ln(2)$, which does not change anything for the properties and inequalities of the von Neumann entropy. Only the upper bound now becomes $\ln(\dim\mathcal{H})$.

where we used that $\mathrm{tr}[\rho] = 1$ in the third line. So we have recast the problem of determining the logarithm of a density matrix into determining the trace of a density matrix to the power n (its logarithm is just the logarithm of a number). This method is called the "replica trick" and is very useful in determining the entanglement entropy in quantum field theories. In chapter 3 we extend this to include gravity.

2.2 The Schwarzschild Black Hole

The Schwarzschild metric is a static solution to the vacuum Einstein field equations and describes spherically symmetric gravitational fields [22]. The Einstein field equations are

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,, \tag{2.21}$$

with G Newton's gravitational constant, $g_{\mu\nu}$ the metric tensor, $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$ the Ricci tensor, $R = R^{\mu}{}_{\mu}$ the Ricci scalar and $T_{\mu\nu} = -2/\sqrt{-g}\ \delta I_M/\delta g^{\mu\nu}$ the energy-momentum tensor, where g is the determinant of the metric and I_M the action describing matter fields. In vacuum $T_{\mu\nu} = 0$ and the field equations can be rewritten as

$$R_{\mu\nu} = 0\,. \tag{2.22}$$

The Schwarzschild metric in four dimensions is

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dt^2 + \frac{1}{1 - r_S/r}\, dr^2 + r^2 \left(d\theta^2 + \sin^2(\theta)\, d\phi^2\right)\,, \tag{2.23}$$

where $r_S = 2GM$ is the Schwarzschild radius with M the mass we associate to the black hole, and the coordinate ranges are $-\infty \leq t \leq \infty$, $r_S \leq r \leq \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. This metric describes the viewpoint of a static observer somewhere outside the black hole. One could, however, move towards or even inside the black hole and the coordinate system can be extended to cover the complete spacetime. At $r = 0$ the Kretschmann curvature scalar diverges, so this point is a curvature singularity. At $r = r_S$ all curvature scalars remain finite, but the $r =$ constant hypersurfaces become null. This means timelike paths in the region $r < r_S$ can never cross this point to infinity, and $r = r_S$ is an event horizon. As nothing can escape once it has crossed this boundary, we fittingly call it a black hole.

Going through a series of coordinate transformations, we can rewrite the maximally extended metric in conformal coordinates and construct the corresponding conformal diagram:

$$ds^2 = \omega^{-2}(r, R, T) \left(-dT^2 + dR^2\right) + r^2 d\Omega_2^2\,, \tag{2.24}$$

with timelike coordinate $T \in (-\pi/2, \pi/2)$, spacelike coordinate $R \in (-\pi, \pi)$ and $d\Omega_2^2 = d\theta^2 + \sin^2(\theta)\, d\phi^2$ the metric on a 2-sphere. The prefactor $\omega^{-2}(r, R, T)$ is the conformal scale factor. The singularity and event horizon are still located at $r = 0$ and $r = 2GM$, respectively. We see the metric is conformally related to Minkowski spacetime at constant angular direction. In fact, the spacetime is asymptotically flat, as they have the same causal structure at conformal future and past null infinity $\mathscr{I}^{\pm}$ and spatial infinity i^0. This can be seen in the conformal diagram (figure 2.1).

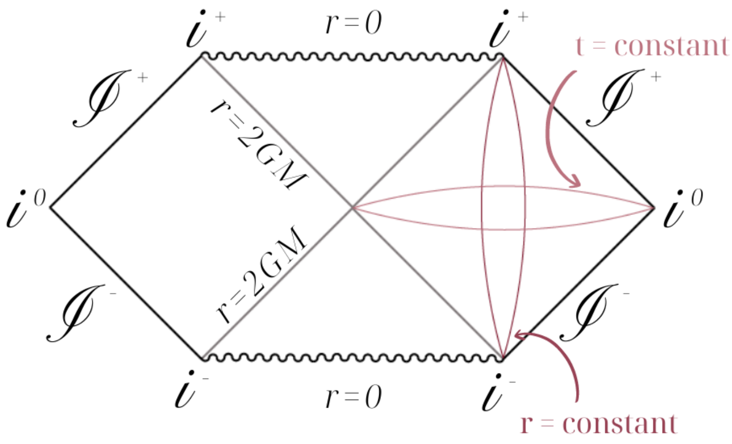


Figure 2.1: The conformal diagram of an eternal Schwarzschild black hole. At $r = 0$ there is a curvature singularity and at $r = 2GM$ is the event horizon. $i^{\pm}$ is future/past timelike infinity, i^0 is spatial infinity and $\mathscr{I}^{\pm}$ is future/past null infinity. Constant t and r curves as in (2.23) are also indicated.

Since the metric of Schwarzschild spacetime is independent of t, it has an isometry generated by the Killing vector $K^\mu = (\partial_t)^\mu$. It is timelike outside of the black hole and can be used to define time evolution and a conserved energy. The event horizon is its Killing horizon, and inside the black hole it becomes spacelike. Together with the Ricci tensor it can be used to construct a conserved current:

$$J_R^\mu = K_\nu R^{\mu\nu}\,, \qquad\qquad \nabla_\mu J^\mu = 0\,. \tag{2.25}$$

Its corresponding conserved energy is

$$E_R = \frac{1}{4\pi G}\int_\Sigma d^3x\sqrt{\gamma}\, n_\mu J_R^\mu\,, \tag{2.26}$$

where Σ is a spacelike hypersurface, γ_{ij} its induced metric, and n^μ its unit normal vector. Then, using the identities $\nabla_\mu\nabla_\nu K^\mu = K^\mu R_{\mu\nu}$ and $\nabla^\mu K^\nu = -\nabla^\nu K^\mu$, we can rewrite E_R with Stokes' theorem to the Komar integral:

$$E_R = \frac{1}{4\pi G}\int_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}\, n_\mu\sigma_\nu\left(\nabla^\mu K^\nu\right). \tag{2.27}$$

$\partial\Sigma$ is the boundary of Σ, with $\gamma_{ij}^{(2)}$ its induced metric, and σ^μ its outward-pointing unit normal vector. Evaluating this integral for the Schwarzschild black hole over a two-sphere at spatial infinity, gives $E_R = M$. Here we chose to normalize the Killing vector as $K_\mu K^\mu(r \to \infty) = -1$.

The Schwarzschild black hole is a solution to the classical theory of general relativity. We know, however, nature fundamentally displays quantum behaviour and we expect it to play a crucial role in extreme situations such as the beginning of the universe or gravitational collapse. To take this into account we would like a theory of quantum gravity. Unfortunately gravity is non-renormalizable, so the usual way of quantizing the theory does not work. The main problem lies in the fact that the metric is now also a dynamical field, and not just a fixed description of the background spacetime. Many paths to quantum gravity have been tried and are still being worked on (see Chapter 14 of Wald [3] for a short overview).

An approximation we can make for now is to couple classical gravity to quantum matter fields. Using this setup, Hawking discovered that black holes create and emit particles at a temperature proportional to their surface gravity [1]. Black holes are thought to be thermal objects that radiate and eventually evaporate completely. This process can be derived and explained in several different ways. An intuitive one considers quantum fields on a fixed black hole background [23]. The vacuum in quantum field theory is an entangled state, which constantly produces entangled particle pairs at every point in space. In the vacuum black hole solution at the event horizon, it can happen one of the particles is lost inside the black hole, while the other one escapes to infinity. This latter one is called Hawking radiation. Since there is global conservation of energy[2], the black hole will evaporate as its radiation carries away positive energy. This alters the conformal diagram to figure 2.2.

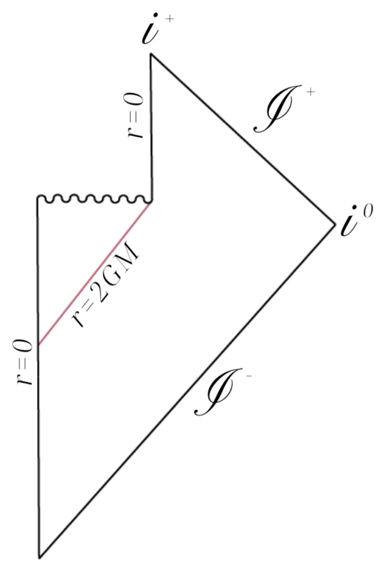


Figure 2.2: Conformal diagram of an evaporating black hole formed by gravitational collapse. At $r = 0$ there is a curvature singularity for some time behind the event horizon at $r = 2GM$. i^+ is future timelike infinity, i^0 is spatial infinity and $\mathscr{I}^\pm$ is future/past null infinity.

Two other ways to derive Hawking radiation are using Euclidean path integrals [18] and Bogolubov transformations. The latter is based on the observation that different observers in quantum field theory in curved spacetime have different notions of the vacuum state and the number of particles present. For instance, we can compare the viewpoint of a static observer somewhere outside the black hole and a freely falling observer near the horizon. Consider a massless scalar field living on a two-dimensional spacetime, that obeys the Klein-Gordon equation. To quantize the field, both observers decompose the field in positive and negative frequency modes. The static observer does so with respect

[2]This is the ADM energy defined with respect to a timelike Killing vector in a asymptotically flat spacetime [22]. It agrees with the Komar integral expression when the metric deviations from flat spacetime at infinity are time-independent.

to the time coordinate t as in (2.23) and defines creation/annihilation operators $\hat{a}_i$, and the freely falling observer uses their proper time τ and defines operators $\hat{b}_i$. Then, both will have their own definition of the vacuum state and neither definition is preferred as they are both physically relevant. The relation between the two sets of modes are the Bogolubov transformations, and can be used to transform between the $\hat{a}$ and $\hat{b}$ operators. Using them to transform between the two vacua, shows one of them can observe an empty universe, while the other sees many particles. See chapter 9 of Carroll [22] for a much more in depth discussion.

In two dimensions the quantum field can be decomposed in in- and outgoing modes (in relation to the black hole) and we can independently choose in which vacuum state they are [24]. There are then four possible vacua:

- *Hartle-Hawking vacuum* [25]
  Both in- and outgoing modes are in vacuum with respect to τ (the b vacuum). In this state the freely falling observer sees an empty spacetime, while the static observer measures an equal in- and outgoing flux of radiation. The state describes an eternal black hole and its corresponding energy-momentum tensor is regular everywhere.

- *Boulware vacuum*
  Both in- and outgoing modes are in vacuum with respect to t (the a vacuum). This describes a situation where no radiation leaves or enters the black hole. However, in this state the enery-momentum tensor is singular on both the future and past horizon, so the state is unphysical.

- *Unruh vacuum* [26]
  The outgoing modes are in the b vacuum, and the ingoing modes are in the a vacuum. This means there is an outgoing thermal flux, and the state describes an evaporating black hole. It is singular on the past horizon, but as we saw in the conformal diagram of an evaporating black hole, there is no past horizon, so the state remains physical.

- *Unruh' vacuum*
  The outgoing modes are in the a vacuum, and the ingoing modes are in the b vacuum. This is the reverse of the previous case, which is singular on the future horizon. This is not a physically relevant vacuum.

Besides Hawking radiation, even more similarities between black hole dynamics and thermodynamics have been found. Actually, this link was established before, and Hawking's discovery put it on solid footing. In analogy these are called the four laws of black hole mechanics [2]:

0. *Zeroth Law*
   The temperature at which a Schwarzschild black hole radiates is

$$T_{BH} = \frac{\hbar\kappa}{2\pi}\,. \tag{2.28}$$

   The surface gravity κ of a stationary black hole is constant over its event horizon, so likewise for its temperature. This statement is similar to the zeroth law in thermodynamics, which says the temperature of a system in thermal equilibrium is constant everywhere. Since the event horizon of a black hole is a Killing horizon, its surface gravity can be defined with the corresponding Killing vector K^μ:

$$\kappa^2 := -\frac{1}{2}(\nabla_\mu K_\nu)(\nabla^\mu K^\nu)\,|_{\text{horizon}}\,. \tag{2.29}$$

For the Schwarzschild black hole $K^\mu = (\partial_t)^\mu$, and $\kappa_{BH} = 1/4GM$. When the black hole is not in thermal equilibrium and evaporates, it losses mass en its temperature rises.

1. *First Law*
   For a rotating black hole, its mass M, horizon area $\mathcal{A}$ and angular momentum J [3] are dynamically related via

   $$dM = \frac{\kappa}{8\pi G}\,d\mathcal{A} + \Omega\,dJ\,, \tag{2.30}$$

   where Ω is the rotational velocity of its horizon. As mentioned before, the surface gravity is related to the black hole temperature. Then, making the assumption that its area is proportional to its entropy (we will justify this in section 3), we find the analog of $dU = dQ + dW$:

   $$dM = T_{BH}\,dS_{BH} + \Omega\,dJ\,. \tag{2.31}$$

   Together with the expression for T_{BH}, this fixes the entropy, called the Bekenstein-Hawking entropy [4]:

   $$S_{BH} = \mathcal{A}/4\hbar G\,. \tag{2.32}$$

   As a Schwarzschild black hole evaporates, the first law then tells its entropy decreases at an accelerating rate.

2. *Second Law*
   The assumed connection between the area and entropy is strengthened by the fact that in the classical theory the horizon area never decreases (since mass can only enter the black hole), like the entropy of a thermally isolated system can never decrease:

   $$d\mathcal{A} \geq 0 \qquad \text{(classically)}\,. \tag{2.33}$$

   This was proved by Hawking assuming cosmic censorship [27], i.e. there are no naked singularities, and the weak energy condition $T_{\mu\nu}N^\mu N^\nu \geq 0$ for every null vector N^μ. However, including quantum effects the black hole radiates and loses mass, leading to a decrease in its entropy. Bekenstein suggested (and it was later proven [28]) that the sum of the entropy of the black and its exterior never decreases [29]. This makes sense as a radiating black hole is not an isolated system. The total is called the generalized entropy and obeys the second law:

   $$dS_{gen} = d(S_{BH} + S_{out}) \geq 0\,. \tag{2.34}$$

   From this law and statistical arguments, Bekenstein derived an entropy bound for any bounded system of radius R and energy E[30]:

   $$S \leq 2\pi RE\,. \tag{2.35}$$

[3] The angular momentum of a rotating black hole is defined by a similar Komar integral as for its mass/energy, but now using the rotational Killing vector $R^\mu = (\partial_\phi)^\mu$.

[4] From now on we will set Planck's constant to $\hbar = 1$.

Thus, a black hole (with $R = \sqrt{\mathcal{A}/4\pi}$ and $E = M = R/G$) is the object that contains the maximal amount of entropy inside a volume of radius R.

3. *Third Law*
   The surface gravity of a black hole can not be reduced to zero in a finite number of steps. Just like in thermodynamics, absolute zero temperature can not be obtained in a finite number of steps. In the case of the Schwarzschild black hole the surface gravity is $\kappa = 1/4M$, so it would take an infinite mass to make it vanish.

Bekenstein suggested to interpret the black hole entropy $S_{BH} = \mathcal{A}/4G$ as a measure of how much information inside the event horizon is inaccessible to an outside observer [31]. This resembles the interpretation of the von Neumann entropy, while the generalized entropy is more like a true thermodynamic entropy as it obeys the second law. A good introduction to black hole thermodynamics and interpretations of its entropy (with many additional references) are the lecture notes by T. Jacobson [32].

2.3 The Black Hole Information Paradox

We saw in the previous section that a black hole can be described as a regular thermal object. This observation led to the following assumption called the central dogma [20]:

> As seen from the outside, a black hole can be described in terms of a quantum system with $\mathcal{A}/4G$ degrees of freedom, which evolves unitarily under time evolution.

This statement means the black hole is described by a unitary Hamiltonian defined on a Hilbert space of finite dimensionality $\exp(\mathcal{A}/4G)$. Considering the black hole in the complete spacetime, there is usually chosen some fictitious surface around it and everything inside it is regarded to be the quantum system as in the central dogma. Its degrees of freedom are then entangled with the ones outside the cutoff surface, and the evolution of the total coupled system should be unitary. The total Hilbert space is then also split into two parts $\mathcal{H} = \mathcal{H}_{in} \times \mathcal{H}_{out}$, or $\mathcal{H} = \mathcal{H}_{BH} \times \mathcal{H}_{rad}$[5]. In the region outside the cut-off surface gravity is expected to be negligible.

Now imagine we create a black hole in a pure quantum state with its surroundings: the total state ρ is pure, and we know $S_{vN}(\rho) = 0$. We want to track the evolution of the von Neumann entropy of the Hawking radiation in time. The state describing the radiation is $\rho_{rad} = \text{tr}_{BH}[\rho]$ and, since the total state is pure, we know $S_{vN}(\rho_{rad}) = S_{vN}(\rho_{BH})$. In the starting situation no Hawking radiation has been created yet, and the total state only describes the black hole: $S_{vN}(\rho_{rad}) = 0$. The black hole then starts to radiate and the entanglement entropy grows as the in- and outgoing Hawking quanta are entangled (see figure 2.3).

[5] This split is not well defined in a theory of gravity as the degrees of freedom are not localized. This leads to UV divergences in expressions for the black hole entropy, since the vacuum is correlated over short distances (see section 3.1).

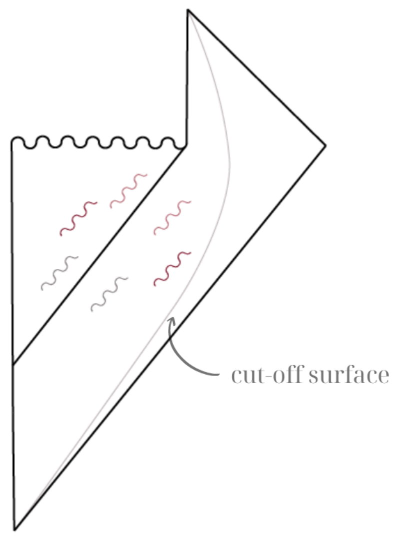


Figure 2.3: The conformal diagram of an evaporating black hole. Hawking radiation of identical colour inside and outside the event horizon are entangled with each other. At constant r a cut-off surface is introduced.

Meanwhile, the black hole losses mass and its horizon area decreases, according to (2.30). When we believe the central dogma, this means the number of black hole degrees of freedom decreases and will eventually become equal to the radiation degrees of freedom: $S_{vN}(\rho_{rad}) = \mathcal{A}_{BH}/4G$. At this time, both subsystems are in a maximally mixed state, and the total state is maximally entangled. If the von Neumann entropy of the radiation would surpass this value, the total state could no longer be pure, as not all radiation can be entangled with the black hole. This is also prohibited by the upper bound property of the von Neumann entropy (see property 2 in 2.1). According to unitarity, however, a pure state remains pure (see property 4 in 2.1), so the radiation entropy would need to start decreasing and eventually get to zero when the black hole is completely evaporated. This evolution in time was first described by Don Page in the so-called Page curve (see figure 2.4) [10]. The moment the radiation entropy is equal to the Bekenstein-Hawking entropy of the black hole is called the Page time.

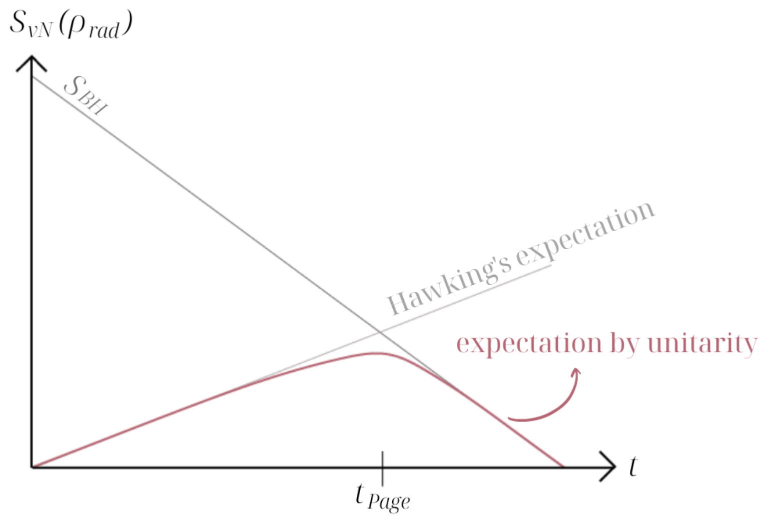


Figure 2.4: The Page curve of the entanglement entropy of an evaporating black hole and its Hawking radiation $S_{vN}(\rho_{rad})$. The dark grey line S_{BH} is the thermodynamic or Bekenstein-Hawking entropy of the black hole, which decreases during evaporation. The light grey line indicates the Hawking calculation for an ever-growing entropy. By unitarity we expect the entropy to follow the pink line, where the entanglement entropy reaches its maximum value at the Page time t_{Page}.

We speak of a paradox because of the discrepancy between Hawking's expectation and the expectation from unitarity. Additionally, it is an information paradox, because we expect no information can be lost in a unitary theory. Therefore, after the black hole has evaporated completely, the radiation should encode the same information. Until the Page time there is no problem. For Sagittarius A* [33], the black hole at the center of the Milky Way, its lifetime is 8.31×10^{86} yr and its Page time 7.27×10^{86} yr [34]. The Page time is estimated to be the time it takes for half of the black hole mass to be evaporated. These times are very long compared to the age of the universe, about 13×10^9 yr. Less massive black holes will evaporate more quickly. Additionally, a black hole will only evaporate when its temperature is above the cosmic microwave background temperature and lighter black holes have a higher temperature. The core problem of the black hole information paradox is finding the physical argument why the entropy of the outgoing radiation starts decreasing after a certain time, and what happens to the information at the end of the evaporation. We will focus mostly on the first question. For the second, various options have been proposed, like a fire wall [35], remnants [36] or loss of information [37].

In the previous section we introduced the Bekenstein-Hawking entropy and the generalized entropy, which is the coarse-grained or thermodynamic entropy of the black hole. When we replace the black hole horizon in the expression for the generalized entropy with an "extremal surface", we have a gravitational expression for the von Neumann entropy of the black hole:

$$S_{vN}(\rho_{BH}) = \min_X \left(\operatorname*{ext}_X \left(\frac{\text{Area}(X)}{4G} + S_{semi-cl}(\Sigma_X) \right) \right) , \tag{2.36}$$

where X is a codimension-2 surface and Σ_X a codimension-1 region between X and the cutoff surface. $S_{semi-cl}(\Sigma_X)$ is the von Neumann entropy of quantum fields living on Σ_X while the geometry is regarded as classical (see section 3.3 for the derivation of this formula). The formula tells you to first find the surface that extremizes the generalized entropy, or more concretely: maximizes in the time direction and minimizes in the spatial direction. When there are multiple options, choose the one that gives the global minimum. X is then called the extremal surface.

The von Neumann entropy of the black hole follows the same Page curve as for the radiation, and (2.36) describes this curve correctly [9]. In the beginning the extremal surface can be moved all the way to the center of the black hole, where its area vanishes and Σ_X is the entire space up to the cutoff surface (see the left diagram in figure 2.5). When the black hole is formed in a pure state, $S_{semi-cl}(\Sigma_X)$ will be zero. Then the black hole will start to radiate and the entanglement entropy for the vanishing surface keeps growing as more and more radiation escapes past the cutoff surface. Meanwhile, another non-vanishing extremal surface appears just inside the black hole, varying in location depending on how much radiation has escaped to infinity (see the middle diagram in figure 2.5). Here its area will not be zero, but $S_{semi-cl}(\Sigma_X)$ will be minimized. Since the black hole shrinks, this extremal surface describes a decreasing von Neumann entropy. Now equation (2.36) tells you to always take the minimum when there are multiple extremal surfaces, so we recover the Page curve (see the right diagram in figure 2.5).

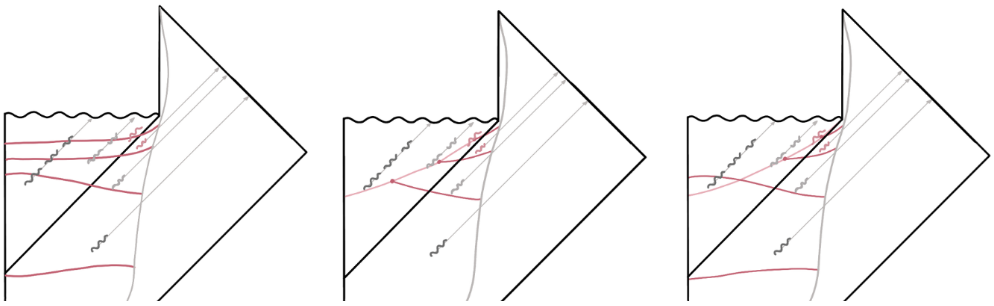


Figure 2.5: In the left diagram a vanishing extremal surface is considered in calculating the entanglement entropy of the black hole. In time more and more Hawking radiation is present on this surface and its entropy contribution keeps growing. In the middle a non-vanishing extremal surface is shown, just behind the event horizon. Its entropy contribution decreases in time. In the extremal surface formulation the one giving minimal entropy should be considered; this is shown in the right diagram. At the Page time the dominating surface switches and this gives rise to the Page curve. Adapted from [20].

Now from the point of view of the radiation, you would expect its von Neumann entropy only grows, since the semi-classical entropy of the region beyond the cutoff surface Σ_{rad} with everything inside increases. To describe the Page curve, however, we need to consider the possibility that Σ_X is disconnected. This gives the so-called island formula:

$$S_{vN}(\rho_{rad}) = \min_{\Sigma_{island}} \left(\underset{\Sigma_{island}}{\text{ext}} \left(\frac{\text{Area}(\partial\Sigma_{island})}{4G} + S_{semi-cl}(\Sigma_{rad} \cup \Sigma_{island}) \right) \right) , \qquad (2.37)$$

where Σ_{island} is a codimension-2 surface which is disconnected from Σ_{rad} (see figure 2.6). There can be any number of islands, including none. Note there is no area term for the boundary of Σ_{rad} as in this part of the spacetime gravitational effects are assumed to be negligible.

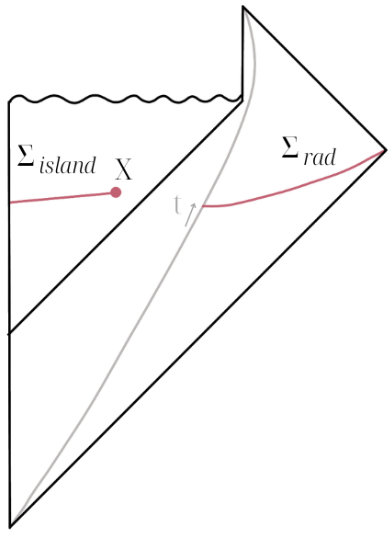


Figure 2.6: Depiction of a possible island Σ_{island} in an evaporating black hole. Here X is the boundary of the island ($\partial\Sigma_{island}$ in the text) and t is the time coordinate as on the cut-off surface. Adapted from [20].

In the case there is no island, the von Neumann entropy is continuously increasing, as mentioned before. When we consider one island inside the black hole, there is an additional area term, but the entanglement entropy decreases since the Hawking quanta in the interior of the black hole purify the outgoing radiation (see the left and middle diagram in figure 2.7). Taking the minimum value at each time gives the Page curve (see the right diagram in figure 2.7). Here, and for the black hole Page curve, the time coordinate we use is the one defined on the cutoff surface.

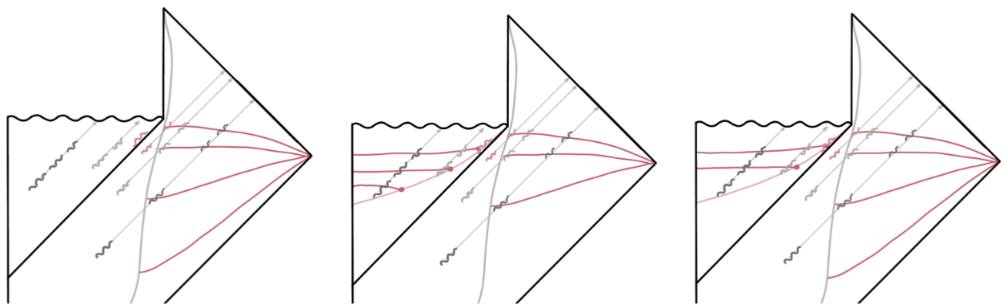


Figure 2.7: In the left diagram the situation with no island is considered in calculating the entanglement entropy of the radiation. In time more and more Hawking radiation is present on this surface and its entropy contribution keeps growing. In the middle a non-vanishing extremal surface is shown, just behind the event horizon. This creates an island inside the black hole. Its entropy contribution decreases as more and more outgoing Hawking radiation gets purified by the ones present inside the black hole. In the extremal surface formulation the one giving minimal entropy should be considered; this is shown in the right diagram. At the Page time the dominating surface switches and this gives rise to the Page curve. Adapted from [20].

The island formula is a very good description of the Page curve, which is not just fabricated to be so, but is derived from a more fundamental basis. In chapter 3 it is derived from holography and in chapter 4 it is derived using the replica trick in the Euclidean path integral formalism. A good introduction to Hawking radiation, the Page curve and the black hole information paradox is "The entropy of Hawking radiation" [20], and also the reviews by Polchinski [37] and Mathur [23].

2.4 De Sitter Spacetime

We say an d-dimensional manifold is maximally symmetric when it has $\frac{1}{2}d(d+1)$ Killing vectors [22]. It also has a constant curvature (constant Ricci scalar) that is the same in every direction, and together with the dimensionality d and metric signature it completely specifies the space. In Lorentzian signature, Minkowski space is the maximally symmetric spacetime with zero curvature. With positive curvature it is de Sitter spacetime, and negative curvature Anti-de Sitter. Here we will focus on de Sitter spacetime, which is a solution to the Einstein equations with a positive vacuum energy or cosmological constant Λ:

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} + \Lambda \, g_{\mu\nu} = 0 \,. \tag{2.38}$$

For general Λ, the maximally symmetric solutions to these equations are the well-known Friedmann-Lemaître-Robertson-Walker spacetimes, described by

$$ds^2 = -dt^2 + a^2(t) \left(\frac{1}{1-kr^2} dr^2 + r^2 d\Omega_2^2 \right) \,, \tag{2.39}$$

where $a(t)$ is the scale factor and $k \in \{-1, 0, 1\}$ indicates constant negative, zero or positive curvature, respectively. The scale factor is related to the Hubble parameter via

$$H(t) = \frac{\partial a(t)/\partial t}{a(t)} \,. \tag{2.40}$$

When space is flat and dominated by the vacuum energy, the scale factor is an exponential $a(t) = \exp(Ht)$, with H a constant. The metric, in planar coordinates, reads:

$$ds^2 = -dt^2 + e^{Ht} \left(dx^2 + dy^2 + dz^2 \right) \,. \tag{2.41}$$

This describes an exponentially expanding universe, and is a solution to the Einstein equations with positive cosmological constant. All solutions are locally unique, so this metric represents de Sitter spacetime (actually part of it, as we will see later).

We can also represent four-dimensional de Sitter space as an embedding in five-dimensional Minkowski space by using the following (hyperbolic) constraint [38]:

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = L^2 \,, \tag{2.42}$$

where $L = \sqrt{3/\Lambda}$ is the characteristic length scale of the space. Additionally, we use the following coordinate transformations:

$$\begin{aligned} x_0 &= L \, \sinh(t/L) \\ x_i &= L \, \omega_i \cosh(t/L) \qquad i = 1, 2, 3, 4 \,, \end{aligned} \tag{2.43}$$

with ω_i the coordinates on a 4-sphere: $\omega_1 = \cos(\theta_1)$, $\omega_2 = \sin(\theta_1)\cos(\theta_2)$, ..., which obey $\sum_i \omega_i^2 = 1$. Applying this condition and transformations to the Minkowski metric we get the de Sitter metric in global coordinates:

$$ds^2 = -dt^2 + L^2 \cosh^2(t/L)\left(d\theta_1^2 + \sin^2(\theta_1)\, d\theta_2^2 + \sin^2(\theta_1)\sin^2(\theta_2)\, d\theta_3^2\right) , \tag{2.44}$$

where $-\infty < t < \infty$, $0 \leq \theta_1, \theta_2 < \pi$ and $0 \leq \theta_3 < 2\pi$. We see in time the metric describes a 3-sphere that initially shrinks, takes on minimal size at $t = 0$ and then expands again. Its topology is $R \times S^3$, so it is a closed space and only has a boundary at past and future null infinity.

From the point-of-view of a static observer, there is a certain radius around them beyond which spacetime expands faster than the speed of light. Signals from beyond that radius can then never reach the observer, so they are limited in how much they can see of the universe. This point is called the cosmological horizon. The situation is the reverse of Schwarzschild black hole case (see figure 2.8). A crucial difference, however, is that multiple observers can agree on the location of a black hole horizon, while the cosmological horizon is observer dependent. This is represented in the metric in static coordinates:

$$ds^2 = -\left(1 - \frac{r^2}{L^2}\right) dt^2 + \frac{1}{1 - r^2/L^2}\, dr^2 + r^2 d\Omega_2^2 , \tag{2.45}$$

where the coordinate ranges are $-\infty \leq t \leq \infty$ and $0 \leq r \leq L$. The topology the metric describes is $R \times B^3$, where B^d is the solid $(d-1)$-sphere or d-ball. The cosmological horizon is located at $r = L$, and is the point where constant r hypersurfaces become null. Note that the time coordinates t are different in all coordinate systems introduced so far.

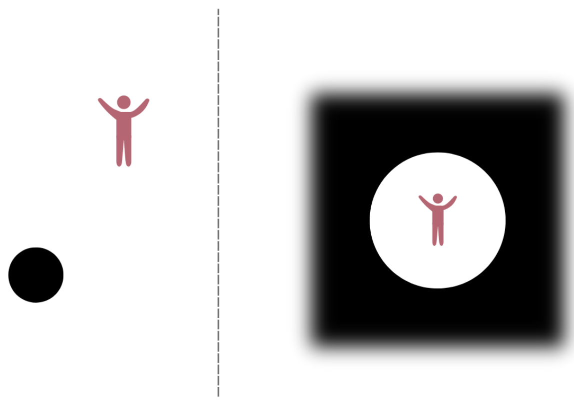


Figure 2.8: On the left the black hole situation is depicted where a distant observer looks at the event horizon of a Schwarzschild black hole. On the right the point-of-view of a static observer in de Sitter spacetime is depicted. They have a cosmological horizon beyond which they can not observe the spacetime.

Starting from the metric in global coordinates we can change to conformal coordinates using

$$\cosh(t/L) = \frac{1}{\cos(T)} , \tag{2.46}$$

with $T \in (-\pi/2, \pi/2)$. The metric becomes:

$$ds^2 = \frac{L^2}{\cos^2(T)}\left(-dT^2 + d\theta_1^2 + \sin^2(\theta_1)\, d\theta_2^2 + \sin^2(\theta_1)\sin^2(\theta_2)\, d\theta_3\right) , \tag{2.47}$$

and the corresponding conformal diagram is figure 2.9.

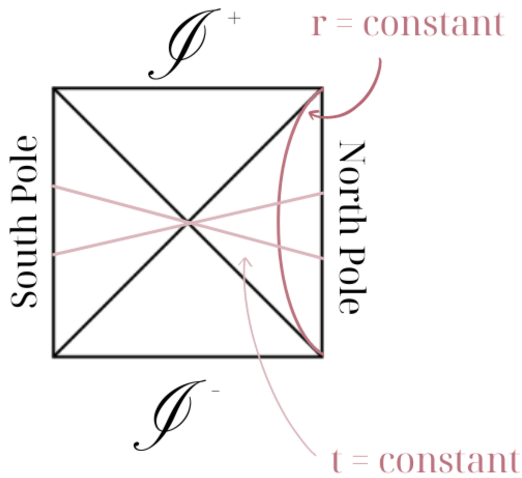


Figure 2.9: The conformal diagram of d-dimensional de Sitter spacetime (with $d > 2$). A horizontal line is a $d-1$-sphere and every point is a $d-2$-sphere. A static observer is located on the north or south pole in the diagram. The diagonal lines are the cosmological horizons for those observers. $\mathscr{I}^{\pm}$ is future/past null infinity. In the right static path lines of constant r and t as in (2.45) are indicated.

Every horizontal constant T-line represents a three-sphere, and every point in the diagram represents a two-sphere. The left and right timelike vertical lines are the south and north pole. The top and bottom horizontal lines are, respectively, future and past null infinity. For a static observer, their world line is the north pole, and their observable universe is the rightmost triangular region, delimited by the past and future horizon (diagonal lines). This region is called the static patch. The planar coordinates cover the causal past of the observer, consisting of the static patch and the triangle containing past null infinity $\mathscr{I}^-$ (see figure 2.10).

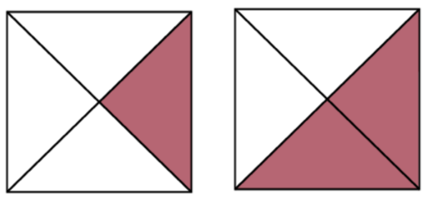


Figure 2.10: On the left the northern static patch covered by metric (2.45) is highlighted. On the right the region of the conformal diagram covered by planar coordinates (2.41) is highlighted.

From the metric (2.45) we see $\partial/\partial t$ is a Killing vector in the static coordinate system. In the static patch it is timelike, and can be used to define time evolution. On the cosmological horizon, it becomes null, and beyond, it becomes spacelike. In the northern static patch it is timelike, but points to the past. In other coordinate systems, $\partial/\partial t$ is not a Killing vector. So, in de Sitter spacetime it is not possible to define a globally conserved energy, as there is no globally well defined timelike killing vector. Restricting our view to the static patch only, we can use the Komar integral (2.27) to define a locally conserved energy:

$$\begin{aligned}
E_R &= \frac{1}{4\pi G}\int_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}\, n_\mu\sigma_\nu\,(\nabla^\mu K^\nu) \\
&= -\frac{1}{4\pi G}\int_{S^2} d\phi d\theta\, L^2\sin(\theta)\,(\nabla^t K^r) \\
&= -\frac{1}{4\pi G}\int_{S^2} d\phi d\theta\, L^2\sin(\theta)\, g^{tt}(\partial_t K^r + \Gamma^r_{t\lambda}K^\lambda) \\
&= -\frac{1}{4\pi G}\int_{S^2} d\phi d\theta\, L^2\sin(\theta)\, g^{tt}(\Gamma^r_{tt}K^t) \\
&= -\frac{1}{4\pi G}\int_{S^2} d\phi d\theta\, L^2\sin(\theta)\,\frac{r}{L^2} = -\frac{2\pi}{4\pi G}\,2\,r\,|_{r=L} = -\frac{L}{G}\,.
\end{aligned} \tag{2.48}$$

The Killing vector is $K^\mu = (\partial_t)^\mu = (1,0,0,0)$, normalized as $K^2(r=0) = -1$. The integral is evaluated over the static patch at a constant time $\Sigma = B^3$. Its boundary $\partial\Sigma = S^2$ is located at the cosmological horizon $r = L$ and has induced metric $ds^2 = L^2(d\theta^2 + \sin^2(\theta)\,d\phi^2)$. The future-pointing normal vector of Σ is $n^\mu = (\sqrt{1/(1-r^2/L^2)},0,0,0)$, and the outward-pointing normal vector of $\partial\Sigma$ is $\sigma^\mu = (0,\sqrt{1-r^2/L^2},0,0)$. For general dimensionality the energy is $E_R = -\frac{\kappa}{4\pi G}\mathcal{A}$, with κ the surface gravity on the horizon and $\mathcal{A}$ the area of the cosmological horizon.

After the discovery that black holes radiate, this idea was soon generalized by Gibbons and Hawking to cosmological horizons [14]. A static observer in de Sitter space observes radiation entering their static patch at a certain temperature. This is derived using Euclidean path integrals in section 4.1.1. Additionally, Gibbons and Hawking generalized the laws of black hole mechanics to event horizons that do not necessarily belong to black holes, or are located in spacetimes that are not necessarily asymptotically flat:

0. *Zeroth Law*
   A static observer feels a thermal bath at temperature

$$T_{dS} = \frac{\kappa}{2\pi}\,. \tag{2.49}$$

   The cosmological horizon of the static patch is a Killing horizon for Killing vector $K^\mu = (\partial_t)^\mu$, so we can define its surface gravity. First we have to normalize K^μ and choose $K^2(r=0) = -1$. The surface gravity is constant over the horizon and has the following value in four dimensions:

$$\kappa_{dS} := \left(-\frac{1}{2}(\nabla_\mu K_\nu)(\nabla^\mu K^\nu)\,|_{\text{horizon}}\right)^{1/2}|_{r=L} \tag{2.50}$$

$$= \frac{1}{L}\,. \tag{2.51}$$

   This immediately leads to a constant temperature $T_{dS} = 1/2\pi L$ on the horizon. It depends on the characteristic length scale of the de Sitter space, or in other words on the cosmological constant. This is not a variable like the mass is for the Schwarzschild black hole.

1. *First Law*
   The dynamical relation between the energy associated with de Sitter spacetime as defined with the Komar integral and the area of the cosmological horizon is
   $$dE = -\frac{\kappa}{8\pi G} d\mathcal{A} \,. \tag{2.52}$$
   The minus sign is usually interpreted as arising from the fact that the energy describes the energy inside the cosmological horizon. In contrast to the Schwarzschild black hole, then, the horizon area decreases, as more energy is introduced. Substituting the expression for the de Sitter temperature, gives the first law:
   $$dE = -T_{dS} dS_{dS} \,, \tag{2.53}$$
   with the following expression for the de Sitter entropy:
   $$S_{dS} = \frac{\mathcal{A}}{4G} \,. \tag{2.54}$$

2. *Second Law*
   Classically, the area of the cosmological horizon can only grow in time, so likewise for the entropy:
   $$dS_{dS} \geq 0 \qquad \text{(classicaly)} \,. \tag{2.55}$$
   Now as radiation enters the static patch, we can no longer consider the static patch to be an isolated system and the horizon area decreases according to the first law. Likewise for Schwarzschild, we can define a generalized second law:
   $$dS_{gen} = d(S_{dS} + S_{out}) \geq 0 \,. \tag{2.56}$$
   The proof by Wall [28] also holds for cosmological horizons.

3. *Third Law*
   As $T_{dS} = 1/2\pi L$ in four dimensions, it would take an zero cosmological constant to reach zero temperature. This means we would no longer be in de Sitter space, but in flat Minkowski space. Or alternatively, since $E \propto L$ it would take an infinite amount of energy. Thus, zero temperature/surface gravity can not be obtained in a finite number of steps.

For a much more rigorous discussion of thermodynamics in de Sitter spacetime, that describes the developments and difficulties in this field up to now, see [39].

Gibbons and Hawking used a Euclidean path integral method to derive that a static observer in de Sitter space observes to be in a heat bath. As in the black hole case, we could also use the semi-classical approximation and compare possible different vacua. We compare the situation for a static observer in the center of their static patch, using coordinates as in (2.45), and a freely falling observer near the cosmological horizon (as seen for the static observer), using their proper time coordinate τ, by quantizing a massless scalar field. There are again four possible vacua (defined in two and generalized to higher dimensions) [24]:

- *Bunch-Davies vacuum* [40]
  Both in- and outgoing modes are in vacuum with respect to τ. This is the analogue of the Hartle-Hawking vacuum. In this state the static observer observes thermal equilibrium.

- *Static vacuum*
  Both in- and outgoing modes are in vacuum with respect to t. This is the analogue of the Boulware vacuum. Now the static observer does not measure any radiation. The energy-momentum tensor is singular on the future and past horizon.

- *Unruh-de Sitter vacuum* [24]
  The ingoing modes are in vacuum with respect to τ, and the outgoing with respect to t. The static observer measures an incoming thermal flux. Note that "in" and "out" are switched compared to the black hole case, as now the observer is surrounded by their horizon. The energy-momentum tensor is singular at the past horizon. However, in planar coordinates this singularity does not show up and this vacuum is an acceptable state in the planar patch (2.41).

- *Unruh-de Sitter' vacuum*
  This state is the reverse of the Unruh-de Sitter vacuum with an outgoing thermal flux. For the remainder this is not a relevant case.

There is no consensus yet on how Gibbons-Hawking radiation backreacts in de Sitter space. Since there is not a unique vacuum, there are multiple possible fates for de Sitter to end up in. It could be stable against radiation (Bunch-Davies vacuum)[15][41], spread out into flat space (static vacuum) [42] or collapse into a singular geometry (Unruh-de Sitter vacuum)[16][24][43].

2.5 The Cosmological Information Paradox

The numerous similarities between the Schwarzschild black hole and de Sitter case entice us to think about a possible "cosmological information paradox". When we consider the Unruh-de Sitter vacuum, a static observer detects an incoming thermal flux. This situation is most similar to the evaporating black hole, but there are many confusing differences [44]. To start, there is no natural analog of the central dogma. The subject is now the whole spacetime except for the static patch, which is also observer-dependent. If we do proceed to view the unobservable universe (or more safely, the cosmological horizon as seen from the inside) as one quantum system, we can place a cut-off surface between the observer and their cosmological horizon. The Hilbert space is then split into two parts $\mathcal{H} = \mathcal{H}_{dS} \times \mathcal{H}_{rad}$. The radiation is collected by the observer in the center. To use the island formula description, however, the radiation needs to be collected in a region where gravity can be neglected, which is not plausible in the center of the static patch.

As the de Sitter entropy is $S_{dS} = \mathcal{A}/4G$, we expect (or hope) the number of degrees of freedom of the horizon is $\mathcal{A}/4G$. Although it has been shown that the de Sitter Hilbert space is finite [45], there is no complete quantum formulation that gives the de Sitter entropy as coming from a Hilbert space of states. It is also not clear what the entropy would describe exactly [46]. Some possibilities are that it is the quantum entanglement of degrees of freedom in- and outside the horizon, or that it counts the number of microscopic states that macroscopically give rise to de Sitter space. Finally, unlike the black hole case it is unclear how you can create de Sitter space from a known initial state. This way we do not have a set initial entropy, like $S_{BH} = 0$ when creating the black hole in a total pure state.

The evolution of de Sitter entropy has been studied in many different set-ups. Sticking to the Unruh-de Sitter vacuum, the conclusion of [24], [43] (pure de Sitter) and [16] (Schwarzschild-de Sitter) is that the horizon shrinks/a singularity forms and that there is no Page curve for the entropy. In [15] and [47] the Bunch-Davies vacuum is considered and a Page curve is found that takes on a constant value after some time. This is in line with the prediction that de Sitter is stable when no radiation is collected. The entropy is determined using islands in de Sitter spacetime.

Chapter 3

Gravitational Entanglement Entropy

In section 2.3 it was mentioned that the island formula gives a correct description of the Page curve of the von Neumann entropy of outgoing Hawking radiation. This formula is a product of a number of alterations, that we will go through in chronological order. It has its origin in holography, and more specifically in the AdS/CFT correspondence. The result is a way to calculate entanglement entropies in strictly gravitational theories, without the use of a dual quantum field theory. Before that, we will introduce the Euclidean path integral formalism of quantum field theory to determine the von Neumann entropy of gravitational systems.

3.1 Entanglement Entropy in QFT

A continuum quantum field theory (QFT) is most conveniently expressed in the (Euclidean) path integral formalism. Using the Rényi-n entropy and the replica trick (2.20), the entanglement entropy is then given by the Euclidean partition function on a certain manifold [21].

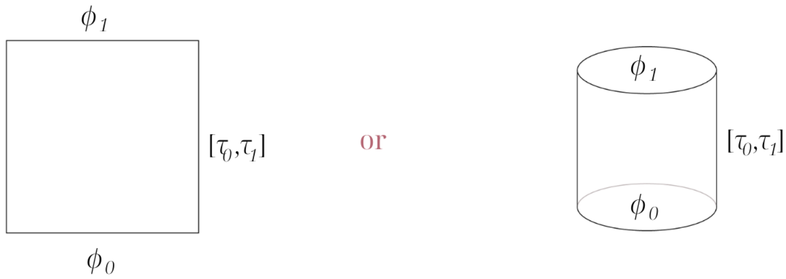
In a QFT the relevant variable is a field $\hat{\phi(x)}$ living on a certain spacetime parametrized by x. Recall that we can rewrite a transition amplitude from one state to another as a path integral by splitting the time interval into infinitely many pieces and inserting a complete eigenbasis of states between every interval:

$$\langle \phi_1(t_1)|\phi_0(t_0)\rangle = \langle \phi_1|\, e^{-i\mathcal{H}(t_1-t_0)}\, |\phi_0\rangle = \int_{\phi(t_0)=\phi_0}^{\phi(t_1)=\phi_1} \mathcal{D}\phi(t)\, e^{iI[\phi(t)]}\,, \tag{3.1}$$

where $|\phi_i\rangle$ is an eigenstate of the field operator $\hat{\phi}$, $\mathcal{H}$ the Hamiltonian or time evolution operator and I its corresponding action [18]. Switching to Euclidean time $\tau = it$, we get the following Euclidean path integral:

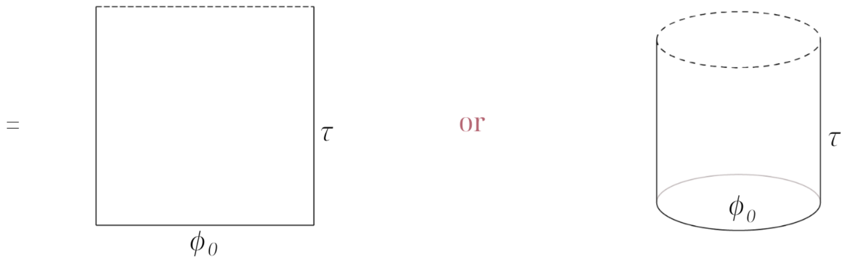
$$\langle \phi_1(\tau_1)|\phi_0(\tau_0)\rangle = \langle \phi_1|\, e^{-\mathcal{H}(\tau_1-\tau_0)}\, |\phi_0\rangle = \int_{\phi(\tau_0)=\phi_0}^{\phi(\tau_1)=\phi_1} \mathcal{D}\phi(\tau)\, e^{-I_E[\phi(\tau)]}\,, \tag{3.2}$$

where I_E now is the Euclidean action. Depending on whether the topology of the space the theory is defined on is a plane or a sphere, this path integral can be seen as integrating over a strip $[\tau_0, \tau_1] \times R^{d-1}$ or cylinder $[\tau_0, \tau_1] \times S^{d-1}$ with the appropriate boundary conditions:



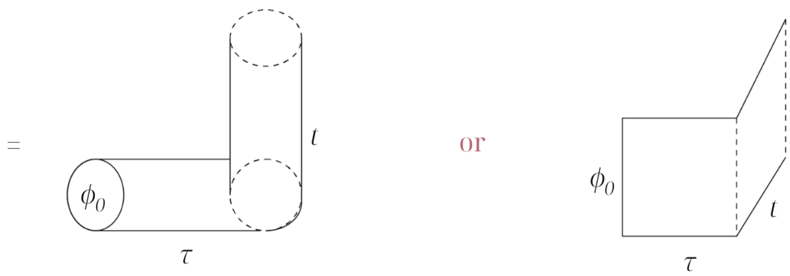
For a transition amplitude we specify boundary conditions on both sides of the path integral, a state vector can then be seen as an integral with one open boundary, or open "cut" in the Euclidean manifold:

$$|\Psi\rangle = e^{-\mathcal{H}\tau}\,|\phi_0\rangle = \int_{\phi(0)=\phi_0}^{\phi(\tau)=?} \mathcal{D}\phi(\tau')\, e^{-I_E[\phi(\tau')]} \tag{3.3}$$



Generally, an open cut Σ in a Euclidean manifold always defines a quantum state on Σ. The state $|\Psi\rangle$ above is defined by a Euclidean path integral, but it is still a state in the Hilbert space of the Lorentzian theory. The state is constructed at a fixed Lorentzian time, and can be evolved therein by the Hamiltonian:

$$|\Psi(t)\rangle = e^{-i\mathcal{H}t}\,|\Psi\rangle \tag{3.4}$$



The ground state can be conveniently constructed by a Euclidean path integral. First consider a generic state decomposed into its energy eigenstates:

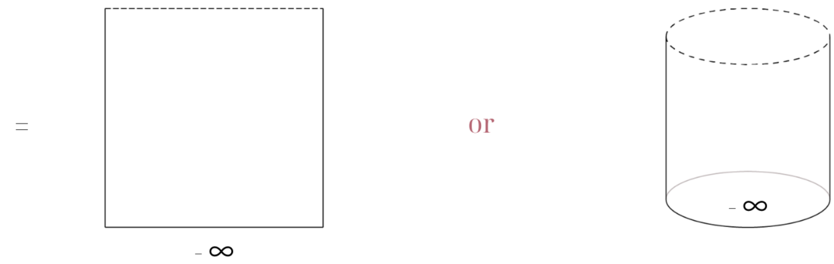
$$|\phi\rangle = \sum_n p_n\,|n\rangle\ , \qquad\qquad \mathcal{H}\,|n\rangle = E_n\,|n\rangle\ . \tag{3.5}$$

Then we see that evolving the state over an infinite Euclidean time selects the ground state:

$$e^{-\mathcal{H}\tau} |\phi\rangle = \sum_n e^{-\mathcal{H}\tau} p_n |n\rangle \overset{\tau\to\infty}{\approx} p_0 e^{-E_0\tau} |0\rangle \ . \tag{3.6}$$

Since E_0 is the smallest energy eigenvalue, this term dominates in the limit. We can conclude that we can construct the ground state in a similar way:

$$|0\rangle = \int_{\tau=-\infty}^{\phi(0)=?} \mathcal{D}\phi(\tau)\, e^{-I_E[\phi(\tau)]} \tag{3.7}$$



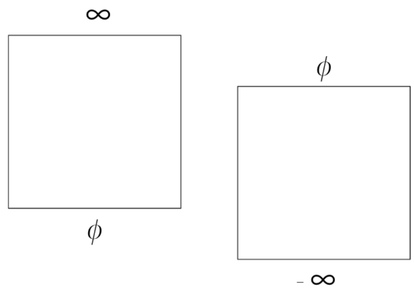
In the same analogy, the density matrix ρ of a state is defined by a path integral with two open cuts:

All figures can be seen as templates for calculating various complex numbers: specifying the open boundary condition of a state $|\Psi\rangle$ gives the wave function $\Psi[\phi] = \langle\phi|\Psi\rangle$, and specifying the boundary conditions of a density matrix ρ specifies its matrix elements $\langle\phi_2|\,\rho\,|\phi_1\rangle$.

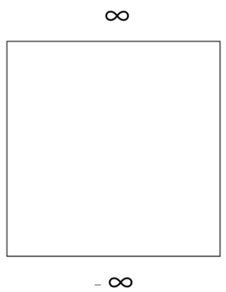
All states and density matrices defined so far were not normalized. The normalization factor of the vacuum density matrix ρ_0 is the vacuum-to-vacuum transition amplitude or vacuum partition function:

$$
\begin{aligned}
\mathcal{Z}_0 = \mathrm{tr}[\rho_0] &= \mathrm{tr}[|0\rangle\langle 0|] = \langle 0|0\rangle \\
&= \sum_\phi \langle 0|\phi\rangle \langle \phi|0\rangle
\end{aligned}
\tag{3.8}
$$

$= \sum_\phi$



$=$



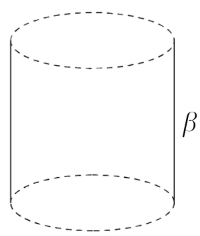
$$
= \int_{\tau=-\infty,\, \phi(\tau=0^-)=\phi(\tau=0^+)}^{\tau=\infty} \mathcal{D}\phi(\tau)\, e^{-I_E[\phi(\tau)]}\,,
$$

where we used the cyclicity of the trace. The + and - superscripts indicate very small deviations from $\tau = 0$, respectively in the positive and negative time direction. We see taking the trace is equivalent to gluing the open edges of the state $|0\rangle$ and its conjugate. The density matrix $\rho_{th} = e^{-\beta\mathcal{H}}$, familiar from statistical thermodynamics for describing an ensemble at temperature $T = 1/\beta$, can be written as a Euclidean path integral for $\tau = \beta$:

$$
\begin{aligned}
\rho_{th} &= e^{-\beta\mathcal{H}} \\
&= \int_{\tau=0}^{\tau=\beta} \mathcal{D}\phi(\tau)\, e^{-I_E[\phi(\tau)]}
\end{aligned}
\tag{3.9}
$$

$=$

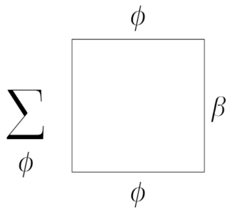
or



And its corresponding normalization factor or thermal partition function is

$$\begin{aligned}\mathcal{Z}_{th} &= \mathrm{tr}[e^{-\beta\mathcal{H}}] \\ &= \sum_{\phi} \langle\phi| \, e^{-\beta\mathcal{H}} \, |\phi\rangle\end{aligned} \tag{3.10}$$

=



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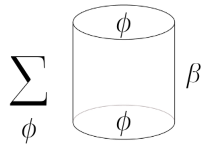

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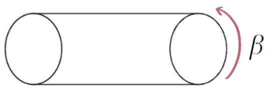

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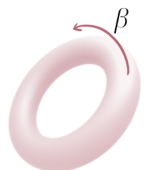


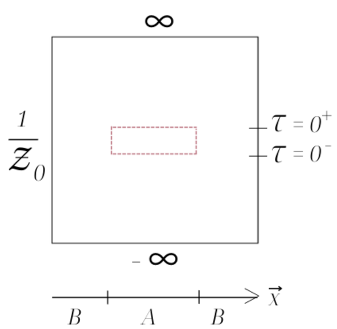
$$= \int_{\phi(\tau=0)=\phi(\tau=\beta)} \mathcal{D}\phi(\tau)\, e^{-I_E[\phi(\tau)]}\,,$$

We know all relevant thermodynamic variables can be distilled from the partition function, so it is a complete description of a thermal system at temperature $T = 1/\beta$ on a $d-1$-dimensional space (R^{d-1} or S^{d-1}). On the other hand, we see it describes a Euclidean quantum field theory in a d-dimensional spacetime with periodic imaginary time ($R^{d-1} \times S^1$ or $S^{d-1} \times S^1$). This correspondence gives us a quick way to derive the temperature of spacetimes, as we will do explicitly in section 4.1.1.

To talk about entanglement entropy, we need to introduce an expression for reduced density matrices in this setting. Let's say the Hilbert space of our theory is bipartite $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Taking the partial trace over system B, or equivalently integrating over all quantum field configurations with support only in subsystem B, gives us the reduced density matrix on system A. For the vacuum and thermal state (now with the appropriate normalization) defined above this looks like [48]:

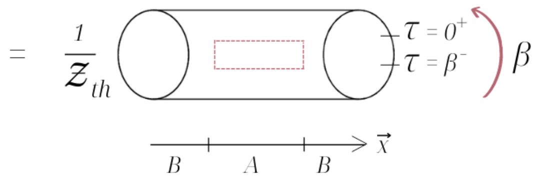
$$\begin{aligned}\rho_{0,A} &= \frac{1}{\mathcal{Z}_0}\,\mathrm{tr}_B[|0\rangle\langle 0|] \\ &= \frac{1}{\mathcal{Z}_0}\int \mathcal{D}\phi^B(0,\vec{x}\in B)\,\langle\phi^B|0\rangle\,\langle 0|\phi^B\rangle\end{aligned} \tag{3.11}$$

=



and

$$\begin{aligned}\rho_{th,A} &= \frac{1}{\mathcal{Z}_{th}} \operatorname{tr}_B[\rho_{th}] \\ &= \frac{1}{\mathcal{Z}_{th}} \int \mathcal{D}\phi^B(0, \vec{x} \in B) \left\langle \phi^B \right| \rho_{th} \left| \phi^B \right\rangle\end{aligned} \tag{3.12}$$



Like the total trace, the partial trace over subsystem B acts as gluing the path integrals along $\tau = 0$ for all $\vec{x} \in B$, such that a slit in the A subsystem remains.

For the remainder we will focus on the reduced density matrix in the vacuum state. Specific matrix elements can be specified by boundary conditions on subsystem A of ϕ_i^A at $\tau = 0^-$ and ϕ_j^A at $\tau = 0^+$:

$$\begin{aligned}[\rho_{0,A}]_{ij} &= \left\langle \phi_i^A \right| \rho_{0,A} \left| \phi_j^A \right\rangle \\ &= \frac{1}{\mathcal{Z}_0} \int \mathcal{D}\phi^B(0, \vec{x} \in B) \left(\left\langle \phi_i^A \right| \otimes \left\langle \phi^B \right| \right) |0\rangle\langle 0|_A \left(\left| \phi_j^A \right\rangle \otimes \left| \phi^B \right\rangle \right) \\ &= \frac{1}{\mathcal{Z}_0} \int_{t=-\infty}^{\tau=\infty} \mathcal{D}\phi(\tau, \vec{x})\, e^{-I_E[\phi]} \prod_{\vec{x} \in A} \delta(\phi(0^-, \vec{x}) - \phi_i^A(\vec{x}))\, \delta(\phi(0^+, \vec{x}) - \phi_j^A(\vec{x}))\,.\end{aligned} \tag{3.13}$$

For the n-th power of the reduced density matrix we then need to consider n copies (or replicas) of the original manifold $\mathcal{M}$ and glue the edges of the n slits systematically together, as we can see from the matrix representation:

$$\begin{aligned}[\rho_{0,A}^n]_{ij} &= \underbrace{[\rho_{0,A}]_{ik}[\rho_{0,A}]_{kl} \dots [\rho_{0,A}]_{mj}}_{n} \\ &= \frac{1}{\mathcal{Z}_0^n} \int \prod_{\beta=1}^{n-1} d\phi_j^{A(\beta)}(\vec{x})\, \delta(\phi_j^{A(\beta)}(\vec{x}) - \phi_i^{A(\beta+1)}(\vec{x})) \int_{t=-\infty}^{\tau=\infty} \prod_{\alpha=1}^{n} \mathcal{D}\phi^{(\alpha)}(\tau, \vec{x})\, e^{-\sum_{\alpha=1}^n I_E[\phi^{(\alpha)}]} \\ &\quad \prod_{\vec{x} \in A} \delta(\phi^{(\alpha)}(0^-, \vec{x}) - \phi_i^{A(\alpha)}(\vec{x}))\, \delta(\phi^{(\alpha)}(0^+, \vec{x}) - \phi_j^{A(\alpha)}(\vec{x}))\,.\end{aligned} \tag{3.14}$$

The first product of integrals acts such that we have repeating indices for the matrix multiplication. For the trace we also need to identify the first and last edges of the slits: $\phi_j^{A(n)}(\vec{x}) = \phi_i^{A(1)}(\vec{x})$. We can do so by including $\beta = n$ in the first product and stating that $\phi_i^{A(n+1)}(\vec{x}) = \phi_i^{A(1)}(\vec{x})$. Considering the big picture, we see that gluing the n sheets together creates something we call the n-fold cover $\mathcal{M}_n$ of the original spacetime (see figure 3.1).

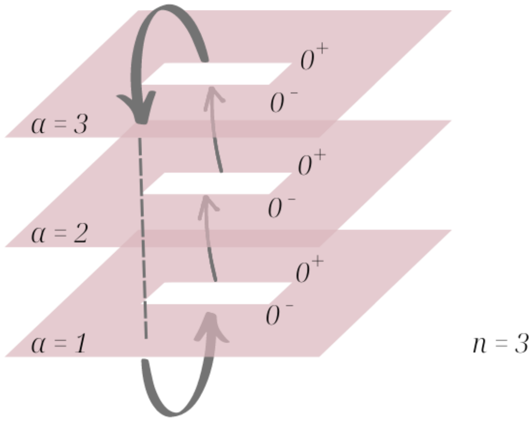


Figure 3.1: An n-fold cover is created by gluing n replicas of some Euclidean manifold together along their slits.

Similar to the partition function of the original manifold being a path integral over the complete space, we can now define the partition function of the n-fold cover $\mathcal{Z}_{0,n}$ as being proportional to the trace of the n-th power of $\rho_{0,A}$:

$$\mathrm{tr}[\rho_{0,A}^n] = \frac{\mathcal{Z}_{0,n}}{\mathcal{Z}_{0,1}^n}\,, \tag{3.15}$$

where we renamed the partition function of the original space $\mathcal{Z}_0$ to $\mathcal{Z}_{0,1}$. Again using the Rényi-n entropy, l'Hôpital's rule and this last equation, we can rewrite the entanglement entropy of our state of interest in terms of partition functions:

$$\begin{aligned} S_{vN}(\rho_{0,A}) &= \lim_{n\to1} S^{(n)}(\rho_{0,A}) \\ &= -\lim_{n\to1} \partial_n \ln\left(\mathrm{tr}\left[\rho_{0,A}^n\right]\right) \\ &= -\lim_{n\to1} \partial_n \left(\ln \mathcal{Z}_{0,n} - n \ln \mathcal{Z}_{0,1}\right)\,. \end{aligned} \tag{3.16}$$

The benefit of this expression is that we can include the effects of gravity directly alongside the contribution of the quantum fields in the expression for the partition function:

$$\mathcal{Z}_{0,n} = \int_{\mathcal{M}_n} \mathcal{D}\phi \mathcal{D}g_n\, e^{-I_{E,QFT}[\phi]-I_{E,HE}[g_n]}\,, \tag{3.17}$$

where $I_{E,QFT}$ is the Euclidean action of the quantum fields ϕ, and $I_{E,HE}$ is the Euclidean Hilbert-Einstein action of the metric g_n.

To end this section we should make some remarks. We assumed we could analytically continue the Rényi-n entropies to non-integer values of n. In the next chapter we will go into more depth on the replica trick in gravitating systems, and we will see we can make this continuation with the use of "cosmic strings" (see section 4.1.2). Also, when we consider a theory with non-trivial time dependence the method introduced here needs to be adapted, since we do not have knowledge of the complete spacetime at every time. The Schwinger-Keldysh formalism solves this by only making use of the causal past of the Cauchy slice under consideration [49]–[51].

Lastly, when splitting the Hilbert space of a continuum QFT, UV (high energy) divergences arise [18][51]. There are degrees of freedom at arbitrarily small scales so it is actually impossible to put in a dividing surface. We can deal with this by implementing a UV cut-off parameter ϵ. The UV behaviour of the gravitational entanglement entropy is

$$S_{vN}(\rho_A) = \begin{cases} a_{d-2}\left(\frac{\ell}{\epsilon}\right)^{d-2} + a_{d-4}\left(\frac{\ell}{\epsilon}\right)^{d-4} + \ldots + a_1\frac{\ell}{\epsilon} + (-1)^{(d-1)/2}\,\tilde{S}_A + \mathcal{O}(\epsilon)\,, & d\,\text{odd}\,, \\ a_{d-2}\left(\frac{\ell}{\epsilon}\right)^{d-2} + a_{d-4}\left(\frac{\ell}{\epsilon}\right)^{d-4} + \ldots + (-1)^{(d-1)/2}\,\tilde{S}_A\,\ln\left(\frac{\ell}{\epsilon}\right) + \mathcal{O}(\epsilon)\,, & d\,\text{even}\,. \end{cases} \tag{3.18}$$

Here ℓ is the size of region A, a_i are real coefficients and $\tilde{S}_A$ a term depending on the theory, the shape and the total state. The leading UV divergence is always proportional to Area(A).

3.2 Holography

Inspired by the Bekenstein bound (2.35) 't Hooft[52] and Susskind[53] developed the holographic principle. It states that the number of degrees of freedom to describe a system must not exceed a quarter of its boundary area. While you would expect the number of degrees of freedom scales with volume, the bound has been shown to hold in a wide range of situations [54]. A holographic theory is, then, one in which the entropy bound is manifest, and currently the best developed one is the AdS/CFT (Anti-de Sitter/conformal field theory) correspondence [55]. This is a specific type of gravity/QFT duality in which quantum gravity in a $(d+1)$-dimensional spacetime (the bulk theory) describes the same system as a d-dimensional quantum field theory without gravity (the boundary theory). They have the same symmetry group and partition function $\mathcal{Z}_{AdS_{d+1}} = \mathcal{Z}_{CFT_d}$, and fields in the bulk are represented by operators in the QFT. Since AdS has a conformal boundary at spatial infinity, it is often said that the dual CFT lives there.

Holography is very useful since the framework of QFTs is much better understood and developed than quantum gravity. Many black hole cases have already been described this way, like the eternal black hole [56]. Unfortunately there is not yet a UV complete theory for quantum gravity in de Sitter spacetime, like certain CFTs are for AdS. De Sitter space is closed, so unfortunately does not have a spatial conformal boundary to define a dual CFT on. It does have conformal boundaries at future and past null infinity, but we do not expect two boundary descriptions. Also, an observer never has access to both, so this description includes more than is physically measurable. Switching to a different viewpoint like the planar or static patch leaves a part of spacetime to be unobservable, but we can still define a dual CFT on past null infinity ([57] compares these two possibilities). Other constructions like the so-called dS/dS correspondence[58] are also being studied.

A conformal field theory[59] is a quantum field theory in which lengths scales are not physically relevant and only angles are. The theory is invariant under conformal transformations, which are coordinate changes $x_\alpha \to \tilde{x}_\alpha(x)$ under which the metric changes as

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x)\,. \tag{3.19}$$

In the remaining chapters we will primarily be interested in using two-dimensional CFTs on Euclidean spacetimes. It is then useful to work in complex coordinates

$$z = x_1 + ix_2 , \qquad \bar{z} = x_1 - ix_2 . \tag{3.20}$$

In two dimensions the possible conformal transformations are translations, complex dilatations and special conformal transformations.

The energy-momentum tensor is the conserved current under translations; $\nabla_\mu T^{\mu\nu} = 0$. In flat spacetime it is traceless:

$$T^\mu{}_\mu = 0 \qquad \text{or} \qquad T_{z\bar{z}} = 0 , \tag{3.21}$$

so the off-diagonal elements vanish. Under conformal transformations it transforms as

$$\tilde{T}_{\tilde{z}\tilde{z}}(\tilde{z}) = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \left[T_{zz}(z) - \frac{c}{12} S(\tilde{z}, z)\right] , \tag{3.22}$$

where $S(\tilde{z}, z)$ is the Schwarzian, defined as

$$S(\tilde{z}, z) = \left(\frac{\partial^3 \tilde{z}}{\partial z^3}\right)\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 \tilde{z}}{\partial z^2}\right)^2 \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} . \tag{3.23}$$

Here, c is the central charge, one of the most important characteristics of the theory and an indication of the number of degrees of freedom. It is also present in the TT operator product expansion (OPE):

$$T_{zz}(z)T_{zz}(w) = \frac{c/2}{(z-w)^2} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots . \tag{3.24}$$

And similarly for $T_{\bar{z}\bar{z}}(\bar{z})T_{\bar{w}\bar{w}}(\bar{w})$ with central charge $\tilde{c}$. The $+\ldots$ indicates additional non-singular terms. The OPE is always assumed to be in time-ordered correlation functions:

$$\langle T_{zz}(z)T_{zz}(w)\rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi e^{-S[\phi]} T_{zz}(z)T_{zz}(w) . \tag{3.25}$$

In curved spacetime the energy-momentum tensor is not traceless and is related to the central charge. This is called the Weyl or trace anomaly:

$$\langle T^\mu{}_\mu \rangle = -\frac{c}{12} R . \tag{3.26}$$

In two-dimensional CFTs there are some standard expressions for entanglement entropy [60][61]. The entanglement entropy of a single interval of length ℓ in a infinitely long system at zero temperature is

$$S_{inf,T=0} = \frac{c}{3} \ln\left(\frac{\ell}{\epsilon}\right) , \tag{3.27}$$

where ϵ is a UV cut-off parameter. In a system of finite length L with periodic boundary conditions, it is

$$S_{periodic\, fin,T=0} = \frac{c}{3} \ln\left(\frac{L}{\pi\epsilon} \sin\left(\frac{\pi\ell}{L}\right)\right) . \tag{3.28}$$

The entropy of the interval in a thermal system at finite temperature β^{-1} is

$$S_{inf,T=1/\beta} = \frac{c}{3}\ln\left(\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi\ell}{\beta}\right)\right). \tag{3.29}$$

For $\ell \ll \beta$ we recover the first result. These expressions are all related by conformal transformations.
When the system is the semi-infinite line $[0,\infty)$ and, thus, has one conformal boundary at $x=0$, the entropy is

$$S_{fin,T=0} = \frac{c}{6}\ln\left(\frac{2\ell}{\epsilon}\right). \tag{3.30}$$

This can also be transformed into other cases. Generally, the entropy of an interval $[z_1, z_2]$ with metric $ds^2 = \Omega^{-2}dzd\bar{z}$ is

$$S_{[z_1,z_2]} = \frac{c}{6}\ln\left(\frac{1}{\epsilon_1\epsilon_2}\frac{|z_1-z_2|^2}{\Omega(z_1,\bar{z}_1)\Omega(z_2,\bar{z}_2)}\right), \tag{3.31}$$

with ϵ_i the UV cut-off at its corresponding boundary point [12].

In the AdS_3/CFT_2 correspondence the Ricci scalar of the gravity theory is related to the central charge of the CFT by

$$c = \frac{3R}{2G_{3d}}, \tag{3.32}$$

where $G^{(3)}$ is gravitational constant in AdS_3 [62].

3.3 From Holographic to Gravitational Entropy

Now we will go through the steps taken to arrive at the island formula introduced in (2.37). In 2006 Ryu and Takayanagi (RT)[6] proposed that the entanglement entropy of a region A in a d-dimensional CFT can be calculated by considering a minimal area surface in the dual $(d+1)$-dimensional AdS spacetime:

$$S_{vN}(A) = \frac{\text{Area}(\gamma_A)}{4G^{(d+1)}}. \tag{3.33}$$

The CFT is defined on $R^{1,d-1}$ or $R\times S^{d-1}$, and A is a subregion of this spacetime with boundary $\partial A \in R^{d-1}$ or S^{d-1}. γ_A is the $(d-1)$-dimensional static minimal area surface in AdS_{d+1} which has the same boundary as A: $\partial\gamma_A = \partial A$. The situation is sketched in figure 3.2. From this definition it immediately follows it obeys $S_{vN}(A) = S_{vN}(A^c)$, with A^c the complement of A, and subadditivity $S_{vN}(A) + S_{vN}(B) \geq S_{vN}(A\cup B)$. Lewkowycz and Maldacena gave some additional arguments for this proposal using the gravitational replica trick method introduced in 3.1[7].

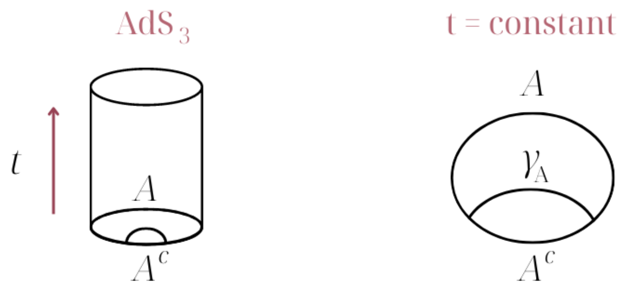


Figure 3.2: A CFT is defined on $R \times S^{d-1}$ (a cylinder), which is dual to an d-dimensional Anti-de Sitter spacetime in the inside. We take A to be a subregion on this AdS boundary and A^c is its complement. γ_A is the minimal area surface which shares its boundaries with those of A and lives in the AdS bulk. On the right the same situation is sketched at a constant time slice.

Ultimately we want to describe the evolution of the entropy of an evaporating black hole or de Sitter space with time. For this the RT proposal needs to be adapted to time dependent geometries. This is what Hubeny, Rangamani and Takayanagi (HRT)[63] did in 2012. The construction is based on light-sheets. A codimension two surface is specified by two constraints

$$\phi_1(x^\mu) = 0 \qquad\qquad \phi_2(x^\mu) = 0\,. \tag{3.34}$$

From this we can construct two null one-forms $\nabla_\mu\phi_1 + \alpha_\pm\nabla_\mu\phi_2$, for two distinct values of $\alpha_\pm$. Then

$$N^\mu_\pm = g^{\mu\nu}(\nabla_\mu\phi_1 + \alpha_\pm\nabla_\mu\phi_2) \tag{3.35}$$

are two null vectors orthogonal to the surface. We can normalize them such that $g_{\mu\nu}N^\mu_+N^\nu_- = -1$ and compute the null extrinsic curvature

$$(\chi_\pm)_{\mu\nu} = h^\alpha{}_\mu h^\beta{}_\nu \nabla_\alpha (N_\pm)_\beta\,, \tag{3.36}$$

where $h_{\mu\nu}$ is the induced metric on the surface. Finally we take the trace to obtain the expansion of the orthogonal null geodesic congruence:

$$\theta_\pm = (\chi_\pm)^\mu{}_\mu\,. \tag{3.37}$$

This is a measure of how well null geodesics remain parallel when moving orthogonal to the surface. The congruence then ends wherever two geodesics cross. It also gives the rate of change of the area of the surface when moving it along the null vectors.

Using this, HRT proposed the entanglement entropy of a region A is given by

$$S_{vN}(A) = \frac{\text{Area}(\mathcal{Y}_A)}{4G^{(d+1)}}\,, \tag{3.38}$$

where $\mathcal{Y}_A$ is a codimension two surface in a $(d+1)$-dimensional spacetime $\mathcal{M}$ with vanishing null geodesic expansions $\theta_\pm = 0$ and $\partial\mathcal{Y}_A = \partial A$. A is a subregion of the boundary of the spacetime $A \subset \partial\mathcal{M}$. Additionally $\mathcal{Y}$ and A are homotopically equivalent, meaning they can be continuously deformed into each other. See figure 3.3 for a sketch of the situation.

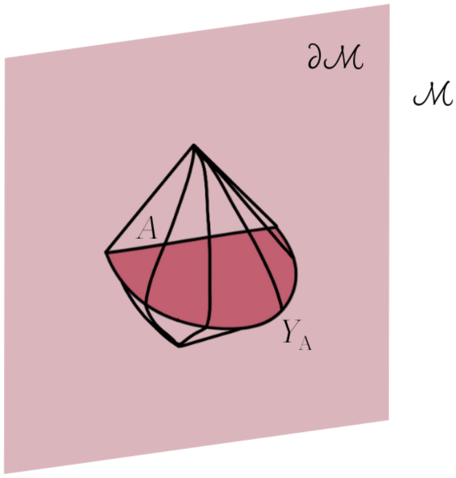


Figure 3.3: On the boundary $\partial\mathcal{M}$ of a bulk theory $\mathcal{M}$ a subregion A is defined. The surface $\mathcal{Y}_A$, living in the bulk, shares its boundaries with those of A and has vanishing null geodesic expansions.

When $\mathcal{Y}$ is not unique, choose the one with minimal area. $\mathcal{Y}$ is called an extremal surface. The condition of vanishing null geodesic expansions can also be rephrased as

$$\frac{\delta \text{Area}(\mathcal{Y})}{\delta \mathcal{Y}^\mu} N^\mu_\pm = 0\,, \tag{3.39}$$

where $\delta\mathcal{Y}^\mu$ is an infinitesimal variation normal to $\mathcal{Y}$ (see figure 3.4). $\mathcal{Y}$ is then said to be a classical marginally trapped surface.

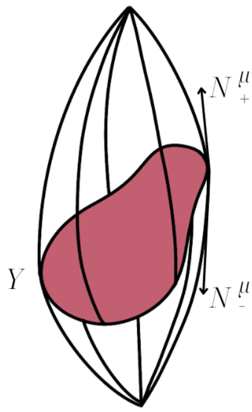


Figure 3.4: The surface $\mathcal{Y}$ is called a classical marginally trapped surface since the variation of its area in the direction of its normal vectors $N^\mu_\pm$ vanishes.

The RT proposal is also purely classical, so to account for the quantum effect of Hawking radiation Faulkner, Lewkowycz and Maldacena (FLM)[64] added a first order quantum correction in 2013. As the minimal surface γ_A divides the bulk into two subsystems, the quantum correction is given by the bulk entanglement entropy of these subsystems. More precisely, denote the bulk region enclosed by A and γ_A A_b and its complement A_b^c (see figure 3.5). The entanglement entropy of A is then

$$\begin{aligned} S_{vN}(A) &= S_{cl}(A) + S_q(A) + \mathcal{O}(G_N) \\ &= \frac{\text{Area}(\gamma_A)}{4G^{(d+1)}} + S_{bulk-ent}(A_b) + \dots \,, \end{aligned} \tag{3.40}$$

where $+\dots$ represents additional quantum corrections. One of them is a correction to the area term since the classical background changes due to the quantum effects. They also include terms that cancel the UV divergencies of $S_{bulk-ent}$, rendering S_q a finite quantity. This first order correction was derived performing the replica trick while including the partition function of the bulk quantum fields on a static spacetime. The total state was assumed to be pure such that $S_{bulk-ent}(A_b) = S_{out}(A_b)$, with S_{out} the von Neumann entropy of matter fields living on A_b. To ensure that A_b is a spacelike region, γ_A has one additional condition. It has to be homologous to A, i.e. the union of A and γ_A is the boundary of a d-dimensional spacelike surface in the bulk.

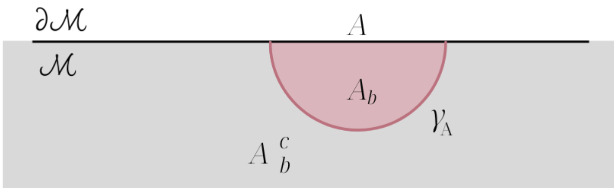


Figure 3.5: A subregion A on the boundary $\partial\mathcal{M}$ of a bulk theory $\mathcal{M}$ is chosen. γ_A is the minimal surface in the bulk homologous to A. It divides the bulk theory into two subregions A_b and A_b^c.

In 2015 Engelhardt and Wall combined the quantum effects and covariant construction into the quantum extremal surface prescription (QES)[8]. The idea is to find a extremal surface that extremizes the total generalized entropy, instead of only the classical area term like in the FLM case. This is then called a quantum extremal surface. More precisely,

$$\begin{aligned} S_{vN}(A) &= S_{gen}(\mathcal{X}_A) \\ &= \frac{\text{Area}(\mathcal{X}_A)}{4G^{(d+1)}} + S_{out}(A_b) + \dots \,, \end{aligned} \tag{3.41}$$

where $\mathcal{X}_A$ is a quantum marginally trapped surface, i.e. it obeys the following condition

$$\frac{\delta S_{gen}(\mathcal{X})}{\delta \mathcal{X}^\mu} N^\mu_\pm = 0 \,. \tag{3.42}$$

Here $\delta\mathcal{X}^\mu$ is an infinitesimal variation normal to $\mathcal{X}$ and $N^\mu_\pm$ two distinct null normal vectors to $\mathcal{X}$. Again, A lives in a field theory on the boundary of a particular spacetime $\mathcal{M}$ that have a holographic duality. $\mathcal{X}_A$ is a codimension two surface in $\mathcal{M}$ which is anchored at A ($\partial\mathcal{X} = \partial A$) and is homologous to A. The region A_b is the part of $\mathcal{M}$ in between A and $\mathcal{X}$. When there are multiple extremal surfaces obeying these condition, the entropy is given by the one that minimizes it. The QES formulation can also be written as

$$S_{vN}(A) = \min_{\mathcal{X}_A} \left(\operatorname{ext}_{\mathcal{X}_A} \left(\frac{\text{Area}(\mathcal{X}_A)}{4G^{(d+1)}} + S_{out}(A_b) \right) \right) . \tag{3.43}$$

The QES and FLM proposals are equivalent up to and including the first order quantum corrections. When spacetime is static, there is a preferred time direction and the extremal surface is minimal on a constant time slice.

3.4 The Island Formula

The QES prescription has been successfully applied to the case of evaporating black holes [9], [65]–[69]. The set-up is usually two-dimensional gravity theory that is asymptotically AdS plus a two-dimensional matter CFT. Additionally, the theory is coupled to a heat bath, such that the black hole can evaporate. This is taken to be the same CFT_2 as describing the matter fields, but now on a fixed flat spacetime "glued" to the conformal boundary of the space. Alternatively, you can consider the boundary to be absorbing, or place a cut-off surface beyond which you neglect gravity as in section 2.3.

The approach used in [66] is particularly convenient for computing the evolution of the black hole entropy. It uses a matter CFT_2 living in the gravity theory that is assumed to have a three-dimensional holographic dual. The generalized entropy is then

$$\begin{aligned} S_{gen}(\mathcal{Y}_A) &= \frac{\text{Area}(\mathcal{Y}_A)}{4G^{(d+1)}} + S_{out}(A_b) \\ &\approx \frac{\text{Area}(\mathcal{Y}_A)}{4G^{(d+1)}} + \frac{\text{Area}(\Sigma_{\mathcal{Y}})}{4G^{(d+2)}} , \end{aligned} \tag{3.44}$$

where we made the approximation that to first order the von Neumann entropy of the matter fields living in the region enclosed by A and $\mathcal{Y}_A$ is given by the area of an extremal surface in the $(d+2)$-dimensional gravity dual. This method is called double-holography. From the $(d+2)$-dimensional point-of-view we have returned to the HRT method.

By applying the double-holography method to calculate the entropy of the radiation, the island formula (2.37) was formulated. We restate it here for convenience:

$$S_{vN}(A) = \min_{\Sigma_{island}} \left(\operatorname*{ext}_{\Sigma_{island}} \left(\frac{\text{Area}(\partial\Sigma_{island})}{4G^{(d+1)}} + S_{out}(A \cup \Sigma_{island}) \right) \right) . \tag{3.45}$$

To compute the entanglement entropy of a region A in a quantum field theory that is entangled with a system in a gravity theory we should use the island formula. Here A and Σ_{island} have the same dimensionality. The extremization is also over the number of islands. This method is considered as the correct way to determine the entropy of any system in or connected with a gravity theory, whether is has a holographic dual description or not. When there is a UV complete dual theory, the state of the system and its entropy can be determined exactly. Otherwise, the island formula gives the entropy in the effective gravity theory, which is a pretty impressive feat since the exact density matrix is unknown and can not be determined. It is also assumed to hold in higher dimensions than the 2D gravity considered in [66].

In the next chapter we will discuss the derivation of the island formula using the gravitational replica trick. First we will review the method used for the eternal black hole, and then apply it to de Sitter spacetime. Since there is no UV complete theory of quantum gravity in de Sitter space, a derivation of the island formula using the replica trick would put its use on a more solid footing.

Chapter 4

Replica Wormholes

In this chapter we will first review the derivation of the classical Bekenstein-Hawking entropy of the Schwarzschild black hole using the replica trick method of Lewkowycz and Maldacena [7], and apply the same method to pure de Sitter spacetime. Then, we will go over the idea of the "replica wormhole" in deriving the island formula, and discuss its application to de Sitter spacetime.

4.1 Bekenstein-Hawking entropy

In section 3.1 we saw we can calculate the von Neumann entropy of a state defined on a Euclidean manifold $\mathcal{M}$ by considering the partition function of its corresponding n-fold cover $\mathcal{M}_n$. To derive the Bekenstein-Hawking entropy ($S = \mathcal{A}/4G$), we only need to take into account the gravitational contribution to the partition function, and we can make the classical approximation to fix the metric in a saddle point $\tilde{g}_n$ [70]. Since $\tilde{g}_n$ is a solution to the classical equations of motion and thus minimizes the action, it is the leading order of the partition function:

$$\begin{aligned} \mathcal{Z}_{th,n} &= \int_{\mathcal{M}_n} \mathcal{D}\phi \mathcal{D}g_n \, e^{-I_{E,HE}[g_n]} \\ &\approx e^{-I_{E,HE}[\tilde{g}_n]} + \text{subleading terms} \,, \end{aligned} \tag{4.1}$$

or

$$\ln \mathcal{Z}_{th,n} \approx -I_{E,HE}[\tilde{g}_n] \,, \tag{4.2}$$

where $S_{E,HE}$ is the Euclidean Hilbert-Einstein action. In this approximation, the entanglement entropy becomes

$$S_{grav} = \lim_{n \to 1} \partial_n \left(I_{E,HE}[\tilde{g}_n] - n \, I_{E,HE}[\tilde{g}_1] \right) \,. \tag{4.3}$$

For now we denote the entropy just by S_{grav} as we make no reference to a quantum field theory or specific quantum state.

4.1.1 Euclidean Schwarzschild and de Sitter Geometry

To see what the n-fold covers of the Schwarzschild black hole and pure de Sitter space look like, let us take a closer look at their Euclidean geometry. This means we take the usual time coordinate and Wick rotate it: $t \to -i\tau$. Then we will see τ becomes periodic with a period β, which differs per geometry.

We start with the Schwarzschild black hole in four dimensions. Its metric is

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dt^2 + \frac{1}{1 - r_S/r}\, dr^2 + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)\,, \tag{4.4}$$

where $r_S = 2GM$ is the Schwarzschild radius, and the coordinate ranges are $-\infty \leq t \leq \infty$, $r_S \leq r \leq \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. In Euclidean time it is

$$ds^2 = \left(1 - \frac{r_S}{r}\right) d\tau^2 + \frac{1}{1 - r_S/r}\, dr^2 + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)\,, \tag{4.5}$$

where now $0 \leq \tau \leq \beta_{BH}$. To determine the period β_{BH}, we consider the near horizon geometry by expanding around r_S [20]:

$$\begin{aligned} ds^2 &\approx \left(1 - \frac{r_S}{\tilde{r} + r_S}\right) d\tau^2 + \frac{1}{1 - r_S/(\tilde{r} + r_S)}\, d\tilde{r}^2 \\ &\approx \frac{\tilde{r}}{r_S} d\tau^2 + \frac{r_S}{\tilde{r}} d\tilde{r}^2\,, \end{aligned} \tag{4.6}$$

where $\tilde{r} = r - r_S \ll 1$ and we ignored the angular directions. If we now redefine $\tilde{r} = z^2/(4r_S)$, we get

$$\begin{aligned} ds^2 &\approx dz^2 + \frac{z^2}{4r_S^2} d\tau^2 \\ &= dz^2 + z^2 d\psi^2\,, \end{aligned} \tag{4.7}$$

where we noted the similarity to the Euclidean plane metric and defined $\psi = \tau/2r_S$ to make this more apparent. To describe the whole plane, and avoid a conical singularity, we know the range of ψ should be $\psi \in [0, 2\pi]$. By its definition, the range of τ then should be $\tau \in [0, 4\pi r_S]$, so its period $\beta_{BH} = 4\pi r_S = 8\pi GM$. As we saw in section 3.1, the period of Euclidean time is related to the inverse temperature of a statistical theory, so the temperature of the Schwarzschild black hole is $T_{BH} = 1/8\pi GM$.

The topology described by the Lorentzian metric (4.4) is $R \times [r_S, \infty) \times S^2$. Going to the Euclidean metric (4.5), the time direction becomes a circle, so it describes $\mathcal{M} = S^1 \times [r_S, \infty) \times S^2$. Suppressing the second angular direction, this represents a solid torus as depicted in figure 4.1.

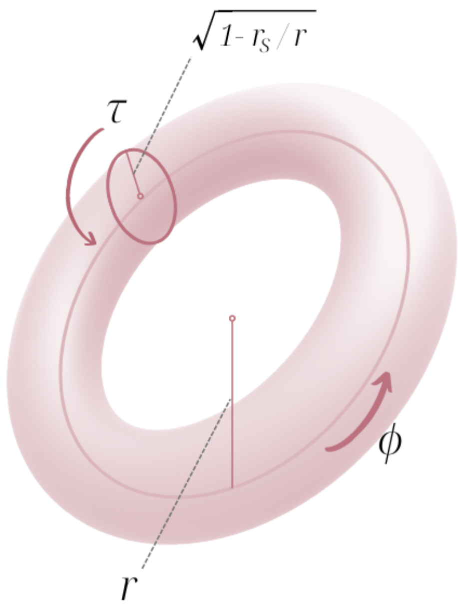


Figure 4.1: Geometry of the four-dimensional Euclidean Schwarzschild black hole (the θ-direction is suppressed). The non-contractible circle of the torus is described by ϕ and has radius r. The contractible circle is described by τ and has radius $\sqrt{1-r_S/r}$.

At the horizon $r = r_S$, the contractible τ-circle vanishes, leaving the ϕ-circle with radius r_S. If we now focus on the τ- and r-direction, we see the manifold takes on a "cigar shape" (see the left figure in 4.2). The tip of the cigar is at the horizon, and the other end stretches to asymptotic infinity. To make the replica manifold $\mathcal{M}_n$, we know from section 3.1 we need to make a cut along $\tau = 0$ and glue multiple of them systematically together. The period of τ then becomes $n\beta_{BH}$. This results in the n-fold cover as illustrated in the right figure of 4.2.

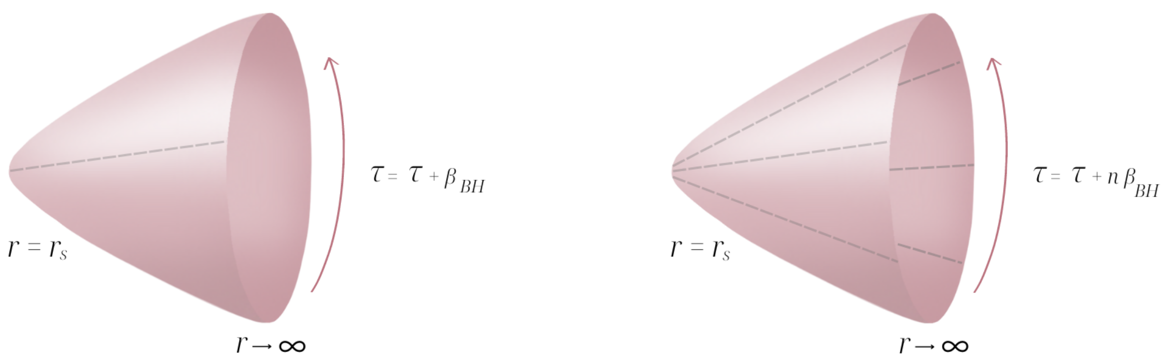


Figure 4.2: Left: The r- and τ-direction of the Euclidean Schwarzschild black hole geometry form a cigar shape. The tip of the cigar is at the event horizon $r = r_S$, while the end is at asymptotic infinity $r \to \infty$. At the end τ has period β_{BH}. Right: The r- and τ-direction of the replica manifold of the Euclidean Schwarzschild black hole. At asymptotic infinity τ has period $n\,\beta_{BH}$. In the figure $n = 6$.

Now for de Sitter spacetime we can follow the same steps. In the end we want to derive the entropy of the cosmological horizon as observed for a static observer, so we need to consider the metric in static coordinates:

$$ds^2 = -\left(1 - \frac{r^2}{L^2}\right) dt^2 + \frac{1}{1 - r^2/L^2} dr^2 + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)\,, \tag{4.8}$$

where $L = \sqrt{3/\Lambda}$ is the characteristic length scale of the space, and $\Lambda > 0$ the cosmological constant. The coordinate ranges are $-\infty \leq t \leq \infty$, $0 \leq r \leq L$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. Its Euclidean metric is

$$ds^2 = \left(1 - \frac{r^2}{L^2}\right) d\tau^2 + \frac{1}{1 - r^2/L^2} dr^2 + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)\,, \tag{4.9}$$

with $0 \leq \tau \leq \beta_{dS}$. Examining the metric near the horizon gives

$$\begin{aligned} ds^2 &\approx \left(1 - \frac{(L-\tilde{r})^2}{L^2}\right) d\tau^2 + \frac{1}{1-(L-\tilde{r})^2/L^2} d\tilde{r}^2 \\ &\approx \frac{2\tilde{r}}{L} d\tau^2 - \frac{L}{2\tilde{r}} d\tilde{r}^2 \\ &= dz^2 + \frac{z^2}{L^2} d\tau^2 \\ &= dz^2 + z^2 d\psi^2\,. \end{aligned} \tag{4.10}$$

We first expanded the radial coordinate close to the horizon: $\tilde{r} = L - r \ll 1$ (chosen such that $0 \leq \tilde{r} \leq L$), while ignoring the angular directions. Then we made the substitutions $\tilde{r} = -z^2/2L$ and $\psi = \tau/L$. Since $\psi \in [0, 2\pi]$, the range for τ is $\tau \in [0, 2\pi L]$ and its period is $\beta_{dS} = 2\pi L$. The temperature of the de Sitter horizon is then $T_{dS} = 1/\beta_{dS} = 1/2\pi L$.

The static patch de Sitter space (4.8) has topology $R \times B^3$, where B^d is the solid $(d-1)$-sphere, or d-ball [71]. After the Wick rotation the topology becomes $\mathcal{M} = S^1 \times B^3$, which describes a solid torus (figure 4.3).

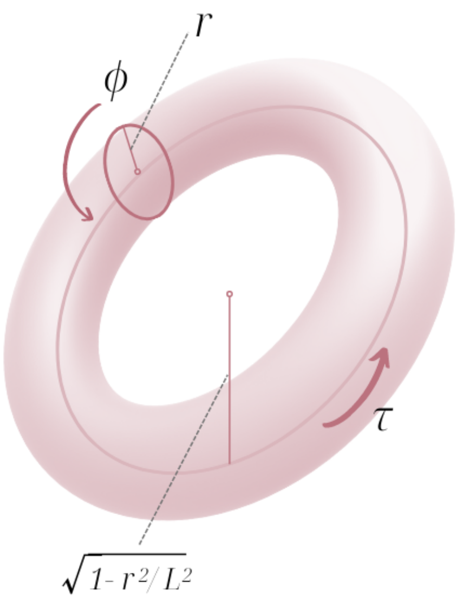


Figure 4.3: Geometry of four-dimensional de Sitter spacetime in static coordinates (the θ-direction is suppressed). The non-contractible circle of the torus is described by τ and has radius $\sqrt{1 - r^2/L^2}$. The contractible circle is described by ϕ and has radius r.

In this case, the non-contractable τ-circle collapses at the horizon $r = L$. The remaining geometry is the solid 2-ball B^2. Looking specifically at the (τ, r)-direction, we see we get a cut-off version of the black hole cigar shape (see the left figure in 4.4). Like in the black hole case, the tip of the cigar is at the horizon, and the other end is located where the observer is. This time this is at the center of the spacetime, and not at asymptotic infinity. We can again construct the replica manifold by making a cut along $\tau = 0$, and gluing multiple of them in between (see the right figure in 4.4).

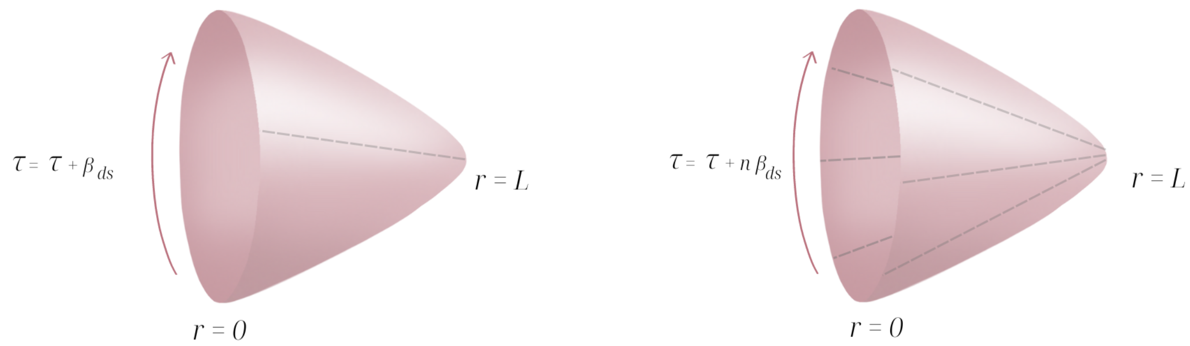


Figure 4.4: Left: The r- and τ-direction of the Euclidean de Sitter geometry form a cigar shape. The tip of the cigar is at the cosmological horizon $r = L$, while the end is at the location of the static observer $r = 0$. At this origin τ has period β_{dS}. Right: The r- and τ-direction of the replica manifold of Euclidean de Sitter spacetime. At the location of the static observer τ has period $n\,\beta_{BH}$. In the figure $n = 6$.

4.1.2 Classical Replica Method

Now that we know what the n-fold covers look like, we can apply (4.3) to calculate their entanglement entropy. In constructing the n-fold covers we made sure the geometry was smooth everywhere and $\tau = \tau + n\beta$. This means it has a replica symmetry Z_n coming from the cyclicity of the trace. This symmetry permutes the replicas and shifts τ over β. Using this symmetry, Lewkowycz and Maldacena argued for a way to analytically continue n to non-integer values needed to take the limit to $n = 1$ [7].

We use the symmetry to take the quotient space of $(\mathcal{M}_n, \tilde{g}_n)$ and construct the orbifold $\hat{\mathcal{M}}_n \equiv \mathcal{M}_n/Z_n$. This results in a geometry topologically equivalent to the original Euclidean space $\mathcal{M}_1$ but including singularities at the fixed points under Z_n (see figure 4.5). For both the Schwarzschild and de Sitter n-fold cover their respective horizon is a fixed point, resulting in a conical singularity of deficit angle:

$$\Delta\varphi = 2\pi\left(1 - \frac{1}{n}\right) . \tag{4.11}$$

Generally the fixed point can be taken to be a codimension two surface in spacetime [51].

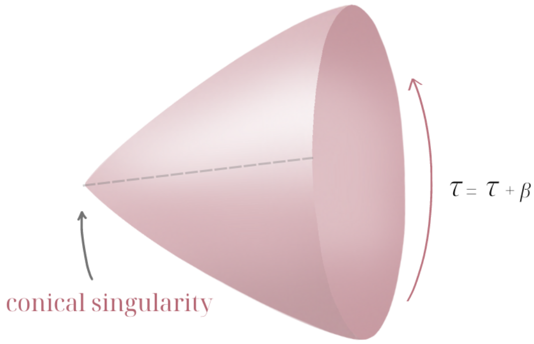


Figure 4.5: When taking the quotient space of the replica manifolds $\mathcal{M}_n$ of the Schwarzschild or de Sitter spacetime with respect to the replica symmetry Z_n a conical singularity arises at the fixed points under this symmetry. That is at the event horizon or cosmological horizon, respectively.

This singularity can be incorporated into the action by treating it as a source of energy-momentum backreacting on $\mathcal{M}_1$ to deform it to $\hat{\mathcal{M}}_n$ [1]. This is usually done by locating a cosmic string at the fixed points of Z_n. Such a cosmic string is a codimension-2 surface with action[2]

$$I_{cosmic\ string} = \frac{1}{4G}\left(1 - \frac{1}{n}\right)\int_{\Sigma_{d-2}} d^{d-2}y\,\sqrt{h}\,, \tag{4.12}$$

where h is the determinant of the induced metric $h_{\mu\nu}$ on the string, and $4GT_n = (1-1/n)$ is the string tension [21]. This allows us to analytically continue n to non-integer values, as this just tunes the strength of the tension.

The metric on the orbifold $\hat{g}_n$ is a solution to the combined action

$$\hat{I}[\hat{\mathcal{M}}_n] = I_{HE}[\mathcal{M}_1] + I_{cosmic\ string}[\mathcal{M}_1]\,. \tag{4.13}$$

Here we switched notation from specifying the metric to the manifold. From this we can determine the action of the n-fold cover by

$$I[\mathcal{M}_n] = n\,\hat{I}[\hat{\mathcal{M}}_n]\,. \tag{4.14}$$

To be clear, we start with the original manifold $\mathcal{M}$ endowed with the saddle point solution of the metric $\tilde{g}$, we replicate it to $(\mathcal{M}_n, \tilde{g}_n)$ and then use the replica symmetry to bring it to $(\hat{\mathcal{M}}_n, \hat{g}_n)$. For the computation, however, we start with the original manifold $\mathcal{M}$ and add the cosmic string action to obtain $\hat{\mathcal{M}}_n$.

[1] Actually, since the induced metric on the codimension-2 surface is fixed, the variation of the cosmic string action with respect to $g_{\mu\nu}$ vanishes and it does not contribute to the Einstein field equations. However, adding this action does change the value of the total action and incorporates the right curvature of a manifold with a conical singularity [72].

[2] From now on I will drop the "E" subscript indicating an action is Euclidean, and specify when this is not the case.

Then, the calculation comes down to determining the effect of varying the replica number n close to one on the action of the n-fold cover:

$$\begin{aligned} S_{grav} &= \lim_{n\to 1} \partial_n \left(I_{HE}[\mathcal{M}_n] - n\, I_{HE}[\mathcal{M}_1]\right) \\ &= \lim_{n\to 1} \partial_n \left(n\, \hat{I}[\hat{\mathcal{M}}_n] - n\, I_{HE}[\mathcal{M}_1]\right) \\ &= \lim_{n\to 1} \left(\hat{I}[\hat{\mathcal{M}}_n] - I_{HE}[\mathcal{M}_1] + n\, \partial_n \hat{I}[\hat{\mathcal{M}}_n]\right) \\ &= \lim_{n\to 1} \left(\partial_n \hat{I}[\hat{\mathcal{M}}_n]\right) . \end{aligned} \tag{4.15}$$

In the last line we used that for $n = 1$ the orbifold action is just the action of the original manifold $\hat{I}[\hat{\mathcal{M}}_1] = I_{HE}[\mathcal{M}_1]$.

When varying the action of the orbifold we note that only perturbations of order $(n-1)$ away from $I_{HE}[\mathcal{M}_n]$ contribute to the entropy. Higher order terms will drop out when taking the limit. We will vary the action with respect to $\delta\hat{g}_n^{\mu\nu} = \partial_n \hat{g}_n^{\mu\nu}$:

$$\partial_n \hat{I}[\hat{\mathcal{M}}_n] = \int_{\hat{\mathcal{M}}_n} d^d x\, \sqrt{\hat{g}_n}\, (EOM)_{\mu\nu}\, \delta\hat{g}_n^{\mu\nu} + \int_{horizon} d^{d-1}y\, \sqrt{\hat{h}_n}\, \Theta(\hat{g}_n^{\mu\nu}, \delta\hat{g}_n^{\mu\nu}) + \mathcal{O}((n-1)^2)\,. \tag{4.16}$$

Here $\hat{g}_n$ and $\hat{h}_n$ are the determinants of $\hat{g}_{n,\mu\nu}$ and of the induced metric on the horizon $\hat{h}_{n,\mu\nu}$, respectively. The first order variation gives the equations of motion (EOM), which vanish since we took $\hat{g}_n^{\mu\nu}$ as a solution to $\hat{I}[\hat{\mathcal{M}}_n]$. However, when varying n we are changing the string tension or opening angle of the singularity at the horizon. This changes the boundary conditions and results in a boundary term $\Theta(\hat{g}_n^{\mu\nu}, \delta\hat{g}_n^{\mu\nu})$ on the horizon. The variation does still satisfy $\delta\hat{g}_n^{\mu\nu}|_{\partial\hat{\mathcal{M}}_n} = 0$, with $\partial\hat{\mathcal{M}}_n$ being asymptotic infinity in the Schwarzschild geometry and the location of the static observer in the de Sitter geometry[3]. The higher order terms come from the adjustment of the metric to this new boundary condition [7]. This leaves us to conclude

$$S_{grav} = \lim_{n\to 1} \int_{horizon} d^{d-1}y\, \sqrt{\hat{h}_n}\, \Theta(\hat{g}_n^{\mu\nu}, \delta\hat{g}_n^{\mu\nu})\,. \tag{4.17}$$

Schwarzschild Black Hole

The Euclidean Hilbert-Einstein action with zero cosmological constant is

$$I_{HE}[\mathcal{M}] = \frac{1}{16\pi G} \int_{\mathcal{M}} d^d x\, \sqrt{g}\, R + \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{d-1}y\, \sqrt{h}\, K\,, \tag{4.18}$$

where the second term is the Gibbons-Hawking-York boundary term with K the second fundamental form or extrinsic curvature of the boundary [18]. It is defined as

$$\begin{aligned} K &= g^{\mu\nu} K_{\mu\nu} \\ &= g^{\mu\nu} \left(\nabla_\mu n_\nu - n^2 n_\mu (n^\lambda \nabla_\lambda n_\nu)\right)\,, \end{aligned} \tag{4.19}$$

[3]This is of course not a real boundary as we know it, but we view it just as a reference location where we fix the metric.

with n^μ a normal vector to the boundary normalized as $n^2 = \pm 1$, for timelike and spacelike boundaries respectively [22]. We included it to make the variational principle well-defined on a manifold with a boundary. It ensures that metric solutions with $\delta g^{\mu\nu} = 0$ on the boundary correspond to an extremum of the action by making the first derivative of the metric variation normal to the boundary vanish [11].

We know the Euclidean Schwarzschild metric (4.5) with geometry 4.1 is a solution to (4.18). When going to the n-fold cover and orbifold we know from the previous sections we should focus on the geometry near the horizon. As in (4.7) we write the near horizon metric as

$$\begin{aligned} ds^2 &= dz^2 + z^2 d\psi^2 \\ &= dz^2 + \frac{z^2}{n^2} d\tau^2 \\ &= n^2 d\tilde{r}^2 + \tilde{r}^2 d\tau^2 \,. \end{aligned} \tag{4.20}$$

We started by noting the geometry locally looks like Euclidean flat space with $\psi \in [0, 2\pi]$. Then, we made some coordinate changes to explicitly incorporate the n dependence of the τ coordinate: $\tilde{r} = r - r_S = z/n$ and $\tau = n\psi$. On $\mathcal{M}_n$ its period is $\tau \in [0, 2\pi n]$ and on $\hat{\mathcal{M}}_n$ it is $\tau \in [0, 2\pi]$. So we see this metric automatically describes the correct conical singularity close to the horizon [51][7]:

$$ds^2 = n^2 d\tilde{r}^2 + \tilde{r}^2 d\tau^2 + r_S^2 d\Omega_2^2 \,. \tag{4.21}$$

For future use, its relevant Christoffel symbols are

$$\Gamma^\tau_{\tau,\tilde{r}} = \Gamma^\tau_{\tilde{r},\tau} = \frac{1}{\tilde{r}} \,, \qquad\qquad \Gamma^{\tilde{r}}_{\tau\tau} = -\frac{\tilde{r}}{n^2} \,. \tag{4.22}$$

Now we will determine the variation of $\hat{I}[\hat{\mathcal{M}}_n]$ under $\delta \hat{g}^{\mu\nu}_n = \partial_n \hat{g}^{\mu\nu}_n$:

$$\begin{aligned} \partial_n \hat{I}[\hat{\mathcal{M}}_n] &= \frac{1}{16\pi G} \int_{\hat{\mathcal{M}}_n} d^d x \sqrt{\hat{g}_n} \nabla_\lambda \left(\hat{g}_{n,\mu\nu} \nabla^\lambda \partial_n \hat{g}^{\mu\nu}_n - \nabla_\alpha \partial_n \hat{g}^{\lambda\alpha}_n \right) \\ &= \frac{1}{16\pi G} \int_{horizon} d^{d-1} y \sqrt{\hat{h}_n}\, n_\lambda \left(\hat{g}_{n,\mu\nu} \nabla^\lambda \partial_n \hat{g}^{\mu\nu}_n - \nabla_\alpha \partial_n \hat{g}^{\lambda\alpha}_n \right) \\ &= \frac{n}{16\pi G} \int_{horizon} d^{d-1} y \sqrt{\hat{h}_n} \left(\hat{g}_{n,\mu\nu} \nabla^{\tilde{r}} \partial_n \hat{g}^{\mu\nu}_n - \nabla_\alpha \partial_n \hat{g}^{\tilde{r}\alpha}_n \right) \\ &= \frac{n}{16\pi G} \int_{horizon} d^{d-1} y \sqrt{\hat{h}_n} \left(\hat{g}_{n,\mu\nu} g^{\tilde{r}\tilde{r}} (\Gamma^\mu_{\tilde{r},\alpha} \partial_n \hat{g}^{\alpha\nu}_n + \Gamma^\nu_{\tilde{r},\alpha} \partial_n \hat{g}^{\mu\alpha}_n) - \Gamma^{\tilde{r}}_{\alpha\beta} \partial_n \hat{g}^{\beta\alpha}_n - \Gamma^\alpha_{\alpha\beta} \partial_n \hat{g}^{\tilde{r}\beta}_n \right) \\ &= \frac{n}{16\pi G} \int_{horizon} d^{d-1} y \sqrt{\hat{h}_n} \left(-\Gamma^\tau_{\tau,\tilde{r}} \partial_n \hat{g}^{\tilde{r}\beta}_n \right) = \frac{1}{n^2} \frac{1}{8\pi G} \frac{1}{\tilde{r}} \int_{horizon} d^{d-1} y \sqrt{\hat{h}_n} \\ &= \frac{1}{n^2} \frac{1}{8\pi G} \frac{1}{\tilde{r}} \int_0^{2\pi} d\tau \int_0^\pi d\theta \int_0^{2\pi} d\phi\, \tilde{r}\, r_S^2 \sin\theta \\ &= \frac{1}{n^2} \frac{\mathcal{A}}{4G} \,. \end{aligned} \tag{4.23}$$

We started with equation (4.65) in Carroll [22] for the variation of the Ricci tensor. Then we applied Stokes' theorem to localize the integral on the horizon with outward pointing normal vector $n^\mu = (0, 1/n, 0, 0)$, normalized as $n^2 = 1$. In the fifth line we used that only $\partial_n \hat{g}_n^{\tilde{r}\tilde{r}} = -2/n^3$ is non-zero in combination with the only non-zero Christoffel symbols (4.22). The induced metric is $ds^2 = \tilde{r}^2 d\tau^2 + r_S^2 d\Omega_2^2$ and we see the $\tilde{r}$ dependence drops out leaving a non-zero finite expression when taking the limit $\tilde{r} \to 0$, or equivalently evaluating r on the horizon. In the last line we recovered the area of the event horizon $\mathcal{A}$.

To finish we take the limit $n \to 1$ and we obtain:

$$S_{grav,\,BH} = \lim_{n\to 1} \frac{1}{n^2}\frac{\mathcal{A}}{4G} = \frac{\mathcal{A}}{4G}\,. \tag{4.24}$$

The classical zeroth order term of the von Neumann entropy of the quantum state generated by the path integral over the Euclidean Schwarzschild geometry with cut at $\tau = 0$ is exactly the Bekenstein-Hawking entropy (2.32).

De Sitter Spacetime

The Euclidean de Sitter metric (4.9) with geometry 4.3 is a solution to

$$I_{HE}[\mathcal{M}] = \frac{1}{16\pi G}\int_{\mathcal{M}} d^d x\,\sqrt{g}\,(R - 2\Lambda)\,, \tag{4.25}$$

where we added the cosmological constant en omitted the Gibbons-Hawking-York boundary term as Euclidean de Sitter space has a spherical topology without boundary [18].

The steps in calculating the entropy are completely analogous to the black hole case. The metric on $\hat{\mathcal{M}}_n$ close to the horizon is

$$ds^2 = n^2 d\tilde{r}^2 + \tilde{r}^2 d\tau^2 + L^2 d\Omega_2^2\,, \tag{4.26}$$

with $\tilde{r} = L - r$ and $\tau = [0, 2\pi]$. The result is

$$S_{grav,\,dS} = \frac{\mathcal{A}}{4G}\,, \tag{4.27}$$

with $\mathcal{A}$ the area of the cosmological horizon. Thus, also the classical zeroth order term of the von Neumann entropy of the quantum state generated by the path integral over the Euclidean de Sitter geometry with cut at $\tau = 0$ is exactly its thermodynamic entropy (2.54).

4.2 The Island Formula

Including quantum matter contributions, two groups [12][13] independently found that applying the replica trick leads to the island formula (2.37). They considered an evaporating black hole described in two-dimensional Jackiw-Teitelboim gravity. In determining the partition functions they only fix the boundary conditions and let the path integral sum over different geometries. There are two different saddle points of the gravitational path integral that dominate at different times during black hole evaporation. In the beginning all replicas are disconnected and lead to the Hawking result of an ever-growing matter entropy. At the Page time, however, a different geometry dominates where all replicas are connected with a so-called replica wormhole. The result for the entropy of radiation is the island formula. Our goal is to discuss whether a replica trick method can also be used to derive the island formula for radiation in de Sitter spacetime and compare it to the quantum extremal surface method in the article by Watse Sybesma [15].

4.2.1 Semi-Classical Replica Method

In this section we first go over the qualitative idea of the calculations performed in [12] and [13]. Both study an evaporating black hole in the stable Hartle-Hawking vacuum. Maldacena found this is equivalent to two disconnected copies of the same CFT entangled in the thermofield double state (2.14). These CFTs are said to live on the asymptotic flat regions of Anti-de Sitter space. This is then where boundary conditions are introduced and the geometry of the spacetime is fixed to be flat, i.e. non-gravitating. The interior is determined by the dominating saddle point of the path integral with the correct boundary conditions. The calculations are done in Jackiw-Teitelboim (JT) gravity combined with a CFT that lives on both the gravitational and non-gravitational part of the spacetime. The path integral on the n-fold cover is, then,

$$\mathcal{Z}_{th,n} = \int_{\mathcal{M}_n} \mathcal{D}\phi \mathcal{D}g_n \, e^{-I_{JT}[g_n] - I_{CFT}[\phi]} \,, \tag{4.28}$$

with the boundary condition that the geometry is flat and has Euclidean time period $\tau = \tau + n\beta$. On the original manifold $\mathcal{M}$ it has period β. The n-fold cover is constructed by only gluing the replicas in the non-gravitational regions. The path integral fills in the gravitational region and has two saddle points that can dominate: a geometry on which all replicas are disconnected in the gravitational region and one where they are all connected and form a replica wormhole (see figure 4.6).

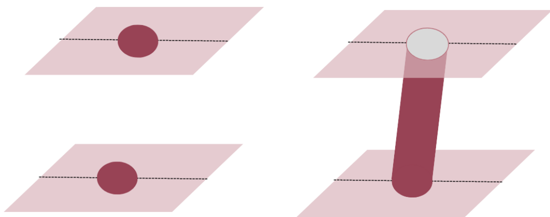


Figure 4.6: Path integral representation of the n-fold cover. The replicas are glued together along the grey lines. The light pink regions are non-gravitational, while the dark pink regions are and are filled in by the path integral. Left is the dominant saddle point where the replicas are disconnected in the gravitational region. On the right the replica wormhole is shown where all replicas are connected. Adapted from [12].

A dominating geometry, then, has the correct boundary conditions and is a solution to

$$I[\mathcal{M}_n] = I_{JT}[\mathcal{M}_n] + I_{CFT}[\mathcal{M}_n] \,, \tag{4.29}$$

where by $I_{JT}[\mathcal{M}_n]$ we mean the JT action of the metric describing $\mathcal{M}_n$ and by $I_{CFT}[\mathcal{M}_n]$ we mean the CFT action of all matter fields living on $\mathcal{M}_n$.

In the case all replicas are disconnected in the gravitational region, taking the quotient space with respect to the replica symmetry $\hat{\mathcal{M}}_n = \mathcal{M}_n / Z_n$ does not result in conical singularities. This means the orbifold metric $\hat{g}_n$ is a solution to

$$\hat{I}[\hat{\mathcal{M}}_n] = I_{JT}[\mathcal{M}_1] \,. \tag{4.30}$$

Then, we add n copies of the CFT originally defined on $\mathcal{M}_1$. Recall that the gravitational part of the action on the n-fold cover is n times that of the orbifold:

$$I_{JT}[\mathcal{M}_n] = n \, I_{JT}[\mathcal{M}_1] \,. \tag{4.31}$$

This means

$$I[\mathcal{M}_n] = n\, I_{JT}[\mathcal{M}_1] + I_{CFT}[\mathcal{M}_n] \tag{4.32}$$

and the von Neumann entropy becomes

$$\begin{aligned} S_{vN}^{dis} &= \lim_{n\to 1} \partial_n \left(I[\mathcal{M}_n] - n\, I[\mathcal{M}_1]\right) \\ &= \lim_{n\to 1} \partial_n \left(I_{CFT}[\mathcal{M}_n] - n\, I_{CFT}[\mathcal{M}_1]\right) \\ &= S_{vN}(\rho_{matter,\, non-grav})\,. \end{aligned} \tag{4.33}$$

Recalling that we constructed the n-fold cover by gluing the replicas along the non-gravitational regions, we see we calculated the entanglement entropy between the matter fields living inside and outside the non-gravitational region. This result is the Hawking expectation of an increasing radiation entropy, when applied to an evaporating black hole.

When all replicas are connected we do need to account for possible conical singularities in the gravitational region of the orbifold. The orbifold action becomes

$$\hat{I}[\hat{\mathcal{M}}_n] = I_{JT}[\mathcal{M}_1] + I_{cosmic\ string}[\mathcal{M}_1]\,. \tag{4.34}$$

The location of the cosmic string (or strings) is fixed by the Einstein equations corresponding to the gravitational part of the action on the n-fold cover

$$I_{JT}[\mathcal{M}_n] = n\left(I_{JT}[\mathcal{M}_1] + I_{cosmic\ string}[\mathcal{M}_1]\right). \tag{4.35}$$

Then the von Neumann entropy can be calculated as

$$\begin{aligned} S_{vN}^{con} &= \lim_{n\to 1} \partial_n \left(I[\mathcal{M}_n] - n\, I[\mathcal{M}_1]\right) \\ &= \lim_{n\to 1} \partial_n \left(n\, \hat{I}[\hat{\mathcal{M}}_n] + I_{CFT}[\mathcal{M}_n] - n\, I_{JT}[\mathcal{M}_1] - n\, I_{CFT}[\mathcal{M}_1]\right) \\ &= \lim_{n\to 1} \left(\hat{I}[\hat{\mathcal{M}}_n] - I_{JT}[\mathcal{M}_1] + n\,\partial_n \hat{I}[\hat{\mathcal{M}}_n] + \partial_n (I_{CFT}[\mathcal{M}_n] - n\, I_{CFT}[\mathcal{M}_1])\right) \\ &= \lim_{n\to 1} \left(\partial_n \hat{I}[\hat{\mathcal{M}}_n] + S_{vN}(\rho_{matter,\, connected})\right) \\ &= \frac{\mathcal{A}}{4G} + S_{vN}(\rho_{matter,\, connected})\,. \end{aligned} \tag{4.36}$$

In the fourth line we used that for $n = 1$ $\hat{I}[\hat{\mathcal{M}}_n] = I_{JT}[\mathcal{M}_1]$ and we introduced the von Neumann entropy of matter fields living in the non-gravitational region plus the region connected through the wormhole. In the last line we followed [12] and our results in the classical replica trick derivation to equate the variation of the orbifold action to the total area of the fixed points. The end result is exactly the generalized entropy. Lastly, since we need to extremize and minimize the action to find a dominating solution to the path integral, this method corresponds to the QES formulation introduced in section 3.3 and we obtain the island formula for the von Neumann entropy of the non-gravitational region (3.45).

4.2.2 Jackiw-Teitelboim Gravity

For the discussion on de Sitter spacetime, we will also work in two-dimensional JT gravity. This way we evade some of the technical difficulties and has the added benefit we can use the two-dimensional CFT set-up introduced in 3.2. In two dimensions the Hilbert-Einstein action becomes trivial as the Ricci tensor only has one degree of freedom left and by the Gauss-Bonnet theorem the action becomes

$$I_{HE} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\,\sqrt{g}R = \frac{1}{2G}(1-\gamma) = \frac{1}{4G}\chi\,, \tag{4.37}$$

where γ is the genus of the manifold $\mathcal{M}$ and χ its Euler characteristic [73][22] [4]. A more physically interesting method is to start with the four-dimensional system of interest and use dimensional reduction to obtain a two-dimensional representation. The result are the dilaton theories of gravity.

Here we are interested in the special case of Jackiw-Teitelboim (JT) gravity [73]. Its action[5] is

$$\begin{aligned} I_{JT} = \frac{1}{16\pi G}&\left(\int_{\mathcal{M}} d^2x\,\sqrt{g}\,\Phi\,(R-2\Lambda) + \Phi_b\int_{\partial\mathcal{M}} dy\,\sqrt{h}\,2K\right) \\ &+ \frac{\Phi_0}{16\pi G}\left(\int_{\mathcal{M}} d^2x\,\sqrt{g}\,\left(R-2\Lambda\right) + \int_{\partial\mathcal{M}} dy\,\sqrt{h}\,2K\right)\,, \end{aligned} \tag{4.38}$$

where Φ is called the dilaton, with Φ_b its value at the boundary. The action describes de Sitter or Anti-de Sitter spacetime depending on the sign of Λ. The second term proportional to a constant Φ_0 is a purely geometrical term proportional to the Euler characteristic, that does not contribute to the dynamics of the system. Also the Gibbons-Hawking-York boundary term is included when relevant. Its equations of motion are

$$R - 2\Lambda = 0\,, \tag{4.39}$$

$$\Phi(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R-2\Lambda)) + g_{\mu\nu}\nabla^2\Phi - \nabla_\mu\nabla_\nu\Phi = 0\,. \tag{4.40}$$

The variation of the action with respect to the dilaton, gives a Lagrange multiplier constraint that fixes the curvature of possible solutions. For a derivation of the equation of motion see appendix A.

In two dimensions we can choose the metric to be in conformal gauge:

$$ds^2 = e^{2\rho(z,\bar{z})}dzd\bar{z}\,, \tag{4.41}$$

where z, $\bar{z}$ are complex coordinates with $-1 \leq |z|^2 \leq 1$. In these coordinates the curvature constraint (4.39) becomes

$$R = -8e^{-2\rho}\partial_z\partial_{\bar{z}}\rho = 2\Lambda\,, \tag{4.42}$$

with one solution being

$$e^{2\rho} = \frac{4}{(1+\Lambda|z|^2)^2}\,. \tag{4.43}$$

The $(z\bar{z})$- and (zz)-components of the equations of motion are then, respectively,

$$\partial_z\partial_{\bar{z}}\Phi + \frac{1}{2}e^{2\rho}\Lambda\Phi = 8\pi G T_{z\bar{z}}\,, \tag{4.44}$$

$$-e^{2\rho}\partial_z\left(e^{-2\rho}\partial_z\Phi\right) = 8\pi G T_{zz}\,. \tag{4.45}$$

[4] The Gauss-Bonnet theorem, actually, holds for compact orientable manifolds, also with boundary.

[5] Actually, there is opposite sign in front of the Λ term in the original JT action, but this is the usual convention in the literature on this topic.

The $(\bar{z}\bar{z})$-component is similar to the (zz)-component. We also included the energy-momentum tensor $T_{\mu\nu}$ coming from a possible addition of a matter action I_{mat}:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{g}}\frac{\delta I_{mat}}{\delta g^{\mu\nu}}\,. \tag{4.46}$$

In vacuum a solution for the dilaton is

$$\Phi = \Phi_h \frac{1+\Lambda|z|^2}{1-\Lambda|z|^2}\,, \tag{4.47}$$

where Φ_h is the value of the dilaton at $|z|^2 = 0$ [6].

4.2.3 Semi-Classical Replica Method: de Sitter Spacetime

As mentioned, the JT action can be derived using dimensional reduction. Characteristics of the two-dimensional metric and dilaton profile can then be linked to the higher dimensional spacetime and given a more concrete physical interpretation. In the end we want to compare our findings using the replica method to the quantum extremal method used in an article by Watse Sybesma [15]. For this we will work with Euclidean JT gravity derived from the three-dimensional spacetime and add a CFT in the Bunch-Davies vacuum.

We follow the same dimensional reduction as in [15] (see appendix B for the explicit derivation) and perform a Wick rotation to obtain the Euclidean metric:

$$\begin{aligned} ds^2 &= -e^{2\rho(x^+x^-)}dx^+dx^- = -\frac{4}{(1-\Lambda x^+x^-)^2}dx^+dx^- \\ &= \frac{4}{(1+\Lambda|w|^2)^2}dwd\bar{w}\,. \end{aligned} \tag{4.48}$$

We started in the Lorentzian metric given by equation (3.10) in [15] in Kruskal coordinates $-1/\Lambda \leq x^+x^- \leq 1/\Lambda$ [38]. In the second line we made the coordinate transformation to complex coordinates in Euclidean time

$$w = -x^-\,, \qquad\qquad \bar{w} = x^+ \tag{4.49}$$

with $-1/\Lambda \leq |w|^2 \leq 1/\Lambda$. We have recovered exactly the Euclidean JT metric solution (4.43), as we should. From the higher dimensional theory we know the static observer is located at $|w|^2 = 1/\Lambda$, the cosmological horizon is at $|w|^2 = 0$ and future and past infinity at $|w|^2 = -1/\Lambda$.

The dimensionless dilaton solution after Wick rotation is

$$\Phi = \frac{S}{2}\frac{1-\Lambda|w|^2}{1+\Lambda|w|^2}\,. \tag{4.50}$$

At $|w|^2 = 1/\Lambda$ the dilaton vanishes, while at $|w|^2 = 0$ it is finite and takes on the value $\Phi_h = S/2$. S is the entropy in both three and two dimensions:

$$S \equiv S_{3d} = \frac{\mathcal{A}_{3d}}{4G_{3d}} = \frac{\pi L}{2G_{3d}} = 2\Phi_h = \frac{1}{4G_{2d}/\Phi}\Big|_{horizon} = \frac{\mathcal{A}_{2d}}{4G_{\text{eff}}} = S_{2d}\,, \tag{4.51}$$

[6] Notice the subtle minus sign differences to the usual expressions for the metric and dilaton [74] because we are considering a Euclidean spacetime with arbitrary cosmological constant, not a Lorentzian one with positive curvature or Euclidean one with negative curvature.

where we used that $G_{2d} = 1/8$ as follows from the dimensional reduction, and the dilaton couples to the gravitational strength as it appears in front of the Ricci scalar in the action.

We now add a CFT and only consider the semi-classical contribution (see appendix C for the derivation of the expressions used here). This is described in the off-diagonal part of its energy-momentum tensor and is completely fixed by the trace anomaly (3.26):

$$\langle T_{w\bar{w}} \rangle = \frac{c}{6} \partial_w \partial_{\bar{w}} \rho \,. \tag{4.52}$$

We can then determine the diagonal entries by conservation of energy $\nabla_\mu \langle T^{\mu\nu} \rangle = 0$ [75]:

$$\begin{aligned} \langle T_{ww} \rangle &= \frac{c}{6} \left(-\partial_w^2 \rho + (\partial_w \rho)^2 \right) + t_w(w) \\ \langle T_{\bar{w}\bar{w}} \rangle &= \frac{c}{6} \left(-\partial_{\bar{w}}^2 \rho + (\partial_{\bar{w}} \rho)^2 \right) + t_{\bar{w}}(\bar{w}) \,. \end{aligned} \tag{4.53}$$

The t_w and $t_{\bar{w}}$ functions arise as integration constants and should be chosen such that the left-hand side transforms according to (3.22) under conformal transformations. Substituting the solution for $e^{2\rho}$, we obtain

$$\begin{aligned} \langle T_{w\bar{w}} \rangle &= \frac{c}{6} \frac{\Lambda}{(1 + \Lambda |w|^2)^2} \\ \langle T_{ww} \rangle &= t_w(w) \,, \end{aligned} \tag{4.54}$$

and similarly for $\langle T_{\bar{w}\bar{w}} \rangle$.

Recall from section 2.4 that the Bunch-Davies (BD) vacuum is a true vacuum for an observer using Kruskal coordinates:

$$\langle T_{ww} \rangle_{BD} = \langle T_{\bar{w}\bar{w}} \rangle_{BD} = 0 \,, \tag{4.55}$$

i.e. $t_w(w) = t_{\bar{w}}(\bar{w}) = 0$. Using the Schwarzian (3.23), we can relate this to the vacuum state in static coordinates $z = r_* - t = r_* + i\tau = \ln(\sqrt{\Lambda} w)/\sqrt{\Lambda}$ with $-\infty < r_* \leq 0$ the tortoise coordinate and $-\infty < t < \infty$:

$$\langle T_{zz} \rangle_{BD} = \langle T_{\bar{z}\bar{z}} \rangle_{BD} = t_z(z) = t_{\bar{z}}(\bar{z}) = -\frac{c}{12} \frac{\Lambda}{2} = -\frac{c\pi^2}{6} T^2 \,. \tag{4.56}$$

The static observer observes thermal equilibrium at temperature $T = 1/2\pi L$ [76]. Returning to Kruskal coordinates, the trace anomaly alters the $(w\bar{w})$-component of the equations of motion for the dilaton to

$$\partial_w \partial_{\bar{w}} \Phi + \frac{1}{2} e^{2\rho} \Lambda \Phi = \pi \frac{c}{6} \frac{\Lambda}{(1 + \Lambda |w|^2)^2} \tag{4.57}$$

and the dilaton gets a semi-classical correction:

$$\Phi_{semi-cl} = \frac{S}{2} \frac{1 - \Lambda |w|^2}{1 + \Lambda |w|^2} + \frac{\pi c}{12} \,. \tag{4.58}$$

The dilaton mediates the gravitational strength, where $\Phi = 0$ corresponds to a strong gravitational field, and $\Phi \to \infty$ a weak one. Now due to the semi-classical correction Φ never vanishes inside the described region $-1/\Lambda \leq |w|^2 \leq 1/\Lambda$. The semi-classical approximation is most valid when $1 \ll c \ll S$, such that higher order quantum corrections are subleading while the classical solution remains unchanged at leading order. We can then define $\epsilon = \pi c/6S$ and rewrite

$$\Phi_{semi-cl} = \frac{S}{2}(1+\epsilon)\frac{1-\frac{1-\epsilon}{1+\epsilon}\Lambda|w|^2}{1+\Lambda|w|^2}\,. \tag{4.59}$$

The de Sitter static patch in the Bunch-Davies vacuum resembles the eternal black hole in the Hartle-Hawking vacuum and together with the similarity between their conformal diagrams drives us to see if we can say the de Sitter static patch is also in a thermofield double state. We could use the proposal made by Susskind [77][78] (and references therin) that, under the assumption there is a holographic description of the static patch, the degrees of freedom should be located on the cosmological horizon [79]. We, then, consider the horizon to be a boundary of the spacetime although it is not a conformal boundary. At the $\tau = 0$ slice its Euclidean geometry consists of two disconnected $d-1$-spheres and τ connects them over an interval $[0, \beta_{dS}/2]$ (see figure 4.3). This is the usual geometry the thermofield double state is defined on [18], but we cannot make the connection concrete as there is no dual theory with a known Hilbert space.

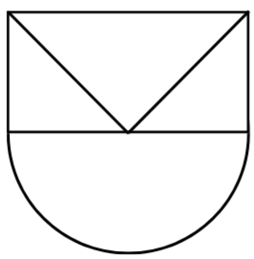


Figure 4.7: Representation of gluing the $\tau = 0$ slice of the Euclidean de Sitter spacetime to the $t = 0$ spatial slice of the Lorentzian spacetime.

A more sound method of creating the vacuum is using the Hartle-Hawking construction [25]. The Lorentzian de Sitter metric (2.45) has a $t \to -t$ symmetry. This means we can glue the $\tau = 0$ slice of the Euclidean metric to the $t = 0$ spatial slice of the Lorentzian metric. A path integral over the Euclidean part then produces de Sitter spacetime suitable for a stable the Bunch-Davies vacuum at the initial time [80]. Afterwards it is evolved over Lorentzian time (see figure 4.7). In this section we will take a different route and not use the cosmological horizon as reference system, but specify boundary conditions at the location of the static observer, just as we did in the classical replica method derivation. This choice was also made and motivated in [81].

Comparing the metric and dilaton solution to the black hole set-up in [12], we see we are again in a reverse situation, where we want to fix the spacetime in the region around the observer. In the conformal diagram this is indicated as region R (see figure 4.8). Its right boundary is at the location of the observer, indicated with o, and its left boundary is labeled with r from now on. We take the cut-off surface to be at $r_{*,r} = -\delta$ with $\delta \ll \epsilon$ and for $(1-\sqrt{\Lambda}\delta)/\sqrt{\Lambda} < |w| \leq 1/\sqrt{\Lambda}$ we impose the metric is flat

$$ds^2 = \frac{1}{1+\Lambda\delta^2}\frac{1}{|w|^2}dwd\bar{w} = \frac{1}{1+4\pi^2\delta^2/\beta^2}\frac{1}{|w|^2}dwd\bar{w}\,. \tag{4.60}$$

The other region $|w| < (1-\sqrt{\Lambda}\delta)/\sqrt{\Lambda}$ [7] can be curved. The constant in front is chosen such that the flat space metric and the Euclidean de Sitter metric are continuously glued

[7] To see the resemblance to [12] we can rewrite the bound as $\sqrt{\Lambda}|w| < 1 - \frac{2\pi}{\beta}\delta$ using $\beta = 2\pi L = 2\pi/\sqrt{\Lambda}$.

at the cut-off surface. The dilaton Φ is only defined in the gravitational region. The CFT is taken to be living on both regions with transparent boundary conditions at the cut-off.

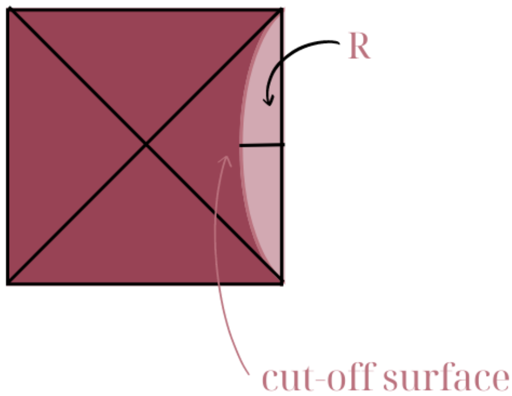


Figure 4.8: Conformal diagram of de Sitter spacetime. In the static patch we introduce a cut-off surface inside of which we neglect gravity and define a region R.

Using this set-up we want to determine the von Neumann entropy of the state ρ_R defined by the gravitational path integral on region R (at constant τ). The main difference to the black hole case is that the state is defined on a cut in a manifold with spherical topology, instead of a disc. The replica trick method for such closed spacetimes is discussed in [82][81]. In contrast to those, however, we do not take our state to be entangled with a disconnected universe or to be living in a gravitating region. The general idea remains the same. In evaluating the gravitational path integral there is one dominating saddle point where all gravitating regions are disconnected, and one where they are all connected through a Euclidean wormhole.

In the disconnected case, n two-spheres are connected only along the cut at region R creating an n-fold cover $\mathcal{M}_n$ that is topologically again a two-sphere. Taking the quotient space with respect to the replica symmetry group results in the orbifold $\hat{\mathcal{M}}_n$ with two conical singularities at the boundaries of R. These do not contribute to the entropy since we fix the metric on R and $\delta\hat{g}_n^{\mu\nu}|_R = 0$. The resulting entropy comes purely from the quantum fields as derived in (4.33):

$$\begin{aligned} S_{vN}^{dis}(\rho_R) &= S_{CFT}(\phi_R) \\ &= \frac{c}{6}\ln\left(\frac{1}{\epsilon_1\epsilon_2}\frac{|w_o - w_r|^2}{e^{-\rho(w_o,\bar{w}_o)}e^{-\rho(w_r,\bar{w}_r)}}\right) \\ &= \frac{c}{12}\ln\left(\frac{1}{(\epsilon_1\epsilon_2)^2}(w_o - w_r)^2(\bar{w}_o - \bar{w}_r)^2 e^{2\rho(w_o,\bar{w}_o)}e^{2\rho(w_r,\bar{w}_r)}\right) \end{aligned} \tag{4.61}$$

Although the quantum fields backreact on the geometry, it only affects the dilaton and not the metric tensor. The CFT entropy can then still be determined in the original Euclidean de Sitter spacetime [82]. We used the standard expression for the von Neumann entropy of an interval in a CFT (3.31), with $\epsilon_{1,2}$ two UV cut-off parameters for each endpoint of R. The conformal factors are as in (4.60) and w_o and $\bar{w}_o$ are positive constants such that $w_o\bar{w}_o = 1/\Lambda$.

When all replica manifolds are connected along region R but also with a Euclidean wormhole, additional conical singularities arise when forming the orbifold. These are located at the boundary of some region I [81]. Then, analogous to the derivation in (4.36), the entropy is

$$\begin{aligned} S_{vN}^{con}(\rho_R) &= \frac{\mathcal{A}}{4G} + S_{CFT}(\phi_{R\cup I}) \\ &= \frac{\text{Area}(\partial I)}{4G} + S_{CFT}(\phi_{R\cup I}) \\ &= \frac{\text{Area}(\partial I)}{4G} + S_{CFT}(\phi_{(R\cup I)^c}) \\ &= S(1+\epsilon)\frac{1-\frac{1-\epsilon}{1+\epsilon}\Lambda|w_{\partial I}|^2}{1+\Lambda|w_{\partial I}|^2} \\ &\quad + \frac{c}{12}\ln\left(\frac{1}{(\epsilon_1\epsilon_2)^2}(w_r - w_{\partial I})^2(\bar{w}_r - \bar{w}_{\partial I})^2 e^{2\rho(w_r,\bar{w}_r)}e^{2\rho(w_{\partial I},\bar{w}_{\partial I})}\right), \end{aligned} \tag{4.62}$$

with the total area of fixed points $\mathcal{A} = \text{Area}(\partial I)$ (I can consist of multiple disconnected parts) and the quantum field entropy is calculated on a manifold both connected on R and I to its replicas. In JT gravity the area of a point is determined by its corresponding value of the dilaton (as mentioned in (4.51)). To compare our result to [15] we make the same assumption that one island region arises with one of its boundary points being the other pole. Then we can equivalently write $S_{CFT}(\phi_{R\cup I}) = S_{CFT}(\phi_{(R\cup I)^c})$ as the state on the total constant τ slice is pure. The conformal factors are as in (4.48).

The von Neumann entropy is now determined firstly by which saddle point dominates and secondly by which island region minimizes the generalized entropy (i.e. the action):

$$S_{vN}(\rho_R)/S = \underset{\text{saddle point}}{\text{ext}} \begin{cases} \epsilon\frac{2}{\pi}\ln\left(\frac{\Lambda}{(1+\Lambda\delta^2)^2}\frac{1}{|w_r|^2}(w_o - w_r)^2(\bar{w}_o - \bar{w}_r)^2\right) \\ \min\limits_{\partial I}\ (1+\epsilon)\frac{1-\frac{1-\epsilon}{1+\epsilon}\Lambda|w_{\partial I}|^2}{1+\Lambda|w_{\partial I}|^2} + \epsilon\frac{2}{\pi}\ln\left(\frac{(w_r - w_{\partial I})^2(\bar{w}_r - \bar{w}_{\partial I})^2}{(1+\Lambda|w_r|^2)^2(1+\Lambda|w_{\partial I}|^2}\right)^2\right) \end{cases} \tag{4.63}$$

This is the same result[8] as derived in [15], now in Euclidean coordinates, and describes a growing entropy that saturates at a maximum value S as a function of the Lorentzian time coordinate on the cut-off surface t_r. Considering replicas of Euclidean de Sitter spacetime has successfully reproduced the island formula.

[8] Minor differences arose because a different convention was chosen in [15] in the expression for the trace anomaly and conformal transformation of the energy-momentum tensor.

Chapter 5

Discussion & Conclusions

We have made a side-by-side comparison of the event horizon of the Schwarzschild black hole and the cosmological horizon in de Sitter spacetime. In the semi-classical approximation both seem to produce radiation and have almost identical temperature, thermodynamic entropy and thermodynamic-like horizon dynamics. However, also the first differences become apparent. To study the cosmological horizon we need to consider de Sitter spacetime from the point of view of a static observer. The cosmological horizon is then manifestly observer dependent and there is also no asymptotic flat region. This complicates the formulation of globally conserved quantities and the set-up of a quantum field theory description.

Our main interest was the von Neumann entropy of radiation in both spacetimes. Considering the black hole in the Unruh vacuum, it will evaporate and the entropy of radiation follows the Page curve. This gives a tangible description of the black hole information paradox. A part of solving the paradox is understanding how the Page curve arises. In de Sitter spacetime an analogous Unruh-de Sitter vacuum can be constructed and its horizon will shrink and the spacetime will collapse. To study the radiation entropy, then, we focused on the stable Hartle-Hawking and Bunch-Davies vacuum, for the black hole and de Sitter respectively. For both the entropy is expected to grow initially and then saturate at a constant value.

The Page curve is successfully described in the island formula. This description was based on holography and the first Ryu-Takayanagi proposal. We discussed its transformation to the island formula. Another method to determine the von Neumann entropy of a state is using the replica method. We showed its application in the Euclidean path integral formulation of quantum field theory including the extension to gravitational effects. First, we calculated the von Neumann entropy of the classical Schwarzschild black hole and de Sitter static patch. We did this only considering their Hilbert-Einstein action in the gravitational path integral and approximating it in their known classical saddle point solutions. For both spacetimes we found the result to be exactly their thermodynamic entropy.

Finally, we discussed the procedure of deriving the island formula using the replica trick for the black hole in the Hartle-Hawking vacuum. We worked in two dimensional Euclidean JT gravity and considered a conformal field theory action alongside it in a semi-classical approximation. The two regimes of the Page curve arise from two alternating dominating saddle points in the gravitational path integral. De Sitter spacetime allows the same replica method. We formulated its solution in JT gravity coming from a dimensional reduction of three dimensional Lorentzian de Sitter spacetime. Next, we added a conformal field theory in the Bunch-Davies vacuum and took into account its backreaction on the geometry. In the von Neumann entropy calculation we studied the completely disconnected and connected saddle points (analogous to the black hole calculation) and recovered the island formula. It was checked the resulting Page curve is qualitatively the same as directly applying the island formula in the same set-up.

We have not checked if the completely disconnected and connected replica manifolds are indeed dominating configurations in the path integral, and whether there could be different ones. The fact, however, that the island formula can be derived from the Euclidean path integral formulation of semi-classical de Sitter spacetime provides more justification to its use. This is especially nice since there is not a complete quantum dual description of de Sitter spacetime yet in which we could otherwise formulate von Neumann entropies. This does mean that the island formula is a way to calculate the von Neumann entropy of some quantum state in an effective gravitational theory, while the exact state is unknown.

In the replica set-up we chose to specify boundary conditions at the location of the static observer. This is preferred over the point of view of some universal observer as we want to study the cosmological horizon and for any physically relevant description we can not include more than we can observe. We also chose to fix spacetime to be flat in a small region surrounding the observer. For future studies this is not needed, I believe, since in the Euclidean path integral formalism a state can be defined on any spacelike cut, including ones with non-zero curvature. This has been discussed for general geometries without asymptotic AdS boundaries in [81].

Interesting directions for further research would be to study a possible replica method application to de Sitter spacetime in the Unruh-de Sitter vacuum. This state has been discussed in [16], [24] and [43], and is found to collapse under the accumulation of radiation. It would be interesting to see if and how this could be reflected in a replica method. For this, the Swinger-Keldysh formalism probably needs to be used as the state does not have time reflection symmetry. On the other end, the formulation of a complete quantum description of de Sitter spacetime would be a great direction into understanding the physical interpretation of its (quantum) entropy.

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Appendix A

Jackiw-Teitelboim action

The action of two-dimensional Jackiw-Teitelboim gravity is

$$
\begin{aligned}
I_{JT} = \frac{1}{16\pi G}\left(\int_{\mathcal{M}} d^2x\,\sqrt{g}\,\Phi\,(R-2\Lambda) + \Phi_b\int_{\partial\mathcal{M}} dy\,\sqrt{h}\,2K\right) \\
+\frac{\Phi_0}{16\pi G}\left(\int_{\mathcal{M}} d^2x\,\sqrt{g}\,(R-2\Lambda) + \int_{\partial\mathcal{M}} dy\,\sqrt{h}\,2K\right)\,,
\end{aligned}
\tag{A.1}
$$

where Φ is called the dilaton, with Φ_b its value at the boundary. The action describes de Sitter or Anti-de Sitter spacetime depending on the sign of Λ. The second term proportional to a constant Φ_0 is a purely geometrical term proportional to the Euler characteristic, that does not contribute to the dynamics of the system. When the action is derived from a higher-dimensional theory this term encodes knowledge of that system. Also the Gibbons-Hawking-York boundary term is included when relevant. It ensures the variational principle is well-defined for a spacetime with a boundary.

First, when we vary the action with respect to the dilaton Φ, we immediately obtain

$$
R - 2\Lambda = 0\,. \tag{A.2}
$$

This acts as a Lagrange multiplier and fixes the curvature of possible solutions.

Now we will vary the action with respect to $g^{\mu\nu}$. First we note that

$$
\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma} \tag{A.3}
$$

independent of metric signature. For Euclidean signature

$$
\delta\sqrt{g} = \frac{1}{2}\frac{1}{\sqrt{g}}\delta g = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}\,. \tag{A.4}
$$

We will also need the following expressions, which can be found in Carroll [22],

$$
\begin{aligned}
\delta R &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\,,\\
\delta R_{\mu\nu} &= \nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})\,,\\
\delta\Gamma^\lambda_{\nu\mu} &= -\frac{1}{2}\left(g_{\alpha\nu}\nabla_\mu(\delta g^{\alpha\lambda}) + g_{\alpha\mu}\nabla_\nu(\delta g^{\alpha\lambda}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\lambda(\delta g^{\alpha\beta})\right)\,.
\end{aligned}
\tag{A.5}
$$

The variation of the action then is

$$\begin{aligned}
\delta I_{JT} &= \frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\,(\delta\sqrt{g})\,\Phi\,(R-2\Lambda)+\sqrt{g}\,\Phi\,(R_{\mu\nu}\delta g^{\mu\nu}+g^{\mu\nu}\delta R_{\mu\nu}) \\
&= \frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\,\sqrt{g}\,\Phi\left(-\frac{1}{2}g_{\mu\nu}(R-2\Lambda)+R_{\mu\nu}\right)\delta g^{\mu\nu}+\sqrt{g}\,\Phi g^{\mu\nu}\left(\nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu})-\nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})\right) \\
&= \frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\,\sqrt{g}\,\Phi\,G_{\mu\nu}\,\delta g^{\mu\nu}+\sqrt{g}\,\Phi\nabla_\rho\left(g^{\mu\nu}(\delta\Gamma^\rho_{\nu\mu})-g^{\mu\rho}(\delta\Gamma^\lambda_{\lambda\mu})\right) \\
&= \frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\,\sqrt{g}\,\Phi\,G_{\mu\nu}\,\delta g^{\mu\nu}+\sqrt{g}\,\Phi\nabla_\rho\left(g_{\mu\nu}(\nabla^\rho\delta g^{\mu\nu})-\nabla_\lambda(\delta g^{\rho\lambda})\right) \\
&= \frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\,\sqrt{g}\,\Phi\,G_{\mu\nu}\,\delta g^{\mu\nu}-\sqrt{g}\,(\nabla_\rho\Phi)\left(g_{\mu\nu}(\nabla^\rho\delta g^{\mu\nu})-\nabla_\lambda(\delta g^{\rho\lambda})\right) \\
&= \frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\,\sqrt{g}\,\Phi\,G_{\mu\nu}\,\delta g^{\mu\nu}+\sqrt{g}\,\left(g_{\mu\nu}(\nabla^\rho\nabla_\rho\Phi)-\nabla_\nu\nabla_\mu\Phi\right)\delta g^{\mu\nu}\,.
\end{aligned} \tag{A.6}$$

In the first and line we plugged in the variation of the metric determinant, Ricci scalar and Ricci tensor. In the third line we introduced the Einstein tensor $G_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}(R - 2\Lambda) + R_{\mu\nu}$ and rewrote the second term using the metric compatibility of the covariant derivative. In fourth line we substituted the variation of the Christoffel symbols. In the fifth and last line we performed an integration by parts, while omitting the boundary contribution. Finding a stationary point $\delta I_{JT}/\delta g^{\mu\nu} = 0$ we get the equations of motion:

$$\Phi(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R-2\Lambda)) + g_{\mu\nu}\nabla^2\Phi - \nabla_\mu\nabla_\nu\Phi = 0\,. \tag{A.7}$$

Appendix B

Dimensional reduction

Here we will explicitly go through the dimensional reduction of three-dimensional de Sitter spacetime to a two-dimensional model as is done in [15]. We begin with the three dimensional Lorentzian action

$$I_{3d} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^3x\,\sqrt{-g}\,(R-2\Lambda) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\,\sqrt{-h}K\,. \tag{B.1}$$

Here, $g_{\mu\nu}$ is the metric tensor of the spacetime and g its determinant, $h_{\mu\nu}$ the induced metric tensor on the boundary of the spacetime and h its determinant, R the Ricci scalar and K the extrinsic curvature of the boundary. When varying this action, we obtain the Einstein equations $R = 6\Lambda$. A solution is de Sitter spacetime with the following metric in Kruskal coordinates:

$$ds^2 = \frac{1}{(1-\Lambda x^+x^-)^2}\left(-4dx^+dx^- + \frac{1}{\Lambda}\left(1+\Lambda x^+x^-\right)^2 d\theta^2\right)\,, \tag{B.2}$$

where $\theta \in [0, 2\pi]$, $-1/\Lambda \leq x^+x^- \leq 1/\Lambda$ and both x^+ and x^- are functions of Euclidean time. Λ is related to the de Sitter length scale via $\Lambda = 1/L^2$. The northern static patch is covered by coordinates $\sigma^\pm = t \pm r_*$ related to the Kruskal coordinates by

$$x^\pm = \pm\frac{1}{\sqrt{\Lambda}}e^{\pm\sqrt{\Lambda}\sigma^\pm}\,, \tag{B.3}$$

with r_* the tortoise coordinate. The location of the observer is at $r_* = 0$ and the cosmological horizon at $r_* = -\infty$. Now, we apply a circular dimensional reduction with the following ansatz [15]:

$$ds^2 = -e^{2\rho(x^+,x^-)}dx^+dx^- + \varphi^2(x^+,x^-)d\theta^2\,. \tag{B.4}$$

The first part will describe the two dimensional theory and is related to the three dimensional theory via

$$\sqrt{-g_{3d}} = \sqrt{-g_{2d}}\varphi\,,\quad \int d^3x = \int d^2x 2\pi\,,\quad R_{3d} = R_{2d} - \frac{2}{\varphi}\Box_{2d}\varphi\,,\quad K_{3d} = K_{2d} + \frac{1}{\varphi}\partial_n\varphi\,, \tag{B.5}$$

where n is the normal direction to the boundary. We will go through their derivation one by one:

$$\sqrt{-g_{3d}} = \sqrt{-(-(-\frac{1}{2}e^{2\rho})(-\frac{1}{2}e^{2\rho})\varphi^2)} = \sqrt{-g_{2d}}\varphi\,,\quad \int d^3x = \int d^2x\int d\theta = \int d^2x 2\pi\,. \tag{B.6}$$

We use the following definitions of the Christoffel symbols, Riemann tensor and Ricci tensor and scalar, respectively:

$$\begin{aligned} \Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}\, g^{\lambda\rho}(\partial_\mu\, g_{\nu\rho} + \partial_\nu\, g_{\rho\mu} - \partial_\rho\, g_{\mu\nu})\,, \\ R^{\rho}{}_{\sigma\mu\nu} &= \partial_\mu\, \Gamma^{\rho}_{\nu\sigma} - \partial_\nu\, \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\, \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\, \Gamma^{\lambda}_{\mu\sigma}\,, \\ R_{\mu\nu} &= R^{\lambda}{}_{\mu\lambda\nu}\,, \\ R &= R^{\mu}{}_{\mu}\,. \end{aligned} \tag{B.7}$$

The two-dimensional metric is $ds^2 = -e^{2\rho(x^+,x^-)}dx^+dx^-$ and the non-zero Christoffel symbols are

$$\Gamma^{\pm}_{\pm\pm} = 2\partial_\pm\rho\,, \tag{B.8}$$

and its Ricci tensor and scalar are

$$\begin{aligned} R^{2d}_{-+} &= -\partial_+\Gamma^{-}_{--} - \Gamma^{+}_{++}\Gamma^{+}_{+-} = -2\partial_-\partial_+\rho\,, \\ R_{2d} &= 2(-2e^{-2\rho})R_{-+} = 8e^{-2\rho}\partial_-\partial_+\rho\,. \end{aligned} \tag{B.9}$$

The only non-zero Christoffel symbols for the three-dimensional metric are

$$\Gamma^{\pm}_{\pm\pm} = 2\partial_\pm\rho\,, \qquad \Gamma^{\pm}_{\theta\theta} = 2e^{-2\rho}\varphi\partial_\mp\varphi\,, \qquad \Gamma^{\theta}_{\pm\theta} = \frac{1}{\varphi}\partial_\pm\varphi\,, \tag{B.10}$$

such that the Ricci tensors are

$$\begin{aligned} R^{3d}_{\pm\pm} &= -\partial_\pm\Gamma^{\theta}_{\pm\theta} + \Gamma^{\theta}_{\pm\theta}\Gamma^{\pm}_{\pm\pm} - \Gamma^{\theta}_{\pm\theta}\Gamma^{\theta}_{\theta\pm} \\ &= \frac{1}{\varphi}\left(2(\partial_\pm\rho)(\partial_\pm\varphi) - \partial_\pm^2\varphi\right)\,, \\ R^{3d}_{\theta\theta} &= \partial_-\Gamma^{-}_{\theta\theta} + \partial_+\Gamma^{+}_{\theta\theta} + \Gamma^{-}_{--}\Gamma^{-}_{\theta\theta} + \Gamma^{+}_{++}\Gamma^{+}_{\theta\theta} - \Gamma^{-}_{\theta\theta}\Gamma^{\theta}_{-\theta} - \Gamma^{+}_{\theta\theta}\Gamma^{\theta}_{+\theta} \\ &= \partial_-(2e^{-2\rho}\varphi\partial_+\varphi) + \partial_+(2e^{-2\rho}\varphi\partial_-\varphi) + 4e^{-2\rho}(\partial_-\rho)(\varphi\partial_+\varphi) \\ &\quad + 4e^{-2\rho}(\partial_+\rho)(\varphi\partial_-\varphi) - 4e^{-2\rho}(\partial_-\varphi)(\partial_+\varphi) \\ &= 4e^{-2\rho}\varphi\partial_-\partial_+\varphi\,, \\ R^{3d}_{-+} &= -\partial_+\Gamma^{\theta}_{-\theta} - \partial_+\Gamma^{-}_{--} + \Gamma^{\lambda}_{\lambda\alpha}\Gamma^{\alpha}_{-+} - \Gamma^{\theta}_{+\theta}\Gamma^{\theta}_{-\theta} \\ &= -2\partial_-\partial_+\rho - \frac{1}{\varphi}\partial_-\partial_+\varphi\,, \end{aligned} \tag{B.11}$$

and

$$\begin{aligned} R_{3d} &= 2(-2e^{-2\rho})(R_{-+}) + \frac{1}{\varphi^2}R_{\theta\theta} \\ &= 8e^{-2\rho}\partial_-\partial_+\rho + 8\frac{1}{\varphi}e^{-2\rho}\varphi\partial_-\partial_+\varphi \\ &= R_{2d} - \frac{2}{\phi}\Box_{2d}\varphi\,. \end{aligned} \tag{B.12}$$

The extrinsic curvature is defined as

$$K = g^{\mu\nu}K_{\mu\nu} = g^{\mu\nu}\left(\nabla_\mu n_\nu - n^2 n_\mu(n^\lambda\nabla_\lambda n_\nu)\right)\,, \tag{B.13}$$

where n^μ is the normal vector to the boundary. For both the two- and three-dimensional theory this boundary is future and past infinity. It is characterized by $x^+x^- = 1/\Lambda$ and is a spacelike hypersurface. Its normal vector is

$$n^\pm = e^{-\rho}\frac{-x^\pm}{\sqrt{-x^-x^+}}\,, \qquad\qquad n^\theta = 0\,, \tag{B.14}$$

normalized as $n^2 = -1$. The extrinsic curvature is

$$\begin{aligned} K_{3d} &= g^{+-}K_{+-} + g^{-+}K_{-+} + g^{\theta\theta}K_{\theta\theta} \\ &= K_{2d} + \frac{1}{\varphi^2}K_{\theta\theta}\,, \end{aligned} \tag{B.15}$$

since K_{+-} and K_{-+} are independent of θ and φ. Then,

$$\begin{aligned} K^{3d}_{\theta\theta} &= -\Gamma^\lambda_{\theta\theta}n_\lambda \\ &= -\varphi e^{-\rho}\frac{1}{\sqrt{-x^-x^+}}(x^-(\partial_-\varphi) + x^+(\partial_+\varphi)) \\ &= \varphi n^\mu\partial_\mu\varphi \equiv \varphi\partial_n\varphi \\ \rightarrow K_{3d} &= K_{2d} + \frac{1}{\varphi}\partial_n\varphi \end{aligned} \tag{B.16}$$

We will now substitute these relations (B.5) in the three dimensional action (B.1) to obtain the two dimensional action:

$$\begin{aligned} I_{2d} =& \frac{1}{16\pi G_{3d}}\int_{\mathcal{M}} d^2x\, 2\pi\,\sqrt{-g_{2d}}\,\varphi\,(R_{2d} - \frac{2}{\varphi}\Box_{2d}\varphi - 2\Lambda) + \frac{1}{8\pi G_{3d}}\int_{\partial\mathcal{M}} dx\, 2\pi\,\sqrt{-h_{2d}}\,\varphi\,(K_{2d} + \frac{1}{\varphi}\partial_n\varphi) \\ =& \frac{1}{8G_{3d}}\int_{\mathcal{M}} d^2x\,\sqrt{-g_{2d}}\,\varphi\,(R_{2d} - 2\Lambda) + \frac{1}{4G_{3d}}\int_{\partial\mathcal{M}} dx\,\sqrt{-h_{2d}}\,\varphi\,K_{2d} \\ & - \frac{1}{4G_{3d}}\int_{\mathcal{M}} d^2x\,\sqrt{-g_{2d}}\Box_{2d}\varphi + \frac{1}{4G_{3d}}\int_{\partial\mathcal{M}} dx\,\sqrt{-h_{2d}}\frac{1}{\varphi}\partial_n\varphi \\ =& \frac{1}{8G_{3d}}\int_{\mathcal{M}} d^2x\,\sqrt{-g_{2d}}\,\varphi\,(R_{2d} - 2\Lambda) + \frac{1}{4G_{3d}}\int_{\partial\mathcal{M}} dx\,\sqrt{-h_{2d}}\,\varphi\,K_{2d} \\ =& \frac{1}{2\pi}\int_{\mathcal{M}} d^2x\,\sqrt{-g_{2d}}\,\Phi\,(R_{2d} - 2\Lambda) + \frac{1}{\pi}\int_{\partial\mathcal{M}} dx\,\sqrt{-h_{2d}}\,\Phi\,K_{2d}\,, \end{aligned} \tag{B.17}$$

where the last two integrals in the second line cancel each other by Stokes' theorem. In the last line we introduced the dimensionless dilaton Φ:

$$\frac{1}{4G_{3d}}\varphi = \frac{1}{\pi}\Phi\,. \tag{B.18}$$

Comparing to the usual prefactor of $1/16\pi G$ to the action, we can see that $G_{2d} = 1/8$. Now for the metric and dilaton solution we can just compare (B.4) to (B.2):

$$ds^2 = -e^{2\rho(x^+,x^-)}dx^+dx^-\,, \quad e^{2\rho(x^+,x^-)} = \frac{4}{(1-\Lambda x^+x^-)^2}\,, \quad \Phi = \frac{S}{2}\frac{1+\Lambda x^+x^-}{1-\Lambda x^+x^-}\,, \tag{B.19}$$

with S the usual three-dimensional entropy

$$S = \frac{\mathcal{A}}{4G_{3d}} = \frac{\pi}{2G_{3d}\sqrt{\Lambda}}\,. \tag{B.20}$$

Appendix C

Semi-classical de Sitter spacetime

To discuss the semi-classical effect of quantum fields on de Sitter spacetime we work in the two-dimensional JT model defined in B and add a CFT in the Bunch-Davies vacuum. The classical metric and dilaton solution in Euclidean signature are (B.19):

$$ds^2 = e^{2\rho(w,\bar{w})}dwd\bar{w}\,, \qquad e^{2\rho(w,\bar{w})} = \frac{4}{(1+\Lambda|w|^2)^2}\,, \qquad \Phi = \frac{S}{2}\frac{1-\Lambda|w|^2}{1+\Lambda|w|^2}\,. \tag{C.1}$$

We use Euclidean Kruskal coordinates $w = -x^-$ and $\bar{w} = x^+$, which are related to the static patch coordinate $z = r_* + i\tau$ via

$$w = \frac{1}{\sqrt{\Lambda}}e^{\sqrt{\Lambda}z}\,. \tag{C.2}$$

Then, we add a CFT action which results in a non-zero energy-momentum tensor in the field equations via

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{g}}\frac{\delta I_{CFT}}{\delta g^{\mu\nu}}\,. \tag{C.3}$$

We assume the equation of motion for the CFT are satisfied and only consider the semi-classical effect coming from the trace anomaly on curved spacetimes(3.26):

$$\left\langle T^{\mu}{}_{\mu}\right\rangle = -\frac{c}{12}R \tag{C.4}$$

$$4e^{-2\rho}\left\langle T_{w\bar{w}}\right\rangle = \frac{c}{12}\,8e^{-2\rho}\partial_w\partial_{\bar{w}}\rho \tag{C.5}$$

$$\left\langle T_{w\bar{w}}\right\rangle = \frac{c}{6}\partial_w\partial_{\bar{w}}\rho = \frac{c}{6}\frac{\Lambda}{(1+\Lambda|w|^2)^2}\,, \tag{C.6}$$

where we used that the Ricci scalar in Euclidean signature is $R = 8e^{-2\rho}\partial_w\partial_{\bar{w}\rho}$ and plugged in the known solution for $e^{2\rho(w,\bar{w})}$ in the last line. We can determine the diagonal elements of the energy-momentum tensor using conservation of energy $\nabla_\mu\left\langle T^{\mu\nu}\right\rangle = 0$:

$$\begin{aligned}
\nabla_\mu T^{\mu w} &= \partial_\mu T^{\mu w} + \Gamma^{\alpha}_{\alpha\beta}T^{\beta w} + \Gamma^{w}_{\alpha\beta}T^{\alpha\beta} \\
&= 4\left(\partial_w T^{ww} + \partial_{\bar{w}}T^{\bar{w}w} + 2\Gamma^{w}_{ww}T^{ww} + \Gamma^{\bar{w}}_{\bar{w}\bar{w}}T^{\bar{w}w}\right) \\
&= 4e^{-4\rho}\left(\partial_w T_{\bar{w}\bar{w}} + \partial_{\bar{w}}T_{w\bar{w}} - 2(\partial_{\bar{w}}\rho)T_{w\bar{w}}\right)\,, \\
\nabla_\mu T^{\mu\bar{w}} &= 4e^{-4\rho}\left(\partial_{\bar{w}}T_{ww} + \partial_w T_{w\bar{w}} - 2(\partial_w\rho)T_{w\bar{w}}\right)\,.
\end{aligned} \tag{C.7}$$

We neglected to write angle brackets around each element of the tensor. Equating these expressions to zero and plugging in $T_{w\bar{w}} = \frac{c}{6}\partial_w\partial_{\bar{w}}\rho$ we get:

$$\begin{aligned} \partial_w T_{\bar{w}\bar{w}} &= -\partial_{\bar{w}} T_{w\bar{w}} + 2(\partial_{\bar{w}}\rho) T_{w\bar{w}} \\ &= \frac{c}{6}\partial_w \left(-\partial_w^2\rho + (\partial_w\rho)^2\right) \,, \\ \partial_{\bar{w}} T_{ww} &= -\partial_w T_{w\bar{w}} + 2(\partial_w\rho) T_{w\bar{w}} \\ &= \frac{c}{6}\partial_{\bar{w}} \left(-\partial_{\bar{w}}^2\rho + (\partial_{\bar{w}}\rho)^2\right) \,. \end{aligned} \tag{C.8}$$

Finally, we can integrate over w and $\bar{w}$ respectively:

$$\begin{aligned} T_{\bar{w}\bar{w}} &= \frac{c}{6}\left(-\partial_w^2\rho + (\partial_w\rho)^2\right) + t_{\bar{w}}(\bar{w}) = t_{\bar{w}}(\bar{w}) \,, \\ T_{ww} &= \frac{c}{6}\left(-\partial_{\bar{w}}^2\rho + (\partial_{\bar{w}}\rho)^2\right) + t_w(w) = t_w(w) \,. \end{aligned} \tag{C.9}$$

We again substituted the known solution for $e^{2\rho(w,\bar{w})}$ in the last step.

Now we want to define the CFT to be in the Bunch-Davies vacuum. We know in this state a static observer, using Euclidean static coordinates $z = \ln(\sqrt{\Lambda}w)/\sqrt{\Lambda}$, measures an equal in- and outgoing energy flux at a temperature $T = 1/2\pi L = \sqrt{\Lambda}/2\pi$ [76]:

$$T_{zz} = T_{\bar{z}\bar{z}} = -\frac{c\pi^2}{6}T^2 = -\frac{c}{24}\Lambda \,. \tag{C.10}$$

We can relate this to the energy-momentum tensor in Euclidean Kruskal coordinates via its transformation under the coordinate change (3.22):

$$\begin{aligned} T_{ww}(w,\bar{w}) = t_w(w) &= \left(\frac{\partial w}{\partial z}\right)^{-2}\left[T_{zz}(z) - \frac{c}{12}S(w,z)\right] \\ &= \left(\frac{\partial w}{\partial z}\right)^{-2}\left[T_{zz}(z) - \frac{c}{12}\left(\left(\frac{\partial^3 w}{\partial z^3}\right)\left(\frac{\partial w}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 w}{\partial z^2}\right)^2\left(\frac{\partial w}{\partial z}\right)^{-2}\right)\right] \\ &= \left(e^{\sqrt{\Lambda}z}\right)^{-2}\left[T_{zz}(z) - \frac{c}{12}\left(\left(\Lambda e^{\sqrt{\Lambda}z}\right)\left(e^{\sqrt{\Lambda}z}\right)^{-1} - \frac{3}{2}\left(\sqrt{\Lambda}e^{\sqrt{\Lambda}z}\right)^2\left(e^{\sqrt{\Lambda}z}\right)^{-2}\right)\right] \\ &= \left(\sqrt{\Lambda}w\right)^{-2}\left[T_{zz}(z) - \frac{c}{12}\left(-\frac{1}{2}\Lambda\right)\right] \,, \\ \rightarrow \Lambda w^2\, t_w(w) &= T_{zz}(z) + \frac{c}{24}\Lambda = 0 \,. \end{aligned} \tag{C.11}$$

We see in Kruskal coordinates the Bunch-Davies vacuum is a true vacuum and the diagonal components of the energy-momentum tensor vanish.

In Kruskal coordinates, then, only the off-diagonal component of the equations of motion (4.44) gets altered to

$$\begin{aligned} \partial_w\partial_{\bar{w}}\Phi + \frac{1}{2}e^{2\rho}\Lambda\Phi &= 8\pi G_{2d}T_{w\bar{w}} = \pi\frac{c}{6}\frac{\Lambda}{(1+\Lambda|w|^2)^2} \\ &= \Lambda\frac{c}{6}\frac{\pi}{4}e^{2\rho} \\ \rightarrow \partial_w\partial_{\bar{w}}\Phi + \frac{1}{2}e^{2\rho}\Lambda\left(\Phi - \frac{\pi c}{12}\right) &= 0 \,, \end{aligned} \tag{C.12}$$

so we immediately see the semi-classical dilaton solution is

$$\Phi_{semi-cl} = \frac{S}{2}\frac{1-\Lambda|w|^2}{1+\Lambda|w|^2} + \frac{\pi c}{12}\,. \tag{C.13}$$